EXTENSION THEOREMS FOR FUNCTIONS OF VANISHING MEAN OSCILLATION

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Extension theorems for functions of vanishing mean oscillation

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Abstract

A locally integrable function is said to be of vanishing mean oscillation (VMO) if its mean oscillation over cubes in $\mathbf{R}^d$ converges to zero with the volume of the cubes. We establish necessary and sufficient conditions for a locally integrable function defined on a bounded measurable set of positive measure to be the restriction to that set of a VMO function.

We consider the similar extension problem pertaining to $BMO(\rho)$ functions; that is, those VMO functions whose mean oscillation over any cube is $O(\rho(\ell(Q)))$ where $\ell(Q)$ is the length of Q and ρ is a positive, non-decreasing function with $\rho(0^+) = 0$.

We apply these results to obtain sufficient conditions for a Blaschke sequence to be the zeros of an analytic $BMO(\rho)$ function on the unit disc.

Contents

1. Introduction

Let F be a locally integrable function on $\mathbf{R}^d$ and let Q be a cube in $\mathbf{R}^d$ with sides parallel to the axes. (We denote the set of all such cubes in $\mathbf{R}^d$ by $\Im'$.) We denote the Lebesgue measure of Q by |Q| and the length of Q by $\ell(Q)$. We denote the average of F on Q by F_Q ; that is $F_Q = \frac{1}{|Q|}\int_Q F\,dt$. We say F is of bounded mean oscillation (abbreviated $BMO(\mathbf{R}^d)$ or simply BMO) if

$$\sup_{Q\in\Im'} \frac{1}{|Q|}\int_Q |F - F_Q| \quad < \infty. \tag{1.1}$$

We denote this supremum by $\|F\|_*$. $\|\ \|_*$ defines a norm on BMO and BMO is a Banach space with respect to this norm. (We identify functions which differ by a constant.) If in (1.1) we restrict the cubes to be dyadic we obtain the space dyadic-BMO. (By a dyadic cube we mean a cube of the form $Q = \{k_j < x_j < (k_j+1)2^{-n}\,; 1 \le j \le d\}$ where n and $k_j, 1 \le j \le d$, are integers .) The function space BMO was introduced in 1961 by John and Nirenberg [7] who proved the following fundamental theorem:

Theorem 1.1

Let F be a locally integrable function on $\mathbf{R}^d$, *and for each* $n \in \mathbf{Z}$ *define:*

$$\overline{\mu}_n(F) = \inf\left\{\frac{1}{\lambda} : \sup_{\ell(Q)\le 2^{-n}} \inf_{a\in\mathbf{R}} \frac{1}{|Q|}\int_Q e^{\lambda|F-a|} \quad < 2\right\}$$

Then,

(1) $F \in BMO$ *if and only if,*

(2) $\sup_{n\in\mathbf{Z}} \overline{\mu}_n(F) < \infty.$

The implication (2)⇒(1) is straightforward while (1)⇒(2) is obtained by means of a Calderon-Zygmund stopping time argument. (This result and other basic results on BMO can be found in [4] and [12].)

A closed subspace of BMO that we will be mainly concerned with, is the space of functions of vanishing mean oscillation (VMO) which was introduced by Sarason in [11] and is defined as:

$$VMO = \{F \in BMO : \lim_{\delta \to 0} (\sup_{\substack{Q \in \Im' \\ \ell(Q) < \delta}} \frac{1}{|Q|} \int_Q |F - F_Q|) \quad = 0\}$$

Equivalently, by the theorem of John and Nirenberg, $F \in VMO$ if and only if $F \in BMO$ and $\lim_{n \to \infty} \overline{\mu}_n(F) = 0$.

A bounded function $\rho : \mathbf{R}^+ \to \mathbf{R}^+$ is called a growth function if ρ is non-decreasing and satisfies $\rho(0^+) = 0$. Furthermore, we will always assume $\rho(2t) \leq 2\rho(t)$.We define

$$BMO(\rho) = \{F \in VMO : \sup_{Q \in \Im'} \frac{1}{|Q|\rho(\ell(Q))} \int_Q |F - F_Q| \quad < \infty\}$$

We define $\widetilde{\rho}(t) = t \int_t^1 \frac{\rho(\theta)}{\theta^2}\, d\theta$ and say ρ is regular if $\exists C > 0$ such that $\widetilde{\rho}(t) \leq C\rho(t)$.

If E is a Lebesgue measurable subset of $\mathbf{R}^d$ of positive measure (throughout we will always assume E has positive measure unless stated otherwise), we can ask for necessary and sufficient conditions for a locally integrable function defined on E to be the restriction to E of a function in $BMO(\mathbf{R}^d)$. This characterization was given by Wolff [14] and is based upon a technique due to Rubio de Francia [10] which generalizes Jones' factorization theorem for A_p -weights [8]. The main result of this dissertation is to obtain a similar characterization for VMO functions and this is the content of the following theorem:

Theorem I

Let E be a bounded measurable subset of $\mathbf{R}^d$ and let f be a locally integrable function defined on E. For each $n \in \mathbf{Z}$ define:

$$\mu_n(f) = \inf \left\{ \frac{1}{\lambda} : \sup_{\ell(Q) \leq 2^{-n}} \inf_{a \in \mathbf{R}} \frac{1}{|Q|} \int_{Q \cap E} e^{\lambda |f-a|} < 2 \right\}$$

Then the following are equivalent:

(1) *f is the restriction of a VMO function on $\mathbf{R}^d$ to E*

(2) $\sup_{n \in \mathbf{Z}} \mu_n(f) < \infty$ *and* $\lim_{n \to \infty} \mu_n(f) = 0$

The proof of this theorem consists of two parts. In the first part we obtain an extension to a function F which is a VMO function relative to a certain net of cubes (in particular, F will be in dyadic $VMO(\mathbf{R}^d)$). We then obtain an extension for each translation of this net of cubes and the second part of the proof consists of averaging these extensions to obtain an extension to $VMO(\mathbf{R}^d)$.

We also obtain a similar characterization for functions in $BMO(\rho)$:

Theorem II

Let E be a bounded measurable subset of $\mathbf{R}^d$. Let f be a locally integrable function defined on E and define $\mu_n(f)$ as in Theorem I. If ρ is a growth function satisfying :

(1) $\mu_n(f) \leq C\,\rho(2^{-n}), \quad \forall n \in \mathbf{Z}$

(2) $\inf_{t>0} \rho(t) |\log t| > 0$

(3) $\exists \lambda > 1$ *such that for all* $m \in Z$ *and for all cubes* $Q, \ell(Q) \leq 2^{-m}$ *with*

$0 < |Q \cap E| < |Q|/\lambda$ *we have*

$$\inf_{n>m} \rho(2^{-n}) \left| \log |\{x \in Q \cap E : \sup_{\substack{x \in Q' \\ \ell(Q') \le 2^{-n}}} \frac{|Q'|}{|Q' \cap E|} > \lambda\}| \right| \ge \rho(2^{-m}) \left| \log \frac{|Q \cap E|}{|Q|} \right|$$

then f *is the restriction to* E *of a function in* $BMO(\tilde{\rho})$. *In particular, if* ρ *is regular then* f *is the restriction to* E *of a function in* $BMO(\rho)$.

Finally we examine some uniqueness properties of these extensions and consider various applications of the above results to Blaschke sequences and the zero sets of analytic $BMO(\rho)$ functions on the unit disc.

Throughout C will denote a positive constant which will be independent of the variables in the equation in which it occurs but which may be different at each occurrence.

2. Preliminary Results

Let E be a measurable subset of $\mathbf{R}^d$ and let $\Im$ be a collection of cubes in $\mathbf{R}^d$ with $\bigcup\{Q : Q \in \Im\} = \mathbf{R}^d$.

Definition:

(1) If F is a locally integrable function on $\mathbf{R}^d$, we define the maximal function of F relative to $\Im$ by

$$(M_\Im F)(x) = \sup_{\substack{x\in Q \\ Q\in\Im}} \frac{1}{|Q|}\int_Q F\,dt \qquad \text{for all } x \in \mathbf{R}^d$$

If $\Im = \Im'$, this is the usual Hardy-Littlewood maximal function.

(2) If f is a locally integrable function on E, we define the maximal function of f relative to $\Im$ by

$$(m_\Im f)(x) = \sup_{\substack{x\in Q \\ Q\in\Im}} \frac{1}{|Q|}\int_{Q\cap E} f\,dt \qquad \text{for all } x \in E.$$

Definition:

(1) Let w be a positive locally integrable function on E and let $1 < p < \infty$. We say w is an $A_p(E)$-weight relative to $\Im$ if

$$\sup_{Q\in\Im}\left(\frac{1}{|Q|}\int_{Q\cap E} w\,dt\right)\left(\frac{1}{|Q|}\int_{Q\cap E}\left(\frac{1}{w}\right)^{\frac{1}{p-1}}dt\right)^{p-1} < \infty \tag{2.1}$$

and we denote the collection of all such weights by $A_p(E,\Im)$. If $E = \mathbf{R}^d$ and $\Im = \Im'$ we abbreviate $A_p(E,\Im)$ by A_p and say w is an A_p-weight.

(2) We say a positive locally integrable function w is an $A_1(E)$-weight relative to $\Im$ if

$$\sup_{Q\in\Im}\left\{\left(\frac{1}{|Q|}\int_{Q\cap E} w\,dt\right)\operatorname{ess\,sup}_{x\in Q}\frac{1}{w(x)}\right\} < \infty.$$

We denote the collection of all such weights by $A_1(E,\Im)$.

We record some properties of $A_p(E,\Im)$-weights in the following proposition

Proposition 2.1

(i) *If* $w \in A_p(E,\Im)$ *then* $w \in A_r(E,\Im)$ *for all* $r > p$ *and* $\left(\frac{1}{w}\right)^{\frac{1}{p-1}} \in A_{\frac{p}{p-1}}$.

(ii) *If* $w_1, w_2 \in A_1(E,\Im)$ *then* $w_1 {w_2}^{1-p} \in A_p(E,\Im)$ *for all* $1 < p < \infty$.

(iii) *If* $w \in A_p$ *then* $F = \log w \in BMO$. *By the theorem of John and Nirenberg (Theorem 1.1), if* $F \in BMO$ *there exists* $\delta > 0$ *such that* $e^{\delta F} \in A_p$.

(iv) *We mention here the following result of Coifmann and Rochberg* [3] :

If $F \in L_1^{loc}(\mathbf{R}^d)$ *and* $Mf(x) < \infty$ a.e., *then for each* $0 < \delta < 1$, $(Mf)^\delta \in A_1$. *(We prove a similar result in lemma 2.1 below).*

Definition:

Let $1 < p < \infty$ and let $w \in A_p(E,\Im)$. We say w satisfies a *reverse Hölder inequality* if there exists $\epsilon > 0$ such that $w^{1+\epsilon} \in A_p(E,\Im)$.

Remark : If $w \in A_p$ then w satisfies a reverse Hölder inequality with ϵ depending on p and the supremum in (2.1). This fact may be deduced from (2.1) by a repeated application of a Calderon-Zygmund stopping time argument. See [1], [9].

The next theorem is a variation of a theorem of Muckenhoupt [9]. The proof is the same and so will be omitted.

Theorem 2.1

Let $1 < p < \infty$ *and let* $w \in A_p(E,\Im)$. *If* w *satisfies a reverse Hölder inequality then* *there exists a constant $C > 0$ such that*

$$\int \{m_{\Im}(f)\}^p \, w \, dx \leq C \int |f|^p w \, dx \tag{2.2}$$

and

$$\int \{m_{\Im}(f)\}^q \left(\frac{1}{w}\right)^{\frac{q}{p}} dx \leq C \int |f|^q \left(\frac{1}{w}\right)^{\frac{q}{p}} w \, dx \tag{2.3}$$

where $q = \dfrac{p}{p-1}$

By a theorem of Rubio de Francia [10],(2.2) and (2.3) imply that there exist $w_1, w_2 \in A_1(E, \Im)$ such that $w = w_1 w_2^{1-p}$. We summarize what we need from the above in the following corollary

Corollary 2.1

If $w \in A_2(E, \Im)$ and w satisfies a reverse Hölder inequality then there exist $w_1, w_2 \in A_2(E, \Im)$ such that $w = \frac{w_1}{w_2}$.

We are now in a position to give the BMO extension theorem of Wolff [14].

Theorem 2.2

If f is measurable on E, then the following are equivalent:

(1) *$f$is the restriction of a BMO function on $\mathbf{R}^d$ to E*

(2) $\exists \lambda > 0$ *such that*

$$\sup_{Q \in \Im'} \frac{1}{|Q|} \int_{Q \cap E} e^{\lambda |f - \frac{1}{|Q \cap E|} \int_{Q \cap E} f|} < \infty$$

(3) $\exists \lambda > 0$ *such that*

$$\sup_{Q \in \Im'} \inf_{a \in \mathbf{R}} \frac{1}{|Q|} \int_{Q \cap E} e^{\lambda |f - a|} < \infty$$

We give the proof as it provides one of the basic steps needed in proving theorem I.

Proof The equivalence of (2) and (3) are straightforward and the implication (1)$\Rightarrow$ (3) is similar to the proof of (1)$\Rightarrow$ (2) in theorem I which we give in §3.

(2)$\Rightarrow$ (1): (2) implies that $w = e^{\frac{\lambda f}{2}} \in A_2(E, \Im_0)$ and satisfies a reverse Hölder inequality. Hence by corollary 2.1, there exist $w_1, w_2 \in A_1(E, \Im_0)$ such that $w = e^{\frac{\lambda f}{2}} = \frac{w_1}{w_2}$. Define $W_i = M_{\Im'}(\chi_E\, w_i)^{\frac{1}{2}}$, $i = 1, 2$. By Proposition 2.1 (iv), $W_1, W_2 \in A_1$, i.e. $\exists C > 0$ such that $W_i \leq M_{\Im'}(W_i) \leq C\, W_i$, $i = 1, 2$. Since $M_{\Im'}(\chi_E\, w_i) = m_{\Im'}(w_i)$, a.e. on E , $(i = 1, 2)$, it follows that $\exists\; g \in L_\infty(\mathbf{R}^d), g > 0$ such that

$$g\left(\frac{W_1}{W_2}\right)^2 = \frac{w_1}{w_2}$$
$$= e^{\frac{\lambda f}{2}} \quad \text{a.e. on } E.$$

Define $F = \frac{2}{\lambda}\left\{\log g + 2\log\left(W_1/W_2\right)\right\}$. Then $F = f$ a.e on E and by Proposition 2.1 (ii) and (iii), $F \in BMO(\mathbf{R}^d)$. ∎

Finally we prove 2 lemmas which are needed in the next section. The first is a variation of the theorem of Coifmann and Rochberg mentioned above while the second is based on lemma 2.2 in [5].

For each $k \in \mathbf{Z}$ we define

$$D_k = \left\{Q \in \Im' : \text{Q dyadic }, \ell(Q) = 2^{-k}\right\}.$$

Lemma 2.1

Let $m, n \in \mathbf{N}$ with $m > n$ and let

$$\Im = \left\{Q : Q = \bigcup\nolimits_{Q_i \in D_m} Q_i \text{ and if } \ell(Q) \leq 2^{-n} \text{ then } Q \subseteq Q_n \in D_n\right\}$$

Then

(1) *Given* $Q \in \Im, \exists C > 0$ *and* $Q_1 \in \Im$ *such that* $Q \subseteq Q_1, |Q_1| \leq C|Q|$ *and whenever* $Q_2 \in \Im$ *satisfies* $|Q_2 \cap Q| > 0$ *and* $|Q_2 \cap Q_1^c| > 0$ *there exist* $Q_3 \in \Im$ *such that* $Q, Q_2 \subseteq Q_3$ *and* $|Q_3| \leq C|Q_2|$. *(The constant* C *depends only on the dimension.)*

(2) $\forall \delta,\ 0 < \delta < 1, \exists\ C_\delta > 0$, *depending only on* δ, *such that*

$$M_\Im\left((M_\Im g)^\delta\right)(x) \leq C_\delta\,(M_\Im g)^\delta\,(x)$$

Proof (1) If $\ell(Q) \geq 2^{-n}$ then we take Q_1 to be that cube with the same center as Q and of length $3\ell(Q)$.

If $\ell(Q) < 2^{-n}$ then $\exists Q_n \in D_n$ with $Q \subseteq Q_n$. If $\text{dist}(Q, \partial Q_n) \geq \ell(Q)$ we can again take Q_1 as above. In the remaining case it is not hard to see that there exists cubes $\{Q_i\}$ in $\Im$ satisfying $|Q \cap Q_i| > 0$ and $\frac{1}{2}\ell(Q) \leq \ell(Q_i) \leq 4\ell(Q)$ and such that if we take Q_1 to be the completion of the Q_i to a cube in $\Im$ then Q_1 satisfies (1).

(2) To prove (2) it suffices to show $\exists C > 0$ such that $\forall Q \in \Im$,

$$\frac{1}{|Q|}\int_Q (M_\Im g)^\delta\ dt \leq C_\delta \inf_{x\in Q} (M_\Im g)^\delta\,.$$

Fix $Q \in \Im$ and let Q_1 be as in (1) and let $g_1 = g\chi_{Q_1}$, $g_2 = (1 - \chi_{Q_1})\,g$ so that $g = g_1 + g_2$.

Claim : $\frac{1}{|Q|}\int_Q (M_\Im g_i)^\delta\ dt \leq C_\delta \inf_{x\in Q} (M_\Im g)^\delta$, $i = 1, 2$.

Proof : $i = 1$: The weak-type estimate for the Hardy-Littlewood maximal function implies

$$\left|\left\{x \in Q_1 : (M_\Im g_1)^\delta > \lambda\right\}\right| \leq C|Q_1|\left(\frac{\lambda_0}{\lambda}\right)^{\frac{1}{\delta}}$$

where C depends only on the dimension and $\lambda_0 = \left(\frac{1}{|Q_1|}\int_{Q_1} g_1\, dt\right)^{\delta}$. This implies

$$\begin{aligned}
\int_{Q_1} (M_{\Im} g_1)^{\delta}\, dt &\le \lambda_0 |Q_1| + C\lambda_0^{\frac{1}{\delta}} |Q_1| \int_{\lambda_0}^{\infty} \lambda^{-\frac{1}{\delta}}\, d\lambda \\
&\le c_\delta \left(\frac{1}{|Q_1|}\int_{Q_1} g_1\, dt\right)^{\delta} \\
&\le c_\delta \left(M_{\Im} g_1\right)^{\delta}(x) \quad \forall x \in Q_1 \\
&\le c_\delta \left(M_{\Im} g\right)^{\delta}(x) \quad \forall x \in Q_1
\end{aligned}$$

$i = 2$: Fix $x \in \text{int}(Q)$.Then (1) of the lemma implies that whenever $Q_2 \in \Im$ contains x and $|\text{supp}(g_2) \cap Q_2| > 0$, $\exists Q_3 \in \Im$ satisfying $Q_1, Q_2 \subseteq Q_3$ and $|Q_3| \le C\,|Q_2|$. This implies

$$\begin{aligned}
\left(M_{\Im} g_2\right)(x) &\le C \inf_{y\in Q} \left(M_{\Im} g_2\right)(y) \\
&\le C \inf_{y\in Q} \left(M_{\Im} g\right)(y) \\
\Rightarrow \quad \frac{1}{|Q|}\int_Q \left(M_{\Im} g_2\right)^{\delta} dt &\le C \inf_{y\in Q} \left(M_{\Im} g\right)^{\delta}(y)
\end{aligned}$$

and this proves the claim in the case $i = 2$. (2) of the lemma now follows from the claim and the fact

$$\left(M_{\Im} g\right)^{\delta} \le C_\delta \left(\left(M_{\Im} g_1\right)^{\delta} + \left(M_{\Im} g_2\right)^{\delta}\right). \qquad \blacksquare$$

Lemma 2.2

Let E be a measurable subset of the unit cube Q_0 with $0 < |E| < 1$. Then if $0 < \beta < \log 1/|E|$, $\exists H \in VMO(Q_0)$, $\|H\|_* \le C_0$ such that :

(1) $0 \le H \le \beta$, *supp*$(H) \subseteq Q$, $H = \beta$ *on* E

(2) $\sup_{Q:\ell(Q)\ge 1} \frac{1}{|Q|}\int_Q H\, dt \le C_0$.

Proof W.l.o.g we may assume $|E| \le 2^{-4d}$ (otherwise we may take H to be constant). Let $\{Q_i\}$ be the maximal subcubes of Q for which $|Q_i \cap E| > \frac{1}{2}|Q_i|$. For each $j \geq 1$ choose n_j so that

$$\Big|\log \sum_{i \geq n_j} |Q_i|\Big| > 2^j \Big|\log |\bigcup Q_i|\Big|$$

and define $G^{(j)} = \{Q_i : n_j \leq i < n_{j+1}\}$ so that $\sum_{Q \in G^{(j)}} |Q| \leq 4^{-\beta_j d}|Q_0|$ where $\beta_j = 2^j \beta_0$, $\beta_0 = \left[\frac{1}{2d}\big|\log|\bigcup Q_i|\,\big|\right]$ and [ ] denotes the greatest integer function.

For each j we now construct a sequence of generations $\left\{G_i^{(j)}\right\}_{i=1}^{\beta_j}$ as follows:

(1) $G_1^{(j)} = G^{(j)}$

(2) Suppose $G_i^{(j)}$ has been defined. For each $Q \in G_i^{(j)}$ let $Q^{(k)}$ denote that dyadic cube of length $2^k \ell(Q)$ containing Q. Choose k minimal so that

$$\sum \left\{|Q_i| : Q_i \in G_i^{(j)}, Q_i \subseteq Q^{(k)}\right\} < 2^{-d}|Q^{(k)}|$$

We define $G_{i+1}^{(j)}$ to be the maximal cubes in $\left\{Q_r^{(k)} : Q_r \in G_i^{(j)}\right\}$. We note that

$$\begin{aligned}4^{-d}\sum\left\{|Q| : Q \in G_{\beta_j - i - 1}^{(j)}\right\} &\leq \sum\left\{|Q| : Q \in G_{\beta_j - i}^{(j)}\right\} \\ &\leq 2^{-d}\sum\left\{|Q| : Q \in G_{\beta_j - i - 1}^{(j)}\right\}\end{aligned}$$

Now fix i, $1 \leq i \leq \beta_j$. Let $G_i^{(j)} = \{Q_k\}_{k=1}^N$ and we assume these cubes are indexed so that $|Q_r| \geq |Q_s|$ whenever $r < s$. Let $r_{k,i} \in C^\infty$ satisfy

(i) $0 \leq r_{k,i} \leq 1$,

(ii) $r_{k,i} = 1$ on $Q_k, \operatorname{supp}(r_{k,i}) \subseteq \widetilde{Q}_k$ where $\widetilde{Q}_k$ denotes that cube with the same center as Q_k and of length $3\ell(Q_k)$.

(iii) $\left|\dfrac{\partial r_{k,i}}{\partial x_l}\right| \leq C/\ell(Q_k)$, $\quad \forall\ 1 \leq l \leq d$.

Now define $\quad A_{1,i} = r_{1,i}$

$$A_{k,i} = A_{k-1,i} + r_{k,i} - r_{k,i}\, A_{k-1,i} \quad 2 \le k \le N,$$

and define $\quad b_{1,i} = r_{1,i}$

$$b_{k,i} = r_{k,i}\,(1 - A_{k-1,i}) \quad 2 \le k \le N.$$

It is clear that $A_N = \sum_{k=1}^{N} b_{k,i}$ and $A_N = 1$ on $\bigcup_{k=1}^{N} Q_k$

Define $a_j = \sum_{i=1}^{\beta_j} \sum_{Q_k \in G_i^{(j)}} b_{k,i}$ and note that $a_j = \beta_j$ on $\bigcup\{Q_k : Q_k \in G_1^{(j)}\}$

We now define $H = \min(\sum_{j \ge 1} \frac{a_j}{2^j}\,,\ \beta_0)$. ∎

Remark : Let $\min\{\ell(Q_i) : Q_i \in G_1^{(j)}\} = 2^{-n}$ and let Q be a cube with $\ell(Q) = 2^{-m}$.

Then for all $m > n$,

$$\frac{1}{|Q|}\int_Q |a_j - (a_j)_Q| \le C\, 2^{n-m}$$

Proof : For any $x_0 \in Q$,

$$\frac{1}{|Q|}\int_Q |a_j - a_j(x_0)| \le \sum_{i=1}^{\beta_j} \sum_{Q_k \in G_i^{(j)}} \frac{1}{|Q|}\int_Q |b_{k,i} - b_{k,i}(x_0)|$$

Now $|\frac{\partial b_{k,i}}{\partial x_l}| \le C/\ell(Q_k)$ —this follows from the definition of the $b_{k,i}$ and the fact $|\frac{\partial A_{k,i}}{\partial x_l}| \le C/\ell(Q_k)$ which can be established by induction. Furthermore there are at most a fixed number of cubes in any $G_i^{(j)}$ which intersect Q. If Q_{k_1} is any such cube and Q_{k_2} is a generation cube containing Q_{k_1} then for all $x \in Q \cap Q_{k_2}$

$$|b_{k,i}(x) - b_{k,i}(x_0)| \le C\,\frac{\ell(Q_{k_1})}{\ell(Q_{k_2})}\,\frac{\ell(Q)}{\ell(Q_{k_1})}$$

and hence

$$\sum_{i=1}^{\beta_j} \sum_{Q_k \in G_i^{(j)}} \frac{1}{|Q|}\int_Q |b_{k,i} - b_{k,i}(x_0)| \le C\, 2^{n-m}.$$

3. Proof of Theorem I

Theorem I

Let E be a bounded measurable subset of $\mathbf{R}^d$ and let f be a locally integrable function defined on E. For each $n \in \mathbf{Z}$ define:

$$\mu_n(f) = \inf\left\{\frac{1}{\lambda} : \sup_{\ell(Q)\le 2^{-n}} \inf_{a\in\mathbf{R}} \frac{1}{|Q|}\int_{Q\cap E} e^{\lambda|f-a|} \quad < 2\right\}$$

Then the following are equivalent:

(1) *f is the restriction of a VMO function on $\mathbf{R}^d$ to E*

(2) $\sup_{n\in\mathbf{Z}} \mu_n(f) < \infty$ *and* $\lim_{n\to\infty} \mu_n(f) = 0$

Proof Without loss of generality we will assume E is contained in the unit cube in $\mathbf{R}^d$.

(1) $\Rightarrow$ (2): Let $F \in VMO$ with $F\chi_E = f$ and for each $n \in \mathbf{Z}$ define

$$\overline{\mu}_n(F) = \inf\left\{\frac{1}{\lambda} : \sup_{\ell(Q)\le 2^{-n}} \inf_{a\in\mathbf{R}} \frac{1}{|Q|}\int_{Q} e^{\lambda|F-a|} \quad < 2\right\}$$

$$\overline{\mu}_n{}^*(F) = \inf\left\{\frac{1}{\lambda} : \sup_{\ell(Q)\le 2^{-n}} \frac{1}{|Q|}\int_{Q} e^{\lambda|F-F_Q|} \quad < 2\right\}$$

$$\|F\|_{*,n} = \sup_{\ell(Q)\le 2^{-n}} \frac{1}{|Q|}\int_{Q} |F-F_Q|$$

Since $F \in BMO$, $\exists C > 0$ such that $\forall n \ge 0$, $\|F\|_{*,n} \le C$ and $\lim_{n\to\infty} \|F\|_{*,n} = 0$.

By Theorem (1.1), $\exists C_1 > 0$ such that whenever $0 < \lambda < C_1/\|F\|_{*,n}$ we have

$$\sup_{\ell(Q)\le 2^{-n}} \frac{1}{|Q|}\int_{Q} e^{\lambda|F-F_Q|} \quad < 2$$

Hence $\overline{\mu}_n{}^*(F) \le \|F\|_{*,n}/C_1$. Since $\overline{\mu}_n(F) \le \overline{\mu}_n{}^*(F)$ and $\mu_n(f) \le \overline{\mu}_n(F)$, it follows that $\mu_n(f) \le C$ for $n = 0, 1, 2, \ldots$ and $\lim_{n\to\infty} \mu_n(f) = 0$ and this proves (1) $\Rightarrow$ (2).

Proof of (2) $\Rightarrow$ (1):

Part (i): Extension to dyadic-VMO.

Let ρ be a bounded growth function satisfying $\rho(2t) \leq 2\rho(t)$, $\forall t > 0$ and $\mu_n(f) \leq \rho(2^{-n})$, $\forall n \geq 0$. Then (2) implies there exists a sequence $\{\lambda_n\}_{n\geq 0}$, $0 < \lambda_n \uparrow \infty$ such that

$$\frac{1}{\lambda_n} \leq C\,\rho(2^{-n}), \quad \forall n \geq 0 \quad \text{and} \quad \sup_{\ell(Q)\leq 2^{-n}} \frac{1}{|Q|}\int_{Q\cap E} e^{\lambda_n|f-f_{Q\cap E}|} < 2$$

Define a sequence $n_k \subseteq \mathbf{N}$ by the condition $\rho(2^{-n}) \leq 2^{-k}$ if and only if $n \geq n_k$. To simplify the notation we will write λ_k for λ_{n_k}. Now define

$$\Im_0 = \left\{Q : Q = \bigcup\{Q_i : Q_i \in D_0\}\right\}$$

and for each $k \geq 1$,

$$\Im_k = \left\{Q : Q = \bigcup\{Q_i : Q_i \in D_{n_{k+1}}\} \text{ and if } \ell(Q) \leq 2^{-n_k} \right.$$
$$\left. \text{then } \exists Q_k \in D_{n_k} \; s.t. \; Q \subseteq Q_k\right\}$$

For each $n = 0, 1, 2, \ldots$ we define $f_n = \sum_{Q\in D_n} f_{Q\cap E}\,\chi_{Q\cap E}$.

Lemma 3.1

There exists $C > 0$, depending only on the dimension, such that for all $k \geq 0$,

$$(1) \quad \sup_{Q\in\Im_{k+1}} \left(\frac{1}{|Q|}\int_{Q\cap E} e^{\lambda_k(f_{n_{k+1}}-f_{n_k})}\right)\left(\frac{1}{|Q|}\int_{Q\cap E} e^{-\lambda_k(f_{n_{k+1}}-f_{n_k})}\right) \leq C$$

$$(2) \quad \sup_{Q\in\Im_0} \left(\frac{1}{|Q|}\int_{Q\cap E} e^{\lambda_0 f_0}\right)\left(\frac{1}{|Q|}\int_{Q\cap E} e^{-\lambda_0 f_0}\right) \leq C$$

Proof Fix $j \in \mathbf{N}$

Claim : $\exists C > 0$ such that for all $\lambda \le \lambda_j$ and for all $Q, \ell(Q) \ge 2^{-n_j}$,

$$\frac{1}{|Q|}\int_{Q\cap E} e^{\lambda(f-f_{n_j})}\ \frac{1}{|Q|}\int_{Q\cap E} e^{-\lambda(f-f_{n_j})} \le C \tag{3.1}$$

Proof : Given $Q, \ell(Q) \ge 2^{-n_j}$, $\exists Q_i \in D_{n_j}$ such that $Q \subseteq \bigcup Q_i$ and $\sum |Q_i| \le 2^d|Q|$.

This implies

$$\begin{aligned}\frac{1}{|Q|}\int_{Q\cap E} e^{\lambda|f-f_{n_j}|} &= \sum \frac{|Q_i|}{|Q|}\left(\frac{1}{|Q_i|}\int_{Q_i\cap E} e^{\lambda|f-f_{n_j}|}\right)\\ &\le \sum \frac{|Q_i|}{|Q|}\left(\frac{1}{|Q_i|}\int_{Q_i\cap E} e^{\lambda|f-f_{Q_i\cap E}|}\right)\\ &\le 2^{d+1}\end{aligned}$$

This implies (3.1) since

$$\frac{1}{|Q|}\int_{Q\cap E} e^{\lambda(f-f_{n_j})}\ \frac{1}{|Q|}\int_{Q\cap E} e^{-\lambda(f-f_{n_j})} \le \left(\frac{1}{|Q|}\int_{Q\cap E} e^{\lambda|f-f_{n_j}|}\right)^2$$

Now $\forall\lambda \le \lambda_j$ and $\forall Q,\ \ell(Q) \le 2^{-n_j},\ Q \subseteq Q_j \in D_{n_j}$ we are given

$$\frac{1}{|Q|}\int_{Q\cap E} e^{\lambda(f-f_{Q\cap E})}\ \frac{1}{|Q|}\int_{Q\cap E} e^{-\lambda(f-f_{Q\cap E})} \le C$$

and hence

$$\left(\frac{1}{|Q|}\int_{Q\cap E} e^{\lambda f}\right)\left(\frac{1}{|Q|}\int_{Q\cap E} e^{-\lambda f}\right) \le C \tag{3.2}$$

Note that if $j = 0$, (3.2) holds for all Q and for all $\lambda \le \lambda_0$. Now (3.2) implies that for all $Q \subseteq Q_j \in D_{n_j}$ and for all $\lambda \le \lambda_j$

$$\frac{1}{|Q|}\int_{Q\cap E} e^{\lambda(f-f_{n_j})}\ \frac{1}{|Q|}\int_{Q\cap E} e^{-\lambda(f-f_{n_j})} \le C \tag{3.3}$$

Since

$$\frac{1}{|Q|}\int_{Q\cap E} e^{\lambda(f_{n_{k+1}}-f_{n_k})} \le \left(\frac{1}{|Q|}\int_{Q\cap E} e^{2\lambda(f-f_{n_k})}\right)^{\frac{1}{2}}\left(\frac{1}{|Q|}\int_{Q\cap E} e^{-2\lambda(f-f_{n_{k+1}})}\right)^{\frac{1}{2}}$$

(and similarly for $\frac{1}{|Q|}\int_{Q\cap E} e^{-\lambda(f_{n_{k+1}}-f_{n_k})}$) we see that (1) follows from (3.1) and (3.3).

Now (3.1) and (3.2) imply that $\forall\lambda \le \lambda_0$, $\forall Q$ with $\ell(Q) \ge 1$,

$$\left(\frac{1}{|Q|}\int_{Q\cap E} e^{\lambda f_0}\right)\left(\frac{1}{|Q|}\int_{Q\cap E} e^{-\lambda f_0}\right) \le$$

$$\le \left(\frac{1}{|Q|}\int_{Q\cap E} e^{\lambda(f-f_0)}\right)\left(\frac{1}{|Q|}\int_{Q\cap E} e^{-\lambda(f-f_0)}\right)\left(\frac{1}{|Q|}\int_{Q\cap E} e^{\lambda f}\right)\left(\frac{1}{|Q|}\int_{Q\cap E} e^{-\lambda f}\right)$$

$$\le C.$$

and this gives (2) which completes the proof of the lemma.

To simplify the notation we set $M_k(g) = M_{\Im_k}(g)$ and $m_k(g) = m_{\Im_k}(g)$. Corollary 2.1 implies the following :

For each $k = 0, 1, 2 \ldots$ there exist functions u_k, v_k such that

(i) $\frac{u_0}{v_0} = \exp\left(\frac{\lambda_0}{2} f_0\right)$

(ii) $\frac{u_k}{v_k} = \exp\left(\frac{\lambda_{k-1}}{2}(f_{n_k} - f_{n_{k-1}})\right) \quad \forall k = 1, 2, \ldots$

(iii) $m_k(u_k) \le C\, u_k$ and $m_k(v_k) \le C\, v_k \quad \forall k = 0, 1, 2, \ldots$

Now for each $k = 0, 1, 2, \ldots$ and for each $x \in \mathbf{R}^d$ we define

$$U_k(x) = M_k(\chi_E u_k)(x)$$

$$V_k(x) = M_k(\chi_E v_k)(x)$$

Then $\quad \exp\left(\frac{\lambda_{k-1}}{2}(f_{n_k} - f_{n_{k-1}})\right) = \frac{U_k}{V_k}\, w_k\,, \quad k = 1, 2, \ldots$

and $\quad \exp\left(\frac{\lambda_0}{2} f_0\right) = \frac{U_0}{V_0}\, w_0$

where $\quad w_k(x) = \frac{u_k}{v_k}\,\frac{m_k(v_k)(x)}{m_k(u_k)(x)} \quad \forall k \ge 0,\ \forall x \in E$

Hence,

$$f_{n_k} - f_{n_{k-1}} = \frac{4}{\lambda_{k-1}} \log \left(\frac{U_k}{V_k}\right)^{\frac{1}{2}} + \frac{2}{\lambda_{k-1}} \log w_k$$

$$\text{and} \qquad f_0 = \frac{4}{\lambda_0} \log \left(\frac{U_0}{V_0}\right)^{\frac{1}{2}} + \frac{2}{\lambda_0} \log w_0 \qquad \text{a.e. on } E.$$

Now lemma 2.1 (2) implies $\exists C > 0$ such that

$$M_k(U_k^{\frac{1}{2}}) \le C\, U_k^{\frac{1}{2}} \quad \text{and} \quad M_k(V_k^{\frac{1}{2}}) \le C\, V_k^{\frac{1}{2}}$$

and so by Proposition 2.1 (ii), $\left(\frac{U_k}{V_k}\right)^{\frac{1}{2}} \in A_2(E, \Im')$.

Then, as in Proposition 2.1 (iii), we conclude that

$$\sup_{Q \in \Im_k} \frac{1}{|Q|} \int_Q \left| \log \left(\frac{U_k}{V_k}\right) - \left(\log \left(\frac{U_k}{V_k}\right)\right)_Q \right| \le C$$

In particular since U_k, V_k are constant on dyadic cubes of length 2^{-n_k}, we have $\log (U_k/V_k) \in$dyadic-VMO.

Claim : For each $k \ge 0$ w_k is the restriction to E of a function W_k where $\log W_k \in$ dyadic-VMO.

Proof :For each $x \in E$ let $Q_k(x)$ denote the dyadic cube of length 2^{-n_k} containing x. If $|Q_k(x) \cap E| > 0$, then

$$u_k(x) \le \frac{|Q_k(x)|}{|Q_k(x) \cap E|}\, m_k(u_k)(x)$$

and

$$v_k(x) \le \frac{|Q_k(x)|}{|Q_k(x) \cap E|}\, m_k(v_k)(x)$$

and hence

$$|\log w_k(x)| \le \log C + \log \frac{|Q_k(x)|}{|Q_k(x) \cap E|}$$

Now lemma 2.2 implies $\exists \widetilde{H}_k(x) \in VMO(Q_k(x))$ satisfying

$$\text{(i)} \quad |\log w_k(x) - \widetilde{H}_k(x)| \le C$$

$$\text{(ii)} \quad \sup_{\ell(Q)\ge \ell(Q_k)} \frac{1}{|Q|}\int_Q \widetilde{H}_k(x)\,dt \le C_0$$

We now define $$H_k(x) = \begin{cases} \widetilde{H}_k & \text{if } |Q_k(x)\cap E| \ne 0 \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check $H_k \in$ dyadic$-VMO$ with $\|H_k\|_* \le C_0$ and $|\log w_k(x) - H_k(x)| \le C$, $\forall x \in E$. This implies $\exists R_k(x) \in L_\infty(\mathbf{R}^d)$ with $\|R_k\|_\infty \le C_0$ and which is constant on dyadic cubes of length 2^{-n_k} and satisfies $R_k(x) = \log w_k(x) - H_k(x)$ a.e. on E. In particular, $R_k \in$ dyadic $- VMO$ with $\|R_k\|_* \le C_0$, $\forall k$. Since H_k is supported on finitely many cubes in D_{n_k} the function $W_k = \exp(R_k + H_k)$ satisfies $\log W_k \in$ dyadic $- VMO$, $\|\log W_k\|_* \le C_0$ and $W_k \chi_E = w_k$ a.e. and the claim now follows.

Now define

$$F = \sum_{k\ge 0} \frac{2}{\lambda_{k-1}} (G_k + R_k + H_k) \tag{3.4}$$

where $G_k = 2\log (U_k/V_k)^{\frac{1}{2}}$ and by λ_{-1} we mean λ_0. Since $G_k + R_k + H_k \in$ dyadic$-VMO$ with $\|G_k + R_k + H_k\|_* \le C_0$ and since $\sum_{k\ge 0} \frac{1}{\lambda_{k-1}} < \infty$, it follows that $F \in$ dyadic $- VMO$ and $\|F\|_* \le C_0$

Furthermore

$$\sum_{k=0}^{\infty} \frac{2}{\lambda_{k-1}} (G_k + R_k + H_k)\,\chi_E = \sum_{k\ge 0} (f_{n_{k+1}} - f_{n_k}) + f_0$$
$$= f \quad \text{a.e. on E .}$$

Hence F is a dyadic-VMO extension of f.

Part (ii): Extension to non-dyadic VMO.

Let Q_0 denote the unit cube in $\mathbf{R}^d$. If $\alpha \in Q_0$ and Q is any cube we define

$$Q^{(\alpha)} = \{x + \alpha : x \in Q\} \qquad \Im_n^{(\alpha)} = \{Q^{(\alpha)} : Q \in \Im_n\},$$

$$D_n^{(\alpha)} = \{Q^{(\alpha)} : Q \in D_n\}, \qquad D^{(\alpha)} = \bigcup D_n^{(\alpha)}$$

The proof of part (i) above applied to each net of dyadic cubes $D^{(\alpha)}$ establishes the following :

$\forall k \geq 0, \ \ \forall \alpha \in Q_0, \ \ \exists$ functions $G_k^{(\alpha)}$, $H_k^{(\alpha)}$ such that

1) $G_k^{(\alpha)}(x-\alpha)$ as a function of x belongs to dyadic-VMO, $\|G_k^{(\alpha)}\|_* \leq C_0$ and $G_k^{(\alpha)}$ is constant on cubes $Q^{(\alpha)} \in D_{n_k}^{(\alpha)}$. (We can assume that the bounded functions $R_k^{(\alpha)}$ are included in the $G_k^{(\alpha)}$.)

2) On each $Q_k^{(\alpha)} \in D_{n_k}^{(\alpha)}$, $|Q_k^{(\alpha)} \cap E| > 0$,

$$H_k^{(\alpha)} = \min\left(\sum_{j\geq 1} \frac{a_{j,k}^{(\alpha)}}{2^j}\ ,\ \beta^{(\alpha)}\right) \qquad \text{where} \qquad a_{j,k}^{(\alpha)} = \sum_{i=1}^{\beta_j^{(\alpha)}} \sum_{G_i^{(j,\alpha)}} b_{l,i} \quad \text{and,}$$

$$\beta_j^{(\alpha)} = 2^j \beta^{(\alpha)}, \quad \beta^{(\alpha)} \leq \left[\frac{1}{2d} \log \frac{|Q_k^{(\alpha)}|}{|Q_k^{(\alpha)} \cap E|}\right]$$

3) If $|Q_k^{(\alpha)} \cap E| = 0$ then $H_k^{(\alpha)} = 0$ and in this case we set $a_{j,k}^{(\alpha)} = \beta^{(\alpha)} = 0$.

4) If we define

$$F^{(\alpha)} = \sum_{k\geq 0} \frac{1}{\lambda_{k-1}} \left(G_k^{(\alpha)} + H_k^{(\alpha)}\right)$$

then $F^{(\alpha)}(x-\alpha) \in$ dyadic $-VMO$ and $F^{(\alpha)}\chi_E = f$ a.e. on E.

We now define $\forall k \geq 0,\ j \geq 1$

$$G_k(x) = \int_{\alpha\in Q_0} G_k^{(\alpha)}(x)\, d\alpha\ , \quad a_{j,k}(x) = \int_{\alpha \in Q_0} a_{j,k}^{(\alpha)}(x)\, d\alpha\ ,$$

$$\beta_j = \int_{\alpha\in Q_0} \beta_j^{(\alpha)}\, d\alpha\,, \quad \text{and} \quad H_k = \min\left(\sum_{j\geq 1} \frac{a_{j,k}}{2^j}\ ,\ \beta_k\right)$$

Lemma 3.2

For all $k \geq 0$, $H_k \in VMO, \|H_k\|_* \leq C_0$ *and*

$$H_k(x) = \int_{\alpha\in Q_0} H_k^{(\alpha)}(x)\, d\alpha \quad \text{a.e. on} \quad E.$$

Proof The last statement in the lemma follows from the fact that $\forall\alpha \in Q_0$,

$$H_k^{(\alpha)}(x) \ = \ \beta_k^{(\alpha)} \leq \sum_{j\geq 1} \frac{a_{j,k}^{(\alpha)}}{2^j} \qquad \text{a.e. on} \quad E$$

and so

$$\int_{\alpha\in Q_0} H_k^{(\alpha)}(x)\, d\alpha = \int_{\alpha\in Q_0} \beta_k^{(\alpha)}\, d\alpha \ = \beta_k \ \leq \ \sum_{j\geq 1} \frac{a_{j,k}}{2^j}$$

To show $H_k(x) \in VMO$, it suffices to show each $a_{j,k} \in VMO$, $\|a_{j,k}\|_* \leq C_0$. Fix $k \geq 0$, $j \geq 1$ and let $\epsilon > 0$. For each $\alpha \in Q_0$, $a_{j,k}^{(\alpha)}(x-\alpha)$, as a function of x, belongs to dyadic-VMO with $\|a_{j,k}^{(\alpha)}(x-\alpha)\|_{*,dyadic} \leq C_0$ and furthermore on each cube $Q_{n_k}^{(\alpha)} \in D_{n_k}^{(\alpha)}$, $a_{j,k}^{(\alpha)} \in VMO(Q_{n_k}^{(\alpha)})$. Hence $\exists n_\alpha \in \mathbf{N}$ such that whenever $Q \subseteq Q_{n_k}^{(\alpha)}$ and $\ell(Q) < 2^{-n_\alpha}$, we have

$$\frac{1}{|Q|}\int_Q |a_{j,k}^{(\alpha)} - (a_{j,k}^{(\alpha)})_Q| \ < \epsilon$$

Choose $N_1 \in \mathbf{N}$ so that the set

$$S_0 = \{\alpha \in Q_0 : \frac{1}{|Q|}\int_Q |a_{j,k}^{(\alpha)} - (a_{j,k}^{(\alpha)})_Q| < \epsilon \text{ whenever } Q \subseteq Q_{n_k}^{(\alpha)} \text{and } \ \ell(Q) \leq 2^{-N_1}\}$$

has measure $\geq (1-\epsilon)$.

Choose N_2 so that $N_2 2^{(n_k - N_2)d} < \epsilon$ and let $N \geq \max(N_1, N_2)$. Let Q be any cube with $2^{-(N+1)} \leq \ell(Q) < 2^{-N}$ and write

$$\overline{a}_{j,k}^{(\alpha)} = \sum_{i=1}^{\beta_j^{(\alpha)}} \sum_{\substack{G_i^{(j,\alpha)} \\ \ell(Q_l) \geq \ell(Q)}} b_{l,i}$$

Then

$$\frac{1}{|Q|} \int_Q |a_{j,k}^{(\alpha)} - \overline{a}_{j,k}^{(\alpha)}| \leq C_0 \quad \text{and} \quad |\overline{a}_{j,k}^{(\alpha)}(x)| \leq \log \frac{1}{\ell(Q)} \leq C.N\,, \quad \forall x \in \mathbf{R}^d.$$

Let $S_1 = \{\alpha \in Q_0 : |Q \cap Q_{n_k}^{(\alpha)}| < |Q|,\ \forall Q_{n_k}^{(\alpha)} \in D_{n_k}^{(\alpha)}\}$ and note that $|S_1| \leq C\,|Q| 2^{n_k d}$. Hence

$$\begin{aligned} \int_{S_1} \left(\frac{1}{|Q|} \int_Q |a_{j,k}^{(\alpha)} - (a_{j,k}^{(\alpha)})_Q|\, dt \right) d\alpha &\leq |S_1|(C\,N + C_0) \\ &\leq C\,|Q| 2^{n_k d} \leq C\,N.2^{-Nd} 2^{n_k d} \\ &\leq C\,\epsilon. \end{aligned}$$

Now

$$\int_{Q_0 \backslash (S_0 \cup S_1)} \left(\frac{1}{|Q|} \int_Q |a_{j,k}^{(\alpha)} - (a_{j,k}^{(\alpha)})_Q|\, dt \right) d\alpha \leq$$

$$\leq 2 \int_{Q_0 \backslash (S_0 \cup S_1)} \left(\frac{1}{|Q|} \int_Q |a_{j,k}^{(\alpha)} - \overline{a}_{j,k}^{(\alpha)}|\, dt + \frac{1}{|Q|} \int_Q |\overline{a}_{j,k}^{(\alpha)} - (\overline{a}_{j,k}^{(\alpha)})_Q|\, dt \right) d\alpha$$

$$\leq C\,|Q_0 \backslash (S_0 \cup S_1)| \leq C\,\epsilon.$$

Also

$$\int_{S_0} \left(\frac{1}{|Q|} \int_Q |a_{j,k}^{(\alpha)} - (a_{j,k}^{(\alpha)})_Q|\, dt \right) d\alpha \leq C\,|S_0|\,\epsilon \leq C\,\epsilon$$

and hence

$$\frac{1}{|Q|}\int_Q |a_{j,k}-(a_{j,k})_Q| < C\,\epsilon \quad \text{whenever} \quad \ell(Q) < 2^{-N}$$

and it follows then that $a_{j,k}\in VMO$, $\|a_{j,k}\|_* \le C_0$ and this completes the the proof of the lemma.

Lemma 3.3

Given n let Q be a cube of length $\le 2^{-n}$ and let k be such that $n_k \le n < n_{k+1}$. Then for all $x, y \in Q$

(1) $|G_j(x) - G_j(y)| \le |x-y| 2^{n_j}, \ \forall\, 0 \le j \le k.$

(2) $\displaystyle\frac{1}{|Q|}\int_Q |G_{k+1} - (G_{k+1})_Q| \le C\,(n-n_k)\,2^{(n_k-n)} + C$

(3) $\displaystyle\forall j > k+1, \ \frac{1}{|Q|}\int_Q |G_j - (G_j)_Q| \le C.$

We first note that lemma 3.3 implies that the function $\displaystyle\sum_{j\ge 0}\frac{G_j}{\lambda_{j-1}} \in BMO(\widetilde{\rho})$. Indeed, given Q as in the statement of the lemma, we have from (3) that

$$\frac{1}{|Q|}\int_Q \Big|\sum_{j\ge k+1}\frac{G_j}{\lambda_{j-1}} - \Big(\sum_{j\ge k+1}\frac{G_j}{\lambda_{j-1}}\Big)_Q\Big| \le \sum_{j\ge k+1}\frac{1}{|Q|}\int_Q \frac{|G_j-(G_j)_Q|}{\lambda_{j-1}}$$

$$\le C \sum_{j\ge k+1}\rho(2^{-n_{j-1}}) \le C\sum_{j\ge k}\rho(2^{-n_j})$$

$$\le C\,\rho(2^{-n_{k+1}})$$

$$\le C\,\rho(2^{-n})$$

(1) and (2) imply,

$$\sum_{j=0}^{k}\frac{1}{|Q|}\int_Q\frac{|G_j-(G_j)_Q|}{\lambda_{j-1}}+\frac{1}{|Q|}\int_Q\frac{|G_{k+1}-(G_{k+1})_Q|}{\lambda_k}$$

$$\le C|x-y|\sum_{j=0}^{k}\rho\left(2^{-n_{j-1}}\right)2^{n_j}+C(n-n_k)2^{(n_k-n)}\rho\left(2^{-n_k}\right)+C\rho\left(2^{-n_k}\right)$$

$$\le C2^{-n}\sum_{j=0}^{k}\rho\left(2^{-n_{j-1}}\right)\left(2^{n_j}-2^{n_{j-1}}\right)+C2^{n_k-n}(n-n_k)\rho\left(2^{-n}\right)+C\rho\left(2^{-n}\right)$$

$$\le C2^{-n}\sum_{j=0}^{k}\rho\left(2^{-n_{j-1}}\right)\int_{2^{-n_j}}^{2^{-n_{j-1}}}\frac{1}{t^2}\,dt+2^{(n_k-n)}\int_{2^{-n}}^{2^{-n_k}}\frac{\rho(t)}{t}\,dt+C\rho\left(2^{-n}\right)$$

$$\le C2^{-n}\left(\sum_{j=0}^{k}\int_{2^{-n_j}}^{2^{-n_{j-1}}}\frac{\rho(t)}{t^2}\,dt+\int_{2^{-n}}^{2^{-n_k}}\frac{\rho(t)}{t^2}\,dt\right)+C\rho\left(2^{-n}\right)$$

$$\le C2^{-n}\int_{2^{-n}}^{1}\frac{\rho(t)}{t^2}\,dt+C\rho\left(2^{-n}\right)$$

$$\le C\widetilde{\rho}\left(2^{-n}\right)+C\rho\left(2^{-n}\right)$$

$$\le C\widetilde{\rho}\left(2^{-n}\right)$$

Proof of Lemma 3.3

(1) Fix $x,y\in Q$ and for $0\le j\le k$, let

$$A_j=\left\{\alpha\in Q_0:\exists Q^{(\alpha)}\in D^{(\alpha)}_{n_j}\quad\text{with } x,y\in Q^{(\alpha)}\right\}\quad\text{and note that}$$

$|A_j^c|\le C\,2^{n_j}\,|x-y|$.

Claim : If $\alpha\in A_{j-1}\cap A_{j-1}^c$ then $|G_j^{(\alpha)}(x)-G_j^{(\alpha)}(y)|\le C_0$

Proof : Without loss of generality take $\alpha=0$. Recall $G_j(x)=\log\dfrac{M_j(\chi_E u_j)}{M_j(\chi_E v_j)}$, and whenever $Q\in\Im_j$ contains x, $\exists Q'\in\Im_j$ containing x and y with $|Q'|\le C\,|Q|$. This implies

$$M_j(\chi_E u_j)(x) \le C M_j(\chi_E u_j)(y) \quad \text{and hence} \quad |G_j(x) - G_j(y)| \le C_0.$$

Claim : If $\alpha \in A_{j-1}^c$ then $|G_j^{(\alpha)}(x) - G_j^{(\alpha)}(y)| \le C(n_j - n_{j-1})$

Proof : Again we can assume $\alpha = 0$. Since $|x-y| < 2^{-n_j}$ whenever $Q \in \Im_j$ contains x, $\exists Q' \in \Im_j$ containing both x and y and which satisfies

$$\ell(Q') < \ell(Q) + 2^{-n_{j-1}} \le \ell(Q)\big(1 + C2^{n_j - n_{j-1}}\big)$$

From this it follows that

$$M_j(\chi_E u_j)(x) \le \big(1 + C2^{n_j - n_{j-1}}\big) M_j(\chi_E u_j)(y)$$

and hence

$$\log M_j(\chi_E u_j)(x) \le C(n_j - n_{j-1}) + \log M_j(\chi_E u_j)(y)$$

Similarly

$$\log M_j(\chi_E v_j)(x) \le C(n_j - n_{j-1}) + \log M_j(\chi_E v_j)(y)$$

and hence $\quad |G_j(x) - G_j(y)| \le C(n_j - n_{j-1})$.

Now fix j, $0 \le j \le k$, and fix $x, y \in Q$. If $i \ge j$ and $\alpha \in A_i$ then $|G_j^{(\alpha)}(x) - G_j^{(\alpha)}(y)| = 0$

This implies

$$|G_j(x) - G_j(y)| \le \int_{\alpha \in A_j^c \cap A_{j-1}} |G_j^{(\alpha)}(x) - G_j^{(\alpha)}(y)| + \int_{\alpha \in A_{j-1}^c} |G_j^{(\alpha)}(x) - G_j^{(\alpha)}(y)|$$

$$\le C(n_j - n_{j-1})|A_{j-1}^c| + C|A_j^c \cap A_{j-1}|$$

$$\le C|x-y|(n_j - n_{j-1})2^{n_{j-1}} + C|x-y|2^{n_j}$$

$$\le C|x-y|2^{n_j} \quad \text{and this proves (1).}$$

Proof of (2): Let $B_k = \{\alpha \in Q_0 : \exists Q^{(\alpha)} \in D_{n_k}^{(\alpha)} \text{ with } Q \subseteq Q^{(\alpha)}\}$ and note that $|B_k^c| \leq C\, 2^{(n_k - n)}$

Claim : If $\alpha \in B_k$ then $\frac{1}{|Q|}\int_Q |G_{k+1}^{(\alpha)} - (G_{k+1}^{(\alpha)})_Q| \leq C.$

Proof : $\exists Q' \in \Im_{k+1}^{(\alpha)}$ containing Q and such that $|Q'| \leq C|Q|$. Furthermore there exists $a_{Q'} \in \mathbf{R}^d$ such that

$$\frac{1}{|Q'|}\int_{Q'} |G_{k+1}^{(\alpha)} - a_{Q'}| \leq C$$

This implies

$$\frac{1}{|Q|}\int_Q |G_{k+1}^{(\alpha)} - a_{Q'}| \leq \frac{C}{|Q'|}\int_{Q'} |G_{k+1}^{(\alpha)} - a_{Q'}| \leq C$$

and the claim now follows.

Claim : If $\alpha \in B_k^c$ then

$$\frac{1}{|Q|}\int_Q |G_{k+1}^{(\alpha)} - (G_{k+1}^{(\alpha)})_Q| \leq C(n - n_k) + C_1.$$

Proof : Without loss of generality we may assume $\alpha = 0$.

Recall,

$$M_{k+1}(U_{k+1}^{\frac{1}{2}})(x) \leq C U_{k+1}^{\frac{1}{2}}(x) \quad \text{where} \quad U_{k+1}^{\frac{1}{2}} = M_{k+1}(\chi_E U_{k+1}^{\frac{1}{2}})(x)$$

Now if $Q' \in \Im_{k+1}^{(0)}$ is that cube of length 2^{-n_k} containing Q, then

$$\begin{aligned}\frac{1}{|Q|}\int_Q U_{k+1}^{\frac{1}{2}}\, dt &\leq \frac{|Q'|}{|Q|}\left(\frac{1}{|Q'|}\int_{Q'} U_{k+1}^{\frac{1}{2}}\, dt\right) \\ &\leq C\, 2^{(n-n_k)d} M_{k+1}(U_{k+1}^{\frac{1}{2}})(x), \\ &\leq \quad C\, 2^{(n-n_k)d}\, U_{k+1}^{\frac{1}{2}}(x), \quad \forall x \in Q'\end{aligned}$$

This implies

$$\frac{1}{|Q|}\int_Q (\log U_{k+1}^{\frac{1}{2}})\,dt \le \inf_{x\in Q}(\log U_{k+1}^{\frac{1}{2}}(x))C\,(n-n_k)+c_1$$

$$\Rightarrow \frac{1}{|Q|}\int_Q\left|\frac{1}{|Q|}\int_Q \log U_{k+1}^{\frac{1}{2}}\,dt - \log U_{k+1}^{\frac{1}{2}}(x)\right|dx \le C\,(n-n_k)+c_1$$

and similarly for $\log V_{k+1}^{\frac{1}{2}}$ and this establishes the claim.

Now

$$\begin{aligned}\int_{\alpha\in Q_0}\frac{1}{|Q|}\int_Q\left|G_{k+1}^{(\alpha)}-(G_{k+1}^{(\alpha)})_Q\right|d\alpha &\le \int_{\alpha\in B_k}\frac{1}{|Q|}\int_Q\left|G_{k+1}^{(\alpha)}-(G_{k+1}^{(\alpha)})_Q\right|d\alpha \\ &\quad+\int_{\alpha\in B_k^c}\frac{1}{|Q|}\int_Q\left|G_{k+1}^{(\alpha)}-(G_{k+1}^{(\alpha)})_Q\right|d\alpha \\ &\le C\,|B_k| + C\,|B_k^c|(1+(n-n_k)) \\ &\le C + C\,2^{(n_k-n)}(n-n_k)\end{aligned}$$

and this proves (2) .

Proof of (3) :

Fix $j>k+1, \alpha\in Q_0$. Then $\exists Q_1\in\Im_j^{(\alpha)}$ and $a_{Q_1}\in\mathbf{R}^d$ such that

$$\frac{1}{|Q_1|}\int_{Q_1}|G_j^{(\alpha)}-a_{Q_1}|\le C \quad\text{and}\quad Q\subseteq Q_1,\ \ |Q_1|\le C\,|Q|$$

This implies

$$\frac{1}{|Q|}\int_Q|G_j^{(\alpha)}-a_{Q_1}|\le C$$

and hence (3) follows.

This completes the proof of lemma 3.3 and theorem I.

A consequence of theorem I which has useful applications is the following corollary:

Corollary 3.1

Let E_1, E_2 *be measurable subsets of the unit cube in* $\mathbf{R}^d$ *and suppose there exists an increasing sequence of positive numbers* $\{\lambda_n\}_{n=0}^{\infty}$ *with* $\lambda_n \to \infty$ *such that for each* $n \in \mathbf{N}$ *and for each cube* Q *with* $\ell(Q) \leq 2^{-n}$ *we have*

$$\min\left(\frac{|Q \cap E_1|}{|Q|}, \frac{|Q \cap E_2|}{|Q|}\right) < 2^{-\lambda_n}.$$

Then there exists $F \in VMO$, $\|F\|_* \leq C\lambda_0$ *with* $F = 0$ *on* E_1 *and* $F = 1$ *on* E_2.

Proof Set $E = E_1 \cup E_2$ in theorem I and define

$$f(x) = \begin{cases} 0 & \text{if } x \in E_1 \\ 1 & \text{if } x \in E_2 \end{cases}$$

and

$$a_Q = \begin{cases} 1 & \text{if } \log \frac{|Q|}{|Q \cap E|} \geq \lambda_{[\log 1/\ell(Q)]+1} \\ 0 & \text{otherwise.} \end{cases}$$

4. Extension to $BMO(\rho)$.

4.1 Proof of Theorem II

The first part of the proof of (2)⇒(1) in theorem I establishes the existence of a dyadic-VMO extension F of f which can be written in the following form :

$$F^{(0)} = \sum_{k\geq 0} \frac{1}{\lambda_{k-1}} \left(G_k^{(0)} + H_k^{(0)} \right)$$

where $\frac{1}{\lambda_k} \leq C\,\rho(2^{-n_k})$ and $G_k^{(0)}$, $H_k^{(0)} \in$ dyadic $- VMO$. The functions $G_k^{(0)}$ are constant on dyadic cubes of length 2^{-n_k} . The functions $H_k^{(0)}$ were obtained from lemma 2.2 from which it is clear that for each k, $\sup_{\ell(Q)\leq t} \frac{1}{|Q|} \int_Q |H_k^{(0)} - (H_k^{(0)})_Q|$ as a function of t, depends only on the geometry of the set E. The hypothesis (3) in theorem II below provides a sufficient condition for the function $\sum_{k\geq 0} \frac{H_k}{\lambda_{k-1}}$ constructed in the proof of theorem I to be in $BMO(\widetilde{\rho})$ for some specified growth function ρ.

Theorem II

Let E be a bounded measurable subset of $\mathbf{R}^d$. Let f be a locally integrable function defined on E and define $\mu_n(f)$ as in Theorem I. If ρ is a growth function satisfying :

(1) $\mu_n(f) \leq C\,\rho(2^{-n}), \quad \forall n \in \mathbf{Z}$

(2) $\inf_{t>0} \rho(t) |\log t| > 0$

(3) $\exists \lambda > 1$ *such that for all* $m \in Z$ *and for all cubes* $Q, \ell(Q) \leq 2^{-m}$ *with* $0 < |Q \cap E| < |Q|/\lambda$ *we have*

$$\inf_{n>m} \rho(2^{-n}) \left| \log \left| \{ x \in Q \cap E : \sup_{\substack{x\in Q' \\ \ell(Q')\leq 2^{-n}}} \frac{|Q'|}{|Q' \cap E|} > \lambda \} \right| \right| \geq \rho(2^{-m}) \left| \log \frac{|Q \cap E|}{|Q|} \right|$$

then f *is the restriction to* E *of a function in* $BMO(\widetilde{\rho})$. *In particular, if* ρ *is regular then* f *is the restriction to* E *of a function in* $BMO(\rho)$.

Proof Without loss of generality we will assume E is contained in the unit cube Q_0 in $\mathbf{R}^d$. It follows from theorem 1.1 that (1) is a necessary condition for f to be the restriction to E of a function in $BMO(\rho)$. We also note that (1) is a sufficient condition for the function $G = \sum_{k\geq 0} \frac{G_k}{\lambda_{k-1}}$ to belong to $BMO(\widetilde{\rho})$.

Fix $\alpha \in Q_0, k \in \mathbf{N}$ and let $Q_k \in D_{n_k}^{(\alpha)}$. If $|Q_k \cap E| \geq |Q_k|/\lambda$ then $H_k^{(\alpha)}$ will satisfy $\|H_k^{(\alpha)}\|_* \leq C \log \lambda$ on Q_k. We assume then that $0 < |Q_k \cap E| < |Q_k|/\lambda$ and for each $n > m$ we define

$$\delta_n(x) = \sup_{\substack{x \in Q \\ \ell(Q) \leq 2^{-n}}} \frac{|Q|}{|Q \cap E|} \quad \text{and} \quad E_n = \{x \in E : \delta_n(x) > \lambda\}$$

Let $\{Q_i\}$ be the maximal dyadic subcubes of Q_k with respect to the property $|Q_i \cap E| > |Q_i|/\lambda$ and note that if $x \in Q_i \cap E$ for some i, then $x \in E_j$ for all $n_k \leq j \leq (\log 1/\ell(Q_i)) - 1$. This implies

$$\begin{aligned} \sum_{\ell(Q_i) < 2^{-n}} |Q_i| &\leq \lambda \sum_{\ell(Q_i) < 2^{-n}} |Q_i \cap E| \\ &\leq \lambda |E_n| \\ &\leq \frac{|Q_k|}{|Q_k \cap E|} 2^{-\left(\frac{\rho(2^{-n_k})}{\rho(2^{-n})} \log \frac{|Q_k|}{|Q_k \cap E|}\right)} \\ &\leq 4^{-cd\left(\frac{\rho(2^{-n_k})}{\rho(2^{-n})} \log \frac{|Q_k|}{|Q_k \cap E|}\right)} \end{aligned}$$

and hence

$$\sum \left\{ |Q_i| : 2^{-n_{j+k+1}} \leq \ell(Q_i) < 2^{-n_{j+k}} \right\} \leq 4^{-cd\left(2^j \log \frac{|Q_k|}{|Q_k \cap E|}\right)}$$

As in the proof of lemma 2.2 , we can find C^∞ functions $\{a_{j,k}^{(\alpha)}\}$ which can be written as $a_{j,k}^{(\alpha)} = \sum_{i=1}^{\beta_j^{(\alpha)}} \sum_{G_i^{(j,\alpha)}} b_{l,i}$ where each $b_{l,i}$ is adapted to cubes of length $\geq 2^{-n_{k+j+1}}$. We then define

$$H_k^{(\alpha)} = \sum_{j\geq 1} \frac{a_{j,k}^{(\alpha)}}{2^j} \tag{4.1}$$

If $|Q_k \cap E| = 0$ then we define $H_k^{(\alpha)} = a_{j,k}^{(\alpha)} = 0$. If $|Q_k \cap E| > |Q_k|/\lambda$, then we may choose the $a_{j,k}^{(\alpha)}$ to be constant and bounded and so that $H_k^{(\alpha)}$ is given by (4.1) .

We note then that in all cases there exists a constant C_λ, depending only on λ, such that $|a_{j,k}^{(\alpha)}| \leq C_\lambda\ (n_{k+j+1} - n_k)$. As in the proof of theorem I, we set

$$H_k(x) = \int_{\alpha\in Q_0} H_k^{(\alpha)}(x)\, d\alpha$$

and

$$H(x) = \sum_{k\geq 0} \frac{1}{\lambda_{k-1}} H_k(x), \quad (\lambda_{-1} = \lambda_0)$$

It remains to show $H \in BMO(\widetilde{\rho})$. Let Q be any cube with $2^{-n_{N+1}} \leq \ell(Q) < 2^{-n_N}$ and let

$$H_1 = \sum_{k=0}^{N} \frac{1}{\lambda_{k-1}} \sum_{j=1}^{N-k-1} \left(\frac{a_{j,k}}{2^j}\right) \quad \text{where} \quad a_{j,k} = \int_{\alpha\in Q_0} a_{j,k}^{(\alpha)}\, d\alpha$$

and define $H_2 = H - H_1$. Lemma 2.2 (2) implies

$$\begin{aligned} \frac{1}{|Q|}\int_Q H_2\, dt &\leq \frac{C}{2^{N+1}} \\ &\leq C\,\rho\left(2^{-n_N}\right) \\ &\leq C\,\widetilde{\rho}\left(2^{-n_N}\right) \end{aligned} \tag{4.2}$$

and so $H_2 \in BMO(\widetilde{\rho})$.

If $\qquad S = \left\{\alpha \in Q_0 : Q \subseteq Q_k \in D_{n_k}^{(\alpha)}\right\}$

then by the remark after lemma 2.2,

$$\frac{1}{|Q|}\int_Q \left|a_{j,k}^{(\alpha)} - \left(a_{j,k}^{(\alpha)}\right)_Q\right| dt \le C\, 2^{n_{k+j+1}-n_N} \quad \text{for all } \alpha \in S$$

If $\alpha \notin S$, then

$$\sup_{x,y\in Q}\left|a_{j,k}^{(\alpha)}(x) - a_{j,k}^{(\alpha)}(y)\right| \le C_\lambda (n_{k+j+1} - n_k)$$

Hence,

$$\begin{aligned}\frac{1}{|Q|}\int_Q \left|a_{j,k}^{(\alpha)} - \left(a_{j,k}^{(\alpha)}\right)_Q\right| dt &\le C\, 2^{n_{k+j+1}-n_N} + C_\lambda (n_{k+j+1} - n_k)|S| \\ &\le C\, 2^{n_{k+j+1}-n_N} + C_\lambda (n_{k+j+1} - n_k) 2^{n_k - n_N} \\ &\le C_\lambda 2^{n_{k+j+1}-n_N}\end{aligned}$$

This implies

$$\begin{aligned}\frac{1}{|Q|}\int_Q |H_1 - (H_1)_Q|\, dt &\le C \sum_{k=0}^{N} \frac{1}{2^k} \sum_{j=1}^{N-k-1} \frac{1}{2^j}\left(\frac{1}{|Q|}\int_Q \left|a_{j,k}^{(\alpha)} - \left(a_{j,k}^{(\alpha)}\right)_Q\right| dt\right) \\ &\le C_\lambda \sum_{k=0}^{N} \frac{1}{2^k} \sum_{j=1}^{N-k-1} \frac{1}{2^j}\left(2^{n_{k+j+1}-n_N}\right) \\ &\le C_\lambda \sum_{k=0}^{N} \frac{1}{2^k} \sum_{j=1}^{N-k-1} \frac{1}{2^k}\left(\rho\left(2^{-n_j}\right) - \rho\left(2^{-n_{j+1}}\right)\right) 2^{n_{k+j+1}-n_N} \\ &\le C_\lambda \sum_{k=0}^{N} \frac{2^{-n_N}}{2^k}\left(\sum_{j=1}^{N-k-1} \rho\left(2^{-n_j}\right)\left(2^{n_{k+j+1}} - 2^{n_{k+j}}\right)\right) \\ &\le C_\lambda \sum_{k=0}^{N} \frac{2^{-n_N}}{2^k}\left(\sum_{j=1}^{N-k-1} \int_{2^{-n_{k+j+1}}}^{2^{-n_{k+j}}} \frac{\rho(t)}{t^2}\, dt\right) \\ &\le C_\lambda \sum_{k=0}^{N} \frac{2^{-n_N}}{2^k}\left(\int_{2^{-n_N}}^{2^{-n_k}} \frac{\rho(t)}{t^2}\, dt\right) \\ &\le C_\lambda \sum_{k=0}^{N} \frac{1}{2^k}\left(2^{-n_N}\int_{2^{-n_N}}^{1} \frac{\rho(t)}{t^2}\, dt\right) \\ &\le C_\lambda\, \widetilde{\rho}\left(2^{-n_N}\right)\end{aligned}$$

Combined with (4.2) we obtain

$$\frac{1}{|Q|}\int_Q \left|H-(H)_Q\right| dt \le C\,\tilde{\rho}\,(\ell(Q))\,.$$

The theorem now follows from the proof of theorem I. ∎

4.2 Uniqueness of the $BMO(\rho)$ extension

Corollary 2.1 implies that the VMO extension is never unique. For $BMO(\rho)$ we have the following:

Theorem 4.1

Let E be a measurable subset of $\mathbf{R}^d$ and ρ a growth function satisfying

$$\limsup_{|Q|\to 0}\left(\rho\,(\ell(Q))\log\frac{|Q|}{|Q\cap E|}\right)=0 \tag{4.3}$$

Then whenever $f\in BMO(\rho)$ satisfies $f\chi_E=0$ we have $f=0$ a.e.

Proof : Without loss of generality we may assume $f\ge 0$. Suppose there exists $\epsilon>0$ such that the set $E_1=\{x\in Q : f>\epsilon\}$ has positive measure. For each $\delta>0$, (4.3) implies there exists n_δ such that,

$$\frac{|Q\cap E|}{|Q|} > 2^{-\delta/\rho(\ell(Q))},\quad \forall Q,\ \ \ell(Q)\le 2^{-n}, n\ge n_\delta$$

For any such Q, theorem 1.1 implies

$$|\{x\in Q : |f-f_Q|>\lambda\}| < C_0\,|Q|2^{-c_1\lambda/\rho(\ell(Q))}$$

$$\text{and hence}\qquad |f_Q|\le C_0\,\delta \tag{4.4}$$

For any n, $\exists Q$, $\ell(Q)\le 2^{-n}$ such that $\dfrac{|Q\cap E_1|}{|Q|}>\dfrac{1}{2}$ and for any such Q, we have $|f_Q|>\dfrac{\epsilon}{2}$ and this contradicts (4.4) for sufficiently small δ.

Remark :

1) Whenever E and ρ satisfy (4.2), the extension to a $BMO(\rho)$ function will be linear. However we do not know if the BMO extension in theorem 2.2 or the VMO extension are linear.

2) Given ρ, it is not difficult to find a set E satisfying (4.2). In the example below we obtain E as the complement of a Cantor set which is constructed using a variable ratio of dissection.

Example :

It suffices to construct E on the unit interval $J^{(0)} = [0,1]$ in $\mathbf{R}$. Fix $N \in \mathbf{N}$. We can find subintervals $\{I_j^{(1)}\}$ of $J^{(0)}$ which are of equal length and satisfy

1) $\sum |I_j^{(1)}| = 2^{-(N+1)}$

2) $J^{(0)} \setminus \{\cup I_j^{(1)}\}$ is the union of intervals $\{J_k^{(1)}\}$ satisfying $\rho(|J_k^{(1)}|) \le \left(\frac{1}{N+3}\right)^2$.

We proceed by induction

Assume $\{J_k^{(n)}\}$ have been defined. On each $J_k^{(n)}$ we remove intervals $I_j^{(n+1)}$ of equal length and satisfying

1) $\sum |I_j^{(n+1)}| = 2^{-(N+n+2)} |J_k^{(n)}|$

2) $J_k^{(n)} \setminus \{\cup I_j^{(n+1)}\}$ is the union of intervals in $\{J_k^{(n+1)}\}$ satisfying $\rho(|J_k^{(n+1)}|) \le 1/(N+n+3)^2$

We define $E = \bigcup_{j,k} I_j^{(k)}$. Let I be an interval and suppose

$$|J_k^{(n+1)}| < |I| \le |J_k^{(n)}|$$

$$\text{Then} \quad |I \cap E| \ge \frac{|J_k^{(n+1)} \cap E|}{2\,|J_k^{(n+1)}|}\,|I|$$

$$
\begin{aligned}
\Rightarrow \quad \log \frac{|I|}{|I \cap E|} &\le N + n + 3 \\
\Rightarrow \quad \rho\,(|I|) \log \frac{|I|}{|I \cap E|} &\le \rho\big(|J_k^{(n)}|\big)\,\big(N + n + 3\big) \\
&\le \frac{1}{N + n + 3} \\
&\longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty
\end{aligned}
$$

5. The zeros of analytic $BMO(\rho)$ functions.

The following definitions and notation will be used in the sequel:

$H^\infty(D)$ = bounded analytic functions on the unit disc $D = \{z : |z| < 1\}$.

H^∞ = boundary values of functions in $H^\infty(D)$.

A sequence $\{z_j\}_{j=1}^\infty \subseteq D$ is called a Blaschke sequence if $\sum_{j=1}^{\infty}(1-|z_j|) < \infty$ and the corresponding analytic function

$$B(z) = \prod_{j=1}^{\infty} \frac{|z_j|}{z_j}\left(\frac{z-z_j}{1-\overline{z}_j z}\right)$$

is called a Blaschke product.

For each $z \in D$ we define

$$I_z = \left\{e^{i\theta} : |\theta - \arg z| < \frac{1}{2}(1-|z|)\right\}$$

$$Q_z = \left\{w : |w| \geq |z|,\ \frac{w}{|w|} \in I_z\right\}$$

A positive measure μ on D is called a Carleson measure if for each $z \in D$,

$\mu\ (Q_z)\ \leq C_0\ |I_z|$

A Blaschke sequence $\{z_j\}$ is called an interpolating sequence if for all $\{\lambda_j\} \in l_\infty$ $\exists F \in H^\infty(D)$ with $F(z_j) = \lambda_j$. Carleson's interpolation theorem (see [4] Chapter 7) states that $\{z_j\}$ is an interpolating sequence if and only if

(1) $\inf_{j \neq k} \left|\frac{z_j - z_k}{1 - \overline{z}_j z_k}\right| > 0$

(2) $\sum(1-|z_j|)\delta_{z_j}$ is a Carleson measure where δ_z denotes the Dirac measure at z.

We will also need the following characterization of $BMO(\rho)$ on the unit circle, $T = \{z : |z| = 1\}$ (see [11])

$f \in BMO(\rho)$ if and only if

$$1) \sup_{|z|>1-\delta} \left(\int_{-\pi}^{\pi} |f - f(z)|^2 \, dP_z(\theta) \right)^{\frac{1}{2}} \leq C \, \rho(\delta) \tag{5.1}$$

where

$$dP_z(\theta) = \frac{1}{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \, d\theta$$

and

$$f(z) = \int_{-\pi}^{\pi} f(e^{i\theta}) dP_z(\theta)$$

or equivalently

$$2) \; \exists C > 0 \text{ such that } \; \forall z_0 \in D$$

$$\int_{Q(z_0)} |\nabla f|^2 (1 - |z|^2) dx \, dy \leq C \, |I_{z_0}| \rho(|I_{z_0}|). \tag{5.2}$$

The purpose of this section is to establish sufficient conditions for a Blaschke sequence $\{z_j\}$ to be the zeros of a function in $H^\infty(D)$ with boundary values in $BMO(\rho)$. Wolff [13] has shown that every Blaschke sequence are the zeros of a function in $H^\infty(D)$ with boundary values in $VMO \cap L_\infty$ and every subset of the unit circle is the zero set of a function in VMO . We note that this result follows directly from corollary 2.1 .

The theorem that we will prove is the following :

Theorem 5.1

Let $\{z_k\}$ *be an interpolating sequence in* D *and suppose* ρ *is a growth function with* ρ^2 *regular . If* $\exists C_0 > 0$ *such that*

$$\inf_{\delta > 0} \frac{\delta^2}{|\log \delta|} \left| \log \sum_{\rho(1-|z_j|)<\delta} (1 - |z_j|) \right| \geq C_0 \tag{5.3}$$

then $\exists f \in BMO\,(\rho) \cap H^\infty$ *with* $f(z_k) = 0$ *for all* z_k.

We note that if B is a Blaschke product, $f \in L_1(T)$ and $f(z) = \int_T f(\theta)\, dP_z(\theta)$ then

$$\int_T |fB - f(z)B(z)|^2\, dP_z(\theta)$$
$$= \int_T |f - f(z)|^2\, dP_z(\theta) + \left(1 - |B(z)|^2\right) |f(z)|^2$$

Hence if $f \in BMO(\rho)$ and if $\exists C > 0$ such that $|z| > 1 - \delta$ implies

$$\left(1 - |B(z)|^2\right) |f(z)|^2 \le C\,\rho^2(\delta)$$

then $Bf \in BMO(\rho)$. The proof of theorem 5.3 consists of obtaining a $BMO(\rho)$ function f satisfying

$$|f(z_k)| \le C\,\rho^2\,(1 - |z_k|) \tag{5.4}$$

and so that the Blaschke product with zeros z_k is sufficiently near to 1 when $|f(z)|$ is large.

Proof of Theorem 5.1

We first show that if $\{z_k\}$ is a Blaschke sequence satisfying (5.3), then there exists $f \in BMO(\rho) \cap H^\infty$ satisfying (5.4). Define a sequence $\{n_k\} \subseteq \mathbf{N}$ by the condition $\rho(2^{-n}) \le 2^{-k}\rho(1)$ if and only if $n \ge n_k$. Then (5.3) implies

$$\sum \left\{|I_{z_j}| : 2^{-n_{k+1}} < |I_{z_j}| \le 2^{-n_k}\right\} < 4^{-cdk2^{2k}}$$

Lemma 2.2 implies there exists $g_k \in C^\infty$ satisfying

$$g_k \ge C\,k2^k \quad \text{on} \quad \bigcup \left\{I_{z_j} : 2^{-n_{k+1}} < |I_{z_j}| \le 2^{-n_k}\right\}$$

$$\text{and} \quad \|f_k\|_*, \|f_k\|_\rho \le C_0$$

$$\text{Set} \quad g = \sum_{k \ge 0} \frac{g_k}{2^{2k}}$$

By the remark after lemma 2.2, $g \in BMO(\rho^2)$ (see also the proof of theorem II) and,

$$g(t) \geq Ck \quad \text{on} \quad \bigcup \{I_{z_j} : 2^{-n_{k+1}} < |I_{z_j}| \leq 2^{-n_k}\}$$

Define

$$f(z) = \exp\left(-\left(g + i\widetilde{g}\right)\right)$$

where $\widetilde{g}$ is the conjugate function of g. (5.2) implies $f \in BMO(\rho^2)$ and $|f(z_j)| \leq C\,2^{-k}$ whenever $|z_j| > 1 - 2^{-n_k}$ and this establishes (5.4).

We note that $f \in BMO(\sigma)$ where

$$\sigma(\delta) = \sup_{\ell(Q) \leq \delta} \frac{1}{|Q|} \int_Q |f - f_Q| \qquad \text{satisfies}$$

$$\sigma(t) \leq \rho^2(t), \;\; \sigma(2t) \leq 2\sigma(t), \;\; \forall t \geq 0$$

It will be convenient to work in the upper half plane $\mathbf{R}^2_+$ though we shall retain the same notation for f and B and note that $BMO(\rho)$ is conformally invariant.

If Q is a cube in $\mathbf{R}^2_+$ of the form $Q = \{(x, y) : 0 < y < a\}$, we define the horizontal projection of Q to be the set

$$Q^* = Q \cap \{(x, 0) : x \in \mathbf{R}\}$$

and the top-half of Q to be the set

$$T(Q) = \{(x, y) \in Q : y > \frac{1}{2}\ell(Q)\}.$$

If $z \in \mathbf{R}^2_+$ we set

$$Q_z = \{(x, y) : |x - \operatorname{Re} z| \leq \frac{\operatorname{Im} z}{2}, \; y \leq \operatorname{Im} z\}$$

$$\text{and} \quad I_z = Q^*_z$$

Claim : If $\left|\dfrac{z-z_j}{z-\bar{z}_j}\right| < \dfrac{1}{6}$ then $|f(z)| \le C\,\sigma\,(1-|z|)$.

Proof : It is clear we must have $|x - x_j| \le 2y_j$ and $y \le 2y_j$. Furthermore if $|x - x_j| < \frac{1}{2}y_j$ then $y \ge \frac{1}{2}y_j$. This implies

$$\frac{1}{2}\,|I_z| \le |I_{z_j}| \le 2\,|I_z|$$

which implies

$$\begin{aligned} |f(z)| &\le |f(z) - f_{I_z}| + |f_{I_z}| \\ &\le C\,\sigma\,(1-|z|) + |f_{I_z} - f_{I_{z_j}}| + |f_{I_{z_j}}| \\ &\le C\,(\sigma\,(1-|z|) + \sigma\,(1-|z_j|)) \\ &\le C\,\sigma\,(1-|z|) \end{aligned}$$

Hence in this case we have

$$\left(1-|B(z)|^2\right)|f(z)|^2 \le C\,\sigma\,(1-|z|)\,.$$

Now suppose $\quad \inf_j \left|\dfrac{z-z_j}{z-\bar{z}_j}\right| \ge \dfrac{1}{6}$.

In this case, the estimate $|\log t| \le (1+2|\log a|)\,(1-t)$ valid for $a^2 < t < 1$ implies

$$\left(1-|B(z)|^2\right) \le C\sum_k \frac{y\,y_k}{|z-\bar{z}_k|^2}$$

Let $A_n = \{z_j : |x - z_j| > 2^n\,y\}$, $n \in \mathbf{Z}$ and choose N so that $2^{-N} < \sigma\,(1-|z|)$.

Then,

$$\begin{aligned} \sum_k \frac{y\,y_k}{|z-\bar{z}_k|^2} &= \sum_{z_k \in A_N^c} \frac{y\,y_k}{|z-\bar{z}_k|^2} + \sum_{n \ge N+1}\ \sum_{z_k \in A_{n-1}\setminus A_n} \frac{y\,y_k}{|z-\bar{z}_k|^2} \\ &= S_1 + S_2, \quad \text{say} \end{aligned}$$

Now $$S_2 \le \sum_{n\ge N+1} \sum_{z_k \in A_{n-1}\setminus A_n} \frac{y\,y_k}{2^{2n}\,y^2}$$

$$\le C \sum_{n\ge N+1} \left(\frac{2^n}{2^{2n}}\right) , \qquad \text{since } \{z_k\} \text{ is interpolating}$$

$$\le C\,\sigma\,(1-|z|)$$

Hence it remains to show $S_1 |f(z)|^2 \le C\,\sigma\,(1-|z|)$.

Let $R = \{(u,v) : |u-x| \le 2^p|I|, 0 < v < 2^{p+1}|I|\}$ where $I = I_z$ and where p is sufficiently large so that $A_N^c \subseteq R$. Subdivide R into dyadic cubes and from the collection with one side along the x-axis, we select those that are maximal with respect to the property of containing some z_j in their top half.We denote this collection of cubes by $\{Q_i\}$. From each Q_i, select a point z_j contained in $T(Q_i)$. To distinguish these points we will denote them by $\{w_j\}$. Since $\{z_j\}$ is an interpolating sequence, we have for each Q_i,

$$\sum_{z_k \in Q_i} \frac{y\,y_k}{|z-\overline{z}_k|^2} \le C\,\frac{y\,\operatorname{Im} w_k}{|z-\overline{w}_k|^2}$$

Let $D_n = \{(u,0) : 2^{n-2}\,|I| \le |u-x| < 2^{n-1}\,|I|\}$ and $J_n = \bigcup_{0\le k\le n+1} D_k$. Let n_1 be the smallest value of n for which there exists a w_j with $\operatorname{Im} w_j \ge y$ and $|x - \operatorname{Re} w_j| < 2^{n_1}\,|I|$. Then,

$$|f(z)| \le C\,\sigma\,(1-|z|) + |f_I - f_{J_{n_1}}|$$
$$+ |f_{I_{w_j}} - f_{J_{n_1}}| + |f_{I_{w_j}}|$$

Furthermore

$$|f_I - f_{J_{n_1}}| \le C \sum_{k=0}^{n_1-1} |f_{J_k} - f_{J_{k+1}}|$$

$$\leq C \sum_{k=0}^{n_1-1} \sigma\left(\left|J_{k+1}\right|\right)$$

$$\leq C \left(\sum_{k=0}^{n_1-1} \frac{\left|J_{k+1}\right|}{|I|} \right) \sigma\left(|I|\right)$$

$$\leq C \, \frac{\left|J_{n_1}\right|}{|I|} \sigma\left(|I|\right)$$

Similarly

$$\left|f_{I_{w_j}} - f_{J_{n_1}}\right| \leq C \, \frac{\left|J_{n_1}\right|}{|I|} \sigma\left(|I|\right)$$

while

$$\left|f_{I_{w_j}}\right| \leq C \, \frac{\left|I_{w_j}\right|}{|I|} \sigma\left(|I|\right) \leq C \, \frac{\left|J_{n_1}\right|}{|I|} \sigma\left(|I|\right)$$

and hence

$$|f(z)| \leq C \, \frac{\left|J_{n_1}\right|}{|I|} \sigma\left(1-|z|\right)$$

Now there exists $\{n_k\} \subseteq \mathbf{N}$ and $C > 0$ such that for all w_k, $\operatorname{Im} w_k \geq y$,

$$\left|\operatorname{Im} w_k\right| \leq C \, \left(\left|J_{n_{k+1}}\right| - \left|J_{n_k}\right|\right)$$

Hence

$$|f(z)|^2 \sum_{\substack{w_k \in R \\ \operatorname{Im} w_k \geq y}} \frac{y \operatorname{Im} w_k}{\left|z - \overline{w}_k\right|^2} \leq C \, |f(z)| \, \sigma\left(1-|z|\right) \frac{\left|J_{n_1}\right|}{|I|} \sum_{k \geq 0}^{N} \frac{|I| \left(\left|J_{n_{k+1}}\right| - \left|J_{n_k}\right|\right)}{\left|J_{n_{k+1}}\right|^2}$$

$$\leq C \, \sigma\left(1-|z|\right), \quad \text{since } f \text{ is bounded}$$

We now prove

$$|f(z)|^2 \sum_{\substack{w_k \in R \\ \operatorname{Im} w_k \leq y}} \frac{y \operatorname{Im} w_k}{\left|z - \overline{w}_k\right|^2} \leq C \, \sigma\left(1-|z|\right) \tag{5.5}$$

Let $\Im_1 = \{w_j : \left|I_{w_j}\right| < |I|, \left|I_{w_j} \cap I\right| \geq \frac{1}{2} \left|I_{w_j}\right|\}$

and $\Im_2 = \{w_j : \left|I_{w_j}\right| < |I|, \left|I_{w_j} \cap I\right| < \frac{1}{2} \left|I_{w_j}\right|\}$

Claim : If $\Im_1 \neq \emptyset$, then

$$|f_I| \leq C\,\sigma\,(|I|)\left(\left|\log \frac{|I|}{\sum_{\Im_1}|I_{w_j}|}\right| + 1\right). \tag{5.6}$$

Proof : For each $w_j \in \Im_1, |f_{I_{w_j}}| \leq \sigma(|I_{w_j}|)$. Theorem 1.1 implies $|f(t)| \leq C.\max_{\Im_1} \sigma(|I_{w_j}|)$ on a subset of $\bigcup\{I_{w_j}\}$ of measure $\geq \frac{1}{2}\sum|I_{w_j}|$. (5.6) now follows from theorem 1.1.

The claim implies

$$|f(z)|^2 \sum_{w_j \in \Im_1} \frac{y \,\operatorname{Im} w_j}{|z - \overline{w}_j|^2}$$

$$\leq \quad C\,\sigma\,(|I|)\left(\left|\log \frac{|I|}{\sum_{\Im_1}|I_{w_j}|}\right| + 1\right) \sum_{I_{w_j}} \frac{|I||I_{w_j}|}{|I|^2}$$

$$\leq C\,\sigma\,(|I|)\,.$$

Finally we consider the contribution from points in $\Im_2$.

Let $m_1 = \min\{n : \exists w_j \in \Im_2 \text{ with } |I_{w_j} \cap D_n| > \frac{1}{2}|I_{w_j}|\}$ and let $\{m_j\} \subseteq \mathbf{N}$ be that sequence of points with the property $|D_{m_j} \cap I_{w_k}| > \frac{1}{2}|I_{w_k}|$ for some $w_k \in \Im_2$

Claim :

$$|f_I| \leq C\,\sigma\,(|I|)\left\{\left|\log \frac{\sum\{|I_{w_j}| : |I_{w_j} \cap D_{m_1}| > \frac{1}{2}|I_{w_j}|\}}{|I|}\right| + \frac{|J_{m_1}|}{|I|}\right\}$$

Proof : Without loss of generality we may assume $|D^+_{m_1} \cap I_{w_j}| > \frac{1}{2}|I_{w_j}|$ where $D^+_{m_1} = D_{m_1} \cap \{(x,0) : x \geq 0\}$ Then the proof of the claim above implies

$$|f_{D^+_{m_1}}| \leq C\,\sigma\,(|I|)\left|\log \frac{\sum\{|I_{w_j}| : |I_{w_j} \cap D_{m_1}| > \frac{1}{2}|I_{w_j}|\}}{|I|}\right|$$

Therefore,

$$|f_I| \leq |f_{D^+_{m_1}}| + |f_I - f_{J_{m_1}}| + |f_{D^+_{m_1}} - f_{J_{m_1}}|$$

$$\leq |f_{D^+_{m_1}}| + C\,\frac{|J_{m_1}|}{|I|}\sigma\,(|I|)$$

which establishes the claim. Hence

$$|f(z)|^2 \sum_{w_j \in \Im_2} \frac{y \operatorname{Im} w_j}{|z-\overline{w}_j|^2} \le C\,(\sigma(|I|)+|f_I|) \sum_{k\ge 1} \left(\sum \frac{|I|\,|I_j|}{|z-\overline{w}_j|^2} \right)$$

where the second sum is taken over those w_j for which $|D_{m_k} \cap I_{w_j}| > \frac{1}{2}|I_{w_j}|$

$$\le C\,(\sigma(|I|)+|f_I|) \frac{\sum |I_j|}{|I|} \left(|I|^2 \sum_{k\ge 1} \frac{1}{|J_{m_k}|^2} \right)$$

$$\le C\,(\sigma(|I|)+|f_I|) \frac{|I|^2}{|J_{m_1}|^2}$$

$$\le C\,\sigma(|I|)\,. \tag{5.7}$$

Now (5.6) and (5.7) imply (5.5) and this completes the proof of the theorem.

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