THEORETICAL INVESTIGATION OF TURBULENT BOUNDARY LAYER OVER A WAVY SURFACE

Thesis by

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This work is dedicated to my parents, and to my wife, Lu, whose loving devotion and encouragement made the completion of the present task possible.

ABSTRACT

The important features of the two-dimensional incompressible turbulent flow over a wavy surface of wavelength comparable with the boundary layer thickness are analyzed.

A turbulent field method using model equation for turbulent shear stress similar to the scheme of Bradshaw, Ferriss and Atwell (1967) is employed with suitable modification to cover the viscous sublayer. The governing differential equations are linearized based on the small but finite amplitude to wavelength ratio. An orthogonal wavy coordinate system, accurate to the second order in the amplitude ratio, is adopted to avoid the severe restriction to the validity of linearization due to the large mean velocity gradient near the wall. Analytic solution up to the second order is obtained by using the method of matched-asymptotic-expansion based on the large Reynolds number and hence the small skin friction coefficient.

In the outer part of the layer, the perturbed flow is practically "inviscid." Solutions for the velocity, Reynolds stress and also the wall pressure distributions agree well with the experimental measurement. In the wall region where the perturbed Reynolds stress plays an important role in the process of momentum transport, only a qualitative agreement is obtained. The results also show that the non-linear second-order effect is negligible for amplitude ratio of 0.03. The discrepancies in the detailed structure of the velocity, shear stress, and skin friction distributions near the wall suggest modifications to the model are required to describe the present problem.

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I. INTRODUCTION

The present theoretical study of the response of a turbulent boundary layer to the cyclic disturbance imposed by a wavy surface is motivated by an interest in the mechanism of cross-hatching ablation. The process of the formation and the growth of the wavy surface is believed to be a closed-loop interaction of perturbations on the ablating surface, coupled with resulting disturbance in the aerodynamic forces, heat and mass transfer.

A study was conducted at GALCIT by Dr. Toshi Kubota in the last few years; the first part covered a linear analysis of small perturbations in incompressible turbulent boundary layer utilizing an eddy-viscosity model,[1] and the second part was a detailed experimental investigation of low-speed turbulent boundary layers over a solid, stationary wavy wall.[2] The results of the study show that the theoretical prediction based on the eddy-viscosity model has a fair agreement with the experimental results in the overall distribution. In the wall region, especially for shear-stress distributions, the disagreement is significant.

The interaction between turbulent boundary layer and a wavy surface is a common phenomenon on the interface of the wind-water waves, on the sandy surface of the desert, and even inside the artery blood vessel. Over the last two decades, the phenomenon of wind generations of waves has drawn more and more interest of study, starting with the work of Miles (1957).[3] Miles assumed a quasi-inviscid flow with a 'turbulent' distribution of mean velocity, i.e. that the Reynolds stresses were not affected by the flow perturbations.

A subsequent contribution by Benjamin (1959)[4] contained the linear formulation which has served as the basis of the most recent investigations of shear flow over a wavy boundary.

It is only recently that the experimental investigation of the structure of turbulent shear layer over waves has been made in sufficient detail to test the various theories. Metzfeld[5] studied flow over five different wave shapes. He found strong changes in velocity profiles, mainly near the crest and the trough. Kendall[6] studied the interaction between turbulent boundary layer and a moving wavy surface. His results and the results obtained by Sigal[2] showed a strong disturbance and an appreciable phase shift of turbulent shear stresses. Experiments with blowing air flow over mechanically generated water waves in a wind-water tunnel has been carried out extensively by the Stanford group (Chambers, Mangarella, Street, Hsu, Yu[7]), also many others.

The eddy-viscosity or mixing-length model are based on local equilibrium between production, dissipation and diffusion of turbulent energy. On the other hand, these experimental findings indicate that the problem cannot be treated as local equilibrium flow, and therefore a more sophisticated model is required to analyze the problem.

The state of art of the computational method was reviewed based on the proceedings of the AFOSR-IFP-Stanford Conference on Computation of Turbulent Boundary Layer.[8] The scheme of Bradshaw, Ferriss and Atwell, which has been proven to yield satisfactory predictions in various cases, was chosen to analyze the problem.

The first part of the research--the linear first-order solution--was completed in December 1972. The result, however, did not show much improvement on the simple eddy-viscosity model for the prediction of shear-stress distribution.

During the period of this research a few articles using linear first-order theory on the wind-wave generation problem have been published in the Journal of Fluid Mechanics. Davis'[9] third modification utilized the BFA model with modifications to include the normal shear stresses. Townsend[10] applied the BFA scheme with the inclusion of gradient vertical diffusion. Their analyses are based on the log asymptotic behavior of the mean velocity profile near the wall. Central attention was directed at the critical layer and the phase shift of the pressure distribution.

Further examination in some parameters involved in the experiments revealed that the pressure gradient velocity u_p and the skin friction velocity u_τ are almost of the same order of magnitude in the experimental data. For an appreciable pressure gradient, the log asymptotic mean velocity profile in the viscous sublayer does not hold any more. The linear analysis was reformulated to include the effect of pressure gradient. In addition, the nonlinear effect is very significant in the experimental results. A tremendous effort has been made to construct the second order expansion, in order to draw a "conclusive" conclusion on the chosen scheme of analysis.

II. FORMULATION OF THE PROBLEM

The following analysis is based on small perturbations upon mean quantities of the undisturbed primary flow, i. e. flow over a flat surface. In Cartesian coordinates, the linearized boundary condition for perturbed velocity $u'(x, y)$ is determined by the derivative of the primary mean flow velocity $U(y)$ at the wall, which is very large in high Reynolds number turbulent boundary layer.

$$u'(x, 0) \simeq \left(\frac{\partial U}{\partial y}\right)_{y=0} y_w$$

$$y_w(x) = a \cos\alpha x, \quad \alpha = \frac{2\pi}{\lambda}$$

$$\left(\frac{u'}{U_\infty}\right)_{y=0} \simeq \frac{1}{2\lambda} R_\lambda C_f y_w, \qquad R_\lambda \equiv \frac{U_\infty \lambda}{\nu}$$

where a is the amplitude and λ is the wavelength of the wavy wall, and C_f is the turbulent skin friction. Therefore, for validity of linearization, it is required that $\frac{a}{\lambda} R_\lambda C_f \ll 1$.

This is not our prime interest. In order to release this strong restriction to make the problem more realistic, we adopted an orthogonal curvilinear coordinate system (Benjamin[4]).

$$\begin{cases} x = \xi - a e^{-\alpha\eta} \sin\alpha\xi \\ y = \eta + a e^{-\alpha\eta} \cos\alpha\xi \end{cases}$$

The linear solution constructed upon these coordinates, however, did not agree well with the experimental results in the wall region. In fact, at $\eta = 0$,

$$\begin{aligned} y &= a \cos\alpha\xi \\ &= y_w(x) - a^2\alpha \sin^2\alpha x + O(a^3\alpha^2) . \end{aligned}$$

Expanding the velocity u in Taylor series about y_w, and applying the non-slip condition at the wall, we see

$$u'(x, y) \simeq \left(\frac{\partial U}{\partial y}\right)_{y_w} (y - y_w) + ------ .$$

In the case of laminar boundary layer, this is of order a^2 as $\eta \to 0$. For turbulent boundary layer, the law of the wall,

$$\frac{U}{u_\tau} = F(y^*), \qquad y^* = \frac{u_\tau (y - y_w)}{\nu}$$

gives

$$\frac{u'}{U_\infty} \simeq \left(\frac{dF}{dy^*}\right) \frac{1}{2} R_\lambda C_f \left(\frac{y - y_w}{\lambda}\right) .$$

It follows that at $\eta = 0$,

$$\left(\frac{u}{U_\infty}\right)_{\eta=0} = O\left(\left(\frac{a}{\lambda}\right)^2 \pi R_\lambda C_f\right)$$

Thus, the first order linear solution with the non-slip condition applied at $\eta = 0$ is valid only for

$$\left(\frac{a}{\lambda}\right)^2 \pi R_\lambda C_f \ll 1 .$$

On the other hand, the experimental data of Kendall,[6] Sigal[2] and others were obtained at $\left(\frac{a}{\lambda}\right)^2 \pi R_\lambda C_f$ about 1.

Therefore, to make the comparison physically meaningful for small but finite a, the coordinates transformation must be such that the deviation of y from y_w at $\eta = 0$ is of order at least a^3. The required second order orthogonal wavy coordinate system as shown in Fig. 1 is given by

$$\begin{cases} \xi = x + ae^{-\alpha y}\sin\alpha x + \frac{1}{2} a^2\alpha e^{-2\alpha y}\sin 2\alpha x \\ \eta = y - ae^{-\alpha y}\cos\alpha x - \frac{1}{2} a^2\alpha(e^{-2\alpha y}\cos 2\alpha x+1) , \end{cases} \tag{2.1a}$$

and

$$\begin{cases} x = \xi - ae^{-\alpha\eta}\sin\alpha\xi + \frac{1}{2} a^2\alpha e^{-2\alpha\eta}\sin 2\alpha\xi \\ y = \eta + ae^{-\alpha\eta}\cos\alpha\xi - \frac{1}{2} a^2\alpha(e^{-2\alpha\eta}\cos 2\alpha\xi-1) . \end{cases} \tag{2.1b}$$

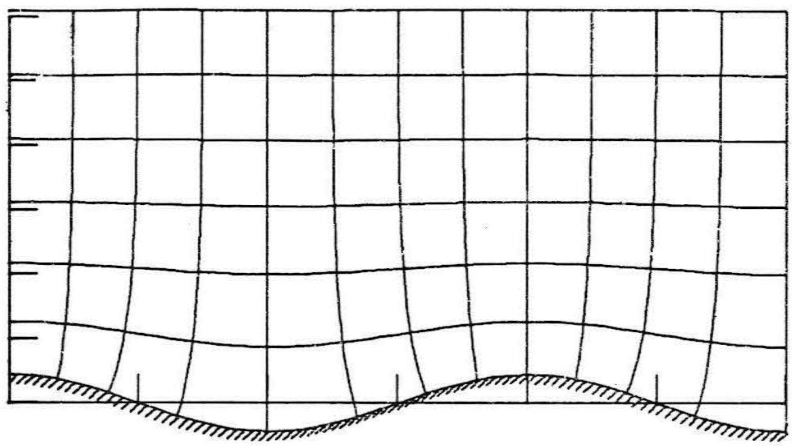

Fig. 1

Orthogonal Wavy Coordinates

which conforms with a sinusoidal wavy surface at $\eta = 0$ up to $O(a^2)$, for $a \ll 1$,

$$y = a\cos\alpha x + O(a^3\alpha^2) \tag{2.2}$$

The elements of length in the direction of increasing ξ, η, z are $hd\xi$, $hd\eta$, dz and

$$h = 1 - a\alpha e^{-\alpha\eta}\cos\alpha\xi + \frac{1}{2}(a\alpha)^2 e^{-2\alpha\eta}(3\cos^2\alpha\xi - 1) \qquad (2.3)$$

Let u, v, w be the velocity in the direction of increasing ξ, η, and z respectively. The governing differential equations for incompressible flow

$$\text{div}\ \vec{v} = 0$$

$$(\vec{v} \times \vec{\omega}) - \text{grad}\left(\frac{P}{\rho} + \frac{1}{2}v^2\right) = \nu\ \text{curl}\ \vec{\omega}$$

are transformed into these new coordinates. The time-averaged mean flow equations are

<u>Continuity:</u>

$$\frac{\partial}{\partial\xi}(h\bar{u}) + \frac{\partial}{\partial\eta}(h\bar{v}) = 0 \qquad (2.4a)$$

<u>Momentum:</u>

$$\rho\left(\frac{\bar{u}}{h}\frac{\partial\bar{u}}{\partial\xi} + \frac{\bar{v}}{h}\frac{\partial\bar{u}}{\partial\eta} + \frac{\bar{u}\bar{v}}{h^2}\frac{\partial h}{\partial\eta} - \frac{\bar{v}^2}{h^2}\frac{\partial h}{\partial\xi}\right)$$

$$= -\frac{1}{h}\frac{\partial\bar{p}}{\partial\xi} + \frac{1}{h}\left(\frac{\partial\sigma_1}{\partial\xi} + \frac{\partial\tau}{\partial\eta}\right) + \frac{1}{h^2}\left(\sigma\frac{\partial h}{\partial\xi} + 2\tau\frac{\partial h}{\partial\eta}\right)$$

$$+ \frac{\mu}{h^2}\left\{\frac{\partial^2\bar{u}}{\partial\xi^2} + \frac{\partial^2\bar{u}}{\partial\eta^2} + \frac{2}{h}\left[\frac{\partial\bar{u}}{\partial\xi}\frac{\partial h}{\partial\xi} + \frac{\partial\bar{v}}{\partial\xi}\frac{\partial h}{\partial\eta} + \left(\bar{v}\frac{\partial h}{\partial\xi} - \bar{u}\frac{\partial h}{\partial\eta}\right)\frac{\partial h}{\partial\eta}\right]\right\}$$

(2.4b)

$$\rho\left(\frac{\bar{u}}{h}\frac{\partial\bar{v}}{\partial\xi} + \frac{\bar{v}}{h}\frac{\partial\bar{v}}{\partial\eta} + \frac{\bar{u}\bar{v}}{h^2}\frac{\partial h}{\partial\xi} - \frac{\bar{u}^2}{h^2}\frac{\partial h}{\partial\eta}\right)$$

$$= -\frac{1}{h}\frac{\partial\bar{p}}{\partial\eta} + \frac{1}{h}\left(\frac{\partial\tau}{\partial\xi} + \frac{\partial\sigma_2}{\partial\eta}\right) + \frac{1}{h^3}\left(2\tau\frac{\partial h}{\partial\xi} - \sigma\frac{\partial h}{\partial\eta}\right)$$

$$+ \frac{\mu}{h}\left\{\frac{\partial^2\bar{v}}{\partial\xi^2} + \frac{\partial^2\bar{v}}{\partial\eta^2} + \frac{2}{h}\left[\frac{\partial\bar{u}}{\partial\eta}\frac{\partial h}{\partial\xi} + \frac{\partial\bar{v}}{\partial\eta}\frac{\partial h}{\partial\eta} + \left(\bar{u}\frac{\partial h}{\partial\eta} - \bar{v}\frac{\partial h}{\partial\xi}\right)\frac{\partial h}{\partial\xi}\right]\right\}$$

(2.4c)

Turbulent Energy:

$$\left(\frac{\bar{u}}{h}\frac{\partial}{\partial\xi}+\frac{\bar{v}}{h}\frac{\partial}{\partial\eta}\right)\frac{1}{2}\rho\overline{q^2}-\frac{1}{h}\left[\sigma_1\frac{\partial\bar{u}}{\partial\xi}+\tau\left(\frac{\partial\bar{u}}{\partial\eta}+\frac{\partial\bar{v}}{\partial\xi}\right)+\sigma_2\frac{\partial\bar{v}}{\partial\eta}\right]$$

$$+\frac{1}{h}\frac{\partial}{\partial\xi}\left(\frac{1}{2}\rho\overline{q^2u'}+\overline{p'u'}\right)+\frac{1}{h}\frac{\partial}{\partial\eta}\left(\frac{1}{2}\rho\overline{q^2v'}+\overline{p'v'}\right)$$

$$+\frac{1}{h^2}\left[\bar{v}\tau-\bar{u}\sigma_2+\left(\frac{1}{2}\rho\overline{q^2u'}+\overline{p'u'}\right)\right]\frac{\partial h}{\partial\xi}$$

$$+\frac{1}{h^2}\left[\bar{u}\tau-\bar{v}\sigma_1+\left(\frac{1}{2}\rho\overline{q^2v'}+\overline{p'v'}\right)\right]\frac{\partial h}{\partial\eta}$$

$$=\mu\left\{\overline{u_i'\nabla^2u_i'}+\frac{2}{h^3}\left[\left(\overline{u'\frac{\partial v'}{\partial\xi}}-\overline{v'\frac{\partial u'}{\partial\xi}}\right)\frac{\partial h}{\partial\eta}+\left(\overline{v'\frac{\partial u'}{\partial\eta}}-\overline{u'\frac{\partial v'}{\partial\eta}}\right)\frac{\partial h}{\partial\xi}\right]\right.$$

$$\left.+\frac{\overline{u'^2}+\overline{v'^2}}{h^3}\left[\left(\frac{\partial^2h}{\partial\xi^2}+\frac{\partial^2h}{\partial\eta^2}\right)-\frac{2}{h}\left(\frac{\partial h}{\partial\xi}\right)^2-\frac{2}{h}\left(\frac{\partial h}{\partial\eta}\right)^2\right]\right\} \qquad (2.4d)$$

where

$$\begin{cases}\sigma_1\equiv-\rho\,\overline{u'^2}\\ \sigma_2\equiv-\rho\,\overline{v'^2}\\ \tau\equiv-\rho\,\overline{u'v'}\end{cases} \qquad (2.5)$$

$$\begin{cases}\sigma\equiv\sigma_1-\sigma_2=-\rho(\overline{u'^2}-\overline{v'^2})\\ q^2\equiv\overline{u'^2}+\overline{v'^2}+\overline{w'^2}\end{cases} \qquad (2.6)$$

$$\nabla^2=\frac{1}{h^2}\frac{\partial^2}{\partial\xi^2}+\frac{1}{h^2}\frac{\partial^2}{\partial\eta^2}+\frac{\partial^2}{\partial z^2}$$

Energy dissipation by viscosity:

To separate the viscous terms in Eq. (2.4d) into the diffusion and dissipation terms, consider the flux of energy due to the working of the fluid stress

$$-\underset{\sim}{\sigma}\,\vec{u}=-(-p\underset{\sim}{I}+\underset{\sim}{\tau})\vec{u} \qquad (2.7)$$

The negative divergence of this gives the rate of energy change

$$\rho\left[\epsilon + \frac{D}{Dt}\left(\tfrac{1}{2}\vec{u}^2\right)\right] = \operatorname{div}(-p\underset{\sim}{I} + \underset{\sim}{\tau})\vec{u}$$

$$= -\vec{u}\cdot\vec{\nabla}p + \vec{u}\cdot\operatorname{div}\underset{\sim}{\tau} + (\underset{\sim}{\tau}\,\vec{\nabla})\cdot\vec{u} \qquad (2.8)$$

By using the equation of motion, it follows that rate of energy dissipation is

$$\begin{aligned} \epsilon &= \frac{1}{\rho}(\underset{\sim}{\tau}\,\vec{\nabla})\cdot\vec{u} \\ \underset{\sim}{\tau} &= \mu\,\underset{\sim}{e} \end{aligned} \qquad (2.9)$$

where

$$\vec{\nabla} \equiv \vec{i}_\xi \frac{1}{h}\frac{\partial}{\partial\xi} + \vec{i}_\eta \frac{1}{h}\frac{\partial}{\partial\eta} + \vec{k}\frac{\partial}{\partial z}\,.$$

The components of the rate of-strain tensor $\underset{\sim}{e}$ are

$$\left\{\begin{aligned} e_{11} &= 2\left(\frac{1}{h}\frac{\partial u}{\partial\xi} + \frac{v}{h^2}\frac{\partial h}{\partial\eta}\right) \\ e_{22} &= 2\left(\frac{1}{h}\frac{\partial v}{\partial\eta} + \frac{u}{h^2}\frac{\partial h}{\partial\xi}\right) \\ e_{33} &= 2\frac{\partial w}{\partial z} \\ e_{12} &= e_{21} = \frac{\partial}{\partial\xi}\left(\frac{v}{h}\right) + \frac{\partial}{\partial\eta}\left(\frac{u}{h}\right) \\ e_{23} &= e_{32} = \frac{1}{h}\frac{\partial w}{\partial\eta} + \frac{\partial v}{\partial z} \\ e_{31} &= e_{13} = \frac{\partial u}{\partial z} + \frac{1}{h}\frac{\partial w}{\partial\xi} \end{aligned}\right. \qquad (2.10)$$

Therefore, the time-averaged viscous dissipation due to turbulent fluctuations is given by

$$\varepsilon' = \nu\left\{\frac{1}{h_j^2}\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} + \frac{1}{h_i}\frac{\partial}{\partial x_i}\left(\frac{1}{h_i h_j^2}\frac{\partial}{\partial x_j} h_i h_j \overline{u_i' u_j'}\right)\right.$$
$$- \frac{1}{h^3}\left[\left(\overline{u'\frac{\partial u'}{\partial \xi}} + \overline{v'\frac{\partial v'}{\partial \xi}}\right)\frac{\partial h}{\partial \xi} + \left(\overline{u'\frac{\partial u'}{\partial \eta}} + \overline{v'\frac{\partial v'}{\partial \eta}}\right)\frac{\partial h}{\partial \eta}\right]$$
$$\left. + \frac{1}{h}\left[\overline{w'\frac{\partial u'}{\partial z}}\frac{\partial h}{\partial \xi} + \overline{w'\frac{\partial v'}{\partial z}}\frac{\partial h}{\partial \eta}\right]\right\} \tag{2.11}$$

where $\begin{cases} x_i = (\xi, \eta, z) \\ h_i = (h, h, 1) \end{cases}$

Eq. (2.4d) and (2.11) contain complicated viscous terms. Viscous effect, however, is significant only in 1) a thin layer near the wall; 2) the small dissipative eddies, i.e. the small-scale structure of the turbulence. The characteristic scales of the small eddies may be represented by the Kolmogorov microscales of length, time and velocity defined as

$$\eta_o \equiv \left(\frac{\nu^3}{\varepsilon'}\right)^{\frac{1}{4}}, \quad t_o \equiv \left(\frac{\nu}{\varepsilon'}\right)^{\frac{1}{2}}, \quad v_o \equiv (\nu\varepsilon')^{\frac{1}{4}} \tag{2.12}$$

which corresponds to $\mathcal{R} \equiv \frac{v_o \eta_o}{\nu} = 1$.

In turbulent shear flow at large Reynolds number, away from the wall, production and dissipation of energy are nearly of the same order of magnitude, though they may not exactly balance. The energy production, on the other hand, is dictated by the deformation of the large-scale structure which has length scale ℓ and velocity scale u_*. For turbulent boundary layer, away from the wall, $\ell \sim \delta$, and $u_* \sim u_\tau$, the friction velocity. Hence

$$\varepsilon' \sim \frac{u_*^3}{\ell} \sim \frac{u_\tau^3}{\delta} \tag{2.13}$$

From Eq. (2.12) and (2.13), we obtain

$$\begin{cases} \dfrac{\eta_o}{\delta} \sim \left(\dfrac{u_\tau \delta}{\nu}\right)^{-\frac{3}{4}} = \left(R\,\dfrac{u_\tau}{U_\infty}\right)^{-\frac{3}{4}} \\ \left(\dfrac{v_o}{u_\tau}\right) \sim \left(\dfrac{u_\tau \delta}{\nu}\right)^{-\frac{1}{4}} = \left(R\,\dfrac{u_\tau}{U_\infty}\right)^{-\frac{1}{4}} \end{cases} \tag{2.14}$$

where

$$R \equiv \frac{U_\infty \delta}{\nu}, \qquad u_\tau = \left(\frac{\tau_w}{\rho}\right)^{\frac{1}{2}}$$

These scale relations show that

$$\frac{\overline{\left(u_i' \dfrac{\partial u_j'}{\partial x_i}\right)} \dfrac{1}{h} \dfrac{\partial h}{\partial x_j}}{\overline{\left(\dfrac{\partial u_i'}{\partial x_j}\right)^2}} \sim \frac{\dfrac{u_\tau}{\delta} \dfrac{v_o}{\eta_o}}{\left(\dfrac{v_o}{\eta_o}\right)^2} = \left(\frac{u_\tau}{v_o}\right)\left(\frac{\eta_o}{\delta}\right) \sim \left(R\,\frac{u_\tau}{U_\infty}\right)^{-\frac{1}{2}}$$

Near the wall, the variations of fluid velocities with respect to the distance from the wall, from the non-slip condition and continuity, are given by

$$\begin{cases} u \sim \eta \\ v \sim \eta^2 \qquad \text{as} \quad \eta \to 0\,. \\ w \sim \eta \end{cases}$$

Thus, in Eq. (2.11), the leading term is $\overline{\left(\dfrac{\partial u_i}{\partial x_j}\right)^2} \sim \overline{\left(\dfrac{\partial u'}{\partial \eta}\right)_o^2}$, while all other terms are at least of order η or higher. In Eq. (2.4d), we may retain the leading term $\overline{u_i' \nabla^2 u_i'}$ only, since all other viscous terms vary at least as η^2.

Therefore, for large Reynolds number, we have

$$\varepsilon' \simeq \frac{\nu}{h_j^2} \overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} \tag{2.15}$$

and hence

$$\nu(\overline{u_i' \nabla^2 u_i'}) = \frac{\nu}{h^2}\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right)\frac{1}{2}\overline{q^2} - \frac{\nu}{h_j^2}\left(\overline{\frac{\partial u_i'}{\partial x_j}}\right)^2$$

$$\simeq \frac{\nu}{h^2}\frac{\partial^2}{\partial\eta^2}\left(\frac{1}{2}\overline{q^2}\right) - \varepsilon' . \qquad (2.16)$$

In the conservation of momentum, viscous effect is important only near the wall. By the same argument of boundary layer approximation, we may retain only the dominant term $\frac{\partial^2 \overline{u}}{\partial \eta^2}$ in ξ momentum equation.

To use the energy equation (2.4d) as an equation for the stresses, Bradshaw's hypothesis that the turbulent motion is uniquely related to the shear stress is applied to define

$$\begin{cases} b \equiv \dfrac{\tau}{\frac{1}{2}\rho\overline{q^2}} \\ \mathcal{L} \equiv \dfrac{(\tau/\rho)^{\frac{3}{2}}}{\varepsilon'} \\ G \equiv \left(\dfrac{\overline{p'v'}}{\rho} + \frac{1}{2}\overline{q^2 v'}\right)\Big/\left(\dfrac{\tau_m}{\rho}\right)^{\frac{1}{2}}\dfrac{\tau}{\rho} \\ H \equiv \left(\dfrac{\overline{p'u'}}{\rho} + \frac{1}{2}\overline{q^2 u'}\right)\Big/\left(\dfrac{\tau_m}{\rho}\right)^{\frac{1}{2}}\dfrac{\tau}{\rho} \end{cases} \qquad (2.17)$$

where τ_m is the maximum of τ. b, G, H are dimensionless functions depending in general on the behavior of the Reynolds stress. $\mathcal{L}$ has the dimension of length, and is the most important of the four functions.

<u>Governing equations in dimensionless form:</u>

By dividing the lengths by the undisturbed boundary-layer thickness δ, the velocity by U_∞, the pressure and stresses by ρU_∞^2, the equations are made dimensionless as

Continuity:

$$\frac{1}{h}\frac{\partial \overline{u}}{\partial \xi} + \frac{1}{h}\frac{\partial \overline{v}}{\partial \eta} + \frac{\overline{u}}{h}\frac{\partial h}{\partial \xi} + \frac{\overline{v}}{h}\frac{\partial h}{\partial \eta} = 0 \tag{2.18a}$$

Momentum:

$$\frac{\overline{u}}{h}\frac{\partial \overline{u}}{\partial \xi} + \frac{\overline{v}}{h}\frac{\partial \overline{u}}{\partial \eta} + \left(\frac{\overline{u}\,\overline{v}}{h^2}\frac{\partial h}{\partial \eta} - \frac{\overline{v}^2}{h^2}\frac{\partial h}{\partial \xi}\right)$$

$$= \frac{1}{h}\frac{\partial \overline{p}}{\partial \xi} + \frac{1}{h}\left(\frac{\partial \sigma_1}{\partial \xi} + \frac{\partial \tau}{\partial \eta}\right) + \frac{1}{h^2}\left(\sigma\frac{\partial h}{\partial \xi} + 2\tau\frac{\partial h}{\partial \eta}\right) + \frac{1}{R}\frac{1}{h^2}\frac{\partial^2 \overline{u}}{\partial \eta^2} \tag{2.18b}$$

$$\frac{\overline{u}}{h}\frac{\partial \overline{v}}{\partial \xi} + \frac{\overline{v}}{h}\frac{\partial \overline{v}}{\partial \eta} + \left(\frac{\overline{u}\,\overline{v}}{h^2}\frac{\partial h}{\partial \xi} - \frac{\overline{u}^2}{h^2}\frac{\partial h}{\partial \eta}\right)$$

$$= -\frac{1}{h}\frac{\partial \overline{p}}{\partial \eta} + \frac{1}{h}\left(\frac{\partial \tau}{\partial \xi} + \frac{\partial \sigma_2}{\partial \eta}\right) + \frac{1}{h^2}\left(2\tau\frac{\partial h}{\partial \xi} - \sigma\frac{\partial h}{\partial \eta}\right) \tag{2.18c}$$

Energy:

$$\frac{\overline{u}}{h}\frac{\partial}{\partial \xi}\left(\frac{\tau}{b}\right) + \frac{\overline{v}}{h}\frac{\partial}{\partial \eta}\left(\frac{\tau}{b}\right) - \frac{1}{h}\left[\tau\left(\frac{\partial \overline{u}}{\partial \eta} + \frac{\partial \overline{v}}{\partial \xi}\right) - \sigma\frac{\partial \overline{v}}{\partial \eta}\right]$$

$$+ \frac{1}{h}\left[\frac{\partial}{\partial \xi}\left(\tau_m^{\frac{1}{2}} H\tau\right) + \frac{\partial}{\partial \eta}\left(\tau_m^{\frac{1}{2}} G\tau\right)\right] + \frac{\tau^{\frac{3}{2}}}{\mathcal{L}}$$

$$+ \frac{1}{h^2}\left[\left(\tau_m^{\frac{1}{2}} H\tau + \overline{u}\sigma + \overline{v}\tau\right)\frac{\partial h}{\partial \xi} + \left(\tau_m^{\frac{1}{2}} G\tau + \overline{u}\tau\right)\frac{\partial h}{\partial \eta}\right]$$

$$= \frac{1}{R}\frac{1}{h^2}\frac{\partial^2}{\partial \eta^2}\left(\frac{\tau}{b}\right) \tag{2.18d}$$

III. SMALL PERTURBATIONS

For a wavy surface of small amplitude given by $y_w = a \cos\alpha x$ with wavelength λ being comparable to the boundary layer thickness, the non-dimensional wall curvature may be approximated by

$$
\begin{aligned}
K \simeq \frac{d^2 y_w}{dx^2} &= -(a\alpha)\alpha \cos\alpha x \\
a\,\alpha &\equiv \frac{2\pi a}{\lambda} \\
\alpha &\equiv \frac{2\pi\delta}{\lambda}
\end{aligned}
\qquad (3.1)
$$

Since $\alpha = O(1)$, K is of order $a\alpha$. In view of a well-known property of boundary layer flow along curved wall (Goldstein 1938), we expect the changes in the flow profile at different positions along the wall are of order $a\alpha$. The flow pattern may then be approximated as the sum of the undisturbed primary flow, and disturbed flows described by perturbations of order $a\alpha$. For a small but finite disturbance, we assume

$$
\left\{
\begin{aligned}
\bar{u} &= U(\eta) + (a\alpha)u' + (a\alpha)^2 u'' \\
\bar{v} &= V(\eta) + (a\alpha)v' + (a\alpha)^2 v'' \\
\bar{p} &= P_o + (a\alpha)p' + (a\alpha)^2 p'' \\
\tau &= T(\eta) + (a\alpha)\tau' + (a\alpha)^2 \tau'' \\
\sigma_1 &= \sigma_1(\eta) + (a\alpha)\sigma_1' + (a\alpha)^2 \sigma_1'' \\
\sigma_2 &= \sigma_2(\eta) + (a\alpha)\sigma_2' + (a\alpha)^2 \sigma_2''
\end{aligned}
\right.
\qquad (3.2)
$$

Following the same argument, we may further assume that, in the first approximation, the stress-energy ratio b, and the diffusion functions G and H are universal functions of η only. The normal velocity $V(\eta)$ may be neglected, since $V << 1$, for $R >> 1$. The shear stress is expected to be a maximum at the edge of viscous sublayer. Thus

$$\begin{aligned} \tau_m^{\frac{1}{2}} &\simeq (T_e + a\alpha\, \tau_e')^{\frac{1}{2}} \\ &\simeq T_m^{\frac{1}{2}} + a\alpha\, \tfrac{1}{2}\, \frac{\tau_e'(\xi)}{\tau_m^{\frac{1}{2}}} \end{aligned} \tag{3.3}$$

In the wall region, we assume that $\mathcal{L}$ is proportional to the normal distance from the wall, i.e.

$$\begin{aligned} \mathcal{L} &= \kappa \int_0^{\eta} h(\xi, \eta')d\eta' \\ &= h\,\kappa\eta + O(a\alpha^2\eta^2) \qquad \text{as} \quad \eta \to 0 \end{aligned}$$

Let $L(\eta)$ be the corresponding dissipation parameter for the primary undisturbed flow. Thus for simplicity, it may be assumed that

$$\mathcal{L} \simeq h\,L(\eta) \tag{3.4}$$

The governing equations (2.18 a, b, c, d) are then linearized as follows:

Linearized Equations: $O(a\alpha)$

Continuity:

$$\frac{\partial u'}{\partial \xi} + \frac{\partial v'}{\partial \eta} = i\alpha U e^{-\alpha\eta}\, e^{i\alpha\xi} \tag{3.5a}$$

Momentum:

$$\xi:\ U\frac{\partial u'}{\partial \xi} + v'\frac{dU}{d\eta} + \frac{\partial p'}{\partial \xi} - \left(\frac{\partial \sigma_1'}{\partial \xi} + \frac{\partial \tau'}{\partial \eta}\right) - \frac{1}{R}\frac{\partial^2 u'}{\partial \eta^2}$$

$$= \left\{-i\alpha(\sigma + 2iT) + \frac{1}{R}\frac{d^2 U}{d\eta^2}\right\} e^{-\alpha\eta}\, e^{i\alpha\xi} \tag{3.5b}$$

$$\eta:\ U\frac{\partial v'}{\partial \xi} + \frac{\partial p'}{\partial \eta} - \left(\frac{\partial \tau'}{\partial \xi} + \frac{\partial \sigma_2'}{\partial \eta}\right)$$

$$= \left\{\alpha U^2 - \alpha(\sigma + 2iT)\right\} e^{-\alpha\eta}\, e^{i\alpha\xi} \tag{3.5c}$$

Energy:

$$\left[U\frac{\partial}{\partial \xi}\left(\frac{\tau'}{b}\right) + v'\frac{d}{d\eta}\left(\frac{T}{b}\right)\right] - \left[T\left(\frac{\partial u'}{\partial \eta} + \frac{\partial v'}{\partial \xi}\right) + \tau'\frac{dU}{d\eta} - \sigma\frac{\partial v'}{\partial \eta}\right]$$

$$+ \frac{\partial}{\partial \xi} H\left(T_m^{\frac{1}{2}}\tau' + \frac{T\tau_e'}{2T_m^{\frac{1}{2}}}\right) + \frac{\partial}{\partial \eta} G\left(T_m^{\frac{1}{2}}\tau' + \frac{T\tau_e'}{2T_m^{\frac{1}{2}}}\right)$$

$$+ \frac{3}{2}\frac{T^{\frac{1}{2}}}{\mathcal{L}}\tau' - \frac{1}{R}\frac{\partial^2}{\partial \eta^2}\left(\frac{\tau'}{b}\right)$$

$$= \left\{i\alpha U(\sigma + iT) - \alpha T_m^{\frac{1}{2}} T(G - iH) + \frac{1}{R}\frac{d^2}{d\eta^2}\left(\frac{T}{b}\right)\right\} e^{-\alpha\eta}\, e^{i\alpha\xi}$$

(3.5d)

Governing Equations for $O((a\alpha)^2)$:

Continuity:

$$\frac{\partial u''}{\partial \xi} + \frac{\partial v''}{\partial \eta} = \alpha U e^{-2\alpha\eta} \sin 2\alpha\xi - \alpha e^{-\alpha\eta}(u' \sin\alpha\xi + v' \cos\alpha\xi) \tag{3.6a}$$

Momentum:

$$\xi:\ U\frac{\partial u''}{\partial \xi} + v''\frac{dU}{d\eta} + \frac{\partial p''}{\partial \xi} - \left(\frac{\partial \sigma_1''}{\partial \xi} + \frac{\partial \tau''}{\partial \eta}\right) - \frac{1}{R}\frac{\partial^2 u''}{\partial \eta^2}$$

$$= -\alpha e^{-2\alpha\eta}\{2T\cos 2\alpha\xi + \sigma \sin 2\alpha\xi\}$$

$$+ \alpha e^{-\alpha\eta}\{2\tau' \cos\alpha\xi + \sigma' \sin\alpha\xi - U v' \cos\alpha\xi\}$$

$$- \left(u'\frac{\partial u'}{\partial \xi} + v'\frac{\partial u'}{\partial \eta}\right)$$

$$+ \frac{1}{R}\left\{\frac{1}{2}\frac{d^2 U}{d\eta^2} e^{-2\alpha\eta}\sin^2\alpha\xi + \frac{\partial^2 u'}{\partial \eta^2} e^{-\alpha\eta}\cos\alpha\xi\right\} \tag{3.6b}$$

$$\eta:\ U\frac{\partial v''}{\partial \xi} + \frac{\partial p''}{\partial \eta} - \left(\frac{\partial \tau''}{\partial \xi} + \frac{\partial \sigma_2''}{\partial \eta}\right)$$

$$= -\alpha e^{-2\alpha\eta}\{U^2\cos 2\alpha\xi + 2T\sin 2\alpha\xi - \sigma\cos 2\alpha\xi\}$$

$$+ \alpha e^{-\alpha\eta}\{2U u' \cos\alpha\xi - U v' \sin\alpha\xi + 2\tau' \sin\alpha\xi$$

$$- \sigma' \cos\alpha\xi\}$$

$$- \left(u'\frac{\partial v'}{\partial \xi} + v'\frac{\partial v'}{\partial \eta}\right) \tag{3.6c}$$

Energy:

$$\left[U\,\frac{\partial}{\partial\xi}\left(\frac{\tau''}{b}\right)+v''\,\frac{d}{d\eta}\left(\frac{T}{b}\right)\right]-\left[T\left(\frac{\partial u''}{\partial\eta}+\frac{\partial v''}{\partial\xi}\right)+\tau''\frac{dU}{d\eta}-\sigma\frac{\partial v''}{\partial\eta}\right]$$

$$+\frac{\partial}{\partial\eta}\left(G\,T_m^{\frac{1}{2}}\,\tau''\right)+\frac{3}{2}\,\frac{T^{\frac{1}{2}}}{\mathscr{L}}\,\tau''-\frac{1}{R}\,\frac{\partial^2}{\partial\eta^2}\left(\frac{\tau''}{b}\right)$$

$$=\alpha e^{-2\alpha\eta}\{UT\cos 2\alpha\xi+U\sigma\sin 2\alpha\xi+G\,T_m^{\frac{1}{2}}\,T\,\cos 2\alpha\xi\}$$

$$-\alpha e^{-\alpha\eta}\{(U\tau'+u'T)\cos\alpha\xi+Tv'\sin\alpha\xi+(U^2\sigma'+u'\sigma)\sin\alpha\xi$$

$$+G\left(G_m^{\frac{1}{2}}\,\tau'+\frac{1}{2}\,\frac{T}{T_m^{\frac{1}{2}}}\,\tau_e'\right)\cos\alpha\xi$$

$$-\left\{\left[u'\,\frac{\partial}{\partial\xi}\left(\frac{\tau'}{b}\right)+v'\,\frac{\partial}{\partial\eta}\left(\frac{\tau'}{b}\right)\right]-\left[\tau'\left(\frac{\partial u'}{\partial\eta}+\frac{\partial v'}{\partial\xi}\right)-\sigma'\,\frac{\partial v'}{\partial\eta}\right]\right.$$

$$\left.+\frac{\partial}{\partial\eta}\,G\left(\frac{\tau_e'\,\tau'}{2\,T_m^{\frac{1}{2}}}-\frac{T\,\tau_e'^2}{8\,T_m^{\frac{3}{2}}}\right)+\frac{3}{8}\,\frac{T'^2}{\mathscr{L}\,T^{\frac{1}{2}}}\right\}$$

$$+\frac{1}{R}\left\{\frac{1}{2}\,\frac{d^2}{d\eta^2}\left(\frac{T}{b}\right)e^{-2\alpha\eta}\sin^2\alpha\xi+\frac{\partial^2}{\partial\eta^2}\left(\frac{\tau'}{b}\right)e^{-\alpha\eta}\cos\alpha\xi\right\}$$

(3.6d)

In order to solve these equations analytically, we need explicit expressions for the mean quantities of the primary flow which appear as coefficients and forcing functions. Not all of them are available. In addition, there is neither first principle nor well-accepted empirical functions to relate the normal Reynolds stresses σ_1 and σ_2, and the longitudinal diffusion H to the mean velocity as for the shear stress. There is still inherent difficulty of having more unknowns than equations.

For large Reynolds number and hence small skin friction turbulent boundary layer, a way out of the difficulty is suggested by the asymptotic-matched method to obtain just the main feature of physical importance. The small skin friction coefficient may be used as a basic parameter.

<u>Primary mean flow</u>:

Following Coles' empirical law of the wake, we assume that the mean flow profile has the following representation

$$U = \frac{\varepsilon}{\kappa}[\kappa F(\zeta) + \tilde{\pi}\, w(\eta)] \tag{3.7}$$

$$T = \varepsilon^2 T^* = \varepsilon^2\left[1 - \frac{dF}{d\zeta} - \frac{1}{\theta}\left(U\int_0^\eta U\,d\eta' - \int_0^\eta U^2 d\eta'\right)\right] \tag{3.8}$$

$$T_{max} \simeq \varepsilon^2 \tag{3.8a}$$

where $\epsilon = \dfrac{u_\tau}{U_\infty} = \left(\dfrac{C_f}{2}\right)^{\frac{1}{2}}$

$\kappa = 0.41,$ Karman constant

$\zeta = R\epsilon\eta,$ scale of viscous sublayer (3.9)

$\theta = \int_0^\infty U(1-U)\,d\eta$, dimensionless momentum thickness

$\tilde{\pi} = 0.55$ for constant-pressure layer.

$F(\zeta)$ is the law-of-the-wall profile, and

$$F(\zeta) \to \zeta \qquad \text{as} \quad \zeta \to 0$$

$$F(\zeta) \sim \frac{1}{\kappa}(\ln\zeta + B), \quad \text{for } \zeta \gg 1, \quad B = 2.05$$

An analytical approximation for this function obtained by Reichardt[11] is given by

$$\begin{aligned} F(\zeta) &= \int_0^\zeta \frac{d\zeta}{1+\kappa\left(\zeta - \zeta_o \tanh\frac{\zeta}{\zeta_o}\right)} \\ &\simeq \frac{1}{\kappa}\ln(\kappa\zeta + 1) + B'\left[1 - e^{-\zeta/\zeta_o} - \frac{\zeta}{\zeta_o}\, e^{-\frac{1}{3}\zeta}\right] \end{aligned} \tag{3.10}$$

where $\zeta_o \simeq 10$, $B' = \frac{1}{\kappa}(B - \ln\kappa)$

The wake function $w(\eta)$ may be approximated by

$$w(\eta) = 1 - \cos(\pi\eta), \qquad 0 \le 1, \tag{3.11}$$

At $\eta = 1$, Eq. (3.7) gives the skin friction law for the flat-surface

$$\frac{\epsilon}{\kappa}(\ln R\epsilon + B + 2\tilde{\pi}) = 1, \tag{3.12}$$

which shows, for large R,

$$\varepsilon \sim \frac{1}{\ln R\varepsilon} : \varepsilon \to 0 \text{ and } R\varepsilon \to \infty, \text{ as } R \to \infty .$$

Combination of Eq. (3.7) and (3.12) gives velocity profile in the defect form

$$U = 1 - \frac{\varepsilon}{\kappa} W^{*}(\eta; R\varepsilon) \tag{3.13}$$

$$W^{*} = -\kappa F(\zeta) + \ln R\varepsilon + B + \tilde{\pi}[2-w(\eta)] \tag{3.14}$$

In the outer region where $\eta = O(1)$,

$$W^{*} \sim W(\eta) = -\ln \eta + \tilde{\pi}(2-w) \tag{3.15}$$

where W is the velocity-defect profile.

In the inner region where $\zeta = O(1)$, we have

$$\begin{cases} U = \varepsilon F(\zeta) \\ T = \varepsilon^2 T^{*} = \varepsilon^2 [1-F'(\zeta)] \\ L = \dfrac{L^{*}}{R\varepsilon}, \end{cases} \tag{3.16}$$

and the energy equation approximated by

$$-T^{*} \frac{dF}{d\zeta} + \frac{T^{*\frac{3}{2}}}{L^{*}} = \frac{d^2}{d\zeta^2} \left(\frac{T^{*}}{b}\right) \tag{3.17}$$

For small ζ, the streamwise turbulent intensity $\overline{u'^2}$ varies as ζ^2. Therefore, the stress-energy ratio may be expressed as

$$b(\zeta) \equiv \frac{\tau}{\frac{1}{2}\rho\overline{q^2}} \simeq \frac{b_1\zeta^3 + O(\zeta^4)}{a_1\zeta^2 + a_2\zeta^3 + O(\zeta^4)} ,$$

then $\quad \frac{d^2}{d\zeta^2}\left(\frac{T^*}{b}\right) \simeq 2a_1 + O(\zeta) .$

Consequently, Eq. (3.17) or the definition of L requires

$$L^* \sim (\zeta^3)^{\frac{3}{2}} .$$

For large ζ, $b \to b_0$ and $L^* \to \kappa\zeta$, thus we assume

$$\begin{cases} b(\zeta) = b_0\left(\frac{\beta\zeta}{1+\beta\zeta}\right), & \beta = 0.03 \\ L^*(\zeta) = \kappa\zeta\,(1 - e^{-\gamma\zeta})^{\frac{7}{2}}, & \gamma = 0.1085 \end{cases} \tag{3.18}$$

where β is obtained by fitting experimental data measured by Klebanoff,[12] Schubauer[13] and Laufer.[14] The value of γ is obtained from Eq. (3.17) by balancing the molecular diffusion with the turbulent dissipation to the zeroth order ζ.

The difference of the normal stress σ appearing in the governing equations is as important as τ only in the outer region. In isotropic turbulence, $\overline{u^2} = \overline{v^2} = \overline{w^2}$, and $\overline{uv} = 0$. The turbulent shear stress $\overline{uv}$ can be produced only if the main flow is not uniform. Thus, the non-vanishing σ as well as τ is a measure of the degree of of anisotropy. Consequently, it is reasonable to relate

$$\sigma = \mathrm{fnc}\left(\tau, \frac{dU}{d\eta}, \Omega\right) \tag{3.19}$$

where Ω is the intermittency factor. Since τ itself is related to the velocity gradient through the energy production, we may simply assume

$$\sigma = \sigma_0(\eta;\ \Omega)\tau \tag{3.20}$$

In the outer region away from the wall,

$$\sigma_0 \simeq -2\,\Omega(\eta) \tag{3.20a}$$

is found to be a good approximation. The distribution of intermittency could be well described by the Gaussian error function

$$\Omega(\eta) = \tfrac{1}{2}(1 - \mathrm{erf}\,\eta^*)$$

$$\text{with} \quad \eta^* = 5(\eta - 0.78) \tag{3.21}$$

Analysis based on skin friction:

According to Bradshaw's[15] hypothesis, the turbulent diffusion of energy is mainly affected by the large eddies, at least in the outer part of the boundary layer. By correlating the vertical diffusion to the entrainment rate, he has shown that

$$G \simeq \left(\frac{\tau_m}{\rho U_\infty^2}\right)^{\frac{1}{2}} G^*(\eta) \simeq \varepsilon\, G^*(\eta) \tag{3.22}$$

where $G^*(\eta) \to 0$ very rapidly as $\eta \to 0$. The streamwise diffusion function H is not known. It is, however, at most of order unity by its definition. Then in the energy equation, we have, for the outer region,

$$U \frac{\partial}{\partial \xi}\left(\frac{\tau'}{b}\right) = O(\tau')$$

$$\frac{\partial}{\partial \xi} H\left(T_m^{\frac{1}{2}}\tau' + \frac{T\,\tau_e'}{2\,T_m^{\frac{1}{2}}}\right) \leq O(\varepsilon) \times O(\tau')$$

$$\frac{\partial}{\partial \eta} G\left(T_m^{\frac{1}{2}}\tau' + \frac{T\,\tau_e'}{2\,T_m^{\frac{1}{2}}}\right) = O(\varepsilon^2) \times O(\tau')$$

$$G\, T_{max}^{\frac{1}{2}}\, T = O(\epsilon^4)$$

In the inner region, on the other hand, the streamwise variation is negligible. Thus, the unknown longitudinal diffusion may be neglected. Furthermore, for a solution to the accuracy of order ϵ, $\epsilon \to 0$, the vertical diffusion term plays no role in the perturbed energy transport processes.

At this stage, it may seem inconsistent to the derivation of G, which is obtained by balancing advection and diffusion. Further experience with the present analysis may make it necessary to describe diffusion processes well inside the boundary layer by inserting a gradient-diffusion term, such as

$$\frac{\overline{pv}}{\rho} + \frac{1}{2}\overline{q^2 v} \propto \eta \frac{\partial}{\partial \eta}\left(\frac{T}{\rho}\right)^{\frac{3}{2}} \quad .$$

Gradient-diffusion is not important, or at most the same order of magnitude as the transport diffusion, in the ordinary boundary layer because well inside the layer but away from the immediate neighborhood of the wall the turbulent shear stress is fairly constant. On the contrary, the presence of the wavy surface is expected to induce an appreciable change to the turbulent stresses due to convection and streamwise pressure gradient which make the state of local equilibrium impossible.

The following analytical solutions are constructed by means of asymptotic-matched expansions up to order of ϵ. Contribution from terms of order ϵ^2 and higher involving the transport diffusion G, the variation of boundary layer thickness δ, etc., will be neglected.

IV. LINEAR FIRST ORDER SOLUTION

The first order linearized governing equations (3.5a, b, c, d) contain forcing functions proportional to $e^{i\alpha\xi}$. The boundary conditions are such that all perturbations vanish as $\eta \to \infty$, and at $\eta = 0$, $u' = v' = \tau' = 0$ by non-slip condition. It may be shown that the homogeneous part of the differential equations does not constitute an eigenvalue problem. Therefore, by the uniqueness theorem of the linear differential equation, we expect

$$\begin{Bmatrix} u' \\ v' \\ p' \\ \tau' \end{Bmatrix} = \begin{Bmatrix} \tilde{u}(\eta) \\ i\tilde{v}(\eta) \\ \tilde{p}(\eta) \\ \tilde{\tau}(\eta) \end{Bmatrix} e^{i\alpha\xi} \qquad \text{etc.} \tag{4.1}$$

Then the governing equations become

$$\alpha\tilde{u} + \frac{d\tilde{v}}{d\eta} = \alpha U e^{-\alpha\eta} \tag{4.2a}$$

$$i\alpha\left(U\tilde{u} + \frac{\tilde{v}}{\alpha}\frac{dU}{d\eta} + \tilde{p} - \tilde{\sigma}_1\right) - \frac{d\tilde{\tau}}{d\eta} - \frac{1}{R}\frac{d^2\tilde{u}}{d\eta^2}$$

$$= \left[-i\alpha(\sigma + 2iT) + \frac{1}{R}\frac{d^2U}{d\eta^2}\right]e^{-\alpha\eta} \tag{4.2b}$$

$$\frac{d\tilde{p}}{d\eta} - \alpha U\tilde{v} - i\alpha\tilde{\tau} - \frac{d\tilde{\sigma}_2}{d\eta} = \alpha[U^2 - (\sigma + 2iT)]e^{-\alpha\eta} \tag{4.2c}$$

$$i\alpha\left[U\left(\frac{\tilde{\tau}}{b}\right) + \frac{\tilde{v}}{\alpha}\frac{d}{d\eta}\left(\frac{T}{b}\right)\right] - \left[T\left(\frac{d\tilde{u}}{d\eta} - \alpha\tilde{v}\right) + \tilde{\tau}\frac{dU}{d\eta} - i\sigma\frac{d\tilde{v}}{d\eta}\right]$$

$$+ \frac{3}{2}\frac{T^{\frac{1}{2}}}{L}\tilde{\tau} - \frac{1}{R}\frac{d^2}{d\eta^2}\left(\frac{\tilde{\tau}}{b}\right)$$

$$= \left[i\alpha U(\sigma + iT) + \frac{1}{R}\frac{d^2}{d\eta^2}\left(\frac{T}{b}\right)\right]e^{-\alpha\eta} \tag{4.2d}$$

1. Outer Layer $\eta = O(1)$:

Using the defect form (3.13) for U, and the relation for σ and τ (3.20), i.e.

$$\begin{cases} U = 1 - \frac{\varepsilon}{\kappa} W(\eta) \\ T = \varepsilon^2 T^*(\eta) \\ \sigma = \sigma_0 T = \varepsilon^2 \sigma_0 T^*(\eta), \end{cases} \tag{4.3}$$

it can be shown from the energy equation (4.2d) that the perturbation Reynolds stress $\tilde{\tau}$ is of order ε^2. The normal stresses $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are expected to be the same order as $\tilde{\tau}$. Therefore, for $R >> 1$, $\varepsilon \to 0$, to the accuracy of order ε, the outer layer velocity and pressure perturbations are practically "inviscid."

$$\begin{cases} \alpha\tilde{u} + \frac{d\tilde{v}}{d\eta} = \alpha(1 - \frac{\varepsilon}{\kappa} W)e^{-\alpha\eta} \\ \tilde{u} + \tilde{p} = \frac{\varepsilon}{\kappa}(W\tilde{u} + \frac{1}{\alpha}\frac{dW}{d\eta}\tilde{v}) + O(\varepsilon^2) \\ \frac{d\tilde{p}}{d\eta} - \alpha\tilde{v} = \alpha e^{-\alpha\eta} - \frac{\varepsilon}{\kappa}(2\alpha e^{-\alpha\eta} + \alpha\tilde{v})W + O(\varepsilon^2) \end{cases} \tag{4.4}$$

Expanding

$$\tilde{u} = \tilde{u}_0(\eta) + \frac{\varepsilon}{\kappa}\tilde{u}_1(\eta) + --- , \text{ etc.} \tag{4.5}$$

then, by substituting Eq. (4.5) into Eq. (4.4) and collecting the like order of ε, we get

$$\begin{cases} \alpha \tilde{u}_i + \dfrac{d\tilde{v}_i}{d\eta} = f_i; \; f_0 = \alpha e^{-\alpha\eta}, \; f_1 = -\alpha W e^{-\alpha\eta} \\ \tilde{u}_i + \tilde{p}_i = g_i; \quad g_0 = 0, \quad g_1 = W\tilde{u}_0 + \dfrac{1}{\alpha}\dfrac{dW}{d\eta}\tilde{v}_0 \\ \dfrac{d\tilde{p}_i}{d\eta} - \alpha\tilde{v}_i = h_i; \; h_0 = \alpha e^{-\alpha\eta}, \; h_1 = -\alpha(2e^{-\alpha\eta} + \tilde{v}_0)W \end{cases} \tag{4.6}$$

Solution to these equations which vanishes as $\eta \to \infty$ is

$$\tilde{u} \simeq e^{-\alpha\eta}\left\{1 + A_0 + \frac{\epsilon}{\kappa}\left[A_1 + (1+A_0)(W - E_1) + \frac{A_0}{\alpha}\frac{dW}{d\eta}\right] + \cdots\right\} \tag{4.7a}$$

$$\tilde{v} \simeq e^{-\alpha\eta}\left\{A_0 + \frac{\epsilon}{\kappa}\left[A_1 - A_0 W + (1+A_0)E_1\right] + \cdots\right\} \tag{4.7b}$$

$$\tilde{p} \simeq -e^{-\alpha\eta}\left\{1 + A_0 + \frac{\epsilon}{\kappa}\left[A_1 - (1+A_0)E_1\right] + \cdots\right\} \tag{4.7c}$$

where

$$E_1(\eta;\alpha) = 2\alpha e^{2\alpha\eta}\int_\eta^\infty W e^{-2\alpha\eta'}\, d\eta' \tag{4.8}$$

In the energy balance (4.2d), the coefficient function of the dissipation term $T^{\frac{1}{2}}/L$ may well be approximated by $U'(\eta)$ in this region. With the velocity perturbations described by Eq. (4.7a, b), it follows:

$$\begin{aligned} \frac{\tilde{T}}{T} \simeq b_0 e^{-\alpha\eta}\Big\{&\sigma_0 + 2i + \frac{\epsilon}{\kappa}\Big[(A_1 + 2W - E_1)\sigma_0 + 2i(A_1 + E_1) \\ &- \frac{ib_0}{2\alpha}\left(\sigma_0 + 2i + \frac{2}{b_0}\right)\frac{dW}{d\eta} - \frac{A_1 + E_1}{b_0\alpha T}\frac{dT}{d\eta}\Big] \\ &+ A_0\left[\sigma_0 + 2i + O\left(\frac{\epsilon}{\kappa}\right)\right] + \cdots\Big\} \end{aligned}$$

2. Intermediate expansions:

Energy and momentum convection dominate the outer region. In the inner region very close to the wall, local production, dissipation and molecular diffusion of turbulent energy balance one another. Somewhere in between, there must be an intermediate region where energy convection, production and dissipation are of the same importance.

If we let

$$\eta = \phi(\epsilon)\hat{\eta}\,; \quad \phi(\epsilon) \to 0, \quad \text{as} \quad \epsilon \to 0$$
$$\tilde{\tau} = g(\epsilon)\,\hat{\tau}\,; \quad g(\epsilon) \to 0, \quad \text{as} \quad \epsilon \to 0 \qquad (4.9)$$

then continuity equation requires $\tilde{v} \sim \phi(\epsilon)$. The dominant terms of the energy equation (4.2d) are

$$g(\epsilon)\left\{\frac{i\alpha}{b} + \frac{\epsilon}{\phi}\,\frac{1}{\kappa}\,\frac{dW}{d\hat{\eta}} + \frac{\epsilon}{\phi}\,\frac{3}{2}\,\frac{T^{*\frac{1}{2}}}{\mathcal{L}}\right\}\hat{\tau} - \frac{\epsilon^2}{\phi}\,T^*\,\frac{d\tilde{u}}{d\hat{\eta}}$$
$$= \epsilon^2\, i\alpha U(\sigma_0 + 2i)T^* \qquad (4.10)$$

Therefore the intermediate scale factor is

$$\phi(\epsilon) = \epsilon, \qquad \text{and} \qquad \eta = \epsilon\,\hat{\eta} \qquad (4.11)$$

If $g(\epsilon) = \epsilon$, convection and pressure gradient would be balanced by Reynolds stress in the momentum equation. It will be clear, however, in the following analysis that in order to match both the outer and inner expansions $g(\epsilon)$ must be ϵ^2.

In terms of $\hat{\eta}$, the primary mean flow profile is

$$U = 1 - \frac{\varepsilon}{\kappa} W(\hat{\eta};\varepsilon) \tag{4.12}$$

$$W \simeq -\ln\varepsilon - \ln\hat{\eta} + 2\widetilde{\pi}\,[1 + O(\varepsilon^2\hat{\eta}^2)] \tag{4.13}$$

$$T^* \simeq 1 - \frac{\varepsilon}{\kappa}\left\{\frac{\Delta}{\alpha}\,\frac{1}{\hat{\eta}} + \frac{\kappa}{\theta^*}\,\hat{\eta}\,[1 - \frac{\varepsilon}{\kappa}(W+2)] + \text{---}\right\} \tag{4.14}$$

$$T^{*\frac{1}{2}}/\hat{L} \simeq -\frac{1}{\kappa}\,\frac{dW}{d\hat{\eta}} \simeq \frac{1}{\kappa\hat{\eta}} \tag{4.15}$$

where

$$\theta^* \equiv \frac{\kappa}{\varepsilon}\,\theta = \int_0^1 W(1 - \frac{\varepsilon}{\kappa}\,W)\,d\eta$$

$$\Delta \equiv \frac{\alpha}{R\varepsilon^3} \leq O(1), \quad \text{(see inner layer Eq. 4.28)}$$

Thus, we let

$$\left.\begin{aligned}
\widetilde{u} &= \hat{u}(\hat{\eta};\varepsilon)\\
\widetilde{v} &= \varepsilon\hat{v}(\hat{\eta};\varepsilon)\\
\widetilde{p} &= \hat{p}(\hat{\eta};\varepsilon)\\
\widetilde{\tau} &= \varepsilon^2\,\hat{\tau}(\hat{\eta};\varepsilon)
\end{aligned}\right\} \tag{4.16}$$

Then the governing equations become

$$\alpha\hat{u} + \frac{d\hat{v}}{d\hat{\eta}} = \alpha\,U\,e^{-\varepsilon\alpha\hat{\eta}} \tag{4.17a}$$

$$i\alpha\left[U\hat{u} + \hat{p} + \frac{\varepsilon}{\kappa}\,\frac{\hat{v}}{\alpha\hat{\eta}}\right] = \varepsilon\frac{d\hat{\tau}}{d\hat{\eta}} + \frac{\varepsilon\Delta}{\alpha}\left[\frac{d^2\hat{u}}{d\hat{\eta}^2} - \frac{\varepsilon}{\kappa}\,W''(\hat{\eta})\right] + O(\varepsilon^2) \tag{4.17b}$$

$$\frac{d\hat{p}}{d\hat{\eta}} = \varepsilon\alpha U^2\,e^{-\varepsilon\alpha\hat{\eta}} + O(\varepsilon^2) \tag{4.17c}$$

$$- \frac{T^*}{\epsilon} \frac{d\hat{u}}{d\hat{\eta}} + \left(\frac{i\alpha U}{b_o} + \frac{1}{2K\hat{\eta}} \right) \hat{\tau} + i\,\sigma_o\, T^* \frac{d\hat{v}}{d\hat{\eta}} + \epsilon\alpha T^* \hat{v}$$

$$= i\alpha U T^* (\sigma_o + i) e^{-\epsilon\alpha\hat{\eta}} + --- \qquad (4.17c)$$

This layer differs from the outer region by the by the participation of the shear stress $\hat{\tau}$ in the momentum balance of order ϵ. Hence we may let

$$\begin{cases} \tilde{u} = \hat{u} \to \tilde{u}_{out} + \frac{\epsilon}{K} \hat{u}(\hat{\eta};\epsilon) \\ \tilde{v} = \epsilon\hat{v} \to \tilde{v}_{out} + \frac{\epsilon^2}{K} \hat{v}(\hat{\eta};\epsilon) \\ \tilde{p} = \hat{p} \to \tilde{p}_{out} \end{cases} \qquad (4.18)$$

Now, the two-term outer solution of $\tilde{v}$(4.7b), as $\eta \to 0$, expressed in terms of the intermediate variable is

$$\tilde{v}_{out} \sim A_o [1 + \frac{\epsilon}{K} \ln \epsilon + \frac{\epsilon}{K} (\ln \hat{\eta} - 2\tilde{\pi} + E_1^o - K\alpha\hat{\eta}) + ---]$$

$$+ \frac{\epsilon}{K} [A_1 + E_1^o + (\epsilon \ln \epsilon) 2\alpha\hat{\eta} + O(\epsilon)]$$

where

$$E_1^o = 2\alpha \int_0^\infty W\, e^{-2\alpha\eta}\, d\eta$$

Therefore, $\tilde{v} = O(\epsilon)$ requires $A_o = 0$ for matching. And the intermediate expansions of the outer solution are

$$\tilde{u}_{out} \sim 1 - \frac{\epsilon}{K} \ln\epsilon - \frac{\epsilon}{K} (\ln \hat{\eta} - 2\hat{\pi} + E_1^o - A_1 + K\alpha\hat{\eta}) + --- \qquad (4.19a)$$

$$\tilde{v}_{out} \sim \frac{\epsilon}{K} \{A_1 + E_1^o + \epsilon \ln \epsilon \;\; 2\alpha\hat{\eta} + \epsilon [2\alpha\hat{\eta} \ln \hat{\eta}$$

$$- (4\tilde{\pi} + 2 - E_1^o + A_1)\alpha\hat{\eta} + ---] + ---\} \qquad (4.19b)$$

$$\tilde{p}_{out} \sim -1 + \frac{\varepsilon}{\kappa}(E_1^o - A_1 + \kappa\alpha\hat{\eta}) + --- \tag{4.19c}$$

$$\tilde{\tau}_{out} \sim \varepsilon^2 \; b_o \; T^*\left[\sigma_o + 2i + \frac{ib_o}{2\kappa\alpha\hat{\eta}}(\sigma_o + 2i + \frac{2}{b_o}) + O(\varepsilon)\right]. \tag{4.19d}$$

The intermediate governing equations are then simplified as

$$\begin{cases} \alpha\hat{u} + \dfrac{d\hat{v}}{d\eta} = 0 \\ U\hat{u} = \dfrac{\kappa}{i\alpha}\dfrac{d\hat{\tau}}{d\hat{\eta}} - \dfrac{C_o}{\alpha\hat{\eta}} + \dfrac{\varepsilon}{\kappa}\left[-\dfrac{\hat{v}}{\alpha\hat{\eta}} + \dfrac{\kappa\Delta}{i\alpha^2}\dfrac{d^2\hat{u}}{d\hat{\eta}^2} + ---\right] \\ -\dfrac{1}{\kappa}\dfrac{d\hat{u}}{d\hat{\eta}} + \left(\dfrac{i\alpha U}{b_o} + \dfrac{1}{\alpha\kappa\hat{\eta}}\right)\dfrac{\hat{\tau}}{T^*} = i\alpha U(\sigma_o + 2i) - \dfrac{1}{\kappa\hat{\eta}} + O(\varepsilon) \end{cases} \tag{4.20}$$

where $C_o = A_1 + E_1^o$. The constant C_o would complicate the representation of the solution to (4.20). However, it will become evident later that in order to match the inner solution of $\tilde{v}$, C_o must vanish. For simplicity, we hence take $C_o = 0$ in advance. Eq. (4.20) forms a system of two-parameter confluent hypergeometric differential equations. Neglecting terms of order ε^2 and higher, the solution is analyzed in Appendix A (pp. 89-95) as

$$\hat{\tau} = -2 + C_1 W_{k,\frac{1}{2}}(z) + De^{-ik\pi}\left\{W_{k,\frac{1}{2}}(z)\int_o^z W_{-k,\frac{1}{2}}(-t)\,dt + W_{-k,\frac{1}{2}}(-z)\int_z^{i\infty} W_{k,\frac{1}{2}}(t)dt\right\} \tag{4.21a}$$

$$\begin{aligned}\hat{u} = -\frac{1}{2k}\frac{d\hat{\tau}}{dz} = \frac{C_1}{2k}\left[\tfrac{1}{2}W_{k,\frac{1}{2}}(z) - \frac{k}{z^{\frac{1}{2}}}W_{(k-\frac{1}{2}),o}(z)\right] \\ + \frac{D}{2k}e^{-ik\pi}\left\{\left[\tfrac{1}{2}W_{k,\frac{1}{2}}(z) - \frac{k}{z^{\frac{1}{2}}}W_{(k-\frac{1}{2}),o}(z)\right]\int_o^z W_{-k,\frac{1}{2}}(-t)dt \right. \\ \left. - \left[\tfrac{1}{2}W_{-k,\frac{1}{2}}(-z) + \frac{k}{(-z)^{\frac{1}{2}}}W_{-(k+\frac{1}{2}),o}(-z)\right]\int_z^{i\infty} W_{k,\frac{1}{2}}(t)dt\right\}\end{aligned} \tag{4.21b}$$

$$\hat{v} = i\kappa(\hat{\tau}' - C_2) \tag{4.21c}$$

where $W_{k,\frac{1}{2}}(z)$, $W_{-k,\frac{1}{2}}(-z)$ are Whittaker's functions and

$$\begin{cases} z = i\dfrac{2\alpha}{\sqrt{b_0}}\displaystyle\int_0^{\hat{\eta}} U(\hat{\eta};\epsilon)d\hat{\eta} \simeq i\dfrac{2\alpha}{\sqrt{b_0}}\, U\,\hat{\eta} \\ k = -\dfrac{\sqrt{b_0}}{4\kappa}, \quad -1 < k < 0 \\ D = \dfrac{b_0}{4}(\sigma_0 + 2i) + \frac{1}{2} \end{cases} \tag{4.22}$$

As $\hat{\eta} \to \infty$, $|z| \to \infty$, the asymptotic expansions are

$$W_{k,\frac{1}{2}}(z) \sim z^k e^{-z/2}\left[1 + \frac{k(1-k)}{z} + O(\frac{1}{z^2})\right]$$

$$W_{-k,\frac{1}{2}}(-z) \sim (-z)^{-k} e^{z/2}\left[1 + \frac{k(1+k)}{z} + O(\frac{1}{z^2})\right]$$

and
$$\hat{\tau} \sim b_0\left[\sigma_0 + 2i + \frac{ib_0}{2\kappa\alpha\hat{\eta}}(\sigma_0 + 2i + \frac{2}{b_0}) + O(\frac{1}{\hat{\eta}^2})\right]$$

$$+ \overline{C}_1\, \hat{\eta}^k e^{-\frac{i\alpha}{\sqrt{b_0}}U\hat{\eta}}\,[1 + O(\frac{1}{\hat{\eta}})] \tag{4.24a}$$

$$\hat{u} = -\frac{1}{2k}\frac{d\hat{\tau}}{dz} \sim \frac{1}{4k}\,\overline{C}_1\,\hat{\eta}^k e^{-i\frac{\alpha}{\sqrt{b_0}}U\hat{\eta}}[1 + O(\frac{1}{\hat{\eta}})] + --- \tag{4.24b}$$

Since $k < 0$, $\hat{\eta}^k \to 0$ as $\hat{\eta} \to \infty$. These expansions match the outer solution. At this stage, the physical behavior of τ' for large $\hat{\eta}$ is evident.

$$\tau'(\xi,\eta) = \tilde{\tau}(\eta)e^{i\alpha\xi}$$

$$\sim \epsilon^2\left\{\overline{C}_1\,\hat{\eta}^k e^{i\alpha(\xi - \frac{U}{\sqrt{b_0}}\hat{\eta})} + b_0(\sigma+2i)e^{i\alpha\xi} + --\right\} \tag{4.25}$$

It represents a downstream <u>wave</u> with amplitude decaying as $\hat{\eta}^{-|k|}$ and frequency changing with $U(\hat{\eta})$, superimposed in a simple streamwise oscillation.

3. <u>Viscous Sublayer</u>:

In the region very close to the wall where

$$\zeta = R\varepsilon\eta = O(1)$$

the primary mean flow velocity and Reynolds stress are given by

$$U = \varepsilon F(\zeta)$$

$$T = \varepsilon^2 T^*(\zeta)$$

$$\sigma = \varepsilon^2 \sigma^*(\zeta)$$

and governed by

$$\begin{cases} F'' + T^{*'} = 0, \quad \text{or} \quad F' + T^* = 1 \\ -T^* \dfrac{dF}{d\zeta} + \dfrac{T^{*\frac{3}{2}}}{L} = \dfrac{d^2}{d\zeta^2}\left(\dfrac{T^*}{b}\right) \end{cases}$$

The stress-energy ratio b and the dissipation parameter L are assumed to be

$$b = b_o\left(\frac{\beta\zeta}{1+\beta\zeta}\right), \qquad \beta = 0.03$$

$$L = \kappa\zeta(1-e^{-\gamma\zeta})^{\frac{7}{2}}, \qquad \gamma = 0.1085$$

Let

$$\begin{cases} \tilde{u} = \varepsilon u^*(\zeta;\varepsilon) \\ \tilde{v} = \dfrac{\alpha}{R} v^*(\zeta;\varepsilon) \\ \tilde{\tau} = \varepsilon^2 \tau^*(\zeta;\varepsilon) \\ \tilde{p} = p^*(\zeta;\varepsilon) \end{cases}$$

The governing equations become

$$\begin{cases} u^* + \dfrac{dv^*}{d\zeta} = F(1 - \dfrac{\alpha}{R\varepsilon}\zeta + ---) & (4.27a) \\ \dfrac{d^2u^*}{d\zeta^2} + \dfrac{d\tau^*}{d\zeta} = -F''(1 - \dfrac{\alpha}{R\varepsilon}\zeta + ---) + \dfrac{i\alpha}{R\varepsilon^3}\{P^* + \varepsilon^2(Fu^* + F'v^* + \sigma^* + 2iT^*)\} & \\ & (4.27b) \\ \dfrac{dp^*}{d\zeta} = \dfrac{\varepsilon}{R}\,[\alpha F^2 + i\alpha\tau^* - \alpha(\sigma^* + 2iT^*) + O(\frac{1}{R})\,] & (4.27c) \end{cases}$$

$$- (T^* \frac{du^*}{d\zeta} + \tau^* \frac{dF}{d\zeta}) + \frac{3}{2}\frac{T^{*\frac{1}{2}}}{L}\tau^* - \frac{d^2}{d\zeta^2}(\frac{\tau^*}{b})$$

$$= \frac{d^2}{d\zeta^2}(\frac{T^*}{b})[1 - \frac{\alpha}{R\varepsilon}\zeta + ---]$$

$$+ \frac{i\alpha}{R\varepsilon}\left\{F(\sigma^* + iT^*) - F(\frac{\tau^*}{b}) - v^*\frac{d}{d\zeta}(\frac{T}{b}) - \sigma^*\frac{dv^*}{d\zeta}\right\} \quad (4.27d)$$

From Eq. (4.27c), we obtain

$$p^* = p(0;\varepsilon) + O(\frac{\varepsilon}{R})$$

Define the pressure gradient and convection parameter

$$\Delta \equiv \frac{\alpha}{R\varepsilon^3} \qquad (4.28)$$

In terms of physical variables

$$\Delta\tilde{p} \equiv \frac{\nu}{\rho u_\tau^3}\left|\frac{\partial p'}{\partial x}\right|_{max.} \qquad (4.28a)$$

For $\alpha \doteq O(1)$, according to the skin friction law

$$\Delta = \frac{\alpha}{\varepsilon^2}\, e^{-(\frac{\kappa}{\varepsilon} - B - 2\tilde{\pi})} \rightarrow (\text{T.S.T.}), \text{ as } R \rightarrow \infty;\ \varepsilon \rightarrow 0.$$

Although the experimental results available for the evaluation of the theory were obtained at $\Delta \simeq 0.3$ and larger, we will assume the magnitude of Δ to be small in conforming with the high Reynolds number approximation.

Let

$$\begin{cases} \tilde{u} = \varepsilon u^* = \varepsilon[(1-\varepsilon^2\Delta\zeta)F + f(\zeta;\varepsilon)] \\ \tilde{v} = \frac{\alpha}{R} v^* = \frac{\alpha}{R} h \\ \tilde{T} = \varepsilon^2 T^* = \varepsilon^2[(1-\varepsilon^2\Delta\zeta)2T^* + g(\zeta;\varepsilon)] \\ \tilde{p} = p^* = p(0) \end{cases} \tag{4.29}$$

and neglecting terms which remains at most $O(\varepsilon^2\Delta)$ as $\zeta >> 1$, Eqs. (4.27 a, b, d) are simplified as (dropping $*$ in T and σ)

$$\begin{cases} f + \frac{dh}{d\zeta} = 0 \\ \frac{d^2f}{d\zeta^2} + \frac{dg}{d\zeta} = i\Delta[p(o) + \varepsilon^2(F^2 + Ff)] \\ T\frac{df}{d\zeta} + \left(\frac{dF}{d\zeta} - \frac{3}{2}\frac{T^{\frac{1}{2}}}{L}\right)g + \frac{d^2}{d\zeta^2}\left(\frac{g}{b}\right) = i\Delta\varepsilon^2 F\left[\frac{2T+g}{b} - (\sigma + 2iT)\right] \end{cases} \tag{4.30}$$

For $\zeta = O(1)$, the convection terms are $O(\varepsilon^2)$ which can be neglected in the solutions up to $O(\varepsilon)$. The pressure gradient $i\Delta p$ then constitutes a particular solution. As for $\zeta >> 1$, $\varepsilon^2 F^2 \sim U^2 = O(1)$, the pressure gradient is balanced by convection up to $O(\frac{\varepsilon}{\kappa}\ell n\,\varepsilon)$. Therefore, for small values of Δ we assume the following form:

$$\begin{cases} f = \bar{f} + i\Delta\tilde{f} \\ g = \bar{g} + i\Delta\tilde{g} \\ h = \bar{h} + i\Delta\tilde{h} \end{cases} \tag{4.31}$$

,

We then have

$$\left\{\begin{array}{l} \dfrac{d^2\overline{f}}{d\zeta^2} + \dfrac{d\overline{g}}{d\zeta} = 0 \\[2ex] T\,\dfrac{d\overline{f}}{d\zeta^2} + \left(\dfrac{dF}{d\zeta} - \dfrac{3}{2}\,\dfrac{T^{\frac{1}{2}}}{L}\right)\overline{g} + \dfrac{d^2}{d\zeta^2}\left(\dfrac{\overline{g}}{b}\right) = 0 \end{array}\right. \tag{4.32}$$

and

$$\left\{\begin{array}{l} \dfrac{d^3\widetilde{f}}{d\zeta^2} + \dfrac{d\widetilde{g}}{d\zeta} = [p(o) + \varepsilon^2(F^2 + F\,\overline{f})] \\[2ex] T\,\dfrac{d\widetilde{f}}{d\zeta} + \left(\dfrac{dF}{d\zeta} - \dfrac{3}{2}\,\dfrac{T^{\frac{1}{2}}}{L}\right)\widetilde{g} + \dfrac{d^2}{d\zeta^2}\left(\dfrac{\widetilde{g}}{b}\right) = \varepsilon^2 F\left[\dfrac{2T+\overline{g}}{b} - (\sigma + 2iT)\right] \end{array}\right. \tag{4.33}$$

3.1 <u>Solution for the constant total stress:</u>

$$\frac{d\overline{f}}{d\zeta} + \overline{g} = \overline{s} \tag{4.34}$$

Let

$$\overline{g} = \overline{s}\,\varphi(\zeta)$$

Then

$$\left.\begin{array}{l} \overline{f} = \overline{s}\left\{\zeta - \displaystyle\int_0^{\zeta} \varphi(t)dt \right\} \\[2ex] \overline{h} = -\,\overline{s}\left\{\frac{1}{2}\,\zeta^2 - \displaystyle\int_0^{\zeta}\int_0^{y} \varphi(t)dt\,dy\right\} \end{array}\right. \tag{4.35}$$

where $\varphi(\zeta)$ is governed by the energy equation,

$$\left.\begin{array}{l} \dfrac{d^2}{d\zeta^2}\left(\dfrac{\varphi}{b}\right) - \left[1 - 2F'(\zeta) + \dfrac{3}{2}\,\dfrac{(1-F')^{\frac{1}{2}}}{L}\right]\varphi = F'(\zeta) - 1 \\[2ex] \varphi(0) = 0 \\[1ex] \varphi \to 1, \quad \zeta \to \infty \end{array}\right. \tag{4.36}$$

Asymptotic solution for large ζ:

$$F(\zeta) \sim \frac{1}{\kappa}(\ln \zeta + B)$$

$$\frac{T^{\frac{1}{2}}}{L} \sim \frac{dF}{d\zeta} \simeq \frac{1}{\kappa\zeta}$$

$$b \sim b_o\left(1 - \frac{1}{\beta\zeta} + \frac{1}{\beta^2\zeta^2} + ---\right)$$

Therefore, as $\zeta \to \infty$, we obtain

$$\frac{d^2}{d\zeta^2}\left(\frac{\varphi}{b}\right) - \left(1 - \frac{1}{2\kappa\zeta}\right)\varphi \simeq -\left(1 - \frac{1}{\kappa\zeta}\right) \tag{4.37}$$

Thus $\varphi(\zeta) \simeq \frac{2\kappa\zeta-2}{2\kappa\zeta-1}\left[1 + O\left(\frac{1}{\zeta^3}\right)\right] + C\,\zeta^{k_1}\, e^{-\sqrt{b_o}\,\zeta}\left[1+O\left(\frac{1}{\zeta}\right)\right]$,

$$\to \frac{2\kappa\zeta-2}{2\kappa\zeta-1}\left[1 + O\left(\frac{1}{\zeta^3}\right)\right] + (\text{T.S.T.}) \tag{4.38}$$

where

$$k_1 = \frac{\sqrt{b_o}}{4\kappa\beta}(2\kappa+\beta)$$

and

$$\bar{f} = \bar{s}\left\{\frac{1}{2\kappa}\ln(2\kappa\zeta-1) - \int_o^{\zeta}\left(\varphi - \frac{2\kappa\zeta'-2}{2\kappa\zeta'-1}\right)d\zeta'\right\}$$

$$\simeq \frac{\bar{s}}{2\kappa}\left\{\ln\zeta + d - \frac{1}{2\kappa\zeta} + O\left(\frac{1}{\zeta^2}\right)\right\}, \quad \text{as } \zeta \to \infty \tag{4.39}$$

$$\bar{h} \simeq -\frac{\bar{s}}{2\kappa}\left\{\zeta(\ln\zeta - 1 + d) - \frac{1}{2\kappa}\ln\zeta + e + O\left(\frac{1}{\zeta}\right)\right\} \tag{4.40}$$

where

$$d = \ln 2\kappa - 2\kappa\int_o^{\infty}\left[\varphi(\zeta) - \frac{2\kappa\zeta-2}{2\kappa\zeta-1}\right]d\zeta$$

3.2 Solution with pressure gradient and connection:

$$\begin{cases} \dfrac{d^2\tilde{f}}{d\zeta^2} + \dfrac{d\tilde{g}}{d\zeta} = p(0) + \epsilon^2(F^2 + F\bar{f}) \\ T\dfrac{d\tilde{f}}{d\zeta} + \left(\dfrac{dF}{d\zeta} - \dfrac{3}{2}\dfrac{T^{\frac{1}{2}}}{L}\right)\tilde{g} + \dfrac{d^2}{d\zeta^2}\left(\dfrac{\tilde{g}}{b}\right) = \epsilon^2 F\left[\dfrac{2T+\bar{g}}{b} - (\sigma+2iT)\right] \end{cases} \tag{4.41}$$

Integrating the momentum equation once, we have

$$\frac{d\tilde{f}}{d\zeta} + \tilde{g} = \tilde{s} + p(0)\zeta + \varepsilon^2 \int_0^\zeta (F^2 + F\bar{f})d\zeta'$$

The contribution of the convection term to the solution up to $O(\varepsilon)$ is important only when $\zeta >> 1$. Let

$$\tilde{f} = \tilde{f}_o + (\frac{\varepsilon}{K})^2 \tilde{f}_1$$

$$\tilde{g} = \tilde{g}_o + (\frac{\varepsilon}{K})^2 \tilde{g}_1$$

Then for $\tilde{f}_1$ and $\tilde{g}_1$, only the asymptotic solution is required for matching with the intermediate expansion. For $\zeta = O(1)$,

$$\begin{cases} \dfrac{d\tilde{f}_o}{d\zeta} + g_o = \tilde{s} + p(0)\zeta \\ T\dfrac{d\tilde{f}_o}{d\zeta} + (\dfrac{dF}{d\zeta} - \dfrac{3}{2}\dfrac{T^{\frac{1}{2}}}{L})\tilde{g}_o + \dfrac{d^z}{dg^2}(\dfrac{\tilde{g}_o}{b}) = 0 \end{cases} \qquad (4.42)$$

Thus, we get

$$\tilde{f}_o = \tilde{s}\zeta + \tfrac{1}{2}p(0)\zeta^2 - \int_0^\zeta \tilde{g}_o \, d\zeta \qquad (4.43)$$

where $\tilde{g}_o$ is governed by

$$\begin{aligned} \frac{d^2}{d\zeta^2}(\frac{\tilde{g}_o}{b}) - \left[1 - 2F'(\zeta) + \frac{3}{2}\frac{(1-F')^{\frac{1}{2}}}{L}\right]\tilde{g}_o &= -T[\tilde{s}+p(0)\zeta] \\ \tilde{g}_o(0) &= 0 \end{aligned} \qquad (4.44)$$

As $\zeta \to \infty$, this reduces to

$$\frac{d^2}{d\zeta^2}(\frac{\tilde{g}_o}{b}) - (1 - \frac{1}{2K\zeta})\tilde{g}_o = -(1 - \frac{1}{K\zeta})[\tilde{s}+p(0)\zeta] \qquad (4.45)$$

which gives

$$\widetilde{g}_o \simeq \frac{2\kappa\zeta-2}{2\kappa\zeta-1}\,[p(0)\zeta + \widetilde{s} + O(\frac{1}{\zeta^3})] + (T.S.T) \qquad (4.46)$$

$$\widetilde{f}_o = \frac{p(0)}{2\kappa}\,\zeta + (\frac{\widetilde{s}}{2} + \frac{p(0)}{4\kappa})\,\frac{1}{\kappa}\ln(2\kappa\zeta-1) - \int_o^{\zeta}(\widetilde{g}_o - \widetilde{g}_{o_a})d\zeta$$

$$\simeq \frac{p(0)}{2\kappa}\,\zeta + (\frac{\widetilde{s}}{2} + \frac{p(0)}{4\kappa})\,\frac{1}{\kappa}\ln\zeta + \frac{\widetilde{d}}{\kappa} \qquad (4.47)$$

where subscript a denotes the asymptotic value, and

$$\widetilde{d} = (\frac{\widetilde{s}}{2} + \frac{p(0)}{4\kappa})\ln 2\kappa - \kappa\int_o^{\infty}(\widetilde{g}_o - \widetilde{g}_{o_a})d\zeta$$

<u>Contribution from the convections for $\zeta >> 1$:</u>

$$\left\{\begin{aligned} &\frac{d\widetilde{f}_1}{d\zeta} + \widetilde{g}_1 = \kappa^2\int_o^{\zeta}(F^2 + F\overline{f}) \\ &T\,\frac{d\widetilde{f}_1}{d\zeta} - \frac{\widetilde{g}_1}{2\kappa\zeta} \simeq \kappa^2 F\left\{\frac{2T+\overline{g}}{b_o} - (\sigma_o + 2i)T\right\} \end{aligned}\right. \qquad (4.48)$$

$$\overline{f} \simeq \frac{\overline{s}}{2\kappa}(\ln\zeta + d)$$

$$(F^2 + F\overline{f}) \simeq \frac{1}{\kappa^2}[(1+\frac{\overline{s}}{2})(\ln\zeta + B)^2 + \frac{\overline{s}}{2}(d-B)(\ln\zeta + B)]$$

Neglecting terms which remain at most $O(\varepsilon^2)$, we obtain

$$\begin{aligned} \widetilde{g}_1 \simeq \frac{2\kappa\zeta-2}{2\kappa\zeta-1}\Bigg\{ &\zeta\Big\{(1+\frac{\overline{s}}{2})[(\ln\zeta + B - 1)^2 + 1] \\ &+ \frac{\overline{s}}{2}(d-B)(\ln\zeta + B - 1)\Big\} \\ &+ D_1(\ln\zeta + B) + O(\frac{1}{\zeta})\Bigg\} \end{aligned} \qquad (4.49)$$

$$\tilde{f}_1 = \frac{\zeta}{2\kappa}\left\{(1+\frac{\bar{s}}{2})\,[\ell n\,\zeta+B-2)^2+2\,]+\frac{\bar{s}}{2}(d-B)(\ell n\zeta+B-2)\right\}$$

$$+\frac{1}{4\kappa^2}\left\{(1+\frac{\bar{s}}{2})[\,\frac{1}{3}(\ell n\zeta+B-1)^3+\ell n\zeta\,]+\frac{\bar{s}}{2}(d-B)(\ell n\zeta+B-2)^2\right\}$$

$$-\,D_1\left\{(\ell n\zeta+B-1)\zeta\;-\;\frac{1}{4\kappa}\,(\ell n\zeta\;+\;B)^2\right\} \tag{4.50}$$

where $\qquad D_1 \;=\; \kappa(\sigma_o+2i\;-\;\frac{2+\bar{s}}{b_o}\;)$

As $\zeta >> 1$, $(\frac{\varepsilon}{\kappa})^2\;(\ell n\zeta)^2 \rightarrow O(1)$. Therefore, by retaining terms which may grow to $O(\varepsilon)$ as $\zeta \rightarrow R\varepsilon^2\,\hat{\eta}$, the asymptotic solution with pressure gradient and convection is given by

$$\tilde{g} \simeq \frac{2\kappa\zeta-2}{2\kappa\zeta-1}\left\{\tilde{s}+\zeta\left\{p(0)+(\frac{\varepsilon}{\kappa})^2[(1+\frac{\bar{s}}{2})(\ell n\zeta+B-1)^2+\frac{\bar{s}}{2}(d-B)\ell n\,\zeta\,]\right\}\right.$$

$$\left.+\;O(\frac{1}{\zeta}\,)\right\} \tag{4.51}$$

$$\tilde{f} \simeq \frac{\zeta}{2\kappa}\left[p(0)+(\frac{\varepsilon}{\kappa})^2\left\{(1+\frac{\bar{s}}{2}\,)(\ell n\zeta+B-2)^2\;\;+[\,\frac{\bar{s}}{2}(d-B)-2\kappa D_1\,](\ell n\zeta+B-1)\right\}\right]$$

$$+\frac{1}{2\kappa}\,\ell n\,\zeta\left[\tilde{s}+\frac{p(0)}{2\kappa}\;+\;(\frac{\varepsilon}{\kappa})^2\left\{\frac{1}{6\kappa}(1+\frac{\bar{s}}{2}\,)\,[\,\ell n^2\zeta+3(B-1)\ell n\zeta\right.\right.$$

$$\left.\left.+\;3(B-1)^2+3\,]\;+\left(\frac{\bar{s}}{4\kappa}\,(d-B)+\,\frac{D_1}{2}\right)(\ell n\zeta+2B)\right\}\right]$$

$$+\;\frac{\tilde{d}}{\kappa}\;+\;O(\frac{1}{\zeta}) \tag{4.52}$$

$$\tilde{h} \;=\; -\;\frac{\zeta^2}{4\kappa}\left\{p(0)+(\frac{\varepsilon}{\kappa})^2\,\ell n\,\zeta\,[(1+\frac{\bar{s}}{2}\,)(\ell n\zeta+2B-5)+\frac{\bar{s}}{2}\,(d-B)-2\kappa D_1\,]\right\}$$

$$-\;\frac{\zeta(\ell n\zeta-1)}{2\kappa}\;\left\{\tilde{s}+\frac{p(0)}{2\kappa}\;+\;(\frac{\varepsilon}{\kappa})^2\,\ell n\zeta\,[\frac{1}{6\kappa}\;(1+\frac{\bar{s}}{2}\,)(\ell n\zeta+3B-5)\right.$$

$$\left.+\;\frac{\bar{s}}{4\kappa}\,(d-B)\;+\;\frac{D_1}{2}\;]\right\}$$

$$+\;\frac{\tilde{d}}{\kappa}\,\zeta\;+\;e_1\;+\;---- \tag{4.53}$$

Intermediate-Inner matching:

Applying the skin friction law for the primary flow

$$\frac{\varepsilon}{\kappa}(\ell n\, R\varepsilon + B + 2\tilde{\pi}) = 1$$

and $$\zeta = R\varepsilon^2 \hat{\eta} .$$

The asymptotic inner expansion expressed in terms of intermediate variable $\hat{\eta}$ becomes

$$\tilde{u}_{inner} \sim (1+\frac{\bar{s}}{2})[1+\frac{\varepsilon}{\kappa}\ell n\,\varepsilon + O(\frac{\varepsilon}{\kappa})]$$

$$+\frac{i\alpha}{2\kappa}\hat{\eta}\left\{p(o;\varepsilon)+(1+\frac{\bar{s}}{2})[1+\frac{\varepsilon}{\kappa}\ell n\varepsilon + O(\varepsilon)]^2\right\}$$

$$+\frac{i\Delta}{2}\left\{\tilde{s}+\frac{p(o;\varepsilon)}{2\kappa}+\frac{1}{6\kappa}(1+\frac{\bar{s}}{2})[1+\frac{\varepsilon}{\kappa}\ell n\,\varepsilon + ---]^2\right\}[1+\frac{\varepsilon}{\kappa}\ell n\varepsilon+--]$$

$$\tilde{v}_{inner} \sim -\varepsilon\left\{\frac{\bar{s}}{2}(\alpha\hat{\eta}+\frac{\varepsilon\Delta}{2\kappa})[1+\frac{\varepsilon}{\kappa}\ell n\,\varepsilon + O(\frac{\varepsilon}{\kappa})]\right.$$

$$+\frac{i\alpha^2}{4\kappa}\hat{\eta}^2[p(o;\varepsilon)+(1+\frac{\bar{s}}{2})(1+\frac{\varepsilon}{\kappa}\ell n\,\varepsilon + --)^2]$$

$$\left.+\frac{i\Delta}{2}\alpha\hat{\eta}[\tilde{s}+\frac{p(o;\varepsilon)}{2\kappa}+\frac{1}{6\kappa}(1+\frac{\bar{s}}{2})(1+\frac{\varepsilon}{\kappa}\ell n\,\varepsilon+--)^2](1+\frac{\varepsilon}{\kappa}\ell n\varepsilon+--)\right\}$$

$$\tilde{p}_{inner} \sim p(o;\varepsilon)$$

$$\tilde{\tau}_{inner} \sim \varepsilon^2\left\{2+\bar{s}+i\Delta\tilde{s}+\frac{i}{\varepsilon}\alpha\hat{\eta}[p+(1+\frac{\bar{s}}{2})(1+\frac{\varepsilon}{\kappa}\ell n\varepsilon+--)^2]\right\} \quad (4.54)$$

On the other hand, the intermediate expansion for small $\hat{\eta}$ (see Appendix A) is

$$\begin{aligned}
\widetilde{u}_{inter} &\simeq 1 - \frac{\epsilon}{\kappa}\, \ell n\, \epsilon - \frac{\epsilon}{\kappa}(\ell n\, \hat{\eta} - 2\widetilde{\pi} + 2E_1^{\,o} + \kappa\alpha\hat{\eta}) + \frac{\epsilon}{\kappa}\hat{u} \\
\widetilde{v}_{inter} &\simeq \epsilon\,\{C_o + (\frac{\epsilon}{\kappa}\ell n\epsilon)2\alpha\hat{\eta} + O(\frac{\epsilon}{\kappa})\} \\
\widetilde{p}_{inter} &\simeq -1 + \frac{\epsilon}{\kappa}(2E_1^{\,o} + \kappa\alpha\hat{\eta}) + --- \\
\hat{\tau}_{inter} &\simeq \epsilon^2\,\{-2 + A(1 - a_1 z - kz\ell nz) + D\widetilde{I}_1 z + ---\}
\end{aligned} \qquad (4.55)$$

Expanding

$$\begin{aligned}
\overline{s} &= \overline{s}_o + \frac{\epsilon}{\kappa}\ell n\,\epsilon\,\overline{s}_1 + \frac{\epsilon}{\kappa}\,\overline{s}_2 \\
\widetilde{s} &= \widetilde{s}_o + \frac{\epsilon}{\kappa}\ell n\,\epsilon\,\widetilde{s}_1 + \frac{\epsilon}{\kappa}\,\widetilde{s}_2
\end{aligned} \qquad (4.56)$$

then matching requires

$C_o = 0$, justifying the previous choice.

$$p(o;\epsilon) = -1 + \frac{\epsilon}{\kappa}\; 2E_1^{\,o}$$

$$\overline{s}_o = 0, \quad \overline{s}_1 = -4,$$

$$s_o = \frac{1}{3\kappa}, \qquad \widetilde{s}_1 = 0$$

$$A = 4 + \frac{i\Delta}{3\kappa}$$

With these values for the constants, the three-term intermediate expansion for small $\hat{\eta}$ is

$$\begin{aligned}
\widetilde{u}_{inter} \simeq 1 - \frac{\epsilon}{\kappa}\,\ell n\epsilon + \frac{\epsilon}{\kappa}\Big\{&\ell n\hat{\eta} + 2(\widetilde{\pi} + 1 - E_1^{\,o} + \frac{a_1}{k} + \ell n\,\frac{2i\alpha}{\sqrt{b_o}} - \frac{D\widetilde{I}_1}{4k}\,) \\
&- \kappa\alpha\hat{\eta} + \frac{i\alpha}{\kappa}\,\hat{\eta}\,(\ell n\hat{\eta} + \ell n\,\frac{2i\alpha}{\sqrt{b_o}} - \frac{2a_2}{k^2} + \frac{1}{2} - \frac{D\widetilde{I}_1}{4k} - \frac{4\kappa^2 D}{b_o}\,) \\
&+ \frac{i\Delta}{6\kappa}\Big[\ell n\,\hat{\eta} + (\ell n\,\frac{2i\alpha}{\sqrt{b_o}} + 1 + \frac{a_1}{k}) + O(\hat{\eta})\Big]\Big\}
\end{aligned} \qquad (4.57a)$$

$$\tilde{v}_{inter} \simeq \epsilon\left\{\left(\frac{\epsilon}{\kappa}\ln\epsilon\right)2\alpha\hat{\eta} + \frac{\epsilon}{\kappa}\left[i\kappa\left(2+\frac{i\Delta}{3\kappa} - C_2\right) - 2\alpha\hat{\eta}\left(2\tilde{\pi} + 1 - E_1^{\,o} + \frac{a_1}{k}\right.\right.\right.$$

$$\left.\left.\left. + \ln\frac{2i\alpha}{\sqrt{b_0}} - \frac{D\tilde{I}_1}{4k}\right) + O(\hat{\eta}^2) + \frac{i\Delta}{3\kappa}\left(O(\hat{\eta}\ln\hat{\eta})\right)\right] + \cdots\right\} \quad (4.57b)$$

$$\tilde{\tau}_{inter} \simeq \epsilon^2\left\{2 + \frac{i\alpha}{\kappa}\hat{\eta}\left[2\ln\hat{\eta} + \left(\frac{2a_1}{k} + 2\ln\frac{2i\alpha}{\sqrt{b_0}} - \frac{D\tilde{I}_1}{2k}\right)\right]\right.$$

$$\left. + \frac{i\Delta}{3\kappa}\left[1 + O(\hat{\eta})\right] + O(\epsilon\ln\epsilon)\right\} \quad (4.57c)$$

and the asymptotic expansion for the inner layer as $\zeta \to \infty$ is

$$\tilde{u}_{inner} \sim 1 - \frac{\epsilon}{\kappa}\ln\epsilon + \frac{\epsilon}{\kappa}\left\{\ln\hat{\eta} - 2\tilde{\pi} + \frac{\bar{s}_2}{2} - \kappa\alpha\hat{\eta}\right.$$

$$+ \frac{i\alpha}{\kappa}\hat{\eta}\left(\ln\hat{\eta} + E_1^{\,o} - 2\tilde{\pi} - 2 - D_1\kappa + \frac{\bar{s}_2}{4}\right)$$

$$+ \frac{i\Delta}{2}\left[\frac{1}{3\kappa}\ln\hat{\eta} + \tilde{s}_2 + \frac{E_1^{\,o}}{\kappa} + \frac{1}{3\kappa}\left(\frac{\bar{s}_2}{4} - 2\tilde{\pi} + \frac{B-3}{2}\right) + \frac{D_1}{2}\right.$$

$$\left.\left. + 2\tilde{d}_0\right]\right\} + \cdots \quad (4.58a)$$

$$\tilde{v}_{inner} \sim \epsilon\left\{\left(\frac{\epsilon}{\kappa}\ln\epsilon\right)2\alpha\hat{\eta} - \left(\frac{\epsilon}{\kappa}\right)\left[\frac{\bar{s}_2}{2}\alpha\hat{\eta} + O(\hat{\eta}^2) + i\Delta\left(O(\hat{\eta}\ln\hat{\eta})\right)\right] + \cdots\right\} \quad (4.58b)$$

$$\tilde{\tau}_{inner} \sim \epsilon^2\left\{2 + \frac{i\Delta}{3\kappa} + \frac{i\alpha}{\kappa}\hat{\eta}\left[2\ln\hat{\eta} - \left(4\tilde{\pi} + 2 - \frac{\bar{s}_2}{2} - 2E_1^{\,o}\right)\right] + O(\epsilon\ln\epsilon)\right\} \quad (4.58c)$$

Intermediate-inner common parts

Eq. (4.57) or Eq. (4.58) is the matched common part. For simplicity, we may take the matched asymptotic inner expansion as a common part.

Therefore, matching of these two expansions in this overlapping region, we obtain

$$\begin{aligned}
\bar{s}_2 &= 4\left(2\tilde{\pi} + 1 - E_1^{\circ} + \frac{a_1}{k} + \ln \frac{2i\alpha}{\sqrt{b_0}} - \frac{D\tilde{I}_1}{4k}\right) \\
\tilde{s}_2 &= \frac{1}{3K}\left(-2\,E_1^{\circ} + \frac{D\tilde{I}_1}{4k} - \frac{B-3}{2}\right) - \frac{D_1}{2} - 2\tilde{d}_0 \qquad (4.59) \\
C_2 &= 2 + \frac{i\Delta}{3K}
\end{aligned}$$

It may be shown that, with these values, the $\hat{\eta}$ terms in $\tilde{u}$, $\tilde{v}$ and $\tilde{\tau}$, and the $\hat{\eta}^2$ terms in $\tilde{v}$ match simultaneously.

V. NONLINEAR EFFECT--SECOND ORDER EXPANSION

The linear first order solution obtained in the previous article is denoted as before by

$$\begin{Bmatrix} u' \\ v' \\ p' \\ \tau' \\ \sigma' \end{Bmatrix} = \begin{Bmatrix} \tilde{u}(\eta) \\ i\tilde{v}(\eta) \\ \tilde{p}(\eta) \\ \tilde{\tau}(\eta) \\ \tilde{\sigma}(\eta) \end{Bmatrix} e^{i\alpha\xi} \tag{5.1}$$

where

$$\begin{cases} \sigma' = \sigma_1' - \sigma_2' \\ \tilde{\sigma} = \tilde{\sigma}_1 - \tilde{\sigma}_2 \end{cases}$$

Substituting the real part of this expression into the second order governing equations (3.6a, b, c, d), and describing $\sin^2$ and $\cos^2$ in terms of the second harmonics, we obtain a system of inhomogeneous equations with forcing functions which contain terms proportional to $e^{i2\alpha\xi}$ and terms depending on η alone (see Appendix B). Again, by the uniqueness theorem, we may let

$$\begin{Bmatrix} u'' \\ v'' \\ p'' \\ \tau'' \end{Bmatrix} = \mathcal{R}e \begin{Bmatrix} \tilde{\tilde{u}}(\eta) \\ i\tilde{\tilde{v}}(\eta) \\ \tilde{\tilde{p}}(\eta) \\ \tilde{\tilde{\tau}}(\eta) \end{Bmatrix} e^{i2\alpha\xi} + \begin{Bmatrix} \bar{\bar{u}}(\eta) \\ \bar{\bar{v}}(\eta) \\ \bar{\bar{p}}(\eta) \\ \bar{\bar{\tau}}(\eta) \end{Bmatrix} \tag{5.2}$$

1. Solution Involving the Second Harmonics

Neglecting terms of order ε^2 and higher, the governing equations are

$$2\alpha\tilde{\tilde{u}} + \frac{d\tilde{\tilde{v}}}{d\eta} = -\alpha\left\{Ue^{-2\alpha\eta} - \tfrac{1}{2}(\tilde{u}-\tilde{v})e^{-\alpha\eta}\right\} \tag{5.3a}$$

$$2i\alpha\left(U\tilde{\tilde{u}} + \frac{\tilde{\tilde{v}}}{2\alpha}\frac{dU}{d\eta} + \tilde{\tilde{p}}\right) - \frac{d\tilde{\tilde{\tau}}}{d\eta} - \frac{1}{R}\frac{d^2\tilde{\tilde{u}}}{d\eta^2}$$

$$= -\frac{i\alpha}{2}\left\{\tilde{u}^2 + U\tilde{v}e^{-\alpha\eta} + \frac{\tilde{v}}{2}\frac{d\tilde{u}}{d\eta}\right\}$$

$$-\frac{1}{R}\left\{\frac{1}{4}\frac{d^2U}{d\eta^2}e^{-2\alpha\eta} - \frac{1}{2}\frac{d^2\tilde{u}}{d\eta^2}e^{-\alpha\eta}\right\} \tag{5.3b}$$

$$\frac{d\tilde{\tilde{p}}}{d\eta} - 2\alpha U\tilde{\tilde{v}} = \alpha\{U\tilde{u}\,e^{-\alpha\eta} - U^2e^{-2\alpha\eta}\} \tag{5.3c}$$

$$\left[2i\alpha U\left(\frac{\tilde{\tilde{\tau}}}{b}\right) + i\,\tilde{\tilde{v}}\frac{d}{d\eta}\left(\frac{T}{b}\right)\right] - \left[T\left(\frac{d\tilde{\tilde{u}}}{d\eta} - 2\alpha\tilde{\tilde{v}}\right) + \tilde{\tilde{\tau}}\frac{dU}{d\eta} - i\sigma\frac{d\tilde{\tilde{v}}}{d\eta}\right]$$

$$+\frac{3}{2}\frac{T^{\frac{1}{2}}}{L}\tilde{\tilde{\tau}} - \frac{1}{R}\frac{d^2}{d\eta^2}\left(\frac{\tilde{\tilde{\tau}}}{b}\right)$$

$$= -i\alpha\left\{U(\sigma+iT)e^{-2\alpha\eta} - \frac{1}{2}[U(\tilde{\sigma}+i\tilde{\tau}) + \tilde{u}(\sigma+iT)\right.$$

$$\left. + i\,T\,\tilde{v}]e^{-\alpha\eta}\right\}$$

$$-\tfrac{1}{2}\left\{i\alpha\tilde{u}\left(\frac{\tilde{\tau}}{b}\right) + i\,\tilde{v}\frac{d}{d\eta}\left(\frac{\tilde{\tau}}{b}\right) - \tilde{\tau}\left(\frac{d\tilde{u}}{d\eta} - \alpha\tilde{v}\right) + i\tilde{\sigma}\frac{d\tilde{v}}{d\eta}\right.$$

$$\left. + \frac{3}{8}\frac{\tilde{\tau}^2}{LT^{\frac{1}{2}}}\right\}$$

$$-\frac{1}{R}\left\{\frac{1}{4}\frac{d^2}{d\eta^2}\left(\frac{T}{b}\right)e^{-2\alpha\eta} - \frac{1}{2}\frac{d^2}{d\eta^2}\left(\frac{\tilde{\tau}}{b}\right)e^{-\alpha\eta}\right\} \tag{5.3d}$$

The first-order perturbation of the normal stress difference $\widetilde{\sigma}$, appeared in the forcing function of Eq. (5.3d), is expected to be the same order as $\widetilde{\tau}$. Based on the same argument discussed in Article III, we further assume

$$\widetilde{\sigma} \simeq \sigma_o \widetilde{\tau} \tag{5.4}$$

to complete the closure of the system.

<u>1.1 Outer Layer $\eta = O(1)$:</u>

By using the expression (4.3) for the primary mean flow and the corresponding first-order outer solution Eq. (4.60), the governing equations for this layer are simplified as

$$\begin{cases} 2\alpha\tilde{\tilde{u}} + \dfrac{d\tilde{\tilde{v}}}{d\eta} = \dfrac{\alpha}{2}\left\{-1 + \dfrac{\epsilon}{\kappa}(3W - 2E_1)\right\} e^{-2\alpha\eta} \\ (1 - \dfrac{\epsilon}{\kappa} W)\tilde{\tilde{u}} - \dfrac{\epsilon}{\kappa}\,\dfrac{W'}{2\alpha}\,\tilde{\tilde{v}} + \tilde{\tilde{p}} = -\dfrac{1}{4}\left\{1 + \dfrac{\epsilon}{\kappa}\,2(W - E_1 - E_1^o)\right\} e^{-2\alpha\eta} \\ \dfrac{d\tilde{\tilde{p}}}{d\eta} - 2\alpha(1 - \dfrac{\epsilon}{\kappa} W)\tilde{\tilde{v}} = \dfrac{\epsilon}{\kappa}\alpha(2W - E_1 - E_1^o)e^{-2\alpha\eta} \end{cases} \tag{5.5}$$

An application of the regular limit process expansion of $\tilde{\tilde{u}}$ etc. in power of ϵ, $\epsilon \to 0$, yields a solution which vanishes as $\eta \to \infty$.

$$\tilde{\tilde{u}} \simeq -\left\{\frac{1}{4} + \frac{\epsilon}{\kappa}\left[\frac{3}{4}W + \frac{1}{2}(E_1 - E_1^o) - E_2 + A_1\right] + \cdots\right\} e^{-2\alpha\eta} \tag{5.6a}$$

$$\tilde{\tilde{v}} \simeq \frac{\epsilon}{\kappa}\left\{\frac{1}{2}(E_1 + E_1^o) - E_2 - A_1 + O(\frac{\epsilon}{\kappa})\right\} e^{-2\alpha\eta} \tag{5.6b}$$

$$\tilde{\tilde{p}} \simeq \frac{\epsilon}{\kappa}\left\{E_1 - E_2 + A_1 + O(\frac{\epsilon}{\kappa})\right\} e^{-2\alpha\eta} \tag{5.6c}$$

where

$$E_2(\eta;\alpha) = 4\alpha e^{4\alpha\eta}\int_\eta^\infty W e^{-4\alpha\eta'}\,d\eta' \tag{5.7}$$

Then, the energy equation (5.3d) together with the assumption (5.4), and the velocity perturbations (5.6 a, b) gives

$$\frac{\approx\atop{T}}{T} \simeq - \left\{\frac{b_0}{2}(\sigma_0+2i)\left[1 - \frac{b_0}{2}(\sigma_0+2i)\right] + O\left(\frac{\varepsilon}{\kappa}\right)\right\} e^{-2\alpha\eta} \tag{5.6d}$$

1.2 Intermediate Expansion:

Similar to the first-order expansion, it may be shown from the energy balance that the intermediate variable is

$$\hat{\eta} = \frac{\eta}{\varepsilon} \tag{5.8}$$

In terms of this variable, the outer expansion of $\approx\atop{v}$ as $\eta \to 0$ becomes

$$\overset{\approx}{v}_{outer} \simeq \varepsilon\left\{\frac{1}{\kappa}(E_1^o - E_2^o - A_1) - \left(\frac{\varepsilon}{\kappa}\ln\varepsilon\right)3\alpha\hat{\eta} + O\left(\frac{\varepsilon}{\kappa}\right)\right\} \tag{5.9}$$

It will be evident in the following analysis that in order to match the inner solution of $\overset{\approx}{v}$, which is of order $\frac{1}{R}$,

$$A_1 = E_1^o - E_2^o \tag{5.10}$$

The intermediate expansions of the two-term outer solution as $\eta \to 0$ are

$$\begin{aligned} \overset{\approx}{u}_{outer} &\simeq -\frac{1}{4} + \left(\frac{\varepsilon}{\kappa}\ln\varepsilon\right)\frac{3}{4} + \frac{\varepsilon}{\kappa}\left\{\frac{3}{4}(\ln\hat{\eta} - 2\tilde{\pi}) - E_1^o + 2E_2^o \right. \\ &\qquad \left. + \frac{1}{2}\kappa\alpha\hat{\eta}\right\} + \cdots \\ \overset{\approx}{v}_{outer} &\simeq -\left(\frac{\varepsilon^2}{\kappa}\ln\varepsilon\right)3\alpha\hat{\eta} + \cdots \\ \overset{\approx}{p}_{outer} &\simeq \frac{\varepsilon}{\kappa}\left\{2(E_1^o - E_2^o) + \cdots\right\} \end{aligned} \tag{5.11}$$

where

$$E_2^o = 4\alpha \int_o^\infty W(\eta) e^{-4\alpha\eta} \, d\eta \qquad (5.12)$$

As discussed in the previous article, this layer differs from the outer region by the involvement of the Reynolds stress $\widetilde{\widetilde{T}}$ in the momentum balance of order ε. If we let

$$\begin{cases} \widetilde{\widetilde{u}} = \left\{ \widetilde{\widetilde{u}}_{outer} - \frac{1}{2}\frac{\varepsilon}{\kappa}\hat{u}(\hat{\eta};\varepsilon) \right\} + \frac{\varepsilon}{\kappa}\hat{\hat{u}}(\hat{\eta};\varepsilon) \\ \widetilde{\widetilde{v}} = \left\{ \widetilde{\widetilde{v}}_{outer} - \frac{3}{2}\frac{\varepsilon^2}{\kappa}\hat{v}(\hat{\eta};\varepsilon) \right\} + \frac{\varepsilon^2}{\kappa}\hat{\hat{v}}(\hat{\eta};\varepsilon) \\ \widetilde{\widetilde{p}} = \widetilde{\widetilde{p}}_{outer} + \hat{\hat{p}} \\ \widetilde{\widetilde{T}} = \varepsilon^2 \hat{\hat{T}} \end{cases} \qquad (5.13)$$

where $\hat{u}$ and $\hat{v}$ are the corresponding first-order (of $a\alpha$) intermediate expansions defined in Eq. (4.18), then by using the differential equations for $\hat{u}$, $\hat{v}$ and $\hat{T}$ given in Eq. (4.20), the complicated governing equations (5.3 a, b, c, d) for this layer are simplified as

$$2\alpha\,\hat{\hat{u}} + \frac{d\hat{\hat{v}}}{d\hat{\eta}} = O + O(\varepsilon) \qquad (5.14a)$$

$$U\,\hat{\hat{u}} = \frac{\kappa}{2i\alpha}\frac{d\hat{\hat{T}}}{d\hat{\eta}} - \frac{\varepsilon}{\kappa}\left\{ \frac{\hat{\hat{v}}}{2\alpha\hat{\eta}} + --- \right\} \qquad (5.14b)$$

$$\frac{d\hat{\hat{p}}}{d\hat{\eta}} = O(\varepsilon^2) \qquad (5.14c)$$

$$-\frac{1}{\kappa}\frac{d\hat{\hat{u}}}{d\hat{\eta}} + \left(\frac{2i\alpha U}{b_o} + \frac{1}{2\kappa\hat{\eta}} \right) \frac{\hat{\hat{T}}}{T^*} = \frac{i\alpha U}{b_o}(\hat{T} - \tfrac{1}{2}\hat{T}^2) + \frac{1}{16\kappa\hat{\eta}}(\hat{T} - 2)^2 + O(\varepsilon) \qquad (5.14d)$$

From Eqs. (5.13) and (5.14c), it follows immediately

$$\tilde{\tilde{p}} = \tilde{\tilde{p}}_{outer} + O(\varepsilon^2) \qquad (5.15)$$

The method of solving the system of equations (5.14 a, b, d) is the same as that of the previous first-order solution which is outlined in Appendix A (p. 89). Define

$$\begin{cases} y \equiv \int_{o^+}^{\hat{\eta}} U(\hat{\eta})d\hat{\eta} \simeq \left\{1 + \frac{\varepsilon}{K}\ln\varepsilon + \frac{\varepsilon}{K}(\ln\hat{\eta} - 2\tilde{\pi} - 1)\right\}\hat{\eta} \\ z_1 \equiv \frac{4i\alpha}{\sqrt{b_o}}\, y\,; \qquad z_1 = 2z \\ k \equiv -\frac{\sqrt{b_o}}{4K} \end{cases} \qquad (5.16)$$

Then neglecting terms of order ε and higher, Eqs. (5.14 b, d) become

$$\begin{cases} \hat{\hat{u}} = -\frac{1}{2k}\frac{d\hat{\hat{\tau}}}{dz_1} & (5.17a) \\ \frac{d^2\hat{\hat{\tau}}}{dz_1} + \left[-\frac{1}{4} + \frac{k}{z_1}\right]\hat{\hat{\tau}} = -G(z) & (5.18b) \end{cases}$$

where

$$G(z) = \frac{1}{16}\left\{(\hat{\tau} - 2)\hat{\tau} - \frac{k}{z}(\hat{\tau}-2)^2\right\} \qquad (5.19)$$

An integral basis for the confluent hypergeometric equation (5.18b) is again formed by a pair of Whittaker's functions

$$W_{k,\frac{1}{2}}(z_1) \quad \text{and} \quad W_{-k,\frac{1}{2}}(-z_1)$$

with the Wronskian given by

$$\tilde{w}\left\{W_{k,\frac{1}{2}}(z_1),\ W_{-k,\frac{1}{2}}(-z_1)\right\} = e^{ik\pi}$$

The forcing function G(z) which characterizes the non-linear effect contains the product of $\hat{\tau}$.

$$\hat{\tau} = \hat{\tau}_0 + i\Delta\,\hat{\tau}_1 \tag{5.19}$$

To be consistent with the first-order expansion, we may treat Δ small, and neglect Δ^2 terms. Let

$$G(z) = G_0(z) + i\Delta\, G_1(z) + O(\Delta^2) \tag{5.20}$$

then

$$\begin{cases} G_0(z) = \dfrac{1}{16}\left\{(\hat{\tau}_0-2)\hat{\tau}_0 - \dfrac{k}{z}(\hat{\tau}_0-2)^2\right\} \\[2ex] G_1(z) = \dfrac{1}{8}\left\{\hat{\tau}_0-1) - \dfrac{k}{z}(\hat{\tau}_0-2)\right\}\hat{\tau}_1 \end{cases} \tag{5.20a}$$

The complete solution for $\hat{\hat{\tau}}$, $\hat{\hat{u}}$ and $\hat{\hat{v}}$ which is finite as $\hat{\eta}\to\infty$, $z_1\to i\infty$ is given by

$$\hat{\hat{\tau}} = A_2\, W_{k,\frac{1}{2}}(z_1) + e^{-ik\pi}\left\{W_{k,\frac{1}{2}}(z_1)\int_0^{z_1} W_{-k,\frac{1}{2}}(-z_1')G(z')dz_1' + W_{-k,\frac{1}{2}}(-z_1)\int_{z_1}^{i\infty} W_{k,\frac{1}{2}}(z_1')G(z')dz_1'\right\} \tag{5.21a}$$

$$\hat{\hat{u}} = \frac{1}{4k}\left[W_{k,\frac{1}{2}}(z_1) - \frac{2k}{z_1^{\frac{1}{2}}}W_{(k-\frac{1}{2}),o}(z_1)\right]\left\{A_2 + e^{-ik\pi}\int_0^{z_1} W_{-k,\frac{1}{2}}(-z_1)G(z')dz_1'\right\}$$
$$-\frac{1}{4k}\left[W_{-k,\frac{1}{2}}(-z_1) + \frac{2k}{(-z_1)^{\frac{1}{2}}}W_{-(k+\frac{1}{2}),o}(-z_1)\right]\left\{e^{-ik\pi}\int_{z_1}^{i\infty} W_{k,\frac{1}{2}}(z_1')G(z')dz_1'\right\} \tag{5.21b}$$

$$\hat{\hat{v}} = i\kappa(\hat{\hat{\tau}} - C_3) \tag{5.21c}$$

As $z \to i\infty$, by using the asymptotic expansion of Whittaker's functions shown in Appendix A, and of $\hat{\tau}$ given by Eq. (4.24a), it can be shown that

$$\hat{\tau} \sim -\frac{b_0}{2}(\sigma_0+2i)\left[1 - \frac{b_0}{2}(\sigma_0+2i)\right]\left\{1 + O\left(\frac{1}{z_1}\right)\right\}$$

$$+ \overline{A}_2 z_1^{-|k|} e^{-\frac{1}{2}z_1}\{1 + O(\frac{1}{z_1})\} \tag{5.22}$$

$$\hat{\hat{u}} \sim \frac{1}{4k} \overline{A}_2 z_1^{-|k|} e^{-\frac{1}{2}z_1}\{1 + O(\frac{1}{z_1})\}$$

where $\overline{A}_2$ is a constant involving A_2 and some constant of integration. The asymptotic expansion of $\hat{\hat{\tau}}$ matches the outer solution (5.6d) automatically. And

$$\tilde{\tilde{u}}_{inter} = \tilde{\tilde{u}}_{outer} + \left(\frac{\varepsilon}{\kappa}\right)\left\{O(z_1^{-|k|})\right\}$$

$$\to \tilde{\tilde{u}}_{outer}, \quad \text{as} \quad \hat{\eta} \to \infty \tag{5.23}$$

As $z \to 0$, by using the power series representation for Whittaker's functions, and let

$$A_2 = A_{20} + i\Delta A_{21}, \tag{5.24}$$

we have

$$\hat{\hat{\tau}} \simeq (\widetilde{A}_{20} + i\Delta\widetilde{A}_{21})\{1 - a_1 z_1 - k z_1 \ln z_1 + --\} + (\widetilde{I}_{20} + i\Delta\widetilde{I}_{21})z_1 + ---$$

$$\hat{\hat{u}} \simeq \tfrac{1}{2}(\widetilde{A}_{20} + i\Delta\widetilde{A}_{21})\{\ln z_1 + 1 + \frac{a_1}{k} + O(z_1 \ln z_1)\}$$

$$- \frac{1}{2k}(\widetilde{I}_{20} + i\Delta\widetilde{I}_{21})[1 - k z_1 + --] \tag{5.25}$$

$$\hat{\hat{v}} \simeq i\kappa\{\widetilde{A}_{20} + i\Delta\widetilde{A}_{21} - C_3 + O(z_1 \ln z_1)\}$$

where

$$\begin{cases} I_{20} = \int_0^{i\infty} W_{k,\frac{1}{2}}(z_1) G_0(z) dz_1 \\ I_{21} = \int_0^{i\infty} W_{k,\frac{1}{2}}(z_1) G_1(z) dz_1 \end{cases} \quad (5.25a)$$

$$\begin{cases} \widetilde{A}_{20} = \dfrac{A_{20}}{\Gamma(1-k)} + \dfrac{I_{20}\, e^{ik\pi}}{\Gamma(1+k)} \\ \widetilde{A}_{21} = \dfrac{A_{21}}{\Gamma(1-k)} + \dfrac{I_{21}\, e^{-ik\pi}}{\Gamma(1+k)} \end{cases} \quad (5.25b)$$

$$\begin{pmatrix} \widetilde{I}_{20} \\ \widetilde{I}_{21} \end{pmatrix} = \Gamma(1-k) \begin{pmatrix} I_{20} \\ I_{21} \end{pmatrix} \quad (5.25c)$$

1.3 Inner Layer:

In the viscous region, the inner variable is

$$\zeta = R\varepsilon\,\eta . \quad (5.26)$$

The primary mean flow velocity and Reynolds stress are

$$U = \varepsilon\, F(\zeta)$$

$$T = \varepsilon^2 T^*(\zeta) ,$$

and the first-order ($a\alpha$) inner-layer solution is described by

$$\begin{cases} \widetilde{u} = \varepsilon\,\{F + f\} \\ \widetilde{v} = \dfrac{\alpha}{R}\, h \\ \widetilde{\tau} = \varepsilon^2\{2T^* + g\} . \end{cases} \quad (4.29)$$

The second-order η momentum equation (5.3c) for this layer becomes

$$\frac{d\tilde{\tilde{p}}}{d\zeta} = o + O(\frac{\varepsilon}{R}) \tag{5.27}$$

which yields

$$\begin{aligned} \tilde{\tilde{p}} &\simeq \tilde{\tilde{p}}_{outer}(o) = (\frac{\epsilon}{K})\tilde{\tilde{p}}_2(o) \\ \text{with} \qquad \tilde{\tilde{p}}_2(o) &= 2(E_1^o - E_2^o) \end{aligned} \tag{5.28}$$

If we let

$$\begin{cases} \tilde{\tilde{u}} = \epsilon\{-\frac{1}{4}F - \frac{1}{2}f + u\} \\ \tilde{\tilde{v}} = \frac{2\alpha}{R}\{-\frac{3}{4}h + v\} \\ \tilde{\tilde{T}} = \epsilon^2 \tau \end{cases} \tag{5.29}$$

and substituting into the governing equations (5.3 a, b, d) expressed in terms of ζ, then by using the relation and energy balance between F and T^*, and the differential equations describing f, g and h given by Eq. (4.30), we obtain a system of simplified equations (dropping * in T)

$$\begin{cases} u + \frac{dv}{d\zeta} = 0 \\ \frac{d^2u}{d\zeta^2} + \frac{d\tau}{d\zeta} = 2i\Delta\left\{\frac{\epsilon}{K}\tilde{\tilde{p}}_2(o) + \epsilon^2(Fu + \frac{1}{4}f^2) + ---\right\} \\ -T\frac{du}{d\zeta} + \left(\frac{3}{2}\frac{T^{\frac{1}{2}}}{L} - \frac{dF}{d\zeta}\right)\tau - \frac{d^2}{d\zeta^2}(\frac{\tau}{b}) = \frac{g}{2}\frac{df}{d\zeta} - \frac{3}{16}\frac{g^2}{LT^{\frac{1}{2}}} + O(\frac{1}{R\epsilon}) \end{cases} \tag{5.30}$$

where in the $i\Delta\epsilon^2$ convection terms, only the part which will be significant as $\zeta >> 1$ is retained.

Now, from the first-order solution, we have

$$\begin{cases} f = \overline{f} + i\Delta \widetilde{f} \\ g = \overline{f} + i\Delta \widetilde{g} \end{cases} \tag{4.31}$$

The leading term of $\overline{f}$ and $\overline{g}$ is $O(\frac{\varepsilon}{K} \ln \varepsilon)$. Therefore, to be consistent with the first-order expansion, terms of order $(\frac{\varepsilon}{K} \ln \varepsilon)^2$ and of order Δ^2 such as $\overline{f}^2$, $\overline{g}^2$ etc. will be neglected in the following. Thus

$$\begin{cases} u + \dfrac{dv}{d\zeta} = 0 \\ \dfrac{d^2u}{d\zeta^2} + \dfrac{d\tau}{d\zeta} = 2i\Delta\left\{\dfrac{\varepsilon}{K}\ \widetilde{\widetilde{p}}_2(o) + \varepsilon^2 F\, u\right\} \\ -T\dfrac{du}{d\zeta} + \left(\dfrac{3}{2}\ \dfrac{T^{\frac{1}{2}}}{L} - \dfrac{dF}{d\zeta}\right)\tau - \dfrac{d^2}{d\zeta^2}\left(\dfrac{\tau}{b}\right) \\ \qquad = \dfrac{i\Delta}{2}\left\{\left(\overline{g}\ \dfrac{d\widetilde{f}}{d\zeta} + \widetilde{g}\ \dfrac{d\overline{f}}{d\zeta}\right) - \dfrac{3}{4}\ \dfrac{\overline{g}\,\widetilde{g}}{L\,T^{\frac{1}{2}}}\right\} \end{cases} \tag{5.31}$$

The process of analysis will be simpler if we pre-match the expression given in Eq. (5.29) with the intermediate expansion and determine the outer condition for u and τ.

$$\begin{aligned} \widetilde{\widetilde{u}}_{\text{inner}} &= \varepsilon\{-\tfrac{1}{4}F - \tfrac{1}{2}f + u\} \\ &= \tfrac{1}{4}\,\varepsilon F - \tfrac{1}{2}\,\varepsilon(F+f) + \varepsilon\, u(\zeta) \qquad (5.32) \\ &\to \tfrac{1}{4}\,U(\hat{\eta}) - \tfrac{1}{2}\,\widetilde{u}_{\text{inner}} + \varepsilon\, u(R\,\varepsilon^2\hat{\eta}), \text{ as } \zeta \to R\varepsilon^2\hat{\eta} \end{aligned}$$

On the other hand, the intermediate expansion for $\widetilde{\widetilde{u}}$ as $\hat{\eta} \to 0$ is

$$\tilde{\tilde{u}}_{\text{inter}} \simeq -\tfrac{1}{4} + \left(\frac{\epsilon}{\kappa}\ell n\,\epsilon\right)\frac{3}{4} + \frac{\epsilon}{\kappa}\left[\frac{3}{4}(\ell n\,\hat{\eta} - 2\tilde{\pi}) - E_1^o + 2E_2^o + \tfrac{1}{2}\,\kappa d\hat{\eta}\right]$$

$$- \tfrac{1}{2}\frac{\epsilon}{\kappa}\hat{u}(\hat{\eta}) + \frac{\epsilon}{\kappa}\hat{\hat{u}}(\hat{\eta})$$

$$= \tfrac{1}{4}U(\hat{\eta}) - \tfrac{1}{2}\tilde{u}_{\text{inter}} + \frac{\epsilon}{\kappa}(\hat{\hat{u}} - 2E_1^o + 2E_2^o) \qquad (5.33)$$

where

$$\begin{cases} U(\hat{\eta}) = 1 + \dfrac{\epsilon}{\kappa}\ell n\,\epsilon + \dfrac{\epsilon}{\kappa}(\ell n\,\hat{\eta} - 2\tilde{\pi}) \\ \tilde{u}_{\text{inter}}(\hat{\eta}) \simeq 1 - \dfrac{\epsilon}{\kappa}\ell n\,\epsilon - \dfrac{\epsilon}{\kappa}(\ell n\,\hat{\eta} - 2\tilde{\pi} + 2E_1^o + \kappa\alpha\hat{\eta}) + \dfrac{\epsilon}{\kappa}\hat{u}(\hat{\eta}) \end{cases}$$

Since the first-order $\tilde{u}_{\text{inter}}$ and $\tilde{u}_{\text{inner}}$ match each other already, therefore the matching condition for u is simply

$$\epsilon\,\{u(\zeta)\}_{\zeta\to\infty} \to \frac{\epsilon}{\kappa}\left\{\hat{\hat{u}}(\hat{\eta}) + 2(E_2^o - E_1^o)\right\}_{\hat{\eta}\to 0} \qquad (5.34)$$

Now, similar to the method employed in the previous article, we assume

$$\begin{cases} u = u_o + i\Delta u_1 \\ v = v_o + i\Delta v_1 \\ \tau = \tau_o + i\Delta\tau_1 \end{cases} \qquad (5.35)$$

The governing equations for u_o, v_o and τ_o are the same as the previous linear case. Hence, the solution must be of the same form as given in Eq. (4.39). In view of the log asymptotic behavior of u_o which grows to a lower order of ϵ for large ζ according to the skin friction law (3.12), it is evident from the matching condition (5.34) that u_o must be of order ϵ. Let

$$u_0 = \frac{\varepsilon}{K} u_{02}$$

$$\left(\frac{du_{02}}{d\zeta}\right)_{\zeta=0} = S_o^* \tag{5.36}$$

For $\zeta >>$, the asymptotic expansion is given by

$$\begin{cases} u_{02} \simeq \dfrac{S_o^*}{2K} \{\ln \zeta + d + O(\frac{1}{\zeta})\} \\ v_{02} \simeq - \dfrac{S_o^*}{2K} \{\zeta(\ln \zeta - 1 + d) - \frac{1}{2K} \ln \zeta + e + O(\frac{1}{\zeta})\} \\ \tau_{02} \simeq \dfrac{2K\zeta - 2}{2K\zeta - 1} \left\{S_o^* + O(\frac{1}{\zeta^2})\right\} + (\text{T.S.T.}) \end{cases} \tag{5.37}$$

Then the equations for $i\Delta$ terms describing the pressure gradient and convection become

$$u_1 + \frac{dv_1}{d\zeta} = 0$$

$$\frac{du_1}{d\zeta} + \tau_1 = S_1^* + 2(\frac{\varepsilon}{K})\left\{\tilde{\tilde{p}}_2(o)\zeta + \varepsilon^2 \int_0^\zeta F u_{02}\, d\zeta\right\} \tag{5.38}$$

$$- T \frac{du_1}{d\zeta} + \left(\frac{3}{2} \frac{T^{\frac{1}{2}}}{L} - \frac{dF}{d\zeta}\right) \tau_1 - \frac{d^2}{d\zeta^2} \left(\frac{\tau_1}{b}\right)$$
$$= \frac{1}{2}\left(\bar{g} \frac{d\tilde{f}}{d\zeta} + \tilde{g} \frac{d\bar{f}}{d\zeta}\right) - \frac{3}{8} \frac{T^{\frac{3}{2}}}{L} \left(\frac{\bar{g}}{T}\frac{\tilde{g}}{T}\right)$$

The integral of the connection term $\varepsilon^3 F u_{02}$ becomes important only when $\zeta >> 1$. We may therefore use the asymptotic function of F and u_{02} for the integrand. To the accuracy of order ε, terms which are at most $O(\varepsilon^2)$ as $\zeta >> 1$ will be neglected.

By using the energy equation for g and f, we obtain an asymptotic solution as

$$\tau_1 \simeq \frac{2\kappa\zeta-2}{2\kappa\zeta-1}\left\{S_1^* + 2\left(\frac{\epsilon}{\kappa}\right)\left[\widetilde{\widetilde{p}}_2(o) + \frac{S_o^*}{2}\left(\frac{\epsilon}{\kappa}\right)^2 \ell n^2 \zeta\right]\zeta\right.$$

$$\left. + \frac{1}{8\kappa\zeta}\left(\frac{\overline{g}}{T}\frac{\widetilde{g}}{T}\right)\right\} + --- \qquad (5.39a)$$

$$u_1 \simeq \frac{1}{2\kappa}\ \ell n\,\zeta\left\{S_1^* + \left(\frac{\epsilon}{\kappa}\right)\widetilde{\widetilde{p}}_2(o) + \left(\frac{\epsilon}{\kappa}\right)^3 \frac{1}{6} S_o^*\ \ell n^2 \zeta\right\}$$

$$+ \left(\frac{\epsilon}{\kappa}\right)\frac{\zeta}{\kappa}\left\{\widetilde{\widetilde{p}}_2(o) + \left(\frac{\epsilon}{\kappa}\right)^2 \frac{S_o^*}{2}\ell n^2 \zeta\right\}$$

$$- \frac{1}{4}\left\{\left(\frac{\epsilon}{\kappa}\ell n\,\epsilon\right)\overline{s}_1 + \frac{\epsilon}{\kappa}\ \overline{s}_2\right\}\widetilde{f} + \text{const} \qquad (5.39b)$$

Applying the skin friction law, then

$$\frac{\epsilon}{\kappa}\ell n\,\zeta = \frac{\epsilon}{\kappa}\ell n\,\mathrm{R}\epsilon^2\hat{\eta} = 1 + \frac{\epsilon}{\kappa}\ \ell n\epsilon + O(\epsilon)\ .$$

The unmatched inner expansion as $\zeta \to \infty$ expressed in terms of $\hat{\eta}$ becomes

$$\epsilon\{u(\zeta)\}_{\zeta\to\infty} \simeq \frac{\epsilon}{\kappa}\left\{\frac{S_o^*}{2}\left(1 + \frac{\epsilon}{\kappa}\ \ell n\ \epsilon + O(\epsilon)\right)\right\}$$

$$+ \frac{i\Delta}{2}\left\{S_1^* + \frac{\epsilon}{\kappa}\left[\widetilde{\widetilde{p}}_2(o) + \frac{1}{6}S_o^*(1+---)\right]\right\}\left(1 + \frac{\epsilon}{\kappa}\ell n\ \epsilon + ---\right)$$

$$+ \left(\frac{\epsilon}{\kappa}\right)\frac{i\alpha}{\kappa}\hat{\eta}\left\{\widetilde{\widetilde{p}}_2(o) + \frac{S_o^*}{2}\left(1 + \frac{\epsilon}{\kappa}\ell n\ \kappa + --\right)\right\}$$

$$- \frac{i\Delta}{4}\left\{\left(\frac{\epsilon}{\kappa}\ell n\ \epsilon\right)\overline{s}_1 + \frac{\epsilon}{\kappa}\overline{s}_2\right\}\epsilon\widetilde{f} + --- \qquad (5.40a)$$

$$\widetilde{\widetilde{\tau}}_{\text{inner}} \simeq \epsilon^2\left\{\frac{\epsilon}{\kappa}S_o^* + \frac{2i\alpha}{\kappa}\hat{\eta}\left[\widetilde{\widetilde{p}}_2(o) + \frac{S_o^*}{2}(1 + ---)\right]\right.$$

$$\left. + i\Delta\ S_1^* + ---\right\} \qquad (5.40b)$$

where $(\epsilon\ \widetilde{f})$ remains at $O(\epsilon)$ as shown in the first-order inner solution.

The intermediate expansion of $\tilde{\tilde{\tau}}$ as $\hat{\eta} \to 0$ is from (5.25) as

$$\tilde{\tilde{\tau}}_{inter} \simeq \varepsilon^2 \Big\{ \tilde{A}_{20} \, [1+O(\hat{\eta} \, \ell n \, \hat{\eta})] - \tilde{I}_{20} \frac{i\alpha}{k} \hat{\eta} \, (1 + ---)$$

$$+ i\Delta \, \tilde{A}_{21} \, [1+O(\hat{\eta} \, \ell n \, \hat{\eta})] + --- \Big\} \tag{5.41a}$$

while the matching condition for εu from (5.34) is

$$\frac{\varepsilon}{\kappa} \left\{ \hat{\hat{u}}(\hat{\eta}) + 2(E_2^o - E_1^o) \right\}_{\eta \to 0}$$

$$= \frac{\varepsilon}{\kappa} \Big\{ \frac{1}{2} \tilde{A}_{20} (\ell n \hat{\eta} + ---) - \frac{1}{2k} \tilde{I}_{20} (1 - \frac{i\alpha}{\kappa} \hat{\eta}) + 2(E_2^o - E_1^o)$$

$$+ \frac{i\Delta}{2} [\tilde{A}_{21} (\ell n \, \hat{\eta} + --) - \frac{1}{k} \tilde{I}_{21} + ----] \Big\} \tag{5.41b}$$

Expanding

$$S_1^* = S_{10}^* + \frac{\varepsilon}{\kappa} \ell n \, \varepsilon \, S_{11}^* + \frac{\varepsilon}{\kappa} S_{12}^*$$

we then obtain

$$\left\{ \begin{aligned} S_{10}^* &= S_{11}^* = 0 \\ \tilde{A}_{20} &= \tilde{A}_{21} = 0 \\ S_o^* &= 4(E_2^o - E_1^o) - \frac{1}{k} \tilde{I}_{20} \\ S_{12}^* &= - \left[\frac{4}{3} (E_2^o - E_1^o) - \frac{1}{6k} \tilde{I}_{20} + \frac{1}{k} \tilde{I}_{21} \right] \end{aligned} \right. \tag{5.42}$$

With these values for constants, the $\hat{\eta}$ terms in the inner solution match the $O(\Delta^o)$ intermediate expansion simultaneously. The constant A_2 for the intermediate expansion is then

$$A_2 = - \frac{\Gamma(1-k)}{\Gamma(1+k)} e^{-ik\pi} [I_{20} + i\Delta \, I_{21}] \tag{5.43}$$

2. Variation to the Mean -- Function of η Alone:

The second order variation to the primary mean flow is purely a non-linear effect due to small but finite disturbance. Within the frame of large Reynolds number and hence small skin friction approximation together with the neglect of the boundary-layer thickness variation, the perturbations are described by the following equations.

$$\frac{d\bar{\bar{v}}}{d\eta} = \mathcal{R}e\left\{-\frac{i\alpha}{2}(\tilde{u}+\tilde{v})e^{-\alpha\eta}\right\} \tag{5.46a}$$

$$\frac{dU}{d\eta}\bar{\bar{v}} - \frac{d\bar{\bar{\tau}}}{d\eta} - \frac{1}{R}\frac{d^2\bar{\bar{u}}}{d\eta^2} = \mathcal{R}e\left\{\frac{\alpha}{2}(2\tilde{\tau}+i\tilde{\sigma}-iU\tilde{v})e^{-\alpha\eta}\right.$$
$$\left. -\frac{i}{2}\tilde{v}\frac{d\tilde{u}^*}{d\eta} + \frac{1}{R}\left[\frac{1}{4}\frac{d^2U}{d\eta^2}e^{-2\alpha\eta} + \frac{1}{2}\frac{d^2\tilde{u}}{d\eta^2}e^{-\alpha\eta}\right]\right\} \tag{5.46b}$$

$$\frac{d\bar{\bar{p}}}{d\eta} = \mathcal{R}e\left\{\alpha e^{-\alpha\eta}(U\tilde{u} - \frac{\tilde{\sigma}}{2} + i\tilde{\tau}) + \alpha\,\tilde{u}\,\tilde{v}^*\right\} \tag{5.46c}$$

$$\frac{d}{d\eta}(\frac{T}{b})\bar{\bar{v}} - \left[T\frac{d\bar{\bar{u}}}{d\eta} + \frac{dU}{d\eta}\bar{\bar{\tau}} - \sigma\frac{d\bar{\bar{v}}}{d\eta}\right] + \frac{3}{2}\frac{T^{\frac{1}{2}}}{L}\bar{\bar{\tau}} - \frac{1}{R}\frac{d^2}{d\eta^2}(\frac{\bar{\bar{\tau}}}{b})$$
$$= \mathcal{R}e\left\{-\frac{\alpha}{2}e^{-\alpha\eta}\left[U(\tilde{\tau}+i\tilde{\sigma}) + T(\tilde{u}-\tilde{v}) + i\sigma\tilde{u}\right]\right.$$
$$+\frac{1}{2}\left[i\alpha\tilde{u}(\frac{\tilde{\tau}^*}{b}) - i\tilde{v}\frac{d}{d\eta}(\frac{\tau^*}{b}) + \tilde{\tau}^*(\frac{d\tilde{u}}{d\eta} - \alpha\tilde{v}) + i\tilde{\sigma}\frac{d\tilde{v}^*}{d\eta} - \frac{3}{8}\frac{\tilde{\tau}\tilde{\tau}^*}{LT^{\frac{1}{2}}}\right]$$
$$\left. +\frac{1}{R}\left[\frac{1}{4}\frac{d^2}{d\eta^2}(\frac{T}{b})e^{-2\alpha\eta} + \frac{1}{2}\frac{d^2}{d\eta^2}(\frac{\tilde{\tau}}{b})e^{-\alpha\eta}\right]\right\} \tag{5.46d}$$

where * quantity signifies its complex conjugate.

The continuity equation (5.46a) can be written as

$$\frac{d\bar{\bar{v}}}{d\eta} = \mathcal{R}e\left\{\frac{i}{2}\frac{d}{d\eta}(\tilde{v}e^{-\alpha\eta}) - \frac{i}{2}(\alpha\,\tilde{u} + \frac{d\tilde{v}}{d\eta})e^{-\alpha\eta}\right\}$$

From the first-order $(a\alpha)$ continuity,

$$\alpha \tilde{u} + \frac{d\tilde{v}}{d\eta} = \alpha U e^{-\alpha\eta} .$$

Since $U(\eta)$ is real, it follows immediately

$$\bar{\bar{v}} = \mathcal{R}e\left(\frac{i}{2}\tilde{v}\, e^{-\alpha\eta}\right) = O(\varepsilon^2) \tag{5.47}$$

Using the outer solution of $\tilde{u}$ and $\tilde{v}$ obtained in the previous article and neglecting terms of order ε^2 and higher, one simple integration of Eq. (5.46c) yields

$$\bar{\bar{p}} \simeq -\tfrac{1}{2}(1 - \frac{\varepsilon}{\kappa}\, 2E_1^0)e^{-2\alpha\eta} + O(\varepsilon^2) \tag{5.48}$$

The streamwise ξ momentum equation (5.46b) is ill-posed in both the outer region $\eta = O(1)$ and the intermediate layer $\hat{\eta} = O(1)$, because

a) The outer solution for $\tilde{u}$ and $\tilde{v}$ is real only up to $O(\varepsilon)$. Hence contribution from the real part of the forcing function is $O(\varepsilon^2)$. Terms of order ε^2, however, involve the variation of the boundary-layer thickness which has been altogether neglected.

b) If we ignore the outer solution of $\bar{\bar{\tau}}$ altogether, then in the intermediate layer $\hat{\eta} = O(1)$, the equation becomes

$$\frac{d\bar{\bar{\tau}}}{d\hat{\eta}} = O(\varepsilon^3)$$

If the constant of integration of $O(\varepsilon^2)$ is determined by the matching with the inner layer expansion, the condition that $\bar{\bar{\tau}}$ vanishes as $\eta \to \infty$ would remain unsatisfied.

Since $\bar{\bar{v}}$ is known, the differential equation for $\bar{\bar{\tau}}$, uncoupled from $\bar{\bar{u}}$ in the outer region, is first order. Only with a typical forcing function, could the solution possibly satisfy two conditions. Aiming at this typical forcing function, we examine

$$\frac{i}{2}\,\tilde{v}\,\frac{d\tilde{u}^*}{d\eta} = \frac{i}{2}\,\frac{d}{d\eta}\,(\tilde{u}^*\tilde{v}) - \frac{i}{2}\,\tilde{u}^*\,\frac{d\tilde{v}}{d\eta}$$

Applying the equation of continuity to replace the derivative of $\tilde{v}$ by U and $\tilde{u}$, and using the relation

$$\mathcal{R}e\,(i\,\tilde{u}\,\tilde{u}^*) \equiv 0$$

$$\mathcal{R}e\,(-\,i\alpha\,\tilde{u}^*) = \mathcal{R}e(i\alpha\,\tilde{u})$$

we then obtain

$$\mathcal{R}e\left(\frac{i}{2}\,\tilde{v}\,\frac{d\tilde{u}^*}{d\eta}\right) = \mathcal{R}e\left\{\frac{i}{2}\,\frac{d}{d\eta}\,(\tilde{u}^*\tilde{v}) + \frac{i}{2}\,\alpha\,U\,\tilde{u}\,e^{-\alpha\eta}\right\}$$

Substituting this expression together with the known $\bar{\bar{v}}$ into Eq. (5.46b), using both the streamwise (ξ) and the normal (η) momentum equations (4.2 b, c) for the first-order perturbations to cancel all the momentumwise balanced terms, we obtain

$$\frac{d\bar{\bar{\tau}}}{d\eta} + \frac{1}{R}\,\frac{d^2\bar{\bar{u}}}{d\eta^2} = \mathcal{R}e\left\{\frac{1}{2}\,\frac{d}{d\eta}(\tilde{\tau}\,e^{-\alpha\eta}) + \frac{1}{R}\,\frac{1}{4}\,\frac{d^2U}{d\eta^2}\,e^{-2\alpha\eta}\right\}$$

$$+\mathcal{R}e\left\{\frac{i}{2}\,\frac{d}{d\eta}\,(\tilde{u}^*\tilde{v}) + \frac{i}{2}\,\frac{d}{d\eta}\,[(\tilde{p}-\tilde{\sigma}_2)e^{-\alpha\eta}]\right\} \quad (5.49)$$

Now the real part of the terms inside the second brace is non-zero only with the $O(\varepsilon^2)$ expansion of $\tilde{u}$, $\tilde{v}$ and $\tilde{p}$ which has been neglected. Therefore, it is consistent in the analysis to take

$$\bar{\bar{\tau}} = \mathcal{R}e(\tfrac{1}{2}\,\tilde{\tau}\,e^{-\alpha\eta})\,, \quad (5.50)$$

valid for both outer and intermediate region.

To construct a solution for $\bar{\bar{u}}$, we again divide the region of interest into three layers. For each layer, the corresponding linear first-order matched solution, together with the governing equations if needed, will be used to simplify the energy equation (5.46d). The processes of simplification are quite involved. In the following, only the simplified version is presented.

2.1 Outer Region:

$$\frac{d\bar{\bar{u}}}{d\eta} = -\frac{\alpha}{2} e^{-2\alpha\eta}\left\{1 + \frac{\varepsilon}{\kappa}\left[W + 4E_1 - 6E_1^o - \frac{1}{\alpha}\frac{dW}{d\eta}\right.\right.$$

$$\left.\left. - \frac{1}{2} b_o^2\left(1 + \frac{\sigma_o^2}{4}\right)\frac{1}{\alpha}\frac{dW}{d\eta}\right] + O(\varepsilon^2)\right\} \qquad (5.51)$$

where $T^{\frac{1}{2}}/L$ has been approximated by $U'(\eta)$. Integrating once, we obtain

$$\bar{\bar{u}} \simeq \frac{1}{4}\left\{1 + \frac{\varepsilon}{\kappa}\left[2W + b_o^2\left(1 + \frac{\sigma_o^2}{4}\right)(W - E_1) - E_1 + 4E_3 - 6E_1^o\right] + --\right\} e^{-2\alpha\eta} \qquad (5.52)$$

where

$$E_3 = 4\alpha^2 e^{2\alpha\eta} \int_\eta^\infty \int_{\eta'}^\infty W(\eta'')e^{-2\alpha\eta''}\, d\eta''\, \eta'$$

$$= 2\alpha e^{2\alpha\eta} \int_\eta^\infty E_1(\eta')e^{-2\alpha\eta'}\, d\eta' \qquad (5.52a)$$

2.2 Intermediate Layer:

$$\frac{d\bar{\bar{u}}}{d\hat{\eta}} = \varepsilon\left\{-\frac{\alpha}{2} + \frac{1}{2\kappa}\frac{d\hat{u}}{d\hat{\eta}} - \frac{1}{2\kappa\hat{\eta}} - \frac{1}{16\kappa\hat{\eta}}\hat{\tau}\hat{\tau}^*\right\} + O(\varepsilon^2) \qquad (5.53)$$

where $\hat{u}$ and $\hat{\tau}$ are functions of z defined in Eq. (4.18) and

$$z = \frac{2i\alpha}{\sqrt{b_o}} U(\hat{\eta})\hat{\eta}$$

$$\hat{u} = \frac{2\kappa}{\sqrt{b_o}} \frac{d\hat{\tau}}{dz}$$

Now, in terms of $\hat{\eta}$, the outer expansion as $\eta \to 0$, becomes

$$\bar{\bar{u}}_{outer} \simeq \tfrac{1}{4} - \frac{\epsilon}{\kappa} \ell n\, \epsilon \left[\tfrac{1}{2} + \frac{b_o^2}{4}\left(1 + \frac{\sigma_o^2}{4}\right)\right]$$

$$- \frac{\epsilon}{\kappa}\left\{\left[\tfrac{1}{2} + \frac{b_o^2}{4}\left(1 + \frac{\sigma_o^2}{4}\right)\right]\ell n\, \hat{\eta} + A_3 + \tfrac{1}{2}\kappa\alpha\hat{\eta}\right\} + \text{---} \quad (5.54)$$

where

$$A_3 = \frac{7}{4} E_1^o - E_3^o - \widetilde{\pi} + \frac{b_o^2}{4}\left(1 + \frac{\sigma_o^2}{4}\right)(E_1^o - 2\widetilde{\pi})$$

$$E_3^o = 2\alpha \int_o^\infty E_1(\eta) e^{-2\alpha\eta}\, d\eta \quad (5.54a)$$

The intermediate solution for $\bar{\bar{u}}$ which matches its outer expansion is

$$\bar{\bar{u}}_{inter} \simeq \tfrac{1}{4} - \frac{\epsilon}{\kappa} \ell n\, \epsilon \left[\tfrac{1}{2} + \frac{b_o^2}{4}\left(1 + \frac{\sigma_o^2}{4}\right)\right]$$

$$+ \frac{\epsilon}{\kappa}\left\{\tfrac{1}{2}\hat{u} - \left(\tfrac{1}{2} + \tfrac{1}{16}\hat{\tau}\hat{\tau}^*\right)\ell n\, \hat{\eta} - \tfrac{1}{2}\kappa\alpha\hat{\eta} - A_3\right.$$

$$\left. - \frac{\sqrt{b_o}}{16\kappa}\int_z^{i\infty}\left(\ell n\, z' - \ell n \frac{2i\alpha}{\sqrt{b_o}}\right)\hat{u}\,\hat{\tau}^*\, dz'\right\} + \text{---} \quad (5.55)$$

where only the real part is to be taken, and

$$\left\{\begin{aligned} \hat{\tau}\hat{\tau}^* &= |\hat{\tau}|^2 \to 4\, b_o^2\left(1 + \frac{\sigma_o^2}{4}\right), \\ \hat{u} &\to -. \end{aligned}\right. \qquad \text{as } \hat{\eta} \to \infty$$

2.3 Inner Layer $\zeta = R\varepsilon\eta$:

If we let

$$\begin{aligned} \bar{\bar{u}} &= \varepsilon\{\frac{F}{4} + u^+\} \\ \bar{\bar{\tau}} &= \varepsilon^2\{\tfrac{1}{2}(2T+g) + \tau^+\} \end{aligned} \tag{5.56}$$

Then the governing equation for the layer is simplified as

$$\begin{aligned} &\frac{d^2u^+}{d\zeta^2} + \frac{d\tau^+}{d\zeta} = 0 + O(i\Delta) \\ &T\frac{du^+}{d\zeta} + \left(\frac{dF}{d\zeta} - \frac{3}{2}\frac{T^{\frac{1}{2}}}{L}\right)\tau^+ - \frac{d^2}{d\zeta^2}\left(\frac{\tau^+}{b}\right) = 0 + O(i\Delta;\Delta^2) \end{aligned} \tag{5.57}$$

The real part of the $i\Delta$ terms is $O(\varepsilon^2\Delta)$. With the neglecting of the Δ^2 terms as before, the asymptotic solution is given by

$$\begin{aligned} u^+ &= \frac{s^+}{2\kappa}\left[\ln\zeta + d + O(\frac{1}{\zeta})\right] + (\text{T.S.T.}) \\ \tau^+ &= \frac{2\kappa\zeta-2}{2\kappa\zeta-1}\left[s^+ + O(\frac{1}{\zeta^2})\right] + (\text{T.S.T.}) \end{aligned} \tag{5.58}$$

where $$s^+ = s_o^+ + \frac{\varepsilon}{\kappa}\ln\varepsilon\, s_1^+ + \frac{\varepsilon}{\kappa}s_2^+ \tag{5.58a}$$

Intermediate-Inner Matching

As $\zeta \to \infty$, in terms of the intermediate variable $\hat{\eta}$, the inner solution becomes

$$\begin{aligned} \bar{\bar{u}}_{inner} &\simeq \tfrac{1}{4}\left\{1 + \frac{\varepsilon}{\kappa}\ln\varepsilon + \frac{\varepsilon}{\kappa}(\ln\hat{\eta} - 2\hat{\pi}\right\} \\ &\quad + \tfrac{1}{2}(s_o^+ + \frac{\varepsilon}{\kappa}\ln\varepsilon\, s_1^+ + \frac{\varepsilon}{\kappa}s_2^+)\ \{1 + \frac{\varepsilon}{\kappa}\ln\varepsilon + O(\frac{\varepsilon}{\kappa})\} \end{aligned} \tag{5.59}$$

$$\bar{\bar{\tau}}_{inner} \simeq \varepsilon^2\{1 + s_o^+ + O(\frac{\varepsilon}{\kappa}\ln\varepsilon)\}$$

On the other hand, the intermediate expansion, as $\hat{\eta} \to 0$

$$u(\hat{\eta}) \to 2 \ln \hat{\eta} + \hat{u}(0)$$

$$\tau(\eta) \to 2 + \frac{i\Delta}{3\kappa} \quad ,$$

becomes

$$\bar{\bar{u}}_{inter} \simeq \tfrac{1}{4} - \frac{\epsilon}{\kappa} \ln \epsilon \left[\tfrac{1}{2} + \frac{b_o^2}{4}\left(1 + \frac{\sigma_o^2}{4}\right)\right]$$

$$+ \frac{\epsilon}{\kappa}\left[\tfrac{1}{4} \ln \hat{\eta} + \tfrac{1}{2}\hat{u}(0) - A_3 - I_3 - \tfrac{1}{2}\kappa\alpha\hat{\eta}\right] \qquad (5.60)$$

$$\bar{\bar{\tau}}_{inter} \simeq \epsilon^2 \{1 + O(\hat{\eta} \ln \hat{\eta})\}$$

where

$$I_3 = \mathcal{R}e \, \frac{\sqrt{b_o}}{16\kappa} \int_0^{i\infty} \left(\ln z - \ln \frac{2i\alpha}{\sqrt{b_o}}\right) \hat{u} \, \hat{\tau}^* \, dz \qquad (5.60a)$$

Therefore matching requires

$$s_o^+ = 0$$

$$s_1^+ = -\tfrac{1}{2}\left[3 + b_o^2\left(1 + \frac{\sigma_o^2}{4}\right)\right] \qquad (5.61)$$

$$s_2^+ = \hat{u}(0) - 2(A_3 + I_3) + \tilde{\pi}$$

The composite solution which is valid uniformly for $0 \le \eta \le \infty$ is then given by

$$\bar{\bar{u}} \simeq \frac{1}{4} e^{-2\alpha\eta} \Bigg\{ U + \frac{\epsilon}{\kappa} \Bigg[3W + b_o^2 \left(1 + \frac{\sigma_o^2}{4}\right)(W - E_1) - E_1 + 4E_3 - 6E_1^o$$

$$+ 2\hat{u}(\hat{\eta}) - \frac{1}{4}\left(|\hat{\tau}(\hat{\eta})|^2 - |\hat{\tau}(\infty)|^2\right) \ln \hat{\eta}$$

$$- \frac{\sqrt{b_o}}{4\kappa} \int_z^{i\infty} \left(\ln z' - \ln \frac{2i\alpha}{\sqrt{b_o}}\right) \hat{u}\, \hat{\tau}^* \, dz'$$

$$- \kappa \left(u^+(\zeta) - u^+_{asymp}\right) \Bigg] + --- \Bigg\} \qquad (5.62a)$$

where $u^+(\zeta)$ for $\zeta = O(1)$ is obtained by the numerical method.

VI. RESULTS AND DISCUSSION

One of the main goals of the present study is to investigate two basic questions: (a) Is the chosen model equation for the turbulent shear stress applicable to this specific problem? (b) How much does the second order non-linear effect contribute? For direct comparison with the experimental data of Sigal, the results of the previous analysis are presented below in Cartesian coordinates (see Appendix C).

1. Velocity profiles:

A comparison of velocity distributions for four stations along the wave is shown in Figures 2-5. Over most of the layer, except in the immediate neighborhood of the wall, the linear first-order theory agrees well with the experiment. Profiles including second-order solution are shown in Figures 3 and 5. A comparison with the first-order theory shows that the non-linear effect is insignificant for wave slope less than about 0.2.

2. Reynolds stress profiles:

Comparisons of the detailed structure of the turbulent shear stress distributions for 8 stations along the wave are shown in Figures 6 and 7. The agreement is only qualitative. The oscillatory behavior of the theoretical solution is the result of the model energy equation which is hyperbolic. Near the wall, the calculated period of oscillation is only about half of that measured.

It is worth noticing that the strong variation of C_T near the wall, as shown in the figures, is not caused by the non-linear effect. It is the result of the presentation in Cartesian coordinates, since $\tau_{xy} = -\rho\,\overline{u'_x u'_y}$ contains the horizontal component of the streamwise normal stress $\rho\,\overline{u'^2}$ which is large compared to $\rho\,\overline{u'v'}$ in the wall region for high Reynolds number turbulent boundary layer.

3. Surface pressure distributions:

$$C_{p_w} \equiv \frac{\bar{P}_w - P_o}{\frac{1}{2}\rho U_\infty^2}$$

$$= -2(a\alpha)\left[1 - \frac{\epsilon}{\kappa} 2 E_1^o + O(\epsilon^2)\right]\cos\alpha x$$

$$- (a\alpha)^2\left[1 + \frac{\epsilon}{\kappa}(4 E_2^o - 6 E_1^o) + ---\right]\cos 2\alpha x$$

Figures 8 and 9 show a comparison of wall pressure measurements with the first and second order theory. The agreement is satisfactory except possibly for a very small phase shift. From the governing equations, it is easy to see that only if the perturbed shear stress τ' is of order ϵ (while the primary $T = O(\epsilon^2)$) somewhere in the boundary layer, could $\tilde{p}(\eta)$ obtain an imaginary solution of order ϵ which would account for the possible phase shift. However, the chosen model equation for the shear stress together with the boundary conditions do not permit τ' to have order of ϵ solution throughout the whole layer.

Contribution from the solution of order ϵ^2 including the variation of the boundary thickness and the normal velocity V is analyzed. The phase shift is found to be

$$\varphi \simeq \frac{2\epsilon^2(1 + E_4^o - E_5^o)}{1 - \frac{\epsilon}{\kappa} 2 E_1^o}$$

where

$$E_4^c = -b_o \alpha \int_0^\infty \sigma(\eta) e^{-2\alpha\eta} d\eta > 0 ,$$

$$E_5^c = \frac{1}{\theta^*} \int_0^\infty (4\alpha\eta - 2\alpha^2\eta^2 - 1) W e^{-2\alpha\eta} d\eta$$

For $\alpha = O(1)$ and $\epsilon^2 \simeq 0.0015$, φ is indeed vanishingly small.

4. Surface shear stress distributions:

The most important part of the present study is to examine the result of the surface shear stress,

$$\tau_w = \frac{\mu}{h} \left(\frac{\partial \bar{u}}{\partial \eta}\right)_{\eta=0} ,$$

which performs a significant role in the process of cross-hatching ablation.

Comparisons of the measured skin friction distributions by Sigal and Kendall with the first and the second order theory are shown in Figures 10-12. Away from the vicinity of maximum slope where

$$\frac{\nu}{\rho u_\tau^3} \left(\frac{\partial \bar{p}}{\partial x}\right) \simeq 0.07 ,$$

Sigal's measurements for WW1 and WW2 with Preston tube must be reliable. Prediction of the phase shift is satisfactory. The predicted magnitude of the perturbed skin friction is twice as large as that measured.

As a reference, a qualitative comparison with Kendall's measurement is made in Fig. 12. Disagreement is profound. In fact, the Reynolds number of his experiment is relatively small, and the pressure gradient parameter Δ is quite large ($\simeq 1.75$), while the theoretical analysis is based on the large Reynolds number approximation. No definite conclusion could be drawn from this comparison.

5. Discussions:

In the outer part of the layer the perturbed flow is practically "inviscid." Predictions for the velocity and Reynolds stress, as well as the wall pressure distributions which are mainly determined by the momentum transport in the outer layer, agree well with the experimental measurements. In the wall region where the perturbed Reynolds stress plays a significant role in the process of momentum transport, only a qualitative agreement is obtained.

In his comment upon Davis' (1972) visco-elastic response of turbulent fluid, Townsend (1972) pointed out that use of the energy equation immediately introduces a visco-elastic response with a relaxation time as large as is plausible. He concluded that for wave slopes greater than 0.1, nonlinear behavior becomes significant, and it seems unlikely that linear theory can describe the observation.

As a matter of fact for small amplitude wave, the flow tends to follow along the wave surface. If the problem is formulated in Cartesian coordinates instead of orthogonal wavy coordinates, we have from the transformation (see Appendix C)

$$\overline{u'_x u'_y} \simeq \overline{u'v'} - a\alpha(\overline{u'^2} - \overline{v'^2})e^{-\alpha\eta} \sin\alpha\xi$$

and

$$\begin{aligned}\tau_{xy} &= T_{xy} + a\alpha\, \tau'_{xy} \\ &= T + a\alpha(\tau' - \sigma e^{-\alpha\eta} \sin\alpha\xi) .\end{aligned}$$

In the wall region the difference of the primary normal stress σ is almost 10 times as large as T for high Reynolds number boundary layer. Therefore, for $a\alpha \simeq 0.1$, the perturbed shear stress will be of the same order of magnitude as the primary undisturbed one if viewed from Cartesian coordinates.

The results of the present study show that the nonlinear and second-order effect is insignificant for amplitude to wavelength ratio of 0.03 (wave slope $a\alpha \simeq 0.2$). The discrepancies in the detailed structure of the velocity, shear stress, and skin friction distributions near the wall suggest modifications to the model are required to describe the present problem.

Formulation of the problem in orthogonal coordinates immediately introduces a curvature effect. Use of the energy equation already incorporates the convective response--a visco-elastic behavior. Consequently, the empirical functions which relate the turbulent motion to the shear stress require a further examination. Particular attention should be paid to the flow close to the surface.

According to the order of magnitude analysis based on the skin friction discussed in Article III, the streamwise and the normal turbulent diffusions are insignificant in the wall region. Figures 13 and 14 show comparisons of the surface shear stress profiles obtained with constant and variable stress-to-energy ratio b in the wall region. The difference is very small. The important questions we may examine are then: How is the dissipation of turbulent energy modified in the presence of pressure gradient and convection?

Laminarization of equilibrium turbulent boundary layers can occur in fairly large favorable pressure gradient. What is the three-dimensional effect associated with flow over wavy boundaries? The strength of streamwise counterrotating vortices in turbulent boundary layer is expected to be amplified on the concave portion of the wave surface because of Taylor-Görtler instability.

Further experimental and theoretical investigations are clearly needed for better understanding of turbulent flows over wavy boundaries.

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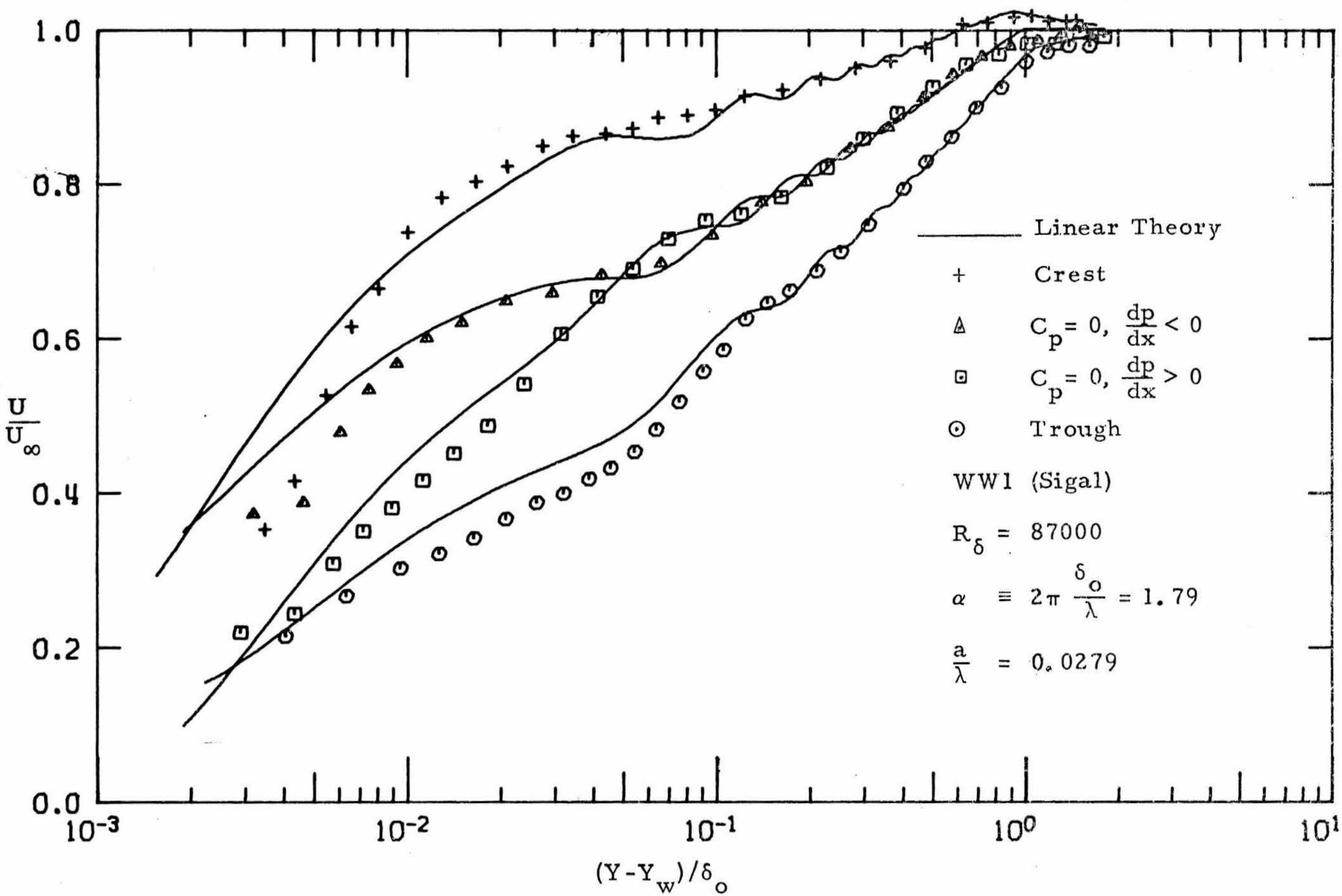


Fig. 2 Velocity Distributions

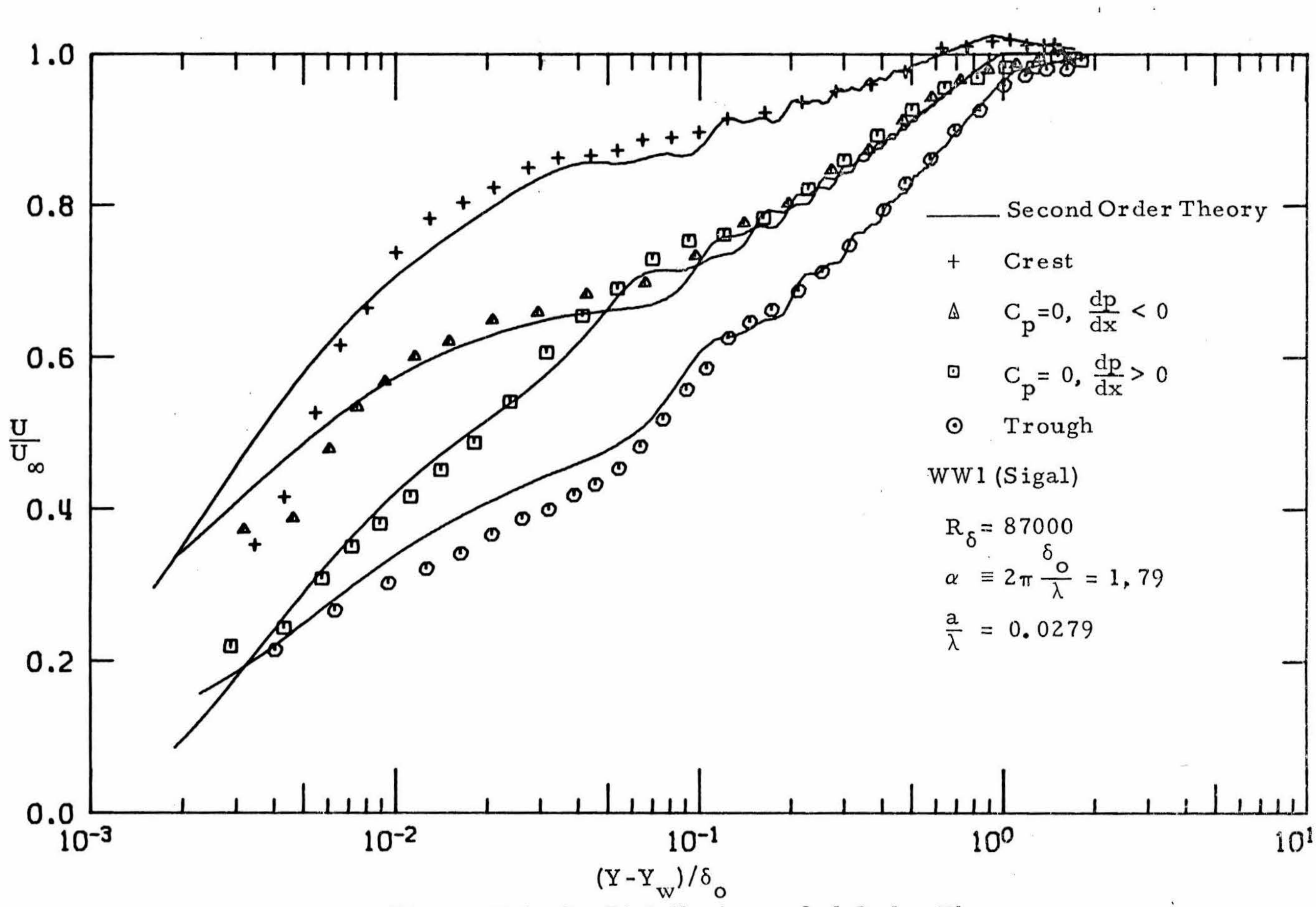


Fig. 3 Velocity Distributions--2nd Order Theory

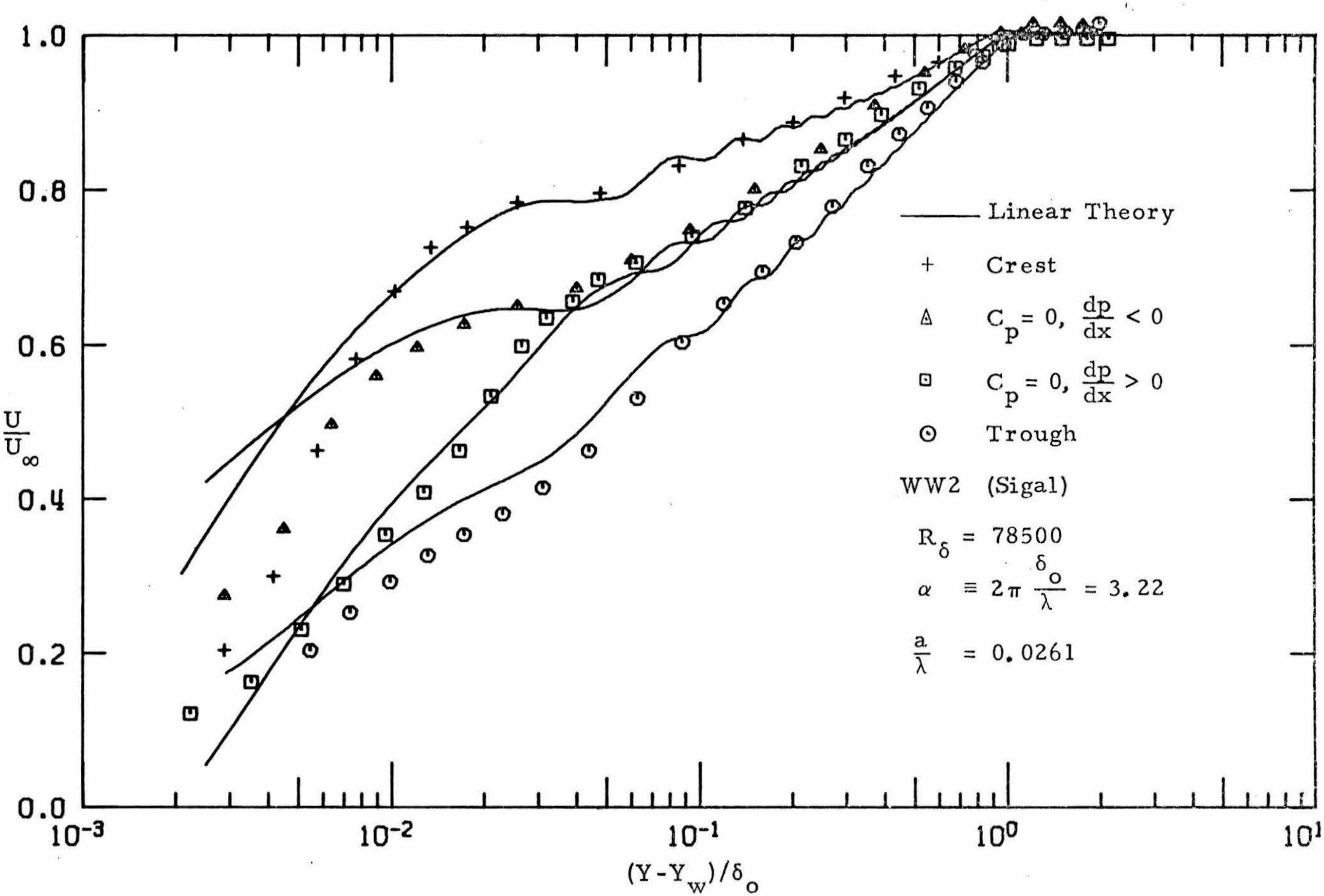


Fig. 4 Velocity Distributions

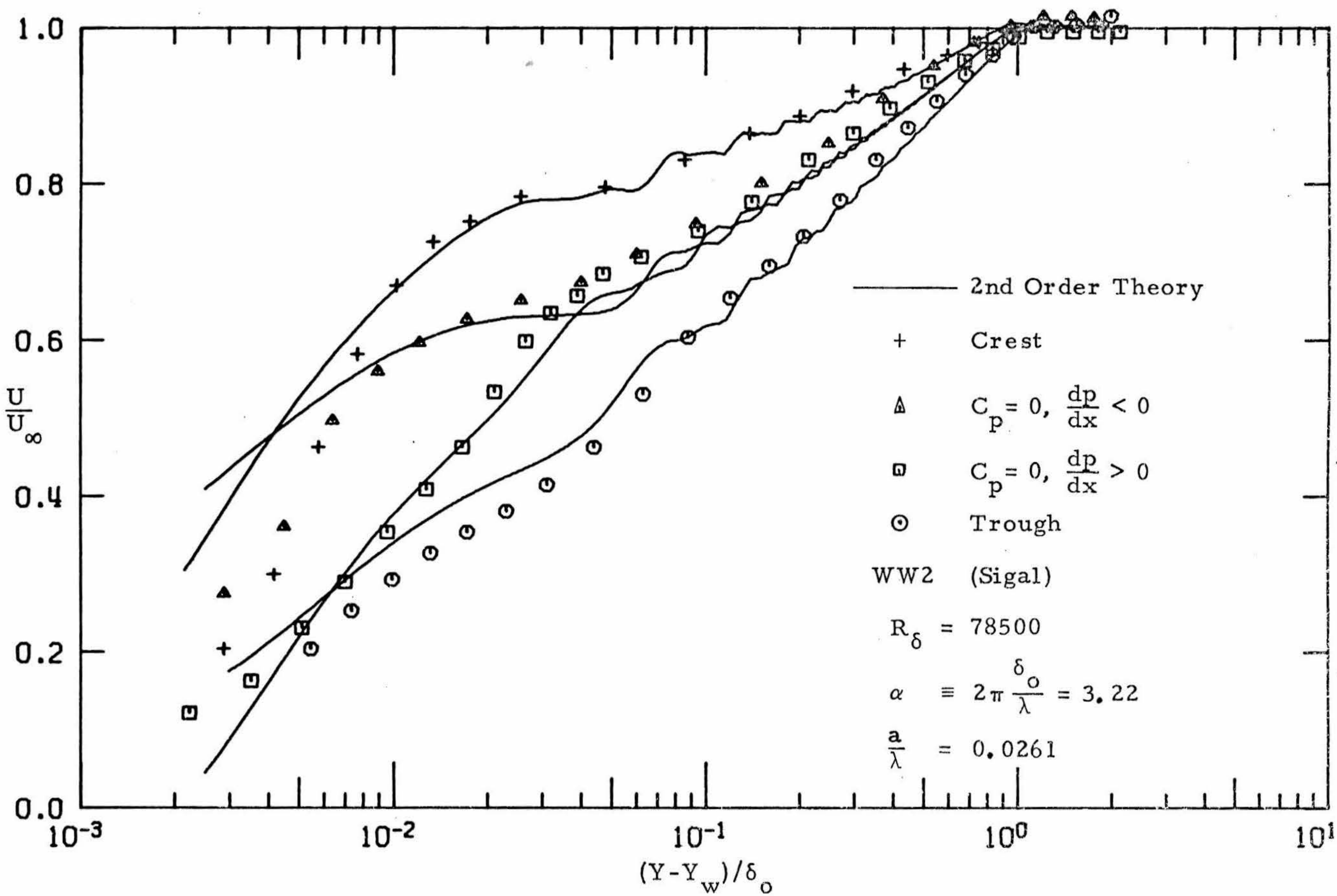


Fig. 5 Velocity Distributions--2nd Order Theory

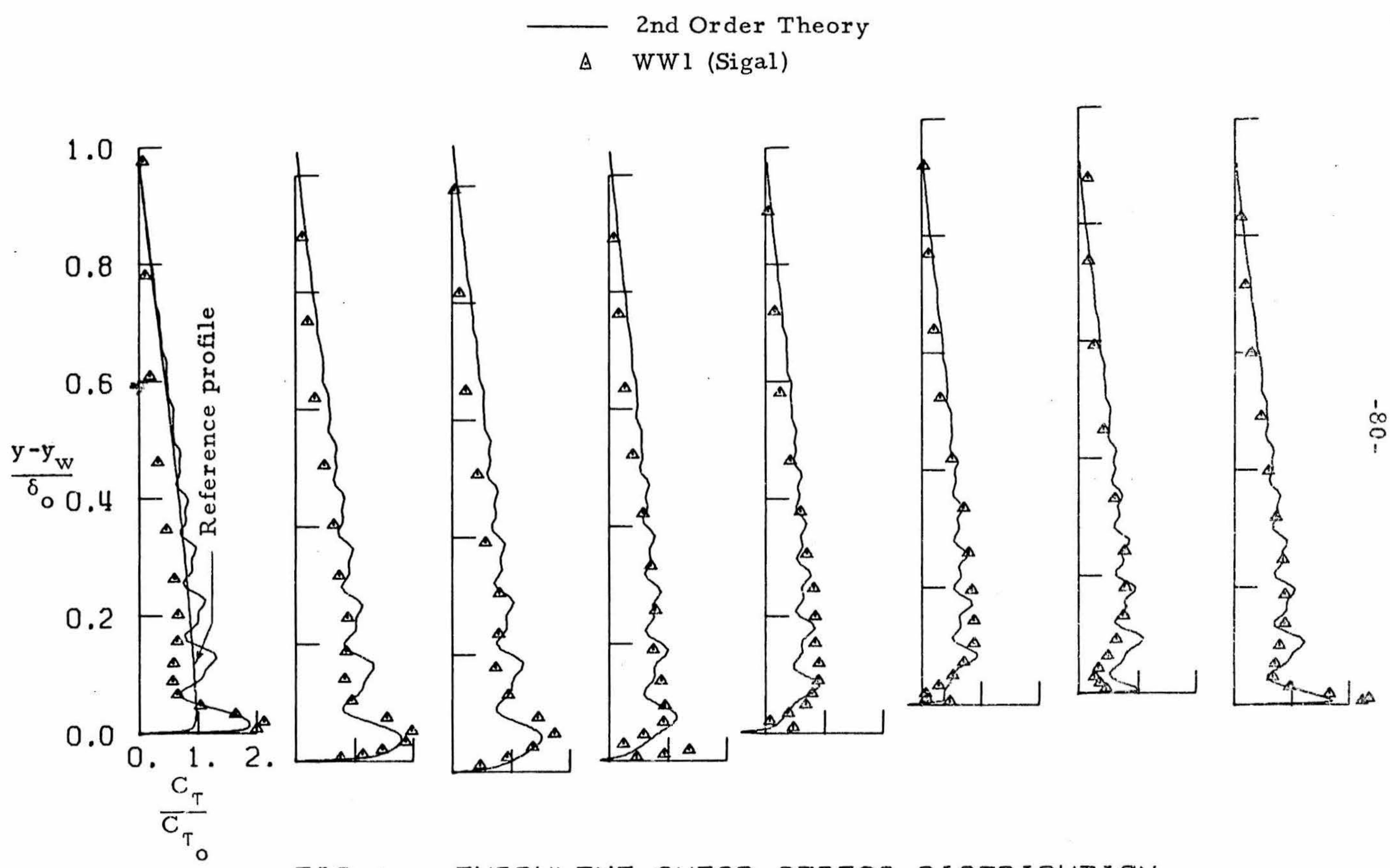


FIG 6 TURBULENT SHEAR STRESS DISTRIBUTION

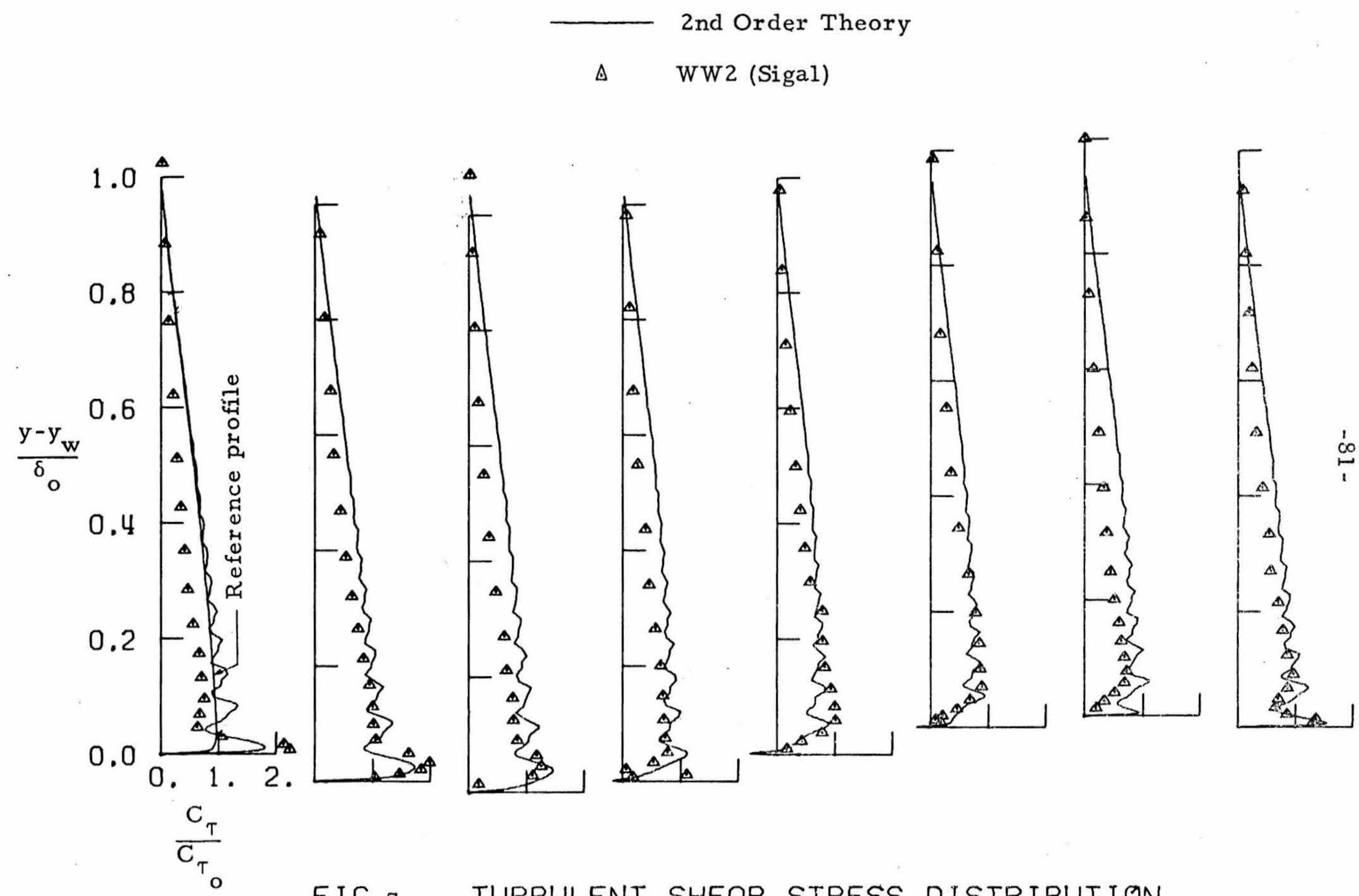


FIG 7 TURBULENT SHEAR STRESS DISTRIBUTION

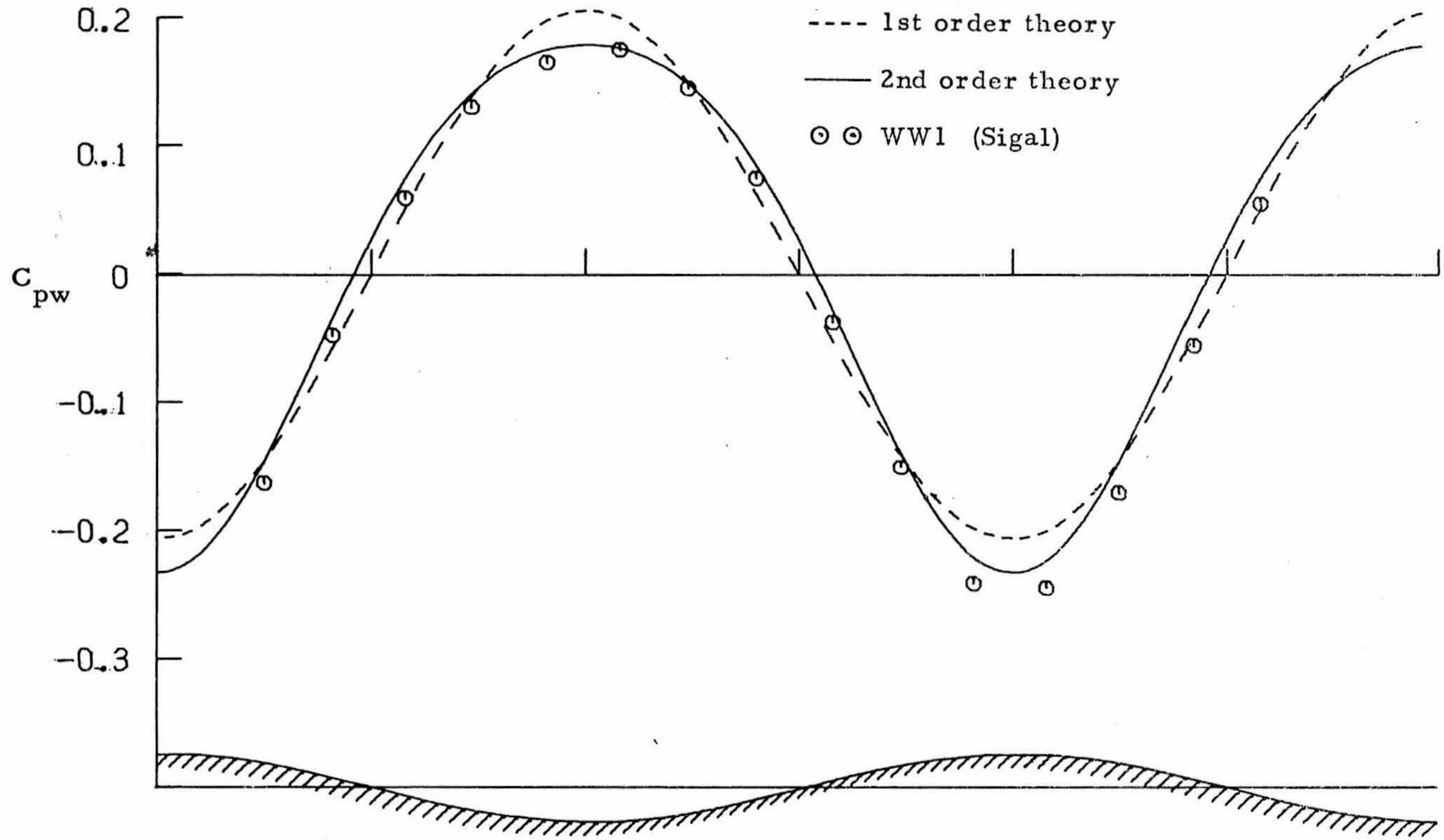


FIG. 8 COMPARISON OF WALL PRESSURE DISTRIBUTIONS

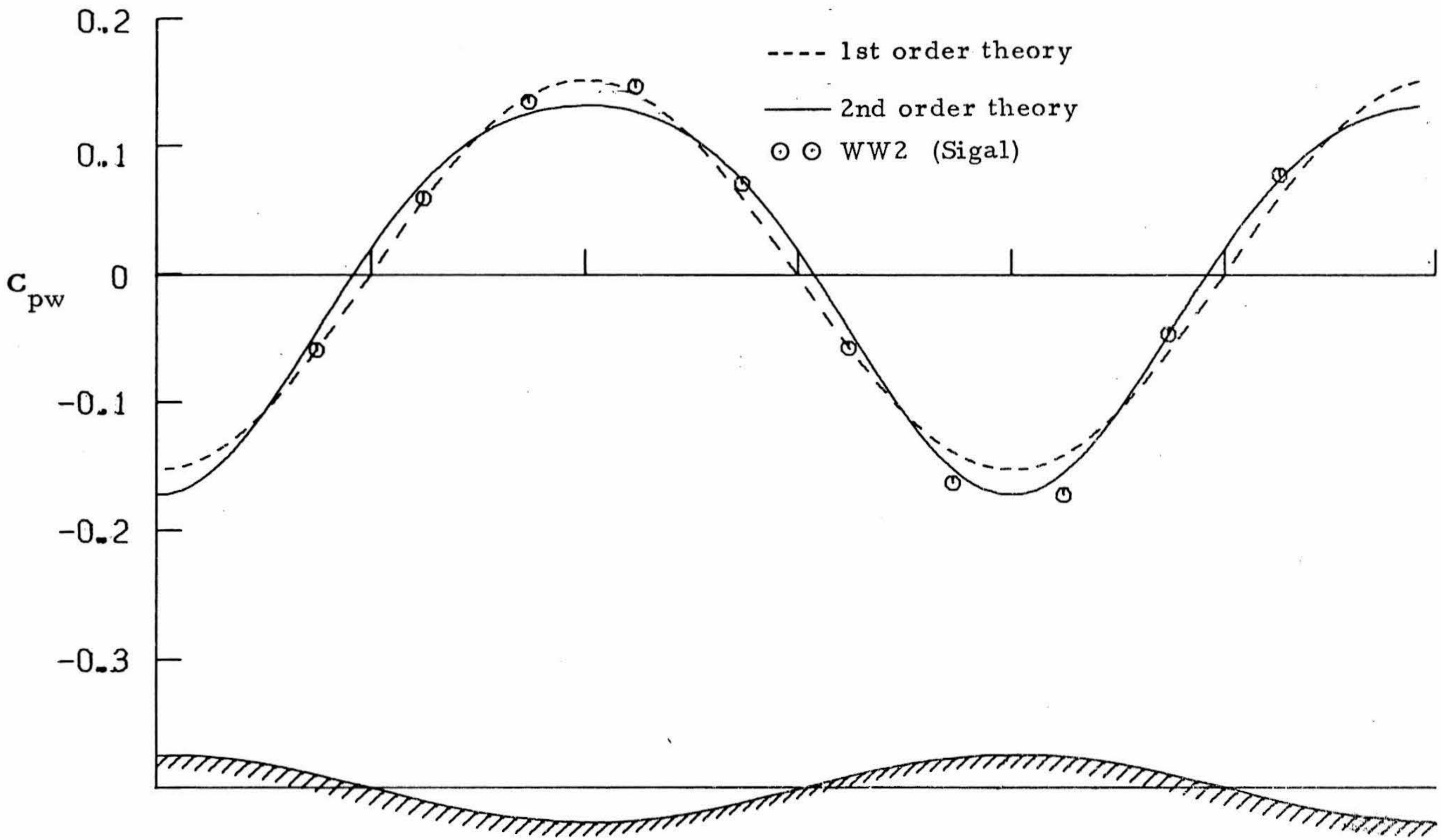


FIG. 9 COMPARISON OF WALL PRESSURE DISTRIBUTIONS

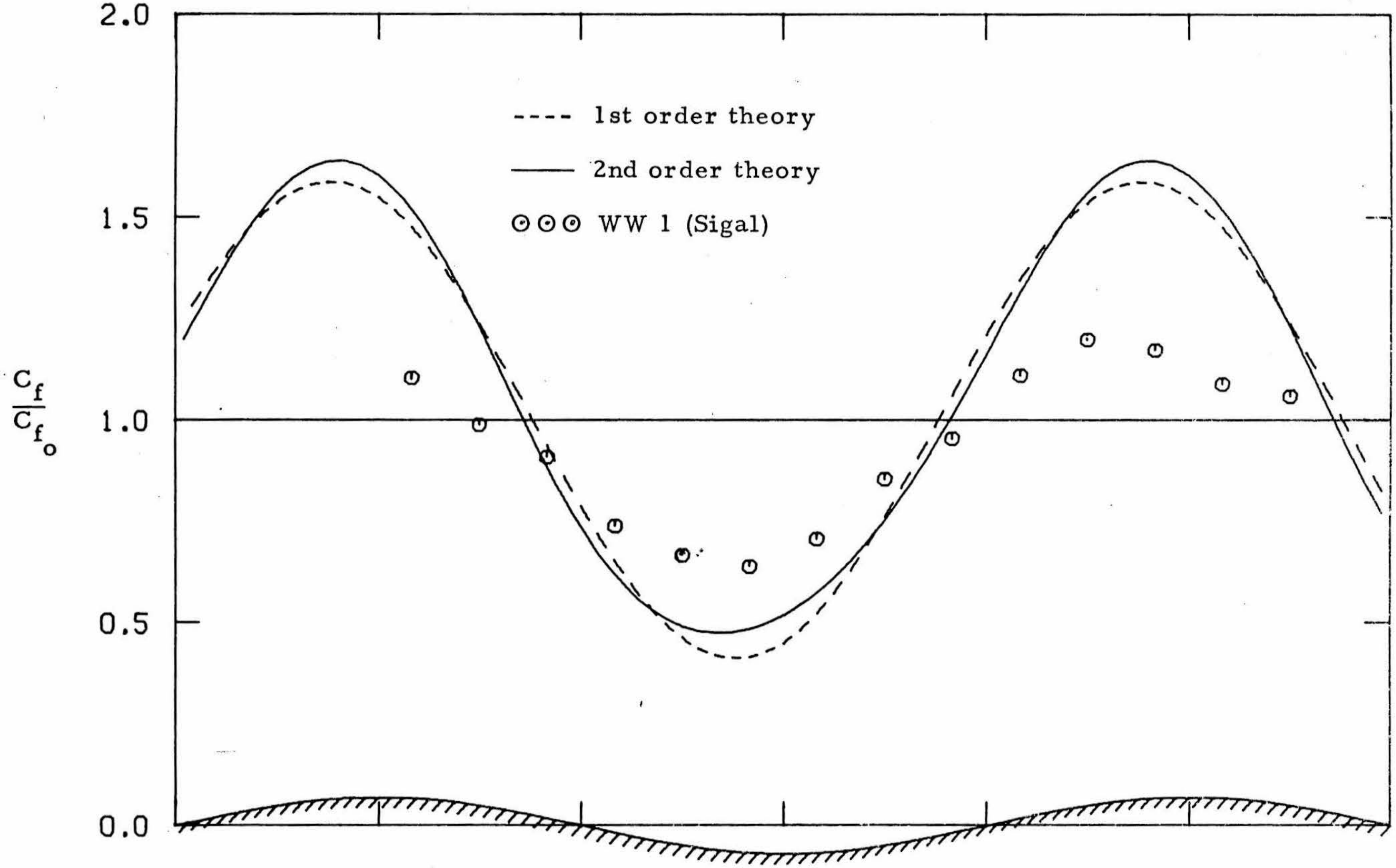


FIG. 10 SURFACE SHEAR STRESS DISTRIBUTIONS

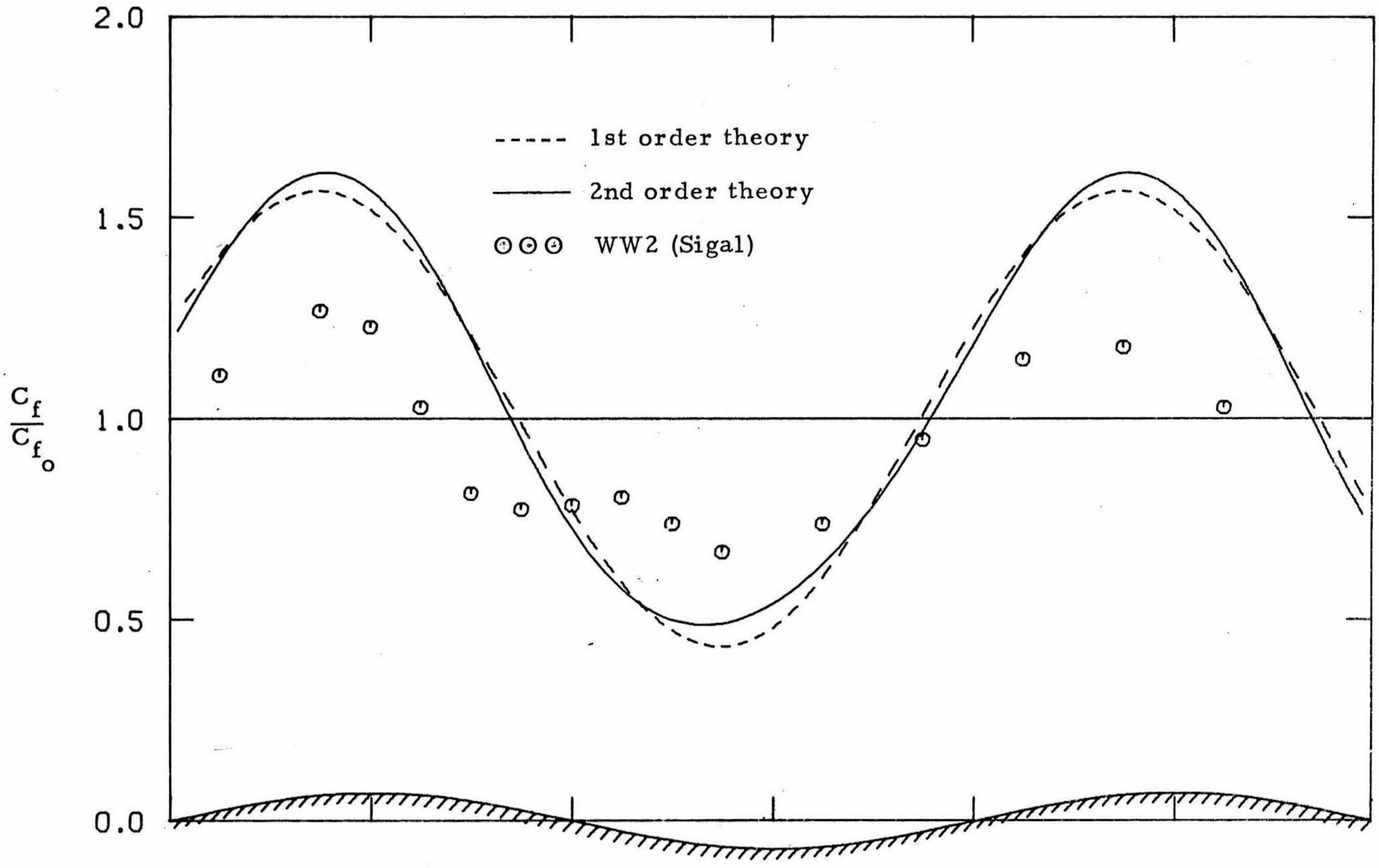


FIG. 11 SURFACE SHEAR STRESS DISTRIBUTIONS

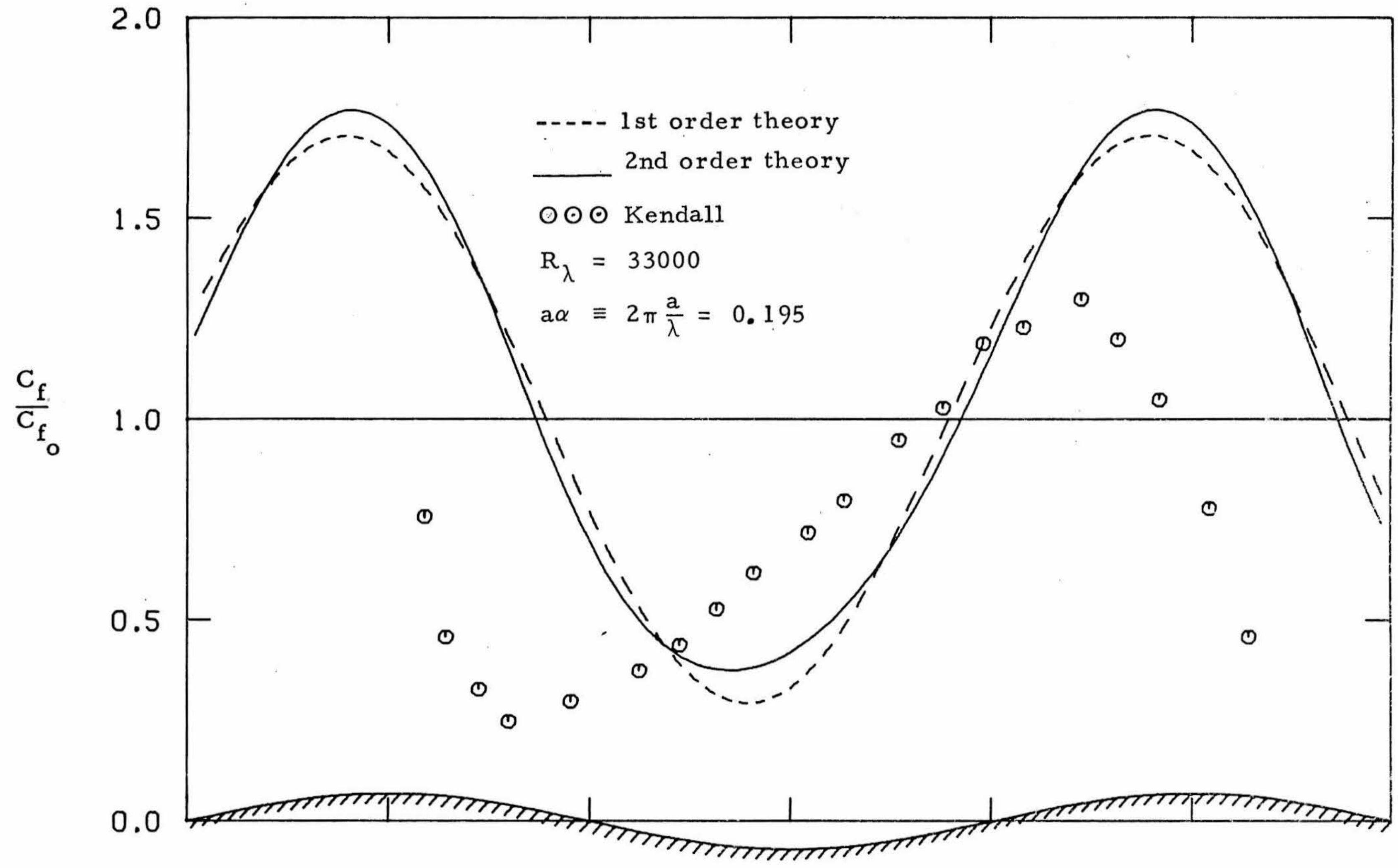


FIG. 12 SURFACE SHEAR STRESS DISTRIBUTIONS

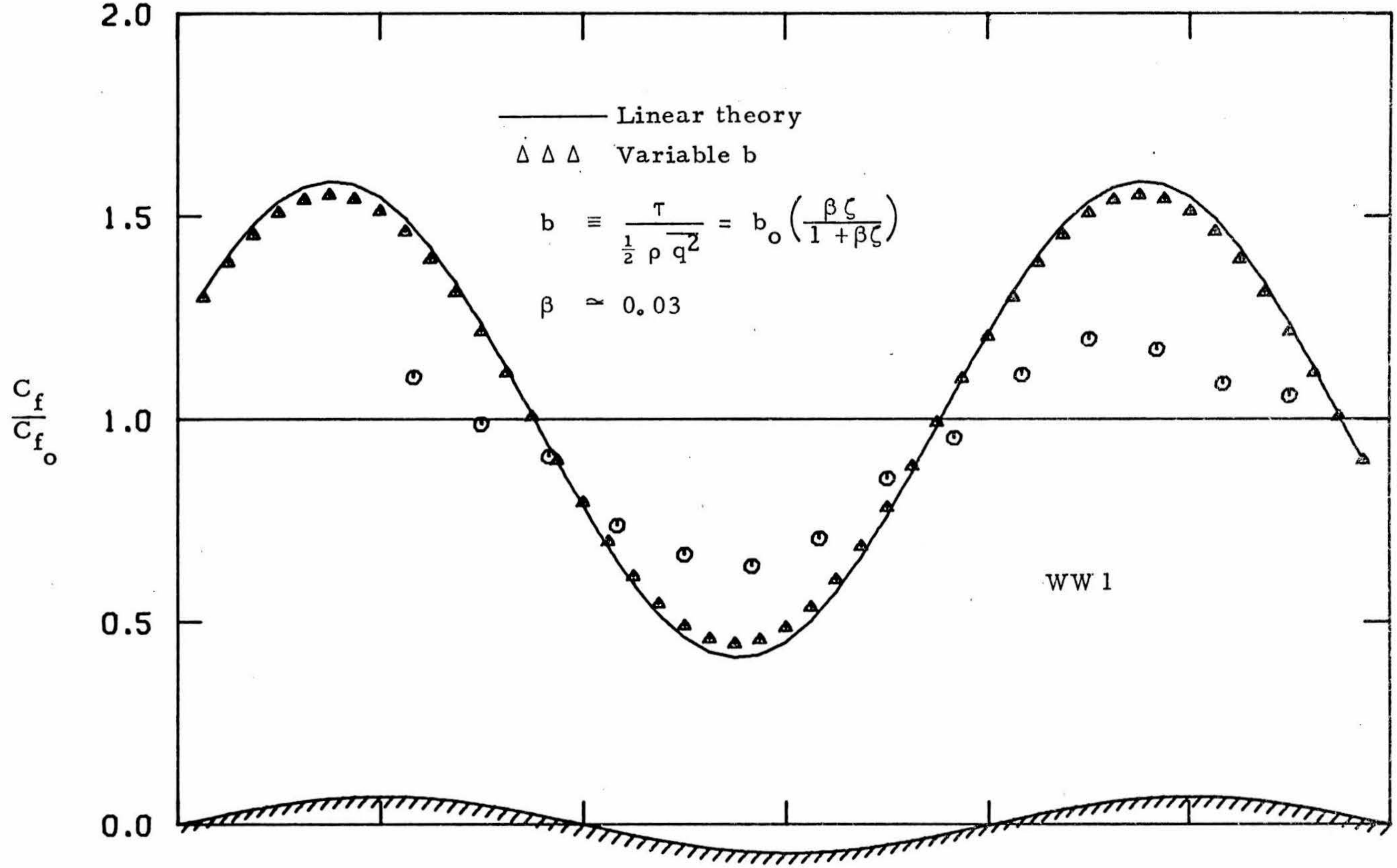


FIG. 13 SURFACE SHEAR STRESS DISTRIBUTIONS

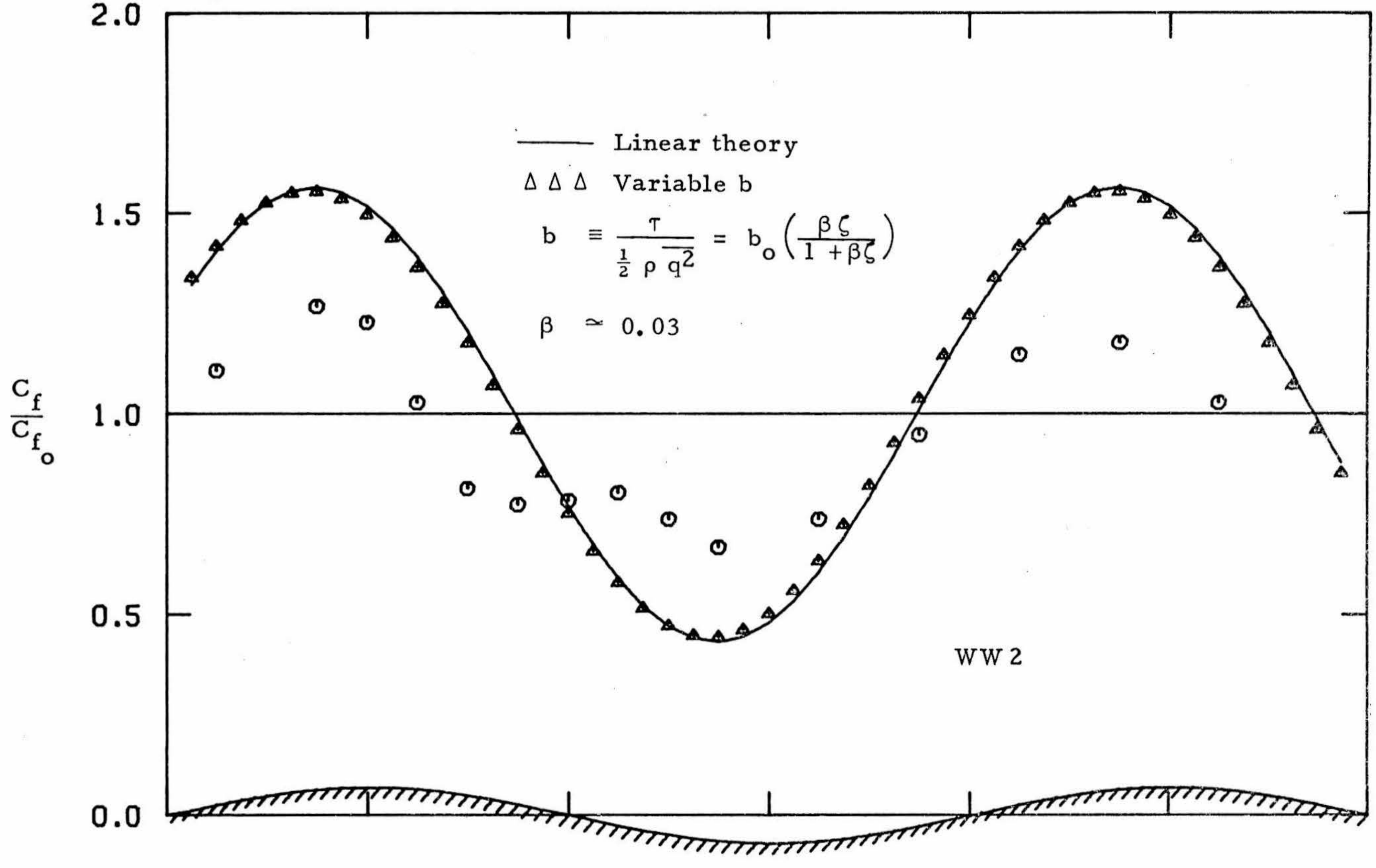


FIG. 14 SURFACE SHEAR STRESS DISTRIBUTIONS

APPENDIX A

SOLUTION OF THE INHOMOGENEOUS CONFLUENT HYPERGEOMETRIC DIFFERENTIAL EQUATION (4.20)

$$\begin{cases} U\hat{u} = \dfrac{\kappa}{i\alpha}\dfrac{d\hat{\tau}}{d\hat{\eta}} \\[2ex] -\dfrac{1}{\kappa}\dfrac{d\hat{u}}{d\hat{\eta}} + \left(\dfrac{i\alpha U}{b_o} + \dfrac{1}{2\kappa\hat{\eta}}\right)\hat{\tau} = i\alpha U(\sigma_o + 2i) - \dfrac{1}{\kappa\hat{\eta}} \end{cases} \tag{A-1}$$

where $U = 1 + \frac{\varepsilon}{\kappa}\ln\varepsilon + \frac{\varepsilon}{\kappa}(\ln\hat{\eta} - 2\tilde{\pi})$. For $\hat{\eta} \gg 1$, the homogeneous part of Eq. (A-1) provides oscillatory solution with variable local frequency proportional the integral of $U(\hat{\eta};\varepsilon)$. Thus, let

$$\begin{aligned} y &= \int_{0^+}^{\hat{\eta}} U(\hat{\eta};\varepsilon)d\hat{\eta}_1 = \hat{\eta}\left[1 + \frac{\varepsilon}{\kappa}\ln\varepsilon + \frac{\varepsilon}{\kappa}(\ln\hat{\eta} - 2\tilde{\pi} - 1)\right] \\ &= \left(U - \frac{\varepsilon}{\kappa}\right)\hat{\eta} \\ &\simeq U\hat{\eta} \end{aligned} \tag{A-2}$$

Define

$$z = i\frac{2\alpha}{\sqrt{b_o}}\, y \tag{A-3a}$$

$$k = -\frac{\sqrt{b_o}}{4\kappa}, \quad -1 < k < 0 \tag{A-3b}$$

Then we obtain a set of degenerate confluent hypergeometric equations

$$\hat{u} = -\frac{1}{2k}\frac{d\hat{\tau}}{dz} \tag{A-4a}$$

$$\frac{d^2\hat{\tau}}{dz^2} + \left(-\frac{1}{4} + \frac{k}{z}\right)\hat{\tau} = -\frac{b_o}{4}(\sigma_o + 2i) - \frac{2k}{z} \tag{A-4b}$$

Solution of $\hat{\tau}$:

The homogeneous part of Eq. (A-4b) is a special case of Whittaker's differential equation in a self-adjoint form

$$\frac{d^2\varphi}{dz^2} + \left\{-\frac{1}{4} + \frac{k}{2z} + \frac{1-\mu^2}{4z^2}\right\}\varphi = 0$$

with $\mu = 1$.

An integral basis is formed by a pair of Whittaker's functions given by

$$W_{k,\frac{1}{2}}(z) = \frac{z^k e^{-z/2}}{\Gamma(1-k)} \int_0^\infty t^{-k}(1+\frac{t}{z})^k e^{-t}\, dt \qquad \text{(A-5a)}$$

$$\mathcal{R}e(1-k) > 0, \quad |\arg z| < \pi$$

$$W_{-k,\frac{1}{2}}(-z) = \frac{(-z)^{-k} e^{z/2}}{\Gamma(1+k)} \int_0^\infty t^k(1-\frac{t}{z})^{-k} e^{-t}\, dt \qquad \text{(A-5b)}$$

$$\mathcal{R}e(1+k) > 0, \quad |\arg(-z)| < \pi$$

Both $W_{k,\frac{1}{2}}(z)$ and $W_{-k,\frac{1}{2}}(-z)$ are finite but not analytic at $z = 0$. A suitable limiting process is taken to obtain their power series representation for $|z| < \infty$.

$$W_{k,\frac{1}{2}}(z) = \frac{1}{\Gamma(-k)}\left\{\mathcal{M}_{k,\frac{1}{2}}(z)\cdot \ell n\, z + H_{k,\frac{1}{2}}(z)\right\}$$

$$W_{-k,\frac{1}{2}}(-z) = \frac{e^{-\pi i}}{\Gamma(k)}\left\{\mathcal{M}_{k,\frac{1}{2}}(z)\left[\ell n z - \frac{\pi e^{ik\pi}}{\sin k\pi}\right] + H_{k,\frac{1}{2}}(z)\right\} \qquad \text{(A-6)}$$

where $\mathcal{M}_{k,\frac{1}{2}}(z)$ is a degenerate confluent hypergeometric series given by

$$\mathcal{M}_{k,\frac{1}{2}}(z) = ze^{-z/2}\,{}_1F_1(1-k;2;z)$$

$$= ze^{-z/2}\sum_{n=0}^{\infty}\frac{(1-k)_n \cdot z^n}{(n+1)!(n!)}\,, \qquad |z| < \infty\,, \tag{A-7}$$

and

$$H_{k,\frac{1}{2}}(z) = \frac{ze^{-z/2}}{\Gamma(1-k)}\left\{\sum_{n=0}^{\infty}\frac{\Gamma(1-k+n)}{(n+1)!(n!)}\,[\psi(1-k+n)\right.$$

$$\left. - \psi(1+n) - \psi(2+n)]z^n + \frac{\Gamma(-k)}{z}\right\} \tag{A-8}$$

where

$$(1-k)_0 = 1, \quad (1-k)_n = (1-k)(2-k) --- (n-k)\ . \tag{A-9}$$

and $\psi(x)$ is Psi (digamma) function

$$\psi(x) \equiv \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma - \frac{1}{x} + x\sum_{n=1}^{\infty}\frac{1}{n(x+n)}$$

$$\gamma = \text{Euler's constant } 0.5772157 \tag{A-10}$$

The Wronskian for the chosen pair of solutions is

$$\widetilde{\mathcal{W}}\left\{W_{k,\frac{1}{2}}(z),\ W_{-k,\frac{1}{2}}(-z)\right\} = (-1)^{-k} \tag{A-11}$$

and the branch line in the z plane is the negative real axis. For positive imaginary z defined by Eq. (A-3), we take

$$(-z) = (ze^{-i\pi})\,; \qquad (-1) = e^{-i\pi}$$

to assure $|\arg(-z)| < \pi$. Thus

$$\widetilde{\mathcal{W}} = e^{ik\pi} \tag{A-12}$$

Then, as $|z| \to \infty$

$$\hat{\tau} \sim -2 + 4D(1 + \frac{4k}{z} + ---) + \overline{C}_1 z^k e^{-z/2}[1+O(\frac{1}{z})]$$

$$= b_0\left[(\sigma_0 + 2i) + \frac{i\, b_0}{2K\alpha y}(\sigma_0 + 2i + \frac{2}{b_0}) + ---\right]$$

$$+ \overline{C}_1 z^k e^{-z/2}[1 + O(\frac{1}{z})] \tag{A-15}$$

<u>Series expansion around z = 0:</u>

Applying the properties of Digamma function

$$\psi(1) = -\gamma ,$$

$$\psi(1+k) = \frac{1}{k} + \psi(k) \tag{A-16}$$

The functions $\mathcal{M}$ and H are expanded near the origin as

$$\mathcal{M}_{k,\frac{1}{2}}(z) \simeq z\{1 - \frac{k}{2}z + \frac{1}{12}(k^2 + \tfrac{1}{2})z^2 + --- \}$$

$$H_{k,\frac{1}{2}}(z) \simeq \frac{\Gamma(-k)}{\Gamma(1-k)}\{1 - a_1 z + a_2 z^2 + ---- \} \tag{A-17}$$

where

$$a_1 = \tfrac{1}{2} + k[\psi(1-k) + 2\gamma - 1]$$

$$a_2 = \tfrac{1}{2} a_1 k - \frac{3}{4}k^2 + \frac{1}{8}$$

The pair of Whittaker's function s is

$$W_{k,\frac{1}{2}}(z) \simeq \frac{1}{\Gamma(1-k)}\{(1 - a_1 z + a_2 z^2 + --) - kz\ell n z(1 - \frac{k}{2}z + --)\}$$

$$W_{-k,\frac{1}{2}}(z) \simeq \frac{1}{\Gamma(1+k)}\left\{(1 - a_1 z + a_2 z^2 + --) - kz(\ell n z - \frac{\pi e^{ik\pi}}{\sin k\pi}((1 - \frac{k}{2}z + --)\right\} \tag{A-18}$$

The series expansion for the integral terms inside the brace of Eq. (A-13) is carried out as

$$\{\quad\} \simeq I_1 W_{-k,\frac{1}{2}}(-z) - \tfrac{1}{2} z^2[1 - \frac{k}{6} z + O(z\ln z)]e^{ik\pi}$$

where $$I_1 = \int_0^\infty W_{k,\frac{1}{2}}(z)dz. \tag{A-19}$$

Therefore, around z = 0, we have

$$\hat{\tau} \simeq -2 + C_1 W_{k,\frac{1}{2}}(z) + DI_1 e^{-ik\pi} W_{-k,\frac{1}{2}}(-z) - \frac{D}{2} z^2[1 + O(z;2\ln z)]$$

or $$\hat{\tau} \simeq -2 + A[(1 - a_1 z + A_2 z^2 + --) - kz\ln z(1 - \frac{k}{2} z + ---)]$$

$$+ D[\tilde{I}_1 z - \tfrac{1}{2}(1 + \tilde{I}_1 k)z^2 + O(z^2\ln z)] \tag{A-20}$$

where

$$A = \frac{C_1}{\Gamma(1-k)} + \frac{DI_1}{\Gamma(1+k)} e^{-ik\pi} \; ; \quad \tilde{I}_1 = I_1\Gamma(1-k).$$

<u>Solution of $\hat{u}$:</u>

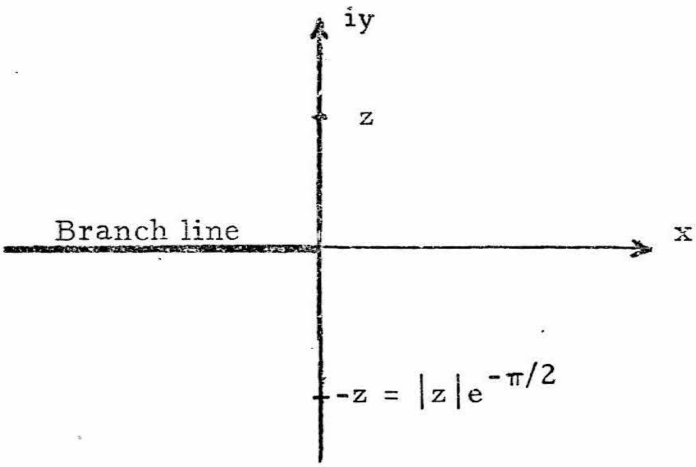
From the integral representation of W, Eq. (A-5a, b), it is immediately seen that

$$\left\{ \begin{aligned} \frac{d}{dz} W_{k,\frac{1}{2}}(z) &= -\tfrac{1}{2} W_{k,\frac{1}{2}}(z) + \frac{k}{z^{\frac{1}{2}}} W_{(k-\frac{1}{2}),o}(z) \\ \frac{d}{dz} W_{-k,\frac{1}{2}}(-z) &= \tfrac{1}{2} W_{-k,\frac{1}{2}}(-z) + \frac{k}{(-z)^{\frac{1}{2}}} W_{-(k+\frac{1}{2}),o}(-z) \end{aligned} \right. \tag{A-21}$$

where

$$W_{(k-\frac{1}{2}),o}(z) = \frac{z^{k-\frac{1}{2}} e^{-z/2}}{\Gamma(1-k)} \int_0^\infty t^{-k} (1+\frac{t}{z})^{k-1} e^{-t} dt \tag{A-22a}$$

$$W_{-(k+\frac{1}{2}),o}(-z) = \frac{(-z)^{-(k+\frac{1}{2})} e^{z/2}}{\Gamma(1+k)} \int_0^\infty t^k (1-\frac{t}{z})^{-(1+k)} e^{-t} dt \tag{A-22b}$$



The complete solution for $\hat{\tau}$ which is finite as $|z| \to \infty$ is

$$\hat{\tau} = -2 + C_1 W_{k,\frac{1}{2}}(z)$$
$$+ De^{-ik\pi}\left\{W_{k,\frac{1}{2}}(z)\int_0^z W_{-k,\frac{1}{2}}(-t) + W_{-k,\frac{1}{2}}(-z)\int_z^{i\infty} W_{k,\frac{1}{2}}(t)dt\right\}. \tag{A-13}$$

where $D = \frac{b_0}{4}(\sigma_0 + 2i) + \frac{1}{2}$.

Asymptotic expansion for large $|z|$:

$$W_{k,\frac{1}{2}}(z) \sim z^k e^{-z/2}\left\{1 + \sum_{n=1}^{N-1} \frac{(-k)_n (1-k)_n}{n!(-z)^n} + O(\frac{1}{z^N})\right\}$$
$$W_{-k,\frac{1}{2}}(-z) \sim (-z)^{-k} e^{z/2}\left\{1 + \sum_{n=1}^{N-1} \frac{(k)_n (1+k)_n}{n!\, z^n} + O(\frac{1}{z^N})\right\} \tag{A-14}$$

It is easy to show that

$$\int_z^{i\infty} W_{k,\frac{1}{2}}(t)dt \sim 2z^k e^{-z/2}\left[1 + \frac{k(3-k)}{z} + ---\right]$$

$$\int_0^z W_{-k,\frac{1}{2}}(-t)dt = \left\{\int_0^{z_0} + \int_{z_0}^z\right\} W_{-k,\frac{1}{2}}(-t)dt; \quad |z| > |z_0| >> 1,$$
$$\sim 2(-z)^{-k} e^{z/2}\left[1 + \frac{k(3+k)}{z} + ---\right] + C$$

Therefore, we obtain

$$\hat{u} = -\frac{1}{2k}\frac{d\hat{T}}{dz}$$

$$= \frac{1}{4k}\left[W_{k,\frac{1}{2}}(z) - \frac{2k}{z^{\frac{1}{2}}} W_{(k-\frac{1}{2}),o}(z)\right]\left\{C_1 + De^{-ik\pi}\int_0^z W_{-k,\frac{1}{2}}(-t)dt\right\}$$

$$-\frac{1}{4k}\left[W_{-k,\frac{1}{2}}(-z) + \frac{2k}{(-z)^{\frac{1}{2}}} W_{-(k+\frac{1}{2}),o}(-z)\right]\left\{De^{-ik\pi}\int_z^{i\infty} W_{k,\frac{1}{2}}(t)dt\right\} \tag{A-23}$$

As $|z| \to \infty$,

$$\hat{u} \sim \frac{\overline{C}_1}{4k} z^k e^{-z/2}\left[1 + O\left(\frac{1}{z}\right)\right] + O\left(\frac{D}{z^2}\right) \tag{A-24}$$

As $z \to 0$,

$$\hat{u} \simeq \frac{A}{2}\left[\ln z(1-kz+--) + \left(1+\frac{a_1}{k}\right) - \left(\frac{2a_2}{k}+\frac{k}{2}\right)z + ---\right]$$

$$-\frac{D}{2k}\left[\tilde{I}_1 - (1+\tilde{I}_1 k)z + O(z^2 \ln z)\right] \tag{A-25}$$

APPENDIX B

SECOND ORDER GOVERNING EQUATIONS IN COMPLEX FORM

First Order Expansions:

$$\begin{Bmatrix} u' \\ v' \\ p' \\ \tau' \\ \sigma' \end{Bmatrix} = \mathcal{R}e \begin{Bmatrix} \tilde{u}(\eta) \\ i\tilde{v}(\eta) \\ \tilde{p}(\eta) \\ \tilde{\tau}(\eta) \\ \tilde{\sigma}(\eta) \end{Bmatrix} e^{i\alpha\xi}$$

Continuity:

$$\frac{\partial u''}{\partial \xi} + \frac{\partial v''}{\partial \eta} = -i\alpha\left\{Ue^{-2\alpha\eta} - \frac{1}{2}(\tilde{u}-\tilde{v})e^{-\alpha\eta}\right\}e^{i2\alpha\xi} - \frac{i\alpha}{2}(\tilde{u}+\tilde{v})e^{-\alpha\eta} \qquad \text{(B-1)}$$

Momentum:*

$$U\frac{\partial u''}{\partial \xi} + v''\frac{dU}{d\eta} + \frac{\partial p''}{\partial \xi} - \left(\frac{\partial \tau''}{\partial \eta} + \frac{\partial \sigma_1''}{\partial \xi}\right) - \frac{1}{R}\frac{\partial^2 u''}{\partial \eta^2}$$

$$= \left\{i\alpha(\sigma + 2iT)e^{-2\alpha\eta} - \frac{i\alpha}{2}(\tilde{\sigma}+2i\tilde{\tau}+U\tilde{v})e^{-\alpha\eta} - \frac{i\alpha}{2}\left(\tilde{u}^2 + \frac{\tilde{v}}{\alpha}\frac{d\tilde{u}}{d\eta}\right) - \frac{1}{R}\left[\frac{1}{4}\frac{d^2U}{d\eta^2}e^{-2\alpha\eta} - \frac{1}{2}\frac{d^2\tilde{u}}{d\eta^2}e^{-\alpha\eta}\right]\right\}e^{i2\alpha\xi}$$

$$+ \left\{\frac{i\alpha}{2}(\tilde{\sigma}-2i\tilde{\tau} - U\tilde{v})e^{-\alpha\eta} - \frac{i}{2}\tilde{v}\frac{d\tilde{u}^*}{d\eta} + \frac{1}{R}\left[\frac{1}{4}\frac{d^2U}{d\eta^2}e^{-2\alpha\eta} + \frac{1}{2}\frac{d^2\tilde{u}}{d\eta^2}e^{-\alpha\eta}\right]\right\} \qquad \text{(B-2)}$$

$$U\frac{\partial v''}{\partial \xi} + \frac{\partial p''}{\partial \eta} - \left(\frac{\partial \tau''}{\partial \xi} + \frac{\partial \sigma_2''}{\partial \eta}\right)$$

$$= \left\{\alpha\left[-U^2+(\sigma+2iT)\right]e^{-2\alpha\eta} + \alpha\left[U\tilde{u} - \frac{1}{2}(\tilde{\sigma}+2i\tilde{\tau})\right]e^{-\alpha\eta}\right\}e^{i2\alpha\xi}$$

$$+ \left\{\alpha(U\tilde{u} - \frac{\tilde{\sigma}}{2} + i\tilde{\tau})e^{-\alpha\eta} + \alpha\tilde{u}\tilde{v}^*\right\} \qquad \text{(B-3)}$$

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* 1. $\tilde{u}^* = \tilde{u}_r - i\tilde{u}_i$ is the complex conjugate of $\tilde{u}$.

2. $\mathcal{R}e(f)\cdot\mathcal{R}e(g) = \frac{1}{2}\mathcal{R}e(fg) + \frac{1}{2}\mathcal{R}e(fg^*)$.

Energy:

$$\left[U \frac{\partial}{\partial \xi}\left(\frac{\tau''}{b}\right) + v'' \frac{d}{d\eta}\left(\frac{T}{b}\right)\right] - \left[T\left(\frac{\partial u''}{\partial \eta} + \frac{\partial v''}{\partial \xi}\right) + \tau'' \frac{dU}{d\eta} - \sigma \frac{\partial v''}{\partial \eta}\right]$$

$$+ \frac{\partial}{\partial \eta}\left(G\, T_m^{\frac{1}{2}} \tau''\right) + \frac{3}{2} \frac{T^{\frac{1}{2}}}{\mathcal{L}} \tau'' - \frac{1}{R} \frac{\partial^2}{\partial \eta^2}\left(\frac{\tau''}{b}\right)$$

$$= \Big\{ \alpha e^{-2\alpha\eta}\left[U(T - i\sigma) + G\, T_m^{\frac{1}{2}} T\right] - \frac{\alpha}{2} e^{-\alpha\eta}\Big[U(\widetilde{\tau} - i\widetilde{\sigma}) - i\sigma\widetilde{u}$$

$$+ T(\widetilde{u} + \widetilde{v}) + G\left(T_m^{\frac{1}{2}} \widetilde{\tau} + \frac{1}{2} \frac{T}{T_m^{\frac{1}{2}}} \widetilde{\tau}_e\right)\Big] - \frac{1}{2}\Big[i\alpha\widetilde{u}\left(\frac{\widetilde{\tau}}{b}\right) + i\widetilde{v} \frac{d}{d\eta}\left(\frac{\widetilde{\tau}}{b}\right)$$

$$- \widetilde{\tau}\left(\frac{d\widetilde{u}}{d\eta} - \alpha\widetilde{v}\right) + i\widetilde{\sigma} \frac{d\widetilde{v}}{d\eta} + \frac{d}{d\eta} G\left(\frac{\widetilde{\tau}_e \widetilde{\tau}}{2T_m^{\frac{1}{2}}} - \frac{T \widetilde{\tau}_e^{\,2}}{8\, T_m^{\frac{3}{2}}}\right) + \frac{3}{8} \frac{\widetilde{\tau}}{\mathcal{L} T^{\frac{1}{2}}}\Big]$$

$$+ \frac{1}{R}\left[- \frac{1}{4} e^{-2\alpha\eta} \frac{d^2}{d\eta^2}\left(\frac{T}{b}\right) + \frac{1}{2} \frac{d^2}{d\eta^2}\left(\frac{\widetilde{\tau}}{b}\right)\right]\Big\} e^{i2\alpha\xi}$$

$$- \Big\{ \frac{\alpha}{2} e^{-\alpha\eta}\left[U(\widetilde{\tau} + i\widetilde{\sigma}) + T(\widetilde{u} - \widetilde{v}) + i\sigma\widetilde{u} + G\left(T_m^{\frac{1}{2}} \widetilde{\tau} + \frac{T\, \widetilde{\tau}_e}{T_m^{\frac{1}{2}}}\right)\right]$$

$$+ \frac{1}{2}\Big[- i\alpha\widetilde{u}\left(\frac{\widetilde{\tau}^*}{b}\right) + i\widetilde{v} \frac{d}{d\eta}\left(\frac{\widetilde{\tau}^*}{b}\right) - \widetilde{\tau}^*\left(\frac{d\widetilde{u}}{d\eta} - \alpha\widetilde{v}\right) - i\widetilde{\sigma} \frac{d\widetilde{v}^*}{d\eta}$$

$$+ \frac{d}{d\eta} G\left(\frac{\widetilde{\tau}_e \widetilde{\tau}}{2\, T_m^{\frac{1}{2}}} - \frac{T\, \widetilde{\tau}_e^{\,*} \widetilde{\tau}_e}{8\, T_m^{\frac{3}{2}}}\right) + \frac{3}{8} \frac{\widetilde{\tau}\, \widetilde{\tau}^*}{\mathcal{L}\, T^{\frac{1}{2}}}\Big]$$

$$- \frac{1}{R}\left[\frac{1}{4} e^{-2\alpha\eta} \frac{d^2}{d\eta^2}\left(\frac{T}{b}\right) + \frac{1}{2} e^{-\alpha\eta} \frac{d^2}{d\eta^2}\left(\frac{\widetilde{\tau}}{b}\right)\right]\Big\} \qquad \text{(B-4)}$$

APPENDIX C

Velocity and Reynolds Stress Profiles Expressed in Cartesian Coordinates

The velocity u_x and u_y in the direction of increasing x and y respectively are given by

$$\begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} F_1 & F_2 \\ -F_2 & F_1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{C-1}$$

where the transformation matrix F is obtained from Eq. (2.1) as

$$\begin{cases} F_1 = 1 - \frac{1}{2}(a\alpha)^2 e^{-2\alpha\eta} \sin^2 \alpha\xi \\ F_2 = (a\alpha) e^{-\alpha\eta} \sin\alpha\xi (1 - a\alpha e^{-\alpha\eta} \cos\alpha\xi) \end{cases} \tag{C-2}$$

1. Velocity:

$$\begin{aligned} \overline{u}_x = {} & U(\eta) + (a\alpha)\tilde{u}(\eta)e^{i\alpha\xi} \\ & + (a\alpha)^2 \Big\{ [\tilde{\tilde{u}}(\eta) + \tfrac{1}{4}U(\eta)e^{-2\alpha\eta}]e^{2i\alpha\xi} \\ & + [\bar{\bar{u}}(\eta) - \tfrac{1}{4}U(\eta)] + i\tilde{v}e^{-\alpha\eta}e^{i\alpha\xi} \sin\alpha\xi \Big\} \end{aligned} \tag{C-3}$$

2. Reynolds stress:

$$\overline{u'_x u'_y} = (F_1^2 - F_2^2)\overline{u'v'} - F_1 F_2 (\overline{u'^2} - \overline{v'^2}) \tag{C-4}$$

$$\begin{aligned} \tau_{xy} \equiv - \rho \overline{u'_x u'_y} = {} & (F_1^2 - F_2^2)\tau + (F_1 F_2)\sigma \\ = {} & T(\eta) + (a\alpha)[\tilde{\tau}(\eta)e^{i\alpha\xi} - \sigma(\eta)e^{-\alpha\eta}\sin\alpha\xi] \\ & + (a\alpha)^2 \Big[\tilde{\tilde{\tau}}(\eta)e^{2i\alpha\xi} + \bar{\bar{\tau}}(\eta) - T e^{-2\alpha\eta} \\ & - \frac{i}{2}(\sigma + 2iT)e^{-2\alpha\eta}e^{2i\alpha\xi} - \tilde{\sigma}(\eta)e^{-\alpha\eta}e^{i\alpha\xi}\sin\alpha\xi \Big] \end{aligned} \tag{C-5}$$