SPECTRAL DENSITY OF FIRST ORDER PIECEWISE LINEAR SYSTEMS EXCITED BY WHITE NOISE

Thesis by

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In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1967

(Submitted May 2, 1967)

ACKNOWLEDGMENTS

The author wishes especially to thank his research advisor, Professor T. K. Caughey, for his suggestions, help and encouragement throughout this investigation.

Most of this work was carried out while he held the Charles Kolling Travelling Scholarship from the University of Sydney. Support was provided also by the Ford Foundation, Inland Steel-Ryerson Foundation, and the California Institute of Technology.

Finally he would like to express his appreciation for the expert typing and patience of Mrs. Madeline Fagergren.

ABSTRACT

The Fokker-Planck (FP) equation is used to develop a general method for finding the spectral density for a class of randomly excited first order systems. This class consists of systems satisfying stochastic differential equations of form $\dot{x} + f(x) = \sum_{j=1}^{m} h_j(x)\, n_j(t)$ where f and the h_j are piecewise linear functions (not necessarily continuous), and the n_j are stationary Gaussian white noise. For such systems, it is shown how the Laplace-transformed FP equation can be solved for the transformed transition probability density. By manipulation of the FP equation and its adjoint, a formula is derived for the transformed autocorrelation function in terms of the transformed transition density. From this, the spectral density is readily obtained. The method generalizes that of Caughey and Dienes, J. Appl. Phys., 32.11.

This method is applied to 4 subclasses: (1) m = 1, h_1 = const. (forcing function excitation); (2) m = 1, $h_1 = f$ (parametric excitation); (3) m = 2, h_1 = const., $h_2 = f$, n_1 and n_2 correlated; (4) the same, uncorrelated. Many special cases, especially in subclass (1), are worked through to obtain explicit formulas for the spectral density, most of which have not been obtained before. Some results are graphed.

Dealing with parametrically excited first order systems leads to two complications. There is some controversy concerning the form of the FP equation involved (see Gray and Caughey, J. Math. Phys., 44.3); and the conditions which apply at irregular points, where the second order coefficient of the FP equation vanishes, are not obvious but require use of the mathematical theory of diffusion processes developed by Feller and others. These points are discussed in the first chapter, relevant results from various sources being summarized and applied. Also discussed is the steady-state density (the limit of the transition density as $t \to \infty$).

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INTRODUCTION

Summary

This thesis is concerned with systems governed by first order stochastic differential equations of the form

$$\frac{dx}{dt} + f(x) = \sum_{j=1}^{m} h_j(x) n_j(t) \quad , \tag{0.1}$$

where each $n_j(t)$ $(j=1,2,\ldots,m)$ represents white noise input. Two special cases are of importance--h_j = const. (forcing function excitation) and $h_j \propto f$ (parametric excitation). Except in Chapter I, f and h_j are assumed to be piecewise linear.

Chapter I summarizes theoretical material concerning (0.1) and the corresponding Fokker-Planck (FP) equation, gathered from various sources, and on which the rest of the thesis is based. In Chapter II, this is applied to the general piecewise linear system, and a formula derived for the spectral density. In the remaining chapters this is applied to special cases -- in Chapter III to systems with only forcing function excitation, in Chapter IV to systems with only parametric excitation, and in Chapter V to some systems with both parametric and forcing function excitation. Some numerical results are presented for the cases treated in Chapter III. In an Appendix, the possibility of application of methods like these of this thesis to second order piecewise linear systems is examined; it appears unlikely that analytical results can be obtained even in the simplest cases.

Physical applications

In (0.1), $x(t)$ is to be considered as the output of a system excited by the inputs $n_j(t)$. Stochastic inputs occur in varied fields. The original application was by Einstein to the Brownian motion. In electrical engineering, thermal noise in electronic circuits is very close to pure white noise. Vibrations in mechanical systems may be generated by a wide variety of more or less random forces; such forces include earthquakes, turbulence in air or water, storm waves, and the force on a vehicle traversing rough terrain.

Unfortunately, the majority of applications require dealing with systems of higher order than the first. In particular, few mechanical systems can be approximated, to any useful order of accuracy, by anything simpler than a single spring and mass--that is, by a second order system. Thus the method developed in this thesis has limited practical utility unless it can be extended to second order systems.

Following the mechanical engineering analog, but sacrificing dimensional accuracy to simplicity of expression, $x(t)$ will henceforth be referred to as "displacement", $f(x)$ as "restoring force", etc.

Previous work

When the stochastic differential equation is linear with only forcing function excitation, the probabilistic properties of $x(t)$ have long been known; this is true of systems of arbitrary order, not just first order.[1] For example, the spectral density may be obtained either directly or by means of the FP equation. If, instead of h_j = const., the h_j are linear in x (so that the excitation may be considered as a combination of forcing function and parametric), then the transition density has been found in only the simplest cases. However, Gray [22, 23] has shown that for linear systems of this type the spectral density is the same as that of an "equivalent" system with constant h_j. Wong and Thomas (see [45, 46]) have worked out the transition density (as an eigenfunction expansion) for two first order systems of this type--in the present thesis, these are example 1 of section 5.3 and the example of section 5.4.

The spectral density of a nonlinear system has been found explicitly in only one simple case, by Caughey and Dienes (see [4, 8] and also Robinson [37]). This is the special case of example 2, section 3.4, in this thesis; our method is a generalization of that used by Caughey and Dienes. The transition density has been found for several others. One of them (example 4, section 3.3 in this thesis) has been known for many years.[2] Two others are obtained by Wong [45] from the two linear parametric cases mentioned above, by the substitutions $y = \ell n\, x$ and $y = \sinh x$ respectively. For these systems, in which the transition density is expressed as an eigenfunction expansion, the autocorrelation and spectral density can be

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[1] See, for example, Wang and Uhlenbeck [40].

[2] See, for example, Chandrasekhar [6] or Kac [28].

found as shown in Payne [36]. These systems are all first order, with forcing function excitation.

Wolaver [43, 44] claims to have obtained the autocorrelation and spectral density of a second order system; however, his method of solution of the FP equation appears to be incorrect, and his formulas do not agree well with his own numerical solutions. See the Appendix for further discussion of his method.

Much work has been done on approximate methods for non-linear systems. The method of equivalent linearization approximates the system by the linear system with the same first and second moments.[1] Various perturbation techniques have also been used.[2] Khazen [31] reduces the FP equation to a system of Volterra-type integral equations to be solved by the usual method of successive approximations.

Autocorrelation and spectral density[3]

Throughout this thesis, well known probabilistic concepts are used with little or no comment. However, some discussion of spectral density seems indicated, since its calculation is the main purpose of this thesis, and since conflicting definitions exist, differing by multiplicative constants.

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[1] For a review of this method and the results obtained by various workers, see Caughey [3]. See also section 3.6 below.

[2] For a review, see Crandall [7].

[3] This paragraph is based on section 1.1 of Karnop [29].

If y(t) is a stationary process, its autocorrelation R(t) is defined as the expectation of $y(t_1)y(t_1+t)$, i.e.,

$$R(t)=E[y(t_1)y(t_1+t)] = \int_\Omega \int_\Omega y_1 y_2 P_2(y_1, t_1; y_2, t_1+t)dy_1 dy_2. \qquad (0.2)$$

Here $P_2(y_1, t_1; y_2, t_2)$ denotes the joint density of y_1 at time t_1 and y_2 at time t_2 -- "density", except in the term "spectral density", will mean "probability density" throughout this thesis. The sample space Ω is the range of all possible values of y. Since the ergodic hypothesis will be assumed, one has also

$$R(t) = \langle y(t_1)y(t_1+t)\rangle = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} y(t_1)y(t_1+t)dt_1 \qquad (0.3)$$

for almost all sample paths. In fact the symbols for ensemble average E[ ] and time average ⟨ ⟩ will be used interchangeably from here on.

The spectral density of y(t) is the Fourier transform of R(t). Since R(t) is real and even, this can be written as a cosine transform (the Wiener-Khinchine relations):

$$\Phi(\omega) = \frac{2}{\pi}\int_0^\infty R(t)\cos\omega t dt \qquad (0.4)$$

$$R(t) = \int_0^\infty \Phi(\omega)\cos\omega t d\omega \;. \qquad (0.5)$$

Since

$$\int_0^\infty \Phi(\omega)d\omega = R(0) = \langle x^2 \rangle \,, \tag{0.6}$$

$\Phi(\omega)\,d\omega$ may be interpreted as the mean square (or power) contained in an infinitesimal band of the sinusoids into which the process has been resolved.

Equation (0.4) defines the <u>one-sided</u> spectral density, which is defined only for $\omega \geq 0$. This will be used exclusively throughout this thesis. The <u>two-sided</u> spectrum, which is the exponential transform of $R(t)$, takes values for both positive and negative ω; it is an even function and for $\omega \geq 0$ is half the one-sided spectrum. Of course negative ω has no physical meaning, but exponential transforms are often mathematically more convenient.

When it is necessary to specify the particular process $y(t)$ to which $R(t)$ and $\Phi(\omega)$ refer, the symbols $R_y(t)$ and $\Phi_y(\omega)$ will be used. The cross-correlation between $y(t)$ and $z(t)$ will be referred to as $R_{yz}(t)$, and the corresponding spectral density as $\Phi_{yz}(\omega)$.

<u>Notation</u>

The following conventions will be adhered to in Chapters II-V, and (occasionally) in Chapter I.

(a) A Greek letter (upper or lower case) denotes a nondimensionalization of the corresponding Roman letter. The only exceptions are Φ and ω, which, following standard usage, denote spectral density and frequency, and ψ, which denotes nondimensionalized q. Quantities not denoted by Roman letters are nondimensionalized by an asterisk (e.g. $\mathscr{P}^*$, Φ^*)--although an asterisk may also denote the adjoint of an operator.

(b) The Laplace transform (with respect to time) of a quantity denoted by an upper case letter (Roman or Greek) is denoted by the corresponding lower case letter. Otherwise a bar over a symbol denotes the transformed symbol.

Several special functions are used in Chapters III-V. The notations used are those of Abramowitz and Stegun [1], except for the parabolic cylinder function, where the more familiar notation $D_\nu(z)$ is used.

CHAPTER I

PROPERTIES OF FIRST ORDER SYSTEMS AND THEIR FOKKER-PLANCK EQUATIONS

The results quoted in this chapter are drawn from various sources and are mostly given without proof. Such proofs and derivations are given in the references cited, often in different form or in more generality than required here. In a few cases, results are expressed with less than mathematical exactitude (e.g., the definitions of stochastic integrals), for reasons of brevity, clarity of exposition, or the writer's ignorance. There is disagreement concerning some results in sections 1.1-2, but this concerns choice of the appropriate mathematical model to represent a physical situation, not the mathematical derivation of results from this model.

1.1 WHITE NOISE AND STOCHASTIC DIFFERENTIAL EQUATIONS

The nth order system

In this and the following section, there is no additional complication if the equation (0.1), with one dependent variable x, is replaced by a set of equations with n dependent variables x_i; i.e., writing x for the n-vector $\{x_i\}$ and summing over repeated suffices (a convention which will be used throughout sections 1.1-2),

$$\frac{dx_i}{dt} + f_i(x) = h_{ij}(x)\, n_j(t) \quad , \quad i = 1, 2, \ldots, n \quad , \quad j = 1, 2, \ldots, m . \qquad (1.1)$$

An nth order equation in one dependent variable can be reduced to this form in the usual way, replacing $x^{(i-1)}$ by x_i.

In all that follows, it will be assumed that the inputs $n_j(t)$ in (1.1) are stationary and Gaussian. This is nearly true in many physical cases and is the simplest possible assumption.

Physical occurrence of "white noise"[1]

Frequently in physical applications a stochastic input has an essentially flat spectrum, up to a frequency so high that it has practically no effect on the system--i.e., much higher than any characteristic frequency of the system. Electronic noise has such a spectrum, and turbulence may approximate it over a considerable range of frequencies. For such disturbing forces as storm waves and strong motion earthquakes it is a very crude approximation, but often justified for design purposes by lack of any exact data on expected disturbances.

Properties of this noise

The drop off of the spectrum for high frequencies is equivalent to non-zero autocorrelation at very small time intervals. However, over any "macroscopic" time interval the signal can be considered uncorrelated. It follows from this that if all the $n_j(t)$ in (1.1) are white in this sense, then the (vector) process x(t), while not truly a Markov process, is effectively Markovian if only "macroscopic" time intervals are to be considered.

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[1] The discussion in this section is largely based on Gray and Caughey [24], where a less condensed treatment is given.

Since it is not convenient to work with processes which are "almost" uncorrelated, "almost" Markovian and with spectra flat "except for very large frequencies", it is desirable to find a stochastic process which does not have these limitations, but which leads to an output x(t) whose properties (e.g., transition density) are limits of those obtained using "physical" white noise. Such an idealized white noise could not occur physically, as it would imply infinite energy in the input.

<u>The stochastic integral equation</u>

Idealized white noise, being uncorrelated, is a pathological function, and (1.1) cannot be interpreted as a set of differential equations in the usual sense. Thus it is often written as a set of integral equations

$$x_i(t) - x_i(t_o) + \int_{t_o}^{t} f_i(x(s))ds = \int_{t_o}^{t} h_{ij}(x(s))\, n_j(s)\, ds$$

$$= \int_{t_o}^{t} h_{ij}(x(s))dw_j(s) \quad . \tag{1.2}$$

Here the $w_j(t)$ are <u>Wiener processes</u>, i.e., $w_j(t)$ is Gaussian, has zero mean and stationary independent (and therefore uncorrelated) increments, and

$$\left\langle \left[w_j(t_2) - w_j(t_1)\right]\left[w_k(t_2) - w_k(t_1)\right]\right\rangle = 2D_{jk}\,|t_2 - t_1| \quad . \tag{1.3}$$

Then $n_j(t)$, the formal derivative of $w_j(t)$, has the required properties, i.e.

$$R_{n_j n_k}(t) = 2D_{jk}\,\delta(t) = 0 \quad \text{for } t \neq 0 \tag{1.4}$$

$$\Phi_{n_j n_k}(\omega) = \frac{2}{\pi} D_{jk} = \text{const.} \tag{1.5}$$

However, since almost all sample paths of $w_j(t)$, though continuous, are not of bounded variation, the integrals $\int_{t_o}^{t} h_{ij}(x(s))dw_j(s)$ cannot be considered as ordinary Riemann-Stieljes (or Lebesque-Stieljes) integrals. So-called stochastic or Ito integrals must be used.

<u>Doob's stochastic integral</u>

There are many possible generalized integrals. The most commonly adapted is Doob's stochastic integral, or the forward Ito integral, which may be defined essentially as follows[1]:

$$\int_a^b g(s)dw(s) = \lim_{\delta \to 0} \sum_{k=0}^{n-1} g(t_k)\left[w(t_{k+1}) - w(t_k)\right] , \tag{1.6}$$

where $a = t_o < t_1 < \ldots < t_n = b$ and $\delta = \max\limits_{0 \le k < n} (t_{k+1} - t_k)$.

From this definition it follows that

$$\left\langle \int_a^b g(s)dw(s) \right\rangle = 0 \tag{1.7}$$

for all g(t), which is convenient. However, the transition probability obtained using this interpretation of (1.1) is not in general the same

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[1] See Doob [9], in particular pages 436-444.

as the limit of that using "physical" white noise.[1] In addition such physically unlikely results are obtained as that increasing white noise excitation may make an unstable system stable.[2]

The symmetric stochastic integral

Gray and Caughey [24] show that if, instead of Doob's stochastic integral, one uses the "symmetric" stochastic integral

$$\int_a^b g(s)dw(s) = \lim_{\delta \to 0} \sum_{k=1}^{n-1} \frac{1}{2}\left[f(t_{k+1})+f(t_k)\right]\left[w(t_{k+1}) - w(t_k)\right] \qquad (1.8)$$

(where t_k and δ are as in (1.6)), then the transition probability obtained is the limit of that for physical white noise, and no physical anomalies occur.

Thus it appears that an appropriate mathematical idealization of physically occurring white noise is as a stochastic process with flat spectrum, such that stochastic differential equations containing it (such as (1.1)) are interpreted as integral equations where the definition (1.8) is used for the integrals. This will be the interpretation throughout this thesis.

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[1] As is shown for the special case of

$$\frac{dx}{dt} + \ell x = x\,n(t)$$

by Caughey and Dienes [5] and (using a different method) Åström [2]. Gray [22] proves it for the system (1.1) assuming a particular kind of "physical" white noise.

[2] See Gray and Caughey [24] for a list of authors who have used this method, or others yielding the same results.

Stochastic difference equations

Another approach (see Gihman [21]) is to define (1.1) as the limit of a set of stochastic difference equations. However, this leads to the same ambiguity, since different difference equations, which would approximate the same differential equation if Wiener processes were well-behaved, give different results--just as, in the integral equation approach, different definitions of stochastic integral, equivalent for well-behaved w(t), give different results. Thus if, following Gihman and most other authors, one takes (1.1) as the limit of the discrete problem

$$\frac{x_i(t_{k+1}) - x_i(t_k)}{t_{k+1} - t_k} + f_i(t_k) = h_{ij}(x(t_k))\, n_j(t_k) \;, \quad t_{k+1} - t_k = \text{const.}, \qquad (1.9)$$

where $n_i(t_k)$ is a one-dimensional random walk whose position at each succeeding t_k changes by jumps whose magnitudes are independent, then the same results are obtained as using Doob's stochastic integral. But if one takes (1.1) as the limit of the more symmetrical discrete problem

$$\frac{x_i(t_{k+1}) - x_i(t_k)}{t_{k+1} - t_k} + \frac{1}{2}\left[f_i(t_k) + f_i(t_{k+1})\right]$$

$$= \frac{1}{2}\left[h_{ij}(x(t_k)) + h_{ij}(x(t_{k+1}))\right] n_j(t_k) \;, \qquad (1.10)$$

then the results obtained are the same as those obtained using the symmetric stochastic integral (and are the limits of those occurring in physical applications).

1.2 THE FOKKER-PLANCK EQUATION

The solution of (1.1)

The system (1.1), interpreted in any of the ways mentioned in the previous section, has a unique solution, $x(t)$, which is a stationary n-dimensional Markov process, whose sample paths are almost all continuous. The existence and uniqueness of this solution follows as in the case of ordinary differential equations for sufficiently well-behaved coefficients $f_i(x)$ and $h_{ij}(x)$.[1]

Form of the FP equation

Since $x(t)$ is a stationary Markov process, it is completely specified by its transition density $P(x, t|x_o)$. (By definition, $P(x, t|x_o)dx_1 \ldots dx_n$ is the probability that a sample path starting at x_o lies in the n-dimensional element $(x_i, x_i + dx_i)$ at time t later.) This transition density, as is well known, satisfies the FP equation (also called the forward Kolmogorov equation), which has the form

$$\frac{\partial P}{\partial t} = \frac{\partial^2}{\partial x_i \partial x_\ell}\left[a_{i\ell}(x)P\right] - \frac{\partial}{\partial x_i}\left[b_i(x)P\right] , \qquad (1.11)$$

with initial condition

$$P(x, t|x_o) = \prod_{i=1}^{n} \delta(x_i - x_{oi}) . \qquad (1.12)$$

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[1] See Doob [9], pages 273-291. Although most treatments require the coefficients to satisfy Lipschitz conditions, a finite number of discontinuities can be allowed for by dividing the space Ω up into sections with the discontinuities on their boundaries, and assuming $x(t)$ continuous across these boundaries.

Boundary conditions may also be necessary, especially if Ω is not the entire n-dimensional space; appropriate conditions are discussed in section 1.4 for the one-dimensional case.

The coefficients of the FP equation

The $a_{i\ell}(x)$ are called the diffusion coefficients of the process, while the $b_i(x)$ are the drift coefficients[1]. They are defined as follows:

$$a_{i\ell}(x) = \lim_{\Delta t \downarrow 0} \frac{E[\Delta x_i \Delta x_\ell, \Delta t \,|\, x]}{2\Delta t} \tag{1.13}$$

$$b_i(x) = \lim_{\Delta t \downarrow 0} \frac{E[\Delta x_i, \Delta t \,|\, x]}{\Delta t} , \tag{1.14}$$

where $E[g(x), t\,|x_o]$ is the conditional expectation of $g(x)$, i.e., the expectation calculated using the transition (or conditional) density $P(x, t\,|x_o)$ as weighting function, and Δx_i is the change in x_i in time Δt along a sample path.

The controversy concerning $b_i(x)$

Formulas (1.13, 1.14) are well established and beyond dispute. However, conflicting results have been obtained in the evaluation of (1.14). As pointed out by Gray and Caughey [24], the differences stem from the various interpretations of the stochastic differential equations (1.1) (as discussed in section 1.1 above). All interpretations give

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[1] See Dynkin [10], page 4.

$$a_{i\ell}(x) = D_{jk} h_{ij} h_{\ell k} , \qquad (1.15)$$

but the use of Doob's stochastic integral gives

$$b_i(x) = - f_i(x) , \qquad (1.16)$$

while the symmetric stochastic integral gives

$$b_i(x) = - f_i(x) + D_{jk} h_{ij} \frac{\partial h_{\ell k}}{\partial x_\ell} . \qquad (1.17)$$

Here D_{jk} is as in (1.4), i.e., $2D_{jk}\,\delta(t_2 - t_1)$ is the cross-correlation of $n_j(t_1)$ and $n_k(t_2)$.

Thus the FP equation in the first interpretation becomes

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x_i}(f_i P) + D_{jk} \frac{\partial}{\partial x_i \partial x_\ell}(h_{ij} h_{\ell k} P) , \qquad (1.18)$$

while in the second it is

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x_i}(f_i P) + D_{jk} \frac{\partial}{\partial x_i}\left[h_{ij} \frac{\partial}{\partial x_\ell}(h_{\ell k} P)\right] . \qquad (1.19)$$

Although (1.19) will be used throughout this thesis, formulas obtained using it (i.e., almost all formulas from this point on) can be transformed into the corresponding formulae obtained using (1.18) by replacing $f_i(x)$ by $f_i(x) + D_{jk} h_{ij} \frac{\partial h_{\ell k}}{\partial x_\ell}$ wherever it occurs. Note that if each $h_{ij}(x)$ is independent of x_i, then the two interpretations yield the same results. An important particular case is the nth order system with the differential equation

$$x^{(n)} + f\left(x, x', \ldots, x^{(n-1)}\right) = h_i\left(x, x', \ldots, x^{(n-2)}\right) n_i(t) , \qquad (1.20)$$

$$i = 1, \ldots, m ,$$

i.e. where the excitation terms do not involve the two highest derivatives. This includes the case of a first order system with only forcing function excitation.

Properties of the coefficients

The behavior of $a_{i\ell}(x)$ and $b_i(x)$ depends fairly directly on that of $f_i(x)$ and $h_{ij}(x)$. For example, $a_{i\ell}$ and b_i are continuous whenever f_i is continuous and h_{ij} continuously differentiable; and if f_i and h_{ij} are linear, $a_{i\ell}$ is quadratic and b_i linear. Thus a well-behaved FP equation depends on well-behaved (i.e., physically reasonable) coefficients in the stochastic differential equation.

An important property is that the matrix $\{a_{i\ell}(x)\}$ is non-negative (positive indefinite). This follows directly from the definition (1.13).

The backwards equation

$P(x, t|x_o)$ is also given, as a function of x_o and t, by the formal adjoint of the FP equation (1.11), namely

$$\frac{\partial P}{\partial t} = a_{i\ell}(x_o) \frac{\partial^2 P}{\partial x_{oi} \partial x_{o\ell}} + b_i(x_o) \frac{\partial P}{\partial x_{oi}} , \qquad (1.21)$$

with the same initial condition (1.12). This equation is called the backwards Kolmogorov equation (henceforth abbreviated to "backwards equation").

1.3 THE ONE-DIMENSIONAL DIFFUSION EQUATION

From this point on, we limit ourselves to the first order system (0.1). Then, putting $a_{11}(x) = a(x) \geq 0$, $b_1(x) = b(x)$, and defining the operators

$$\mathcal{F}(x)(\cdot) = a(x)\frac{\partial^2}{\partial x^2} + b(x)\frac{\partial}{\partial x} \tag{1.22}$$

$$\mathcal{F}^*(x)(\cdot) = \frac{\partial}{\partial x}\left\{\frac{\partial}{\partial x}[a(x)\cdot] - b(x)(\cdot)\right\}, \tag{1.23}$$

the FP and backwards equations for $P(x,t|x_o)$ can be written (for $x \in \Omega = (y_1, y_2)$ say, where y_1, y_2 may be infinite)

$$\frac{\partial P}{\partial t} = \mathcal{F}^*(x)P \tag{1.24}$$

and

$$\frac{\partial P}{\partial t} = \mathcal{F}(x_o)P \tag{1.25}$$

respectively.

The following discussion (in sections 1.3-5) of the one-dimensional diffusion equation (1.25) and its formal adjoint (1.24) presents results mostly due to Feller and published by him in a series of papers.[1] All but the last two of these make use of the theory of semigroups, but this is avoided here. Except in [14], the coefficients a(x) and b(x) may be rather general functions--in particular they need not be continuous, and so the piecewise linear a

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[1] [14] through [19]. Other authors who have done related work on one-dimensional diffusion include Hille [25], Yosida, Ito, Dynkin, McKean. [10] and [26] are general texts. Åström [2] has applied Feller's results to first order linear systems.

and b of Chapter II are certainly permissible.

The domains of $\mathfrak{F}$, $\mathfrak{F}^*$

Physical reasoning can be used to determine properties which the solution $P(x, t|x_o)$ of (1.24) and (1.25) must have to be the transition density of a process x(t) satisfying (0.1). (We will write $P \in D[\mathfrak{F}(x_o)]$, $P \in D[\mathfrak{F}^*(x)]$.)

(a) The probability of ending in (x, x+dx) having started at x_o must be almost the same as the probability of ending in (x, x+dx) having started at any point near x_o, unless x_o is an exceptional point at which there is some sort of barrier. Thus, writing $z_1, z_2, \ldots$ for all such exceptional points (where $y_1 < z_1 < z_2 < \ldots < y_2$), $P(x, t|x_o)$ is continuous with respect to x_o in closed intervals not containing z_i in their interiors.

(b) Since $P(x, t|x_o)$ is a probability density,

$$\int_\Omega P(x, t|x_o)dx = 1 . \qquad (1.26)$$

However, the space of functions of integral 1 is not linear, so the domain of $\mathfrak{F}^*$ is taken instead as the space of all integrable functions. In addition it is assumed that there is no accumulation of probability mass at individual points, i.e. that $\int^x P(x, t|x_o)dx$ is continuous with respect to x. Again we do not eliminate the possibility that there may be exceptional points where, by the nature of the process, probability must be absorbed.

Further restrictions on $P(x, t|x_o)$ can be obtained using (1.24) and (1.25). We define

$$W(x) = \exp\left(-\int_c^x \frac{b(\xi)}{a(\xi)}\, d\xi\right), \tag{1.27}$$

where c is close enough to x for the integral to converge; if there is no such c, $W(x) = \infty$. Then one has

$$\mathfrak{F} = aW \frac{\partial}{\partial x}\left(W^{-1} \frac{\partial}{\partial x}\right) \tag{1.28}$$

$$\mathfrak{F}^* = \frac{\partial}{\partial x}\left[W^{-1} \frac{\partial}{\partial x}(aW\,\cdot)\right]. \tag{1.29}$$

(c) Integrating $\frac{\partial P}{\partial t} = \mathfrak{F}(x_o)P$ from x_o- to x_o+, one obtains

$$\left[W^{-1}(x_o)\frac{\partial P}{\partial x_o}\right]_{x_o=x_o-}^{x_o+} = \frac{\partial}{\partial t}\int_{x_o-}^{x_o+} \frac{P(x, t\,|x_o)}{a(x_o)W(x_o)}\, dx_o = 0\,, \tag{1.30}$$

since the integral disappears. Thus $W^{-1}\frac{\partial P}{\partial x_o}$ is continuous with respect to x_o. Then the continuity of $W(x_o)$ implies that $\frac{\partial P}{\partial x_o}$ is continuous wherever $W(x_o)$ is not infinite.

(d) Integrating $\frac{\partial P}{\partial t} = \mathfrak{F}^*(x)P$ from x- to x+, and using the continuity of $\int^x P dx$, one obtains

$$\left[W^{-1}(x)\frac{\partial}{\partial x}\Big(a(x)W(x)P\Big)\right]_{x=x-}^{x+} = \frac{\partial}{\partial t}\int_{x-}^{x+} P(x, t\,|x_o)dx = 0\,. \tag{1.31}$$

Thus $Q(x, t\,|x_o)$ is continuous with respect to x, where

$$Q(x, t\,|x_o) = W^{-1}\frac{\partial}{\partial x}(aWP) = \frac{\partial}{\partial x}[a(x)P] - b(x)P\,. \tag{1.32}$$

(e) Integrating $\frac{\partial P}{\partial t} = \mathfrak{F}^*(x)P$ twice, firstly from z to x, then from x- to x+, one obtains

$$\left[a(x)W(x)P\right]_{x=x-}^{x+} = \frac{\partial}{\partial t}\int_{x-}^{x+} W^{-1}(x)\int_{z}^{x} P(\xi, t|x_o)d\xi\, dx = 0 . \qquad (1.33)$$

Thus aWP is continuous with respect to x. Continuity of W then implies continuity of aP, for $W(x) \neq 0$.

Thus aWP and Q are continuous with respect to x, and P and $W^{-1}\frac{\partial P}{\partial x_o}$ are continuous with respect to x_o. These properties will be used in the following chapters where the FP equation is solved separately in various segments and these solutions connected together.

The canonical form of the diffusion equation

From this point until section 1.5, it will be assumed that a(x) is bounded away from zero in any interval interior to the (open) interval Ω. (It is permitted that $a(x) \to 0$ as $x \to y_1$ or y_2.) We will speak of (y_1, y_2) as a regular interval. Feller [17, 18, 19] has shown that the differential operator $\mathfrak{F}$, defined on $C(\Omega)$, can be written in the form

$$\mathfrak{F} = D_m D_s \quad \text{where } D_y = \frac{d}{dy} , \qquad (1.34)$$

where the monotonic increasing functions s(x) and m(x) are the so-called canonical scale and canonical measure. As shown in [17], page 95, s is uniquely defined up to an arbitrary linear transformation, and, for given s, m is determined up to an additive constant. If $W(x)$ is defined as in (1.27) one may take

$$s(x) = \int_z^x W(\xi)d\xi \tag{1.35}$$

$$m(x) = \int_z^x a^{-1}(\xi)W^{-1}(\xi)d\xi . \tag{1.36}$$

1.4 BOUNDARY CONDITIONS

Accessible and inaccessible boundaries

Consider y_j ($j = 1$ or 2), one of the endpoints of the interval Ω. Then the boundary y_j is defined to be accessible if there is nonzero probability of a path from a given point in Ω reaching y_j in a finite time. According to Feller [15], theorem 3, y_j is accessible if and only if all solutions of $\mathfrak{F}z - \lambda z = 0$ are bounded near y_j. Comparing with pages 487-88 of Feller [14], it is seen that this is the case if and only if $W(x)\int^x a^{-1}(\xi)W^{-1}(\xi)d\xi$ is integrable in some (and therefore every) neighborhood of y_j; that is,

$$\left| \int^{y_j} m(x)ds(x) \right| < \infty . \tag{1.37}$$

Regular, exit, entrance and natural boundaries

The classification into accessible and inaccessible boundary points can be further subdivided. According to [14], page 487, a regular boundary y_j has

$$|s(y_j)| < \infty , \quad |m(y_j)| < \infty ; \tag{1.38}$$

an <u>exit</u> boundary has

$$|m(y_j)| = \infty \ , \quad \left|\int^{y_j} mds\right| < \infty \ ; \tag{1.39}$$

an <u>entrance</u> boundary has

$$|s(y_j)| = \infty \ , \quad \left|\int^{y_j} sdm\right| < \infty \ ; \tag{1.40}$$

a <u>natural</u> boundary has

$$\left|\int^{y_j} mds\right| = \infty \ , \quad \left|\int^{y_j} sdm\right| = \infty \ . \tag{1.41}$$

The monotonicity of s and m leads to the following:

$$\left|\int mds\right| < \infty \Rightarrow |s| < \infty \tag{1.42}$$

$$\left|\int sdm\right| < \infty \Rightarrow |m| < \infty \tag{1.43}$$

$$\text{both } |s| \ , \ |m| < \infty \Rightarrow \text{both } \left|\int mds\right| \ , \ \left|\int sdm\right| < \infty \ . \tag{1.44}$$

Thus both regular and exit boundaries are accessible, while entrance and natural boundaries are inaccessible, and there are no other possibilities.

<u>Admissible boundary conditions</u>

Feller [14, 18] has determined the most general permissible boundary conditions for the FP and backwards equations, given that the backwards equation has the form (1.25) and that the process is stationary Markovian with continuous paths.[1] These conditions are somewhat too general for our purposes. For example, they include conditions which can be interpreted as the absorption of a path at a boundary for a finite (random) time, and then its return to a random point in Ω. Also, the appropriate FP equation adjoint to (1.25) is not (1.24), nor is it in general a differential equation.

We will restrict ourselves to the classical elastic boundary conditions, which are as follows at y_j a regular boundary (see [15], section 11). For $u(x,t)$ satisfying $u_t = \mathfrak{F}(x)u$,

$$c_j u(x,t) = (-1)^{j-1} \lim_{x\to y_j} W^{-1}(x) u_x(x,t) \; ; \qquad (1.45)$$

for $v(x,t)$ satisfying $v_t = \mathfrak{F}^*(x)v$,

$$c_j \lim_{x\to y_j} a(x)W(x)v(x,t) = (-1)^j \left\{ [a(x)v(x,t)]_x - b(x)v(x,t) \right\} . \qquad (1.46)$$

Here $0 \le c_j \le \infty$. If $c_j = 0$, the boundary is reflecting. If $c_j = \infty$, it is absorbing.

Appropriate boundary conditions

Suppose both boundaries are regular. Then unless both boundaries are reflecting, $P(x,t|x_o) \to 0$ as $t \to \infty$ for $y_1 < x, x_o < y_2$ (see section 1.6). Then both autocorrelation and spectrum become trivial, for $P_2(x_1, t_1, x_2, t_2)$ is identically zero for $y_1 < x_1, x_2 < y_2$.

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[1] Wentzell [41, 42] has done the same for diffusion in n dimensions. See Dynkin [10], pages 6-7.

According to whether paths are considered to disappear on being absorbed at y_j, or to remain constant at y_j for all later times (which means essentially that the y_j are being considered as points in the process), one gets from (0.2)

$$R(t) = 0 \tag{1.47}$$

or

$$R(t) = A_1 y_1^2 + A_2 y_2^2 \tag{1.48}$$

where A_j is the probability of reaching y_j (so $A_1 + A_2 = 1$). To exclude this case, we will consider only reflecting conditions at regular outer boundaries.

For convenience of calculation, the process on Ω may be divided up into interdependent subprocesses, each on a subinterval of Ω. The boundary conditions at points of division will be given by the properties of $P(x, t|x_o)$ deduced above (continuity of P and $W^{-1} \frac{dP}{dx_o}$ with respect to x_o, and of Q, aWP with respect to x). Such conditions must be admissible by their derivation.

Conditions at irregular boundaries

If y_j is not a regular boundary, the conditions (1.45, 46) may be superfluous. As shown in [14], page 488, if y_j is a natural boundary, then both $u_t = \mathfrak{J} u$ and $v_t = \mathfrak{J}^* v$ have only one independent solution (belonging to $D(\mathfrak{J})$ and $D(\mathfrak{J}^*)$ respectively), and if u, v are these solutions, u, $W^{-1}u'$, Wav, $(av)'$-bv all approach 0 as $y \to y_j$. Thus no boundary conditions are required--although it may be convenient to use, e.g., $(av)' - bv \to 0$ to determine which solution of

$v_t = \mathfrak{F}^* v$ belongs to $D(\mathfrak{F}^*)$, rather than checking its integrability directly. Similarly, no condition need be imposed at an entrance boundary for $u_t = \mathfrak{F} u$--the only solution satisfies $W^{-1}u' \to 0$, so the boundary is automatically reflecting--or at an exit boundary for $v_t = \mathfrak{F}^* v$--the only solution satisfies $aWv \to 0$, so the boundary is automatically absorbing. This is reasonable, since no paths can reach a natural or an entrance boundary in a finite time, while none can arrive from a natural or an exit boundary.

Types of natural boundaries

It is seen from (1.41, 44) that there are three types of natural boundary y_j:

$$\text{(a)} \quad |s(y_j)| = \infty \ , \quad |m(y_j)| < \infty \tag{1.49}$$

$$\text{(b)} \quad |s(y_j)| = \infty \ , \quad |m(y_j)| = \infty \tag{1.50}$$

$$\text{(c)} \quad |s(y_j)| < \infty \ , \quad |m(y_j)| = \infty \ . \tag{1.51}$$

Before interpreting these conditions in terms of the behavior of sample paths, some definitions will be given. Let $f(x, y)$ be the probability that the first passage time from x to y is finite; then the process is recurrent if $f(x, y) = f(y, x) = 1$ and transient if $f(x, y) f(y, x) < 1$, for all $y_1 < x < y < y_2$. By problem 4.5.4, page 124 of Ito and McKean [26], every process of the type being considered is either recurrent or transient. Suppose a reflecting boundary be placed at x ($y_1 < x_o < x < y_2$); then y_1 will be called an attracting boundary if a path starting at x_o, while never reaching y_1 in finite time, is, for arbitrarily small δ, within $(y_1, y_1 + \delta)$ with probability greater than $1 - \delta$ if $t \geq$ some $T(=T(\delta, x_o))$; similarly for the boundary y_2. (Note that this use of the term "attracting" is completely different from that of Dynkin [10].)

According to problem 4.7.6, page 134 of [26], a process is recurrent if, for $j = 1$ and 2, either $|s(y_j)| = \infty$ -- i.e., y_j is either entrance or natural type (a) or (b)--or y_j is a regular reflecting boundary. Otherwise the process is transient. Thus it is seen that the distinguishing characteristic of a natural boundary of type (c) is that there is a positive probability that a path, starting from any point x_o, will eventually pass into any given neighborhood of the boundary and remain there. Such a boundary will be called strongly attracting.

It is possible that the set of paths from x_o tends to y_j in probability, but that no individual path tends to y_j. That is, a process may be recurrent but have one or two attracting boundaries. An example is the process on $(-\infty, \infty)$ governed by the stochastic differential equation $\frac{dx}{dt} = n(t)$. Here $|m(\pm\infty)| = |s(\pm\infty)| = \infty$, so the process is natural of type (b). In fact, it is shown in section 1.6 that a natural boundary y_j is attracting if and only if $|m(y_j)| = \infty$. Thus the distinguishing characteristic of a natural boundary of type (a) is that it is non-attracting; type (b) is attracting but not strongly attracting.

1.5 REGULAR AND IRREGULAR POINTS

In this section, the assumption that Ω is a regular interval (i.e., that $a(x) \neq 0$ for x interior to Ω), made above, is dropped.

Essentially, a regular point x is one which paths can cross continuously in either direction. Let e^+ [e^-] be the probability that the crossing time from x+ [x-] to x is zero. (According to Blumenthal's zero-one law, it is either zero or one.) Then, according to Ito and McKean [26], page 91, x is a regular point if both e^+ and e^- are zero. Irregular[1] points are classified as follows: a left shunt has $e^+ = 0$, $e^- = 1$; a right shunt has $e^+ = 1$, $e^- = 0$; a trap has $e^+ = e^- = 0$.[2]

Irregular points as boundaries

If the interval $\Omega = (y_1, y_2)$ is divided at x, and subprocesses on each of the intervals (y_1, x), (x, y_2) are considered, then the following are apparent.

(a) If x is regular, it is a regular boundary for both (y_1, x) and (x, y_2).

(b) If x is a left shunt, then it is either an exit boundary for (x, y_2) and an entrance boundary for (y_1, x), or an exit boundary for (x, y_2) and a regular boundary for (y_1, x), or a regular boundary for (x, y_2) and an entrance boundary for (x, y_2). For a right shunt, interchange (x, y_2) and (y_1, x) throughout the above statement.

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[1] The term singular will be reserved for the singular points of a differential operator. (It will be shown that real singular points of $\mathfrak{F}$ are irregular, and vice versa.)

[2] Feller [16] calls traps and shunts absorption and translation points respectively, and gives different, but equivalent, definitions.

(c) If x is a trap, then it is either an inaccessible boundary on both sides, or an exit boundary on both sides, or an accessible boundary on one side and a natural boundary on the other. We will speak of an inaccessible trap, an accessible trap, or a trap accessible from one side, respectively.

Zeros of a and b

A shunt can be considered as a point where the second order diffusion operator $\mathfrak{F}$ degenerates to a first order operator, and a trap a point where it degenerates to a zeroth order operator. From the discussion above, together with conditions (1.37-41), specific results can be proved to justify these statements. One can show that, for an irregular point of any kind, it is necessary and sufficient that $a(x) \to 0$ on at least one side. Thus, a regular interval consists entirely of regular points. Also, if $a(x)$ and $b(x)$ are continuous at z, z is a trap if $a(z) = 0$ and $b(x)$ crosses the x axis at z, while $a(z) = 0$, $b(z) < 0$ [>0] implies a left [right] shunt. A large number of similar results can also be proved, for example when $a(x)$ and/or $b(x)$ are discontinuous at z and tend to zero on one side only. Pathological cases, as when z is a limit point of zeros of $a(x)$ or $b(x)$, are excluded from consideration.

An example

To illustrate the method, one of the above results is proved. Let $a(z) = 0$, $b(z) < 0$, a and b continuous at z. Then, for z'

just greater than z,

$$m(z')-m(z) = \int_z^{z'} a^{-1}(x)W^{-1}(x)dx$$

$$\doteqdot \exp\left[-\int_{z'}^{c} b(x)a^{-1}(x)dx\right] \int_z^{z'} a^{-1}(x)\exp\left[-b(z)\int_x^{z'} a^{-1}(\xi)d\xi\right] dx$$

$$= \infty \tag{1.52}$$

$$\int_z^{z'} mds = \int_z^{z'} W(x)\int_x^{z'} a^{-1}(\xi)W^{-1}(\xi)d\xi\, dx$$

$$\doteqdot \int_z^{z'} dx \int_x^{z'} a^{-1}(\xi)\exp\left[b(z)\int_x^{z} a^{-1}(\eta)d\eta\right] d\xi$$

$$\to 0 \quad \text{as } z' \to z . \tag{1.53}$$

Thus z is an exit boundary from above, by (1.39). Similarly, for z' just less than z,

$$s(z) - s(z') = \infty , \quad \int_{z'}^{z} sdm \to 0 \quad \text{as } z' \to z , \tag{1.54}$$

so that by (1.40), z is an entrance boundary from below. Thus z is a left shunt, as was to be proved.

1.6 THE STEADY-STATE DENSITY

In a time-independent physical system excited by stationary random noise, it is reasonable to expect that, as time $t \to \infty$, the transition density $P(x, t | x_o)$ should tend to a value independent of t, the so-called steady-state density $P_o(x)$, and that if P satisfies a FP equation then P_o satisfies the same equation. In addition, it is apparent that, for some systems, P_o should be independent of the initial position x_o.

Justification of the above statements for one-dimensional diffusion was provided by Maruyama and Tanaka (see [32, 39]). This was extended to n dimensions by the same authors [33] (see also Yaglom [47] and Gray [22]) and to more general metric spaces by Khas'minskii [30]. Despite its generality, the treatment in this last paper is the most easily adapted to our purposes.

Some results of Khas'minskii

In the one-dimensional case, the functions of x $m(x)$ and $P(x, t | x_o)$ can be replaced by set functions (measures) as follows. If $E \subset \Omega$, put

$$m(E) = \int_E dm(x) \tag{1.55}$$

$$P(E, t | x_o) = \int_E P(x, t | x_o)\, dx \,. \tag{1.56}$$

These set functions are more general than the corresponding functions of x. Consider a diffusion on an arbitrary σ-compact complete metric space Ω. That is (following Khas'minskii), specify a Markov transition function $P(E, t|x_o)$ for all measurable $E \subset \Omega$, $t \geq 0$, and $x_o \in \Omega$, with

$$P(\Omega, t|x_o) = 1 \tag{1.57}$$

and satisfying certain other conditions (e.g. 1^o-3^o, p. 179 of [30]). In certain circumstances a canonical measure $m(E)$ can be defined and it is unique (see theorems 2.1, 3.2 of [30]). The following results are relevant.

(a) If $m(\Omega) < \infty$, then for any measurable E,

$$\lim_{t\to\infty} P(E, t|x_o) = \frac{m(E)}{m(\Omega)} \tag{1.58}$$

(see Khas'minskii's theorem 3.4).

(b) For a non-recurrent process and compact E,

$$\int_0^\infty P(E, t|x_o)\, dt < \infty \quad . \tag{1.59}$$

Hence

$$\lim_{t\to\infty} P(E, t|x_o) = 0 \tag{1.60}$$

(see lemma 3.1 and its corollary).

(c) For a recurrent process with $m(\Omega) = \infty$, $m(E) < \infty$,

$$\lim_{t\to\infty} \frac{1}{T} \int_0^T P(E, t|x_o)\, dt = 0 \tag{1.61}$$

(see the corollary to theorem 3.1). Khas'minskii hypothesizes that (1.61) may be strengthened to (1.60), and has a proof for the one-dimensional case (see remark 2, page 191).

Application to one-dimensional diffusion

In the type of process dealt with in previous sections, the above results lead to the following

Theorem: Let $\Omega = (y_1, y_2)$ be a regular interval. If y_1 and y_2 are reflecting or non-attracting natural boundaries, then, for any $x \in \Omega$,

$$\lim_{t \to \infty} P(x, t | x_o) = P_o(x) \quad , \tag{1.62}$$

where $P_o(x)$ is the solution of $\mathfrak{F}^* P_o = 0$ with boundary conditions $Q_o(y_1) = Q_o(y_2) = 0$ and

$$\int_\Omega P_o(x) dx = 1 \quad . \tag{1.63}$$

In every other case,

$$\lim_{t \to \infty} P(x, t | x_o) = 0 \quad . \tag{1.64}$$

Here Q_o denotes $\frac{d}{dx}(a P_o) - bP_o$, so $Q_o = 0$ is the steady-state analog of the reflecting boundary condition $Q = 0$. By a reflecting boundary is meant a regular boundary with $Q = 0$, or an entrance boundary. (One has $Q = 0$ automatically at an entrance or natural boundary.) The restriction to a regular interval is equivalent to $^o3$ on page 179 of [30]. The restriction to reflecting or non-attracting natural boundaries is equivalent to $m(\Omega) < \infty$. *Integrating*

$\mathfrak{F}^* P_o = 0$ once and using the boundary conditions, one gets $Q_o(x) = 0$ for all $x \in \Omega$. Integrating again and using (1.63) it is seen that

$$P_o(x) = \frac{a^{-1}(x)\exp\left[\int_z^x \frac{b(\xi)}{a(\xi)}\,d\xi\right]}{\int_{y_2}^{y_1} a^{-1}(x)\exp\left[\int_z^x \frac{b(\xi)}{a(\xi)}\,d\xi\right]} = \frac{\frac{dm}{dx}}{m(y_2) - m(y_1)}\ , \quad z \in \Omega\ . \tag{1.65}$$

It is apparent from (1.56) that this, together with (1.62), is the same as (1.58); note that m(x) is differentiable on account of (1.36) for $x \in \Omega$.

The second result, (1.64), follows immediately from (b) and (c). Although (1.57) does not hold if a boundary (y_1 say) is exit or regular non-reflecting, it is likely that result (b) does not require this pre-condition. In any case, it can be made to hold by adjoining y_1 to the process $\left(\text{so } \Omega = [y_1, y_2) \text{ instead of } \Omega = (y_1, y_2)\right)$.

Irregular points

If there are irregular points interior to (y_1, y_2), the theorem above cannot be applied directly. In particular, the steady-state density may depend on the initial condition x_o, and so will be written $P_o(x|x_o)$. Types of irregular points are defined in section 1.5. The arguments used below are based on the idea of a shunt as a point which allows passage in one direction but not the other, while a trap is the end of every path reaching it (if any do). The term absorbing

<u>boundary</u> is used below for exit and non-reflecting regular boundaries (i.e., accessible traps and shunts directed away from the interval under consideration).

The following cases occur.

(a) Suppose x lies to the right [left] of x_o and is separated from it by a left [right] shunt or by a trap. Then $P(x, t | x_o) = 0$ for all t (we will say that x is <u>inaccessible from x_o</u>), and $P_o(x | x_o) = 0$.

(b) Suppose that (a) does not hold, i.e., x is accessible from x_o, and that an absorbing or attracting boundary is accessible from x. Then, although $P(x, t | x_o) \neq 0$ for all finite t, $P_o(x | x_o) = 0$.

(c) In all other cases, both $P(x, t | x_o)$ and $P_o(x | x_o)$ are non-zero.

<u>Determination of P_o in the general case</u>

From the above, it is seen that two cases can occur, according to whether or not x_o lies in a regular interval bounded on both sides by absorbing or attracting boundaries. If not (case I), all paths from x_o will end in a single regular interval, or all will tend (at least in probability) to the same trap, and $P_o(x | x_o)$ will be zero outside this region. If so (case II), all paths will eventually divide between two such regions, one on either side of x_o, and neither containing x_o.

Once a path has passed into one of these regions, it stays there, so that the process in such a region (S say) can be treated independently of that on the remainder of Ω. If S is an attracting boundary, no path actually reaches S in finite time, so the above statement must be modified somewhat, but the result as far as the steady-state density is concerned is the same. Let $P_o^S(x)$ be steady-state density for $x_o \in S$. Then, if S is a single finite point y (i.e., an accessible trap or an attracting boundary),

$$P_o^S(x) = \delta(x-y) \quad . \tag{1.66}$$

If S is the point $\pm\infty$,

$$P_o^S(x) = 0 \tag{1.67}$$

for all finite x. If S is a regular interval, then $P_o^S(x)$ can be found as described earlier, using (1.62, 63). Note that in every case $P_o^S(x)$ is independent of x_o (so long as $x_o \in S$), and is zero for $x \notin S$.

If, for given x_o, there is only one region S (case I above), then

$$P_o(x|x_o) = P_o^S(x) \ . \tag{1.68}$$

If there are two regions, S_1 and S_2 say (case II), then

$$P_o(x|x_o) = A_{S_1}(x_o)P_o^{S_1}(x) + A_{S_2}(x_o)P_o^{S_2}(x), \tag{1.69}$$

where $A_{S_1}(x_o)$ is the probability of a path from x_o reaching (or tending to) S_1 rather than S_2; and $A_{S_2}(x_o) = 1 - A_{S_1}(x_o)$.

Determination of $A_{S_i}(x_o)$

This is a first passage problem. Let $S_1 = (x_2, x_1)$ and $S_2 = (x_4, x_3)$, where $x_1 \geq x_2 > x_o > x_3 \geq x_4$. In the case where at least one of x_2, x_3 (say x_2) is absorbing, $A_{S_1}(x_o)$ is given by theorem 5 of Feller [17]. It appears that this theorem is also true for x_2 a strongly attracting natural boundary (i.e., satisfying (1.51)). According to this theorem

$$A_{S_1}(x_o) = \frac{\int_{x_3}^{x_o} \exp\left[-\int_z^x \frac{b(\xi)}{a(\xi)}\, d\xi\right] dx}{\int_{x_3}^{x_2} \exp\left[-\int_z^x \frac{b(\xi)}{a(\xi)}\, d\xi\right] dx} = \frac{s(x_o) - s(x_3)}{s(x_2) - s(x_3)}, \quad z \in (x_3, x_2) \qquad (1.70)$$

for x_3 absorbing or strongly attracting, and

$$A_{S_1}(x_o) = 1 \qquad (1.71)$$

otherwise. When both $s(x_2)$ and $s(x_3)$ are infinite (both boundaries attracting but not strongly attracting) this theorem breaks down. In the case $\frac{dx}{dt} = n(t)$, $S_1 = +\infty$, $S_2 = -\infty$, it is apparent from symmetry that $A_{S_1}(x_o) = \frac{1}{2}$. In general one would expect that $A_{S_1}(x_o) = \text{const.}$, where the constant must be found by letting $t \to \infty$ in $P(E, t\,|\,x_o)$, with $E = (x, x_2)$.

Variation of $P_o(x\,|\,x_o)$ with x_o

In Case I, it is apparent from (1.68) that $P_o(x\,|\,x_o)$ is independent of x_o, provided x_o remains within the same regular interval, together with any interval which is accessible from this interval. However, in Case II, since $A_{S_1}(x_o)$, $A_{S_2}(x_o)$ vary continuously with x_o, (1.69) shows that (in most cases) $P_o(x|x_o)$ varies continuously as x_o varies in the interval concerned.

For P_o to be independent of x_o for all $x_o \in \Omega$, the most complex pattern of irregular points possible is illustrated in the next page. Note that any of the irregular points shown may be omitted and P_o will still be independent of x_o.

$\underline{P_o}$ as Abelian limit of P

In the following chapters, instead of $P(x, t|x_o)$ one first finds its Laplace transform

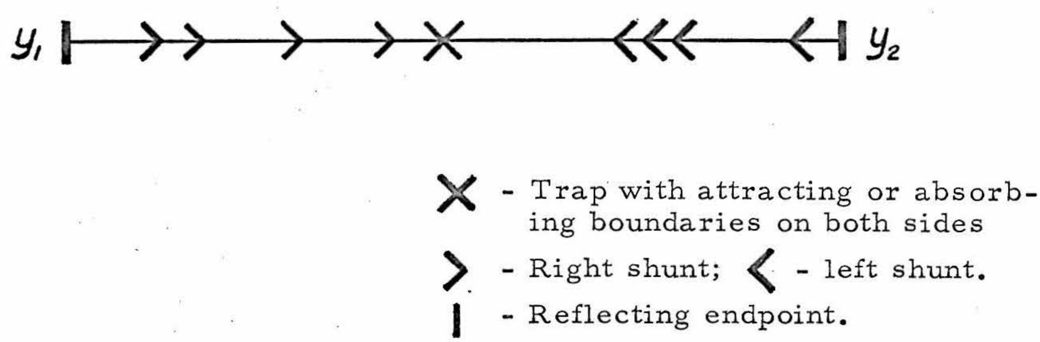
$$p(x, s|x_o) = \int_0^\infty P(x, t|x_o)e^{-st}dt \,. \tag{1.72}$$

Thus it is of interest to show that

$$P_o(x) = \lim_{s\to o+} sp(x, s|x_o) \;, \tag{1.73}$$

the Abelian limit of $P(x, t|x_o)$. The existence of $p(x, s|x_o)$ for all s with Re s > 0 follows immediately for all x where $P_o(x) < \infty$ -- i.e., everywhere except accessible traps. In addition, $p(x, s|x_o)$ exists for Re s = 0 if and only if the process is transient on the regular interval containing x, on account of (1.59)[1]. Zero belongs to the

[1] That this condition is both necessary and sufficient for transience is shown in problem 4.11.8, page 159, Ito and McKean [26].



Most complex possible system of irregular points having P_o independent of x_o.

point spectrum of $\mathfrak{F}^*$ on a regular interval (y_1, y_2) if and only if $m(y_2) - m(y_1) < \infty$. Thus for x lying in such an interval, $\lim_{s \to o+} sp(x, s | x_o)$ is proportional to the corresponding eigenfunction,[1] and is thus equal to $P_o(x)$ when normalized. Otherwise, one must have $\lim_{s \to o+} sp(x, s | x_o) = 0 = P_o(x)$.

Suppose that there are no attracting traps interior to Ω. Then the existence of the Laplace transform for all $x \in \Omega$, all $\mathrm{Re}\, s > 0$ shows that the eigenvalues of $\mathfrak{F}^*$ must lie entirely in the non-positive half plane. If there are such traps, consider the independent processes on the various subintervals to obtain the same result.

Types of stability

This term will be used in three ways in this thesis. A process on (y_1, y_2) will be called stable if neither y_1 nor y_2 is an infinite attracting boundary. It will be called stable in mean [or in mean square] if $\langle x \rangle$ [or $\langle x^2 \rangle$] is finite.

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[1] See Åström [2], section 5.

CHAPTER II

SPECTRUM OF AN ARBITRARY PIECEWISE LINEAR SYSTEM

2.1 SOLUTION OF THE FOKKER-PLANCK EQUATION

Piecewise linear systems

In this chapter a method is derived to obtain the spectral density and other properties of the system (0.1), where the functions $f(x)$, $h_j(x)$ are piecewise linear. By this is meant that they are linear in each of a finite number (say n-1) of segments (x_{i+1}, x_i); the end-points x_1, x_n of the interval in which the process occurs may or may not be finite, and f and h_j need not be continuous at the points x_i. In the ith segment, (x_{i+1}, x_i), we put

$$f(x) = \ell_{oi}\, x + k_{oi} \tag{2.1}$$

$$h_j(x) = \ell_{ji}\, x + k_{ji} \quad . \tag{2.2}$$

The FP equation

For the first order system (0.1), the FP equation (1.11) for $P(x, t|x_o)$ becomes

$$\frac{\partial P}{\partial t} = \frac{\partial^2}{\partial x^2}\left(a(x)P\right) - \frac{\partial}{\partial x}\left(b(x)P\right) , \tag{2.3}$$

where, substituting (2.1), (2.2) into (1.15), (1.17),

$$a(x) = \sum_{j,k} D_{jk}(\ell_{ji}x + k_{ji})(\ell_{ki}x + k_{ki}) \tag{2.4}$$

$$b(x) = -\ell_{oi}x - k_{oi} + \sum_{j,k} D_{jk}\ell_{ji}(\ell_{ki}x + k_{ki}) \tag{2.5}$$

for $x \in (x_{i+1}, x_i)$. Thus a(x), b(x) can be written

$$a(x) = a_i(x) = A_i x^2 + 2B_i x + C_i \tag{2.6}$$

$$b(x) = b_i(x) = D_i x + E_i \tag{2.7}$$

for $x \in (x_{i+1}, x_i)$, where A_i, B_i, C_i, D_i, E_i $(i = 1, \ldots, n-1)$ are constants. Since the matrix D_{jk} is non-negative, (2.4) gives

$$A_i C_i \geq B_i^2 \quad \text{for every } i \quad . \tag{2.8}$$

It is convenient to include all irregular points--i.e. where $a(x) \to 0$ on one or both sides--among the endpoints x_i of intervals.

The backwards equation

This is the formal adjoint of (2.3), namely

$$\frac{\partial P}{\partial t} = a(x_o)\frac{\partial^2 P}{\partial x_o^2} + b(x_o)\frac{\partial P}{\partial x_o} \quad . \tag{2.9}$$

Initial conditions

The initial condition for both (2.3) and (2.9) is, from (1.12),

$$P(x, 0 | x_o) = \delta(x - x_o) \quad . \tag{2.10}$$

Boundary conditions

At x_1 and x_n, a reflecting boundary is assumed (see section 1.4). Thus the appropriate boundary conditions for (2.3) are

$$Q(x_1, t \mid x_o) = Q(x_n, t \mid x_o) = 0 , \tag{2.11}$$

where

$$Q = \frac{\partial}{\partial x}(aP) - bP , \tag{2.12}$$

and the appropriate boundary conditions for (2.9) are

$$\frac{\partial P}{\partial x_o}(x, t \mid x_1) = \frac{\partial P}{\partial x_o}(x, t \mid x_n) = 0 . \tag{2.13}$$

Conditions (2.11) and (2.13) are both necessary only if x_1 and x_n are regular boundaries. If x_1 or x_n is infinite, it is a natural boundary, and (2.11) and (2.13) are both automatically satisfied. If x_1 or x_n is finite but irregular, one or the other condition, or both, will be redundant (according to whether it is an entrance, exit or natural boundary). Note that one is restricted to integrable solutions of (2.3) and bounded solutions of (2.9).

Since the method of solution is to find the general solution in each interval of linearity, and to piece these together, boundary conditions are also needed at all points x_i $(i=2, \ldots, n-1)$. These are provided for (2.3) by the continuity of Q and aP with respect to x, and for (2.9) by the continuity of P and $\frac{\partial P}{\partial x_o}$ with respect to x_o. Again some conditions become redundant when the x_i are irregular points.

The Laplace-transformed FP equation

By applying a Laplace transform with respect to t to (2.3), one obtains a second order ordinary differential equation for $p(x, s \mid x_o)$, the Laplace transform of $P(x, t \mid x_o)$. This is (using the initial condition)

$$\frac{d^2}{dx^2}\Big(a(x)p\Big) - \frac{d}{dx}\Big(b(x)p\Big) - sp = -\,\delta(x-x_o)\ . \qquad (2.14)$$

Similarly the Laplace transform of the backwards equation (2.9) is

$$a(x_o)\frac{d^2p}{dx_o^2} + b(x_o)\frac{dp}{dx_o} - sp = -\,\delta(x-x_o)\ . \qquad (2.15)$$

The boundary conditions remain the same as in the untransformed case, except that P and Q are replaced by their transforms, namely p and q.

Singularities of the transformed FP operator

Since (for $x \in (x_{i+1}, x_i)$) a(x) is a quadratic in x while b(x) is linear, the transformed FP operator

$$\overline{\mathfrak{F}^*} = a(x)\frac{d^2}{dx^2} + \Big[2a'(x) - b(x)\Big]\frac{d}{dx} + \Big[a''(x) - b'(x) - s\Big](\cdot) \qquad (2.16)$$

will have 3 regular singular points in the complex plane, at ∞ and at

$$\alpha_{1,2} = \frac{-B_i \pm \Big[B_i^2 - A_iC_i\Big]^{\frac{1}{2}}}{A_i}\ , \qquad (2.17)$$

except when 2 or 3 of these points coalesce to form an irregular singularity. Thus 2 linearly independent solutions for the homogeneous equation $\overline{\mathfrak{F}^*}p = 0$ can be found in terms of known special functions. The following cases occur.

(a) $A_i \neq 0$, $B_i^2 < A_i C_i$. Then α_1 and α_2 are distinct and finite. By a linear transformation of the independent variable which transforms α_1 to 0 and α_2 to 1, the equation becomes the hypergeometric equation.[1]

(b) $A_i \neq 0$, $B_i^2 = A_i C_i$. Then α_1 and α_2 coalesce to form an irregular (double) singularity. By replacing the independent variable x by $(x + B_i)^{-1}$, the equation becomes a confluent hypergeometric equation.[2]

(c) $A_i = B_i = 0$. Then α_1 and α_2 both coalesce with ∞, forming an irregular (triple) singularity. The solution can be expressed in terms of parabolic cylinder functions.[3]

The case $A_i = 0$, $B_i \neq 0$ cannot occur, on account of (2.8). This also shows that α_1 and α_2 are not real unless they are equal, so that the standard solutions for case (a) are complex-valued and must be combined to form 2 independent real solutions (since only real solutions are of interest in what follows). Except in special cases, this leads to considerable algebraic complication. One such special case is dealt with in section 5.4. Case (b) is dealt with in Chapter IV

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1 For solutions of this equation, see Abramowitz and Stegun [1], page 563.

2 For solutions see Erdelyi <u>et al</u>. [11], page 251.

3 See [1], page 686.

(a special case) and sections 5.2-3 (a less special case), while case (c) (forcing function excitation only) is dealt with in Chapter III.

Although the above deals only with the FP operator, it is also true of the transformed backwards operator $\overline{\mathfrak{F}}$. This remark applies also to the rest of this section, mutatis mutandis.

General solution in any sub-interval

As indicated above, it is possible to find, in terms of known special functions, two linearly independent functions $p_1^i(x)$, $p_2^i(x)$ which satisfy the homogeneous equation $\overline{\mathfrak{F}^*}p = 0$ for $x \in (x_{i+1}, x_i)$. It will be assumed that $p_1^i(x)$ and $p_2^i(x)$ are real for real x. Since the coefficients of the differential equation are real, two such real solutions do exist; if the standard solutions are not real, two linear combinations of them will be.

Then if $x_o \notin (x_i, x_{i+1})$ the general solution for x in this interval will be

$$p(x, s\,|x_o) = c_1^i\, p_1^i(x) + c_2^i p_2^i(x) \quad , \tag{2.18}$$

where c_1^i and c_2^i are constant with respect to x. If $x_o \in (x_k, x_{k+1})$, then for x in this interval a particular solution is

$$\int_{x_{k+1}}^{x} \frac{p_1^k(x)p_2^k(z) - p_2^k(x)p_1^k(z)}{a_k(z)\,w_k(z)}\,\delta(z-x_o)dz$$

$$= \begin{cases} 0 & \text{for } x \le x_o \\ \dfrac{p_1^k(x)p_2^k(x_o) - p_2^k(x)p_1^k(x_o)}{a_k(x_o)w_k(x_o)} & \text{for } x \ge x_o\,. \end{cases} \tag{2.19}$$

Here

$$w_k(x) = p_1^k(x)\frac{dp_2^k(x)}{dx} - p_2^k(x)\frac{dp_1^k(x)}{dx} \tag{2.20}$$

is the Wronskian of the solutions p_1^k, p_2^k. Then (2.18) applies also in the interval (x_{k+1}, x_k), except that the constants c_1^k, c_2^k have different values for $x > x_o$ and $x < x_o$, say c_1^{k+}, c_2^{k+} and c_1^{k-}, c_2^{k-}, respectively, for which

$$c_1^{k-} - c_1^{k+} = -\frac{p_2^k(x_o)}{a_k(x_o)w_k(x_o)} \tag{2.21}$$

$$c_2^{k-} - c_2^{k+} = \frac{p_1^k(x_o)}{a_k(x_o)w_k(x_o)} \quad . \tag{2.22}$$

When $x_o = x_i$, no difficulty is experienced; because of the continuity of $p(x, s\,|x_o)$ with respect to x_o, this can be considered as the limit of either of the two cases just dealt with. Cases when x_o is an irregular point can be dealt with as they arise.

Boundary conditions

On account of (2.11), one has at the end points x_1 and x_n

$$c_1^1 q_1^1(x_1) + c_2^1 q_2^1(x_1) = 0 \tag{2.23}$$

$$c_1^{n-1} q_1^{n-1}(x_n) + c_2^{n-1} q_2^{n-1}(x_n) = 0 \quad . \tag{2.24}$$

(Here $q_j^i = \frac{d}{dx}(a_i p_j^i) - b_i p_j^i$.) At the junction points of intervals, $x_i (i = 2, \ldots, n-1)$, the continuity of Q and aP (and hence q and ap) shows that

$$c_1^i q_1^i(x_i) + c_2^i q_2^i(x_i) = c_1^{i-1} q_1^{i-1}(x_i) + c_2^{i-1} q_2^{i-1}(x_i) \tag{2.25}$$

$$a_i(x_i)\left[c_1^i p_1^i(x_i) + c_2^i p_2^i(x_i)\right] = a_{i-1}(x_i)\left[c_1^{i-1} p_1^{i-1}(x_i) + c_2^{i-1} p_2^{i-1}(x_i)\right] . \tag{2.26}$$

Existence of a solution to the transformed FP equation

Equations (2.21-26) form a set of 2n linear algebraic equations for the 2n otherwise undetermined coefficients c_1^1, c_2^1, c_1^2, ..., c_1^{k+}, c_2^{k+}, c_1^{k-}, c_2^{k-}, ..., c_1^{n-1}, c_2^{n-1}. This nonhomogeneous set will have a unique nontrivial solution provided the corresponding homogeneous set has no nontrivial solution (the Fredholm alternative). The only nonhomogeneous terms are those on the right of (2.21) and (2.22). Putting these equal to zero is equivalent to omitting the term $\delta(x-x_o)$ on the right of (2.14), i.e., to solving the homogeneous differential equation (with the same boundary conditions). This will have a nontrivial solution only if $s = \lambda$, where λ is an eigenvalue of the original FP operator, $\mathfrak{F}^* = \frac{\partial^2}{\partial x^2}(a\,\cdot) - \frac{\partial}{\partial x}(b\,\cdot)$. But this can only occur for values of s for which $p(x, s\,|x_o)$ does not exist. Thus the nonhomogeneous set will have a unique nontrivial solution whenever $p(x, s\,|x_o)$ exists; which is of course to be expected. In particular it will have a solution for all Re $s > 0$.

The transformed transition density

Having solved (2.21-26), $p(x, s\,|x_o)$, the Laplace transform of $P(x, t\,|x_o)$, is immediately found using (2.18). Although the inverse transform has been found in only a few special cases (when it can usually be found more readily without transforming in the first place),

$p(x, s|x_o)$ is of considerable use as it is.

For example, it can be verified that

$$\lim_{s\to 0} sp(x, s|x_o) = P_o(x) = \lim_{t\to\infty} P(x, t|x_o) , \qquad (2.27)$$

where $P_o(x)$ is found as in section 1.6. See the following section and the special cases of Chapters III - V.

As shown in section 2.3, the Laplace transform of the auto-correlation can be explicitly formulated in terms of p and its derivatives with respect to x and x_o. This can then be used to obtain the spectral density. This is the principal result in this thesis.

2.2 THE STEADY-STATE DENSITY

Determination of P_o

The method of determination of $P_o(x)$ is described in section 1.6. If (x_n, x_1) is a regular interval with reflecting or nonattracting natural boundaries, then, from (1.65),

$$P_o(x) = \frac{C_{oi}}{A_i x^2 + 2B_i x + C_i} \exp\left[\int \left(\frac{D_i x + E_i}{A_i x^2 + 2B_i x + C_i}\right) dx\right] . \qquad (2.28)$$

The integral here can be evaluated by elementary methods. The continuity of aP_o allows the determination of the constants C_{oi} up to a multiplicative factor, which can be found using (1.63), i.e.

$$\int_{x_n}^{x_1} P_o(x)dx = 1 \quad . \qquad (2.29)$$

For the evaluation of this integral, see below.

If (x_1, x_n) contains irregular points, and/or has absorbing or attracting natural boundaries, then the situation becomes more involved. The steady-state density may depend on x_o, contain delta functions, or be identically zero. For its evaluation, see section 1.6.

<u>The integral of P_o</u>

In almost all cases, the determination of $P_o(x)$ requires the calculation of the indefinite integral

$$I(x) = \int \left(A_i x^2 + 2B_i x + C_i\right)^{-1} \exp\left[\int^x \left(\frac{D_i z + E_i}{A_i z^2 + 2B_i z + C_i}\right) dz\right] dx \,. \qquad (2.30)$$

In some cases, this integral can be found by elementary methods. In general, one notes that $\mathfrak{F}^* P_o = 0$ can be written

$$\frac{d}{dx}\left[a(x)\frac{dI}{dx}\right] - b(x)\frac{dI}{dx} = 0 \;, \qquad (2.31)$$

which is a second order equation of the same type as the homogeneous Laplace-transformed FP equation solved in section 2.1. One solution to (2.31) is I = const. Pick a solution linearly independent to this and multiply it by a constant so that its derivative is equal to the integrand in equation (2.30). Then this is the required I(x).

Integrating the Laplace-transformed FP equation (2.14) and letting $s \to 0$,

$$\int_{x_n}^{x} P_o(z|x_o)dz = \lim_{s \to 0} q(x, s|x_o) + H(x-x_o) , \qquad (2.32)$$

where H(x) is the Heaviside unit function. This is an alternative method of finding I--but only useful if $p(x, s|x_o)$, and hence $q(x, s|x_o)$, has been found already.

2.3 THE LAPLACE-TRANSFORMED AUTOCORRELATION

By the definition (0.2), for a stationary process,

$$R(t) = \langle x_o x \rangle = \int_{x_n}^{x_1} \int_{x_n}^{x_1} x_o x P_2(x_o, 0; x, t) dx_o dx . \qquad (2.33)$$

If $P_o(x)$ is identically zero, except perhaps for delta functions, the same is true of $P_2(x_o, 0;x, t)$, and the integral (2.33) is trivial and R(t) is constant. We assume this is not the case. Instead of R(t) we will find its Laplace transform r(s), whence (as shown in (2.64)) $\Phi(\omega)$ can be directly obtained.

The Laplace-transformed autocorrelation

Since $\{x(t)\}$ is a Markov process,

$$P_2(x_o, 0;x, t) = P_o(x_o)P(x, t|x_o) . \qquad (2.34)$$

Thus

$$R(t) = \int_{x_n}^{x_1} x_o P_o(x_o) \left[\int_{x_n}^{x_1} x P(x, t|x_o) dx \right] dx_o , \qquad (2.35)$$

so that, if r(s) is the Laplace transform of R(t),

$$r(s) = \int_{x_n}^{x_1} x_o P_o(x_o) \left[\int_{x_n}^{x_1} xp(x, s\,|x_o) dx \right] dx_o \quad . \tag{2.36}$$

Determination of $\int xp\,dx$

The Laplace-transformed FP equation (2.14) can be written

$$\frac{dq}{dx} - sp = -\delta(x - x_o) \quad , \tag{2.37}$$

where $q = q(x, s\,|x_o)$. Multiplying this equation by x and integrating from x_{i+1} to x_i ,

$$s \int_{x_{i+1}}^{x_i} xp\,dx = \int_{x_{i+1}}^{x_i} x \frac{dq}{dx} dx + x_o \delta_{ik} \quad (\text{where } x_o \in (x_{k+1}, x_k))$$

$$= \Big[xq \Big]_{x_{i+1}}^{x_i} - \int_{x_{i+1}}^{x_i} q\,dx + x_o \delta_{ik}$$

$$= \Big[xq - a(x)p \Big]_{x_{i+1}}^{x_i} + \int_{x_{i+1}}^{x_i} (D_i x + E_i) p\,dx + x_o \delta_{ik} \quad , \tag{2.38}$$

from the definition of q. But, integrating (2.37) from x_{i+1} to x_i ,

$$s \int_{x_{i+1}}^{x_i} p\,dx = \Big[q \Big]_{x_{i+1}}^{x_i} + \delta_{ik} \quad . \tag{2.39}$$

Eliminating $\int p\,dx$ from (2.38) using (2.39),

$$\int_{x_{i+1}}^{x_i} xp\,dx = \frac{1}{s-D_i}\left\{\left[x+\frac{E_i}{s}\ q-ap\right]_{x_{i+1}}^{x_i} + \left(x_o+\frac{E_i}{s}\right)\delta_{ik}\right\} . \qquad (2.40)$$

Adding over all intervals (x_{i+1}, x_i) ,

$$\int_{x_n}^{x_1} xp\,dx = \frac{1}{s-D_k}\left(x_o+\frac{E_k}{s}\right)+\sum_{i=1}^{n-1}\left\{\frac{1}{s-D_i}\left[\left(x+\frac{E_i}{s}\right)q-ap\right]_{x=x_{i+1}}^{x_i}\right\} . \qquad (2.41)$$

According to (2.36), this must be multiplied by $x_oP_o(x_o)$ and integrated with respect to x_o. That is, it is required to find the following quantities:

$$\int_{x_{k+1}}^{x_k}\left(x_o+\frac{E_k}{s}\right)x_oP_o(x_o)dx_o = \mathcal{J}_k \text{ say ,} \qquad (2.42)$$

$$a(x_i)\int_{x_{k+1}}^{x_k} x_op(x_i, s\,|x_o)P_o(x_o)dx_o = \mathcal{P}_k(x_i) \text{ say,} \qquad (2.43)$$

$$\int_{x_{k+1}}^{x_k} x_oq(x_i s\,|x_o)P_o(x_o)dx_o = \mathcal{Q}_k(x_i) \text{ say ,} \qquad (2.44)$$

for k = 1, 2, ...n-1 and i = 1, 2, ..., n.

Determination of $\mathcal{I}_k$

Using $Q_o(x) = 0$ and integrating by parts, (2.42) can be reduced to a combination of P_o and $\int P_o(x_o)dx_o$, which can be found either directly (see section 2.2), or by the formulas (2.27) and (2.32).

We will perform this reduction explicitly. Integrating $Q_o(x)=0$ (itself the integral of $\mathfrak{F}^*P_o=0$), one gets

$$aP_o - D_k\int xP_o(x)dx - E_k\int P_o(x)dx = 0 \quad ; \tag{2.45}$$

while if $Q_o(x) = 0$ is multiplied by x and then integrated, noting that

$$x\frac{d}{dx}(aP_o) = \frac{d}{dx}(xaP_o) - aP_o \quad , \tag{2.46}$$

one gets

$$xaP_o - (A_k+D_k)\int x^2P_odx - (2B_k+E_k)\int xP_odx - C_k\int P_odx = 0 \ . \tag{2.47}$$

Combining (2.45) and (2.47),

$$\int\left(x + \frac{E_k}{s}\right)xP_odx = aP_o\left[\frac{D_kx-2B_k-E_k}{D_k(A_k+D_k)} + \frac{E_k}{D_ks}\right]$$

$$+\int P_odx\left[\frac{(2B_k+E_k)E_k-C_kD_k}{D_k(A_k+D_k)} - \frac{E_k^2}{D_ks}\right] \ . \tag{2.48}$$

Thus

$$\mathcal{I}_k = \left[a(x_o)P_o(x_o)\left\{\frac{D_kx_o-2B_k-E_k}{D_k(A_k+D_k)} + \frac{E_k}{D_ks}\right\}\right]_{x_o=x_{k+1}}^{x_k}$$

$$+\int_{x_{k+1}}^{x_k} P_o(x_o)dx_o\left\{\frac{(2B_k+E_k)E_k-C_kD_k}{D_k(A_k+D_k)} - \frac{E_k^2}{D_ks}\right\} . \tag{2.49}$$

Determination of $\mathcal{O}_k$

The backwards equation (2.15) for $p(x, s|x_o)$ can be written

$$a(x_o)\frac{d^2p}{dx_o^2} + (D_k x_o + E_k)\frac{dp}{dx_o} - sp = -\delta(x-x_o) \,. \tag{2.50}$$

Also, $P_o(x_o)$ satisfies $\mathcal{F}^* P_o = 0$, or

$$\frac{d^2}{dx_o^2}\Big[a(x_o)P_o\Big] - \frac{d}{dx_o}\Big[(D_k x_o + E_k)P_o\Big] = 0 \;, \tag{2.51}$$

and also the integrated form of this, $Q_o = 0$, or

$$\frac{d}{dx_o}\Big[a(x_o)P_o\Big] - (D_k x_o + E_k)P_o = 0 \;. \tag{2.52}$$

Forming $p \times (2.51) - P_o \times (2.50)$ and simplifying slightly,

$$\frac{d}{dx_o}\Big[p\frac{d}{dx_o}(aP_o) - aP_o\frac{dp}{dx_o} - (D_k x_o + E_k)pP_o\Big] + spP_o = P_o\,\delta(x-x_o) \,. \tag{2.53}$$

Then, using (2.52),

$$spP_o = \frac{d}{dx_o}\left(aP_o\frac{dp}{dx_o}\right) + P_o\,\delta(x-x_o) \;. \tag{2.54}$$

Thus, integrating by parts,

$$s\int_{x_{k+1}}^{x_k} x_o p P_o\, dx_o = \left[x_o a P_o \frac{dp}{dx_o}\right]_{x_o = x_{k+1}}^{x_k} - \int_{x_{k+1}}^{x_k} aP_o\frac{dp}{dx_o}\,dx_o + xP_o(x)\,\delta_{ik} \,. \tag{2.55}$$

(Throughout this derivation P_o and p without variables indicated denote $P_o(x_o)$ and $p(x, s\,|x_o)$.) Integrating by parts again and using (2.52),

$$s\int_{x_{k+1}}^{x_k} x_o p P_o dx_o = \left[aP_o\left(x_o \frac{dp}{dx_o} - p\right)\right]_{x_o=x_{k+1}}^{x_k} + D_k\int_{x_{k+1}}^{x_k} x_o p P_o dx_o + E_k\int_{x_{k+1}}^{x_k} pP_o dx_o$$

$$+ xP_o(x)\,\delta_{ik}\,, \qquad (2.56)$$

and so, integrating (2.54) to obtain $\int_{x_{k+1}}^{x_k} pP_o dx_o$,

$$\int_{x_{k+1}}^{x_k} x_o p P_o dx_o = \frac{1}{s-D_k}\left\{\left[aP_o\left\{\left(x_o + \frac{E_k}{s}\right)\frac{dp}{dx_o} - p\right\}\right]_{x_o=x_{k+1}}^{x_k} + \left(x + \frac{E_k}{s}\right)P_o(x)\,\delta_{ik}\right\}. \qquad (2.57)$$

Thus $a(x)\int_{x_{k+1}}^{x_k} x_o p(x, s\,|x_o)P_o(x_o)dx_o$ can be evaluated for any interval (x_k, x_{k+1}). In (2.43) $x = x_i$, the endpoint of an interval; but $a(x)p(x, s\,|x_o)$ is continuous with respect to x, so that it makes no difference whether x_i is considered to belong to the interval (x_i, x_{i-1}) or the interval (x_{i+1}, x_i) -- unless i = 1 or n (i.e., x_i is an endpoint). One can thus in most cases consider x_i not to lie in (x_k, x_{k+1}), so that the last term in (2.57) is zero. This is also the case for i = k = 1 or n when x_i is infinite--in fact then all terms in (2.57) are zero. In the remaining case, note that since in general $\frac{dp}{dx_o}(x, s\,|x_o)$ has different limits as $x \to x_o$ from above or below, it can be seen from (2.43) that one must let $x \to x_i$ before letting $x_o \to x_i$ -- $\frac{dp}{dx_o}$ is not continuous at

$x = x_i$ (for any i), so the first limiting process is necessary. The same remarks will also apply to $\frac{dq}{dx_o}$ in the formula for $\mathcal{Q}_k(x_i)$ derived below (2.62).

Taking into consideration the above points, the formula for $\mathcal{P}_k(x_i)$ can be written

$$\mathcal{P}_k(x_i) = \frac{1}{s-D_k}\left\{\left[a(x_o)P_o(x_o)\left\{\left(x_o+\frac{E_k}{s}\right)D^*\left[a(x_i)p(x_i, s\,|x_o)\right]\right.\right.\right.$$
$$\left.\left.\left. -a(x_i)p(x_i, s\,|x_o)\right\}\right]_{x_o=x_{k+1}}^{x_k} + \left(x_i+\frac{E_k}{s}\right)a(x_i)P_o(x_i)\,\delta_{jk}\right\},$$

(2.58)

where j = i-1 or i and

$$D^*f(x_i, x_o) = \lim_{\substack{x'\in(x_{k+1}, x_k)\\ x'\to x_o}}\ \lim_{\substack{x\in(x_{j+1}, x_j)\\ x\to x_i}} \frac{df}{dx'}(x, x') . \qquad (2.59)$$

<u>Determination of $\mathcal{Q}_k$</u>

Operate on (2.57) by

$$\frac{d}{dx}\left[a(x)(\cdot)\right] - b(x)(\cdot) . \qquad (2.60)$$

The first term does not contain x explicitly and is linear and homogeneous in p; thus the operation (2.60) simply replaces p by q. The second term becomes

$$a(x)P_o(x) + \left(x + \frac{E_k}{s}\right)Q_o(x) \ . \tag{2.61}$$

But $Q_o(x) = 0$. Thus, as in the case of $\mathcal{P}_{ik}$,

$$\mathcal{Q}_k(x_i) = \frac{1}{s-D_k}\left\{\left[a(x_o)P_o(x_o)\left\{\left(x_o + \frac{E_k}{s}\right)D^* q(x_i, s|x_o) - q(x_i, s|x_o)\right\}\right]_{x_o = x_{k+1}}^{x_k} + a(x_i)P_o(x_o)\delta_{jk}\right\} \ . \tag{2.62}$$

<u>The formula for r(s)</u>

Summarizing, the Laplace-transformed autocorrelation is (substituting (2.41) into (2.36))

$$r(s) = \sum_{k=1}^{n-1}\left[\frac{\mathcal{J}_k}{s-D_k} + \sum_{i=1}^{n-1}\left\{\frac{1}{s-D_i}\left[\left(x + \frac{E_i}{s}\right)\mathcal{Q}_k(x) - \mathcal{P}_k(x)\right]_{x=x_{i+1}}^{x_i}\right\}\right] \ , \tag{2.63}$$

where $\mathcal{J}_k$, $\mathcal{P}_k$, $\mathcal{Q}_k$ are given by (2.49, 58, 62) respectively. Actually the last term in (2.58, 62) can always be omitted, and thus $\mathcal{P}_k$, $\mathcal{Q}_k$ somewhat simplified. It has been explained above how this can be done for $i = 2, 3, \ldots, n-1$ by suitably choosing $j \neq k$. For $i = k = 1$, the last term in (2.58) becomes $\frac{1}{s-D_1}\left(x_1 + \frac{E_1}{s}\right)a(x_1)P_o(x_1)$, and the last term in (2.62) becomes $\frac{1}{s-D_1}a(x_1)P_o(x_1)$. Substituting in (2.63) it is seen that the corresponding terms cancel each other out. The same happens for $i = k+1 = n$, the other possible exception. These slightly revised versions of $\mathcal{P}_k$, $\mathcal{Q}_k$ will be used in future.

2.4 THE SPECTRAL DENSITY

Spectral density

The Weiner-Khinchine relation (0.4) can be written

$$\Phi(\omega) = \frac{1}{\pi}\int_0^\infty R(t)\left(e^{i\omega t}+e^{-i\omega t}\right)dt$$

$$= \frac{1}{\pi}\left[\overline{r(i\omega)} + r(i\omega)\right]$$

$$= \frac{2}{\pi}\,\mathrm{Re}\; r(i\omega) \quad , \qquad (2.64)$$

where r(s) is the Laplace-transformed autocorrelation obtained in the last section.

This formula (which also applies to multidimensional systems) was used by Caughey and Dienes [4, 8] to find the spectrum of the system governed by the stochastic differential equation

$$\dot{x} + k \;\mathrm{sgn}\; x = n(t) \quad , \quad -\infty < x < \infty \;. \qquad (2.65)$$

Here r(s) can be found by direct integration of (2.36).

Variance

A formula for $\langle x^2 \rangle$ is easily obtained in terms of known quantities. For

$$\langle x^2 \rangle = \int_{x_1}^{x_n} x^2 P_o(x)\,dx$$

$$= \sum_{k=1}^{n-1} \lim_{s\to\infty} \mathcal{J}_k \quad , \qquad (2.66)$$

where $\mathcal{J}_k$ is defined by (2.42). This formula can also be obtained by using

$$\langle x^2 \rangle = R(0) = \lim_{s \to \infty} sr(s) \quad , \tag{2.67}$$

r(s) being given by (2.63), and noting that both $\mathcal{P}_k$ and $\mathcal{Q}_k$ remain finite as $s \to \infty$. Using the formula (2.49) for $\mathcal{J}_k$,

$$\langle x^2 \rangle = \sum_{k=1}^{n-1} \frac{1}{D_k(A_k+D_k)} \left\{ \Big[a(x)P_o(x)(D_k x - 2B_k - E_k) \Big]_{x=x_{k+1}}^{x_k} + \Big[(2B_k+E_k)E_k - C_k D_k \Big] \int_{x_{k+1}}^{x_k} P_o(x)dx \right\} . \tag{2.68}$$

Mean

It is evident from (2.45) that

$$\langle x \rangle = \sum_{k=1}^{n-1} \frac{1}{D_k} \left\{ \Big[a(x)P_o(x) \Big]_{x=x_{k+1}}^{x_k} - E_k \int_{x_{k+1}}^{x_k} P_o(x)dx \right\} . \tag{2.69}$$

This could also be obtained by a limiting process on r(s), since

$$\langle x \rangle^2 = R(\infty) = \lim_{s \to 0} sr(s) \quad . \tag{2.70}$$

However this involves finding the limits as $s \to 0$ of $\mathcal{P}_k$, $\mathcal{Q}_k$.

Translation of displacement

It is sometimes convenient to replace x by a new displacement y, where

$$y = x - z \tag{2.71}$$

where z is constant. For example, with asymmetrical f(x), it may be more convenient in given circumstances to define x either so that f(0) = 0 or so that $\langle x \rangle = 0$. It is apparent from (2.35) that

$$R_y(t) = R_x(t) + z(z - \langle x \rangle) - z \int_{x_n}^{x_1} P(x_o) \left[\int_{x_n}^{x_1} xP(x, t|x_o)dx \right] dx_o \,, \tag{2.72}$$

so that

$$r_y(s) = r_x(s) + \frac{z}{s}(z - \langle x \rangle) - z\,g(s) \,, \tag{2.73}$$

where

$$g(s) = \int_{x_n}^{x_1} P(x_o) \left[\int_{x_n}^{x_1} xp(x, s|x_o)dx \right] dx_o \,. \tag{2.74}$$

Evaluation of g(s)

The quantity $\int_{x_n}^{x_1} xp\,dx$ is given by (2.41). Thus g(s) is given by the same formula (2.63) as r(s), but with $\mathcal{J}_k$, $\mathcal{P}_k(x_i)$, $\mathcal{Q}_k(x_i)$ replaced by $\mathcal{J}_k'$, $\mathcal{P}_k'(x_i)$, $\mathcal{Q}_k'(x_i)$, where

$$\mathcal{J}_k' = \int_{x_{k+1}}^{x_k} \left(x_o + \frac{E_k}{s}\right) P_o(x_o)\,dx_o \tag{2.75}$$

$$= \left[\frac{a(x_o)P(x_o)}{D_k}\right]_{x_o=x_{k+1}}^{x_k} + E_k\left(\frac{1}{s} - \frac{1}{D_k}\right)\int_{x_{k+1}}^{x_k} P_o(x_o)\,dx_o \tag{2.76}$$

(using (2.45));

$$\mathcal{P}_k'(x_i) = a(x_i)\int_{x_{k+1}}^{x_k} p(x_i, s\,|x_o)P_o(x_o)\,dx_o \tag{2.77}$$

$$= \frac{1}{s}\left[a(x_o)P_o(x_o)D^*\left[a(x_i)p(x_i, s|x_o)\right]\right]_{x_o=x_{k+1}}^{x_k} + a(x_i)P_o(x_i)\,\delta_{jk} \tag{2.78}$$

(obtained by integrating (2.54)); and

$$\mathcal{Q}_k'(x_i) = \int_{x_{k+1}}^{x_k} q(x_i, s|x_o)P_o(x_o)\,dx_o \tag{2.79}$$

$$= \frac{1}{s}\left[a(x_o)P_o(x_o)D^* q(x_i, s|x_o)\right]_{x_o=x_{k+1}}^{x_k} . \tag{2.80}$$

Here D^* and j are as in (2.59)--but note that one cannot always omit the δ_{ik} term in (2.78), as one could in (2.58, 62).

Spectral density at $\omega = 0$

From (2.70) together with the analyticity of r(s) for s > 0, it is seen that

$$r_x(s) = \frac{\langle x \rangle^2}{s} + b(s) \quad , \tag{2.81}$$

where b(s) is analytic at s = 0. In fact, putting $z = \langle x \rangle$ in (2.73),

$$b(s) = r_y(x) + \langle x \rangle \left[g(s) - \frac{\langle x \rangle}{s} \right] . \tag{2.82}$$

Since $r_x(0)$ is infinite for $\langle x \rangle \neq 0$, (2.64) cannot be used to obtain $\Phi(0)$. However, using the definition (0.4) and noting that

$$\int_0^\infty \cos \omega t \, dt = \frac{1}{2} \int_{-\infty}^\infty e^{i\omega t} \, dt = \frac{\pi}{2} \delta(\omega) \quad , \tag{2.83}$$

one gets

$$\Phi_x(\omega) = \langle x \rangle^2 \delta(\omega) + \frac{2}{\pi} \operatorname{Re} b(i\omega) \quad , \tag{2.84}$$

where $\operatorname{Re} b(i\omega) = \operatorname{Re} r_x(i\omega)$ for $\omega > 0$.

CHAPTER III

SYSTEMS WITH WHITE FORCING FUNCTION EXCITATION

3.1 THE GENERAL CASE--SOLUTION OF THE FOKKER-PLANCK EQUATION

In this chapter we deal with the first order piecewise linear system where the only excitation is a single white noise forcing function. Excitation of this type has been the case most widely dealt with in previous work. The linear case has long been completely solved. The only nonlinear case for which the spectrum has been worked out is that with a bang-bang restoring force ($f(x)=k \operatorname{sgn} x$), which has been dealt with by Caughey and Dienes [4, 8] by a method similar to that used in this thesis.

The formulas for the general case, on an interval containing an arbitrary number of linear segments, are worked out in this and the following section. Special cases are worked out in the next three sections, including numerical results. Approximation of more general nonlinear forcing functions is discussed in section 3.6.

The stochastic differential equation

For the systems to be dealt with in this chapter, (0.1) reduces to

$$\dot{x} + f(x) = n(t) \tag{3.1}$$

(i.e., $m=1$ and $h_1(x) = 1$), where

$$f(x) = \ell_i x + k_i \quad \text{for} \quad x \in (x_{i+1}, x_i) \,, \quad i = 1, 2, \ldots, n-1 \tag{3.2}$$

(so that there are n-1 linear segments), and n(t) is white noise satisfying

$$\langle n(t_1)n(t_2)\rangle = 2D\,\delta(t_1 - t_2) \quad , \quad \langle n(t_1)\rangle = 0 \quad . \tag{3.3}$$

For simplicity of exposition, it will be assumed throughout the general derivations of this and the next section that $\ell_i \neq 0$ for all i. The method when one or more $\ell_i = 0$ is indicated at the end of this section.

Nondimensionalization

Provided $\ell_i \neq 0$, the following convenient dimensionless quantities can be defined:

$$\xi^i = \frac{\ell_i x + k_i}{|\ell_i D|^{\frac{1}{2}}} \qquad \text{for } x \in (x_{i+1}, x_i) \tag{3.4}$$

$$\sigma_i = \frac{s}{\ell_i} \tag{3.5}$$

$$\pi^i(\xi, \sigma | \xi_o) = |\ell_i D|^{\frac{1}{2}} p(x, s|x_o) \quad \text{for } x \in (x_{i+1}, x_i) \tag{3.6}$$

$$\Pi_o^i(\xi) = \frac{|\ell_i D|^{\frac{1}{2}}}{\ell_i} P_o(x) \qquad \text{for } x \in (x_{i+1}, x_i) \tag{3.7}$$

$$\psi^i(\xi, \sigma | \xi_o) = \operatorname{sgn} \ell_i \frac{d\pi^i}{d\xi^i} + \xi^i \pi^i = q(x, s|x_o) \quad \text{for } x \in (x_{i+1}, x_i) \tag{3.8}$$

$$\kappa_i = \frac{k_i}{|\ell_i D|^{\frac{1}{2}}} \quad . \tag{3.9}$$

A suffix on ξ, π, ψ will correspond to the same suffix on x, p, q, respectively. Both suffixes and superfixes may be omitted when no confusion would result. Note that ℓ_i may be negative for some i; thus to avoid imaginary quantities, $|\ell_i|^{\frac{1}{2}}$ is used in the nondimensionalization.

FP and backwards equations

The Laplace-transformed FP equation (2.14) in this case becomes

$$D\frac{d^2p}{dx^2} + \frac{d}{dx}\left\{(\ell_i x + k_i)p\right\} - sp = -\delta(x - x_o) \quad \text{for } x \in (x_{i+1}, x_i) \ , \tag{3.10}$$

or, using the dimensionless variables defined above,

$$\frac{d^2\pi}{d\xi^2} + \operatorname{sgn}\ell\left\{\frac{d}{d\xi}(\xi\pi) - \sigma\pi\right\} = -\delta(\xi - \xi_o) \ . \tag{3.11}$$

Similarly, the backwards equation becomes

$$\frac{d^2\pi}{d\xi_o^2} - \operatorname{sgn}\ell\left\{\xi_o\frac{d\pi}{d\xi_o} + \sigma\pi\right\} = -\delta(\xi - \xi_o) \ . \tag{3.12}$$

Since q and ap (and therefore p) are continuous with respect to x, the transition conditions for (3.11) at $\xi = \xi_i$ $(i = 2, 3, \ldots, n-1)$ become

$$|\ell_{i-1}|^{-\frac{1}{2}}\pi^{i-1}(\xi_i, \sigma|\xi_o) = |\ell_i|^{-\frac{1}{2}}\pi^i(\xi_i, \sigma|\xi_o) \tag{3.13}$$

$$\psi^{i-1}(\xi_i, \sigma|\xi_o) = \psi^i(\xi_i, \sigma|\xi_o) \ . \tag{3.14}$$

Fundamental sets of solutions

Omitting the non-homogeneous term, (3.11) can be written

$$\frac{d^2}{d\xi^2}\left(e^{\frac{1}{2}\zeta}\pi\right) + \left\{(\tfrac{1}{2} - \sigma)\operatorname{sgn}\ell - \tfrac{1}{4}\xi^2\right\}\left(e^{\frac{1}{2}\zeta}\pi\right) = 0 \; , \qquad (3.15)$$

where

$$\zeta^i = \frac{1}{2}\left(\operatorname{sgn}\ell_i\right)\left(\xi^i\right)^2 \; . \qquad (3.16)$$

But this is the defining equation for the parabolic cylinder function $D_\nu(z)$.[1] Two solutions are thus

$$e^{-\frac{1}{2}\zeta} D_{-\sigma}(\xi) \quad , \quad e^{-\frac{1}{2}\zeta} D_{-\sigma}(-\xi) \qquad (3.17)$$

for $\ell > 0$; for $\ell < 0$ replace $D_{-\sigma}$ by $D_{\sigma-1}$. An alternative pair of solutions are

$$\pi_1(\xi) = e^{-\zeta} M\left(\tfrac{1}{2}\sigma, \tfrac{1}{2}, \zeta\right) \qquad (3.18)$$

$$\pi_2(\xi) = \xi e^{-\zeta} M\left(\tfrac{1}{2} + \tfrac{1}{2}\sigma, \tfrac{3}{2}, \zeta\right) \; , \qquad (3.19)$$

where $M(\alpha, \gamma, z)$ is Kummer's form of the confluent hypergeometric function.[2] Each of these pairs forms a fundamental set--i.e., is linearly independent--provided σ is not a non-positive integer. The relationship between (3.17) and (3.18-19) is (from (8.2.4) in [11]):

---

[1] See Abramowitz and Stegun [1], Chapter 19, or Erdélyi *et al.* [12] Chapter 8.

[2] See [1], Chapter 15, or Erdélyi *et al.* [11], Chapter 6.

$$\pi_1(\xi) = 2^{\frac{1}{2}\sigma-1}\,\frac{\Gamma(\frac{1}{2}\sigma+\frac{1}{2})}{\Gamma(\frac{1}{2})}\,e^{-\frac{1}{2}\zeta}\Big(D_{-\sigma}(\xi)+D_{-\sigma}(-\xi)\Big) \quad \text{for } \ell>0$$

$$= 2^{-\frac{1}{2}\sigma-\frac{3}{2}}\,\frac{\Gamma(-\frac{1}{2}\sigma)}{\Gamma(\frac{1}{2})}\,e^{-\frac{1}{2}\zeta}\Big(D_{\sigma-1}(\xi)+D_{\sigma-1}(-\xi)\Big) \quad \text{for } \ell<0 \tag{3.20}$$

$$\pi_2(\xi) = 2^{\frac{1}{2}\sigma-\frac{1}{2}}\,\frac{\Gamma(\frac{1}{2}\sigma)}{\Gamma(-\frac{1}{2})}\,e^{-\frac{1}{2}\zeta}\Big(D_{-\sigma}(\xi)-D_{-\sigma}(-\xi)\Big) \quad \text{for } \ell>0$$

$$= 2^{-\frac{1}{2}\sigma-1}\,\frac{\Gamma(-\frac{1}{2}\sigma+\frac{1}{2})}{\Gamma(-\frac{1}{2})}\,e^{-\frac{1}{2}\zeta}\Big(D_{\sigma-1}(\xi)-D_{\sigma-1}(-\xi)\Big) \quad \text{for } \ell<0\,. \tag{3.21}$$

The first and second solutions (3.17) approach zero as $\xi \to \pm\infty$, respectively, which is convenient if x_1 and x_n are infinite. In addition, they remain linearly independent as $\ell \to 0$. However, the even and odd solutions (3.18, 19) are more convenient in other respects, in particular in their behavior as $\sigma \to 0$, i.e., $t \to \infty$, and are simpler to compute numerically. They will be used for the general derivations in sections 3.1-2, but in the special cases dealt with in sections 3.3-5 results will be expressed in whichever formulation is most compact.

Related quantities

Using (3.8) and the recurrence relationships for $M(\alpha, \gamma, z)$, one has

$$\psi_1(\xi) = \sigma\xi e^{-\zeta} M\left(1+\tfrac{1}{2}\sigma, \tfrac{3}{2}, \zeta\right) \tag{3.22}$$

$$\psi_2(\xi) = (\operatorname{sgn}\ell)\, e^{-\zeta} M\left(\tfrac{1}{2}+\tfrac{1}{2}\sigma, \tfrac{1}{2}, \zeta\right). \tag{3.23}$$

(The corresponding expressions using (3.17) are

$$\begin{aligned} &\mp\sigma e^{-\frac{1}{2}\zeta} D_{-\sigma-1}(\pm\xi) && \text{for } \ell > 0 \\ &\pm e^{-\frac{1}{2}\zeta} D_{\sigma}(\pm\xi) && \text{for } \ell < 0\ .\,) \end{aligned} \tag{3.24}$$

The Wronskian of π_1 and π_2 is, using 13.1.20 in [1],

$$w(\xi) = e^{-\zeta}\ . \tag{3.25}$$

A fundamental set of solutions to the backwards equation (3.12) is similarly found to be

$$M\left(\tfrac{1}{2}\sigma, \tfrac{1}{2}, \zeta_o\right), \quad \xi_o M\left(\tfrac{1}{2}+\tfrac{1}{2}\sigma, \tfrac{3}{2}, \zeta_o\right). \tag{3.26}$$

<u>The Laplace-transformed transition density</u>

The dimensionless form of $p(x, s|x_o)$ is the solution of (3.11), which can be expressed in the form

$$\pi^i(\xi, \sigma|\xi_o) = \gamma_1^i \pi_1^i(\xi) + \gamma_2^i \pi_2^i(\xi) \tag{3.27}$$

(compare with (2.18)). Here the γ_j^i are functions of ξ_o and σ, constant with respect to ξ except for $i = k$, where $x_o \in (x_{k+1}, x_k)$; the γ_j^k take different values, γ_j^{k+} and γ_j^{k-}, respectively, according as $x \gtrless x_o$. According to (2.21, 22),

$$\gamma_1^{k-} = \gamma_1^{k+} - \xi_o^k M\left(\tfrac{1}{2}\sigma_k + \tfrac{1}{2},\ \tfrac{3}{2},\ \zeta_o^k\right) \operatorname{sgn} \ell_k \tag{3.28}$$

$$\gamma_2^{k-} = \gamma_2^{k+} + M\left(\tfrac{1}{2}\sigma_k,\ \tfrac{1}{2},\ \zeta_o^k\right) \operatorname{sgn} \ell_k \ . \tag{3.29}$$

At $x = x_2, x_3, \ldots, x_{n-1}$, one has, from (3.13, 14),

$$|\ell_{i-1}|^{-\frac{1}{2}}\left(\gamma_1^{i-1}\pi_1^{i-1} + \gamma_2^{i-1}\pi_2^{i-1}\right) = |\ell_i|^{-\frac{1}{2}}\left(\gamma_1^i\pi_1^i + \gamma_2^i\pi_2^i\right) \tag{3.30}$$

$$\gamma_1^{i-1}\ \psi_1^{i-1} + \gamma_2^{i-1}\ \psi_2^{i-1} = \gamma_1^i\ \psi_1^i + \gamma_2^i\ \psi_2^i \tag{3.31}$$

(all π's and ψ's having argument ξ_i). Solving these equations for γ_1^i, γ_2^i in terms of γ_1^{i-1}, γ_2^{i-1}, noting that

$$\pi_1\psi_2 - \pi_2\psi_1 = (\operatorname{sgn}\ell)\, w(\xi) = (\operatorname{sgn}\ell)\, e^{-\zeta} , \tag{3.32}$$

one gets

$$\gamma_1^i = (\operatorname{sgn}\ell_i) e^{\zeta_i^i}\left[\gamma_1^{i-1}\left(\left|\frac{\ell_i}{\ell_{i-1}}\right|^{\frac{1}{2}}\pi_1^{i-1}\psi_2^i - \psi_1^{i-1}\pi_2^i\right)\right.$$
$$\left. + \gamma_2^{i-1}\left(\left|\frac{\ell_i}{\ell_{i-1}}\right|^{\frac{1}{2}}\pi_2^{i-1}\psi_2^i - \psi_2^{i-1}\pi_2^i\right)\right] \tag{3.33}$$

$$\gamma_2^i = -(\operatorname{sgn}\ell_i) e^{\zeta_i^i}\left[\gamma_1^{i-1}\left(\left|\frac{\ell_i}{\ell_{i-1}}\right|^{\frac{1}{2}}\pi_1^{i-1}\psi_1^i - \psi_1^{i-1}\pi_1^i\right)\right.$$
$$\left. + \gamma_2^{i-1}\left(\left|\frac{\ell_i}{\ell_{i-1}}\right|^{\frac{1}{2}}\pi_2^{i-1}\psi_1^i - \psi_2^{i-1}\pi_1^i\right)\right] . \tag{3.34}$$

If x_1 and x_n are finite, they are regular boundaries, and, by (2.23, 24),

$$\gamma_1^1 \psi_1^1(\xi_1) + \gamma_2^1 \psi_2^1(\xi_1) = 0 \tag{3.35}$$

$$\gamma_1^{n-1} \psi_1^{n-1}(\xi_n) + \gamma_2^{n-1} \psi_2^{n-1}(\xi_n) = 0 \,. \tag{3.36}$$

If $x_1 = \infty$, it is a natural boundary, and the only condition which π must satisfy is integrability. However, the integrable solution must satisfy $\psi \to 0$ as $x \to \infty$, and it is seen, using

$$M(\alpha, \gamma, z) \sim \frac{\Gamma(\gamma)}{\Gamma(\alpha)} \frac{e^z}{z^{\gamma-\alpha}} \qquad \text{as } z \to \infty \,, \tag{3.37}$$

that there is only one such solution. Thus the condition (3.35) is equivalent to integrability; for $\ell_1 > 0$, it becomes, using (3.37),

$$\sqrt{2}\,\Gamma\left(\frac{1}{2}\sigma_1 + \frac{1}{2}\right)\gamma_1^1 + \Gamma\left(\frac{1}{2}\sigma_1\right)\gamma_2^1 = 0 \,. \tag{3.38}$$

If $x_n = -\infty$, the condition corresponding to (3.36) is, for $\ell_n > 0$,

$$\sqrt{2}\,\Gamma\left(\frac{1}{2}\sigma_{n-1} + \frac{1}{2}\right)\gamma_1^{n-1} - \Gamma\left(\frac{1}{2}\sigma_{n-1}\right)\gamma_2^{n-1} = 0. \tag{3.39}$$

If ℓ_1 or ℓ_n is negative, then similar formulas can be found. However, then the corresponding boundary is attracting, and the system is unstable and not of much interest (having steady-state density zero and no spectral density).

<u>Case $\ell_i = 0$ for some i</u>

In this case the nondimensionalization of (3.4-9) breaks down. However, this leads to no significant difficulty; in fact the FP equation becomes much simpler, as it has constant coefficients. Let us consider the limits of the solutions π_1 and π_2 to (3.11) as $\ell \to 0$.

Using Darwin's asymptotic expansions--see Abramowitz and Stegun [1],19.10, case (i) --it can be shown that, provided $s > -\frac{k^2}{4D}$,

$$D_{-\sigma}(\pm\xi) \sim K_{1,2} \exp\left\{\mp \frac{kx}{2D}\left(1+\frac{4sD}{k^2}\right)^{\frac{1}{2}}\right\} \text{ as } \ell \downarrow 0. \tag{3.40}$$

Similarly, $D_{\sigma-1}(\pm\xi)$ are asymptotic to the same expressions as $\ell \uparrow 0$. Here K_1 and K_2 are complicated functions of $\frac{s}{\ell}$ and $\frac{4sD}{k^2}$. Thus, the fundamental set of solutions

$$p_{1,2}(x) = \exp\left[-\frac{kx}{2D}\left\{1 \pm \left(1+\frac{4sD}{k^2}\right)^{\frac{1}{2}}\right\}\right] \tag{3.41}$$

to the FP equation with $\ell = 0$ are, up to multiplicative constants, the limits of the fundamental set of solutions (3.17) to the FP equation with $\ell \neq 0$. Thus formulas derived in cases where all $\ell_i \neq 0$ can be used when $\ell_i = 0$, simply by substituting from (3.41), provided they are expressed in such a way that the multiplicative constants mentioned above cancel out. For examples, see sections 3.3-5. Note that since one of these multiplicative constants is infinitely larger than the other--which one depends on whether k, s $\gtrless 0$ --the solutions $\pi_{1,2}$ given in (3.18, 19) become proportional in the limit as $\ell \to 0$. This means that formulas must be expressed in terms of (3.17), not (3.18, 19), before substituting from (3.41).

If both ℓ_i and k_i are zero in some interval, a fundamental set of solutions is

$$p_{1,2}(x) = \exp\left\{\mp\left(\frac{s}{D}\right)^{\frac{1}{2}} x\right\}. \tag{3.42}$$

These are the limits of (3.41) as $k \to 0$. In this case, the solutions (3.18, 19) remain linearly independent and finite in the limit, and it is easily seen from the power series for $M(\alpha, \gamma, z)$ that

$$\pi_1 \sim \cosh\left[\left(\frac{s}{D}\right)^{\frac{1}{2}} x\right] , \quad \pi_2 \sim \left(\frac{|\ell|}{s}\right)^{\frac{1}{2}} \sinh\left[\left(\frac{s}{D}\right)^{\frac{1}{2}} x\right] . \qquad (3.43)$$

3.2 THE STEADY-STATE DENSITY AND SPECTRAL DENSITY

Steady-state density

The interval (x_n, x_1) is regular ($a(x) = \text{const.} \neq 0$). If either x_1 or x_n is infinite and ℓ_1 or ℓ_n respectively negative, then this point is an attracting natural boundary and $P_o \equiv 0$. If not, boundaries are regular reflecting (if finite) or nonattracting (otherwise). Thus, from (2.28) and (3.7),

$$\Pi_o^i(\xi) = \Gamma_i e^{-\zeta^i} \quad \text{for } x \in (x_{i+1}, x_i) , \qquad (3.44)$$

where the Γ_i are constants. Then the continuity of P_o gives

$$\frac{\Gamma_i}{\Gamma_j} = \operatorname{sgn}\left(\frac{\ell_j}{\ell_i}\right) \left|\frac{\ell_j}{\ell_i}\right|^{\frac{1}{2}} \exp\left(\zeta_i^i - \zeta_i^{i-1} + \zeta_{i-1}^{i-1} - \zeta_{i-1}^{i-2} + \ldots + \zeta_{j+1}^{j+1} - \zeta_{j+1}^{j}\right) , \qquad (3.45)$$

where $n-1 \geq i > j \geq 1$. Also, (2.29) gives

$$\int_{x_n}^{x_1} P_o dx = \int_{\xi_n}^{\xi_1} \Pi_o d\xi = \sum_{i=1}^{n-1} \Gamma_i \int_{\xi_{i+1}^i}^{\xi_i^i} e^{-\zeta^i} d\xi^i = 1 . \qquad (3.46)$$

Thus, eliminating $\Gamma_2, \ldots, \Gamma_n$ from (3.46) by means of (3.45) (with j = 1),

$$\Gamma_1 = \left[\sum_{i=1}^{n-1} \operatorname{sgn}\left(\frac{\ell_i}{\ell_1}\right) \left|\frac{\ell_1}{\ell_i}\right|^{\frac{1}{2}} \exp\left(\zeta_i^i - \zeta_i^{i-1} + \ldots - \zeta_2^1\right) \int_{\xi_{i+1}}^{\xi_i} e^{-\zeta^i} d\xi \right]^{-1} , \qquad (3.47)$$

and similarly for $\Gamma_2, \ldots, \Gamma_n$. In particular, for $\ell_i > 0$,

$$\int_{\xi_{i+1}}^{\xi_i} e^{-\zeta^i} d\xi = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \left(\operatorname{erf} \frac{\xi_i^i}{\sqrt{2}} - \operatorname{erf} \frac{\xi_{i+1}^i}{\sqrt{2}} \right) . \qquad (3.48)$$

If some $\ell_i < 0$, (3.46) involves the integral $\int e^{\frac{1}{2}\xi^2} d\xi$. Tables are available from which this can be obtained.[1]

$\underline{P_o \text{ as limit of } P}$

The same $P_o(x)$ is obtained by finding the Abelian limit of $P(x, t | x_o)$ for large time, i.e., $\lim_{s \to 0} sp(x, s | x_o)$. As $s \to 0$, $\pi_j(\xi)$ and $\psi_j(\xi)$ behave as follows:

$$\pi_1 \sim e^{-\zeta} \quad , \quad \pi_2 \sim e^{-\zeta} \int_0^{\xi} e^{\zeta} d\xi \ ,$$
$$(3.49)$$
$$\psi_1 \sim \sigma \int_0^{\xi} e^{-\zeta} d\xi \quad , \quad \psi_2 \sim \operatorname{sgn} \ell \ .$$

---

[1] For example, Abramowitz and Stegun [1], table 7.5 gives Dawson's integral, $e^{-x^2} \int_0^x e^{t^2} dt$.

Thus (3.28, 29, 33-36) become, to first order as $s \to 0$,

$$\gamma_1^{k-} = \gamma_1^{k+} - \operatorname{sgn} \ell_k \int_0^{\xi_o^k} e^{\zeta^k} d\xi^k \tag{3.50}$$

$$\gamma_2^{k-} = \gamma_2^{k+} + \operatorname{sgn} \ell_k \tag{3.51}$$

$$\gamma_1^i = \gamma_1^{i-1} \left| \frac{\ell_i}{\ell_{i-1}} \right|^{\frac{1}{2}} e^{\zeta_i^i - \zeta_i^{i-1}} + \gamma_2^{i-1} \left[\left| \frac{\ell_i}{\ell_{i-1}} \right|^{\frac{1}{2}} e^{\zeta_i^i - \zeta_i^{i-1}} \int_0^{\xi_i^{i-1}} e^{\zeta^{i-1}} d\xi^{i-1} \right.$$

$$\left. - \operatorname{sgn}\left(\frac{\ell_i}{\ell_{i-1}}\right) \int_0^{\xi_i^i} e^{\zeta^i} d\xi^i \right] \tag{3.52}$$

$$\gamma_2^i = -\gamma_1^{i-1} (\operatorname{sgn} \ell_i) \left[\sigma_i \left| \frac{\ell_i}{\ell_{i-1}} \right|^{\frac{1}{2}} e^{\zeta_i^i - \zeta_i^{i-1}} \int_0^{\xi_i^i} e^{-\zeta^i} d\xi^i - \sigma_{i-1} \int_0^{\xi_i^{i-1}} e^{-\zeta^{i-1}} d\xi^{i-1} \right]$$

$$+ \gamma_2^{i-1} \operatorname{sgn}\left(\frac{\ell_i}{\ell_{i-1}}\right) \tag{3.53}$$

$$\gamma_1^1 \sigma_1 \int_0^{\xi_1^1} e^{-\zeta^1} d\xi^1 + \gamma_2^1 \operatorname{sgn} \ell_1 = 0 \tag{3.54}$$

$$\gamma_1^{n-1} \sigma_{n-1} \int_0^{\xi_n^{n-1}} e^{-\zeta^{n-1}} d\xi^{n-1} + \gamma_2^{n-1} \operatorname{sgn} \ell_{n-1} = 0 \ . \tag{3.55}$$

From (3.53-55) it is seen that the coefficients γ_2^i are an order of magnitude smaller than the coefficients γ_1^i. But (3.51) shows that $\gamma_2^i = 0(1)$. Thus $\gamma_1^i = 0(\frac{1}{\sigma})$, and (3.50, 52) reduce to

$$\gamma_1^{k-} = \gamma_1^{k+} \tag{3.56}$$

$$\gamma_1^i = \gamma_1^{i-1} \left|\frac{\ell_i}{\ell_{i-1}}\right|^{\frac{1}{2}} e^{\zeta_i^i - \zeta_i^{i-1}} \tag{3.57}$$

to first order. Using (3.57) on (3.53),

$$\gamma_2^i = -\operatorname{sgn} \ell_i \left\{ \gamma_1^i \sigma_i \int_0^{\xi_i^i} e^{-\zeta^i} d\xi^i - \gamma_1^{i-1} \sigma_{i-1} \int_0^{\xi_i^{i-1}} e^{-\zeta^{i-1}} d\xi^{i-1} \right\} + \gamma_2^{i-1} \operatorname{sgn}\left(\frac{\ell_i}{\ell_{i-1}}\right). \tag{3.58}$$

(This is of course just (3.31) to first order.) Successive application of this formula gives γ_2^{k+} in terms of $\gamma_1^i (i < k)$:

$$\gamma_2^{k+} = -\operatorname{sgn} \ell_k \left\{ \gamma_1^k \sigma_k \int_0^{\xi_k^k} e^{-\zeta^k} d\xi^k + \sum_{i=2}^{k-1} \gamma_1^i \sigma_i \int_{\xi_{i=1}^i}^{\xi_i^i} e^{-\zeta^i} d\xi^i \right.$$

$$\left. - \gamma_1^1 \sigma_1 \int_0^{\xi_2^1} e^{-\zeta^1} d\xi^1 - \gamma_2^1 \operatorname{sgn} \ell_1 \right\}$$

$$= -\operatorname{sgn} \ell_k \left\{ \sum_{i=1}^{k-1} \gamma_1^i \sigma_i \int_{\xi_{i+1}^i}^{\xi_i^i} e^{-\zeta^i} d\xi^i + \gamma_1^k \sigma_k \int_0^{\xi_k^k} e^{-\zeta^k} d\xi^k \right\}, \tag{3.59}$$

using (3.54). Similarly,

$$\gamma_2^{k-} = \operatorname{sgn} \ell_k \left\{ -\gamma_1^k \sigma_k \int_0^{\xi_{k+1}^k} e^{-\zeta^k} d\xi^k + \sum_{i=k+1}^{n-1} \gamma_1^i \sigma_i \int_{\xi_{i+1}^i}^{\xi_i^i} e^{-\zeta^i} d\xi^i \right\} \tag{3.60}$$

gives γ_2^{k-} in terms of γ_1^i $(i > k)$. Substituting (3.59) and (3.60) into (3.51), one gets

$$\sum_{i=1}^{n-1} \gamma_1^i \sigma_i \int_{\xi_{i+1}^i}^{\xi_i^i} e^{-\zeta^i} d\xi^i = 1 \quad . \tag{3.61}$$

Thus we have n equations (3.57, 61) which on solution will give γ_1^i for small s. Comparing them with (3.45, 46) for Γ_i, we see that

$$\Gamma_i = \lim_{s \to 0} \sigma_i \gamma_1^i \quad , \tag{3.62}$$

so that

$$P_o(x) = \lim_{s \to 0} sp(x, s \,|x_o) \quad , \tag{3.63}$$

as was required.

Spectral density

To obtain the Laplace transformed autocorrelation, use is made of (2.49, 58, 62, 63), which in this case take the form

$$r(s) = \sum_{k=1}^{n-1} \frac{D}{\ell_k^2} \left[\frac{\vartheta_k^*}{\sigma_{k+1}} + \sum_{i=1}^{n-1} \left\{ \frac{1}{\sigma_{i+1}} \left[\left| \frac{\ell_k}{\ell_i} \right|^{\frac{3}{2}} \left(\xi^i - \varkappa_i (1+\sigma_i^{-1}) \right) \mathcal{Q}_k^*(\xi) \right. \right. \right.$$

$$\left. \left. \left. - \frac{\ell_k}{\ell_i} \theta_k^*(\xi) \right]_{x=x_{i+1}}^{x_i} \right\} \right] , \qquad (3.64)$$

where

$$\vartheta_k^* = \left[\Pi_o^k(\xi_o) \left(\varkappa_k (2+\sigma_k^{-1}) - \xi_o^k \right) \right]_{x_o = x_{k+1}}^{x_k}$$

$$+ \left[1 + \varkappa_k^2 \operatorname{sgn} \ell_k (1+\sigma_k^{-1}) \right] \int_{\xi_{k+1}^k}^{\xi_k^k} \Pi_o(\xi_o) d\xi_o^k \qquad (3.65)$$

$$\theta_k^*(\xi_i) = \frac{1}{\sigma_{k+1}} \left[\Pi_o^k(\xi_o) \left\{ \left[\xi_o^k - \varkappa_k (1+\sigma_k^{-1}) \right] D^* \pi^j(\xi_i, \sigma | \xi_o) \right. \right.$$

$$\left. \left. - \pi^j(\xi_i, \sigma | \xi_o) \right\} \right]_{x_o = x_{k+1}}^{x_k} \left| \frac{\ell_k}{\ell_j} \right|^{\frac{1}{2}} \operatorname{sgn} \ell_k \qquad (3.66)$$

$$\mathcal{Q}_k^*(\xi_i) = \frac{1}{\sigma_{k+1}} \left[\Pi_o^k(\xi_o) \left\{ \left[\xi_o^k - \varkappa_k (1+\sigma_k^{-1}) \right] D^* \psi^j(\xi_i, \sigma | \xi_o) - \psi^j(\xi_i, \sigma | \xi_o) \right\} \right]_{x_o = x_{k+1}}^{x_k} . \quad (3.67)$$

Here $i = 1, 2, \ldots, n$; $k = 1, 2, \ldots, n-1$; $j = i$ or $i-1$ ($j=i$ for $i=1, k+1, \ldots, n-1$ and $j = i-1$ for $i = 2, 3, \ldots, k, n$ is a suitable choice, satisfying the requirement that $j = k$ is not allowed except when $i = 1$ or n); and

$$D^* f(\xi_i, \xi_o) = \lim_{\substack{x' \in (x_{k+1} x_k) \\ x' \to x_o}} \; \lim_{\substack{x \in (x_{j+1}, x_j) \\ x \to x_i}} \frac{df(\xi, \xi')}{d\xi'} . \quad (3.68)$$

Having found r(s), the spectral density $\Phi(\omega)$ follows immediately from (2.64).

Variance and mean

For the case treated in this chapter, (2.68) and (2.69) become, respectively,

$$\langle x^2 \rangle = \sum_{k=1}^{n-1} \frac{D}{\ell_k} \left\{ \left[\left(2\varkappa_k - \xi^k \right) \Pi_o(\xi) \right]_{x=x_{k+1}}^{x_k} + \left(1 + \varkappa_k^2 \operatorname{sgn} \ell_k \right) \int_{\xi_{k+1}^k}^{\xi_k^k} \Pi_o(\xi) d\xi \right\}$$

(3.69)

$$\langle x \rangle = \sum_{k=1}^{n-1} \left| \frac{D}{\ell_k} \right|^{\frac{1}{2}} \left\{ \varkappa_k \operatorname{sgn} \ell_k \int_{\xi_{k+1}^k}^{\xi_k^k} \Pi_o(\xi) d\xi - \left[\Pi_o(\xi) \right]_{x=x_{k+1}}^{x_k} \right\} . \quad (3.70)$$

3.3 ONE-INTERVAL CASES

In the following 3 sections, the methods developed in the preceding section are used to obtain the Laplace transform of the transition probability, and hence the spectral density, for various examples.

A general one-interval case

Here $f(x) = \ell x + k$ for all x in the interval (x_1, x_2). By a suitable translation of x, we assume $k = 0$.

If the dimensionless transformed transition probability density is (from (3.27))

$$\pi(\xi, \sigma | \xi_o) = \gamma_1^{\pm} \pi_1(\xi) + \gamma_2^{\pm} \pi_2(\xi) \tag{3.71}$$

($\pm$ depending on whether $x \gtrless x_o$), then on account of the reflection condition $q = 0$ at $x = x_1, x_2$, (3.35, 36) gives

$$\gamma_1^{+} \psi_1(\xi_1) + \gamma_2^{+} \psi_2(\xi_1) = 0 \tag{3.72}$$

$$\gamma_1^{-} \psi_1(\xi_2) + \gamma_2^{-} \psi_2(\xi_2) = 0 \ , \tag{3.73}$$

and the delta function initial condition gives, by (3.28, 29),

$$\gamma_1^{-} = \gamma_1^{+} - e^{\zeta_o} \pi_2(\xi_o) \operatorname{sgn} \ell \tag{3.74}$$

$$\gamma_2^{-} = \gamma_2^{+} + e^{\zeta_o} \pi_1(\xi_o) \operatorname{sgn} \ell \ . \tag{3.75}$$

Solving these equations,

$$\gamma_1^+ = (\mathrm{sgn}\,\ell)\, e^{\zeta_o} \frac{\psi_2(\xi_2)\pi_1(\xi_o) - \psi_1(\xi_2)\pi_2(\xi_o)}{\psi_1(\xi_1)\psi_2(\xi_2) - \psi_1(\xi_2)\psi_2(\xi_1)} \tag{3.76}$$

$$\gamma_1^- = (\mathrm{sgn}\,\ell)\, e^{\zeta_o} \frac{\psi_2(\xi_1)\pi_1(\xi_o) - \psi_1(\xi_1)\pi_2(\xi_o)}{\psi_1(\xi_1)\psi_2(\xi_2) - \psi_1(\xi_2)\psi_2(\xi_1)} \tag{3.77}$$

$$\gamma_2^+ = -(\mathrm{sgn}\,\ell)\, e^{\zeta_o} \frac{\psi_2(\xi_2)\pi_1(\xi_o) - \psi_1(\xi_2)\pi_2(\xi_o)}{\psi_1(\xi_1)\psi_2(\xi_2) - \psi_1(\xi_2)\psi_2(\xi_1)} \tag{3.78}$$

$$\gamma_2^- = -(\mathrm{sgn}\,\ell)\, e^{\zeta_o} \frac{\psi_2(\xi_1)\pi_1(\xi_o) - \psi_1(\xi_1)\pi_2(\xi_o)}{\psi_1(\xi_1)\psi_2(\xi_2) - \psi_1(\xi_2)\psi_2(\xi_1)} \quad . \tag{3.79}$$

Thus the transformed probability density is

$$\pi(\xi, \sigma | \xi_o) = (\mathrm{sgn}\,\ell) e^{\zeta_o} \frac{\Big((\psi_2(\xi_1)\pi_1(\xi) - \psi_1(\xi_1)\pi_2(\xi)\Big)\Big((\psi_2(\xi_2)\pi_1(\xi_o) - \psi_1(\xi_2)\pi_2(\xi_o)\Big)}{\psi_1(\xi_1)\psi_2(\xi_2) - \psi_1(\xi_2)\psi_2(\xi_1)} \tag{3.80}$$

for $x \geq x_o$; and for $x \leq x_o$ interchange ξ and ξ_o in the π_i's.

The steady-state density $\Pi_o(x)$ is, from (3.47),

$$\Pi_o(\xi) = \left(e^{\zeta} \int_{\xi_2}^{\xi_1} e^{-\zeta}\, d\xi \right)^{-1} \tag{3.81}$$

$$= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{e^{-\frac{1}{2}\xi^2}}{\mathrm{erf}\,\frac{\xi_1}{\sqrt{2}} - \mathrm{erf}\,\frac{\xi_2}{\sqrt{2}}} \quad \text{if } \ell > 0 \; . \tag{3.82}$$

It is easily verified that this is the Abelian limit of (3.80).

To obtain r(s), one proceeds as follows. From (3.65),

$$\mathcal{J}^* = 1 - \frac{\xi_1 e^{-\zeta_1} - \xi_2 e^{-\zeta_2}}{\int_{\xi_2}^{\xi_1} e^{-\zeta}\, d\xi} \quad . \tag{3.83}$$

Also, from (3.66),

$$\vartheta^*(\xi_1) = \frac{1}{\sigma+1}\left[(\operatorname{sgn}\ell)\,\Pi_o(\xi_o)\left\{\xi_o D^*\pi(\xi_1,\sigma|\xi_o)-\pi(\xi_1,\sigma|\xi_o)\right\}\right]_{x_o=x_2}^{x_1} \quad . \tag{3.84}$$

But, applying (3.32) to (3.80),

$$\pi(\xi_1,\sigma|\xi_1) = \frac{\pi_1(\xi_1)\psi_2(\xi_2)-\pi_2(\xi_1)\psi_1(\xi_2)}{\psi_1(\xi_1)\psi_2(\xi_2)-\psi_2(\xi_1)\psi_1(\xi_2)} \tag{3.85}$$

$$\pi(\xi_1,\sigma|\xi_2) = \frac{(\operatorname{sgn}\ell)e^{-\zeta_1}}{\psi_1(\xi_1)\psi_2(\xi_2)-\psi_2(\xi_1)\psi_1(\xi_2)} \quad , \tag{3.86}$$

while, noting that

$$\frac{d}{d\xi}\left[(\operatorname{sgn}\ell)\,e^{\zeta}\,\pi_i(\xi)\right] = e^{\zeta}\,\psi_i(\xi) \quad , \tag{3.87}$$

it is seen that

$$D^*\pi(\xi_1,\sigma|\xi_1) = \operatorname{sgn}\ell \quad , \quad D^*\pi(\xi_1,\sigma|\xi_2) = 0 \quad . \tag{3.88}$$

Thus

$$\vartheta^*(\xi_1) = \frac{1}{\sigma+1}\left(\int_{\xi_2}^{\xi_1} e^{-\zeta}\,d\xi\right)^{-1} e^{-\zeta_1}\left\{\frac{e^{-\zeta_2}-\operatorname{sgn}\ell\left[\pi_1(\xi_1)\psi_2(\xi_2)-\pi_2(\xi_1)\psi_1(\xi_2)\right]}{\psi_1(\xi_1)\psi_2(\xi_2)-\psi_2(\xi_1)\psi_1(\xi_2)} + \xi_1\right\} . \tag{3.89}$$

Similarly,

$$\vartheta^*(\xi_2) = \frac{1}{\sigma+1}\left(\int_{\xi_2}^{\xi_1} e^{-\zeta}\,d\xi\right)^{-1} e^{-\zeta_2}\left\{\frac{e^{-\zeta_1}-\operatorname{sgn}\ell\left[\pi_1(\xi_2)\psi_2(\xi_1)-\pi_2(\xi_2)\psi_1(\xi_1)\right]}{\psi_1(\xi_1)\psi_2(\xi_2)-\psi_2(\xi_1)\psi_1(\xi_2)} + \xi_2\right\} . \tag{3.90}$$

Noting that $\psi(\xi_i, \sigma | \xi_o) \equiv 0 \quad (i=1,2)$, (3.67) gives

$$\mathcal{Q}^*(\xi_1) = \mathcal{Q}^*(\xi_2) = 0 \quad . \tag{3.91}$$

Thus (3.64) gives

$$r(s) = \frac{D}{\ell^2}\frac{1}{\sigma+1}\left[1 - \left(\int_{\xi_2}^{\xi_1} e^{-\zeta} d\xi\right)^{-1} \frac{1}{\sigma+1}\left\{\left(\xi_1 e^{-\zeta_1} - \xi_2 e^{-\zeta_2}\right)(\sigma+2)\right.\right.$$

$$\left.\left. + \frac{2e^{-(\zeta_1+\zeta_2)} - \operatorname{sgn}\ell\left\{e^{-\zeta_1}\left[\pi_1(\xi_1)\psi_2(\xi_2) - \pi_2(\xi_1)\psi_1(\xi_2)\right] + e^{-\zeta_2}\left[\pi_1(\xi_2)\psi_2(\xi_1) - \pi_2(\xi_2)\psi_1(\xi_1)\right]\right\}}{\psi_1(\xi_1)\psi_2(\xi_2) - \psi_2(\xi_1)\psi_1(\xi_2)}\right\}\right] . \tag{3.92}$$

Then, by (2.64),

$$\Phi(\omega) = \frac{2D}{\pi}\left[\frac{1}{\ell^2+\omega^2} - \left(\int_{\xi_2}^{\xi_1} e^{-\zeta} d\xi\right)^{-1}\left\{\frac{2\ell^2}{(\ell^2+\omega^2)^2}\left(\xi_1 e^{-\zeta_1} - \xi_2 e^{-\zeta_2}\right)\right.\right.$$

$$+\mathrm{Re}\left[\frac{|\ell|}{i\omega(\ell+i\omega)^2}\left\{2 - e^{-\zeta_1}\left[M\left(\frac{i\omega}{2\ell}, \frac{1}{2}, \zeta_1\right)M\left(\frac{1}{2}+\frac{i\omega}{2\ell}, \frac{1}{2}, \zeta_2\right)\right.\right.\right.$$

$$\left. - \frac{i\omega}{|\ell|}\xi_1\xi_2 M\left(1+\frac{i\omega}{2\ell}, \frac{3}{2}, \zeta_1\right)M\left(\frac{1}{2}+\frac{i\omega}{2\ell}, \frac{3}{2}, \zeta_2\right)\right] - e^{-\zeta_2}\left[M\left(\frac{i\omega}{2\ell}, \frac{1}{2}, \zeta_2\right)M\left(\frac{1}{2}+\frac{i\omega}{2\ell}, \frac{1}{2}, \zeta_2\right)\right.$$

$$\left.\left. - \frac{i\omega}{|\ell|}\xi_1\xi_2 M\left(1+\frac{i\omega}{2\ell}, \frac{3}{2}, \zeta_2\right)M\left(\frac{1}{2}+\frac{i\omega}{2\ell}, \frac{3}{2}, \zeta_1\right)\right)\right\} \div \left\{\xi_1 M\left(1+\frac{i\omega}{2\ell}, \frac{3}{2}, \zeta_1\right)M\left(\frac{1}{2}+\frac{i\omega}{2\ell}, \frac{1}{2}, \zeta_2\right)\right.$$

$$\left.\left.\left.\left. - \xi_2 M\left(1+\frac{i\omega}{2\ell}, \frac{3}{2}, \zeta_2\right)M\left(\frac{1}{2}+\frac{i\omega}{2\ell}, \frac{1}{2}, \zeta_1\right)\right\}\right]\right\}\right] . \tag{3.93}$$

This is the required formula for the spectral density. Finally, (3.69) gives

$$\langle x^2 \rangle = \frac{D}{\ell}\left[1 - \frac{\xi_1 e^{-\zeta_1} - \xi_2 e^{-\zeta_2}}{\int_{\xi_2}^{\xi_1} e^{-\xi}\, d\xi}\right], \tag{3.94}$$

while, from (3.70),

$$\langle x \rangle = \left(\frac{D}{\ell}\right)^{\frac{1}{2}} \frac{e^{-\zeta_2} - e^{-\zeta_1}}{\int_{\xi_2}^{\xi_1} e^{-\zeta} d\xi} . \tag{3.95}$$

The derivation for this fairly simple case has been given in considerable detail, most of which will be omitted in future cases. Firstly, several special cases of the one-interval case will be dealt with. The form of the restoring force f(x) in each of these cases is illustrated on the next page.

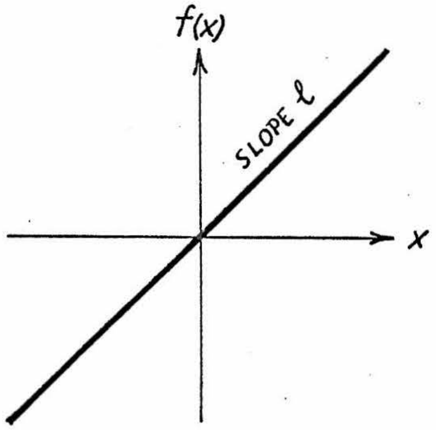
Example (1)--the linear system

$x_1 = \infty$, $x_2 = -\infty$. Since the boundaries are infinite, ℓ must be positive for stability. As $\xi_1 \to \infty$, $\xi_2 \to -\infty$,

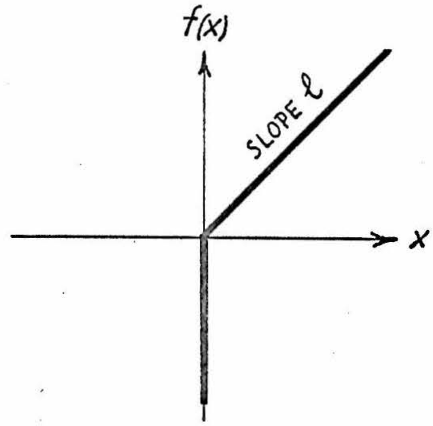
$$\pi_1(\xi_1) = \pi_1(\xi_2) \sim \frac{\sqrt{\pi}}{\Gamma(\frac{1}{2}\sigma)} \zeta_1^{\frac{1}{2}(\sigma-1)} \tag{3.96}$$

$$\pi_2(\xi_1) = -\pi_2(\xi_2) \sim \frac{\sqrt{\frac{\pi}{2}}}{\Gamma(\frac{1}{2}\sigma+\frac{1}{2})} \zeta_1^{\frac{1}{2}(\sigma-1)} \tag{3.97}$$

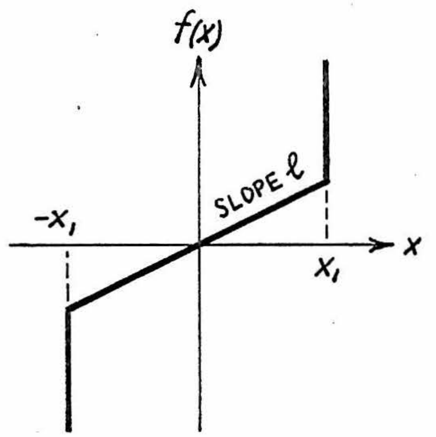
$$\psi_1(\xi_1) = -\psi_1(\xi_2) \sim \frac{\sqrt{2\pi}}{\Gamma(\frac{1}{2}\sigma)} \zeta_1^{\frac{1}{2}\sigma} \tag{3.98}$$



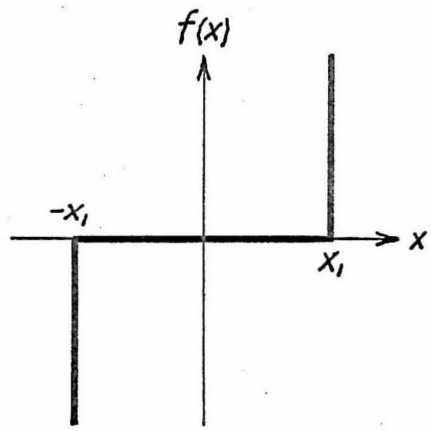
Section 3.3, example 1 — linear system



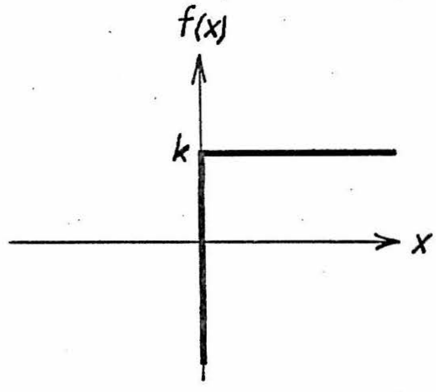
Example 2 — rectifier



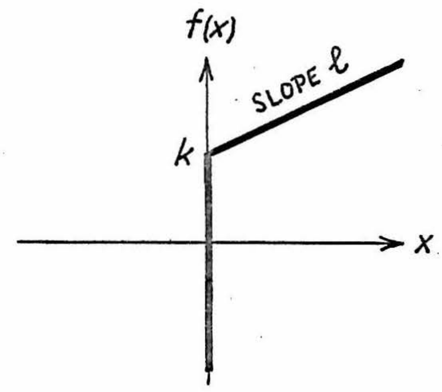
Example 3 — hard limiter



Special case, limit of example 3



Example 4 — another rectifier



Example 4 is obtained by letting $\ell \to 0$ in this case.

$$\psi_2(\xi_1) = \psi_2(\xi_2) \sim \frac{\sqrt{\pi}}{\Gamma(\frac{1}{2}\sigma+\frac{1}{2})}\,\zeta_1^{\frac{1}{2}\sigma} \quad . \tag{3.99}$$

Therefore equation (3.80) for the transformed transition density becomes

$$\pi(\xi, \sigma\,|\,\xi_o) = (2\pi)^{-\frac{1}{2}}\,\Gamma(\sigma)\,e^{\frac{1}{2}(\zeta_o-\zeta)}\,D_{-\sigma}(\pm\xi)D_{-\sigma}(\mp\xi_o) \quad \text{for } x \gtrless x_o\,, \tag{3.100}$$

confluent hypergeometric functions being replaced here by parabolic cylinder functions using (3.20, 21).

In this simple case, the transition density is known to be[1]

$$P(x, t\,|x_o) = \left[\frac{\ell}{2\pi D(1-e^{-2\ell t})}\right]^{\frac{1}{2}} \exp\left[-\frac{\ell(x-x_o e^{-\ell t})^2}{D(1-e^{-2\ell t})}\right] \quad . \tag{3.101}$$

However, the writer has been unable to derive this directly by inverse transformation of (3.100).

All terms except the first in (3.92) disappear. Thus (3.82, 92-94) become

$$\Pi_o(\xi) = \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} e^{-\frac{1}{2}\xi^2} \tag{3.102}$$

$$r(s) = \frac{D}{\ell(\ell+s)} \quad , \qquad \text{so } R(t) = \frac{D}{\ell}\,e^{-\ell t} \tag{3.103}$$

$$\Phi(\omega) = \frac{2D}{\pi(\ell^2+\omega^2)} \tag{3.104}$$

$$\langle x^2 \rangle = \frac{D}{\ell} \;; \tag{3.105}$$

---

[1] See, for example, Wang and Uhlenbeck [40].

all of which results are well known from other methods.

Example (2)--rectifier

$x_1 = \infty$, $x_2 = 0$, $\ell > 0$. Here $\pi_i(\xi_1)$, $\psi_i(\xi_1)$ have the same form as the previous case, while

$$\pi_1(\xi_2) = \psi_2(\xi_2) = 1 \quad , \quad \pi_2(\xi_2) = \psi_1(\xi_2) = 0 \quad . \tag{3.106}$$

Thus (3.80, 82, 92-95) reduce to

$$\begin{aligned} \pi(\xi, \sigma \mid \xi_o) &= 2^{\frac{1}{2}(\sigma-1)}\pi^{-\frac{1}{2}}\Gamma(\tfrac{1}{2}\sigma)e^{-\frac{1}{2}\zeta}D_{-\sigma}(\xi)M(\tfrac{1}{2}\sigma, \tfrac{1}{2}, \zeta_o) \quad , \quad x \geq x_o \\ &= 2^{\frac{1}{2}(\sigma-1)}\pi^{-\frac{1}{2}}\Gamma(\tfrac{1}{2}\sigma)e^{-\zeta+\frac{1}{2}\zeta_o}D_{-\sigma}(\xi_o)M(\tfrac{1}{2}\sigma, \tfrac{1}{2}, \zeta) \; , \quad x \leq x_o \end{aligned} \tag{3.107}$$

$$\Pi_o(\xi) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} e^{-\frac{1}{2}\xi^2} \tag{3.108}$$

$$r(s) = \frac{D}{\ell^2}\,\frac{1}{\sigma+1}\left[1 + \frac{\pi^{-\frac{1}{2}}}{\sigma+1}\,\frac{\Gamma(\frac{1}{2}\sigma)}{\Gamma(\frac{1}{2}\sigma+\frac{1}{2})}\right] \tag{3.109}$$

$$\Phi(\omega) = \frac{2D}{\pi}\quad \frac{1}{\ell^2+\omega^2} + \mathrm{Re}\left[\frac{\pi^{-\frac{1}{2}}}{(\ell+i\omega)^2}\,\frac{\Gamma\left(\frac{i\omega}{2\ell}\right)}{\Gamma\left(\frac{i\omega}{2\ell}+\frac{1}{2}\right)}\right] \quad , \quad \omega \neq 0 \tag{3.110}$$

$$\langle x^2 \rangle = \frac{D}{\ell} \tag{3.111}$$

$$\langle x \rangle = \left(\frac{2D}{\pi\ell}\right)^{\frac{1}{2}} \quad . \tag{3.112}$$

The graph of the spectral density in this case is given as one of the limiting cases $\left(\frac{\ell_2}{\ell_1} = \infty\right)$ in Fig. (1) on page 117.

Example (3)--hard limiter

$x_1 = -x_2 < \infty$. In this case, since

$$\pi_1(\xi_2) = \pi_1(\xi_1) \quad , \quad \pi_2(\xi_2) = -\pi_2(\xi_1) \; ,$$
$$\psi_1(\xi_2) = -\psi_1(\xi_1) \quad , \quad \psi_2(\xi_2) = \psi_2(\xi_1) \; , \tag{3.113}$$

(3.80) becomes

$$\pi(\xi, \sigma \,|\, \xi_o) = (\operatorname{sgn} \ell) e^{\zeta_o} \frac{\left[\psi_2(\xi_1)\psi_1(\xi) \mp \psi_1(\xi_1)\psi_2(\xi)\right]\left[\psi_2(\xi_2)\pi_1(\xi_o) \pm \psi_1(\xi_1)\pi_2(\xi_o)\right]}{2\psi_1(\xi_1)\psi_2(\xi_2)} \tag{3.114}$$

for $x \gtrless x_o$, respectively. Then, from (3.81),

$$\Pi_o(\xi) = \tfrac{1}{2} e^{\mp\frac{1}{2}\xi^2} \left(\int_0^{\xi_1} e^{\mp\frac{1}{2}\xi^2} d\xi \right)^{-1} \qquad \text{for } \ell \gtrless 0 \; , \tag{3.115}$$

and, from (3.92-94), using (3.32),

$$r(s) = \frac{D}{\ell^2} \frac{1}{\sigma+1} \left\{ 1 - \left(\int_0^{\xi_1} e^{-\zeta} d\xi \right)^{-1} \frac{e^{-\zeta_1}}{\sigma+1} \left[(\sigma+2)\xi_1 - \operatorname{sgn} \ell \frac{\pi_2(\xi_1)}{\psi_2(\xi_1)} \right] \right\} \tag{3.116}$$

$$\Phi(\omega) = \frac{2D}{\pi} \left[\frac{1}{\ell^2+\omega^2} - \left(\int_0^{\xi_1} e^{-\zeta} d\xi \right)^{-1} \xi_1 e^{-\zeta_1} \left\{ \frac{2\ell^2}{(\ell^2+\omega^2)^2} - \operatorname{Re} \left[\frac{1}{(\ell+i\omega)^2} \frac{M\left(\frac{1}{2}+\frac{i\omega}{2\ell}, \frac{3}{2}, \zeta_1\right)}{M\left(\frac{1}{2}+\frac{i\omega}{2\ell}, \frac{1}{2}, \zeta_1\right)} \right] \right\} \right] \tag{3.117}$$

$$\langle x^2 \rangle = \frac{D}{\ell} \left(1 - \frac{\xi_1 e^{-\zeta_1}}{\int_0^{\xi_1} e^{-\zeta} d\xi} \right) . \tag{3.118}$$

In Fig. (2) on page 118, $\Phi(\omega)$ is plotted to show the variation of the shape of the spectrum as ξ_1 varies. Variation of this dimensionless parameter can be considered to represent variation of either slope ℓ or cutoff point x_1. Note the behavior for negative ξ_1 (i.e., negative slope). Both Φ and ω are nondimensionalized using D and $\langle x^2\rangle - \langle x\rangle^2$; as shown in section 3.6, this means that the "equivalent linear" system in each case is the same. These dimensionless forms of ω and Φ are

$$\omega^* = \frac{\langle x^2\rangle - \langle x\rangle^2}{D}\,\omega \tag{3.119}$$

$$\Phi^*(\omega^*) = \frac{D\Phi(\omega)}{\left(\langle x^2\rangle - \langle x\rangle^2\right)^2} \,. \tag{3.120}$$

In this example, $\langle x\rangle = 0$. However, the nondimensionalization (3.119-20) will also be used when this is not the case (e.g. example (1), section 3.4).

A special case

$\ell = 0$. This is a system which limits the displacement at $\pm x_1$, but otherwise exerts no restoring force. In the limit as $\xi_1 \to 0$ (i.e., as $\ell \to 0$), one has, for $|\xi| \le \xi_1$,

$$\pi_1(\xi) \sim \cosh\left[\left(\frac{s}{D}\right)^{\frac{1}{2}} x\right] \;,\quad \pi_2(\xi) \sim \left(\frac{|\ell|}{s}\right)^{\frac{1}{2}} \sinh\left[\left(\frac{s}{D}\right)^{\frac{1}{2}} x\right]$$
$$\psi_1(\xi) \sim \left(\frac{s}{|\ell|}\right)^{\frac{1}{2}} \sinh\left[\left(\frac{s}{D}\right)^{\frac{1}{2}} x\right] \;,\quad \psi_2(\xi) \sim \cosh\left[\left(\frac{s}{D}\right)^{\frac{1}{2}} x\right] \tag{3.121}$$

$$\int_0^{\xi} e^{-\zeta}\, d\xi \sim \xi\left(1 - \frac{\zeta}{1!3} + \frac{\zeta^2}{2!5} - \ldots .\right) \,. \tag{3.122}$$

Thus one obtains

$$p(x, s\,|x_o) = \left(\frac{1}{sD}\right)^{\frac{1}{2}} \frac{\cosh\left[\left(\frac{s}{D}\right)^{\frac{1}{2}}(x_1 \mp x)\right]\cosh\left[\left(\frac{s}{D}\right)^{\frac{1}{2}}(x_1 \pm x_o)\right]}{\sinh\left[2\left(\frac{s}{D}\right)^{\frac{1}{2}}x_1\right]} \tag{3.123}$$

for $x \gtrless x_o$ respectively, and

$$P_o(x) = \frac{1}{2x_1} \tag{3.124}$$

$$r(s) = \frac{x_1^{\,2}}{3s} - \frac{D}{s^2}\left\{1 - \frac{\tanh\left[\left(\frac{s}{D}\right)^{\frac{1}{2}}x_1\right]}{\left(\frac{s}{D}\right)^{\frac{1}{2}}x_1}\right\} \tag{3.125}$$

$$\Phi(\omega) = \frac{2D}{\pi\omega^2}\left[1 - \mathrm{Re}\left\{\frac{\tanh\left[\left(\frac{i\omega}{D}\right)^{\frac{1}{2}}x_1\right]}{\left(\frac{i\omega}{D}\right)^{\frac{1}{2}}x_1}\right\}\right] \tag{3.126}$$

$$\langle x^2 \rangle = \frac{1}{3}x_1^{\,2} \quad . \tag{3.127}$$

<u>Example (4)--another rectifier</u>

$x_1 = \infty$, $x_2 = 0$, $\ell = 0$, $k > 0$. Brownian motion in a constant force-field (e.g. gravity) with a reflecting barrier also leads to this process. Its transition density has been found by different methods by Smoluchowski (see [6]), Kac [28], Wong [45]. Here it will be considered as the limit as $\ell \to 0$ of the case with the same x_1, x_2, k, but with $\ell > 0$. In this case, $\xi_1 = \infty$, $\xi_2 = \varkappa = \frac{k}{\sqrt{\ell D}}$.

The general one-interval case worked out above has $k = 0$. The formula (3.80) for $\pi(\xi, \sigma\,|\,\xi_o)$ is, however, unchanged for $k \neq 0$.

Substituting for ξ_1 and ξ_2, it becomes

$$\pi(\xi,\sigma\,|\,\xi_o) = e^{-\frac{1}{2}\zeta+\zeta_o+\frac{1}{4}\varkappa^2}\frac{D_{-\sigma}(\xi)}{\sigma D_{-\sigma-1}(\varkappa)}\left[\psi_2(\varkappa)\pi_1(\xi_o)-\psi_1(\varkappa)\pi_2(\xi_o)\right],\quad x\ge x_o$$

(3.128)

$$= e^{\zeta_o+\frac{1}{4}\varkappa^2}\frac{D_{-\sigma}(\xi)}{\sigma D_{-\sigma-1}(\varkappa)}\left[\psi_2(\varkappa)\pi_1(\xi)-\psi_1(\varkappa)\pi_2(\xi)\right],\quad x\le x_o\,.$$

(3.129)

Evaluating $\mathcal{J}^*(\xi_2)$, $\mathcal{P}^*(\xi_2)$, $\mathcal{Q}^*(\xi_2)$ as before (but with $k\neq 0$), and substituting in (3.64),

$$r(s) = \frac{D}{\ell^2}\left[\frac{1}{\sigma+1}+\frac{\varkappa^2}{\sigma}-\frac{e^{-\frac{1}{2}\varkappa^2}}{\int_\varkappa^\infty e^{-\zeta}d\xi}\left\{\frac{\varkappa}{\sigma}\left[1+\frac{1}{(\sigma+1)^2}\right]-\frac{1}{(\sigma+1)^2}\frac{D_{-\sigma}(\varkappa)}{\sigma D_{-\sigma-1}(\varkappa)}\right\}\right]\,.$$

(3.130)

Now let $\ell\to 0$ as explained at the end of the section 3.1. To find the limiting transformed transition density $p(x, s\,|x_o)$, it is necessary to find the constants K_1, K_2 of (3.40); otherwise p can be found only up to a multiplicative constant. The inverse transform of p can be found and agrees with the transition density found by Smoluchowski, etc. The limit of (3.130) is more readily found. If $p_{1,2}$ are the solutions in (3.41)--and $q_{1,2}$ correspondingly--then, as $\ell\downarrow 0$,

$$-\frac{s}{(\ell D)^{\frac{1}{2}}}\frac{D_{-\sigma-1}(\pm\xi)}{D_{-\sigma}(\pm\xi)}\sim\frac{q_{1,2}(x)}{p_{1,2}(x)}=\frac{k}{2D}\left[1\mp\left(1+\frac{4sD}{k^2}\right)^{\frac{1}{2}}\right]\,. \qquad (3.131)$$

Substituting into (3.130), the limiting autocorrelation is

$$r(s) = \frac{2D^2}{ks} - \frac{D}{s^2} - \frac{k^2}{2s^3}\left[1 - \left(1 + \frac{4sD}{k^2}\right)^{\frac{1}{2}}\right] . \qquad (3.132)$$

Note that here we have used

$$\int_{\xi_2}^{\infty} e^{-\zeta}\, d\xi \sim \frac{e^{-\zeta_2}}{\xi_2}\left(1 - \xi_2^{-2} + 3\xi_2^{-4} \ldots\right) \quad \text{as } \xi_2 \to \infty , \qquad (3.133)$$

which follows immediately from the asymptotic expansion for the error function. From r(s) one obtains, using (2.64, 67, 70),

$$\Phi(\omega) = \frac{2}{\pi}\left\{\frac{D}{\omega^2} + \frac{k^2}{2\sqrt{2}\,\omega^3}\left[\left\{1 + \left(\frac{4\omega D}{k^2}\right)^2\right\}^{\frac{1}{2}} - 1\right]^{\frac{1}{2}}\right\} \qquad (3.134)$$

$$\langle x^2 \rangle = \frac{2D^2}{k^2} \qquad (3.135)$$

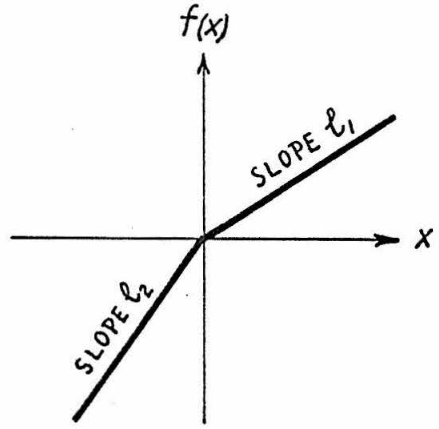
$$\langle x \rangle = \frac{D}{k} . \qquad (3.136)$$

3.4 SOME TWO-INTERVAL CASES

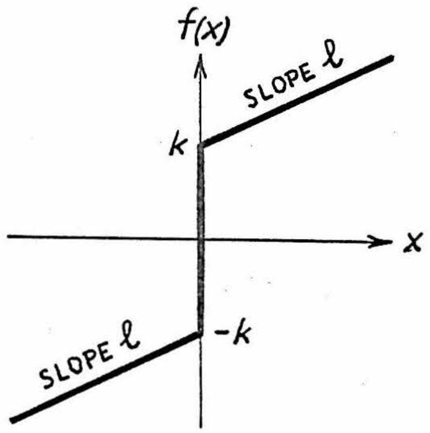
Only two of the simplest cases will be worked out. The form of the restoring force f(x) in each of these cases is illustrated on the next page.

Example (1)--continuous bilinear device

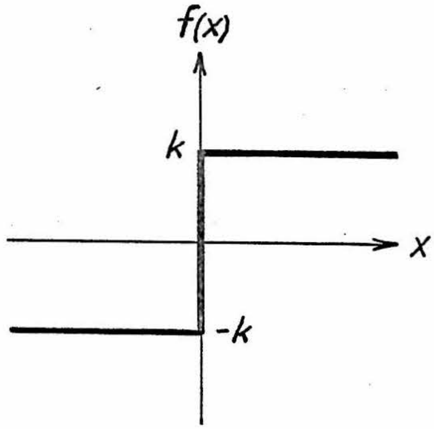
$x_1 = \infty$, $x_2 = 0$, $x_3 = -\infty$, $k_1 = k_2 = 0$. For stability, ℓ_1 and ℓ_2 must be positive. Using (3.28, 29, 33, 34, 38, 39), one obtains, for $x_o > 0$,



Section 3.4, example 1 — continuous bilinear system



Example 2 — preloaded spring



Special case (bang-bang), limit of example 2

$$\gamma_1^{1-} = \gamma_1^{1+} - \xi_o^1 M\left(\frac{1}{2}\sigma_1 + \frac{1}{2}, \frac{3}{2}, \zeta_o^1\right) \tag{3.137}$$

$$\gamma_2^{1-} = \gamma_2^{1+} + M\left(\frac{1}{2}\sigma_1, \frac{1}{2}, \zeta_o^1\right) \tag{3.138}$$

$$\gamma_1^2 = \left(\frac{\ell_2}{\ell_1}\right)^{\frac{1}{2}} \gamma_1^{1-} \tag{3.139}$$

$$\gamma_2^2 = \gamma_2^{1-} \tag{3.140}$$

$$\sqrt{2}\,\Gamma\left(\frac{1}{2}\sigma_1 + \frac{1}{2}\right)\gamma_1^{1+} + \Gamma\left(\frac{1}{2}\sigma_1\right)\gamma_2^{1+} = 0 \tag{3.141}$$

$$\sqrt{2}\,\Gamma\left(\frac{1}{2}\sigma_2 + \frac{1}{2}\right)\gamma_1^2 - \Gamma\left(\frac{1}{2}\sigma_2\right)\gamma_2^2 = 0\ , \tag{3.142}$$

and similarly for $x_o < 0$. Solving these, one gets

$$\pi^i(\xi, \sigma | \xi_o) = e^{\zeta_o} \frac{\left(\pi_1^i(\xi) - a_i \pi_2^i(\xi)\right)\left(\pi_1^j(\xi_o) + b_j \pi_2^j(\xi_o)\right)}{a_i + b_i}\ , \tag{3.143}$$

where i, j = 1, 2 and

$$a_i = \left(2\frac{\ell_1}{\ell_i}\right)^{\frac{1}{2}} \frac{\Gamma(\frac{1}{2}\sigma_1 + \frac{1}{2})}{\Gamma(\frac{1}{2}\sigma_1)}\ , \quad b_i = \left(2\frac{\ell_2}{\ell_i}\right)^{\frac{1}{2}} \frac{\Gamma(\frac{1}{2}\sigma_2 + \frac{1}{2})}{\Gamma(\frac{1}{2}\sigma_2)}\ , \tag{3.144}$$

and where $x \geq x_o$; for $x \leq x_o$, interchange ξ and ξ_o in $\pi_j^i(\cdot)$. Then as in the previous section, one gets

$$\Pi_o^i(\xi) = \Gamma_i e^{-\zeta^i} = \frac{\left(\frac{2}{\pi \ell_i}\right)^{\frac{1}{2}}}{\ell_1^{-\frac{1}{2}} + \ell_2^{-\frac{1}{2}}} e^{-\zeta^i}\ , \quad i = 1, 2 \tag{3.145}$$

$$\mathcal{J}_k^* = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \Gamma_k \tag{3.146}$$

$$\mathcal{P}_k^*(\xi_1) = \mathcal{P}_k^*(\xi_3) = 0 \tag{3.147}$$

$$\mathcal{P}_k^*(\xi_2) = (-)^{k-1} \frac{\Gamma k}{(\sigma_k+1)} \pi^k(0, \sigma|0) , \tag{3.148}$$

and similarly for $\mathcal{Q}_k^*$; however, the coefficients of $\mathcal{Q}_k^*(\xi_2)$ in (3.64) are zero. Thus one gets

$$r(s) = \frac{D}{\ell_1^{-\frac{1}{2}}+\ell_2^{-\frac{1}{2}}}\left[\frac{\ell_1^{-\frac{3}{2}}}{\ell_1+s} + \frac{\ell_2^{-\frac{3}{2}}}{\ell_2+s} + \pi^{-\frac{1}{2}} \frac{\left(\frac{1}{\ell_1+s} - \frac{1}{\ell_2+s}\right)^2}{\ell_1^{\frac{1}{2}}\frac{\Gamma(\frac{1}{2}\sigma_1+\frac{1}{2})}{\Gamma(\frac{1}{2}\sigma_1)} + \ell_2^{\frac{1}{2}}\frac{\Gamma(\frac{1}{2}\sigma_2+\frac{1}{2})}{\Gamma(\frac{1}{2}\sigma_2)}}\right] \tag{3.149}$$

$$\Phi(\omega) = \frac{2D}{\pi\left(\ell_1^{-\frac{1}{2}}+\ell_2^{-\frac{1}{2}}\right)}\left[\frac{\ell_1^{-\frac{1}{2}}}{\ell_1^2+\omega^2} + \frac{\ell_2^{-\frac{1}{2}}}{\ell_2^2+\omega^2} + \pi^{-\frac{1}{2}}\mathrm{Re}\left\{\frac{\left(\frac{1}{\ell_1+i\omega} - \frac{1}{\ell_2+i\omega}\right)^2}{\ell_1^{\frac{1}{2}}\frac{\Gamma(\frac{i\omega}{2\ell_1}+\frac{1}{2})}{\Gamma(\frac{i\omega}{2\ell_1})} + \ell_2^{\frac{1}{2}}\frac{\Gamma(\frac{i\omega}{2\ell_2}+\frac{1}{2})}{\Gamma(\frac{i\omega}{2\ell_2})}}\right\}\right] \tag{3.150}$$

$$\langle x^2 \rangle = D\left(\frac{\ell_1^{-\frac{3}{2}} + \ell_2^{-\frac{3}{2}}}{\ell_1^{-\frac{1}{2}} + \ell_2^{-\frac{1}{2}}}\right) \tag{3.151}$$

$$\langle x \rangle = \left(\frac{2D}{\pi}\right)^{\frac{1}{2}}\left(\ell_1^{-\frac{1}{2}} - \ell_2^{-\frac{1}{2}}\right) . \tag{3.152}$$

Note that putting $\ell_1 = \ell_2$ in the above leads to the results obtained above for the linear system, while $\ell_1 = \ell$, $\ell_2 = \infty$ leads to the rectifier device dealt with above.

In Fig. (1) on page 117, $\Phi(\omega)$ is plotted--in dimensionless form, Φ^* against ω^*--for various values of $\frac{\ell_2}{\ell_1}$.

Example (2)--pre-loaded spring

$x_1 = \infty$, $x_2 = 0$, $x_3 = -\infty$, $\ell_1 = \ell_2 = \ell > 0$, $k_1 = -k_2 = k$. The restoring force in this case resembles an idealized type of friction, where the result is obtained by adding a linear term to a Coulomb friction term.

By symmetry one has

$$\begin{aligned} \pi_1^2(-\xi) &= \pi_1^1(\xi) \;, \quad \pi_2^2(-\xi) = -\pi_2^1(\xi) \\ \psi_1^2(-\xi) &= -\psi_1^1(\xi) \;, \quad \psi_2^2(-\xi) = \psi_2^1(\xi) \;. \end{aligned} \tag{3.153}$$

Using this to simplify the solution of equations (3.28, 29, 33, 34, 38, 39), one obtains

$\pi(\xi, \sigma | \xi_o)$

$$= \mp \tfrac{1}{2} e^{\zeta_o} \left[\pi_1(\xi) \mp a\pi_2(\xi)\right] \left[\frac{\pi_1^1(\varkappa)\pi_2(\xi_o) \mp \pi_2^1(\varkappa)\pi_1(\xi_o)}{\pi_1^1(\varkappa) - a\pi_2^1(\varkappa)} + \frac{\psi_1^1(\varkappa)\pi_2(\xi_o) \mp \psi_2^1(\varkappa)\pi_1^1(\xi_o)}{\psi_1^1(\varkappa) - a\psi_2^1(\varkappa)}\right]$$

$$\text{for } x \gtrless x_o \gtrless 0 \tag{3.154}$$

$$= \mp \tfrac{1}{2} e^{\zeta_o} \left[\pi_1(\xi_o) \mp a\pi_2(\xi_o)\right] \left[\frac{\pi_1^1(\varkappa)\pi_2(\xi) \mp \pi_2^1(\varkappa)\pi_1(\xi)}{\pi_1^1(\varkappa) - a\pi_2^1(\varkappa)} + \frac{\psi_1^1(\varkappa)\pi_2(\xi) \mp \psi_2^1(\varkappa)\pi_1(\xi)}{\psi_1^1(\varkappa) - a\psi_2^1(\varkappa)}\right]$$

$$\text{for } x_o \gtrless x \gtrless 0 \tag{3.155}$$

$$= \tfrac{1}{2} e^{\zeta_o - \frac{1}{2}\varkappa^2} \frac{\left[\pi_1(\xi_o) \pm a\pi_2(\xi_o)\right]\left[\pi_1(\xi) \mp a\pi_2(\xi)\right]}{\left[\pi_1^1(\varkappa) - a\pi_2^1(\varkappa)\right]\left[\psi_1^1(\varkappa) - a\psi_2^1(\varkappa)\right]}$$

$$\text{for } x \gtrless 0 \gtrless x_o \ , \qquad (3.156)$$

where

$$a = \sqrt{2}\, \frac{\Gamma(\frac{1}{2}\sigma + \frac{1}{2})}{\Gamma(\frac{1}{2}\sigma)} \ . \qquad (3.157)$$

Then

$$\Pi_o(\xi) = \Gamma e^{-\zeta} = \frac{\left(\frac{1}{2\pi}\right)^{\frac{1}{2}}}{\operatorname{erfc} \frac{\varkappa}{\sqrt{2}}} e^{-\zeta} \qquad (3.158)$$

$$\mathcal{J}_1^* = \mathcal{J}_2^* = \tfrac{1}{2}\left\{1 + (1 + \sigma^{-1})\left[\varkappa^2 - \frac{\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\varkappa}{\operatorname{erfc} \frac{\varkappa}{\sqrt{2}}} e^{-\frac{1}{2}\varkappa^2}\right]\right\} \qquad (3.159)$$

$$\mathcal{Q}_1^*(\xi_2) = \mathcal{Q}_2^*(\xi_2) = \frac{1}{\sigma+1} \frac{\frac{1}{2}\left(\frac{1}{2\pi}\right)^{\frac{1}{2}}}{\operatorname{erfc} \frac{\varkappa}{\sqrt{2}}} e^{-\frac{1}{2}\varkappa^2}\left[1 + \frac{\varkappa}{\sigma} \frac{\psi_1^1(\varkappa) - a\psi_2^1(\varkappa)}{\pi_1^1(\varkappa) - a\pi_2^1(\varkappa)}\right] . \qquad (3.160)$$

The coefficients of all other $\mathcal{P}_k^*(\xi_i)$, $\mathcal{Q}_k^*(\xi_i)$ in (3.64) are zero. Thus one gets

$$r(s) = \frac{D}{\ell^2}\left[\frac{1}{\sigma+1} + \frac{\varkappa^2}{\sigma} - \frac{\varkappa}{\sigma} \frac{\left(\frac{2}{\pi}\right)^{\frac{1}{2}} e^{-\frac{1}{2}\varkappa^2}}{\operatorname{erfc} \frac{\varkappa}{\sqrt{2}}}\left\{1 - \frac{1}{(\sigma+1)^2}\left[1 - \varkappa \frac{D_{-\sigma-1}(\varkappa)}{D_{-\sigma}(\varkappa)}\right]\right\}\right]$$

$$(3.161)$$

$$\Phi(\omega) = \frac{2D}{\pi}\left\{\frac{1}{\ell^2+\omega^2} - \frac{(\frac{2}{\pi})^{\frac{1}{2}}\varkappa e^{-\frac{1}{2}\varkappa^2}}{\operatorname{erfc}\frac{\varkappa}{\sqrt{2}}}\left[\frac{2\ell^2}{(\ell^2+\omega^2)^2}\right.\right.$$

$$\left.\left. + \varkappa \operatorname{Re}\left\{\frac{\ell}{i\omega(\ell+i\omega)^2}\cdot\frac{D_{-\frac{i\omega}{\ell}-1}(\varkappa)}{D_{-\frac{i\omega}{\ell}}(\varkappa)}\right\}\right]\right\} \qquad (3.162)$$

$$\langle x^2\rangle = \frac{D}{\ell}\left[1+\varkappa^2 - \frac{(\frac{2}{\pi})^{\frac{1}{2}}\varkappa e^{-\frac{1}{2}\varkappa^2}}{\operatorname{erfc}\frac{\varkappa}{\sqrt{2}}}\right] . \qquad (3.163)$$

In Fig. (3) on page 119, $\Phi^*(\omega^*)$ is plotted for various values of $\varkappa$, which represents, in dimensionless form, the size of the jump at zero. The behavior as $\varkappa \to -\infty$ is of some interest.

A special case

As $\varkappa \to \infty$ (i.e., $\ell \to 0$ with positive k) we approach the case with bang-bang restoring force dealt with by Caughey and Dienes [4, 8] and Robinson [37]. Letting $\ell \to 0$ in (3.154-6, 158, 161) and using (3.131, 133), one gets $p(x, s\,|\,x_o)$ -- see [4] for the form of this function and its inverse transform--and also

$$P_o(x) = \frac{k}{2D}\, e^{-\frac{kx}{D}} \qquad (3.164)$$

$$r(s) = \frac{2D^2}{k^2 s} - \frac{D}{s^2} + \frac{k^4}{4s^4 D}\left[1 - \left(1 + \frac{4sD}{k^2}\right)^{\frac{1}{2}}\right]^2 \qquad (3.165)$$

and, using (2.64, 67),

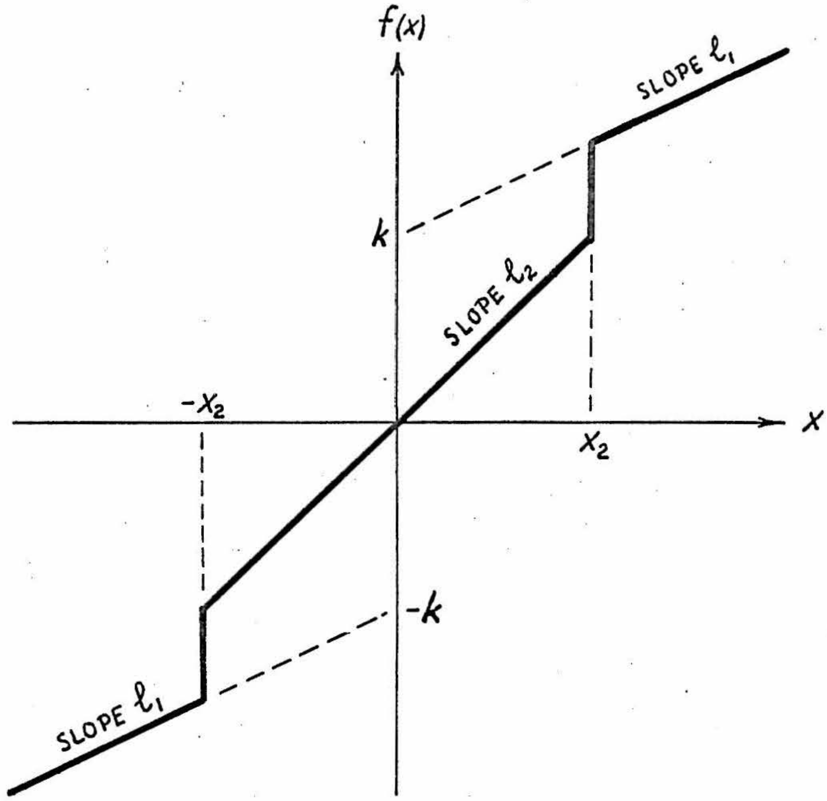
$$\Phi(\omega) = \frac{2D}{\pi}\left\{\frac{1}{\omega^2} + \frac{k^4}{2\omega^4 D^2}\left[1 - \left(\frac{1}{2}\left\{1 + \left[1 + \left(\frac{4\omega D}{k^2}\right)^2\right]^{\frac{1}{2}}\right\}\right)^{\frac{1}{2}}\right]\right\} \tag{3.166}$$

$$\langle x^2 \rangle = \frac{2D^2}{k^2} \; . \tag{3.167}$$

These results agree with [4] except for a factor of 2 in (3.166), which is explained if the result in [4] is taken to be the two-sided spectral density. (See the discussion in the Introduction of the distinction between two- and one-sided spectra.)

3.5 THE SYMMETRIC THREE-INTERVAL CASE

The only three-interval case which will be worked out is the symmetric case--$\ell_1 = \ell_3 > 0$, $k_1 = -k_3 = k$ say, $k_2 = 0$, $x_3 = -x_2$ --with in addition $x_1 = \infty$, $x_4 = -\infty$. (See the sketch on the next page.) Continuity of f(x) at $\pm x_2$ will not be assumed, as this produces no significant simplification in the results; those for continuous f(x) can be obtained by simply replacing k by $(\ell_2 - \ell_1)x_2$ wherever it appears. Such a trilinear characteristic can be used as a reasonable approximation to almost any continuous symmetric nonlinear characteristic with infinite endpoints. A method of approximation is suggested in section 3.6. As is seen below--e.g. (3.192)--the formulas even for the symmetric trilinear case are not simple, and the increased accuracy of, say, a five-interval approximation would probably not justify the trouble of working it out.



Typical restoring force for the symmetric trilinear system of section 3.5.

The transition density

For the symmetric three-interval case, the set of simultaneous equations can be simplified slightly by noting that

$$\pi_1^1(\xi_2) = \pi_1^3(-\xi_2) \quad , \quad \pi_2^1(\xi_2) = -\pi_2^3(-\xi_2) \ ,$$
$$\psi_1^1(\xi_2) = -\psi_1^3(-\xi_2) \ , \quad \psi_2^1(\xi_2) = \psi_2^3(-\xi_2) \ . \tag{3.168}$$

Then solving the set of simultaneous equations, the Laplace-transformed transition density is given for various x, x_o by the following table:

Position of x, x_o	$\pi(\xi, \sigma \mid \xi_o)$	Eqn. no.
$x \gtrless x_o \gtrless \pm x_2$	$\frac{1}{2} e^{\zeta_o - \frac{1}{2}\zeta + \frac{1}{2}\zeta_2^1} D_{-\sigma_1}(\pm\xi) \left[\frac{F(\xi_o)}{F} + \frac{G(\xi_o)}{G}\right]$	3.169
$x_o \gtrless x \gtrless \pm x_2$	$\frac{1}{2} e^{\frac{1}{2}\zeta_o + \frac{1}{2}\zeta_2^1} D_{-\sigma_1}(\pm\xi_o) \left[\frac{F(\xi)}{F} + \frac{G(\xi)}{G}\right]$	3.170
$x \gtrless \pm x_2 \gtrless x_o \gtrless \mp x_2$	$\frac{1}{2} e^{\zeta_o - \frac{1}{2}\zeta - \zeta_2^2 + \frac{1}{2}\zeta_2^1} D_{-\sigma_1}(\pm\xi) \left[\frac{\pi_1^2(\xi_o)}{F} \pm \frac{\pi_2^2(\xi_o)}{G}\right]$	3.171
$x_o \gtrless \pm x_2 \gtrless x \gtrless \mp x_2$	$\frac{1}{2} \left\lvert\frac{\ell_2}{\ell_1}\right\rvert^{\frac{1}{2}} e^{\frac{1}{2}\zeta_o - \frac{1}{2}\zeta_2^1} D_{-\sigma_1}(\pm\xi_o) \left[\frac{\pi_1^2(\xi)}{F} \pm \frac{\pi_2^2(\xi)}{G}\right]$	3.172
$\pm x_2 \gtrless x \gtrless x_o \gtrless \mp x_2$	$\frac{1}{2}(\operatorname{sgn} \ell_2) e^{\zeta_o} \frac{F}{G} \left[\frac{\pi_1^2(\xi)}{F} \mp \frac{\pi_2^2(\xi)}{G}\right] \left[\frac{\pi_1^2(\xi_o)}{F} \pm \frac{\pi_2^2(\xi_o)}{G}\right]$	3.173
$x \gtrless \pm x_2$; $x_o \gtrless \mp x_2$	$\frac{1}{2}(\operatorname{sgn} \ell_2) \left\lvert\frac{\ell_2}{\ell_1}\right\rvert^{\frac{1}{2}} e^{\frac{1}{2}(\zeta_o - \zeta) + \zeta_2^1 - \zeta_2^2} \left[\frac{D_{-\sigma_1}(\pm\xi) D_{-\sigma_1}(\mp\xi_o)}{FG}\right]$	3.174

Here for simplicity of expression we have written

$$F = \left|\frac{\ell_2}{\ell_1}\right|^{\frac{1}{2}} D_{-\sigma_1}(\xi_2^1)\psi_1^2(\xi_2) + \sigma_1 D_{-\sigma_1-1}(\xi_2^1)\pi_1^2(\xi_2) \tag{3.175}$$

$$G = \left|\frac{\ell_2}{\ell_1}\right|^{\frac{1}{2}} D_{-\sigma_1}(\xi_2^1)\psi_2^2(\xi_2) + \sigma_1 D_{-\sigma_1-1}(\xi_2^1)\pi_2^2(\xi_2) \tag{3.176}$$

$$F(\xi) = \pi_1(\xi)\left[\pi_1^2(\xi_2)\psi_2^1(\xi_2) - \left|\frac{\ell_2}{\ell_1}\right|^{\frac{1}{2}} \psi_1^2(\xi_2)\pi_2^1(\xi_2)\right]$$

$$\mp\pi_2(\xi)\left[\pi_1^2(\xi_2)\psi_1^1(\xi_2) - \left|\frac{\ell_2}{\ell_1}\right|^{\frac{1}{2}} \psi_1^2(\xi_2)\pi_1^1(\xi_2)\right] \tag{3.177}$$

$$G(\xi) = \pi_1(\xi)\left[\pi_2^2(\xi_2)\psi_2^1(\xi_2) - \left|\frac{\ell_2}{\ell_1}\right|^{\frac{1}{2}} \psi_2^2(\xi_2)\pi_2^1(\xi_2)\right]$$

$$\mp\pi_2(\xi)\left[\pi_2^2(\xi_2)\psi_1^1(\xi_2) - \left|\frac{\ell_2}{\ell_1}\right|^{\frac{1}{2}} \psi_2^2(\xi_2)\pi_1^1(\xi_2)\right] \; ; \tag{3.178}$$

in these last two expressions the $\mp$ sign is taken according as $x > x_2$ or $x < -x_2$.

The steady-state density

According to (3.44),

$$\Pi_o^i(\xi) = \Gamma_i e^{-\zeta^i} \qquad (i = 1, 2, 3, \quad \Gamma_3 = \Gamma_1) \; , \tag{3.179}$$

where Γ_i can be found from (3.45, 47), which give

$$\Gamma_1 e^{-\zeta_2^1} = \operatorname{sgn} \ell_2 \left|\frac{\ell_2}{\ell_1}\right|^{\frac{1}{2}} \Gamma_2 \ell^{-\zeta_2^2} = \tfrac{1}{2}\left[e^{\zeta_2^1} \int_{\xi_2^1}^{\infty} e^{-\zeta^1} d\xi^1 + \operatorname{sgn} \ell_2 \left|\frac{\ell_1}{\ell_2}\right|^{\frac{1}{2}} e^{\zeta_2^2} \int_0^{\xi_2^2} e^{-\zeta^2} d\xi^2 \right]^{-1} . \tag{3.180}$$

<u>The transformed autocorrelation</u>

By symmetry,

$$\pi(-\xi,\ \sigma | -\xi_o) = \pi(\xi,\ \sigma | \xi_o)$$

$$\psi(-\xi,\ \sigma | -\xi_o) = -\psi(\xi,\ \sigma | \xi_o) \ , \tag{3.181}$$

and conversely for the derivatives of these quantities. Thus (3.65-67) give

$$\mathscr{I}_1^* = \mathscr{I}_3^* = \Gamma_1 \left\{ e^{-\zeta_2^1} \left[\xi_2^1 - \varkappa(2 + \frac{1}{\sigma_1}) \right] + \left[1 + \varkappa^2 (1 + \frac{1}{\sigma_1}) \right] \int_{\xi_2^1}^{\infty} e^{-\zeta^1} d\xi \right\} \tag{3.182}$$

$$\mathscr{I}_2^* = 2\Gamma_2 \left\{ -\xi_2^2\, e^{-\zeta_2^2} + \int_0^{\xi_2^2} e^{-\zeta^2} d\xi \right\} \tag{3.183}$$

$$\mathscr{P}_1^*(\pm\xi_2) = -\mathscr{P}_3^*(\mp\xi_2) = \frac{\Gamma_1 e^{-\zeta_2^1}}{\sigma_1 + 1} \left\{ \pm\left[\varkappa(1 + \frac{1}{\sigma_1}) - \xi_2^1 \right] D^* \pi^2(\xi_2, \sigma | \pm\xi_2) + \pi^2(\xi_2, \sigma | \pm\xi_2) \right\} \tag{3.184}$$

$$\mathscr{P}_2^*(\xi_2) = -\mathscr{P}_2^*(-\xi_2) = \frac{\Gamma_2 e^{-\zeta_2^2}}{\sigma_2 + 1} \left\{ \xi_2^2 \left[D^* \pi^1(\xi_2,\ \sigma | \xi_2) + D^* \pi^1(\xi_2,\ \sigma | -\xi_2) \right] - \pi^1(\xi_2,\ \sigma | \xi_2) + \pi^1(\xi_2,\ \sigma | -\xi_2) \right\} \tag{3.185}$$

$$\mathcal{Z}_1^*(\pm\xi_2) = \mathcal{Z}_3^*(\mp\xi_2) = \frac{\Gamma_1 e^{-\zeta_2^1}}{\sigma_1+1}\left\{\left[\varkappa(1+\frac{1}{\sigma_1}) - \xi_2^1\right]D^*\psi^2(\xi_2, \sigma|\pm\xi_2) \pm \psi^2(\xi_2, \sigma|\pm\xi_2)\right\}$$

(3.186)

$$\mathcal{Z}_2^*(\xi_2) = \mathcal{Z}_2^*(-\xi_2) = \frac{\Gamma_2 e^{-\zeta_2^2}}{\sigma_2+1}\left\{\xi_2^2\left[D^*\psi^1(\xi_2, \sigma|\xi_2) + D^*\psi^1(\xi_2, \sigma|-\xi_2)\right]\right.$$

$$\left. - \psi^1(\xi_2, \sigma|\xi_2) + \psi^1(\xi_2, \sigma|-\xi_2)\right\} . \qquad (3.187)$$

Here, from (3.171, 172),

$$\pi^1(\xi_2, \sigma|\pm\xi_2) = \left|\frac{\ell_1}{\ell_2}\right|^{\frac{1}{2}}\pi^2(\xi_2, \sigma|\pm\xi_2) = \tfrac{1}{2}D_{-\sigma_1}(\xi_2^1)\left[\frac{\pi_1^2(\xi_2)}{F} \pm \frac{\pi_2^2(\xi_2)}{G}\right] \qquad (3.188)$$

$$\psi^1(\xi_2, \sigma|\pm\xi_2) = -\left|\frac{\ell_1}{\ell_2}\right|^{\frac{1}{2}}D^*\pi^2(\xi_2, \sigma|\pm\xi_2) = -\tfrac{1}{2}\sigma_1 D_{-\sigma_1-1}(\xi_2^1)\left[\frac{\pi_1^2(\xi_2)}{F} \pm \frac{\pi_2^2(\xi_2)}{G}\right]$$

(3.189)

$$D^*\pi^1(\xi_2, \sigma|\pm\xi_2) = -(\mathrm{sgn}\,\ell_2)\left|\frac{\ell_1}{\ell_2}\right|^{\frac{1}{2}}\psi^2(\xi_2, \sigma|\pm\xi_2) = -\tfrac{1}{2}D_{-\sigma_1}(\xi_2^1)\left[\frac{\psi_1(\xi_2)}{F} \pm \frac{\psi_2^2(\xi_2)}{G}\right]$$

(3.190)

$$D^*\psi^1(\xi_2, \sigma|\pm\xi_2) = (\mathrm{sgn}\,\ell_2)\left|\frac{\ell_1}{\ell_2}\right|^{\frac{1}{2}}D^*\psi^2(\xi_2, \sigma|\pm\xi_2) = \tfrac{1}{2}\sigma_1 D_{-\sigma_1-1}(\xi_2^1)\left[\frac{\psi_1^2(\xi_2)}{F} \pm \frac{\psi_2^2(\xi_2)}{G}\right].$$

(3.191)

Substituting into (3.64), the following expression is obtained for r(s):

$$r(s) = \frac{D}{\ell_1^2}\left[e^{\zeta_2^1}\int_{\xi_2^1}^{\infty} e^{-\zeta^1}\,d\xi^1 + (\operatorname{sgn}\ell_2)\left|\frac{\ell_1}{\ell_2}\right|^{\frac{1}{2}} e^{\zeta_2^2}\int_0^{\xi_2^2} e^{-\zeta^2}\,d\xi^2\right]^{-1}$$

$$\times\left\{\frac{1}{1+\sigma_1}\left[e^{\zeta_2^1}\int_{\xi_2^1}^{\infty} e^{-\zeta^1}\,d\xi^1 + \xi_2^1 - \varkappa\right] + \frac{\operatorname{sgn}\ell_2}{1+\sigma_2}\left|\frac{\ell_1}{\ell_2}\right|^{\frac{5}{2}}\left[e^{\zeta_2^2}\int_0^{\xi_2^2} e^{-\zeta^2}\,d\xi^2 - \xi_2^2\right]\right.$$

$$-\frac{\varkappa}{\sigma_1}\left[1 - \varkappa e^{\zeta_2^1}\int_{\xi_2^1}^{\infty} e^{-\zeta^1}\,d\xi^1\right]$$

$$-\left[\left(\frac{1}{1+\sigma_1}\right)^2\left\{\left[\xi_2^1 - \varkappa\left(1+\frac{1}{\sigma_1}\right)\right]\frac{\sigma_1 D_{-\sigma_1-1}(\xi_2^1)}{D_{-\sigma_1}(\xi_2^1)} + 1\right\}\left\{\left[\xi_2^2 - \left|\frac{\ell_1}{\ell_2}\right|^{\frac{1}{2}}\frac{\varkappa}{\sigma_2}\right]\frac{\psi_2^2(\xi_2)}{\pi_2^2(\xi_2)} - 1\right\}\right.$$

$$-2\,\frac{\ell_1}{\ell_2}\left(\frac{1}{1+\sigma_1}\right)\left(\frac{1}{1+\sigma_2}\right)\left\{\left[\xi_2^1 - \varkappa\left(1+\frac{1}{\sigma_1}\right)\right]\frac{\sigma_1 D_{-\sigma_1-1}(\xi_2^1)}{D_{-\sigma_1}(\xi_2^1)} + 1\right\}\left\{\xi_2^2\,\frac{\psi_2^2(\xi_2)}{\pi_2^2(\xi_2)} - 1\right\}$$

$$\left.+\left(\frac{\ell_1}{\ell_2}\right)^2\left(\frac{1}{1+\sigma_2}\right)^2\left\{\left(\xi_2^1 - \varkappa\right)\frac{\sigma_1 D_{-\sigma_1-1}(\xi_2^1)}{D_{-\sigma_1}(\xi_2^1)} + 1\right\}\left\{\xi_2^2\,\frac{\psi_2^2(\xi_2)}{\pi_2^2(\xi_2)} - 1\right\}\right]$$

$$\left.\div\left[\frac{\sigma_1 D_{-\sigma_1-1}(\xi_2^1)}{D_{-\sigma_1}(\xi_2^1)} + (\operatorname{sgn}\ell_2)\left|\frac{\ell_2}{\ell_1}\right|^{\frac{1}{2}}\frac{\psi_2^2(\xi_2)}{\pi_2^2(\xi_2)}\right]\right\}. \qquad (3.192)$$

Spectral density

From (3.192), $\Phi(\omega)$ is obtained using (2.64). Thus

$$\Phi(\omega) = \frac{2D}{\pi}\left[e^{\zeta_2^1}\int_{\xi_2^1}^{\infty} e^{-\zeta^1} d\xi^1 + (\mathrm{sgn}\,\ell_2)\left|\frac{\ell_1}{\ell_2}\right|^{\frac{1}{2}} e^{\zeta_2^2}\int_0^{\xi_2^2} e^{-\zeta^2} d\xi^2\right]^{-1}$$

$$\times\left[\frac{1}{\ell_1^2+\omega^2}\left[e^{\zeta_2^1}\int_{\xi_2^1}^{\infty} e^{-\zeta^1} d\xi^1 + \xi_2^1 - \varkappa\right] + \frac{\mathrm{sgn}\,\ell_2}{\ell_2^2+\omega^2}\left|\frac{\ell_1}{\ell_2}\right|^{\frac{1}{2}}\left[e^{\zeta_2^2}\int_0^{\xi_2^2} e^{-\zeta^2} d\xi^2 - \xi_2^2\right]\right.$$

$$- \mathrm{Re}\left\{\left[\left(\frac{1}{\ell_1+i\omega}\right)^2\left\{\left[\xi_2^1 - \varkappa\left(1+\frac{\ell_1}{i\omega}\right)\right]A+1\right\}\left\{\left[\xi_2^2 - (\mathrm{sgn}\,\ell_2)|\ell_1\ell_2|^{\frac{1}{2}}\frac{\varkappa}{i\omega}\right]B-1\right\}\right.\right.$$

$$- 2\left(\frac{1}{\ell_1+i\omega}\right)\left(\frac{1}{\ell_2+i\omega}\right)\left\{\left[\xi_2^1 - \varkappa\left(1+\frac{\ell_1}{i\omega}\right)\right]A+1\right\}\left(\xi_2^2 B-1\right)$$

$$\left.\left.\left. + \left(\frac{1}{\ell_2+i\omega}\right)^2\left[\left(\xi_2^1-\varkappa\right)A+1\right]\left(\xi_2^2 B-1\right)\right] \div \left[A+(\mathrm{sgn}\,\ell_2)\left|\frac{\ell_2}{\ell_1}\right|^{\frac{1}{2}} B\right]\right\}\right], \tag{3.193}$$

where

$$A = \frac{i\omega D_{-\frac{i\omega}{\ell_1}-1}(\xi_2^1)}{\ell_1 D_{-\frac{i\omega}{\ell_1}}(\xi_2^1)} \tag{3.194}$$

$$B = \frac{(\operatorname{sgn} \ell_2) M\left(\frac{1}{2} + \frac{i\omega}{2\ell_2}, \frac{1}{2}, \zeta_2^2\right)}{\xi_2^2 M\left(\frac{1}{2} + \frac{i\omega}{2\ell_2}, \frac{3}{2}, \zeta_2^2\right)} \quad . \tag{3.195}$$

Variance

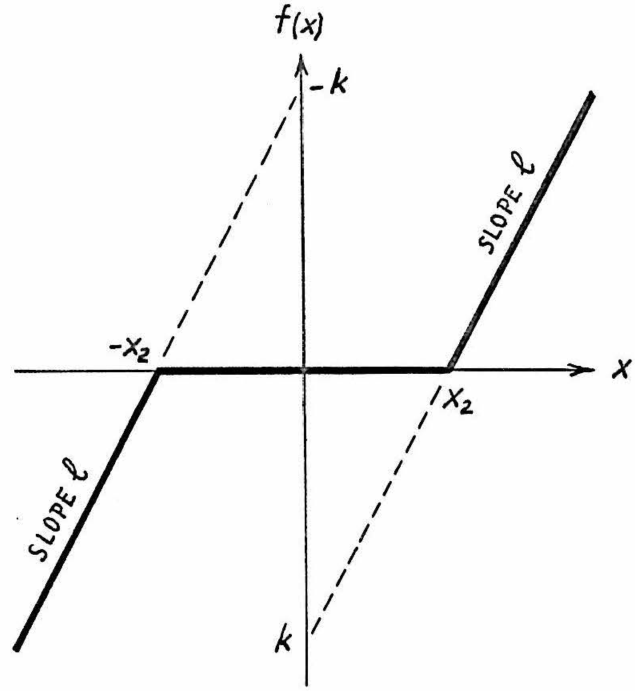
From (3.69),

$$\langle x^2 \rangle = \frac{D}{\ell_1} \; \frac{e^{\zeta_2^1} \int_{\xi_2^1}^{\infty} e^{-\zeta^1} d\xi^1 (1+\kappa^2) + \left|\frac{\ell_1}{\ell_2}\right|^{\frac{3}{2}} e^{\zeta_2^2} \int_0^{\xi_2^2} e^{-\zeta^2} d\xi^2 + \xi_2^1 - 2\kappa - \left|\frac{\ell_1}{\ell_2}\right|^{\frac{3}{2}} \xi_2^2}{e^{\zeta_2^1} \int_{\xi_2^1}^{\infty} e^{-\zeta^1} d\xi^1 + (\operatorname{sgn} \ell_2) \left|\frac{\ell_1}{\ell_2}\right|^{\frac{1}{2}} e^{\zeta_2^2} \int_0^{\xi_2^2} e^{-\zeta^2} d\xi^2} \; . \tag{3.196}$$

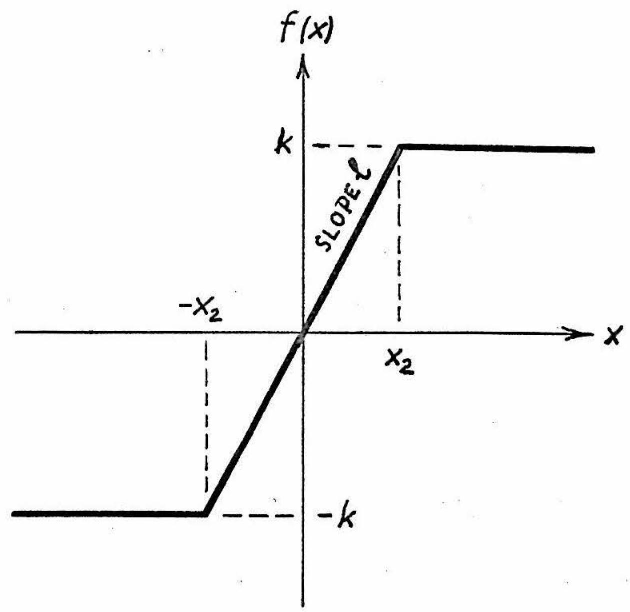
Special cases

A check of the above formulas is obtained by noting that two of the cases dealt with in previous sections can be obtained by limiting processes on the symmetric three-interval case. The hard limiter device (example (3), section 3.3) is obtained by letting $\ell_1 \to \infty$, noting that then $A \to 0$. The friction-type device (example (2) of section 3.4) is obtained by putting $x_2 = 0$.

Two other limiting cases, those with continuous characteristic and ℓ_2 and ℓ_1, respectively, zero, will now be worked out. The restoring force f(x) for those cases is illustrated on the next page.



Section 3.5, example 1 — slack spring



Example 2 — soft limiter

<u>Example (1) -- slack spring</u>

Put $\ell_2 = 0$, $k = -\ell_1 x_2$. Write $\ell_1 = \ell$. Since $k = -\ell x_2$, $\xi_2^1 = 0$ and $A = \dfrac{\sqrt{2}\,\Gamma(\frac{1}{2}\sigma_1 + \frac{1}{2})}{\Gamma(\frac{1}{2}\sigma_1)}$. In the second interval, $(-x_2, x_2)$, (3.121) and (3.122)apply. Thus

$$r(s) = \frac{D}{\ell^2}\left[\frac{1}{1+\sigma} + \frac{1}{\varkappa - (\frac{\pi}{2})^{\frac{1}{2}}}\left\{\frac{\varkappa}{\sigma}\left[1 - (\frac{\pi}{2})^{\frac{1}{2}}\varkappa + \frac{1}{3}\varkappa^2 - \frac{1}{\sigma(1+\sigma)}\right]\right.\right.$$

$$\left.\left. - \left[\left(\frac{1}{1+\sigma}\right)^2 + \frac{2}{\sigma(1+\sigma)} - \frac{1}{\sigma^2}\right] \div \left[\frac{\sqrt{\sigma}}{\tanh(\sqrt{\sigma}\,\varkappa)} - \frac{\sqrt{2}\,\Gamma(\frac{1}{2}\sigma + \frac{1}{2})}{\Gamma(\frac{1}{2}\sigma)}\right]\right\}\right] \tag{3.197}$$

$$\Phi(\omega) = \frac{2D}{\pi}\left\{\frac{1}{\ell^2 + \omega^2} + \frac{1}{\varkappa - (\frac{\pi}{2})^{\frac{1}{2}}}\left[\frac{\ell^2}{\omega^2(\ell^2 + \omega^2)}\right.\right.$$

$$\left.\left. - \mathrm{Re}\left\{\left[\left(\frac{1}{\ell + i\omega}\right)^2 + \frac{2}{i\omega(\ell + i\omega)} + \frac{1}{\omega^2}\right] \div \left[\frac{(\frac{i\omega}{\ell})^{\frac{1}{2}}}{\tanh\left\{(\frac{i\omega}{\ell})^{\frac{1}{2}}\varkappa\right\}} - \frac{\sqrt{2}\,\Gamma(\frac{i\omega}{2\ell} + \frac{1}{2})}{\Gamma(\frac{i\omega}{2\ell})}\right]\right\}\right]\right\} \tag{3.198}$$

$$\langle x^2 \rangle = \frac{D}{\ell}\;\frac{\varkappa(1 + \frac{1}{3}\varkappa^2) - (\frac{\pi}{2})^{\frac{1}{2}}(1 + \varkappa^2)}{\varkappa - (\frac{\pi}{2})^{\frac{1}{2}}}\quad . \tag{3.199}$$

As $\ell \to \infty$ ($\varkappa \to -\infty$), these reduce to (3.125-127). In Fig. (4) on page 120, $\Phi^*(\omega^*)$ is plotted for various values of $\varkappa$. (Note that $\varkappa$ in this figure is equivalent to $-\varkappa$ here.)

Example (2) -- soft limiter

Put $\ell_1 = 0$, $k = \ell_2 x_2$. Write $\ell_2 = \ell$. Then $k = \ell x_2$, and, in the interval (x_2, ∞), (3.131) and (3.133) hold. Thus

$$r(s) = \frac{D}{\ell^2}\left[1 + \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \xi_2^2 e^{\zeta_2^2} \operatorname{erf}\frac{\xi_2^2}{\sqrt{2}}\right]^{-1} \left[\frac{1}{1+\sigma}\left\{\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \xi_2^2 e^{\zeta_2^2} \operatorname{erf}\frac{\xi_2^2}{\sqrt{2}} - 2\zeta_2^2\right\}\right.$$
$$+ \frac{1}{\sigma}\left(2\zeta_2^2 + 2 + \frac{1}{\zeta_2^2}\right) - \frac{1}{\sigma^2}(1+2\zeta_2^2) - 2\zeta_2^2 \left\{\frac{1}{\sigma^2}\left[(1-\frac{1}{\sigma})A-1\right]\left[(1-\frac{1}{\sigma})B-1\right]\right.$$
$$\left.\left. - \frac{2}{\sigma}\left(\frac{1}{1+\sigma}\right)\left[(1-\frac{1}{\sigma})A-1\right](B-1) + \left(\frac{1}{1+\sigma}\right)^2 (A-1)(B-1)\right\} \div (A-B)\right], \tag{3.200}$$

where

$$A = \zeta_2^2\left[1 - \left(1 + \frac{4sD}{k^2}\right)^{\frac{1}{2}}\right], \quad B = \frac{M(\frac{1}{2}+\frac{1}{2}\sigma, \frac{1}{2}, \zeta_2^2)}{M(\frac{1}{2}+\frac{1}{2}\sigma, \frac{3}{2}, \zeta_2^2)}; \tag{3.201}$$

$$\Phi(\omega) = \frac{2D}{\pi}\left[1 + \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \xi_2^2 e^{\zeta_2^2} \operatorname{erf}\frac{\xi_2^2}{\sqrt{2}}\right]^{-1} \left[\frac{1}{\ell^2+\omega^2}\left\{\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \xi_2^2 e^{\zeta_2^2} \operatorname{erf}\frac{\xi_2^2}{\sqrt{2}} - 2\zeta_2^2\right\}\right.$$
$$+ \frac{1}{\omega^2}(1+2\zeta_2^2) + 2\zeta_2^2 \operatorname{Re}\left\{\frac{1}{\omega^2}\left[(1+i\frac{\ell}{\omega})A-1\right]\left[(1+i\frac{\ell}{\omega})B-1\right]\right.$$
$$\left.\left. + \frac{2i}{\omega}\left(\frac{1}{\ell+i\omega}\right)\left[(1+i\frac{\ell}{\omega})A-1\right](B-1) + \left(\frac{1}{\ell+i\omega}\right)^2 (A-1)(B-1)\right\} \div (A-B)\right], \tag{3.202}$$

where A and B are as in (3.201), but with σ replaced by $\frac{i\omega}{\ell}$; and

$$\langle x^2 \rangle = \frac{D}{\ell}\,\frac{2 + \frac{1}{\zeta_2^2} + \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \xi_2^2 e^{\zeta_2^2} \operatorname{erf}\frac{\xi_2^2}{\sqrt{2}}}{1 + \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \xi_2^2 e^{\zeta_2^2} \operatorname{erf}\frac{\xi_2}{\sqrt{2}}}. \tag{3.203}$$

As $\ell \to \infty$, k remaining finite, (i.e., $\xi_2^2 \to \infty$), (3.200, 202, 203) reduce to (3.165-67). In Fig. (5) on page 121, $\Phi^*(\omega^*)$ is plotted for various values of ξ_2^2.

Computation of $\Phi(\omega)$

The formulas obtained in this chapter were found suitable for computation with little modification. Confluent hypergeometric functions and error functions can be evaluated using their rapidly convergent power series. The gamma functions $\Gamma(ix)$ and $\Gamma(\frac{1}{2}+ix)$ are best evaluated by adding a suitable positive integer n to the argument, finding the gamma function of the new argument by Sterling's asymptotic series for log $\Gamma(z)$, then using $\Gamma(z) = z^{-1}\Gamma(z+1)$ n times. Choose n the smallest integer which gives sufficient accuracy to Sterling's series--n = 4 gives nine figures. The results given in Figs. (1)-(5), (8) at the end of this chapter were obtained in this manner on the IBM 7094/7040 at California Institute of Technology.

3.6 APPROXIMATION BY EQUIVALENT PIECEWISE LINEARIZATION

The method of equivalent linearization is a useful method of approximating the properties of a nonlinear system. It gives good results for the first and second moments, but no indication of variation in the shape of the autocorrelation and spectral density curves for various nonlinear terms. A more accurate method of approximation would be to replace the nonlinear system, not by a linear system, but by a piecewise linear system optimized in the same way.

The method is discussed somewhat sketchily in this section, and an example worked out. It should be possible to extend it, for example, to parametrically excited systems. It would be interesting to find whether the autocorrelation or spectrum of an arbitrary nonlinear system can be approximated arbitrarily closely (in some sense) by that of a piecewise linear system with a sufficient number of segments.

Equivalent linearization[1]

We consider the application of this method to the first order nonlinear system

$$\dot{x} + f(x) = n(t) \,, \quad -\infty < x < \infty \,. \tag{3.204}$$

It is replaced by the linear system

$$\dot{x} + \ell_{eq}(x - \langle x \rangle) = n(t) \,, \quad -\infty < x < \infty \,, \tag{3.205}$$

where ℓ_{eq} is chosen so as to minimize

$$E\left[\left\{f(x) - \ell_{eq}(x - \langle x \rangle)\right\}\right] \,. \tag{3.206}$$

This gives

$$\ell_{eq} = \frac{\langle x f(x) \rangle - \langle x \rangle \langle f(x) \rangle}{\langle x^2 \rangle - \langle x \rangle^2} \tag{3.207}$$

$$= \frac{D}{\langle x^2 \rangle - \langle x \rangle^2} \,. \tag{3.208}$$

Thus the mean and mean square are the same for (3.204) and the equivalent system (3.205).

---

[1] Caughey in [3] reviews this method.

If the range of x in (3.204) is less than $(-\infty, \infty)$, (3.207) and (3.208) are not equal. It can be shown that (3.208) gives a more reasonable approximation. For example, if $f(x) = 0$, $-x_1 < x < x_1$, (the special case of example (3), section 3.3), then (3.207) gives $\ell_{eq} = 0$, while (3.208) gives a positive value for ℓ_{eq} which is the limit as $\ell \to \infty$ of that for example (1), section 3.5.

Since the steady-state density corresponding to (3.204) usually involves integrals which cannot be evaluated, the expectations in (3.208) are often evaluated using the equivalent density. This introduces small discrepancies between the actual and equivalent linear mean and mean square. The equation (3.208) will involve ℓ_{eq} on both sides, and must usually be solved by some approximation technique.

The piecewise linear approximation

We proceed in the same manner as above. In the system

$$\dot{x} + f(x) = n(t) \quad , \qquad x \in (x_n, x_1) \quad , \tag{3.209}$$

nonlinear f(x) is to be replaced by $\ell_i x + k_i$, $x \in (x_{i+1}, x_i)$, where $i = 1, 2, \ldots, n-1$, where the 3n-4 unspecified variables ℓ_i, k_i, x_i are to be determined by minimizing

$$I = E\left[\left\{f(x) - \ell_i x - k_i\right\}^2\right] = \sum_{i=1}^{n-1} F_i\left\{\left[f(x) - \ell_i x - k_i\right]^2\right\} , \tag{3.210}$$

where

$$F_k(z) = \int_{x_{k+1}}^{x_k} z \, P_o(x) \, dx \quad , \tag{3.211}$$

$P_o(x)$ being the steady-state density corresponding to (3.209). To minimize I, it is necessary that

$$\frac{\partial I}{\partial \ell_i} = \frac{\partial I}{\partial k_i} = 0 \ , \quad i = 1, 2, \ldots, n-1; \quad \frac{\partial I}{\partial x_i} = 0 \ , \quad i = 2, 3, \ldots, n-1; \tag{3.212}$$

whence

$$\ell_i = \frac{F_i(1)F_i[xf(x)] - F_i(x)F_i[f(x)]}{F_i(1)F_i(x^2) - [F_i(x)]^2} \tag{3.213}$$

$$k_i = \frac{F_i(x)F_i[xf(x)] - F_i(x^2)F_i[f(x)]}{F_i(1)F_i(x^2) - [F_i(x)]^2} \tag{3.214}$$

$$f(x_i-) - \ell_i x_i - k_i = \pm \left[f(x_i+) - \ell_{i-1} x_i - k_{i-1} \right] \ . \tag{3.215}$$

<u>The ambiguity in (3.215)</u>

If f(x) is continuous, the positive sign in (3.215) implies the continuity of the equivalent restoring force $\ell x + k$; the negative sign implies that $\ell_i x_i + k_i$ is as far above $f(x_i)$ as $\ell_{i-1} x_i + k_{i-1}$ is below it (or vice versa). To find which sign represents a true minimum of I, it would be necessary to examine the second derivatives of I. In some cases it is apparent which is to be used. For example, if $f(x) = x^3$, the best two-piece approximation (which must be symmetrical) has $x_2 = 0$, $k_2 = -k_1 > 0$ (i.e., the negative sign in (3.215)), while the best three-piece approximation has both $\ell_1 x_1 + k_1 < f(x_1)$, $\ell_2 x_1 + k_2 < f(x_1)$ (and thus the positive sign in (3.215)). For simplicity from now on, the positive sign will be assumed, and continuous f(x), so that (3.215) is

$$x_i = -\frac{k_i - k_{i-1}}{\ell_i - \ell_{i-1}} \quad . \tag{3.216}$$

Approximating the expectations

Since $P_o(x)$ is in general not known explicitly, it will be replaced by the steady-state density corresponding to the equivalent system. Doing this, $F_i(\cdot)$ in (3.213, 214) becomes $G_i(\cdot)$, where

$$G_i(z) = \int_{x_{i+1}}^{x_i} z\, e^{-\zeta^i} dx \ , \tag{3.217}$$

ζ^i being given by (3.16). Then $G_i(1)$, $G_i(x)$, $G_i(x^2)$ can be expressed analytically, but $G_i[f(x)]$, $G_i[xf(x)]$ must in general be evaluated numerically. Exceptional cases occur when f(x) is a polynomial in x or in e^x (e.g., sinh x).

Obtaining ℓ_i, k_i, x_i

Substitute for x_i in (3.213, 214) using (3.216). Then the right-hand sides of (3.213, 214) will be fairly complicated functions of ℓ_j, k_j $(j = i, i\pm1)$. Even with the replacement of F_i by G_i, it is not to be expected that these equations can be solved explicitly, and a numerical method is indicated. Since the right-hand sides appear to vary slowly with variation of the ℓ_j and k_j, iteration should converge. The method is to choose a piecewise linear approximation to f(x) by eye, substitute into the right-hand sides of (3.213, 214) to obtain improved values, and repeat until no further improvement is obtained.

<u>Example -- $f(x) = x^3$, $D = 1$ -- three piece approximation</u>

By symmetry, $\ell_3 = \ell_1$, $k_3 = -k_1 = -k$ say, $k_2 = 0$, $G_2(x) = G_2[f(x)] = 0$. By (3.216), $x_2 = \frac{k}{\ell_2 - \ell_1}$. Thus, once ℓ_1, ℓ_2, k are evaluated, the equivalent system can be treated as in section 3.5, and the spectral density obtained by (3.193).

From Fig. (6) on page 122, it is seen that $\ell_1 = 6$, $\ell_2 = 1$, $k = -5$ (and so $x_2 = 1$) gives a good first approximation to x^3. Substituting into (3.213, 214) (with F_i replaced by G_i) and using symmetry, higher approximations can be obtained. Thus one gets:

Approx. no.	ℓ_1	ℓ_2	k	x_2
1	6.0	1.0	-5.0	1.0
2	5.926	.5651	-5.457	1.0179
3	6.3712	.59862	-5.8788	1.01840

It is seen that the iteration is converging. In Fig. (6) these successive approximations to the forcing function are shown, together with the corresponding steady-state probabilities; the values corresponding to the exact and the equivalent linear systems are shown for comparison in Fig. (7) on page 123. In Fig. (8), page 124, the spectral density for the best three-piece approximation is compared to that of the equivalent linear system.

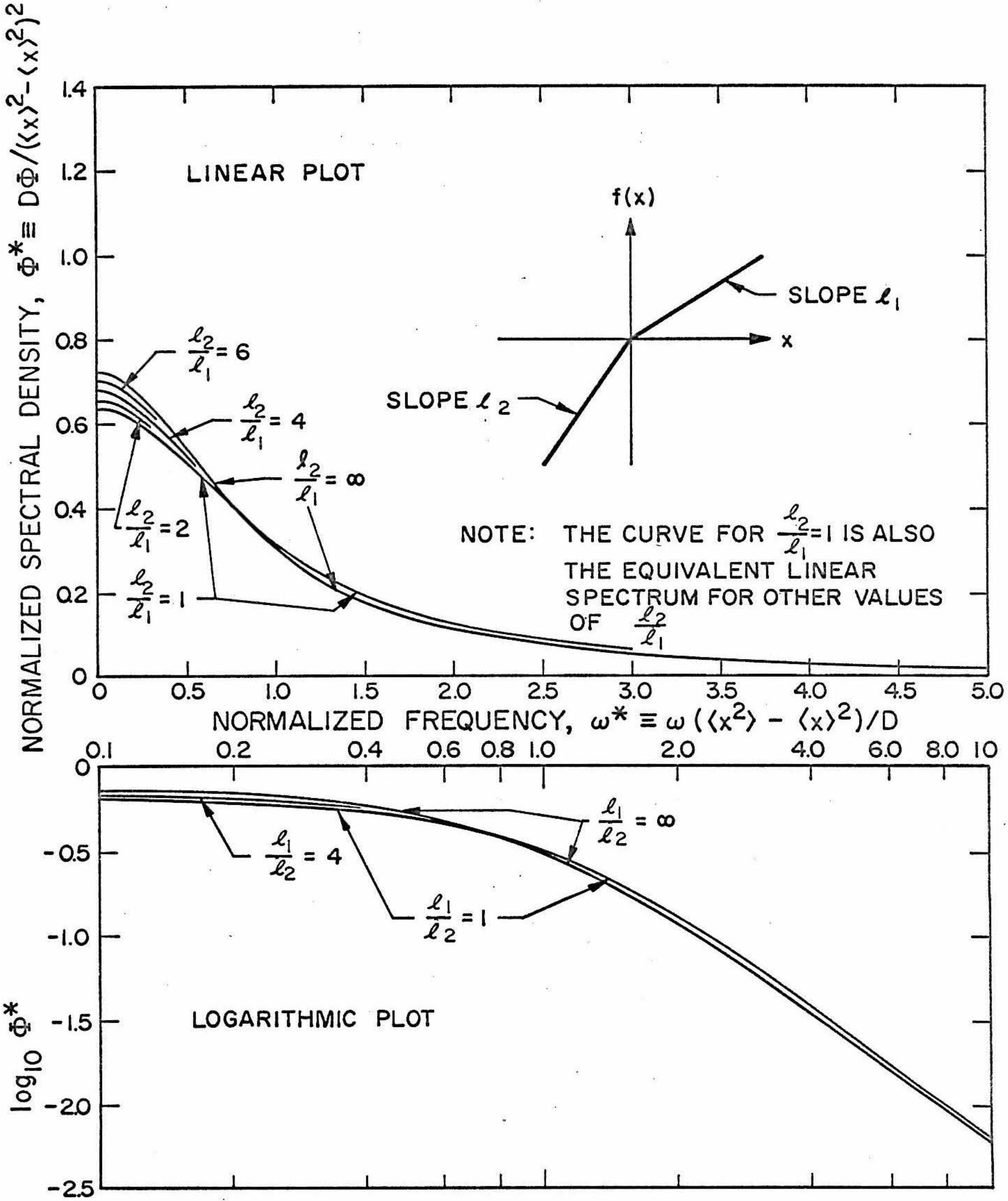


Fig. 1. Normalized spectral density of system with continuous bilinear restoring force f(x) — example 1, section 3.4.

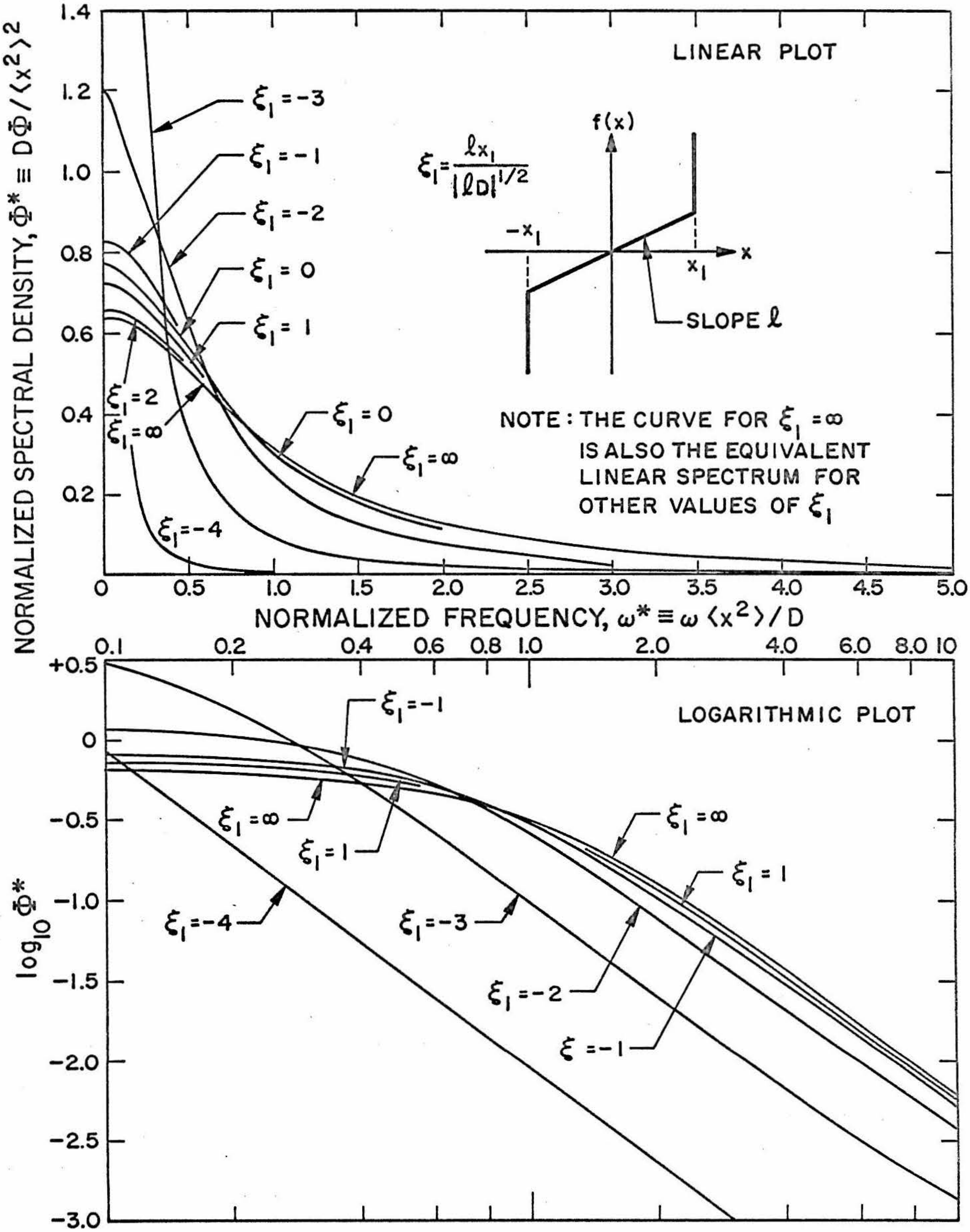


Fig. 2. Normalized spectral density of system with hard limiter type restoring force f(x) — example 3, section 3.3.

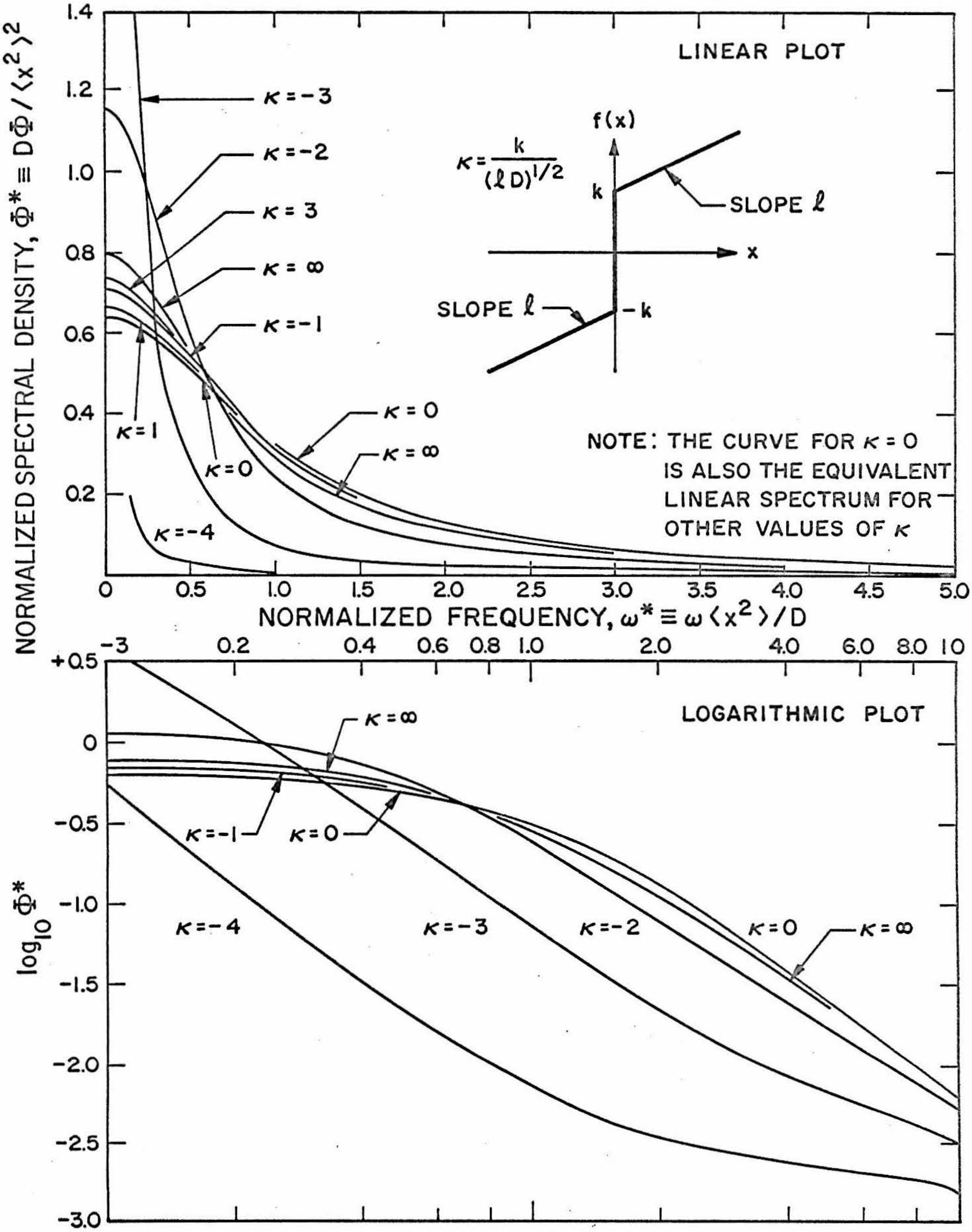


Fig. 3. Normalized spectral density of system with preloaded spring type restoring force f(x) — example 2, section 3.4.

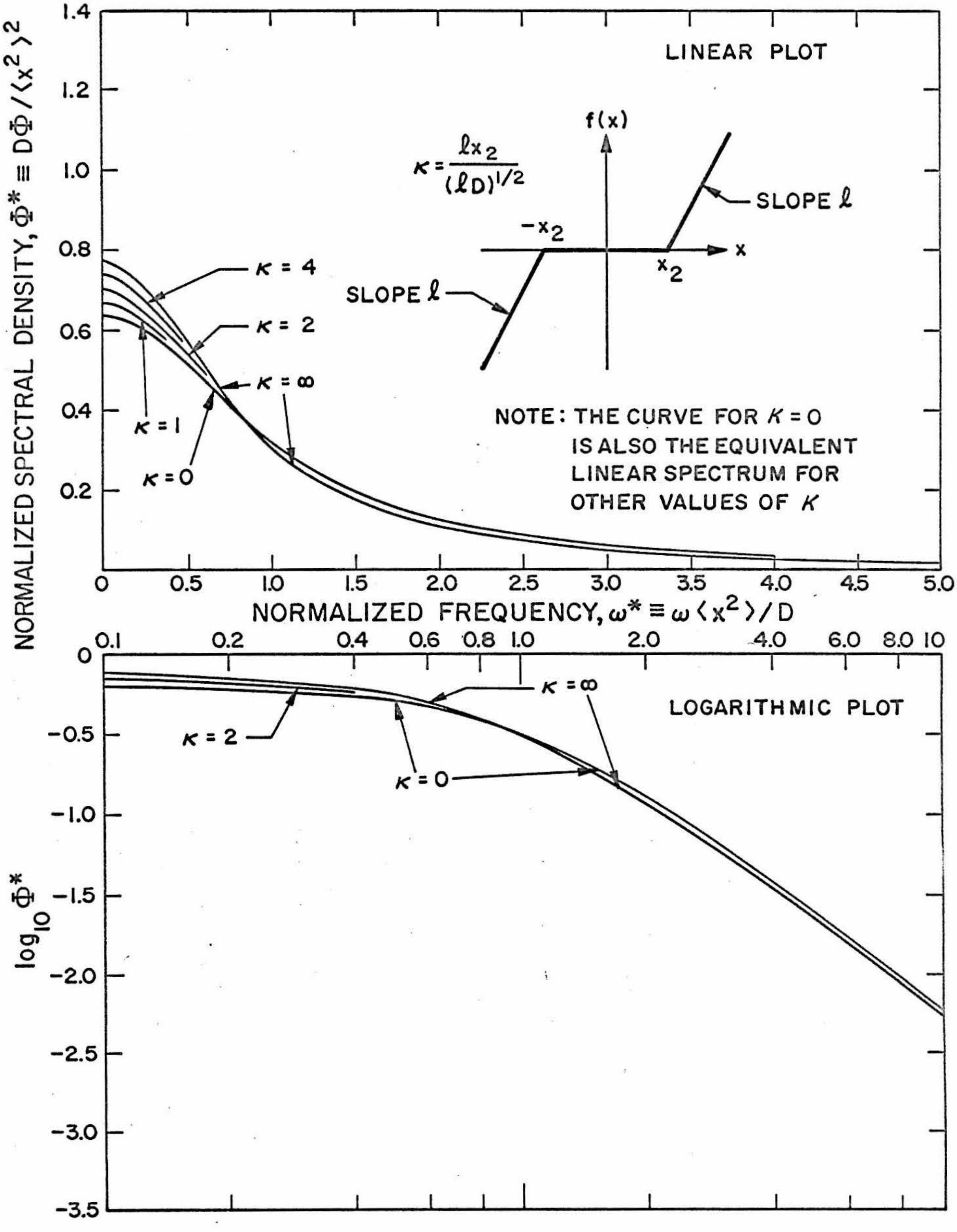


Fig. 4. Normalized spectral density of system with slack spring type restoring force f(x) — example 1, section 3.5.

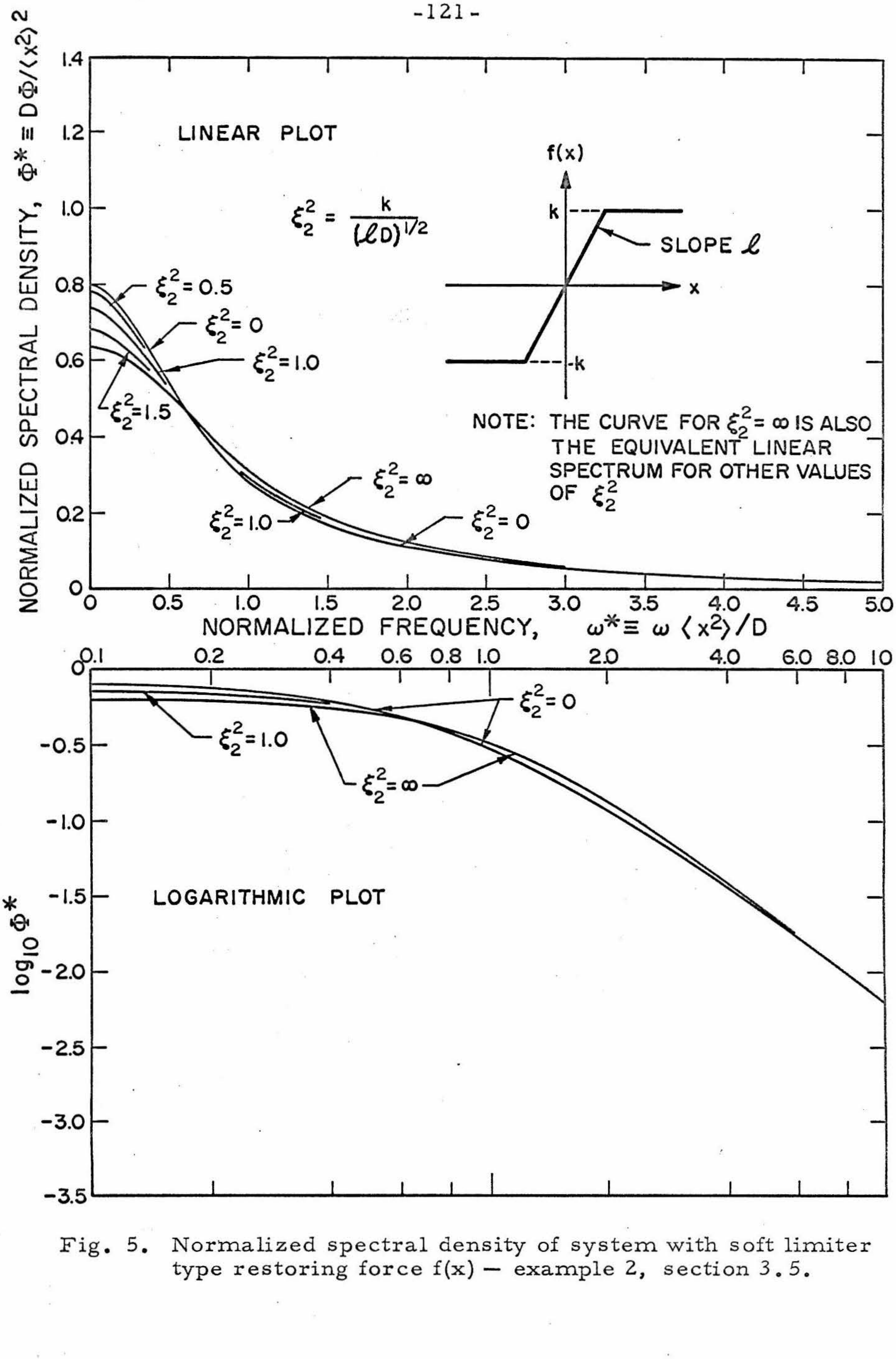


Fig. 5. Normalized spectral density of system with soft limiter type restoring force f(x) — example 2, section 3.5.

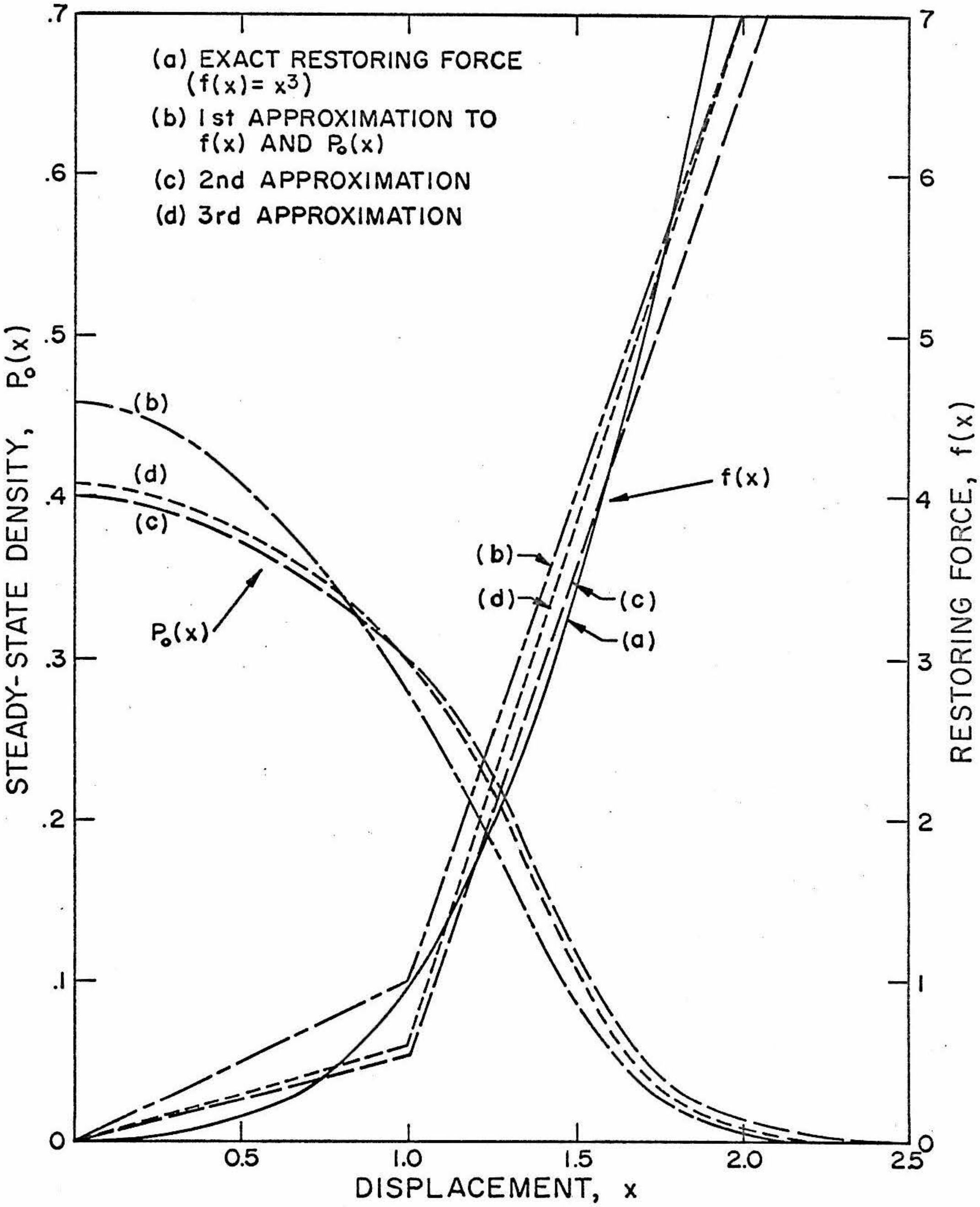


Fig. 6. Successive approximations to the equivalent 3-piece restoring force for the system $\frac{dx}{dt} + x^3 = n(t)$, $\langle n(t_1)n(t_2)\rangle = 2\,\delta(t_1 - t_2)$, and the corresponding steady-state densities.

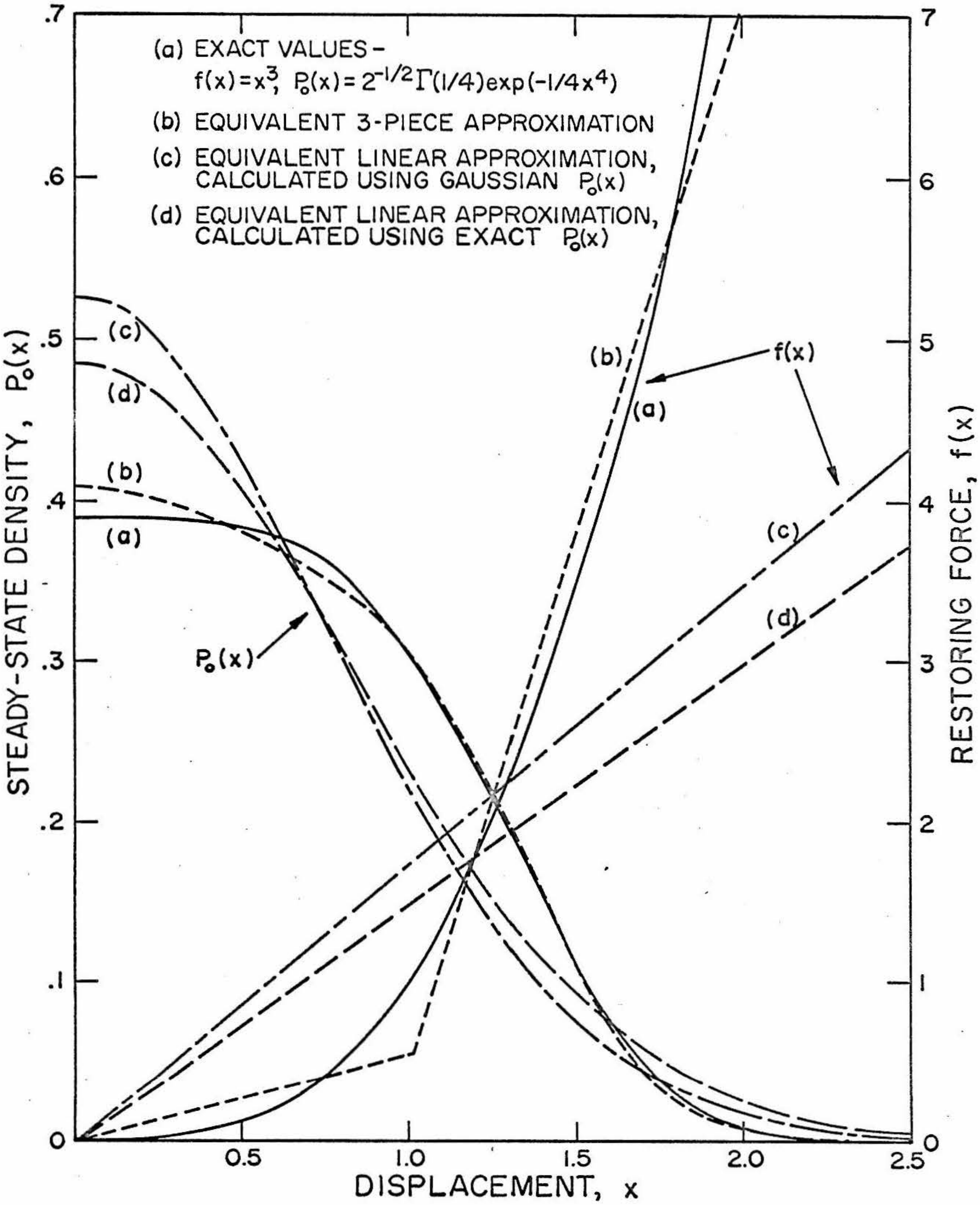


Fig. 7. Comparison of the exact restoring force and steady-state density for the system $\frac{dx}{dt} + x^3 = n(t)$, $\langle n(t_1)n(t_2)\rangle = 2\delta(t_1-t_2)$, with the equivalent 3-piece and equivalent linear approximations.

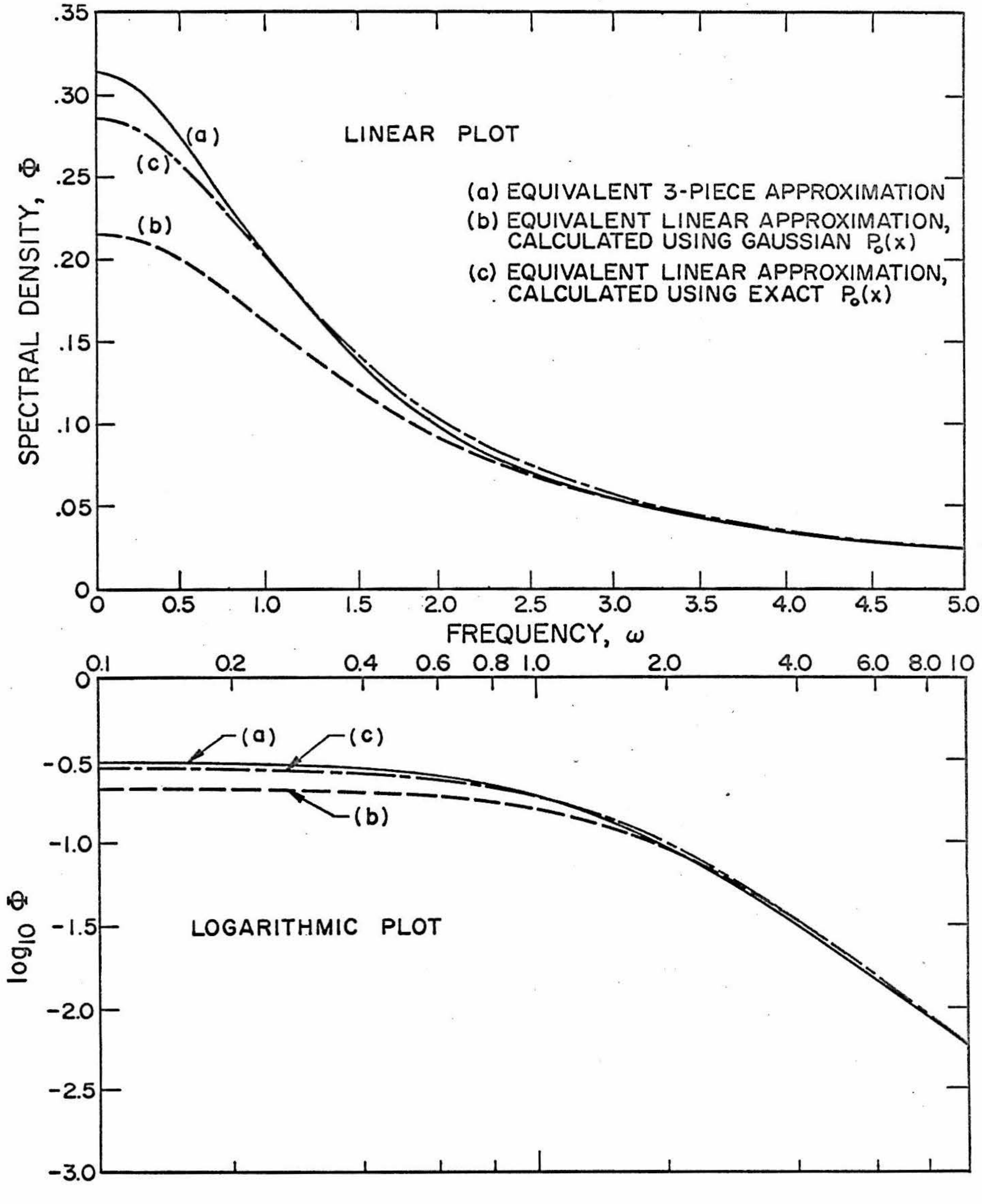


Fig. 8. The various approximations to the spectral density of the system $\frac{dx}{dt} + x^3 = n(t)$, $\langle n(t_1)n(t_2)\rangle = 2\delta(t_1 - t_2)$.

CHAPTER IV

SYSTEMS WITH WHITE PARAMETRIC EXCITATION

4.1 THE GENERAL CASE

In this chapter we consider the first order piecewise linear system with parametric excitation consisting of a single white noise function, and with no forcing function excitation as in Chapter III. Such systems lead to a FP equation whose solution does not involve special functions, so that the formulas for spectral density, etc., are quite simple. On the other hand, interesting combinations of irregular points can occur in these systems. Previous work on such systems appears to have been limited to the linear case.[1]

The stochastic differential equation

For the system dealt with in this chapter, (0.1) can be written

$$\dot{x} + f(x)[1 + m(t)] = 0 \qquad (4.1)$$

(i.e. $m = 1$ and $h_1(x) = -f(x)$), where

$$f(x) = k_i x + \ell_i \text{ for } x \in (x_{i+1}, x_i), \quad i = 1, 2, \ldots, n-1 \qquad (4.2)$$

(so that there are $n - 1$ linear segments), and $m(t)$ is white noise satisfying

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[1] For example in Gray [22], section 2.

$$\langle m(t_1)m(t_2)\rangle = 2D\,\delta(t_1 - t_2)\ , \quad \langle m(t)\rangle = 0\ . \tag{4.3}$$

It will be assumed throughout the general derivations below that no $\ell_i = 0$. If some ℓ_i were zero, the power type solutions of the FP equation found below would be replaced by exponentials in the ith interval; it is easily seen that these are the limits as $\ell_i \to 0$ of the case where $\ell_i \neq 0$. The following two dimensionless quantities will be useful:

$$\lambda_i = \frac{1}{2\ell_i D} \tag{4.4}$$

$$\sigma_i = \frac{s}{\ell_i^2 D}\ . \tag{4.5}$$

Suffixes on ℓ, λ, k, σ, etc., will be omitted whenever no confusion would result therefrom.

<u>FP and backwards equations</u>

The Laplace-transformed FP equation (2.14) in this case becomes

$$D\frac{d^2}{dx^2}\left[(\ell x + k)^2 p\right] + (1 - \ell D)\frac{d}{dx}\left[(\ell x + k)p\right] - sp = -\delta(x - x_o)\ , \tag{4.6}$$

while the corresponding backwards equation (2.15) is

$$D(\ell x_o + k)^2\frac{d^2 p}{dx_o^2} - (1 - \ell D)(\ell x_o + k)\frac{dp}{dx_o} - sp = -\delta(x - x_o)\ . \tag{4.7}$$

Thus the diffusion coefficient and the drift coefficient are

$$a(x) = D(\ell x + k)^2\ , \quad b(x) = -(1 - \ell D)(\ell x + k)\ , \tag{4.8}$$

respectively.

Irregular points

These consist of all points where $\ell x + k \to 0$ on one or both sides. From (1.27, 35, 36) one gets, neglecting constant terms which must be added to make W, s, m continuous at x_i,

$$W(x) = |\ell x + k|^{-1+2\lambda} \tag{4.9}$$

$$s(x) = D \operatorname{sgn}(\ell x + k) \, |\ell x + k|^{2\lambda} \tag{4.10}$$

$$m(x) = -\operatorname{sgn}(\ell x + k) \, |\ell x + k|^{-2\lambda} . \tag{4.11}$$

Thus, if $\ell x + k \to 0$ as $x \to z \neq \pm\infty$ on one side,

$$\int m ds = -\frac{1}{\ell} \log |\ell x + k| \to \pm \infty \tag{4.12}$$

$$\int s dm = \frac{1}{\ell} \log |\ell x + k| \to \pm \infty , \tag{4.13}$$

so that, by (1.41), z is a natural boundary on this side. Also, by (1.49, 51), z is attracting if and only if $\ell > 0$. (On the other hand, if $z = \pm\infty$, z is a natural boundary which is attracting if and only if $\ell < 0$; or $\ell = 0$, $k \operatorname{sgn} x < 0$.)

The boundary on the other side of the irregular point z will be regular or natural, depending on whether or not $\ell x + k \to 0$ on this side as well. If it is regular, the appropriate boundary condition here is $Q(z, t|x_o) = 0$, since Q is continuous at $x = z$ (although aP need not be) and $Q \to 0$ at a natural boundary. Since all irregular points have a natural boundary on at least one side, no path can cross from one side to the other, and one can consider these points as dividing the process into independent subprocesses, one on each regular interval, and consider each of them separately. From now on, therefore, it will be assumed that x_1 and x_n are the endpoints of such a regular interval.

A fundamental set of solutions

It is easily seen that a pair of independent real solutions to the homogeneous form of (4.6) are, provided $\ell_i \neq 0$,

$$p_1^i(x) = |\ell_i x + k_i|^{-\beta_1^i} \tag{4.14}$$

$$p_2^i(x) = |\ell_i x + k_i|^{-\beta_2^i}, \tag{4.15}$$

where

$$\beta_{1,2}^i = 1 + \lambda_i \pm (\lambda_i^2 + \sigma_i)^{\frac{1}{2}} . \tag{4.16}$$

If $\ell_i = 0$, a fundamental set of solutions would be

$$p_{1,2}^i = \exp\left\{ - \frac{x}{2k_i D} \left[1 \pm (1 + 4sD)^{\frac{1}{2}} \right] \right\} . \tag{4.17}$$

(However, it is assumed in what follows that this is not the case.)

The following quantities will be required:

$$q_{1,2}(x) = \frac{d}{dx}\left[a(x)p_{1,2}(x) \right] - b(x)p_{1,2}(x) = \left(\frac{\beta_{2,1}-1}{2\lambda} \right) (\ell x+k) \, |\ell x+k|^{-\beta_{1,2}} \tag{4.18}$$

$$w(x) = 2\ell(\lambda^2+\sigma)^{\frac{1}{2}} \, |\ell x+k|^{-4-2\lambda} \, (\ell x+k) \tag{4.19}$$

-- the latter being the Wronskian of p_1 and p_2.

The Laplace-transformed transition density

Putting

$$p(x, s \mid x_o) = c_1^i p_1^i(x) + c_2^i p_2^i(x) \quad , \tag{4.20}$$

the coefficients c_1^i, c_2^i are given by the set of simultaneous equations (2.21, 22, 25, 26), which become, in this case,

$$c_1^{k-} - c_1^{k+} = - \frac{\lambda_k \left| \ell_k x_o + k_k \right|^{-2+\beta_1^k} (\ell_k x_o + k_k)}{\left(\lambda_k^2 + \sigma_k\right)^{\frac{1}{2}}} \tag{4.21}$$

$$c_2^{k-} - c_2^{k+} = \frac{\lambda_k \left| \ell_k x_o + k_k \right|^{-2+\beta_2^k} (\ell_k x_o + k_k)}{\left(\lambda_k^2 + \sigma_k\right)^{\frac{1}{2}}} \tag{4.22}$$

$$c_1^i = \frac{(\beta_1^i - 1)}{2(\lambda_i^2 + \sigma_i)^{\frac{1}{2}}} \left| \ell_i x_i + k_i \right|^{-2+\beta_1^i} \left\{ \left[1 - \frac{\lambda_i (\beta_2^{i-1} - 1)(\ell_i x_i + k_i)}{\lambda_{i-1} (\beta_1^i - 1)(\ell_{i-1} x_i + k_{i-1})} \right] \left| \ell_{i-1} x_i + k_{i-1} \right|^{2-\beta_1^{i-1}} c_1^{i-1} \right.$$
$$\left. + \left[1 - \frac{\lambda_i (\beta_1^{i-1} - 1)(\ell_i x_i + k_i)}{\lambda_{i-1} (\beta_1^i - 1)(\ell_{i-1} x_i + k_{i-1})} \right] \left| \ell_{i-1} x_i + k_{i-1} \right|^{2-\beta_2^{i-1}} c_2^{i-1} \right\}$$

(4.23)

$$c_2^i = \frac{(1 - \beta_2^i)}{2(\lambda_i^2 + \sigma_i)^{\frac{1}{2}}} \left| \ell_i x_i + k_i \right|^{-2+\beta_2^i} \left\{ \left[1 - \frac{\lambda_i (\beta_2^{i-1} - 1)(\ell_i x_i + k_i)}{\lambda_{i-1} (\beta_2^i - 1)(\ell_{i-1} x_i + k_{i-1})} \right] \left| \ell_{i-1} x_i + k_{i-1} \right|^{2-\beta_1^{i-1}} c_1^{i-1} \right.$$
$$\left. + \left[1 - \frac{\lambda_i (\beta_1^{i-1} - 1)(\ell_i x_i + k_i)}{\lambda_{i-1} (\beta_2^i - 1)(\ell_{i-1} x_i + k_{i-1})} \right] \left| \ell_{i-1} x_i + k_{i-1} \right|^{2-\beta_2^{i-1}} c_2^{i-1} \right\} ,$$

(4.24)

for $i = 2, \ldots, n-1$; together with the boundary conditions at x_1 and x_n, which take the following forms: if x_1 is infinite,

$$c_2^1 = 0 \ ; \tag{4.25}$$

if x_1 is finite and $\ell_1 x_1 + k_1 \neq 0$,

$$c_1^1 \left(\beta_2^1 - 1\right) |\ell_1 x_1 + k_1|^{-\beta_1^1} + c_2^1 \left(\beta_1^1 - 1\right) |\ell_1 x_1 + k_1|^{-\beta_2^1} = 0 \ ; \tag{4.26}$$

if $\ell_1 x_1 + k_1 = 0$,

$$c_1^1 = 0 \ . \tag{4.27}$$

The corresponding equations at x_n are

$$c_2^{n-1} = 0 \tag{4.28}$$

$$c_1^{n-1}\left(\beta_2^{n-1} - 1\right) |\ell_{n-1} x_n + k_{n-1}|^{-\beta_1^{n-1}} + c_2^{n-1}\left(\beta_1^{n-1} - 1\right) |\ell_{n-1} x_n - k_{n-1}|^{-\beta_2^{n-1}} = 0 \tag{4.29}$$

$$c_1^{n-1} = 0 \ . \tag{4.30}$$

Steady-state probability

Assume neither boundary is an attracting natural boundary. That is, if either x_1 or x_n is infinite, then ℓ_1 or ℓ_n respectively is positive; if x_1 or x_n is finite, either $\ell x + k \neq 0$ at this point, or ℓ_1 or ℓ_n respectively is negative. For then $m(x_1) - m(x_n) < \infty$. Then, by (2.28),

$$P_o(x) = C_i |\ell_i x + k_i|^{-1-2\lambda_i} \ . \tag{4.31}$$

The continuity of aP_o then gives, for $n-1 \geq i > j \geq 1$,

$$\frac{C_i}{C_j} = |\ell_i x_i + k_i|^{-1+2\lambda_i} \left|\frac{\ell_{i-1}x_i + k_{i-1}}{\ell_{i-1}x_{i-1}+k_{i-1}}\right|^{1-2\lambda_{i-1}} \cdots \left|\frac{\ell_{j+1}x_{j+2}+k_{j+1}}{\ell_{j+1}x_{j+1}+k_{j+1}}\right|^{1-2\lambda_{j+1}} |\ell_j x_{j+1}+k_j|^{1-2\lambda_j} . \tag{4.32}$$

The C_i are thus determined up to an arbitrary constant. To determine this, one uses (2.29), which gives

$$C_j = \left[D \sum_{i=1}^{n-1} \frac{C_i}{C_j} \ell_i \left(|\ell_i x_{i+1} + k_i|^{-2\lambda_i} - |\ell_i x_i + k_i|^{-2\lambda_i} \right) \right]^{-1} , \tag{4.33}$$

where $\frac{C_i}{C_j}$ is given by (4.32).

If either x_1 or x_n is an attracting natural boundary, then $P_o(x)$ is zero for all x interior to (x_1, x_n). If there is only one such boundary, say x_j (j = i or n),

$$\begin{aligned} P_o(x) &= \delta(x-x_j) \qquad \text{for finite } x_j \\ &= 0 \qquad\qquad\quad \text{for infinite } x_j \end{aligned} \tag{4.34}$$

(see 1.66-68). If both boundaries are attracting and finite,

$$P_o(x) = P_o(x|x_o) = \frac{s(x_1)-s(x_o)}{s(x_1)-s(x_n)}\,\delta(x_1-x) + \frac{s(x_o)-s(x_n)}{s(x_1)-s(x_n)}\,\delta(x-x_n) \tag{4.35}$$

(see 1.66, 69, 70); note that by (4.10, 11), $s(x_1)$ and $s(x_n)$ are finite whenever $m(x_1)$ and $m(x_n)$ are infinite--i.e., whenever both boundaries are attracting. If one or both boundaries are infinite, the corresponding terms in (4.35) vanish.

$P_o(x)$ as Abelian limit of $P(x, t \mid x_o)$

As in Chapter III, it can be verified that

$$P_o(x) = \lim_{s \to 0} sp(x_1 s \mid x_o) \quad . \tag{4.36}$$

Suppose, for implicity, that all ℓ_i are positive, and that x_1, x_n are both regular boundaries. Then (4.21-24, 26, 29) become, to first order as $s \to 0$,

$$c_1^{k-} - c_1^{k+} = - \left| \ell_k x_o + k_k \right|^{2\lambda_k - 1} (\ell_k x_o + k_k) \tag{4.37}$$

$$c_2^{k-} - c_2^{k+} = \operatorname{sgn} (\ell_k x_o + k_k) \tag{4.38}$$

$$c_1^i = \left| \ell_i x_i + k_i \right|^{2\lambda_i - 1} \left\{ \left| \ell_{i-1} x_i + k_{i-1} \right|^{1-2\lambda_{i-1}} c_1^{i-1} + \left[1 - \frac{\ell_i x_i + k_i}{\ell_{i-1} x_i + k_{i-1}} \right] \left| \ell_{i-1} x_i + k_{i-1} \right| c_2^{i-1} \right\} \tag{4.39}$$

$$c_2^i = \left| \ell_i x_i + k_i \right|^{-1} \left[1 - \frac{\ell_i x_i + k_i}{\ell_{i-1} x_i + k_{i-1}} \right] \left| \ell_{i-1} x_i + k_{i-1} \right|^{1-2\lambda_{i-1}} sDc_1^{i-1} + \operatorname{sgn}\left(\frac{\ell_i x_i + k_i}{\ell_{i-1} x_i + k_{i-1}} \right) c_2^{i-1} \tag{4.40}$$

$$sD\, c_1^1 = \left| \ell_1 x_1 + k_1 \right|^{2\lambda_1} c_2^1 \tag{4.41}$$

$$sD\, c_1^{n-1} = \left| \ell_{n-1} x_n + k_{n-1} \right|^{2\lambda_{n-1}} c_2^{n-1} \quad . \tag{4.42}$$

From (4.40-42) it is seen that the coefficients c_2^i are an order of magnitude smaller than the coefficients c_1^i. But from (4.38) the c_2^i are at least of order unity. Therefore $c_1^i = O(\frac{1}{s})$ and (4.37, 39) become

$$c_1^{k-} = c_1^{k+} \tag{4.43}$$

$$\frac{c_1^i}{c_1^{i-1}} = |\ell_i x_i + k_i|^{-1+2\lambda_i} |\ell_{i-1} x_i + k_{i-1}|^{1-2\lambda_{i-1}} . \tag{4.44}$$

Successive use of (4.44) and (4.40) allows the determination of c_2^{k+} in terms of c_1^i $(i < k)$ and c_2^{k-} in terms of c_1^i $(i > k)$. Substituting into (4.38),

$$sD \sum_{i=1}^{n-1} c_1^i \ell_i \left(|\ell_i x_{i+1} + k_i|^{-2\lambda_i} - |\ell_i x_i + k_i|^{-2\lambda_i} \right) = 1 . \tag{4.45}$$

Comparing (4.44, 45) with (4.32, 33) shows that one must have

$$C_i = \lim_{s \to 0} s \, c_1^i , \tag{4.46}$$

so that (4.36) has been verified. Similarly this can be proved in the more general case, with arbitrary ℓ_i and possible natural boundaries. (An attracting boundary gives $\lim_{s \to 0} sp = 0$ for all interior x.) This method is the same as that used for the same purpose in section 3.2.

Spectral density

If neither x_1 nor x_n is an attracting natural boundary, the Laplace-transformed autocorrelation is given by (2.49, 58, 62, 63), which in this case take the form

$$r(s) = \sum_{k=1}^{n-1} \left\{ \frac{\mathscr{I}_k}{s+\ell_k(1-\ell_k D)} + \sum_{i=1}^{n-1} \frac{1}{s+\ell_i(1-\ell_i D)} \left[\left\{ x - \frac{k_i}{s}(1-\ell_i D) \right\} \mathscr{Q}_k(x) - \mathscr{P}_k(x) \right]_{x=x_{i+1}}^{x_i} \right\} \quad (4.47)$$

$$\mathscr{I}_k = \frac{D}{\ell_k^2} C_k \left[|\ell_k x_o + k_k|^{1-2\lambda_k} \left\{ \frac{\ell_k x_o + k_k}{2\ell_k D - 1} - k_k \left(\frac{2}{\ell_k D - 1} - \frac{\ell_k}{s} \right) - \frac{k_k^2}{\ell_k x_o + k_k} \left[1 - \ell_k s(\ell_k D - 1) \right] \right\} \right]_{x_o = x_{k+1}}^{x_k}$$

(4.48)

$$\mathscr{P}_k(x_i) = \frac{C_k D}{s+\ell_k(1-\ell_k D)} \left[|\ell_k x_o + k_k|^{1-2\lambda_k} \left\{ \left[x_o - \frac{k_k}{s}(1-\ell_k D) \right] D^* \left[a(x_i) p(x_i, s|x_o) \right] - a(x_i) p(x_i, s|x_o) \right\} \right]_{x_o = x_{k+1}}^{x_k} \quad (4.49)$$

$$\mathscr{Q}_k(x_i) = \frac{C_k D}{s+\ell_k(1-\ell_k D)} \left[|\ell_k x_o + k_k|^{1-2\lambda_k} \left[x_o - \frac{k_k}{s}(1-\ell_k D) \right] D^* q(x_i, s|x_o) - q(x_i, s|x_o) \right]_{x_o = x_{k+1}}^{x_k} . \quad (4.50)$$

Here D^* is given by (2.59). The spectral density is then obtained using (2.64).

If there is at least one attracting boundary, the auto-correlation can be determined more directly, by direct substitution into (2.33). If there is only one such boundary, say x_j ($j = 1$ or n), then

$$P_2(x_o, 0; x, t) = \delta(x - x_j)\,\delta(x_o - x_j) \quad , \tag{4.51}$$

so that

$$R(t) = x_j^2 \quad , \quad \Phi(\omega) = 0 \text{ for } \omega > 0 \; . \tag{4.52}$$

Similarly, if both x_1 and x_n are attracting,

$$R(t) = \frac{s(x_1) - s(x_o)}{s(x_1) - s(x_n)}\, x_1^2 + \frac{s(x_o) - s(x_n)}{s(x_1) - s(x_n)}\, x_n^2 \quad , \tag{4.53}$$

using (4.35), where x_o is the original starting point (at large negative time). Again $\Phi(\omega) = 0$ for all $\omega > 0$. This zero spectral density is to be expected, for once the system reaches the steady state, it remains stationary, and so the only frequency occurring is $\omega = 0$. See (2.84) for $\Phi(0)$.

4.2 SOME EXAMPLES

The three examples dealt with below have the same restoring forces $f(x)$ as three of the simpler cases dealt with in Chapter III. However, the possible occurrence of irregular points in this chapter often allows a smaller interval (x_1, x_n) to be considered than in Chapter III.

Example (1)--the linear case

Assume $k = 0$. Then there is a trap at 0, and, if the initial position x_o is positive, it is necessary to consider only the interval $(0, \infty) = (x_2, x_1)$; $x_o < 0$ is similar. Then if

$$p(x, s\,|x_o) = c_1^+ p_1(x) + c_2^+ p_2(x) \quad \text{for } x \ge x_o$$
$$= c_1^- p_1(x) + c_2^- p_2(x) \quad \text{for } x \le x_o \;, \tag{4.54}$$

where $p_{1,2}(x)$ are given by (4.14, 15), then $c_1^{\pm}$, $c_2^{\pm}$ satisfy (4.21, 22, 25, 30), since 0 and ∞ are both natural boundaries. So

$$c_1^- - c_1^+ = - \,|\ell x_o|^{-1+\beta_1} \lambda(\lambda^2+\sigma)^{-\frac{1}{2}} \tag{4.55}$$

$$c_2^- - c_2^+ = |\ell x_o|^{-1+\beta_2} \lambda(\lambda^2+\sigma)^{-\frac{1}{2}} \tag{4.56}$$

$$c_2^+ = 0 \tag{4.57}$$

$$c_1^- = 0 \;. \tag{4.58}$$

Solving these, (4.54) becomes

$$p(x, s\,|x_o) = \frac{\lambda\left(\frac{x}{x_o}\right)^{-\beta_1}}{\ell x_o(\lambda^2+\sigma)^{\frac{1}{2}}} \;, \qquad x \gtrless x_o$$
$$= \frac{\lambda\left(\frac{x}{x_o}\right)^{-\beta_2}}{\ell x_o(\lambda^2+\sigma)^{\frac{1}{2}}} \;, \qquad x \lessgtr x_o \;. \tag{4.59}$$

The inverse Laplace transform $P(x, t\,|x_o)$ can be found, using

Erdélyi, et al. [14], p. 246, no. (6). It is

$$P(x, t \,|\, x_o) = \frac{e^{-\frac{\ell^2 t}{4D}}}{2x\sqrt{\pi D t}} \left(\frac{x_o}{x}\right)^{\frac{\ell}{2D} + \frac{1}{4Dt}\ln\left(\frac{x}{x_o}\right)} . \tag{4.60}$$

It is easily verified that this result is the same as that obtained by Gray [22], equation 2.19, by a variable change in the Langevin equation $\frac{dx}{dt} = n(t)$.

If $\ell > 0$, it is seen that $x = 0$ is an attracting boundary, and $x = \infty$ a nonattracting boundary. Thus in this case, by (4.34),

$$P_o(x) = \delta(x) . \tag{4.61}$$

If $\ell < 0$, $x = 0$ is nonattracting while $x = \infty$ is attracting, so the process is unstable, i.e.,

$$P_o(x) \equiv 0 . \tag{4.62}$$

These results can be verified by taking the Abelian limit of (4.60), since

$$P_o(x) = \lim_{s \to 0} sp(x, s \,|\, x_o) = 0 , \quad x \neq 0 , \tag{4.63}$$

while, for small positive δ,

$$\begin{aligned} \int_0^\delta P_o(x)dx = \lim_{s\to\infty} \int_0^\delta sp(x, s \,|\, x_o)\, dx &= 1 , \quad \ell > 0 \\ &= 0 , \quad \ell < 0 . \end{aligned} \tag{4.64}$$

The autocorrelation and spectral density are trivial in this case. When $\ell > 0$, both are identically zero, as is to be expected since all paths tend towards zero. When $\ell < 0$, neither exist.

Example (2)

1 interval, x_1 positive and finite, $x_2 = k = 0$. This restoring force is essentially that of example (3) of section 3.3 (the hard limiter), since this produces a trap at $x = 0$, so that the process can be divided into two independent subprocesses, on $(-x_1, 0)$ and $(0, x_1)$. If $x_o > 0$, the following analysis holds; if $x_o < 0$, corresponding results can be obtained by symmetry.

Equations (4.55, 56, 58) still hold in this case, while the condition at the regular boundary x_1 becomes (4.26), i.e.,

$$c_1^+(1-\beta_2)|\ell x_1|^{-\beta_1} + c_2^+(1-\beta_1)|\ell x_1|^{-\beta_2} = 0 \quad . \tag{4.65}$$

Solving these four equations and substituting in (4.54),

$$\begin{aligned} p(x, s|x_o) &= \frac{\left(\frac{x}{x_o}\right)^{-\frac{\beta_1}{\lambda}}}{\ell x_o(\lambda^2+\sigma)^{\frac{1}{2}}}\left[1-\frac{\beta_2-1}{\beta_1-1}\left(\frac{x}{x_1}\right)^{\beta_1-\beta_2}\right], \quad x \geq x_o \\ &= \frac{\left(\frac{x}{x_o}\right)^{-\frac{\beta_2}{\lambda}}}{\ell x_o(\lambda^2+\sigma)^{\frac{1}{2}}}\left[1-\frac{\beta_2-1}{\beta_1-1}\left(\frac{x_o}{x_1}\right)^{\beta_1-\beta_2}\right], \quad x \leq x_o \quad . \end{aligned} \tag{4.66}$$

If $\ell > 0$, $x = 0$ is an attracting boundary, and

$$P_o(x) = \delta(x) \quad . \tag{4.67}$$

If $\ell < 0$, $x = 0$ is nonattracting, and by (4.31, 33),

$$P_o(x) = \frac{1}{|\ell| x_1 D} \left(\frac{x}{x_1}\right)^{-(1+2\lambda)} \quad . \tag{4.68}$$

As in the previous case, these results can be verified by taking limits of sp and $\int$ sp dx.

For $\ell > 0$, trivially $r(s) = \Phi(\omega) = 0$. For $\ell < 0$, by (4.48-50),

$$\mathcal{J} = \frac{x_1^2}{1-2\ell D} \quad , \quad \mathcal{P}(x_1) = \left(\frac{\beta_1 - 2}{\beta_1 - 1}\right) \frac{\ell x_1^2}{s+\ell(1-\ell D)} \quad ,$$
$$\mathcal{P}(0) = \mathcal{Q}(x_1) = \mathcal{Q}(0) = 0 \quad . \tag{4.69}$$

Thus, by (4.47),

$$r(s) = \frac{x_1^2}{s+\ell(1-\ell D)} \left[\frac{1}{1-2\ell D} + \frac{\ell}{s+\ell(1-\ell D)} \left(\frac{\beta_1 - 2}{\beta_1 - 1}\right)\right] . \tag{4.70}$$

This can be checked by noting that

$$\langle x^2 \rangle = \lim_{s\to\infty} sr(s) = \frac{x_1^2}{1-2\ell D} \tag{4.71}$$

$$\langle x \rangle^2 = \lim_{s\to 0} sr(s) = \frac{x_1^2}{(1-\ell D)^2} \quad , \tag{4.72}$$

both results which can be easily verified directly. Using (2.64) one has, for $\ell < 0$,

$$\Phi(\omega) = \frac{\ell x_1^2}{\omega^2+\ell^2(1-\ell D)^2}\left[\frac{1-\ell D}{1-2\ell D} + \frac{1}{\omega^2+\ell^2(1-\ell D)^2}\left\{\ell^2(1-\ell D)(2-\ell D) - \omega^2\right.\right.$$

$$\left.\left. + \frac{\ell}{2\sqrt{2}\,\omega}\left[\ell^2(1-\ell D)^2 - \omega^2\right]\left[(1+16\omega^2 D^2)^{\frac{1}{2}}-1\right] + \frac{\ell^2}{\sqrt{2}}\left[(1+16\omega^2 D^2)^{\frac{1}{2}}+1\right]^{\frac{1}{2}}\right\}\right]. \tag{4.73}$$

Example (3)

2 intervals, $x_1 = \infty$, $x_2 = 0$, $x_3 = -\infty$, $\ell_1 = \ell_2 = \ell$, $k_2 = -k_1 = -k \neq 0$. There are four somewhat different cases in this example, depending on the relative signs of ℓ and k. In no case is $x = 0$ irregular. When ℓk is positive, there are no irregular points at all. When ℓk is negative, $\pm \frac{k}{\ell}$ are traps, and we restrict ourselves to the process on $\left(+\frac{k}{\ell}, -\frac{k}{\ell}\right)$, redefining x_1 and x_3 as $\mp\frac{k}{\ell}$. (The processes on $\left(-\frac{k}{\ell}, \infty\right)$ and $\left(-\infty, \frac{k}{\ell}\right)$ are essentially that of example (1).) See the sketches on the next page for the four cases.

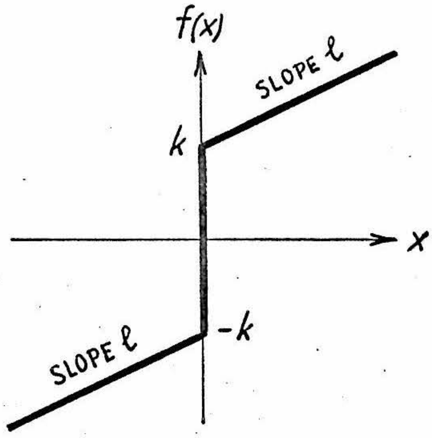
In cases (a) and (b), the coefficients c_i^j are obtained by solving (4.21-25, 28), which gives

$$p(x, s|x_o) = \left|\frac{\lambda}{k}\right| \frac{1}{(\lambda^2+\sigma)^{\frac{1}{2}}}\left(1+\frac{\ell|x|}{k}\right)^{-\beta_1}\left[\left(1+\frac{\ell|x_o|}{k}\right)^{-1+\beta_1} + \frac{\lambda}{1-\beta_2}\left(1+\frac{\ell|x_o|}{k}\right)^{-1+\beta_2}\right],$$

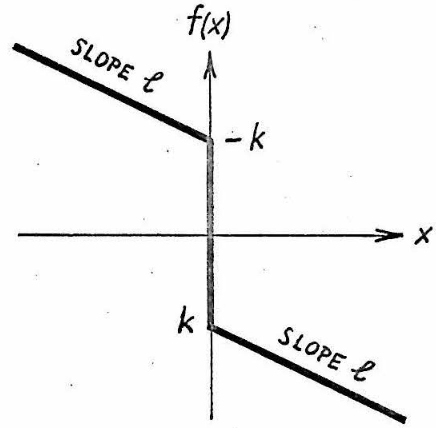
$$x \gtrless x_o \gtrless 0 \tag{4.74}$$

$$= \left|\frac{\lambda}{k}\right| \frac{1}{(\lambda^2+\sigma)^{\frac{1}{2}}}\left(1+\frac{\ell|x_o|}{k}\right)^{-1+\beta_2}\left[\left(1+\frac{\ell|x|}{k}\right)^{-\beta_2} + \frac{\lambda}{1-\beta_2}\left(1+\frac{\ell|x|}{k}\right)^{-\beta_1}\right],$$

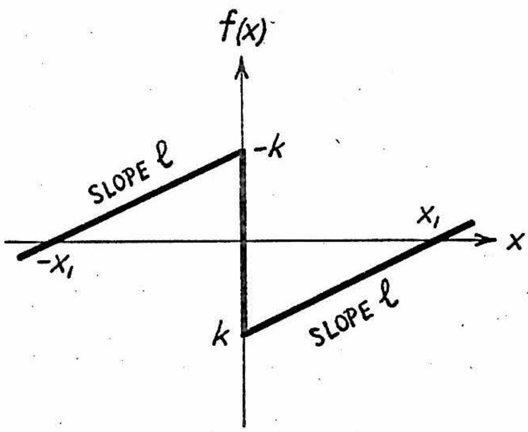
$$x_o \gtrless x \gtrless 0 \tag{4.75}$$



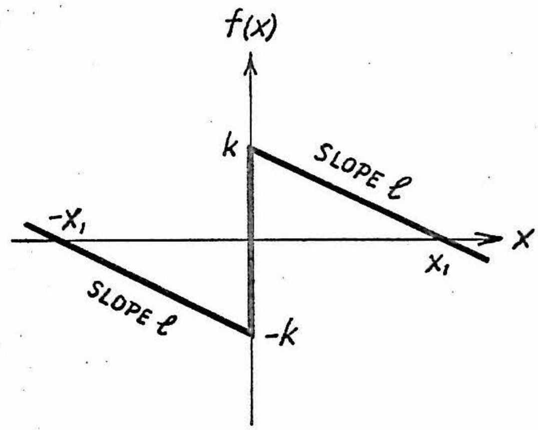
Case (a) — $\ell > 0$, $k > 0$



Case (b) — $\ell < 0$, $k < 0$



Case (c) — $\ell > 0$, $k < 0$



Case (d) — $\ell < 0$, $k > 0$

The 4 cases of example 3, section 4.2.

$$= \left|\frac{\lambda}{k}\right| \frac{1}{1-\beta_2} \left(1 + \frac{\ell|x_o|}{k}\right)^{-1+\beta_2} \left(1 + \frac{\ell|x|}{k}\right)^{-\beta_1}, \quad x \gtrless 0, \; x_o \lesseqgtr 0 . \tag{4.76}$$

In cases (c) and (d), (4.25, 28) are replaced by (4.27, 30), but the results obtained have the same form, except that β_1 and β_2 are interchanged throughout.

In case (a), since $\pm\infty$ are natural nonattracting boundaries, (4.31-33) give

$$P_o(x) = \frac{1}{2|k|D} \left(1 + \frac{\ell}{k}|x|\right)^{-1-2\lambda} . \tag{4.77}$$

Case (d) gives the same result, since $\mp\frac{k}{\ell}$ are nonattracting. In case (b), $\pm\infty$ are attracting, so $P_o(x) \equiv 0$. In case (c), $\mp\frac{k}{\ell}$ are attracting, so, by (4.35),

$$P_o(x) = A(x_o)\delta(-\frac{k}{\ell}) + (1 - A(x_o))\delta(\frac{k}{\ell}) , \tag{4.78}$$

where

$$\begin{aligned} A(x_o) &= 1 - \tfrac{1}{2}\left(1 + \frac{\ell x_o}{k}\right)^{2\lambda} , \quad 0 \le x_o \le -\frac{k}{\ell} \\ &= \tfrac{1}{2}\left(1 - \frac{\ell x_o}{k}\right)^{2\lambda} , \quad \frac{k}{\ell} \le x_o \le 0 . \end{aligned} \tag{4.79}$$

The same results are obtained by considering limits of sp and $\int$ sp dx.

In case (a), using (4.48-50),

$$\mathcal{J}_1 = \mathcal{J}_2 = \frac{k^2}{2\ell^2}\left[\frac{2}{\ell D-1} - \frac{1}{2\ell D-1} + 1 - \frac{\ell^2 D}{s}\right] \tag{4.80}$$

$$\mathcal{Q}_1(0) = \mathcal{Q}_2(0) = \frac{\frac{1}{4}k}{s+\ell(1-\ell D)}\left[\frac{\ell}{s}(1-\ell D)(1-\beta_2) - 1\right] ; \qquad (4.81)$$

all other $\mathcal{P}$'s and $\mathcal{Q}$'s are either zero or cancel out in the formula (4.47) for r(s), which leads to

$$r(s) = \frac{k^2D^2}{s+\ell(1-\ell D)}\left\{\frac{2}{(1-\ell D)(1-2\ell D)} - \frac{1}{sD} - \frac{1}{\ell D^2}\,\frac{1-\ell D}{s+\ell(1-\ell D)}\cdot\frac{\ell}{s}\left[\frac{\ell}{s}(1-\ell D)(1-\beta_2) - 1\right]\right\}. \qquad (4.82)$$

In case (d), the same result is obtained, except that β_2 is replaced by β_1 in (4.82). In both cases

$$\langle x^2 \rangle = \frac{2k^2D^2}{(1-\ell D)(1-2\ell D)} , \qquad (4.83)$$

while $\Phi(\omega)$ can be obtained in an explicit (but complicated) form using (2.64). It is seen that the denominator of (4.83) disappears for $2\ell D = 1, 2$. In fact case (a) is unstable in mean square for $D \geq \frac{1}{2\ell}$,[1] so that (4.82, 83) are meaningless in this range. In case (d), ℓ is negative, so this does not arise.

In case (b), there are attracting natural boundaries at $\pm\infty$, which means that the system is unstable, and that autocorrelation and spectral density do not exist. In case (c), since all paths tend to $\pm\frac{k}{\ell}$ as $t \to \infty$,

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[1] See discussion by Gray in [22], part 2.

$$R(t) = \left(\frac{k}{\ell}\right)^2 \qquad (4.84)$$

$$\Phi(\omega) = 0 \qquad \text{for } \omega > 0. \qquad (4.85)$$

If $x_1^2 = \left(-\frac{k}{\ell}\right)^2$ were not equal to $x_3^2 = \left(\frac{k}{\ell}\right)^2$, it would be necessary to use (4.53) to obtain $R(t)$, which would depend on x_o.

CHAPTER V

SYSTEMS WITH BOTH PARAMETRIC AND FORCING FUNCTION EXCITATION

5.1 PRELIMINARIES

In this chapter we deal with some piecewise linear systems excited by both the forcing function excitation dealt with in Chapter III and the parametric excitation of Chapter IV. No work appears to have been published concerning nonlinear systems of this type. We deal in detail only with the two extreme cases of no correlation and of perfect correlation between the two white excitation functions.

The stochastic differential equation

We consider systems for which (0.1) can be written

$$\dot{x} + f(x)(1 + m(t)) = n(t) \tag{5.1}$$

(i.e., $m = 2$, $h_1 = -f$, $h_2 = 1$), where

$$f(x) = k_i x + \ell_i \quad \text{for } x \in (x_{i+1}, x_i) \;, \quad i = 1, 2, \ldots, n-1, \tag{5.2}$$

so that there are n-1 linear intervals. The white noise forcing function, $n(t)$, and parametric excitation, $m(t)$, satisfy

$$\langle m(t_1) m(t_2) \rangle = 2D_1 \; \delta(t_1 - t_2) \tag{5.3}$$

$$\langle n(t_1) n(t_2) \rangle = 2D_2 \, \delta(t_1 - t_2) \tag{5.4}$$

$$\langle m(t_1)\, n(t_2)\rangle = 2D_{12}\,\delta(t_1 - t_2) \tag{5.5}$$

$$\langle m(t)\rangle = \langle n(t)\rangle = 0 \quad . \tag{5.6}$$

Here D_1, D_2 are used instead of the D_{11}, D_{22} of Chapter II. Note that (5.6) represents no restriction, as it can always be obtained by a suitable linear transformation of f(x), m(t) and n(t).

Special cases to be dealt with

The following cases will be worked through, as outlined in Chapter II, and the spectral density, etc., found.

(a) $D_{12} = (D_1 D_2)^{\frac{1}{2}} \neq 0$ -- i.e., perfect positive correlation between forcing function and parametric excitation. This case is dealt with in sections 5.2-3. Note that it can be obtained by putting

$$m(t) = \left(\frac{D_1}{D_2}\right)^{\frac{1}{2}} n(t) \quad , \tag{5.7}$$

so that there is essentially only one white excitation in this case. The case of perfect negative correlation is no different.

(b) $D_{12} = 0$, $D_1, D_2 \neq 0$ -- i.e., no correlation between forcing function and parametric excitation; dealt with in section 5.4.

Two other special cases, $D_1 = 0$ and $D_2 = 0$, have been dealt with in the preceding two chapters.

The case of partial correlation

The special cases (a) and (b) are chosen primarily for their mathematical simplicity. In each case one can obtain fairly simple solutions to the Laplace-transformed FP equation in terms of known special functions, which are real for real x, x_o and s. In the more general case of partial correlation between m(t) and n(t) $\left(\text{i.e., } 0 < |D_{12}| < (D_1 D_2)^{\frac{1}{2}}\right)$, the simplest solutions consist of hypergeometric functions in which both argument and parameters are complex. The results for partial correlation (i.e., transition density, spectrum, etc.) would be expected to lie between these for the two extreme cases. There are no irregular points in the case of partial correlation. (It resembles (b) rather than (a) in this respect.)

Nondimensionalization

It is convenient to define the following dimensionless quantities:

$$\xi^i = \left(\frac{D_1}{D_2}\right)^{\frac{1}{2}} (\ell_i x + k_i) \quad \text{for } x \in (x_{i+1}, x_i) \tag{5.8}$$

$$\sigma_i = \frac{s}{\ell_i^2 D_1} \tag{5.9}$$

$$\pi^i(\xi, \sigma | \xi_o) = \ell_i (D_1 D_2)^{\frac{1}{2}} p(x, s | x_o) \text{ for } x \in (x_{i+1}, x_i) \tag{5.10}$$

$$\rho = \frac{D_{12}}{(D_1 D_2)^{\frac{1}{2}}} \tag{5.11}$$

$$\lambda_i = \frac{1}{2\ell_i D_1} \tag{5.12}$$

$$\varkappa_i = \left(\frac{D_1}{D_2}\right)^{\frac{1}{2}} k_i \tag{5.13}$$

$$\alpha_i(\xi) = (\xi^i)^2 - 2\rho\xi^i + 1 = \frac{a(x)}{D_2} \qquad \text{for } x \in (x_{i+1}, x_i) \tag{5.14}$$

$$\psi^i(\xi, \sigma | \xi_o) = \frac{d}{d\xi^i}\left[\alpha_i(\xi)\pi^i\right] + \left[(2\lambda_i - 1)\xi^i + \rho\right]\pi^i = q(x, s|x_o) \text{ for } x \in (x_{i+1}, x_i) \ . \tag{5.15}$$

A suffix on ξ, π, ψ will correspond to the same suffix on x, p, q, respectively. Both suffixes and superfixes may be omitted when no confusion would result.

The FP equation

The Laplace-transformed FP equation, (2.14), becomes for the system (5.1),

$$\frac{d^2}{dx^2}\left\{\left[D_1(\ell_i x + k_i)^2 - 2D_{12}(\ell_i x + k_i) + D_2\right]p\right\} + \frac{d}{dx}\left\{\left[(\ell_i x + k_i)(1 - \ell_i D_1) + \ell_i D_{12}\right]p\right\} - sp = -\delta(x - x_o)$$
$$\text{for } x \in (x_{i+1}, x_i) \ , \tag{5.16}$$

or, using the dimensionless variables defined above,

$$\frac{d^2}{d\xi^2}\left[\alpha(\xi)\pi\right] + \frac{d}{d\xi}\left\{\left[(2\lambda - 1)\xi + \rho\right]\pi\right\} - \sigma\pi = -\delta(\xi - \xi_o) \ . \tag{5.17}$$

Similarly, the backwards equation is

$$\alpha(\xi_o)\frac{d^2\pi}{d\xi_o^2} - \left[(2\lambda-1)\xi_o+\rho\right]\frac{d\pi}{d\xi_o} - \sigma\pi = -\delta(\xi-\xi_o) \quad . \tag{5.18}$$

The continuity of ap and q at the points x_i leads to the following when nondimensionalized:

$$\lambda_i\alpha_i(\xi_i)\pi^i(\xi_i, \sigma|\xi_o) = \lambda_{i-1}\alpha_{i-1}(\xi_i)\pi^{i-1}(\xi_i, \sigma|\xi_o) \tag{5.19}$$

$$\psi^i(\xi_i, \sigma|\xi_o) = \psi^{i-1}(\xi_i, \sigma|\xi_o) \quad . \tag{5.20}$$

5.2 PERFECT CORRELATION--GENERAL CASE

In this section we deal with the stochastic differential equation

$$\dot{x} + (\ell_i x + k_i)(1+m(t)) = n(t) \quad , \tag{5.21}$$

where m(t), n(t) satisfy (5.3-6) with

$$D_{12} = (D_1D_2)^{\frac{1}{2}}, \text{ or } \rho = 1 \quad . \tag{5.22}$$

The dimensionless variables of (5.8-15) can be used for all intervals where $\ell_i \neq 0$. As before, in the general derivation below it will be assumed that no $\ell_i = 0$.

Irregular points

The FP equation is given by (5.17) with $\rho = 1$. That is

$$\frac{d^2}{d\xi^2}\left[(\xi-1)^2\pi\right] + \frac{d}{d\xi}\left\{\left[(2\lambda-1)\xi+1\right]\pi\right\} - \sigma\pi = -\delta(\xi-\xi_o) \quad . \tag{5.23}$$

Thus it is seen that x is an irregular point only when $\xi \to 1$ from one or both sides. The corresponding limiting value of the drift coefficient b(x) is $-\left(\frac{D_2}{D_1}\right)^{\frac{1}{2}}$, so that the irregular point is always a left shunt (see section 1.5). This is the case whether $\xi \to 1$ on both sides, so that the point is an exit boundary for the process on the right and an entrance boundary for that on the left, or whether $\xi \to 1$ only on one side, the point being a regular boundary on the other side. (This occurs when $\ell x + k$ is discontinuous at the point.)

A fundamental set of solutions

The differential equation (5.23) has an irregular (double) singular point at $\xi = 1$, and a regular singular point at $\xi = \infty$. Kummer's confluent hypergeometric equation[1] has the same singularities, but at ∞ and 0, respectively. Thus we attempt to find solutions in terms of confluent hypergeometric functions. Putting

$$\xi = \frac{2\lambda}{\zeta} + 1 \quad , \quad \pi = \zeta^{1+\lambda+(\lambda^2+\sigma)^{\frac{1}{2}}}\Psi \, , \tag{5.24}$$

the homogeneous part of (5.23) becomes

$$\zeta \frac{d^2\Psi}{d\zeta^2} + \left[1 + 2(\lambda^2+\sigma)^{\frac{1}{2}} - \zeta\right]\frac{d\Psi}{d\zeta} - \left[1 + \lambda + (\lambda^2+\sigma)^{\frac{1}{2}}\right]\Psi = 0 \, , \tag{5.25}$$

which is in Kummer's form. Thus a fundamental set of solutions is

$$\pi_1(\xi) = |\zeta|^{\beta_1} M(\beta_1, 1+\beta_1-\beta_2, \zeta) \tag{5.26}$$

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[1] See Abramowitz and Stegun [1], Chapter 13, or Erdelyi et al. [11], Chapter 6.

$$\pi_2(\xi) = \zeta^{\beta_1} U(\beta_1, 1+\beta_1-\beta_2, \zeta) \quad \text{for } \zeta = \frac{2\lambda}{\xi-1} > 0$$
$$= (-\zeta)^{\beta_1} e^{\zeta} U(1-\beta_2, 1+\beta_1-\beta_2, -\zeta) \quad \text{for } \zeta < 0 , \tag{5.27}$$

where (as in the previous chapter)

$$\beta_{1,2} = 1 + \lambda \pm (\lambda^2 + \sigma)^{\frac{1}{2}} , \tag{5.28}$$

and M, U denote Kummer's confluent hypergeometric functions of the first and second kinds, respectively. An alternative to π_2 is

$$|\zeta|^{\beta_2} M(\beta_2, 1+\beta_2-\beta_1, \zeta) . \tag{5.29}$$

However, this becomes infinite when $(\lambda^2+\sigma)^{\frac{1}{2}}$ is a positive integer; (5.26, 27) remain independent and finite for all λ and σ .

Related quantities

Using (5.15) and equations 13.4.11, 23, 26 in [1],

$$\psi_1 = -(1-\beta_2)(\xi-1)|\zeta|^{\beta_1} M(\beta_1-1, 1+\beta_1-\beta_2, \zeta) \tag{5.30}$$

$$\psi_2 = (\xi-1)\zeta^{\beta_1} U(\beta_1-1, 1+\beta_1-\beta_2, \zeta) \quad \text{for } \zeta > 0$$
$$= \sigma(\xi-1)(-\zeta)^{\beta_1} e^{\zeta} U(2-\beta_2, 1+\beta_1-\beta_2, -\zeta) \quad \text{for } \zeta < 0 . \tag{5.31}$$

The Wronskian of π_1 and π_2 is (using equations 13.1.22, 23 in [1])

$$w(\xi) = \frac{\Gamma(1+\beta_1-\beta_2)}{\Gamma(\beta_1)} (\xi-1)^{-1} \zeta^{\beta_1+\beta_2} e^{\zeta} \quad \text{for } \zeta > 0$$

$$= - \frac{\Gamma(1+\beta_1-\beta_2)}{\Gamma(1-\beta_2)} (1-\xi)^{-1} (-\zeta)^{\beta_1+\beta_2} e^{\zeta} \quad \text{for } \zeta < 0 . \tag{5.32}$$

A fundamental set of solutions to the transformed backwards equation is obtained similarly to the forward equation:

$$|\zeta_o|^{1-\beta_1} e^{-\zeta_o} M(\beta_1, 1+\beta_1-\beta_2, \zeta_o)$$

$$\zeta_o^{1-\beta_1} e^{-\zeta_o} U(\beta_1, 1+\beta_1-\beta_2, \zeta_o) \text{ or } (-\zeta_o)^{1-\beta_1} U(1-\beta_2, 1+\beta_1-\beta_2, -\zeta_o) \tag{5.33}$$

$$\text{for } \zeta_o \gtrless 0 .$$

<u>The Laplace-transformed transition density</u>

Putting

$$\pi^i(\xi, \sigma | \xi_o) = \gamma_1^i \pi_1^i(\xi) + \gamma_2^i \pi_2^i(\xi) \ , \tag{5.34}$$

the coefficients γ_1^i, γ_2^i satisfy the usual conditions at regular points x_i $(i = 1, \ldots, n)$ (see (5.19, 20)) and x_o (see (2.21, 22)). The conditions which follow are for $\zeta > 0$; for $\zeta < 0$ the appropriate changes can be seen from the previous paragraph. The conditions are:

$$\gamma_1^{k-} = \gamma_1^{k+} - \frac{\Gamma(\beta_1^k)}{\Gamma(1+\beta_1^k-\beta_2^k)} \frac{\zeta_o^{-\beta_2} e^{-\zeta_o}}{(\xi_o^k-1)} U\left(\beta_1^k, 1+\beta_1^k-\beta_2^k, \zeta_o^k\right) \tag{5.35}$$

$$\gamma_2^{k-} = \gamma_2^{k+} + \frac{\Gamma(\beta_1^k)}{\Gamma(1+\beta_1^k-\beta_2^k)} \frac{\zeta_o^{-\beta_2} e^{-\zeta_o}}{(\xi_o^k-1)} M\left(\beta_1^k, 1+\beta_1^k-\beta_2^k, \zeta_o^k\right), \tag{5.36}$$

where $x_o \in (x_{k+1}, x_k)$;

$$\lambda_i^3(\zeta_i^i)^{-2+\beta_1^i}\left[\gamma_1^i M\left(\beta_1^i,\ 1+\beta_1^i-\beta_2^i,\ \zeta_i^i\right)+\gamma_2^i U\left(\beta_1^i,\ 1+\beta_1^i-\beta_2^i,\ \zeta_i^i\right)\right]$$

$$= \lambda_{i-1}^3(\zeta_i^{i-1})^{-2+\beta_1^{i-1}}\left[\gamma_1^{i-1} M\left(\beta_1^{i-1},\ 1+\beta_1^{i-1}-\beta_2^{i-1},\ \zeta_i^{i-1}\right) + \gamma_2^{i-1} U\left(\beta_1^{i-1},\ 1+\beta_1^{i-1}-\beta_2^{i-1},\ \zeta_i^{i-1}\right)\right] \quad (5.37)$$

$$\lambda_i(\zeta_i^i)^{-1+\beta_1^i}\left[\gamma_1^i(1-\beta_2^i) M\left(\beta_1^i-1,\ 1+\beta_1^i-\beta_2^i,\ \zeta_i^i\right)-\gamma_2^i U\left(\beta_1^i-1,\ 1+\beta_1^i-\beta_2^i,\ \zeta_i^i\right)\right]$$

$$= \lambda_{i-1}(\zeta_i^{i-1})^{-1+\beta_1^{i-1}}\left[\gamma_1^{i-1}(1-\beta_2^{i-1}) M\left(\beta_1^{i-1}-1,\ 1+\beta_1^{i-1}-\beta_2^{i-1},\ \zeta_i^{i-1}\right) - \gamma_2^{i-1} U\left(\beta_1^{i-1}-1,\ 1+\beta_1^{i-1}-\beta_2^{i-1},\ \zeta_i^{i-1}\right)\right] \quad (5.38)$$

for $i = 2, \ldots, n-1$; together with boundary conditions at x_1 and x_n. At x_1 the condition is the first or second of

$$\gamma_2^1 = 0 \quad (5.39)$$

$$\gamma_1^1(1-\beta_2^1) M\left(\beta_1^1-1,\ 1+\beta_1^1-\beta_2^1,\ \zeta_1^1\right) = \gamma_2^1 U\left(\beta_1^1-1,\ 1+\beta_1^1-\beta_2^1,\ \zeta_1^1\right), \quad (5.40)$$

depending on whether x_1 is infinite or finite and regular; and similarly at x_n .

Conditions at irregular points

Conditions (5.37, 38, 40) must be modified if the points concerned are not regular. If x_i is an exit boundary from above, then, since $W(x) = \exp\left\{-\int^x \frac{b(x)}{a(x)}\,dx\right\} \to 0$ as $x \downarrow x_i$, aP need not be continuous at x_i (see section 1.3). Otherwise both aP and Q must be continuous. If x_i is an entrance boundary from below, then since

$$M(\alpha, \gamma, z) \sim \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)}(-z)^{-\alpha}\ , \quad U(\alpha, \gamma, -z) \sim (-z)^{-\alpha} \quad \text{as } z \to -\infty\ , \tag{5.41}$$

one has $a(x_i-)P(x_i-, t\,|x_o) = 0$. This implies that, if x_i is a regular boundary from above, $a(x_i+)P(x_i+, t\,|x_o) = 0$. This absorption condition is to be expected on the right of a left shunt.

According to section 1.4, no boundary conditions are required for an exit boundary; however, the integrability of P requires that $\gamma_1^{i-1} = 0$, since

$$M(\alpha, \gamma, z) \sim \frac{\Gamma(\gamma)}{\Gamma(\alpha)}\frac{e^z}{z^{\gamma-\alpha}}\ , \quad U(\alpha, \gamma, z) \sim z^{-\alpha} \quad \text{as } z \to \infty\ . \tag{5.42}$$

Incidentally, this shows that $a(x_i+)P(x_i+, t\,|x_o) = 0$ in this case also.

The above considerations lead to the following conditions relating the coefficients on either side of the three kinds of left shunts which occur.

(a) Exit boundary above, entrance boundary below:

$$\gamma_1^{i-1} = 0 \quad , \quad \lambda_i \frac{\Gamma(1+\beta_1^i-\beta_2^i)}{\Gamma(1-\beta_2^i)} \gamma_1^i = \lambda_{i-1}\gamma_2^{i-1} \tag{5.43}$$

(the latter following from the continuity of ψ, using the asymptotic formulas (5.41-42)).

(b) Exit boundary above, regular boundary below:

$$\gamma_1^{i-1} = 0 \quad , \quad \gamma_1^i \psi_1^i(\xi_i) + \gamma_2^i \psi_2^i(\xi_i) = 2\lambda_{i-1}\, \gamma_2^{i-1} \quad , \tag{5.44}$$

where ψ_1, ψ_2 are given by (5.30-31).

(c) Regular boundary above, entrance boundary below:

$$\begin{aligned} &\gamma_1^{i-1}\pi_1^{i-1}(\xi_i) + \gamma_2^{i-1}\pi_2^{i-1}(\xi_i) = 0 \\ &2\lambda_i \frac{\Gamma(1+\beta_1^i-\beta_2^i)}{\Gamma(1-\beta_2^i)} \gamma_1^i = \gamma_1^{i-1}\psi_1^{i-1}(\xi_i) + \gamma_2^{i-1}\psi_2^{i-1}(\xi_i) \; . \end{aligned} \tag{5.45}$$

Similarly, if $\xi \to 1$ as $x \to x_1$ (i.e., if x_1 is an entrance boundary), (5.40) is replaced by

$$\gamma_1^1 = 0 \quad , \tag{5.46}$$

and if $\xi \to 1$ as $x \to x_n$ (i.e., x_n an exit boundary)

$$\gamma_1^{n-1} = 0 \; . \tag{5.47}$$

As might be expected, if x_i is irregular, γ_1^j , γ_2^j ($j < i$) can be completely determined independently of γ_1^j, γ_2^j ($j \geq i$), but not vice versa. That is, the probability density to the left of a left shunt depends on that to the right, but that to the right does not depend on that to the left. In particular, if $x_o < x_i$, $\pi(x, s|x_o) = 0$ for all $x > x_i$.

Steady-state probability

All x to the right of the least irregular point lie in a regular interval bounded below by either an exit boundary or a regular absorbing boundary. Thus, according to the discussion of section 1.6, the steady-state density $P_o(x)$ is zero except in the region to the left of the least irregular point.

Let the least irregular point be x_ℓ. Then (2.28) gives, for $x_n < x_\ell$,

$$P_o(x) = C_i |\zeta^i|^{1+2\lambda_i} e^{\zeta^i} . \tag{5.48}$$

The continuity of aP_o throughout the interval (x_n, x_ℓ) shows that, for $n-1 \geq i > j \geq \ell$,

$$\frac{C_i}{C_j} = \left(\frac{\lambda_j}{\lambda_i}\right)^2 |\zeta_i^i|^{-2\lambda_i - 1} \left(\zeta_i^{i-1}/\zeta_{i-1}^{i-1}\right)^{2\lambda_{i-1}+1} \cdots \left(\zeta_{j+2}^{j+1}/\zeta_{j+1}^{j+1}\right)^{2\lambda_{j+1}+1} |\zeta_{j+1}^j|^{2\lambda_j+1}$$

$$\times \exp\left(-\zeta_i^i + \zeta_i^{i-1} - \zeta_{i-1}^{i-1} + \ldots + \zeta_{j+1}^j\right) . \tag{5.49}$$

The C_i are thus determined up to an arbitrary constant. To determine this, use (2.29), noting that

$$\int_{-\infty}^{\xi} \left(\frac{2\lambda}{1-\xi'}\right)^{1+2\lambda} \exp\left(\frac{2\lambda}{\xi'-1}\right) d\xi' = 2\lambda \int_0^{\frac{2\lambda}{1-\xi}} t^{2\lambda-1} e^{-t} dz = 2\lambda\, \gamma\left(2\lambda, \frac{2\lambda}{1-\xi}\right) , \tag{5.50}$$

where $\gamma(\alpha, z) = \int_0^z t^{\alpha-1} e^{-t} dt$ is the incomplete gamma function. Thus one gets, for $j \geq \ell$,

$$C_j = (D_1 D_2)^{-\frac{1}{2}} \left\{ \sum_{i=\ell}^{n} \frac{C_i}{C_j} (2\lambda_i)^2 e^{-2\lambda \epsilon_i \pi i} \left[\gamma(2\lambda_i, -\zeta_i^i) - \gamma(2\lambda_i, -\zeta_{i+1}^i) \right] \right\}^{-1} , \quad (5.51)$$

where $\frac{C_i}{C_j}$ is given by (5.49) and $\epsilon_i = \text{sgn}(-\zeta_i^i)$.

Exceptional cases

If x_n is an exit boundary, $\ell = n$ in the above, and it is apparent that all paths eventually reach x_n and stay there. That is, all $C_i = 0$, and

$$P_o(x) = \delta(x - x_n) \quad . \quad (5.52)$$

If $x_n = -\infty$ and $\ell_{n-1} < 0$, then x_n is natural attracting and $P_o(x) \equiv 0$.

$P_o(x)$ as Abelian limit of $P(x, t | x_o)$

As in previous cases, it can be verified that

$$P_o(x) = \lim_{s \to 0} sp(x, s | x_o) \quad , \quad (5.53)$$

for all x_o. In fact, letting $s \to 0$ in (5.35-40, 43-45) it can be shown that

$$\gamma_1^i = O(\frac{1}{\sigma}) \quad , \quad \gamma_2^i = O(1) \quad , \quad \gamma_1^{k-} - \gamma_1^{k+} = O(1) \quad , \quad (5.54)$$

so that, to the first order,

$$\pi^i(\xi, \sigma | \xi_o) \sim \gamma_1^i |\zeta^i|^{2\lambda_i + 1} M(1+2\lambda^i, 1+2\lambda^i, \zeta^i) = \gamma_1^i |\zeta^i|^{2\lambda_i+1} e^{\zeta^i} . \quad (5.55)$$

It can also be shown that

$$\gamma_1^i \sim \frac{\ell_i}{s}(D_1 D_2)^{\frac{1}{2}} C_i \quad . \tag{5.56}$$

The method is the same as that used for the same purpose in section 3.2, and the details are easily filled in.

Spectral density

The Laplace-transformed autocorrelation is given by (2.49, 58, 62, 63), which in this case take the form

$$r(s) = D_1^2 D_2 \sum_{k=\ell}^{n-1} \left\{ \frac{4\lambda_k^2}{\sigma_k + 2\lambda_k - 1} \mathcal{J}_k^* \right.$$

$$\left. + \sum_{i=\ell}^{n-1} \frac{4\lambda_i^2}{\sigma_i + 2\lambda_i - 1} \left[2\lambda_i \left\{ \xi - \varkappa_i - \frac{1}{\sigma_i} + \frac{\varkappa_i}{\sigma_i}(1-2\lambda_i) \right\} \mathcal{Q}_k^*(\xi) - \mathcal{P}_k^*(\xi) \right]_{x=x_i+1}^{x_i} \right\} , \tag{5.57}$$

where

$$\mathcal{J}_k^* = 4\lambda_k^3 \left[\left(\frac{D_1}{D_2}\right)^{\frac{1}{2}} a(x_o) P_o(x_o) \left\{ \frac{\xi_o}{1-\lambda_k} - \frac{4\varkappa_k}{1-2\lambda_k} + \frac{3}{(1-2\lambda_k)(1-\lambda_k)} \right. \right.$$

$$\left. \left. + \frac{2}{\sigma_k}\left(\varkappa_k - \frac{1}{1-2\lambda_k} \right) \right\} \right]_{x_o = x_{k+1}}^{x_k}$$

$$+ 4\lambda_k^2 \int_{x_{k+1}}^{x_k} P_o(x_o)\,dx_o \left[\varkappa_k^2 - \frac{2\varkappa_k}{1-2\lambda_k} + \frac{\lambda_k + 1}{(1-2\lambda_k)(1-\lambda_k)} \right.$$

$$\left. - \frac{1}{\sigma_k}\left(\varkappa_k - \frac{1}{1-2\lambda_k} \right)^2 (1-2\lambda_k) \right] \tag{5.58}$$

$$\mathcal{P}_k^*(\xi_i) = \frac{8\lambda_k^3}{\sigma_k+2\lambda_k-1}\left[\left(\frac{D_1}{D_2}\right)^{\frac{1}{2}} a(x_o)P_o(x_o)\left\{\left[\xi_o - \varkappa_k - \frac{1}{\sigma_k}\right.\right.\right.$$

$$\left.\left.\left. + \frac{\varkappa_k}{\sigma_k}(1-2\lambda_k)\right]D^*\left[\alpha(\xi_i)\pi(\xi_i, \sigma|\xi_o)\right] - \alpha(\xi_i)\pi(\xi_i, \sigma|\xi_o)\right\}\right]_{x_o=x_{k+1}}^{x_k}$$

(5.59)

$$\mathcal{Q}_k^*(\xi_i) = \frac{4\lambda_k^2}{\sigma_k+2\lambda_k-1}\left[\left(\frac{D_1}{D_2}\right)^{\frac{1}{2}} a(x_o)P_o(x_o)\left\{\left[\xi_o - \varkappa_k - \frac{1}{\sigma_k}\right.\right.\right.$$

$$\left.\left.\left. + \frac{\varkappa_k}{\sigma_k}(1-2\lambda_k)\right]D^*\psi(\xi_i, \sigma|\xi_o) - \psi(\xi_i, \sigma|\xi_o)\right\}\right]_{x_o=x_{k+1}}^{x_k},$$

(5.60)

where D^* is given by (3.68), and x_ℓ is the leftmost irregular point. The spectral density $\Phi(\omega)$ is then given by (2.64).

<u>Variance and mean</u>

For the case treated in this section, (2.68) and (2.69) become, respectively,

$$\langle x^2\rangle = 4D_1D_2\sum_{k=\ell}^{n-1}\lambda_k^2\left\{\lambda_k\left[\left(\frac{D_1}{D_2}\right)^{\frac{1}{2}} a(x)P_o(x)\left\{\frac{\xi}{1-\lambda_k} - \frac{4\varkappa_k}{1-2\lambda_k} + \frac{3}{(1-2\lambda_k)(1-\lambda_k)}\right\}\right]_{x=x_{k+1}}^{x_k}\right.$$

$$\left. + \int_{x_{k+1}}^{x_k} P_o(x)dx\left[\varkappa_k^2 - \frac{2\varkappa_k}{1-2\lambda_k} + \frac{\lambda_k+1}{(1-2\lambda_k)(1-\lambda_k)}\right]\right\} \qquad (5.61)$$

$$\langle x\rangle = 2(D_1D_2)^{\frac{1}{2}}\sum_{k=\ell}^{n-1}\frac{\lambda_k}{1-2\lambda_k}\left\{2\lambda_k\left[\left(\frac{D_1}{D_2}\right)^{\frac{1}{2}}a(x)P_o(x)\right]_{x=x_{k+1}}^{x_k}\right.$$

$$\left.-\int_{x_{k+1}}^{x_k}P_o(x)dx\left[\varkappa(1-2\lambda_k)-1\right]\right\}. \qquad (5.62)$$

<u>Case $\ell_i = 0$ for some i</u>

In all the above it is assumed that $\ell_i \neq 0$, for all i; otherwise the nondimensionalization breaks down. In an interval (x_{i+1}, x_i) where $\ell_i = 0$, the FP equation takes the form

$$\left(D_1^{\frac{1}{2}}k_i - D_2^{\frac{1}{2}}\right)^2\frac{d^2p}{dx^2} + k\frac{dp}{dx} - sp = -\delta(x-x_o)\ . \qquad (5.63)$$

Except that D is replaced by $\left(D_1^{\frac{1}{2}}k_i - D_2^{\frac{1}{2}}\right)^2$, this equation is the same as that of Chapter III, with $\ell_i = 0$, which has been dealt with at the end of section 3.1.

As in this previous case, it appears possible to show, using asymptotic expansions for $M(\alpha, \gamma, z)$ and $U(\alpha, \gamma, z)$[1], that the solutions for $\ell_i = 0$ are the limits of those as $\ell_i \to 0$, so that all results can be obtained as limits. However, the writer has not worked through such a proof in detail.

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[1] See Abramowitz and Stegun [1], section 13.5, or (more detailed) Slater [38].

The case $\ell_i = 0, \quad 1 - \left(\frac{D_1}{D_2}\right)^{\frac{1}{2}} k_i = 0$, eliminates all stochastic terms in (5.21) for x in the ith interval, which can be considered composed entirely of left shunts.

5.3 PERFECT CORRELATION--EXAMPLES

Only two cases will be worked out in detail.

Example (1)--the linear case

Assume $\ell > 0$, $k = 0$. The only irregular point is $x = \frac{1}{\ell}\left(\frac{D_2}{D_1}\right)^{\frac{1}{2}}$. Thus there are two intervals, $(x_2, x_1) = \left(\frac{1}{\ell}\left(\frac{D_2}{D_1}\right)^{\frac{1}{2}}, \infty\right)$ and $(x_3, x_2) = \left(-\infty, \frac{1}{\ell}\left(\frac{D_2}{D_1}\right)^{\frac{1}{2}}\right)$. Wong [45] has found the transition density for $x_o \in (x_3, x_2)$ in the form of an eigenfunction expansion (case F, page 271). Since x_2 is an exit boundary above and an entrance boundary below, (5.35, 36, 39, 43) give, for $x_o > x_2$,

$$\gamma_2^{1+} = 0 \tag{5.64}$$

$$\gamma_1^{1-} = \gamma_1^{1+} - \frac{\Gamma(\beta_1)}{\Gamma(1+\beta_1-\beta_2)} \frac{\zeta_o^{-\beta_2} e^{-\zeta_o}}{\xi_o - 1} U(\beta_1, 1+\beta_1-\beta_2, \zeta_o) \tag{5.65}$$

$$\gamma_2^{1-} = \gamma_2^{1+} + \frac{\Gamma(\beta_1)}{\Gamma(1+\beta_1-\beta_2)} \frac{\zeta_o^{-\beta_2} e^{-\zeta_o}}{\xi_o - 1} M(\beta_1, 1+\beta_1-\beta_2, \zeta_o) \tag{5.66}$$

$$\gamma_1^{1-} = 0 \tag{5.67}$$

$$\frac{\Gamma(1+\beta_1-\beta_2)}{\Gamma(1-\beta_2)} \gamma_1^2 = \gamma_2^{1-} \tag{5.68}$$

$$\gamma_2^2 = 0 \ . \tag{5.69}$$

Solving these,

$$\pi(\xi, \sigma | \xi_o) = \frac{\Gamma(\beta_1)e^{-\zeta_o}}{\Gamma(1+\beta_1-\beta_2)} \frac{\zeta_o^{-\beta_2}\zeta^{\beta_1}}{\xi_o - 1} M(\beta_1, 1+\beta_1-\beta_2, \zeta)U(\beta_1, 1+\beta_1-\beta_2, \zeta_o)$$

$$\text{for } x > x_o > x_2 \qquad (5.70)$$

$$= \frac{\Gamma(\beta_1)e^{-\zeta_o}}{\Gamma(1+\beta_1-\beta_2)} \frac{\zeta_o^{-\beta_2}\zeta^{\beta_1}}{\xi_o - 1} M(\beta_1, 1+\beta_1-\beta_2, \zeta_o)U(\beta_1, 1+\beta_1-\beta_2, \zeta)$$

$$\text{for } x_o > x > x_2 \qquad (5.71)$$

$$= \frac{\Gamma(\beta_1)\,\Gamma(1-\beta_2)e^{-\zeta_o}}{[\Gamma(1+\beta_1-\beta_2)]^2} \frac{\zeta_o^{-\beta_2}(-\zeta)^{\beta_1}}{\xi_o - 1} M(\beta_1, 1+\beta_1-\beta_2, \zeta)M(\beta_1, 1+\beta_1-\beta_2, \zeta_o)$$

$$\text{for } x_o > x_2 > x \; . \qquad (5.72)$$

Forming the corresponding set of simultaneous equations for $x_o < x_2$ and solving,

$$\pi(\xi, \sigma | \xi_o) = 0 \qquad \text{for } x > x_2 > x_o \qquad (5.73)$$

$$= \frac{\Gamma(1-\beta_2)}{\Gamma(1+\beta_1-\beta_2)} \frac{(-\zeta_o)^{-\beta_2}(-\zeta)^{\beta_1}}{1-\xi_o} e^{\zeta-\zeta_o} M(\beta_1, 1+\beta_1-\beta_2, \zeta_o)U(1-\beta_2, 1+\beta_1-\beta_2, -\zeta)$$

$$\text{for } x_2 > x > x_o \qquad (5.74)$$

$$= \frac{\Gamma(1-\beta_2)}{\Gamma(1+\beta_1-\beta_2)} \frac{(-\zeta_o)^{-\beta_2}(-\zeta)^{\beta_1}}{1-\xi_o} M(\beta_1, 1+\beta_1-\beta_2, \zeta)U(1-\beta_2, 1+\beta_1-\beta_2, -\zeta_o)$$

$$\text{for } x_2 > x_o > x \; . \qquad (5.75)$$

The steady-state density $P_o(x)$ is, from (5.48, 51),

$$P_o(x) = \frac{(D_1 D_2)^{-\frac{1}{2}}}{2\lambda\Gamma(2\lambda+1)}(-\zeta)^{1+2\lambda} e^{\zeta} \qquad \text{for } x < x_2$$

$$= 0 \qquad \text{for } x > x_2 \; . \tag{5.76}$$

It is easily verified that, for all x_o, $sp(x, s|x_o)$ tends to this same $P_o(x)$ as $s \to 0$.

To obtain the spectral density, note that

$$\mathcal{I}_2^* = \frac{4\lambda^2}{2\lambda-1}\left[\frac{\lambda+1}{\lambda-1} + \frac{1}{\sigma}\right] , \tag{5.77}$$

while all $\mathcal{P}_k$, $\mathcal{Q}_k$ are zero. Thus

$$r(s) = \frac{D_2}{\ell(1-\ell D_1)[s+\ell(1-\ell D_1)]}\left[\frac{1+2\ell D_1}{1-2\ell D_1} + \frac{\ell^2 D_1}{s}\right] , \tag{5.78}$$

so that

$$\Phi(\omega) = \frac{2}{\pi}\,\mathrm{Re}[r(i\omega)] = \frac{2}{\pi}D_2\left\{(1-\ell D_1)(1-2\ell D_2)\left[\omega^2+\ell^2(1-\ell D_1)^2\right]\right\}^{-1} \tag{5.79}$$

$$\langle x^2 \rangle = \lim_{s\to\infty} sr(s) = \frac{D_2}{\ell}\;\frac{1+2\ell D_1}{(1-\ell D_1)(1-2\ell D_1)} \tag{5.80}$$

$$\langle x \rangle = \left[\lim_{s\to 0} sr(s)\right]^{\frac{1}{2}} = -\frac{D_1^{\frac{1}{2}} D_2^{\frac{1}{2}}}{1-\ell D_1} \; . \tag{5.81}$$

These last two results can, of course, be easily found directly using (5.76).

Note that according to Gray [22, 23] our system (being linear) should have the same mean, autocorrelation and spectrum as the system

$$\frac{dy}{dt} + \ell(1-\ell D_1)y = a(t) - \ell D_1^{\frac{1}{2}} D_2^{\frac{1}{2}} , \qquad (5.82)$$

where a(t) is white noise of magnitude such that $\langle y^2 \rangle = \langle x^2 \rangle$, so that, using (5.80),

$$\langle a(t_1) a(t_2) \rangle = \frac{2D_2 \delta(t_1 - t_2)}{(1-\ell D_1)(1-2\ell D_1)} . \qquad (5.83)$$

Since (5.82) is just the linear system treated in section 3.3 (except that $k \neq 0$), the results of (5.78, 79, 81) are easily verified.

Note that the system is unstable in mean for $D_1 \geq \frac{1}{\ell}$, and unstable in mean square for $D_1 \geq \frac{1}{2\ell}$, so that equations (5.78, 79) hold only for $D_1 < \frac{1}{2\ell}$.

Example (2)

2 intervals, $x_1 = \infty$, $x_2 = 0$, $x_3 = -\infty$, $\ell_1 = \ell_2 = \ell > 0$, $k_2 = -k_1 = -k \neq 0$. There are several different cases, depending on the value of k. For $k > \left(\frac{D_2}{D_1}\right)^{\frac{1}{2}}$, there are no irregular points. For $k = \left(\frac{D_2}{D_1}\right)^{\frac{1}{2}}$, there is one left shunt at $x = 0$, consisting of a regular boundary below and an exit boundary above. For $|k| < \left(\frac{D_2}{D_1}\right)^{\frac{1}{2}}$ there is one left shunt at $x = \ell^{-1}\left[\left(\frac{D_2}{D_1}\right)^{\frac{1}{2}} - k\right]$, consisting of an entrance boundary below and an exit boundary above, and the situation is qualitatively the same as in the previous example. For $k = -\left(\frac{D_2}{D_1}\right)^{\frac{1}{2}}$, there is an additional left shunt at $x = 0$, consisting of an entrance boundary below and a regular boundary above. For $k < -\left(\frac{D_2}{D_1}\right)^{\frac{1}{2}}$ there are two shunts, both of entrance-exit type, at

$$x = \ell^{-1}\left[\left(\frac{D_2}{D_1}\right)^{\frac{1}{2}} \mp k\right].$$

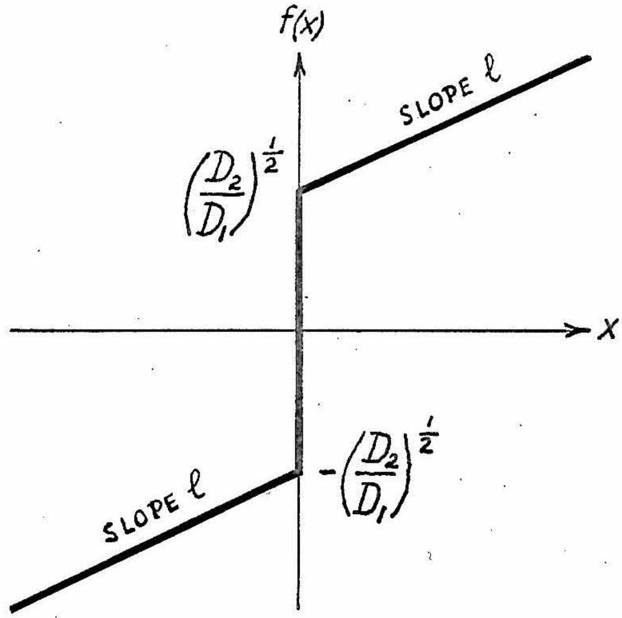
Only one case will be worked through, that where $k = \left(\frac{D_2}{D_1}\right)^{\frac{1}{2}}$. The restoring force f(x) for this case is shown on the next page. Since the shunt lies at $x_2 = 0$, no new points of division need be defined. Equations (5.35, 36, 39) give, for $x_o > 0$, the same equations (5.64-67) for γ_1^{1+}, γ_2^{1+}, γ_1^{1-}, γ_2^{1-} as in the linear case. Thus above the shunt the transition density is again given by (5.70-71). However, (5.43) is replaced by (5.44), which becomes

$$\lambda^{\beta_1}\left[(1-\beta_2)M(\beta_1-1, 1+\beta_1-\beta_2, -\lambda)\gamma_1^2 - \sigma e^{-\lambda}U(2-\beta_2, 1+\beta_1-\beta_2, \lambda)\gamma_2^2\right] = \lambda\gamma_2^{1-} \tag{5.84}$$

for $x_o > 0$, and similarly for $x_o < 0$, so that

$$\pi(\xi, \sigma | \xi_o) = \frac{\lambda^{1-\beta_1}}{1-\beta_2} \frac{\Gamma(\beta_1)}{\Gamma(1+\beta_1-\beta_2)} \frac{e^{-\zeta_o}\zeta_o^{-\beta_2}(-\zeta)^{\beta_1}}{\xi_o - 1} \times \frac{M(\beta_1, 1+\beta_1-\beta_2, \zeta)\, M(\beta_1, 1+\beta_1-\beta_2, \zeta_o)}{M(\beta_1-1, 1+\beta_1-\beta_2, -\lambda)} \quad \text{for } x_o > 0 > x \tag{5.85}$$

$$= \frac{\Gamma(1-\beta_2)}{\Gamma(1+\beta_1-\beta_2)} \frac{e^{-\zeta_o}(-\zeta_o)^{-\beta_2}(-\zeta)^{\beta_1}}{1-\xi_o} M(\beta_1, 1+\beta_1-\beta_2, \zeta_o) \times \left[\frac{\sigma e^{-\lambda}}{1-\beta_2} \frac{U(2-\beta_2, 1+\beta_1-\beta_2, \lambda)}{M(\beta_1-1, 1+\beta_1-\beta_2, -\lambda)} M(\beta_1, 1+\beta_1-\beta_2, \zeta) + e^{\zeta}U(1-\beta_2, 1+\beta_1-\beta_2, -\zeta)\right] \quad \text{for } 0 > x > x_o \tag{5.86}$$



The special case of example 2, section 5.3.

$$= \frac{\Gamma(1-\beta_2)}{\Gamma(1+\beta_1-\beta_2)} \frac{e^{-\zeta_o}(-\zeta_o)^{-\beta_2}(-\zeta)^{\beta_1}}{1-\xi_o} M(\beta_1, 1+\beta_1-\beta_2, \zeta)$$

$$\times \left[\frac{\sigma e^{-\lambda}}{1-\beta_2} \frac{U(2-\beta_2, 1+\beta_1-\beta_2, \lambda)}{M(\beta_1-1, 1+\beta_1-\beta_2, -\lambda)} M(\beta_1, 1+\beta_1-\beta_2, \zeta_o) + e^{\zeta_o} U(1-\beta_2, 1+\beta_1-\beta_2, -\zeta_o)\right]$$

for $0 > x_o > x$.

(5.87)

The steady-state density $P_o(x)$ is, from (5.48, 51),

$$P_o(x) = \frac{(D_1 D_2)^{-\frac{1}{2}}(-\zeta)^{1+2\lambda} e^{\zeta}}{4\lambda^2 \gamma(2\lambda, \lambda)} \qquad \text{for } x < 0$$

$$= 0 \qquad \text{for } x > 0 \quad . \tag{5.88}$$

To obtain the spectral density, one finds that

$$\mathcal{I}_2^* = \frac{8\lambda^2}{2\lambda-1}\left\{\frac{\lambda^2-2\lambda+2}{\lambda-1} + \frac{2}{\sigma}(\lambda-1)^2 - \frac{\lambda^{2\lambda}e^{-\lambda}}{\gamma(2\lambda,\lambda)}\left[\frac{\lambda-3}{\lambda-1} + \frac{2}{\sigma}(\lambda-1)\right]\right\} \tag{5.89}$$

$$\mathcal{P}_2^*(\xi_2) = \frac{16\lambda^{2+2\lambda}e^{-\lambda}}{(\sigma+2\lambda-1)\gamma(2\lambda,\lambda)}\left[\frac{\lambda-1}{\sigma} - \frac{M(\beta_1, 1+\beta_1-\beta_2, -\lambda)}{(1-\beta_2)M(\beta_1-1, 1+\beta_1-\beta_2, -\lambda)}\right], \tag{5.90}$$

while $\mathcal{P}_2^*(\xi_3)$, $\mathcal{Q}_2^*(\xi_2)$, $\mathcal{Q}_2^*(\xi_3)$ are all zero. Note that on account of the discontinuity of $a(x)P(x, t|x_o)$ at $x = 0$, the quantity $\alpha(\xi_2)\pi(\xi_2, \sigma|\xi_2)$ in (5.59) must be considered as the limit of $\alpha(\xi_o)\pi(\xi, \sigma|\xi_o)$ as x, $x_o \to 0$, both from below. That is, $x = 0$ must be treated as though it were an end-point, rather than a point of discontinuity. This is permissible, since $Q_o(0-) = 0$, so the argument at the end of section 2.3 holds. Thus, using (5.86) and (5.32),

$$\alpha(\xi_2)\pi(\xi_2,\sigma|\xi_2) = \frac{2M(\beta_1, 1+\beta_1-\beta_2, -\lambda)}{(1-\beta_2)M(\beta_1-1, 1+\beta_1-\beta_2, -\lambda)} \quad . \tag{5.91}$$

Similarly,

$$D^*\left[\alpha(\xi_2)\pi(\xi_2,\sigma|\xi_2)\right] = 1. \tag{5.92}$$

Thus (5.90) for $\mathscr{P}^*(\xi_2)$ is obtained from (5.59). The corresponding evaluation of $\mathscr{Q}_2^*(\xi_2)$ is straightforward, since $Q(x,t|x_o)$ is continuous at $x = 0$. Substituting (5.89, 90) into (5.57),

$$r(s) = \frac{D_2}{\ell(1-\ell D_1)[s+\ell(1-\ell D_1)]}\left[\frac{1-4\ell D_1+8\ell^2D_1^2}{\ell D_1(1-2\ell D_1)} + \frac{(1-2\ell D_1)^2}{sD_1}\right.$$
$$- \frac{2\lambda^{2\lambda}e^{-\lambda}}{\gamma(2\lambda,\lambda)}\left\{\frac{1-6\ell D_1}{1-2\ell D_1} + \frac{\ell}{s}(1-2\ell D_1)\right.$$
$$\left.\left. + \frac{\ell(1-\ell D_1)}{s+\ell(1-\ell D_1)}\left[\frac{\ell}{s}(1-2\ell D_1) - \frac{2M(\beta_1, 1+\beta_1-\beta_2, -\lambda)}{(1-\beta_2)M(\beta_1-1, 1+\beta_1-\beta_2, -\lambda)}\right]\right\}\right] . \tag{5.93}$$

Thus the spectral density $\Phi(\omega)$ is obtained by (2.64). Also,

$$\langle x^2\rangle = \frac{D_1D_2\left[1-4\ell D_1+8\ell^2D_1^2 - \frac{2\lambda^{2\lambda}e^{-\lambda}}{\gamma(2\lambda,\lambda)}\ell D_1(1-6\ell D_1)\right]}{\ell^2D_1^2(1-\ell D_1)(1-2\ell D_1)} \tag{5.94}$$

$$\langle x\rangle = -\frac{(D_1D_2)^{\frac{1}{2}}}{1-\ell D_1}\left[\frac{2\lambda e^{-\lambda}}{\gamma(2\lambda,\lambda)} - \frac{1-2\ell D_1}{\ell D_1}\right] \quad . \tag{5.95}$$

As in the previous case, the process is unstable in mean square for $D_1 \geq \frac{1}{2\ell}$, and in mean for $D_1 \geq \frac{1}{\ell}$.

5.4 ZERO CORRELATION CASE

In this section we deal with the stochastic differential equation

$$\dot{x} + \left(\ell_i x + k_i\right)(1+m(t)\Big) = n(t) \quad , \quad x \in (x_{i+1}, x_i) \quad , \tag{5.96}$$

where

$$\langle m(t_1)m(t_2)\rangle = 2D_1\,\delta(t_1-t_2) \tag{5.97}$$

$$\langle n(t_1)\,n(t_2)\rangle = 2D_2\,\delta(t_1-t_2) \tag{5.98}$$

$$\langle m(t_1)\,n(t_2)\rangle = \langle m(t)\rangle = \langle n(t)\rangle = 0 \quad . \tag{5.99}$$

The dimensionless variables of (5.8-15) can be used for all intervals for which $\ell_i \neq 0$. In the general derivation below it will be assumed that no $\ell_i = 0$.

<u>Solutions to the FP equation</u>

The FP equation is given by (5.17) with $\rho = 0$. That is

$$\frac{d^2}{d\xi^2}\left[(\xi^2+1)\pi\right] + \frac{d}{d\xi}\left[(2\lambda-1)\xi\pi\right] - \sigma\pi = -\delta(\xi-\xi_o) \ . \tag{5.100}$$

Since $\alpha(\xi) \geq 1$, there can be no irregular points; in this respect the system resembles that of Chapter III.

By making the substitutions

$$\zeta = \xi(\xi^2+1)^{-\frac{1}{2}} \quad , \quad \Psi = (1-\zeta^2)^{-\frac{1}{2}(1+\lambda)}\,\pi \quad , \tag{5.101}$$

the homogeneous form of (5.100) becomes

$$(1-\zeta^2)\frac{d^2\Psi}{d\zeta^2} - 2\zeta\frac{d\Psi}{d\zeta} + \left[\lambda(\lambda+1) - \frac{\lambda^2+\sigma}{1-\zeta^2}\right]\Psi = 0 \ . \tag{5.102}$$

This is Legendre's equation of degree λ and order μ,[1] where

$$\mu = (\lambda^2 + \sigma)^{\frac{1}{2}} . \tag{5.103}$$

Thus two solutions are

$$\pi_1(\xi) = (\xi^2+1)^{-\frac{1}{2}(1+\lambda)} P_\lambda^{-\mu}(\zeta) \tag{5.104}$$

$$\pi_2(\xi) = (\xi^2+1)^{-\frac{1}{2}(1+\lambda)} P_\lambda^{\mu}(\zeta) , \tag{5.105}$$

where $P_\lambda^\mu(x)$ denotes the associated Legendre function of the first kind of degree λ and order μ. These solutions are independent (form a fundamental set) provided μ is not an integer. When this is the case, P_λ^μ could be replaced by Q_λ^μ in π_2.

Related quantities

Using (5.15) and equation 8.5.4 in [1],

$$\psi_{1,2}(\xi) = (\beta_{2,1}-1)(\xi^2+1)^{-\frac{1}{2}\lambda} P_{\lambda-1}^{\mp\mu}(\zeta) . \tag{5.106}$$

The Wronskian of π_1 and π_2 is

$$w(\xi) = \frac{2}{\pi}(\xi^2+1)^{-\frac{3}{2}-\lambda} \sin \mu\pi . \tag{5.107}$$

(This result can be obtained by substituting equation 3.4.17 in [11] into equation 3.4.25.) A fundamental set of solutions of the backwards

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[1] See Erdélyi et al. [11], Chapter 3, or Abramowitz and Stegun [1], chapter 8. Since we are concerned with real ζ, $-1<\zeta<1$, the slightly modified Legendre functions dealt with in [11], section 3.4, and [1], section 8.3, are required, and will be used throughout.

equation is obtained similarly to the forward equation, giving

$$\left(\xi_o^2+1\right)^{\frac{1}{2}\lambda} P_\lambda^{\mp\mu}(\zeta_o) \ . \tag{5.108}$$

The Laplace-transformed transition density

Putting

$$\pi^i(\xi,\sigma|\xi_o) = \gamma_1^i \pi_1^i(\xi) + \gamma_2^i \pi_2^i(\xi) \ , \tag{5.109}$$

the coefficients γ_1^i, γ_2^i satisfy the usual conditions at x_o (see (2.21, 22)) and x_i, $i = 1, 2, \ldots n$ (see (5.19, 20)). These become:

$$\gamma_1^{k-} = \gamma_2^{k+} - \frac{\pi}{2\sin\pi\mu k}\left(\xi_o^2+1\right)^{\frac{1}{2}\lambda_k} P_{\lambda_k}^{\mu_k}(\zeta_o) \tag{5.110}$$

$$\gamma_2^{k-} = \gamma_2^{k+} + \frac{\pi}{2\sin\pi\mu k}\left(\xi_o^2+1\right)^{\frac{1}{2}\lambda_k} P_{\lambda_k}^{-\mu_k}(\zeta_o) \ , \tag{5.111}$$

where $x_o \in (x_{k+1}, x_k)$;

$$\lambda_i\left[\left(\xi_i^i\right)^2+1\right]^{\frac{1}{2}-\frac{1}{2}\lambda_i}\left[\gamma_1^i P_{\lambda_i}^{-\mu_i}(\zeta_i^i) + \gamma_2^i P_{\lambda_i}^{\mu_i}(\zeta_i^i)\right]$$

$$=\lambda_{i-1}\left[\left(\xi_i^{i-1}\right)^2+1\right]^{\frac{1}{2}-\frac{1}{2}\lambda_{i-1}}\left[\gamma_1^{i-1} P_{\lambda_{i-1}}^{-\mu_{i-1}}\left(\zeta_i^{i-1}\right) + \gamma_2^{i-1} P_{\lambda_{i-1}}^{\mu_{i-1}}\left(\zeta_i^{i-1}\right)\right] \tag{5.112}$$

$$\left[\left(\xi_i^{\,i}\right)^2+1\right]^{-\frac{1}{2}\lambda_i}\left[\gamma_1^{\,i}\left(\beta_2^{\,i}-1\right)P_{\lambda_i-1}^{-\mu_i}\left(\zeta_i^{\,i}\right)+\gamma_2^{\,i}\left(\beta_1^{\,i}-1\right)P_{\lambda_i-1}^{-\mu_i}\left(\zeta_i^{\,i}\right)\right]$$

$$=\left[\left(\xi_i^{\,i-1}\right)^2+1\right]^{-\frac{1}{2}\lambda_{i-1}}\left[\gamma_1^{\,i-1}\left(\beta_2^{\,i-1}-1\right)P_{\lambda_{i-1}-1}^{-\mu_{i-1}}\left(\zeta_i^{\,i-1}\right)+\gamma_2^{\,i-1}\left(\beta_1^{\,i-1}-1\right)P_{\lambda_{i-1}-1}^{\mu_{i-1}}\left(\zeta_i^{\,i-1}\right)\right] \tag{5.113}$$

for $i = 2, 3, \ldots, n-1$; and

$$\gamma_1^{\,1}\left(\beta_2^{\,1}-1\right)P_{\lambda_1-1}^{-\mu_1}\left(\zeta_1^{\,1}\right)+\gamma_2^{\,1}\left(\beta_1^{\,1}-1\right)P_{\lambda_1-1}^{\mu_1}\left(\zeta_1^{\,1}\right)=0 \tag{5.114}$$

$$\gamma_1^{\,n-1}\left(\beta_2^{\,n-1}-1\right)P_{\lambda_{n-1}-1}^{-\mu_{n-1}}\left(\zeta_n^{\,n-1}\right)+\gamma_2^{\,n-1}\left(\beta_1^{\,n-1}-1\right)P_{\lambda_{n-1}-1}^{\mu_{n-1}}\left(\zeta_n^{\,n-1}\right)=0. \tag{5.115}$$

If x_1 or x_n respectively is infinite, (5.114) or (5.115) respectively becomes

$$\gamma_2^{\,1} = 0 \tag{5.116}$$

$$\frac{\gamma_1^{\,n-1}}{\gamma_2^{\,n-1}} = \frac{\sin \lambda_{n-1}\pi}{\pi}\,\Gamma(\beta_1^{\,n-1})\,\Gamma(1-\beta_2^{\,n-1}), \tag{5.117}$$

since, as $z \to 1$,

$$P_\lambda^\mu(z) \sim \frac{2^{\frac{1}{2}\mu}(1-z)^{-\frac{1}{2}\mu}}{\Gamma(1-\mu)}, \qquad \text{all } \mu \neq 1, 2, 3, \ldots \tag{5.118}$$

$$\begin{aligned} P_\lambda^\mu(-z) &\sim 2^{\frac{1}{2}\mu}\,\frac{\sin \pi\lambda}{\pi}\,\Gamma(\mu)(1-z)^{-\frac{1}{2}\mu}, \qquad \mu > 0 \\ &\sim \frac{2^{-\frac{1}{2}\mu}\,\Gamma(-\mu)(1-z)^{\frac{1}{2}\mu}}{\Gamma(1+\lambda-\mu)\Gamma(-\lambda-\mu)}, \qquad \mu < 0 \end{aligned} \tag{5.119}$$

(see [11], section 3.9.2).

Steady-state probability

From (2.28),

$$P_o(x) = C_i\left[\left(\xi^i\right)^2 + 1\right]^{-\frac{1}{2}-\lambda_i} . \qquad (5.120)$$

Since aP_o is continuous, this gives, for $n-1 \geq i > j \geq 1$,

$$\frac{C_i}{C_j} = \left[\left(\xi_i^i\right)^2+1\right]^{-\frac{1}{2}+\lambda_i} \left[\frac{\left(\xi_i^{i-1}\right)^2+1}{\left(\xi_{i-1}^{i-1}\right)^2+1}\right]^{\frac{1}{2}-\lambda_{i-1}} \cdots \left[\frac{\left(\xi_{j+2}^{j+1}\right)^2+1}{\left(\xi_{j+1}^{j+1}\right)^2+1}\right]^{\frac{1}{2}-\lambda_{j+1}} \left[\left(\xi_{j+1}^{j}\right)^2+1\right]^{\frac{1}{2}-\lambda_j} .$$

(5.121)

The C_i are thus determined up to an arbitrary constant. To determine this, use (2.29), noting that, for $\xi \geq 0$,

$$\int_0^{\xi} (z^2+1)^{-\frac{1}{2}-\lambda} dz = \frac{1}{2}\int_0^{\xi^2(\xi^2+1)^{-1}} t^{-\frac{1}{2}}(1-t)^{\lambda-1}dt = \frac{1}{2}B_{\xi^2(\xi^2+1)^{-1}}(\tfrac{1}{2}, \lambda) ,$$

(5.122)

where $B_z(a, b) = \int_0^z t^{a-1}(1-t)^{b-1}dt$ is the incomplete beta function. Thus one gets

$$C_j = \left(D_1 D_2\right)^{-\frac{1}{2}}\left\{\sum_{i=1}^{n-1} \frac{C_i}{C_j}\lambda_i\left[\varepsilon_i^i B_{(\zeta_i^i)^2}(\tfrac{1}{2}, \lambda_i) - \varepsilon_{i+1}^i B_{(\zeta_{i+1}^i)^2}(\tfrac{1}{2}, \lambda_i)\right]\right\}^{-1} , \qquad (5.123)$$

where $\frac{C_i}{C_j}$ is given by (5.121) and $\varepsilon_k^i = \operatorname{sgn} \xi_k^i$.

$P_o(x)$ as Abelian limit of $P(x, t|x_o)$

We verify that

$$P_o(x) = \lim_{s \to 0} sp(x, s|x_o) \quad . \tag{5.124}$$

Letting $s \to 0$ in (5.110-117), it can be shown that

$$\gamma_1^i = O(\frac{1}{\sigma}) \quad , \quad \gamma_2^i = O(1) \quad , \quad \gamma_1^{k-} - \gamma_1^{k+} = O(1) \quad , \tag{5.125}$$

so that, to first order,

$$\pi^i(\xi, \sigma|\xi_o) \sim \gamma_1^i \frac{2^{-\lambda}}{\Gamma(\lambda+1)} \left[(\xi^i)^2 + 1\right]^{-\frac{1}{2} - \lambda_i} , \tag{5.126}$$

since (see [1], 8.6.10)

$$P_\nu^{-\nu}(z) = \frac{2^{-\nu}(1-z^2)^{\frac{1}{2}\nu}}{\Gamma(\nu+1)} \quad . \tag{5.127}$$

It can also be shown that

$$\gamma_1^i \sim \frac{\ell_i}{s}(D_1 D_2)^{\frac{1}{2}} 2^\lambda \Gamma(\lambda+1) C_i \quad . \tag{5.128}$$

The method is the same as that used for the same purpose in section 3.2.

Spectral density

The Laplace-transformed autocorrelation is given by (2.49, 58, 62, 63), which in this case take the form

$$r(s) = D_1^2 D_2 \sum_{k=1}^{n-1} \left\{ \frac{4\lambda_k^2}{\sigma_k + 2\lambda_k - 1}\, \vartheta_k^* \right.$$

$$\left. + \sum_{i=1}^{n-1} \frac{4\lambda_i^2}{\sigma_i + 2\lambda_i - 1} \left[2\lambda_i \left\{ \xi - \varkappa_i + \frac{\varkappa_i}{\sigma_i}(1 - 2\lambda_i) \right\} \mathcal{Q}_k^*(\xi) - \theta_k^*(\xi) \right]_{x=x_{i+1}}^{x_i} \right\},$$

(5.129)

where

$$\vartheta_k^* = 4\lambda_k^3 \left[\left(\frac{D_1}{D_2} \right)^{\frac{1}{2}} a(x_o) P_o(x_o) \left(\frac{\xi_o}{1-\lambda_k} - \frac{4\varkappa_k}{1-2\lambda_k} + \frac{2\varkappa_k}{\sigma_k} \right) \right]_{x=x_{k+1}}^{x_k}$$

$$+ 2\lambda_k^2 \int_{x_{k+1}}^{x_k} P_o(x_o)\, dx_o \left[2\varkappa_k^2 - \frac{1}{1-\lambda_k} - \frac{2\varkappa_k^2}{\sigma_k}(1 - 2\lambda_k) \right] \qquad (5.130)$$

$$\theta_k^*(\xi_i) = \frac{8\lambda_k^3}{\sigma_k + 2\lambda_k - 1} \left[\left(\frac{D_1}{D_2} \right)^{\frac{1}{2}} a(x_o) P_o(x_o) \left\{ \left[\xi_o - \varkappa_k + \frac{\varkappa_k}{\sigma_k}(1-2\lambda_k) \right] D^* \left[\alpha(\xi_i) \pi(\xi_i, \sigma \mid \xi_o) \right] \right. \right.$$

$$\left. \left. - \alpha(\xi_i) \pi(\xi_i, \sigma \mid \xi_o) \right\} \right]_{x_o = x_{k+1}}^{x_k} \qquad (5.131)$$

$$\mathcal{Q}_k^*(\xi_i) = \frac{4\lambda_k^2}{\sigma_k + 2\lambda_k - 1} \left[\left(\frac{D_1}{D_2} \right)^{\frac{1}{2}} a(x_o) P(x_o) \left\{ \left[\xi_o - \varkappa_k + \frac{\varkappa_k}{\sigma_k}(1 - 2\lambda_k) \right] D^* \psi(\xi_i, \sigma \mid \xi_o) \right. \right.$$

$$\left. \left. - \psi(\xi_i, \sigma \mid \xi_o) \right\} \right]_{x_o = x_{k+1}}^{x_k}, \qquad (5.132)$$

where D^* is given by (3.68). The spectral density $\Phi(\omega)$ is then given by (2.64).

Variance and Mean

For the case treated in this section, (2.68) and (2.69) become, respectively,

$$\langle x^2 \rangle = 2D_1 D_2 \sum_{k=1}^{n-1} \lambda_k^2 \left\{ 2\lambda_k \left[\left(\frac{D_1}{D_2}\right)^{\frac{1}{2}} a(x) P_o(x) \left(\frac{\xi}{1-\lambda_k} - \frac{4\varkappa_k}{1-2\lambda_k} \right) \right]_{x=x_{k+1}}^{x_k} \right.$$

$$\left. - \int_{x_{k+1}}^{x_k} P_o(x)\, dx \left(2\varkappa_k^2 - \frac{1}{1-\lambda_k} \right) \right\} \qquad (5.133)$$

$$\langle x \rangle = 2(D_1 D_2)^{\frac{1}{2}} \sum_{k=1}^{n-1} \lambda_k \left\{ \frac{2\lambda_k}{1-2\lambda_k} \left[\left(\frac{D_1}{D_2}\right)^{\frac{1}{2}} a(x) P_o(x) \right]_{x=x_{k+1}}^{x_k} - \varkappa \int_{x_{k+1}}^{x_k} P_o(x) dx \right\} .$$

(5.134)

Case $\ell_i = 0$ for some i

In all the above it is assumed that $\ell_i \neq 0$, for all i. In the exceptional case, the FP equation takes the form

$$(D_1 k_i^2 + D_2) \frac{d^2 p}{dx^2} + k_i \frac{dp}{dx} - sp = -\delta(x-x_o) \quad . \qquad (5.135)$$

Except that D is replaced by $D_1 k_i^2 + D_2$, this is identical to that of Chapter III, with $\ell_i = 0$; see section 3.1 for solution.

Alternatively, it appears to be possible, as before, to treat this case as the limit of the general case letting $\ell_i \to 0$.[1]

Example --the linear case

Since there can be no irregular points in these uncorrelated systems, any system can be handled exactly the same as the system with the same restoring function dealt with in Chapter III. Thus only one example will be worked, that of the linear system (with k = 0). Wong [45] has found the transition probability in this case in the form of an eigenfunction expansion (case E, p. 269).

Equations (5.110, 111, 116, 117) become

$$\gamma_2^+ = 0 \tag{5.136}$$

$$\gamma_1^- = \gamma_1^+ - \frac{\pi}{2 \sin \pi\mu} (\xi_o^2 + 1)^{\frac{1}{2}\lambda} P_\lambda^\mu(\zeta_o) \tag{5.137}$$

$$\gamma_2^- = \gamma_2^+ + \frac{\pi}{2 \sin \pi\mu} (\xi_o^2 + 1)^{\frac{1}{2}\lambda} P_\lambda^{-\mu}(\zeta_o) \tag{5.138}$$

$$\gamma_1^- = \gamma_2^- \frac{\sin \pi\lambda}{\pi} \Gamma(\beta_1)\Gamma(1-\beta_2) , \tag{5.139}$$

whence

---

[1] Appropriate asymptotic formulas for the Legendre function are given on p. 658 of Jeffries and Jeffries [27].

$$\pi(\xi, \sigma | \xi_o) = \frac{\left(\xi_o^2+1\right)^{\frac{1}{2}\lambda}\left(\xi^2+1\right)^{-\frac{1}{2}(1+\lambda)}}{2 \sin \mu\pi} P_\lambda^{-\mu}(\zeta)\Big[\sin \lambda\pi\,\Gamma(\beta_1)\Gamma(1-\beta_2)P_\lambda^{-\mu}(\zeta_o)$$

$$+ \pi\, P_\lambda^{\mu}(\zeta_o)\Big] \quad \text{for } x \geq x_o \quad (5.140)$$

$$= \frac{\left(\xi_o^2+1\right)^{\frac{1}{2}\lambda}\left(\xi^2+1\right)^{-\frac{1}{2}(1+\lambda)}}{2 \sin \mu\pi} P_\lambda^{-\mu}(\zeta_o)\Big[\sin \lambda\pi\,\Gamma(\beta_1)\Gamma(1-\beta_2)P_\lambda^{-\mu}(\zeta)$$

$$+\pi\, P_\lambda^{\mu}(\zeta)\Big] \quad \text{for } x \leq x_o\,. \quad (5.141)$$

The steady-state probability $P_o(x)$ is, from (5.120, 123),

$$P_o(x) = \frac{(D_1D_2)^{\frac{1}{2}}}{2\lambda B(\frac{1}{2}, \lambda)}(\xi^2+1)^{-\frac{1}{2}-\lambda} = \frac{\pi^{\frac{1}{2}}(D_1D_2)^{-\frac{1}{2}}\Gamma(\lambda-1)}{2\Gamma(\lambda+\frac{1}{2})}(\xi^2+1)^{-\frac{1}{2}-\lambda}\,. \tag{5.142}$$

Using (5.127), it easily verified that this equals $\lim_{s\to 0} sp(x, s\,|x_o)$.

To obtain the spectral density, note that

$$\mathscr{I}^* = \frac{1}{\ell D_1(1-2\ell D_1)}\,, \tag{5.143}$$

while all $\mathscr{P}$, $\mathscr{Q}$ are zero. Thus

$$r(s) = \frac{D_2}{\ell(1-2\ell D_1)[s+\ell(1-\ell D_1)]} \tag{5.144}$$

$$\Phi(\omega) = \frac{2}{\pi} D_2(1-\ell D_1)\left\{(1-2\ell D_1)\left[\omega^2+\ell^2(1-\ell D_1)^2\right]\right\}^{-1} \tag{5.145}$$

$$\langle x^2\rangle = \frac{D_2}{\ell}\,\frac{1}{1-2\ell D_1}\,. \tag{5.146}$$

Note that the system is unstable in mean square for $D_1 \geq \frac{1}{2\ell}$, so these equations hold only for $D_1 < \frac{1}{2\ell}$.

According to Gray [22, 23] this linear system should have the same autocorrelation and spectrum as the system

$$\frac{dy}{dt} + \ell(1-\ell D_1)y = a(t) \quad , \tag{5.147}$$

where a(t) is white noise such that $\langle y^2 \rangle = \langle x^2 \rangle$. This is easily verified.

APPENDIX

SOME NOTES ON SECOND ORDER SYSTEMS

The FP equation

Consider the system with stochastic differential equation

$$\ddot{x} + f(x, \dot{x}) = \sum_{j=1}^{m} h_j(x, \dot{x}) n_j(t) , \tag{A1}$$

where the $n_j(t)$ represent white noise. This can be written as a pair of first order equations, thus:

$$\dot{x} - y = 0 \tag{A2}$$

$$\dot{y} + f(x, y) = \sum_{j=1}^{m} h_j(x, y) n_j(t) \quad . \tag{A3}$$

Then, by (1.18), the FP equation is

$$\frac{\partial P}{\partial t} = \sum_{j,k=1}^{m} D_{jk} \frac{\partial}{\partial y} \left[h_j \frac{\partial}{\partial y} (h_k P) \right] - y \frac{\partial P}{\partial x} + \frac{\partial}{\partial y} (fP) \tag{A4}$$

$$= \frac{\partial^2}{\partial y^2} \left[a(x, y) P \right] - \frac{\partial}{\partial y} \left[b(x, y) P \right] - y \frac{\partial P}{\partial x} \tag{A5}$$

say. In particular, if $h_j(x, y)$, $f(x, y)$ are piecewise linear in x, y, so is $b(x, y)$, and $a(x, y)$ is piecewise quadratic.

Appropriate boundary conditions

The right hand side of (A5) does not contain a term in $\frac{\partial^2 P}{\partial x^2}$, so that, considered as a parabolic equation, (A5) is degenerate. Fichera [20] has developed, and Oleinik [34, 35] has extended, a theory of boundary value problems for so-called elliptic-parabolic equations, i.e., equations of form

$$\sum_{i,j} a_{ij} u_{x_i x_j} + \sum_i b_i u_{x_i} + cu = f \ , \qquad \text{(A6)}$$

where the matrix $[a_{ij}]$ is non-negative, but not necessarily positive definite. Fichera shows that the boundary can be divided into three regions: $\Sigma^{(1)}$, where no boundary conditions can be imposed; $\Sigma^{(2)}$, where the Dirichlet condition u = const. can be imposed; and $\Sigma^{(3)}$, where either the Dirichlet condition or the Neumann condition $a_{ij} u_{x_i} n_j$ = const. ($\{n_j\}$ being the unit normal) can be imposed. Obviously, linear combinations of these two conditions can also be imposed on $\Sigma^{(3)}$.

Consider (A5) in the domain $\{t > 0,\ x_2 < x < x_1,\ y_2 < y < y_1\}$, where y_1 and y_2 are finite and a(x, y) is not zero for $y = y_1$ or y_2. Then $\Sigma^{(1)}$ consists of the surfaces $t = \infty$, $x = x_1$ for $y \geq 0$, and $x = x_2$ for $y \leq 0$; $\Sigma^{(2)}$ consists of the surfaces $t = 0$, $x = x_1$ for $y < 0$, and $x = x_2$ for $y > 0$; and $\Sigma^{(3)}$ consists of the surfaces $y = y_1, y_2$. It is apparent that sample paths can not enter through $\Sigma^{(1)}$, can enter but not exit through $\Sigma^{(2)}$, and can do both through $\Sigma^{(3)}$. The permissible boundary conditions on $\Sigma^{(3)}$ may be compared with the elastic boundary condition (1.46) in the one-dimensional case.

More general domains

If y_1 or y_2 is infinite, or if $a(x, y) \to 0$ as $y \to y_1$ or y_2, then this part of $\Sigma^{(3)}$ may degenerate to $\Sigma^{(1)}$ or $\Sigma^{(2)}$--compare with the contrast between regular and irregular boundaries in section 1.4. If instead of a rectangle in the x, y plane a more general region is chosen, $\Sigma^{(3)}$ will consist of all parts of its boundary where the tangent is not perpendicular to the x axis. Similarly for time dependent boundaries the same thing is true.

Difficulties encountered

In attempting to deal with second order systems by methods similar to those used in Chapter II for first order systems, the following problems occur. Firstly, analytical solutions are not known for the FP equation (A5) with delta function initial condition, or its Laplace transform, except in the simplest linear cases. Secondly, the piecing together of solutions in different linear subdomains involves integral equations rather than algebraic equations, since their common boundaries are lines (or curves) in the x, y plane, rather than points on the x axis. Thirdly, the writer has not been able to generalize the method of section 2.3, whereby the Laplace-transformed autocorrelation is expressed in terms of the Laplace-transformed transition density. It appears unlikely that all these difficulties can be eliminated.

The method of Robinson

The output (x, y) of the second order system (A2-3) forms a two dimensional Markov process, so that the formula (0.2) for the autocorrelation can be written

$$R_x(t) = \iiiint x x_o P(x, y, t | x_o, y_o) P_o(x_o, y_o) dx\, dy\, dx_o\, dy_o \quad . \tag{A7}$$

Thus, once P and P_o have been found by solving the FP equation, four integrations are necessary to obtain $R_x(t)$ (unless the third difficulty mentioned above can be overcome). Robinson [37] has shown that this can be reduced to two. In fact

$$R_x(t) = \iint x\, v(x, y, t)\, dx\, dy \quad , \tag{A8}$$

where

$$v(x, y, t) = \iint x_o P(x, y, t | x_o, y_o) P_o(x_o, y_o) dx_o\, dy_o \quad , \tag{A9}$$

so that v satisfies the FP equation (A5) with initial condition

$$v(x, y, 0) = x\, P_o(x, y) \quad . \tag{A10}$$

The steady-state FP equation can be solved to obtain P_o in several important cases. However, the time dependent equation, with initial condition (A10)--or any other initial condition--has been solved in only one (linear) case.

The work of Wolaver

Wolaver in [43, 44] claims to have solved the FP equation for v in the two special cases where (A1) takes the form

(a) $\ddot{x} + c\dot{x} + k \operatorname{sgn} x = n(t)$, $\quad -\infty < x < \infty$. (A11)

(b) $\ddot{x} + c\dot{x} + \ell x + k \operatorname{sgn} x = n(t)$, $\quad -\infty < x < \infty$. (A12)

However, as Professor Caughey pointed out to the writer, the solutions Wolaver obtains do not satisfy the boundary conditions he assumes, which are themselves incorrect.

Wolaver's method is to solve the FP equation by Fourier transform methods in each of the regions $x \gtrless 0$, and to match up the solutions. He assumes--see his equation (II-7)--that $v(o, y, t) \equiv 0$. This is not so, and in fact it is seen from the discussion above of appropriate boundary conditions that when solving in the region $x > 0$, one can specify $v(o, y, t)$ for $y < 0$, but not for $y \geq 0$; and conversely when solving in the region $x < 0$. Thus it is to be expected that any "solutions" obtained by Wolaver to his ill-posed problems will be incorrect, since there is no reason to believe that any solutions exist.

Correction to Wolaver

Case (a) above--bang-bang restoring force--is worked through here, without making the invalid assumption $v(o, y, t) \equiv 0$. The treatment is incomplete, since it leads to an integral equation for $v(o, y, t)$ which the writer has not been able to solve. Case (b) can be treated in the same way.

Statement of the problem

For x > 0, the FP equation (A4) becomes

$$D \frac{\partial^2 v}{\partial y^2} + c \frac{\partial}{\partial y}(yv) + k \frac{\partial v}{\partial y} - y \frac{\partial v}{\partial x} - \frac{\partial v}{\partial t} = 0 \quad , \tag{A13}$$

and the initial condition (A10) is

$$v(x, y, o) = \frac{kx}{\sqrt{\pi}} \left(\frac{c}{2D}\right)^{\frac{3}{2}} \exp\left[-\frac{c}{D}\left(kx + \tfrac{1}{2} y^2\right)\right] . \tag{A14}$$

According to the discussion above, for a well-posed problem one can specify v(o, y, t) for y > 0, and upon solution one will obtain v(o, y, t) for y < 0. However, by symmetry one has

$$v(o, y, t) = -v(o, -y, t) \quad , \tag{A15}$$

so that an obvious method is to solve the problem for arbitrary v(o, y, t), y > 0, find the corresponding value of v(o, -y, t), and then substitute into (A15), which should lead to an equation for v(o, y, t), y > 0, with a unique solution.

Partial solution

Following Wolaver, Appendix II, one performs on (A13, 14) a two-sided Fourier transform with respect to y and a one-sided Fourier transform with respect to x, so that, if

$$v_1(x, \eta, t) = \int_{-\infty}^{\infty} v\, e^{-i\eta y}\, dy = \mathfrak{F}_y(v) \tag{A16}$$

$$v_2(\xi, \eta, t) = \int_0^\infty v_1 e^{-i\xi x}\, dx = \mathfrak{F}_x(v_1) \quad , \tag{A17}$$

one gets

$$\frac{\partial v_2}{\partial t} + (c\eta - \xi)\frac{\partial v_2}{\partial \eta} = (i\eta k - D\eta^2)v^2 + g(\eta, t) \tag{A18}$$

$$v_2(\xi, \eta, 0) = \frac{ck}{2D}\left(\frac{ck}{D} + i\xi\right)^{-2} e^{-\frac{D\eta^2}{2c}} \quad , \tag{A19}$$

where

$$g(\eta, t) = -\mathfrak{F}_y[yv(0, y, t)] \quad . \tag{A20}$$

Solving (A18-19) by the method of subsidiary equations (or by Laplace transformation with respect to t) one obtains

$$\begin{aligned} v_2(\xi, \eta, t) = {} & \frac{ck}{2D}\left(\frac{ck}{D} + i\xi\right)^{-2} \exp\left\{-\frac{D}{c^3}\left[\tfrac{1}{2}c^2\eta^2 + (c\eta - \xi)\left(\xi - \frac{ick}{D}\right)\left(1 - e^{-ct}\right)\right.\right. \\ & \left.\left. + c\xi\left(\xi - \frac{ick}{D}\right)\right]\right\} + \int_0^t g\left[(c\eta - \xi)e^{-c(t-s)} + \xi, s\right] \exp\left\{-\frac{D}{c^3}\left[\tfrac{1}{2}(c\eta - \xi)^2\left(1 - e^{-2c(t-s)}\right)\right.\right. \\ & \left.\left. + (c\eta - \xi)\left(\xi - \frac{ick}{D}\right)\left(1 - e^{-c(t-s)}\right) + c\xi\left(\xi - \frac{ick}{D}\right)(t-s)\right]\right\} ds \quad . \end{aligned} \tag{A21}$$

Carrying out the inverse transform with respect to x,

$$v_1(x,\eta,t) = v_o(x,\eta,t) + \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi e^{i\xi x}\int_0^t g\left[(c\eta-\xi)e^{-c(t-s)}+\xi, s\right]$$

$$\times \exp\left\{-\frac{D}{c^3}\left[\tfrac{1}{2}(c\eta-\xi)^2\left(1-e^{-2c(t-s)}\right)+(c\eta-\xi)\left(\xi-\frac{ick}{D}\right)\left(1-e^{-c(t-s)}\right)\right.\right.$$

$$\left.\left.+\, c\xi\left(\xi-\frac{ick}{D}\right)(t-s)\right]\right\} ds \quad , \tag{A22}$$

where

$$v_o(x,\eta,t) = \left[\frac{1}{2\pi}\left(ct-1+e^{-ct}\right)\right]^{\frac{1}{2}}\exp\left\{-\frac{c\left[kt-c^{-1}(k-i\eta D)\left(1-e^{-ct}\right)-cx\right]^2}{D\left(ct-1+e^{-ct}\right)}\right\}$$

$$-\left(\frac{c}{2D}\right)^{\frac{1}{2}}\left[kt-\frac{1}{c}(k+i\eta D)\left(1-e^{-ct}\right)-cx\right]\exp\left[-\frac{i\eta k}{c}\left(1-e^{-ct}\right)\right]$$

$$\times\,\mathrm{erfc}\quad\left(\frac{c}{D}\right)^{\frac{1}{2}}\frac{\left[kt-c^{-1}(k+i\eta D)\left(1-e^{-ct}\right)-cx\right]}{\left(ct-1+e^{-ct}\right)^{\frac{1}{2}}} \quad . \tag{A23}$$

But, from (A15),

$$v_1(0,\eta,t) = v_1(0,-\eta,t) = 0 \tag{A24}$$

$$g(\eta,t) = g(-\eta,t) \quad . \tag{A25}$$

Thus, substituting (A22) into (A24) and using (A25),

$$\frac{1}{\pi}\int_{-\infty}^{\infty} d\xi \int_{0}^{t} g\left[(c\eta-\xi)e^{-c(t-s)}+\xi, s\right]\cos\left[\frac{k}{c^2}(c\eta-\xi)\left(1-e^{-c(t-s)}\right)+\frac{k\xi}{c}(t-s)\right]$$

$$\times \exp\left\{-\frac{D}{c^3}\left[\tfrac{1}{2}(c\eta-\xi)^2+\xi(c\eta-\xi)\left(1-e^{-c(t-s)}\right)+c\xi^2(t-s)\right]\right\} ds$$

$$= -v_o(0,-\eta,t) - v_o(0,\eta,t) \quad . \qquad \text{(A26)}$$

It is now necessary to solve this integral equation for g, substitute into (A22), and then invert with respect to y to obtain v (x, y, t). This the writer has not been able to do.

A problem of Wang and Uhlenbeck

On page 338 of [40] the problem is stated of Brownian motion in a constant field of force (gravity) above a reflecting surface. The FP equation involved is identical to (A13), and P_o, and therefore v(x, y, 0), is the same except for a factor of 2 (since only the region $x > 0$ is considered). The reflecting condition at $x = 0$ leads to $v(0, y, t) = v(0, -y, t)$ instead of (A15). It is easily seen that this problem leads to the same integral equation (A26), with a somewhat different term on the right hand side.

Conclusions

The writer is not hopeful that useful results for second order systems can be obtained by the method of Robinson, or by any similar method related to those used in the body of this thesis for first order systems. It is unlikely that exact solutions can be obtained for the integral equation (A26), or any integral equations similarly obtained for more complicated systems. It is possible that approximate solutions can be obtained, but these would have to be simple enough to be twice integrable to obtain $R_x(t)$ by (A8). A numerical method of solution might be adopted, but the multiple numerical quadratures necessary would require a large amount of computation.

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