ON THE TRANSPORT PROPERTIES OF FLUID - PARTICLE FLOW

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ABSTRACT

The hydrodynamic forces acting on a solid particle in a viscous, incompressible fluid medium at low Reynolds number flow is investigated mathematically as a prerequisite to the understanding of transport processes in two-phase flow involving solid particles and fluid. Viscous interaction between a small number of spherical particles and continuous solid boundaries as well as fluid interface are analyzed under a "point-force" approximation. Non-spherical and elastic spherical particles in simple shear flow are then considered. Non-steady motion of a spherical particle is briefly touched upon to illustrate the transient effect of particle motion.

A macroscopic continuum description of particle-fluid flow is formulated in terms of spatial averages yielding a set of particle continuum and bulk fluid equations. Phenomenological formulas describing the transport processes in a fluid medium are extended to cases where the volume concentration of solid particles is sufficiently high to exert an important influence. Hydrodynamic forces acting on a spherical solid particle in such a system, e. g. drag, torque, rotational coupling force, and viscous collision force between streams of different sized particles moving relative to each other are obtained. Phenomenological constants, such as the shear viscosity coefficient, the thermal conductivity coefficient, and the diffusion coefficient of the bulk fluid, are found as a function of the material properties of the constituents of the two-phase system and the volume concentration of solid. For transient heat conduction phenomena, it is found that the introduction of a complex conductivity for the bulk fluid permits a simple mathematical description of this otherwise complicated process. The rate of heat transfer between particle continuum and bulk fluid is also investigated by means of an Oseen-type approximation to the energy equation.

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1. INTRODUCTION

The subject of fluid-particle flow has focussed the attention of both engineers and scientists for many decades. Such two-phase flow phenomena occur almost in every aspect of daily life from the flow of blood to the motion of dusty air, and are important in many industrial applications from sedimentation processes to fluidized beds. In the fields of propulsion and combustion, new interest has recently been directed to the transport phenomena of fluid-solid particle flow, which is the main theme of this thesis.

To understand such a complicated phenomenon involving a cloud of solid particles in a viscous fluid medium, it is essential that the phenomenon involving a single particle be first fairly well understood. Part II of this thesis is devoted entirely to this aspect. However, due to grave mathematical difficulty inherent in the structure of the Navier-Stokes equations, general treatment of this problem is so far not possible. As a result, the present treatment will be entirely restricted to low Reynolds number flow where simplification in the flow field equation is permissible.

The fundamental solution to Stokes equation is first examined. This forms the basis of the approximation that it is sufficiently accurate to replace particles in a fluid flow field by point forces. The viscous drag on a class of axially symmetric particles in a uniform flow is then investigated by means of the matching technique. Viscous interaction between a small number of spherical solid particles in a fluid medium is next being considered using a "point force" approximation. This serves to pave the way for the introduction of the "screened" Stokes equation in Part III of this thesis. Viscous interaction between a spherical particle and continuous solid boundaries, as well as fluid interface, is analyzed. This permits an estimate of the importance of the influence of continuous boundaries on the hydrodynamic forces acting on a solid particle moving relative to them. The shape of the particle too has a significant effect on the drag and lift it would experience when moving through a fluid medium. This is treated in a fairly general manner for particles the shapes of which do not deviate markedly from a sphere. Hydrodynamic forces arising from the elastic deformation of a spherical particle in a shear flow is then taken into account. It is shown that a particle will drift sideways so as to decrease the slip velocity between the fluid medium and the particle. Non-steady motion of a spherical particle in a viscous fluid medium is briefly touched upon. The fundamental solution to the linearized time-dependent Navier-Stokes equation is obtained. The motion of a particle starting from rest under externally applied force is then dealt with. This makes it possible to estimate the relative importance of transient effect on particle motion.

Part III of this thesis deals with the macroscopic continuum description of solid particle-fluid flow. There, particles are assumed to be small compared with any scale of phenomenon of macroscopic interest. Also, the particles will be assumed to be numerous so that the concept of continuum applies. For very dilute volume concentration of solid particles, the general practice is to modify the fluid field equations slightly by merely adding a body force term, taken as the particle fluid interaction force[1-6]. For moderate particle volume concentration, this procedure tends to obliterate many significant changes in the transport properties of the system. This part of the thesis is primarily devoted to a critical examination of the importance of this effect. On starting from the Navier-Stokes equation, and by assuming certain properties possessed by spatial averages of physical quantities of the two-phase system, a set of bulk fluid equations are obtained. This is supplemented by a set of phenomenological formulas characterizing the transport properties of the bulk fluid, thus reducing the problem to the determination of the phenomenological constants involved.

The hydrodynamic forces acting between particle cloud and bulk fluid are analyzed based on a "smoothed field" and "detailed field" consideration. A "screened" Stokes equation is introduced to describe the disturbances produced in such a particle suspension. The existence of a rotational coupling force is demonstrated and its formula obtained. It is to be noted that this force has no analogue in single particle consideration and should be regarded as an effect arising from particles as a cloud.

The exchange of momentum between streams of particles of different sizes moving relative to each other, resulting from viscous disturbances generated in the fluid medium, called viscous collision, is investigated quantitatively. At low relative velocity, this mechanism of momentum exchange is shown to be far more important than that arising from direct contact collision between streams of particles.

Shear viscosity coefficient of the bulk fluid as a function of volume concentration of solid particles is obtained. At moderate volume concentration, this differs significantly from that of the pure fluid. At low volume concentration, the well known Einstein's formula is recovered[7].

Thermal transmission property of the bulk fluid is then examined. The steady-state thermal conductivity of the bulk fluid is expressed as a function of the thermal conductivities of the pure fluid and solid and the volume concentration of solid. The same idea is extended to the case of transient heat conduction. It is found that the introduction of a complex thermal conductivity coefficient permits a simple description of this otherwise complicated process. Similar to the thermal conductivity coefficient, diffusion coefficient in a particle suspension is obtained by a simple analogy. When the mean particle temperature and mean bulk fluid temperature are unequal, heat transfer between the two phases takes place. By means of an Oseen type approximation to the energy equation, the rate of heat transfer is found to depend on the temperature difference, the thermal conductivities of the two phases, and the volume concentration of solid particles. Explicit expression of Nusselt's number is then given for cases where Peclet's number is small.

PART II.

The Hydrodynamic Forces Acting on a Solid Particle in an Incompressible Fluid Medium at Low Reynolds Number Flow

1. Fundamental Solution of Stokes Equation

For slow motion of a viscous incompressible fluid flowing past a finite solid object, it is sufficiently accurate to take Stokes equation as the governing field equation for the fluid. The object resists the movement of the fluid and exerts on the fluid field a drag force, $\vec{D}$. If the object is small, it is possible to regard it as a point force. The velocity and pressure field produced by a point force in a Stokes flow is referred to as the fundamental solution of Stokes equation. Mathematically, this is given by the solution $\vec{u}$, fluid velocity, and p , fluid pressure of the following boundary value problem.

$$\nabla p = \mu \nabla^2 \vec{u} + D \hat{x} \, \delta(\vec{x}) \tag{1.1}$$

$$\nabla \cdot \vec{u} = 0 \tag{1.2}$$

$$\vec{u}, \; p \longrightarrow 0 \qquad \text{as} \qquad r \longrightarrow \infty \tag{1.3}$$

where $\delta(\vec{x})$ is the Dirac delta function and the point force has been taken to be applied in the x -direction. μ is the viscosity of the fluid.

To solve for $\vec{u}$ and p , take divergence of equation (1.1) and use equation (1.2):

$$\nabla^2 p = D \frac{\partial}{\partial x} (\delta(\vec{x})) \tag{1.4}$$

It is well known that the solution of

$$\nabla^2 \phi = \delta(\vec{x}) \quad , \quad \phi \longrightarrow 0 \quad r \longrightarrow \infty$$

is

$$\phi = - \frac{1}{4\pi r}$$

Therefore,

$$p = -\frac{D}{4\pi}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \frac{Dx}{4\pi r^3} \tag{1.5}$$

$\frac{1}{r}$ being even in x .

Substitute equation (1. 5) into equation (1. 1)

$$\nabla^2\vec{u} = \frac{D}{4\pi\mu}\nabla\left(\frac{x}{r^3}\right) - \frac{D}{\mu}\hat{x}\,\delta(\vec{x}) = \vec{E} \tag{1.6}$$

where $\vec{E}$ has the property $\nabla\cdot\vec{E} = 0$ from equation (1. 4). The solution of (1. 6) is

$$\vec{u} = -\int\frac{\vec{E}(\vec{x}')}{4\pi|\vec{x}-\vec{x}'|}\,d^3x' \tag{1.7}$$

To see that $\vec{u}$ as given by (1. 7) satisfies equation (1.2),take div. of (1. 7)

$$\nabla\cdot\vec{u} = -\nabla\cdot\int\frac{\vec{E}(\vec{x}')\,d^3x'}{4\pi|\vec{x}-\vec{x}'|} = -\frac{1}{4\pi}\int\left(\nabla\frac{1}{|\vec{x}-\vec{x}'|}\right)\cdot\vec{E}(\vec{x}')\,d^3x'$$

$$= \frac{1}{4\pi}\int\left(\nabla'\frac{1}{|\vec{x}-\vec{x}'|}\right)\cdot\vec{E}(\vec{x}')\,d^3x'$$

Integrate by parts

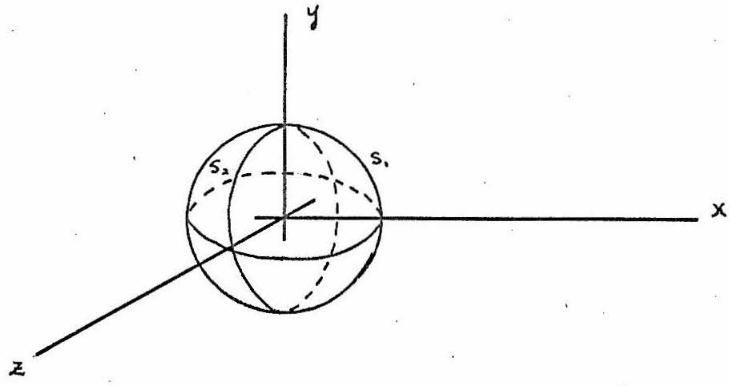
$$\nabla\cdot\vec{u} = -\frac{1}{4\pi}\int\frac{1}{|\vec{x}-\vec{x}'|}\,\nabla'\cdot\vec{E}(\vec{x}')\,d^3x' = 0$$

since $\nabla\cdot\vec{E} = 0$ by equation (1. 4).

Now to obtain explicit form of equation (1.7),it is necessary first to evaluate $\vec{E}$.

$$\vec{E} = \frac{\nabla p}{\mu} - \frac{D}{\mu}\hat{x}\,\delta(\vec{x})$$

It is to be noted that p as given by equation (1.5) is a discontinuous function, being singular at $r = 0$. To take the gradient of p one should add the proper magnitude of the "jump value."



To find the proper "jump value", Δ_x, in the x -direction, one may integrate the pressure force acting on a sphere of radius r in the x direction. Let S_1 and S_2 denote the two hemispheres formed with the $y - z$ plane. Then

$$\Delta_x = \int_{S_1} p \, | \cos\theta \, ds | - \int_{S_2} p \, | \cos\theta \, ds |$$

$$= \frac{D}{4\pi} \cdot 2 \int_{S_1} \frac{\cos^2\theta}{r^2} \, ds = \frac{D}{3}$$

Similarly,

$$\Delta_y = \Delta_z = 0$$

The above consideration gives the following formulas

$$\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) = \frac{1}{r^3} - \frac{3x^2}{r^5} + \frac{4\pi}{3} \delta(\vec{x}) \tag{1.8}$$

$$\frac{\partial}{\partial y}\left(\frac{x}{r^3}\right) = - \frac{3xy}{r^5} \tag{1.9}$$

Hence

$$\nabla \phi = \hat{x}\left[\frac{D}{4\pi}\left(\frac{1}{r^3} - \frac{3x^2}{r^5}\right) + \frac{D}{3}\,\delta(\vec{x})\right]$$

$$+ \hat{y}\left[\frac{D}{4\pi}\left(-\frac{3xy}{r^5}\right)\right] + \hat{z}\left[\frac{D}{4\pi}\left(-\frac{3xy}{r^5}\right)\right] \tag{1.10}$$

On using equations (1. 8), (1. 9), and (1. 10), it is easy to verify that equation (1. 4) is indeed satisfied.

From equation (1. 10)

$$\vec{E} = \frac{D}{4\pi\mu}\left\{-\hat{x}\left[\frac{2P_2(\cos\theta)}{r^3} + \frac{8\pi}{3}\,\delta(\vec{x})\right] + \hat{y}\,\frac{P_2^1(\cos\theta)\cos\phi}{r^3}\right.$$

$$\left. + \hat{z}\,\frac{P_2^1(\cos\theta)\sin\phi}{r^3}\right\} \tag{1.11}$$

where the x -axis has been taken as the polar axis. P_l^m denotes the associated Legendre function.

Substituting equation (1. 11) into equation (1. 7) and using the following representation for $\frac{1}{|\vec{x}-\vec{x}'|}$ and the orthogonal properties of $P_l^m(\cos\theta)$, $\sin m\phi$, $\cos m\phi$ (see Appendix IIA)

$$\frac{1}{4\pi|\vec{x}-\vec{x}'|} = \frac{1}{2\pi}\sum_{l=0}^{\infty}\sum_{m=0}^{l} \epsilon_m \frac{(l-m)!}{(l+m)!}\left[P_l^m(\cos\theta)\,P_l^m(\cos\theta')\left(\cos m\phi\cos m\phi'\right.\right.$$

$$\left.\left. + \sin m\phi\,\sin m\phi'\right)\right]\left[\frac{r_<^l}{r_>^{l+1}}\right] \tag{1.12}$$

where $\epsilon_m = 1 \quad m > 0$

$\qquad = \frac{1}{2} \quad m = 0$

$r_<$ ($r_>$) is the smaller (larger) of $|\vec{x}|$ and $|\vec{x}'|$

Integrate over $d\Omega'$ and then dr' ; equation (1. 7) becomes

$$\vec{U} = \frac{D}{4\pi\mu}\left\{\left[\frac{2}{3r}+\frac{P_2(\cos\theta)}{3r}\right]\hat{x} - \left[\frac{P_2'(\cos\theta)\cos\phi}{6r}\right]\hat{y} - \left[\frac{P_2'(\cos\theta)\sin\phi}{6r}\right]\hat{z}\right\}$$

Therefore, the fundamental solution is

$$\vec{U} = \frac{D}{8\pi\mu}\left[\frac{1}{r}+\frac{x^2}{r^3}\right]\hat{x} + \frac{D}{8\pi\mu}\left[\frac{xy}{r^3}\right]\hat{y} + \frac{D}{8\pi\mu}\left[\frac{xz}{r^3}\right]\hat{z}$$

$$p = \frac{D}{4\pi}\,\frac{x}{r^3} \tag{1.13}$$

The above solution can also be obtained in a less direct way, a technique which will again be used in later sections. Stokes original solution of uniform flow past a sphere, after subtracting out the velocity at infinity, can be written as[8]

$$p = \frac{D}{4\pi}\,\frac{x}{r^3}$$

$$\vec{U} = \frac{D}{8\pi\mu}\left[-\frac{x^2}{r^3}\left(\frac{a^2}{r^2}-1\right)+\frac{1}{3r}\left(3+\frac{a^2}{r^2}\right)\right]\hat{x}$$
$$+\frac{D}{8\pi\mu}\left[\frac{xy}{r^3}\left(1-\frac{a^2}{r^2}\right)\right]\hat{y} + \frac{D}{8\pi\mu}\left[\frac{xz}{r^3}\left(1-\frac{a^2}{r^2}\right)\right]\hat{z} \tag{1.14}$$

where $D = -6a\pi\mu U_\infty$, $a =$ radius of sphere.

Now keep D fixed and let $a \to 0$; equation (1.13) is recovered.

2. Drag Formula for a Class of Axially Symmetric Particles in a Uniform Flow

Consider a uniform stream of fluid flowing past an axially symmetric particle, the axis of symmetry being in the direction of the unperturbed flow. Let the fluid be incompressible with density ρ

and viscosity μ . If $\vec{U}'$ and p' are the velocity and pressure of the fluid, then $\vec{U}'$ and p' satisfy the time independent Navier-Stokes equation and continuity equation together with the no slip condition on the surface of the particle and appropriate conditions at infinity. These equations and conditions are

$$\rho \vec{U}' \cdot \nabla' \vec{U}' = -\nabla' p' + \mu \nabla'^2 \vec{U}' \tag{2.1}$$

$$\nabla' \cdot \vec{U}' = 0 \tag{2.2}$$

$$\vec{U}' = U_\infty \hat{x} \qquad r = \infty \tag{2.3}$$

$$p' = 0 \qquad r = \infty \tag{2.4}$$

$$\vec{U}' = 0 \qquad \text{on surface of particle.} \tag{2.5}$$

Take a coordinate system x', y', z' centered at the particle with the x'- axis coinciding with the axis of symmetry of the particle. Upon introducing the following dimensionless variables, equations (2.1) through (2.5) can be put into a non-dimensional form

$$x = \frac{x'}{a} \qquad y = \frac{y'}{a} \qquad z = \frac{z'}{a} \qquad r = \frac{r'}{a}$$

$$p = \frac{a p'}{\mu U_\infty} \qquad \vec{U} = \frac{\vec{U}'}{U_\infty} \qquad R = \frac{\rho U_\infty a}{\mu} << 1$$

where a is a characteristic length of the particle and R is the Reynolds number, so that

$$R \vec{U} \cdot \nabla \vec{U} = -\nabla p + \nabla^2 \vec{U} \tag{2.6}$$

$$\nabla \cdot \vec{U} = 0 \tag{2.7}$$

$$\vec{U} = \hat{x} \, , \quad p = 0 \qquad r = \infty \tag{2.8}$$

$$\vec{u} = 0 \qquad \text{on surface of particle.} \tag{2.9}$$

To solve equations (2.6), (2.7) satisfying conditions (2.8) and (2.9), the matching technique as explained in references 9, 10, 11, 12, and 13 will be used.

Stokes Expansion. First, seek a solution to equation (2.6) and (2.7) in the form of Stokes expansion.

$$\vec{u} = \vec{u}_0 + R\vec{u}_1 + o(R) \tag{2.10}$$

$$p = p_0 + Rp_1 + o(R) \tag{2.11}$$

Substitute (2.10) and (2.11) into equations (2.6) and (2.7) and equate coefficients according to powers of R to obtain

$$\nabla p_0 = \nabla^2 \vec{u}_0 \tag{2.12}$$

$$\nabla \cdot \vec{u}_0 = 0 \tag{2.13}$$

$$\vec{u}_0 = 0 \qquad \text{on surface of particle}$$

$$\nabla p_1 + \vec{u}_0 \cdot \nabla \vec{u}_0 = \nabla^2 \vec{u}_1 \tag{2.14}$$

$$\nabla \cdot \vec{u}_1 = 0 \tag{2.15}$$

$$\vec{u}_1 = 0 \qquad \text{on surface of particle.}$$

Also, require $\vec{u}_0$, $\vec{u}_1$, etc., to match the asymptotic solution of equations (2.6) and (2.7).

Asymptotic or Oseen Expansion. Suppose the drag force acting on the particle is $D'\hat{x}$; then an asymptotic solution of equations (2.6) and (2.7) would be given by the solution of

$$R\,\vec{u} \cdot \nabla \vec{u} = -\nabla p + \nabla^2 \vec{u} - D\hat{x}\,\delta(\vec{x}) \tag{2.16}$$

$$\nabla \cdot \vec{u} = 0 \tag{2.17}$$

$$\vec{u} = \hat{x} \quad , \quad p = 0 \qquad r = \infty \tag{2.18}$$

where $\quad D = \dfrac{D'}{a\mu U_\infty}$

Solution of (2.16) - (2.18) will approach the exact solution for large r or when Rr is of order unity where $R << 1$. This suggests the use of stretched variables defined by

$$\begin{aligned} &\xi = Rx \qquad \eta = Ry \qquad \zeta = Rz \qquad \rho = Rr \\ &\vec{W}(\xi, \eta, \zeta, R) = \vec{u}(x, y, z, R) \\ &P(\xi, \eta, \zeta, R) = R^{-1} p(x, y, z, R) \end{aligned} \tag{2.19}$$

In terms of the stretched variables, equations (2.16) and (2.17) become

$$\vec{W} \cdot \nabla \vec{W} = -\nabla p + \nabla^2 \vec{W} - R D \hat{x} \delta(\vec{\rho}) \tag{2.20}$$

$$\nabla \cdot \vec{W} = 0 \tag{2.21}$$

$$\vec{W} = \hat{x} \quad , \quad P = 0 \qquad \rho = \infty \tag{2.22}$$

Now, seek an asymptotic solution in the form of

$$\begin{aligned} \vec{W} &= \vec{W}_0 + R\vec{W}_1 + o(R) \\ P &= P_0 + R P_1 + o(R) \\ D &= D_0 + R D_1 + o(R) \end{aligned} \tag{2.23}$$

Substituting (2.23) into equations (2.20) and (2.21) and on equating coefficients of powers of R, one obtains

$$\begin{aligned} &\vec{W}_0 \cdot \nabla \vec{W}_0 = -\nabla P_0 + \nabla^2 \vec{W}_0 \\ &\qquad \nabla \cdot \vec{W}_0 = 0 \\ &\vec{W}_0 = \hat{x} \qquad P_0 = 0 \qquad \rho = \infty \end{aligned} \tag{2.24}$$

$$\vec{W}_0 \cdot \nabla \vec{W}_1 + \vec{W}_1 \cdot \nabla \vec{W}_0 = -\nabla P_1 + \nabla^2 \vec{W}_1 - D_0 \hat{x}\, \delta(\vec{\rho})$$

$$\nabla \cdot \vec{W}_1 = 0 \qquad (2.25)$$

$$\vec{W}_1 = 0 \qquad P_1 = 0 \qquad \rho = \infty$$

To obtain the asymptotic solution, it is to be noted that the only unknown in equations (2.24) and (2.25) is D_0, which, of course, can be found using the inner solution or the Stokes expansion. The relation between the Stokes expansion and the asymptotic expansion is that they must match to all terms of unity with respect to Rr or ρ in their respective expansions in R.

<u>Zeroth Order Asymptotic Solution.</u> The solution of equation (2.24) satisfying the appropriate boundary conditions is

$$\vec{W}_0 = \hat{x} \qquad P_0 = 0 \qquad (2.26)$$

Thus, from equation (2.25), the first order asymptotic solution must satisfy

$$\frac{\partial \vec{W}_1}{\partial \xi} = -\nabla P_1 + \nabla^2 \vec{W}_1 - D_0 \hat{x}\, \delta(\vec{\rho}) \qquad (2.27)$$

$$\nabla \cdot \vec{W}_1 = 0 \qquad (2.28)$$

$$\vec{W}_1 = 0 \qquad P_1 = 0 \qquad \rho = \infty \qquad (2.29)$$

<u>First Order Asymptotic Solution (Fundamental Solution of Oseen Equation.</u> To solve $\vec{W}_1$ and P_1, take divergence of equation (2.27) and use equation (2.28) to obtain

$$\nabla^2 P_1 = -D_0 \frac{\partial}{\partial \xi} \delta(\vec{\rho}) \qquad (2.30)$$

Therefore, as in Section 1,

$$P_1 = -\frac{D_0}{4\pi}\frac{\xi}{\rho^3} \tag{2.31}$$

and

$$\nabla P_1 + D_0\,\hat{x}\,\delta(\vec{\rho}) = \frac{D_0}{4\pi}\left\{\left[\frac{2P_1(\cos\theta)}{\rho^3} + \frac{8\pi}{3}\delta(\vec{\rho})\right]\hat{x}\right.$$

$$\left. + \left[\frac{P_1'(\cos\theta)\cos\phi}{\rho^3}\right]\hat{y} + \left[\frac{P_1'(\cos\theta)\sin\phi}{\rho^3}\right]\hat{z}\right\} \tag{2.32}$$

where the x -axis has been chosen as the polar axis.

Equation (2. 27) can be written as

$$\nabla^2\vec{W}_1 - \frac{\partial\vec{W}_1}{\partial\xi} = \nabla P_1 + D_0\,\hat{x}\,\delta(\vec{\rho}) \tag{2.33}$$

Let $\vec{W}_1 = e^{\xi/2}\vec{V}$ and substitute into equation (2. 33). Then $\vec{V}$ satisfies

$$\nabla^2\vec{V} - \frac{1}{4}\vec{V} = e^{-\xi/2}\left[\nabla P_1 + D_0\,\hat{x}\,\delta(\vec{\rho})\right] \tag{2.34}$$

Using the Green's function as given by equation (A-4) of Appendix A, the solution of equation (2. 34) is

$$\vec{V} = -\frac{1}{4\pi}\int\frac{e^{-\frac{1}{2}|\vec{\rho}-\vec{\rho}'|}}{|\vec{\rho}-\vec{\rho}'|}e^{-\xi'/2}\left[\nabla' P_1(\vec{\rho}') + D_0\,\hat{x}\,\delta(\vec{\rho}')\right]d^3\rho'$$

Therefore,

$$\vec{W}_1 = -\frac{1}{4\pi}\int\frac{e^{-\frac{1}{2}|\vec{\rho}-\vec{\rho}'|}}{|\vec{\rho}-\vec{\rho}'|}e^{\frac{1}{2}(\xi-\xi')}\left[\nabla' P_1(\vec{\rho}') + D_0\,\hat{x}\,\delta(\vec{\rho}')\right]d^3\xi' \tag{2.35}$$

In order to show that equation (2. 35) is the required solution, it is necessary to demonstrate that

$$\nabla \cdot \vec{W}_1 = 0 \qquad \text{and} \qquad \vec{W}_1 \to 0 \qquad \text{as} \quad \rho \to \infty$$

Take divergence of equation (2. 35):

$$\nabla \cdot \vec{W}_1 = -\frac{1}{4\pi} \int \left(\nabla \frac{e^{-\frac{1}{2}|\vec{\rho}-\vec{\rho}'|}}{|\vec{\rho}-\vec{\rho}'|} e^{\frac{1}{2}(\xi-\xi')} \right) \cdot \left[\nabla' P_1 + D_0 \hat{x}\, \delta(\vec{\rho}') \right] d^3\xi'$$

$$= \frac{1}{4\pi} \int \nabla' \left(\frac{e^{-\frac{1}{2}|\vec{\rho}-\vec{\rho}'|}}{|\vec{\rho}-\vec{\rho}'|} e^{\frac{1}{2}(\xi-\xi')} \right) \cdot \left[\nabla' P_1 + D_0 \hat{x}\, \delta(\vec{\rho}') \right] d^3\xi'$$

Integrate by Parts

$$= -\frac{1}{4\pi} \int \frac{e^{-\frac{1}{2}|\vec{\rho}-\vec{\rho}'|}}{|\vec{\rho}'-\vec{\rho}|} e^{\frac{1}{2}(\xi-\xi')} \nabla' \cdot \left[\nabla' P_1 + D_0 \hat{x}\, \delta(\vec{\rho}') \right] d^3\xi'$$

But $\nabla' \cdot \left[\nabla' P_1 + D_0 \hat{x}\, \delta(\vec{\rho}') \right] = 0$ by equation (2. 30). Hence, the continuity equation is satisfied. Also, since

$$|\vec{\rho}-\vec{\rho}'| \geq (\xi-\xi')$$

from equation (2. 35),

$$|\vec{W}_1| \leq \frac{1}{4\pi} \left| \int \frac{\left[\nabla' P(\vec{\rho}') + D_0 \hat{x}\, \delta(\vec{\rho}') \right]}{|\vec{\rho}-\vec{\rho}'|} d^3\xi' \right|$$

The right hand side was shown to be the fundamental solution of Stokes equation (Section 1) which vanishes as $\rho \to \infty$

$$\therefore \quad \vec{W}_1 \to 0 \qquad \text{as} \quad \rho \to \infty$$

Hence equation (2. 35) is indeed the required solution.

To facilitate the application of the matching requirement, it is useful to expand equation (2. 35) as a power series in ρ

The function $\quad -\dfrac{1}{4\pi} \dfrac{e^{\frac{1}{2}|\vec{\rho}-\vec{\rho}'|}}{|\vec{\rho}-\vec{\rho}'|}$

has the representation (see equation (A-1), Appendix A)

$$-\frac{1}{4\pi}\frac{e^{-\frac{1}{2}|\vec{\rho}-\vec{\rho}'|}}{|\vec{\rho}-\vec{\rho}'|} = -\sum_{l=0}^{\infty}\sum_{m=0}^{l}\left[\frac{K_{l+\frac{1}{2}}(\frac{\rho_>}{2})}{\sqrt{\rho_>}}\;\frac{I_{l+\frac{1}{2}}(\frac{\rho_<}{2})}{\sqrt{\rho_<}}\right]\frac{\epsilon_m(2l+1)}{2\pi}\;\frac{(l-m)!}{(l+m)!}$$

$$\times P_l^m(\cos\theta)\,P_l^m(\cos\theta')\left[\cos m\phi\cos m\phi' + \sin m\phi\sin m\phi'\right] \qquad (2.36)$$

Denote $\vec{W}_1 = u_1\hat{x} + v_1\hat{y} + w_1\hat{z}$. Substitute equations (2.32) and (2.36) into equation (2.35) and integrate out the δ -function and ϕ' . Then

$$u_1 = -\frac{D_0}{6\pi}\frac{e^{-\frac{(\rho-\xi)}{2}}}{\rho} - \frac{D_0 e^{\xi/2}}{2\pi}\int\int\left[\frac{2P_2(\cos\theta')}{\rho'^3}\right]\sum_{l=0}^{\infty}\left[\frac{K_{l+\frac{1}{2}}(\frac{\rho_>}{2})}{\sqrt{\rho_>}}\;\frac{I_{l+\frac{1}{2}}(\frac{\rho_<}{2})}{\sqrt{\rho_<}}\right]\frac{d\rho'}{\rho'}$$

$$\times\left[\frac{(2l+1)}{2}P_2(\cos\theta)\,e^{-\frac{\rho'}{2}\cos\theta'}P_l(\cos\theta)\,P_l(\cos\theta')\right]d\cos\theta'$$

To integrate θ' , consider the integral

$$K_l = \int_{-1}^{+1} P_2(x)\,P_l(x)\,e^{-\alpha x}\,dx$$

$$= \int_{-1}^{+1}\left\{\frac{3(l+1)(l+2)}{2(2l+1)(2l+3)}P_{l+2}(x) + \frac{3}{2(2l+1)}\left[\frac{(l+1)^2}{(2l+3)} + \frac{l^2}{(2l-1)} - \frac{(2l+1)}{3}\right]P_l(x) + \frac{3l(l-1)}{2(4l^2-1)}P_{l-2}(x)\right\}e^{-\alpha x}\,dx$$

Using the orthogonal property of the Legendre functions and the representation

$$e^{-\alpha x} = \sqrt{\frac{\pi}{2\alpha}}\sum_{n=0}^{\infty}(-1)^n(2n+1)\,I_{n+\frac{1}{2}}(\alpha)\,P_n(x)$$

one obtains

$$K_l = \sqrt{\frac{\pi}{2\alpha}}(-1)^l\left\{\frac{3(l+1)(l+2)}{2(2l+1)(2l+3)}I_{l+\frac{5}{2}}(\alpha) + \frac{3}{(2l+1)}\left[\frac{(l+1)^2}{(2l+3)} + \frac{l^2}{(2l-1)} - \frac{2l+1}{3}\right]I_{l+\frac{1}{2}}(\alpha) + \frac{3l(l-1)}{(4l^2-1)}I_{l-\frac{3}{2}}(\alpha)\right\}$$

$$(\,I_n = 0 \qquad n < 0\,)$$

Therefore,

$$U_1 = -\frac{D_0 e^{-\frac{1}{2}(\rho-\xi)}}{6\pi\rho} - \frac{D_0 e^{\xi/2}}{2\pi} \sum_{\ell=0}^{\infty} f_\ell(\rho)\, P_\ell(\cos\theta) \tag{2.37}$$

where

$$f_\ell(\rho) = (-1)^\ell \frac{(2\ell+1)}{2} \left\{ \int_0^\infty \sqrt{\frac{\pi}{\rho'}} \left[\frac{K_{\ell+1/2}\left(\frac{\rho_>}{2}\right)}{\sqrt{\rho_>}} \frac{I_{\ell+1/2}\left(\frac{\rho_<}{2}\right)}{\sqrt{\rho_<}} \right] \left[\frac{3(\ell+1)(\ell+2)}{(2\ell+1)(2\ell+3)} I_{\ell+5/2}\left(\frac{\rho'}{2}\right) \right.\right.$$
$$\left.\left. + \frac{3}{(2\ell+1)} \left(\frac{(\ell+1)^2}{(2\ell+3)} + \frac{\ell^2}{(2\ell-1)} - \frac{(2\ell+1)}{3} \right) I_{\ell+1/2}\left(\frac{\rho'}{2}\right) + \frac{3\ell(\ell-1)}{(4\ell^2-1)} I_{\ell-3/2}\left(\frac{\rho'}{2}\right) \right] \frac{d\rho'}{\rho'} \right\} \tag{2.38}$$

Similarly, using integration by parts on the following integral,

$$F_\ell = \int_{-1}^{+1} e^{-\alpha x} P_2'(x)\, P_\ell'(x)\, dx$$
$$= -3 \int_{-1}^{+1} (1 - \alpha x - 3x^2 + \alpha x^3)\, P_\ell(x)\, e^{-\alpha x}\, dx$$

it is straightforward to obtain

$$V_1 = -\frac{D_0 e^{\xi/2}}{4\pi} \sum_{\ell=1}^{\infty} g_\ell(\rho)\, P_\ell'(\cos\theta) \cos\phi$$

$$W_1 = -\frac{D_0 e^{\xi/2}}{4\pi} \sum_{\ell=1}^{\infty} g_\ell(\rho)\, P_\ell'(\cos\theta) \sin\phi \tag{2.39}$$

where

$$g_\ell(\rho) = (-1)^{\ell+1} \frac{6(2\ell+1)}{2\ell(\ell+1)} \int_0^\infty \left[\frac{K_{\ell+1/2}\left(\frac{\rho_>}{2}\right)}{\sqrt{\rho_>}} \frac{I_{\ell+1/2}\left(\frac{\rho_<}{2}\right)}{\sqrt{\rho_<}} \right] \sqrt{\frac{\pi}{\rho'}} \left\{ \frac{\rho'(\ell+1)(\ell+2)(\ell+3)}{2(2\ell+1)(2\ell+3)(2\ell+5)} I_{\ell+7/2}\left(\frac{\rho'}{2}\right) \right.$$

$$+ \frac{3(\ell+1)(\ell+2)}{(2\ell+1)(2\ell+3)} I_{\ell+5/2}\left(\frac{\rho'}{2}\right) + \left[-\frac{\rho'(\ell+1)}{2(2\ell+1)} + \frac{\rho'(\ell+1)(\ell+2)^2}{2(2\ell+1)(2\ell+3)(2\ell+5)} + \frac{\rho'(\ell+1)^3}{2(2\ell+1)^2(2\ell+3)} \right.$$

$$\left. + \frac{\rho'\ell^2(\ell+1)}{2(2\ell+1)^2(2\ell-1)} \right] I_{\ell+3/2}\left(\frac{\rho'}{2}\right) - \left[1 - \frac{3(\ell+1)^2}{(2\ell+1)(2\ell+3)} - \frac{3\ell^2}{(2\ell+1)(2\ell-1)} \right] I_{\ell+1/2}\left(\frac{\rho'}{2}\right)$$

$$+ \left[\frac{-\rho'\ell}{2(2\ell+1)} + \frac{\rho'\ell(\ell+1)^2}{2(2\ell+1)^2(2\ell+3)} + \frac{\rho'\ell^3}{2(2\ell+1)^2(2\ell-1)} + \frac{\rho'\ell(\ell-1)^2}{2(2\ell+1)(2\ell-1)(2\ell-3)} \right] I_{\ell-1/2}\left(\frac{\rho'}{2}\right)$$

$$\left. + \frac{3\ell(\ell-1)}{(2\ell+1)(2\ell-1)} I_{\ell-3/2}\left(\frac{\rho'}{2}\right) + \frac{\rho'\ell(\ell-1)(\ell-2)}{2(2\ell+1)(2\ell-1)(2\ell-3)} I_{\ell-5/2}\left(\frac{\rho'}{2}\right) \right\} \frac{d\rho'}{\rho'} \qquad (2.40)$$

From equations (2.38) and (2.40), to terms of order ρ

$$\begin{aligned}
f_0(\rho) &= \frac{1}{24} + o(\rho) \\
f_1(\rho) &= -\frac{1}{10} + o(\rho) \\
f_2(\rho) &= \frac{1}{6\rho} + o(\rho) \\
f_3(\rho) &= -\frac{1}{40} + o(\rho) \\
f_n(\rho) &= o(\rho) \qquad n > 3 \\
g_1(\rho) &= \frac{3}{20} + o(\rho) \\
g_2(\rho) &= -\frac{1}{6\rho} + o(\rho) \\
g_3(\rho) &= \frac{1}{60} + o(\rho)
\end{aligned} \qquad (2.41)$$

From equations (2. 26), (2. 37), (2. 39), and (2. 41), by the matching hypothesis, the terms that have to be matched by the Stokes expansion are:

to order unity $\quad \vec{U} = \hat{x}$

to order R

$$\vec{U} = \hat{x}\left[-\frac{D_0}{8\pi}\left(\frac{1}{\rho}+\frac{\xi^2}{\rho^3}\right)+\frac{D_0}{16\pi}-\frac{D_0}{32\pi}\left(\frac{\xi}{\rho}+\frac{\xi^3}{\rho^3}\right)\right]$$
$$+\hat{y}\left[-\frac{D_0\,\xi\eta}{8\pi\,\rho^3}-\frac{D_0}{32\pi}\left(-\frac{\eta}{\rho}+\frac{\xi^2\eta}{\rho^3}\right)\right]$$
$$+\hat{z}\left[-\frac{D_0\,\xi\zeta}{8\pi\,\rho^3}-\frac{D_0}{32\pi}\left(-\frac{\zeta}{\rho}+\frac{\xi^2\zeta}{\rho^3}\right)\right] \qquad (2.\ 42)$$

<u>Zeroth Order Stokes Expansion.</u> From equations (2. 12), (2.13), and (2. 42), the solution of the zeroth order Stokes expansion must satisfy

$$\nabla p_0 = \nabla^2 \vec{u}_0 \qquad (2.\ 12)$$

$$\nabla \cdot \vec{u}_0 = 0 \qquad (2.\ 13)$$

$$\vec{W}_0 = 0 \qquad \text{on surface of body}$$

$$\vec{W}_0 = \hat{x} \qquad r = \infty$$

However, in the matching hypothesis, only the leading terms in $\frac{1}{\rho}$ are of importance. The leading terms of the solution of equations (2. 12) and (2. 13) can be obtained by considering the body as a point force of magnitude $-D_0\hat{x}$. This is the fundamental solution of the Stokes equation which has been obtained in Section 1. Hence

$$\vec{U}_0 \sim \hat{x}\left[1-\frac{D_0}{8\pi}\left(\frac{1}{r}+\frac{x^2}{r^3}\right)\right]-\hat{y}\left[\frac{D_0}{8\pi}\,\frac{xy}{r^3}\right]-\hat{z}\left[\frac{D_0}{8\pi}\,\frac{xz}{r^3}\right]$$

$$p_0 \sim -\frac{D_0\,x}{4\pi r^3} \qquad (2.\ 43)$$

On comparing terms of equations (2. 42) and (2. 43), it is clear that the terms that have to be matched by $\vec{U}_1$ are

$$\hat{x}\left[\frac{D_0}{16\pi}\right] - \hat{x}\left[\frac{D_0}{32\pi}\left(\frac{\xi}{\rho} + \frac{\xi^3}{\rho^3}\right)\right]$$
$$- \hat{y}\left[\frac{D_0}{32\pi}\left(-\frac{\eta}{\rho} + \frac{\xi^2\eta}{\rho^3}\right)\right]$$
$$- \hat{z}\left[\frac{D_0}{32\pi}\left(-\frac{\zeta}{\rho} + \frac{\xi^2\zeta}{\rho^3}\right)\right] \tag{2.44}$$

<u>First Order Stokes Expansion.</u> From equations (2. 14) and (2. 15), the first order Stokes expansion satisfies

$$\nabla p_1 + \vec{u}_0 \cdot \nabla \vec{u}_0 = \nabla^2 \vec{u}_1 \tag{2.45}$$

$$\nabla \cdot \vec{u}_1 = 0 \tag{2.46}$$

Using equation (2. 43), the leading terms of $\vec{u}_0 \cdot \nabla \vec{u}_0$ are

$$\vec{u}_0 \cdot \nabla \vec{u}_0 \sim \hat{x}\left\{-\frac{D_0}{8\pi}\left[\frac{x}{r^3} - \frac{3x^3}{r^5}\right] + \frac{D_0^2}{64\pi^2}\left[-\frac{4x^3}{r^6}\right]\right\}$$
$$+ \hat{y}\left\{-\frac{D_0}{8\pi}\left[\frac{y}{r^3} - \frac{3x^2y}{r^5}\right] + \frac{D_0^2}{64\pi^2}\left[\frac{y}{r^4} - \frac{4x^2y}{r^6}\right]\right\}$$
$$+ \hat{z}\left\{-\frac{D_0}{8\pi}\left[\frac{z}{r^3} - \frac{3x^2z}{r^5}\right] + \frac{D_0^2}{64\pi^2}\left[\frac{z}{r^4} - \frac{4x^2z}{r^6}\right]\right\} \tag{2.47}$$

A particular solution of equations (2. 45) and (2. 46) corresponding to terms of (2. 47) is

$$p_1 = \frac{D_0^2}{64\pi^2}\left[-\frac{1}{r^2} + \frac{2x^2}{r^4}\right]$$

$$\vec{U}_p = \hat{x}\left\{-\frac{D_0}{8\pi}\left[\frac{x}{4r}+\frac{x^3}{4r^3}\right]+\frac{D_0^2}{64\pi^2}\left[\frac{x^3}{r^4}\right]\right\}$$
$$+\hat{y}\left\{-\frac{D_0}{8\pi}\left[-\frac{y}{4r}+\frac{x^2 y}{4r^3}\right]+\frac{D_0^2}{64\pi^2}\left[-\frac{y}{2r^2}+\frac{x^2 y}{r^4}\right]\right\}$$
$$+\hat{z}\left\{-\frac{D_0}{8\pi}\left[-\frac{z}{4r}+\frac{x^2 z}{4r^3}\right]+\frac{D_0^2}{64\pi^2}\left[-\frac{z}{2r^2}+\frac{x^2 z}{r^4}\right]\right\} \tag{2.48}$$

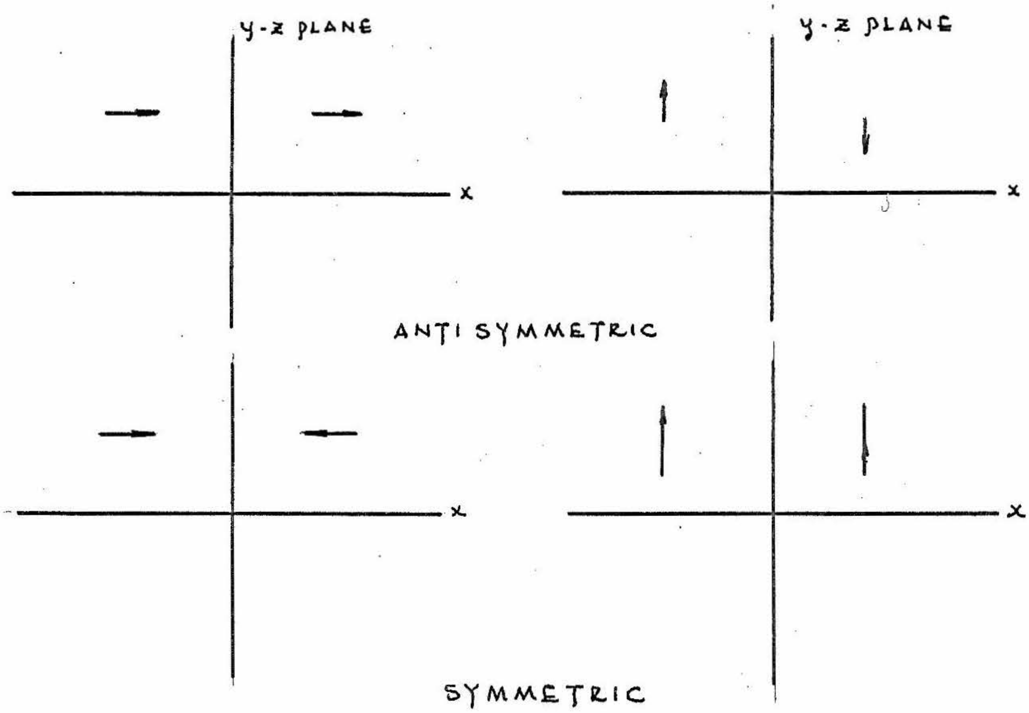
On comparing equations (2. 44) and (2. 48), it is clear that all the terms are matched except for the constant $\frac{D_0}{16\pi}\hat{x}$. As far as the Stokes expansion is concerned, this term corresponds to a uniform flow at infinity. The drag given rise by the requirement that this term be matched is therefore equal to

$$D_1 = \frac{D_0^2}{16\pi} \tag{2.49}$$

Now consider only the class of axially symmetric bodies which are also symmetric with respect to the $y-z$ plane. It is intended to show that had it not been for the requirement imposed by the matching hypothesis that the term $\frac{D_0}{16\pi}\hat{x}$ be matched, this class of bodies would have $D_1 = 0$. The proof will be based on a symmetry argument. As vector quantities will be dealt with, "symmetry" and "antisymmetry" of a vector quantity with respect to the $y-z$ plane will first be defined.

A vector quantity will be called "symmetric" or "antisymmetric" with respect to the $y-z$ plane if its magnitudes at the mirror image points with respect to the $y-z$ plane are the same while the directions are as indicated in the diagram.

It is clear from the definition above that the unit vector $\hat{x}$ is antisymmetric, while $\hat{\rho} = \hat{y}\cos\phi + \hat{z}\sin\phi$ is symmetric.



Because of the linearity of Stokes equation, for the problem under consideration, it is obvious that if the direction of flow at infinity is reversed, the velocity at any point will reverse in direction but the magnitude will remain unchanged. Since the body is symmetric about the y-z plane, it follows that

$u_x \hat{x}$ is antisymmetric,

$u_\rho \hat{\rho}$ is antisymmetric.

u_x, u_ρ are the velocity components in the $\hat{x}$ and $\hat{\rho}$ directions.

As $\hat{x}$ is antisymmetric, $\hat{\rho}$ is symmetric; therefore, u_x must be an even function of x and u_ρ is an odd function of x .

Now consider the function $\vec{u}_0 \cdot \nabla \vec{u}_0$ in equation (2.14).

$$\vec{u}_0 \cdot \nabla \vec{u}_0 = \left(u_\rho \frac{\partial u_\rho}{\partial \rho} + u_x \frac{\partial u_\rho}{\partial x} \right) \hat{\rho}$$

$$+ \left(u_\rho \frac{\partial u_x}{\partial \rho} + u_x \frac{\partial u_x}{\partial x} \right) \hat{x}$$ in cylindrical coordinates.

From the conclusion that u_x is an even function of x and u_ρ is an odd function of x , it is clear that $\vec{u}_0 \cdot \nabla \vec{u}_0$ is symmetric with respect to the $y-z$ plane. Now consider equation (2.14):

$$\nabla^2 \vec{w}_1 - \nabla p_1 = \vec{u}_0 \cdot \nabla \vec{u}_0 \qquad (2.14)$$

The term $\vec{u}_0 \cdot \nabla \vec{u}_0$ can be regarded as a body force acting on the fluid. However, a symmetric body force cannot produce any drag on a symmetrical body. Hence, the proof is completed. From equation (2.49) the drag on the class of bodies under consideration to order R is given by

$$D \hat{x} = D_0 \left(1 + \frac{D_0}{16\pi} R \right) \hat{x} \qquad (2.50)$$

where $\mu U_\infty a D_0$ = Stokes drag on body.

From equation (2.48) and the fundamental Stokes solution which yields a drag force given by equation (2.49), the leading terms of $\vec{u}_1$ are

$$\vec{U}_1 \sim \hat{x}\left[\frac{D_0}{16\pi} - \frac{D_0^2}{128\pi^2}\left(\frac{1}{r} + \frac{x^2}{r^3}\right) - \frac{D_0}{8\pi}\left(\frac{x}{4r} + \frac{x^3}{4r^3}\right) + \frac{D_0^2}{64\pi^2}\left(\frac{x^3}{r^4}\right)\right]$$
$$+ \hat{y}\left[-\frac{D_0^2}{128\pi^2}\left(\frac{xy}{r^3}\right) - \frac{D_0}{8\pi}\left(-\frac{y}{4r} + \frac{x^2y}{4r^3}\right) + \frac{D_0^2}{64\pi^2}\left(-\frac{y}{2r^2} + \frac{x^2y}{r^4}\right)\right]$$
$$+ \hat{z}\left[-\frac{D_0^2}{128\pi^2}\left(\frac{xz}{r^3}\right) - \frac{D_0}{8\pi}\left(-\frac{z}{4r} + \frac{x^2z}{4r^3}\right) + \frac{D_0^2}{64\pi^2}\left(-\frac{z}{2r^2} + \frac{x^2z}{r^4}\right)\right] \qquad (2.51)$$

To compute the higher order solutions of the Stokes expansion, it is necessary to calculate $\vec{U}_0 \cdot \nabla \vec{U}_1 + \vec{U}_1 \cdot \nabla \vec{U}_0$ which has as its leading terms

$$[\vec{U}_0 \cdot \nabla \vec{U}_1 + \vec{U}_1 \cdot \nabla \vec{U}_0]$$
$$\sim \left\{\left[\frac{D_0}{8\pi}\left(\frac{1}{4r} + \frac{x^2}{2r^3} - \frac{3x^4}{4r^5}\right) + \frac{D_0^2}{128\pi^2}\left(\frac{1}{r^2} - \frac{2x}{r^3} + \frac{8x^2}{r^4} + \frac{6x^3}{r^5} - \frac{13x^4}{r^6}\right)\right]\hat{x}\right.$$
$$+ \left[\frac{D_0}{8\pi}\left(\frac{3xy}{4r^3} - \frac{3x^3y}{4r^5}\right) + \frac{D_0^2}{128\pi^2}\left(-\frac{2y}{r^3} + \frac{9xy}{r^4} + \frac{6x^2y}{r^5} - \frac{13x^3y}{r^6}\right)\right]\hat{y}$$
$$+ \left[\frac{D_0}{8\pi}\left(\frac{3xz}{4r^3} - \frac{3x^3z}{4r^5}\right) + \frac{D_0^2}{128\pi^2}\left(-\frac{2z}{r^3} + \frac{9xz}{r^4} + \frac{6x^2z}{r^5} - \frac{13x^3z}{r^6}\right)\right]\hat{z}$$

On computing the particular solution corresponding to these terms, it is found, as was first pointed out by Proudman and Pearson[12], a term of the form

$$\vec{U}_{2p} \sim -\frac{D_0^2}{160\pi^2}\ln r$$

would appear. To match this term, by the matching hypothesis, a term $-\frac{D_0^2}{160\pi^2} R^2 \ln R$ would appear. Since the outer solution does not seem to possess such a term, it was suggested by Proudman and Person that a Stokes zeroth-order solution multiplied by $-\frac{D_0^2}{160\pi^2} R^2 \ln R$ should be added to the Stokes expansion so that the two constant terms involving $R^2 \ln R$ can be cancelled out in the process of matching. In so doing, the drag on the particle would be increased by

$$\frac{D_0^3}{160\pi^2} R^2 \ln R$$

On combining with equation (2.50), the drag formula becomes

$$D = D_0 \left[1 + \frac{D_0}{16\pi} R + \frac{D_0^2}{160\pi^2} R^2 \ln R + \cdots \right] \tag{2.52}$$

Equation (2.52) is in agreement with the result of Brenner and Cox[14].

3. Viscous Interaction Between a Small Number of Spherical Particles

The main aim of this section is to investigate the modification on the drag experienced by a spherical particle in a uniform flow field due to the presence of a small number of spherical particles in its vicinity. Here, the fluid velocity is assumed to be small so that Stokes equation applies. However, even with this simplification, the full mathematical problem is still too involved to admit a closed solution. In the following, this problem will be dealt with by means of a "point force" approximation.

(a) Two Spheres in a Uniform Flow. Consider the case of two spheres A and B of radius a and b, respectively, in a uniform flow, the line joining the centers of A and B being parallel to the unperturbed fluid velocity. Imagine two observers, A and B, stationed on sphere A and B, respectively. Consider observer A . If d , the distance between the centers of spheres A and B is much greater than a, b , then to A sphere, B is approximately a point. Thus, if D_B is the drag of sphere B , observer A can replace sphere B by $-\hat{x}\, D_B\, \delta(x-d)\,\delta(y)\,\delta(z)$. Therefore, observer A will seek a solution to the following problem

$$\nabla p = \mu \nabla^2 \vec{U} - D_B\, \hat{x}\, \delta(x-d)\,\delta(y)\,\delta(z) \tag{3.1}$$

$$\nabla \cdot \vec{U} = 0 \tag{3.2}$$

with the boundary conditions

$$\vec{U} = U_\infty \hat{x} \qquad r = \infty \tag{3.3}$$

$$p = 0 \qquad r = \infty$$

$$\vec{U} = 0 \qquad r = a \tag{3.4}$$

Similarly, observer B will seek a solution to the following problem:

$$\nabla p = \mu \nabla^2 \vec{u} - D_A\, \hat{x}\, \delta(x+d)\,\delta(y)\,\delta(z) \tag{3.5}$$

$$\nabla \cdot \vec{u} = 0 \tag{3.6}$$

with the boundary conditions

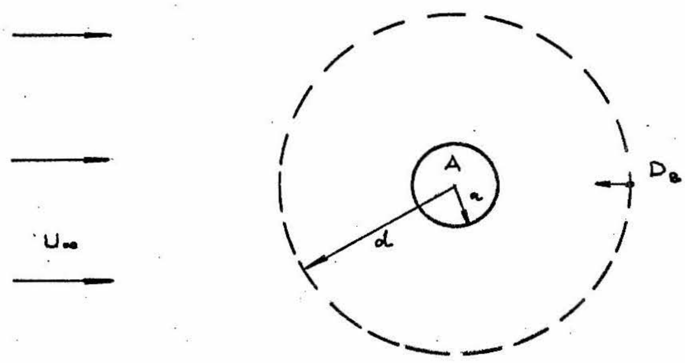
$$\vec{U} = U_\infty \hat{x}\ , \quad p = 0 \qquad r = \infty \tag{3.7}$$

$$\vec{U} = 0 \qquad r = b \tag{3.8}$$

The present method is to solve problems A and B separately to yield two linear algebraic equations for D_B and D_A , the solution of which gives the drag experienced by each sphere.

<u>Solution of Problem A</u>

In polar coordinates, the governing equations and boundary



conditions are

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, u_\theta) = 0 \tag{3.9}$$

$$\frac{\partial p}{\partial r} + D_B \frac{\delta(r-d)\,\delta(\cos\theta-1)}{2\pi d^2} = \mu\left[\nabla^2 u_r - \frac{2u_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, u_\theta)\right] \tag{3.10}$$

$$\frac{1}{r}\frac{\partial p}{\partial\theta} = \mu\left[\nabla^2 u_\theta - \frac{u_\theta}{r^2\sin^2\theta} + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta}\right] \tag{3.11}$$

$$u_r = U_\infty\cos\theta \;;\; u_\theta = -U_\infty\sin\theta \;,\; p = 0 \qquad r = \infty \tag{3.12}$$

$$u_r = u_\theta = 0 \qquad r = a \tag{3.13}$$

Now divide the space into two regions according to $r > d$ and $r < d$. For $r < d$, the solution of equations (3. 9), (3.10), (3. 11) satisfying the boundary condition (3. 12) is (using Lamb's solution, see Appendix B)

$$p = \sum_{l=0}^{\infty} \left(C_l r^l + \frac{D_l}{r^{l+1}} \right) P_l(\cos\theta)$$

$$u_r = \frac{1}{\mu} \sum_{l=0}^{\infty} \left[\left(\frac{l C_l}{2(2l+3)} r^{l+1} + \frac{(l+1)}{2(2l+1)} \frac{D_l}{r^l} \right) P_l(\cos\theta) \right.$$

$$\left. + \left(l E_l r^{l-1} - \frac{(l+1) F_l}{r^{l+2}} \right) P_l(\cos\theta) \right]$$

$$u_\theta = \frac{1}{\mu} \sum_{l=0}^{\infty} \left[\left(\frac{(l+3) C_l}{2(l+1)(2l+3)} r^{l+1} - \frac{(l-2)}{2l(2l-1)} \frac{D_l}{r^l} \right) P_l'(\cos\theta) \right.$$

$$\left. + \left(E_l r^{l-1} + \frac{F_l}{r^{l+2}} \right) P_l'(\cos\theta) \right] \qquad (3.14)$$

For $r > d$, the solution can be written as

$$p = \sum_{l=1}^{\infty} \frac{A_l}{r^{l+1}} P_l(\cos\theta)$$

$$u_r = U_\infty P_1(\cos\theta) + \sum_{l=1}^{\infty} \frac{(l+1)}{\mu\, 2(2l+1)} \frac{A_l}{r^l} P_l(\cos\theta)$$

$$- \sum_{l=0}^{\infty} \frac{(l+1) B_l}{r^{l+2}} P_l(\cos\theta)$$

$$u_\theta = U_\infty P_1'(\cos\theta) - \frac{1}{\mu} \sum_{l=1}^{\infty} \frac{(l-2)}{2l(2l-1)} \frac{A_l}{r^l} P_l'(\cos\theta)$$

$$+ \sum_{l=0}^{\infty} \frac{B_l}{r^{l+2}} P_l'(\cos\theta) \qquad (3.15)$$

To satisfy boundary condition (3. 13),

$$\frac{1}{\mu}\left[\frac{l C_l a^{l+1}}{2(2l+3)} + \frac{(l+1)}{2(2l-1)}\frac{D_l}{a^l}\right] + \left[l E_l a^{l-1} - \frac{(l+1)F_l}{a^{l+2}}\right] = 0 \qquad (3.16)$$

$$\frac{1}{\mu}\left[\frac{(l+3) C_l a^{l+1}}{2(l+1)(2l+3)} - \frac{(l-2)}{2l(2l-1)}\frac{D_l}{a^l}\right] + \left[E_l a^{l-1} + \frac{F_l}{a^{l+2}}\right] = 0 \qquad (3.17)$$

The remaining step to the solution of this problem is to match the two parts of solution given by equations (3.14) and (3.15) adequately on the surface of the sphere $r = d$.

Matching Consideration

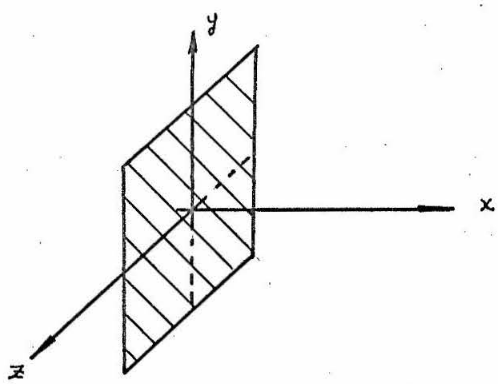
It is clear that the functions u_r, u_θ, p must be continuous everywhere on $r = d$ except possibly at the singularity B . However, before resorting to this "physically obvious (?)" continuity criterion, it is advantageous and indeed proved to be important to investigate the situation more thoroughly. Use will be made of the fundamental solution of Stokes equation obtained in Section 1. In cylindrical coordinates it can be written as

$$u_x = \frac{D}{8\pi\mu}\left[\frac{x^2}{r^3} + \frac{1}{r}\right]$$

$$u_\rho = \frac{D}{8\pi\mu}\left[\frac{x\rho}{r^3}\right]$$

$$p = \frac{D x}{4\pi r^3} \qquad (3.18)$$

Consider the $y-z$ plane to be the surface on which matching of solution is to be made. From equation (3.18) u_x is even in x ; hence, on approaching the origin from any direction, u_x always



tends to $+\infty$. Therefore, on the matching surface, u_x is continuous.

Now consider the pressure. It is an odd function in x . Hence $Lt._{\substack{x^+\to 0\\ r\to 0}}: p \to +\infty$ while $Lt_{\substack{x^-\to 0\\ r\to 0}}: p \to -\infty$. Therefore, although p is continuous everywhere on the matching surface, it has singular limits of opposite sign at the origin. However, the function

$$\frac{\partial p}{\partial x} = \frac{D}{4\pi}\left(\frac{1}{r^3} - \frac{x^2}{r^5}\right)$$

is even in x , so that the singular behavior of this function does not depend on the path. Thus, it is clear that the continuity of $\frac{\partial p}{\partial r}$ should be specified as the matching condition and not p in problem A.

Similarly, the continuity of $\frac{\partial u_\theta}{\partial r}$ will be taken instead of the continuity of u_θ .

The above consideration provides three matching conditions. The fourth matching condition must, however, be derived from the behavior of the singularity.

From equation (3. 10),

$$\frac{\partial p}{\partial r} + D_B \frac{\delta(r-d)\,\delta(\cos\theta-1)}{2\pi d^2} = \mu \left[\nabla^2 u_r - \frac{2u_r}{r^2} - \frac{2}{r^2 \sin\theta} \frac{\partial}{\partial\theta}(\sin\theta\, u_\theta) \right]$$

Integrate this equation over r from $r = d-\epsilon$ to $d+\epsilon$ and let $\epsilon \to 0$. Since u_r and $\int u_\theta\, dr$ are continuous, the last two terms vanish as $\epsilon \to 0$. Hence

$$\underset{\epsilon\to 0}{Lt} \int_{d-\epsilon}^{d+\epsilon} r^2 \frac{\partial p}{\partial r}\, dr + D_B \frac{\delta(\cos\theta-1)}{2\pi} = \mu \int_{d-\epsilon}^{d+\epsilon} \frac{\partial}{\partial r}\left(r^2 \frac{\partial u_r}{\partial r}\right) dr$$

From the continuity equation, $\frac{\partial u_r}{\partial r}$ must behave as u_θ and therefore is discontinuous at d. Hence

$$d^2 \left[p \right]_{d-\epsilon}^{d+\epsilon} + D_B \frac{\delta(\cos\theta-1)}{2\pi} = \mu\, d^2 \left[\frac{\partial u_r}{\partial r} \right]_{d-\epsilon}^{d+\epsilon} \qquad \epsilon \to 0$$

Summing up, the matching conditions are

$$\epsilon \to 0 \qquad \left[u_r \right]_{d-\epsilon}^{d+\epsilon} = 0 \tag{3.19}$$

$$\left[\frac{\partial p}{\partial r} \right]_{d-\epsilon}^{d+\epsilon} = 0 \tag{3.20}$$

$$\left[\frac{\partial u_\theta}{\partial r} \right]_{d-\epsilon}^{d+\epsilon} = 0 \tag{3.21}$$

$$\left[p \right]_{d-\epsilon}^{d+\epsilon} + \frac{D_B\, \delta(\cos\theta-1)}{2\pi d^2} = \mu \left[\frac{\partial u_r}{\partial r} \right]_{d-\epsilon}^{d+\epsilon} \tag{3.22}$$

Equations (3.16), (3.17), (3.19), (3.20), (3.21), and (3.22) provide six equations for the determination of the six sets of unknowns A, B, C, D, E, and F. Writing equations (3.19) - (3.22) out in full,

$$
\left.
\begin{aligned}
&-\frac{B_0}{d^2} = -\frac{F_0}{d^2} \\
&U_\infty + \frac{A_1}{\mu d} - \frac{2B_1}{d^3} = \frac{C_1 d^2}{10\mu} + \frac{D_1}{\mu d} + E_1 - \frac{2F_1}{d^3} \\
&\frac{(l+1)A_l}{\mu\, 2(2l-1)d^l} - \frac{(l+1)B_l}{d^{l+2}} = \frac{l\, C_l\, d^{l+1}}{2(2l+3)\mu} + \frac{(l+1)D_l}{2(2l-1)\mu d^l} \\
&\qquad + l\, E_l\, d^{l-1} - \frac{(l+1)F_l}{d^{l+2}} \qquad l = 2, 3 \cdots
\end{aligned}
\right\} \tag{3.23}
$$

$$
-(l+1)\frac{A_l}{d^{l+2}} = l\, C_l\, d^{l-1} - (l+1)\frac{D_l}{d^{l+2}} \qquad l = 1, 2 \cdots \tag{3.24}
$$

$$
\begin{aligned}
\frac{(l-2)}{2(2l-1)\mu}\frac{A_l}{d^{l+1}} - \frac{(l+2)B_l}{d^{l+3}} &= \frac{(l+3)C_l d^l}{\mu\, 2(2l+3)} + \frac{(l-2)}{2(2l-1)\mu}\frac{D_l}{d^{l+1}} \\
&\quad + (l-1)E_l d^{l-2} - \frac{(l+2)F_l}{d^{l+3}} \qquad l = 1, 2, \cdots
\end{aligned} \tag{3.25}
$$

$$
\begin{aligned}
&\left[\sum_{l=1}^{\infty} \frac{A_l}{d^{l+1}} P_l(\cos\theta) - \sum_{l=0}^{\infty}\left(C_l d^l + \frac{D_l}{d^{l+1}}\right)P_l(\cos\theta)\right] + \frac{D_0\, \delta(\cos\theta - 1)}{2\pi d^2} \\
&= \mu\left[-\sum_{l=1}^{\infty} \frac{(l+1)l}{\mu\, 2(2l-1)}\frac{A_l}{d^{l+1}} P_l(\cos\theta) + \sum_{l=0}^{\infty} \frac{(l+1)(l+2)B_l}{d^{l+3}} P_l(\cos\theta)\right. \\
&\quad - \left[\sum_{l=0}^{\infty}\left(\frac{l(l+1)C_l d^l}{2(2l+3)} - \frac{l(l+1)}{2(2l-1)}\frac{D_l}{d^{l+1}}\right)P_l(\cos\theta)\right. \\
&\quad \left. + \mu\sum_{l=0}^{\infty}\left(l(l-1)E_l d^{l-2} + \frac{(l+1)(l+2)}{d^{l+3}}F_l\right)P_l(\cos\theta)\right]
\end{aligned} \tag{3.26}
$$

In equation (3. 26), to obtain the equation for the coefficients of index ℓ, multiply the whole equation by $P_\ell(\cos\theta)$ and integrate $d\cos\theta$ from -1 to $+1$. Solving for the set corresponding to $\ell = 1$, one obtains

$$D_1 = -\frac{3a\mu U_\infty}{2} - \frac{3}{8\pi}\left(\frac{a^3}{3d^3} - \frac{a}{d}\right)$$

using equation B-3 of Appendix B.

Drag on sphere $A = D_A = -4\pi D_1$. Therefore,

$$D_A = 6\pi\mu a U_\infty + \frac{3}{2}\left(\frac{a^3}{3d^3} - \frac{a}{d}\right) D_B \tag{3. 27}$$

Proceed exactly the same as above. It is easy then to obtain from Problem B

$$D_B = 6\pi\mu b U_\infty + \frac{3}{2}\left(\frac{b^3}{3d^3} - \frac{b}{d}\right) D_A \tag{3. 28}$$

Solving equations (3. 27) and (3. 28) simultaneously,

$$D_A = 6\pi\mu a U_\infty \frac{\left[1 - \frac{3}{2}\frac{b}{a}\left(\frac{a}{d} - \frac{a^3}{3d^3}\right)\right]}{\left[1 - \frac{9}{4}\left(\frac{a}{d} - \frac{a^3}{3d^3}\right)\left(\frac{b}{d} - \frac{b^3}{3d^3}\right)\right]}$$

$$D_B = 6\pi\mu b U_\infty \frac{\left[1 - \frac{3}{2}\frac{a}{b}\left(\frac{b}{d} - \frac{b^3}{3d^3}\right)\right]}{\left[1 - \frac{9}{4}\left(\frac{a}{d} - \frac{a^3}{3d^3}\right)\left(\frac{b}{d} - \frac{b^3}{3d^3}\right)\right]} \tag{3. 29}$$

If $a = b$, then

$$D_A = D_B = \frac{6\pi\mu a U_\infty}{1 + \frac{3}{2}\left(\frac{a}{d} - \frac{a^3}{3d^3}\right)} \tag{3. 30}$$

The correction factor to Stokes drag formula is

$$\lambda' = \frac{1}{1 + \frac{3}{2}\left(\frac{a}{d} - \frac{a^3}{3d^3}\right)}$$

An exact solution of the above two spheres problem has been given by Stimson and Jefferys[17]. The exact solution yields a correction factor

$$\lambda_{exact} = \frac{4}{3} \sinh\alpha \sum_{n=1}^{\infty} \frac{n(n+1)}{(2n-1)(2n+3)} \left[1 - \frac{4\sinh^2(n+\frac{1}{2})\alpha - (2n+1)^2 \sinh^2\alpha}{2\sinh(2n+1)\alpha + (2n+1)\sinh 2\alpha} \right]$$

where α is given by $\cosh\alpha = \frac{d}{2a}$. Comparing

$\frac{\text{center to center distance}}{\text{diameter}} = \frac{d}{2a}$	λ_{exact}	λ'	λ''
1	0.645	0.593	0.571
1.543	0.702	0.680	0.673
2.352	0.768	0.760	0.758
3.762	0.836	0.835	0.834
6.132	0.892	0.890	0.890
10.068	0.931	0.930	0.930
∞	1	1	1

Linear Superposition.

When there are more than two spheres in a row, no exact solution is available. However, using the "point force" approximation above, the problem can be solved by simple linear superposition. In the case of three spheres, A, B, and C, using the result of equation (3.23),

$$D_A = 6\pi\mu a U_\infty + \frac{3}{2}\left[\frac{a^3}{3d_{AB}^3} - \frac{a}{d_{AB}}\right] D_B + \frac{3}{2}\left[\frac{a^3}{3d_{AC}^3} - \frac{a}{d_{AC}}\right] D_C$$

$$D_B = 6\pi\mu b U_\infty + \frac{3}{2}\left[\frac{b^3}{3 d_{BC}^3} - \frac{b}{d_{BC}}\right] D_C + \frac{3}{2}\left[\frac{b^3}{3 d_{AB}^3} - \frac{b}{d_{AB}}\right] D_A$$

$$D_C = 6\pi\mu c U_\infty + \frac{3}{2}\left[\frac{c^3}{3 d_{AC}^3} - \frac{c}{d_{AC}}\right] D_A + \frac{3}{2}\left[\frac{c^3}{3 d_{BC}^3} - \frac{c}{d_{BC}}\right] D_B$$

Solving

$$\begin{aligned} D_A = 6\pi\mu a U_\infty \Big[& 1 + \frac{3b}{2a}\left(\frac{a^3}{3 d_{AB}^3} - \frac{a}{d_{AB}}\right) + \frac{3c}{2a}\left(\frac{a^3}{3 d_{AC}^3} - \frac{a}{d_{AC}}\right) + \frac{9b}{4a}\left(\frac{a^3}{3 d_{AC}^3} - \frac{a}{d_{AC}}\right)\left(\frac{c^3}{3 d_{BC}^3} - \frac{c}{d_{BC}}\right) \\ & + \frac{9c}{4a}\left(\frac{a^3}{3 d_{AB}^3} - \frac{a}{d_{AB}}\right)\left(\frac{b^3}{3 d_{AC}^3} - \frac{b}{d_{BC}}\right) - \frac{9}{4}\left(\frac{b^3}{d_{BC}^3} - \frac{b}{d_{BC}}\right)\left(\frac{c^3}{d_{BC}^3} - \frac{c}{d_{BC}}\right)\Big] \times \\ & \Big[1 - \frac{9}{4}\left(\frac{a^3}{3 d_{AB}^3} - \frac{a}{d_{AB}}\right)\left(\frac{b^3}{3 d_{AB}^3} - \frac{b}{d_{AB}}\right) - \frac{9}{4}\left(\frac{a^3}{3 d_{AC}^3} - \frac{a}{d_{AC}}\right)\left(\frac{c^3}{3 d_{AC}^3} - \frac{c}{d_{AC}}\right) \\ & - \frac{9}{4}\left(\frac{b^3}{3 d_{BC}^3} - \frac{b}{d_{BC}}\right)\left(\frac{c^3}{3 d_{BC}^3} - \frac{c}{d_{BC}}\right) - \frac{27}{8}\left(\frac{a^3}{3 d_{AB}^3} - \frac{a}{d_{AB}}\right)\left(\frac{b^3}{3 d_{BC}^3} - \frac{b}{d_{BC}}\right)\left(\frac{c^3}{3 d_{AC}^3} - \frac{c}{d_{AC}}\right) \\ & - \frac{27}{8}\left(\frac{a^3}{3 d_{AC}^3} - \frac{a}{d_{AC}}\right)\left(\frac{b^3}{3 d_{AB}^3} - \frac{b}{d_{AB}}\right)\left(\frac{c^3}{3 d_{BC}^3} - \frac{c}{d_{BC}}\right)\Big]^{-1} \end{aligned}$$

and D_B, D_C can be obtained from equation (3.31) by cyclic permutation.

If $d_{AB}, d_{AC}, d_{BC} >> a, b, c$, then equation (3.31) can be approximately written as

$$D_A \simeq 6\pi\mu a U_\infty \left[1 - \frac{3b}{2 d_{AB}} - \frac{3c}{2 d_{AC}}\right] + O\left[\left(\frac{b}{d_{AB}}\right)^2, \left(\frac{c}{d_{AC}}\right)^2\right] \qquad (3.32)$$

(b) <u>A More General Method.</u> It is quite clear that the method of solution used in (a) makes full use of the axial symmetry of the problem. Here it is intended to approach the problem in a more general manner and to obtain a few results that are of use in a later section.

Consider a sphere in an infinite space filled with an incompressible fluid save for some point or line singularities (e.g., point force, etc.). If the sphere is not present, let $\vec{W}$ be the velocity of the fluid. Take a coordinate system with its origin coinciding with the center of the sphere. With respect to this coordinate system,

$\vec{W}$ on the surface of the sphere can be written as:

denoting

$$Y^{e}_{l,m}(\theta,\phi) = P^{m}_{l}(\cos\theta)\cos m\phi$$

$$Y^{o}_{l,m}(\theta,\phi) = P^{m}_{l}(\cos\theta)\sin m\phi$$

$$W_r = \sum_{l=0}^{\infty}\sum_{m=0}^{l}\left[K^{e}_{lm}\,Y^{e}_{lm}(\theta,\phi) + K^{o}_{lm}\,Y^{o}_{lm}(\theta,\phi)\right]$$

$$W_\theta = \sum_{l=0}^{\infty}\sum_{m=0}^{l}\left[M^{e}_{lm}\,\frac{Y^{e}_{lm}(\theta,\phi)}{\sin\theta} + M^{o}_{lm}\,\frac{Y^{o}_{lm}(\theta,\phi)}{\sin\theta}\right]$$

$$W_\phi = \sum_{l=0}^{\infty}\sum_{m=0}^{l}\left[N^{e}_{lm}\,\frac{Y^{e}_{lm}(\theta,\phi)}{\sin\theta} + N^{o}_{lm}\,\frac{Y^{o}_{lm}(\theta,\phi)}{\sin\theta}\right] \qquad (3.33)$$

where the coefficients are given by

$$K^{e,o}_{lm} = \frac{(2l+1)(l-m)!}{2\pi\,(l+m)!}\int_{\text{sphere surface}} W_r\,Y^{e,o}_{lm}(\theta,\phi)\,d\Omega$$

$$M^{e,o}_{lm} = \frac{(2l+1)(l-m)!}{2\pi\,(l+m)!}\int W_\theta\,Y^{e,o}_{lm}(\theta,\phi)\sin\theta\,d\Omega$$

$$N^{e,o}_{lm} = \frac{(2l+1)(l-m)!}{2\pi\,(l+m)!}\int W_\phi\,Y^{e,o}_{lm}(\theta,\phi)\sin\theta\,d\Omega \qquad (3.34)$$

The perturbation velocity and pressure field due to the presence of the sphere can be obtained using the general solution of Appendix B. The condition that the perturbation velocity and pressure vanish at infinity requires that the coefficients E, A, and C of equation (B-1), Appendix B, be set equal to zero. The other coefficients are determined by the no-slip condition on the sphere. Writing out the equations for $l = 1$, $m = 0$ from equation (3.33) (a = radius of sphere)

$$\frac{F_{10}^{o}}{\mu a} - 2\frac{B_{10}^{o}}{a^3} = -K_{10}^{o}$$

$$-\frac{F_{10}^{o}}{3a\mu} - \frac{2B_{10}^{o}}{3a^3} = -M_{00}^{o}$$

Solving

$$F_{10}^{o} = \frac{a\mu}{2}\left[3M_{00}^{o} - K_{10}^{o}\right] \qquad (3.35)$$

But from equation (B-3) of Appendix B , the drag in the direction of the polar axis is given by

$$\text{Drag} = -4\pi F_{10}^{o}$$

Therefore, by equation (3.35),

$$\text{Drag (polar axis)} = -2\pi\mu a\left[3M_{00}^{o} - K_{10}^{o}\right]$$

But from the definition of M_{00}^{o} , K_{10}^{o} , equation (3.34),

$$\text{Drag}_{\text{(polar axis)}} = -2\pi\mu a\left[\frac{3}{4\pi}\int_{\text{sphere surface}} W_\theta \sin\theta \, d\Omega - \frac{3}{4\pi}\int_{\text{sphere surface}} W_r \cos\theta \, d\Omega\right]$$

$$= \frac{3\mu a}{2}\int_{\text{sphere}} W_{\text{polar axis}} \, d\Omega \qquad (3.36)$$

Let the mean unperturbated velocity be denoted by

$$W_m = \frac{1}{4\pi}\int_{\text{sphere}} W \, d\Omega$$

Hence,

$$\text{Drag (polar axis)} = 6\pi\mu a \, W_{m,\,\text{polar axis}} \qquad (3.37)$$

Since the polar axis is arbitrary, therefore

Drag on sphere = $6\pi\mu a \cdot$ mean unperturbed velocity over surface of sphere in corresponding direction (3.38)

Let the x- axis be the polar axis and L be the scale of length. x', y', z', r' are the dimensionless variables:

$$x' = \frac{x}{L} \qquad y' = \frac{y}{L} \qquad z' = \frac{z}{L} \qquad r' = \frac{r}{L}$$

Expand W_x as a power series about the center of the sphere so that

$$W_x = W_o + \sum_{l,m,n=1}^{\infty} A_{lmn}\, x'^{l} y'^{m} z'^{n} \tag{3.39}$$

where

$$A_{lmn} = \frac{1}{n!\,m!\,l!}\left[\frac{\partial^{l+m+n}}{\partial x'^{l}\partial y'^{m}\partial z'^{n}}(W_x)\right]_o$$

Substitute equation (3.39) into (3.36)

$$D_x = \frac{3\mu a}{2}\int_{sphere}\left[W_o + \sum_{l,m,n=1}^{\infty} A_{lmn}\left(\frac{a}{L}\right)^{l+m+n}\cos^{l}\theta\,\sin^{m+n}\theta\,\sin^{n}\phi\,\cos^{m}\phi\right]\sin\theta\, d\theta\, d\phi$$

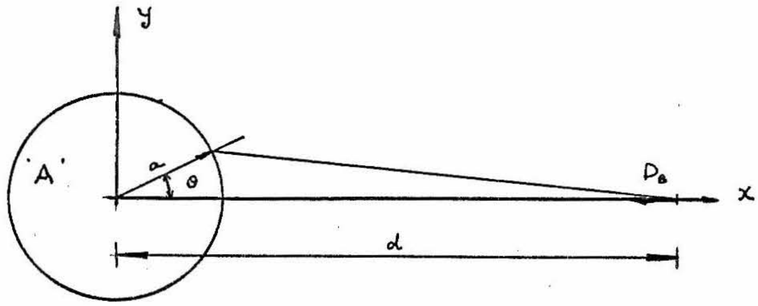
$$= \frac{3\mu a}{2}\left[4\pi W_o + \sum_{\substack{l,m,n\\ \neq\, odd}} \frac{2}{n!\,m!\,l!} A_{lmn}\left(\frac{a}{L}\right)^{l+m+n} B\left(\frac{l+1}{2},\frac{m+n+2}{2}\right)B\left(\frac{n+1}{2},\frac{m+1}{2}\right)\right] \tag{3.40}$$

where $B(n,m)$ is the beta function.

From equation (3.40), the following formula holds:

$$\overrightarrow{\text{drag}} = 6\pi\mu a\,\overrightarrow{V}_{ext} + O\left(\frac{a^2}{L^2}\right) \tag{3.41}$$

where $\overrightarrow{V}_{ext.}$ is the velocity of fluid at the center of the sphere if the sphere is removed.



Now consider Problem A above once again. Using equation (1. 13) of Section 1, the velocity due to a point force D_B as shown on the surface of sphere A is

$$W_x = -\frac{D_B}{8\pi\mu}\left[\frac{(d-a\cos\theta)^2}{(a^2+d^2-2ad\cos\theta)^{3/2}}+\frac{1}{(a^2+d^2-2ad\cos\theta)^{1/2}}\right] \qquad (3.42)$$

Substituting equation (3. 42) into equation (3. 36), the drag on sphere A due to point force D_B is

$$D_A = \frac{3\mu a}{2}\int_{sphere} -\frac{D_B}{8\pi\mu}\left[\frac{(d-a\cos\theta)^2}{(a^2+d^2-2ad\cos\theta)^{3/2}}+\frac{1}{(a^2+d^2-2ad\cos\theta)^{1/2}}\right] d\Omega$$

$$\therefore \quad D_A = \frac{3}{2} D_B \left[\frac{a^3}{3d^3} - \frac{a}{d}\right]$$

Hence equation (3. 27) is being recovered.

If equation (3. 41) is used instead of equation (3. 36), then in this case,

$$\vec{V}_{ext} = \left(U_\infty - \frac{D_B}{4\pi\mu d}\right)\hat{x}$$

Therefore,

$$D_A = 6\pi\mu a\left[U_\infty - \frac{D_B}{4\pi\mu d}\right]$$

Similarly,

$$D_B = 6\pi\mu b \left[U_\infty - \frac{D_A}{4\pi\mu d} \right]$$

In case $a = b$

$$D_A = D_B = \frac{6\pi\mu a U_\infty}{1 + \frac{3}{2}\frac{a}{d}} = 6\pi\mu a \lambda'' U_\infty \qquad (3.43)$$

the correction factor λ'' has been shown on page 35. It agrees fairly well with the exact solution.

4. The Effect of Continuous Solid Boundaries and Fluid Interface on the Drag of a Solid Particle

When a small solid particle moves through a viscous fluid adjacent to continuous solid boundaries or fluid interface, the drag force which the particle encounters depends on its proximity to such discontinuities. Since the work of H. Lorentz (1907), this so-called "wall effect" on the drag of a particle has been the subject of much research[18-28]. However, most of the work done so far, with the exception of reference 27, is only approximate. Here, the problem will also be attacked in an approximate fashion by means of the "point force" technique employed in previous sections.

(a) Potential Functions for Stokes Equation. Stokes equation is

$$\nabla p = \mu \nabla^2 \vec{u} \qquad (4.1)$$

$$\nabla \cdot \vec{u} = 0 \qquad (4.2)$$

The structure of equations (4.1) and (4.2) permits one to construct a scalar and a vector potential such that the scalar potential for $\vec{u}$, satisfies the inhomogeneous equation while the vector potential satisfies the homogeneous equation. It is easy to verify that scalar potential Φ , defined by the following equation, will satisfy equations (4. 1) and (4. 2) identically

$$\vec{u} = \frac{1}{\mu}\left[\nabla^2\Phi\,\hat{x} - \nabla\frac{\partial\Phi}{\partial x}\right]$$

$$p = -\nabla^2\frac{\partial\Phi}{\partial x} \tag{4. 3}$$

where

$$\nabla^4\Phi = 0 \tag{4. 4}$$

Let $\vec{u}$ be given by vector potential $\vec{A}$ as

$$\vec{u} = \nabla\times\vec{A} \tag{4. 5}$$

then the continuity equation is satisfied identically. Substitute into the homogeneous part of equation (4. 1)

$$\nabla^2\vec{u} = \nabla^2(\nabla\times\vec{A}) = \nabla\times(\nabla^2\vec{A}) = 0$$

Hence

$$\vec{A} = \phi_1\hat{x} + \phi_2\hat{y} + \phi_3\hat{z} \tag{4. 6}$$

The homogeneous Stokes equation will be satisfied if ϕ_1 , ϕ_2 , and ϕ_3 are harmonic functions, i. e.,

$$\nabla^2\phi_1 = 0\ ,\quad \nabla^2\phi_2 = 0\ ,\quad \nabla^2\phi_3 = 0 \tag{4. 7}$$

(b) <u>The "Point Force" Approximation.</u> Consider an axially symmetric solid particle moving close to a solid wall in a viscous fluid with its velocity vector in the direction of its axis of symmetry. Let $-\alpha\vec{u}$ be the drag such a particle would experience on moving through a stationary fluid of infinite extent with velocity $\vec{U}$. The exact mathematical problem in this case is to seek a solution to Stokes equation satisfying the no-slip boundary condition on the wall and the surface of the particle. The "point force" approximation involves the following two steps:

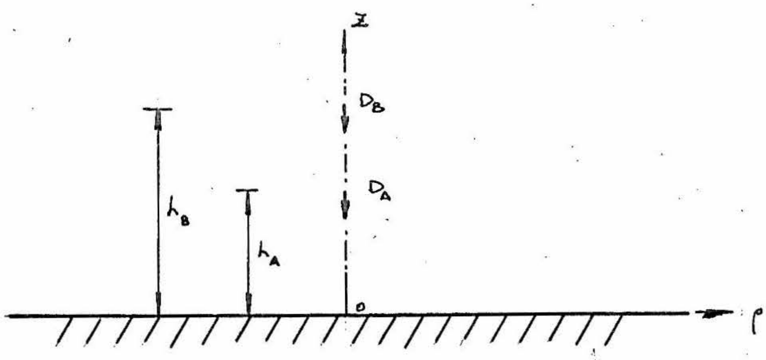
(1) Replace the particle by a point force of strength $\vec{D}$ where $-\vec{D}$ is the drag on the particle.

(2) Replace the no-slip condition on the surface of the particle by the following equation which is a direct generalization of equation (3.41).

$$\vec{D} = -\alpha \vec{V}_{ext.} \tag{4.8}$$

where $\vec{V}_{ext.}$ is the velocity of fluid as seen by the point force.

(c) <u>Two Axially Symmetric Particles Falling Towards a Plane Wall.</u> Consider two axially symmetric particles A and B characterized by geometric factor α_a, α_b (defined in (b)) at distance h_a and h_b falling towards a plane wall. Let D_A and D_B be the drag on the particles. On using the point force approximation outlined in the field equation and boundary conditions become



$$\nabla p = \mu \nabla^2 \vec{u} - D_A \hat{z}\, \delta(z-h_a)\, \delta(\rho) - D_B \hat{z}\, \delta(z-h_b)\, \delta(\rho)$$

$$\nabla \cdot \vec{u} = 0 \qquad (4.9)$$

$$\vec{u} = 0 \qquad p = 0 \qquad \text{at infinity} \qquad (4.10)$$

$$\vec{u} = 0 \qquad z = 0 \qquad (4.11)$$

The present problem is linear, and hence it is possible to deal with each point force separately and then superimpose the solutions together. Consider particle A only. Let

$$\vec{u} = \vec{u}_c + \vec{u}_p$$
$$p = p_c + p_p \qquad (4.12)$$

so that the particular solution $\vec{u}_p$, p_p solves the inhomogeneous part of equation (4.9). Here, to facilitate calculation, the particular solution will be taken as the one corresponding to a point force $\vec{D}_A$ at A and an image force at $z = -h_a$ with the wall removed. $\vec{u}_p$ and p_p can be calculated easily by means of equation (1.13) so that the boundary conditions for $\vec{u}_c$ and p_c are (the subscript c will be dropped):

$$\vec{u} = 0 \qquad p = 0 \qquad \text{at infinity} \qquad (4.13)$$

$$u_z = 0 \qquad z = 0$$

$$u_\rho = -\frac{D_A}{4\pi\mu}\left[\frac{h_a \rho}{(h_a^2 + \rho^2)^{3/2}}\right] \quad , \; z = 0 \qquad (4.14)$$

To solve for $\vec{u}$ and p , one can make use of scalar potential Φ introduced in (a). Hence, let

$$u_z = \frac{1}{\mu}\left[\frac{\partial^2 \Phi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \Phi}{\partial \rho}\right]$$

$$u_\rho = -\frac{1}{\mu}\,\frac{\partial^2 \Phi}{\partial \rho\,\partial z}$$

$$p = -\frac{1}{\rho}\,\frac{\partial}{\partial \rho}\left[\rho\,\frac{\partial^2 \Phi}{\partial \rho\,\partial z}\right] - \frac{\partial^3 \Phi}{\partial z^3} \tag{4.15}$$

where

$$\left[\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\,\frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2}\right]^2 \Phi = 0 \tag{4.16}$$

and from equations (4. 13) and (4. 14), the boundary conditions for Φ are

$$\Phi = 0 \qquad z = \infty \;,\; \rho = \infty \tag{4.17}$$

$$\frac{\partial^2 \Phi}{\partial \rho^2} + \frac{1}{\rho}\,\frac{\partial \Phi}{\partial \rho} = 0 \qquad \text{at } z = 0 \tag{4.18}$$

$$\frac{\partial \Phi}{\partial z} = -\frac{D_A}{4\pi}\,\frac{h_a}{(h_a^2 + \rho^2)^{1/2}} \;, \qquad z = 0 \tag{4.19}$$

where equation (4. 19) has been obtained after one integration with respect to ρ .

Define the zeroth order Hankel transform by

$$\bar{f}(\xi) = \int_0^\infty \rho\, f(\rho)\, J_0(\xi\rho)\, d\rho$$

From equations (4. 16) through (4. 19), Φ satisfies the following equation and boundary value problem:

$$\xi^4 \bar{\Phi} - 2\xi^2 \frac{d^2 \bar{\Phi}}{dz^2} + \frac{d^4 \bar{\Phi}}{dz^4} = 0 \tag{4.20}$$

$$\bar{\Phi} = 0 \qquad z = \infty \tag{4.21}$$

$$\bar{\Phi} = 0 \qquad z = 0 \tag{4.22}$$

$$\frac{d\bar{\Phi}}{dZ} = -\frac{D_A h_A}{4\pi}\frac{e^{-h_a \xi}}{\xi} \qquad Z = 0 \tag{4.23}$$

The solution of equation (4.20) satisfying (4.21), (4.22), and (4.23) is

$$\bar{\Phi} = -\frac{D_A h_a}{4\pi\xi} Z e^{-(h_a+Z)\xi} \tag{4.24}$$

Therefore

$$\Phi = -\frac{D_A h_a Z}{4\pi\left[\rho^2+(h_a+Z)^2\right]^{1/2}} \tag{4.25}$$

Substituting into equation (4.15),

$$U_z = \frac{1}{\mu}\left[\frac{D_A h_a Z}{2\pi\left[\rho^2+(h+Z)^2\right]^{3/2}} - \frac{3 D_A h_a Z \rho^2}{4\pi\left[\rho^2+(h_a+Z)^2\right]^{5/2}}\right] \tag{4.26}$$

Similarly, an equation similar to equation (4.26) can be obtained for particle B .

To determine D_A and D_B , apply equation (4.8), using $\vec{U}_f$ and (4.26).

For particle A :

$$\vec{V}_{ext\,A} = \left[U_A + \frac{D_A}{8\pi\mu h_a} + \frac{D_A}{16\pi\mu h_a} + \frac{D_B}{4\pi\mu(h_a+h_b)} + \frac{D_B h_a h_b}{2\pi\mu(h_a+h_b)^3} - \frac{D_B}{4\pi\mu|h_b-h_a|}\right]\hat{Z} \tag{4.27}$$

For particle B :

$$\vec{V}_{ext\,B} = \left[U_B + \frac{3D_B}{16\pi\mu h_b} + \frac{D_A}{4\pi\mu(h_a+h_b)} + \frac{D_A h_a h_b}{2\pi\mu(h_a+h_b)^3} - \frac{D_A}{4\pi\mu|h_b-h_a|}\right]\hat{Z}$$

Therefore,

$$D_A = \alpha_A\left[U_A + \frac{3D_A}{16\pi\mu h_a} + \frac{D_B}{4\pi\mu(h_a+h_b)} + \frac{D_B h_a h_b}{2\pi\mu(h_a+h_b)^3} - \frac{D_B}{4\pi\mu|h_b-h_a|}\right] \tag{4.28}$$

$$D_B = \alpha_b \left[U_B + \frac{3 D_B}{16\pi\mu h_b} + \frac{D_A}{4\pi\mu(h_a+h_b)} + \frac{D_A h_a h_b}{2\pi\mu(h_a+h_b)^3} - \frac{D_A}{4\pi\mu|h_a-h_b|} \right] \qquad (4.29)$$

D_A and D_B can be obtained by solving equations (4.28) and (4.29) simultaneously.

If the particles are spheres $\alpha_a = 6\pi\mu a$, $\alpha_b = 6\pi\mu b$, a and b are the radii of A and B , D_A and D_B are given by

$$D_A = \frac{6\pi\mu a U_A \left[1 - \frac{9b}{8h_b}\right] + 6\pi\mu U_B \left[\frac{3 a h_a h_b}{(h_a+h_b)^3} - \frac{3 a h_a}{|h_b^2 - h_a^2|}\right]}{\left[1 - \frac{9a}{8h_a}\right]\left[1 - \frac{9b}{8h_b}\right] - \left[\frac{3 a h_a h_b}{(h_a+h_b)^3} - \frac{3 a h_a}{|h_a^2 - h_b^2|}\right]\left[\frac{3 b h_a h_b}{(h_a+h_b)^3} - \frac{3 b h_b}{|h_b^2 - h_a^2|}\right]} \qquad (4.30)$$

and D_B by interchanging a and b .

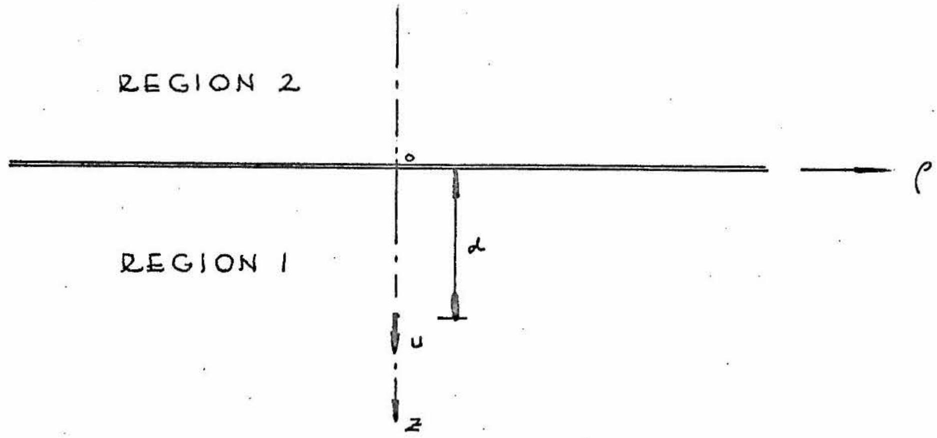
For a single sphere case, put $b = 0$, $U_B = 0$ into equation (4.30):

$$D = \frac{6\pi\mu a U}{1 - \frac{9a}{8h}} \qquad (4.31)$$

Equation (4.31) agrees fairly well with the exact solution given by Maude[27].

(d) <u>Particle Moving Perpendicularly to a Fluid Interface.</u> The proper macroscopic boundary conditions at the interface of two fluids have been discussed in great detail in many books, notably in Levich's[29]. Here, for simplicity, the fluids involved will be assumed to be isothermal and homogeneous and the interface free from absorbed material. Furthermore, the surface tension at the interface will be assumed to be very large so that the continuity of normal stress is guaranteed. Consider a spherical particle moving perpendicular to a fluid interface as shown. Upon assuming that the interface remains practically plane, the relevant boundary conditions are

$$\vec{u} \, , \; p = 0 \qquad \text{at infinity}$$



At $z = 0$

$$u_{1\rho} = u_{2\rho} \qquad \text{(continuity of tangential velocity)} \tag{4.32}$$

$$u_{1z} = u_{2z} \simeq 0 \tag{4.33}$$

(continuity of tangential stress)

$$\mu_1 \left[\frac{\partial u_{1z}}{\partial \rho} + \frac{\partial u_{1\rho}}{\partial z} \right] = \mu_2 \left[\frac{\partial u_{2z}}{\partial \rho} + \frac{\partial u_{2\rho}}{\partial z} \right] \tag{4.34}$$

Split the solution into a homogeneous and a particular solution as in (c) and use the fundamental solution of Stokes equation obtained in Section 1, so that:

in region (2), let

$$u_{2z} = \frac{1}{\mu_2}\left[\frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho}\right]$$

$$u_{2\rho} = -\frac{1}{\mu_2}\frac{\partial^2 f}{\partial \rho \partial z} \qquad (4.35)$$

$$p_2 = -\left[\frac{\partial^2}{\partial \rho^2}\left(\frac{\partial f}{\partial z}\right) + \frac{1}{\rho}\frac{\partial^2 f}{\partial z \partial \rho}\right] - \frac{\partial^3 f}{\partial z^3}$$

and

$$\nabla^4 f = 0 \qquad (4.36)$$

in region (1) (- D = drag on particle), let

$$u_{1z} = \frac{1}{\mu_1}\left[\frac{\partial^2 \Phi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \Phi}{\partial \rho}\right] + \frac{D}{8\pi\mu_1}\left[\frac{(z-d)^2}{[(z-d)^2+\rho^2]^{3/2}} + \frac{1}{[(z-d)^2+\rho^2]^{1/2}}\right]$$

$$u_{1\rho} = -\frac{1}{\mu_1}\frac{\partial^2 \Phi}{\partial \rho \partial z} + \frac{D}{8\pi\mu_1}\left[\frac{\rho(z-d)}{[(z-d)^2+\rho^2]^{3/2}}\right] \qquad (4.37)$$

$$p_1 = -\left[\frac{\partial^3 \Phi}{\partial \rho^2 \partial z} + \frac{1}{\rho}\frac{\partial^2 \Phi}{\partial \rho \partial z}\right] - \frac{\partial^3 \Phi}{\partial z^3} + \frac{D(z-d)}{4\pi[(z-d)^2+\rho^2]^{3/2}}$$

and

$$\nabla^4 \Phi = 0 \qquad (4.38)$$

Substituting (4.35) and (4.37) into equations (4.32), (4.33), and (4.34), the boundary conditions become

at $z = 0$

$$\left[\frac{\partial^2 \Phi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \Phi}{\partial \rho}\right] - \frac{\partial^2 \Phi}{\partial z^2} + \frac{D}{4\pi}\frac{d^2}{(d^2+\rho^2)^{3/2}} = \left[\frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho} - \frac{\partial^2 f}{\partial z^2}\right] \qquad (4.39)$$

$$\frac{1}{\mu_1}\frac{\partial \Phi}{\partial z} - \frac{D d}{8\pi\mu_1(d^2+\rho^2)^{1/2}} = \frac{1}{\mu_2}\frac{\partial f}{\partial z} \qquad (4.40)$$

$$\frac{1}{\mu_1}\left[\frac{\partial^2 \Phi}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial \Phi}{\partial \rho}\right]+\frac{D}{8\pi\mu_1}\left[\frac{d^2}{[d^2+\rho^2]^{3/2}}+\frac{1}{(d^2+\rho^2)^{1/2}}\right]=\frac{1}{\mu_2}\left[\frac{\partial^2 \psi}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial \psi}{\partial \rho}\right]=0 \quad (4.41)$$

where equation (4.39) has been integrated once with respect to ρ .

Let $\bar{\Phi}$, $\bar{\psi}$ be the zeroth order Hankel transform of Φ and ψ as defined in (c); then from equations (4.36), (4.38), (4.39), (4.40), and (4.41), $\bar{\Phi}$ and $\bar{\psi}$ satisfy the following differential equation and boundary conditions:

$$\left(-\xi^2+\frac{d^2}{dz^2}\right)^2\begin{bmatrix}\bar{\Phi}\\ \bar{\psi}\end{bmatrix}=0 \quad (4.42)$$

at $z=0$

$$-\xi^2\bar{\Phi}-\frac{d^2\bar{\Phi}}{dz^2}+\frac{D d e^{-d\xi}}{4\pi}=-\xi^2\bar{\psi}-\frac{d^2\bar{\psi}}{dz^2} \quad (4.43)$$

$$\frac{1}{\mu_1}\frac{d\bar{\Phi}}{dz}-\frac{D d e^{-d\xi}}{8\pi\mu_1\xi}=\frac{1}{\mu_2}\frac{d\bar{\psi}}{dz} \quad (4.44)$$

$$-\frac{\xi^2}{\mu_1}\bar{\Phi}+\frac{D}{8\pi\mu_1}\left[d e^{-\xi d}+\frac{e^{-\xi d}}{\xi}\right]=-\frac{\xi^2}{\mu_2}\bar{\psi}=0 \quad (4.45)$$

The solution of equation (4.42) satisfying the boundary conditions is

$$\bar{\psi}=F z e^{\xi z} \quad (4.46)$$

$$\bar{\Phi}=(A+Bz)e^{-\xi z} \quad (4.47)$$

where

$$A=\frac{D}{8\pi}\left[\frac{d e^{-\xi d}}{\xi^2}+\frac{e^{-\xi d}}{\xi^3}\right]$$

$$B = \frac{D}{8\pi} \frac{\mu_1 \mu_2}{\mu_1 + \mu_2} \left[\frac{2d}{\xi \mu_1} + \frac{(\mu_1 + \mu_2)}{\mu_1 \mu_2 \xi^2} \right] e^{-\xi d}$$

$$F = -\frac{D}{4\pi} \left[\frac{\mu_2 d}{(\mu_1 + \mu_2)\xi} \right] e^{-\xi d}$$

From equation (4.37) the reflected velocity is given by

$$\overline{U}_{1z} \text{ reflected} = -\frac{\xi^2}{\mu_1} \overline{\Phi} = \frac{-D}{8\pi\mu_1} \left[d + \frac{1}{\xi} + \frac{2d\mu_2 z\xi}{\mu_1 + \mu_2} - z \right] e^{-(d+z)\xi}$$

Hence

$$\overline{U}_{1z\,ref}(\rho, z) = \int_0^\infty \xi J_0(\xi\rho) \overline{U}_{1z}(\xi, z)\, d\xi$$

Therefore,

$$U_{1z\,ref}(0, d) = \int_0^\infty \xi \overline{U}_{1z}(\xi, d)\, d\xi$$

$$= -\frac{D}{8\pi\mu_1 d} \left[1 + \frac{\mu_2}{2(\mu_1 + \mu_2)} \right] \qquad (4.48)$$

Using equation (4.8),

$$V_{ext} = -U - \frac{D}{8\pi\mu_1 d} \left[1 + \frac{\mu_2}{2(\mu_1 + \mu_2)} \right]$$

so that

$$D = -6\pi\mu_1 a V_{ext} = 6\pi\mu_1 a U + \frac{3aD}{4d} \left[1 + \frac{\mu_2}{2(\mu_1 + \mu_2)} \right]$$

$$= \frac{6\pi\mu_1 a U}{1 - \frac{3a}{4d} \left[1 + \frac{\mu_2}{2(\mu_1 + \mu_2)} \right]} \qquad (4.49)$$

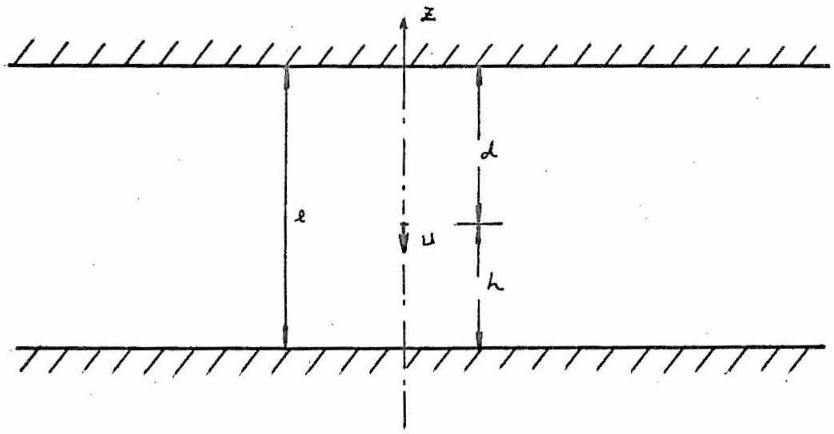
If $\mu_2 \rightarrow \infty$ (solid wall),

$$D = \frac{6\pi\mu a U}{1 - \frac{9a}{8d}} \qquad \text{as equation (4.31)}$$

If $\mu_1 \to 0$ (free surface)

$$D = \frac{6\pi\mu a U}{1 - \frac{3a}{4d}} \tag{4.50}$$

(e) <u>Particle Moving Between Two Parallel Walls.</u> For the case of an axially symmetric particle moving perpendicularly to two walls, as shown, the technique used in (c) and (d) applies. By means



of Hankel transform on ρ and imposing the no-slip condition on the two walls, it is straightforward to obtain the following drag formula.

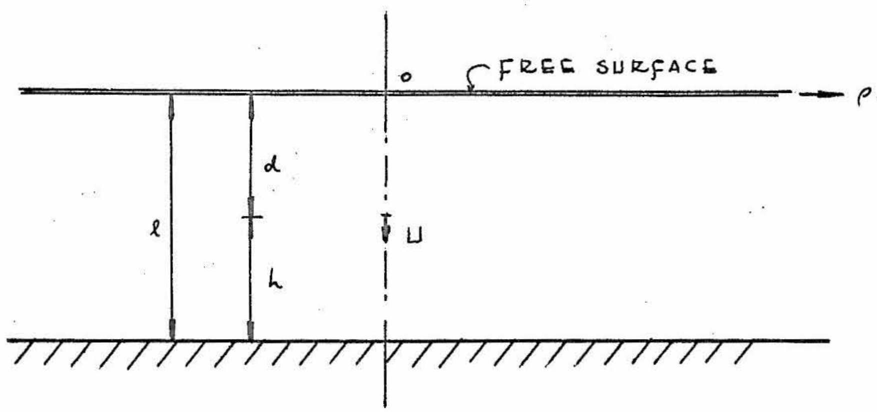
α geometric factor of particle ($6\pi\mu a$ for a sphere)

$$\sigma = \frac{l}{h}$$

$$I(\sigma) = \int_0^\infty \Big[\frac{e^{(2\sigma-2)\eta}}{\eta} + \frac{e^{(2\sigma-2)\eta}}{\eta^2} + \frac{e^{(2\sigma-2)\eta}}{2\eta^3} + \frac{(\sigma-1)^2 e^{2\eta}}{\eta} + \frac{(\sigma-1)e^{2\eta}}{\eta^2}$$
$$+ \frac{e^{2\eta}}{2\eta^3} - 2\sigma(\sigma-1) - \frac{2\sigma^2}{\eta} - \frac{(4\sigma-1)}{2\eta^2} - \frac{1}{\eta^3} - \frac{(\sigma^2-\sigma+1)e^{-2\eta}}{\eta}$$
$$+ \frac{\sigma-1}{\eta^2} e^{-2\eta} - \frac{e^{-2\eta}}{2\eta^3} - \frac{e^{-(2\sigma-2)\eta}}{\eta} + \frac{e^{-(2\sigma-2)\eta}}{\eta^2} - \frac{e^{-(2\sigma-2)\eta}}{2\eta^3} + \frac{\sigma e^{-2\eta}}{\eta} + \frac{e^{-2\sigma\eta}}{\eta^3} \Big] \times$$
$$\Big[\frac{e^{2\sigma\eta}}{2} - 2\sigma\eta^2 - 1 + \frac{e^{-2\sigma\eta}}{2} \Big]^{-1} \eta^3 \, d\eta$$

$$\text{drag} = D = \frac{\alpha U}{1 - \frac{3a}{4h} I(\sigma)} \qquad (4.51)$$

(f) Particle Moving Perpendicular to a Solid Wall and Free Surface. If, instead of two solid walls as in (e), the upper wall is replaced by a free surface, the same analysis as before can be carried out without difficulty. The only necessary modification is to



change the no-slip condition at the upper wall into a stress-free condition. On carrying out the necessary changes, the following drag formula can easily be obtained:

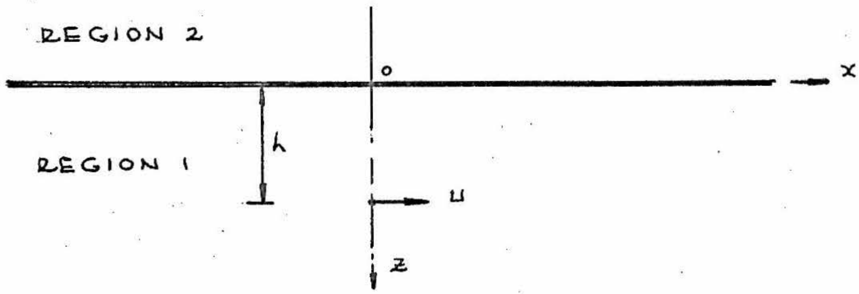
α = geometric factor of particle (equal to $6\pi\mu a$ for a sphere

$\sigma = \dfrac{l}{h}$

$$J(\sigma) = \int_0^\infty \Big[2\eta^2 e^{2(\sigma-1)\eta} + 2\eta e^{2(\sigma-1)\eta} + e^{2(\sigma-1)\eta} + 2(\sigma-1)\eta e^{2\eta} + e^{2\eta}$$
$$+ 4(1-2\sigma)\eta^2 - 4\sigma\eta - 2 - 2(\sigma-1)\eta e^{-2\eta} + e^{-2\eta} + 2\eta^2 e^{-2(\sigma-1)\eta}$$
$$- 2\eta e^{-2(\sigma-1)\eta} + e^{-2(\sigma-1)\eta} - 2e^{-2\sigma\eta} \Big] \Big[e^{2\sigma\eta} - e^{-2\sigma\eta} - 4\sigma\eta \Big]^{-1} d\eta$$

$$\text{drag} = D = \frac{\alpha U}{1 - \frac{3a}{4h} J(\sigma)} \qquad (4.52)$$

(g) Particle Moving Parallel to a Fluid Interface or Plane Wall. Consider a sphere of radius a moving parallel to an interface with velocity U in the x - direction, as shown. Let $-D$ be



the drag experienced by the particle. The Boundary conditions at the interface are $z = 0$

$$\mu_1 \left[\frac{\partial U_{1z}}{\partial x} + \frac{\partial U_{1x}}{\partial z} \right] = \mu_2 \left[\frac{\partial U_{2z}}{\partial x} + \frac{\partial U_{2x}}{\partial z} \right] \tag{4.53}$$

$$\mu_1 \left[\frac{\partial U_{1y}}{\partial z} + \frac{\partial U_{1z}}{\partial y} \right] = \mu_2 \left[\frac{\partial U_{2z}}{\partial y} + \frac{\partial U_{2y}}{\partial z} \right] \tag{4.54}$$

$$U_{1x} = U_{2x} \tag{4.55}$$

$$U_{1y} = U_{2y} \tag{4.56}$$

$$U_{1z} = U_{2z} \simeq 0 \tag{4.57}$$

$$\vec{U}_{1,2} = 0 \qquad \text{at infinity}$$

Here, using the scalar function Φ alone is insufficient, and it is necessary to add one component of vector potential $\vec{A}$ defined in (a) in order that the boundary conditions can be satisfied.

In region (1), let

$$u_{1x} = \frac{1}{\mu_1}\left[\frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2}\right] + \frac{D}{8\pi\mu_1}\left[\frac{1}{[x^2+y^2+(z-h)^2]^{1/2}} + \frac{x^2}{[x^2+y^2+(z-h)^2]^{3/2}}\right]$$

$$u_{1y} = -\frac{1}{\mu_1}\left[\frac{\partial^2 \Phi}{\partial x \partial y} - \frac{\partial \phi}{\partial z}\right] + \frac{D}{8\pi\mu_1}\,\frac{x\,y}{[x^2+y^2+(z-h)^2]^{3/2}} \tag{4.58}$$

$$u_{1z} = -\frac{1}{\mu_1}\left[\frac{\partial^2 \Phi}{\partial x \partial z} + \frac{\partial \phi}{\partial y}\right] + \frac{D}{8\pi\mu_1}\,\frac{x\,(z-h)}{[x^2+y^2+(z-h)^2]^{3/2}}$$

$$p_1 = -\nabla^2 \frac{\partial \Phi}{\partial x} + \frac{D}{4\pi}\,\frac{x}{[x^2+y^2+(z-h)^2]^{3/2}}$$

where

$$\nabla^4 \Phi = 0 \tag{4.59}$$

$$\nabla^2 \phi = 0 \tag{4.60}$$

In region (2), let

$$\begin{aligned} u_{2x} &= \frac{1}{\mu_2}\left[\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}\right] \\ u_{2y} &= -\frac{1}{\mu_2}\left[\frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial \chi}{\partial z}\right] \\ u_{2z} &= -\frac{1}{\mu_2}\left[\frac{\partial^2 \psi}{\partial x \partial z} + \frac{\partial \chi}{\partial y}\right] \\ p_2 &= -\nabla^2 \frac{\partial \psi}{\partial x} \end{aligned} \tag{4.61}$$

where

$$\nabla^4 \psi = 0 \tag{4.62}$$

$$\nabla^2 \chi = 0 \tag{4.63}$$

Substituting equations (4.58) and (4.61) into (4.53) to (4.57), the boundary conditions become:

At $z=0$

$$-\frac{\partial^3 \Phi}{\partial x^2 \partial z} - \frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^3 \Phi}{\partial y^2 \partial z} + \frac{\partial^3 \Phi}{\partial z^3} + \frac{3D}{4\pi} \frac{x^2 h}{(x^2+y^2+h^2)^{5/2}}$$

$$= -\frac{\partial^3 \psi}{\partial x^2 \partial z} - \frac{\partial^2 \chi}{\partial x \partial y} + \frac{\partial^3 \psi}{\partial y^2 \partial z} + \frac{\partial^3 \psi}{\partial z^3} \tag{4.64}$$

$$-2\frac{\partial^3 \Phi}{\partial x \partial y \partial z} + \frac{\partial^2 \phi}{\partial z^2} - \frac{\partial^2 \phi}{\partial y^2} + \frac{3D}{4\pi} \frac{x y h}{(x^2+y^2+h^2)^{5/2}} = -2\frac{\partial^3 \psi}{\partial x \partial y \partial z}$$

$$+ \frac{\partial^2 \chi}{\partial z^2} - \frac{\partial^2 \chi}{\partial y^2} \tag{4.65}$$

$$\frac{1}{\mu_1}\left[\frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2}\right] + \frac{D}{8\pi\mu_1}\left[\frac{1}{(x^2+y^2+h^2)^{3/2}} + \frac{x^2}{(x^2+y^2+h^2)^{5/2}}\right] = \frac{1}{\mu_2}\left[\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}\right] \tag{4.66}$$

$$-\frac{1}{\mu_1}\frac{\partial^2 \Phi}{\partial x \partial y} + \frac{1}{\mu_1}\frac{\partial \phi}{\partial z} + \frac{D}{8\pi\mu_1}\frac{x y}{(x^2+y^2+h^2)^{5/2}} = -\frac{1}{\mu_2}\frac{\partial^2 \psi}{\partial x \partial y} + \frac{1}{\mu_2}\frac{\partial \chi}{\partial z} \tag{4.67}$$

$$-\frac{1}{\mu_1}\frac{\partial^2 \Phi}{\partial x \partial z} - \frac{1}{\mu_1}\frac{\partial \phi}{\partial y} - \frac{D}{8\pi\mu_1}\frac{x h}{(x^2+y^2+h^2)^{3/2}} = -\frac{1}{\mu_2}\frac{\partial^2 \psi}{\partial x \partial z} - \frac{1}{\mu_2}\frac{\partial \chi}{\partial y} = 0 \tag{4.68}$$

Denote the Fourier transform of $f(x,y)$ by

$$\bar{f} = \iint_{-\infty}^{+\infty} f(x,y)\, e^{-i(\lambda x + \alpha y)}\, dx\, dy$$

Then from (4.64) to (4.68), the boundary conditions for $\bar{\Phi}$, $\bar{\phi}$, $\bar{\psi}$, and $\bar{\chi}$ are

$$\lambda^2 \frac{\partial \bar{\Phi}}{\partial z} + \lambda\alpha\bar{\phi} - \alpha^2\frac{\partial \bar{\Phi}}{\partial z} + \frac{\partial^3 \bar{\Phi}}{\partial z^3} + \frac{D}{2}\left[1 - \frac{\lambda^2 h}{(\lambda^2+\alpha^2)^{1/2}}\right] e^{-(\alpha^2+\lambda^2)^{1/2} h}$$

$$= \lambda^2\frac{\partial \bar{\psi}}{\partial z} + \alpha\lambda\bar{\chi} - \alpha^2\frac{\partial \bar{\psi}}{\partial z} + \frac{\partial^3 \bar{\psi}}{\partial z^3} \tag{4.69}$$

$$2\alpha\lambda\frac{\partial\bar{\Phi}}{\partial z}+\frac{\partial^2\bar{\phi}}{\partial z^2}+\alpha^2\bar{\phi}-\frac{Dh\alpha\lambda}{2(\alpha^2+\lambda^2)^{1/2}}e^{-(\alpha^2+\lambda^2)^{1/2}h}=2\alpha\lambda\frac{\partial\bar{\Psi}}{\partial z}+\frac{\partial^2\bar{\chi}}{\partial z^2}+\alpha^2\bar{\chi} \tag{4.70}$$

$$\frac{1}{\mu_1}\left[-\alpha^2\bar{\Phi}+\frac{\partial^2\bar{\Phi}}{\partial z^2}\right]+\frac{D}{4\mu_1}\left[\frac{2}{(\alpha^2+\lambda^2)^{1/2}}-\frac{h\lambda^2}{(\alpha^2+\lambda^2)}-\frac{\lambda^2}{(\alpha^2+\lambda^2)^{3/2}}\right]e^{-h(\alpha^2+\lambda^2)^{1/2}}$$

$$=\frac{1}{\mu_2}\left[-\alpha^2\bar{\Psi}+\frac{\partial^2\bar{\Psi}}{\partial z^2}\right] \tag{4.71}$$

$$\frac{1}{\mu_1}\alpha\lambda\bar{\Phi}+\frac{1}{\mu_1}\frac{\partial\bar{\phi}}{\partial z}-\frac{D}{4\mu_1}\left[\frac{h\alpha\lambda}{(\alpha^2+\lambda^2)}+\frac{\alpha\lambda}{(\alpha^2+\lambda^2)^{3/2}}\right]e^{-(\alpha^2+\lambda^2)^{1/2}h}$$

$$=\frac{1}{\mu_2}\alpha\lambda\bar{\Psi}+\frac{1}{\mu_2}\frac{\partial\bar{\chi}}{\partial z} \tag{4.72}$$

$$\frac{\lambda}{\mu_1}\frac{\partial\bar{\Phi}}{\partial z}+\frac{\alpha}{\mu_1}\bar{\phi}-\frac{D\lambda h}{4\mu_1(\lambda^2+\alpha^2)^{1/2}}e^{-h(\lambda^2+\alpha^2)^{1/2}}=\frac{\lambda}{\mu_2}\frac{\partial\bar{\Psi}}{\partial z}+\frac{\alpha}{\mu_2}\bar{\chi}\simeq 0 \tag{4.73}$$

Also, from equations (4.59), (4.60), (4.62), and (4.63), the differential equations for $\bar{\Phi}$, $\bar{\phi}$, $\bar{\chi}$, $\bar{\Psi}$ are

$$\left[\frac{d^2}{dz^2}-(\alpha^2+\lambda^2)\right]^2\begin{bmatrix}\bar{\Phi}\\ \bar{\Psi}\end{bmatrix}=0 \tag{4.74}$$

$$\left\{\frac{d^2}{dz^2}-(\alpha^2+\lambda^2)\right\}\begin{bmatrix}\bar{\phi}\\ \bar{\chi}\end{bmatrix}=0 \tag{4.75}$$

The solution of equations (4.74) and (4.75) satisfying the boundary conditions at infinity is

$$\begin{aligned}\bar{\Phi}&=A\,e^{-(\lambda^2+\alpha^2)^{1/2}z}+Bz\,e^{-(\lambda^2+\alpha^2)^{1/2}z}\\ \bar{\phi}&=C\,e^{-(\lambda^2+\alpha^2)^{1/2}z}\\ \bar{\Psi}&=E\,e^{(\lambda^2+\alpha^2)^{1/2}z}+Fz\,e^{(\lambda^2+\alpha^2)^{1/2}z}\\ \bar{\chi}&=H\,e^{(\lambda^2+\alpha^2)^{1/2}z}\end{aligned} \tag{4.76}$$

Substitute equation (4. 76) into equations (4. 69) to (4. 73) to yield six equations for the six unknown coefficients. Solving

$$\Delta = -2\left(1+\frac{\mu_2}{\mu_1}\right)^2 (\lambda^2+\alpha^2)^{3/2}$$

$$A = \frac{D}{4\Delta}\left(1+\frac{\mu_2}{\mu_1}\right)\left\{\left[2+2h(\lambda^2+\alpha^2)^{1/2}\right] e^{-(\lambda^2+\alpha^2)^{1/2}h} + \frac{\mu_2}{\mu_1}\left[\frac{3\lambda^2 h}{(\alpha^2+\lambda^2)^{1/2}} - 2 + \frac{(3\lambda^4+14\alpha^2\lambda^2+8\alpha^4)h}{\lambda^2(\alpha^2+\lambda^2)^{1/2}}\right] e^{-(\lambda^2+\alpha^2)^{1/2}h}\right\}$$

$$B = \frac{D}{4\Delta}\left(1+\frac{\mu_2}{\mu_1}\right)\left\{\left[2(\lambda^2+\alpha^2)^{1/2}\right] e^{-(\alpha^2+\lambda^2)^{1/2}h} + \frac{\mu_2}{\mu_1}\left[(4\lambda^2+4\alpha^2)h - 2(\lambda^2+\alpha^2)^{1/2}\right] e^{-(\alpha^2+\lambda^2)^{1/2}h}\right\}$$

From equation (4. 58), the reflected velocity is

$$\overline{U}_{1x\,\text{reflected}}(\lambda,\alpha,z) = \frac{1}{\mu_1}\left(-\alpha^2\overline{\Phi} + \frac{d^2\overline{\Phi}}{dz^2}\right)$$

$$= \frac{1}{\mu_1}\left[A\lambda^2 e^{-(\lambda^2+\alpha^2)^{1/2}z} + B\left[-2(\alpha^2+\lambda^2)^{1/2} + \lambda^2 z\right] e^{-(\alpha^2+\lambda^2)^{1/2}z}\right]$$

$$\overline{U}_{1x}(\lambda,\alpha,h) = \frac{1}{\mu_1}\left(1+\frac{\mu_2}{\mu_1}\right)\frac{D}{4\Delta}\left\{\left[4\lambda^2 h(\alpha^2+\lambda^2)^{1/2} - 2\lambda^2 - 4\alpha^2\right] e^{-2h(\alpha^2+\lambda^2)^{1/2}}\right.$$

$$+\frac{\mu_2}{\mu_1}\left[\frac{3\lambda^4 h}{(\alpha^2+\lambda^2)^{1/2}} + 2\lambda^2 + 4\alpha^2 + \frac{(3\lambda^4+14\alpha^2\lambda^2+8\alpha^4)h}{(\lambda^2+\alpha^2)^{1/2}}\right.$$

$$\left.\left. - h(10\lambda^2+8\alpha^2)(\lambda^2+\alpha^2)^{1/2} + \lambda^2 h^2(4\lambda^2+4\alpha^2)\right] e^{-2(\lambda^2+\alpha^2)^{1/2}h}\right\} \qquad (4.77)$$

Hence

$$U_{1x\,\text{reflected}}(x,y,h) = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} e^{i(\lambda x+\alpha y)}\,\overline{U}_{1x}(\lambda,\alpha,h)\,d\alpha\,d\lambda$$

Therefore

$$U_{1x\,\text{reflected}}(0,0,h) = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} \overline{U}_{1x}(\lambda,\alpha,h)\,d\alpha\,d\lambda$$

The last double integral can be performed by using polar coordinates

$$\lambda = \rho\cos\theta \qquad \alpha = \rho\sin\theta$$

$$U_{1x}(0,0,h) = \frac{1}{(2\pi)^2}\int_0^{\infty}\int_0^{2\pi} - \frac{D}{8\left(1+\frac{\mu_2}{\mu_1}\right)\rho^2\mu_1}\Big\{\left[4\rho^3 h\cos^2\theta - 2\rho^2\cos^2\theta - 4\rho^2\sin^2\theta\right]e^{-2h\rho}$$

$$+\frac{\mu_2}{\mu_1}\Big[3\rho^3 h\cos^4\theta + 2\rho^2\cos^2\theta + 4\rho^2\sin^2\theta + (3\rho^3\cos^4\theta + 14\rho^3\cos^2\theta\sin^2\theta + 8\rho^3\sin^3\theta)h$$

$$- h\,(10\cos^2\theta + 8\sin^2\theta)\rho^3 + 4h^2\rho^4\cos^2\theta\Big]e^{-2\rho h}\Big\}\rho^2\,d\rho\,d\theta$$

$$= \frac{D}{16\pi\mu_1 h}\left[\frac{1}{1+\frac{\mu_2}{\mu_1}}\right]\left[1-\frac{3\mu_2}{2\mu_1}\right] \tag{4.78}$$

using equation (4. 8),

$$V_{ext} = -U + \frac{D}{16\pi\mu_1 h}\left[\frac{1}{1+\frac{\mu_2}{\mu_1}}\right]\left[1-\frac{3\mu_2}{2\mu_1}\right]$$

Therefore

$$D = 6\pi\mu_1 a\left[U - \frac{D}{16\pi\mu_1 h}\left(\frac{1}{1+\frac{\mu_2}{\mu_1}}\right)\left(1-\frac{3\mu_2}{2\mu_1}\right)\right]$$

Hence

$$D = \frac{6\pi\mu_1 a U}{1+\frac{1}{1+\frac{\mu_2}{\mu_1}}\left[\frac{3a}{8h}-\frac{9a}{16h}\frac{\mu_2}{\mu_1}\right]} \tag{4.79}$$

In the case of a solid wall,

$$\mu_2 \to \infty$$

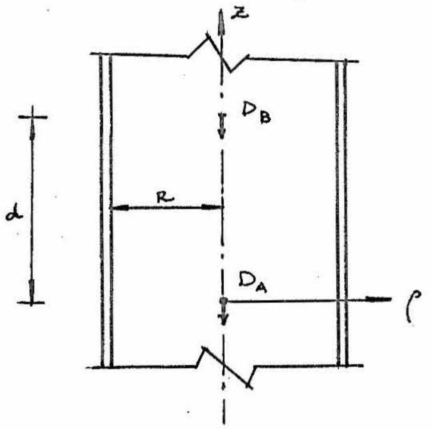
$$D = \frac{6\pi\mu_1 a U}{1-\frac{9a}{16h}} \tag{4.80}$$

In the case of a free surface,

$\mu_{\iota} \to 0$

$$D = \frac{6\pi\mu a U}{1 + \sqrt{\frac{3a}{8h}}} \tag{4.81}$$

(h) <u>Long Circular Cylinder.</u> Consider two spherical particles A and B falling along the axis of an infinite circular cylinder of radius R. Because of the linearity of the problem, it is possible to consider the flow field due to each particle separately. For



particle A, using cylindrical coordinates as shown, let

$$U_z = \frac{1}{\mu}\left[\frac{\partial^2\Phi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\Phi}{\partial\rho}\right] - \frac{D}{8\pi\mu}\left[\frac{1}{(z^2+\rho^2)^{1/2}} + \frac{z^2}{(z^2+\rho^2)^{3/2}}\right] \tag{4.82}$$

$$U_\rho = -\frac{1}{\mu}\frac{\partial^2\Phi}{\partial\rho\,\partial z} - \frac{D z \rho}{8\pi\mu(z^2+\rho^2)^{3/2}} \tag{4.83}$$

Then Stokes equation would be satisfied if

$$\nabla^4\Phi = 0 \tag{4.84}$$

The boundary condition at infinity and the no-slip condition at $\rho = R$ require

$$\Phi = 0 \qquad\qquad z = \pm\infty$$

$$\frac{\partial^2 \Phi}{\partial \rho^2} + \frac{1}{R}\frac{\partial \Phi}{\partial \rho} = \frac{D}{8\pi}\left[\frac{z^2}{(z^2+R^2)^{3/2}} + \frac{1}{(z^2+R^2)^{1/2}}\right] \qquad \rho = R \tag{4.85}$$

$$\frac{\partial^2 \Phi}{\partial \rho \partial z} = -\frac{D z R}{8\pi (z^2+\rho^2)^{3/2}} \qquad \text{or} \qquad \frac{\partial \Phi}{\partial \rho} = \frac{D R}{8\pi (z^2+\rho^2)} \tag{4.86}$$

Combining (4.85) and (4.86),

$$\frac{\partial^2 \Phi}{\partial \rho^2} = \frac{D z^2}{8\pi (z^2+R^2)^{3/2}} \tag{4.87}$$

The equation for Φ (equation (4.86))as well as the boundary conditions, equations (4.86), (4.87), are all even in z. Therefore, the solution must be even in z, i.e.,

$$\left.\frac{\partial^n \Phi}{\partial z^n} = 0\right|_{z=0} \qquad n \quad \text{odd integer}$$

Define the Fourier cosine transform of $f(z)$ by

$$\bar{f}(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(z) \cos \alpha z \, dz$$

Then $\bar{\Phi}$ satisfies the following differential equation and boundary conditions.

$$\left[\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} - \alpha^2\right]^2 \bar{\Phi} = 0 \tag{4.88}$$

$\rho = R$

$$\frac{\partial \bar{\Phi}}{\partial \rho} = \sqrt{\frac{2}{\pi}} \frac{DR}{8\pi} K_0(\alpha R) \tag{4.89}$$

$$\frac{\partial^2 \bar{\Phi}}{\partial \rho^2} = \sqrt{\frac{2}{\pi}} \frac{D}{8\pi}\left[K_0(\alpha R) - \alpha R\, K_1(\alpha R)\right] \tag{4.90}$$

A solution of equation (4. 88) which is finite at $\rho = 0$ is

$$\bar{\Phi} = \frac{A\rho}{2\alpha} I_1(\alpha\rho) + B\, I_0(\alpha\rho) \tag{4. 91}$$

Substitute equation (4. 91) into equations (4. 89) and (4. 90) and solve for A and B :

$$A = \frac{D}{8\pi}\sqrt{\frac{2}{\pi}}\;\frac{\left[(\alpha R I_0(\alpha R) - I_1(\alpha R))K_0(\alpha R) - I_1(\alpha R)K_0(\alpha R) + \alpha R\, I_1(\alpha R)\, K_1(\alpha R)\right]}{\left[\frac{1}{2} I_0(\alpha R)\left(\alpha R\, I_0(\alpha R) - I_1(\alpha R)\right) - \frac{1}{2} I_1(\alpha R) I_0(\alpha R) - \frac{1}{2}(\alpha R)\, I_1^2(\alpha R)\right]}$$

$$B = \frac{DR}{8\pi\alpha}\sqrt{\frac{2}{\pi}}\;\frac{\left[\alpha R\, I_0(\alpha R)\, K_1(\alpha R) + \alpha R\, K_0(\alpha R)\, I_1(\alpha R)\right]}{\left[I_0(\alpha R)\, I_1(\alpha R) + \alpha R\, I_1^2(\alpha R) - \alpha R\, I_0^2(\alpha R)\right]} \tag{4. 92}$$

From equation (4. 82) the reflected velocity is

$$\bar{U}_z = \frac{1}{\mu}\left[\frac{\partial^2 \bar{\Phi}}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \bar{\Phi}}{\partial \rho}\right]$$

Therefore, using (4. 91),

$$U_{z\,refl.}(\rho, z) = \sqrt{\frac{2}{\pi}}\,\frac{1}{\mu}\int_0^\infty \left\{ \frac{A}{2}\left[2 I_0(\alpha\rho) + \alpha\rho\, I_1(\alpha\rho)\right] + \frac{B\alpha I_1(\alpha\rho)}{\rho} \right.$$

$$\left. + \frac{B\alpha^2}{2}\left[I_2(\alpha\rho) + I_0(\alpha\rho)\right] \right\} \cos\alpha z \; d\alpha$$

Hence

$$U_{z\,refl.}(0, z) = \sqrt{\frac{2}{\pi}}\,\frac{1}{\mu}\int_0^\infty (A + \alpha^2 B)\cos\alpha z\; d\alpha \tag{4. 93}$$

From equation (4. 92),

$$A + \alpha^2 B = \frac{D}{8\pi}\sqrt{\frac{2}{\pi}}\left[\frac{s I_0(s) K_0(s) - 2 I_1(s) K_0(s) + s I_1(s) K_1(s) - \frac{s^2}{2} I_0(s) K_1(s) - \frac{1}{2} s^2 K_0(s) I_1(s)}{\frac{s}{2} I_0^2(s) - I_0(s) I_1(s) - \frac{s}{2} I_1^2(s)}\right]$$

$$= \frac{D}{8\pi}\sqrt{\frac{2}{\pi}}\; \psi(s) \tag{4. 94}$$

where $s = \alpha R$

Therefore,

$$U_{A}(0, z)_{\to fl.} = \frac{D}{4\pi^2 \mu R} \int_0^\infty f(s) \cos \frac{sz}{R} \, ds$$

$$= \frac{D}{4\pi^2 \mu R} \; H\left(\frac{z}{R}\right) \tag{4.95}$$

If $z >> R$, by repeat integration by parts,

$$H\left(\frac{z}{R}\right) \sim \sum_{n=0}^{\infty} (-1)^{n+1} f^{2n+1}(0) \left(\frac{R}{z}\right)^{2n+2} \tag{4.96}$$

where
$$f^{2n+1}(0) = \frac{d^{2n+1}}{ds^{2n+1}} f(0)$$

Now apply equation (4. 8):

$$V_{ext.\,A} = - U_A - \frac{D_A}{4\pi^2 \mu R} H(0) - \frac{D_B}{4\pi^2 \mu R} H\left(\frac{d}{R}\right) + \frac{D_B}{4\pi \mu d}$$

$$V_{ext\,A} = - U_B - \frac{D_B}{4\pi^2 \mu R} H(0) - \frac{D_A}{4\pi^2 \mu R} H\left(\frac{d}{R}\right) + \frac{D_A}{4\pi \mu d}$$

Therefore,

$$D_A = 6\pi\mu a \left[U_A + \frac{D_A}{4\pi^2 \mu R} H(0) + \frac{D_B}{4\pi^2 \mu R} H\left(\frac{d}{R}\right) - \frac{D_B}{4\pi \mu d} \right] \tag{4.97}$$

$$D_B = 6\pi\mu b \left[U_B + \frac{D_B}{4\pi^2 \mu R} H(0) + \frac{D_B}{4\pi^2 \mu R} H\left(\frac{d}{R}\right) - \frac{D_A}{4\pi \mu d} \right] \tag{4.98}$$

Solving (4. 97) and (4. 98),

$$D_A = \frac{6\pi\mu\left[a\,U_A\left(1-\frac{3b}{2\pi R}H(0)\right) - b\,U_B\left(\frac{3a}{2d}-\frac{3a\,H(\frac{d}{R})}{2\pi R}\right)\right]}{\left(1-\frac{3b\,H(0)}{2\pi R}\right)\left(1-\frac{3a\,H(0)}{2\pi R}\right)-\left(\frac{3b}{2d}-\frac{3b\,H(\frac{d}{R})}{2\pi R}\right)\left(\frac{3a}{2d}-\frac{3a\,H(\frac{d}{R})}{2\pi R}\right)} \tag{4.99}$$

Similarly, for D_B with a interchanged with b ,

If $a = b$, $U_A = U_B$, then

$$D_A = D_B = \frac{6\pi\mu a U}{\left(1+\frac{3a}{2d}-\frac{3a}{2\pi R}H(0)-\frac{3a}{2\pi R}H(\frac{d}{R})\right)} \tag{4.100}$$

5. Non-Spherical Particle

It is quite clear that the hydrodynamic forces a particle would experience when moving slowly through a viscous fluid depends very much on its geometric shape. A general treatment of this "shape factor" problem is mathematically formidable. In the literature on Stokes flow, only particles with certain well-defined regular geometry have been dealt with. Here, an attempt is made to treat approximately, but in a fairly general manner, this problem for particles the shape of which does not deviate very much from a sphere. The resulting formula is necessarily not exact, yet its simplicity and generality do seem to serve as a first step towards the ultimate goal.

In general, the shape of a particle can be represented by

$$r = 1 + f(\theta, \phi) \tag{5.1}$$

where the expression has been normalized with respect to the mean radius

$$a = \frac{1}{4\pi}\int r' \, d\Omega \qquad ; \qquad r = \frac{r'}{a}$$

Now $f(\theta, \phi)$ can be represented by

$$f(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=0}^{l}\left[\epsilon_{lm} P_l^m(\cos\theta)\cos m\phi + \delta_{lm} P_l^m(\cos\theta)\sin m\phi\right] \tag{5.2}$$

It will be assumed in all subsequent analysis of this section that

$$|\epsilon_{lm}| << 1 \quad ; \quad |\delta_{lm}| << 1 \qquad \text{for all } l, m \tag{5.3}$$

Conditions imposed by (5.3) permit one to neglect terms of order ϵ_{lm}^2 and δ_{lm}^2 in comparison with unity. Further, the decomposition of $f(\theta, \phi)$ into spherical harmonics allows each mode to be considered separately.

Consider a solid particle in a shear flow. Let the unperturbed velocity be given by $\vec{U}_\infty = \hat{x}\,(1+\beta y)$. If the particle is a sphere of radius unity, then the velocity field is given by

$$\begin{aligned}
\vec{U} = \hat{e}_r &\left\{ \left[1 - \frac{3}{2r} + \frac{1}{2r^3}\right]\cos\theta - \left[-\frac{\beta r}{3} + \frac{5\beta}{6r^2} - \frac{\beta}{2r^4}\right] 3\sin\theta\cos\theta\cos\phi \right\} \\
+ \frac{\hat{e}_\theta}{\sin\theta} &\left\{ \left[-1 + \frac{3}{4r} + \frac{1}{4r^3}\right]\sin^2\theta + \left[\frac{2\beta}{15r^4} - \frac{2\beta r}{15}\right]\left(-\frac{3}{2}\sin\theta\right)(5\cos^2\theta - 1)\cos\phi \right. \\
&\left. + \left[\frac{-3\beta}{10r^4} - \frac{\beta}{2r^2} + \frac{4\beta r}{5}\right](-\sin\theta\cos\phi) \right\} \\
+ \frac{\hat{e}_\phi}{\sin\theta} &\left\{ \left[\frac{\beta}{6r^4} - \frac{\beta}{6r^2}\right] 3\sin\theta\cos\theta\sin\phi \right\}
\end{aligned} \tag{5.4}$$

where the x - axis has been taken as the polar axis.

If the particle shape is given by

$$r = 1 + \epsilon_{lm} P_l^m(\cos\theta)\cos m\phi + \delta_{lm} P_l^m(\cos\theta)\sin m\phi$$

then the flow field will differ from that given by (5.4) only to order ϵ_{lm} or δ_{lm}. The necessary correction can be obtained by making use of Lamb's general solution (Appendix B) and the relations of associated Legendre functions. It is straightforward, although rather lengthy, to show that to order ϵ_{lm} and δ_{lm} the following is the required solution satisfying the no-slip condition on the surface of the particle.

Denoting

$$Y^e_{lm}(\theta, \phi) = P_l^m(\cos\theta)\cos \phi m$$
$$Y^o_{lm}(\theta, \phi) = P_l^m(\cos\theta)\sin m\phi$$

$$p = -\frac{3}{2r^2} Y^e_{10} + \left[\frac{3(l-m+1)}{2}\right]\left[-\frac{\epsilon_{lm}}{r^{l+2}} Y^e_{l+1,m} - \frac{\delta_{lm}}{r^{l+2}} Y^o_{l+1,m}\right] + \frac{5\beta}{3r^3} Y^e_{21}$$

$$+ \left[\frac{3(l-1)(2l-3)(l+m)}{2l(2l+1)}\right]\left[\frac{\epsilon_{lm}}{r^l} Y^e_{l-1,m} + \frac{\delta_{lm}}{r^l} Y^o_{l-1,m}\right]$$

$$+ \frac{F^e_{l+2,m-1}}{r^{l+3}} Y^e_{l+2,m-1} + \frac{F^e_{l,m-1}}{r^{l+1}} Y^e_{l,m-1} + \frac{F^e_{l-2,m-1}}{r^{l-1}} Y^e_{l-2,m-1} + \frac{F^e_{l+2,m+1}}{r^{l+3}} Y^e_{l+2,m+1}$$

$$+ \frac{F^e_{l,m+1}}{r^{l+1}} Y^e_{l,m+1} + \frac{F^e_{l-2,m+1}}{r^{l-1}} Y^e_{l-2,m+1} + \frac{F^o_{l+2,m-1}}{r^{l+3}} Y^o_{l+2,m-1} + \frac{F^o_{l,m-1}}{r^{l+1}} Y^o_{l,m-1}$$

$$+ \frac{F^o_{l-2,m-1}}{r^{l-1}} Y^o_{l-2,m-1} + \frac{F^o_{l+2,m+1}}{r^{l+3}} Y^o_{l+2,m+1} + \frac{F^o_{l,m+1}}{r^{l+1}} Y^o_{l,m+1} + \frac{F^o_{l-2,m+1}}{r^{l-1}} Y^o_{l-2,m+1}$$

$$U_r = \left[1 - \frac{3}{2r} + \frac{1}{2r^3} \right] Y^e_{10} - \left[-\frac{\beta r}{3} + \frac{5\beta}{6r^2} - \frac{\beta}{2r^4} \right] (3 \sin\theta \cos\theta \cos\phi)$$

$$+ \left[-\frac{3(l-m+1)(l+2)}{4(2l+1)r^{l+1}} + \frac{3(l-m+1)(l+2)}{4(2l+1)r^{l+3}} \right] \left[\epsilon_{lm} Y^e_{l+1,m} + \delta_{lm} Y^o_{l+1,m} \right]$$

$$+ \left[\frac{3(l-1)(l+m)}{4(2l+1)r^{l-1}} - \frac{3(l-1)(l+m)}{4(2l+1)r^{l+1}} \right] \left[\epsilon_{lm} Y^e_{l-1,m} + \delta_{lm} Y^o_{l-1,m} \right]$$

$$+ \left[\frac{(l+3) F^e_{l+2,m-1}}{2(2l+3)r^{l+2}} - \frac{(l+3) B^e_{l+2,m-1}}{r^{l+4}} \right] Y^e_{l+2,m-1} + \left[\frac{(l+3) F^o_{l+2,m-1}}{2(2l+3)r^{l+2}} - \frac{B^o_{l+2,m-1}(l+3)}{r^{l+4}} \right] Y^o_{l+2,m-1}$$

$$+ \left[\frac{(l+1) F^e_{l,m-1}}{2(2l-1)r^{l}} - \frac{B^e_{l,m-1}(l+1)}{r^{l+2}} \right] Y^e_{l,m-1} + \left[\frac{(l+1) F^o_{l,m-1}}{2(2l-1)r^{l}} - \frac{B^o_{l,m-1}(l+1)}{r^{l+2}} \right] Y^o_{l,m-1}$$

$$+ \left[\frac{(l-1) F^e_{l-2,m-1}}{2(2l-5)r^{l-2}} - \frac{B^e_{l-2,m-1}(l-1)}{r^{l}} \right] Y^e_{l-2,m-1} + \left[\frac{(l-1) F^o_{l-2,m-1}}{2(2l-5)r^{l-2}} - \frac{B^o_{l-2,m-1}(l-1)}{r^{l}} \right] Y^o_{l-2,m-1}$$

$$+ \left[\frac{(l+3) F^e_{l+2,m+1}}{2(2l+3)r^{l+2}} - \frac{B^e_{l+2,m+1}(l+3)}{r^{l+4}} \right] Y^e_{l+2,m+1} + \left[\frac{(l+3) F^o_{l+2,m+1}}{2(2l+3)r^{l+2}} - \frac{B^o_{l+2,m+1}(l+3)}{r^{l+4}} \right] Y^o_{l+2,m+1}$$

$$+ \left[\frac{(l+1) F^e_{l,m+1}}{2(2l-1)r^{l}} - \frac{B^e_{l,m-1}(l+1)}{r^{l+2}} \right] Y^e_{l,m+1} + \left[\frac{(l+1) F^o_{l,m+1}}{2(2l-1)r^{l}} - \frac{B^o_{l,m-1}(l+1)}{r^{l+2}} \right] Y^o_{l,m+1}$$

$$+ \left[\frac{(l-1) F^e_{l-2,m+1}}{2(2l-5)r^{l-2}} - \frac{B^e_{l-2,m-1}(l-1)}{r^{l}} \right] Y^e_{l-2,m+1} + \left[\frac{(l-1) F^o_{l-2,m+1}}{2(2l-5)r^{l-2}} - \frac{B^o_{l-2,m-1}(l-1)}{r^{l}} \right] Y^o_{l-2,m+1}$$

$$U_\theta = -\sin\Theta + \frac{2}{3\sin\Theta}\left(\frac{3}{4r} + \frac{1}{4r^3}\right)\left(1 - Y^e_{2,0}\right) - \left(\frac{2\beta}{15\sqrt{r^4}} - \frac{2\beta r}{\sqrt{15}}\right)\left(-\frac{3}{2}\sin\Theta\cos\phi\right)\times$$

$$\times\left(5\cos^2\Theta - 1\right) + \left(\frac{-3\beta}{10\sqrt{r^4}} - \frac{\beta}{2\sqrt{r^2}} + \frac{4\beta r}{\sqrt{5}}\right)\left(-\sin\Theta\cos\phi\right)$$

$$+ \frac{1}{\sin\Theta}\left\{\left[\frac{3(l-1)(l-m+1)\epsilon_{lm}}{4(l+1)(2l+1)r^{l+1}} - \frac{3(l-m+1)\epsilon_{lm}}{4(2l+1)r^{l+3}}\right]\left[\frac{(l+1)(l-m+2)}{(2l+3)}Y^e_{l+1,m} - \frac{(l+m+1)(l+2)}{(2l+3)}Y^e_{l,m}\right]\right.$$

$$+\left[\frac{3(l-1)(l-m+1)\delta_{lm}}{4(l+1)(2l+1)r^{l+1}} - \frac{3(l-m+1)\delta_{lm}}{4(2l+1)r^{l+3}}\right]\left[\frac{(l+1)(l-m+2)}{(2l+3)}Y^o_{l+1,m} - \frac{(l+m+1)(l+2)}{(2l+3)}Y^o_{l,m}\right]$$

$$+\left[-\frac{3(l+m)(l-3)\epsilon_{lm}}{4l(2l+1)r^{l-1}} + \frac{3(l-1)(l+m)\epsilon_{lm}}{4l(2l+1)r^{l+1}}\right]\left[\frac{(l-1)(l-m)}{(2l-1)}Y^e_{l,m} - \frac{l(l+m-1)}{(2l-1)}Y^e_{l-1,m}\right]$$

$$+\left[-\frac{3(l+m)(l-3)\delta_{lm}}{4l(2l+1)r^{l-1}} + \frac{3(l-1)(l+m)\delta_{lm}}{4l(2l+1)r^{l+1}}\right]\left[\frac{(l-1)(l-m)}{(2l-1)}Y^o_{l,m} - \frac{(l+m-1)l}{(2l-1)}Y^o_{l-2,m}\right]$$

$$+\left[\frac{3m^2\epsilon_{lm}}{2l(l+1)r^{l+1}}Y^e_{lm} + \frac{3m^2\delta_{lm}}{2l(l+1)r^{l+1}}Y^o_{lm}\right]$$

$$+\left[\frac{-l\,F^e_{l+2,m-1}}{2(l+2)(2l+3)r^{l+2}} + \frac{B^e_{l+2,m-1}}{r^{l+4}}\right]\left[\frac{(l+2)(l-m+4)}{(2l+5)}Y^e_{l+3,m-1} - \frac{(l+m+1)(l+3)}{(2l+5)}Y^e_{l+1,m-1}\right]$$

$$+\left[-\frac{(l-2)F^e_{l,m-1}}{2l(2l-1)r^{l}} + \frac{B^e_{l,m-1}}{r^{l+2}}\right]\left[\frac{l(l-m+2)}{(2l+1)}Y^e_{l+1,m-1} - \frac{(l+m-1)(l+1)}{(2l+1)}Y^e_{l-1,m-1}\right]$$

$$+\left[-\frac{(l-4)F^e_{l-2,m-1}}{2(l-2)(2l-5)r^{l-2}} + \frac{B^e_{l-2,m-1}}{r^{l}}\right]\left[\frac{(l-2)(l-m)}{(2l-3)}Y^e_{l-1,m-1} - \frac{(l+m-3)(l-1)}{(2l-3)}Y^e_{l-3,m-1}\right]$$

$$+\left[\frac{(m-1)D^o_{l+1,m-1}}{r^{l+2}}Y^e_{l+1,m-1} + \frac{(m-1)D^o_{l-1,m-1}}{r^{l}}Y^e_{l-1,m-1}\right]$$

$$+\left[-\frac{l\,F^e_{l+2,m+1}}{2(l+2)(2l+3)r^{l+2}} + \frac{B^e_{l+2,m+1}}{r^{l+4}}\right]\left[\frac{(l+2)(l-m+2)}{(2l+5)}Y^e_{l+3,m+1} - \frac{(l+m+3)(l+3)}{(2l+7)}Y^e_{l+1,m+1}\right] +$$

cont.

$$+\left[-\frac{(\ell-2)\,F^{\ell}_{\ell,m+1}}{2(\ell-2)(2\ell-5)\,r^{\ell-2}}+\frac{B^{\ell}_{\ell,m+1}}{r^{\ell+2}}\right]\left[\frac{\ell(\ell-m)}{(2\ell+1)}Y^{\ell}_{\ell+1,m+1}-\frac{(\ell+m+1)(\ell+1)}{(2\ell+1)}Y^{\ell}_{\ell-1,m+1}\right]$$

$$+\left[-\frac{(\ell-4)\,F^{\ell}_{\ell-2,m+1}}{2(\ell-2)(2\ell-5)\,r^{\ell-2}}+\frac{B^{\ell}_{\ell-2,m+1}}{r^{\ell}}\right]\left[\frac{(\ell-2)(\ell-m-2)}{(2\ell-3)}Y^{\ell}_{\ell-1,m+1}-\frac{(\ell+m-1)(\ell-1)}{(2\ell-3)}Y^{\ell}_{\ell-3,m+1}\right]$$

$$+\left[\frac{(m+1)\,D^{0}_{\ell+1,m+1}}{r^{\ell+2}}Y^{\ell}_{\ell+1,m+1}+\frac{(m+1)\,D^{0}_{\ell-1,m+1}}{r^{\ell}}Y^{\ell}_{\ell-1,m+1}\right]$$

$$+\left[-\frac{\ell\,F^{0}_{\ell+2,m-1}}{2(\ell+2)(2\ell+3)\,r^{\ell+2}}+\frac{B^{0}_{\ell+2,m-1}}{r^{\ell+4}}\right]\left[\frac{(\ell+2)(\ell-m+4)}{(2\ell+5)}Y^{0}_{\ell+3,m-1}-\frac{(\ell+m+1)(\ell+3)}{(2\ell+3)}Y^{0}_{\ell+1,m-1}\right]$$

$$+\left[-\frac{(\ell-2)\,F^{0}_{\ell,m-1}}{2\ell(2\ell-1)\,r^{\ell}}+\frac{B^{0}_{\ell,m-1}}{r^{\ell+2}}\right]\left[\frac{\ell(\ell-m+2)}{(2\ell+1)}Y^{0}_{\ell+1,m-1}-\frac{(\ell+m-1)(\ell+1)}{(2\ell+1)}Y^{0}_{\ell-1,m-1}\right]$$

$$+\left[-\frac{(\ell-4)\,F^{0}_{\ell-2,m-1}}{2(\ell-2)(2\ell-5)\,r^{\ell-2}}+\frac{B^{0}_{\ell-2,m-1}}{r^{\ell}}\right]\left[\frac{(\ell-2)(\ell-m)}{(2\ell-3)}Y^{0}_{\ell-1,m-1}-\frac{(\ell+m-3)(\ell-1)}{(2\ell-3)}Y^{0}_{\ell-3,m-1}\right]$$

$$+\left[-\frac{(m-1)\,D^{\ell}_{\ell+1,m-1}}{r^{\ell+2}}Y^{0}_{\ell+1,m-1}-\frac{(m-1)\,D^{\ell}_{\ell-1,m-1}}{r^{\ell}}Y^{0}_{\ell-1,m-1}\right]$$

$$+\left[\frac{-\ell\,F^{0}_{\ell+2,m+1}}{2(\ell+2)(2\ell+3)\,r^{\ell+2}}+\frac{B^{0}_{\ell+2,m+1}}{r^{\ell+4}}\right]\left[\frac{(\ell+2)(\ell-m+2)}{(2\ell+5)}Y^{0}_{\ell+3,m+1}-\frac{(\ell+m+3)(\ell+3)}{(2\ell+3)}Y^{0}_{\ell+1,m+1}\right]$$

$$+\left[-\frac{(\ell-2)\,F^{0}_{\ell,m+1}}{2\ell(2\ell-1)\,r^{\ell}}+\frac{B^{0}_{\ell,m+1}}{r^{\ell+2}}\right]\left[\frac{\ell(\ell-m)}{(2\ell+1)}Y^{0}_{\ell+1,m+1}-\frac{(\ell+m+1)(\ell+1)}{(2\ell+1)}Y^{0}_{\ell-1,m+1}\right]$$

$$+\left[-\frac{(\ell-4)\,F^{0}_{\ell-2,m+1}}{2(\ell-2)(2\ell-5)\,r^{\ell-2}}+\frac{B^{0}_{\ell-2,m+1}}{r^{\ell}}\right]\left[\frac{(\ell-2)(\ell-m-2)}{(2\ell-3)}Y^{0}_{\ell-1,m+1}-\frac{(\ell+m-1)(\ell-1)}{(2\ell-3)}Y^{0}_{\ell-3,m+1}\right]$$

$$+\left[-\frac{(m+1)\,D^{\ell}_{\ell+1,m+1}}{r^{\ell+2}}Y^{0}_{\ell+1,m+1}-\frac{(m+1)\,D^{\ell}_{\ell-1,m+1}}{r^{\ell}}Y^{0}_{\ell-1,m+1}\right]\Bigg\}$$

$$U_\phi = \frac{1}{\sin\theta}\left\{\left[-\frac{\beta}{6r^4}+\frac{\beta}{6r^2}\right]\left[-3\sin\theta\cos\theta\sin\phi\right]\right.$$

$$+\left[-\frac{3m(l-m+1)(l-1)\epsilon_{lm}}{4(l+1)(2l+1)r^{l+1}}+\frac{3m(l-m+1)\epsilon_{lm}}{4(2l+1)r^{l+3}}\right]Y^{o}_{l+1,m}$$

$$-\left[-\frac{3m(l-m+1)(l-1)\delta_{lm}}{4(l+1)(2l+1)r^{l+1}}+\frac{3m(l-m+1)\delta_{lm}}{4(2l+1)r^{l+3}}\right]Y^{e}_{l+1,m}$$

$$+\left[\frac{3m(l+m)(l-3)\epsilon_{lm}}{4l(2l+1)r^{l-1}}-\frac{3(l-1)(l+m)\epsilon_{lm}}{4(2l+1)l\,r^{l+1}}\right]Y^{o}_{l-1,m}$$

$$-\left[\frac{3m(l+m)(l-3)\delta_{lm}}{4l(2l+1)r^{l-1}}-\frac{3(l-1)(l+m)\delta_{lm}}{4(2l+1)l\,r^{l+1}}\right]Y^{e}_{l-1,m}$$

$$+\left[\frac{3m\,\epsilon_{lm}}{2l(l+1)r^{l+1}}\right]\left[\frac{(l+1)(l+m)}{(2l+1)}Y^{o}_{l-1,m}-\frac{l(l-m+1)}{(2l+1)}Y^{o}_{l+1,m}\right]$$

$$+\left[-\frac{3m\,\delta_{lm}}{2l(l+1)r^{l+1}}\right]\left[\frac{(l+1)(l+m)}{(2l+1)}Y^{e}_{l-1,m}-\frac{l(l-m+1)}{(2l+1)}Y^{e}_{l+1,m}\right]$$

$$+\left[\frac{(m-1)\,l\,F^{e}_{l+2,m-1}}{2(l+2)(2l+3)r^{l+2}}-\frac{(m-1)B^{e}_{l+2,m-1}}{r^{l+4}}\right]Y^{o}_{l+2,m-1}$$

$$+\left[\frac{(m-1)(l-2)F^{e}_{l,m-1}}{2l(2l-1)r^{l}}-\frac{(m-1)B^{e}_{l,m-1}}{r^{l+2}}\right]Y^{o}_{l,m-1}$$

$$+\left[\frac{(m-1)(l-4)F^{e}_{l-2,m-1}}{2(l-2)(2l-5)r^{l-2}}-\frac{(m-1)B^{e}_{l-2,m-1}}{r^{l}}\right]Y^{o}_{l-2,m-1}$$

$$-\frac{D^{o}_{l+1,m-1}}{r^{l+2}}\left[\frac{(l+1)(l-m+3)}{(2l+3)}Y^{o}_{l+2,m-1}-\frac{(l+2)(l+m)}{(2l+3)}Y^{o}_{l,m-1}\right]$$

$$-\frac{D^{o}_{l-1,m-1}}{r^{l}}\left[\frac{(l-1)(l-m+1)}{(2l-1)}Y^{o}_{l,m-1}-\frac{l(l+m-2)}{(2l-1)}Y^{o}_{l-2,m-1}\right]$$

$$+\left[\frac{(m+1)\,l\,F^{e}_{l+2,m+1}}{2(l+2)(2l+3)r^{l+2}}-\frac{(m+1)B^{e}_{l+2,m+1}}{r^{l+4}}\right]Y^{o}_{l+2,m+1}$$

$$+\left[\frac{(m+1)(l-2)F^{e}_{l,m+1}}{2l(2l-1)r^{l}}-\frac{(m+1)B^{e}_{l,m+1}}{r^{l+2}}\right]Y^{o}_{l,m+1}$$

$$+\left[\frac{(m+1)(l-4)F^{e}_{l-2,m+1}}{2(l-2)(2l-5)r^{l-2}}-\frac{(m+1)B^{e}_{l-2,m+1}}{r^{l}}\right]Y^{o}_{l-2,m+1}+$$

cont.

$$
\begin{aligned}
&- \frac{D^{0}_{\ell+1,m+1}}{r^{\ell+2}}\left[\frac{(\ell+1)(\ell-m+1)}{(2\ell+3)}Y^{0}_{\ell+2,m+1} - \frac{(\ell+2)(\ell+m+2)}{(2\ell+3)}Y^{0}_{\ell,m+1}\right] \\
&- \frac{D^{0}_{\ell-1,m+1}}{r^{\ell}}\left[\frac{(\ell-1)(\ell-m-1)}{(2\ell-1)}Y^{0}_{\ell,m+1} - \frac{\ell(\ell+m)}{(2\ell-1)}Y^{0}_{\ell-2,m+1}\right] \\
&- \left[\frac{(m-1)\ell F^{0}_{\ell+2,m-1}}{2(\ell+2)(2\ell+3)r^{\ell+2}} - \frac{(m-1)B^{0}_{\ell+2,m-1}}{r^{\ell+4}}\right]Y^{\ell}_{\ell+2,m-1} \\
&- \left[\frac{(m-1)(\ell-2)F^{0}_{\ell,m-1}}{2\ell(2\ell-1)r^{\ell}} - \frac{(m-1)B^{0}_{\ell,m-1}}{r^{\ell+2}}\right]Y^{\ell}_{\ell,m-1} \\
&- \left[\frac{(m-1)(\ell-4)F^{0}_{\ell-2,m-1}}{2(\ell-2)(2\ell-5)r^{\ell-2}} - \frac{(m-1)B^{0}_{\ell-2,m-1}}{r^{\ell}}\right]Y^{\ell}_{\ell-2,m-1} \\
&- \frac{D^{\ell}_{\ell+1,m-1}}{r^{\ell+2}}\left[\frac{(\ell+1)(\ell-m+3)}{(2\ell+3)}Y^{\ell}_{\ell+2,m-1} - \frac{(\ell+2)(\ell+m)}{(2\ell+3)}Y^{\ell}_{\ell,m-1}\right] \\
&- \frac{D^{\ell}_{\ell-1,m-1}}{r^{\ell}}\left[\frac{(\ell-1)(\ell-m+1)}{(2\ell-1)}Y^{\ell}_{\ell,m-1} - \frac{\ell(\ell+m-2)}{(2\ell-1)}Y^{\ell}_{\ell-2,m-1}\right] \\
&- \left[\frac{(m+1)\ell F^{0}_{\ell+2,m+1}}{2(\ell+2)(2\ell+3)r^{\ell+2}} - \frac{(m+1)B^{0}_{\ell+2,m+1}}{r^{\ell+4}}\right]Y^{\ell}_{\ell+2,m+1} \\
&- \left[\frac{(m+1)(\ell-2)F^{0}_{\ell,m+1}}{2\ell(2\ell-1)r^{\ell}} - \frac{(m+1)B^{0}_{\ell,m+1}}{r^{\ell+2}}\right]Y^{\ell}_{\ell,m+1} \\
&- \left[\frac{(m+1)(\ell-4)F^{0}_{\ell-2,m+1}}{2(\ell-2)(2\ell-5)r^{\ell-2}} - \frac{(m+1)B^{0}_{\ell-2,m+1}}{r^{\ell}}\right]Y^{\ell}_{\ell-2,m+1} \\
&- \frac{D^{\ell}_{\ell+1,m+1}}{r^{\ell+2}}\left[\frac{(\ell+1)(\ell-m+1)}{(2\ell+3)}Y^{\ell}_{\ell+1,m+1} - \frac{(\ell+2)(\ell+m+2)}{(2\ell+3)}Y^{\ell}_{\ell,m+1}\right] \\
&- \frac{D^{\ell}_{\ell-1,m+1}}{r^{\ell}}\left[\frac{(\ell-1)(\ell-m-1)}{(2\ell-1)}Y^{\ell}_{\ell,m+1} - \frac{\ell(\ell+m)}{(2\ell-1)}Y^{\ell}_{\ell-2,m-1}\right]\Bigg\}
\end{aligned}
\tag{5.5}
$$

where $B^{\ell}_{\ell+2,m-1} = \dfrac{F^{\ell}_{\ell+2,m-1}}{2(2\ell+3)}$; $B^{\ell}_{\ell+2,m-1} = \dfrac{F^{\ell}_{\ell+2,m+1}}{2(2\ell+3)}$

$$B^{e}_{\ell,m-1} = \frac{F^{e}_{\ell,m-1}}{2(2\ell-1)} \quad ; \quad B^{e}_{\ell,m+1} = \frac{F^{e}_{\ell,m+1}}{2(2\ell-1)}$$

$$B^{e}_{\ell-2,m-1} = \frac{F^{e}_{\ell-2,m-1}}{2(2\ell-5)} \quad ; \quad B^{o}_{\ell-2,m+1} = \frac{F^{e}_{\ell-2,m+1}}{2(2\ell-5)}$$

$$F^{e}_{\ell+2,m-1} = \frac{-5(\ell-m+1)(\ell-m+2)(\ell-m+3)}{2(2\ell+1)}\,\epsilon_{\ell m}\,\beta$$

$$F^{e}_{\ell-2,m-1} = \frac{-5(\ell-2)(2\ell-5)(\ell+m)(\ell+m-1)(\ell+m-2)}{2(\ell-1)(2\ell+1)(2\ell-1)}\,\epsilon_{\ell m}\,\beta$$

$$F^{e}_{\ell,m-1} = \frac{(\ell+m)(\ell-m+1)(12\ell^{3}+18\ell^{2}+30\ell m-18\ell+15m-12)}{2(2\ell+1)(2\ell+3)(\ell+1)}\,\epsilon_{\ell m}\,\beta$$

$$D^{o}_{\ell+1,m-1} = \frac{(\ell-m+1)(\ell-m+2)(\ell+5m-3)}{2(2\ell+1)(\ell+1)(\ell+2)}\,\epsilon_{\ell m}\,\beta$$

$$D^{o}_{\ell-1,m-1} = \frac{(\ell+m)(\ell+m-1)(\ell-5m+4)}{2\ell(2\ell+1)(\ell-1)}\,\epsilon_{\ell m}\,\beta$$

$$F^{e}_{\ell+2,m+1} = \frac{5(\ell-m+1)}{2(2\ell+1)}\,\epsilon_{\ell m}\,\beta$$

$$F^{e}_{\ell-2,m+1} = \frac{5(\ell+m)(\ell-2)(2\ell-5)}{2(\ell-1)(2\ell+1)(2\ell-1)}\,\epsilon_{\ell m}\,\beta$$

$$F^{e}_{\ell,m+1} = \frac{(-12\ell^{3}-18\ell^{2}+30\ell m+18\ell+15m+12)}{2(2\ell+1)(2\ell+3)(\ell+1)}\,\epsilon_{\ell m}\,\beta$$

$$D^{o}_{\ell+1,m+1} = \frac{(\ell-5m-3)}{2(2\ell+1)(\ell+1)(\ell+2)}\,\epsilon_{\ell m}\,\beta$$

$$D^{o}_{\ell-1,m+1} = \frac{(\ell+5m+4)}{2\ell(2\ell+1)(\ell-1)}\,\epsilon_{\ell m}\,\beta$$

Also, with $\epsilon_{\ell m}$ replaced by $\delta_{\ell m}$

$$
\begin{array}{llll}
F^{0}_{\ell+2,m-1} \to F^{\ell}_{\ell+2,m-1} & \qquad & F^{0}_{\ell+2,m+1} \to F^{\ell}_{\ell+2,m+1} \\
F^{0}_{\ell,m-1} \to F^{\ell}_{\ell,m-1} & & F^{0}_{\ell,m+1} \to F^{\ell}_{\ell,m+1} \\
F^{0}_{\ell-2,m-1} \to F^{\ell}_{\ell-2,m-1} & & F^{0}_{\ell-2,m+1} \to F^{\ell}_{\ell-2,m+1} \\
B^{0}_{\ell+2,m-1} \to B^{\ell}_{\ell+2,m-1} & & B^{0}_{\ell+2,m+1} \to B^{\ell}_{\ell+2,m+1} \\
B^{0}_{\ell,m-1} \to B^{\ell}_{\ell,m-1} & & B^{0}_{\ell,m+1} \to B^{\ell}_{\ell,m+1} \\
B^{0}_{\ell-2,m-1} \to B^{\ell}_{\ell-2,m-1} & & B^{0}_{\ell-2,m+1} \to B^{\ell}_{\ell-2,m+1} \\
D^{0}_{\ell+1,m-1} \to -D^{\ell}_{\ell+1,m-1} & & D^{0}_{\ell+1,m+1} \to -D^{\ell}_{\ell+1,m+1} \\
D^{0}_{\ell-1,m-1} \to -D^{\ell}_{\ell-1,m-1} & & D^{0}_{\ell-1,m+1} \to -D^{\ell}_{\ell-1,m+1}
\end{array}
$$

For a particle, the shape of which is given by equations (5.1) and (5.2), the corresponding solution can be obtained by summing over the appropriate index ℓ and m.

To find the forces acting on the particle, it is only necessary to find the forces acting on a sphere of radius R and let $R \to \infty$. By means of equation (5.5) one obtains, after carrying out the necessary computation, the dimensionless force acting on the particle.

$$
\begin{aligned}
\vec{F} = \; & 6\pi \left[1 - \frac{\epsilon_{20}}{5} - \epsilon_{11}\beta + \frac{4}{7}\epsilon_{31}\beta \right] \hat{x} \\
& + 12\pi \left[\frac{3}{20}\epsilon_{21} - \frac{5}{7}\epsilon_{32}\beta + \frac{1}{28}\epsilon_{30}\beta \right] \hat{y} \\
& + 12\pi \left[\frac{3}{20}\delta_{21} - \frac{5}{7}\delta_{32}\beta \right] \hat{z}
\end{aligned}
\tag{5.6}
$$

The dimensional (primed) and dimensionless (unprimed) quantities are related by

$$\vec{U}_\infty = U_\infty \hat{x} + \beta' y \qquad \text{at infinity}$$

$$\beta = \frac{\beta' a}{U_\infty}$$

$$F = \frac{F'}{\mu a U_\infty}$$

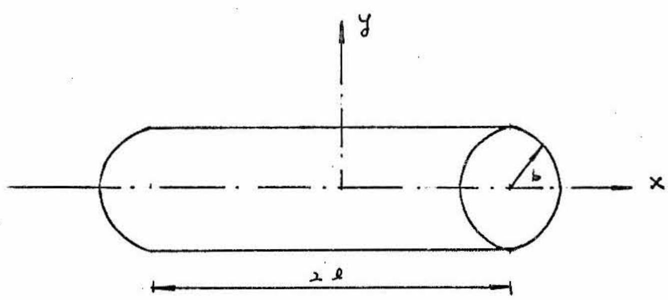
$$a = \frac{1}{4\pi} \int r'(\theta, \phi)\, d\Omega$$

$$\epsilon_{\ell m} = \left[\frac{1}{a} \int r'(\theta, \phi)\, P_\ell^m(\cos\theta) \cos m\phi \, d\Omega\right]\left[\frac{(2\ell+1)}{2\pi}\, \frac{(\ell-m)!}{(\ell+m)!}\, \epsilon_m\right]$$

$$\delta_{\ell m} = \left[\frac{1}{a} \int r'(\theta, \phi)\, P_\ell^m(\cos\theta) \sin m\phi \, d\Omega\right]\left[\frac{(2\ell+1)}{2\pi}\, \frac{(\ell-m)!}{(\ell+m)!}\right] \qquad (5.7)$$

$$\epsilon_m = 1, \quad m > 0$$
$$= \tfrac{1}{2} \quad m = 0$$

Example



Consider the case of a finite cylinder with axis of symmetry pointing in the x - direction in a simple shear flow $\vec{U}_\infty = (u_\infty + \beta y)\hat{x}$. Let $\ell \simeq b$; then the geometry of the particle is defined by

$$\tan \alpha = b/\ell$$

$$r = \begin{cases} \dfrac{l}{\cos\theta} & 0 \le \theta \le \alpha \\ \dfrac{b}{\sin\theta} & \alpha < \theta \le \dfrac{\pi}{2} \end{cases}$$

Hence, as in (5.7),

$$a = \int_0^{\alpha} \frac{l}{\cos\theta} \sin\theta \, d\theta + \int_{\alpha}^{\pi/2} \frac{b}{\sin\theta} \sin\theta \, d\theta$$

$$= \left[-l \ln(\cos\alpha) + b\left(\frac{\pi}{2} - \alpha\right) \right]$$

$$= l \ln\left(\frac{\sqrt{l^2+b^2}}{l}\right) + b\left(\frac{\pi}{2} - \tan^{-1}\frac{b}{l}\right) \tag{5.8}$$

$$\epsilon_{20} = \frac{5}{a} \left[\int_0^{\alpha} \frac{l}{\cos\theta} \left(\frac{3\cos^2\theta}{2} - \frac{1}{2} \right) \sin\theta \, d\theta \right.$$

$$\left. + \int_{\alpha}^{\pi/2} \frac{b}{\sin\theta} \left(\frac{3\cos^2\theta}{2} - \frac{1}{2} \right) \sin\theta \, d\theta \right]$$

$$= \frac{5}{a} \left\{ l \left[\frac{3}{8}(1 - \cos 2\alpha) + \frac{1}{2} \ln(\cos\alpha) \right] + b \left[\frac{\pi}{2} - \alpha - \frac{3}{4} \sin 2\alpha \right] \right\}$$

By equation (5.6), the drag is approximately given by

$$D_x = 6\pi\mu U_\infty \left[\frac{l}{2} \ln\left(\frac{\sqrt{l^2+b^2}}{l}\right) + \frac{3lb^2}{4(l^2+b^2)} \right] \tag{5.9}$$

6. Elastic Spherical Particle in Shear Flow

In this section, the additional hydrodynamic forces acting on a spherical particle due to elastic deformation are being investigated.

Here, the particle will be assumed to be small compared with the characteristic length of the flow field. Thus, it is sufficient to consider the particle as existing in a simple shear flow with velocity $\vec{U}_\infty = (U_\infty + \beta y)\hat{x}$ relative to the sphere.

It is well known that a rigid sphere in a simple shear flow will experience a drag and a torque due to hydrodynamic forces. Therefore, if the particle is not subjected to other external forces, it will undergo a translational and an angular acceleration. Let the angular velocity of the particle be $\omega\hat{z}$. Then the drag and torque acting on such a rigid particle of radius a is

(drag) $\quad \vec{D} = 6\pi\mu a U_\infty \hat{x}$

(torque) $\quad \vec{T} = -8\pi\mu a^3 (\omega + \frac{\beta}{2})\hat{z}$

Here, it will be assumed that the sphere undergoes linear and angular acceleration so that the drag may be considered to be balanced by a constant body force equal to $-\frac{9\mu U_\infty}{2a^2}\hat{x}$ and the torque by a body force in the form $\vec{f} = A\hat{z}\times\vec{r}$. Also, deformation due to centrifugal force arising from the angular velocity of the sphere will be neglected.

To determine the constant A:

$$\text{torque} = 8\pi\mu a^3 (\omega + \frac{\beta}{2})\hat{z} = -\int_{\text{sphere}} A(\hat{z}\times\vec{r})\times\vec{r}\,dv$$

$$\therefore \quad A = \frac{15\mu}{a^2}\left(\frac{\beta}{2} + \omega\right)$$

Hence

$$\vec{f} = \frac{15\mu}{a^2}\left(\frac{\beta}{2} + \omega\right)\hat{z}\times\vec{r} \tag{6.1}$$

For slow motion of the fluid, Stokes equation applies. From the general solution of Stokes equation (Appendix B), the flow field near the sphere is found to be

$$
\begin{aligned}
\vec{U} = \hat{e}_r \Big\{ & U_\infty \left[1 - \frac{3a}{2r} + \frac{a^3}{2r^3} \right] \cos\theta + \beta \left[-\frac{r}{3} + \frac{5a^3}{6r^2} - \frac{a^5}{2r^4} \right] (-3\sin\theta\cos\theta\cos\phi) \Big\} \\
+ \hat{e}_\theta \Big\{ & U_\infty \left[-1 + \frac{3a}{4r} + \frac{a^3}{4r^3} \right] \sin\theta + \beta \left[\frac{r}{5} - \frac{a^5}{5r^4} \right] (5\cos^2\theta - 1)\cos\phi \\
& + \beta \left[\frac{3a^5}{10r^4} + \frac{a^3}{2r^2} - \frac{4r}{5} \right] \cos\phi + \frac{a^3\omega}{r^2}\cos\phi \Big\} \\
+ \hat{e}_\phi \Big\{ & \beta \left[\frac{a^5}{2r^4} - \frac{a^3}{2r^2} \right] \cos\theta \sin\phi - \frac{a^3\omega}{r^2} \cos\theta \sin\phi \Big\}
\end{aligned}
$$

$$
p = -\frac{3\mu a U_\infty}{2r^2}\cos\theta - \frac{5\beta\mu a^2}{r^2}\sin\theta\cos\theta\cos\phi \qquad (6.2)
$$

(the x - axis is taken as the polar axis)

Let $\overline{\overline{\sigma}}$ denote the stress tensor. Then, the surface stresses acting on the sphere corresponding to the velocity and pressure field given by (6. 2) are

$$
\begin{aligned}
\overline{\overline{\sigma}} \cdot \hat{e}_r = \hat{e}_r & \left[\frac{3\mu U_\infty}{2a} P_1(\cos\theta) - \frac{5\beta\mu}{\sqrt{3}} P_2'(\cos\theta)\cos\phi \right] \\
& + \frac{\hat{e}_\theta}{\sin\theta} \left[-\frac{\mu U_\infty}{a} + \frac{\mu U_\infty}{a} P_2(\cos\theta) + 3\mu(\beta+\omega) P_1'(\cos\theta)\cos\phi - \frac{2\beta\mu}{\sqrt{3}} P_3'(\cos\theta)\cos\phi \right] \\
& + \frac{\hat{e}_\phi}{\sin\theta} \left[\frac{\beta\mu}{\sqrt{3}} P_2'(\cos\theta)\sin\phi - \omega\mu P_1'(\cos\theta)\sin\phi \right]
\end{aligned}
$$

(6. 3)

To describe the deformation of the spherical particle, it is assumed that the elastic displacement is small, so that linear elasticity theory applies. Also, the material of the sphere will be taken as homogeneous and isotropic, characterized by the stress-strain relation

$$\sigma_{ij} = \lambda e_{kk} \delta_{ij} + 2 G e_{ij}$$

where

$$e_{ij} = \left(\frac{\partial S_i}{\partial x_j} + \frac{\partial S_j}{\partial x_i} \right)$$

and $\vec{S}$ is the displacement vector.

The displacement field is governed by the Navier equation

$$(\lambda + G) \nabla (\nabla \cdot \vec{S}) + G \nabla^2 \vec{S} + \vec{F} = 0 \qquad (6.4)$$

where $\vec{F}$ is the body force, and in the present case,

$$\vec{F} = -\frac{9 \mu U_\infty}{2 a^2} \hat{x} + \frac{15 \mu}{a^2} \left(\frac{\beta}{2} + \omega \right) \hat{z} \times \vec{r}$$

The particular solution to equation (6.4) corresponding to above is

$$\vec{S}_p = \hat{e}_r \left\{ \frac{9 \mu U_\infty r^2}{4 a^2 (\lambda + 2G)} \left[\frac{2}{5} P_3(\cos\theta) + \frac{3}{5} P_1(\cos\theta) \right] \right.$$

$$\left. + \frac{5\mu}{2 a^2 G} \left(\frac{\beta}{2} + \omega \right) r^3 \left[-\frac{1}{420} P_4^3(\cos\theta) \cos 3\phi + \frac{1}{10} P_4^1(\cos\theta) \cos\phi \right] \right\}$$

$+$

cont.

$$+ \frac{\hat{e}_\theta}{\sin\theta}\left\{\frac{9\mu U_\infty r^2}{4a^2(\lambda+2G)}\left[\frac{2}{15}P_3'(\cos\theta)+\frac{1}{5}P_1'(\cos\theta)\right]\sin\phi\right.$$

$$-\frac{5\mu}{2a^2G}\left(\frac{\beta}{2}+\omega\right)r^3\left[\frac{1}{1890}P_5^3(\cos\theta)\cos3\phi-\frac{2}{135}P_3^3(\cos\theta)\cos3\phi\right.$$

$$\left.\left.-\frac{2}{45}P_5'(\cos\theta)\cos\phi+\frac{2}{45}P_3'(\cos\theta)\cos\phi-\frac{3}{5}P_1'(\cos\theta)\cos\phi\right]\right\}$$

$$+\frac{\hat{e}_\phi}{\sin\theta}\left\{\frac{5\mu}{2a^2G}\left(\frac{\beta}{2}+\omega\right)r^3\left[-\frac{2}{35}P_4'(\cos\theta)-\frac{1}{7}P_2'(\cos\theta)\right]\sin\phi\right\}$$

(6. 5)

The stress components corresponding to (6. 5) are

$$\bar{\bar{\sigma}}\cdot\hat{e}_r = \hat{e}_r\left\{\frac{18\mu U_\infty G r}{5a^2(\lambda+2G)}P_3(\cos\theta)+\frac{9\mu U_\infty r}{a^2(\lambda+2G)}\left(\frac{6G+5\lambda}{10}\right)P_1(\cos\theta)\right.$$

$$\left.+\frac{15\mu r^2}{a^2}\left(\frac{\beta}{2}+\omega\right)\left[-\frac{1}{420}P_4^3(\cos\theta)\cos3\phi+\frac{1}{10}P_4'(\cos\theta)\cos\phi\right]\right\}$$

$$+\frac{\hat{e}_\theta}{\sin\theta}\left\{\frac{9\mu U_\infty G r}{a^2(\lambda+2G)}\left[\frac{8}{35}P_4(\cos\theta)-\frac{2}{21}P_2(\cos\theta)-\frac{2}{15}\right]\right.$$

$$+\frac{15\mu r^2}{2a^2}\left(\frac{\beta}{2}+\omega\right)\left\{-\frac{1}{945}P_5^3(\cos\theta)\cos3\phi+\frac{7}{540}P_3^3(\cos\theta)\cos3\phi\right.$$

$$\left.\left.+\frac{4}{45}P_5'(\cos\theta)\cos\phi-\frac{11}{20}P_3'(\cos\theta)\cos\phi+\frac{2}{5}P_1'(\cos\theta)\cos\phi\right]\right\}$$

(6. 6)

$$+\frac{\hat{e}_\phi}{\sin\theta}\left\{\frac{15\mu r^2}{2a^2}\left(\frac{\beta}{2}+\omega\right)\left[\frac{1}{420}P_4^3\sin3\phi-\frac{1}{14}P_4'\sin\phi-\frac{2}{21}P_2'\sin\phi\right]\right\}$$

From equations (6. 3) and (6. 6), the boundary condition to be satisfied by the homogeneous solution of equation (6.4) is

At $r = a$

$$\bar{\sigma}\cdot\hat{e}_r = \hat{e}_r\Big[-\frac{3\mu U_\infty(5\lambda+4G)}{5a(\lambda+2G)}P_1(\cos\theta) - \frac{18\mu U_\infty G}{5a(\lambda+2G)}P_3(\cos\theta) - \frac{5\beta\mu}{\sqrt{3}}P_2'(\cos\theta)\cos\phi$$

$$-\frac{3\mu}{2a}\left(\frac{\beta}{\sqrt{2}}+\omega\right)P_4'(\cos\theta)\cos\phi + \frac{\mu}{2\sqrt{8}a}\left(\frac{\beta}{\sqrt{2}}+\omega\right)P_4^3(\cos\theta)\cos 3\phi\Big]$$

$$+\frac{\hat{e}_\theta}{\sin\theta}\Big[-\frac{\mu U_\infty(5\lambda+4G)}{5a(\lambda+2G)} + \frac{\mu U_\infty(7\lambda+20G)}{7a(\lambda+2G)}P_2(\cos\theta) - \frac{72\mu U_\infty G}{35a(\lambda+2G)}P_4(\cos\theta)$$

$$+\frac{3\mu\beta}{\sqrt{2}}P_1'(\cos\theta)\cos\phi + \mu\left(-\frac{5\beta}{2\sqrt{4}}+\frac{11\omega}{12}\right)P_3'(\cos\theta)\cos\phi$$

$$-\frac{2}{3}\mu\left(\frac{\beta}{\sqrt{2}}+\omega\right)P_5'(\cos\theta)\cos\phi - \frac{7}{72}\mu\left(\frac{\beta}{\sqrt{2}}+\omega\right)P_3^3(\cos\theta)\cos 3\phi$$

$$+\frac{\mu}{6\sqrt{3}a}\left(\frac{\beta}{\sqrt{2}}+\omega\right)P_5^3(\cos\theta)\cos 3\phi\Big]$$

$$+\frac{\hat{e}_\phi}{\sin\theta}\Big[\mu\left(\frac{29\beta}{42}-\frac{2\omega}{7}\right)P_2'(\cos\theta)\sin\phi + \frac{15\mu}{2\sqrt{8}}\left(\frac{\beta}{\sqrt{2}}+\omega\right)P_4'(\cos\theta)\sin\phi$$

$$-\frac{\mu}{\sqrt{56}}\left(\frac{\beta}{\sqrt{2}}+\omega\right)P_4^3(\cos\theta)\sin 3\phi\Big] \tag{6.7}$$

The homogeneous solution of equation (6.4) can be obtained by using equations (C-1) and (C-2) of Appendix C and the boundary condition provided by equation (6.7). Writing out explicitly the equations for coefficients B_{10}^a, B_{30}^a, C_{30}^a, B_{21}^a, C_{21}^a

$$-B_{10}^a\left[\frac{3\lambda+2G}{\lambda-G}\right]2Ga = -\frac{3\mu U_\infty(5\lambda+4G)}{5a(\lambda+2G)}$$

$$B_{30}^a\left[\frac{3\lambda+4G}{3\lambda+G}\right]2Ga^3 + C_{30}^a(4Ga) = -\frac{18\mu U_\infty G}{5a(\lambda+2G)}$$

$$B_{10}^a\left[\frac{3\lambda+2G}{\lambda-G}\right]\frac{2}{3}Ga = \frac{\mu U_\infty(5\lambda+4G)}{5a(\lambda+2G)}$$

$$B_{10}^{e}\, G a \frac{2}{3}\left[\frac{3\lambda+2G}{\lambda-G}\right] - C_{30}^{e}\left[\frac{16aG}{7}\right] - \frac{12Ga^3}{7}\left[\frac{15\lambda+14G}{2(3\lambda+G)}\right] B_{30}^{e} = \frac{\mu U_\infty (7\lambda+20G)}{7a(\lambda+2G)}$$

$$B_{21}^{o}\, a^2 G - 2G\, C_{21}^{e} = \frac{5\mu\beta}{\sqrt{3}}$$

$$B_{21}^{o}\, \frac{9Ga^2}{5}\left[\frac{8\lambda+7G}{3\lambda}\right] + \frac{9}{5} G\, C_{21}^{e} = -\frac{3\mu\beta}{\sqrt{2}}$$

Solving

$$B_{30}^{e} = 0 \quad , \quad B_{21}^{o} = 0$$

$$C_{30}^{e} = -\frac{9\mu U_\infty}{10a^2(\lambda+2G)} \quad ; \quad C_{21}^{e} = -\frac{5\mu\beta}{\sqrt{6}\,G}$$

Therefore,

$$\vec{S}_c\cdot\hat{e}_r = \frac{-9\mu U_\infty r^2}{10a^2(\lambda+2G)} P_3(\cos\theta) - \frac{5\mu\beta}{\sqrt{6}\,G} r P_2'(\cos\theta)\cos\phi + \cdots \tag{6.8}$$

Combining equations (6. 5) and (6. 8),

$$\vec{S} = \vec{S}_p + \vec{S}_c = \hat{e}_r\left[-\frac{5\mu\beta r}{\sqrt{6}\,G} P_2'(\cos\theta)\cos\phi + \cdots\right] + \cdots \tag{6.9}$$

In order to find the additional hydrodynamic forces acting on the particle, it is necessary to find the deformed shape of the particle. Since by assumption $\frac{|\vec{S}|}{a} \ll 1$

$$\vec{r}\left(a,\ \theta + \frac{S_\theta}{a+S_r},\ \phi + \frac{S_\phi}{a+S_r}\right) = a\hat{e}_r + \vec{S}(a,\theta,\phi) \tag{6.10}$$

where $\vec{r}(a,\theta,\phi)$ gives the deformed shape of the particle. From (6. 10),

$$\vec{r}(a,\theta\,\phi) \simeq a\hat{e}_r + \vec{S}\left(a;\ \theta - \frac{S_\theta}{a+S_r};\ \phi - \frac{S_\phi}{a+S_r}\right)$$

$$\frac{\vec{r}}{a}(a,\theta,\phi) = \hat{e}_r + \frac{\vec{S}(a,\theta,\phi)}{a} + O(S^2/a^2)$$

Hence

$$\frac{r^2}{a^2} = \left(1 + \frac{S_r}{a}\right)^2 + \left(\frac{S_\theta}{a}\right)^2 + \left(\frac{S_\phi}{a}\right)^2$$

Therefore,

$$\frac{r(a,\theta,\phi)}{a} \simeq 1 + \frac{S_r(a,\theta,\phi)}{a}$$

From equation (6.9) above and equation (5.7),

$$\epsilon_{21} = -\frac{5\mu\beta}{6G}$$

and

$$F_y = \frac{-3\pi\mu^2 U_\infty a\beta}{2G} \qquad (6.11)$$

Therefore, a transverse force acts on the particle as a result of elastic deformation, even though the drag remains unaltered to the first approximation. It is to be noted that the transverse force is in such a direction as to cause the particle to drift to the side where the velocity difference between fluid and particle would be smaller.

7. Non-Stationary Motion

This section concerns primarily the non-stationary motion of a small, spherical particle in a viscous, incompressible fluid under the action of external forces. The prime objective is to obtain an estimate of the importance of transient effects and to illustrate the complicated nature of transient motion even in very simple particle systems.

Consider a sphere of radius a suddenly being set into constant motion with velocity $U_\infty \hat{x}$. It will be assumed that the motion is slow enough that the linearized, time-dependent Navier-Stokes equation is quite adequate for the purpose of describing the flow field. Thus, the fluid field is given by the solution of (using a coordinate system fixed to the center of the sphere):

$$\nabla \cdot \vec{u} = 0 \tag{7.1}$$

$$\rho \frac{\partial \vec{u}}{\partial t} + \nabla p = \mu \nabla^2 \vec{u} \tag{7.2}$$

$$\vec{u} = 0 \qquad r = a \tag{7.3}$$

$$\vec{u} = u_\infty H(t)\, \hat{x} \ , \ p = 0 \qquad r = \infty \tag{7.4}$$

where $H(t)$ is the unit step function.

To find $\vec{u}$ and p, take Laplace transform of (7.1) to (7.4). Then

$$\nabla \cdot \vec{\bar{u}} = 0 \tag{7.5}$$

$$\rho s \vec{\bar{u}} - \rho u_\infty \hat{x} + \nabla \bar{p} = \mu \nabla^2 \vec{\bar{u}} \tag{7.6}$$

and the boundary conditions

$$\vec{\bar{u}} = 0 \qquad r = a \tag{7.7}$$

$$\vec{\bar{u}} = \frac{u_\infty}{s} \hat{x} \ , \ \bar{p} = 0 \ , \ r = \infty \tag{7.8}$$

where s is the transformed variable.

A general solution of equations (7.5) and (7.6) satisfying the condition at infinity is

$$\bar{p} = \sum_{l=0}^{\infty} \frac{A_l}{r^{l+1}} P_l(\cos\theta)$$

$$\bar{u}_r = \frac{U_\infty}{s} P_1(\cos\theta) + \frac{1}{s\rho} \sum_{\ell=0}^{\infty} \frac{(\ell+1) A_\ell}{r^{\ell+2}} P_\ell(\cos\theta)$$

$$+ \sum_{\ell=0}^{\infty} B_\ell\, \ell(\ell+1) \left[\frac{K_{\ell-\frac{1}{2}}(\sqrt{\frac{s}{\nu}}r)}{\sqrt{r}} - \frac{K_{\ell+\frac{3}{2}}(\sqrt{\frac{s}{\nu}}r)}{\sqrt{r}} \right] P_\ell(\cos\theta)$$

$$\bar{u}_\theta = \frac{U_\infty}{s} P_1'(\cos\theta) - \frac{1}{s\rho} \sum_{\ell=0}^{\infty} \frac{A_\ell}{r^{\ell+2}} P_\ell'(\cos\theta)$$

$$+ \sum_{\ell=0}^{\infty} B_\ell \left[(\ell+1) \frac{K_{\ell-\frac{1}{2}}(\sqrt{\frac{s}{\nu}}r)}{\sqrt{r}} + \ell\, \frac{K_{\ell+\frac{3}{2}}(\sqrt{\frac{s}{\nu}}r)}{\sqrt{r}} \right] P_\ell'(\cos\theta) \qquad (7.9)$$

where $\nu/\mu = 1/\rho$, and the x axis has been taken as the polar axis.

Impose condition (7.7) on (7.9) to obtain

$$\bar{p} = -\frac{\rho a^3 U_\infty}{2} \left[\frac{K_{5/2}(\sqrt{\frac{s}{\nu}}a)}{K_{1/2}(\sqrt{\frac{s}{\nu}}a)} \right] \frac{x}{r^3}$$

$$\bar{u}_r = \frac{U_\infty}{s} \cos\theta - \frac{a^3 U_\infty}{s} \left[\frac{K_{5/2}(\sqrt{\frac{s}{\nu}}a)}{K_{1/2}(\sqrt{\frac{s}{\nu}}a)} \right] \frac{\cos\theta}{r^3}$$

$$- \frac{\sqrt{a}\, U_\infty}{s\, K_{1/2}(\sqrt{\frac{s}{\nu}}a)} \left[\frac{K_{1/2}(\sqrt{\frac{s}{\nu}}r)}{\sqrt{r}} - \frac{K_{5/2}(\sqrt{\frac{s}{\nu}}r)}{\sqrt{r}} \right] \cos\theta$$

$$\bar{u}_\theta = -\frac{U_\infty}{s} \sin\theta - \frac{a^3 U_\infty}{2s} \left[\frac{K_{5/2}(\sqrt{\frac{s}{\nu}}a)}{K_{1/2}(\sqrt{\frac{s}{\nu}}a)} \right] \frac{\sin\theta}{r^3}$$

$$+ \frac{\sqrt{a}\, U_\infty}{2s} \frac{1}{K_{1/2}(\sqrt{\frac{s}{\nu}}a)} \left[\frac{2K_{1/2}(\sqrt{\frac{s}{\nu}}r)}{\sqrt{r}} + \frac{K_{5/2}(\sqrt{\frac{s}{\nu}}r)}{\sqrt{r}} \right] \qquad (7.10)$$

The inverse transform of (7.10) yields

$$p = -\frac{\rho a^3 \cos\theta}{2r^2} \left[\delta(t) + \frac{3}{a}\sqrt{\frac{\nu}{\pi t}} + \frac{3\nu}{a^2} \right] U_\infty$$

$$u_r = \left[1 - \frac{a^3}{r^3} - \frac{6a^2}{r^3}\sqrt{\frac{\nu t}{\pi}} - \frac{3\nu a t}{r^3} + \frac{3a}{r^2}\sqrt{\frac{\nu t}{\pi}} \left(1 + \frac{a}{r}\right) e^{-\frac{(r-a)^2}{4\nu t}} \right.$$

$$\left. - \frac{3a}{2r}\left(1 - \frac{2\nu t}{r^2} - \frac{a^2}{r^2}\right) \operatorname{erfc}\left(\frac{r-a}{2\sqrt{\nu t}}\right) \right] U_\infty \cos\theta$$

$$u_\theta = -\left[1+\frac{a^3}{2r^3}+\frac{3a^2}{r^3}\sqrt{\frac{\nu t}{\pi}}+\frac{3a\nu t}{2r^3}-\frac{3a}{2r^2}\sqrt{\frac{\nu t}{\pi}}\left(1+\frac{a}{r}\right)e^{-\frac{(r-a)^2}{4\nu t}}\right.$$

$$\left.-\frac{3a}{2r}\left(\frac{a}{r}+\frac{\nu t}{r^2}+\frac{(r-a)^2}{2r^2}\right)\operatorname{erfc}\left(\frac{r-a}{2\sqrt{\nu t}}\right)\right]U_\infty \sin\theta \qquad (7.11)$$

From equation (7. 11) the drag on the particle can be found to be

$$\text{drag} = \left[6\pi\mu a+\frac{2\pi\rho a^3}{3}\delta(t)+6a^2\mu\sqrt{\frac{\pi}{\nu t}}\right]U_\infty \qquad (7.12)$$

For an arbitrary motion of the particle, the drag can be obtained from equation (7. 12) by means of the Duhamel integral.

$$\text{drag} = \int_0^t\left[\frac{2\pi\rho a^2}{3}\delta(t-\tau)\frac{du}{d\tau}+6\pi\mu a\left(1+\frac{a}{\sqrt{\pi\nu(t-\tau)}}\right)\frac{du}{d\tau}\right]d\tau$$

Therefore,

$$\text{drag} = \frac{2\pi\rho a^3}{3}\frac{du}{dt}+6\pi\mu a\left[u(t)+\frac{a}{\sqrt{\pi\nu}}\int_0^t\frac{du}{d\tau}\frac{d\tau}{\sqrt{t-\tau}}\right] \qquad (7.13)$$

Equation (7. 13) is the Boussinesq's formula[8, 20].

Now consider an external force $\vec{F}(t)$, e. g. gravity, acting on the sphere which is at rest at time $t=0$. The motion of the particle is described by

$$m_p\frac{d\vec{u}_p}{dt} = \vec{F}(t) - \overrightarrow{drag} \qquad (7.14)$$

where m_p is the mass of the particle.

By means of equation (7. 13) and on writing $M = m_p + \frac{2\pi\rho a^3}{3}$ each Cartesian component of (7. 14) can be written as

$$M \frac{du_p}{dt} = f(t) - 6\pi\mu a U_p - \frac{6\pi\mu a^2}{\sqrt{\pi}\sqrt{\nu}} \int_0^t \frac{du_p}{d\tau} \frac{d\tau}{\sqrt{t-\tau}} \tag{7.15}$$

To solve equation (7. 15), take Laplace transform of the whole equation and use the convolution theorem for the integral:

$$M s \bar{U}_p = \bar{f}(s) - 6\pi\mu a \bar{U}_p - \frac{6\pi\mu a^2}{\sqrt{\nu}} \sqrt{s}\, \bar{U}_p$$

or

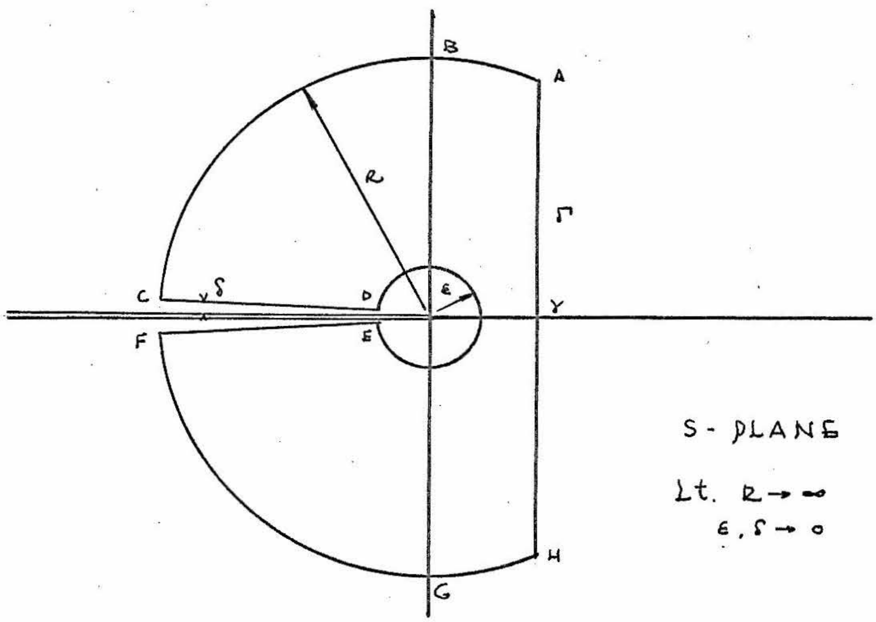
$$\bar{U}_p = \frac{\bar{f}(s)}{\left(Ms + 6\pi\mu a + 6\pi\mu a^2 \sqrt{\frac{s}{\nu}} \right)}$$

Therefore,

$$U_p = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\bar{f}(s)\, e^{st}}{M\left[s + \frac{6\pi\mu a^2 \sqrt{s}}{M\sqrt{\nu}} + \frac{6\pi\mu a}{M} \right]} ds \tag{7.16}$$

To integrate (7. 16), $f(t)$ will be assumed to be a physically reasonable forcing function so that $\bar{f}(s)$ is an analytic function save for some isolated poles. The denominator in (7. 16) depends on $\sqrt{s}$ and so, to insure single-valuedness, introduce a cut in the s - plane along the negative real axis. To evaluate the integral, take the contour as shown below.

Consider the function $(s + A\sqrt{s} + B)$ where A and B are real, positive constants. The imaginary part of $(s + A\sqrt{s} + B)$ is $(r \sin\theta + A\sqrt{r} \sin \theta/2)$ which has no zero in the s plane except on the positive real axis. The real part of $(s + A\sqrt{s} + B)$ is $(r\cos\theta + A \cos \theta/2 + B)$ which has no zero on the positive real axis. Therefore, the function $\left(s + \frac{6\pi\mu a^2\sqrt{s}}{M\sqrt{\nu}} + \frac{6\pi\mu a}{M} \right)^{-1}$ has no



pole within the contour Γ shown.

If α_i $(i=1,\dots n)$ are the poles of $\bar{f}(s)$ and f_{α_i}, the residues then by Cauchy's theorem,

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\bar{f}(s)\,e^{st}}{M\left[s+\frac{6\pi\mu a^2\sqrt{s}}{M\sqrt{\nu}}+\frac{6\pi\mu a}{M}\right]}\,ds = -\frac{1}{2\pi i}\left[\int_{ABC}+\int_{CD}+\int_{DE}+\int_{EF}+\int_{FGH}\right]$$

$$+\sum_{i=1}^{n}\frac{f_{\alpha_i}\,e^{\alpha_i t}}{M\left[\frac{6\pi\mu a}{M}+\alpha_i+\frac{6\pi\mu a^2\sqrt{\alpha_i}}{M\sqrt{\nu}}\right]} \tag{7.17}$$

For a reasonable function $f(t)$, it is easy to show that the integral over ABC and FGH gives no contribution.

The integral over path DE yields as $\epsilon \to 0$ the contribution

$$\underset{\epsilon\to 0}{\mathrm{Lt}}\left[-\frac{\epsilon\,\bar{f}(\epsilon)}{6\pi\mu a}\right]$$

Integral over CD gives

$$\frac{1}{2\pi i}\int_0^\infty \frac{\bar{f}(-r)\, e^{-rt}\, dr}{M\left[\frac{6\pi\mu a}{M} - r + \frac{6\pi\mu a^2\sqrt{r}}{M\sqrt{\nu}}\, i\right]}$$

Integral over EF gives

$$-\frac{1}{2\pi i}\int_0^\infty \frac{\bar{f}(-r)\, e^{-rt}\, dr}{M\left[-r + \frac{6\pi\mu a}{M} - \frac{6\pi\mu a^2\sqrt{r}}{M\sqrt{\nu}}\, i\right]}$$

Summing up all the contributions,

$$U_p(t) = \sum_{i=1}^{n} \frac{f_{\alpha_i}\, e^{\alpha_i t}}{M\left[\frac{6\pi\mu a}{M} + \alpha_i + \frac{6\pi\mu a^2\sqrt{\alpha_i}}{M\sqrt{\nu}}\right]} + \underset{\epsilon\to 0}{Lt}\left[\frac{\epsilon\, \bar{f}(\epsilon)}{6\pi\mu a}\right]$$

$$+ \frac{6a^2\mu}{M^2\sqrt{\nu}}\int_0^\infty \frac{\sqrt{x}\;\bar{f}(-x)\, e^{-xt}}{\left[\left(\frac{6\pi\mu a}{M} - x\right)^2 + \frac{36\pi^2\mu^2a^4x}{M^2\nu}\right]}\, dx \qquad (7.18)$$

The integral in equation (7.18) represents the transient solution.

Special Cases.

(a) Constant force, e.g. gravity. In this case

$$f(t) = K \qquad \bar{f}(s) = \frac{K}{s}$$

From (7.18),

$$u_p(t) = \frac{K}{6\pi\mu a} - \frac{6\mu a^2 K}{M^2\sqrt{\nu}}\int_0^\infty \frac{e^{-xt}}{\sqrt{x}\left[\left(\frac{6\pi\mu a}{M} - x\right)^2 + \frac{36\pi^2\mu^2a^4x}{M^2\nu}\right]}\, dx \qquad (7.19)$$

For large t, (7.19) becomes

$$U_p(t) \sim \frac{K}{6\pi\mu a} - \frac{K}{6\sqrt{\nu}\,\pi^{3/2}\mu\sqrt{t}} + \frac{6\mu a^2 K}{M^2\sqrt{\nu}}\left(\frac{a^2}{\nu} - \frac{M}{3\pi\mu a}\right)\frac{\sqrt{\pi}}{2t^{3/2}} + \cdots$$

(b) Sinusoidal force. In this case,

$$f(t) = \sin \lambda t \qquad : \quad \bar{f}(s) = \frac{\lambda}{s^2+\lambda^2}$$

$$\alpha_i = \pm i\lambda \qquad : \quad f_{\pm i\lambda} = \pm \frac{1}{2i}$$

$$\therefore \quad u_p(t) = \frac{\left(\frac{6\pi\mu a}{M} + \frac{6\pi\mu a^2\sqrt{\lambda}}{M\sqrt{2\nu}}\right)\sin\lambda t - \left(2 + \frac{6\pi\mu a^2\sqrt{\lambda}}{M\sqrt{2\nu}}\right)\cos\lambda t}{M\left[\left(\frac{6\pi\mu a}{M} + \frac{6\pi\mu a^2}{M}\sqrt{\frac{\lambda}{2\nu}}\right)^2 + \left(2 + \frac{6\pi\mu a^2}{M}\sqrt{\frac{\lambda}{2\nu}}\right)^2\right]}$$

$$+ \frac{6\mu a^2\lambda}{M^2\sqrt{\nu}}\int_0^\infty \frac{e^{xt}\sqrt{x}}{(x^2+\lambda^2)\left[\left(\frac{6\pi\mu a}{M} - x\right)^2 + \frac{36\pi^2\mu^2a^4x}{M^2\nu}\right]}\,dx \tag{7.20}$$

Fundamental Solution

Consider a force pulse of intensity D applied to the fluid at time t_0 at the origin in the x-direction. It is clear that disturbances will be initiated and propagate in all directions. The velocity and pressure field resulting from such a pulse are given by the fundamental solution of the linearized time-dependent Navier-Stokes equation, i.e., $\vec{u}$ and p satisfy the following equations:

$$\rho\frac{\partial\vec{u}}{\partial t} + \nabla p = \mu\nabla^2\vec{u} + D\hat{x}\,\delta(\vec{x})\,\delta(t-t_0) \tag{7.21}$$

$$\nabla\cdot\vec{u} = 0 \tag{7.22}$$

$$\vec{u},\ p = 0 \qquad \text{at infinity.}$$

To solve for $\vec{u}$ and p, take Laplace transform of equations (7.21) and (7.22); then

$$\rho s\bar{\vec{u}} + \nabla\bar{p} = \mu\nabla^2\bar{\vec{u}} + D\hat{x}\,\delta(\vec{x})\,e^{-st_0} \tag{7.23}$$

$$\nabla\cdot\bar{\vec{u}} = 0 \tag{7.24}$$

Take divergence of equation (7.23) and use (7.24)

$$\nabla^2 p = D e^{-st_0} \frac{\partial}{\partial x} (\delta(\vec{x}))$$

Hence

$$p = \frac{D}{4\pi} \frac{x}{r^3} e^{-st_0} \quad \text{or} \quad p = \frac{D}{4\pi} \frac{x}{r^3} \delta(t-t_0) \tag{7.25}$$

From equation (7.23),

$$\vec{u} = -\frac{1}{4\pi\mu} \int \frac{e^{-\sqrt{\frac{s}{\nu}}|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} \left[\nabla' p - D\hat{x}\, \delta(\vec{x}')\, e^{-st_0} \right] d^3x'$$

On computing ∇p as in Section 1,

$$\vec{U} = -\frac{1}{4\pi} \int \frac{e^{-\sqrt{\frac{s}{\nu}}|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} d^3x' \frac{D}{4\pi\mu} \left\{ -\hat{x} \left[\frac{2P_2(\cos\theta')}{r'^3} + \frac{8\pi}{3} \delta(\vec{x}') \right] + \hat{y} \left[\frac{P_2'(\cos\theta')\cos\phi'}{r'^3} \right] + \hat{z} \left[\frac{P_2'(\cos\theta')\sin\phi'}{r'^3} \right] \right\} e^{-st_0} \tag{7.26}$$

Inverting the Laplace transform,

$$\vec{u}(\vec{x},t) = -\frac{1}{4\pi} \int d^3x' \left\{ \frac{e^{-\frac{|\vec{x}-\vec{x}'|^2}{4\nu(t-t_0)}}}{2\sqrt{\pi\nu}\,(t-t_0)^{3/2}} H(t-t_0) \frac{D}{4\pi\mu} \left[-\hat{x} \frac{P_2(\cos\theta')}{r'^3} + \hat{y} \frac{P_2'(\cos\theta')\cos\phi'}{r'^3} + \hat{z} \frac{P_2'(\cos\theta')\sin\phi'}{r'^3} \right] \right\} + \frac{D}{6\pi\mu} \frac{e^{-\frac{r^2}{4(t-t_0)\nu}}}{\sqrt{\nu}\,(t-t_0)^{3/2}} \frac{H(t-t_0)}{2\sqrt{\pi}} \hat{x} \tag{7.27}$$

If, instead of a pulse force, a constant force is applied, then, integrating over all time, equation (7.27) becomes

$$\vec{u}(\vec{x}) = -\frac{1}{4\pi} \int \frac{d^3x'}{|\vec{x}-\vec{x}'|} \frac{D}{4\pi\mu} \left[-\hat{x} \frac{P_2(\cos\theta')}{r'^3} + \hat{y} \frac{P_2'(\cos\theta')\cos\phi'}{r'^3} + \hat{z} \frac{P_2'(\cos\theta')\sin\phi'}{r'^3} \right] + \frac{D}{6\pi\mu r} \hat{x} \tag{7.28}$$

Equation (7.28) is the fundamental solution of Stokes equation as obtained in Section 1.

PART III.

Macroscopic Continuum Description of Particle - Fluid Flow

8. Particle Continuum Equations

The particle-fluid system that is being considered in this thesis consists of solid spheres and viscous incompressible fluid. The particles are assumed to be small and numerous. Further, they would be regarded as having the same size unless otherwise specified.

Owing to the existence of viscous fluid intervening between particles, a form of viscous interaction exists between neighboring spheres. This interaction tends to smooth out any possible velocity dispersion among adjacent particles. In general, as a first approximation, the velocity difference can be neglected. The consequence of this lack of dispersion is that particle-particle collision is infrequent. In fact, for slow motion in which viscous forces dominate, direct contact collision between particles during which significant momentum exchange takes place is rather rare. Thus, the particle cloud may be regarded as "collisionless."

To describe the motion of the particle cloud, it is possible to define a distribution function $f(\vec{x}, \vec{v}, t)$ for the particles and to obtain the conservation equations by taking moments of the collisionless Boltzmann equation as was done by Marble[4]. Here, such a procedure will not be undertaken as the force acting on each particle is still unknown (a major subject of investigation in later sections). Instead, for simplicity, the following assumptions would be made to achieve the same goal.

(1) The number density of the particles is large.

(2) The radius of the spheres is very small compared with any macroscopic length of interest.

(3) Velocity dispersion is negligible.

Assumptions (1) and (2) assure that the particle cloud may be considered as a continuum, while assumption (3) permits the use of a particle continuum velocity $\vec{u}_p$ with sufficient precision. With the above assumptions, the following field equations for the particle continuum can be written down immediately.

Continuity

$$\frac{\partial n}{\partial t} + \nabla \cdot (n \vec{u}_p) = 0 \tag{8.1}$$

Conservation of Linear Momentum

$$m_p n \left(\frac{\partial \vec{u}_p}{\partial t} + \vec{u}_p \cdot \nabla \vec{u}_p \right) = \vec{F} \tag{8.2}$$

Conservation of Angular Momentum

$$I_p \left(\frac{\partial \vec{\omega}_p}{\partial t} + \vec{u}_p \cdot \nabla \vec{\omega}_p \right) = \vec{T} \tag{8.3}$$

Conservation of Energy

$$\frac{\partial}{\partial t}\left[n \left(e_p + \frac{m_p}{2} \vec{u}_p^{\,2} + \frac{I_p}{2} \vec{\omega}_p^{\,2} \right)\right] + \nabla \cdot \left[n \vec{u}_p \left(e_p + \frac{m_p}{2} \vec{u}_p^{\,2} + \frac{I_p}{2} \vec{\omega}_p^{\,2} \right)\right]$$

$$= \vec{F} \cdot \vec{u}_p + n \vec{T} \cdot \vec{\omega}_p + Q \tag{8.4}$$

where

n = number density of particle

$\vec{u}_p$ = velocity of particle continuum

$\vec{\omega}_p$ = angular velocity of particle

m_p = mass of a single particle

I_p = moment of inertia of a single particle

e_p = internal energy of particle

$\vec{F}$ = force exerted on particle continuum by surrounding fluid per unit volume of space

$\vec{T}$ = torque exerted on a particle by its surrounding fluid

Q = heat given to particle continuum by surrounding fluid per unit volume

9. Bulk Fluid Equations

The fluid phase of the particle-fluid system is assumed to consist of a viscous incompressible fluid, the motion of which is described by the Navier-Stokes equation. As the fluid does not occupy all space, for the region exterior to all spheres, the following equations apply.

Continuity

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0 \tag{9.1}$$

Momentum

$$\frac{\partial \rho \vec{u}}{\partial t} + \nabla \cdot (\rho \vec{u} \vec{u}) = \nabla \cdot \bar{\bar{\sigma}} \tag{9.2}$$

$$\sigma_{ij} = - p \, \delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

(Navier-Stokes relation)

Energy

$$\frac{\partial}{\partial t}\left[\rho (e + \tfrac{1}{2} \vec{u}^2) \right] + \nabla \cdot \left[\rho \vec{u} (e + \tfrac{1}{2} \vec{u}^2) \right] = \nabla \cdot (\bar{\bar{\sigma}} \cdot \vec{u}) - \nabla \cdot \vec{q} \tag{9.3}$$

$$\vec{q} = - k \nabla T$$

(Fourier law of conduction)

where

ρ = density of fluid

μ = viscosity of fluid

k = thermal conductivity of fluid

$\vec{u}$ = velocity of fluid

$\bar{\bar{\sigma}}$ = stress tensor

$\vec{q}$ = heat flux vector

e = internal energy of fluid

T = temperature of fluid

p = pressure of fluid

In principle, the particle continuum equations of Section 1, together with equations (9.1), (9.2), and (9.3) above, give a complete description of the two-phase system. A solution of these sets of equations satisfying appropriate boundary conditions would, in effect, yield all available information. However, a moment of thought would easily convince anyone that such is an impossible and forbidding task. Imagine the particles to be of the size of 10 microns or 10^{-3} cm, and assume that they are about 100 microns apart. Thus, in a cubic millimeter, there are about 1000 particles, giving 1000 isolated surfaces on which the solution has to satisfy the no-slip boundary condition (a rather hopeless endeavor!). Of course, the difficulty comes from the deterministic approach to the problem, a course that is purposely avoided in all subsequent sections of this thesis.

For many practical purposes, full detailed information concerning the behavior of the fluid-particle system is not required. In fact, a gross averaged description is sufficient to meet the need.

With this in mind, it should be quite clear that a "macroscopic description" in terms of averaged quantities not only would satisfy practical requirements but also would circumvent the dilemma of the deterministic approach. To be sure, a "microscopic description" involving a detailed account of the behavior of a typical particle and its surrounding fluid is indispensable in determining the transport properties of the two-phase system. These two descriptions will be taken as complementary to each other in the analysis to follow.

By the assumptions of section 8, it is to be noted that there are two basic scales in the two-phase system in consideration. The first scale length is the one that measures the influence of a single particle on the fluid, which is of the order of the size of the sphere. The second scale length corresponds to the measure of macroscopic changes that are of direct interest. It will be assumed from now on that the two scales differ by several orders of magnitude.

To describe the system macroscopically, it is necessary first to define bulk or averaged quantities of the fluid. For this purpose, let $\langle \ \rangle$ denote an averaging operation involving the following steps:

(a) divide space into small cubic volume V, very large compared with the size of a particle but very small in relation to the macro-scale;

(b) take averages of the physical quantities of the fluid over the volume occupied by the fluid V_f in V. Thus, for a quantity Q,

$$\langle Q \rangle \equiv \frac{1}{V} \int_{V_f} Q \, dv$$

By means of operator $\langle \ \rangle$, the following "bulk fluid" quantities can be defined.

Bulk fluid density $\qquad \rho_f = \langle \rho \rangle$

Bulk fluid velocity $\qquad \vec{U}_f = \frac{1}{\rho_f} \langle \rho \vec{u} \rangle$

Bulk fluid internal energy $\qquad e_f = \frac{1}{\rho_f} \langle \rho e \rangle$

Bulk fluid pressure $\qquad p_f = \frac{1}{1-c} \langle p \rangle$

Bulk fluid temperature $\qquad T_f = \frac{1}{1-c} \langle T \rangle$

where $\quad c = \frac{4\pi}{3} a^3 n =$ volume concentration of solid particles.

For systems in which the particle volume concentration is relatively low, it is clear that the presence of particles introduces only small local variations in the fluid field. With respect to the macro-scale, these local variations can be ignored, so that variations in the macro-scale can be represented adequately by variations of the averages. Naturally, the above statement is not exact, yet its accuracy would improve if the relative size of the two scales increases. Here, for clarity, the following explicit assumption would be made.

<u>Bulk Fluid Assumption.</u>

If ∇ is the gradient operator, then, with respect to the macro-scale, the two operators ∇ and $\langle \ \rangle$ commute, or

$$\langle \nabla \quad \rangle \simeq \nabla \langle \quad \rangle$$

The assumption assumes that the average of the derivative is equal to the derivative of the average. This is exact in the limit $v \to \infty$.

Granted that the bulk fluid assumption holds, then a set of bulk fluid equations can be derived from equations (9.1), (9.2), and (9.3).

Apply $< >$ to (9.1) and use the assumption of commutation, then

$$\frac{\partial}{\partial t}\langle \rho \rangle + \nabla \cdot \langle \rho \vec{u} \rangle = 0$$

$$\frac{\partial \rho_f}{\partial t} + \nabla \cdot (\rho_f \vec{u}_f) = 0 \qquad (9.4)$$

which is the continuity equation of bulk fluid.

Applying $< >$ to (9.2), one obtains

$$\frac{\partial}{\partial t}(\rho_f \vec{u}_f) + \nabla \cdot (\rho_f \vec{u}_f \vec{u}_f) = \nabla \cdot \bar{\bar{\sigma}}_f + \vec{F}_f \qquad (9.5)$$

where

$$\nabla \cdot \bar{\bar{\sigma}}_f + \vec{F}_f = \nabla \cdot (\rho_f \vec{u}_f \vec{u}_f) - \langle \nabla \cdot (\rho \vec{u} \vec{u}) \rangle + \langle \nabla \cdot \bar{\bar{\sigma}} \rangle$$

The difference of the terms $\nabla \cdot (\rho_f \vec{u}_f \vec{u}_f) - \langle \nabla \cdot (\rho \vec{u} \vec{u}) \rangle + \langle \nabla \cdot \bar{\bar{\sigma}} \rangle$ in (9.5) has been separated into two parts for the following reasons. Consider the case in which the particles have negligible inertia. Then the particles and fluid will move together and so behave as more or less a homogeneous fluid to which one can assign a stress tensor $\bar{\bar{\sigma}}_f$ to describe its mechanical behavior. When the particles do possess inertia and hence slip occurs, it is still meaningful to retain the concept $\bar{\bar{\sigma}}_f$ and to account for the difference by the term $\vec{F}_f$. $\vec{F}_f$, meanwhile, signifies the body force arising from particle-fluid interaction, part of which can be seen by considering the term $\langle \nabla \cdot \bar{\bar{\sigma}} \rangle$

$$\langle \nabla \cdot \bar{\bar{\sigma}} \rangle = \frac{1}{v} \int_{v_f} \nabla \cdot \bar{\bar{\sigma}} \, dv = \frac{1}{v} \int_{s_f} \hat{n} \cdot \bar{\bar{\sigma}} \, ds$$

The contribution to the last integral from the surfaces of the spheres gives the total force exerted on the fluid by the particles in volume v .

Moreover, the difference between the terms $\nabla \cdot (\rho_f \vec{u}_f \vec{u}_f)$ and $< \nabla \cdot (\rho \vec{u} \vec{u}) >$ represents the momentum associated with the disturbances produced in the fluid due to the presence of the particles. The existence of this momentum difference makes it clear that, in general,

$$\bar{\bar{\sigma}}_f \neq \bar{\bar{\sigma}}$$
$$\vec{F}_f \neq n \vec{F}_p$$

where $\vec{F}_p$ is the force exerted on the fluid by a single particle.

Similarly, applying $< \ >$ to equation (9.3), one obtains

$$\frac{\partial}{\partial t}\left[\rho_f (e_f + \tfrac{1}{2}\vec{u}_f^{\,2})\right] + \nabla\cdot\left[\rho_f \vec{u}_f (e_f + \tfrac{1}{2}\vec{u}_f^{\,2})\right] = \nabla\cdot(\bar{\bar{\sigma}}_f \cdot \vec{u}_f) - \nabla\cdot\vec{q}_f + Q_f \qquad (9.6)$$

where

$$-\nabla\cdot\vec{q}_f + Q_f = -<\nabla\cdot\vec{q}> + <\nabla\cdot(\bar{\bar{\sigma}}\cdot\vec{u})> - \nabla\cdot(\bar{\bar{\sigma}}_f\cdot\vec{u}_f) + \nabla\cdot\left[\rho_f \vec{u}_f (e_f - \tfrac{1}{2}\vec{u}_f^{\,2})\right] - <\nabla\cdot[\rho\vec{u}(e + \tfrac{1}{2}\vec{u}^{\,2})]> + \frac{\partial}{\partial t}\left(\rho_f \frac{\vec{u}_f^{\,2}}{2}\right) - \frac{\partial}{\partial t}<\tfrac{1}{2}\rho\vec{u}^{\,2}>$$

The term Q denotes the heat exchange between particles and fluid which arises from the term $< \nabla\cdot\vec{q} >$

$$<\nabla\cdot\vec{q}> = \frac{1}{V}\int_{V_f} \nabla\cdot\vec{q}\, dv - \frac{1}{V}\int_{S_f} \hat{n}\cdot\vec{q}\, ds$$

The contribution to the last integral from the surfaces of the particles gives the heat flux from particle to fluid.

The generalized heat flux vector $\vec{q}_f$ is quite complicated. However, when motion is sufficiently low, the heat conduction part dominates. It is accurate enough to regard $\vec{q}_f \approx \vec{q}_{conduction}$.

Finally, summarizing, the field equations for the bulk fluid are:

$$\frac{\partial \rho_f}{\partial t} + \nabla \cdot (\rho_f \vec{u}_f) = 0 \tag{9.4}$$

$$\frac{\partial \rho_f \vec{u}_f}{\partial t} + \nabla \cdot (\rho_f \vec{u}_f \vec{u}_f) = \nabla \cdot \bar{\bar{\sigma}}_f + \vec{F}_f \tag{9.5}$$

$$\frac{\partial}{\partial t}\left[\rho_f (e_f + \tfrac{1}{2} \vec{u}_f^{\,2})\right] + \nabla \cdot \left[\rho_f \vec{u}_f (e_f + \tfrac{1}{2} \vec{u}_f^{\,2})\right] = \nabla \cdot (\bar{\bar{\sigma}}_f \cdot \vec{u}_f) - \nabla \cdot \vec{q}_f + Q_f \tag{9.6}$$

10. Approximate Phenomenological Formulas of Bulk Fluid

Equations (9.4), (9.5), and (9.6) provide the field equations for the description of the bulk fluid. However, by themselves alone, they do not form a closed set. To make the set of equations complete, it is necessary to supplement them with phenomenological formulas relating $\bar{\bar{\sigma}}_f$, $\vec{F}_f$, $\vec{q}_f$ and Q_f to the field variables. As mentioned in Section 2, the stress tensor $\bar{\bar{\sigma}}_f$ describes the mechanical behavior of a homogeneous suspension; hence, to first approximation, one may postulate that

$$\sigma_{fij} = -p_f \delta_{ij} + \mu_f \left(\frac{\partial u_{fi}}{\partial x_j} + \frac{\partial u_{fj}}{\partial x_i} \right) + \zeta \delta_{ij} \frac{\partial u_{fk}}{\partial x_k}$$

where μ_f is the coefficient of viscosity of bulk fluid depending on μ and c .

Note: In general, it is easy to show that ζ depends on concentration gradient and therefore is small compared with μ_f . It will be ignored in subsequent consideration.

Similarly, the vector $\vec{q}_f = \vec{q}_{conduction}$ can, in the steady state case, be postulated as given by

$$\vec{q}_f = -k_f \nabla T_f$$

where k_f is the steady-state thermal conductivity of the bulk fluid.

It is the goal of this thesis to determine the phenomenological constants μ_f and k_f, the force of interaction $\vec{F}_f$ and the heat transfer rate between solid particles and bulk fluid Q_f. In this way, it is believed that a first step towards the understanding of the transport properties of particle-fluid flow might be achieved.

11. The "Screened" Stokes Equation and the Drag Formula

Consider a homogeneous cloud of spherical particles of radius a and number density n moving steadily in an otherwise stationary, viscous incompressible fluid with velocity $u_p \hat{x}$. Let the drag experienced by a typical particle be $-D\hat{x}$. To obtain an overall picture or a macroscopic description of the flow field, one may use the bulk fluid equations of Section 9. With $\vec{F}_f = nD\hat{x}$, they are

$$\nabla \cdot \vec{u}_f = 0 \tag{11.1}$$

$$\vec{u}_f \cdot \nabla \vec{u}_f = -\nabla p_f + \mu_f \nabla^2 \vec{u}_f + nD\hat{x} \tag{11.2}$$

It is obvious that the solution of the bulk fluid equations does not in itself provide any information concerning the drag on the spheres. To obtain a drag formula, it is necessary to examine in great detail the actual fluid motion near a sphere. In fact, it is only through the knowledge of this that the hydrodynamic forces acting on the spheres can be found and hence the overall fluid motion known.

To do so, a "detailed" or "microscopic" description is required. This will be carried out by the following consideration.

"Screened" Stokes Equation

The solution of equations (11.1) and (11.2) gives an averaged value of the fluid field at any point. Near any sphere, the velocity and pressure field would differ appreciably from $\vec{u}_f$ and p_f . However, if one fixes one's attention on a typical sphere A , one may regard A as existing in an environment with a smoothed-out background fluid velocity and pressure $\vec{u}_f$ and p_f . Let $\vec{u}_\alpha$ and p_α be the velocity and pressure disturbance produced by particle A upon the gross fluid flow field, so that the actual fluid velocity and pressure near A are

$$\begin{aligned} \vec{u} &= \vec{u}_f + \vec{u}_\alpha \\ p &= p_f + p_\alpha \end{aligned} \tag{11.3}$$

To determine $\vec{u}_\alpha$ and p_α , it is necessary to examine the propagation of disturbance in a particle suspension. Consider first the particles are infinitely apart. Then, in the Stokes flow regime, the disturbance satisfies the Stokes equation, or

$$\begin{aligned} \nabla \cdot \vec{u}_\alpha &= 0 \\ \nabla p_\alpha &= \mu \nabla^2 \vec{u}_\alpha \end{aligned}$$

Now, let the interparticle distance be decreased. It is clear that the presence of other particles will significantly alter the disturbance $\vec{u}_\alpha$ and p_α of the particle A . The effect of the other particles will tend to confine $\vec{u}_\alpha$ to the vicinity of A . A form of screening action is in operation, so that the disturbance produced by particle A would not be felt by particles at some distance from

A . From the consideration on the drag of a finite number of particles in Part II, it is clear that a good simulation of the modification on $\vec{u}_a$ by the presence of other particles is to regard each of the other particles as exerting a point force proportional to $\vec{u}_a$ at the locations of the particles. For very large interparticle distance, this is approximately given by $-6\pi\mu a\,\vec{u}_a$. However, for a sufficiently dense cloud, the mutual influence of all particles must be considered and hence, in general, one can assume that the other particles exert a body force $-\lambda^2\vec{u}_a$ on the fluid opposing the spread of $\vec{u}_a$. λ^2 will be referred to as the screening constant and is an inherent property of the suspension. As a result of the above discussion, the disturbance $\vec{u}_a$ and p_a associated with particle A should be found as a solution of the "screened" Stokes equation*

$$\nabla\cdot\vec{u}_a = 0$$

$$\nabla p_a = \mu\nabla^2\vec{u}_a - \lambda^2\vec{u}_a \tag{11.4}$$

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* The "screened" Stokes equation can also be obtained by the following argument.

The only possible effect of the presence of other particles is to exert a body force on the fluid so as to confine $\vec{u}_a$. Let this force be $\vec{F} = \vec{F}(\vec{u}_a)$. Develop $\vec{F}$ into a series

$$\vec{F} = \vec{A} + \text{const.}\,\vec{u}_a + O(\vec{u}_a^2)$$

Since $\vec{u}_a = 0$, $\vec{F} = 0$, therefore $\vec{A} = 0$. Also, Stokes equation was obtained by linearizing the Navier-Stokes equation; hence, to be consistent, the quadratic and higher order terms in $\vec{u}_a$ should be dropped. So

$$\vec{F} = -\lambda^2\vec{u}_a$$

The negative sign indicates that the force tends to oppose the spread of the disturbance.

Drag Formula

Now we return to the drag problem. The solution of the bulk fluid equations (11. 1) and (11. 2) is

$$\vec{u}_f = -u_p\,\hat{x} \qquad \text{(with respect to coordinate system fixed to center of sphere } A\text{)}$$

$$p_f = n D x \tag{11.5}$$

Thus, macroscopically, the particles literally pierce through the fluid without causing perceptible fluid motion, while the force exerted on the fluid by the particles is balanced by an overall pressure gradient.

Near sphere A, however, the velocity and pressure disturbance is given by the solution of the "screened" Stokes equation.

$$\nabla \cdot \vec{u}_a = 0$$

$$\nabla p_a = \mu \nabla^2 \vec{u}_a - \lambda^2 \vec{u}_a \tag{11.4}$$

The term $\lambda^2 \vec{u}_a$ represents the body force exerted by the particle cloud on the fluid if the relative velocity between fluid and particles is $\vec{u}_a$. However, for the problem under consideration, a particle exerts a force equal to D when the relative velocity is u_p. Therefore, $\lambda^2 = \frac{nD}{u_p}$. Hence, equation (11. 4) becomes

$$\nabla \cdot \vec{u}_a = 0$$

$$\nabla p_a = \mu \nabla^2 \vec{u}_a - \frac{nD}{u_p} \vec{u}_a \tag{11.6}$$

The boundary conditions on $\vec{u}_a$, p_a are:

(1) disturbances vanish at large distance from sphere, i. e.,

$$\vec{u}_a = 0 \qquad p_a = 0 \qquad r = \infty$$

(2) the no-slip boundary condition on the sphere must be satisfied, i.e.,

$$\vec{u} = \vec{u}_f + \vec{u}_a = 0 \qquad \text{or} \qquad \vec{u}_a = u_p \hat{x} \qquad \text{at} \qquad r = a$$

To solve equation (11.6), let $\alpha^2 = \frac{\lambda^2}{\mu}$. Then, by means of the general solution of "screened" Stokes equation of Appendix D, the solution satisfying the required boundary condition is found to be

$$p_a = \frac{\lambda^2 a^3}{2}\left[\frac{3}{\alpha^2 a^2} + \frac{3}{\alpha a} + 1\right] u_p \frac{\cos\theta}{r^2}$$

$$\vec{W}_a = \hat{e}_r\left[\frac{a^3}{r^3}\left(\frac{3}{\alpha^2 a^2} + \frac{3}{\alpha a} + 1\right) - \frac{3a}{\alpha}\left(\frac{1}{r^2} + \frac{1}{\alpha r^3}\right)e^{-\alpha(r-a)}\right] u_p \cos\theta$$

$$+ \hat{e}_\theta\left[\frac{a^3}{2r^3}\left(\frac{3}{\alpha^2 a^2} + \frac{3}{\alpha a} + 1\right) - \frac{3a}{2\alpha}\left(\frac{1}{r^2} + \frac{1}{\alpha r^3} + \frac{\alpha}{r}\right)e^{-\alpha(r-a)}\right] u_p \sin\theta \qquad (11.7)$$

From equations (11.5) and (11.7), the stress components on the surface of sphere A are

$$\sigma_{rr} = -\frac{3\mu\alpha u_p}{2}\left[\frac{1}{\alpha a} + 1 + \alpha a\right]\cos\theta$$

$$\sigma_{r\theta} = \frac{3\mu\alpha u_p}{2}\left[1 + \frac{1}{\alpha a}\right]\sin\theta \qquad (11.8)$$

Hence, using (11.8),

$$D = \text{drag} = 6\pi\mu a U_p \left(1 + \alpha a + \frac{\alpha^2 a^2}{3}\right) \qquad (11.9a)$$

or

$$\alpha a = \frac{9c + 3\sqrt{8c - 3c^2}}{4 - 6c}$$

$$\alpha^2 a^2 = \frac{9c}{2}\left(1 + \alpha a + \frac{\alpha^2 a^2}{3}\right)$$

where c volume concentration of solid. Therefore,

$$\text{drag on sphere} = \frac{\mu U_p}{n a^2}\left[\frac{9c + 3\sqrt{8c - 3c^2}}{4 - 6c}\right]^2 \tag{11.9}$$

Also,

$$\text{(screening constant)} \quad \alpha^2 = \frac{\mu}{a^2}\left[\frac{9c + 3\sqrt{8c - 3c^2}}{4 - 6c}\right]^2 \tag{11.10}$$

The drag formula, equation (11.9), was obtained in 1947 by Brinkman[31] (see also references 32 and 33) through a different argument. Brinkman's original treatment in many respects, however, was rather obscure. In addition, he was uncertain whether the viscosity in equation (11.9) should be the viscosity of the pure fluid or Einstein's viscosity coefficient $\mu^* = \mu(1 + 2.5c)$

Recent experiments performed by Hoppel and Epstein[34] confirmed that equation (11.9) is valid for the whole range $c < 0.5$ even though they rejected Brinkman's equation as arbitrary and nonrigorous.

Motion of a Single Particle

The drag formula, as given by equation (11.9), was derived by considering the motion of a cloud of particles. Now suppose all particles, except one, are stationary, say, sphere A. In this case, let $u\hat{x}$ be the steady velocity of A. Then, with respect to a coordinate system fixed to A, the solution of the bulk fluid equation is

$$\vec{U}_f = -U\hat{x}$$
$$p_f = 0 \qquad (11.11)$$

Again, if $\vec{u}_a, p_a$ is the disturbance associated with sphere A, such that near A the actual fluid velocity and pressure are

$$\vec{u} = \vec{u}_f + \vec{u}_a$$
$$p = p_f + p_a \qquad (11.12)$$

$\vec{u}_a$ and p_a are to be found as the solution of "screened" Stokes equation

$$\nabla \cdot \vec{u}_a = 0$$
$$\nabla p_a = \mu \nabla^2 \vec{u}_a - \mu \alpha^2 \vec{u}_a \qquad (11.13)$$

where α is given by (11.10).

The boundary conditions on $\vec{u}_a$ and p_a are

(a) $\vec{U}_a = 0 \qquad p_a = 0 \qquad r = \infty$

(b) $\vec{u} = \vec{u}_f + \vec{u}_a = 0 \qquad r = a$

or $\vec{U}_a = u\hat{x}$ (11.14)

The solution of equation (11.13) satisfying (11.14) is the same as given by (11.7). The stresses acting on the surface of the sphere are

$$\sigma_{rr} = -\frac{3\alpha\mu U}{2}\left(\frac{1}{\alpha a} + 1 + \frac{\alpha a}{3}\right)\cos\theta$$

$$\sigma_{r\theta} = \frac{3\mu\alpha U}{2}\left(1 + \frac{1}{\alpha a}\right)\sin\theta \qquad (11.15)$$

From (11.15), the drag on sphere A is

$$D = 6\pi\mu a U\left(1 + \alpha a + \frac{\alpha^2 a^2}{9}\right)$$
$$= \frac{\mu U}{n a^2}\left[\frac{9c + 3\sqrt{8c - 3c^2}}{4 - 6c}\right]^2 (1-c)$$

(11.16)

On comparing equations (11.9) and (11.16), it is quite clear that the drag given by (11.16) is smaller. The difference between the two formulas is physically obvious and is the direct result of the overall pressure gradient which exists when a cloud of particles moves through a fluid. Here, it should be pointed out that this difference clearly illustrates the necessity of an overall or bulk fluid consideration in the determination of hydrodynamic forces acting on spheres in particle-fluid flow.

Fundamental Solution of "Screened" Stokes Equation

The "screened" Stokes equation formulated above shall again be used in subsequent sections. Because of this, it is meaningful to examine the nature of this equation by obtaining its fundamental solution. Consider a disturbance characterized by a point force acting at the origin in the x - direction. Then

$$\nabla p = \mu \nabla^2 \vec{u} - \mu \alpha^2 \vec{u} + D \hat{x}\, \delta(\vec{x}) \tag{11.17}$$

$$\nabla \cdot \vec{u} = 0 \tag{11.18}$$

Take divergence of equation (11.17) and use (11.18),

$$\nabla^2 p = D \frac{\partial}{\partial x} \delta(\vec{x})$$

Therefore, as in Section 1 of Part II A, one obtains

$$\vec{E} = \nabla p - D \hat{x}\, \delta(\vec{x}) = \frac{D}{4\pi} \left\{ -\hat{x} \left[\frac{2 P_2(\cos\theta)}{r^3} + \frac{8\pi}{3} \delta(\vec{x}) \right] + \hat{y} \left[\frac{P_2'(\cos\theta) \cos\phi}{r^3} \right] + \hat{z} \left[\frac{P_2'(\cos\theta) \sin\phi}{r^3} \right] \right\} \tag{11.19}$$

From equation (11.11)

$$\nabla^2 \vec{u} - \alpha^2 \vec{u} = \frac{\vec{E}}{\mu}$$

Therefore, using the Green's function of Appendix A ,

$$\vec{u} = -\frac{1}{4\pi} \int \frac{e^{-\lambda|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} \vec{E}(\vec{x}')\, d^3x' \tag{11.20}$$

It is easy to verify that $\vec{u}$ given by (11. 20) satisfies (11. 18).

To integrate (11. 20), the representation of

$$-\frac{1}{4\pi} \frac{e^{-\lambda|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}$$

in series form, as given in Appendix A , may be used. Also, for integration over the radial part, formulas provided on pages 79-80 of reference 35 are useful. After integration, one obtains

$$\begin{aligned} \vec{U} = {} & \hat{x} \frac{D}{2\pi\mu} \left\{ \frac{e^{-\alpha r}}{3r} + \left[\frac{1}{\alpha^2 r^3} - \frac{e^{-\alpha r}}{3r}\left(1 + \frac{3}{\alpha r} + \frac{3}{\alpha^2 r^2}\right)\right] P_2(\cos\theta) \right\} \\ & + \hat{y} \frac{D}{4\pi\mu} \left\{ -\frac{1}{\alpha^2 r^3} + \frac{e^{-\alpha r}}{3r}\left[1 + \frac{3}{\alpha r} + \frac{3}{\alpha^2 r^2}\right] \right\} P_2^1(\cos\theta) \cos\phi \\ & + \hat{z} \frac{D}{4\pi\mu} \left\{ -\frac{1}{\alpha^2 r^3} + \frac{e^{-\alpha r}}{3r}\left[1 + \frac{3}{\alpha r} + \frac{3}{\alpha^2 r^2}\right] \right\} P_2^1(\cos\theta) \sin\phi \end{aligned} \tag{11.21}$$

Equation (11. 21) shows clearly the manner in which the disturbance dies off. The longitudinal component of the disturbance decays at a rate of $\frac{1}{r^3}$ as compared to $\frac{1}{r}$ in the case of Stokes equation. The transverse components are exponentially damped. The screening length is approximately equal to $\frac{1}{\alpha}$. Therefore, in a particle suspension, disturbances are localized. They are screened by the particles and tend to decay at a fast rate when propagating away from the source.

Particle Cloud with a Distributed Size of Particles

In the derivation of (11. 9) above, a particle cloud of one size particles was assumed. Now consider the case of a particle cloud consisting of particles of a range of particle radius such that the number of particles with radius σ is $f(\sigma)\,d\sigma$, $a_1 \leq \sigma \leq a_2$

Let $D(\sigma)$ be the drag on the particle with radius σ . Then the screening constant of the particle cloud is

$$\lambda_1^2 = \frac{1}{u_p} \int_{a_1}^{a_2} D(\sigma) f(\sigma)\, d\sigma$$

where u_p is the steady velocity of the particle cloud.

The body force applied to the fluid by the particle cloud is

$$\vec{F} = \hat{x} \int_{a_1}^{a_2} D(\sigma) f(\sigma)\, d\sigma = u_p \lambda_1^2 \hat{x} \tag{11.22}$$

Hence the solution of the bulk fluid equations is

$\vec{u}_f = -u_p \hat{x}$ (with respect to coordinate system fixed to particle of radius σ)

$$p_f = \hat{x} \int_{a_1}^{a_2} D(\sigma) f(\sigma)\, d\sigma = u_p \lambda_1^2 \hat{x}$$

The "screened" Stokes equation governing the disturbance of a particle of radius σ is

$$\begin{aligned} \nabla p_\sigma &= \mu \nabla^2 \vec{u}_\sigma - \lambda_1^2 \vec{u}_\sigma \\ \nabla \cdot \vec{u}_\sigma &= 0 \end{aligned} \tag{11.23}$$

The corresponding boundary conditions on $\vec{u}_\sigma$, p_σ are

$$\begin{aligned} &\text{(a)} \quad \vec{u}_\sigma = 0 \quad p_\sigma = 0 \quad r = \infty \\ &\text{(b)} \quad \vec{u} = \vec{u}_f + \vec{u}_\sigma = 0 \quad r = \sigma \end{aligned} \tag{11.24}$$

Equation (11.23) and boundary conditions (11.24) are similar to $\vec{u}_a$ and p_a in equation (11.6). By changing the appropriate parameters of equations (11.9a) one obtains

$$D(\sigma) = 6\pi\mu U_p \sigma \left(1 + \alpha_1 \sigma + \frac{\alpha_1^2 \sigma^2}{3}\right) \qquad (11.25)$$

where

$$\alpha_1^2 = \frac{\lambda_1^2}{\mu} = \frac{1}{\mu u_p} \int_{a_1}^{a_2} D(\sigma) f(\sigma) d\sigma$$

Multiply both sides of equation (11.25) by $f(\sigma) d\sigma$ and integrate over the range of σ ; then

$$\alpha_1^2 = 6\pi \left[M_1 + \alpha_1 M_2 + \frac{C \alpha_1^2}{4\pi} \right] \qquad (11.26)$$

where

$$M_1 = \int_{a_1}^{a_2} f(\sigma) \sigma d\sigma$$

$$M_2 = \int_{a_1}^{a_2} f(\sigma) \sigma^2 d\sigma$$

$$c = \int_{a_1}^{a_2} \frac{4\pi}{3} \sigma^3 f(\sigma) d\sigma \qquad (11.27)$$

Solving equation (11.26),

$$\alpha_1 = \frac{6\pi M_2 + \sqrt{36\pi^2 M_2^2 + 24\pi M_1 (1 - \frac{3}{2}c)}}{2 - 3c} \qquad (11.28)$$

Therefore, the drag formula is

$$D(\sigma) = 6\pi\mu \sigma u_p \left(1 + \alpha_1 \sigma + \frac{\alpha_1^2 \sigma^2}{3}\right) \qquad (11.29)$$

where α_1 is given by (11.26).

For the particular case where there are only two types of particles with radius a and b , then

$$f(\sigma) = n_a \delta(\sigma - a) + n_b \delta(\sigma - b)$$

From (11.21),

$$M_1 = a N_a + b N_b$$

$$M_2 = a^2 N_a + b^2 N_b$$

$$c = \frac{4\pi}{3} (a^3 N_a + b^3 N_b)$$

Hence, by (11.22),

$$\alpha_1 = \frac{6\pi(a^2N_a+b^2N_b)+\sqrt{36\pi^2(a^2N_a+b^2N_b)^2+24\pi(aN_a+bN_b)(1-\frac{3}{2}c)}}{2-3c} \tag{11.30}$$

Also,

$$\begin{aligned} D_a &= 6\pi\mu a U_p\left(1+\alpha_1 a+\frac{\alpha_1^2a^2}{3}\right) \\ D_b &= 6\pi\mu b U_p\left(1+\alpha_1 b+\frac{\alpha_1^2b^2}{3}\right) \end{aligned} \tag{11.31}$$

12. The Torque Formula and Rotational Coupling Force

Consider a particle cloud of spheres of radius a and volume concentration c suspended in a viscous incompressible fluid. Let A be one of the particles of this homogeneous suspension. If A rotates steadily with an angular velocity $\vec{\omega}$, then the disturbance produced by A is given by the solution of the "screened" Stokes equation.

$$\nabla p = \mu\nabla^2\vec{u} - \mu\alpha^2\vec{u} \tag{12.1}$$

$$\nabla\cdot\vec{u} = 0 \tag{12.2}$$

For the particle cloud under consideration, α is given by (Section 11)

$$\alpha = \frac{1}{a}\left[\frac{9c+3\sqrt{8c-3c^2}}{4-6c}\right]$$

The boundary conditions on the disturbance are

(a) $\quad p = 0 \qquad \vec{u} = 0 \qquad r = \infty$

(b) $\quad \vec{u} = \vec{\omega}\times\vec{x} \qquad r = a$

To find the disturbance produced, let

$$\begin{aligned} \vec{u} &= \operatorname{curl}[f(r)\vec{\omega}], \quad p = 0 \\ &= [\nabla f(r)]\times\vec{\omega} \end{aligned} \tag{12.3}$$

Then equation (12.2) is satisfied identically.

Substitute (12.3) into (12.1) and obtain the following equation to be satisfied by $f(r)$:

$$\nabla^2 f(r) - \alpha^2 f(r) = 0$$

or

$$\frac{d^2 f}{d r^2} + \frac{2}{r}\frac{d f}{d r} - \alpha^2 f = 0 \qquad (12.4)$$

The solution of (11.4) satisfying the boundary condition at infinity is

$$f = \frac{B e^{-\alpha r}}{r}$$

Therefore,

$$\vec{u} = B\left[\frac{1}{r^2} + \frac{\alpha}{r}\right] e^{-\alpha r}\, \vec{\omega} \times \hat{e}_r$$

From the boundary condition at $r = a$

$$B = \frac{a^3 e^{\alpha a}}{1 + \alpha a}$$

Hence,

$$\vec{u} = \frac{a^3 e^{\alpha a}}{1 + \alpha a}\left(\frac{\alpha}{r} + \frac{1}{r^2}\right) e^{-\alpha r}\, \vec{\omega} \times \hat{e}_r$$

$$= \frac{a^3 e^{\alpha a}}{1 + \alpha a}\left(\frac{\alpha}{r} + \frac{1}{r^2}\right) e^{-\alpha a} \sin\theta\, \omega\, \hat{e}_\phi \qquad (12.5)$$

where $\vec{\omega}$ has been chosen as the polar axis.

From equation (12.5) the stress component on the surface of the sphere is

$$\sigma_{r\phi} = \mu\left(\frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}\right)_{r=a}$$

$$= -\frac{\mu}{1 + \alpha a}\left(3 + 3\alpha a + \alpha^2 a^2\right) \sin\theta\, \Omega$$

Total moment acting on the sphere is

$$T = \int_{sphere} \sigma_{r\phi} \, a \sin\theta \, ds$$

Hence

$$\text{torque} = \vec{T} = -8\pi\mu a^3 \left(1 + \frac{\alpha^2 a^2}{3(1+\alpha a)}\right)\vec{\omega} \tag{12.6}$$

Equation (12.6) gives the torque a spinning sphere will experience in an otherwise stationary particle suspension. For very dilute suspension, $\alpha \to 0$, (12.6) reduces to the well-known Kirchhoff formula

$$\vec{T} = -8\pi\mu a^3 \vec{\omega}$$

Combining equations (12.5) and (12.6), if $\vec{T}$ is the torque applied on the fluid by a spinning sphere, the velocity field is given by

$$\vec{U} = \frac{3e^{\alpha a}}{8\pi\mu} \frac{1}{3 + 3\alpha a + \alpha^2 a^2}\left(\frac{\alpha}{r^2} + \frac{1}{r^3}\right) e^{-\alpha r} \vec{T} \times \vec{x} \tag{12.7}$$

Now, in equation (12.7), keep $\vec{T}$ fixed and let the sphere go to zero, i.e., $a \to 0$. Then the fundamental solution of the "screened" Stokes equation corresponding to a point torque $\vec{T}$ applied at the origin in obtained.

$$\vec{u} = \frac{1}{8\pi\mu}\left(\frac{\alpha}{r^2} + \frac{1}{r^3}\right) e^{-\alpha r} \vec{T} \times \vec{x} \tag{12.8}$$

Rotational Coupling Force

Consider now a body force $\vec{f}(\vec{x})$ applied to the fluid in the homogeneous suspension. Then the "screened" Stokes equation becomes

$$\frac{1}{\mu}\nabla p = \nabla^2 \vec{u} - \alpha^2 \vec{u} + \frac{1}{\mu}\vec{f}(\vec{x}) \tag{12.9}$$

By means of Appendix II A, the fluid velocity induced by $\vec{f}(\vec{x})$ is

$$\vec{u}_f = \frac{1}{4\pi\mu}\int \frac{e^{-\alpha|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}\vec{f}(\vec{x}')\, d^3x' \tag{12.10}$$

Now consider in the homogeneous suspension that there is a spatial distribution of moment $\vec{T}(\vec{x})$. Each torque element will induce fluid motion in the suspension. The overall fluid flow field due to this spatial distribution of torque can be obtained by means of the fundamental solution given by equation (12. 8). Thus,

$$\vec{U}_T = \frac{1}{8\pi\mu}\int\left(\frac{\alpha}{|\vec{x}-\vec{x}'|^2}+\frac{1}{|\vec{x}-\vec{x}'|^3}\right)e^{-\alpha|\vec{x}-\vec{x}'|}\,\vec{T}(\vec{x}')\times(\vec{x}-\vec{x}')\,d^3x'$$

$$= \frac{1}{8\pi\mu}\int -\nabla'\left(\frac{e^{-\alpha|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}\right)\times\vec{T}(\vec{x}')\,d^3x'$$

$$= \frac{1}{8\pi\mu}\int\frac{e^{-\alpha|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}\nabla'\times\vec{T}(\vec{x}')\,d^3x' - \int\nabla'\times\left(\frac{e^{-\alpha|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}\vec{T}(\vec{x}')\right)d^3x'$$

The last integral can be converted into a surface integral of

$$\frac{e^{-\alpha|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}\,\hat{n}\times\vec{T}$$

which vanishes if $\vec{T}$ is assumed localized. Therefore,

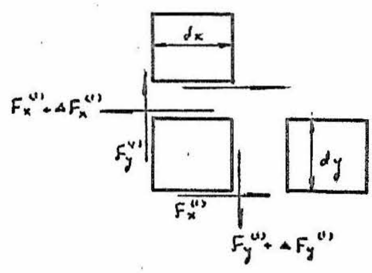
$$\vec{U}_T = \frac{1}{4\pi\mu}\int\frac{e^{-\alpha|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}\left(\frac{1}{2}\nabla'\times\vec{T}(\vec{x}')\right)d^3x' \tag{12. 11}$$

On comparing equations (12. 10) and (12. 11), it is clear that a spatial distribution of moments is equivalent to a body force. This will be referred to as the rotational coupling force, given by

$$\vec{F}(\vec{x}) = \frac{1}{2}\nabla\times\vec{T}(\vec{x}) \tag{12. 12}$$

A cloud of spinning particles exerts a spatial distribution of torque on the fluid. By equation (12. 12), the rotational coupling force can be calculated. Equation (12. 12) has no analogue in the single particle motion considered in Part II. This rotational coupling force arises solely as a result of collective behavior of the particles.

Equation (12. 12) above was derived entirely from a mathematical consideration. However, the same formula may be obtained by a more physical argument. Neglect the screening effect of particles and consider a torque field being applied on a fluid in a domain V . Divide V into many small elements $dv = dx\,dy\,dz$. In general, a point torque can be replaced by a pair of equal but opposite forces acting at an infinitesimal distance apart.



Consider the fluid element shown. If T_z , the z - component of the torque field varies from element to element. F_x and F_y , the force components, too, must vary from element to element. The relation of this variation is given by

$$d T_z = \Delta F_x^{(1)} dy - \Delta F_y^{(1)} dx \tag{12.13}$$

Similarly, by considering the other components of $\vec{T}$, one obtains

$$d T_x = \Delta F_y^{(2)} dz - \Delta F_z^{(2)} dy$$

$$d T_y = \Delta F_z^{(3)} dx - \Delta F_x^{(3)} dz$$

or

$$\Delta F_x^{(1)} = \frac{\partial T_z}{\partial y} \qquad \Delta F_x^{(3)} = -\frac{\partial T_y}{\partial z}$$

$$\Delta F_y^{(2)} = \frac{\partial T_x}{\partial z} \qquad \Delta F_y^{(1)} = -\frac{\partial T_z}{\partial x}$$

$$\Delta F_z^{(3)} = \frac{\partial T_y}{\partial x} \qquad \Delta F_z^{(2)} = -\frac{\partial T_x}{\partial y}$$

Now there is a net body force acting on the element dv. The component of this force is equal to

$$\frac{1}{2}\left(\Delta F_x^{(1)} + \Delta F_x^{(2)}\right) = \frac{1}{2}\left(\frac{\partial T_z}{\partial y} - \frac{\partial T_y}{\partial z}\right)$$

Similarly, the y and z components are

$$\frac{1}{2}\left(\frac{\partial T_x}{\partial z} - \frac{\partial T_z}{\partial x}\right) \quad \text{and} \quad \frac{1}{2}\left(\frac{\partial T_y}{\partial x} - \frac{\partial T_x}{\partial y}\right)$$

or, in general,

$$\vec{F} = \frac{1}{2}\,\mathrm{curl}\,\vec{T}$$

as equation (12.12).

The factor $1/2$ comes from the fact that $\Delta F_x^{(1)}$ etc. varies continuously. Such a factor should be included when replaced by a completely localized point force.

<u>Example.</u>

Consider a fluid particle system. Assume the particles are spheres of radius a and are somehow fixed in space against translation and rotation. For the purpose of this example, it will be assumed that the drag and torque on a sphere of the particle cloud are given approximately by

$$\begin{aligned} \vec{D} &\simeq 6\pi\mu a \vec{u}_f \\ \vec{T} &\simeq 8\pi\mu a^3 \left(\tfrac{1}{2}\,\mathrm{curl}\,\vec{u}_f\right) \end{aligned} \qquad (12.14)$$

Essentially, in (12.14), the mutual interference effect of the particles has been neglected.

Therefore, with respect to the bulk fluid, the particle cloud exerts:

a drag force $= -6\pi\mu a n \vec{u}_f$

a rotational coupling force $= -2\pi\mu a^3 n \nabla \times (\nabla \times \vec{u}_f)$

$$= -\frac{3c}{2}\mu \text{ curl curl } \vec{u}_f$$

Hence,

$$\vec{F}_f = -6\pi\mu a n \vec{u}_f - \frac{3}{2} c\mu \text{ curl curl } \vec{u}_f \qquad (12.15).$$

Substitute (12. 15) into the "bulk fluid" equations of Section 9; then, for steady flow and homogeneous particle cloud (n = constant)

$$\nabla \cdot \vec{u}_f = 0 \qquad (12.16)$$

$$\vec{u}_f \cdot \nabla \vec{u}_f = -\nabla p_f + \mu_f \nabla^2 \vec{u}_f + \vec{F}_f \qquad (12.17)$$

Equation (12. 17) becomes

$$\vec{u}_f \cdot \nabla \vec{u}_f = -\nabla p_f + \mu_f \nabla^2 \vec{u}_f - 6\pi\mu a n \vec{u}_f - \frac{3}{2} \nabla \times \nabla \times \vec{u}_f \, c\mu \qquad (12.18)$$

But

$$\text{Curl curl } \vec{u}_f = \nabla(\nabla \cdot \vec{u}_f) - \nabla^2 \vec{u}_f = -\nabla^2 \vec{u}_f \qquad (12.19)$$

by means of equation (12. 16).

Substitute (12. 19) into (12. 17):

$$\vec{u}_f \cdot \nabla \vec{u}_f = -\nabla p_f + (\mu_f + \frac{3}{2} c\mu) \nabla^2 \vec{u}_f - 6\pi\mu a n \vec{u}_f \qquad (12.20)$$

From (12. 20), it is clear that the effective viscosity of the bulk fluid is now equal to

$$\mu^* = \mu_f + \frac{3}{2} c\mu$$

However, if the particles are fixed but free to rotate, then $\vec{T} = 0$ and hence the rotational coupling force is zero, and $\mu^* = \mu_f$.

The physical interpretation of this apparent increase in viscosity is rather obvious. The fluid will find it harder to flow through a particle cloud if the particles are fixed against rotation than if they are free to turn. Particles locked against rotation generate additional disturbances in the fluid and hinder the general fluid flow. At the same time, they increase the dissipation rate. Macroscopically, this is represented by an apparent increase in the bulk fluid viscosity which equals $1.5\, c\, \mu$ as a zeroth order approximation.

13. Viscosity Coefficient of Bulk Fluid

In Section 3, the bulk fluid stress-strain rate relation was postulated in analogy to the well-known Navier-Stokes relation to be

$$\sigma_{f\,ij} = -p_f\, \delta_{ij} + \mu_f \left(\frac{\partial u_{fi}}{\partial x_j} + \frac{\partial u_{fj}}{\partial x_i} \right) \tag{13.1}$$

Substituting relation (13. 1) into the bulk fluid momentum and continuity equation, for a homogeneous fluid-particle system,

$$\nabla \cdot \vec{u}_f = 0 \tag{13.2}$$

$$\rho_f\, \vec{u}_f \cdot \nabla \vec{u}_f = -\nabla p_f + \mu_f\, \nabla^2 \vec{u}_f + \vec{F}_f \tag{13.3}$$

The bulk fluid viscosity coefficient μ_f in the above equations is still unknown and is the subject of investigation in this section.

The subject "viscosity of a suspension" has been the goal of many researches ever since the work of Einstein[7] in 1906. In rheology as well as in physical chemistry, a good deal of work has been published since that time. However, the so-called "viscosity of a suspension" has never been clearly defined, and hence its relation to the description of the flow field remains rather obscure. To avoid confusion and ambiguity, the term "bulk fluid viscosity" shall be used throughout this section to designate a material constant of the bulk fluid which characterizes the mechanical behavior of the bulk fluid by relation (13.1). Thus, μ_f is a quantity which has a meaning only in the macroscopic sense, as described in Sections 9 and 10.

Review on "Viscosity of a Suspension"

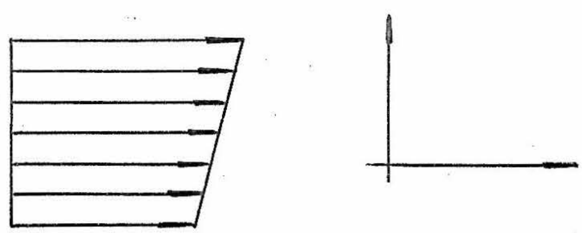
The hydrodynamical treatment of viscosity of a suspension, as initiated by Einstein[7], has been extended by many research workers since 1906. Among them, Jeffery[40] was the first to apply the same technique to ellipsoidal particles. Later, in 1932, Taylor[41] employed an essentially similar argument to deal with the problem of the viscosity of a fluid containing small drops of another liquid. Since then, particles of various shapes have been treated by other authors, yet the mathematical treatment and physical idea are essentially that of Einstein. This so-called "method of Stokes-Einstein" has been outlined in great detail by Sadron[42] and Frisch and Simha[43]. In 1938, Burgers took a substantially different approach to this problem (Burgers' method is also explained in references 42 and 43). To a first approximation in c, the volume concentration of solid particles, the above two methods do agree. For suspensions of relatively high concentration, many formulas were proposed, e.g. references 45 and 46. However, the methods used were invariably based on small perturbations of some basic hydrodynamic flow field. The resulting viscosity formula is generally expressed in a power series of c and is seldom given beyond the term involving c^2. Also, the coefficients of the various terms, as proposed by various authors, do not agree. Indeed, even the very basic physical mechanism of dissipation involved is still a matter lacking general agreement and hence making "viscosity of a suspension" very much a subject of present day research.

Viscosity of Bulk Fluid

Consider a fluid-particle system in which the particles are spheres of radius a . Assume that the inertia of the particles is very small so that the particles always translate and rotate with the same linear and angular velocity of the bulk fluid. Thus, the fluid-particle system is essentially homogeneous. For such a system, the gross particle-fluid interaction force, $\vec{F}_f$, is equal to zero. Hence the bulk fluid equations of (13.2) and (13.3) reduce to

$$\nabla \cdot \vec{u}_f = 0 \qquad (13.4)$$

$$\rho_f \, \vec{u}_f \cdot \nabla \vec{u}_f = - \nabla p_f + \mu_f \nabla^2 \vec{u}_f \qquad (13.5)$$



Consider the case of simple parallel shear flow. This is given by

$$p_f = 0$$

$$\vec{u}_f = \beta y \hat{x}$$ (with respect to a coordinate system fixed to a sphere) (13.6)

which is a solution of equations (13.4) and (13.5). From equation (13.6), the angular velocity of the bulk fluid is

$$\vec{\omega}_f = \frac{1}{2} \operatorname{curl} \vec{u}_f = - \frac{\beta}{2} \hat{z}$$

By assumption, this is also the angular velocity of the particles.

$$\vec{\omega}_p = -\frac{\beta}{\sqrt{2}}\,\hat{z} \tag{13.7}$$

Let A be a particle of the two-phase system. Then, near sphere A, let $\vec{u}_a$ and p_a be the disturbance produced by A on the gross fluid field. As in Section 4, the actual fluid velocity and pressure in the vicinity of A are given by

$$\begin{aligned} \vec{u} &= \vec{u}_f + \vec{u}_a \\ p &= p_f + p_a \end{aligned} \tag{13.8}$$

Now $\vec{u}_a$ and p_a are described by the "screened" Stokes equation of Section 11:

$$\begin{aligned} \nabla p_a &= \mu\,\nabla^2\vec{u}_a - \mu\alpha^2\vec{u}_a \\ \nabla\cdot\vec{u}_a &= 0 \end{aligned} \tag{13.9}$$

where

$$\alpha = \frac{1}{a}\left[\frac{9c + 3\sqrt{8c - 3c^2}}{4 - 6c}\right] \tag{13.10}$$

The boundary conditions on the disturbance are

$$\vec{u}_a = 0 \qquad p_a = 0 \qquad r = \infty \tag{13.11}$$

$$\vec{u} = \vec{u}_a + \vec{u}_f = \vec{\omega}_p \times \vec{r} \qquad r = a \tag{13.12}$$

Using (13.6) and (13.7), equation (13.12) becomes

$r = a$

$$\begin{aligned} \vec{u}_a = \hat{e}_r\left[-\beta a \sin\theta\cos\theta\cos\phi\right] &+ \hat{e}_\theta\left[\beta a\left(\sin^2\theta\cos\phi - \tfrac{1}{2}\cos\phi\right)\right] \\ &+ \hat{e}_\phi\left[\frac{\beta a}{\sqrt{2}}\cos\theta\sin\phi\right] \end{aligned} \tag{13.13}$$

or

$$\vec{u}_a = \frac{\beta a}{3} P_1'(\cos\theta)\cos\phi\, \hat{e}_r + \frac{\beta a}{6}\frac{\partial}{\partial\theta}\left[P_2'(\cos\theta)\right]\cos\phi\, \hat{e}_\theta - \frac{\beta a}{6\sin\theta} P_2'(\cos\theta)\sin\phi\, \hat{e}_\phi$$

To solve for $\vec{u}_a$, p_a, the general solution provided in Appendix D may be used. Therefore, let

$$\begin{aligned}\vec{u}_a = {} & \hat{e}_r\left\{B\left[\frac{6K_{5/2}(\alpha r)}{r^{3/2}}P_2'(\cos\theta)\cos\phi\right] - A\left[\frac{3}{r^4}P_2'(\cos\theta)\cos\phi\right]\right\} \\ & + \hat{e}_\theta\left\{B\left[\frac{1}{r}\frac{d}{dr}\left(\sqrt{r}\,K_{5/2}(\alpha r)\right)\frac{\partial}{\partial\theta}P_2'(\cos\theta)\cos\phi\right] + A\left[\frac{1}{r^4}\frac{\partial}{\partial\theta}P_2'(\cos\theta)\cos\phi\right]\right\} \\ & + \hat{e}_\phi\left\{-B\left[\frac{1}{r\sin\theta}\frac{d}{dr}\left(\sqrt{r}\,K_{5/2}(\alpha r)\right)P_2'(\cos\theta)\sin\phi\right] - A\left[\frac{1}{r^4\sin\theta}P_2'(\cos\theta)\sin\phi\right]\right\}\end{aligned} \tag{13.14}$$

Impose boundary condition (13.13)

$$\begin{aligned} B &= \frac{5\beta a^{5/2}}{9\left[5K_{5/2}(\alpha a) - \alpha a\left(K_{3/2}(\alpha a) + K_{7/2}(\alpha a)\right)\right]} \\ A &= \frac{a^4}{3}\left[\frac{6K_{5/2}(\alpha a)\,B}{a^{3/2}} - \frac{\beta a}{3}\right]\end{aligned} \tag{13.15}$$

In polar coordinates, the stress components are

$$\sigma_{rr} = 2\mu\frac{\partial u_r}{\partial r}$$

$$\sigma_{\theta\theta} = 2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}\right)$$

$$\sigma_{\phi\phi} = 2\mu\left(\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} + \frac{u_r}{r} + \frac{u_\theta\cot\theta}{r}\right)$$

$$\sigma_{r\theta} = \mu\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$$

$$\sigma_{\theta\phi} = \mu \left(\frac{1}{r\sin\theta} \frac{\partial u_\theta}{\partial\phi} + \frac{1}{r} \frac{\partial u_\phi}{\partial\theta} - \frac{u_\phi \cot\theta}{r} \right)$$

$$\sigma_{\phi r} = \mu \left(\frac{\partial u_\phi}{\partial r} + \frac{1}{r\sin\theta} \frac{\partial u_r}{\partial\phi} - \frac{u_\phi}{r} \right) \tag{13.16}$$

From equations (13.6), (13.14), (13.15), and (13.16), the stress components of bulk fluid and disturbance are

$$\sigma_{rr}^{f} = 2\mu_f \beta \sin\theta \cos\theta \cos\phi$$

$$\sigma_{\theta\theta}^{f} = -2\mu_f \beta \sin\theta \cos\theta \cos\phi$$

$$\sigma_{\phi\phi}^{f} = 0$$

$$\sigma_{r\theta}^{f} = \mu_f \beta (\cos^2\theta - \sin^2\theta) \cos\phi$$

$$\sigma_{\theta\phi}^{f} = \mu_f \beta \sin\theta \sin\phi$$

$$\sigma_{\phi r}^{f} = -\mu_f \beta \cos\theta \sin\phi$$

with $\eta = \frac{r}{a}$ $\quad \gamma = -\frac{5}{18}\sqrt{\frac{2\alpha a}{\pi}}\, e^{\alpha a}\left(\frac{1}{1+\alpha a}\right)$

$$\sigma = -\frac{\alpha a}{9(1+\alpha a)} \left[1 + \frac{6}{\alpha a} + \frac{15}{(\alpha a)^2} + \frac{15}{(\alpha a)^3} \right]$$

$$\sigma_{rr}^{a} = 2\mu \left\{ 18\gamma\beta\sqrt{\frac{\pi}{2\alpha a}}\, e^{-\alpha a\eta} \left[\frac{\alpha a}{\eta^2} + \frac{5}{\eta^3} + \frac{12}{\alpha a\eta^4} + \frac{12}{(\alpha a)^2\eta^5} \right] - \frac{36\sigma\beta}{\eta^5} \right\} \sin\theta\cos\theta\cos\phi$$

$$\sigma_{\theta\theta}^{a} = 2\mu \left\{ -\gamma\beta\sqrt{\frac{\pi}{2\alpha a}}\, e^{-\alpha a\eta} \left[\frac{12\alpha a}{\eta^2} + \frac{54}{\eta^3} + \frac{126}{\alpha a\eta^4} + \frac{126}{(\alpha a)^2\eta^5} \right] + \frac{21\sigma\beta}{\eta^5} \right\} \sin\theta\cos\theta\cos\phi$$

$$\sigma_{\phi\phi}^{a} = 2\mu \left\{ -\gamma\beta\sqrt{\frac{\pi}{2\alpha a}}\, e^{-\alpha a\eta} \left[\frac{6\alpha a}{\eta^2} + \frac{36}{\eta^3} + \frac{90}{\alpha a\eta^4} + \frac{90}{(\alpha a)^2\eta^5} \right] + \frac{15\sigma\beta}{\eta^5} \right\} \sin\theta\cos\theta\cos\phi$$

$$\sigma_{r\theta}^{a} = \mu \left\{ -\gamma\beta\sqrt{\frac{\pi}{2\alpha a}}\, e^{-\alpha a\eta} \left[\frac{3(\alpha a)^2}{\eta} + \frac{15(\alpha a)}{\eta^2} + \frac{63}{\eta^3} + \frac{144}{\alpha a\eta^4} + \frac{144}{(\alpha a)^2\eta^5} \right] + \frac{240\sigma\beta}{\eta^5} \right\} \cos 2\theta \cos\phi$$

$$\sigma_{\theta\phi}^{a} = \mu \left\{ 6\gamma\beta\sqrt{\frac{\pi}{2\alpha a}}\, e^{-\alpha a\eta} \left[\frac{\alpha a}{\eta^2} + \frac{3}{\eta^3} + \frac{6}{\alpha a\eta^4} + \frac{6}{(\alpha a)^2\eta^5} \right] - \frac{6\sigma\beta}{\eta^5} \right\} \sin\theta\sin\phi$$

$$\sigma_{r\phi}^{a} = \mu \left\{ \gamma\beta\sqrt{\frac{\pi}{2\alpha a}}\, e^{-\alpha a\eta} \left[\frac{3(\alpha a)^2}{\eta} + \frac{15\alpha a}{\eta^2} + \frac{63}{\eta^3} + \frac{144}{\alpha a\eta^4} + \frac{144}{(\alpha a)^2\eta^5} \right] - \frac{240\sigma\beta}{\eta^5} \right\} \times \tag{13.17}$$

$$\times \cos\theta \sin\phi$$

where superscript f and a denote that of the bulk fluid and disturbance, respectively.

The dissipation function in spherical coordinates is

$$\Phi = \frac{1}{2\mu}\left[\sigma_{rr}^2 + \sigma_{\theta\theta}^2 + \sigma_{\phi\phi}^2 + 2\sigma_{r\theta}^2 + 2\sigma_{\theta\phi}^2 + 2\sigma_{\phi r}^2\right] \tag{13.18}$$

The dissipation function due to the presence of particle A is

$$\Phi_A = \frac{1}{2\mu}\Big[(\sigma_{rr}^f + \sigma_{rr}^a)^2 + (\sigma_{\theta\theta}^f + \sigma_{\theta\theta}^a)^2 + (\sigma_{\phi\phi}^f + \sigma_{\phi\phi}^a)^2 + 2(\sigma_{r\theta}^f + \sigma_{r\theta}^a)^2$$
$$+ 2(\sigma_{\theta\phi}^f + \sigma_{\theta\phi}^a)^2 + 2(\sigma_{\phi r}^f + \sigma_{\phi r}^a)^2 - (\sigma_{rr}^f)^2 - (\sigma_{\theta\theta}^f)^2 - (\sigma_{\phi\phi}^f)^2$$
$$- 2(\sigma_{r\theta}^f)^2 - 2(\sigma_{\theta\phi}^f)^2 - 2(\sigma_{\phi r}^f)^2\Big]$$

Total dissipation due to the presence of particle A

$$= \int_{V_0} \Phi_A \, dv \qquad (V_0 = \text{volume outside sphere A})$$

$$= \frac{1}{2\pi}\iint_a^\infty \Big[(\sigma_{rr}^a)^2 + (\sigma_{\theta\theta}^a)^2 + (\sigma_{\phi\phi}^a)^2 + 2(\sigma_{r\theta}^a)^2 + 2(\sigma_{\phi r}^a)^2$$
$$+ 2\sigma_{rr}^f\sigma_{rr}^a + 2\sigma_{\theta\theta}^f\sigma_{\theta\theta}^a + 2\sigma_{\phi\phi}^f\sigma_{\phi\phi}^a + 4\sigma_{r\theta}^f\sigma_{r\theta}^a$$
$$+ 4\sigma_{\theta\phi}^f\sigma_{\theta\phi}^a + 4\sigma_{\phi r}^f\sigma_{\phi r}^a\Big]\, r^2 dr\, d\Omega$$

using (13.17),

$$\int_{V_0} \Phi_A \, dv = \frac{\pi\mu\beta^2 a^3}{9(1+\alpha a)^2}\left[5(\alpha a)^3 + 18(\alpha a)^2 + 36(\alpha a) + 18\right] + \frac{8\pi\mu a^3\beta^2}{3} \tag{13.19}$$

Since near sphere A actual field velocity is given by

$$\vec{U} = \vec{U}_f + \vec{U}_A$$

Dissipation per unit volume due to $\vec{U}_f$ part of $\vec{U}$

$$\Phi = \frac{1}{2\mu}(\sigma_{ij})^2(1-c) = \mu\beta^2(1-c)$$

actual dissipation rate per unit volume of bulk fluid

$$= \mu \beta^2 (1-c) + \frac{8\pi \mu a^3 n \beta^2}{3} + \frac{\pi \mu a^3 \beta^2 n}{9(1+\alpha a)^2}\left[18 + 36\alpha a + 18(\alpha a)^2 + 5(\alpha a)^3\right]$$

$$= \mu \beta^2 \left[1 + c\left(2.5 + \frac{5(\alpha a)^3}{12(1+\alpha a)^2}\right)\right] \qquad (13.20)$$

However, the actual dissipation rate can also be calculated by just considering the bulk fluid.

$$\Phi_f = \mu_f \beta^2 \qquad \text{per unit volume} \qquad (13.21)$$

Equate (13.21) and (13.20)

$$\frac{\mu_f}{\mu} = 1 + c\left[2.5 + \frac{5(\alpha a)^3}{12(1+\alpha a)^2}\right]$$

But

$$\alpha a = \epsilon = \frac{9c + 3\sqrt{8c - 3c^2}}{4 - 6c}$$

Therefore,

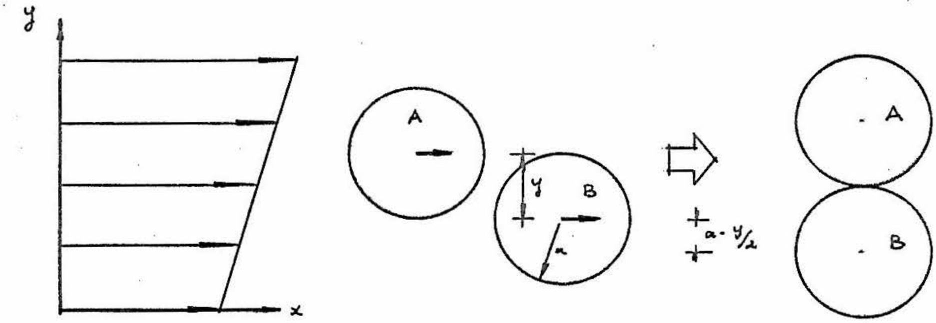
$$\left(\frac{\mu_f}{\mu}\right)_{ideal} = 1 + c\left(2.5 + \frac{5\epsilon^3}{12(1+\epsilon)^2}\right) \qquad (13.22)$$

In the above analysis, the particles and fluid elements are assumed to be in a state where both relative angular and translational velocities are zero. This corresponds to the condition of minimum deviation from dynamic equilibrium and hence minimum dissipation. If Brownian motion and possible particle collision, etc., are taken into consideration, the rate of dissipation would then be higher, resulting in a higher μ_f. Thus, the expression obtained above (13.22) can be regarded as the ideal viscosity of the bulk fluid corresponding to the lowest value possible. It should be pointed out here, however, that the inclusion of Brownian motion, etc., should be done in such a way that the bulk fluid remains homogeneous and isotropic, for otherwise the concept of viscosity of bulk fluid would be rendered meaningless. In actual experiments, the ideal state of zero relative angular and translational velocities can never be achieved, due to particle collisions, wall effect, etc. As a consequence, experimental data of μ_f should be higher than that of the ideal value.

The Effect of Collision

In the above analysis, the particles are assumed to rotate and translate with the same velocity of the fluid. However, for particles of finite size, such a state cannot be attained. Owing to the difference in velocities between adjacent layers of fluid, and therefore the particles, collisions are inevitable. The direct result of these collisions is to make the particle-fluid system deviate from its ideal state of minimum dissipation, giving rise to an increase in the value of μ_f .

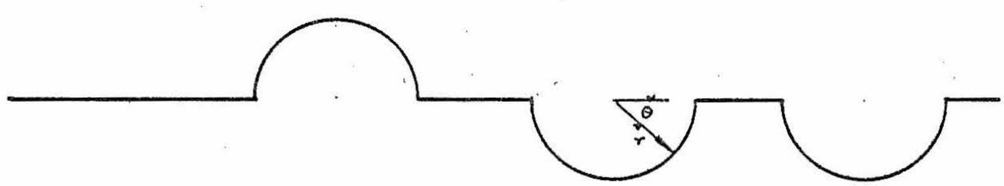
To account for the increase in μ_f due to collision, a detailed analysis of the process of particle-particle encounters is necessary, but for sufficiently viscous fluid and light particles, it is observed in many experiments that particles do not actually collide but just slip past by one another. In the following, this slip-by model will be adopted. To carry out the analysis, it is necessary, first, to find out the average deviation of a particle from its dynamic equilibrium state with respect to the fluid at which relative rotational and translational velocities are both zero.



Slip-by Model

To reduce the complexity of the collision process, the following simplifying assumptions will be made:

(a) the displaced path of a sphere during an encounter will be taken as a semi-circle;

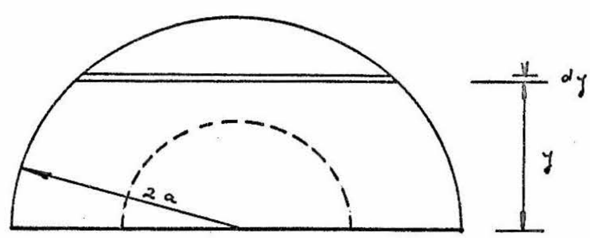


(b) during a collision, a sphere will be displaced sideways in the z - direction as well as vertically in the y - direction. A sphere moving sideways, however, will have a higher probability of collision. On the average, it will be assumed that, for the purpose of calculating the mean displacement of a sphere from its dynamic equilibrium position due to collision between adjacent layers of spheres, one may consider the spheres of each layer as long cylinders. However, the collision cross-section for each sphere will be taken as a circle of radius $2a$.

By assumption (a), the average displacement over the displaced path is

$$= \frac{\int r \sin\theta \, dl}{\int dl} = \frac{\int_0^\pi r^2 \sin\theta \, d\theta}{\int_0^\pi r \, d\theta} = \frac{2r}{\pi} \qquad (13.23)$$

Consider a particle moving in a straight path. Let X be the probability of a collision. If the particle moves in some curved path, it will spend more time per unit distance in the x - direction, and hence the probability of collision increases. The relative ratio must be proportional to the time the particle will take to traverse unit distance in the x - direction. Over a path in the form of a semicircle, a particle will take $\frac{\pi r}{U}$ seconds to travel a distance which will otherwise take $\frac{2r}{U}$ seconds. This gives a ratio of $\frac{\pi}{2}$. On combining this with equation (12.23), it is clear that the average displacement due to a collision can be computed from the maximum displacement over the collision path if the time spent in collision is calculated on the basis that the particle is travelling in a straight line.



Collision Cross-section

By assumption (b), a particle will suffer a maximum displacement over the collision path equal to $x = a - y/2$ due to a collision with impact parameter y. The probability of collision with impact parameter y is proportional to the volumetric rate of flow through the strip dy relative to the particle, i.e.,

$$P_r(y)\,dy \propto y\sqrt{2a^2-y^2}\,dy$$

or, in terms of x,

$$P_r(x)\,dx = A(a-x)\sqrt{2ax-x^2}\,dx \qquad 0 < x < a$$

normalizing

$$P_r(x)\,dx = \frac{3(a-x)}{a^3}\sqrt{2ax-x^2}\,dx$$

From this it follows that the mean displacement is

$$\bar{x} = \int_0^a x\,P_r(x)\,dx = \left(1-\frac{3\pi}{16}\right)a \qquad (13.24)$$

A particle which is displaced a distance $\bar{x}$ from its dynamic equilibrium position will experience a slip velocity $\beta\bar{x}$ with respect to the fluid and, by formula (11.16), a drag force of magnitude

$$F = \frac{\mu\beta\bar{x}}{n a^2}\left[\frac{9c+3\sqrt{8c-3c^2}}{4-6c}\right]^2(1-c) \qquad (13.25)$$

This gives rise to a rate of dissipation of

$$\text{force} \times \text{slip velocity} = \frac{\mu\beta^2\bar{x}^2}{n a^2}\left[\frac{9c+3\sqrt{8c-3c^2}}{4-6c}\right]^2(1-c)$$

On the other hand, a particle travelling in a straight line will spend a certain fraction of its time in collision with a particle moving through the upper half of its collision cross section. This fraction is equal to the volumetric ratio of particle to fluid flowing through it, or

c for a homogeneous particle-fluid system. After accounting for collision with particles above and below, the extra dissipation rate per unit volume of bulk fluid is

$$\Phi_{coll.} = \mu \beta^2 \, 2\left(1-\frac{3\pi}{16}\right)\left[\frac{9c+3\sqrt{8c-3c^2}}{4-6c}\right]^2 (1-c)\, c \tag{13.26}$$

Adding this dissipation rate to equation (13.20), one obtains

$$\frac{\mu_f}{\mu} = 1 + \left(2.5 + \frac{5\,\epsilon^3}{12\, r(1+\epsilon)^2}\right)c + 2\left(1-\frac{3\pi}{16}\right)^2 c\,(1-c)\,\epsilon^2 \tag{13.27}$$

where

$$\epsilon = \left[\frac{9c+3\sqrt{8c-3c^2}}{4-6c}\right]$$

14. <u>Momentum Exchange Between Two Streams of Particles of Different Size Moving Relative to One Another in a Viscous Fluid</u>

The purpose of this section is to give a very elementary account of the dynamic viscous interactions when two different streams of particles flow past each other in a viscous fluid in slow motion. As the goal is qualitative in nature and serves only to lay down the fundamental concepts of the phenomenon, an attempt will not be made to give a full and detailed quantitative analysis of this complicated process.

<u>Viscous Collision</u>

Consider two spherical particles A and B with radii a and b moving with velocities U_A and U_B, respectively in a viscous fluid as shown. To a first approximation, assume the drag on the particles to be in the x - direction. This is a fairly good

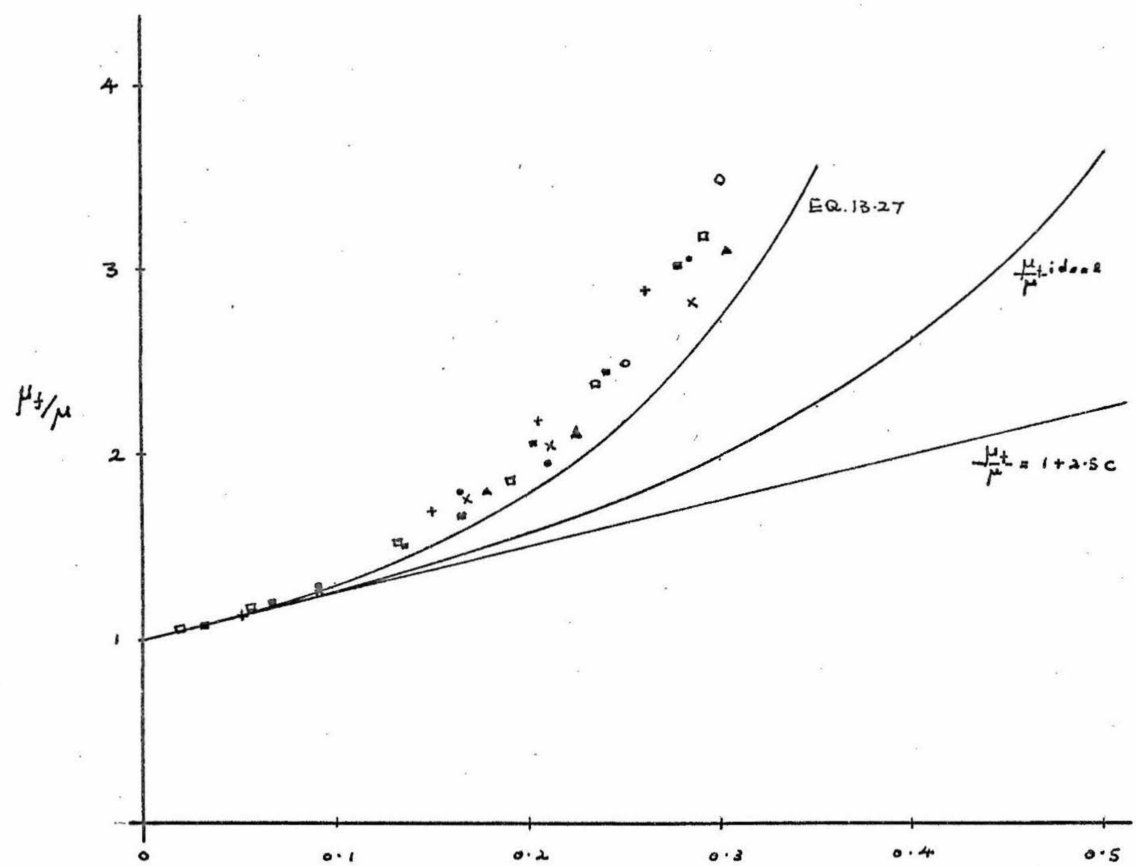
+ } Ward and Whitmore (1950, Brit. J. App. Phys. 1, 286)
o }

Robinson (1949, J. Phy. Colloid Chem. 53, 1042)

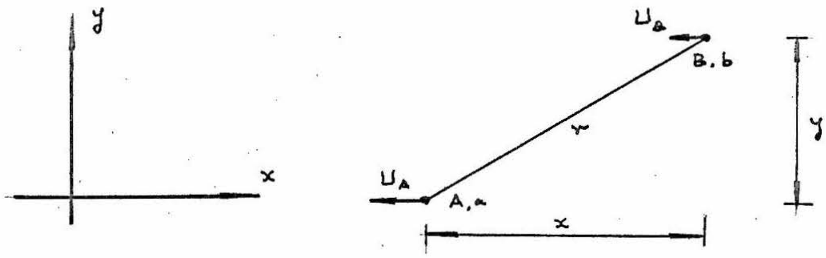
- • glass spheres in S. A. E. No. 30 motor oil
- × glass spheres in S. A. E. No. 50 motor oil
- ▲ glass spheres in castor oil

Vand (1948, J. Phy. Colloid Chem. 52, 277)

- ◘ glass spheres with No. 3 Ostwald viscosimeter
- ▪ glass spheres with No. 4 Ostwald viscosimeter



μ_t/μ vs. C DIAGRAM



approximation to the actual case of two streams of particles moving in the x - direction, for in this case, on the average, the drag in the y - direction is zero.

By means of the "point force" approximation of Part II, the drag experienced by particles A and B can be found.

For particle A

$$V_{ext} = U_A + \frac{D_B}{8\pi\mu}\left[\frac{x^2}{r^3} + \frac{1}{r}\right]$$

For particle B

$$V_{ext.} = U_B + \frac{D_A}{8\pi\mu}\left[\frac{x^2}{r^3} + \frac{1}{r}\right]$$

Hence

$$D_A = -6\pi\mu a U_A - \frac{3 D_B a}{4}\left[\frac{x^2}{r^3} + \frac{1}{r}\right] \tag{14.1}$$

$$D_B = -6\pi\mu b U_B - \frac{3 D_A b}{4}\left[\frac{x^2}{r^3} + \frac{1}{r}\right] \tag{14.2}$$

Solving (14.1) and (14.2),

$$D_A = \frac{-6\pi\mu a U_A + \frac{9}{2} a b \pi\mu U_B\left(\frac{x^2}{r^3} + \frac{1}{r}\right)}{1 - \frac{9ab}{16}\left[\frac{x^2}{r^3} + \frac{1}{r}\right]^2} \tag{14.3}$$

$$D_B = \frac{-6\pi\mu b U_B + \frac{9}{2} a b \pi \mu U_A \left(\frac{X^2}{Y^3} + \frac{1}{Y}\right)}{1 - \frac{9ab}{16}\left[\frac{X^2}{Y^3} + \frac{1}{Y}\right]^2} \tag{14.4}$$

Equations (14.3) and (14.4) can be rewritten as

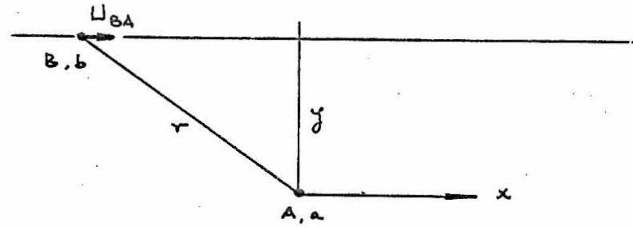
$$D_A = \frac{-6\pi\mu a U_A + \frac{9}{2}\pi\mu a b U_A \left(\frac{X^2}{Y^3} + \frac{1}{Y}\right)}{1 - \frac{9ab}{16}\left(\frac{X^2}{Y^3} + \frac{1}{Y}\right)^2} + \frac{\frac{9}{2}\pi\mu a b (U_B - U_A)\left(\frac{X^2}{Y^3} + \frac{1}{Y}\right)}{1 - \frac{9ab}{16}\left(\frac{X^2}{Y^3} + \frac{1}{Y}\right)^2} \tag{14.5}$$

$$D_B = \frac{-6\pi\mu b U_B + \frac{9}{2}\pi\mu a b U_B \left(\frac{X^2}{Y^3} + \frac{1}{Y}\right)}{1 - \frac{9ab}{16}\left(\frac{X^2}{Y^3} + \frac{1}{Y}\right)^2} + \frac{\frac{9}{2}\pi\mu a b (U_A - U_B)\left(\frac{X^2}{Y^3} + \frac{1}{Y}\right)}{1 - \frac{9ab}{16}\left(\frac{X^2}{Y^3} + \frac{1}{Y}\right)^2} \tag{14.6}$$

Equations (14.5) and (14.6) have been purposely partitioned into two terms. The first term will be called the "stationary interaction term," for it represents the drag that would be exerted on A or B if the other particle is moving at the same velocity. In the case of two streams of particles flowing past each other, the "stationary interaction term" depends on the number density of the particles alone. The second term represents the dynamic effect due to the relative motion of the particles and shall be referred to as the "viscous collision" term.

The "Impulse Function"

Consider two streams of particles flowing past each other. Let A and B be two particles of the two streams. To a first approximation, assume that the path of B relative to A is straight and also that the relative velocity U_{BA} is constant. (This approximation is justified if one is interested in the overall effect.) Further from



equation (14.6), the collision force is significant only when A and B are relatively close together, and during this span of time one may consider the path of B to be sufficiently straight.) If f_A denotes the force acting on A due to "viscous collision" between A and B, then, from equation (14.6),

$$f_A = \frac{\frac{9}{2}\pi \mu a b U_{BA}\left(\frac{x^2}{r^3} + \frac{1}{r}\right)}{1 - \frac{9ab}{16}\left(\frac{x^2}{r^3} + \frac{1}{r}\right)^2} \qquad (14.7)$$

In the course of time, B will move past A and vanish into infinity. The impulse given to A at time interval dt is

$$dI_A = f_A \, dt \qquad (14.8)$$

From (14.7) and (14.8),

$$I_A = \int_{-T}^{T} \frac{\frac{9}{2} a b \pi \mu U_{BA}\left[\frac{x^2}{(x^2+y^2)^{3/2}} + \frac{1}{(x^2+y^2)^{1/2}}\right]}{1 - \frac{9ab}{16}\left[\frac{x^2}{(x^2+y^2)^{3/2}} + \frac{1}{(x^2+y^2)^{1/2}}\right]^2} dt$$

where y is the impact parameter and $2T$ is the collision time.

But

$$U_{BA} = \frac{dx}{dt}$$

Hence

$$I_A = \int_{-\xi(y)}^{\xi(y)} \frac{\frac{9}{2} a b \pi \mu (2x^2+y^2)(x^2+y^2)^{3/2}}{(x^2+y^2)^3 - \frac{9ab}{16}(2x^2+y^2)^2} dx \qquad (14.9)$$

Now if the number density of particles B is N_B, then the collision frequency with impact parameter y is

$$d\, N(y) = 2\pi y N_B U_{BA} dy$$

Therefore, the total force acting on particle A is

$$dF_A = I_A(a, b, y)\, dN(y)$$

$$F_A = 18 a b \pi^2 \mu N_B U_{BA} \int_{y_{min}}^{y_{max}} y\, dy \int_0^{\xi(y)} \frac{(2x^2+y^2)(x^2+y^2)^{3/2}}{(x^2+y^2)^3 - \frac{9ab}{16}(2x^2+y^2)^2} dx$$

If particles A can be regarded as a continuum with number density N_A, then the force per unit volume acting on particles due to viscous collision with B is

$$F_{BA} = 18 a b \pi^2 \mu N_A N_B U_{BA} \int_{y_{min}}^{y_{max}} y\, dy \int_0^{\xi(y)} \frac{(2x^2+y^2)(x^2+y^2)^{3/2}}{(x^2+y^2)^3 - \frac{9ab}{16}(2x^2+y^2)^2} dx \qquad (14.10)$$

where the cutoff distance may be taken as

$y_{max} \simeq$ viscous screening distance, Section 11,

$\xi^2 \simeq$ (viscous screening distance)$^2 - y^2$.

Equation (14.10) was obtained from consideration of particle pairs only. It provides a rough estimate on the effect of viscous collision between two streams of particles moving relative to each other. That F_{BA} depends linearly on U_{BA} is characteristic of viscous force.

Density Effect

One of the important phenomena of a particle cloud, as has been analyzed in Section 11, is the effect of viscous screening or density effect. In the derivation of equation (14.10), this was, however, neglected in order to show clearly the physical phenomenon involved. Here, it would be assumed that the number densities of the two streams of particles are so large that single particle-particle encounter, as dealt with above, is predominated by density effect. With this assumption, the "screened" Stokes equation can once again be used.

Consider a stream of particles A moving through an otherwise stationary, viscous incompressible fluid and a stationary cloud of particles B. Let $-D_A\hat{x}$ be the drag on a particle A and $-D_B\hat{x}$ be the induced viscous force acting on a particle B due to the motion of particles A. In this case, the solution of the bulk fluid equations with respect to particles A is

$$
\begin{aligned}
p_f &= (N_A D_A + N_B D_B)\,x \\
\vec{u}_f &= -U_{AB}\,\hat{x}
\end{aligned}
\qquad (14.11)
$$

The disturbance p_a, $\vec{u}_a$ associated with a particle A is given by the solution of the "screened" Stokes equation and the following boundary conditions:

$$
\begin{aligned}
\nabla p_a &= \mu \nabla^2 \vec{u}_a - \mu \alpha^2 \vec{u}_a \\
\nabla\cdot\vec{u}_a &= 0
\end{aligned}
\qquad (14.12)
$$

$$
\begin{aligned}
&\vec{u}_a = 0 && r = \infty \\
&\vec{u} = \vec{u}_a + \vec{u}_f = 0 && r = a
\end{aligned}
$$

where, from equation (11.30),

$$\alpha = \frac{6\pi(a^2N_a + b^2N_b) + \sqrt{36\pi^2(a^2N_a+b^2N_b)^2 + 24\pi(aN_a+bN_b)(1-\frac{3c}{2})}}{2-3c} \tag{14.13}$$

The solution of equation (14. 12) is

$$\vec{U}_a = \hat{e}_r\left\{\frac{a^3}{r^3}\left[1+\frac{3}{\alpha a}+\frac{3}{\alpha^2a^2}\right] - \frac{3a}{\alpha}e^{\alpha a}\left[\frac{1}{r^2}+\frac{1}{\alpha r^3}\right]e^{-\alpha r}\right\}U_{AB}\cos\theta$$

$$+\hat{e}_\theta\left\{\frac{a^3}{2r^3}\left[1+\frac{3}{\alpha a}+\frac{3}{\alpha^2a^2}\right] - \frac{3a}{2\alpha}e^{\alpha a}\left[\frac{\alpha}{r}+\frac{1}{r^2}+\frac{1}{\alpha r^3}\right]e^{-\alpha r}\right\}U_{AB}\sin\theta \tag{14.14}$$

so that

$$U_{ax} = U_r\cos\theta - U_\theta\sin\theta$$

$$= \frac{a^3}{r^3}\left[\left(1+\frac{3}{\alpha a}+\frac{3}{\alpha^2a^2}\right) - \frac{3a}{\alpha}e^{-\alpha(r-a)}\left(\frac{1}{r^2}+\frac{1}{\alpha r^3}\right)\right]U_{AB}(3\cos^2\theta - 1)$$

$$+\frac{3a}{2r}e^{-\alpha(r-a)}U_{AB}\sin^2\theta \tag{14.15}$$

By means of the approximation of equation (3. 41) of Part II, the force acting on particles B outside the volume of sphere ξ is

$$f_{BA} = 6\pi\mu bN_b\int U_{ax}\,dv$$

$$= \frac{24\pi^2\mu abN_bU_{AB}e^{\alpha(a-\xi)}}{\alpha^2}(1+\alpha\xi)$$

let $\xi \to a$

$$f_{BA} = \frac{24\pi^2\mu abN_bU_{AB}}{\alpha^2}(1+\alpha\xi) \tag{14.16}$$

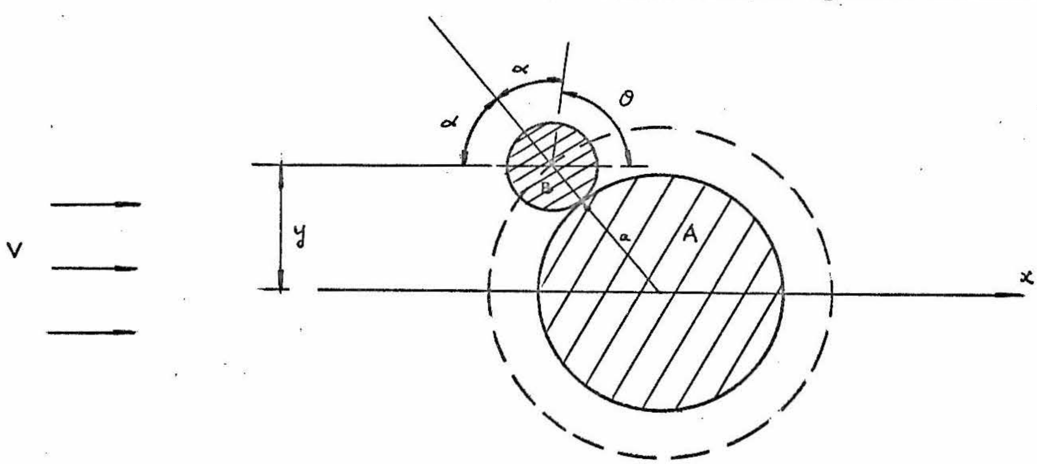
Hence, the force acting on particles B per unit volume is

$$F_{BA} = \frac{24\pi^2 \mu a b N_a N_b U_{AB}}{\alpha^2}\left[1 + \frac{\alpha(a+b)}{2}\right] \qquad (14.17)$$

where, in equation (14.17), ξ has been put equal to $\frac{a+b}{2}$ so that F_{BA} is completely symmetric in a, b.

Direct Contact Collision

In order to compare the importance of momentum exchange due to relative motion of two streams of particles through viscous forces and direct contact collision, in the following, calculation would be made based on the classical hard elastic sphere model.



Let $U_b > U_a$, then, in the center of mass system, the geometry of collision would be as shown. The velocity of the center of mass system is

$$U_{cm} = \frac{m_a U_a + m_b U_b}{m_a + m_b} \qquad (14.18)$$

The relation between impact parameter y and scattering angle θ is

$$\sin\alpha = \frac{y}{a+b} = \cos\frac{\theta}{2} \qquad (14.19)$$

Number of particles with impact parameter between y and $y + dy$ striking A per second is

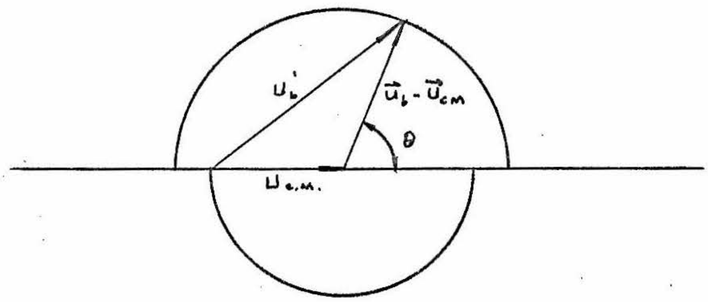
$$d\,N(y) = 2\pi\, y\, dy\; N_b\, |U_b - U_a|$$

Therefore,

$$d\,N(\theta) = \pi\, N_b\, |U_b - U_a|\, (a+b)^2 \cos\frac{\theta}{2} \sin\frac{\theta}{2}\, d\theta$$

Hence

$$\frac{d\,N(\theta)}{d\,\Omega_{cm}} = \frac{N_b\, |U_b - U_a|\, (a+b)^2}{4} \tag{14.20}$$



For a B - particle scattered into the angle θ, the change in forward momentum is

$$\Delta p_b = m_b \left[U_b - (U_b - U_{cm}) \cos\theta - U_{cm} \right]$$

$$= \frac{m_a m_b\, (U_b - U_a)}{m_a + m_b}\, (1 - \cos\theta)$$

Therefore, force acting on particle A is

$$f_a = \int \left| \frac{d\,N(\theta)}{d\,\Omega} \right| \Delta p_b\, d\Omega$$

$$= \frac{\pi\, (a+b)^2\, N_b\, m_a m_b\, |U_b - U_a|\, (U_b - U_a)}{m_a + m_b} \tag{14.21}$$

From (14.21), the force exerted on particles A per unit volume due to contact collision is

$$F_{ab} = \frac{\pi\, (a+b)^2\, N_a N_b\, m_a m_b\, |U_b - U_a|\, (U_b - U_a)}{m_a + m_b} \tag{14.22}$$

15. Thermal Conductivity of Bulk Fluid

The aim of this section is to describe the thermal transmission property of the bulk fluid from the known thermal properties of the fluid and particles. It will be shown later that in the treatment of transient heat transfer, the introducetion of a complex thermal conductivity would simplify the mathematical description. However, to build up an understanding of the problem, the steady-state case will first be treated.

Steady-State Thermal Conductivity

Consider the case of a homogeneous particle - fluid system at rest. Then, from Section 9, the bulk fluid equations reduce to

$$\nabla \cdot \vec{q}_f = 0 \tag{15.1}$$

Now, the steady-state thermal conductivity of the bulk fluid will be defined by

$$\vec{q}_{f_{cond.}} = \frac{\int_v \vec{q}_{actual}\, dv}{V} = -k_f \nabla T_f \tag{15.2}$$

where V is the volume defined in Section 9.

Substituting (15.2) into (15.1), the governing field equation for T_f is

$$\nabla^2 T_f = 0 \tag{15.3}$$

Let

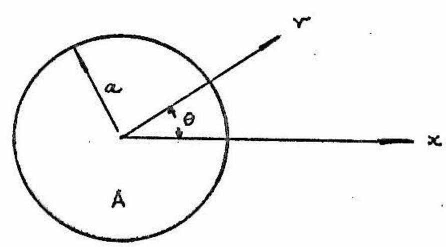
k_p = thermal conductivity of solid particle

k = thermal conductivity of fluid

$c = \frac{4}{3}\pi a^3 n$ volume concentration of particles

n = number density of particles

Take a coordinate system with origin at the center of a particle A. Consider the case in which the bulk fluid temperature distribution is linear, i.e., $T_f = T_0 + \beta x$, which satisfies (15.3).



Assume that the number density of the particles is such that, as a first approximation, the mutual influence effect among the particles can be neglected.

Let

T_p = actual temperature of particle A

$T = T_f + T_a$ = actual temperature of fluid near A

Here, T_a may be considered as the thermal disturbance produced on the fluid associated with the presence of particle A.

From Fourier's law of heat conduction, T_p and T_a satisfy

$$k_p \nabla^2 T_p = 0 \qquad r < a \tag{15.4}$$

$$k \nabla^2 T_a = 0 \qquad r > a \tag{15.5}$$

To determine T_p and T_a, one needs to impose proper boundary conditions on them. Since T_a represents a thermal disturbance, it must die out far away from A . Let $T_a = 0$ at $r = \rho$ where $\rho >> a$. Thus, the required set of boundary conditions is

$$\left.\begin{array}{ll} \text{(a)} & T_a = 0 \qquad r = \rho \\ \text{(b)} & T_p \quad \text{is bounded} \\ \text{(c)} & \text{continuity of temperature field} \\ & \qquad T = T_p \quad , \quad r = a \\ \text{(d)} & \text{continuity of normal component of} \\ & \text{heat flux vector at surface of sphere,} \\ & \qquad q_r \ \text{continuous at} \ r = a \end{array}\right\} \qquad (15.6)$$

The general solution of (15.4) and (15.5) is

$$T_a = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

$$T_p = \sum_{l=0}^{\infty} \left(C_l r^l + \frac{D_l}{r^{l+1}} \right) P_l(\cos\theta)$$

Impose the conditions of (a) to (d); obtain

$$A_l = B_l = C_l = D_l = 0 \qquad l > 1$$

Also,

$$A_0 = 0$$

$$B_0 = 0$$

$$C_0 = T_0$$

$$B_1 = \frac{\beta \left(\frac{k}{k_p} - 1 \right)}{\frac{1}{a^3}\left(1 + \frac{2k}{k_p} \right) + \frac{1}{\rho^3}\left(\frac{k}{k_p} - 1 \right)}$$

$$A_1 = \frac{\beta\left(1-\frac{k}{k_p}\right)}{\frac{\xi^3}{a^3}\left(1+\frac{2k}{k_p}\right)+\left(\frac{k}{k_p}-1\right)}$$

$$C_1 = \beta + \frac{\beta\left(\frac{k}{k_p}-1\right)\left(\frac{1}{a^3}-\frac{1}{\xi^3}\right)}{\frac{1}{a^3}\left(1+\frac{2k}{k_p}\right)+\frac{1}{\xi^3}\left(\frac{k}{k_p}-1\right)}$$

Therefore,

$$T_a = \frac{\beta\left(1-\frac{k}{k_p}\right) r\cos\theta}{\frac{\xi^3}{a^3}\left(1+\frac{2k}{k_p}\right)+\left(\frac{k}{k_p}-1\right)} + \frac{\beta\left(\frac{k}{k_p}-1\right)}{\frac{1}{a^3}\left(1+\frac{2k}{k_p}\right)+\frac{1}{\xi^3}\left(\frac{k}{k_p}-1\right)}\,\frac{\cos\theta}{r^2} \qquad (15.7)$$

$$T_p = T_0 + \left[\beta + \frac{\beta\left(\frac{k}{k_p}-1\right)\left(\frac{1}{a^3}-\frac{1}{\xi^3}\right)}{\frac{1}{a^3}\left(1+\frac{2k}{k_p}\right)+\frac{1}{\xi^3}\left(\frac{k}{k_p}-1\right)}\right] r\cos\theta \qquad (15.8)$$

It is clear from (15.7) that thermal disturbance T_a distorts the isotherms of the temperature field. The total heat transport due to this distortion is

$$q_x = -k\int_{|\vec{\xi}| - a} \frac{\partial T_a}{\partial x}\, dv = \frac{-\frac{4}{3}\pi(\xi^3 - a^3)\, k\left(1-\frac{k}{k_p}\right)\beta}{\frac{\xi^3}{a^3}\left(1+\frac{2k}{k_p}\right)+\left(\frac{k}{k_p}-1\right)}$$

Let $\xi \to \infty$

$$q_x(T_a) \longrightarrow \frac{-k\beta\left(1-\frac{k}{k_p}\right)\frac{4}{3}\pi a^3}{\left(1+\frac{2k}{k_p}\right)} \tag{15.9}$$

$$T_a = \frac{\beta\left(\frac{k}{k_p}-1\right)\cos\theta}{\frac{1}{a^3}\left(1+\frac{2k}{k_p}\right)r^2} \tag{15.10}$$

$$T_p = T_0 + \frac{3\beta\frac{k}{k_p}}{1+2\frac{k}{k_p}}\, r\cos\theta \tag{15.11}$$

On using (15. 10) and (15. 11), the integral of equation (15. 2) can be evaluated.

$$\frac{\int_V q_x\, dv}{V} = \frac{\int_{V_f} q_x\, dv}{V} + \frac{\int_{V_p} q_x\, dv}{V}, \quad (V_f + V_p = V)$$

$$= \frac{-k\int_{V_f}\frac{\partial T}{\partial x}dv}{V} - \frac{k_p\int_{V_p}\frac{\partial T_p}{\partial x}dv}{V}$$

Therefore,

$$-\beta k_f = \frac{-k\beta\left(1-\frac{k}{k_p}\right)c}{1+\frac{2k}{k_p}} - k\beta(1-c) - \frac{k_p c\beta\frac{3k}{k_p}}{1+\frac{2k}{k_p}} \tag{15.12}$$

Hence,

$$k_f = k\left[1 - \frac{3c\,\frac{k}{k_p}}{1 + 2\,\frac{k}{k_p}}\right] + \frac{3c\,k_p}{2 + \frac{k_p}{k}} \qquad (15.13)$$

Secondary Polarization

The value of k_f obtained above, of course, holds good only if c is small for that the mutual interference effect among particles has completely been neglected. However, before an attempt is made to extend the above analysis to account for this, it is found that it is helpful first to summarize some of the aspects of the present problem demonstrated in the preceding section.

From equation (15.10) it is clear that the insertion of a particle into an otherwise homogeneous fluid medium produces a thermal disturbance radiating from the surface of the particle. The pattern of the disturbance is typical of a dipole field. For simplicity, one can regard a particle as being transformed into a thermal dipole. Such a phenomenon will be referred to as thermal polarization. The strength of the induced dipole is proportional to the local temperature gradient, β . Further, the field associated with the induced dipole distorts the original temperature field, resulting in an enhancement or reduction in heat transfer. The amount of increase or decrease in heat transport is proportional to the local temperature gradient or the dipole strength.

Now consider a cloud of particles, each being polarized into a dipole. The thermal disturbances associated with these dipoles will set up thermal gradients which tend to further polarize other particles. This further polarization, which will be referred to as secondary polarization, is the mutual interference effect of the particles, and is now going to be accounted for. In order to reduce complication, a "dipole approximation" will be made in the following analysis to the effect that a particle can only be polarized into a dipole, i.e., all higher order poles can be neglected.

To see the necessary modifications which have to be made to account for secondary polarization, first consider the particles to be infinitely apart. Then equations (15.4) and (15.5) will give a complete description of the temperature field. Now bring the particles together. It is clear that this will not affect equation (15.4); indeed, the only necessary change that is required is in equation (15.5), which describes T_a .

From the remarks made in the preceding paragraphs, it is known that T_a will tend to polarize all other particles, which will in turn tend to enhance or oppose the flow of heat associated with A . Therefore, in calculating $q_x(T_a)$, equation (15.9), one should use k_f instead of k , i.e., one may regard that outside the sphere is a homogeneous medium characterized by a thermal conductivity k_f.

$$q_x(T_a) = -k_f \int_{|\vec{\xi}| - \text{sphere } A} \frac{\partial T_a}{\partial x} \, dv$$

Hence, instead of equation (15.12), one has

$$-k_f = \frac{-k_f\left(1-\frac{k}{k_p}\right)c}{1+2\frac{k}{k_p}} - k(1-c) - \frac{k_p c \frac{3k}{k_p}}{1+2\frac{k}{k_p}} \qquad (15.14)$$

Solving for k_f ,

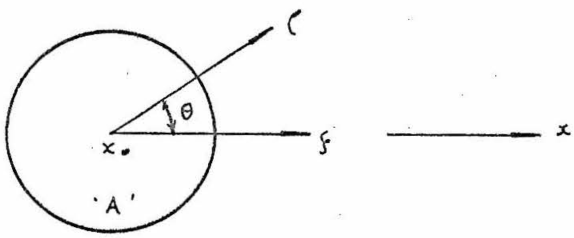
$$k_f = \frac{k\,[\,2k + k_p + 2c\,(k_p - k)\,]}{[\,k_p + 2k - c\,(k_p - k)\,]} \tag{15.15}$$ *

Equation (15.15)* agrees very well with experimental data obtained by Sugawara and Yoshizawa[48]. Comparison with their experimental results of glass balls in air and water is shown on the following page.

The Complex Thermal Conductivity

The case of steady-state thermal conductivity of bulk fluid has been dealt with. Here, it is intended to generalize the result above to unsteady thermal conduction. It will be found, as will be shown below, that it is advantageous to introduce a complex thermal conductivity and to carry out the solution of this type of problem in the Fourier transform plane.

As in the steady state case, consider a fluid-particle system of vast extent and fix a coordinate system with origin at the center of a sphere A which is located at x_0 as shown. As far as



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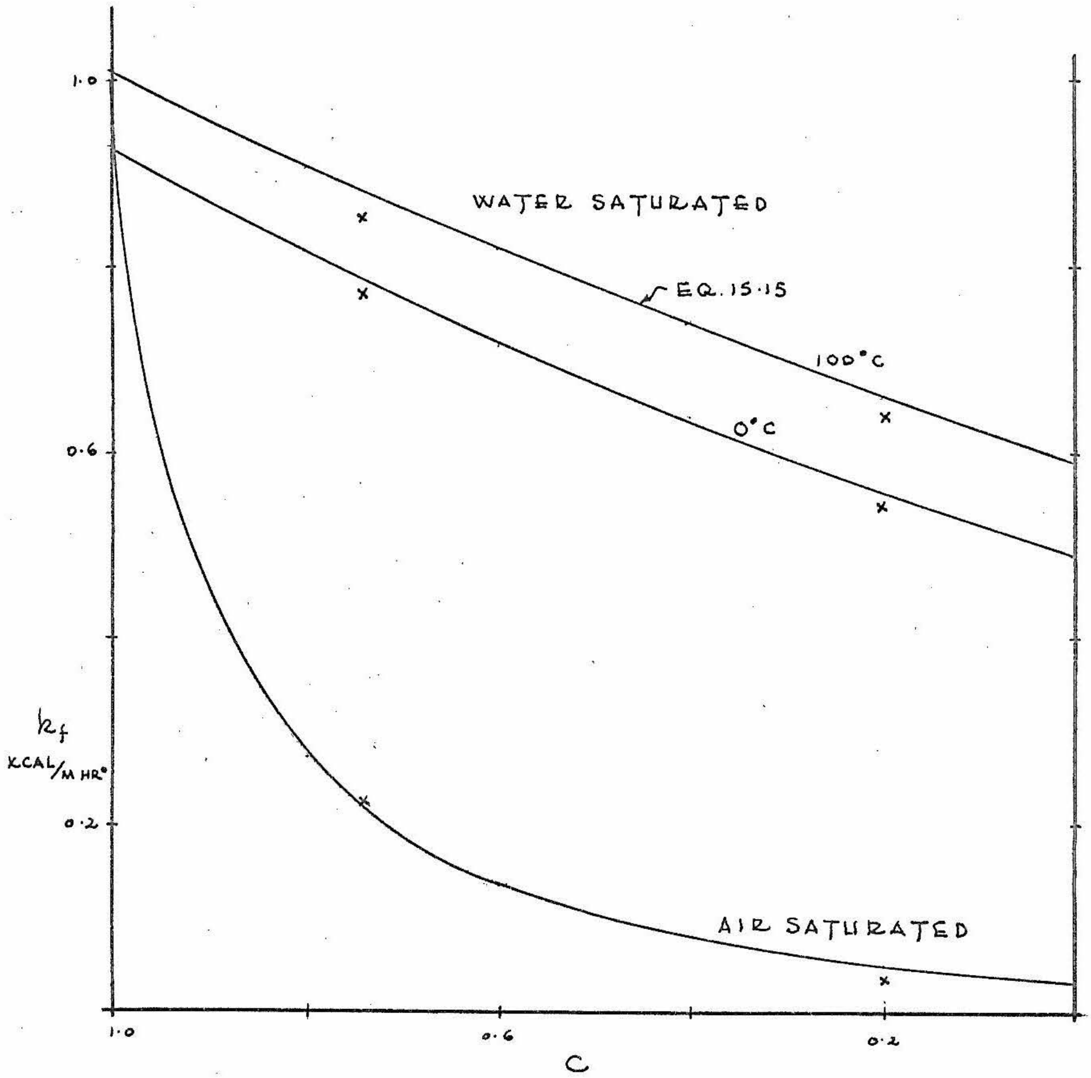
* Equation (15.15) is similar in form to Maxwell's formula for the electrical conductivity of a heterogeneous media in which mutual influence among spherical particles has been omitted.[47]



k_f vs. C DIAGRAM

× EXPT. RESULT OF SUGAWARA AND YOSHIZAWA (41)
(GLASS BALL SAMPLE)

sphere A is concerned, one may regard it as situated in an averaged thermal field described by T_f, the bulk fluid temperature. Here, c, the volume concentration of solid particles will be assumed to be such that the secondary polarization effect can be neglected as a first approximation.

Denote

$T_p(\vec{r},t)$ = temperature of particle A

$T_a(\vec{r},t)$ = temperature of thermal disturbance associated with particle A

so that T_p and T_a satisfy the Fourier heat conduction equation,

$$\kappa \frac{\partial T_a}{\partial t} = \nabla^2 T_a \tag{15.16}$$

$$\kappa_p \frac{\partial T_p}{\partial t} = \nabla^2 T_p \tag{15.17}$$

where
$$\kappa_p = \frac{\rho_p c_p}{h_p} \quad ; \quad \kappa = \frac{\rho \bar{c}}{h}$$

ρ, ρ_p = density of fluid, particle

$\bar{c}$, c_p = heat capacity of fluid, particle

h, h_p = thermal conductivity of fluid, particle.

As it is generally possible to decompose a heat pulse into Fourier components without loss of generality, let the bulk fluid temperature be given by $T_f = T_f(\vec{r},\omega)\, e^{i\omega t}$ where $T_f(\vec{r},\omega)$ and ω may be complex, and T_f satisfies

$$-i\omega \rho_f c_f T_f = h_f \nabla^2 T_f \tag{15.18}$$

where h_f is the complex thermal conductivity of the bulk fluid.

Since T_f varies as $e^{i\omega t}$, one expects all quantities to vary as $e^{i\omega t}$. Hence, let

$$T_a = T_a(\vec{\rho})\, e^{i\omega t}$$
$$T_p = T_p(\vec{\rho})\, e^{i\omega t} \qquad (15.19)$$

$Re\ \omega > 0$

so that equations (15.16) and (15.17) become

$$\nabla^2 T_a - \frac{(1+i)^2}{2} \kappa \omega T_a = 0 \qquad (15.20)$$

$$\nabla^2 T_p - \frac{(1+i)^2}{2} \kappa_p \omega T_p = 0 \qquad (15.21)$$

As in the steady state case, the boundary conditions on T_p and T_a will be taken as (15.6), i.e., the conditions of continuity and boundedness.

Consider the case where T_f depends on x only. Then near sphere A,

$$T_f(\vec{r},\omega) = T_f(x_0,\omega) + \left(\frac{\partial T_f}{\partial x}\right)_{x_0} \xi + \cdots$$

Here, the sphere will be considered as so small that in the macroscopic scale one can make a "dipole approximation" and take

$$T_f(\vec{r},\omega) = T_f(x_0,\omega) + \left(\frac{\partial T_f}{\partial x}\right)_{x_0} \xi + \cdots \qquad (15.22)$$

The general solution of (15.20) and (15.21) satisfying the boundedness conditions at infinity and the origin is

$$T_a = \sum_{l=0}^{\infty} A_l\, H_l^{(1)}\left[(1+i)\sqrt{\frac{\kappa\omega}{2}}\,\rho\right] P_l(\cos\theta) \qquad (15.23)$$

$$T_p = \sum_{l=0}^{\infty} B_l\, j_l\left[(1+i)\sqrt{\frac{\kappa_p\omega}{2}}\,\rho\right] P_l(\cos\theta) \qquad (15.24)$$

where $H_\ell^{(1)}$ and j_ℓ are the spherical Hankel and Bessel functions, respectively.

The conditions of continuity of temperature and heat flux on the surface of A require

$$A_\ell = B_\ell = 0 \qquad \ell > 1$$

$$A_0 H_0^{(1)}\left[(1+i)\sqrt{\frac{\kappa\omega}{2}}\,a\right] + T_f(x_0) = B_0 j_0\left[(1+i)\sqrt{\frac{\kappa_p\omega}{2}}\,a\right]$$

$$A_1 H_1^{(1)}\left[(1+i)\sqrt{\frac{\kappa\omega}{2}}\,a\right] + \left(\frac{\partial T_f}{\partial x}\right)_{x_0} a = B_1 j_1\left[(1+i)\sqrt{\frac{\kappa_p\omega}{2}}\,a\right]$$

$$k A_0 \left(\frac{dH_0^{(1)}}{d\rho}\right)_a = k_p B_0 \left(\frac{dj_0}{d\rho}\right)_a \tag{15.25}$$

$$k A_1 \left(\frac{dH_1^{(1)}}{d\rho}\right)_a + k\left(\frac{\partial T_f}{\partial x}\right)_{x_0} = k_p B_1 \left(\frac{dj_1}{d\rho}\right)_a$$

Solving equation (15.25),

$$A_0 = \frac{T_f(x_0)}{\left\{\dfrac{k\left(\dfrac{dH_0^{(1)}}{d\rho}\right)_a}{k_p\left(\dfrac{dj_0}{d\rho}\right)_a} - H_0^{(1)}\left[(1+i)\sqrt{\frac{\kappa\omega}{2}}\,a\right]\right\}}$$

$$B_0 = \frac{T_f(x_0)}{\left\{j_0\left[(1+i)\sqrt{\frac{\kappa_p\omega}{2}}\,a\right] - \dfrac{k_p\left(\dfrac{dj_0}{d\rho}\right)_a}{k\left(\dfrac{dH_0^{(1)}}{d\rho}\right)_a}\right\}}$$

$$A_1 = \left(\frac{\partial T_f}{\partial x}\right)_{x_0} \left\{ \frac{\frac{k}{k_\rho} j_1\left[(1+i)\sqrt{\frac{\kappa_\rho \omega}{2}}\, a\right] - a\left(\frac{dj_1}{d\rho}\right)_a}{H_1^{(1)}\left(\frac{dj_1}{d\rho}\right)_a - \frac{k}{k_\rho} j_1\left(\frac{dH_1^{(1)}}{d\rho}\right)_a} \right\}$$

$$B_1 = \left(\frac{\partial T_f}{\partial x}\right)_{x_0} \left\{ \frac{a\left(\frac{dH_1^{(1)}}{d\rho}\right)_a - H_1^{(1)}\left[(1+i)\sqrt{\frac{\kappa\omega}{2}}\, a\right]}{j_1\left(\frac{dH_1^{(1)}}{d\rho}\right)_a - \frac{k_\rho}{k} H_1^{(1)}\left(\frac{dj_1}{d\rho}\right)_a} \right\}$$

(15. 26)

and

$$T_a = A_0 H_0^{(1)}\left[(1+i)\sqrt{\frac{\kappa\omega}{2}}\,\rho\right] + A_1 H_1^{(1)}\left[(1+i)\sqrt{\frac{\kappa\omega}{2}}\,\rho\right]\cos\theta$$

$$T_\rho = B_0 j_0\left[(1+i)\sqrt{\frac{\kappa_\rho\omega}{2}}\,\rho\right] + B_1 j_1\left[(1+i)\sqrt{\frac{\kappa_\rho\omega}{2}}\,\rho\right]\cos\theta \qquad (15.27)$$

By means of (15. 27), the heat transport due to T_a is (omitting the factor $e^{i\omega t}$)

$$q_x(T_a) = -k \iint_a^\infty \frac{\partial T_a}{\partial \rho}\, \rho^2\, d\rho\, d\Omega$$

$$= \frac{4\pi}{3} a^2 A_1 H_1^{(1)}\left[(1+i)\sqrt{\frac{\kappa\omega}{2}}\, a\right] k \qquad (15.28)$$

Also,

$$q_x(T_p) = -k_p \iint_{sphere} \frac{\partial T_p}{\partial \xi}\, dv$$

$$= -\frac{4\pi}{3} a^2 B_1 j_1'\left[(1+i)\sqrt{\frac{\kappa_p \omega}{2}}\, a\right] k_p \tag{15.29}$$

From equation (15.2), suppressing the factor $e^{i\omega t}$,

$$-k_f \int_v \left(\frac{\partial T_f}{\partial x}\right) dv = \int_v q_x\, dv$$

$$= \int_v \frac{4\pi}{3} a^2 N \left\{ A_1 H_1^{(1)'}\left[(1+i)\sqrt{\frac{\kappa\omega}{2}}\, a\right] k - B_1 j_1'\left[(1+i)\sqrt{\frac{\kappa_p\omega}{2}}\, a\right] k_p \right\} dv$$

$$+ \int_v -k\left(\frac{\partial T_f}{\partial x}\right)(1-c)\, dv$$

Therefore,

$$k_f^+ = \frac{c}{a}\left\{ k_p j_1 \left[\frac{a\dfrac{dH_1^{(1)}}{d\rho} - H_1^{(1)}}{j_1 \dfrac{dH_1^{(1)}}{d\rho} - \dfrac{k_p}{k} H_1^{(1)} \dfrac{dj_1}{d\rho}}\right] - k H_1^{(1)} \left[\frac{\dfrac{k}{k_p} j_1 - a\dfrac{dj_1}{d\rho}}{H_1^{(1)}\dfrac{dj_1}{d\rho} - \dfrac{k}{k_p} j_1 \dfrac{dH_1^{(1)}}{d\rho}}\right]\right\}_{\rho=a} + k(1-c) \tag{15.30}$$

where $^+$ indicates $R_e\, \omega > 0$.

In the limit $\omega \to 0$, (15.30) reduces to

$$k_f = k\left[1 - \frac{3c\dfrac{k}{k_p}}{1+2\dfrac{k}{k_p}}\right] - \frac{3c\, k_p}{2+\dfrac{k_p}{k}}$$

which is equation (15.13), corresponding to the steady state case.

For $R_e\, \omega < 0$, equations (15.20), (15.21) can be rewritten as

$$\nabla^2 T_a - \frac{(i-1)^2}{2} \varkappa(-\omega) T_a = 0$$

$$\nabla^2 T_p - \frac{(i-1)^2}{2} \varkappa_p(-\omega) T_p = 0$$

On carrying out calculations as above, it is easy to obtain

$$k_f^-(\omega) = k_f^+(-\omega)^* \tag{15.31}$$

where $*$ signifies the complex conjugate.

If $|\varkappa_p \omega a^2| \ll 1$, $|\varkappa \omega a^2| \ll 1$, then (15.30) is approximately given by

$$k_f^+ \simeq k_{f\,\text{steady}} + i\omega k_1 \tag{15.32}$$

where

$$k_1 = a^2 \varkappa c \left[\frac{\left(1 - \frac{k}{k_p}\right) k \left(\frac{k}{k_p} - 1 + \frac{3\varkappa_p}{5\varkappa} \right)}{\left(1 + 2\frac{k}{k_p}\right)^2} \right] \tag{15.33}$$

To account for the effect of secondary polarization, one can proceed as in the steady state case. Equation (15.30) becomes

$$k_f^+ = \frac{c}{a}\left\{ k_p j_1 \left[\frac{a\frac{dH_1^{(1)}}{d\rho} - H_1^{(1)}}{j_1\frac{dH_1^{(1)}}{d\rho} - \frac{k_p}{k} H_1^{(1)} \frac{dj_1}{d\rho}} \right] - k_f^+ H_1^{(1)} \left[\frac{\frac{k}{k_p} j_1 - a\frac{dj_1}{d\rho}}{H_1^{(1)}\frac{dj_1}{d\rho} - \frac{k}{k_p} j_1 \frac{dH_1^{(1)}}{d\rho}} \right] \right\}_{\rho=a} + k(1-c)$$

Hence

$$k_f^+ = \frac{k(1-c) + \frac{c k_p}{a} j_1 \left[\frac{a\frac{dH_1^{(1)}}{d\rho} - H_1^{(1)}}{j_1 \frac{dH_1^{(1)}}{d\rho} - \frac{k_p}{k} H_1^{(1)} \frac{dj_1}{d\rho}} \right]_{\rho=a}}{1 + \frac{c}{a} H_1^{(1)} \left[\frac{\frac{k}{k_p} j_1 - a\frac{dj_1}{d\rho}}{H_1^{(1)}\frac{dj_1}{d\rho} - \frac{k}{k_p} j_1 \frac{dH_1^{(1)}}{d\rho}} \right]_{\rho=a}} \tag{15.34}$$

If $|\alpha_f \omega a^2| << 1 \quad ; \quad |\alpha \omega a^2| << 1$

$$k_f^+ = k\left[\frac{2k + k_p + 2c(k_p - k)}{2k + k_p - c(k_p - k)}\right] + i\omega c a^2\left\{\left[\frac{k_p}{(k_p+2k)(k_p+2k-c(k_p-k))}\right.\right. \tag{15.35}$$
$$\left.\left.\times\left(1 - \frac{k}{k_p}\,\frac{2k + k_p + 2c(k_p - k)}{2k + k_p - c(k_p - k)}\right)\right]\left[\frac{3\alpha_p k}{5} + \alpha(k - k_p)\right]\right\}$$

<u>Complex Fourier Conduction Equation</u>

The analysis shown above reveals that k_f depends on ω as well as on a, the radius of solid particle, in addition to the thermal properties of the constituents of thep article-fluid system. Here, it is intended to outline a method for solving transient heat conduction problems for such a system. For this purpose, the Fourier transform technique will be employed. Denoting $T_\omega(\vec{r}, \omega)$ as the Fourier transform of T_f, i.e.,

$$T_\omega(\vec{r}, \omega) = \int_{-\infty}^{\infty} T_f(\vec{r}, t)\, e^{-i\omega t}\, dt$$

so that

$$H(t)\frac{\partial T_f}{\partial t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} i\omega\, T_\omega(\vec{r}, \omega)\, e^{i\omega t}\, d\omega - \frac{1}{2\pi}\int_{-\infty}^{\infty} T_f(\vec{r}, 0)\, e^{i\omega t}\, d\omega$$

Consider a motionless particle-fluid system. The particle continuum and bulk fluid equations of Sections 8 and 9 reduce to

$$R_e\,\omega > 0 \qquad i\omega T_\omega^+(\vec{r}, \omega) - T_f^+(\vec{r}, 0) = \frac{k_f^+(\omega)}{\rho_s c_s}\nabla^2 T_\omega^+(\vec{r}, \omega)$$
$$\tag{15.36}$$
$$R_e\,\omega < 0 \qquad i\omega T_\omega^-(\vec{r}, \omega) - T_f^-(\vec{r}, 0) = \frac{k_f^-(\omega)}{\rho_s c_s}\nabla^2 T_\omega^-(\vec{r}, \omega)$$

where $$\rho_s C_s = \rho_f C_f + c \rho_p C_p$$

and

$$T_f(\vec{r}, t) = \frac{1}{2\pi}\int_0^{\infty} T_\omega^{+}(\vec{r}, \omega) e^{i\omega t} d\omega + \frac{1}{2\pi}\int_{-\infty}^{0} T_\omega^{-}(\vec{r}, \omega) e^{i\omega t} d\omega \tag{15.37}$$

Equation (15.36) is the complex Fourier conduction equation which describes the heat transmission phenomenon under consideration, while (15.37) gives the inversion back to the time variable t .

For problems which vary slowly with time, one may use the approximated value of k_f given by (15.35), so that making use of (15.31), (15.36) becomes

$$i\omega T_\omega(\vec{r}, \omega) - T_f(\vec{r}, 0) = \frac{k_{f\,steady} + i\omega k_1}{\rho_s C_s} \nabla^2 T_\omega(\vec{r}, \omega) \tag{15.38}$$

By inverting (15.38) one obtains

$$\rho_s C_s \frac{\partial T_f}{\partial t} = k_{f\,steady} \nabla^2 T_f + k_1 \nabla^2 \left(\frac{\partial T_f}{\partial t} + T_f(\vec{r}, 0)\,\delta(t) \right)$$

If $\nabla^2 T_f(\vec{r}, 0) = 0$, the equation reduces to

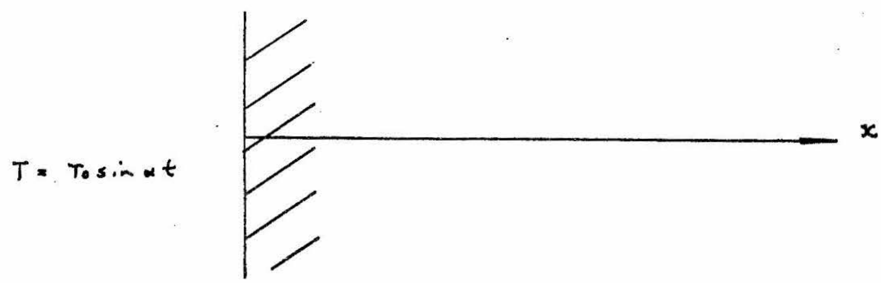
$$\rho_s C_s \frac{\partial T_f}{\partial t} = k_{f\,steady} \nabla^2 T_f + k_1 \nabla^2 \left(\frac{\partial T_f}{\partial t}\right) \tag{15.39}$$

The term $k_1 \nabla^2 \frac{\partial T_f}{\partial t}$ can be interpreted as follows. As has been seen earlier, a spherical particle behaves as a dipole in a temperature field and the dipole strength is proportional to the temperature gradient, i.e., $\vec{P} = \text{const.}\ \nabla T_f$. If Q denotes the amount of heat stored in the dipoles, then $Q = \text{const.}\ \nabla \cdot \vec{P} = \text{const.}\ \nabla^2 T_f$ (This is exactly analogous to the relation between electric charge and dipole where excess charge $\rho = \text{const.}\ \nabla \cdot \vec{P}_{electric}$.)

Thus the term $k_1 \nabla^2 \frac{\partial T_f}{\partial t}$ is actually $\frac{\partial Q}{\partial t}$ or the rate of change of heat associated with the heat dipoles. Hence (15.39) merely accounts for this extra heat flux which normally does not arise in a homogeneous medium.

Example

To illustrate the use of (15.36), consider a semi-infinite slab of metal B containing infinitely many small, spherical particles of metal A. Let one face of the alloy slab be kept at a temperature $T = T_0 \sin \alpha t$. The present problem is to inquire into the temperature variation after the transient period.



For the present problem, equation (15.36) becomes

$$\begin{aligned} i\omega T_\omega^+ &= \frac{k_f^+(\omega)}{\rho_s c_s} \frac{d^2 T_\omega^+}{dx^2} \\ i\omega T_\omega^- &= \frac{k_f^-(\omega)}{\rho_s c_s} \frac{d^2 T_\omega^-}{dx^2} \end{aligned} \qquad (15.40)$$

Assume α is such that one can use the approximate value of k_f given by (15.35), i.e.,

$$k_f = k_{f_{steady}} + i\omega k_1 \qquad , \; k_1 > 0$$

The boundary conditions on T_ω are

(a) $T_\omega = 0 \qquad x = \infty$

(b) $T_\omega = \int_{-\infty}^{\infty} T_0 H(t) \sin \alpha t \, e^{-i\omega t} \, dt = \frac{1}{2}\left(\frac{1}{\omega+\alpha} - \frac{1}{\omega-\alpha}\right) T_0$

$x = 0$

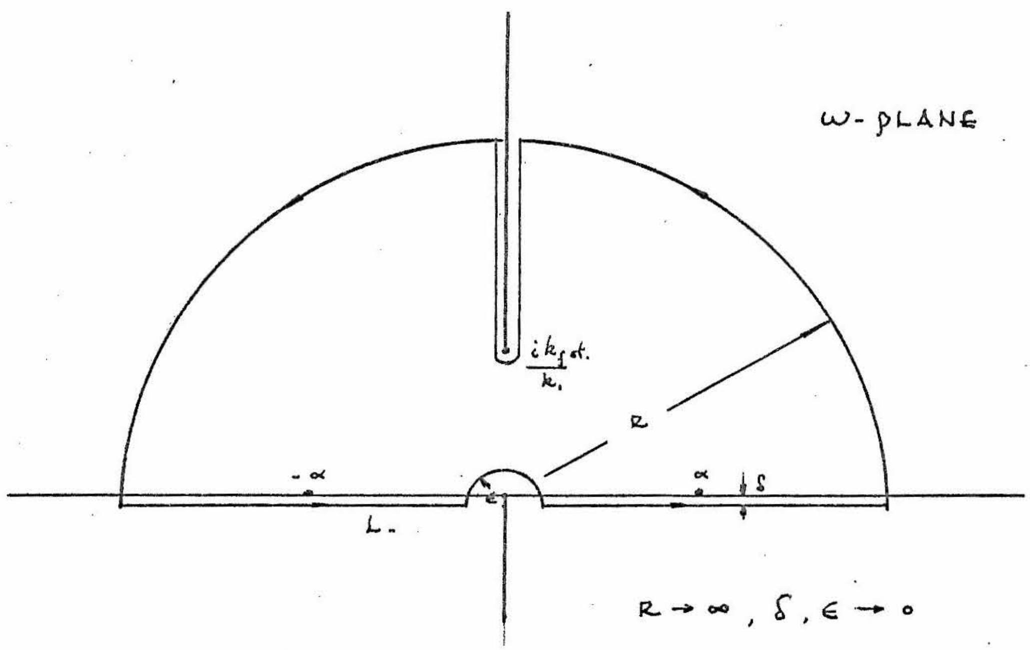
(ω has a small negative imaginary part).

The solution of (15.40) satisfying the boundary conditions is

$$T_\omega = \frac{T_0}{2}\left(\frac{1}{\omega+\alpha} - \frac{1}{\omega-\alpha}\right) e^{-\sqrt{\frac{i\omega(k_{f\,st.} - i\omega k_1)}{h_{f\,st.}^2 + \omega^2 h_1^2}}\sqrt{\rho_1 c_1}\, x} \qquad (15.41)$$

Substitute (15.41) into the inversion formula, equation (15.37):

$$T_f(x,t) = \frac{1}{2\pi}\int_{L_-} \frac{T_0}{2}\left(\frac{1}{\omega+\alpha} - \frac{1}{\omega-\alpha}\right) e^{-\sqrt{\frac{\rho_1 c_1 \omega(\omega k_1 + i k_{f\,st.})}{h_{f\,st.}^2 + \omega^2 h_1^2}}\, x + i\omega t} \, d\omega \qquad (15.42)$$



The integral of (15. 42) can be evaluated by completing the contour as shown. By Jordan lemma, the integral over the semi-circle does not contribute. Hence, by Cauchy's theorem, the integral of (15. 42) is given by the contribution from the poles at $\omega = \pm\alpha$ and the integral over the cut. However, the long time response is given by the residues of the poles.

$$
\begin{aligned}
T_f(x,t) &= 2\pi i \frac{T_0}{4\pi}\left[\exp\left(-\sqrt{\frac{\rho_s c_s (\alpha^2 k_1 - i\alpha k_{fst})}{k_{fst}^2 + \alpha^2 k_1^2}}\, x - i\alpha t \right)\right. \\
&\qquad \left. - \exp\left(-\sqrt{\frac{\rho_s c_s (\alpha^2 k_1 + i\alpha k_{fst})}{k_{fst}^2 + \alpha^2 k_1^2}}\, x + i\alpha t \right)\right] \\
&= T_0 \exp\left(-\sqrt{\frac{\rho_s c_s \alpha}{(k_{fst}^2 + \alpha^2 k_1^2)^{1/2}}}\cos\beta x \right) \cdot \\
&\qquad \sin\left[\alpha t - \sqrt{\frac{\rho_s c_s \alpha}{(k_{fst}^2 + \alpha^2 k_1^2)^{1/2}}}\sin\beta x \right]
\end{aligned}
\qquad (15.43)
$$

where

$$\tan 2\beta = \frac{k_{fst}}{\alpha k_1}$$

The solution given by (15. 43) shows that, as compared with a homogeneous medium, the inclusion of foreign small spherical objects tends to alter the phase at any specific location. Also, the penetration depth, as characterized by the value of the exponent in front of x, depends more strongly on the frequency. The physical explanation of this is quite clear if one imagines each particle to behave as an oscillating dipole out of phase with the driving frequency. The heat flux associated with the dipoles causes the phase shift as well as alters the decay rate.

16. Diffusion Coefficient of a Substance in a Particle-Fluid Suspension

The process of diffusion of a substance through a static, homogeneous fluid is described macroscopically by the following conservation and phenomenological equations

$$\frac{\partial Y}{\partial t} + \nabla \cdot \vec{J} = 0 \tag{16.1}$$

$$\vec{J} = - D \nabla Y \tag{16.2}$$

where

Y is the concentration of the substance,

$\vec{J}$ is the diffusional flux,

D is the diffusion coefficient.

If solid spherical particles are introduced into the fluid, the rate of diffusion is changed. On assuming that no chemical reaction takes place, then, on the surfaces of the solid

$$\vec{J} \cdot \hat{n} = 0$$

or

$$\frac{\partial Y}{\partial n} = 0 \tag{16.3}$$

where $\hat{n}$ is the normal to the sphere.

It is easy to see that equations (16.1), (16.2), and (16.3) are exactly similar to the process of heat diffusion through a particle suspension. Hence, for an overall macroscopic description of this phenomenon, one may define

$$- D_e \nabla Y_s = \vec{J_s} = \frac{\int_v \vec{J}_{actual}\, dv}{V}$$

in analogy to equation (15. 2) for heat flux.

The boundary condition given by (16. 3) corresponds exactly to to the case of insulated spheres. Therefore, one can obtain the diffusion coefficient corresponding to a solid particle suspension by putting $k \rightarrow D$ and $k_p = 0$ in equation (15. 15). Thus,

$$D_e = D \frac{2(1-c)}{(2+c)} \tag{16. 4}$$

where c is the volume concentration of solid particles.

17. The Rate of Heat Transfer Between Particles and Bulk Fluid

The rate of heat transfer from a solid sphere to a moving fluid has been investigated theoretically and experimentally by many authors[49-54] in the past. In this section, the heat transfer rate from solid spherical particles to the surrounding fluid in a fluid-particle system when there is relative motion between them is being calculated.

Let T_p, T_f and $\vec{U}_f = u\hat{z}$ be the particle temperature, bulk fluid temperature, and bulk fluid velocity with respect to a coordinate system fixed to a solid sphere A as obtained from the solution of the bulk fluid equations and particle continuum equations. Near sphere A, let the actual fluid temperature be $(T_f + T_a)$ and the actual temperature of the particle be $(T_p + T_i)$, so that T_a and T_i represent the thermal disturbances associated with particle A outside and inside the sphere. To determine T_a and T_i so as to find the heat transfer rate, the following three assumptions will be made.

(1) All time-varying effects are small and can be ignored.

(2) Heat is lost uniformly from the whole of the sphere; let the heat transfer rate between a particle and fluid be

$$\frac{4}{3}\pi a^3 Q$$

(3) The convective heat transfer term $\vec{u} \cdot \nabla (c_f T_f)$ can be approximated by $\vec{U_f} \cdot \nabla (c_f T_f)$

(Approximation in (3) is valid if $\frac{u c_f a}{k}$ is small.)

By means of assumptions (1) and (2), the governing equation for T_i can be written as

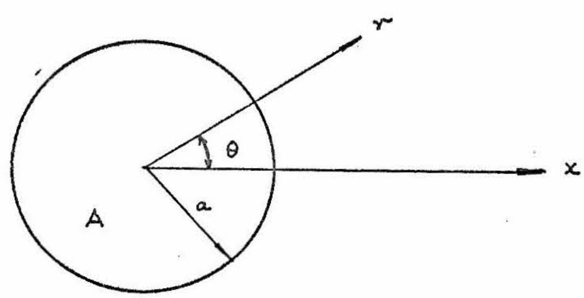
$$k_p \nabla^2 T_i + Q = 0 \tag{17.1}$$

Also, on neglecting mutual interference effect of particles as a first approximation (this will be accounted for later), the equation governing T_a is

$$\vec{u_f} \cdot \nabla (\rho \bar{c} T_a) - k \nabla^2 T_a = 0$$

On using assumption (3) this becomes

$$\alpha \frac{\partial T_a}{\partial x} - \nabla^2 T_a = 0 \quad ; \quad \alpha = \frac{U \bar{c} \rho}{k} \tag{17.2}$$



The conditions that have to be imposed on T_i and T_a for a unique determination of the present problem are

(a) The disturbance goes to zero away from the sphere, i.e.,

$$T_a = 0 \qquad r = \infty$$

(b) The disturbance must be bounded, i.e., T_i bounded.

(c) The mean temperature of the particle is equal to T_p

(d) Temperature is continuous on the surface of the sphere, i.e., $r = a$, $T_i + T_p = T_f + T_a$

(e) The heat flux vector normal to the surface of the sphere must be continuous

$$k_p \frac{\partial}{\partial r}(T_i + T_p) = k \frac{\partial}{\partial r}(T_a + T_f) \qquad , \ r = a$$

A particular solution of equation (17.1) is

$$T = -\frac{Q r^2}{6 k_p}$$

Thus, the complete solution of equation (17.1) satisfying condition (b) is

$$T_i = \frac{-Q r^2}{6 k_p} + \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) \tag{17.3}$$

By condition (c),

$$A_0 = \frac{Q a^2}{10 k_p}$$

Hence (17.3) becomes

$$T_i = \frac{Q}{k_p}\left(\frac{a^2}{10} - \frac{r^2}{6}\right) + \sum_{\ell=1}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) \tag{17.4}$$

A complete solution of (17.2) satisfying condition (a) can be written as

$$T_a = e^{\frac{\alpha x}{2}} \sum_{\ell=0}^{\infty} B_\ell \frac{K_{\ell+\frac{1}{2}}\left(\frac{\alpha r}{2}\right)}{\sqrt{r}} P_\ell(\cos\theta) \tag{17.5}$$

Now conditions (d) and (e) provide a unique determination of the coefficients $\mathcal{Q}$, A_ℓ and B_ℓ. That is,

$$e^{-\frac{\alpha a \cos\theta}{2}}\left[-\frac{\mathcal{Q}a^2}{15k_p}+(T_p-T_f)+\sum_{\ell=1}^{\infty}A_\ell a^\ell P_\ell(\cos\theta)\right]=\sum_{\ell=0}^{\infty}B_\ell\frac{K_{\ell+1/2}(\frac{\alpha a}{2})}{\sqrt{a}}P_\ell(\cos\theta) \qquad (17.6)$$

$$e^{-\frac{\alpha a\cos\theta}{2}}k_p\left[-\frac{\mathcal{Q}a}{3k_p}+\sum_{\ell=1}^{\infty}\ell A_\ell a^{\ell-1}P_\ell(\cos\theta)\right]=k\sum_{\ell=0}^{\infty}B_\ell\left[\frac{\alpha\cos\theta}{2}\frac{K_{\ell+1/2}(\frac{\alpha a}{2})}{\sqrt{a}}\right. \qquad (17.7)$$

$$\left.+\frac{d}{dr}\left(\frac{K_{\ell+1/2}(\frac{\alpha r}{2})}{\sqrt{r}}\right)_{r=a}\right]P_\ell(\cos\theta)$$

On using the relation

$$e^{-\frac{\alpha a}{2}\cos\theta}=\sqrt{\frac{\pi}{\alpha a}}\sum_{n=0}^{\infty}(-1)^n(2n+1)I_{n+1/2}\left(\frac{\alpha a}{2}\right)P_n(\cos\theta)$$

the coefficients in (17.6) and (17.7) can be solved by solving an infinite determinant. However, on restricting to small value of αa, one can easily obtain the heat transfer from particle to fluid:

$$Q_p=\frac{4}{3}\pi a^3\mathcal{Q}$$

$$=4\pi a k(T_p-T_f)\left[\frac{5}{5+\eta}+\frac{5(5+5\eta-\eta^2)}{2(1+2\eta)(5+\eta)^2}\alpha a\right. \qquad (17.8)$$

$$\left.-\frac{(125+1060\eta+12\eta^2-3926\eta^3-1540\eta^4-240\eta^5)}{12(5+11\eta+2\eta^2)}\alpha^2a^2+\cdots\right]$$

where $\eta=\frac{k}{k_p}$. Or if

Nusselt number, $Nu=\dfrac{Q_p}{2\pi a k(T_p-T_f)}$

$$Nu=\frac{10}{5+\eta}+\frac{5(5+5\eta-\eta^2)}{(1+2\eta)(5+\eta)^2}\alpha a-\frac{(125+1060\eta+12\eta^2-3926\eta^3-1540\eta^4-240\eta^5)}{6(5+11\eta+2\eta^2)^3}\alpha^2a^2+\cdots \qquad (17.9)$$

In the limit $k_p \to \infty$ or $\eta \to 0$

$$Nu = 2 + \alpha a - \frac{1}{6}\alpha^2 a^2 + \cdots$$

If the volume concentration of the solid particles is sufficiently large, it is necessary to modify the above analysis to account for this density effect. To do so, the method that is going to be used is still based on the approximation that if one fixes one's attention on a specific particle one may regard this particle as situated in an overall averaged fluid medium constituted by the fluid and other particles. To see this more clearly, first assume that the particles are far apart. Then, as shown above, the thermal disturbances associated with particle A are governed by the equations

$$k_p \nabla^2 T_i + Q = 0 \quad , \quad r < a \tag{17.1}$$

$$\alpha \frac{\partial T_a}{\partial x} - \nabla^2 T_a = 0 \quad , \quad r > a \tag{17.2}$$

Now let the particles be brought closer together, but assume that A is the only particle that has a higher temperature than the fluid. It is clear that nothing has really changed except the thermal conductivity and fluid density of the medium outside A . Thus, one should account for this change in thermal transmission property by using k_f (bulk fluid thermal conductivity) instead of k and ρ_f in place of ρ , so that equations (17.1) and (17.2) become

$$k_p \nabla^2 T_i + Q = 0 \quad , \quad r < a \tag{17.10}$$

$$\frac{u C_f \rho_f}{k_f} \frac{\partial T_a}{\partial x} - \nabla^2 T_a = 0 \quad , \quad r > a \tag{17.11}$$

Let the temperature of the other particles be raised to T_f . Heat is now being given to the fluid everywhere. Locally, near sphere

A , the change from the earlier picture can be obtained from the bulk fluid equation. In this gross description, it is possible to regard each particle as a heat source of strength $\frac{4}{3}\pi a^3 Q$, or in effect, the bulk fluid is flowing through a region where there is a heat source of strength $\frac{4}{3}\pi a^3 N Q = c Q$ per unit volume. Thus

$$\rho_f C_f U \frac{\partial T_f}{\partial x} = k_f \nabla^2 T_f + c Q \tag{17.12}$$

which yields a solution

$$T_f = T_{f0} + \frac{c Q}{\alpha_f k_f} x \tag{17.13}$$

where

$$\alpha_f = \frac{U C_f \rho_f}{k_f} \tag{17.14}$$

T_{f0} = mean bulk fluid temperature near A

Hence, the averaged background temperature with respect to particle A is

$$T_s = c T_p + (1-c) T_{f0} \tag{17.15}$$

With the above modification, one may now seek a solution to T_a and T_i , equations (17.10) and (17.11) satisfying the boundedness and continuity conditions as before.

$$T_i = \frac{Q}{k_p}\left[\frac{a^2}{10} - \frac{r^2}{6}\right] + \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) \tag{17.16}$$

$$T_a = e^{\frac{\alpha_f}{2} x} \sum_{\ell=0}^{\infty} B_\ell \frac{K_{\ell+\frac{1}{2}}\left(\frac{\alpha_f r}{2}\right)}{\sqrt{r}} P_\ell(\cos\theta) \tag{17.17}$$

Imposing the continuity conditions

$$e^{-\frac{\alpha_f a}{2}\cos\theta}\left[-\frac{Q a^2}{15 k_p} + (T_p - T_s) - \frac{c a Q}{\alpha_f k_f}\cos\theta + \sum_{l=1}^{\infty} A_l a^l P_l(\cos\theta)\right] = \sum_{l=0}^{\infty} \frac{B_l K_{l+\frac{1}{2}}\left(\frac{\alpha_f a}{2}\right)}{\sqrt{a}} P_l(\cos\theta) \tag{17.18}$$

$$e^{-\frac{\alpha_f a}{2}\cos\theta} k_p \left[-\frac{Q a}{3 k_p} + \sum_{l=1}^{\infty} A_l\, l\, a^{l-1} P_l(\cos\theta) - \frac{c Q}{\alpha_f k_f}\cos\theta\right] = k_f \sum_{l=0}^{\infty} B_l \left[\frac{\alpha_f \cos\theta}{2}\, \frac{K_{l+\frac{1}{2}}\left(\frac{\alpha_f a}{2}\right)}{\sqrt{a}} + \frac{d}{dr}\left(\frac{K_{l+\frac{1}{2}}\left(\frac{\alpha_f r}{2}\right)}{\sqrt{r}}\right)_{r=a}\right] \cdot P_l(\cos\theta) \tag{17.19}$$

On solving equations (17.18) and (17.19), denoting $\zeta = \frac{k_f}{k_p}$ the heat transfer from a particle to fluid

$$Q_p = \frac{4}{3}\pi a^3 Q$$

$$= 4\pi a k_f \left[T_p(1-c) - T_f(1-c)\right]\left[\frac{15}{5+\zeta} + \frac{15(5+5\zeta-\zeta^2)}{2(1+2\zeta)(5+\zeta)^2}\alpha a - \frac{(125+1060\zeta+12\zeta^2-3926\zeta^3-1540\zeta^4-240\zeta^5)}{4(5+11\zeta+2\zeta^2)^3}\alpha^2 a^2 + \cdots\right] \tag{17.20}$$

where

α_f is given by (17.14),

$$k_f = \frac{k\left[2k + k_p + 2c(k_p - k)\right]}{\left[k_p + 2k - c(k_p - k)\right]}$$ from Section 15.

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APPENDIX A.

On Green's Function of a Certain Differential Equation

In this appendix, the Green's function for the following differential equation is constructed for the full physical space:

$$(\nabla^2 - \alpha^2)\, G(\vec{x}, \vec{x}') = \delta(\vec{x} - \vec{x}') \qquad \text{(A-1)}$$

where $\delta(\vec{x})$ is the Dirac δ- function.

Method 1.

Denote the Fourier transform of $G(\vec{x}, \vec{x}')$ by

$$\bar{G} = \int_{-\infty}^{\infty} G(\vec{x}, \vec{x}')\, e^{-i\vec{k}\cdot\vec{x}}\, d^3x$$

Then, from equation (A-1), $\bar{G}$ satisfies

$$-(k^2 + \alpha^2)\, \bar{G} = e^{-i\vec{k}\cdot\vec{x}'} \qquad \text{(A-2)}$$

or

$$\bar{G} = -\frac{e^{-i\vec{k}\cdot\vec{x}'}}{k^2 + \alpha^2}$$

Therefore,

$$G(\vec{x}, \vec{x}') = -\frac{1}{(2\pi)^3} \int \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}}{(k^2 + \alpha^2)}\, d^3k \qquad \text{(A-3)}$$

Let $\vec{x} - \vec{x}'$ be the polar axis; then equation (A-3) becomes

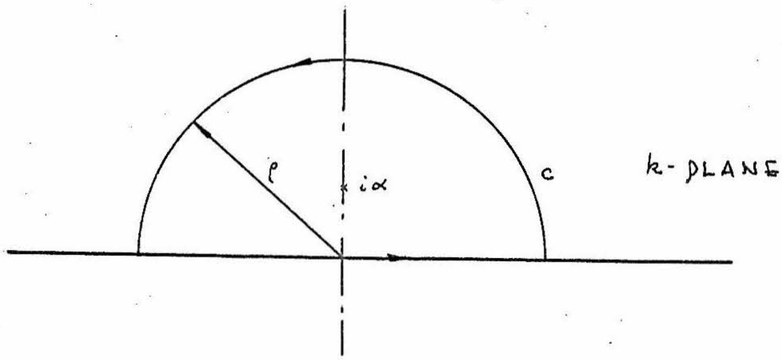
$$G(\vec{x}, \vec{x}') = \frac{-1}{(2\pi)^3} \iiint \frac{e^{iRk\cos\theta}}{k^2 + \alpha^2}\, k^2 \sin\theta\, dk\, d\theta\, d\phi$$

where $R = |\vec{x} - \vec{x}'|$

Integrate over ϕ and θ

$$G(\vec{x}, \vec{x}') = -\frac{1}{(2\pi)^2 iR} \int_{-\infty}^{\infty} \frac{e^{ikR}}{k^2 + \alpha^2}\, k\, dk$$

The last integral can be evaluated in the complex k plane by completing the contour in the upper half plane.



By Jordan lemma, there is no contribution to the integral

$$\int_C \frac{e^{ikR}}{k^2+\alpha^2}\, k\, dk$$

over the semi-circle as $\rho \to \infty$. The only contribution comes from the pole $i\alpha$. Hence

$$G(\vec{x},\vec{x}\,') = -\frac{e^{-\alpha R}}{4\pi R} = -\frac{e^{-\alpha|\vec{x}-\vec{x}\,'|}}{4\pi|\vec{x}-\vec{x}\,'|} \qquad \text{(A-4)}$$

<u>Method 2.</u>

Equation (A-4) above is not in a convenient form for the purpose of computation. An alternative representation of the Green's function can be constructed as follows.

In polar coordinates

$$\delta(\vec{x}-\vec{x}\,') = \frac{\delta(r-r')\,\delta(\cos\theta-\cos\theta')\,\delta(\phi-\phi')}{r^2} \qquad \text{(A-5)}$$

Since $P_l^m(\cos\theta)\begin{Bmatrix}\cos m\phi\\ \sin m\phi\end{Bmatrix}$ form a complete set, let

$$G(\vec{x},\vec{x}\,') = \sum_{l=0}^{\infty}\sum_{m=0}^{l} g_l(r)\left[C_{lm}^{1} P_l^m(\cos\theta) P_l^m(\cos\theta')\cos m\phi\cos m\phi'\right] + f_l(r)\left[C_{lm}^{2} P_l^m(\cos\theta) P_l^m(\cos\theta')\sin m\phi\sin m\phi'\right] \qquad \text{(A-6)}$$

where $P_l^m(\cos\theta)$ denotes the associated Legendre function. Also,

$$\nabla^2 g_\ell(r) P_\ell^m(\cos\theta) \cos m\phi = \left[\frac{d^2 g_\ell}{dr^2} + \frac{2}{r}\frac{dg_\ell}{dr} - \frac{\ell(\ell+1)}{r^2} g_\ell \right] P_\ell^m(\cos\theta) \cos m\phi$$

Substitute (A-5) and (A-6) into (A-1):

$$\sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell} \left[\frac{d^2 g_\ell}{dr^2} + \frac{2}{r}\frac{dg_\ell}{dr} - \frac{\ell(\ell+1)}{r^2} g_\ell - \alpha^2 g_\ell \right] C_{\ell m}^2 \, P_\ell^m(\cos\theta) \, P_\ell^m(\cos\theta') \cos m\phi \cos m\phi'$$

$$+ \sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell} \left[\frac{d^2 f_\ell}{dr^2} + \frac{2}{r}\frac{df_\ell}{dr} - \frac{\ell(\ell+1)}{r^2} f_\ell - \alpha^2 f_\ell \right] C_{\ell m}^2 \, P_\ell^m(\cos\theta) \, P_\ell^m(\cos\theta') \sin m\phi \sin m\phi'$$

$$= \frac{\delta(r-r')\,\delta(\cos\theta - \cos\theta')\,\delta(\phi-\phi')}{r^2} \tag{A-7}$$

Take

$$C_{\ell m}^2 = \epsilon_m \frac{(2\ell+1)}{2\pi} \frac{(\ell-m)!}{(\ell+m)!} \qquad \begin{array}{ll} \epsilon_m = 1/2 & m = 0 \\ \quad\; = 1 & m > 0 \end{array}$$

Using the orthogonal property of $P_\ell^m(\cos\theta)$, $\cos m\phi$, $\sin m\phi$ from (A-7),

$$\frac{d^2 g_\ell}{dr^2} + \frac{2}{r}\frac{dg_\ell}{dr} - \frac{\ell(\ell+1)}{r^2} g_\ell - \alpha^2 g_\ell = \frac{\delta(r-r')}{r^2} \tag{A-8}$$

and a similar equation for $f_\ell(r)$.

The homogeneous part of equation (A-8) has the solutions

$$g_\ell = k_\ell = \sqrt{\frac{\pi}{2r\alpha}}\, K_{\ell+1/2}(\alpha r)$$

$$i_\ell = \sqrt{\frac{\pi}{2r\alpha}}\, I_{\ell+1/2}(\alpha r)$$

where K and I are modified Bessel functions.

The discontinuity in slope is

$$\lim_{\epsilon\to 0}\left[\frac{dg_\ell}{dr}\right]_{r'-\epsilon}^{r'+\epsilon} = \frac{1}{r'^2} \tag{A-9}$$

Therefore, $g_\ell(r)$, being continuous, may be taken as

$$g_\ell = \begin{cases} A\,\dfrac{\pi}{2k}\,\dfrac{K_{\ell+\frac{1}{2}}(\alpha r)}{\sqrt{r}}\,\dfrac{I_{\ell+\frac{1}{2}}(\alpha r')}{\sqrt{r'}} & r > r' \\[2ex] A\,\dfrac{\pi}{2k}\,\dfrac{K_{\ell+\frac{1}{2}}(\alpha r')}{\sqrt{r'}}\,\dfrac{I_{\ell+\frac{1}{2}}(\alpha r)}{\sqrt{r}} & r < r' \end{cases} \tag{A-10}$$

The Wronskian of k_ℓ and i_ℓ is

$$\frac{2k}{\pi} W[k_\ell' i_\ell - k_\ell i_\ell'] = -\frac{1}{r^2} \tag{A-11}$$

Hence, on substituting (A-10) into (A-9) and using (A-11),

$$A = -\frac{2k}{\pi}$$

Therefore,

$$G(\vec{x},\vec{x}') = -\sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell}\left[\frac{K_{\ell+\frac{1}{2}}(\alpha r_>)}{\sqrt{r_>}}\,\frac{I_{\ell+\frac{1}{2}}(\alpha r_<)}{\sqrt{r_<}}\right] C_{\ell m}^2\, P_\ell^m(\cos\theta)\, P_\ell^m(\cos\theta')$$
$$\times(\cos m\phi\cos m\phi' + \sin m\phi\sin m\phi') \tag{A-12}$$

Since the Green's function is unique, from equations (A-4) and (A-12)

$$\frac{e^{-\alpha|\vec{x}-\vec{x}'|}}{4\pi|\vec{x}-\vec{x}'|} = \sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell}\left[\frac{K_{\ell+\frac{1}{2}}(\alpha r_>)}{\sqrt{r_>}}\,\frac{I_{\ell+\frac{1}{2}}(\alpha r_<)}{\sqrt{r_<}}\right] C_{\ell m}^2\, P_\ell^m(\cos\theta)\, P_\ell^m(\cos\theta')$$
$$\times(\cos m\phi\cos m\phi' + \sin m\phi\sin m\phi') \tag{A-13}$$

where $r_<$ ($r_>$) is the smaller (larger) of $|\vec{x}|$, $|\vec{x}'|$

For $\alpha = 0$, equation (A-13) reduces to

$$\frac{1}{4\pi|\vec{x}-\vec{x}'|} = \sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell}\frac{\epsilon_m}{2\pi}\,\frac{(\ell-m)!}{(\ell+m)!}\Big[P_\ell^m(\cos\theta)\,P_\ell^m(\cos\theta')\,(\cos m\phi\cos m\phi'$$
$$+\sin m\phi\sin m\phi')\Big]\left[\frac{r_<^\ell}{r_>^{\ell+1}}\right] \tag{A-14}$$

APPENDIX B.

Lamb's General Solution of Stokes' Equation

Lamb[15] has provided a general solution of the Stokes Equation in polar coordinates. Kaufman[16] has put this into a vector form. Here, Lamb's solution will be written out explicitly together with some of its properties.

Stokes Equation

$$\nabla p = \mu \nabla^2 \vec{u}$$

$$\nabla \cdot \vec{u} = 0$$

Lamb's Solution

Let $Y^e_{l,m}(\theta,\phi) = P_l^m(\cos\theta)\cos m\phi$, $Y^o_{lm}(\theta,\phi) = P_l^m(\cos\theta)\sin m\phi$

The general solution can be written as

$$p = \sum_{l=0}^{\infty}\sum_{m=0}^{l} \left[E^e_{lm} r^l + \frac{F^e_{lm}}{r^{l+1}}\right] Y^e_{l,m}(\theta,\phi) + \left[E^o_{lm} r^l + \frac{F^o_{lm}}{r^{l+1}}\right] Y^o_{lm}(\theta,\phi)$$

$$u_r = \frac{1}{\mu}\sum_{l=0}^{\infty}\sum_{m=0}^{l}\left[\frac{l\, r^{l+1} E^e_{lm}}{2(2l+3)} + \frac{(l+1)F^e_{lm}}{2(2l-1)r^l}\right] Y^e_{lm}(\theta,\phi) + \left[\frac{l\, r^{l+1} E^o_{lm}}{2(2l+3)} + \frac{(l+1)F^o_{lm}}{2(2l-1)r^l}\right] Y^o_{lm}$$

$$+ \sum_{l=0}^{\infty}\sum_{m=0}^{l}\left[A^e_{lm}\, l\, r^{l-1} - B^e_{lm}\frac{(l+1)}{r^{l+2}}\right] Y^e_{lm}(\theta,\phi) + \left[A^o_{lm}\, l\, r^{l-1} - B^o_{lm}\frac{(l+1)}{r^{l+2}}\right] Y^o_{lm}(\theta,\phi)$$

$$u_\theta = \frac{1}{\mu\sin\theta}\sum_{l=0}^{\infty}\sum_{m=0}^{l}\left[\frac{(l+3)E^e_{lm} r^{l+1}}{2(l+1)(2l+3)} - \frac{(l-2)F^e_{lm}}{2l(2l-1)r^l}\right]\left[\frac{l(l-m+1)}{(2l+1)}Y^e_{l+1,m} - \frac{(l+m)(l+1)}{(2l+1)}Y^e_{l-1,m}\right]$$

$$+ \left[\frac{(l+3)E^o_{lm} r^{l+1}}{2(l+1)(2l+3)} - \frac{(l-2)F^o_{lm}}{2l(2l-1)r^l}\right]\left[\frac{l(l-m+1)}{(2l+1)}Y^o_{l+1,m} - \frac{(l+m)(l+1)}{(2l+1)}Y^o_{l-1,m}\right]$$

$$+ \frac{1}{\sin\theta}\sum_{l=0}^{\infty}\sum_{m=0}^{l}\left[A^e_{lm} r^{l-1} + \frac{B^e_{lm}}{r^{l+2}}\right]\left[\frac{l(l-m+1)}{(2l+1)}Y^e_{l+1,m} - \frac{(l+m)(l+1)}{(2l+1)}Y^e_{l-1,m}\right]$$

$$+ \left[A^o_{lm} r^{l-1} + \frac{B^o_{lm}}{r^{l+2}}\right]\left[\frac{l(l-m+1)}{(2l+1)}Y^o_{l+1,m} - \frac{(l+m)(l+1)}{(2l+1)}Y^o_{l-1,m}\right]$$

$$+ \frac{1}{\sin\theta}\sum_{l=0}^{\infty}\sum_{m=1}^{l} m\left[C^e_{lm} r^l + \frac{D^e_{lm}}{r^{l+1}}\right]\left[-Y^o_{lm}(\theta,\phi)\right] + m\left[C^o_{lm} r^l + \frac{D^o_{lm}}{r^{l+1}}\right] Y^e_{lm}$$

$$U_\phi = \frac{1}{\mu \sin\theta} \sum_{l=0}^{\infty} \sum_{m=0}^{l} -m \left[\frac{(l+3)\, r^{l+1} E^{e}_{lm}}{2(l+1)(2l+3)} - \frac{(l-2)\, F^{e}_{lm}}{2l(2l-1)\, r^{l}} \right] Y^{o}_{lm}(\theta,\phi)$$

$$+ m \left[\frac{(l+3)\, r^{l+1} E^{o}_{lm}}{2(l+1)(2l+3)} - \frac{(l-2)\, F^{o}_{lm}}{2l(2l-1)\, r^{l}} \right] Y^{e}_{lm}(\theta,\phi)$$

$$+ \frac{1}{\sin\theta} \sum_{l=0}^{\infty} \sum_{m=1}^{l} -m \left[A^{e}_{lm}\, r^{l-1} + \frac{B^{e}_{lm}}{r^{l+2}} \right] Y^{o}_{lm} + m \left[A^{o}_{lm}\, r^{l-1} + \frac{B^{o}_{lm}}{r^{l+2}} \right] Y^{e}_{lm}$$

$$- \frac{1}{\sin\theta} \sum_{l=0}^{\infty} \sum_{m=0}^{l} \left[C^{e}_{lm}\, r^{l} + \frac{D^{e}_{lm}}{r^{l+1}} \right] \left[\frac{l(l-m+1)}{(2l+1)} Y^{e}_{l+1,m} - \frac{(l+1)(l+m)}{(2l+1)} Y^{e}_{l-1,m} \right]$$

$$+ \left[C^{o}_{lm}\, r^{l} + \frac{D^{o}_{lm}}{r^{l+1}} \right] \left[\frac{l(l-m+1)}{(2l+1)} Y^{o}_{l+1,m} - \frac{(l+1)(l+m)}{(2l+1)} Y^{o}_{l-1,m} \right] \qquad \text{(B-1)}$$

Let x- axis be the polar axis. Assume that a flow field is given by the general solution above. Consider a sphere of radius r and denote the stress acting onthis sphere in the x direction by P_{rx}. Then

$$\text{drag } (x\text{-direction}) = \int_{\substack{\text{surface of} \\ \text{sphere } r}} P_{rx}\, ds \qquad \text{(B-2)}$$

Substitute (B-1) into (B-2) and, by the orthogonal properties of the angular functions, the following formula holds:

$$D_x = -4\pi F^{e}_{10} \qquad \text{(B-3)}$$

where F^{e}_{10} is the coefficient defined in (B-1).

APPENDIX C.

Elastic Distortion of a Sphere

(Part of this appendix is based on Chapter 13 of reference 30.)

Navier Equation

$$(\lambda + G)\,\nabla(\nabla\cdot\vec{S}) + G\,\nabla^2\vec{S} = 0$$

A general solution of the homogeneous Navier equation (finite at $r = 0$) can be written as

$$\vec{S} = \hat{e}_r \left\{ \sum_{l=0}^{\infty}\sum_{m=0}^{l} \left[B^e_{lm}\, r^{l+1} P_l^m(\cos\theta)\cos m\phi + B^o_{lm}\, r^{l+1} P_l^m(\cos\theta)\sin m\phi \right] \right.$$

$$\left. + \sum_{l=1}^{\infty}\sum_{m=0}^{l} \left[C^e_{lm}\, r^{l-1} P_l^m(\cos\theta)\cos m\phi + C^o_{lm}\, r^{l-1} P_l^m(\cos\theta)\sin m\phi \right] \right\}$$

$$+ \frac{\hat{e}_\theta}{\sin\theta} \left\{ \sum_{l=0}^{\infty}\sum_{m=0}^{l} \left[A^e_{lm}\, r^l \left(-\frac{m(2l+1)}{l(l+1)} P_l^m(\cos\theta)\sin m\phi \right) + A^o_{lm}\, r^l \left(\frac{m(2l+1)}{l(l+1)} P_l^m(\cos\theta)\cos m\phi \right) \right] \right.$$

$$+ \left[\frac{B^e_{lm}\, l\, r^{l+1}}{(2l+1)} \left(\frac{\lambda(l+3)+G(l+5)}{\lambda l + G(l-2)} \right) \left(\frac{(l-m+1)}{l+1} P^m_{l+1} - \frac{(l+m)}{l} P^m_{l-1} \right) \cos m\phi \right]$$

$$+ \left[\frac{B^o_{lm}\, l\, r^{l+1}}{2l+1} \left(\frac{\lambda(l+3)+G(l+5)}{\lambda l + G(l-2)} \right) \left(\frac{l-m+1}{l+1} P^m_{l+1} - \frac{l+m}{l} P^m_{l-1} \right) \sin m\phi \right]$$

$$+ \sum_{l=1}^{\infty}\sum_{m=0}^{l} \left[\frac{C^e_{lm}(l+1)\, r^{l-1}}{2l+1} \left(\frac{l-m+1}{l+1} P^m_{l+1}(\cos\theta) - \frac{(l+m)}{l} P^m_{l-1}(\cos\theta) \right) \cos m\phi \right]$$

$$\left. + \left[\frac{C^o_{lm}(l+1)\, r^{l-1}}{2l+1} \left(\frac{l-m+1}{l+1} P^m_{l+1}(\cos\theta) - \frac{(l+m)}{l} P^m_{l-1}(\cos\theta) \right) \sin m\phi \right] \right\}$$

$$+ \frac{\hat{e}_\phi}{\sin\theta} \left\{ \sum_{l=0}^{\infty}\sum_{m=0}^{l} \left[A^e_{lm} \left(-\frac{(l-m+1)}{l+1} P^m_{l+1}(\cos\theta) + \frac{l+m}{l} P^m_{l-1}(\cos\theta) \right) \cos m\phi \right] + \right.$$

cont.

$$+\left[A^{o}_{\ell m}\left(-\frac{\ell-m+1}{\ell+1}P^{m}_{\ell+1}(\cos\theta)+\frac{\ell+m}{\ell}P^{m}_{\ell-1}(\cos\theta)\right)\sin m\phi\right]$$

$$+\left[-\frac{B^{e}_{\ell m}\, m\, r^{\ell+1}}{\ell+1}\left(\frac{\lambda(\ell+3)+G(\ell+5)}{\lambda\ell+G(\ell-2)}\right)P^{m}_{\ell}(\cos\theta)\sin m\phi\right]$$

$$+\left[\frac{B^{o}_{\ell m}\, m\, r^{\ell+1}}{\ell+1}\left(\frac{\lambda(\ell+3)+G(\ell+5)}{\lambda\ell+G(\ell-2)}\right)P^{m}_{\ell}(\cos\theta)\cos m\phi\right] \qquad \text{(C-1)}$$

$$+\sum_{\ell=1}^{\infty}\sum_{m=0}^{\ell}\left[-\frac{C^{e}_{\ell m}\, m\, r^{\ell-1}}{\ell}P^{m}_{\ell}(\cos\theta)\sin m\phi+\frac{C^{o}_{\ell m}\, m\, r^{\ell-1}}{\ell}P^{m}_{\ell}(\cos\theta)\cos m\phi\right]\Bigg]\Bigg\}$$

The corresponding stress components are

$$\bar{\bar{\sigma}}\cdot\hat{e}_r=\hat{e}_r\Bigg\{\sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell}\left[B^{e}_{\ell m}\left(\frac{\lambda G(2\ell^2-2\ell-6)+2G^2(\ell+1)(\ell-2)}{\lambda\ell+G(\ell-2)}\right)r^{\ell}P^{m}_{\ell}(\cos\theta)\cos m\phi\right]$$

$$+\left[B^{o}_{\ell m}\left(\frac{\lambda G(2\ell^2-2\ell-6)+2G^2(\ell+1)(\ell-2)}{\lambda\ell+G(\ell-2)}\right)r^{\ell}P^{m}_{\ell}(\cos\theta)\sin m\phi\right]$$

$$+\sum_{\ell=1}^{\infty}\sum_{m=0}^{\ell}\left[2G(\ell-1)r^{\ell-2}P^{m}_{\ell}(\cos\theta)\right]\left[C^{e}_{\ell m}\cos m\phi+C^{o}_{\ell m}\sin m\phi\right]\Bigg\}$$

$$+\frac{\hat{e}_\theta}{\sin\theta}\Bigg\{\sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell}\left[-A^{e}_{\ell m}\sin m\phi+A^{o}_{\ell m}\cos m\phi\right]\left[G(\ell-1)r^{\ell-1}\left(\frac{m(2\ell+1)}{\ell(\ell+1)}P^{m}_{\ell}(\cos\theta)\right)\right]$$

$$+\left[B^{e}_{\ell m}\cos m\phi+B^{o}_{\ell m}\sin m\phi\right]\left[G r^{\ell}\left(1+\frac{\ell}{(\ell+1)}\frac{\lambda(\ell+3)+G(\ell+5)}{\lambda\ell+G(\ell-2)}\right)\right]\times$$

$$\times\left[\frac{\ell(\ell-m+1)}{2\ell+1}P^{m}_{\ell+1}(\cos\theta)-\frac{(\ell+m)(\ell+1)}{(2\ell+1)}P^{m}_{\ell-1}(\cos\theta)\right]$$

$$+\sum_{\ell=1}^{\infty}\sum_{m=0}^{\ell}\left[C^{e}_{\ell m}\cos m\phi+C^{o}_{\ell m}\sin m\phi\right]\left[\frac{2G(\ell-1)r^{\ell-2}}{2\ell+1}\right]\times$$

$$\times\left[(\ell-m+1)P^{m}_{\ell+1}(\cos\theta)-\frac{(\ell+m)(\ell+1)}{\ell}P^{m}_{\ell-1}(\cos\theta)\right]\Bigg\}$$

$+$

cont.

$$
\begin{aligned}
&+ \frac{\hat{e}_\phi}{\sin\theta} \left\{ \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} \left[A_{\ell m}^{e} \cos m\phi + A_{\ell m}^{o} \sin m\phi \right] \left[G(\ell-1) r^{\ell-1} \right] \times \right. \\
&\qquad \times \left[-\frac{(\ell-m+1)}{(\ell+1)} P_{\ell+1}^{m}(\cos\theta) + \frac{(\ell+m)}{\ell} P_{\ell-1}^{m}(\cos\theta) \right] \\
&\qquad + m G r^{\ell} \left[-B_{\ell m}^{e} \sin m\phi + B_{\ell m}^{o} \cos m\phi \right] \left[1 + \frac{\ell}{\ell+1} \, \frac{\lambda(\ell+3) + G(\ell+5)}{\lambda\ell + G(\ell-2)} \right] \\
&\quad \left. + \sum_{\ell=1}^{\infty} \sum_{m=0}^{\ell} \left[-C_{\ell m}^{e} \sin m\phi + C_{\ell m}^{o} \cos m\phi \right] \left[\frac{2m(\ell-1)}{\ell} G r^{\ell-2} P_{\ell}^{m}(\cos\theta) \right] \right\}
\end{aligned}
\tag{C-2}
$$

APPENDIX D.

General Solution of "Screened" Stokes' Equation

The "screened" Stokes equation is

$$\nabla \cdot \vec{u} = 0$$
$$\nabla p = \mu \nabla^2 \vec{u} - \mu \alpha^2 \vec{u} \qquad \text{(D-1)}$$

Let the solution be

$$\vec{u} = \nabla \phi + \nabla \times \vec{A} \qquad \text{(D-2)}$$

$$p = -\mu \alpha^2 \phi \qquad \text{(D-3)}$$

Then equation (D-1) will be satisfied identically if

$$\nabla^2 \phi = 0 \qquad \text{(D-4)}$$

$$\nabla^2 \vec{A} - \alpha^2 \vec{A} = 0 \qquad \text{(D-5)}$$

The complete solution of (D-4) which vanishes at infinity is

$$\phi = \sum_{l,m} \frac{B_{lm}}{r^{l+1}} X_l^m(\theta, \phi) \qquad \text{(D-6)}$$

where

$$X_l^m(\theta, \phi) = P_l^m(\cos\theta)\, e^{im\phi}$$

Therefore, the pressure and longitudinal velocity components are

$$p = -\frac{\mu \alpha^2 B_{lm}}{r^{l+1}} X_l^m(\theta, \phi)$$

$$\vec{U}_{long.} = \hat{e}_r \left[-\frac{(l+1) B_{lm}}{r^{l+2}} X_l^m(\theta, \phi) \right] + \hat{e}_\theta \left[\frac{B_{lm}}{r^{l+2}} \frac{\partial}{\partial \theta} X_l^m(\theta, \phi) \right]$$
$$+ \hat{e}_\phi \left[\frac{im\, B_{lm}}{r^{l+2} \sin\theta} X_l^m(\theta, \phi) \right] \qquad \text{(D-7)}$$

Equation (D-5) has two linearly independent solutions. If

Z_ℓ = modified spherical Bessel function

$$= \sqrt{\frac{\pi}{2\alpha r}}\; K_{\ell+\frac{1}{2}}(\alpha r)$$

the two transverse velocity vectors are

$$\vec{U}^{(1)}_{trans.} = \hat{e}_r \left[\frac{\ell(\ell+1)}{r}\, Z_\ell(\alpha r)\, X^m_\ell(\theta,\phi) \right]$$

$$+ \hat{e}_\theta \left[\frac{1}{r}\, \frac{d}{dr}\Big(r\, Z_\ell(\alpha r) \Big) \frac{\partial}{\partial\theta} X^m_\ell(\theta,\phi) \right]$$

$$+ \hat{e}_\phi \left[\frac{im}{r\sin\theta}\, \frac{d}{dr}\Big(r\, Z_\ell(\alpha r) \Big) X^m_\ell(\theta,\phi) \right] \qquad \text{(D-8)}$$

$$\vec{U}^{(1)}_{trans.} = \hat{e}_\theta \left[\frac{m}{\sin\theta}\, Z_\ell(\alpha r)\, X^m_\ell(\theta,\phi) \right]$$

$$+ \hat{e}_\phi \left[i\, Z_\ell(\alpha r)\, \frac{\partial}{\partial\theta} X^m_\ell(\theta,\phi) \right] \qquad \text{(D-9)}$$