MULTIPLICATION IN RIESZ SPACES

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Abstract.

A. G. Vulih has shown how an essentially unique intrinsic multiplication can be defined in certain types of Riesz spaces (vector lattices) L. In general, the multiplication is not universally defined in L, but L can always be imbedded in a larger space $L^{\#}$ in which multiplication is universally defined.

If φ is a normal integral on L, then φ can be extended to a normal integral on a larger space $L_1(\varphi)$ in $L^{\#}$, and $L_1(\varphi)$ may be regarded as an abstract integral space. A very general form of the Radon-Nikodym theorem can be proved in $L_1(\varphi)$, and this can be used to give a relatively simple proof of a theorem of Segal giving a necessary and sufficient condition that the Radon-Nikodym theorem hold in a measure space.

In another application, the multiplication is used to give a representation of certain Riesz spaces as rings of operators on a Hilbert space.

CONTENTS

Introduction.

A Riesz space (or, vector lattice) is a real linear space L (with elements x, y, z,...) partially ordered by $\leq$ such that

(i) $x \leq y$ implies $x + z \leq y + z$ for all $z \in L$.

(ii) $x \geq 0$ implies $ax \geq 0$ for every real number $a \geq 0$.

(iii) L is a lattice under $\leq$.

For the basic properties of Riesz spaces we refer to N. Bourbaki ([2], Ch.II), H. Nakano ([10],[11]), or W. A. J. Luxemburg and A. C. Zaanen ([8], Note VI). In particular we recall that the linear structure of L forces the lattice structure to be distributive, i.e. if $x_\alpha \in L$, $\alpha \in \{\alpha\}$, and $\sup(x_\alpha)$ exists in L, then for any y L we have $\inf(\sup_\alpha(x_\alpha), y) = \sup_\alpha \inf(x_\alpha, y)$.

We recall the usual notations : $L^+ = \{x \in L : x \geq 0\}$, $x_+ = \sup(x, 0) \in L^+$, $x_- = \sup(-x, 0) \in L^+$, $|x| = \sup(x, -x)$. If $\inf(|x|, |y|) = 0$, then x and y are said to be disjoint, and this is denoted by $x \perp y$.

The sequence $\{x_n : n \in N\}$ of elements of L is <u>increasing</u> if $x_1 \leq x_2 \leq \ldots$, and this is denoted by $x_n \uparrow$. If $x_n \uparrow$ and $x = \sup(x_n)$ exists in L, we write $x_n \uparrow x$. An indexed set $\{x_\alpha : \alpha \in \{\alpha\}\}$ in L is said to be directed upwards if for every pair α, $\alpha' \in \{\alpha\}$ there exists $\alpha'' \in \{\alpha\}$ such that $x_{\alpha''}$ $\sup(x_\alpha, x_{\alpha'})$. Again this is denoted by $x_\alpha \uparrow$, and if $x = \sup(x_\alpha)$ exists in L we write $x_\alpha \uparrow x$.

A linear subspace K of L is called an (order) ideal if $x \in K$ and $|y| \leq |x|$ implies $y \in K$. K is called a normal subspace if in addition $x_\alpha \uparrow x$ and $x_\alpha \in K$ implies $x \in K$.

A Riesz space L is called Dedekind complete if every non-empty subset of L which is bounded above has a supremum. L is called σ-Dedekind complete if every countable subset of L which is bounded above has a supremum. L is called Archimedean if $0 \leq x, y \in L$, $0 \leq nx \leq y$ for all $n \in N$ implies that $x = 0$. A property implied by σ-Dedekind completeness and which implies the Archimedean property is that for all $0 \leq x, y \in L$ the element $\sup_n \inf(y, nx)$ exist in L (c.f. [8], example 29.11 in Note IX).

In a Riesz space L there is no a priori method of multiplying elements together, but it is well known that there exist many representations of various types of Riesz spaces as function spaces (c.f. D. G. Johnson and J.E. Kist, [6]), and so it is natural to ask if a multiplication can be introduced into a Riesz space corresponding to pointwise multiplication of functions. The answer, as given by B. Z. Vulih [19], is that this can indeed be done for Dedekind complete spaces with a weak order unit, and in addition such a multiplication is unique up to a scale factor (determined by the choice of unit). (c.f. also E. Hewitt [5] ; here will be found references to earlier notes of Vulih.) It cannot be expected that such a multiplication will in general be universally defined in L (indeed, this is not even true for pointwise multiplication in $L_1(0, 1)$), but Vulih shows that by a construction due to A. G. Pinsker L may be imbedded as an order dense ideal in a larger space $L^\#$, and that multiplication may be universally defined in $L^{\#}$. The imbedding of L in $L^{\#}$ is in many ways analogous to the imbedding of a space of integrable functions, say, in the space of measurable functions.

Most of the results in part I, dealing with the definition and basic properties of multiplication in Riesz spaces, are due to Vulih, and specific references are not usually given. Many of the proofs, however, have been revised and shortened; in particular, theorem 4.2, which is new, allows a considerable simplification of theorem 4.3.

Vulih considers only Dedekind complete spaces, but it is seen in section §1 that the basic properties of unitary elements (including Freudenthal's important theorem 1.5) depend only on the existence of projections into normal subspaces. In addition, theorem 1.6 (a strengthened form of the corresponding theorem of Vulih which is essentially contained in corollary 1.6.1) allows some remarks about multiplication in σ-Dedekind complete spaces.

Finally, Vulih shows that in a Riesz space L every element has an inverse if and only if $L = L^{\#}$, but he does not isolate the implicit necessary and sufficient condition P_2, which (in its various equivalent forms) plays an important role in parts II, III, IV, and V.

In these later parts II, III, IV, and V, we include references in every case (hopefully) where the result is not new. In part II we discuss the definition and properties of $L^{\#}$. It turns out that any positive linear functional defined on all of $L^{\#}$ must have a very simple nature, and so a normal integral φ on L cannot in general be extended to a normal integral on all of $L^{\#}$. However, in part III we see that φ can always be extended to a certain space $L_1(\varphi)$ between L and $L^{\#}$, and $L_1(\varphi)$ is very similar to the usual L_1 integration spaces. In part IV we prove a strong form of the Radon-Nikodym theorem for Riesz spaces, and in part V we apply it to prove Segal's theorem giving a necessary and sufficient condition for the Radon-Nikodym theorem to hold in a measure space. Also in part V we include a discussion of rings of bounded self-adjoint operators on a Hilbert space, viewing the rings of operators as Riesz spaces.

I MULTIPLICATION IN RIESZ SPACES.

§1. Units and unitary elements.

Let L be a Riesz space which has the property that for $0 \leq x, y \in L$, $\sup_n \inf(nx, y)$ exists in L. We further suppose that L has a (weak order) unit, i.e. an element $1 \in L$ such that $\inf(x, 1) > 0$ whenever $x > 0$.

An element $e \in L$ is called unitary (with respect to 1) if $\inf(e, 1 - e) = 0$. We will denote by $U(L, 1) = U(L)$ the set of all unitary elements. We agree that e will always denote a unitary elememt.

The following theorem shows, among other things, that if L happens to be Dedekind complete then U(L) is a complete Boolean algebra.

Theorem 1.1 (i) If E is an arbitrary set of unitary elements, then E is bounded; and if sup(E) or inf(E) exist then they are unitary.

(ii) If $\inf(e, e') = 0$, then $e + e' \in U(L)$.

(iii) If $e \geq e'$, then $e - e' \in U(L)$.

(iv) If a and a' are real numbers ≥ 0, then $\inf(ae, a'e') = \min(a, a') \cdot \inf(e, e')$.

Proof (i) If $e \in U(L)$, then clearly $0 \leq e \leq 1$; hence E is bounded. Suppose sup(E) exists. Now, $\inf(\sup(E), 1 - \sup(E)) =$

$$= \sup_{e \in E} \inf(e, 1 - \sup(E)) \leq \sup_{e \in E} \inf(e, 1 - e) = 0; \text{ hence } \sup(E) \in U(L).$$

The fact that inf(E) is unitary (if it exists) follows by considering the set $\{1 - e : e \in E\}$.

(ii) If $\inf(e, e') = 0$, then $e + e' = \sup(e, e') \in U(L)$.

(iii) $0 \leq \inf(e', 1 - e) \leq \inf(e', 1 - e') = 0$, so $\inf(e', 1 - e) = 0$.

Therefore $e - e' = e - e' + \inf(e', 1 - e) = \inf(e, 1 - e') \in U(L)$.

(iv) Let $e'' = \inf(e, e')$. e'', $e - e''$, and $e' - e''$ are pairwise disjoint; hence $\inf(ae, a'e') = \inf(ae'' + a(e-e''), a'e'' + a'(e'-e'')) =$ $= \inf(ae'', a'e'') + \inf(ae'', a'(e'-e'')) + \inf(a(e-e''), a'e'') +$ $+ \inf(a(e-e''), a'(e'-e'')) = \min(a, a')e''$.

<u>Definition</u> For any $x \in L$ we define the <u>characteristic element of x</u> (or, <u>support of x</u>) to be $s(x) = \sup_n \inf(n|x|, 1)$.

We note that $s(x)$ is always a unitary element. Indeed, $\sup_n \inf(n|x|, 1)$ is the projection of 1 into $[x]$, the normal subspace of L generated by x, so that $s(x) \perp (1 - s(x))$ (c.f. [8], Corollary 29.7). Alternatively, we may observe that $2s(x) = \sup_n \inf(n\,x, 2\cdot 1)$, so that $\inf(2s(x), 1) = \sup_n \inf(n\,x, 1) = s(x)$, and hence that $\inf(s(x), 1-s(x))$ $= \inf(2s(x), 1) - s(x) = 0$. It is clear that $s(x) = 0$ if and only if $x = 0$.

<u>Note</u> Vulih [19] defines the characteristic element of x to be the smallest unitary element e for which $\inf(x, 1 - e) = 0$, and then proves that this is equivalent to the definition given above. In the course of this he also shows that $s(x)$ is the smallest unitary element e satisfying $e \geqslant \inf(x, 1)$. However, we will not require either of these characterizations.

<u>Theorem 1.2</u> Let $x_\alpha \geqslant 0$, $\alpha \in \{\alpha\}$, $x = \sup(x_\alpha)$, $y = \inf(x_\alpha)$. Then

(i) $s(x) = \sup_\alpha s(x_\alpha)$.

(ii) $s(y) \leq \inf_\alpha s(x_\alpha)$, and equality holds if $\{x_\alpha\}$ is a finite set.

<u>Proof</u> (i) $s(x) = \sup_n \inf(nx, 1) = \sup_n \inf(\sup_\alpha nx_\alpha, 1) =$
$\sup_n \sup_\alpha \inf(nx_\alpha, 1) = \sup_\alpha \sup_n \inf(nx_\alpha, 1) = \sup_\alpha s(x_\alpha)$.

(ii) Since $0 \leq y \leq x$, we have $s(y) \leq s(x\ )$ for each $\alpha \in \{\alpha\}$, and hence $s(y) \leq \inf s(x_\alpha)$. For the remaining part, it is sufficient to consider the case where $\{x_\alpha\} = \{x_1, x_2\}$. In this case $s(y) =$
$\sup_n \inf(ny, 1) = \sup_n \inf(nx_1, nx_2, 1) = \sup_n \inf[\inf(nx_1,1), \inf(nx_2,1)]$.
But $0 \leq \inf(nx_i, 1) \uparrow_n$, $n = 1,2$; hence it follows that $s(y) =$
$= \inf[\sup_n \inf(nx_1, 1), \sup_n \inf(nx_2, 1)] = \inf(s(x_1), s(x_2))$.

<u>Theorem 1.3</u> (i) For arbitrary $x,y \in L$, $\inf(|x|,|y|) = 0$ if and only if $\inf(s(x), s(y)) = 0$.

(ii) For arbitrary $x,y \in L$, $s(x+y) \leq \sup(s(x), s(y))$. Equality holds if $x,y \geq 0$.

<u>Proof</u> (i) This follows immediately from theorem 1.2(ii) since for any $x \in L$ $x = 0$ if and only if $s(x) = 0$.

(ii) First of all, if $0 \leq x \leq y$, then $s(y) \leq s(x + y) \leq s(2y) =$
$= s(y)$, so that $s(x + y) = s(y)$. Then for any $x,y \geq 0$, write $x + y =$
$\inf(x, y) + \sup(x, y)$, and the preceeding implies that $s(x + y) =$
$= s[\sup(x, y)] = \sup(s(x), s(y))$. In general then $s(x + y) = s(|s + y|) \leq$
$\leq s(|x| + |y|) = \sup(s(|x|), s(|y|))$.

The next theorem, which will turn out to be very important, is due to Freudenthal [4]. The proof given below (a modification of one due to Vulih [19]) depends on the following lemma.

<u>Lemma 1.4</u> Let $0 \leq x \in L$, $a > 0$, and $e = s[(x - a1)_+]$. Then $ae \leq x$.

<u>Proof</u> Since $s[(x - a1)_+] = s[(a^{-1}x - 1)_+]$, it is sufficient to consider the case where $a = 1$. Let $x' = \inf(x, 1)$. Then $\inf(x', 1) = x'$, and $x' - 1 = \inf(x, 1) - 1 = \inf(x - 1, 0) = -(x - 1)_-$. Thus $\inf(x', e) - e = (x' - 1) - [\inf(x', 1 - e) - (1 - e)] = -(x - 1)_- - [\inf(x', 1 - e) - (1 - e)]$. But the support of the left side of this equation is $\leq e$, while the support of the right side is $\leq (1 - e)$. Hence each side is zero, and in particular $e = \inf(x', e) \leq x' \leq x$ as required.

<u>Theorem 1.5</u> If $0 < x \in L$, then there exists a number $a > 0$ and a unitary element $e > 0$ such that $0 < ae \leq x$.

<u>Proof</u> Since L is Archimedean, $\inf_{a>0} (a1) = 0$, and so there is some $a > 0$ such that $x \nleq a1$. Thus $(x - a1)_+ > 0$; hence $e = s[(x - a1)_+] > 0$ and the preceding lemma implies immediately that $ae \leq x$.

For any real number a, we define $s_a(x)$ to be $s[(x - a1)_+]$. Note that $s_a(x) = s_1(a^{-1}x)$. As a first important application of theorem 1.5

we have the following theorem.

Theorem 1.6 If $0 \leqslant x \in L$, then $x = \sup\left\{rs_r(x) : 0 \quad r = \text{rational}\right\}$.

Proof As a preliminary, we prove that if $0 < be \leqslant x$ and $0 < c < b$, then $e \leqslant s_{b-c}(x)$. First of all, if $a > 1$ and $e > 0$, then $(ae - 1)_+ = \sup(ae, 1) - 1 = ae + 1 - \inf(ae, 1) - 1 = ae - e = (a - 1)e$, and hence $s\left[(ae - 1)_+\right] = e$. Next, if $0 < e \leqslant x$ and $a > 1$, then $s_1(ax) = s\left[(ax - 1)_+\right] \geqslant s\left[(ae - 1)_+\right] = e$. Finally, if $0 < be \leqslant x$ and $0 < c < b$, then $e \leqslant b^{-1}x$ and $\frac{b}{b - c} > 1$, so $e \leqslant s_1(\frac{b}{b - c} b^{-1}x) =$ $= s_{b-c}(x)$.

Now, by lemma 1.4, $as_a(x) \leqslant x$ for all $a \geqslant 0$. If x is not the required supremum, then there exists $z < x$ such that $rs_r(x) \leqslant z$ for all $r \geqslant 0$. But then, by theorem 1.5, there exists $r > 0$ and $e > 0$ such that $z + 3re \leqslant x$. Let $a = \sup\left\{a' : 0 \leqslant a'e \leqslant x\right\} \geqslant 3r$, and let r' be such that $a - r \leqslant r' \leqslant a$. Then $0 < r'e \leqslant x$ and $0 < r < r'$, so by the previous paragraph we have $e \leqslant s_{r'-r}(x)$, and hence $(r' - r)e$ $\leqslant (r' - r)s_{r'-r}(x) \leqslant z$. But then $(a + r)e = (a - 2r)e + 3re \leqslant$ $\leqslant (r' - r)e + 3re \leqslant z + 3re \leqslant x$, contradicting the maximality of a.

For most of our work we will not need the full strength of theorem 1.6, but only the following immediate corollary.

Corollary 1.6.1 If $0 \leqslant x \in L$, then $x = \sup\left\{ae : 0 \leqslant ae \leqslant x\right\}$.

Note Corollary 1.6.1 is equivalent to the statement that $0 \leq x \in L$ can be attained as the supremum of those finite positive linear combinations of unitary elements which are $\leq x$, i.e. $x =$ $= \sup \left\{x' : 0 \leq x' = \sum_{i=1}^{n} a_i e_i \leq x\right\}$. This follows from the fact that $\sum a_i e_i$ may be assumed to have disjoint summands (in which case $\sum a_i e_i = \sup(a_i e_i)$), and the fact that if a set of elements in a Riesz space is enlarged by including supremums of finite subsets then the two sets have the same supremum (if any).

§2. Definition and properties of multiplication in spaces with a unit.

We now suppose that L is a Dedekind complete Riesz space with a weak order unit 1.

Definition (i) If $e, e' \in U(L)$, the product ee' is defined by $ee' = \inf(e, e')$.

(ii) If $x \geq 0$ and $y \geq 0$, the product xy is defined by $xy = \sup\left\{abee' : 0 \leq ae \leq x, 0 \leq be' \leq y\right\}$ if this supremum exists. xy is not defined if the supremum does not exist.

(iii) In general, the product xy is defined by $xy = x_+y_+ - x_+y_- - x_-y_+ + x_-y_-$ if all the products on the right exist.

Note In Vulih's original notes (see references in [19] or [5]), he bases the multiplication on a representation of positive elements as transfinite linear combinations of unitary elements. In [19] he changes the definition to make it depend on a representation of positive elements as supremums of finite linear combinations of unitary elements (c.f. Note after Corollary 1.6.1). More precisely, for $x, y \geqslant 0$, if $0 \leqslant x' = \sum a_\lambda e_\lambda \leqslant x$ and $0 \leqslant y' = \sum b_\mu e_\mu \leqslant y$ are two finite sums, he defines $x'y'$ to be $\sum_{\lambda,\mu} a_\lambda b_\mu e_\lambda e_\mu$. He then defines xy to be $\sup \{x'y' : 0 \leqslant x' \leqslant x, 0 \leqslant y' \leqslant y\}$ if this supremum exists. He shows that the particular representation of x' as a finite sum does not affect the product $x'y'$; and with this observation it is easy to see that his definition of xy coincides with the definition given above, for we may write x' and y' in such a way that they have disjoint summands, so that $\sum a_\lambda b_\mu e_\lambda e_\mu$ has disjoint summands and hence equals $\sup_{\lambda,\mu} \{a_\lambda b_\mu e_\lambda e_\mu\}$.

In a more recent note [20] Vulih apparently describes a simplification of the multiplication give in [19], but I have been unable as yet to obtain a copy of this note.

The following properties of the multiplication are obvious:

(i) If xy exists, then yx exists and $yx = xy$.

(ii) If $x \geqslant 0$ and $y \geqslant 0$ and xy exists, then $xy \geqslant 0$.

(iii) If xy exists and a is a real number, then $(ax)y$ exists and $(ax)y = a(xy)$.

(iv) If $0 \leqslant x \leqslant y$, $0 \leqslant z$, and yz exists, then xz exists and $xz \leqslant yz$.

<u>Theorem 2.1</u> Let $x \geqslant 0$, $y_\alpha \geqslant 0$, $\alpha \in \{\alpha\}$, $y = \sup(y_\alpha)$. If xy_α exists for each α and the set $\{xy_\alpha\}$ is bounded, then xy exists and $xy =$

$= \sup(xy_\alpha)$.

<u>Proof</u> First consider the special case where each y_α is of the form $y_\alpha = b_\alpha e_\alpha$. Suppose $0 \leq ae' \leq x$ and $0 \leq be \leq y$. Then $be = \inf(be, y) =$
$= \inf(be, \sup_\alpha(b_\alpha e_\alpha)) = \sup_\alpha \inf(be, be_\alpha) = \sup_\alpha(\min(b, b_\alpha)ee_\alpha)$; hence
$e = \sup_\alpha(\min(1, b/b_\alpha)ee_\alpha)$. Therefore $abee' = ab \inf(e', e) =$
$= ab \inf(e', \sup_\alpha(\min(1, b/b_\alpha)ee_\alpha)) = ab \sup_\alpha(\min(1, b/b_\alpha)e'ee_\alpha) =$
$= \sup_\alpha(\min(b, b_\alpha)ee_\alpha(ae')) \leq \sup_\alpha(b_\alpha e_\alpha x)$. Thus xy exists and $xy \leq$
$\leq \sup(xb_\alpha e_\alpha)$.

Now consider the general case. $y = \sup\{be: 0 \leq be \leq y_\alpha$ for some $\alpha\}$; hence, by the preceeding, there exists $xy \leq$
$\leq \sup_\alpha \sup\{xbe : 0 \leq be \leq y_\alpha\} \leq \sup_\alpha(xy_\alpha)$. But clearly $\sup_\alpha(xy_\alpha) \leq xy$; hence $xy = \sup(xy_\alpha)$.

<u>Corollary 2.1.1</u> Let $x_\alpha \geq 0$, $y_\alpha \geq 0$, $\alpha \in \{\alpha\}$, $x = \sup(x_\alpha)$, $y = \sup(y_\alpha)$. If $x_\alpha y_\beta$ exists for every $\alpha, \beta \in \{\alpha\}$ and the set $\{x_\alpha y_\beta\}$ is bounded, then xy exists and $xy = \sup_{\alpha,\beta}(x_\alpha y_\beta)$.

The above theorem shows that multiplication is, in sense, continuous. We will see later (c.f. theorem 10.3) that in fact it enjoys a very strong order-continuity property.

<u>Theorem 2.2</u> If xy exists, then $s(xy) = \inf(s(x), s(y))$.

<u>Proof</u> First suppose $x, y \geq 0$. Then $s(xy) =$

$= s[\sup(abee' : 0 < ae \leq x,\ 0 < be' \leq y)] =$

$= \sup(s(abee') : 0 < ae \leq x,\ 0 < be' \leq y) =$

$= \sup(\inf(e, e') : 0 < ae \leq x,\ 0 < be' \leq y) =$

$= \inf[\sup(e : 0 < ae \leq x), \sup(e' : 0 < be' \leq y)] = \inf(s(x), s(y)).$

In general, $xy = x_+y_+ - x_-y_+ - x_+y_- + x_-y_-$. By the preceeding, the summands are disjoint; hence $(xy)_+ = x_+y_+ + x_-y_-$, and $(xy)_- = x_+y_- + x_-y_+$. Then, by theorem 1.3(ii), $s(xy) = s(|xy|) =$

$= s((xy)_+ + (xy)_-) = s(x_+y_+) + s(x_-y_-) + s(x_+y_-) + s(x_-y_+) =$

$= s(x_+)s(y_+) + s(x_-)s(y_-) + s(x_+)s(y_-) + s(x_-)s(y_+)$. Now, if $\inf(e', e'') = 0$, then $ee' + ee'' = \sup(ee', ee'') = \sup(\inf(e, e'), \inf(e, e'')) = \inf(e, \sup(e', e'')) = e(e' + e'')$. Hence it follows that $s(xy) =$

$= s(x_+)(s(y_+) + s(y_-)) + s(x_-)(s(y_+) + s(y_-)) = (s(x_+) + s(x_-))s(y) =$

$= s(x)s(y)$.

In the following straightforward but rather long theorem we gather together the basic properties of the multiplication.

<u>Theorem 2.3</u> (i) xe always exists, and $x1 = x$ and $x0 = 0$.

(ii) If xy exists, and $|x'| \leq x$ and $|y'| \leq y$, then $x'y'$ exists.

(iii) If xy and xz exist, then $x(y + z)$ exists, and $x(y + z) = xy + xz$.

(iv) $xe = x$ if and only if $e \geq s(x)$.

(v) If $xy = 0$ then $x \perp y$, and if $x \perp y$ then xy exists and $xy = 0$.

(vi) If xy, yz, and $(xy)z$ all exist, then $x(yz)$ exists and

$x(yz) = (xy)z$.

(vii) $|xy| = |x||y|$.

<u>Proof</u> (i) Clearly we may suppose $x \geqslant 0$. Then $x = \sup\{ae' : 0 \leq ae' \leq x\}$ $\geqslant \sup\{ae'e : 0 \leq ae' \leq x\} = xe$ by theorem 2.1. The rest is clear.

(ii) If $0 \leq ae \leq x'_+ \leq |x'| \leq x$ and $0 \leq be' \leq y'_+ \leq y$, then $abee' \leq xy$. Thus $\sup\{abee' : 0 \leq ae \leq x'_+, \ 0 \leq be' \leq y'_+\}$ exists, and hence, by definition, $x'_+y'_+$ exists. Similarly $x'_+y'_-$, $x'_-y'_+$, and $x'_-y'_-$ all exist, so $x'y'$ exists by definition.

(iii) First suppose that $x, y, z \geqslant 0$. We can see immediately that $x(y + z)$ exists; for $y + z \leqslant 2\sup(y, z)$, and $x\cdot\sup(y, z)$ exists (in fact equals $\sup(xy, xz)$ by theorem 2.1), so by part (ii) $x(y + z)$ exists. In order to prove $x(y + z) = xy + xz$ we first prove $x(ae + be') = xae + xbe'$. Since $ae + be'$ may be written as a disjoint sum (namely $ae + be' = (a + b)ee' + a(e - ee') + b(e' - ee')$), we may suppose without restriction of generality that $e \perp e'$. In this case $x(ae + be') = x\cdot\sup(ae, be') = \sup(xae, xbe') = xae + xbe'$ (since $xae \perp xbe'$ by theorem 2.2). Then for any $x, y, z \geqslant 0$ we have

$x(y + z) = x\cdot\sup(ae + be' : 0 \leq ae \leq y, 0 \leq be' \leq z) =$

$= \sup(xae + xbe' : 0 \leq ae \leq y, 0 \leq be' \leq z) =$

$= \sup(xae : 0 \leq ae \leq y) + \sup(xbe' : 0 \leq be' \leq z) = xy + xz.$

For general x, y, z, let $0 \leqslant u = y_+ + z_+ - (y + z)_+ =$ $= y_- + z_- - (y + z)_-$. (u is $\geqslant 0$ because of the minimality of the Jordan decomposition of an element into its positive and negative parts.) Then by the preceeding we have $xy + xz =$

$= x_+(y_+ + z_+) - x_+(y_- + z_-) - x_-(y_+ + z_+) + x_-(y_- + z_-) =$

$= x_+((y + z)_+ + u) - x_+((y + z)_- + u) - x_-((y + z)_+ + u) + x_-((y + z)_- + u) =$

$= x_+(y + z)_+ - x_+(y + z)_- - x_-(y + z)_+ + x_-(y + z)_- = x(y + z)$ by definition.

(iv) If $e \geq s(x)$, then $e \geq s(x_+)$. Thus if $0 < ae' \leq x_+$, then $e' \leq s(x_+) \leq e$, and hence $x_+e = \sup(ae'e : 0 < ae' \leq x_+) =$ $= \sup(ae' : 0 < ae' \leq x_+) = x_+$. Similarly $x_-e = x_-$, and hence $xe = x_+e - x_-e = x_+ - x_- = x$.

Conversely, if $xe = x$, then $s(x) = s(xe) = \inf(s(x), e) \leq e$.

(v) If $xy = 0$, then $0 = s(xy) = \inf(s(x), s(y))$, and hence $x \perp y$ by theorem 1.3(i). Conversely, suppose $x \perp y$. Then $x_+ \perp y_+$, so $0 = \sup(abee' : 0 < ae \leq x_+, 0 < be' \leq y_+) = x_+y_+$. Similarly x_+y_-, x_-y_+, and x_-y_- all exist and equal 0; hence xy exists and equals 0.

(vi) In view of part (iii), we may suppose without restriction of generality that $x, y \geq 0$. The result then follows immediately, in view of the fact that, by theorem 2.1, $(xy)z =$ $= \sup(abcee'e'' : 0 \leq ae \leq x, 0 \leq be' \leq y, 0 \leq ce'' \leq z)$.

(vii) This follows immediately from parts (ii) and (iii), in view of the fact that $(xy)_+ = x_+y_+ + x_-y_-$ and $(xy)_- = x_+y_- + x_-y_+$.

<u>Corollary 2.3.1</u> If $x \not\leq y$, then there exist $e > 0$ and $a > 0$ such that $xe \geq ye + ae$.

Proof First note that for any $z \in L$, $z \cdot s(z_+) = z_+ s(z_+) - z_- s(z_+) = z_+$. Now, if $x \nleq y$, then $(x - y)_+ \quad 0$, and hence $e = s\left[(x - y)_+\right] > 0$. Then $xe - ye = (x - y)e = (x - y)_+ > 0$, and the result now follows immediately from theorem 1.5.

Corollary 2.3.2 xe is the projection of x into $[e]$.

Proof It is sufficient to consider the case where $x \geqslant 0$. But then the projection of x into $[e]$ equals $\sup_n \inf(x, ne) =$
$= \sup_n \left[\inf(xe, ne) + \inf(x(1 - e), ne)\right] = \sup_n \inf(xe, ne) =$
$= \sup_n \inf(xe, n \cdot 1) = xe$.

Remark Hewitt has suggested (in [5]) that one might try to weaken the condition of Dedekind completeness. The definition of multiplication makes sense, of course, in any space with a unit, but for the multiplication to enjoy reasonable properties (such as those expressed in theorems 2.1, 2.2, and 2.3), it seems that some sort of lattice-completeness is necessary. However, in view of the fact (c.f. theorem 1.6) that every positive element can be achieved as a supemum of a countable set of multiples of unitary elements, it can be seen that in a σ-Dedekind complete space the results of section §2 continue to hold (with some modifications, mainly the replacing of some sets of elements by appropriate countable sets - for example, in theorem 2.1). (c.f. also [21].)

§3. Powers and roots.

It is easy to see that in general not every element in L has a square. Indeed, if for $x, y \in L$ x^2 and y^2 exist, then xy exists; for (supposing $x, y \geq 0$) if $0 \leq ae \leq x$ and $0 \leq be' \leq y$, then $abee' \leq \leq \frac{1}{2}(a^2 + b^2)ee' \leq \frac{1}{2}a^2e + \frac{1}{2}b^2e' \leq \frac{1}{2}x^2 + \frac{1}{2}y^2$, so that xy exists.

It is also easy to see (e.g. by induction) that if x^n does exist, then $(x_+)^n$ and $(x_-)^n$ also exist and $x^n = (x_+)^n + (-1)^n(x_-)^n$.

In the other direction we have the following theorem. (An element $y \in L$ is called an n^{th} root of x if $y^n = x$.)

<u>Theorem 3.1</u> For any element $x \geq 0$ and any integer $n > 0$ there exists a unique positive n^{th} root $x^{1/n}$.

<u>Proof</u> We will prove the theorem for $n = 2$, and it will be seen that the proof easily extends to any n. We have $x = \sup\limits_{a \geq 0}(as_a(x))$; and for any real number $a \geq 0$, $a^{\frac{1}{2}} \leq \max(a, 1)$; hence we may define

$y = \sup\limits_{a \geq 0}(a^{\frac{1}{2}}s_a(x)) \leq \sup(x, 1)$. Then $x = \sup\limits_{a \geq 0}(as_a(x)) =$

$= \sup\limits_{b,c \geq 0}(b^{\frac{1}{2}}c^{\frac{1}{2}}s_a(x) : a = \max(b, c)) = \sup\limits_{b,c \geq 0}(b^{\frac{1}{2}}c^{\frac{1}{2}}s_b(x)s_c(x))$ (since $a = \max(b, c)$ implies $s_a(x) = \inf(s_b(x), s_c(x))$, and this equals y^2 by corollary 2.1.1.

Suppose there also exists $0 \leq z \neq y$ such that $z^2 = x$. We may suppose for definiteness that $z \not\geq y$. Then there exists $e > 0$ and $a > 0$ such that $ze + ae \leq ye$. But then $xe = y^2e \geq (ze + ae)^2 = = z^2e + 2aze + a^2e > z^2e$, a contradiction.

§4. Inverse elements.

Definition Let $x \in L$. If there exists $y \in L$ such that $s(y) \leqslant s(x)$, and xy exists and equals $s(x)$, then y is the inverse of x, denoted by x^{-1}.

We note immediately that the inverse, if it exists, is unique. For suppose also $y'x = s(x)$ and $s(y') \leqslant s(x)$; then $s(y - y') \leqslant s(x)$, so $s(y - y') = \inf(s(y - y'), s(x)) = s((y - y')x) = 0$; i.e. $y - y' = 0$.

We also note that $s(x^{-1}) = \inf(s(x^{-1}), s(x)) = s(x^{-1}x) = s(x)$.

The following remarks are obvious:

(i) If $xy = s(x)$, then x^{-1} exists and $x^{-1} = ys(x)$.

(ii) If x^{-1} exists, and $a \neq 0$, then $(ax)^{-1}$ exists and equals $a^{-1}x^{-1}$.

(iii) If xy, x^{-1}, y^{-1}, $x^{-1}y^{-1}$ all exist, then $(xy)^{-1}$ exists and equals $x^{-1}y^{-1}$.

(iv) If x^{-1} exists, and $e \in U(L)$, then $(xe)^{-1}$ exists and equals $x^{-1}e$.

Theorem 4.1 (i) If $x \geqslant 0$ and x^{-1} exists, then $x^{-1} \geqslant 0$.

(ii) Let $x = y + z$, where $y \perp z$. If x^{-1} exists, then y^{-1} and z^{-1} exist, and $x^{-1} = y^{-1} + z^{-1}$. Conversely, if y^{-1} and z^{-1} exist, then x^{-1} exists.

(iii) If x^{-1} exists, then $(x_+)^{-1}$, $(x_-)^{-1}$, and $|x|^{-1}$ exist, and equal $(x^{-1})_+$, $(x^{-1})_-$, and $|x^{-1}|$ respectively. Conversely, if $(x_+)^{-1}$

and $(x_-)^{-1}$ exist, or if $|x|^{-1}$ exists, then x^{-1} exists.

<u>Proof</u> (i) $s(x) = xx^{-1} = x(x^{-1})_+ - x(x^{-1})_-$. Thus $s(x)$ has been written as the difference of two disjoint positive elements, and so $x(x^{-1})_- = 0$. But, since $s\left[(x^{-1})_-\right] \leq s(x^{-1}) \leq s(x)$, this implies $s\left[(x^{-1})_-\right] =$ $= s\left[x(x^{-1})_-\right] = 0$, i.e. $(x^{-1})_- = 0$. Thus $x^{-1} \geqslant 0$.

(ii) Note that $s(y) = s(x)s(y) = x^{-1}xs(y) = x^{-1}((y + z)s(y)) =$ $= x^{-1}(ys(y)) = x^{-1}y$. Thus y^{-1} exists (equal to $x^{-1}s(y)$). Similarly z^{-1} exists. It is clear that $s(y^{-1} + z^{-1}) = s(x)$ and that $(y^{-1} + z^{-1})(y + z) = s(x)$. The converse part is clear.

(iii) This follows immediately from (i) and (ii).

In general, not every element will have an inverse. However, we do have the following useful criterion. (c.f. also §7.)

<u>Theorem 4.2</u> For $x \geqslant 0$, let $S = \left\{y \quad 0 : s(y) \leqslant s(x), \text{ and } xy \leqslant s(x)\right\}$. Then x^{-1} exists if and only if $\sup(S)$ exists, and in this case $x^{-1} = \sup(S)$.

<u>Proof</u> First note that S is not empty, since always $0 \in S$. If x^{-1} exists, then $y \in S$ implies $y = ys(y) = yxx^{-1} \leqslant s(x)x^{-1} = x^{-1}$, so $\sup(S)$ exists.

Conversely, suppose there exists $\sup(S) = z$. Clearly $s(z) \leqslant s(x)$, and by theorem 2.1 the product xz exists and $xz \leqslant s(x)$. Suppose $xz < s(x)$. Then by theorem 1.5 there exists $0 < e \leqslant s(x)$ and $1 > a > 0$ such that $xz + 2ae \leqslant s(x)$, i.e. $xze \leqslant (1 - 2a)e$. Now, since $0 < e \leq s(x)$, there exists $0 < e' \leq e$ and $b' > 0$ such that $b'e' \leq x$. Let $b = \sup(b'' : b''e' \leq x) \geq b' > 0$. Let $c = ab(1 - a)^{-1} > 0$, and note that $b(b + c)^{-1} = 1 - a$. Now $(b + c)e' \nleq x$, so by corollary 2.3.1 there exists $0 < e'' \leq e'$ such that $(b + c)e'' \geq xe''$. Therefore $(b + c)^{-1}e''x \leq e'' \leq s(x)$, so $(b + c)^{-1}e'' \in S$, i.e. $(b + c)^{-1}e'' \leq ze''$. But then $(xe'')(ze'') \geq (be'')((b + c)^{-1}e'')$ = $(1 - a)e'' > (1 - 2a)e'' \geq xze''$, a contradiction. Hence $xz = s(x)$, and so $z = x^{-1}$ as required.

As an immediate application of this theorem we have the following.

<u>Theorem 4.3</u> Suppose x^{-1} exists, $|y| \geq |x|$, and $s(y) = s(x)$. Then y^{-1} exists, and $|y|^{-1} \leq |x|^{-1}$.

<u>Proof</u> In view of theorem 4.1, we may suppose $y \geq x \geq 0$. Let $S = \{z \geq 0 : s(z) \leq s(y), \text{ and } zy \leq s(y)\}$. If $z \in S$, then zx exists, since zy exists, so $z = zs(x) = zxx^{-1} \leq zyx^{-1} \leq s(y)x^{-1} = x^{-1}$. Thus $\sup(S)$ exists, and hence y^{-1} exists by theorem 4.2.

To complete the theorem, we observe that $y^{-1} = y^{-1}s(x) = y^{-1}xx^{-1} \leq$ $\leq y^{-1}yx^{-1} = x^{-1}$.

§5. <u>Uniqueness of the product.</u>

The main theorem in this section shows that, once a unit 1 is fixed in the Dedekind complete Riesz space L, then any multiplication with certain natural properties is uniquely determined by the Riesz space structure.

Theorem 5.1 Let xy denote (as usual) the product of x and y as defined in §2. Suppose in L another multiplication $x*y$ is defined for some pairs x, y, such that:

(i) $x*1$ always exists and equals x.

(ii) If $x*y$ exists, then $y*x$ exists and equals $x*y$.

(iii) If $x*y$, $(x*y)*z$, and $y*z$ all exist, then $x*(y*z)$ exists and equals $(x*y)*z$.

(iv) If $x*y$ and $x*z$ exist, then $x*(y + z)$ exists and equals $x*y + x*z$.

(v) If $x*y$ exists, and a is real, then $(ax)*y$ exists and equals $a(x*y)$.

(vi) If $x, y \geqslant 0$ and $x*y$ exists, then $x*y \geqslant 0$.

(vii) If $x*y$ exists, and $|x'| \leqslant x$ and $|y'| \leqslant y$, then $x'*y'$ exists.

(viii) If $x \perp y$, then $x*y$ exists and equals 0; and if $x*y = 0$ then $x \perp y$.

Then $x*y$ exists if and only xy exists, and $x*y = xy$.

Proof First, for any $e \in U(L)$, $e*e$ exists since $0 \leqslant e \leqslant 1$; and $e*e = e*e + e*(1 - e) = e*(e + (1 - e)) = e*1 = e$. Next, if $e'' = \inf(e, e')$ then $e*e' = \left[e'' + (e - e'')\right]*\left[e'' + (e' - e'')\right] =$

$= e''*e'' + e''*(e' - e'') + (e - e'')*e'' + (e - e'')*(e' - e'') =$

$= e''*e'' = \inf(e, e')$. Now let $x \geqslant 0$; $x*e$ exists since $|e| \leqslant 1$, and

$x{*}e = (xe){*}e + [x(1-e)]{*}e = (xe){*}e = (xe){*}e + (xe){*}(1-e) =$
$= (xe){*}1 = xe.$

Now suppose $x, y \geqslant 0$ and $x{*}y$ exists. Then, if $0 \leqslant ae \leqslant x$ and $0 \leqslant be' \leqslant y$, $abee' = (ae){*}(be')$ $x{*}y$. Hence there exists $xy =$ $= \sup(abee' : 0 \leqslant ae \leqslant x,\ 0 \leqslant be' \leqslant y) \leqslant x{*}y$. Suppose $xy < x{*}y$. Then there exists $ae > 0$ such that $x{*}y \geqslant xy + 2ae$. Since L is Archimedean, there exists c such that $ce \nleqslant xe$, and hence there exists $0 < e' \leqslant e$ such that $ce' \geqslant xe'$. Let $d = a/c > 0$, and let $b = \sup(b' : 0 \leqslant b'e' \leqslant y)$. Then $be' \leqslant y$; but $(b+d)e' \nleqslant y$, so there exists $0 < e'' \leqslant e'$ such that $(b+d)e'' > ye''$. But then $x(b+d)e'' = x{*}(b+d)e'' \geqslant x{*}ye'' \geqslant xye'' + 2ae'' \geqslant xbe'' + 2ae''$, and hence $xde'' \geqslant 2ae''$. But $xe'' \leqslant ce''$, so this implies $2ae'' \leqslant dxe'' \leqslant$ $\leqslant dce'' = ae''$, a contradiction. Thus if $x, y \geqslant 0$ and $x{*}y$ exists, then xy exists and $xy = x{*}y$.

Conversely, suppose $x, y \geqslant 0$ and xy exists. Let $z = x + s(x)$, so that $z \geqslant s(x)$, and hence there exists $z^{-1} \leqslant s(x)$. Clearly yz exists; and since $0 \leqslant z^{-1} \leqslant s(x)$, there exists $(yz){*}z^{-1} = (yz)z^{-1} =$ $= ys(x) = y{*}s(x)$. Also $z^{-1}{*}z$ exists, and equals $z^{-1}z = s(x)$, so $(yz){*}(z^{-1}{*}z)$ exists. But then there exists $((yz){*}z^{-1}){*}z = (y{*}s(x)){*}z =$ $= y{*}(s(x){*}z) = y{*}z$. Thus, since $0 \leqslant x \leqslant z$, $y{*}x$ exists.

The theorem now follows immediately.

The uniqueness of multiplication in the theorem above depends critically on the unit chosen. In general, two elements which have a certain product with respect to one unit will have a different product (or none at all) with respect to another unit. The next theorem relates multiplication with respect to different units. In the course of it we will use the following lemma.

<u>Lemma 5.2</u> Let 1 and 1' be two units of L. Then $U(L, 1') = \{1'e : e \in U(L, 1)\}$ (where multiplication is with respect to 1).

<u>Proof</u> e and e' will denote elements of $U(L, 1)$ and $U(L, 1')$ respectively. First we show that any element of the form $1'e$ is unitary with respect to 1'. This is immediate, since $\inf(1'e, 1' - 1'e) = 1'\inf(e, 1 - e) = 0$. Conversely, suppose $e' \in U(L, 1')$. Let $e = s(e') \in U(L, 1)$. e' is disjoint from $1' - e'$; hence $1'e = 1's(e') = [e' + (1' - e')]s(e') = e's(e') = e'$ as required.

<u>Theorem 5.3</u> Let 1 and 1' be units of L. Denote the product of x and y with respect to 1 by xy, and the product with respect to 1' by $x*y$. If xy and $x*y$ exist, then $1'(x*y) = xy$.

<u>Proof</u> First note that $1'(e_1'*e_2') = 1'(1'e_1*1'e_2) = 1'\cdot\inf(1'e_1, 1'e_2) = 1'1'\cdot\inf(e_1, e_2) = (1'e_1)(1'e_2) = e_1'e_2'$. Now suppose $x, y \geqslant 0$. Then by corollary 1.6.1 $x = \sup(ae_1' : 0 \leqslant ae_1' \leqslant x)$ and $y = \sup(be_2' : 0 \leqslant be_2' \leqslant y)$; hence, by corollary 2.1.1, $xy = \sup(abe_1'e_2' : 0 \leqslant ae_1' \leqslant x, 0 \leqslant be_2' \leqslant y) = \sup(1'(abe_1'*e_2') : 0 \leqslant ae_1' \leqslant x, 0 \leqslant be_2' \leqslant y) = 1'(x*y)$.

The proof for general x, y now follows immediately.

§6. Property P.

In this section we discuss a certain property which a Riesz space may possess. We will see later (c.f. section §7) that a Dedekind complete Riesz space has this property if and only if every two elements have a product and every element has an inverse. We will also see (c.f. section §8) that every Dedekind complete Riesz space can be imbedded as an order - dense ideal in a space having this property.

Definition We list below three properties (called P_1, P_2, P_3) which a Riesz space may have:

P_1 : There exists a unit $1 \in L$; and a subset $S \subset L^+$ has a supremum if for every $0 < e \in U(L)$ there exists $0 < e' \leq e$ and a real number b such that $xe' \leq be'$ for all $x \in S$.

P_2 : A subset $S \subset L^+$ has a supremum if for every $0 < y \in L$ there exists a real number b such that $\sup_{x \in S} \inf(by, x) \neq by$.

P_3 : If the elements of the subset $S \subset L^+$ are mutually disjoint, then $\sup(S)$ exists.

Theorem 6.1 In a Dedekind complete Riesz space L, P_1, P_2, and P_3 are mutually equivalent.

Proof We will prove $P_1 \Rightarrow P_2 \Rightarrow P_3 \Rightarrow P_1$.

(i) Suppose P_1 holds, and suppose $S \subset L^+$ is such that for every $0 < y \in L$ there exists b such that $\sup_{x \in S} \inf(by, x) \neq by$. In particular if $e > 0$, there exists b such that $\sup_{x \in S} \inf(be, x) < be$, and hence, by theorem 1.5, there exist $0 < e' \leq e$ and $c > 0$ such that $\sup_{x \in S} \inf(be, x) \leq be - ce'$. Then it follows that $xe' \leq be'$ for every $x \in S$; for if $xe' \nleq be'$, then there exists $0 < e'' \leq e'$ such that $xe'' \geq be''$, and then $be'' > (b - c)e'' \geq \inf(be'', xe'') = be''$, a contradiction. Hence, by P_1, $\sup(S)$ exists, and so P_2 holds.

(ii) Suppose P_2 holds, and suppose that $S \subset L^+$ is a set of mutually disjoint elements. For $0 < y \in L$, we want to find b such that $\sup_{x \in S} \inf(by, x) \neq by$. If y is disjoint from every $x \in S$, then $b = 1$ will do, so suppose that for some $z \in S$, $y' = \inf(y, z) > 0$. Then there exists b such that $by' \nleq z$, i.e. $\inf(by', z) \neq by'$, and since y' is disjoint from every other $x \in S$, $\sup_{x \in S} \inf(by', x) < by'$. But then, since $y \geq y'$, $\sup_{x \in S} \inf(by, x) < by$ (for if equality held, then $\sup_x \inf(by', x) = \sup_x \inf(by', by, x) = \inf(by', \sup_x \inf(by, x)) =$ $= \inf(by', by) = by'$, a contradiction). Hence, by P_2, $\sup(S)$ exists, and so P_3 holds.

(iii) Suppose that P_3 holds. We first show that L then has a unit. In fact, let $\{x_\alpha\}$ be a collection of positive elements, maximal with respect to the property that its elements are mutually disjoint. By property P_3 it follows immediately that $1 = \sup(x_\alpha)$ exists, and it is clear that 1 is a weak order unit (for otherwise there would exist $x > 0$ such that $x \perp x_\alpha$ for all α, and then $\{x_\alpha\}$ could be enlarged).

Now let $S \subset L^+$ be such that for every $0 < e \in U(L)$ there exists $0 < e' \leq e$ and b such that $xe' \leq be'$ for all $x \in S$. We shall say (for the moment) that a set E of unitary elements is <u>admissible</u> if its elements are mutually disjoint and for each $e \in E$ there exists a_e

such that $xe \leq a_e e$ for every $x \in S$. Let A be the collection of admissible sets. A is inductively ordered by inclusion, so there is a maximal admissible set E_o, and we can see, by the assumption on S, that $\sup(e : e \in E_o) = 1$. Now, since E_o is admissible, its elements are mutually disjoint, so by property P_3 there exists $y = \sup(a_e e : e \in E_o)$. We can see that y is an upper bound for S; for if not, then there is an $x \in S$ such that $x \nleq y$, so there exists $e' > 0$ and $b > 0$ such that $xe' \geqslant ye' + be'$. But since $\sup(e : e \in E_o) = 1$, there exists $e \in E_o$ such that $e'' = ee' \neq 0$, and then $ye'' = ye \cdot e' = a_e e \cdot e' \geqslant xe \cdot e' = xe'' \geqslant ye'' + be''$, a contradiction. Thus y is an upper bound for S, and so, since L is Dedekind complete, $\sup(S)$ exists. Hence P_1 holds.

By virtue of this theorem, we may refer to any of the properties P_1, P_2, P_3 as simply property P, so long as we are working in a Dedekind complete space.

§7. Perfect rings and fields.

We again suppose that L is a Dedekind complete Riesz space and has a unit 1. L is called a perfect ring if the product xy exists for every $x, y \in L$. L is called a field if it is a perfect ring and, in addition, every element has an inverse.

Theorem 7.1 L is a field if every element has an inverse.

Proof Given $x, y \in L$, we must show xy exists. Let $e = s\left[(1 - |x|)_+\right]$. Then $|xe| = |x|e \leq e$, and hence $(xe)y$ exists. On the other hand, $|x|(1 - e) \geqslant 1 - e$, so $|x|^{-1}(1 - e) \leq 1 - e$, and hence $x^{-1}(1 - e)y^{-1}$ exists, which implies that $xy(1 - e) = \left[x^{-1}y^{-1}(1 - e)\right]^{-1}$ exists. But then $xy = xye + xy(1 - e)$ exists.

Theorem 7.2 L is a field if and only if it has property P.

Proof Suppose L is a field. Let $S \subset L^+$ be a set whose elements are mutually disjoint. For each $x \in S$, $(x + s(x))^{-1} \leq 1$; hence there exists $y = \sup((x + s(x))^{-1} : x \in S)$. Now, since each $x \in S$ is disjoint from all the other elements of S, we have $ys(x) = (x + s(x))^{-1}$, and hence $y^{-1} \geqslant y^{-1}s(x) = x + s(x) \geqslant x$. Thus y^{-1} is an upper bound for S, so $\sup(S)$ exists. Hence property P_3 holds.

Conversely, suppose L has property P_1. By theorem 7.1 it is sufficient to show that every $x \in L$ has an inverse. Clearly it is sufficient to show this for every $x > 0$. Let
$S = \left\{y \geqslant 0 : s(y) \leq s(x), \text{ and } xy \leq s(x)\right\} \subset L^+$, and let $0 < e \in U(L)$. We want to show that there exists $0 < e' \leq e$ and b such that $ye' \leq be'$ for all $y \in S$. Clearly it is sufficient to consider e such that $es(x) > 0$. Then, by theorem 1.5 there exists $0 < e' \leq es(x)$ and $c > 0$ such that $ce' \leq xe'$. Let $b = c^{-1}$. Then for all $y \in S$, $ye' = bcye' =$ $= b(ce')y \leq b(xe')y = b(xy)e' \leq bs(x)e' = be'$. Therefore, by property P_1

sup(S) exists, and hence, by theorem 4.2, x^{-1} exists.

Since property P_2 does not refer to any particular unit of L, theorem 7.2 implies that if L has property P, then it is a perfect ring under multiplication with respect to any unit of L. On the other hand, it is not true that if L is a perfect ring under multiplication with respect to some particular unit then L necessarily has property P (c.f. theorem 7.4 below). However, we do have the following theorem.

Theorem 7.3 L is a perfect ring under multiplication with respect to every unit in L if and only if L has property P.

Proof It is sufficient to prove the "only if" part, and to do this it is sufficient, by theorem 7.2, to prove that every element $0 \leq x \in L$ has an inverse with respect to 1. Let $1' = x + (1 - s(x))$. Clearly $1'$ is a unit of L, so, denoting multiplication with respect to $1'$ by $x*y$, and taking $x = y = 1$ in theorem 5.3, we have $1'(1*1) = 1$. Thus $1'$ has an inverse $1*1$ with respect to 1, and so, since $x \perp (1 - s(x))$, x^{-1} exists by theorem 4.1(ii).

In order to state the following simple result, we recall that an element $1 \in L$ is called a strong order unit if for every $x \in L$ there exists a number a such that $|x| \leq a \cdot 1$; clearly any strong order unit is a weak order unit.

<u>Theorem 7.4</u> Suppose 1 is a strong order unit of L. Then

(i) L is a perfect ring under multiplication with respect to 1.

(ii) L is a field if and only if it is isomorphic to R_n (where R_n is the direct sum of n copies of the real numbers R).

<u>Proof</u> (i) This follows immediately from theorem 2.3(ii).

(ii) It is easy to see that $R_n = \overbrace{R \oplus \cdots \oplus R}^{n}$ is a field with (1, 1, ..., 1) as a strong order unit, and that every element in R_n is bounded with respect to (1, 1, ..., 1).

Conversely, suppose L is a field. We can see that there are at most a finite number of disjoint unitary elements. For if there is an infinite sequence $\{e_k\}$ of disjoint unitary elements then there exists in L an element $y = \sum_{k=1}^{\infty} k^{-1}e_k = \sup_k(e_1 + \cdots + k^{-1}e_k) \leq 1$, but y does not have an inverse which is bounded with respect to 1. Thus the boolean algebra U(L, 1) consists of just a finite number n of atoms, and hence $L = \overbrace{R \oplus \cdots \oplus R}^{n}$.

<u>Note</u> For any Riesz space L with unit 1, the ideal L_o is defined by $L_o = \{x \in L : |x| \leq a \cdot 1 \text{ for some } a\}$. Theorem 7.4(i) then says that L_o, considered as a space in itself, is a perfect ring.

II EXTENSIONS OF RIESZ SPACES.

§8. Pinsker's extension to spaces with property P.

Let L be a Dedekind complete Riesz space, not necessarily having a unit. In this section we discuss a construction (due to A. G. Pinsker [12], [13]) of a space $L^{\#}$ which will turn out to have property P, and into which L may be imbedded as an order dense ideal. Essentially, the extension consists in adjoining to L the formal supremums of subsets $S \subset L^+$ satisfying the conditions of property P_2. However, it is technically easier to proceed somewhat differently.

We will call a subset X L^+ a section if $y \in X$ whenever $0 \leq y \leq x \in X$ and if X is closed in the sense that $\{x_\alpha\} \subset X$ and $x_\alpha \leq x \in L$ for all α implies $\sup(x_\alpha) \in X$. Let $\overline{L}$ be the collection of sections of L. Define an order in $\overline{L}$ by $X \leq Y$ if $X \subset Y$. Denote $0 = \{0\}$, so always $X \geq 0$. Every collection $\{X_\alpha\}$ of elements of $\overline{L}$ has an inf and a sup, namely, $\inf(X_\alpha) = \bigcap X_\alpha$ and $\sup(X_\alpha) =$ the smallest section containing $\bigcup X_\alpha$. For two elements $X, Y \in \overline{L}$, we can see that $\sup(X, Y) = \{\sup(x, y) : x \in X, y \in Y\}$. For $a \geq 0$ we may define $aX = \{ax : x \in X\}$, and $X + Y$ is defined by $X + Y = \{x + y : x \in X, y \in Y\}$. (It is easy to see that these sets are indeed sections.) We note that $X \geq Y$ implies $X + Z \geq Y + Z$.

Lemma 8.1 $\sup(X + Z, Y + Z) = \sup(X, Y) + Z$.

Proof Firstly, $\sup(X, Y) + Z \geq X + Z$, and similarly $\geq Y + Z$, so $\sup(X, Y) + Z \geq \sup(X + Z, Y + Z)$. On the other hand, $\sup(X + Z, Y + Z) =$
$\{\sup(x + z, y + z') : x \in X, y \in Y, z, z' \in Z\} \geq$
$\geq \{\sup(x + z, y + z) : x \in X, y \in Y, z \in Z\} =$
$= \{\sup(x, y) + z : x \in X, y \in Y, z \in Z\} = \sup(X, Y) + Z$.

Imbed $L^+ \longrightarrow \overline{L}$ by $0 \leq x \longrightarrow \{y : 0 \leq y \leq x\}$. Thus we may consider L^+ a subset of $\overline{L}$. It is easy to see that the imbedding preserves lattice and vector operations.

For $X, Y, Z \in \overline{L}$, it is not necessarily true that $X + Z = Y + Z$ implies $X = Y$. (e.g. consider $Z = L^+$.) However, this is true if we restrict ourselves to an appropriate subspace of $\overline{L}$ as follows.

We say that $X \in \overline{L}$ is finite if for every $0 < x \in L$ there exists b such that $bx \not\leq X$ (i.e. $bx \notin X$). Let $L^{\#+}$ be the set of finite sections. We note that

(i) $X \leq Y \in L^{\#+}$ implies $X \in L^{\#+}$;

(ii) $L^{\#+}$ is closed under vector and finite lattice operations;

(iii) $L^{\#+}$ is closed under bounded supremums (i.e. $\{X_\alpha\} \subset L^{\#+}$, $X_\alpha \leq X \in L^{\#+}$ for all α implies $\sup(X_\alpha) \in L^{\#+}$).

Lemma 8.2 If $X, Y, Z \in L^{\#+}$ and $X + Z = Y + Z$, then $X = Y$.

Proof Since, by lemma 8.1, $\sup(X, Y) + Z = \sup(X + Z, Y + Z) = X + Z$, we may suppose (by replacing Y with $\sup(X, Y)$) that $Y \geq X$. Suppose $Y > X$. Then there exists $y \in Y$ such that $y \notin X$. But then, letting $y^* = \sup_{x \in X} \inf(y, x) \in X$, we have $y^* < y$. Then, for any $x \in X$, we have $x + y - y^* = \sup(x, y) + \inf(x, y) - y^* \leq \sup(x, y) \in Y$, and hence, for any element $x + z \in X + Z$ we have $x + z + y - y^* \in Y + Z = X + Z$. But then, by induction, $x + z + n(y - y^*) \in X + Y$ for all n, and this a contradiction since $X + Z$ is a finite section.

<u>Definition</u> For $Y \leq Z \in L^{\#+}$, define $(Z - Y)$ to be the smallest section containing $\left\{ z - y^* : z \in Z,\ y^* = \sup_{y \in Y} \inf(z, y) \right\}$.

<u>Lemma 8.3</u> If $X, Y, Z \in L^{\#+}$ and $X + Y = Z$, then $X = (Z - Y)$.

<u>Proof</u> By the previous lemma, it is sufficient to show that $(Z - Y) + Y = Z$. Now, any $z \in Z$ can be written $z = (z - y^*) + y^* \in (Z - Y) + Y$; hence $Z \leq (Z - Y) + Y$. Conversely, if $z - y^* \in (Z - Y)$ and $y \in Y$, then, as before, $z - y^* + y = \sup(z, y) + \inf(z,y) - y^* \leq \sup(z, y) \in Z$. But then, since any $x \in (Z - Y)$ is a supremum of elements of the form $z - y^*$, it follows that $x + y \in Z$ for any $x \in (Z - Y)$ and $y \in Y$.

We now want to show that $L^{\#+}$ is actually the positive part of a Riesz space $L^{\#}$. We define $L^{\#}$ to be the set of ordered pairs (X, Y) of finite sections, with the usual condition that $(X, Y) = (X', Y')$ if $X + Y' = X' + Y$. Vector operations are defined componentwise, and order is defined by $(X, Y) \geq 0$ if $X \geq Y$. Since $Y \leq X \in L^{\#+}$ implies $(X, Y) = (X - Y, 0)$, we may write without confusion $(X, Y) = X - Y$ for every $X, Y \in L^{\#+}$.

We will use f, g, h, to denote elements of $L^{\#}$.

Theorem 8.4 (i) $L^{\#}$ is a Dedekind complete Riesz space with property P.

(ii) L is an order - dense ideal in $L^{\#}$.

(iii) If L has a unit 1, then 1 is also a unit for $L^{\#}$, and $U(L^{\#}, 1) = U(L, 1)$.

Proof (i) To show that $L^{\#}$ is a Riesz space, we only have to show that for $f, g \in L^{\#}$, $\sup(f, g)$ exists. But for $f \in L^{\#}$ the sets $\{X \in L^{\#+} : X \geqslant f\}$ and $\{Y \in L^{\#+} : Y \geqslant -f\}$ are non-empty (since, by definition f can be written $f = X - Y$ with $X \geqslant 0$, $Y \geqslant 0$), and hence we may write $f_+ = \sup(f, 0) = \inf(X : X \geqslant 0, X \geqslant f) \in L^{\#+}$, and $f_- = \sup(-f, 0) = \inf(Y : Y \geqslant 0, Y \geqslant -f) \in L^{\#+}$. Then, in the usual way, we have $\sup(f, g) = \frac{1}{2}(f + g + |f - g|)$, where $|f| = f_+ + f_-$.

The fact that $L^{\#}$ is Dedkind complete follows from the fact that $L^{\#+}$ is closed under bounded supremums, and the fact that it has property P follows immediately from the definition of finite section.

The remaining parts (ii) and (iii) are clear.

The next theorem shows that $L^{\#}$ is, in a sense, a maximal extension.

Theorem 8.5 Let L be a Dedekind complete Riesz space. Then

(i) If L has property P, and is an order dense ideal in an Archimedean Riesz space E, then $L = E$. In particular $L = L^{\#}$ if L has property P.

(ii) $L^{\#} = (L^{\#})^{\#}$.

(iii) If L is an order dense ideal in a Dedekind complete Riesz space E with property P, then $L^{\#} = E$. In particular, if $L \subset E \subset L^{\#}$, then $L^{\#} = E$ if E has property P.

Proof (i) Let $0 \leq x \in E$. The fact that x is then in L follows immediately by applying property P_2 to the set $\{y \in L : 0 \leq y \leq x\}$.

(ii) Follows immediately from (i) since $L^{\#}$ has property P.

(iii) $L \subset E$ implies $L^{\#} \subset E^{\#} = E$; hence $L \subset L^{\#} \subset E$. But since L is order dense in E, $L^{\#}$ is also order dense in E, and hence, by (i), $L^{\#} = E$.

§9. Definition of multiplication in spaces without unit.

Let L be a Dedekind complete Riesz space, not necessarily having a unit. Its extension $L^{\#}$ does have a unit however, and in fact, by theorem 7.2, multiplication in $L^{\#}$ with respect to this unit is universally defined. We can use this to induce a multiplication in L by saying: for $x, y \in L$, if xy (which exists in $L^{\#}$) is in L, then the product of x and y is defined and equals xy.

It is easy to verify that the multiplication thus defined in L satisfies all the properties described in theorem 5.1 (except, of course, part (i)). In particular, part (vii) is satisfied since L is an ideal in $L^{\#}$.

Remark The same multiplication could be defined in L without reference to $L^{\#}$ by considering L as the Riesz direct sum of the disjoint normal subspaces $[x_\alpha]$, where $\{x_\alpha\}$ is a collection of positive elements of L maximal with respect to the property of having its elements mutually disjoint (c.f. [8], p. 111 in Note VIII). Each component $[x_\alpha]$ has a unit (namely x_α), so multiplication can be defined in each $[x_\alpha]$, and then multiplication can be defined in L componentwise.

§10. Order-continuity of the product.

Let L be a Dedekind complete Riesz space with a unit. Recall that if $\{x_\alpha\}$, $\alpha \in \{\alpha\}$, is a net in L, then $\{x_\alpha\}$ is said to order converge to x (written (o) - lim(x) = x) if $\inf_{\beta} \sup_{\alpha \geq \beta}(x_\alpha) = \sup_{\beta} \inf_{\alpha \geq \beta}(x_\alpha) = x$. Vulih [19] derives the following criterion for convergence to 0 from the work of Steen [18].

Theorem 10.1 $(o)\text{-}\lim(x_\alpha) = 0$ if and only if there exists $z \in L$ and $\gamma \in \{\alpha\}$ such that $|x_\alpha| \leq z$ for all $\alpha \geq \gamma$, and for every $a > 0$ and $0 < e \in U(L)$ there exists $0 < e' \leq e$ and $\beta \in \{\alpha\}$ such that $|x_\alpha e'| \leq ae'$ for all $\alpha \geq \beta$.

Proof Suppose $(o)\text{-}\lim(x_\alpha) = 0$. Then, firstly, for some γ, $\sup_{\alpha \geq \gamma}(|x_\alpha|)$ must exist; and secondly, for any $a > 0$ and $e > 0$ there must exist β such that $\sup_{\alpha \geq \beta}(|x_\alpha|) \not\geq ae$. But then there exists $0 < e' \leq e$ such that $(\sup_{\alpha \geq \beta}(|x_\alpha|)e' \leq ae'$, and so $\sup_{\alpha \geq \beta}(|x_\alpha e'|) \leq ae'$, since $\sup_{\alpha \geq \beta}(|x_\alpha e|) =$ $= \sup_{\alpha \geq \beta}(|x_\alpha| e) = (\sup_{\alpha \geq \beta}(|x_\alpha|)e$, by theorem 2.1.

Conversely, suppose $|x_\alpha| \leq z$ for all $\alpha \geq \gamma$, but $\{x_\alpha\}$ does not converge to 0. Then $\inf_\beta \sup_{\alpha \geq \beta}(|x_\alpha|) > 0$, so there exists $a > 0$ and $e > 0$ such that $\inf_\beta \sup_{\alpha \geq \beta}(|x_\alpha|) \geq 2ae$. Hence for any $0 < e' \leq e$ and any β, $\sup_{\alpha \geq \beta}(|x_\alpha e'|) = (\sup_{\alpha \geq \beta}(|x_\alpha|)e' \geq 2ae' > ae'$, contradicting the second part of the criterion.

(Note : The uniform boundedness part of the above crierion is necessary to insure the existence of $\inf_\beta \sup_{\alpha \geq \beta}(|x_\alpha|)$.)

In the next theorem we demonstrate a continuity property in $L^\#$, and deduce as a corollary a similar continuity property in L. The theorem depends on the following lemma.

<u>Lemma 10.2</u> Suppose $\{f_\alpha\}$ is a net in $L^\#$ and $(o)\text{-}\lim(f_\alpha) = 0$. Then for any $g \in L^\#$, $(o)\text{-}\lim(f_\alpha g) = 0$.

<u>Proof</u> We apply the criterion of the preceeding theorem. For the uniform boundedness we simply note that $\sup_{\alpha \geq \beta}(|f_\alpha g|) = (\sup_{\alpha \geq \beta}(|f_\alpha|)|g|$. Next, for $a > 0$ and $e > 0$ there exists $0 < e' \leq e$ and $b > 0$ such that $|ge'| < be'$, and then there exists $0 < e'' \leq e'$ and β such that $|f_\alpha e''| \leq ab^{-1}e''$ for $\alpha \geq \beta$. But then, for $\alpha \geq \beta$ we have $|f_\alpha ge''| = |f_\alpha e''||ge''| \leq ae''$, as required.

<u>Theorem 10.3</u> Let $\{f_\alpha\}$ and $\{g_\alpha\}$ be two nets in $L^\#$ indexed by the

same directed set $\{\alpha\}$. If (o)-lim$(f_\alpha) = f$ and (o)-lim$(g_\alpha) = g$, then (o)-lim$(f_\alpha g_\alpha) = fg$.

<u>Proof</u> $|f_\alpha g_\alpha - fg| = |f_\alpha g_\alpha - f_\alpha g + f_\alpha g - fg| \leq$
$\leq |f_\alpha||g_\alpha - g| + |g||f_\alpha - f| \leq (\sup(|f_\alpha|)|g_\alpha - g| + |g||f_\alpha - f|$.
Now, each term of the right side converges to 0 by the preceeding lemma, and hence their sum does; so the left side also converges to 0 as required.

<u>Corollary 10.3.1</u> Let $\{x_\alpha\}$ and $\{y_\alpha\}$ be two nets in L indexed by the same directed set. Suppose (o)-lim$(x_\alpha) = x$, (o)-lim$(y_\alpha) = y$, $x_\alpha y_\alpha$ exists for each α, and there exists $z \in L$ such that $|x_\alpha y_\alpha| \leq z$ for all α. Then the product xy exists in L, and (o)-lim$(x_\alpha y_\alpha) = xy$.

<u>Proof</u> The fact that the products $x_\alpha y_\alpha$ are uniformly bounded implies that $\inf_\beta \sup_{\alpha \geq \beta}(x_\alpha y_\alpha)$ and $\sup_\beta \inf_{\alpha \geq \beta}(x_\alpha y_\alpha)$ both exist in L. The theorem implies that they are both equal to xy $(\in L^\#)$, which must then also be in L.

<u>Remark</u> Corollary 10.3.1 can be proved directly without referring to $L^\#$ (indeed Vulih does this in [19]). However, technical difficulties then are encountered since many of the products used in the proofs of lemma 10.2 and theorem 10.3 may not exist in L.

§11. Positive linear functionals on spaces with property P.

In this section we will see that many spaces with property P have positive linear functionals of only very special types. In this respect they behave very much like spaces of all real-valued functions on a given set (c.f. [8], example 20.8 in Note VI, and example 27.3(iii) in Note VIII).

Recall that a positive linear functional φ on L is called an integral if $0 \leq x_n \downarrow 0$ implies $\varphi(x_n) \to 0$ (c.f. [8], §20. in Note VI).

Theorem 11.1 Let L be a Dedekind complete Riesz space with property P. Then any positive linear functional φ on L is an integral.

Proof By normalization we may suppose that $\varphi(1) \leq 1$ (it may be zero). Let $0 \leq x_n \downarrow 0$, and suppose $\varphi(x_n) \not\to 0$. By multiplying each x_n by a suitable fixed constant, we may assume that $\varphi(x_n) \geq 2$ for all n. Define $y_n = (x_n - 1)_+$, and note that $y_n \downarrow 0$ and $\varphi(y_n) \geq \varphi(x_n - 1) = \varphi(x_n) - \varphi(1) \geq 2 - 1 = 1$.

Now, for every $e > 0$ there exists, by theorem 10.1, $0 < e' \leq e$ and N such that $x_n e' \leq \frac{1}{2}e'$ for all $n \geq N$; hence $y_n e' = 0$ for $n \geq N$. But then, letting $S = \left\{ z_n : 0 \leq z_n = \sum_{k=1}^{n} y_k, \ n = 1, 2, \ldots \right\}$, we can see (taking account of property P_1) that sup(S) exists. It then follows that for every n $\varphi(\sup(S)) \geq \varphi(z_n) = \varphi(\sum_{k=1}^{n} y_k) =$

$= \sum_{k=1}^{n} \varphi(y_k) \geq n$, a contradiction since $\varphi(\sup(S))$ must be finite.

<u>Lemma 11.2</u> Let L be a Dedekind complete Riesz space with a unit 1, and φ a non-negative integral on L which is not identically zero. Then $\varphi(1) > 0$.

<u>Proof</u> For some $0 < x \in L$ we have $\varphi(x) > 0$. By corollary 1.6.2, $\inf(x, n\cdot 1) \uparrow_n x$; so for some n $\varphi(\inf(x, n\cdot 1)) > 0$. Then $\varphi(1) = n^{-1}\varphi(n\cdot 1) \geq n^{-1}\varphi(\inf(x, n\cdot 1)) > 0$.

If L is a Riesz space with a unit 1, then by a <u>decomposition of e</u> $\in U(L)$ we mean a collection $\{e_\alpha\}$, $\alpha \in \{\alpha\}$, of unitary elements such that $e_\alpha > 0$ for all $\alpha \in \{\alpha\}$, $e_\alpha \perp e_\beta$ for $\alpha \neq \beta$, and $\sup(e_\alpha) = e$. The cardinal of the index set $\{\alpha\}$ is called the <u>cardinal of the decomposition</u>.

Recall that a set X is said to have a <u>measurable cardinal</u> if there exists a countably additive measure ν on the collection of all subsets of X such that $\nu(X) = 1$ and $\nu(F) = 0$ for every finite subset F of X. If such a measure ν does not exist, then X is said to have a <u>non-measurable cardinal</u> (c.f. [8], p.697 in Note VII).

Recall that a positive linear functional φ on L is said to be a <u>normal integral</u> if $0 \leq x_\alpha \downarrow 0$ implies $\inf \varphi(x_\alpha) = 0$ for every set $\{x_\alpha\}$ directed downwards to zero (c.f. [8], §27. in Note VIII).

<u>Theorem 11.3</u> Let L be a Dedekind complete Riesz space with property P,

and suppose that every decomposition of 1 has a non-measurable cardinal. Then every positive linear functional φ on L is concentrated on a finite number of atoms of U(L, 1), and consequently is a normal integral.

Proof By theorem 11.1 we may assume that φ is an integral; and we may as well assume that φ is not identically zero, so that by lemma 11.2 $\varphi(1) > 0$.

Let A be the family of collections $\{e_\alpha\}$ of mutually disjoint unitary elements such that $\varphi(e_\alpha) = 0$. A is inductively ordered by inclusion, so there is a maximal such collection D. Let $e' = \sup(e : e \in D)$. We can see that $\varphi(e') = 0$, for otherwise φ would induce a non-zero measure on the cardinal of D which is zero on finite subsets, whereas by assumption D has non-measurable cardinal. It follows, by lemma 11.2, that φ is zero on the normal subspace generated by e'.

Thus $\varphi(1 - e') = \varphi(1) > 0$, so $1 - e' > 0$ and φ is concentrated on $[1 - e']$. If $1 - e'$ is an atom we are done; if it is not, then it can be decomposed into two parts, and φ must be greater than zero on each of them or else D could be enlarged. If one of these parts is not an atom then it can be further decomposed, and again φ must be greater than zero on each of the parts; etc. This process of decomposition must stop after a finite number of steps, for property P_3 makes it impossible to have an infinite number of mutually disjoint elements on each of which φ is greater than zero.

Thus $1 - e'$ decomposes into a finite number of atoms, and φ is concentrated on them.

III ABSTRACT INTEGRAL SPACES.

§12. Extension of φ to $L^{\#+}$.

Let L be Riesz space (Dedekind complete), not necessarily having a unit. Let φ be a non-negative normal integral on L. (Hereafter, all integrals will be non-negative, so we will not always mention this explicitly in the future.) As usual, x, y, z, will denote elements of L and f, g, h, will denote elements of $L^{\#}$.

We define a new functional $\varphi^{\#}$ on $L^{\#+}$ as follows: for $0 \leq f \in L^{\#+}$ define $\varphi^{\#}(f) = \sup(\varphi(x) : x \in L, 0 \leq x \leq f)$. $\varphi^{\#}(f)$ may equal $+\infty$, but for $0 \leq x \in L$ $\varphi^{\#}(x) = \varphi(x)$.

Theorem 12.1 (c.f. [8], theorem 30.6 in Note IX) If $0 \leq f_\alpha \uparrow f \in L^{\#+}$, then $\varphi^{\#}(f) = \sup \varphi^{\#}(f_\alpha)$.

Proof Assume first that $\varphi^{\#}(f) < \infty$. Then, given $\varepsilon > 0$, there exists $x \in L$ such that $\varphi^{\#}(f) \leq \varphi(x) + \varepsilon$. Let $x_\alpha = \inf(f_\alpha, x) \leq f_\alpha$. $x_\alpha \in L$ and $x_\alpha \uparrow x$, so $\varphi(x_\alpha) \uparrow \varphi(x)$. Then $\sup \varphi^{\#}(f_\alpha) + \varepsilon \geqslant \varphi^{\#}(f)$.

If $^{\#}(f) = \infty$, then for any N there exists $x \leq f$ such that $\varphi(x) > N$. Now, $\inf(x, f_\alpha) \uparrow x$, so $\sup \varphi^{\#}(f_\alpha) \geqslant \varphi(x) \geqslant N$. Hence $\varphi^{\#}(f_\alpha) \uparrow \infty$.

Theorem 12.2 $\varphi^{\#}$ is a positive linear functional on $L^{\#+}$, and is strictly positive if φ is strictly positive.

<u>Proof</u> We only need to verify that $\varphi^{\#}$ is additive. Let $f, g \in L^{\#+}$. Every $z \in L^+$, $z \leq f + g$ can be written $z = x + y$ with $f \geq x \in L^+$ and $g \geq y \in L^+$, and so $\varphi^{\#}(f + g) = \sup(\varphi(x + y) : 0 \leq x \leq f,\ 0 \leq y \leq g) =$ $= \sup(\varphi(x) : 0 \leq x \leq f) + \sup(\varphi(y) : 0 \leq y \leq g) = \varphi^{\#}(f) + \varphi^{\#}(g)$.

Since $\varphi^{\#}$ is an extension of φ, we may (when confusion does not result) write φ for $\varphi^{\#}$.

§13. $L_1(\varphi, L)$ and $L_2(\varphi, L)$.

We continue to suppose that L is a Dedekind complete Riesz space with a normal integral φ. Let us suppose for the moment that φ is strictly positive.

<u>Definition</u> (i) If φ is strictly positive, then $L_1(\varphi, L) = L_1(\varphi) =$ $L_1 = \{f \in L^{\#} : \varphi(|f|) < \infty\}$ (where we are writing φ for $\varphi^{\#}$).

(ii) A norm is defined on $L_1(\varphi)$ by $\|f\|_1 = \varphi(|f|)$. (This is a norm rather than a seminorm since φ is strictly positive.)

(iii) φ is extended to all of $L_1(\varphi)$ by defining $\varphi(f) =$ $\varphi(f_+) - \varphi(f_-)$.

We note that L_1 is an ideal in $L^{\#}$, and that, by theorems 12.1 and 12.2, φ (i.e. $\varphi^{\#}$) is a strictly positive normal integral on L_1.

The next theorem is the key to showing that $L_1(\varphi)$ (and later $L_2(\varphi)$) is complete.

<u>Theorem 13.1</u> If $0 \leq f_\alpha \uparrow \in L_1(\varphi)$ and $\sup\|f_\alpha\|_1 < \infty$, then there exists $\sup(f_\alpha) \in L_1(\varphi)$.

<u>Proof</u> First we use property P_2 to show that there exists $\sup(f_\alpha) \in L^\#$. Let $0 < g \in L^\#$, and suppose that for every b $\sup_\alpha \inf(bg, f_\alpha) = bg$. Then $b\varphi(g) = \varphi(bg) = \varphi(\sup_\alpha \inf(bg, f_\alpha)) = \sup_\alpha \varphi(\inf(bg, f_\alpha)) \leq$ $\leq \sup_\alpha \varphi(f_\alpha) < \infty$. But since $\varphi(g) > 0$, this cannot be true for every b, i.e. there must exist b such that $\sup_\alpha \inf(bg, f_\alpha) \neq bg$. But then since $L^\#$ has property P_2, there exists $f = \sup(f_\alpha) \in L^\#$.

Then to show $f \in L_1(\varphi)$, we only have to notice that, by theorem 12.1, $\varphi(f) = \sup \varphi(f_\alpha) < \infty$.

<u>Corollary 13.1.1</u> $L_1(\varphi)$ is complete (in the norm $\|\cdot\|_1$).

<u>Proof</u> Suppose $0 \leq f_n \uparrow \in L_1$, and $\sup\|f_n\|_1 < \infty$. Then the theorem implies that $\sup(f_n)$ exists in L_1. But this is exactly the criterion of Amemiya [1] that a normed Riesz space be complete. (c.f. also [8] theorem 5.3 in Note II, and theorem 26.3 in Note VIII.)

Suppose now that φ is <u>not</u> strictly positive. Decompose $L = C_\varphi \oplus N_\varphi$ (where N_φ is the null ideal of φ, and $C_\varphi = N_\varphi^\perp$ is the carrier or support of φ; c.f. [8], pp 107 - 108 in Note VIII.) This decomposition induces a decomposition $L^\# = C_\varphi^\# \oplus N_\varphi^\#$; for $C_\varphi^\#$ and $N_\varphi^\#$ are disjoint normal subspaces of $L^\#$, and since $C_\varphi^\# + N_\varphi^\#$ has property P_3 and $L = C_\varphi + N_\varphi \subset C_\varphi^\# + N_\varphi^\# \subset L^\#$, theorem 8.5(iii) implies $L^\# = C_\varphi^\# + N_\varphi^\#$.

φ is strictly positive on C_φ, so we may define $L_1(\varphi, L)$ in general to be $L_1(\varphi, C_\varphi)$. By an abuse of language we will sometimes say that $f \in L_1(\varphi, L)$ if the component of f in $C_\varphi^\#$ is in $L_1(\varphi, C_\varphi)$.

<u>Definition</u> $L_2(\varphi, L) = L_2(\varphi, C_\varphi) = L_2(\varphi) = L_2 = \{ f \in C_\varphi^\# : \varphi(f^2) < \infty \}$.

It is easy to see that L_2 is an ideal in $L^\#$; for

(i) $f \in L_2$ implies $af \in L_2$;

(ii) $(f + g)^2 = f^2 + g^2 + 2fg \leq 2(f^2 + g^2)$, so $\varphi((f + g)^2) \leq \leq 2(\varphi(f^2) + \varphi(g^2))$, and hence $f, g \in L_2$ implies $(f + g) \in L_2$;

(iii) $|f| \leq |g|$ and $g \in L_2$ implies $\varphi(f^2) \leq \varphi(g^2) < \infty$, so that $f \in L_2$.

Now, $fg \leq \frac{1}{2}(f^2 + g^2)$, so for $f, g \in L_2$, $|\varphi(fg)| \leq \frac{1}{2}(\varphi(f^2) + \varphi(g^2))$ and this is $< \infty$. Hence we may define in L_2 an <u>inner product</u> $(f, g) = \varphi(fg)$, and a <u>norm</u> $\|f\|_2 = (f, f)^{\frac{1}{2}} = [\varphi(f^2)]^{\frac{1}{2}} = \|f^2\|_1^{\frac{1}{2}}$. ($\|\cdot\|_2$ is a norm rather than a seminorm since $\|f\|_2 = 0$ implies $\varphi(f^2) = 0$, so that $f^2 = 0$ and hence $f = 0$.) Note that $f \in L_2$ if and only if $f^2 \in L_1$.

For the proof of the next theorem we will use the following lemma.

<u>Lemma 13.2</u> If $0 \leq f_n \in L^\#$ and $f_n^2 \uparrow g$, then $0 \leq f_n \uparrow g^{\frac{1}{2}} \in L^\#$.

<u>Proof</u> $f_n \leq \sup(f_n^2, 1) \leq \sup(g, 1)$; hence there exists $0 \leq h = \sup(f_n)$. Then $g = \sup(f_n^2) = h^2$, by theorem 10.3, so by the uniqueness of positive square roots we have $h = g^{\frac{1}{2}}$ as required.

<u>Theorem 13.3</u> $L_2(\varphi)$ is a Hilbert space.

<u>Proof</u> We only have to prove that L_2 is complete in the norm $\|\cdot\|_2$. Suppose $0 \leq f_n \uparrow \in L_2$ and $\sup\|f_n\|_2 < \infty$. Then $0 \leq f_n^2 \uparrow \in L_1$ and $\sup\|f_n^2\|_1 = \sup\|f_n\|_2^2 < \infty$, so by thoerem 13.1 there exists $g = \sup(f_n^2) \in L_1$. But then, by the preceeding lemma, there exists $\sup(f_n) = g^{\frac{1}{2}} \in L_2$. Thus, by Amemiya's theorem [1], L_2 is complete.

The next theorem will be useful later on (c.f. theorem 17.3). Recall that if L has a unit 1, then $L_o = \{x \in L : |x| \leq a\cdot 1 \text{ for some } a\}$.

<u>Theorem 13.4</u> If L has a unit, then L_o is norm dense in $L_1(\varphi)$ and $L_2(\varphi)$.

<u>Proof</u> Consider first $L_2(\varphi)$. Since L_o contains all elements of the form ae, L_o is order dense in $L^{\#}$. Suppose $f \in L_2$. Given $\varepsilon > 0$ we want to find $x \in L_o$ such that $\|f - x\|_2 \leq \varepsilon$. It is sufficient to consider the case where $f \geqslant 0$. Then $f = \sup(x : x \in L_o, 0 \leq x \leq f)$, and hence $f^2 = \sup(x^2 : x \in L_o, 0 \leq x \leq f)$. But then, by theorem 12.1, $\varphi(f^2) = \sup(\varphi(x^2) : x \in L_o, 0 \leq x \leq f)$, and hence there exists $x \in L_o$, $0 \leq x \leq f$, such that $\varphi(f^2) \leq \varphi(x^2) + \varepsilon^2/2$, i.e. $\|f\|_2^2 - \|x\|_2^2 \leq \varepsilon^2/2$. Now, $f \geqslant x \geqslant 0$ implies $f + x \geqslant 2x$, so $\|f + x\|_2^2 \geqslant 4\|x\|_2^2$. Then, by the parallelogram law, $\|f - x\|_2^2 = 2\|f\|_2^2 + 2\|x\|_2^2 - \|f + x\|_2^2 \leq$

$\leq 2\|f\|_2^2 + 2\|x\|_2^2 - 4\|x\|_2^2 = 2(\|f\|_2^2 - \|x\|_2^2) \leq \varepsilon^2$ as required.

The fact that L_o is norm dense in L_1 follows by a similar but easier argument.

IV THE RADON-NIKODYM THEOREM.

§14. Absolute continuity.

Let φ be a normal integral on the Dedekind complete Riesz sapce L, and let ψ be a normal integral on some Riesz subspace $E \subset L^{\#}$. Then ψ is said to be absolutely continuous with respect to φ if $L_1(\psi) \oplus N_\psi$ is order dense in $L^{\#}$, and, for $0 \leq f \in L^{\#}$, $\varphi(f) = 0$ implies $\psi(f) = 0$.

Note Requiring that $L_1(\psi) \oplus N_\psi$ be dense in $L^{\#}$ is equivalent to the more usual condition (c.f. [23], p.134) that φ and ψ be initially defined on the same space, for we may regard $(L_1(\varphi) \oplus N_\varphi) \cap (L_1(\psi) \oplus N_\psi)$ as the initial domain of φ and ψ , and this is order dense in $L^{\#}$.

Theorem 14.1 Let φ be a normal integral on L, and let $0 \leq g \in L^{\#}$. Define ψ on $L^{\#+}$ by $\psi(f) = \varphi(fg)$ for all $0 \leq f \in L^{\#+}$, and then on some Riesz subspace $E \subset L^{\#}$ by $\psi(f) = \psi(f_+) - \psi(f_-)$ whenever $\psi(f_+)$ and $\psi(f_-)$ are finite. Then ψ is a normal integral absolutely continuous with respect to φ .

Proof Since φ is normal and multiplication is (o)-continuous, ψ is a normal integral on $L_1(\psi) \oplus N_\psi = \{f \in L^{\#} : \psi(|f|) < \infty\}$.

Next, if $\varphi(f) = 0$ then $f \in N_\varphi^{\#} = (C_\varphi^{\#})^{\perp}$, and hence $fg \in (C_\varphi^{\#})^{\perp} = N_\varphi^{\#}$, i.e. $\psi(f) = \varphi(fg) = 0$.

Finally, we must show that, given $0 < f \in L^{\#}$, there exists $0 < h \in L_1(\psi) \oplus N_\psi$ such that $h \leq f$. But if $f > 0$, then there exist $e > 0$, $a > 0$, and $0 \leq b < \infty$ such that $0 < ae \leq f$ and $ge \leq be$. It follows that $\psi(ae) = \varphi(aeg) \leq \varphi(abe) \leq b\varphi(f) < \infty$, and hence ae is the required element in $L_1(\psi) \oplus N_\psi$.

The converse of this theorem will be the subject of the next section.

§15. The Radon-Nikodym theorem.

In this section we prove a very general form of the Radon-Nikodym theorem (c.f. theorem 15.2). But first we prove a special case. (The proof parallels very closely that given in [23] for measure spaces.)

Theorem 15.1 Let L be a Dedekind complete Riesz space with a unit 1. Let φ be a strictly positive normal integral on L, and let $0 \leq \psi$ be any normal integral on L. Then there exists a unique $0 \leq g \in L_1(\varphi)$ such that $f \in L_1(\psi)$ if and only if $fg \in L_1(\varphi)$, and $\psi(f) = \varphi(fg)$ for every $f \in L_1(\psi)$.

Proof (i) Define ω on L by $\omega = \varphi + \psi$. ω is clearly a strictly positive normal integral on L. We must verify that $\omega^{\#} = \varphi^{\#} + \psi^{\#}$. For $0 \leq f \in L^{\#}$, $(\varphi + \psi)^{\#}(f) = \sup((\varphi + \psi)(x) : 0 \leq x \leq f) =$ $= \sup(\varphi(x) + \psi(x) : 0 \leq x \leq f) \leq \varphi^{\#}(f) + \psi^{\#}(f)$. On the other hand, $0 \leq x, y \leq f$ implies $z = \sup(x, y) \leq f$; thus $\varphi(x) + \psi(y) \leq (\varphi + \psi)(z)$, so $\varphi^{\#}(f) + \psi^{\#}(f) = \sup(\varphi(x) + \psi(y) : 0 \leq x, y \leq f) \leq$ $\leq \sup((\varphi + \psi)(z) : 0 \leq z \leq f) = (\varphi + \psi)^{\#}(f)$. Since, then, $\omega^{\#} = \varphi^{\#} + \psi^{\#}$, we will henceforth omit the $\#$ on φ, ψ, ω.

(ii) Consider the Hilbert space $L_2(\omega)$. For $f \in L_2(\omega)$ we have $|\psi(f)| \leq \psi(|f|) \leq \omega(|f|) = (|f|, 1) \leq \|f\|_2 \|1\|_2$ by the Schwarz inequality. Thus ψ is a bounded linear functional on $L_2(\omega)$, and so there exists $h \in L_2(\omega)$ such that $\psi(f) = (f, h) = \omega(fh)$ for all $f \in L_2(\omega)$.

(iii) We prove now several facts about h. First of all, $h \geq 0$ (for, taking $f = s(h_-)$ in the above we have $0 \leq \psi(s(h_-)) = \omega(s(h_-)h) =$ $= \omega(-h_-) \leq 0$, and hence $h_- = 0$).

Secondly, $s\left[(1 - h)_+\right] = 1$. For if not, then there exists $e > 0$ such that $e \perp s\left[(1 - h)_+\right]$, and then $e - he = (1 - h)e \leq 0$, i.e. $he \geq e$. But then $\psi(e) = \omega(eh) \geq \omega(e) = \varphi(e) + \psi(e) \geq \psi(e)$; hence equality holds throughout, and so $\varphi(e) = 0$, a contradiction since φ is strictly positive. Note that it follows immediately from $s\left[(1 - h)_+\right] = 1$ that $1 - h = (1 - h)_+ \geq 0$, i.e. $h \leq 1$; but this is a weaker statement.

Next we use property P_1 to show that there exists in $L^{\#}$ $0 \leq g = \sum_{n=1}^{\infty} h^n = \sup_n(h + \cdots + h^n)$. For $e > 0$ we have $e \leq s\left[(1 - h)_+\right]$ by the preceeding; hence there exists $0 < e' \leq e$ and $1 > a > 0$ such that $ae' \leq (1 - h)_+ e' = (1 - h)e' = e' - he'$, i.e. $0 \leq he' \leq (1 - a)e'$. Then, for every n, $h^n e' \leq (1 - a)^n e'$, and hence $he' + \cdots + h^n e' \leq Ke'$, where $K = \sum_{k=1}^{\infty} (1 - a)^k$. Therefore $\sup_n(h + \cdots + h^n)e' \leq Ke'$, and hence, by property P_1, there exists $g = \sum h^n \in L^{\#+}$. Note that the preceeding implies $h^n \downarrow 0$.

(iv) For any $0 \leq f \in L^{\#}$, the preceeding shows that $f \sum_{k=1}^{n} h^k \uparrow fg$, so $\lim_{n\to\infty} \varphi(f \sum_{k=1}^{n} h^k) = \varphi(fg)$; and also $fh^n \downarrow 0$, so $\lim_{n\to\infty} \psi(fh^n) = 0$ for every $0 \leq f \in L_1(\psi)$. Now, we have $\psi(f) = \omega(fh)$ for all $f \in L_2(\omega)$; but since $L_2(\omega)$ is order dense in $L^{\#}$ and $h \geqslant 0$, this implies that in fact $\psi(f) = \omega(fh)$ for all $0 \leq f \in L^{\#+}$. Thus for every $0 \leq f \in L_1(\psi)$ we have $\psi(f) = \omega(fh) = \varphi(fh) + \psi(fh) = \varphi(fh) + \omega(fh^2) = \varphi(fh) + \varphi(fh^2) + \psi(fh^2) = \dots =$ $= \varphi(f \sum_{k=1}^{n} h^k) + \psi(fh^n) \longrightarrow \varphi(fg)$. Therefore $\psi(f) = \varphi(fg)$ for every $0 \leq f \in L_1(\psi)$, and hence the same equation holds for every $0 \leq f \in L^{\#}$.

The equation also shows that $f \in L_1(\psi)$ if and only if $fg \in L_1(\varphi)$, and in particular, taking $f = 1$, we see that $g \in L_1(\varphi)$.

(v) Finally, we show that g is unique. Suppose there also exists g' such that $\psi(f) = \varphi(fg')$ for $f \in L_1(\psi)$. Let $e = s[(g - g')_+]$. Then $\varphi(eg') = \psi(e) = \varphi(eg)$, so $0 = \varphi(ge - g'e) = \varphi((g - g')e) = \varphi((g - g')_+)$, and hence $(g - g')_+ = 0$, i.e. $g \leq g'$. Similarly $g' \leq g$, and hence $g' = g$.

<u>Theorem 15.2</u> Let L be a Dedekind complete Riesz space, φ a (non-negative) normal integral on L, and ψ a (non-negative) normal integral absolutely continuous with respect to φ. Fix a unit $1 \in L^{\#}$. Then there exists an element $0 \leq g \in L^{\#}$ such that $f \in L_1(\psi)$ if and only if $fg \in L_1(\varphi)$, and $\psi(f) = \varphi(fg)$ for every $f \in L_1(\psi)$. g is unique in the sense that its component in $C^{\#}_{\varphi}$ is uniquely determined.

<u>Proof</u> Write $L = C_\varphi \oplus N_\varphi$. φ is zero on N_φ, so by absolute continuity ψ is also zero on N_φ, i.e. we may consider ψ simply as a normal integral on C_φ. And φ is strictly positive on C_φ.

The element g that we require depends on the particular unit chosen for $L^\#$, but theorem 5.3 (giving the formula for change of units) shows that if the theorem is true for any unit then it is true for every unit. Consequently we may let $\{x_\alpha\}$ be a maximal collection of mutually disjoint positive elements of C_φ, and take $\sup(x_\alpha)$ as a unit for $C_\varphi^\#$. We have $C_\varphi = \bigcup\oplus [x_\alpha]$ and $C_\varphi^\# = \bigcup\oplus [x_\alpha]^\#$ (where $\bigcup\oplus [x_\alpha]$ denotes the Riesz direct sum of the $[x_\alpha]$, i.e. the smallest normal sub-space of C_φ containing all the $[x_\alpha]$. c.f. [8], p.111 in Note VIII.)

For each α, $[x_\alpha]$ is a Dedekind complete Riesz space with a unit x_α, and on $[x_\alpha]$ φ acts as a strictly positive normal integral. Thus, by theorem 15.1 there exists a unique $0 \leq g_\alpha \in [x_\alpha]^\#$ such that $\psi(f_\alpha) = \varphi(f_\alpha g_\alpha)$ for every $0 \leq f \in [x_\alpha]^\#$.

Let $0 \leq g = \sup(g_\alpha) \in C_\varphi^\#$. (Such an element exists since $C_\varphi^\#$ has property P_3.) For any $0 \leq f \in C_\varphi^\#$ (whose component in $[x_\alpha]$ is f_α), the component of fg in $[x_\alpha]$ is $f_\alpha g_\alpha$, for $(fg)_\alpha = fgx_\alpha = (fx_\alpha)(gx_\alpha) = f_\alpha g_\alpha$. But then $\psi(f) = \sum_\alpha \psi(f_\alpha) = \sum_\alpha \varphi(f_\alpha g_\alpha) = \sum_\alpha \varphi[(fg)_\alpha] = \varphi(fg)$. The theorem now follows immediately; in particular the uniqueness of g follows from the uniqueness of each g_α.

V APPLICATIONS

§16. Segal's theorem.

It is interesting to note that in theorem 15.2 no condition such as σ-finiteness is required. In this section we use this fact to examine the Radon-Nikodym theorem for measure spaces. In particular we give a relatively simple proof of the theorem of Segal [17] which gives a necessary and sufficient condition that the Radon-Nikodym hold in a given measure space. (The Radon-Nikodym theorem is said to hold in a given measure space (X, S, μ) if for any integral ψ absolutely continuous with respect to the integral $\int \cdot d\mu$ there exists a μ-unique measurable function g such that $\psi(f) = \int fg\, d\mu$ for every ψ-integrable f.)

Briefly, our proof will proceed in the following steps :

(i) Given the measure space (X, S, μ) with no purely infinite sets, consider the measure algebra (B, μ) of measurable sets modulo null sets. μ is said to be <u>localizable</u> if B is complete as a lattice.

(ii) The space $L = L_1(X, S, \mu)$ of integrable functions modulo null functions is a Dedekind complete Riesz space with a strictly positive normal integral $\varphi(f^*) = \int f\, d\mu$ (where f^* denotes the class of functions equivalent to f a.e.).

(iii) Imbed $L \subset L^{\#}$, and pick a unit $1 \in L^{\#}$ corresponding to χ_X. Consider the complete Boolean algebra $U(L^{\#}, 1)$.

(iv) There exists a measure-preserving Boolean-algebra-isomorphism of B into $U(L^{\#})$, which is onto if and only if μ is localizable.

(v) The isomorphism of (iv) generates an isomorphism ρ^* of M (equivalence classes of measurable functions) into $L^\#$ which maps $L_1(X, S, \mu)$ onto $L_1(\varphi, L)$. ρ^* preserves the multiplicative structure.

(vi) ρ^* maps M onto $L^\#$ if and only if μ is localizable.

(vii) It follows from (vi) that the Radon-Nikodym theorem holds in (X, S, μ) if and only if μ is localizable. This is Segal's theorem.

We now give the details :

(i) Let (X, S, μ) be a measure space, i.e. S a σ-ring of subsets of X, and μ a (not necessarily finite) countably additive measure on S. We may suppose that μ is already extended by the Carathéodory procedure, so that S is the σ-algebra of measurable sets. Let S_0 be the subring of measurable sets with finite measure. We will <u>assume</u> that there are no purely infinite sets, i.e. if E is a measurable set with $\mu(E) > 0$ then there exists a measurable set $K \subset E$ such that $0 < \mu(K) < \infty$. It follows immediately from this that if $F \subset X$ is such that $\mu(F \cap K) = 0$ for all $K \in S_0$, then $F \in S$ and $\mu(F) = 0$.

Two sets $E, F \in S$ are said to be equivalent if their symmetric difference $(E - F) \cup (F - E)$ is a null set. We will denote by E^* the equivalence class of sets equivalent to E, and by B the collection of equivalence classes. It is easy to see that B is a σ-algebra and that the mapping $E \to E^*$ is a σ-algebra homomorphism. μ may be considered as a measure on B by setting $\mu(E^*) = \mu(E)$. The system (B, μ) is called the <u>measure algebra</u> of the measure space (X, S, μ). We repeat that μ is called localizable if B is complete as a lattice.

Let B_0 be the subalgebra of B consisting of those elements which have finite measure. Since X has no purely infinite sets, we can see that for any $E^* \in B$, $E^* = \sup(K^* : K^* \in B_0, K^* \leq E^*)$; indeed, E^* is certainly an upper bound for all such K^*, and if F^* is also an upper bound then $F^* \geqslant E^* \cap K^*$ for all $K^* \in B_0$, so that $(E^* - F^*) \cap K^* = (E^* \cap K^*) - F^* = 0$ for all $K^* \in B_0$, and hence $E^* - F^* = 0$, i.e. $E^* \leqslant F^*$ as required. Thus B_0 is order-dense in B.

(ii) Let $L = L_1(X, S, \mu)$ = equivalence classes of integrable functions modulo null functions. Denote by f^* the equivalence class of functions equal to f a.e.. L is a σ-Dedekind complete Riesz space with an integral φ defined by $\varphi(f^*) = \int f \, d\mu$ for $f^* \in L$. φ is strictly positive on L, hence L is Dedekind complete (in fact, super-Dedekind complete), and φ is a normal integral (c.f. [8], theorem 27.16 in Note VIII).

(iii) Imbed $L \subset L^\#$. For a unit in $L^\#$, let $1 = \sup(e_\alpha)$, where e_α is the element of L determined by the characteristic function of E_α for $E_\alpha \in S_0$. Recall that $U(L^\#, 1)$ is a complete Boolean algebra (theorem 1.1).

(iv) We want to define a measure-preserving isomorphism ρ of B into $U(L^\#)$. For $E^* \in B_0$, define $\rho(E^*)$ to be the element in L determined by χ_E. Let $U(L^\#)_0 = \{e \in U(L^\#) : \varphi(e) < \infty\}$. We now show that ρ maps B_0 <u>onto</u> $U(L^\#)_0$. Let $e \in U(L^\#)_0$. Now, $e = \sup(x : 0 \leq x \leq e, x \in L)$, and for any such x we have $0 \leq x \leq s(x) \leq$ $\leq e$, and also $s(x) \in L$ since $\varphi(s(x)) \leq \varphi(e) < \infty$; hence $e = \sup(e' : 0 \leqslant e' \leqslant e, e' \in L \cap U(L^\#))$. Since $\varphi(e) < \infty$ and φ is strictly positive, it follows that there exists a sequence $\{e'_n\}$ in

$L \cap U(L^{\#})$ such that $e'_n \uparrow e$. Since each e'_n is determined by the characteristic function of some measurable set, it follows that e is also determined by the characteristic function of some measurable set E, i.e. $\rho(E^*) = e$. Finally, since $\varphi(e) < \infty$, $\mu(E^*) < \infty$, so $e \in \rho(B_o)$. Thus indeed $\rho(B_o) = U(L^{\#})_o$.

Since B_o is order dense in the Boolean algebra B, and $U(L^{\#})_o$ is order dense in the complete Boolean algebra $U(L^{\#})$, ρ can be extended uniquely to an (algebraic) isomorphism of B into $U(L^{\#})$, and the extension (again denoted by ρ) maps B onto $U(L^{\#})$ if and only if B is complete. (c.f. [17], lemma 3.3.2.)

It is easy to see that ρ is measure-preserving, i.e. $\varphi(\rho(E^*)) = \mu(E^*)$ for all $E^* \in B$. Thus ρ is a measure-preserving algebraic isomorphism of B into $U(L^{\#})$, and ρ maps B onto $U(L^{\#})$ if and only if μ is localizable.

(v) Let $M =$ equivalence classes of measurable functions modulo null functions. We can see that $\rho : B \rightarrow U(L^{\#})$ induces in a natural way an algebraic isomorphism ρ^* of M into $L^{\#}$. Indeed, for every measurable function $f \geq 0$ we have $f^* = \sup(a\chi_E^* : 0 \leq a\chi_E^* \leq f^*)$. The collection $\{a\rho(E^*) : 0 \leq a\chi_E^* \leq f^*\} \subset L^{\#+}$ satisfies the conditions of property P_2, so we may define, for $0 \leq f^* \in M$, $\rho^*(f^*) = \sup(a\rho(E^*) : 0 \leq a\chi_E^* \leq f^*) \in L^{\#}$. In general, we define $\rho^*(f^*) = \rho^*(f_+^*) - \rho^*(f_-^*)$.

It is clear that ρ^* is measure-preserving in the sense that, for $0 \leq f^* \in M$, $\varphi(\rho^*(f^*)) = \int f \, d\mu$. In fact, ρ^* is an extension of the identity map of $L \rightarrow L$. We can even see that ρ^* maps $L_1(X, S, \mu)$

onto $L_1(\varphi, L)$; for, given $0 \leq f^{\#} \in L_1(\varphi, L)$, we have $f^{\#} = \sup(r \cdot s_r(f^{\#}) : 0 < r = \text{rational})$ by theorem 1.6. But $\varphi(s_r(f^{\#})) \leq$ $\leq r^{-1}\varphi(f^{\#}) < \infty$, so there exists $E^* \in B_0$ such that $\rho(E^*) = s_r(f^{\#})$, and hence $\rho^*(r\chi_E^*) = r \cdot s_r(f^{\#})$. The set $\{r \cdot s_r(f^{\#}) : 0 < r = \text{rational}\}$ is countable, so there exists $f^* = \sup\{\rho^{*-1}[r \cdot s_r(f^{\#})]\} \in M$, and $\rho^*(f^*) = f^{\#}$. In addition, $\int f^* d\mu = \varphi(f^{\#}) < \infty$, so $f^* \in L_1(X, S, \mu)$.

Thus $L_1(X, S, \mu)$ and $L_1(\varphi, L)$ are identical, so integrals can be considered as exactly the same on either space, and in particular there is no confusion in saying that one integral is absolutely continuous with respect to another without specifying which space is being considered.

Note that $\rho(X^*) = 1$, and hence ρ^* is also an isomorphism of the multiplicative structure (since, by theorem 5.1, multiplication is determined by the linear and order structure).

(vi) We have in general $\rho^*(M) \subset L^{\#}$, and we want to show that equality holds if and only if μ is localizable. In one direction this is clear, for if ρ^* maps M onto $L^{\#}$ then ρ maps B onto $U(L^{\#})$ and hence μ is localizable.

Conversely, suppose μ is localizable, so that ρ maps B onto $U(L^{\#})$. Then for any $e \in U(L^{\#})$ there exists $\rho^{-1}(e) = E^* \in B$, so for any element of the form $ae \in L^{\#}$ there exists $\rho^{*-1}(ae) = a\chi_E^* \in M$. Now suppose $0 \leq f^{\#} \in L^{\#}$. Again we have $f^{\#} = \sup(r \cdot s_r(f^{\#}) : 0 < r)$ and the collection $\{r \cdot s_r(f^{\#}) : 0 < r = \text{rational}\}$ is countable, so there exists $f^* = \sup\{\rho^{*-1}[r \cdot s_r(f^{\#})]\} \in M$, and $\rho^*(f^*) = f^{\#}$. Thus ρ^* maps M onto $L^{\#}$ as required.

(vii) Now suppose μ is localizable. Then M is isomorphic to $L^{\#}$. But the Radon-Nikodym theorem holds for $L^{\#}$, and therefore it holds for M (i.e. it holds in the original measure space).

Conversely, suppose the Radon-Nikodym theorem holds in the measure space. For any $0 \leq g^{\#} \in L^{\#}$ we want to find $g^* \in M$ such that $\rho^*(g^*) = g^{\#}$. To do this, define the normal integral ψ by $\psi(f^{\#}) = \varphi(f^{\#}g^{\#})$. ψ is absolutely continuous with respect to φ, and so, considering ψ and φ as integrals on $L_1(X, S, \mu)$, the Radon-Nikodym theorem implies that there exists $g^* \in M$ such that $\psi(f^*) = \varphi(f^*g^*)$ for all $f^* \in L_1(X, S, \mu)$. Then, considering φ and ψ as integrals on $L_1(\varphi, L)$ again, we have $\psi(f^{\#}) = \varphi(f^{\#} \cdot \rho^*(g^*))$ for all $f \in L_1(\varphi, L)$, and hence, by the uniqueness of the Radon-Nikodym derivative $\rho^*(g^*) = g^{\#}$ as required. Thus ρ^* maps M onto $L^{\#}$, and hence, by (vi), μ is localizable.

Note For a discussion of Segal's theorem along somewhat different lines, see Zaanen [24]. Here is also found a generalization of Segal's theorem to the case where there may exist purely infinite sets. It turns out (c.f. [24], theorem 10.2) that the Radon-Nikodym theorem holds if and only if the contracted measure is localizable.

§17. Rings of operators on a Hilbert space.

Let H be a Hilbert space, and let $B(H)$ be the set of bounded self-adjoint operators on H. $B(H)$ is (partially) ordered by :

$0 \leq A \in B(H)$ if $(Ah, h) \geq 0$ for all $h \in H$. In general, two elements $A, B \in B(H)$ will not have a supremum or infimum; in fact (c.f. Kadison [7]) $\sup(A, B)$ exists in $B(H)$ if and only if $A \geq B$ or $B \geq A$. However, it is known that certain subsets of $B(H)$ are isomorphic in themselves to Riesz spaces; indeed we have the following theorem. (Recall that the sequence A_n is said to converge strongly to A if $\|(A_n - A)h\|$ converges to zero for every $h \in H$.

Theorem 17.1 (c.f. Vulih [22]) Every strongly closed ring L of bounded self-adjoint operators is isomorphic to a Dedekind complete Riesz space.

Proof The proof consists mainly in collecting the appropriate facts from Riesz - Nagy [15]. First of all we observe that L must be commutative since the product of self-adjoint operators is self-adjoint only if the operators commute. (This follows immediately from the formula $(AB)^* = B^*A^*$.) Next we see that $A \in L$ implies $A^{\frac{1}{2}} \in L$ since $A^{\frac{1}{2}}$ is the strong limit of a sequence of polynomials in A (c.f. [15], p.265; note that the use of I, which may not be in L, can be avoided by using for successive approximations to $A^{\frac{1}{2}}$ $X_{n+1} = X_n + \frac{1}{2}(A - X_n^2)$.) From this it follows that, for $A, B \in L$, $\sup(A, B) = \frac{1}{2}(A + B + |A - B|) \in L$ (c.f.[15], p.279). Finally, any set of elements of L directed upwards and bounded has a supremum (c.f. [15], p.263; here this is proved only for monotone increasing sequences, but the same proof holds for directed sets. It is also shown here that the directed set actually converges strongly to its supremum).

Recall that the pricipal identity of $S \subset B(H)$ is the projection onto $H \ominus N$, where $N = \bigcap \{N(A) : N(A) = \text{the null space of } A, A \in S\}$ (c.f. Naimark [9], p.445). A strongly closed ring L may not contain the identity operator I, but we can use theorem 17.1 to show that L does contain its principal identity E (c.f. Vulih [22]). First we prove the following lemma.

<u>Lemma 17.2</u> Let $A \in B(H)$, $0 \leq A \leq I$, and let $P = P_A$ be the projection onto $H \ominus N(A) = \overline{R(A)}$ (where $R(A)$ = the range of A). Let $B_n = A^{1/2^n}$, $n \geq 0$. Then $B_n \uparrow P$.

<u>Proof</u> It is easy to see that $0 \leq B_n \uparrow \leq I$, and hence there exists $B = \sup(B_n) \in L$. B is self-adjoint, and since $B_n = B_{n+1}^2$, $B = B^2$. Hence B is a projection, and we will show $B = P$ by showing that $N(B) = N(A)$.

On the one hand, if $h \in N(A)$, then $\|B_1 h\|^2 = (A^{\frac{1}{2}}h, A^{\frac{1}{2}}h) = (Ah, h) =$ $= 0$, so that $B_1 h = 0$, which implies, by induction, that $B_n h = 0$ for all n. Hence $Bh = 0$, i.e. $h \in N(B)$.

On the other hand, since $0 \leq A = B_0 \leq B$, $h \in N(B)$ implies $0 \leq (Ah, h) \leq (Bh, h) = 0$, so that $Ah = 0$, i.e. $h \in N(A)$.

<u>Theorem 17.3</u> L contains its principal identity E. E acts as a strong order unit in L (considered as a Riesz space), and $U(L, E) =$ $= \{P_A : A \in L\}$.

Proof It follows immediately from the lemma that for any $A \in L$ we also have $P_A \in L$. To show $E \in L$ it is thus sufficient to show $E = \sup\{P_A : A \in L\}$. Since $0 \leq P_A \leq I$ for all $A \in L$, there exists $P = \sup\{P_A : A \in L\} \in L$, and it is sufficient to show that P is a projection (for clearly $N(P) = \bigcap\{N(P_A) : A \in L\} = N(E)$).

Now, for any two projections P' and P'', it is easy to verify that $\sup(P', P'') = \frac{1}{2}(P' + P'' + |P' - P''|) = P' + P'' - P'P''$, and from this it follows immediately that $[\sup(P', P'')]^2 = \sup(P', P'')$, i.e. that $\sup(P', P'')$ is a projection. Thus the sup of any finite collection of projections is again a projection, and so, by adding to the set $\{P_A\}$ the supremums of all finite subsets, we may suppose that we have a set $\{P_\alpha\}$ of projections which is directed upwards and hence approaches P strongly. But then, since $P_\alpha^2 = P_\alpha$, we have $P^2 = P$, i.e. P is a projection as required, and hence $E \in L$.

The rest of the theorem is clear.

A strongly closed ring L has its usual operator multiplication, but considering it as a Dedekind complete Riesz space we may also introduce the abstract multiplication with respect to the strong order unit E. Happily it turns out that these two multiplications are identical. Indeed, it is clear that operator multiplication satisfies conditions (i),...,(vii) of theorem 5.1 (on the uniqueness of multiplication); and we show in the following lemma that it also satisfies the remaining condition (viii).

Lemma 17.4 Let $A, B \in L$. Then their operator product AB equals 0 if and only if $\inf(|A|, |B|) = 0$.

<u>Proof</u> Since $|A|^2 = A^2$, we may conclude, by the uniqueness of positive square roots (c.f. [15], p.265), that $|A||B| = |AB|$. Also, $A = 0$ if and only if $|A| = 0$. Hence it is sufficient to consider the case where $A \geqslant 0$ and $B \geqslant 0$.

Suppose $\inf(A, B) = 0$. Let $m = \max(\|A\|, \|B\|)$. Then $0 \leqslant AB \leqslant (mI)B = mB$, and $0 \leqslant AB \leqslant mA$, so $0 \leqslant AB \leqslant m \cdot \inf(A, B) = 0$, i.e. $AB = 0$.

Conversely, suppose $AB = 0$. Decompose $H = N(A) \oplus \overline{R(A)}$, and let $0 \leqslant C = \inf(A, B)$. Then for $h \in N(A)$, $0 \leqslant (Ch, h) \leqslant (Ah, h) = 0$, so $Ch = 0$. For $h \in R(A)$, $h = Ag$, $0 \leqslant (Ch, h) \leqslant (Bh, h) = (BAg, h) = 0$, so $Ch = 0$, and hence $Ch = 0$ for all $h \in \overline{R(A)}$. Thus $C = 0$ as required.

Thus we have

<u>Theorem 17.5</u> In a strongly closed ring $L \subset B(H)$, operator multiplication coincides with the abstract multiplication determined by considering L as a Riesz space with unit E.

We can use theorem 17.5 to prove the following variation of the Radon-Nikodym theorem.

<u>Theorem 17.6</u> Let L be a strongly closed ring of bounded self-adjoint operators. Let φ and ψ be two normal integrals on L such that $\psi \leq \varphi$. Then there exists $A \in L$, $0 \leq A \leq E$, such that $\psi(B) = \varphi(AB)$ for all $B \in L$.

Proof Taking account of the previous theorems of this section, we may immediately apply theorem 15.2 and conclude that there exists $A \geq 0$ in the extension $L^{\#}$ such that $\psi(B) = \varphi(AB)$ for all $B \in L$. To conclude that in fact $A \in L$, we observe that $\psi \leq \varphi$ implies $A \leq E$ (and recall that L is an ideal in $L^{\#}$).

Note The above theorem is a special case of a theorem of S. Sakai [16].

Let $S \subset B(H)$. We define S' to be the set of bounded self-adjoint operators which commute with every operator in S, and define $S'' = (S')'$. We will say that a set $S \subset B(H)$ is of the type S'' if $S = S''$. As usual we have $S_1' \subset S_2'$ if $S_1 \supset S_2$, and $S' = S'''$. From this it follows that if S is commutative then $S' \supset S$, so $S'' \subset S' = S'''$, and hence S'' is also commutative. In this case S'' is a ring, since the product of commuting self-adjoint operators is again self-adjoint. Conversely, if S'' is a ring then, as above, S'' is commutative.

It is clear that a ring S of the type S'' is strongly closed, and hence is isomorphic to a Dedekind complete Riesz space, with the principal identity E acting as a strong unit. It is also true that for every $0 \neq A \in S$ there is a normal integral φ on S such that $\varphi(A) \neq 0$; this follows from the fact (c.f. Dixmier [3], p.54) that for $h \in H$ the functional φ defined by $\varphi(A) = (Ah, h)$ is a normal integral. In the next theorem we prove the converse.

Theorem 17.7 (c.f. Pinsker [14]) Let L be a Dedekind complete

Riesz space with a strong order unit 1, and suppose that for every $0 \neq x \in L$ there is a normal integral φ such that $\varphi(x) \neq 0$. Then L is isomorphic to a ring of operators of the type S''.

<u>Proof</u> Since there are a sufficient number of normal integrals, L can be written as the Riesz direct sum $\bigcup \oplus L_\alpha$ of normal subspaces L_α each of which possesses a strictly positive normal integral (c.f. [8], theorem 27.17 in Note VIII). If each L_α can be represented as a ring S_α of the type S'' on some Hilbert space H_α, then it is easy to see that L may be represented as a ring of the type S'' on the direct sum $\oplus H_\alpha$ (c.f. [14]). Thus it is sufficient to consider the case where L possesses a strictly positive normal integral φ.

Let H be the Hilbert space $L_2(\varphi, L)$. For every $x \in L$ define the operator A_x on H by $A_x(f) = xf$ for all $f \in H$. It is easy to see that A_x is bounded and self-adjoint (bounded since 1 is a strong unit), and that the mapping $x \longrightarrow A_x$ is a Riesz-space-isomorphism of L onto $S = \{A_x : x \in L\}$.

To complete the proof we must show that $S = S''$. Let $B \in S'$. Then for $y \in L$ we have $B(y) = B(y1) = B(A_y(1)) = A_y(B(1)) = y \cdot B(1)$. Now, $|B(1)| \leq \|B\|1$, for if $B(1) \nleq \|B\|1$ then there exists $e > 0$ and $a > 0$ such that $B(1)e \geqslant (\|B\| + a)e$, and then (taking $y = e$ in the above) $B(e) = e \cdot B(1) \geqslant (\|B\| + a)e$, so that $\|B(e)\|_2 \geqslant (\|B\| + a)\|e\|_2$, a contradiction. Hence $B(1) \in L$, and so the above shows that $B(y) = A_{B(1)}(y)$ for all $y \in L$. But by theorem 13.4 $L = L_0$ is norm-dense in $H = L_2(\varphi)$, so $B = A_{B(1)} \in S$. Thus $S' \subset S$; and since S is commutative $S \subset S'$. Therefore $S = S'$, and so $S = S''$.

<u>Note</u> The above proof follows Pinsker's proof in [14], except that he considers a representation of L as a function space with φ acting as an integral on it, and uses as his Hilbert space the L_2 space determined by φ in this function space.

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