SOME CENTRAL LIMIT THEOREMS

FOR DOUBLY RESTRICTED PARTITIONS

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ABSTRACT

Let $P_{\bar{K},\bar{L}}(N)$ be the number of <u>unordered</u> partitions of a positive integer N into K or fewer positive integer parts, each part not exceeding L. A distribution of the form

$$\sum_{N \leq x} P_{\bar{K},\bar{L}}(N)$$

is considered first. For any fixed K, this distribution approaches a piecewise polynomial function as L increases to infinity. As both K and L approach infinity, this distribution is asymptotically normal. These results are proved by studying the convergence of the characteristic function.

The main result is the asymptotic behavior of $P_{\bar{K},\bar{K}}(N)$ itself, for certain large K and N. This is obtained by studying a contour integral of the generating function taken along the unit circle. The bulk of the estimate comes from integrating along a small arc near the point 1. Diophantine approximation is used to show that the integral along the rest of the circle is much smaller.

1. INTRODUCTION

We shall consider four partition enumerating functions. If K, L, and N are positive integers, the function $P_{K,L}(N)$ is defined to be the number of unordered partitions of N into K parts, the largest of which is L. For example, 22 has 11 partitions into 5 parts with 6 as a largest part. These are

6 + 6 + 6 + 3 + 1,	6 + 6 + 6 + 2 + 2,
6 + 6 + 5 + 4 + 1,	6 + 6 + 5 + 3 + 2,
6 + 6 + 4 + 4 + 2,	6 + 6 + 4 + 3 + 3,
6 + 5 + 5 + 5 + 1,	6 + 5 + 5 + 4 + 2,
6 + 5 + 5 + 3 + 3,	6 + 5 + 4 + 4 + 3,
6 + 4 + 4 + 4 + 4.	

Therefore $P_{5,6}(22) = 11$.

The function $P_{\bar{K},L}(N)$ is similar to $P_{K,L}(N)$ except that the number of parts of each enumerated partition is K or less. Thus, we have

$$P_{\bar{K},L}(N) = \sum_{r=1}^{K} P_{r,L}(N).$$

The function $P_{K,\bar{L}}(N)$ differs from $P_{K,L}(N)$ in another way. Each enumerated partition has exactly K parts, but the largest part may be equal to L or any smaller integer. Hence we have

$$P_{K,\bar{L}}(N) = \sum_{s=1}^{L} P_{K,s}(N).$$

The function $P_{\bar{K},\bar{L}}(N)$ represents both of these modifications jointly, enumerating partitions with not more than K parts, each part not being greater than L. We may write

$$P_{\bar{K},\bar{L}}(N) = \sum_{r=1}^{K} \sum_{s=1}^{L} P_{r,s}(N).$$

These functions are natural refinements of the more familiar functions

$$P_K(N) = \lim_{L \to \infty} P_{K,\bar{L}}(N) \quad \text{and}$$

$$P(N) = \lim_{\substack{K \to \infty \\ L \to \infty}} P_{\bar{K},\bar{L}}(N)$$

which enumerate simply restricted and unrestricted partitions respectively.

Most of the classical lore of these functions can be found in accessible introductions to combinatorial analysis or number theory. In particular, a comprehensive treatment of nineteenth century algebraic methods can be found in MacMahon ([5], Volume II, Section VII). Hardy and Wright ([4], pp. 273-96) give a more concise treatment, as does Riordan ([8], pp. 107-62).

In the next section we shall cite the elementary properties we need. One of these, Lemma 2-5, is a relationship connecting the functions $P_{K,L}(N)$, $P_{\bar{K},L}(N)$, $P_{K,\bar{L}}(N)$ and $P_{\bar{K},\bar{L}}(N)$. Consequently it will suffice to discuss $P_{\bar{K},\bar{L}}(N)$ only.

It will be convenient to define $P_{\bar{K},\bar{L}}(N) = 0$ for all negative integers N. We set $P_{\bar{K},\bar{L}}(0) = 1$.

Before stating Theorem 1, we introduce some more notation.

Let $g_1(x) = 1$ if $0 \leq x \leq 1$ and let $g_1(x) = 0$ elsewhere. For the integers $K \geq 2$, $g_K(x)$ is defined recursively by the formula

$$g_K(x) = \int_{x-1}^{x} g_{K-1}(t)dt.$$

The function $g_K(x)$ is the K-fold convolution of $g_1(x)$, and it can be written in the form

$$g_K(x) = \frac{1}{(K-1)!} \sum_{u=0}^{[x]} \binom{K}{u}(-1)^u(x-u)^{K-1}.$$

We shall be interested in the following normalization of $g_K(x)$:

$$h_K(x) = \sqrt{K/12}\, g_K(\sqrt{K/12}x + (K/2)).$$

We now state the simplest result of this dissertation.

Theorem 1. Let K be a constant and L and B increase to infinity in such a way that B/L approaches a limit a, and let $b = \sqrt{12/K}(a - K/2)$. Then we have

$$\binom{K+L}{K}^{-1} \sum_{N=0}^{B} P_{\bar{K},\bar{L}}(N) \longrightarrow \int_{-\infty}^{b} h_K(x)dx.$$

This result was suggested heuristically in 1928 by Tricomi [10], who noticed a correspondence between the partitions of an integer S into n unequal parts not exceeding N and the ways that a sum equal to S could result when n balls are drawn from an urn containing a set of N balls numbered from 1 to N. He assumed that as S, n and N grew large the inequality restriction for the parts would become relatively insignificant, so that the probability density for S would approximate the convolution of identically distributed random processes. The actual number of such partitions is equal to $P_{\bar{n},\overline{N-n}}(S - \frac{n(n+1)}{2})$ ([8], p. 113), so Tricomi's result would be a special case of Theorem 1.

The proof of Theorem 1 appears in Section 4 and uses a general theorem of Levy and Cramér on probability distributions. The next result is similarly proved:

<u>Theorem 2</u>. If K, L and B increase to infinity in such a way that $\sqrt{12}(B-\frac{KL}{2})(KL(K+L+1))^{\frac{1}{2}}$ approaches a limit b, then

$$\binom{K+L}{K}^{-1} \sum_{N=0}^{B} P_{\bar{K},\bar{L}}(N) \longrightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{b} e^{-x^2/2} dx.$$

Tricomi [11] and Castelnuovo [1] have proved similar results for ordered partitions. Their methods deal with identically distributed independent random variables, and these appear to be applicable only to ordered partitions or to partitions with unequal parts.

In 1941 Erdös and Lehner [3] obtained a similar result for simply restricted partitions:

> If, for some fixed x, K and N approach infinity in such a way that $K = [\sqrt{N}(x+(\log n)/C)]$, where $C = \pi\sqrt{2/3}$, then

$$\lim \frac{P_{\bar{K}}(N)}{P(N)} = \exp[-(2/C)\exp(-Cx/2)].$$

Their proof uses an inclusion-exclusion process and the asymptotic formula $P(N) = e^{C\sqrt{N}}/(4N\sqrt{3})$.

The next theorem is the principal result in this dissertation:

Theorem 3. If K and N approach infinity in such a way that $\sqrt{6}(N - \frac{K^2}{2})K^{-3/2}$ approaches a limit b, then we have

$$\frac{1}{6} K^{3/2}\binom{2K}{K}^{-1} P_{\bar{K},\bar{K}}(N) \longrightarrow \frac{e^{-b^2/2}}{\sqrt{2\pi}}.$$

The proof of this theorem is based on contour integration of the generating function $F_{K,K}(z) = \sum_N P_{\bar{K},\bar{K}}(N)z^N$ along a path surrounding the origin. In the usual treatments of the partition function P(N), which is the limiting case as K increases to infinity, it is necessary to confine the contour of integration to the interior of the unit circle. A case in point is the recent approach by Newman [6] in which the contour is a circle of radius $1 - (\pi/\sqrt{6n})$. He isolates a short segment close to 1 which contributes the bulk of his estimate. He then shows that the generating function is of a smaller exponential order on the remainder of the circle. Since our function $F_{K,K}(z)$ is a polynomial, it is possible to use the unit circle itself. The major part of the estimate is done on an arc centered on the point z = 1. Diophantine approximation is used to obtain a bound on the polynomial elsewhere on the circle.

2. ELEMENTARY PROPERTIES OF $P_{\bar{K},\bar{L}}(N)$

Unless stated otherwise, each of the following lemmas is a direct consequence of the definitions.

Lemma 2-1. If either $N < 0$ or $N > KL$ holds, then we have

$$P_{\bar{K},\bar{L}}(N) = 0.$$

Lemma 2-2. For all positive K and L we have

$$P_{\bar{K},\bar{1}}(N) = 1 \qquad \text{if } 0 \leq N \leq K, \text{ and}$$

$$P_{\bar{1},\bar{L}}(N) = 1 \qquad \text{if } 0 \leq N \leq L.$$

Lemma 2-3. For $K \geq 2$, $L \geq 2$ and all N we have

$$P_{\bar{K},\bar{L}}(N) = P_{\bar{K},\overline{L-1}}(N) + P_{\overline{K-1},\bar{L}}(N-L).$$

Lemma 2-4. $P_{\bar{K},\bar{L}}(N) = P_{\bar{L},\bar{K}}(N)$.

This is best proven with a Ferrer's graph, as in [4], pp. 273-74, or in [8], pp. 113-14.

Lemma 2-5. $P_{\bar{K},\bar{L}}(N) = P_{\overline{K+1},L}(N+L) = P_{K,\overline{L+1}}(N+K)$

$$= P_{K+1,L+1}(N+K+L+1).$$

This follows from Lemmas 2-3 and 2-4 and the formulas appearing with the definitions.

We now introduce the generating function

$$F_{K,L}(z) = \sum_{N=0}^{KL} P_{\bar{K},\bar{L}}(N) z^N .$$

Lemma 2-6. Except wherever the denominator vanishes we have

$$F_{K,L}(z) = \frac{\prod_{r=1}^{K+L} (1-z^r)}{\prod_{r=1}^{K} (1-z^r) \prod_{r=1}^{L} (1-z^r)} .$$

This can be deduced from Lemmas 2-1, 2-2 and 2-3. Other derivations of this formula appear in [5], p. 5, and in [8], p. 153, problem 5. This lemma shows that the polynomial $F_{K,L}(z)$ has all its zeroes on the circle $|z| = 1$.

Lemma 2-7.

$$\sum_{N=0}^{KL} P_{\bar{K},\bar{L}}(N) = \lim_{z \to 1} F_{K,L}(z) = \binom{K+L}{K} .$$

3. THE DISTRIBUTION $Q_{K,L}(x)$ AND ITS CHARACTERISTIC FUNCTION $q_{K,L}(t)$

The weak asymptotic behavior of $P_{\bar{K},\bar{L}}(N)$ is easily investigated in terms of the following function:

$$Q_{K,L}(x) = \binom{K+L}{K}^{-1} \sum P_{\bar{K},\bar{L}}(N),$$

where the sum is taken over all integers $N \leq (KL/2) + C_{K,L}x$ and where $C_{K,L}$ is defined to be $\sqrt{KL(K+L+1)/12}$. (Whenever there is no danger of ambiguity, the subscripts K and L are omitted.) Since we have $Q(x) = 0$ for $x < -KL/(2C)$, and $Q(x) = 1$ for $x > KL/(2C)$, and since Q is a nondecreasing step function for intermediate values of x, we can treat Q as a distribution. It can be shown easily, using Lemmas 3-3 and 3-4, that Q has mean 0 and variance 1.

We now introduce its characteristic function

$$q(t) = q_{K,L}(t) = \int_{-\infty}^{\infty} e^{ixt} dQ_{K,L}(x),$$

which is defined for every real value of t. A simple calculation yields the following:

<u>Lemma 3-1</u>. $$q(t) = \binom{K+L}{K}^{-1} \sum_{N=0}^{KL} P_{\bar{K},\bar{L}}(N)\exp(((N-(KL/2))/C)it)$$

$$= \binom{K+L}{K}^{-1} \exp(-KLit/(2C))F(e^{it/C}).$$

This can be carried further with Lemma 2-6 and the exponential formula for the sine.

Lemma 3-2. Wherever the denominator does not vanish we have

$$q(t) = \frac{\prod_{r=1}^{K+L} \frac{\sin(rt/2C)}{(rt/2C)}}{\prod_{r=1}^{K} \frac{\sin(rt/2C)}{(rt/2C)} \prod_{r=1}^{L} \frac{\sin(rt/2C)}{(rt/2C)}}.$$

The following lemmas are needed only for the proofs of Theorems 2 and 3.

Lemma 3-3. Within the interval $\{|t| < (2\pi C/(K+L))\}$, $q(t)$ has an analytic logarithm with a power series expansion of the form

$$\log q(t) = \sum_{n=1}^{\infty} a_{2n} t^{2n},$$

where

$$a_{2n} = -\frac{\zeta(2n)}{n(2\pi C)^{2n}} \left\{ \sum_{r=1}^{K+L} r^{2n} - \sum_{r=1}^{K} r^{2n} - \sum_{r=1}^{L} r^{2n} \right\}.$$

Proof. For any complex z such that $|z| < 1$, the quotient $\frac{\sin \pi z}{\pi z}$ is nonzero and analytic. It has a logarithm which can be found from the Weierstrass product for the sine:

$$\log \frac{\sin \pi z}{\pi z} = \sum_{s=1}^{\infty} \log(1-(z/s)^2) = -\sum_{s=1}^{\infty}\sum_{n=1}^{\infty} (z/s)^{2n}/n$$

$$= -\sum_{n=1}^{\infty} (z^{2n}/n)\left(\sum_{s=1}^{\infty} s^{-2n}\right) = -\sum_{n=1}^{\infty} \zeta(2n) z^{2n}/n.$$

For $|t| < (2\pi C/(K+L))$, Lemma 3-2 gives us

$$\log q(t) = \sum_{r=1}^{K+L} \log \frac{\sin (rt/2C)}{(rt/2C)} - \sum_{r=1}^{K} \log \frac{\sin (rt/2C)}{(rt/2C)} - \sum_{r=1}^{L} \log \frac{\sin (rt/2C)}{(rt/2C)}.$$

Now, using the expansion of $\log \frac{\sin \pi z}{\pi z}$, we have

$$\log q(t) = -\sum_{n=1}^{\infty} \zeta(2n)\left\{\sum_{r=1}^{K+L}\left(\frac{rt}{2\pi C}\right)^{2n} - \sum_{r=1}^{K}\left(\frac{rt}{2\pi C}\right)^{2n} - \sum_{r=1}^{L}\left(\frac{rt}{2\pi C}\right)^{2n}\right\}/n$$

$$= -\sum_{n=1}^{\infty} \zeta(2n)(t/(2\pi C))^{2n}\left\{\sum_{r=1}^{K+L} r^{2n} - \sum_{r=1}^{K} r^{2n} - \sum_{r=1}^{L} r^{2n}\right\}/n.$$

and the lemma is proved.

Lemma 3-4. In the previous lemma, a_2 is equal to $-1/2$.

Proof. Since we have

$$\sum_{r=1}^{K+L} r^2 - \sum_{r=1}^{K} r^2 - \sum_{r=1}^{L} r^2$$

$$= \frac{1}{6}\left\{ (K+L)(K+L+1)(2K+2L+1) - K(K+1)(2K+1) - L(L+1)(2L+1) \right\}$$

$$= K^2L + KL + KL^2,$$

it is easy to show that

$$a_2 = -\zeta(2)(2\pi C)^{-2}KL(K+L+1) = -1/2.$$

Lemma 3-5. Within the interval $\left\{|t| = (2\pi C/(K+L))\right\}$ we have

$$q(t) \le e^{-t^2/2}.$$

Proof. Lemmas 3-3 and 3-4 imply that

$$\log e^{-t^2/2} - \log q(t) = -\sum_{n=2}^{\infty} a_{2n} t^{2n},$$

which is nonnegative since all the a_{2n}'s are negative.

Lemma 3-6. Within the interval $\left\{|t| \le (\pi C/(K+L))\right\}$ we have

$$\log e^{-t^2/2} - \log q(t) \le (2t^4/15)(K^{-1}+L^{-1}).$$

Proof. As above, we have

$$\log e^{-t^2/2} - \log q(t)$$

$$\le \sum_{n=2}^{\infty} \zeta(4)(t/(2\pi C))^{2n} \left\{ \sum_{r=1}^{K+L} r^{2n} - \sum_{r=1}^{K} r^{2n} - \sum_{r=1}^{L} r^{2n} \right\} /n$$

$$= \zeta(4)\left\{\sum_{n=1}^{\infty} n^{-1}(t/(2\pi C))^{2n}\sum_{r=1}^{K}\left\{(r+L)^{2n}-r^{2n}\right\}\right.$$

$$\left.-(t/(2\pi C))^2\sum_{r=1}^{K}\left\{(r+L)^2-r^2\right\}\right\}$$

$$= \zeta(4)\left\{\sum_{r=1}^{K}\left(\log\left(1-\left(\frac{rt}{2\pi C}\right)^2\right)-\log\left(1-\left(\frac{r+L}{2\pi C}t\right)^2\right)\right)\right.$$

$$\left.-(t/(2\pi C))^2 KL(K+L+1)\right\}$$

$$= \frac{\pi^4}{90}\left\{\sum_{r=1}^{K}\int\frac{du}{u+1} - (t/(2\pi C))^2 KL(K+L+1)\right\},$$

where the integration is taken in the interval

$$[-((r+L)t/(2\pi C))^2, -(rt/2\pi C)^2].$$

Since $|t| < (\pi C/K+L)$, we have $(1+u)^{-1} < 1 - 4u/3$ in the integrand. Using this estimate, integrating and summing we find

$$\log e^{-t^2/2} - \log q(t)$$

$$\leq (2/15)\pi^4(t/(2\pi C))^4 KL(K+L+1)(K^2+L^2+KL+K+L)$$

$$= (2t^4/15)(K^{-1}+L^{-1}-(K+L+1)^{-1}).$$

and the lemma is proved.

Lemma 3-7. For all t satisfying $|t| \leq (\pi C/K+L)$ we have

$$0 \leq e^{-t^2/2} - q(t) = (2/15)(K^{-1}+L^{-1})t^4e^{-t^2/2}.$$

Proof. The leftmost inequality comes from Lemma 3-5. Lemma 3-6 gives us

$$q(t) \geq \exp(-(t^2/2)-(2t^4/15)(K^{-1}+L^{-1})).$$

Hence we have

$$\begin{aligned} e^{-t^2/2} - q(t) &\leq e^{-t^2/2}(1-\exp(-(2t^4/15)(K^{-1}+L^{-1}))) \\ &= (2/15)(K^{-1}+L^{-1})t^4e^{-t^2/2}, \end{aligned}$$

since $1 - e^{-u} \leq u$ if $u \geq 0$.

4. THE PROOF OF THEOREM 1

Since $C_{K,L}$ is defined to be $\sqrt{KL(K+L+1)/12}$, we see at once that if K is fixed as L increases to infinity then C also increases to infinity and that

$$L/C = \sqrt{12/K} + O(L^{-1}).$$

For any finite t and r, $1 \leq r \leq K$, it is possible to take L so large that

$$\frac{\sin (rt/2C)}{(rt/2C)}$$

is then nonzero and increasing to 1 as L increases to infinity. From Lemma 3-2 we have immediately that $\lim\limits_{L \to \infty} q_{K,L}(t)$ exists and equals

$$\lim_{L \to \infty} \prod_{r=1}^{K} \frac{\dfrac{\sin ((r+L)t/2C)}{(r+L)t/2C}}{\dfrac{\sin (rt/2C)}{(rt/2C)}} = \left(\frac{\sin \sqrt{3/K}t}{\sqrt{3/K}t} \right)^K,$$

which is the Fourier transform of $h_K(x)$.

We now refer to the following theorem of Levy and Cramér:

> Given a sequence of distributions $F_1(x), F_2(x), \ldots$ and the characteristic functions $\phi_1(t), \phi_2(t), \ldots$. A necessary and sufficient condition for the convergence of the sequence $F_n(x)$ to a distribution function $F^*(x)$ is that, for every t, the sequence $\phi_n(t)$ converges to a limit $\phi(t)$, which is continuous for the special value $t = 0$.

When this condition is satisfied, the limit $\phi(t)$ is identical with the characteristic function of the limiting distribution $F^*(x)$. ([2], p. 96)

We shall take the distribution functions to be $F_L(x) = Q_{K,L}(x)$. Since the characteristic function is identical to the Fourier transform, we also have $\phi_L(t) = q_{K,L}(t)$ and

$$\phi(t) = \left(\frac{\sin \sqrt{3/K}t}{\sqrt{3/K}t}\right)^K .$$

This immediately yields $Q_{K,L}(c) \longrightarrow \int_{-\infty}^{c} h_K(x)dx$ for any real value of c.

Now for any $\epsilon > 0$ the hypothesis of Theorem 1 implies that there is an L_ϵ such that if $L \geq L_\epsilon$ then we have

$$\left|\frac{B-(KL/2)+(1/2)}{C_{K,L}} - b\right| < \epsilon .$$

Since Q is nondecreasing, we have

$$Q(b-\epsilon) \leq Q\left(\frac{B-(KL/2)+(1/2)}{C_{K,L}}\right) \leq Q(b+\epsilon)$$

for all $L \geq L_\epsilon$. As this L increases to infinity, we see that

$$\int_{-\infty}^{b-\epsilon} h_K(x)dx \leq \liminf_{L \to \infty} Q\left(\frac{B-(KL/2)+(1/2)}{C_{K,L}}\right)$$
$$\leq \limsup_{L \to \infty} Q\left(\frac{B-(KL/2)+(1/2)}{C_{K,L}}\right) \leq \int_{-\infty}^{b+\epsilon} h_K(x)dx .$$

The continuity of the integral implies that the extreme right and left members of this inequality will approach $\int_{-\infty}^{b} h_K(x)dx$ as $\epsilon \to 0$. Thus

$$\lim_{L \to \infty} Q\left(\frac{B-(KL/2)+(1/2)}{C_{K,L}}\right)$$

exists and equals $\int_{-\infty}^{b} h_K(x)dx$. The definition of Q shows that this is equivalent to Theorem 1.

5. THE PROOF OF THEOREM 2

As K and L both increase to infinity, it is apparent that $C_{K,L}/(K+L)$ does also. Thus Lemma 3-7 implies that $q_{K,L}(t)$ converges pointwise to $e^{-t^2/2}$. The rest of the proof of Theorem 2 follows from this fact in the same way that the proof of Theorem 1 followed from the convergence of $q_{K,L}(t)$ in Section 4.

6. THE PROOF OF THEOREM 3

The Cauchy residue theorem gives us

$$P_{\bar{K},\bar{K}}(N) = (2\pi i)^{-1}\int z^{-(N+1)}F_{K,K}(z)dz,$$

where the contour is the circle $|z| = 1$, traversed counterclockwise. F is the generating function introduced in Section 2.

If we write $z = e^{2\pi i\theta}$, $-\frac{1}{2} \leq \theta \leq \frac{1}{2}$, this becomes

$$P_{\bar{K},\bar{K}}(N) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi iN\theta}F_{K,K}(e^{2\pi i\theta})d\theta$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{Re}\left\{e^{-2\pi iN\theta}F_{K,K}(e^{2\pi i\theta})\right\}d\theta$$

$$= 2\int_{0}^{\frac{1}{2}} \binom{2K}{K}\cos(2\pi((K^2/2)-N)\theta)q_{K,K}(2\pi C_{K,K}\theta)d\theta,$$

since $F_{K,K}(e^{2\pi i\theta}) = \binom{2K}{K}e^{K^2\pi i\theta}q_{K,K}(2\pi C_{K,K}\theta)$, and since q is an even real function.

Now we write

$$P_{\bar{K},\bar{K}}(N) - \binom{2K}{K}(\sqrt{2\pi}C)^{-1}\exp(((K^2/2)-N)^2/2C^2)$$

$$= 2\binom{2K}{K}\int_{0}^{\frac{1}{2K}}\cos(2\pi((K^2/2)-N)\theta)q_{K,K}(2\pi C_{K,K}\theta)d\theta$$

$$+ 2\int_{\frac{1}{2K}}^{\frac{1}{2}} \mathrm{Re}\left\{e^{2\pi i N\theta} F_{K,K}(e^{2\pi i\theta})\right\} d\theta$$

$$- 2\binom{2K}{K}\int_0^{\infty} \cos(2\pi((K^2/2)-N)\theta)e^{-(2\pi C\theta)^2/2} d\theta$$

$$= 2\binom{2K}{K}\int_0^{\frac{1}{2K}} \cos(2\pi((K^2/2)-N)\theta)\left\{q(2\pi C\theta)-e^{-(2\pi C\theta)^2/2}\right\} d\theta$$

$$+ 2\int_{\frac{1}{2K}}^{\frac{1}{2}} \mathrm{Re}\left\{e^{2\pi i N\theta} F(e^{2\pi i\theta})\right\} d\theta$$

$$- 2\binom{2K}{K}\int_{\frac{1}{2K}}^{\infty} \cos(2\pi((K^2/2)-N)\theta)e^{-(2\pi C\theta)^2/2} d\theta.$$

Thus we have

$$\left|P_{K,K}(N) - \binom{2K}{K}(\sqrt{2\pi}C)^{-1}\exp(((K^2/2)-N)^2/2C^2)\right|$$

$$\leq 2\binom{2K}{K}\int_0^{\frac{1}{4K}} \cos(2\pi((K^2/2)-N)\theta)\left|q(2\pi C\theta)-e^{-(2\pi C\theta)^2/2}\right| d\theta$$

$$+\ 2\binom{2K}{K}\int_{\frac{1}{4K}}^{\frac{1}{2K}}\cos(2\pi((K^2/2)-N)\theta\left|q(2\pi C\theta)-e^{-(2\pi C\theta)^2/2}\right|d\theta$$

$$+\quad 2\int_{\frac{1}{2K}}^{\frac{1}{2}}|F(e^{2\pi i\theta})|d\theta$$

$$+\ 2\binom{2K}{K}\int_{\frac{1}{2K}}^{\infty}\cos(2\pi((K^2/2)-N)\theta)e^{-(2\pi C\theta)^2/2}d\theta$$

$$= I_1 + I_2 + I_3 + I_4, \tag{6-1}$$

say. As a consequence of Theorem 3-7, we can write

$$I_1 \leq 2\binom{2K}{K}\int_0^{\frac{1}{4K}}|q(2\pi C\theta)-e^{-(2\pi C\theta)^2/2}|d\theta$$

$$\leq (8/15K)\binom{2K}{K}\int_0^{\frac{1}{4K}}(2\pi C\theta)^4e^{-(2\pi C\theta)^2/2}d\theta$$

$$< (8/15K)\binom{2K}{K}\int_0^{\infty}t^4e^{-t^2/2}dt(2\pi C)^{-1}$$

$$= 4\binom{2K}{K}/(5CK\sqrt{2\pi}). \tag{6-2}$$

From Lemma 3-5 we deduce that

$$I_2 \le 2\binom{2K}{K}\int_{\frac{1}{4K}}^{\frac{1}{2K}} |q(2\pi C\theta) - e^{-(2\pi C\theta)^2/2}|\,d\theta$$

$$\le 2\binom{2K}{K}\int_{\frac{1}{4K}}^{\frac{1}{2K}} e^{-(2\pi C\theta)^2/2}\,d\theta.$$

Thus we have

$$I_2 + I_4 \le 2\binom{2K}{K}\int_{\frac{1}{4K}}^{\infty} e^{-(2\pi C\theta)^2/2}\,d\theta$$

$$= 2\binom{2K}{K}(4\pi C)^{-1}\int_{\frac{\pi^2C^2}{8K^2}}^{\infty} e^{-u}u^{-1/2}\,du$$

$$\le 2K\binom{2K}{K}e^{-\pi^2C^2/8K^2}(\sqrt{2\pi}^2C^2)^{-1} \qquad (6\text{-}3)$$

The estimation of I_3 is made possible by the following lemma, which is proved in Section 7.

<u>Lemma 6-1</u>. For $\frac{1}{2K} \le \theta \le \frac{1}{2}$, we have

$$\left|F_{K,K}(e^{2\pi i\theta})\right| < 55K^{9\frac{1}{2}}(K+1)^3 2^{-K}\binom{2K}{K}.$$

This yields immediately that

$$I_3 < 55K^{9\frac{1}{2}}(K+1)^3 2^{-K}\binom{2K}{K}. \qquad (6\text{-}4)$$

The estimates expressed in inequalities (6-2), (6-3) and (6-4) are all of smaller order than

$$\binom{2K}{K}/C_{K,K}$$

as K increases to infinity. Theorem 3 now follows from inequality (6-1).

7. ESTIMATING F(z) ON AN ARC OF THE CIRCLE $|z| = -1$

The proof of Lemma 6-1 will be worked out in terms of the function $G_K(\theta) = \left|F_{K,K}(e^{2\pi i\theta})\right|$. We shall derive an upper bound for G on the interval $\frac{1}{2K} \leq \theta \leq \frac{1}{2}$. For large K, this bound will be much smaller than the maximum for G, $\binom{2K}{K}$, attained at $\theta = 0$.

We begin with the following well-known lemma from the theory of Diophantine approximation.

Lemma 7-1. If K is a positive integer and if θ is a real number on the interval [0,1], then there are relatively prime integers, p and q, which satisfy

$$0 \leq p \leq q \leq 2K, \quad \text{and} \quad |\theta - p/q| < \frac{1}{q(2K+1)}.$$

We have taken $n = 2K$ in Niven's statement and proof of this result ([7], pp. 3-4).

We shall choose any particular θ_1 in the interval $\left[\frac{1}{2K}, \frac{1}{2}\right]$. For this θ_1 each fraction p/q determined by Lemma 7-1 must have $q \geq 2$. For any such choice of p/q we shall obtain an estimate for $G(\theta)$ for all θ in the interval $|\theta - p/q| < \frac{1}{q(2K+1)}$. This bound, which depends only on K, is given by the following lemma:

Lemma 7-2. If $\frac{1}{2K} \leq \theta \leq \frac{1}{2}$, and if $D = (2\pi)^{3/2} 3^{-9} \pi \exp\left(\frac{9\pi^2}{48} + 5\right)$, we have

$$G_K(\theta) < DK^9(K+1)^3 2^K < 55K^{9\frac{1}{2}}(K+1)^3 2^{-K}\binom{2K}{K}.$$

Lemma 7-2 directly implies Lemma 6-1. It is easily verified for the case $K = 1$. We shall give the proof for $K \geq 2$ by mathematical induction on K. Given K, we assume that the lemma is true for all smaller integers. The proof splits naturally into two cases: (1) $K < q \leq 2K$, and (2) $q \leq K$.

CASE 1: $2 \leq K < q \leq 2K$

Within the interval $|\theta - p/q| < \frac{1}{q(2K+1)}$, the only factor of

$$G(\theta) = \frac{\prod_{r=K+1}^{2K} |1-e^{2\pi i r\theta}|}{\prod_{r=1}^{K} |1-e^{2\pi i r\theta}|}$$

which vanishes at all is

$$|1-e^{2\pi i q\theta}| = 2|\sin q\pi\theta| = 2 \sin q\pi|\theta - p/q| < \pi/K. \tag{7-1}$$

The following result will be convenient in estimating the rest of the quotient above.

Lemma 7-3. For any θ in the interval $\left[0,\frac{1}{2}\right]$, and for any $n \geq 1$, we have

$$\prod_{s=1}^{n} |\sin s\pi\theta| < \sqrt{2\pi n^3}\, 2^{-n/2} \tag{7-2}$$

Equivalently, when $x = e^{2\pi i\theta}$ is considered, we have

$$\prod_{s=1}^{n} |1-x^s| < \sqrt{2\pi n^3}\, 2^{-n/2} \tag{7-3}$$

Proof. If $n = 1$, the lemma may be verified by inspection. If $n \geq 2$, we have, if $0 < \theta < 1/n$,

$$\log \frac{\prod_{s=1}^{n} \sin s\pi\theta}{(n\pi\theta)^n} = \log \prod_{s=1}^{n} \frac{\sin s\pi\theta}{s\pi\theta} + \log \frac{n!(\pi\theta)^n}{(n\pi\theta)^n}$$

$$= \sum_{s=1}^{n} \sum_{u=1}^{\infty} \log(1-(s\theta/u)^2) + \log \frac{n!}{n^n}$$

$$< -\sum_{s=1}^{n} \sum_{u=1}^{\infty} \sum_{t=1}^{\infty} (s\theta/u)^{2t}/t + \log\left\{\sqrt{2\pi}e^{-n}n^{n+1/2}(1+(1/4n))/n^n\right\}$$

$$< \log\left\{\sqrt{2\pi n}e^{-n}(1+(1/4n))\right\} - \sum_{s=1}^{n} (s\theta)^2 \sum_{u=1}^{\infty} (1/u^2)$$

$$< \log\left\{\sqrt{2\pi n}e^{-n}(1+(1/4n))\right\} - n^3\theta^2\pi^2/18.$$

Thus $\log \sum_{s=1}^{n} \sin s\pi\theta$ is bounded above by

$$n \log n\pi\theta + \log\left\{\sqrt{2\pi n}e^{-n}(1+(1/4n))\right\} - n^3\theta^2\pi^2/18,$$

which attains its maximum,

$$n(\log 3-3/2) + \log\left\{\sqrt{2\pi n}(1+(1/4n))\right\},$$

when θ is equal to $3/\pi n$. Since $\log 3 - 3/2 < -(\log 2)/2$, the lemma follows for the case $0 < \theta < 1/n$.

For the remaining case, $\theta \geq 1/n$, it is sufficient to cite a lemma in a recent paper by Sudler, ([9], pp. 4-7). The proof of Lemma 7-3 is then complete.

Now we can bound

$$G^*(p/q) = \frac{\prod_{r=1}^{q-1} f(rp/q) \prod_{r=q+1}^{2K} f(rp/q)}{\prod_{r=1}^{K} f(rp/q)^2} = \frac{\prod_{r=q+1}^{2K} f(rp/q) \prod_{r=K+1}^{q-1} f(rp/q)^2}{\prod_{r=1}^{q-1} f(rp/q)}$$

$$= \frac{\prod_{r=1}^{2K-q} f(rp/q) \prod_{r=1}^{q-1-K} f(rp/q)^2}{\prod_{r-1}^{q-1} f(rp/q)},$$

where $f(v) = |1-e^{2\pi i v}|$.

The previous lemma implies that the numerator is less than $(\sqrt{2\pi})^3(2K-q)^3(q-1-K)^6 2^{(q/2)-1}$ if $K + 1 < q < 2K$. Since we have $(2K-q)(q-1-K)^2 < 4K^3/27$, this bound becomes $(\sqrt{2\pi})^3(2^6/3^9)K^9 2^{K-1}$. If $q = K + 1$, the numerator is less than $\sqrt{2\pi}K^3 2^{K/2}$. If $q = 2K$, the numerator is less than $2\pi(K-1)^6 2^{K-1}$. Since the denominator of G^* is equal to q, we can write

$$G^*(p/q) < (2\pi)^{3/2}K^8 2^{K+5}3^{-9} \qquad \text{if } K + 1 < q < 2K,$$

$$G^*(p/q) < \sqrt{2\pi}K^2 2^{K/2} \qquad \text{if } q = K + 1, \text{ and}$$

$$G^*(p/q) < (\pi/2K)(K-1)^6 2^K \qquad \text{if } q = 2K.$$

From these inequalities, along with the fact that we are considering $K \geq 2$, we obtain, for $K < q \leq 2K$,

$$G^*(p/q) < (2\pi)^{3/2} K^8 2^{K+5} 3^{-9}. \tag{7-4}$$

Now we estimate

$$\frac{G^*(\theta)}{G^*(p/q)} = \frac{\prod_{\substack{r=K+1 \\ r\neq q}}^{2K} \left|\frac{1-e^{2\pi i r\theta}}{1-e^{2\pi i r p/q}}\right|}{\prod_{r=1}^{K} \left|\frac{1-e^{2\pi i r\theta}}{1-e^{2\pi i r p/q}}\right|}$$

for $0 < |\theta - p/q| < \frac{1}{q(2K+1)}$. A typical factor $\left|\frac{1-e^{2\pi i r\theta}}{1-e^{2\pi i r p/q}}\right| = \left|\frac{\sin \pi r\theta}{\sin(\pi r p/q)}\right|$ is close to 1. Hence we consider

$$\frac{\sin \pi r\theta}{\sin(\pi r p/q)} - 1 = \cot(\pi r p/q)\sin \pi r u + \cos \pi r u - 1,$$

where $u = \theta - p/q$.

For the numerator of $G^*(\theta)/G^*(p/q)$ we have

$$\log \sum_{\substack{r=K+1 \\ r\neq q}}^{2K} \frac{\sin \pi r\theta}{\sin(\pi r p/q)}$$

$$\leq \sum_{\substack{r=K+1 \\ r\neq q}}^{2K} |\cos \pi r u - 1| + |\cot(\pi r p/q)||\sin \pi r u|$$

$$\leq \sum_{\substack{r=K+1 \\ r\neq q}}^{2K} ((\pi r u)^2/2+\pi r|u||\cot(\pi rp/q)|)$$

$$\leq \sum_{\substack{r=K+1 \\ r\neq q}}^{2K} r^2(\pi u)^2/2 + \sum_{\substack{r=K+1 \\ r\neq q}}^{2K} \pi r|u||\cot(\pi rp/q)|$$

$$< 5\pi^2u^2K^3/4 + 2\pi K|u| \sum_{\substack{r=K+1 \\ r\neq q}}^{2K} |\cot(\pi rp/q)|$$

$$< 5\pi^2/(16q) + (\pi/q) \sum_{\substack{r=K+1 \\ r\neq q}}^{2K} |\cot(\pi rp/q)|,$$

because $|u| < \frac{1}{2Kq}$.

Since $K < q$, no more than one value of r, $K + 1 \leq r \leq 2K$, can fall in any residue class modulo q, and we have

$$\prod_{\substack{r=K+1 \\ r\neq q}}^{2K} |\cot(\pi rp/q)| \leq 2 \prod_{r=1}^{[K/2]} \cot(\pi r1/q) = 2 \prod_{r=1}^{[K/2]} q/(\pi r)$$

$$< (2q/\pi)(1+\log(K/2)).$$

We can then write

$$\log \prod_{\substack{r=K+1 \\ r\neq q}}^{2K} \left|\frac{\sin \pi r\theta}{\sin(\pi rp/q)}\right| = 5\pi^2/(16q) + 2 + 2 \log K - 2 \log 2. \qquad (7\text{-}5)$$

We now show that each of the factors of the denominator is bounded away from 0. Since $r \le K$ here, we have $r|u| < \frac{1}{2q}$, and

$$\left|\frac{\sin \pi r\theta}{\sin(\pi rp/q)}\right| \ge 1 - |1-\cos \pi ru| - |\cot(\pi rp/q)||\sin \pi ru|$$

$$= 1 - (\pi Ku)^2/2 - \pi K|u|\cot(\pi/q)$$

$$> 1 - \pi^2/(8q^2) - K(q/\pi)/(2Kq)$$

$$> 1/2 - \pi^2/(8q^2) > 1/3.$$

Since $\log v \ge (v-1)/v$ for $0 < v \le 1$, we have

$$-\log \prod_{r=1}^{K} \left|\frac{\sin \pi r\theta}{\sin(\pi rp/q)}\right| < 3 \sum_{r=1}^{K}\left(\left|\frac{\sin \pi r\theta}{\sin(\pi rp/q)}\right| - 1\right)$$

$$\le (3\pi^2u^2)/2 \sum_{r=1}^{K} r^2 + 3 \sum_{r=1}^{K} \pi r|u||\cot(\pi rp/q)|$$

$$< \pi^2u^2K^3 + 3\pi|u|K \sum_{r=1}^{K} |\cot(\pi rp/q)|$$

$$< \pi^2/(4q) + 3\pi/q \sum_{r=1}^{[K+1/2]} \cot(\pi r1/q)$$

$$< \pi^2/(4q) + 3 + 3\log(K+1) - 3\log 2.$$

Combining this inequality with inequality (7-5), we have

$$\log G^*(\theta)/G^*(p/q) \leq 9\pi^2/(16q) + 5 - 5\log 2 + 2\log K + 3\log(K+1).$$

From inequalities (7-1) and (7-4) we obtain

$$G_K(\theta) = |1-e^{2\pi iq\theta}|(G^*(\theta)/G^*(p/q))G^*(p/q)$$
$$< (\pi/K)\exp\left\{9\pi^2/(16q)+5-5\log 2\right\}K^2(K+1)^3(2\pi)^{3/2}2^5 3^{-9}K^8 2^K.$$

And we obtain

$$G_K(\theta) \leq DK^9(K+1)^3 2^K, \qquad (7\text{-}6)$$

where $D = \pi\exp\left\{9\pi^2/48+5\right\}(2\pi)^{3/2}3^{-9} > 2$.

Stirling's formula for factorials yields

$$2^K < \binom{2K}{K}2^{-K}\sqrt{K\pi}\left(1+\frac{1}{4K}\right)^2 = \binom{2K}{K}2^{-K}\sqrt{K\pi}(9/8)^2.$$

Since $D(9/8)^2\sqrt{\pi} < 55$, both formulas in Lemma 7-2 hold for Case 1.

CASE 2: $q \leq K$

There are unique integers K_1 and K_2 which satisfy

$$K = K_1 q + K_2, \qquad 0 \leq K_2 < q.$$

In connection with this decomposition, a very interesting relation holds:

<u>Lemma 7-4</u>. If $|\theta-p/q| < \frac{1}{q(2K+1)}$, then we have

$$G_K(\theta) \leq \binom{2K_1}{K_1} \qquad \text{if } K_2 = 0, \text{ and}$$

$$G_K(\theta) \leq \binom{2K_1}{K_1} G_{K_2}(\theta) \qquad \text{if } 0 < K_2 < q.$$

<u>Proof</u>. For any $\theta \neq p/q$ we can write

$$\frac{\prod_{r=2K_2+1}^{2K} |\sin \pi r\theta|}{\prod_{r=K_2+1}^{K} \sin^2 \pi r\theta} = \prod_{t=1+2K_2}^{q+2K_2} \frac{\prod_{s=0}^{2K_1-1} |\sin \pi((pt/q)+(t+sq)u)|}{\prod_{s=0}^{K_1-1} \sin^2 \pi((pt/q)+(t+sq)u)}, \qquad (7\text{-}7)$$

where $u = \theta - p/q$. This product represents $G_K(\theta)$ if $K_2 = 0$ and $G_K(\theta)/G_{K_2}(\theta)$ if $K_2 > 0$.

The index t in this product assumes q consecutive integer values, exactly one of which, say t_o, is divisible by q.

If $K_2 < q/2$, we have $t_o = q$. The corresponding part of the product is

$$\frac{\prod_{s=0}^{2K_1-1} |\sin \pi(p+(s+1)qu)|}{\prod_{s=0}^{K_1-1} \sin^2 \pi(p+(s+1)qu)} = \frac{\prod_{s=1}^{2K_1} |\sin \pi squ|}{\prod_{s=1}^{K_1} \sin^2 \pi squ}$$

$$= \binom{2K_1}{K_1} \prod_{s=1}^{2K_1} \frac{\sin \pi squ}{\pi squ} \Bigg/ \prod_{s=1}^{2K_1} \left(\frac{\sin \pi squ}{\pi squ}\right)^2,$$

which is bounded away from both infinity and zero in the given interval. As in the proof of Lemma 3-3, we have

$$\log \left\{ \prod_{s=1}^{2K_1} \frac{\sin \pi squ}{\pi squ} \Bigg/ \prod_{s=1}^{2K_1} \left(\frac{\sin \pi squ}{\pi squ}\right)^2 \right\}$$

$$= -\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (qu/n)^{2m} m^{-1} \left\{ \sum_{s=1}^{2K_1} s^{2m} - 2 \sum_{s=1}^{K_1} s^{2m} \right\} \leq 0.$$

If $K_2 \geq q/2$, we have $t_o = 2q$. The corresponding part of the product has the same bound as above, as can be shown in the same way.

Thus to complete the proof of Lemma 7-4 it is sufficient to show that the rest of (7-7),

$$f(u) = \prod_{\substack{t=1+2K_2 \\ q \nmid t}}^{q+2K_2} \frac{\prod_{s=0}^{2K_1-1} |\sin \pi((pt/q)+(t+sq)u|}{\prod_{s=0}^{K_1-1} \sin^2 \pi((pt/q)+(t+sq)u)},$$

is not greater than 1 in absolute value. All factors of f(u) are nonvanishing in the given interval. As u approaches 0, f(u) approaches 1. Thus it suffices to show that the logarithmic derivative below is nonpositive for positive u. (It follows in the same way that this derivative is nonnegative for negative u.) We have

$$\frac{d}{du}(\log f(u)) = \sum_{\substack{t=1+2K_2 \\ q\nmid t}}^{q+2K_2} \left(\sum_{s=0}^{2K_1-1} (t+sq)\pi \cot \pi((pt/q)+(t+sq)u) \right.$$

$$\left. - 2\sum_{s=0}^{K_1-1} (t+sq)\pi \cot \pi((pt/q)+(t+sq)u) \right)$$

$$= \sum_{\substack{t=1+2K_2 \\ q\nmid t}}^{q+2K_2} \sum_{s=0}^{K_1-1} ((t+(s+K_1)q)\pi \cot \pi((pt/q)+(t+(s+K_1)q)u)$$

$$- (t+sq)\pi \cot \pi((pt/q)+(t+sq)u))$$

$$= \sum_{\substack{t=1+2K_2 \\ q\nmid t}}^{q+2K_2} \sum_{s=0}^{K_1-1} (t+sq)\pi(\cot \pi((pt/q)+(t+(s+K_1)q)u)$$

$$- \cot \pi((pt/q)+(t+sq)u))$$

$$+ \sum_{s=0}^{K_1-1} \pi K_1 q \left(\sum_{\substack{t=1+2K_2 \\ q\nmid t}}^{q+2K_2} \cot \pi((pt/q)+(t+(s+K_1)q)u) \right).$$

In the first summation term of the last expansion, the "cotangent minus cotangent" factors are nonpositive, because the cotangent is monotonically decreasing on each open interval between multiples of π. Since both arguments are on the same such interval, and since u is nonnegative, the whole term is nonnegative. In the other summation term we have

$$\sum_{\substack{t=1+2K_2 \\ q\nmid t}}^{q+2K_2} \cot \pi((pt/q)+(t+(s+K_1)q)u) \leq \sum_{\substack{t=1+2K_2 \\ q\nmid t}}^{q+2K_2} \cot \pi(pt/q) = 0,$$

and the proof of Lemma 7-4 is complete.

The proof of Lemma 7-2 for Case 2 now subdivides into the following cases: (a) $K_2 = 0$; (b) $K_2 > 0$ and $1/(2K_2+1) < \theta \leq 1/2$; (c) $K_2 = 1$ and $\theta \leq 1/(2K_2+1)$; (d) $1 < K_2 \leq q/2$ and $\theta \leq 1/(2K_2+1)$; and $q/2 < K_2 < q$ and $\theta \leq 1/(2K_2+1)$.

CASE 2a: $K_2 = 0$

Stirling's approximation yields $\binom{2n}{n} < 2^{2n}$. Lemma 7-4 now yields

$$G_K(\theta) = \binom{2K_1}{K_1} < 2^{2K_1} \leq 2^K < DK^9(K+1)^3 2^K,$$

and the second formula in Lemma 7-2 follows from this in the same way it did from relation (7-6).

CASE 2b: $0 < K_2$ and $1/(2K_2+1) < \theta \leq 1/2$

Recalling that Lemma 7-2 is assumed for integers less than K, including K_2 in particular, we have

$$G_K(\theta) \le \binom{2K_1}{K_1} G_{K_2}(\theta) < 2^{2K_1} DK_2^9(K_2+1)^3 2^{K_2}$$

$$< DK^9(K+1)^3 2^{K_1 q+K_2} = DK^9(K+1)^3 2^K.$$

CASE 2c: $K_2 = 1$ and $\theta \le 1/(2K_2+1)$

It is sufficient to write

$$G_K(\theta) \le \binom{2K_1}{K_1} G_1(\theta) < 2^{2K_1+1} < DK^9(K+1)^3 2^K.$$

CASE 2d: $1 < K_2 = q/2$ and $\theta < 1/(2K_2+1)$

Since $K_1 \ge 1$ and $K_2 \ge 2$, we have $2K_1 + 2K_2 \le 2K_1K_2 + K_2$. Our estimate is then

$$G_K(\theta) = \binom{2K_1}{K_1} G_{K_2}(\theta) = \binom{2K_1}{K_1}\binom{2K_2}{K_2} < 2^{2K_1+2K_2} \le 2^K.$$

CASE 2e: $q/2 < K_2 < q$ and $\theta < 1/(2K_2+1)$

Lemma 3-5 tells us that when $|\theta| < 1/(2K_2+1)$ we have

$$G_{K_2}(\theta) = \binom{2K_2}{K_2} e^{-\pi^2\theta^2K_2^3/3} < 2^{2K_2-(\pi^2\theta^2K_2^3)/(3 \log 2)}$$

$$< 2^{2K_2-4\theta^2K^3},$$

since $4 < \pi^2/(3 \log 2)$. The hypothesis of Lemma 7-2 yields

$$\theta < (p/q) - 1/(2K+1) \ge 1/q - 1/(q(2K+1)) = 2K/(q(2K+1)),$$

so that we now have

$$G_K(\theta) = \binom{2K_1}{K_1}\binom{2K_2}{K_2}2^{-4K_2^3(2K/(q(2K+1)))^2}$$

$$< 2^{2K_1 + 2K_2 - 4K_2(2K)^2(2K+1)^{-2}q^{-2}((q/2) + (1/2))^2}$$

$$< 2^{qK_1 + K_2} = 2^K.$$

Thus the discussion for Case 2 is completed, and--with it--the proof of Lemma 7-2.

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