A SLENDER CONE STARTING IMPULSIVELY

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ABSTRACT

The problem of a semi-infinite slender cone which starts impulsively from rest so that it suddenly has a constant supersonic velocity is considered. It is treated by using the acoustic wave equation for the air at rest at infinity. The problem is reduced to that of dealing with the radial velocity in two conical variables in space-time.

It is shown that there are three fundamental regions from the physical or mathematical standpoint. The boundary conditions and equations for each of these regions are developed so that a numerical solution of the problem may be obtained for a given Mach number and cone angle. From the solution of the radial velocity the potential and thence the pressure on the cone are obtained.

An approximation to the pressure far back on the cone where the curvature is small is obtained as an improvement on the piston value for zero curvature. This is done by suppressing variations in the axial direction and solving the resulting equation by Riemann's integration method.

An attempt to solve the problem by distributing sources on the axis with resulting difficulties is discussed.

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I. INTRODUCTION

Recently a number of problems in non-stationary, or time dependent, supersonic flow have been considered and solved. A minor number of these have dealt with motion which is non-stationary in the streamwise direction, that is, acceleration and deceleration. Biot (Reference 1) has presented a method for finding the pressure or drag on an accelerating two-dimensional planar body at the speed of sound. He works out the pressure and drag for a wedge. Gardner and Ludloff (Reference 2) have worked out the drag for a decelerating double wedge airfoil at supersonic and transonic Mach numbers. In addition to these the problem of an axially symmetric body having a non-steady motion in the axial direction is one which still requires study. A preliminary treatment of this problem was made by Frankl (Reference 3), but this was merely an order of magnitude investigation of drag forces. A more complete solution appears to be difficult.

Thus the simplified problem of a semi-infinite cone starting impulsively from rest has been chosen to give some insight into the more general problem. This form of the problem is convenient since it allows the use of conical flow methods first introduced in aerodynamics by Busemann (References 4 and 5).

II. ACOUSTIC EQUATIONS

If assumptions are made about the compression of the air being small due to a body moving through it, the basic equations of the non-stationary air flow can be derived as acoustic equations as is done, for example, by Baker and Copson (Reference 6).

Consider the air is initially at rest everywhere with pressure and density of p_o and ρ_o, respectively. Any disturbance which arises is described by a velocity vector $\vec{q}(x,y,z,t)$. Also at the point (x,y,z,t) the condition of the air is described by a density ρ which can be written in terms of the initial density ρ_o and a condensation s.

$$\rho = \rho_o (1 + s) \tag{1}$$

This equation defines s. It is assumed that disturbances are small enough so that the square of s is negligible with respect to unity.

To describe the motion of the fluid at a point the equations of continuity and momentum, respectively, can be written in the forms

$$\rho_t + div(\rho \vec{q}) = 0 \tag{2}$$

$$\rho \vec{q}_t = -grad\ p \tag{3}$$

if there is no external applied force. In the continuity equation the

the divergence term can be expanded as

$$div(\rho\vec{q}) = \rho_o(1+s)\, div\, \vec{q} + \vec{q} \cdot grad\, \rho_o(1+s)$$

Then further inserting the assumptions that s << 1 and grad s $\doteq$ 0, the continuity equation is written as

$$\rho_t + \rho_o\, div\, \vec{q} = 0 \tag{4}$$

or

$$s_t + div\, \vec{q} = 0 \tag{5}$$

If it is further assumed that pressure is a function of density only the momentum equation can be written

$$\rho_o \vec{q}_t = -\left(\frac{dp}{d\rho}\right)_o grad\, \rho$$

or

$$\vec{q}_t = -\left(\frac{dp}{d\rho}\right)_o grad\, s \tag{6}$$

Since $\left(\frac{dp}{d\rho}\right)_o$ has the dimensions of velocity squared, let $c^2 = \left(\frac{dp}{d\rho}\right)_o$ where c is a velocity, then

$$\vec{q}_t = -c^2\, grad\, s$$

If heat conduction is neglected because of the assumed small values of s and because the movement is rapid, the adiabatic equation

$$\frac{p}{p_o} = \left(\frac{\rho}{\rho_o}\right)^{\gamma} \tag{7}$$

can be used for the pressure-density relation of the gas at a point. In differential form this is

$$p = p_0 + \left(\frac{dp}{d\rho}\right)_0 d\rho$$

or using the definitions of c and s

$$\frac{p - p_0}{\rho_0} = c^2 s \tag{8}$$

Since the fluid is originally irrotational it will remain irrotational if no mechanism exists to produce rotation. This is the case since viscous forces and external forces are neglected. The assumption concerning external forces only requires a conservative field. With irrotationality a velocity potential, ϕ , exists defined by

$$\vec{q}(x,y,z,t) = \operatorname{grad} \phi(x,y,z,t) \tag{9}$$

From Equations (5), (6), and (8) the following acoustical equations can now be written

$$\phi_t = -c^2 s \tag{10}$$

$$s_t + \operatorname{div} \vec{q} = 0 \tag{11}$$

$$\frac{p - p_0}{\rho_0} = c^2 s \tag{12}$$

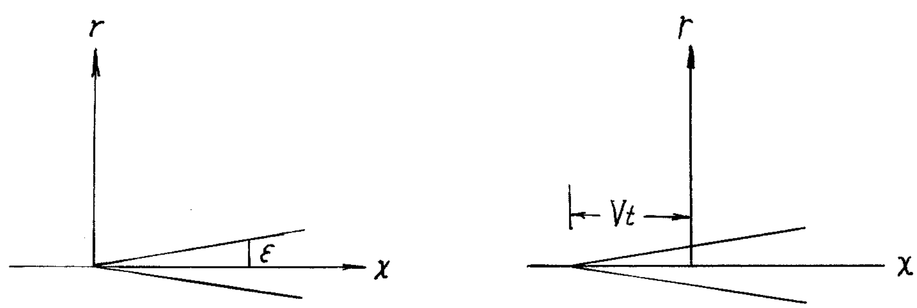
Now using Equation (9) the above equations can be combined to give the wave equation

$$\nabla^2 \phi - \frac{1}{c^2} \phi_{tt} = 0 \tag{13}$$

III. THE PROBLEM OF THE IMPULSIVELY STARTING CONE

3.1 In Physical Coordinates

For the problem of a semi-infinite slender cone starting impulsively from rest the previously derived acoustic equations or wave equation can be specialized. The coordinate system is fixed with respect to the air at time $t < 0$ and remains fixed with respect to the air infinitely far ahead of the cone for time $t > 0$. The cone starts impulsively at the time $t = 0$ and has an axial velocity V for all time $t > 0$. The x-axis extends along the axis of the cone with the positive direction being toward the rear of the cone. The radial coordinate r is measured perpendicular to the x-axis. The angle which the generatrix of the cone makes with the x-axis is ε.



$t \leqq 0$ $\qquad\qquad$ $t > 0$

The velocities of the air at any point in this coordinate system are

$$u(x,r,t) = \phi_x(x,r,t)$$

$$v(x,r,t) = \phi_r(x,r,t)$$

and by rotational symmetry all derivatives with respect to ω, the angle of rotation about the x-axis, are zero. In particular

$$\phi_\omega = 0$$

at all times.

The wave equation in cylindrical coordinates has the form

$$\Box_{x,r,t}\,\phi = \phi_{rr} + \frac{1}{r}\phi_r + \phi_{xx} - \frac{1}{c^2}\phi_{tt} = 0 \tag{14}$$

If this is differentiated with respect to x, r, and t the following equations are obtained:

$$\Box_{x,r,t}\,u = 0 \tag{15}$$

$$\Box_{x,r,t}\,v = \frac{v}{r^2} \tag{16}$$

$$\Box_{x,r,t}\,\phi_t = 0 \tag{17}$$

3.2 Reduction to Conical Coordinates

It can be argued that the flow of the problem is conical, or that u, v, and ϕ_t are functions only of $\frac{x}{ct}$ and $\frac{r}{ct}$ which will be called X and R, respectively. The argument is similar to that made in the case of stationary conical flow, for example by Lagerstrom (Reference 7). There is no characteristic length in the problem. The only characteristic values are V, ρ_o, and c. Thus $\frac{u}{V}$ and $\frac{v}{V}$ must be expressed as functions of $\frac{x}{ct}$, $\frac{r}{ct}$, and $\frac{V}{c}$

or M. It may be assumed then that for any given V the potential ϕ can be given by

$$\phi(x,r,t) = ct \cdot \psi(X,R) \tag{18}$$

In connection with the discussion of the conical property of the problem it is interesting to consider Figure 1 which pictures the problem by suppressing one space dimension to allow the time variation to be shown.

Now following the same procedure as was used by Lagerstrom a dependent variable w(x,r,t) is considered where w is either u, v, or ϕ_t. Letting the coordinates (x,r,t) be denoted (x_1, x_2, x_3), it is easily seen that the above equations ((15,), (16), and (17)) are invariant under the transformation $x_i = \kappa x_i$, where κ is any constant. To say that w is invariant under this transformation is the same as saying that w is homogeneous of order zero in x, r, and t.

Now a new set of coordinates (ρ, θ, t) is introduced where

$$\begin{aligned} \rho &= \sqrt{x^2 + r^2} \\ \theta &= \tan^{-1} \frac{r}{x} \\ t &= t \end{aligned} \tag{19}$$

The coordinates ρ and θ are homogeneous of order one and zero, respectively, in x and r. Euler's theorem for homogeneous functions states that if $w(x_1, x_2, x_3)$ is a homogeneous function of order n, then

$$n \cdot w = \sum_{i=1}^{3} (x_i\, w_{x_i})$$

Since w is homogeneous of order zero,

$$x_3 \cdot w_{x_3} = -\sum_{1,2} x_i\, w_{x_i} = -\sum_{1,2} (x_i\, w_\rho \rho_{x_i} + x_i\, w_\theta \theta_{x_i})$$

giving

$$t \cdot w_t = -\rho \cdot w_\rho$$

By a repeated application of the theorem

$$t(t \cdot w_t)_t = \rho(\rho \cdot w_\rho)_\rho$$

These can be combined in the following form:

$$t^2 \cdot w_{tt} = \rho(\rho \cdot w_\rho)_\rho + \rho \cdot w_\rho \tag{20}$$

The terms $w_{xx} + w_{rr}$ of the wave operator become under the transformation (19).

$$w_{xx} + w_{rr} = \frac{1}{\rho}(\rho \cdot w_\rho)_\rho + \frac{1}{\rho^2} w_{\theta\theta} \tag{21}$$

and the remaining term of the operator becomes

$$\frac{1}{r} w_{rr} = \frac{1}{\rho} w_\rho + \frac{\cot\theta}{\rho^2} w_\theta \tag{22}$$

If w is conical, that is, if it can be expressed in the variables $\frac{x}{ct}$ and $\frac{r}{ct}$, then it is sufficient to solve for the case where ct = 1.

In this case then, the operator $\Box_{x,r,t}$ can be written in conical coordinates

$$\Box_{x,r,t} W = \left(\frac{1}{\rho} - \rho\right)\left(\rho \cdot W_\rho + W\right)_\rho + \frac{\cot\theta}{\rho^2} W_\theta + \frac{1}{\rho^2} W_{\theta\theta} = \Theta_{\rho,\theta} W$$

and the Equations (15), (16), and (17) for the velocities and ϕ_t become

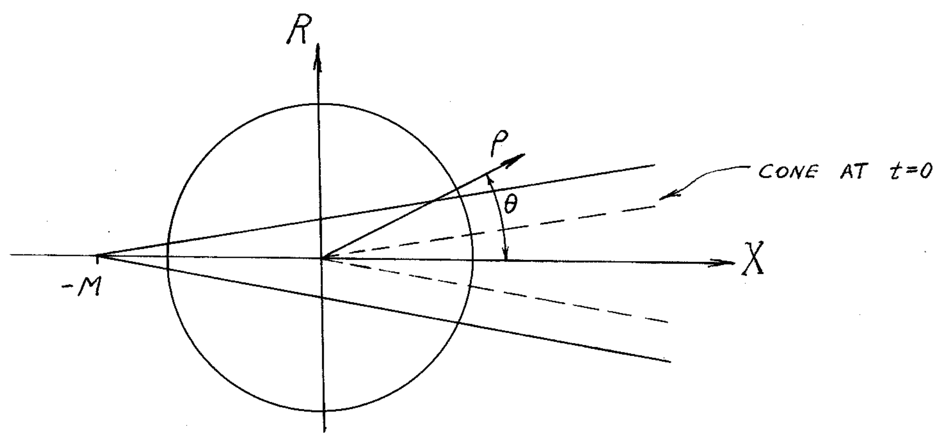
$$\Theta_{\rho,\theta}\, u = 0 \tag{23}$$

$$\Theta_{\rho,\theta}\, v = \frac{v}{\rho^2 \sin^2\theta} \tag{24}$$

$$\Theta_{\rho\theta}\, \phi_t = 0 \tag{25}$$

The coordinates (ρ, θ) are now in the $\left(\frac{x}{ct}, \frac{r}{ct}\right)$, or (X,R), coordinate plane since ct = 1.

Since the problem is conical it may, as stated before, be solved in the plane ct = 1. This means that the (ρ,θ) coordinates can be considered in this plane or in the XR plane at any time. For any time t > 0 the following figure shows the position of the cone with respect to the XR-axes. The figure is drawn for the case when M,

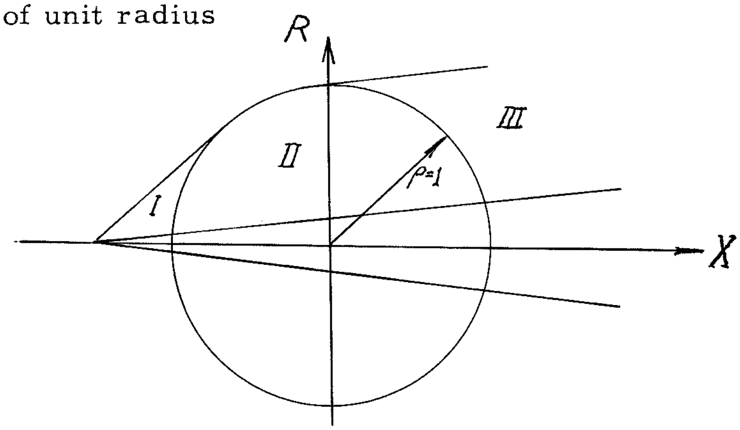


Cone at t > 0, M > 1

the Mach number at which the cone is travelling, is greater than one.

3.3 The Three Fundamental Regions

Some interesting aspects of the problem can be considered on the basis of this figure. It is redrawn for this purpose. The circle of unit radius



is the signal from the apex of the cone at t = 0. This will from time to time be called the Mach circle because of its close analogy to the Mach circle in stationary conical problems. It serves to separate three fundamental regions of the problem.

Region I is bounded by the cone, the circle, and the wave front originating from the apex at time $t > 0$. Within this region it can be said that the air doesn't know that the cone was at rest at time $t \leqq 0$. The solution within and on the boundary of this region is the same as that for the stationary cone problem and can be considered already solved.

Region III is bounded by the cone, the circle, and the wave front or signal front of the cone at time t = 0. This region extends to infinity out along the infinite cone. Within this region the air doesn't know that there is an apex on the cone, but it does know that the cone started at t = 0.

Region II is bounded by the cone and the circle. The air within this region knows both that there is an apex on the cone and that the cone started at t = 0.

3.31 The Characteristics in Region III

Equations (23), (24), and (25) are hyperbolic in Region III and elliptic in Region II. In Region III then it is of interest to consider the characteristics. This can be done with the original equation for the velocity potential as well as with the velocity equations. This equation is

$$\phi_{rr} - \frac{1}{r}\phi_r + \phi_{xx} - \frac{1}{c^2}\phi_{tt} = 0 \tag{14}$$

With the potential being of the form

$$\phi(x,r,t) = ct \cdot \psi(X,R) \tag{18}$$

the equation becomes

$$\psi_{RR}(1-R^2) - 2RX\,\psi_{RX} + \psi_{XX}(1-X^2) + \frac{1}{R}\psi_R = 0 \tag{27}$$

The slopes of the characteristics of this equation in the XR plane are

$$\frac{dR}{dX} = -\frac{RX}{1-X^2} \pm\sqrt{R^2+X^2-1} \tag{28}$$

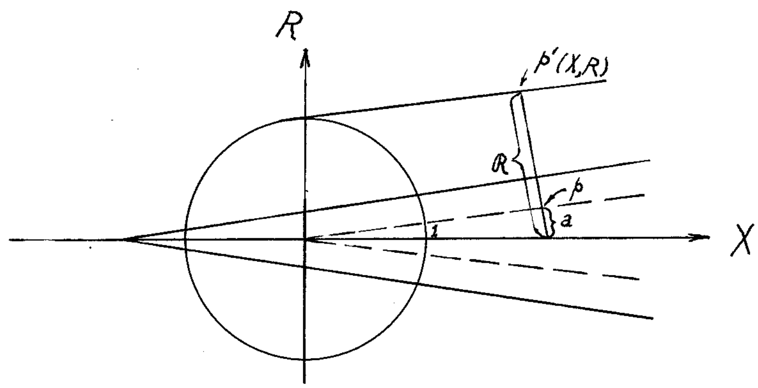
Thus the characteristics in Region III are straight lines tangent to the Mach circle. This is the same situation as Busemann showed (Reference 4) for stationary conical flow.

3.32 The Boundary Conditions for Each Region

The problem will be solved by solving for the radial velocity v and integrating it to get the potential. This is done because the boundary condition can be given in terms of v. To solve the problem in Region III it is necessary to have the value of v given along the cone and along the wave front. On the cone the approximation is used that $v = V \tan \varepsilon$. In the calculations this is normalized so that $v = 1$.

For the value of v on the wave front it is necessary to make an approximation. For this purpose a result due to G. N. Ward (Reference 8) is used. If air is flowing past a circular tube in the direction of the axis of the tube there is no disturbance in the air until at some point in the direction of flow there is a change in the shape of the tube. Suppose the tube suddenly begins to expand in the streamwise direction so that the rate of increase in the radius is constant. Then the radial velocity of the fluid at a point on the wave front varies as the square root of the radius of the tube at the point where the wave originated and inversely as the square root of the radial distance to the point.

It is now argued that an analogous situation exists in the problem of the impulsively starting cone. At the time $t = 0$ the radius of curvature of the surface of the cone at point p in the plane perpendicular to the generatrix through p is a. This corresponds



to the radius of the cylindrical tube in the above stationary problem. At t = 0 the radius at p which was a suddenly starts to increase. Also at t = 0 a wave front starts which at some time later is at p'. This point p' corresponds to any point on the wave front in the above stationary problem. Arguing physically the point p' can only know that a surface of radius a started to expand at t = 0 just as the point on the wave front in the stationary problem only knows that the cylinder started to expand. The above argument includes a slight approximation since in the cone problem there is a small rate of change of a in the x direction.

Writing the boundary condition analytically with reference to the figure

$$\frac{v}{V \tan \varepsilon} = \sqrt{\frac{a}{\mathcal{R}}} \tag{29}$$

Using the normalized value of v, and at the time ct = 1

$$v = \sqrt{\frac{a}{1+a}} \tag{30}$$

To solve the problem in Region II where Equation (24) is elliptic it is sufficient to have the value of v around the boundary of the region. As in Region III the approximation is made that $v = V \tan \varepsilon$ on the cone. On the common boundary with Region III the value is obtained from the solution for Region III. On the portion of the circle $\rho = 1$ which is not a boundary of either Region I or Region III the value of v is zero. On the common boundary with Region I the value is taken from the solution of Region I.

As previously stated, the solution to the problem in Region I is the same as that for a cone in stationary supersonic flow. In the linearized solution to this problem the radial velocity can be written as

$$v = kB\sqrt{\left(\frac{s}{Br}\right)^2 - 1} \tag{31}$$

where s is the distance in the x direction from the apex of the cone and $B = \sqrt{M^2-1}$. The approximation is then made that on the surface of the cone

$$v = V \tan \varepsilon$$

This is consistent with the boundary condition on the cone used in Regions II and III. It gives

$$k = \frac{V \tan \varepsilon}{\sqrt{\cot^2 \varepsilon - B^2}}$$

Thus the normalized value for v to be used in calculations is

$$v = \frac{\sqrt{\left(\frac{s}{r}\right)^2 - B^2}}{\sqrt{\cot^2 \varepsilon - B^2}} \tag{32}$$

3.4 Solution of v in Region III

3.41 Transformation to Normal Form

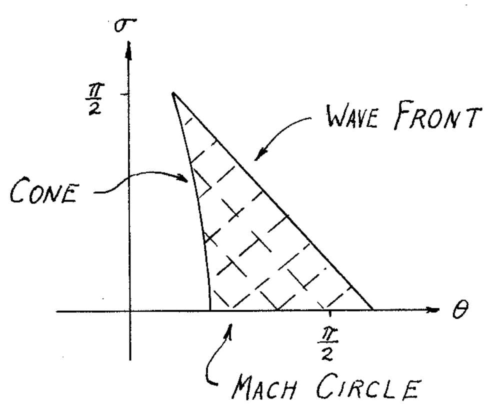
It is now convenient to transform the differential Equation (24) for v into more convenient form for obtaining a solution in Region III. Consider the Chaplygin transformation in the form used by Goldstein and Ward (Reference 9) where

$$\begin{aligned} \sigma &= \sec^{-1}\rho \\ \theta &= \theta \end{aligned} \tag{33}$$

This form is used because its analogy is readily written for the case where the equation is elliptic. Equation (24) for v then becomes

$$v_{\theta\theta} - v_{\sigma\sigma} + \cot\sigma \cdot v_{\sigma} + \cot\theta \cdot v_{\theta} - \frac{v}{\sin^2\theta} = 0 \tag{34}$$

Under the transformation (33) Region III in the XR, or $\rho\theta$, plane becomes that shown in the figure in the $\sigma\theta$ plane. The directions of the



characteristics are shown by the dashed lines. These are evident from the form of Equation (34). The point at an infinite distance back along the cone is transformed to the point $(\varepsilon, \frac{\pi}{2})$. This is also the limit of all points in the region as X becomes infinite. Thus it is the point at which all points on the cone, except the apex, started at t = 0. The equations of the wave front and cone are, respectively

$$\sigma + \theta = \frac{\pi}{2} + \varepsilon$$
$$\sigma = \sec^{-1}\left[\frac{M \sin\varepsilon}{\sin(\theta-\varepsilon)}\right]$$

The boundary condition on the wave front from Equation (30) can be written in terms of the new coordinates (σ, θ). From the geometry

$$a = \tan\varepsilon\,(X \sec\varepsilon + \tan\varepsilon)$$

and using $X = \rho \cos\theta$, and since on the boundary $\sigma + \theta = \frac{\pi}{2} + \varepsilon$,

$$V = \frac{1}{\sqrt{1 + \dfrac{\cot\varepsilon}{\cos\theta \cdot \csc(\theta-\varepsilon) + \sin\varepsilon}}}$$

As mentioned before the normalized value of v on the cone is unity.

3.42 Transformation to Characteristic Coordinates

Equation (34) can now be transformed to normal form where the independent variables are the characteristic coordinates. Let the characteristic coordinates be

$$\xi = \varphi(\theta,\sigma)$$
$$\eta = \psi(\theta,\sigma) \tag{35}$$

Then Equation (34) becomes

$$V_{\xi\xi}\,Q(\varphi,\varphi) + 2V_{\xi\eta}\,Q(\varphi,\psi) + V_{\eta\eta}\,Q(\psi,\psi)$$
$$+ V_\xi\left[\varphi_{\theta\theta} - \varphi_{\sigma\sigma} + \cot\theta\cdot\varphi_\theta + \cot\sigma\cdot\varphi_\sigma\right]$$
$$+ V_\eta\left[\psi_{\theta\theta} - \psi_{\sigma\sigma} + \cot\theta\cdot\psi_\theta + \cot\sigma\cdot\psi_\sigma\right] - \frac{V}{\sin^2\theta} = 0 \tag{36}$$

To get the normal form it is necessary that

$$Q(\varphi,\varphi) = Q(\psi,\psi) = 0$$

that is,

$$Q(\varphi,\varphi) = \varphi_\theta^2 - \varphi_\sigma^2 = 0$$

$$Q(\psi,\psi) = \psi_\theta^2 - \psi_\sigma^2 = 0$$

These give

$$\varphi_\theta = \pm \varphi_\sigma$$

and (37)

$$\psi_\theta = \pm \psi_\sigma$$

Choosing $\varphi_\theta = \varphi_\sigma$, then $\psi_\theta = -\psi_\sigma$. From Equation (35) these conditions are satisfied by

$$\xi = \sigma + \theta \quad \text{and} \quad \eta = \sigma - \theta \tag{38}$$

or

$$\sigma = \frac{\xi + \eta}{2} \quad \text{and} \quad \theta = \frac{\xi - \eta}{2}$$

With this change of variables, Equation (36) becomes

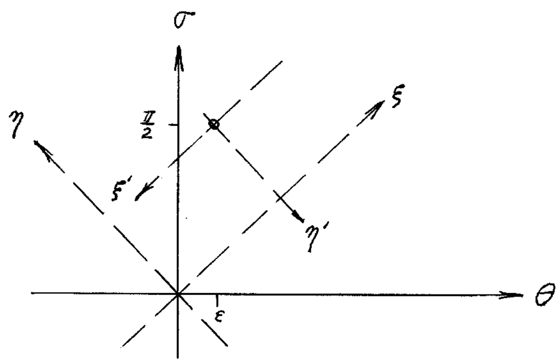
$$V_{\xi\eta} - \frac{1}{2}\left[\frac{\sin\xi}{\cos\eta - \cos\xi}\right]V_\xi + \frac{1}{2}\left[\frac{\sin\eta}{\cos\eta - \cos\xi}\right]V_\eta + \frac{V}{4\sin^2\left(\frac{\xi-\eta}{2}\right)} = 0 \tag{39}$$

This equation is now in characteristic coordinates. In this form it can be changed to a difference equation for the purpose of numerical solution of the problem.

With this in view it is convenient to make a further transformation involving a rotation, translation and a contraction. The rotation and translation is given by

$$\eta' = \eta + \frac{\pi}{2} - \varepsilon$$
$$\xi' = -\xi + \frac{\pi}{2} + \varepsilon \qquad (40)$$

as shown in the figure. Then the contraction



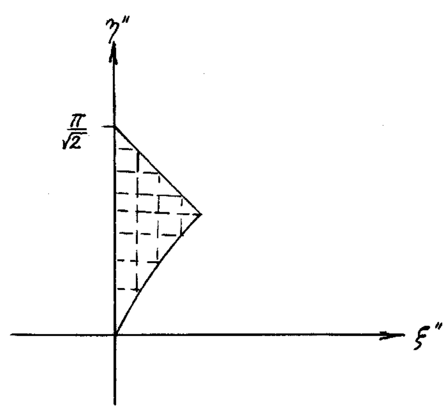
to (ξ'', η'') is made:

$$\eta'' = \frac{1}{\sqrt{2}}$$
$$\xi'' = \frac{1}{\sqrt{2}}$$

This finally puts the equation in the form

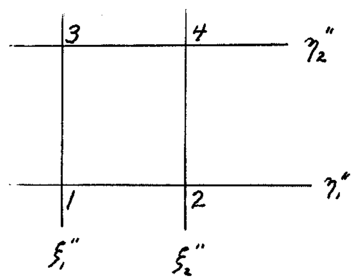
$$V_{\xi''\eta''} + \frac{1}{\sqrt{2}}\left[\frac{\cos(\sqrt{2}\,\xi'' - \varepsilon)}{\sin(\sqrt{2}\,\eta'' + \varepsilon) - \sin(\sqrt{2}\,\xi'' - \varepsilon)}\right] V_{\xi''}$$
$$- \frac{1}{\sqrt{2}}\left[\frac{\cos\sqrt{2}(\eta'' + \varepsilon)}{\sin(\sqrt{2}\,\eta'' + \varepsilon) - \sin(\sqrt{2}\,\xi'' - \varepsilon)}\right] V_{\eta''} + \frac{V}{2\sin^2\left(\frac{\eta'' - \xi''}{\sqrt{2}} + \varepsilon\right)} = 0 \qquad (41)$$

Region III is now as shown in the figure.



3.43 Formation of Difference Equation

The region can be covered by a mesh of characteristics. Considering a sample rectangle in this mesh, as in the figure, the terms in the difference equation can be constructed.



The following difference forms are used as approximations for the derivatives:

$$V_{\xi''}(\eta_1) \doteq \frac{V_2 - V_1}{\Delta \xi''} \quad ; \quad V_{\xi''}(\eta_2) \doteq \frac{V_4 - V_3}{\Delta \xi''}$$

$$V_{\xi''\eta''} \doteq \frac{V_4 - V_3 - V_2 + V_1}{\Delta \xi'' \, \Delta \eta''}$$

$$V_{\xi''} \doteq \frac{V_2 - V_1 + V_4 - V_3}{2 \Delta \xi''}$$

$$V_{\eta''} \doteq \frac{V_3 - V_1 + V_4 - V_2}{2 \Delta \eta''}$$

Letting

$$A = \frac{1}{\sqrt{2}} \left[\frac{\cos(\sqrt{2}\,\eta'' + \varepsilon)}{\sin(\sqrt{2}\,\eta'' + \varepsilon) - \sin(\sqrt{2}\,\xi'' - \varepsilon)} \right]$$

$$B = \frac{1}{\sqrt{2}} \left[\frac{\cos(\sqrt{2}\,\xi'' - \varepsilon)}{\sin(\sqrt{2}\,\eta'' + \varepsilon) - \sin(\sqrt{2}\,\xi'' - \varepsilon)} \right] \qquad (42)$$

$$C = \frac{}{\sqrt{2}} \left[\frac{1}{2 \sin^2\left(\frac{\eta'' - \xi''}{\sqrt{2}} + \varepsilon\right)} \right]$$

the equation becomes

$$v_{\xi''\eta''} + B v_{\xi''} - A v_{\eta''} + C v = 0$$

The difference equation which approximates this equation is then

$$\frac{v_4 - v_3 - v_2 + v_1}{\Delta\xi''\Delta\eta''} + B\,\frac{v_2 - v_1 + v_4 - v_3}{2\Delta\xi''} - A\,\frac{v_3 - v_1 + v_4 - v_2}{2\Delta\eta''} + C\,\frac{v_1 + v_2 + v_3 + v_4}{4} = 0 \tag{43}$$

Solving for v_4,

$$v_4 = \left[\frac{1}{\frac{1}{\Delta\xi''\Delta\eta''} + \frac{B}{2\Delta\xi''} - \frac{A}{2\Delta\eta''} + \frac{C}{4}}\right]$$

$$\times\left\{\frac{1}{\Delta\xi''\Delta\eta''}\left[-v_1 + v_2 + v_3\right] + \frac{B}{2\Delta\xi''}\left[v_1 - v_2 + v_3\right] + \frac{A}{2\Delta\eta''}\left[-v_1 - v_2 + v_3\right] + \frac{C}{4}\left[-v_1 - v_2 - v_3\right]\right\} \tag{44}$$

With this equation it is then possible to solve for the value of v at each point of the characteristics mesh by working a step at a time from the boundaries on which the boundary conditions are given. The division of the region is shown in Figure 2. The mesh size is taken smaller for the original steps since the variation in v is more rapid. This is of course done to make the approximation of the differential equation by the difference equation more uniform over the whole region.

The numerical example shown in the figures is for a cone half angle ε of ten degrees and for a Mach number of $\sqrt{2}$.

3.5 Solution of v in Region II

3.51 Transformation to Normal Form

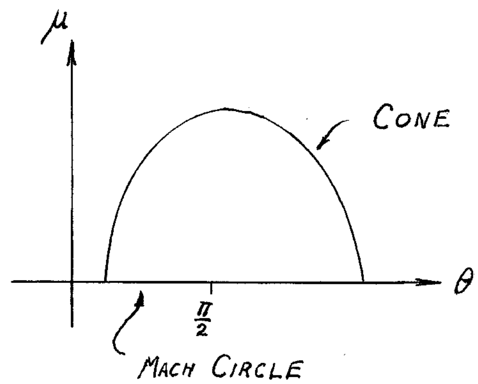
For the purpose of solving Equation (24) in Region II, where it is elliptic, it is transformed by a transformation which is analogous to the transformation (33) which is

$$\mu = \operatorname{sech}^{-1}\rho \qquad\qquad (45)$$
$$\theta = \theta$$

Equation (24) then becomes

$$V_{\mu\mu} - \coth\mu \cdot V_{\mu} + \coth\theta \cdot V_{\theta} + V_{\theta\theta} - \frac{V}{\sin^2\theta} = 0 \qquad (46)$$

Under the transformation (45) Region II in the XR, or $\rho\theta$, plane becomes that shown in the figure in the $\mu\theta$ plane. The equation



of the circle and the cone bounding Region II are, respectively,

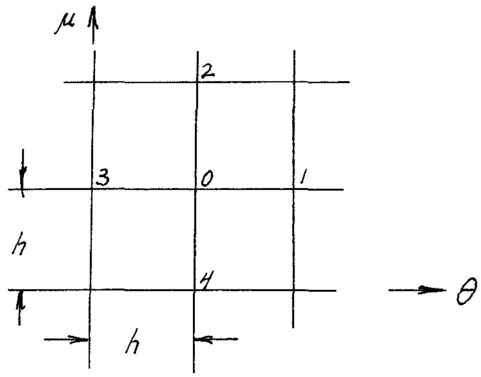
$$\mu = 0 \qquad\qquad (47)$$
$$\mu = \operatorname{sech}^{-1}\left[\frac{M\sin\varepsilon}{\sin(\theta-\varepsilon)}\right]$$

The boundary conditions are now used as they were discussed in Section 3.32. On the θ axis for $\frac{\pi}{2} + \varepsilon \geq \theta \geq \varepsilon + \sin^{-1}(M\sin\varepsilon)$ the value of v is that obtained from the solution of Region III. For $\frac{\pi}{2} - \cos^{-1}\left(\frac{1}{M}\right) \geq \theta \geq \frac{\pi}{2} + \varepsilon$, v = 0. For $\varepsilon + \sin^{-1}(M\sin\varepsilon) \geq \theta \geq \frac{\pi}{2} - \cos^{-1}\left(\frac{1}{M}\right)$

the value of v is obtained from the solution in Region I.

3.52 Formation of Difference Equation

The boundary conditions on Region II now having been determined, it is possible to solve Equation (46) throughout the region by using the relaxation method. Let the region in the $\mu\,\theta$ plane be divided by a mesh of lines parallel to the axes and a distance h apart. A sample set of points at the intersections of this mesh is shown in the figure. Finite difference approximations to the



derivatives at the point 0 are

$$
\begin{aligned}
V_\theta &\doteq \frac{V_1 - V_3}{2h} \\
V_\mu &\doteq \frac{V_2 - V_4}{2h} \\
V_{\theta\theta} &\doteq \frac{V_1 + V_3 - 2V_0}{h^2} \\
V_{\mu\mu} &\doteq \frac{V_2 + V_4 - 2V_0}{h^2}
\end{aligned}
\qquad (48)
$$

Thus the difference equation approximating the differential equation (46) at the point 0 is

$$\frac{V_2 + V_4 - 2V_0}{h^2} - \coth\mu_0 \cdot \frac{V_2 - V_4}{2h} + \cot\theta_0 \cdot \frac{V_1 - V_3}{2h} + \frac{V_1 + V_3 - 2V_0}{h^2} - \frac{V_0}{\sin^2\theta_0} = 0$$

which can be written

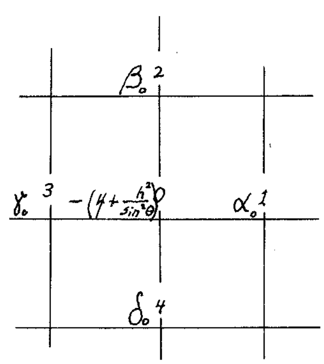
$$V_1\left(1+\frac{h}{2}\cot\theta_0\right)+V_2\left(1-\frac{h}{2}\coth\mu_0\right)+V_3\left(1-\frac{h}{2}\cot\theta_0\right)+V_4\left(1+\frac{h}{2}\coth\mu_0\right)-V_0\left(4+\frac{h^2}{\sin^2\theta_0}\right)=0 \qquad (49)$$

or

$$\alpha_0 V_1+\beta_0 V_2+\gamma_0 V_3+\delta_0 V_4-\left(4+\frac{h^2}{\sin^2\theta_0}\right)V_0=0 \qquad (50)$$

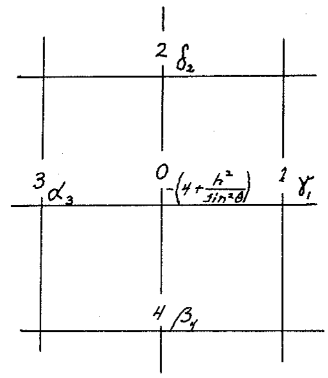
3.53 Relaxation Procedure

Briefly, then, the relaxation method proceeds as follows (Reference 10). Initial values of v are chosen for each point, or intersection, of the mesh. These values are substituted in Equation (50) and an error or residual is found for the point 0. Let this be E_o. If the equation were satisfied this residual would be zero. For carrying out this calculation of residuals each point on the mesh can be considered to have a residual operator which, for the point 0, can be diagrammed as



Residual Operator

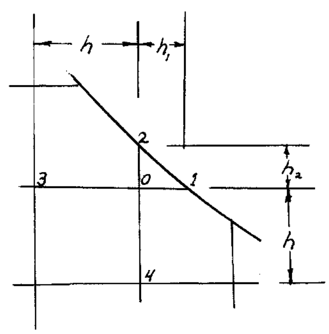
To reduce these residuals to zero the values of v at each point are altered or relaxed one at a time. If the value of v at point 0 is altered, then its residual and the residuals of the adjacent points are altered according to the Equation (50) for each point. These changes in the residuals at each point for a unit change in v at the point 0 can be diagrammed as a relaxation operator.



Relaxation Operator

The diagram shows the relaxation operator for the point 0.

For points adjacent to the boundary of the region, where the boundary does not pass through intersections of the mesh, the above equations and operators must be modified. The most general case involved in the problem was that where two adjacent sides of the star were truncated as shown in the figure. If the



approximation of using two terms of a Taylor expansion is employed about the point 0, then the approximations to the derivatives at the point 0 are

$$
\begin{aligned}
V_{\theta\theta} &\doteq 2\frac{h(V_1-V_0)+h_1(V_3-V_0)}{hh_1(h+h_1)} \\
V_{\theta} &\doteq \frac{h^2(V_1-V_0)-h_1^2(V_3-V_0)}{hh_1(h+h_1)} \\
V_{\mu\mu} &\doteq 2\frac{h(V_2-V_0)+h_2(V_4-V_0)}{hh_2(h+h_2)} \\
V_{\mu} &\doteq \frac{h^2(V_2-V_0)-h^2(V_4-V_0)}{hh_2(h+h_2)}
\end{aligned}
\tag{51}
$$

These, when substituted in the differential equation, give the difference equation

$$
\alpha_0'V_1+\beta_0'V_2+\gamma_0'V_3+\delta_0'V_4-\left[\frac{2}{\lambda_1}+\frac{2}{\lambda_2}-h\coth\mu_0\left(\frac{1-\lambda_2}{\lambda_2}\right)+h\cot\theta_0\left(\frac{1-\lambda_1}{\lambda_1}\right)+\frac{h^2}{\sin^2\theta_0}\right]V_0=0 \tag{52}
$$

where

$$
\begin{aligned}
\alpha_0' &= \frac{2}{\lambda_1(1+\lambda_1)} + h\cot\theta_0\,\frac{1}{\lambda_1(1+\lambda_1)} \\
\beta_0' &= \frac{2}{\lambda_2(1+\lambda_2)} - h\coth\mu_0\,\frac{1}{\lambda_2(1+\lambda_2)} \\
\gamma_0' &= \frac{2}{1+\lambda_1} - h\cot\theta_0\,\frac{\lambda_1}{1+\lambda_1} \\
\delta_0' &= \frac{2}{1+\lambda_2} + h\coth\mu_0\,\frac{\lambda_2}{1+\lambda_2}
\end{aligned}
\tag{53}
$$

in which $\lambda_1 = h_1/h$ and $\lambda_2 = h_2/h$. If the boundary truncates sides 2 and 3 of the star instead of 1 and 2, the difference equation is

$$
\alpha_0''V_1+\beta_0''V_2+\gamma_0''V_3+\delta_0''V_4-\left[\frac{2}{\lambda_1}+\frac{2}{\lambda_2}-h\coth\mu_0\left(\frac{1-\lambda_2}{\lambda_2}\right)-h\cot\theta_0\left(\frac{1-\lambda_1}{\lambda_1}\right)+\frac{h^2}{\sin^2\theta_0}\right]V_0=0 \tag{54}
$$

where

$$\begin{aligned}
\alpha_0'' &= \frac{2}{1+\lambda_1} + h \cot\theta_0 \frac{\lambda_1}{1+\lambda_1} \\
\beta_0'' &= \frac{2}{\lambda_2(1+\lambda_2)} - h \coth\mu_0 \frac{1}{\lambda_2(1+\lambda_2)} \\
\gamma_0'' &= \frac{2}{\lambda_1(1+\lambda_1)} - h \cot\theta_0 \frac{1}{\lambda_1(1+\lambda_1)} \\
\delta_0'' &= \frac{2}{1+\lambda_2} + h \cot\mu_0 \frac{\lambda_2}{1+\lambda_2}
\end{aligned} \tag{55}$$

In this case h_1 is the length of the truncated side to point 3.

Residual and relaxation operators are now available for all points of the mesh. The numerical values for the residual operators used are shown in Figure 3. The values for the relaxation operators are easily obtained therefrom, and are shown in Figure 4.

The mesh size is also shown by Figure 3. The accuracy of the value of v at each point in the mesh is dependent on the mesh size because of the approximations made in going from the differential equation to the difference equation. In this particular problem the accuracy of the boundary condition on the boundary with Region III is already limited by the mesh size of the characteristic mesh used in that region. It is considered that the mesh size chosen for Region II is commensurate with this consideration and yet sufficiently small to show the essential characteristics of the desired results.

The resulting values of v are shown in Figure 5 together with the boundary conditions.

3.6 Potential and Pressure on the Cone

The value of v having now been determined everywhere in the XR plane it is possible by integration to find the potential at any point. These values of v are shown in Figure 6. Use is made of the fact that the potential is zero everywhere outside the region influenced by signals from the cone. In the XR plane the value of a function F(X,R) is obtained by

$$F(X,R) = \int_{\infty}^{R} v \cdot dR \tag{56}$$

where v is the normalized value obtained in the calculations. Then the potential at any time is

$$\phi(x,r,t) = ct\ V \tan\epsilon\ F(X,R) = ct \cdot \psi(X,R) \tag{57}$$

and the time derivative is

$$\phi_t(x,r,t) = cV \tan\epsilon \left[tF_X X_t + tF_R R_t + F \right]$$

or

$$\phi_t(x,r,t) = cV \tan\epsilon \left[-XF_X - RF_R + F \right] \tag{58}$$

The values of F which are calculated and given in Figure 7 are the values on the cone, that is,

$$R = (X + M)\tan\epsilon \tag{59}$$

If this is substituted in Equation (58)

$$\phi_t = cV\tan\epsilon\left[-XF_X(X,R(X)) - \tan\epsilon(X+M)F_R(X,R(X)) + F(X,R(X))\right] \tag{60}$$

If Z is the distance along the surface of the cone measured from the apex in the XR plane, then

$$\phi_t = cV\tan\epsilon\left[-X\sec\epsilon F_Z(X,R(X)) - M\tan\epsilon F_R(X,R(X)) + F(X,R(X))\right] \tag{61}$$

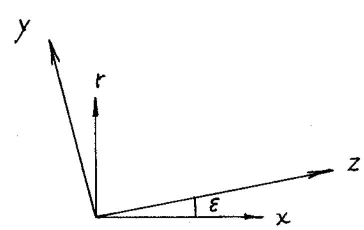
On the plot of F(X,R(X)) as a function of X the slope of the curve is $\sec\epsilon F_Z$. Since F_R is always unity on the cone, it is now possible to calculate ϕ_t and thus the pressure from Figure 7.

A plot of $-\dfrac{\phi_t}{cV\tan\epsilon}$ as a function of X is shown in Figures 8 and 9. This is essentially a plot of pressure since $\dfrac{p-p_o}{\rho_o} = -\phi_t$. With the scale of pressure used the piston value is unity.

IV. AN APPROXIMATE SOLUTION FAR BACK ON THE CONE

It seems from a consideration of the physical problem of the cone starting impulsively that an approximation could be made for the rear portion of the cone. The first approximation is to consider the case of being so far back that the surface of the cone is flat with respect to the thickness of the wave sent out since time $t = 0$. This is merely the one dimensional piston problem. A next approximation is to take into account the curvature of the surface. This is the purpose of the approximation presently being made. A more exact statement of what can be considered the rear portion of the cone will be arrived at in the subsequent development of the problem.

Suppose a coordinate system (y,z) is placed so that the z-axis is along the surface of the cone and the y-axis is perpendicular to the surface. The yz plane contains the axis of the cone, and the yz origin is at the vertex of the cone. The relation between the (y,z) coordinates and the (x,r) coordinates are as shown



in the figure.

The acoustic wave equation was written before as

$$\phi_{rr} + \frac{1}{r}\phi_r + \phi_{xx} - \frac{1}{c^2}\phi_{tt} = 0 \tag{14}$$

Using the transformation according to the above figure of

$$z = x\cos\varepsilon + r\sin\varepsilon$$

$$y = -x\sin\varepsilon + r\cos\varepsilon$$

the equation becomes in the (y, z) coordinates

$$\phi_{yy} + \phi_{zz} + \frac{\phi_y \cos\varepsilon + \phi_z \sin\varepsilon}{z\sin\varepsilon + y\cos\varepsilon} - \frac{1}{c^2}\phi_{tt} = 0 \tag{62}$$

The assumption is now made that the derivatives with respect to z are negligible compared to the other quantities. This means that gradients along the cone are small compared to those perpendicular to the surface. This seems physically reasonable. The equation can now be written

$$\phi_{yy} + \frac{\phi_y}{z\tan\varepsilon + y} - \frac{1}{c^2}\phi_{tt} = 0 \tag{63}$$

The condition is now imposed that $y \ll z\tan\varepsilon$. This condition states that the problem is being solved for a point at a distance from the surface of the cone which is small compared to the radius of the cone. This is of course always true on the surface of the cone, but it is also true at a point in the wave from the cone if it is far enough back on the cone. It may be noted that $z\tan\varepsilon = r\sec\varepsilon$, if r is the radius of the cone. Let $z\tan\varepsilon = \zeta$, and the equation becomes

$$\phi_{yy} + \frac{\phi_y}{\zeta} - \frac{1}{c^2}\phi_{tt} = 0 \tag{64}$$

The quantity ς may be thought of as the radius of a cylinder fitting the surface of the cone at the point considered. It is thus a parameter in the problem determined by the distance back on the cone at which the problem is being solved.

For the cone starting impulsively the problem may now be written formally as

$$\begin{gathered}
\phi_{yy} + \frac{\phi_y}{\varsigma} - \frac{1}{c^2}\phi_{tt} = 0 \\
\text{at } y=0 \begin{cases} \phi_y = 0 \; ; & t \leq 0 \\ \phi_y = 1 \; ; & t > 0 \end{cases} \\
\phi = 0 \text{ for all } y \text{ at } t \leq 0
\end{gathered} \tag{65}$$

where ϕ_y is normalized by a factor $\frac{1}{\sqrt{\sin \varepsilon}}$. To eliminate the first derivative term let

$$\phi(y,t) = e^{-\frac{y}{2\varsigma}} \varphi(y,t) \tag{66}$$

then

$$\phi_y = -\frac{1}{2\varsigma} e^{-\frac{y}{2\varsigma}} \varphi + e^{-\frac{y}{2\varsigma}} \varphi_y$$

and at y = 0

$$1 = -\frac{\varphi}{2\varsigma} + \varphi_y$$

The equation then becomes

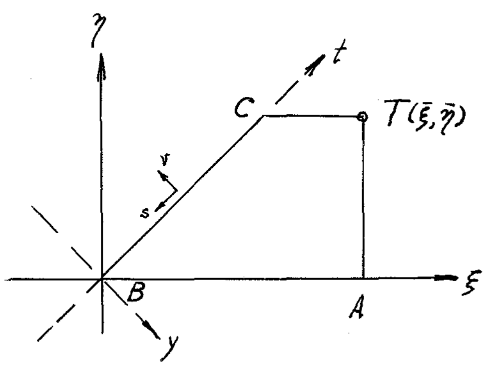
$$\varphi_{yy} - \frac{1}{c^2}\varphi_{tt} = \frac{1}{4\varsigma^2}\varphi \tag{67}$$

This hyperbolic equation can be put in normal form with the characteristic coordinates as independent variables by the transformation $\xi = t + y$ and $\eta = t - y$. Letting c = 1, this gives

$$\varphi_{\xi\eta}(\xi,\eta) + \frac{1}{16\xi^2}\varphi(\xi,\eta) = 0 \qquad (68)$$

This is the telegraph equation and can be treated with Riemann's integration method.

Consider the point $T(\bar{\xi},\bar{\eta})$ in the $\xi\eta$ plane.



For the above telegraph equation the Riemann function is (Reference 11)

$$p(\xi,\eta;\bar{\xi},\bar{\eta}) = J_0\left[\frac{1}{2\xi}\sqrt{(\xi-\bar{\xi})(\eta-\bar{\eta})}\right] \qquad (69)$$

Since the equation is homogeneous, Green's formula can be written for the contour TCBA as

$$0 = \oint P\,d\eta - \oint Q\,d\xi$$

where

$$P = \frac{1}{2}(p\varphi_\eta - \varphi p_\eta)$$

$$Q = \frac{1}{2}(p\varphi_\xi - \varphi p_\xi)$$

More explicitly, this becomes

$$0 = \frac{1}{2}\int_{AT+CB} p\,\varphi_\eta\,d\eta - \frac{1}{2}\int_{AT+CB} \varphi\,p_\eta\,d\eta - \frac{1}{2}\int_{TC+CB+BA} p\,\varphi_\xi\,d\xi + \frac{1}{2}\int_{TC+CB+BA} \varphi\,p_\xi\,d\xi$$

Since p = 1 on AT and TC some of these integrals are easily integrated. Also the integrals along CB involving derivatives of φ can be combined to form a line integral. Rewriting then

$$0 = \varphi(T) - \frac{1}{2}\varphi(C) + \frac{1}{2}\int_{CB} p\,\varphi_\nu\,ds + \frac{1}{2}\int_{CB} \varphi\,p_\xi\,d\xi - \frac{1}{2}\int_{CB} \varphi\,p_\eta\,d\eta \tag{70}$$

where ν is the outward normal and s is taken along the line CB.

The derivatives of p are

$$\begin{aligned} p_\xi &= \frac{\bar{\eta} - \eta}{4\xi\sqrt{(\xi-\bar{\xi})(\eta-\bar{\eta})}}\, J_1\left[\frac{1}{2\xi}\sqrt{(\xi-\bar{\xi})(\eta-\bar{\eta})}\right] \\ p_\eta &= \frac{\bar{\xi} - \xi}{4\xi\sqrt{(\xi-\bar{\xi})(\eta-\bar{\eta})}}\, J_1\left[\frac{1}{2\xi}\sqrt{(\xi-\bar{\xi})(\eta-\bar{\eta})}\right] \end{aligned} \tag{71}$$

Using the fact that $\xi = \eta$ on CB the equation becomes

$$0 = \varphi(T) - \frac{1}{2}\varphi(C) + \frac{1}{2}\int_{CB} p\,\varphi_\nu\,ds + \frac{1}{2}\int_{CB} \varphi\,\frac{\bar{\eta}-\bar{\xi}}{4\xi\sqrt{(\xi-\bar{\xi})(\xi-\bar{\eta})}}\, J_1\left[\frac{1}{2\xi}\sqrt{(\xi-\bar{\xi})(\xi-\bar{\eta})}\right] d\xi \tag{72}$$

This is an equation giving $\varphi(y,t)$ as a function of $\varphi(0,t)$. Actually to find the pressure on the cone it is only necessary to find $\varphi(0,t)$. This is done by solving an integral equation.

If the point T is taken on CB, then $\bar{\xi} = \bar{\eta}$, and the last integral vanishes. Also the points T and C coincide. This gives the integral equation

$$0 = \varphi(T) + \int_{CB} p\,\varphi_\nu\,ds \tag{73}$$

Because of the coordinate transformation, $\sqrt{2}\,\varphi_\nu = -\varphi_y$ and $ds = \sqrt{2}\,dt$

As stated before, at y = 0,

$$\varphi_y = \phi_y + \frac{1}{2\varsigma}\varphi = 1 + \frac{1}{2\varsigma}\varphi$$

Thus the integral equation becomes, at y = 0,

$$0 = \varphi(0,t) + \int_0^t J_0\left(\frac{t-\tau}{2\varsigma}\right)d\tau + \frac{1}{2\varsigma}\int_0^t \varphi(0,\tau)\, J_0\left(\frac{t-\tau}{2\varsigma}\right)d\tau \tag{74}$$

Assuming that $\varphi(0,t)$ is of the form

$$\varphi(0,t) = a_0 t + a_1 t\left(\frac{t}{2\varsigma}\right) + a_2 t\left(\frac{t}{2\varsigma}\right)^2 + \cdots$$

the coefficients come out to give the series

$$\varphi(0,t) = -t + \frac{1}{2}t\left(\frac{t}{2\varsigma}\right) - \frac{1}{12}t\left(\frac{t}{2\varsigma}\right)^2 + \cdots \tag{75}$$

Relaxing the condition that c = 1, the potential at y = 0 is then

$$\phi(0,t) = -ct + \frac{1}{2}ct\left(\frac{ct}{2\varsigma}\right) - \frac{1}{12}ct\left(\frac{ct}{2\varsigma}\right)^2 + \cdots \tag{76}$$

If the velocity perpendicular to the surface of the cone is not normalized to unity but is taken as $q = V \sin\varepsilon$, then the potential is

$$\phi(0,t) = V\sin\varepsilon\left[-ct + \frac{1}{2}ct\left(\frac{ct}{2\varsigma}\right) - \frac{1}{12}ct\left(\frac{ct}{2\varsigma}\right)^2 + \cdots\right] \tag{77}$$

and

$$\phi_t(0,t) = cV\sin\varepsilon\left[-1 + \left(\frac{ct}{2\varsigma}\right) - \frac{1}{4}\left(\frac{ct}{2\varsigma}\right)^2 + \cdots\right] \tag{78}$$

This gives the pressure as

$$p - p_0 = \rho_0 c\, V \sin\varepsilon \left[1 - \left(\frac{ct}{2\varsigma}\right) + \frac{1}{4}\left(\frac{ct}{2\varsigma}\right)^2 + \cdots \right] \tag{79}$$

The first term of this series is the one dimensional piston value where the piston is considered moving perpendicular to the surface of the cone.

It is of course necessary to consider the matter of convergence of the above series. Because of the complication of the process of evaluating coefficients it doesn't appear possible to evaluate the coefficient of the general term. Therefore no rigorous argument on convergence can be made. However, from an engineering standpoint, if $\frac{ct}{2\varsigma}$ is sufficiently small the second two terms can be considered as corrections on the piston value of the pressure to account for the curvature of the cone. Since $\frac{ct}{2\varsigma}$ is the ratio of the thickness of the wave to the diameter of the cylinder fitting the cone, the condition that $\frac{ct}{2\varsigma}$ be small is the condition that says this solution is only good far back on the cone. Figure 9 gives a plot of the pressure.

V. DISCUSSION OF RESULTS

By combining the resulting pressures of Section III for the front part of the cone and of Section IV for the rear part of the cone a picture of the pressure distribution for the entire cone at any time is obtained. This is shown in Figure 9 for the example worked out of a cone with a half angle of ten degrees starting at a Mach number of $\sqrt{2}$.

The pressure curve for the rear of the cone when plotted with $p-p_0/\rho_0 c V \sin\epsilon$ as a function of $\frac{ct}{2\zeta}$ is good for any cone and any Mach number. When plotted as a function of X as in Figure 9 it is dependent on the values in the particular problem.

The pressure curve for the front portion of the cone resulting from the calculations of Section III is not so general. It is for the particular cone angle and Mach number used in the calculations.

However, for cases where the Mach number is greater than one the constant pressure value on the very tip of the cone is obtained from the results of the stationary problem. With this as a starting point and the general curve at the rear of the cone determined, the example worked out should serve as a guide as to how they should be joined for an engineering approximation.

VI. SOLUTION BY INTEGRAL EQUATION

The most direct way of solving the problem of the impulsively starting cone would seem to be to integrate the fundamental solution of the wave equation, which can be considered as sources on the axis of the cone, over the proper range of the variables. An integral equation for the source strength is obtained through the known radial velocity on the surface of the cone. Knowing the source strength it is then possible to obtain the potential and pressure. However, except for Region I, the kernel of the integral equation becomes so complicated that even numerical methods seem impracticable.

The problem can be stated as

$$\phi_{rr} + \frac{1}{r}\phi_r + \phi_{xx} - \frac{1}{c^2}\phi_{tt} = 0 \tag{14}$$

$\phi = 0$ for $t \leq 0$

$\phi = 0$ for $x < -Vt$

$\phi_r = V \tan \varepsilon$ for $r = (x + Vt)\tan\varepsilon$, $t > 0$

where the last condition is an approximation to the physical problem.

The potential ϕ can be written as

$$\phi(x,r,t) = -\frac{1}{4\pi}\int_0^t \int_{-\infty}^{\tau} \frac{\delta(t-\tau-\frac{R}{c})\, S(\xi,\tau)}{R}\, d\xi\, d\tau \tag{80}$$

where

$$\delta(t-\tau-\tfrac{R}{c}) = 1 \quad \text{when} \quad t = \tau + \tfrac{R}{c}$$

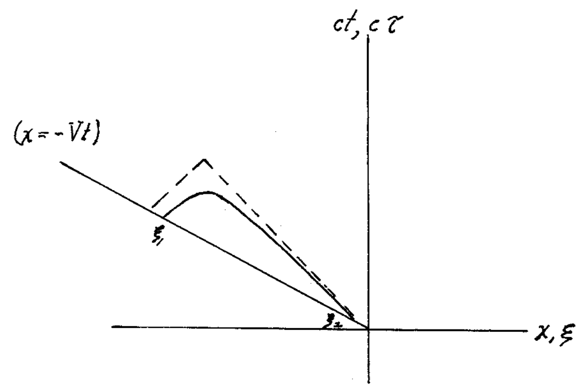
$$= 0 \quad \text{when} \quad t \neq \tau + \tfrac{R}{c}$$

$$R = \sqrt{(x-\xi)^2 + r^2} = c(t-\tau)$$

and $S(\xi,\tau)$ is a source strength to be determined. Then

$$\phi(x,r,t) = -\frac{1}{4\pi}\int_{\xi_1(x,r,t)}^{\xi_2(x,r,t)} \frac{S(\xi,\tau(\xi))}{R}\, d\xi \qquad (81)$$

where $\tau(\xi) = t - \frac{1}{c}\sqrt{(x-\xi)^2+r^2}$. Thus the integration is made for values of $S(\xi,\tau)$ along a hyperbola in the $\xi\tau$ plane, or xt plane. For the point (x,r,t) in Region I the limits of integration, ξ_1 and ξ_2, are the intersections of the hyperbola and the line x = - Vt. For Region II $\xi_1 < 0$ and $\xi_2 > 0$, and for Region III $\xi_1 > 0$ and



$\xi_2 > 0$. Now solving for the limits in Region I

$$x - \xi_1 = -\frac{s}{B^2} + \frac{Ms}{B^2}\sqrt{1-\lambda^2}$$

$$x - \xi_2 = -\frac{s}{B^2} - \frac{Ms}{B^2}\sqrt{1-\lambda^2}$$

where $s = x + Vt$, $B^2 = M^2 - 1$, and $\lambda = \frac{Br}{s}$

With reference to the distribution of sources on the axis for the case of the cone in stationary supersonic flow, it is assumed that for Region I

$$S(\xi,\tau) = e(\xi + V\tau) \tag{82}$$

where e is a constant to be determined. This gives

$$\phi(x,r,t) = -\frac{e}{4\pi}\int_{\xi_1}^{\xi_2} \frac{\left[\xi + Vt - \frac{V}{c}\sqrt{(x-\xi)^2 + r^2}\,\right]}{\sqrt{(x-\xi)^2 + r^2}}\, d\xi$$

which upon integration is

$$\phi(x,r,t) = -\frac{es}{4\pi}\left\{\log \frac{-1 + M\sqrt{1-\lambda^2} + \sqrt{(M\sqrt{1-\lambda^2}-1)^2 + B^2\lambda^2}}{-1 - M\sqrt{1-\lambda^2} + \sqrt{(M\sqrt{1-\lambda^2}+1)^2 + B^2\lambda^2}}\right.$$

$$\left. + \frac{1}{B^2}\left[\sqrt{(M\sqrt{1-\lambda^2}+1)^2 + B^2\lambda^2} - \sqrt{(M\sqrt{1-\lambda^2}-1)^2 + B^2\lambda^2} - 2M^2\sqrt{1-\lambda^2}\right]\right\} \tag{83}$$

To evaluate e it is now necessary to find ϕ_r which is found by differentiating where $\phi_r = \phi_\lambda \cdot \lambda_r$. Taking $\lambda \ll 1$ and neglecting squares of it with respect to unity, the simple form is obtained

$$\phi_r = \frac{eB}{2\pi\lambda} \tag{84}$$

Thus the boundary condition is closely approximated by letting

$$e = 2\pi\varepsilon^2 V$$

If now the condition that $\lambda \ll 1$ is used again in Equation (83), and the value of e is substituted, then

$$\phi(x,r,t) = -\varepsilon^2 V s\left(\log\frac{2}{\lambda} - 1\right) \tag{85}$$

Recalling that $s = x + Vt$ and differentiating with respect to t

$$\phi_t = -\varepsilon^2 V^2 \log \frac{2}{\lambda} \tag{86}$$

which gives

$$C_p = 2\varepsilon^2 \log \frac{2}{\lambda} \tag{87}$$

and on the cone

$$C_p = 2\varepsilon^2 \log \frac{2}{B\varepsilon} \tag{88}$$

This is the value obtained in the linearized stationary cone problem when the same approximating assumptions are made.

In proceeding on to Regions II and III there is no reason for assuming that $S(\xi, \tau)$ changes linearly with $\xi + V\tau$ so it must be held in the integral as $S(\xi, \tau)$. By dimensional arguments made earlier $S(\xi, \tau)d\xi$ can be written as $S\left(\frac{\xi}{ct}\right) \, d\left(\frac{\xi}{ct}\right)$. If ϕ is differentiated with respect to r and ϕ_r is evaluated at the surface of the cone, an integral equation for $S\left(\frac{\xi}{ct}\right)$ is obtained. The kernels of these integral equations are unfortunately complicated and it doesn't seem practicable to solve the equations even by numerical methods. It was for this reason that the problem was attacked more indirectly through the differential equation.

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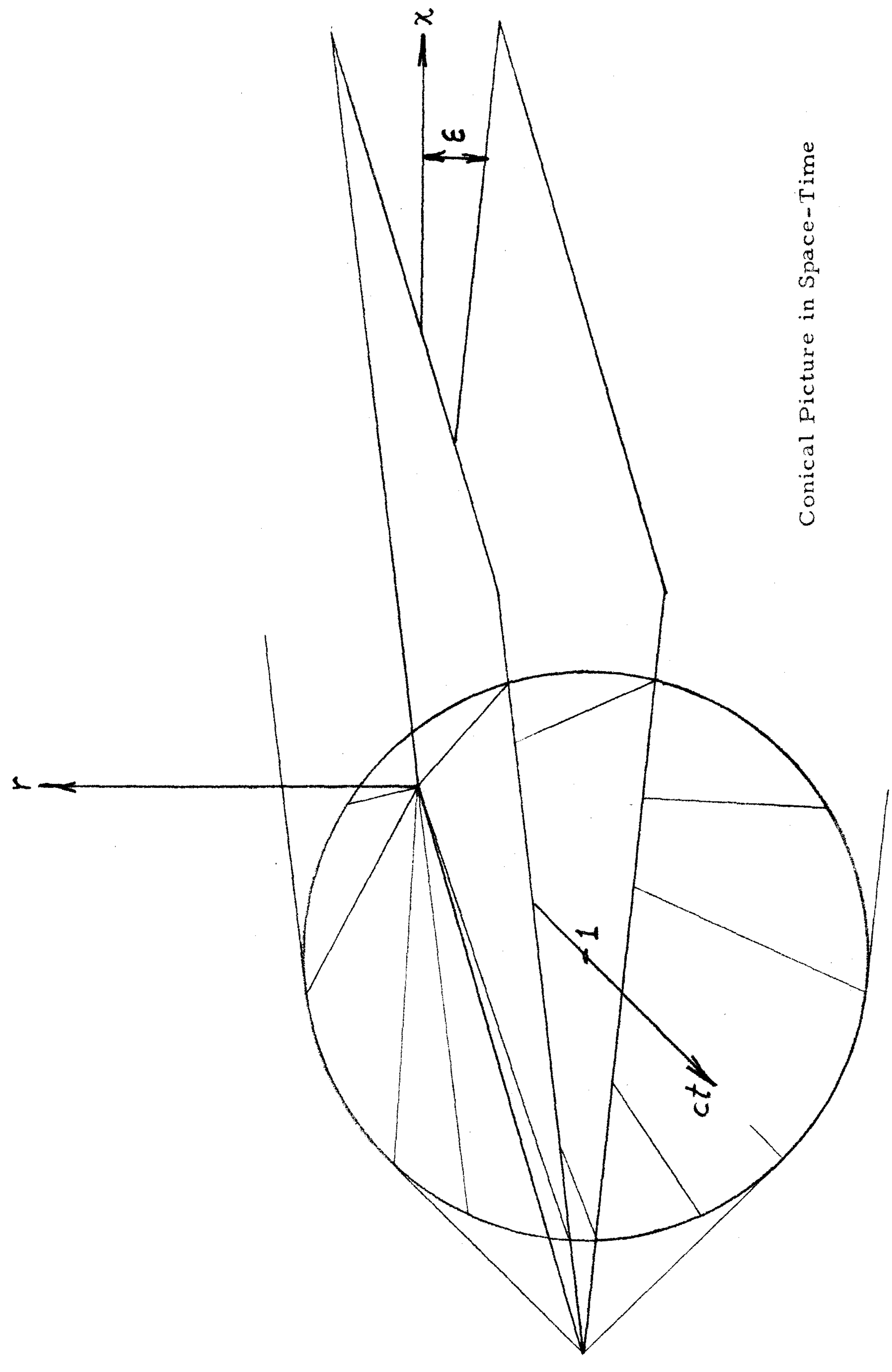
NOTATION

a	defined on page 13
B	$\sqrt{M^2-1}$
c	velocity of sound
F	defined in Equation (56)
h	mesh spacing defined on page 22
M	Mach number
p	pressure, also used for Riemann function
P,Q	defined on page 32
q	vector velocity
r	radial coordinate measured perpendicular to x-axis
R	$\frac{r}{ct}$, also used briefly as $\sqrt{(x-\xi)^2+r^2}$
$\mathcal{R}$	defined on page 13
s	condensation, also distance in x-direction from apex of cone
t	time
u,v	velocities in x- and v-directions, respectively
V	velocity of cone in negative x-direction
w	general notation for either u, v, or ϕ_t
x	coordinate in direction of cone axis at rest in fluid
X	$\frac{x}{ct}$
y,z	coordinate defined on page 29
Z_1	distance along surface of cone, defined on page 28

$\alpha, \beta, \gamma, \delta$	coefficients as defined in Equation (50)
γ	ratio of specific heats
ϵ	half-angle of cone
ζ	z tan ε
ξ, η	characteristic coordinates as defined in Equation (38)
ξ', η'	defined in Equation (40)
ξ'', η''	defined on page 18
θ	conical coordinate defined by Equation (19)
$\Theta_{\rho,\theta}$	operator defined on page 9
λ	$\frac{\beta r}{s}$
μ	transformed coordinate defined in Equation (45)
σ	transformed coordinate defined in Equation (33)
τ	time
φ	briefly used as defined in Equation (35), also used as defined in Equation (66)
ϕ	velocity potential
ψ	defined by Equation (18), also briefly used as defined in Equation (35)
ω	angle of rotation about x-axis
$\Box_{x,r,t}$	wave operator defined by Equation (14)
0,1,2,3,4	used as subscripts to designate mesh points

Figure 1



Conical Picture in Space-Time

FIGURE 2

CHARACTERISTIC MESH IN REGION III
WITH CALCULATED VALUES OF v

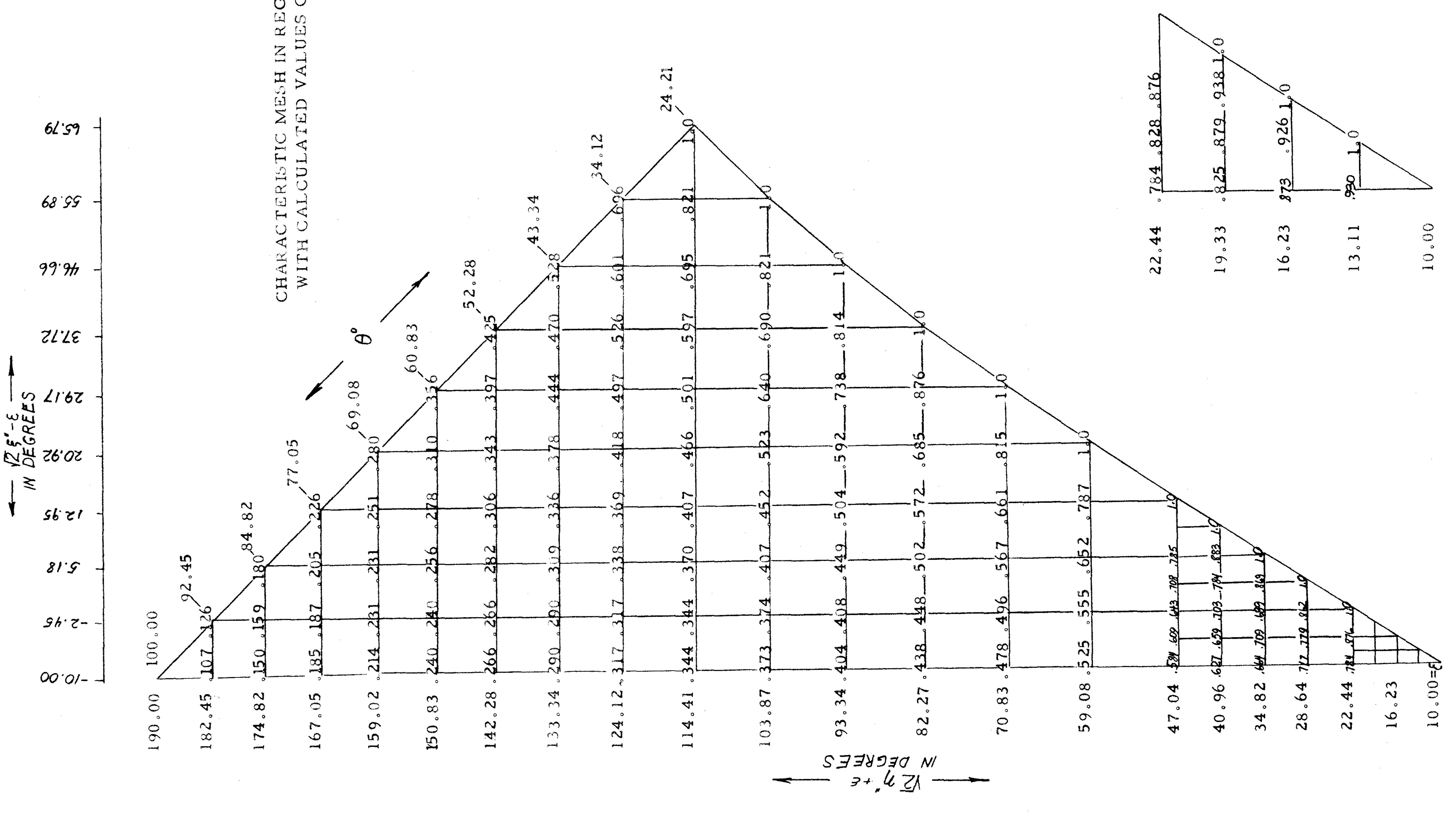
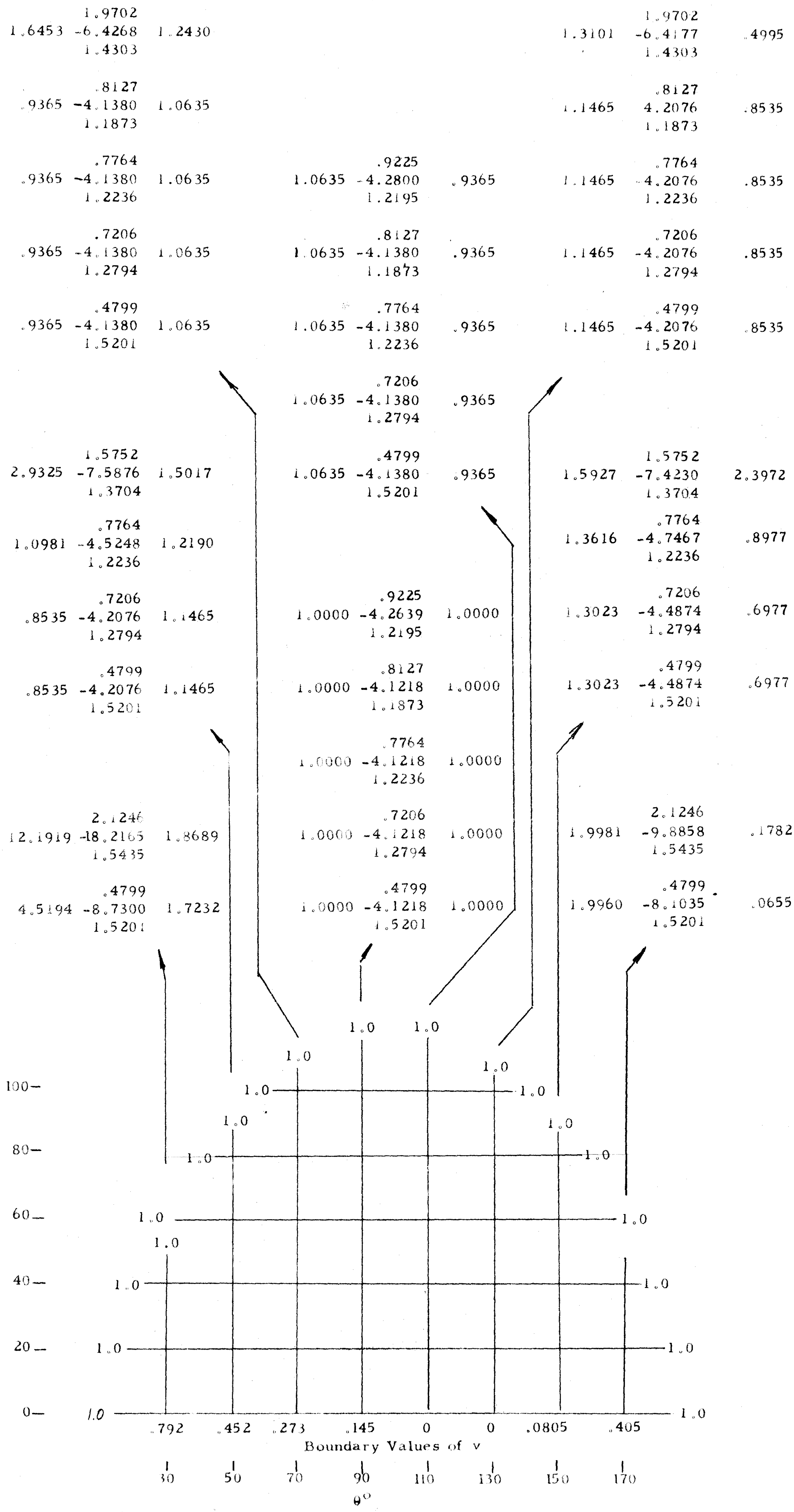


FIGURE 3



RESIDUAL OPERATORS

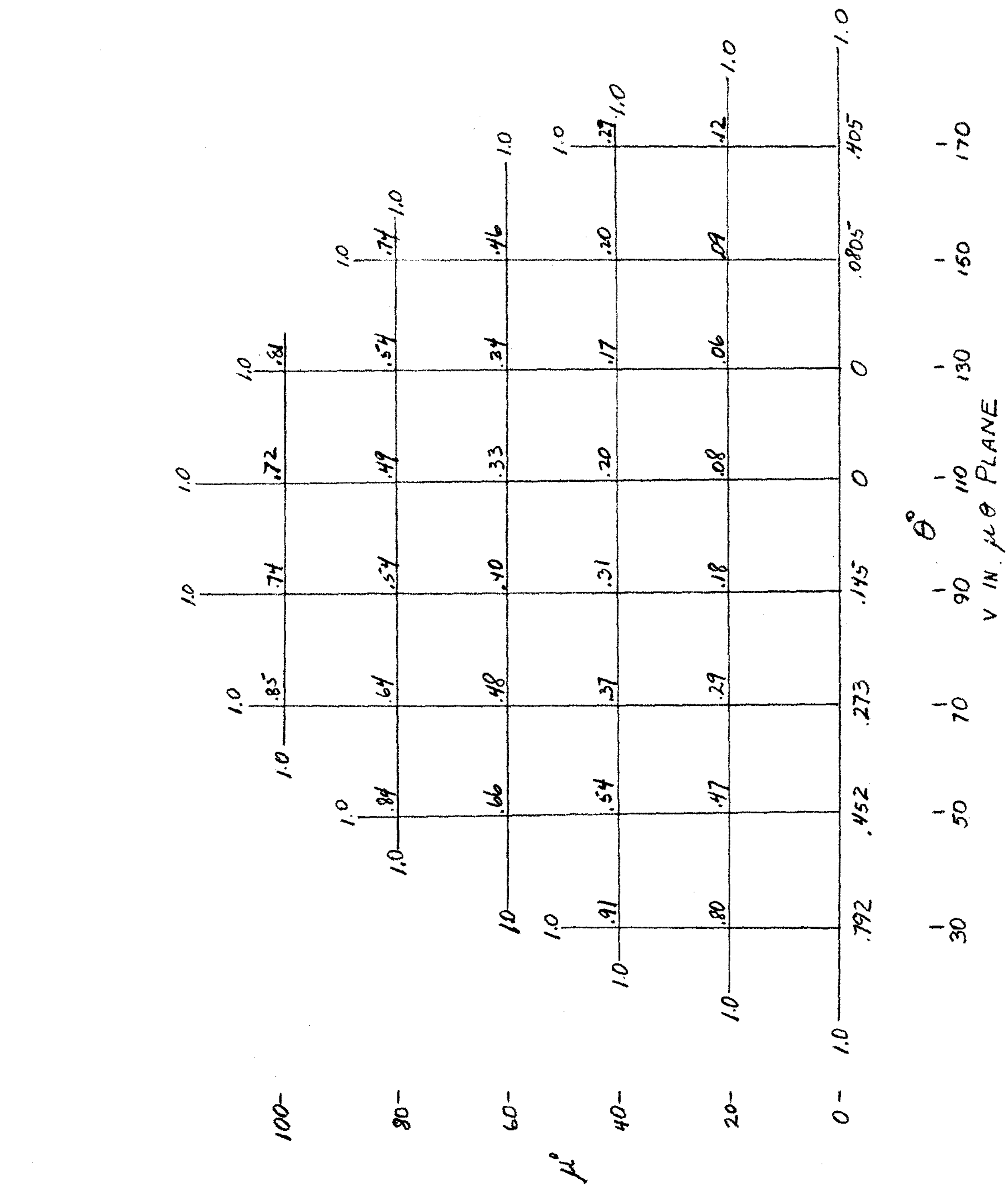
FIGURE 4

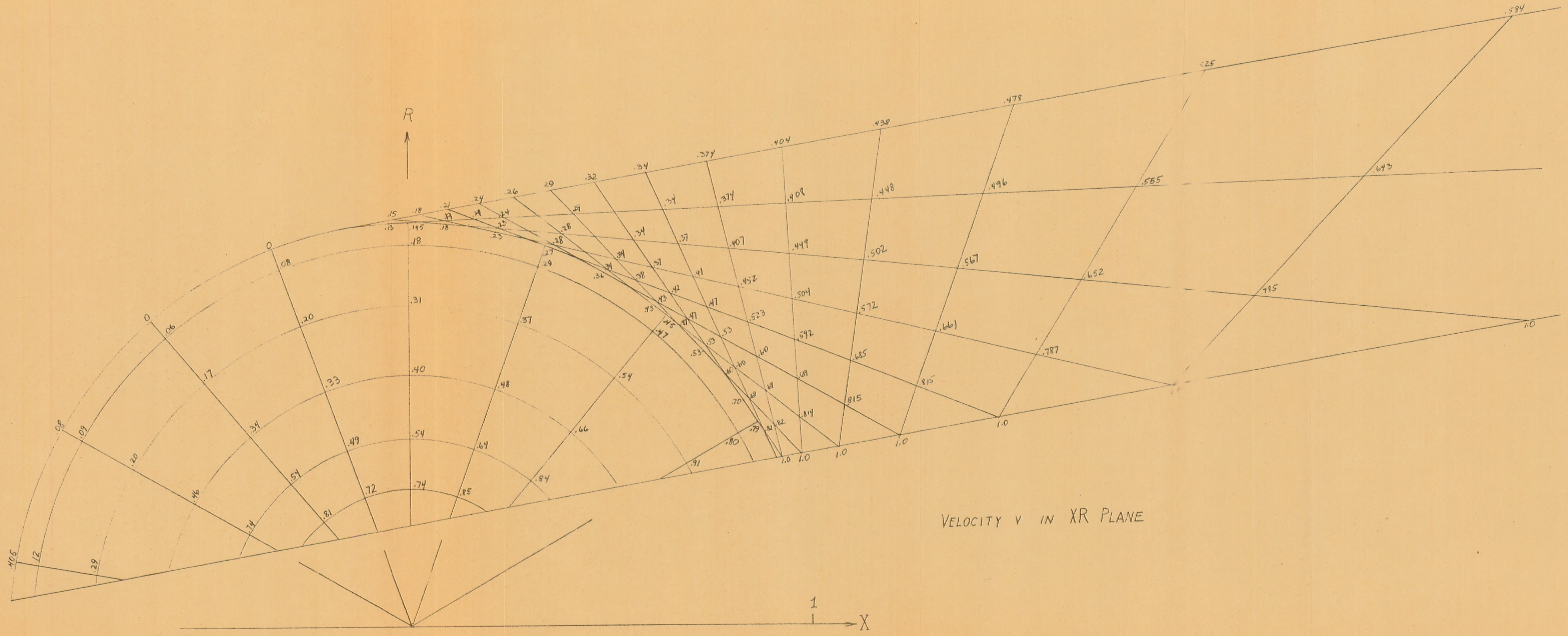
	-6.4268	1.0000				.9365	-6.4177	
	.8127						.8127	
	1.4303						1.4303	
1.5017	-4.1380	1.0000				.9365	-4.2076	1.5927
	.7764						.7764	
	1.1873						1.1873	
1.2190	-4.1380	1.0000	1.0000	-4.2800	1.3101	.9365	-4.2072	1.3616
	.7206			.8127			.7206	
	1.2236			1.2195			1.2236	
1.1465	-4.1380	1.0000	1.0000	-4.1380	1.1465	.9365	-4.2076	1.3023
	.4799			.7764			.4799	
	1.2794			1.1873			1.2794	
1.1465	-4.1380	1.0000	1.0000	-4.1380	1.1465	.9365	-4.2076	1.3023
				.7206				
				1.2236				
			1.0000	-4.1380	1.1465			
				.4799				
				1.2794				
	-7.5876	.9365	1.0000	-4.1380	1.1465	.8535	-7.4230	
	.7764						.7764	
	1.3764						1.3704	
	-4.5248	.9365				.8535	-4.7467	
	.7206						.7206	
	1.2236						1.2236	
1.8689	-4.2076	.9365	1.2430	-4.2639	1.0635	.8535	-4.4874	1.9981
	.4799			.8127			.4799	
	1.2794			1.2195			1.2794	
1.7232	-4.2076	.9365	1.0635	-4.1218	1.0635	.8535	-4.4874	1.9960
				.7764				
				1.1873				
			1.0635	-4.1218	1.0635			
				.7206				
				1.2236				
	-18.2165	.8535	1.0635	-4.1218	1.0635	.6977	-9.8858	
	.4799			.4799			.4799	
	1.5435			1.2794			1.5435	
	-8.7300	.8535	1.0635	-4.1218	1.0635	.6977	-8.1035	

RELAXATION OPERATORS

(Arrangement same as Figure 3)

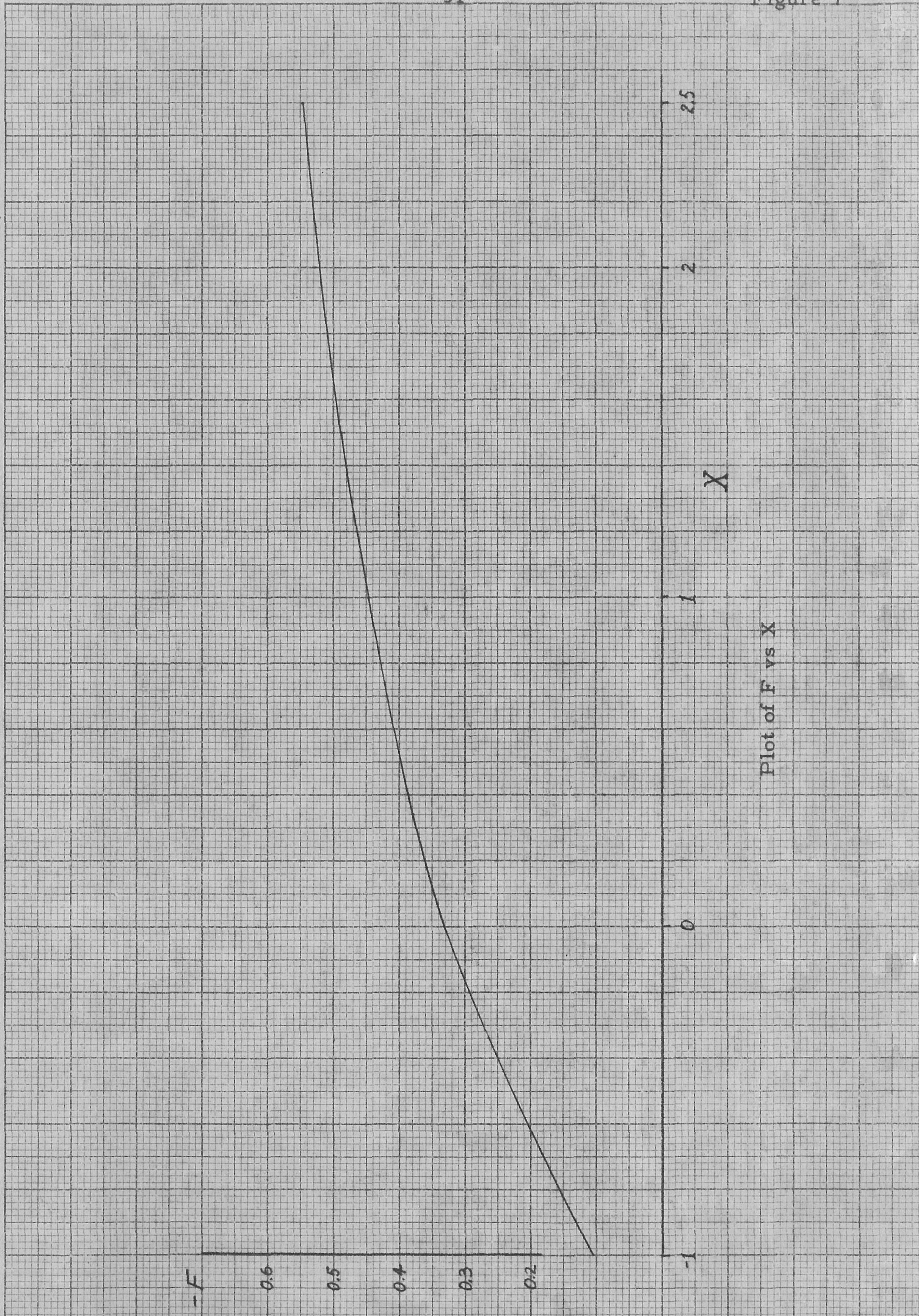
FIGURE 5





VELOCITY v IN XR PLANE

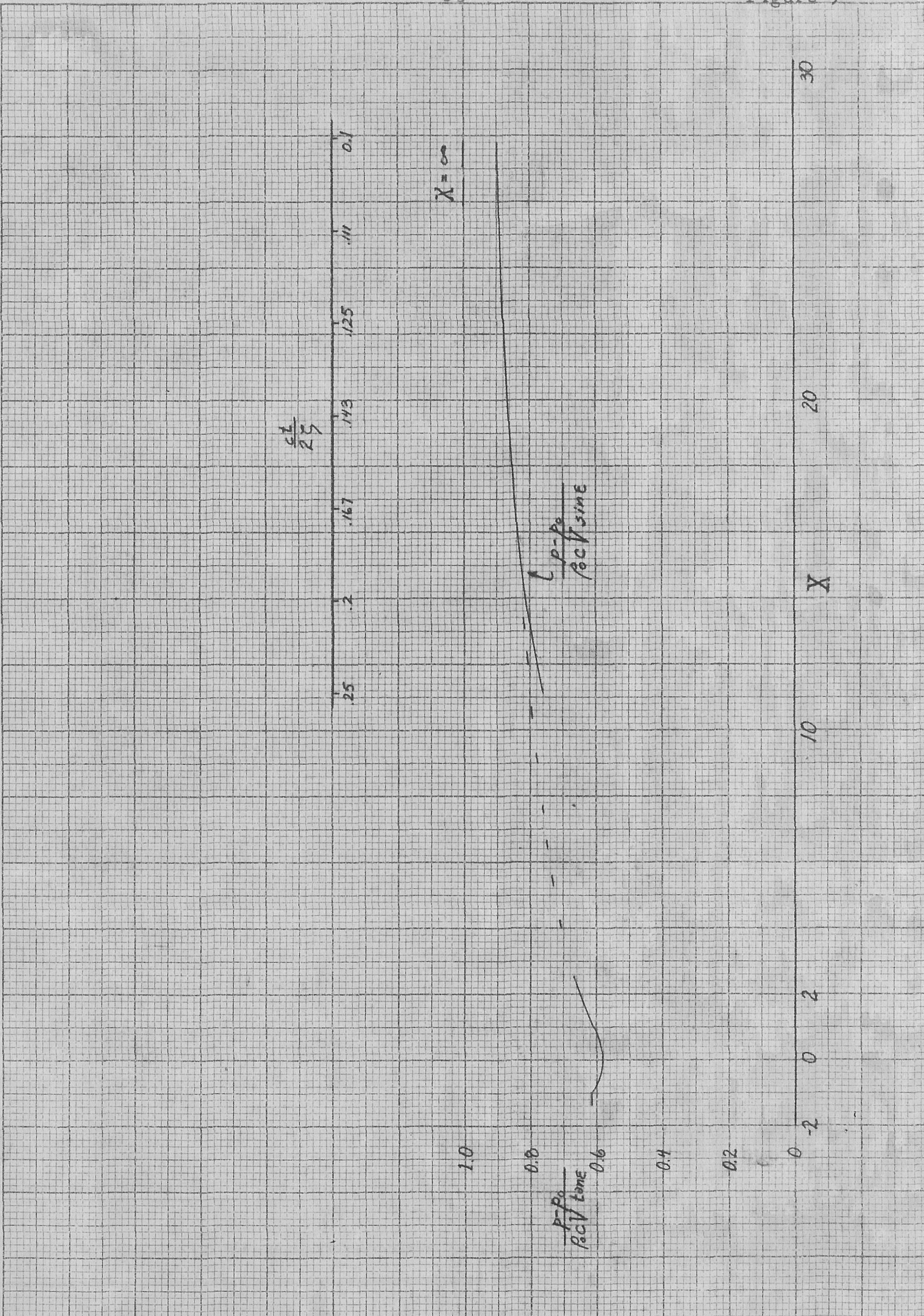
Figure 7



Plot of F vs X

Figure 8


at X = ∞
1.0
0.8
0.6
0.4
0.2
0
$\frac{p-p_0}{\rho_0 c V \tan\epsilon}$
OR
$\frac{-\phi_t}{c V \tan\epsilon}$
-1
0
1
2
3
X
Pressure vs X


X
0
0.2
0.4
0.6
0.8
1.0
-2
0
2
10
20
30
.25
.2
.167
.143
.125
.111
0.1