LOCALLY CONVEX RIESZ SPACES AND ARCHIMEDEAN QUOTIENT SPACES

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ABSTRACT

A Riesz space with a Hausdorff, locally convex topology determined by Riesz seminorms is called a locally convex Riesz space. A sequence $\{x_n\}$ in a locally convex Riesz space L is said to converge locally to $x \in L$ if for some topologically bounded set B and every real $r > 0$ there exists N (r) and $n \geq N(r)$ implies $x - x_n \in rB$. Local Cauchy sequences are defined analogously, and L is said to be locally complete if every local Cauchy sequence converges locally. Then L is locally complete if and only if every monotone local Cauchy sequence has a least upper bound. This is a somewhat more general form of the completeness criterion for Riesz - normed Riesz spaces given by Luxemburg and Zaanen. Locally complete, bound, locally convex Riesz spaces are barrelled. If the space is metrizable, local completeness and topological completeness are equivalent.

Two measures of the non - archimedean character of a non - archimedean Riesz space L are the smallest ideal $A_o(L)$ such that quotient space is archimedean and the ideal $I(L) = \{ x \in L:$ for some $0 \leq v \in L$, $n|x| \leq v$ for $n = 1, 2, \ldots\}$. In general $A_o(L) \supset I(L)$. If L is itself a quotient space, a necessary and sufficient condition that $A_o(L) = I(L)$ is given. There is an example where $A_o(L) \neq I(L)$.

A necessary and sufficient condition that a Riesz space L have every quotient space archimedean is that for every $0 \leq u, v \in L$

there exist $u_1 = \sup(\inf(n v, u): n = 1, 2, \ldots)$, $v_1 = \sup(\inf(n u, v): n = 1, 2, \ldots)$, and real numbers m_1 and m_2 such that $m_1 u_1 \geq v_1$ and $m_2 v_1 \geq u_1$. If, in addition, L is Dedekind σ - complete, then L may be represented as the space of all functions which vanish off finite subsets of some non- empty set.

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Introduction.

A subset S of a Riesz space L is said to be solid if $x \in L$, $y \in S$, $|x| \leq |y|$ implies $x \in S$. (For the definition and basic properties of Riesz spaces, also called vector lattices, cf. W. A. J. Luxemburg and A. C. Zaanen [4] and H. Nakano [6].) We call a Hausdorff locally convex topology $\mathcal{T}$ on L a locally convex Riesz topology and the pair $(L, \mathcal{T})$ a locally convex Riesz space if there exists a neighborhood basis for zero consisting of solid sets.

Locally convex topologies on partially ordered vector spaces have been studies by several authors; including H. Nakano [6], I. Namioka [7], and I. Kawai [2]. (In particular our definition of a locally convex Riesz space is the same as Kawai's definition of a "locally convex lattice.") A seminorm ρ on a Riesz space L is said to be a Riesz seminorm if $x, y \in L$ and $|x| \leq |y|$ implies $\rho(x) \leq \rho(y)$. A Riesz seminorm which is also a norm is called a Riesz norm. Riesz spaces equipped with Riesz norms have been studied extensively by Luxemburg and Zaanen [4].

We show in Chapter 1 that a Hausdorff locally convex topology $\mathcal{T}$ on a Riesz space L defines a locally convex Riesz space if and only if there exists a family of Riesz seminorms Ω such that $\{\{x \in L: \rho(x) \leq 1\} : \rho \in \Omega\}$ is a $\mathcal{T}$ - neighborhood basis for zero. In addition locally convex Riesz spaces are archimedean and the lattice operations are continuous. In Chapters 2 and 3 we investigate the space of continuous linear functionals and the space of topologically bounded linear functionals of a given locally convex Riesz space. We show that both these spaces are ideals in the Riesz space $L^{\sim}$ of all order bounded linear functionals, and obtain a characterization of the weakest and strongest locally convex Riesz topologies with a given topological dual.

Chapter 4 introduces the concept of relative uniform convergence. Let L be a Riesz space and $\{x_n\}$ a sequence in L. Then x_n is said to converge <u>relatively uniformly</u> to $x \in L$ if for some $0 \leq v \in L$ and for every real $a > 0$ there exists $N(a)$ such that $n \geq N(a)$ implies $|x - x_n| \leq aV$. For an arbitrary set S in L denote by S'^{ru} the set $\{x \in L:$ for some sequence $\{x_n\} \subset S$, $x_n \longrightarrow x$ (relatively uniformly)$\}$. A set S is called <u>relatively uniformly closed</u> if $S'^{ru} = S$. In general, relative uniform convergence does not define a topology and thus S'^{ru} is not necessarily relatively uniformly closed.

A sequence $\{x_n\}$ in a Riesz space L is said to be <u>monotone increasing</u>, written $x_n \uparrow$, if $x_n \leq x_{n+1}$, $n = 1, 2, \ldots$. A net $\{x_\lambda : \lambda \in \Lambda\}$ is said to be <u>monotone increasing</u>, written $x_\lambda \uparrow$, if $\lambda, \lambda' \in \Lambda$, $\lambda \leq \lambda'$, implies $x_\lambda \leq x_{\lambda'}$. An arbitrary system S in L is said to be <u>directed upwards</u> if $x, y \in S$ implies that there exists $z \in S$ such that $\sup(x, y) \leq z$. A system which is directed upwards may always be considered a monotone increasing net defined on S itself. Now Luxemburg and Zaanen [4] have given a characterization of norm complete Riesz - normed spaces. In particular their characterization shows that a

Riesz - normed space is topologically complete if and only if monotone Cauchy sequences have "sup's." In seeking to fit this into the general theory of locally convex Riesz spaces we are led to the concept of local convergence (introduced by H. Gordon [1]).

Let $(L, \mathcal{T})$ be a locally convex Riesz space. A sequence $\{x_n\}$ is said to converge $\mathcal{T}$ - locally to $x \in L$ if for some $\mathcal{T}$ - bounded set B and for every $a > 0$ there exists $N(a)$ such that $n \geq N(a)$ implies $x - x_n \in aB$. In an analogous manner we define $\mathcal{T}$ - local Cauchy sequence. The Riesz space L is said to be $\mathcal{T}$ - locally complete if every $\mathcal{T}$ - local Cauchy sequence converges $\mathcal{T}$ - locally to an element of L.

In Chapter 6 we show that L is $\mathcal{T}$ - locally complete if and only if every monotone $\mathcal{T}$ - local Cauchy sequence has a least upper bound. Further, given two locally convex Riesz topologies, $\mathcal{T}' \geq \mathcal{T}$, if L is $\mathcal{T}$ - locally complete, it is also $\mathcal{T}'$ - locally complete. Finally if L is $\mathcal{T}$ - locally complete, then the space of $\mathcal{T}$ - bounded linear functionals equals the space of order bounded linear functionals.

A locally convex space is <u>bound</u> if every convex circled set, which absorbs each topologically bounded set is a neighborhood of zero. It is <u>barrelled</u> if closed, convex, circled sets which absorb every point are neighborhoods of zero. In Chapter 7 we show that a $\mathcal{T}$ - locally complete, bound, locally convex Riesz space $(L, \mathcal{T})$ is barrelled and $\mathcal{T} = m(L, L^{\sim})$.

For metrizable locally convex Riesz spaces, $\mathcal{T}$ - convergence and

τ - local convergence agree. Thus a metrizable locally convex Riesz space is complete if and only if every monotone Cauchy sequence has a least upper bound.

The concept of relative uniform convergence may also be applied to the study of quotient spaces. Luxemburg [5] has shown that if L is a Riesz space and A is an ideal of L, L/A is archimedean if and only if A is relatively uniformly closed. In Chapter 9 we consider two measures of the non-archimedean character of a Riesz space L which is not archimedean -- $A_o(L)$, the smallest ideal such that the quotient space is archimedean, and $I(L) = \{x \in L:$ for some $0 \leq v \in L,\ n|x| \leq v,\ n = 1, 2, \ldots\}$. In general $I(L) \subset A_o(L)$. For spaces L which are themselves quotient spaces, we give a necessary and sufficient condition that $I(L) = A_o(L)$.

In Chapter 10 we give an example to show that $I(L)$ is not always $A_o(L)$. We also use this example to show that, even if L is archimedean, for an ideal A of L, A'^{ru} is not necessarily relatively uniformly closed.

Next we consider Riesz spaces such that every quotient space is archimedean. In Chapter 11 we show that a Riesz space L is of this type if and only if for every $0 \leq u,\ v \in L$ there exist $u_1 = \sup(\inf(n v, u): n = 1, 2, \ldots)$, $v_1 = \sup(\inf(n u, v): n = 1, 2, \ldots)$ and real numbers m_1 and m_2 such that $m_1 u_1 \geq v_1$ and $m_2 v_1 \geq u_1$. In Chapter 12 we see that if, in addition, we require that L be Dedekind σ - complete (or even relatively uniformly complete) then for some set X, L is isomorphic to the space of all real - valued functions on X which vanish off finite sets.

Chapter 13 characterizes locally convex Riesz spaces such that the positive cone has an interior point as spaces with a strong unit and the uniform topology. In Chapter 14 we give some special results for locally convex Riesz spaces such that order convergence implies topological convergence -- (A, ii) spaces.

I LOCALLY CONVEX RIESZ SPACES

1. Definition

Let L be a Riesz space and $\mathcal{T}$ be a locally convex, Hausdorff topology on L. Then $\mathcal{T}$ is called a locally convex Riesz topology and the pair $(L, \mathcal{T})$ is called a locally convex Riesz space if there exists a neighborhood basis of zero for $\mathcal{T}$ composed of solid sets. There are several alternate forms for this definition (I. Kawai in [2] calls such spaces locally convex lattices and proves (ii) implies (iii) in the theorem below.)

1.1 Theorem Let L be a Riesz space and $\mathcal{T}$ a linear topology on L. Then the following statements are equivalent:

(i) $(L, \mathcal{T})$ is a locally convex Riesz space.

(ii) The topology $\mathcal{T}$ is locally convex and Hausdorff and if $\{x_\lambda : \lambda \in \Lambda\}$ and $\{y_\lambda : \lambda \in \Lambda\}$ are nets in L defined on the same directed set Λ such that $|y_\lambda| \leq |x_\lambda|$ for each $\lambda \in \Lambda$ and $x_\lambda \to 0\ (\mathcal{T})$, then $y_\lambda \to 0\ (\mathcal{T})$.

(iii) The topology $\mathcal{T}$ is Hausdorff, and there exists a family Ω of Riesz seminorms such that $x_\lambda \to 0\ (\mathcal{T})$ if and only if $\rho(x_\lambda) \to 0$ for each $\rho \in \Omega$.

(iv) There is a neighborhood basis $\mathcal{U}$ of zero for $\mathcal{T}$ consisting of solid, convex, $\mathcal{T}$-closed sets and $\bigcap \{U : U \in \mathcal{U}\} = 0$.

Proof: To show that (i) implies (ii), let U be a solid $\mathcal{T}$-neighborhood of zero. There exists $\lambda_0 \in \Lambda$ such that $\lambda \geq \lambda_0$ implies $x_\lambda \in U$. Then since $0 \leq |y_\lambda| \leq |x_\lambda|$ and U is solid,

$y_\lambda \in U$. Thus $y_\lambda \to 0(\mathcal{T})$.

To show that (ii) implies (iii), let Ω be a family of seminorms (not necessarily Riesz seminorms) such that $x_\lambda \to 0(\mathcal{T})$ if and only if $\rho(x_\lambda) \to 0$. Let $\rho \in \Omega$ and $x \in L$. If $\{\rho(u): 0 \leq u \leq |x|, u \in L\}$ is unbounded, we may select a sequence $0 \leq u_n \in L$, $\rho(u_n) \geq n$, $0 \leq u_n \leq |x|$, $n = 1, 2, \ldots$. Then $\frac{1}{n} x \to 0 (\mathcal{T})$ and $\frac{1}{n} u_n \leq |\frac{1}{n} x|$, but $\rho(\frac{1}{n} u_n) \geq 1$, which contradicts the fact that by (ii) $u_n \to 0 (\mathcal{T})$. Thus we may define a Riesz seminorm $\bar{\rho}$ by $\bar{\rho}(x) = \sup(\rho(u): 0 \leq u \leq |x|)$, and we have $\rho(x) \leq \rho(x^+) + \rho(x^-) \leq 2\bar{\rho}(x)$.

Now we show that $\bar{\rho}$ is also $\mathcal{T}$-continuous. Assume, to the contrary, that there exists a net $\{x_\lambda : \lambda \in \Lambda\}$ such that $x_\lambda \to 0 (\mathcal{T})$, but $\bar{\rho}(x_\lambda) \geq a > 0$. Then by the definition of $\bar{\rho}$, for each $\lambda \in \Lambda$ there exists u_λ such that $0 \leq u_\lambda \leq |x_\lambda|$ and $\rho(u_\lambda) \geq \frac{a}{2}$. But by (ii) $u_\lambda \to 0 (\mathcal{T})$, which is a contradiction. Hence $\{\bar{\rho} : \rho \in \Omega\}$ is a family of Riesz seminorms which satisfies the condition of (iii).

Finally, that (iii) implies (iv) and (iv) implies (i) is obvious.

We now give two immediate consequences of the definition.

<u>1.2</u> <u>Theorem</u> If $(L, \mathcal{T})$ is a locally convex Riesz space, L is archimedean.

<u>Proof</u>: Suppose there exist $u, v \in L$ such that $0 \leq nu \leq v$ for every $n = 1, 2, \ldots$. Then $0 \leq u \leq \frac{1}{n} v$. Now $\frac{1}{n} v \to 0 (\mathcal{T})$ and by the previous theorem the constant sequence $u \to 0 (\mathcal{T})$. Since

$\mathcal{T}$ is Hausdorff, $u = 0$.

<u>1.3</u> <u>Theorem</u> If $(L, \mathcal{T})$ is a locally convex Riesz space, the lattice operations are $\mathcal{T}$-uniformly continuous.

<u>Proof</u>: Since $\inf(x,y) = -\sup(-x, -y)$, it is sufficient to show this for the "sup" operation. Let $\mathcal{U}$ be a $\mathcal{T}$-neighborhood basis of zero consisting of solid sets. Let $V \in \mathcal{U}$. Then there exists $U \in \mathcal{U}$ such that $U + U \subset V$. Suppose $x - x_o \in U$ and $y - y_o \in U$. Then $|\sup(x,y) - \sup(x_o, y_o)| \leq |\sup(x,y) - \sup(x_o, y)| +$

$$+ |\sup(x_o, y) - \sup(x_o, y_o)| \leq |x - x_o| + |y - y_o| .$$

Now $|x - x_o|$ and $|y - y_o| \in U$, since U is solid, and thus $|x - x_o| + |y - y_o| \in V$. Since V is solid, $\sup(x, y) - \sup(x_o, y_o) \in V$. Thus the "sup" operation is uniformly continuous.

We conclude this section with a fundamental lemma. It is proved by Luxemburg and Zaanen for Riesz normed spaces in [4].

<u>1.4</u> <u>Lemma</u> Let $(L, \mathcal{T})$ be a locally convex Riesz space. If $\{x_\lambda : \lambda \in \Lambda\}$ is a net in L, $x_\lambda \uparrow$, and $x_\lambda \to y$ $(\mathcal{T})$, $y \in L$, then $y = \sup\{x_\lambda : \lambda \in \Lambda\}$.

<u>Proof</u>: Fix $\lambda \in \Lambda$. If $\lambda \leq \lambda'$, then $|x_\lambda - \inf(x_\lambda, y)| = |\inf(x_\lambda, x_{\lambda'}) - \inf(x_\lambda, y)| \leq |x_{\lambda'} - y|$. By the theorem in 1.3 $|x_{\lambda'} - y| \to 0$ $(\mathcal{T})$, and thus by 1.1 (ii), $x_\lambda = \inf(x_\lambda, y)$ or $x_\lambda \leq y$ for each $\lambda \in \Lambda$. Suppose z is another upper bound, $z \leq y$. Then $|z - x_\lambda| \leq |y - x_\lambda|$. Again by 1.1 (ii)

$|z - x_\lambda| \to 0$ $(\mathcal{T})$ or $x_\lambda \to z$ $(\mathcal{T})$. Since $\mathcal{T}$ is Hausdorff $z = y$.

2. Dual Spaces

If L is a Riesz space, let $L^{\sim}$ be the space of all order-bounded linear functionals (cf. [4]). The space $L^{\sim}$ is again a Riesz space under the ordering $0 \leq \phi$ if and only if $\phi(u) \geq 0$ for every $0 \leq u \in L$. Indeed $L^{\sim}$ is Dedekind complete. Now if L possesses a locally convex Riesz topology $\mathcal{T}$, let $(L, \mathcal{T})^*$ be the topological dual space. These two dual spaces are related as follows:

2.1 Theorem If $(L, \mathcal{T})$ is a locally convex Riesz space, then $(L, \mathcal{T})^*$ is an ideal of $L^{\sim}$, and hence $(L, \mathcal{T})^*$ is Dedekind complete.

Proof: Let $\phi \in (L, \mathcal{T})^*$. Then ϕ is bounded on a solid neighborhood U of zero. To show $\phi \in L^{\sim}$, we need only show that for any $0 \leq u \in L$, ϕ is bounded on $\{x \in L: |x| \leq u\}$. There exists a real number $a > 0$ such that $u \in aU$. Since U is solid $\{x \in L: |x| \leq u\} \subset aU$ and ϕ is bounded on $\{x \in L: |x| \leq u\}$. Thus $(L, \mathcal{T})^* \subset L^{\sim}$.

Now we show that $\phi \in (L, \mathcal{T})^*$ implies $|\phi| \in (L, \mathcal{T})^*$. Let $\phi \in (L, \mathcal{T})^*$ and U be a solid neighborhood of zero such that $x \in U$ implies $|\phi(x)| \leq 1$. We know then that $|\phi|(|x|) =$ $= \sup(|\phi(u)| : 0 \leq u \leq |x|) \leq 1$. Since $||\phi|(x)| \leq |\phi|(|x|)$, $|\phi|$ is bounded by 1 on U and thus $|\phi| \in (L, \mathcal{T})^*$.

Now if $\phi \in (L, \mathcal{T})^*$ and $\psi \in L^{\sim}$, $0 \leq |\psi| \leq |\phi|$, and if $|\phi|$ is bounded on the solid neighborhood U by 1, as above, then $x \in U$ implies $|\psi(x)| \leq |\psi|(|x|) \leq |\phi|(|x|) \leq 1$ and $\psi \in (L, \mathcal{T})^*$. Thus $(L, \mathcal{T})^*$ is an ideal of $L^{\sim}$ and, since $L^{\sim}$ is

Dedekind complete, $(L, \mathcal{T})^*$ is Dedekind complete.

Let L be a Riesz space and $0 \leq \phi \in L$. Then we may define a Riesz seminorm ρ_ϕ on L by $\rho_\phi(x) = \phi(|x|)$. A collection of Riesz seminorms Ω is said to <u>generate</u> a locally convex Riesz topology if the collection of sets $\{\{x: \rho(x) \leq 1\}: \rho \in \Omega\}$ is a neighborhood basis for such a topology.

In the next two theorems we characterize locally convex Riesz topologies on a given Riesz space L with the same topological dual. Here if F is an ideal of $L^\sim$, $F^\circ = \{x \in L: |\phi(x)| \leq 1$ for every $\phi \in F\}$ and $\sigma(F, L)$ is the weak topology on F determined by L.

<u>2.2</u> <u>Theorem</u> Let L be a Riesz space and F an ideal of $L^\sim$ such that $F^\circ = \{0\}$. Then

(i) $\{\rho_\phi : 0 \leq \phi \in F\}$ generates a locally convex Riesz topology on L, call it $|\sigma|(L, F)$.

(ii) $(L, |\sigma|(L, F))^* = F$.

(iii) The topology of uniform convergence on $\sigma(F, L)$ - compact, convex, solid subsets of F generates a locally convex Riesz topology on L, call it $|m|(L, F)$.

<u>Proof</u>: (i) To show this we note that if $0 \leq \phi_1, \phi_2 \in L^\sim$ then $\{x: \rho_{\phi_1}(x) \leq 1\} \cap \{x: \rho_{\phi_2}(x) \leq 1\} \supset \{x: \rho_{\sup(\phi_1, \phi_2)}(x) \leq 1\}$. Also if $\rho_\phi(x) = 0$ for every $\phi \in F$, then $x = 0$ since $F^\circ = \{0\}$. Thus $|\sigma|(L, F)$ is a locally convex Riesz topology.

(ii) Now if $\phi \in F$, then $|\phi(x)| \leq \rho_{|\phi|}(x)$ for every $x \in L$

and $\phi \in (L, |\sigma|(L,F))^*$. If $\phi \in (L, |\sigma|(L, F))^*$ then there exists $0 \leq \psi \in F$ and a real $k > 0$, $|\phi(x)| \leq k\rho_\psi(x)$. For $0 \leq u \in L$, $|\phi|(u) = \sup(|\phi(v)| : 0 \leq v \leq u) \leq k\,\psi(u)$. Thus $|\phi| \leq k\psi$ and $\phi \in F$, since F is an ideal.

(iii) Let $\mathscr{S}$ be the collection of all $\sigma(F, L)$ compact, convex, solid subsets of F. Now if $0 \leq \phi \in F$ and $V_\phi = \{x \in L: \phi(|x|) \leq 1\}$, then $(V_\phi)^{\circ_F}$ is convex and compact by (ii) and the Šmulian compactness criterion. In addition V_ϕ is solid. To see this let $\psi \in V_\phi{}^\circ$ and $x \in V_\phi$. Then $||\psi|(x)| \leq |\psi|(|x|) = \sup(|\psi(u)| : 0 \leq u \leq x) \leq 1$, since V_ϕ is solid, and $|\psi| \in V_\phi{}^\circ$. Now if $|\theta| \leq |\psi|$ and $x \in V_\phi$, $|\theta(x)| \leq |\theta|(|x|) \leq |\psi|(|x|) \leq 1$ and $\theta \in V_\phi$. Thus V_ϕ is solid and $\mathscr{S} \supset \{V_\phi : 0 \leq \phi \in F\}$.

Now a neighborhood basis of zero for $|m|(L, F)$ consists of $\{S_1{}^\circ \cap S_2{}^\circ \cap \ldots \cap S_n{}^\circ : S_1, S_2, \ldots, S_n \in \mathscr{S}\}$. Again since $S \in \mathscr{S}$ implies S is solid, S° is also solid and hence the neighborhood basis consists of solid, convex sets. That $|m|(L, F)$ is Hausdorff follows since it is stronger than $|\sigma|(L, F)$.

<u>2.3</u> <u>Theorem</u> Let L be a Riesz space and F an ideal in $L^\sim$ such that $F^\circ = \{0\}$. Then if $\mathcal{T}$ is a locally convex Riesz topology on L, $(L,\mathcal{T})^* = F$ if and only if $|\sigma|(L, F) \leq \mathcal{T} \leq |m|(L, F)$.

<u>Proof</u>: Suppose $(L,\mathcal{T})^* = F$ and let Λ be a family of Riesz seminorms generating $\mathcal{T}$. Let $0 \leq \phi \in F$. Then there exists $\rho \in \Lambda$ and a real number $a > 0$ such that $|\phi(x)| \leq a\,\rho(x)$,

for every $x \in L$. Then $\{x: \phi(x) \leq 1\} \supset \{x: \rho(x) \leq \frac{1}{a}\}$ and $|\sigma|(L, F) \leq \mathcal{T}$.

Now let U be a $\mathcal{T}$-closed, convex, solid neighborhood of zero. Then U° is $\sigma(F, L)$ - compact, convex, and solid. Since $U = U^{\circ\circ}$, U is an $|m|(L, F)$ neighborhood of zero and $\mathcal{T} \leq |m|(L, F)$.

Suppose $|\sigma|(L, F) \leq \mathcal{T} \leq |m|(L, F)$. Let $\phi \in F$. By (ii) of the previous theorem ϕ is $|\sigma|(L, F)$ - continuous and hence $\mathcal{T}$-continuous. Suppose $\phi \in (L, \mathcal{T})^*$. Then there exists a $\mathcal{T}$-closed, convex, solid $\mathcal{T}$-neighborhood U of zero on which ϕ is bounded. Since $\mathcal{T} \leq |m|(L, F)$ there exists a $\sigma(F,L)$ - compact, convex, solid set S such that $S^\circ \subset U$. Then ϕ is bounded on S° and, again by Šmulian's criterion, $\phi \in F$.

We have the following simple corollary:

<u>2.4</u> <u>Corollary</u> A Riesz space L possesses a locally convex Riesz topology if and only if $(L^\sim)^\circ = \{0\}$.

<u>Proof</u>: If $(L^\sim)^\circ = \{0\}$, the topology $|\sigma|(L, L^\sim)$ is a locally convex Riesz topology.

If $(L, \mathcal{T})$ is a locally convex Riesz space and $0 \neq x \in L$, then there exists a $\mathcal{T}$-continuous Riesz seminorm ρ such that $\rho(x) \neq 0$. In this case Luxemburg and Zaanen [4] have shown that there exists $0 \leq \phi \in L^\sim$ such that $\phi(x) \neq 0$, i.e. $(L^\sim)^\circ = \{0\}$.

Next we prove a useful lemma.

<u>2.5</u> <u>Lemma</u> Let (L,τ) be a locally convex Riesz space. If $\{u_\lambda\}$ is a net, $0 \le u_\lambda \uparrow$, and $u_\lambda \to u$ (weakly), then $u_\lambda \to u$ (τ) and $u = \sup_\lambda u_\lambda$.

<u>Proof</u>: Let $\mathcal{U}$ be a τ-neighborhood basis for zero composed of τ-closed, solid, convex sets. Designate $(L,\tau)^*$ by L^*. Suppose $0 \le u_\lambda \uparrow$ and $u_\lambda \to u$ (weakly). Then if $U \in \mathcal{U}$, U^0 is $\sigma(L^*, L)$ - compact, convex, and solid. Let $H(U) = \{\phi \in U^0 : \phi \ge 0\}$. Since $\{\phi : 0 \le \phi \in L^*\}$ is $\sigma(L^*, L)$ closed, $H(U)$ is also $\sigma(L^*, L)$ compact. Now considering u_λ and u as continuous functions on $H(U)$, we have $(u - u_\lambda)\downarrow$, $(u - u_\lambda) \to 0$ (pointwise) and thus $(u - u_\lambda) \to 0$ (uniformly). So there exists λ_0 such that $\phi \in U^0$ implies, for $\lambda \ge \lambda_0$,

$$|\phi(u - u_\lambda)| \le |\phi|(u - u_\lambda) \le 1, \text{ i. e., } (u - u_\lambda) \in U^{00} = U$$

and thus $u_\lambda \to u$ (τ). That $u = \sup_\lambda u_\lambda$ follows from 1.4.

Now let L be a Riesz space. If $\phi \in L^\sim$, ϕ is called an <u>integral</u> if for any sequence $u_n \downarrow 0$ in L, $\phi(u_n) \to 0$. The set of all integrals on L is designated by $L_c^\sim$ and is a band (normal subspace) of $L^\sim$. If $\phi \in L^\sim$ and for any directed system $u_\lambda \downarrow 0$, $\inf_\lambda |\phi(u_\lambda)| = 0$, then ϕ is called a <u>normal integral</u>. The set of all normal integrals on L is designated by $L_n^\sim$ and is also a band of $L^\sim$. The following theorem gives a characterization of locally convex Riesz spaces such that every continuous linear functional is an integral and those such that every continuous linear functional is a normal integral.

2.6 Theorem Let $(L, \mathcal{T})$ be a locally convex Riesz space. Then

(i) $(L, \mathcal{T})^* \subset L_c^\sim$ if and only if for every sequence $u_n \downarrow 0$ in L, $u_n \to 0$ $(\mathcal{T})$;

(ii) $(L, \mathcal{T})^* \subset L_n^\sim$ if and only if for every net $u_\lambda \downarrow 0$ in L, $u_\lambda \to 0$ $(\mathcal{T})$.

Proof: We shall prove (ii). The proof of (i) is similar. Suppose that for any net $u_\lambda \downarrow 0$ in L, $u_\lambda \to 0$ $(\mathcal{T})$. A directed system $u_\lambda \downarrow 0$ may be considered as a net under the Riesz space ordering. Then $u_\lambda \to 0$ $(\mathcal{T})$ and $\inf_\lambda |\phi(u_\lambda)| = 0$ for any $\phi \in (L, \mathcal{T})^*$. Thus $(L, \mathcal{T})^* \subset L_n^\sim$.

Suppose $(L, \mathcal{T})^* \subset L_n^\sim$ and $\{u_\lambda\}$ is a net such that $u_\lambda \downarrow 0$. Now if $\phi \in (L, \mathcal{T})^*$, $\phi(u_\lambda) \to 0$ and $u_\lambda \to 0$ (weakly). But then by 2.5, $u_\lambda \to 0$ $(\mathcal{T})$.

Paralleling the notation introduced by Luxemburg and Zaanen [4], we say that a locally convex Riesz space $(L, \mathcal{T})$ has property (A, i) if $(L, \mathcal{T})^* \subset L_c^\sim$ and property (A, ii) if $(L, \mathcal{T})^* \subset L_n^\sim$. (These spaces are called o - continuous and M. S. o - continuous respectively by Kawai [2].) If a locally convex Riesz space has property (A, ii), order convergence implies topological convergence. For this reason many special theorems may be proved about them, (cf. section 13).

3. Topological Boundedness

We begin this section with a lemma on topologically bounded sets.

3.1 Lemma Let $(L, \mathcal{T})$ be a locally convex Riesz space. If A is a $\mathcal{T}$-bounded subset of L, then there exists a solid set B such that $A \subset B$ and B is $\mathcal{T}$-bounded.

Proof: Let $B = \{x \in L: \text{ for some } y \in A,\ 0 \leq |x| \leq |y|\}$. Certainly B is solid and $A \subset B$. To show that B is $\mathcal{T}$-bounded, we have only to show that B is absorbed by solid $\mathcal{T}$-neighborhoods of zero. Let U be such a neighborhood. Then for some real $a > 0$ $A \subset aU$. If $x \in B$, for some $y \in A$, $|x| \leq |y|$. Since $y \in aU$ and aU is solid, $x \in aU$, i. e., $B \subset aU$. Thus B is $\mathcal{T}$-bounded.

If $(L, \mathcal{T})$ is a locally convex Riesz space, let $(L, \mathcal{T})^b$ be the space of linear functionals on L which are bounded on $\mathcal{T}$-bounded sets. We have:

3.2 Theorem $(L, \mathcal{T})^b$ is an ideal in $L^{\sim}$.

Proof: Since order intervals are $\mathcal{T}$-bounded, $(L, \mathcal{T})^b \subset L^{\sim}$. Let $\phi \in (L, \mathcal{T})^b$ and $\psi \in L$ such that $|\psi| \leq |\phi|$. By 3.1 to show that $\psi \in (L, \mathcal{T})^b$ it is sufficient to show that ψ is bounded on solid, $\mathcal{T}$-bounded sets. Let B be such a set. If $x \in B$

$$|\psi(x)| \leq |\psi|(|x|) \leq |\phi|(x) = \sup(|\phi(u)|: 0 \leq u \leq x) \leq$$
$$\leq \sup(|\phi(y)| : y \in B).$$

Thus ψ is bounded on B and $\psi \in (L, \mathcal{T})^b$.

Now from a given locally convex Riesz topology $\mathcal{T}$ on a Riesz space L, we can construct a stronger locally convex Riesz topology $\mathcal{T}_b$ by letting a neighborhood basis of zero consist of all convex, solid subsets of L which absorb every $\mathcal{T}$-bounded set.

3.3 Theorem Let $\mathcal{T}$ be a locally convex Riesz topology on a Riesz space L. Then, designating $(L,\mathcal{T})^b$ by L^b,

(i) $\mathcal{T}_b = |m|(L, L^b) = m(L, L^b)$ where $m(L,L^b)$ is the Mackey topology;

(ii) $(L,\mathcal{T}_b)^* = L^b$.

Proof: We have only to show that any convex subset A of L which absorbs every $\mathcal{T}$-bounded subset of L contains a solid, convex set B which absorbs every $\mathcal{T}$-bounded set. Then $\mathcal{T}_b = m(L,L^b)$ and the other results follow immediately.

Let $B = \{x \in A: \{y: |y| \leq |x|\} \subset A\}$. By 3.1 it is sufficient to show that B absorbs solid $\mathcal{T}$-bounded sets. Let S be such a set. Then if $aA \supset S$ it is easy to see that $aB \supset S$.

As a corollary we obtain a result of Namioka [7].

3.4 Corollary Let L be a Riesz space such that $(L^\sim)^\circ = \{0\}$. Then $m(L,L^\sim)$ is the strongest locally convex Riesz topology on L.

Proof: We have $L^\sim = (L,|\sigma|(L, L^\sim))^* \subset (L, |\sigma|(L, L^\sim))^b \subset L^\sim$. Then $(L, |\sigma|(L, L^\sim))^b = L^\sim$ and the result follows from 3.3.

4. Relative Uniform Convergence

We now introduce one of the central concepts of the present work. A sequence $\{x_n\}$ of elements of a Riesz space L is said to converge to $x \in L$ relatively uniformly, written $x_n \to x$ (r. u.), if there exists $0 \leq v \in L$ such that for every real $r > 0$ there exists a real number N (r) and $n \geq N(r)$ implies $|x - x_n| \leq rv$. If S is a subset of L, then designate by S'^{ru} the set $\{x \in L$: for some sequence $\{x_n\}$ in S, $x_n \to x$ (r.u.)$\}$. A set S is said to be relatively uniformly closed if $S = S'^{ru}$.

In general relative uniform convergence does not correspond to topological convergence, i. e., there are sets S in L such that S'^{ru} is not relatively uniformly closed (cf. Chapter 10). We do have the following:

4.1 Theorem Let L be a Riesz space.

(i) The strongest locally convex topology $\mathcal{T}$ on L such that $x_n \to 0$ (r. u.) implies $x_n \to x$ $(\mathcal{T})$ is generated by all the Riesz seminorms on L.

(ii) If $(L^{\sim})^{\circ} = \{0\}$, this topology is a locally convex Riesz topology, m (L, $L^{\sim}$).

Proof: Certainly if ρ is a Riesz seminorm, $x_n \to 0$ (r. u.) implies $\rho(x_n) \to 0$. If ρ is an arbitrary seminorm such that $x_n \to 0$ (r. u.) implies $\rho(x_n) \to 0$, then we may define a Riesz seminorm by $\tilde{\rho}(x) = \sup\, (\rho(u)\colon 0 \leq u \leq |x|)$. If for some element x, $(\rho(u)\colon 0 \leq u \leq |x|)$ were not bounded, we would have a sequence

$\{u_n\}$ such that $0 \leq u_n \leq |x|$ and $\rho(u_n) \geq n$ for each $n = 1, 2, \ldots$. But then $\frac{1}{n} u_n \to 0$ (r. u.) and $\rho(u_n) \not\to 0$. This proves (i), and (ii) follows from 3.4.

We may also discuss completeness with respect to relative uniform convergence. A sequence $\{x_n\}$ in a Riesz space L is said to be a <u>relative uniform Cauchy sequence</u> if for some $0 \leq v \in L$ and every real $r > 0$ there exists $N(r)$ and $n, m \geq N(r)$ implies $|x_n - x_m| \leq r\,v$. The space L is said to be <u>relatively uniformly complete</u> if for every relative uniform Cauchy sequence $\{x_n\}$ in L there exists $x \in L$ such that $x_n \to x$ (r. u.).

II LOCAL COMPLETENESS

5. Definition

Let $(L, \mathcal{T})$ be a locally convex Riesz space. Then a sequence $\{x_n\}$ in L is said to converge $\mathcal{T}$-locally to $x \in L$, written $x_n \to x$ ($\mathcal{T}$-local), if there exists a $\mathcal{T}$-bounded set B such that for every real $r > 0$ there exists $N(r)$ and $n \geq N(r)$ implies $|x - x_n| \in rB$. (This concept is introduced in a somewhat more general setting by H. Gordon [1].) Local convergence is very similar to relative uniform convergence. In fact, if we replace the requirement that B be $\mathcal{T}$-bounded by order bounded in the above definition, we obtain the definition of relative uniform convergence.

It is convenient at this point to extend the concept of a Riesz norm. Let $\overline{R}^+$ be the set of extended non-negative real numbers, i. e., $\{r: r \text{ is real and } r \geq 0\} \cup \{\infty\}$. This set is given the obvious ordering and the algebraic operations are extended in the following manner:

(i) for any real $r \geq 0$, $r + \infty = \infty$, and $\infty + \infty = \infty$,

(ii) for any real $r > 0$, $r\infty = \infty$, but $0\infty = 0$.

If L is a Riesz space and λ is a map from L to $\overline{R}^+$, then λ is called an extended Riesz norm if

(i) $\lambda(x) = 0$ implies $x = 0$,

(ii) for any real r and $x \in L$, $\lambda(rx) = |r| \lambda(x)$,

(iii) for every $x, y \in L$, $\lambda(x+y) \leq \lambda(x) + \lambda(y)$,

(iv) if $x, y \in L$ and $|x| \leq |y|$, then $\lambda(x) \leq \lambda(y)$.

If λ is an extended Riesz norm on a Riesz space L, let $L_\lambda = \{x \in L: \lambda(x) < \infty\}$. Then we may easily verify that L_λ is an ideal of L and λ is a Riesz norm on L_λ.

Now the $\mathcal{T}$-bounded set B in the definition of local convergence may be selected so as to be convex and solid. We call $\mathcal{T}$-bounded, convex, solid sets <u>$\mathcal{T}$-admissible</u>. If B is such a set we may define an extended Riesz norm λ by

$\lambda(x) = \inf\{r: 0 < r$ is real and $\frac{1}{r}|x| \notin B\}$ ($= \infty$, if for every real $r > 0$, $\frac{1}{r}|x| \notin B$). Extended Riesz norms defined in this manner are called <u>$\mathcal{T}$-admissible extended Riesz norms</u>. If, in addition, the set B is $\mathcal{T}$-closed, then λ is called a <u>$\mathcal{T}$-closed extended Riesz norm</u>.

<u>5.1</u> <u>Lemma</u> Let $(L, \mathcal{T})$ be a locally convex Riesz space.

(i) An extended Riesz norm λ is $\mathcal{T}$-admissible if and only if for every $\mathcal{T}$-continuous Riesz seminorm ρ there exists a real number a_ρ such that $\rho(x) \leq a_\rho \lambda(x)$ for every $x \in L$.

(ii) A sequence $x_n \to x$ ($\mathcal{T}$-local) if and only if for some $\mathcal{T}$-closed extended Riesz norm λ, $\lambda(x - x_n) \to 0$.

<u>Proof</u>: (i) Suppose λ is $\mathcal{T}$-admissible, defined on the $\mathcal{T}$-admissible set B. Then if ρ is $\mathcal{T}$-continuous Riesz seminorm there exists $a > 0$ such that $\rho(x) \leq a$ for every $x \in B$. It follows that $\rho(x) \leq a\lambda(x)$ for every $x \in L$.

If for every $\mathcal{T}$-continuous Riesz seminorm ρ, there exists a_ρ such that $\rho(x) \leq a_\rho \lambda(x)$, then the set $B = \{x: \lambda(x) \leq 1\}$

is a τ-admissible set which determines λ. Thus λ is τ-admissible.

(ii) To see this we need only note that the τ-closure of a τ-admissible set is again τ-admissible.

As an example we consider order intervals. Let (L,τ) be an arbitrary locally convex Riesz space and $0 < u \in L$. Then since order intervals are τ-bounded and τ-closed, λ_u defined by $\lambda_u(x) = \inf\{r\colon ru \geq |x|\}$, is a τ-closed extended Riesz norm. In this case the ideal L_{λ_u} is the principal ideal generated by u in L -- $\{x \in L\colon |x| \leq mu \text{ for some real } m\}$.

Now we consider completeness with respect to τ-local convergence. A sequence $\{x_n\}$ in (L,τ) is said to be a <u>τ-local Cauchy</u> sequence if for some τ-bounded set B and every $r > 0$ there exists $N(r)$ and $m, n \geq N(r)$ implies $x_m - x_n \in rB$. The space L is said to be <u>τ-locally complete</u> if for every τ-local Cauchy sequence $\{x_n\}$ there exists $x \in L$ such that $x_n \rightarrow x$ (τ), (and thus since we may assume B is τ-closed, $x_n \rightarrow x$ (τ-local)).

6. $\mathcal{T}$-local Completeness

For Riesz normed Riesz spaces, local convergence and topological convergence agree. Now if $0 \leq u_n \in L$ for $n = 1, 2, \ldots$, and there exists $u \in L$ such that $u = \sup_n \{ \sum_{k=1}^{n} u_k : 1, 2, \ldots \}$, designate u by $\sum u_n$. Using this notation we state below a theorem of Luxemburg and Zaanen [4].

6.1 Theorem Let L be a Riesz space and p be a Riesz norm on L. Then the following conditions are equivalent.

(i) L is p - complete.

(ii) If $0 \leq u_n \in L$ for $n = 1, 2, \ldots$, and $\sum p(u_n) < \infty$, then $\sum u_n$ exists and $p(\sum u_n) \leq \sum p(u_n)$.

(iii) If $0 \leq u_n \in L$ for $n = 1, 2, \ldots$, and $\sum p(u_n) < \infty$, then $\sum u_n$ exists.

We generalize this to the case of local completeness in the following theorem.

6.2 Theorem Let $(L, \mathcal{T})$ be a locally convex Riesz space. Then the following conditions are equivalent.

(i) L is $\mathcal{T}$-locally complete.

(ii) For every $\mathcal{T}$-closed extended Riesz norm λ, L_λ is λ-complete.

(iii) If $0 \leq u_n \uparrow$ and $\{u_n\}$ is a $\mathcal{T}$-local Cauchy sequence then $\sup_n u_n$ exists.

(iv) If for any $\mathcal{T}$-closed extended Riesz norm λ and

$0 \leq u_n \in L$, $n = 1, 2, \ldots$, $\sum \lambda(u_n) < \infty$, then $\sum u_n$ exists.

(v) L is relatively uniformly complete; and if for any $\mathcal{T}$ - closed extended Riesz norm λ and $0 \leq u_n \in L$, $n = 1, 2, \ldots$, $\sum \lambda(u_n) < \infty$, then there exists $v \in L$ such that $u_n \leq v$, $n = 1, 2, \ldots$.

Proof: ((i) implies (ii)) Let λ be a $\mathcal{T}$-closed extended Riesz norm and $\{x_n\}$ a λ-Cauchy sequence in L_λ. Then $\{x_n\}$ is a $\mathcal{T}$-local Cauchy sequence in L and, by (i), for some $x \in L$, $x_n \to x$ $(\mathcal{T})$. Since λ is $\mathcal{T}$-closed $\lambda(x - x_n) \to 0$ and $x \in L_\lambda$.

((ii) implies (iii)) If $0 \leq u_n \uparrow$ and $\{u_n\}$ is a $\mathcal{T}$-local Cauchy sequence, then for some $\mathcal{T}$-closed λ and integer n_o, $\{u_n - u_{n_o}\}$ is a λ-Cauchy sequence in L_λ. Then for some $v \in L_\lambda$, $\lambda(u_n - u_{n_o} - v) \to 0$ and $u_n \to v + u_{n_o}$ ($\mathcal{T}$-local). By 1.4 $v + u_{n_o} = \sup_n u_n$.

((iii) implies (iv)) This is immediate since $\sum \lambda(u_n) < \infty$ implies $\{u_1 + \ldots + u_n\}$ is a $\mathcal{T}$-local Cauchy sequence.

((iv) implies (v)) If for some $\mathcal{T}$-closed λ, $\sum \lambda(u_n) < \infty$, then by (iv) $\sum u_n$ exists and $u_n \leq \sum u_n$ for every $n = 1, 2, \ldots$.

Now let $\{x_n\}$ be a relative uniform Cauchy sequence. Then for some $0 \leq v \in L$, $\{x_n\}$ is a λ_v Cauchy sequence, where $\lambda_v(x) = \inf\{r : rv \geq |x|\}$. Pick a subsequence $\{y_n\}$ such that $\lambda_v(y_{n+1} - y_n) \leq 2^{-n}$. Hence $\sum \lambda_v(y_{n+1} - y_n) < \infty$. Then since we have shown that λ_v is a $\mathcal{T}$-closed extended Riesz norm, by (iv), $w = \sum_{n=1}^{\infty} |y_{n+1} - y_n|$ exists. In addition, for each n $|y_{n+1} - y_n| + (y_{n+1} - y_n) \geq 0$ and

$$\lambda_v(|y_{n+1} - y_n| + (y_{n+1} - y_n)) \leq 2\lambda(|y_{n+1} - y_n|),\ n = 1, 2, \ldots .$$

Then again by (iv), $z = \sum_{n=1}^{\infty} |y_{n+1} - y_n| + (y_{n+1} - y_n)$ exists.

Now $\sum_{k=n+1}^{m} |y_{k+1} - y_k| \uparrow_m w - \sum_{k=1}^{n} |y_{k+1} - y_k|$. But

$$\sum_{k=n+1}^{m} |y_{k+1} - y_k| \leq \sum_{k=n+1}^{m} 2^{-k} v \leq 2^{-n} v.$$ Thus

$$\left| w - \sum_{k=1}^{n} |y_{k+1} - y_k| \right| \leq 2^{-n} v \quad \text{and} \quad \sum_{k=1}^{n} |y_{k+1} - y_k| \rightarrow w \ (\text{r. u.}).$$

We have similarly $\left| z - \left[\sum_{k=1}^{n} |y_{k+1} - y_k| + (y_{k+1} - y_k) \right] \right| =$

$$= \left| z - w + y_1 - y_{n+1} - \left(w - \sum_{k=1}^{n} |y_{k+1} - y_k|\right) \right| \leq 2^{n-1} v.$$

Thus $y_n \rightarrow z - w + y_1$ (r. u.) and then $x_n \rightarrow z - w + y_1$ (r. u.). Hence L is relatively uniformly complete.

((v) implies (i)) Let $\{x_n\}$ be a $\mathcal{T}$-local Cauchy sequence. Then for some $\mathcal{T}$-closed λ, $\{x_n\}$ is a λ-Cauchy sequence. Pick a subsequence $\{y_n\}$ such that $\lambda(y_{n+1} - y_n) \leq n^{-4}$, $n = 1, 2, \ldots$. Now $\sum_{n=1}^{\infty} \lambda(n^2 |y_{n+1} - y_n|) < \infty$ and by (iv) there exists v such that $n^2 |y_{n+1} - y_n| \leq v$, $n = 1, 2, \ldots$. If $m \geq n$

$$|y_m - y_n| \leq \sum_{k=n}^{m-1} |y_{k+1} - y_k| \leq \left(\sum_{k=n}^{m-1} 2^{-k} \right) v.$$ Thus $\{y_n\}$ is a relative uniform Cauchy sequence. By (iv) there exists $y \in L$ such that $y_n \rightarrow y$ (r. u.). Then $y_n \rightarrow y$ ($\mathcal{T}$) and thus $x_n \rightarrow y$ ($\mathcal{T}$). Hence L is $\mathcal{T}$-locally complete.

<u>6.3 Corollary</u> Let $(L, \mathcal{T})$ be a locally convex Riesz space such that L is $\mathcal{T}$-locally complete. If $\mathcal{T}'$ is another locally convex Riesz topology on L such that $\mathcal{T}' \geq \mathcal{T}$, L is also $\mathcal{T}'$-locally complete.

<u>Proof</u>: Suppose for some $\mathcal{T}'$-closed extended Riesz norm λ and $0 \leq u_n \in L$, $n = 1, 2, \ldots$, $\sum \lambda(u_n) < \infty$. Then λ is also $\mathcal{T}$-closed and by 6.2 (iv) $\sum u_n$ exists. Hence, again by 6.2 (iv), L is $\mathcal{T}'$-locally complete.

We have also:

<u>6.4 Theorem</u> Let $(L, \mathcal{T})$ be a locally convex Riesz space such that L is $\mathcal{T}$-locally complete. Then $(L, \mathcal{T})^b = L^\sim$.

<u>Proof</u>: Suppose there exists $0 \leq \phi \in L^\sim$ such that $\phi \notin (L, \mathcal{T})^b$. Then for some $\mathcal{T}$-closed, $\mathcal{T}$-admissible set B, ϕ is unbounded on B. Let λ be the corresponding $\mathcal{T}$-closed extended Riesz norm. Then there exists $0 \leq u_n \in L$ such that $\lambda(u_n) \leq 1$ and $\phi(u_n) \geq n^3$ for $n = 1, 2, \ldots$. Then $\sum \lambda(n^{-2} u_n) < \infty$ and by 6.2 (v) there exists $v \in L$ such that $n^{-2} u_n \leq v$ for $n = 1, 2, \ldots$. Then $\phi(v) \geq n$ for $n = 1, 2, \ldots$, which is impossible. Thus $\phi \in (L, \mathcal{T})^b$ and $(L, \mathcal{T})^b = L^\sim$.

7. Applications to Bound and Metrizable Spaces

We recall that a locally convex space is <u>bound</u> if every convex, circled set which absorbs each topologically bounded set is a neighborhood of zero. It is said to be <u>barrelled</u> if closed, convex, circled sets which absorb every point are neighborhoods of zero. Kelley and Namioka have shown that if a bound space is sequentially complete, it is barrelled [3]. For locally convex Riesz spaces this may be improved.

<u>7.1</u> <u>Theorem</u> Let $(L, \mathcal{T})$ be a bounded locally convex Riesz space. If L is $\mathcal{T}$-locally complete, then it is a barrelled space and $\mathcal{T} = m\ (L, L^{\sim})$.

<u>Proof</u>: Since L is $\mathcal{T}$-locally complete, by 6.4, $(L, \mathcal{T})^b = L^{\sim}$. Now for a bound space $(L, \mathcal{T})^* = (L, \mathcal{T})^b$ and $\mathcal{T} = m\ (L, (L, \mathcal{T})^*)$. Thus $\mathcal{T} = m\ (L, L^{\sim})$.

Now we must show that if A is a $\mathcal{T}$-barrel (a $\mathcal{T}$-closed, convex, circled set which absorbs every point), then A absorbs any $\mathcal{T}$-bounded set B. We may assume that B is $\mathcal{T}$-closed, convex, and solid. Let λ be the corresponding $\mathcal{T}$-closed extended Riesz norm. Then by 6.2 (ii), we know that L_{λ} is λ-complete and thus of the second category in itself with respect to λ. Further let $\mathcal{T}_{\lambda}$ be the topology $\mathcal{T}$ restricted to L_{λ}. Then since $\mathcal{T}_{\lambda}$ is weaker than the λ-topology, L_{λ} is of the second category in itself with respect to $\mathcal{T}_{\lambda}$.

If we let $A_{\lambda} = A \cap L_{\lambda}$, then A_{λ} is $\mathcal{T}$-closed and by assumption $\bigcup\{n A_\lambda : n = 1, 2, \ldots\} = L_\lambda$. Then for some n, $n A_\lambda$ has a τ_λ interior point, and thus A_λ has a τ_λ interior point. Thus for some $x \in L_\lambda$ and some solid, convex, τ_λ-neighborhood of zero U, $A_\lambda \supset x + U$. Since A_λ and U are both circled, $A_\lambda \supset -x + U$. Now since A_λ and U are convex $A_\lambda \supset \frac{1}{2}(x+U) + \frac{1}{2}(-x+U) = \frac{1}{2}U + \frac{1}{2}U = U$. Finally since B is τ_λ bounded and A_λ contains a τ_λ neighborhood of zero, A_λ absorbs B. Thus A absorbs B and (L, τ) is barrelled.

<u>7.2</u> <u>Corollary</u> If (L, τ) is a bound locally convex Riesz space such that $0 \leq u_n \uparrow$, $\{u_n\}$ a τ-Cauchy sequence in L, implies $\sup_n u_n$ exists, then $\tau = m(L, L^\sim)$ and the space is barrelled.

<u>Proof:</u> By 6.2 (iii), L is τ-locally complete and the result follows from 7.1.

We now consider metrizable locally convex Riesz spaces. The following lemma shows that in this case τ-convergence and τ-local convergence agree.

<u>7.3</u> <u>Lemma</u> If (L, τ) is a metrizable locally convex Riesz space, then a τ-Cauchy sequence is also a τ-local Cauchy sequence.

<u>Proof:</u> If τ is metrizable there exists a countable set of Riesz seminorms $\rho_1 \leq \rho_2 \leq \ldots$, which generate τ. Let $\{x_n\}$ be a τ-Cauchy sequence. Then for each k,

$\alpha_k = \sup\{\rho_k(x_m - x_n): m, n = 1, 2, \ldots,\} < \infty$. Let $B = \{x: \rho_k(x) \leq k\alpha_k, k = 1, 2, \ldots\}$. Then B is

$\mathcal{T}$-bounded. Now given a real $a > 0$ pick an integer k_o such that $\frac{1}{k_o} < a$. Then there exists n_o such that $m, n \geq n_o$ implies $\rho_k (x_m - x_n) \leq a\ (k\alpha_k)$, $k = 1, 2, \ldots, k_o$. If $k > k_o$, we have $\rho_k (x_m - x_n) \leq \alpha_k \leq a\ (k\,\alpha_k)$. Thus $m, n \geq n_o$ implies $|x_m - x_n| \in aB$ and thus $\{x_n\}$ is a $\mathcal{T}$-local Cauchy sequence.

7.4 Theorem Let $(L, \mathcal{T})$ be a metrizable locally convex Riesz space. Then the following statements are equivalent.

(i) L is $\mathcal{T}$-complete.

(ii) L is $\mathcal{T}$-locally complete.

(iii) If $0 \leq u_n \uparrow$ and $\{u_n\}$ is a $\mathcal{T}$-Cauchy sequence, then $\sup_n u_n$ exists.

(iv) Let $\rho_1 \leq \rho_2 \leq \ldots$, be a countable family of Riesz seminorms generating $\mathcal{T}$. If $0 \leq u_n \in L$, $n = 1, 2, \ldots$, and $\sum \rho_n (u_n) < \infty$, then $\sum u_n$ exists.

Proof: The equivalence of (i) and (ii) follows from 7.3. Then (ii) is equivalent to (iii) by 7.3 and 6.2 (iii). Certainly (iii) implies (iv). We have only to show that (iv) implies (iii).

Suppose $\rho_1 \leq \rho_2 \leq \ldots$, is a countable family of Riesz seminorms generating $\mathcal{T}$ and $0 \leq u_n \uparrow$ where $\{u_n\}$ is a $\mathcal{T}$-Cauchy sequence. Then we may select a subsequence $0 \leq v_n \uparrow$ such that $\rho_n (v_{n+1} - v_n) \leq 2^{-n}$. Hence $\sum \rho_n (v_{n+1} - v_n) < \infty$ and $\sum (v_{n+1} - v_n) = \sup_n (v_n - v_1)$ exists. Then $\sup_n v_n = \sup_n u_n$ exists and (ii) holds.

We have the following corollary.

7.5 Corollary Let $(L, \mathcal{T})$ be a metrizable locally convex Riesz space. If for some weaker locally convex Riesz topology $\mathcal{T}'$, every monotone $\mathcal{T}'$ Cauchy sequence has a least upper bound, then $(L, \mathcal{T})$ is $\mathcal{T}$ - complete and $\mathcal{T} = m(L, L^{\sim})$.

Proof: By 6.2, the condition on $\mathcal{T}'$ implies L is $\mathcal{T}'$ - locally complete. Then by 6.3, L is $\mathcal{T}$ - locally complete and, by 7.4, $\mathcal{T}$ - complete. Since metrizable locally convex spaces are bound, 7.1 implies $\mathcal{T} = m(L, L\ \ )$.

We conclude this chapter with an example to show that a $\mathcal{T}$ - locally complete space is not necessarily sequentially complete.

7.6 Example Let L be the Riesz space of all sequences which converge to zero. Let p be the "sup" norm, i.e. $p(x) = \sup \{|x(k)| : k = 1, 2, \ldots\}$. It is well known that L is p-complete and $(L, p)^* \cong \ell_1$, where ℓ_1 is the Riesz space of all absolutely convergent series. Now by 7.5, the p - topology is $m(L, L^{\sim})$. Then by 3.4 and 2.3 $L^{\sim} = (L, p)^* \cong \ell_1$.

Let $\mathcal{T} = |\sigma|(L, L^{\sim})$. Since $L^{\sim} = (L, \mathcal{T})^* \subset (L, \mathcal{T})^b \subset L^{\sim}$, $(L, \mathcal{T})^b = (L, p)^b = L^{\sim}$, and $\mathcal{T}$ and p define the same bounded sets and hence determine the same local Cauchy sequences. Since L is p-complete, L is p-locally complete and thus $\mathcal{T}$ - locally complete.

Now let e_n be the sequence $e_n(n) = 1$, but $e_n(k) = 0$ for $k \neq n$, $n = 1, 2, \ldots$. Then $\left\{ \sum_{m=1}^{n} e_m \right\}$ is a τ-Cauchy sequence, which does not converge. Thus L is τ-locally complete, but not τ-sequentially complete.

III ARCHIMEDEAN QUOTIENT SPACES

8. Relatively Uniformly Closed Ideals

The concept of relative uniform convergence finds further application in the study of quotient spaces. If L is a Riesz space, A is an ideal of L, and π is the quotient map of L onto L/A, then L/A is again a Riesz space under the ordering -- $\pi(x) \geq 0$ if and only if $x + y \geq 0$ for some $y \in A$.

Ideals of a given Riesz space such that the quotient space is archimedean have been characterized by Luxemburg [5] in the following manner:

8.1 Theorem Let L be a Riesz space and let A be an ideal of L. Then the following conditions are equivalent.

(i) L/A is archimedean.

(ii) For all $0 \leq u, v \in L$, $(u - \frac{1}{n} v)^+ \in A$ for $n = 1, 2, \ldots$, then $u \in A$.

(iii) A is relatively uniformly closed.

Proof: We show (i) implies (ii). Suppose L/A is archimedean and for $0 \leq u, v \in L$, $(u - \frac{1}{n} v)^+ \in A$, $n = 1, 2, \ldots$. Then for each n, let $y_n = (u - \frac{1}{n} v)^+ = \sup (u - \frac{1}{n} v, 0)$, $n = 1, 2, \ldots$. Now if π is the quotient map of L onto L/A, $u \leq y_n + \frac{1}{n} v$ and $0 \leq \pi(u) \leq \frac{1}{n} \pi(v)$, $n = 1, 2, \ldots$. Since L/A is archimedean, $\pi(u) = 0$ and $u \in A$.

Now we show that (ii) implies (iii). Suppose $x \in A^{\prime ru}$. We may assume for some sequence $\{x_n\}$ in A and $0 \leq v \in L$,

$|x - x_n| \leq \frac{1}{n} v$ for $n = 1, 2, \ldots$. Now $|x| \leq |x - x_n| + |x_n| \leq \frac{1}{n} v + |x_n|$ and $|x| - \frac{1}{n} v \leq |x_n|$, $n = 1, 2, \ldots$. Thus $(|x| - \frac{1}{n} v)^+ \leq |x_n|$ and since A is an ideal $(|x| - \frac{1}{n} v)^+ \in A$, $n = 1, 2, \ldots$. Hence by (ii), $|x| \in A$ and also $x \in A$, and A is relatively uniformly closed.

To show that (iii) implies (i), assume A is relatively uniformly closed and let π be the quotient map of L onto L/A. Suppose for some $0 \leq u, v \in L$, $0 \leq n\pi(u) \leq \pi(v)$, $n = 1, 2, \ldots$. Then there exists $0 \leq w_n \in A$ and $u \leq \frac{1}{n} v + w_n$, for $n = 1, 2, \ldots$. By the lemma of Riesz we have $u = u_n + z_n$, where $0 \leq u_n \leq \frac{1}{n} v$ and $0 \leq z_n \leq w_n$, $n = 1, 2, \ldots$. Then the sequence $\{z_n\} \subset A$ and $|u - z_n| \leq \frac{1}{n} v$ or $z_n \rightarrow u$ (r. u.). Thus by (iii), $u \in A$ and L/A is archimedean.

9. Non-archimedean Riesz Spaces

If a Riesz space L is non-archimedean, then there exist some $0 < u \in L$ and $0 \leq v \in L$ such that $nu \leq v$ for $n = 1, 2, \ldots$. For an arbitrary Riesz space L we define $I(L) = \{x \in L:$ for some $0 \leq v \in L,\ n|x| \leq v,\ n = 1, 2, \ldots\}$. It is easy to see that $I(L)$ is an ideal and L is archimedean if and only if $I(L) = \{0\}$.

Again let L be an arbitrary Riesz space. If A is an ideal of L and L/A is archimedean, or equivalently, A is relatively uniformly closed, then it is obvious that $A \supset I(L)$. Now let $\mathfrak{A}(L) = \{A: A$ is an ideal of L and L/A is archimedean$\}$. Then certainly $L \in \mathfrak{A}$. Let $A_0(L) = \bigcap\{A: A \in \mathfrak{A}(L)\}$. We have:

9.1 Theorem Let L be an arbitrary Riesz space. Then $A_0(L) \in \mathfrak{A}(L)$ and $A_0(L) \supset I(L)$.

Proof: Certainly $A_0(L) \supset I(L)$. Further, since it is obtained as an intersection of relatively uniformly closed ideals, it is both an ideal and relatively uniformly closed. Thus $A_0(L) \in \mathfrak{A}$.

It is not true that $I(L) = A_0(L)$ for all Riesz spaces. We shall give an example in Chapter 10.

First we examine Riesz spaces which are themselves obtained as quotient spaces.

<u>9.2</u> <u>Theorem</u> Let E be a Riesz space and B be an ideal in E. Let $L = E/B$ and π be the natural map of E onto L. If A is an ideal of L, then $B(A) = \{x \in E: \pi(x) \in A\}$ is an ideal of E and $L/A \cong E/B(A)$.

<u>Proof</u>: Certainly $B(A)$ is an ideal. Let η_E be the natural map of E onto $E/B(A)$ and η_L be the natural map of L onto L/A. Then the desired isomorphism μ is given by

$$\mu(\eta_E(x)) = \eta_L(\pi(x)).$$

<u>9.3</u> <u>Theorem</u> Let E be a Riesz space and B be an ideal in E. Let $L = E/B$. Then $I(L) = A_o(L)$ if and only if B'^{ru} is relatively uniformly closed.

<u>Proof</u>: Using the notation of the previous theorem,

$$B[I(L)] = \{x \in E: \pi(x) \in I(L)\} = \{x \in E: \text{ for some } 0 \leq v \in E, |\pi(x)| \leq \tfrac{1}{n}\pi(v), n = 1, 2, \ldots\} = \{x \in E: \text{ for some } v \geq 0$$
$$\text{and sequence } \{b_n\} \subset B, |x - b_n| \leq \tfrac{1}{n} v, n = 1, 2, \ldots\} = B'^{ru}.$$

Now since by 9.1 $A_o(L) \supset I(L)$, $A_o(L) = I(L)$ if and only if $L/I(L)$ is archimedean. By 9.2 this is equivalent to E/B'^{ru} being archimedean. Finally by 8.1 this holds if and only if B'^{ru} is relatively uniformly closed.

<u>9.4</u> <u>Corollary</u> Let E be a Riesz space with a strong unit or a Riesz space possessing a complete, metrizable, locally convex Riesz topology. Then if B is an ideal of E and $L = E/B$, $I(L) = A_o(L)$.

Proof: We have to show that B'^{ru} is relatively uniformly closed. In the case that E has a strong unit, this is obvious.

Let $\mathcal{T}$ be a complete, metrizable topology on E and B an ideal in E. Then we claim $B'^{ru} = \overline{B}^{\mathcal{T}}$. From this it follows that B'^{ru} is relatively uniformly closed.

Certainly $B'^{ru} \subset \overline{B}^{\mathcal{T}}$. Let $x \in \overline{B}^{\mathcal{T}}$. Then there exists a sequence $\{x_n\} \subset B$ such that $x_n \to x(\mathcal{T})$. Let $\rho_1 \leq \rho_2 \leq \ldots$ be a countable set of Riesz seminorms generating $\mathcal{T}$. Pick a subsequence $\{y_n\}$ of $\{x_n\}$ such that $\rho_n(x - y_n) \leq n^{-3}$. Now $\sum \rho_n(n\,|x - y_n|) < \infty$, and by 7.4 (iv) there exists $v = \sum n\,|x - y_n|$. Then $|x - y_n| \leq \frac{1}{n} v$ for $n = 1, 2, \ldots$ and $y_n \to x$ (r. u.), i. e., $x \in B'^{ru}$ and $B'^{ru} = \overline{B}^{\mathcal{T}}$.

It is not true that for any ideal B of a Riesz space E, B'^{ru} is relatively uniformly closed. We shall give an example in the next chapter.

10. An Example

The following example was given by K. Yosida [8]. We use it first to show that $I(L)$ is not always $A_o(L)$.

(i) Let $F = \prod_{n=1}^{\infty} E_n$, where E_n is the space $R \times R$ ordered lexicographically for $n = 1, 2, \ldots$. We order F by the product ordering. Thus elements of F are of the form
$x = [(x_1, y_1), (x_2, y_2), \ldots]$ and $x \geq 0$ if for each i
$x_i > 0$ or $x_i = 0$ and $y_i \geq 0$, $i = 1, 2, \ldots$. Let L be the subspace of F consisting of all $x \in F$ such that $\{i: x_i \neq 0\}$ is finite. Certainly L is a Riesz space.

(ii) $I(L) = \{x \in L: x_i = 0, \ i = 1, 2, \ldots$ and $\{i: y_i \neq 0\}$ is finite$\}$.

Proof: Suppose $0 \leq x$, $\tilde{x} \in L$ and $nx \leq \tilde{x}$, $n = 1, 2, \ldots$. If for some i, $x_i > 0$, then $nx_i \leq \tilde{x}_i$ for $n = 1, 2, \ldots$, and this is impossible. Thus $x_i = 0$, $i = 1, 2, \ldots$. Suppose for some i, $y_i > 0$. If $\tilde{x}_i = 0$, then $ny_i \leq \tilde{y}_i$ for $n = 1, 2, \ldots$, and this is impossible. Thus

$\{i: y_i \neq 0\} \subset \{i: \tilde{x}_i \neq 0\}$ and thus $\{i: y_i \neq 0\}$ is finite.

Now let $x \in L$ such that $x_i = 0$, $i = 1, 2, \ldots$ and $\{i: y_i \neq 0\}$ is finite. Define $\tilde{x}$ by

$$\tilde{x}_i = 1 \quad \text{if } y_i \neq 0,$$
$$\tilde{x}_i = 0 \quad \text{if } y_i = 0,$$
$$\tilde{y}_i = 0 \quad i = 1, 2, \ldots .$$

Then $n|x| \leq \tilde{x}$ for $n = 1, 2, \ldots$, and $x \in I(L)$.

(iii) $A_o(L) = \{x \in L: \; x_i = 0, \; i = 1, 2, \ldots\}$.

Proof: Let $x \in L$ and $x_i = 0$, $i = 1, 2, \ldots$. Define $\tilde{x}$ by $\tilde{x}_i = 0$, $i = 1, 2, \ldots$, and $\tilde{y}_i = i|y_i|$, $i = 1, 2, \ldots$. For each n define $x^{(n)}$ by

$$x_i^{(n)} = 0 \qquad i = 1, 2, \ldots$$

$$y_i^{(n)} = |y_i| \qquad 1 \leq i \leq n$$

$$y_i^{(n)} = 0 \qquad n < i.$$

Then $x^{(n)} \in I(L) \subset A_o(L)$ and $|x - x^{(n)}| \leq \frac{1}{n}\tilde{x}$, $n = 1, 2, \ldots$. Since $A_o(L)$ is relatively uniformly closed, $x \in A_o(L)$.

Now suppose $x \in L$ and $x_i \neq 0$ for some i. Let $M = \{x \in L: \; x_i = 0\}$. It is easy to see that M is maximal. Then L/M is isomorphic to the real numbers and $M \in \mathfrak{A}$. Hence $M \supset A_o(L)$ and, since $x \notin M$, $x \notin A_o(L)$.

Thus in this case $A_o(L) \neq I(L)$. If furthermore, for some archimedean space E there exists an ideal B of E such that $L \cong E/B$, then we also have an example of an ideal B in an archimedean Riesz space such that $B^{'ru}$ is not relatively uniformly closed (by 9.3).

(iv) Let T_2 be the space $R \times R$ ordered lexicographically. First we shall represent T_2 as a quotient space of an archimedean space. Let E_2 be the subspace of the Riesz space of all sequences $\{s(n)\}$ such that the following limits exist:

$$\text{(a)} \quad \lim_{n \to \infty} s(n) = s_0 \quad \text{and}$$

$$\text{(b)} \quad \lim_{n \to \infty} n\,(s(n) - s_0) = s_1 .$$

(v) E_2 is a Riesz space.

<u>Proof</u>: Certainly E_2 is a linear subspace of the space of all sequences. We have only to show that if $s \in E_2$ then $|s| \in E_2$. Let $s \in E_2$. Then $\lim_{n \to \infty} |s|(n) = \lim_{n \to \infty} |s(n)| = |s_0|$. Now if $s_0 = 0$, we have $\lim_{n \to \infty} n\,|s|(n) = \lim_{n \to \infty} |n\, s(n)| = |s_1|$. If $s_0 \neq 0$, we may assume $s_0 > 0$ since $|s| = |-s|$. Then $\lim_{n \to \infty} n\,(|s|(n) - |s_0|) = \lim_{n \to \infty} n\,(s(n) - s_0) = s_1$. Thus $|s| \in E_2$ and E_2 is a Riesz space.

(vi) Let $I_2 = \{s \in E_2 : s_0 = s_1 = 0\}$. It is clear that I_2 is an ideal of E_2. Let v and u be the sequences $v(n) = 1$, $n = 1, 2, \ldots$, $u(n) = \frac{1}{n}$, $n = 1, 2, \ldots$. Then $u, v \in E_2$ and for any $s \in E_2$, $s - s_0 v - s_1 u \in I_2$.

<u>Proof</u>: Certainly $u, v \in E_2$. If $s \in E_2$, let $t = s - s_0 v - s_1 u$. Then $\lim_{n \to \infty} t(n) = \lim_{n \to \infty} (s(n) - s_0 - \frac{1}{n} s_1) = \lim s(n) - s_0 = 0$.

Also $n\,t(n) = n\,(s(n) - s_0) - s_1$. Thus $\lim_{n \to \infty} n\,t(n) =$
$= \lim_{n \to \infty} n\,(s(n) - s_0) - s_1 = 0$.

(vii) $T_2 \cong E_2/I_2$

<u>Proof</u>: Let π be the quotient map of E_2 onto E_2/I_2. Then we may define a map of E_2/I_2 onto $R \times R$ by $\eta(\pi(s)) = \eta[\pi(s_0 v + s_1 u)] = (s_0, s_1)$. Certainly η is 1 - 1 and preserves the linear operations. To complete the proof we need only show that $\pi(s) \geq 0$ if and only if $s_0 > 0$ or $s_0 = 0$ and $s_1 \geq 0$.

Since $u \notin I_2$ and $v \notin I_2$, $\pi(u) \neq 0$ and $\pi(v) \neq 0$. We now show that $\pi(u) \leq n\pi(v)$ for $n = 1, 2, \ldots$. For each n, let $i^{(n)} \in I_2$ be defined by $i^{(n)}(k) = n$ for $1 \leq k \leq n$, and $i^{(n)}(k) = 0$ for $k > n$, $n = 1, 2, \ldots$. Then we have $n u \leq v + i^{(n)}$ and $n\pi(u) \leq \pi(v)$, $n = 1, 2, \ldots$.

Now if for some $\pi(s) \in E_2/I_2$, $s_0 > 0$, then
$0 < s_0 \left(\pi(v) - \frac{|s_1|}{s_0}\pi(u)\right) \leq s_1\pi(u) + s_0\pi(v) = \pi(s)$.
If $s_0 = 0$ and $s_1 \geq 0$, then $0 \leq s_1\pi(u) = \pi(s)$.

On the other hand if $s_0 < 0$, then
$\pi(s) = s_0\left(\pi(v) + \frac{s_1}{s_0}\pi(u)\right) < 0$. If $s_0 = 0$ and $s_1 < 0$, then $\pi(s) = s_1\pi(u) < 0$. Thus η is a Riesz isomorphism of E_2/I_2 onto T_2.

(viii) Now let $G = \prod_{n=1}^{\infty} E_2$ with the product ordering. Let p_k be the projection into the k'th component, $k = 1, 2, \ldots$. If $x \in G$, $p_k(x)$ is a sequence in E_2 for $k = 1, 2, \ldots$. Let $\lim_{n\to\infty} p_k(x)(n) = x_{k,o}$ and $\lim_{n\to\infty} n(p_k(x)(n) - x_{k,o}) = x_{k,1}$.

Finally let $E = \{x \in G\colon \{k\colon x_{k,o} \neq 0\}$ is finite$\}$. Now E is a Riesz subspace of G and hence is archimedean. Let $I = \{x \in E\colon\ x_{k,o} = x_{k,1} = 0,\ k = 1, 2, \ldots\}$. It is easy to see that I is an ideal of E. Let π be the quotient map of E onto E/I. We may define a map of E/I onto L by $\eta(\pi(x)) = [(x_{1,0}, x_{1,1}), (x_{2,0}, x_{2,1}), \ldots]$. Now it is clear that η is an isomorphism.

Thus since $E/I \cong L$, and $I(L) \neq A_o(L)$, I'^{ru} is not relatively uniformly closed.

IV RIESZ SPACES SUCH THAT EVERY QUOTIENT SPACE IS ARCHIMEDEAN

11. A General Characterization

As we have seen, an archimedean Riesz space has, in general, many quotient spaces which are not archimedean. Thus a Riesz space such that every quotient space is archimedean must be of a special type. Some examples are finite dimensional spaces and the space of all sequences such that all but a finite number of terms are constant.

Let L be a Riesz space and B be a band in L. Then B is said to be a projection band if for every $0 \leq v \in L$, there exists $u_B = \sup \{0 \leq u \leq v : u \in B\}$ and $u_B \in B$.

11.1 Theorem Let L be a Riesz space such that for every ideal A the quotient space L/A is archimedean. Then for every $0 \leq u, v \in L$ there exists a real number $m \geq 0$ such that $\sup(\inf(nv, u) : n = 1, 2, \ldots) = \inf(mu, v)$, and thus every principal ideal in L is a projection band.

Proof: Assume, to the contrary, that the set $\{\inf(nv, u) : n = 1, 2, \ldots\}$ contains infinitely many distinct elements. Let A be the ideal generated by $\{nv - \inf(nv, u) : n = 1, 2, \ldots\}$. Then $A = \{x \in L : |x| \leq k(nv - \inf(nv, u))$ for some positive integers k and $n\}$.

Now $v \notin A$, since $v \in A$ implies that for some k and n, $v \leq k(nv - \inf(nv, u))$. Then

$(kn - 1)\, v \geq \inf (kn\, v, k\, u) \geq \inf (kn\, v, u)$ and

$kn\, v \geq \inf (kn\, v, u) + v \geq \inf ((kn+1)\, v, u)$. Indeed for

$j \geq kn$, $j\, v \geq \inf ((j + 1)\, v, u)$ and

$\inf (j\, v, u) = \inf ((j + 1)\, v, u)$. Thus

$\sup (\inf (j\, v, u): \ j = 1, 2, \ldots) = \inf (kn\, v, u)$, contrary to our assumption.

Let π be the quotient map of L onto L/A. Since for every integer n, $n\, v \leq u + (n\, v - \inf (n\, v, u))$, we have $n\, \pi (v) \leq \pi (u)$, $n = 1, 2, \ldots$. Thus L/A is not archimedean, which is a contradiction.

Hence for some m, $\sup (\inf (n\, v, u): \ n = 1, 2, \ldots) =$
$= \inf (m\, v, u)$.

<u>11.2</u> <u>Lemma</u> Let L be a Riesz space and $0 \leq u, v \in L$. If $u_1 = \sup (\inf (n\, v, u): \ n = 1, 2, \ldots)$ exists in L and $u_2 = u - u_1$, then $u_1 \perp u_2$ and $u_2 \perp v$.

Proof: We have $\inf (u_2, u_1) = \inf (u - u_1, u_1) =$
$= \inf (u, 2u_1) - u_1$. Now $\inf (n\, v, u) \uparrow u_1$ implies
$2 \inf (n\, v, u) = \inf (2n\, v, 2u) \uparrow 2u_1$ and
$\inf (2n\, v, 2u, u) = \inf (2n\, v, u) \uparrow \inf (u, 2u_1) = u_1$. Thus
$\inf (u_2, u_1) = 0$.

Also we have $\inf (v, u_2) = \inf (v, u - u_1) =$
$= \inf (v + u_1, u) - u_1$. Then $\inf (n\, v, u) + v =$
$= \inf ((n+1)\, v, u + v) \uparrow v + u_1$. Thus
$\inf ((n+1)\, v, u + v, u) = \inf ((n+1)\, v, u) \uparrow \inf (v + u_1, u) = u_1$.

It then follows that $\inf(v, u_2) = 0$.

11.3 Theorem Let L be a Riesz space. Then the following conditions are equivalent:

(i) For every ideal A of L the quotient space L/A is archimedean.

(ii) For every $0 \leq u$, $v \in L$ there exist
$v_1 = \sup(\inf(n u, v): n = 1, 2, \ldots)$ and
$u_1 = \sup(\inf(n v, u): n = 1, 2, \ldots)$ in L and real numbers m_1 and m_2 such that $m_1 v_1 \geq u_1$ and $m_2 u_1 \geq v_1$.

Proof: (i) implies (ii). By 11.1 there exist integers m_1 and m_2 such that $u_1 = \inf(m_1 v, u)$ and $v_1 = \inf(m_2 u, v)$. Let $v_2 = v - v_1$. Now $u_1 \leq m_1 v = m_1 v_1 + m_1 v_2$. By 11.2 $v_2 \perp u$ and hence $v_2 \perp u_1$. Thus $u_1 \leq m_1 v_1$. Similarly $v_1 \leq m_2 u_1$.

(ii) implies (i). Let A be an ideal of L and π be the quotient map of L onto L/A. Suppose for some $0 \leq u$, $v \in L$, $0 \leq n \pi(u) \leq \pi(v)$, $n = 1, 2, \ldots$. We may assume there exists $0 \leq w_n \in A$ such that $n u \leq v + w_n$, $n = 1, 2, \ldots$. We are assuming $u_1 = \sup(\inf(n v, u): n = 1, 2, \ldots)$ and $v_1 = \sup(\inf(n u, v): n = 1, 2, \ldots)$ exist. Let $u_2 = u - u_1$ and $v_2 = v - v_1$. Then $n u_1 + n u_2 \leq v + w_n$, $n = 1, 2, \ldots$. Since $u_2 \perp v$ by 11.2, $u_2 \leq w_1$ and $u_2 \in A$. Now since $v_2 \perp u$, $n u_1 \leq v_1 + w_n$ for each $n = 1, 2, \ldots$. But there exists an integer m such that $m u_1 \geq v_1$. Then $n u_1 \leq m u_1 + w_n$ and $0 \leq n \pi(u_1) \leq m \pi(u_1)$ for every $n = 1, 2, \ldots$. Thus

$\pi(u_1) = 0$, $u \in A$, and L/A is archimedean.

12. Dedekind σ - complete Case

A Riesz space such that every quotient space is archimedean is not necessarily Dedekind σ- complete. An example is the space of all sequences which are constant except for a finite number of terms. We shall show that if a Riesz space has every quotient space archimedean and is Dedekind σ- complete (or even relatively uniformly complete) then the space is simply a Riesz space of all functions which vanish off finite sets on some non-empty set.

12.1 Theorem Let L be a Dedekind σ- complete Riesz space. Then L is relatively uniformly complete.

Proof: Let $\{x_n\}$ be a relative uniform Cauchy sequence. Then there exists $0 \leq v \in L$ and a subsequence $\{y_n\}$ such that $|y_{n+1} - y_n| \leq 2^{-n} v$. We have $\sum_{k=1}^{n} |y_{k+1} - y_k| \leq \sum 2^{-k} v \leq v$. Then $u = \sup_n \sum_{k=1}^{n} |y_{k+1} - y_k|$ exists by assumption. Also $\sum_{k=1}^{n} |y_{k+1} - y_k| + (y_{k+1} - y_k) \uparrow \sum_{k=1}^{n} 2\,(2^{-k} v) \leq 2 v$. Let $w = \sup_n \sum_{k=1}^{n} |y_{k+1} - y_k| + (y_{k+1} - y_k)$. Now if $z = w - u + y_1$,

$$|z - y_{n+1}| \leq \left| w - \sum_{k=1}^{n} |y_{k+1} - y_k| + (y_{k+1} - y_k) \right| + \left| u - \sum_{k=1}^{n} |y_{k+1} - y_k| \right| \leq$$

$$\leq \sum_{k=n+1}^{\infty} 2\,(2^{-k} v) + \sum_{k=n+1}^{\infty} 2^{-k} v = 3\,(2^{-n} v).$$ Thus $y_n \rightarrow z$ (r. u.) and then $x_n \rightarrow z$ (r. u.). Hence L is relatively uniformly complete.

We digress to discuss atomic Riesz spaces. Let L be a Riesz space and $a \in L$. Then a is called an atom of L if the principal ideal generated by a,

$I(a) = \{x \in L: |x| \leq m\, a$ for some real $m\}$, is one-dimensional. The space L is said to be <u>atomic</u> if for every $0 < v \in L$ there exists an atom a such that $0 \leq a \leq v$.

<u>12.2</u> <u>Lemma</u> Let L be a Riesz space.

(i) If a is an atom of L, either $a > 0$ or $a < 0$.

(ii) If a_1 and a_2 are atoms of L, then either $a_1 = \lambda a_2$ for some real λ or $a_1 \perp a_2$.

<u>Proof</u>: (i) If $a = 0$, $I(a) = \{0\}$, thus $a \neq 0$. Now $a = a^+ - a^-$. If both a^+ and a^- are non-zero, they are linearly independent since they are orthogonal. Thus either a^+ or a^- is zero.

(ii) We may assume $a_1 > 0$, $a_2 > 0$. If $\inf(a_1, a_2) \neq 0$, then $\inf(a_1, a_2) \in I(a_1)$ and $\inf(a_1, a_2) \in I(a_2)$. Thus $\inf(a_1, a_2) = \alpha\, a_1 = \beta\, a_2$ where α and β are positive real numbers. Hence $a_1 = \frac{\beta}{\alpha}\, a_2$.

We return to the problem at hand and prove:

<u>12.3</u> <u>Lemma</u> Let L be a Riesz space which is relatively uniformly complete and such that every quotient space is archimedean. Let I be a principal ideal in L. Then I does not contain a countable, orthogonal set of positive elements.

<u>Proof</u>: Suppose to the contrary that we have $0 < v \in L$ and $I(v)$ contains a countable set $\{0 < u_n: n = 1, 2, \ldots\}$ such

that $u_i \perp u_j$, $i \neq j$ $i, j = 1, 2, \ldots$. We may also assume $u_n \leq 2^{-n} v$, $n = 1, 2, \ldots$. Let A be the ideal in L generated by $\{u_n : n = 1, 2, \ldots\}$.
$A = \{x \in L: |x| \leq m(\sum_{k=1}^{n} u_k)$ for some real $m > 0$ and integer $n\}$.
Note that $A \subset I$. Now $\sum_{k=n}^{m} u_k \leq (\sum_{k=n}^{m} 2^{-k})\, v$ and $\{\sum_{k=1}^{n} u_k\}$ is a relative uniform Cauchy sequence. Then for some $u \in L$, $\sum_{k=1}^{n} u_k \to u$ (r. u.). The limit is unique since L is archimedean and thus $u = \sup_n \sum_{k=1}^{n} u_k$. Since $u \notin A$ and $u \leq v$, $v \notin A$. Now $u - \sum_{k=1}^{n} u_k \leq \sum_{k=n+1}^{\infty} 2^{-k} v = 2^{-n} v$. But this implies L/A is not archimedean. Thus I does not contain a countable orthogonal set of positive elements.

We have now:

12.4 Theorem Let L be a Riesz space. Then the following conditions are equivalent:

(i) Every quotient space of L is archimedean and L is Dedekind σ - complete.

(ii) Every quotient space of L is archimedean and L is relatively uniformly complete.

(iii) Every principal ideal of L is finite dimensional.

(iv) There exists a non-empty set X such that L is Riesz isomorphic to the Riesz space of all real functions on X which vanish off finite sets.

Proof: ((i) implies (ii)) This follows from 12.1.

((ii) implies (iii)) Let $I = I(v)$, $0 \leq v \in L$, be a principal ideal. We may assume $v \neq 0$. We shall show first that I is atomic. Let $0 < w \in I$ and J be the principal ideal generated by w. If J is one - dimensional, w is itself an atom. If not there exists $0 \neq x \in J$ such that x and w are linearly independent. Then either $\{x^+, w\}$ or $\{x^-, w\}$ is independent. Thus we may assume $0 \leq u \leq w$ and u and w are linearly independent. Finally since L is archimedean, we may assume that $\lambda > 1$ implies $w \ngeq \lambda u$.

Now let $w_1 = \sup_n \inf (n (w - u), w)$. We claim $w_1 \neq w$. Suppose $w_1 = w$. Then by 11.1 there exists $m > 1$ such that $w = w_1 = \inf (m (w - u), w)$. This implies $\inf (m (w - u) - w, 0) = 0$, i. e., $m (w - u) - w \geq 0$ or $w \geq \frac{m}{m-1} u$, which is a contradiction. Let $0 < z_1 = w - w_1$. We now repeat the above process for w_1. Either w_1 is an atom or $w_1 = z_2 + w_2$, $0 < z_2$, $0 < w_2$, $z_2 \perp w_2$. After a finite number of steps we must obtain a w_n which is an atom. Otherwise I contains a countable subset of positive orthogonal elements -- $\{z_n : n = 1, 2, \dots\}$, which contradicts 12.3. Thus I is atomic.

Now we shall show that I is finite dimensional. We define an atom of I to be <u>normalized</u> if $0 \leq a \leq v$, but $\lambda > 1$ implies $v \ngeq \lambda a$. By the previous paragraph, I contains at least one normalized atom. On the other hand by 12.3, the set of all normalized atoms of I must be finite, say $\{a_1, a_2, \dots, a_n\}$. Let $u = \sup \{a_1, a_2, \dots, a_n\} = \sum_{k=1}^{n} a_k$. If $u \neq v$, $v - u$ contains an atom $a > 0$. Then by 12.2 for some real $\lambda > 0$

and $1 \leq k \leq n$, $a = \lambda a_i$. But then $v = u + (v - u) \geq a_i + \lambda a_i \geq (1+\lambda) a_i$, which is a contradiction. Thus $v = \sum_{k=1}^{n} a_k$ and $\{a_1 \dots a_n\}$ is a basis for I.

(iii) implies (iv). Since every principal ideal of L is finite dimensional, it follows that L is atomic. If X is a maximal orthogonal set of atoms of L, then it is also clear that L is Riesz isomorphic to the space of all functions which vanish off finite subsets of X.

(iv) implies (i). Suppose for some set X, L is the space of all functions which vanish off finite subsets of X. Then it is clear that L is Dedekind σ- complete. Let $0 \leq f$, $g \in L$ and $X_f = \{x \in X: f(x) \neq 0\}$ and $X_g = \{x \in X: g(x) \neq 0\}$. Then $g_1 = \sup_n \inf (n f, g)$ is the function given by

$$g_1(x) = g(x) \text{ if } x \in X_f \cap X_g \text{ and } 0 \text{ otherwise,}$$

and $f_1 = \sup_n \inf (n g, f)$ is the function

$$f_1(x) = f(x) \text{ if } x \in X_f \cap X_g \text{ and } 0 \text{ otherwise.}$$

It is clear that since $X_f \cap X_g$ is finite, we may select real numbers m_1 and m_2 such that $m_1 f_1 \geq g_1$ and $m_2 g_1 \geq f_1$. Thus by 11.3, every quotient space of L is archimedean.

V OTHER RESULTS

13. Spaces with Property (A, ii)

For locally convex Riesz spaces with property (A, ii), one can give completeness results which parallel those for $\mathcal{T}$- local completeness obtained in 6.2 and 6.3.

13.1 Theorem Let $(L, \mathcal{T})$ be a locally convex Riesz space with property (A, ii). If $0 \leq u_\lambda \uparrow$, $\{u_\lambda\}$ a $\mathcal{T}$- Cauchy net, implies that for some $u \in L$, $u_\lambda \longrightarrow u \ (\mathcal{T})$, then L is $\mathcal{T}$- complete.

Proof: We show first that L is Dedekind complete. Suppose $0 \leq u_\lambda \uparrow \leq v$ is an order - bounded, directed system in L. Then, under the Riesz space ordering, $\{u_\lambda : \lambda \in \Lambda\}$ may be considered a net. Let $W = \{w \in L: \ 0 \leq u_\lambda \leq w$ for every $\lambda \in \Lambda\}$. Then W and $\{w - u_\lambda : \ w \in W, \lambda \in \Lambda\}$ are monotone decreasing nets. Suppose for some $0 \leq z \in L$, $z \leq w - u_\lambda$ for every $w \in W$ and $\lambda \in \Lambda$. Then if $w \in W$, $u_\lambda \leq w - z$ for every $\lambda \in \Lambda$ and $w - z \in W$. Since $v \in W$, it follows that $v - nz \in W$ for $n = 1, 2, \ldots$. In particular $0 \leq nz \leq v$ for $n = 1, 2, \ldots$, and since L is archimedean, by 1.2, $z = 0$. Thus $\inf \{w - u : \ w \in W$ and $\lambda \in \Lambda\} = 0$. Since $(L, \mathcal{T})$ has property (A, ii), the net $\{w - u_\lambda : \ w \in W$ and $\lambda \in \Lambda\}$ converges to zero in $\mathcal{T}$. Now if ρ is a $\mathcal{T}$- continuous Riesz seminorm and r is a positive real number, there exist

$\lambda_o \in \Lambda$ and $w_o \in W$ such that $\rho(w_o - u_{\lambda_o}) < r$. If $u_\lambda \geq u_{\lambda_o}$, then $u_\lambda - u_{\lambda_o} \leq w_o - u_{\lambda_o}$ and $\rho(u_\lambda - u_{\lambda_o}) < r$. So $u_\lambda \uparrow$ is a monotone Cauchy net, and by assumption, there exists $u \in L$ such that $u_\lambda \longrightarrow 0\ (\mathcal{T})$. Then by 1.4, $u_\lambda \uparrow u$ and L is Dedekind complete.

Since L is Dedekind complete, by a theorem of Nakano [6], for every $0 \leq u \in L$, $\{x\colon 0 \leq x \leq u\}$ is $\mathcal{T}$- complete. Now suppose $\{x_\lambda\colon \lambda \in \Lambda\}$ is a $\mathcal{T}$- Cauchy net. By 1.3 $\{x_\lambda^+ : \lambda \in \Lambda\}$ and $\{x_\lambda^-\colon \lambda \in \Lambda\}$ are also $\mathcal{T}$- Cauchy nets. If $x_\lambda^+ \longrightarrow u_1\ (\mathcal{T})$ and $x_\lambda^- \longrightarrow u_2\ (\mathcal{T})$, then $x_\lambda \longrightarrow u_1 - u_2$. Thus we need only show that positive Cauchy nets converge.

Let $\{u_\lambda\}$ be a positive $\mathcal{T}$- Cauchy net. Then for each $0 \leq v \in L$, $\{\inf(u_\lambda, v)\}$ is a $\mathcal{T}$- Cauchy net and, by the above, for some $0 \leq u_v \leq v$, $\inf(u_\lambda, v) \rightarrow u_v\ (\mathcal{T})$. Now consider the net $\{u_v\colon 0 \leq v \in L\}$. It is certainly monotone. Let ρ be a $\mathcal{T}$- continuous Riesz seminorm and r a positive real number. Then there exists λ_o such that $\lambda \geq \lambda_o$ implies $\rho(u_\lambda - u_{\lambda_o}) < r$. Let $w = u_{\lambda_o}$. If $v \geq w$ and $\lambda \geq \lambda_o$ then

$$|u_v - u_w| \leq |u_v - \inf(u_\lambda, v)| +$$
$$+ |\inf(u_\lambda, v) - \inf(u_\lambda, w)| + |\inf(u_\lambda, w) - u_\lambda| \leq$$
$$\leq |u_v - \inf(u_\lambda, v)| + |u_\lambda - w| + |w - u_\lambda| \text{ and}$$

$\rho(u_v - u_w) < 2r$. Thus $\{u_v\colon 0 \leq v \in L\}$ is a $\mathcal{T}$- Cauchy net and converges to say $u \in L$.

Now we have only to show that $u_\lambda \longrightarrow u\ (\mathcal{T})$. Again let ρ be a $\mathcal{T}$- continuous Riesz seminorm and r a positive real number.

Then there exists λ_0 such that $\lambda \geq \lambda_0$ implies $\rho(u_\lambda - u_{\lambda_0}) < r$. Pick $w \geq u_{\lambda_0}$ such that $\rho(u - u_w) < r$. Then if $\lambda \geq \lambda_0$

$$|u - u_{\lambda_0}| \leq |u - u_w| + |u_w - \inf(w, u_\lambda)| + |\inf(w, u_\lambda) - u_{\lambda_0}| \leq$$

$$\leq |u - u_w| + |u_w - \inf(w, u_\lambda)| + |u_\lambda - u_{\lambda_0}| \text{ and}$$

$\rho(u - u_{\lambda_0}) < 2r$. Thus $u_\lambda \to u\ (\mathcal{T})$ and L is $\mathcal{T}$- complete.

<u>13.2</u> <u>Theorem</u> Let L be a Riesz space possessing an (A, ii) locally convex lattice topology $\mathcal{T}$ such that L is $\mathcal{T}$- complete. If $\mathcal{T}'$ is another (A, ii) locally convex Riesz topology on L such that $\mathcal{T}' \geq \mathcal{T}$, then L is also $\mathcal{T}'$ complete.

<u>Proof</u>: By 13.1 we need only show that monotone $\mathcal{T}'$- Cauchy nets converge. Let $0 \leq u_\lambda \uparrow$, $\{u_\lambda\}$ a $\mathcal{T}'$- Cauchy net on L. Then $\{u_\lambda\}$ is a $\mathcal{T}$- Cauchy net and for some $u \in L$ $u_\lambda \to u\ (\mathcal{T})$. By 1.4 $u = \sup_\lambda u_\lambda$ and since $\mathcal{T}'$ has property (A, ii), $u_\lambda \to u\ (\mathcal{T}')$.

14. Positive Cones with Interior Points

In general the positive cone of a locally convex Riesz space does not have an interior point. Indeed we have:

14.1 Theorem Let $(L, \mathcal{T})$ be a locally convex Riesz space. Then the positive cone $L^{+} = \{u \in L: u \geq 0\}$ has a $\mathcal{T}$- interior point if and only if L has a strong unit $v \geq 0$ and the topology $\mathcal{T}$ is generated by the Riesz norm p, where
$p_v(x) = \inf \{\alpha : \alpha v \geq |x|\}$.

Proof: Assume L has a $\mathcal{T}$- interior point $v \geq 0$. Then there exists a $\mathcal{T}$-continuous Riesz seminorm ρ_o such that $\rho_o(v - x) < 1$ implies $x \geq 0$. Let $x \in L$. Then there exists a real number $\alpha > 0$ such that $\rho_o(\alpha x) < 1$. Hence $\rho_o(v - (v - \alpha|x|)) = \rho_o(\alpha x) < 1$ and $v - \alpha|x| \geq 0$ or $\frac{1}{\alpha} v \geq |x|$. Thus v is a strong unit.

Now let ρ be any $\mathcal{T}$- continuous Riesz seminorm. Then if $\alpha v \geq |x|$, we have $\rho(x) \leq \alpha \rho(v)$. Thus $\rho(x) \leq p_v(x)\, \rho(v)$ and the p_v- topology is stronger than $\mathcal{T}$. On the other hand if $\rho_o(x) < 1$, then $\rho_o(v - (v - |x|)) < 1$ and $v \geq |x|$. Thus $\{x: \rho_o(x) < 1\} \subset \{x: p_v(x) \leq 1\}$. This shows that the p_v-topology is weaker than $\mathcal{T}$ and hence they are the same.

If L has a strong unit $v \geq 0$ and $\mathcal{T}$ is generated by p_v, then certainly v is an interior point L . In fact if $p_v(v - x) < \frac{1}{2}$, then $v - x \leq \frac{1}{2} v$ or $0 \leq \frac{1}{2} v \leq x$.

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