L_q ESTIMATES FOR REARRANGEMENTS OF FOURIER SERIES

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ABSTRACT

This work is concerned with estimating the upper envelopes S^* of the absolute values of the partial sums of rearranged trigonometric sums. A. M. Garsia [Annals of Math. 79 (1964), 634-9] gave an estimate for the L_2 norms of the S^*, averaged over all rearrangements of the original (finite) sum. This estimate enabled him to prove that the Fourier series of any function in L_2 can be rearranged so that it converges a.e. The main result of this thesis is a similar estimate of the L_q norms of the S^*, for all even integers q. This holds for finite linear combinations of functions which satisfy a condition which is a generalization of orthonormality in the L_2 case. This estimate for finite sums is extended to Fourier series of L_q functions; it is shown that there are functions to which the Men'shov-Paley Theorem does not apply, but whose Fourier series can nevertheless be rearranged so that the S^* of the rearranged series is in L_q.

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I. Presentation of Results

A. M. Garsia [2] has recently proved that every L_2 Fourier series can be rearranged so that it converges almost everywhere. His proof rests on an estimate for the upper envelopes of the absolute values of the partial sums of a rearranged finite sum $T = \sum_{\nu=1}^{n} a_\nu \phi_\nu$ where $\phi_1, \cdots, \phi_n$ are orthonormal functions on a measure space, and $a_1, \cdots, a_n$ are real numbers. If $\sigma = (\sigma_1, \cdots, \sigma_n)$ is a permutation of $\{1, \cdots, n\}$ write

$$S^*(x,\sigma,T) = S^*(x,\sigma) = \max_{1 \le m \le n} \left| \sum_{\nu=1}^{m} a_{\sigma_\nu} \phi_{\sigma_\nu}(x) \right|. \tag{1.1}$$

His estimate is

$$\frac{1}{n!} \sum_{\sigma} \int [S^*(x,\sigma,T)]^2 dx \le A \int |T|^2 \tag{1.2}$$

for all $a_1, \cdots, a_n$. Here A is a constant not depending on n or the particular system (ϕ_ν) used. (We will use the notation $A_r, A(r,s,\cdots)$ for positive constants, not necessarily the same at each occurrence, which depend only on the indicated parameters.) The sum is over all $n!$ permutations. Garsia uses (1.2) to show that the Fourier series of any f in L_2 can be partitioned into finite blocks of terms, each of which can be rearranged in such a way that the entire rearranged series converges a.e.

Theorem 1 of this work is an estimate of the L_q norms of the

$S^*(x,\sigma)$ for all even q. It holds for a different class of ϕ_ν than Garsia's result, and the right-hand side of (1.2) is replaced by a stronger norm of T than $\|T\|_q$. The property of the ϕ_ν that we need is the following generalization of orthonormality:

(1.3) Definition. Let r be a positive integer. Let (ϕ_ν) be a system of complex-valued functions on a measure space. We will say (ϕ_ν) is a P-system for r if for all indices $\nu_1, \cdots, \nu_r, \mu_1, \cdots, \mu_r$ we have

$$\int \phi_{\nu_1} \cdots \phi_{\nu_r} \bar{\phi}_{\mu_1} \cdots \bar{\phi}_{\mu_r} \geq 0. \tag{1.4}$$

(Here and elsewhere in this work we assume that all integrals written down exist.) Any orthonormal system is a P-system for 1. The systems

$$\text{a)} \quad \phi_1 = 1, \ \phi_2(x) = e^{ix}, \ \phi_3(x) = e^{-ix}, \ \phi_4(x) = e^{2ix}, \cdots \quad (0 \leq x \leq 2\pi)$$

$$\text{b)} \quad \phi_\nu(x) = \cos(\nu-1)x \quad (\nu = 1, 2, \cdots) \tag{1.5}$$

c) the Rademacher functions $R_\nu(x) = \operatorname{sgn} \sin 2^\nu \pi x \ (\nu = 1, 2, \cdots; \ 0 \leq x \leq 1)$

are P-systems for all r. In example (a), the integrals (1.4) are either 0 or 2π. Then write $\cos \nu x = (e^{i\nu x} + e^{-i\nu x})/2$ and use that (a) is a P-system to see that (1.4) holds for (b). Example (c) is taken care of by the facts that the Rademacher functions are independent random variables and that $\int R_\nu^\alpha = 1$ or 0 if α is even or odd, respectively.

However, $(\sin \nu x : \nu = 1, 2, \cdots)$ is not a P-system for any $r > 1$ since

$$\int_0^{2\pi} \sin^{2r-1} x \ \sin(2r-1)\,dx \quad = \quad -\pi/4^{r-1} \ (r=2,4,6,\cdots),$$

$$\int_0^{2\pi} (\sin 2x)^{2r-2} \sin x \ \sin(4r-5)x \ dx = -\pi/4^{r-1} \ (r=3,5,7,\cdots).$$

From now on we will refer to (1.5)(a) and (b), with this particular indexing, as the exponential and cosine systems.

Theorem 1. Let $\phi_1,\cdots,\phi_n$ be a P-system for r. Let $T=\sum_1^n a_\nu \phi_\nu$, $\hat{T}=\sum_1^n |a_\nu|\phi_\nu$, where $a_1,\cdots,a_n$ are complex numbers. Then

$$\frac{1}{n!}\sum_{\sigma} \int [S^*(x,\sigma,T)]^{2r} dx \le A_r \int |\hat{T}|^{2r} \qquad (1.6)$$

for all n and $a_1,\cdots,a_n$.

Curiously enough, no other special property of the ϕ_ν is needed. Theorem 1 includes Garsia's L_2 result (1.2) since $\int|T|^2=\int|\hat{T}|^2$.

Corollary 1. There is a σ (which may depend on T) such that

$$\int [S^*(x,\sigma,T)]^{2r} dx \le A_r \int |\hat{T}|^{2r}. \qquad (1.7)$$

The spaces X_{2r}. Theorem 2 will be a statement analogous to (1.6) about trigonometric Fourier series of certain functions in $L_{2r}(0,2\pi)$. For a given positive integer r, the functions we will consider are those $f \sim \sum_{-\infty}^{\infty} a_\nu e^{i\nu x}$ such that $\sum_{-\infty}^{\infty} |a_\nu| e^{i\nu x}$ is the Fourier series of a function $\hat{f}$ in L_{2r}. This class of functions will be denoted by X_{2r}. In this definition we may replace

$(e^{i\nu x})$ by another complete orthonormal system (ϕ_ν), in which case we will write $X_{2r}(\phi)$.

Observe first that $X_2 = L_2$, for if $f \in X_2$ then its Fourier coefficients are square-summable, and vice versa. Also $\|f\|_2 = \|\hat{f}\|_2$. (Thus $\sum |a_\nu| e^{i\nu x} \sim g \in L_{2r}$ implies that $\sum a_\nu e^{i\nu x}$ is an L_2 Fourier series.) We will establish the following facts about the classes X_{2r}.

(1.7) Theorem. a) For $r = 2, 3, \cdots$ the class X_{2r} is strictly contained in L_{2r} and

$$\|f\|_{2r} \le \|\hat{f}\|_{2r} \tag{1.8}$$

for all $f \in X_{2r}$.

b) For all r the functional $N(f) = \|\hat{f}\|_{2r}$ is a norm on X_{2r}, and X_{2r} is a Banach space under this norm.

Part (b) is proved in Appendix 1. Part (a) is a special case of a well-known result, which we now state. First let us adopt a notation of Hardy and Littlewood [3]. We will say that the series $\sum_{-\infty}^{\infty} A_\nu e^{i\nu x}$ is a majorant of the series $\sum_{-\infty}^{\infty} a_\nu e^{i\nu x}$ if $|a_\nu| \le A_\nu$ for all v, and we will write $\sum a_\nu e^{i\nu x} \prec \sum A_\nu e^{i\nu x}$. If these are Fourier series we use the same terminology for the respective functions. In Appendix 1 we use the same notation for more general series $\sum a_\nu \phi_\nu$.

(1.9) <u>If</u> $\sum_{-\infty}^{\infty} A_\nu e^{i\nu x} \sim F \in L_{2r}$ <u>and</u> $\sum_{-\infty}^{\infty} a_\nu e^{i\nu x} \prec \sum_{-\infty}^{\infty} A_\nu e^{i\nu x}$

<u>then</u> $\sum a_\nu e^{i\nu x} \sim f \in L_{2r}$ <u>and</u>

$$\int_0^{2\pi} |f|^{2r} \leq \int_0^{2\pi} |F|^{2r}. \tag{1.10}$$

We present a proof of (1.9) which gives a simple example of calculation with P-systems. First we establish (1.9) for trigonometric polynomials:

$$\int_0^{2\pi} \left|\sum_{\nu=-n}^{n} a_\nu e^{i\nu x}\right|^{2r} dx$$

$$= \sum_{\nu_1 \cdots \nu_r \mu_1 \cdots \mu_r = -n}^{n} a_{\nu_1} \cdots a_{\nu_r} \bar{a}_{\mu_1} \cdots \bar{a}_{\mu_r} \int \exp[i(\nu_1 + \cdots + \nu_r - \mu_1 - \cdots - \mu_r)]\, dx$$

$$\leq \sum A_{\nu_1} \cdots A_{\nu_r} A_{\mu_1} \cdots A_{\mu_r} \int \exp[i(\nu_1 + \cdots + \nu_r - \mu_1 - \cdots - \mu_r)]\, dx$$

$$= \int \left|\sum_{\nu=1}^{n} A_\nu e^{i\nu x}\right|^{2r} dx. \tag{1.11}$$

Here we have used only that the exponential system is a P-system for r. Passing to the case of infinite series, we note that $F \in L_{2r}$ implies $F \in L_2$. Hence $\sum |a_\nu|^2 \leq \sum A_\nu^2 < \infty$, and so $\sum a_\nu e^{i\nu x} \sim f \in L_2$. Let $\sigma_n(x,f)$ and $\sigma_n(x,F)$ be the Cesaro means of f and F. The relation $f \prec F$ implies $\sigma_n(\cdot,f) \prec \sigma_n(\cdot,F)$ for all n. Thus by (1.11),

$$\int_0^{2\pi} |\sigma_n(x,f)|^{2r} dx \leq \int_0^{2\pi} |\sigma_n(x,F)|^{2r} dx. \tag{1.12}$$

But $\sigma_n(x,F) \to F(x)$ strongly in L_{2r} and $\sigma_n(x,f) \to f(x)$ a.e., from which Fatou's Lemma implies that $f \in L_{2r}$ and (1.10) holds.

To prove $X_{2r} \subset L_{2r}$ strictly, for $r = 2, 3, \cdots,$ we choose $g \in L_2 - L_{2r}$ with positive coefficients (for example, see (5.11). There are sign changes of these coefficients which put the resulting series in L_q for all $q < \infty$ [1, vol. I, p. 215].

Statement of Theorem 2. We will consider rearrangements of $\sum_1^\infty a_\nu \phi_\nu$, where $\phi_1, \phi_2, \cdots$ is the exponential or cosine system (1.5)(a) or (b). Let $0 = n_0 < n_1 < n_2 < \cdots$ be a sequence of integers such that $n_{k+1}/n_k > \alpha > 1$ $(k = 1, 2, \cdots)$ for some fixed α. We will make a probability space $(\Omega, d\sigma)$ from the permutations $\sigma = (\sigma_1, \sigma_2, \cdots)$ of $\{1, 2, \cdots\}$ which permute the indices in each block $\{\nu : n_k + 1 \le \nu \le n_{k+1}\}$ $(k = 0, 1, 2, \cdots)$ among themselves. The permutations in each block will be independent random variables, with each permutation of the k^{th} block having probability $1/(n_{k+1} - n_k)!$. Given $f \sim \sum_1^\infty a_\nu \phi_\nu$ and a $\sigma \in \Omega$ set

$$S^*(x, \sigma, f) = S^*(x, \sigma) = \sup_n \left| \sum_{\nu=1}^n a_{\sigma_\nu} \phi_{\sigma_\nu}(x) \right|.$$

Theorem 2. Let $f \in X_{2r}$. Then $S^*(\cdot, \cdot, f) \in L_{2r}(\Omega \times [0, 2\pi])$ and

$$\int_\Omega \int_0^{2\pi} [S^*(x, \sigma, f)]^{2r} dx\, d\sigma \le A(\alpha, r) \int_0^{2\pi} |\hat{f}|^{2r}. \tag{1.13}$$

<u>Corollary 2</u>. <u>Let</u> $f \in X_{2r}$. <u>The function</u> $S^{*}(\cdot, \sigma, f)$ <u>belongs to</u> $L_{2r}(0, 2\pi)$ <u>for almost all</u> σ <u>in</u> Ω , <u>and</u>

$$\int_0^{2\pi} [S^{*}(x,\sigma)]^{2r} dx \le A(\alpha, r) \int_0^{2\pi} |\hat{f}|^{2r} \qquad (1.14)$$

<u>holds for all</u> σ <u>in a set of positive measure in</u> Ω .

<u>Comparison with the Men'shov-Paley Theorem</u>. This result [1, vol. II, p. 189-92] gives a condition which guarantees that $S^{*} \in L_q$ without rearrangement. Let $\phi = (\phi_1, \phi_2, \cdots)$ be a uniformly bounded orthonormal system $(|\phi_\nu| \le M \ (\nu = 1, 2, \cdots))$. If $a = (a_1, a_2, \cdots)$ is a sequence of complex numbers tending to zero let $a_1^{*}, a_2^{*}, \cdots$ denote the sequence obtained by rearranging the sequence $|a_1|, |a_2|, \cdots$ (excluding the a_ν which are zero) in nonascending order. If several $|a_\nu|$ are equal then there is a corresponding repetition in the a_ν^{*} . Then write

$$B_q^{*}(a) = \left(\sum_{\nu=1}^{\infty} a_\nu^{*q} \nu^{q-2} \right)^{1/q} .$$

If $q \ge 2$ we always have $\sum a_\nu^q \nu^{q-2} \ge \sum a_\nu^{*q} \nu^{q-2}$. If $f \sim \sum_1^\infty a_\nu \phi_\nu$ write $S^{*}(x,f) = S^{*}(x) = \sup_n \left| \sum_{\nu=1}^{n} a_\nu \phi_\nu(x) \right|$.

Denote by $HL_q(\phi) = HL_q$ (for Hardy and Littlewood) the set of $f \sim \sum_1^\infty a_\nu \phi_\nu$ such that $B_q^{*}(a) < \infty$. We will then write $B_q^{*}(f)$ for $B_q^{*}(a)$. (For $q \ge 2$ it can easily be shown that if $B_q^{*}(a) < \infty$ then $\sum |a_\nu|^2 < \infty$ and hence $\sum a_\nu \phi_\nu$ is an L_2 Fourier series.

We will suppose that (ϕ_ν) is complete in L_2 .) The Men'shov-Paley Theorem is stated in (1.15)(b).

(1.15)a) If $q \geq 2$ then $HL_q \subseteq L_q$ and

$$\|f\|_q \leq A(q,M)\, B_q^*(f) \tag{1.16}$$

for all $f \in HL_q$.

b) If $q > 2$ and $f \in HL_q$ then $S^*(\cdot, f) \in L_q$ and

$$\|S^*\|_q \leq A(q,M)\, B_q^*(f). \tag{1.17}$$

(1.15)(a) is due to Hardy, Littlewood, and Paley [1, vol. II, p. 121], and is a weaker form of (1.15)(b). Applying (1.15)(a) to $\hat{f}$ for f in $HL_{2r}(\phi)$ gives

(1.18) Let (ϕ_ν) be a uniformly bounded $(|\phi_\nu| \leq M)$ orthonormal system. Then $HL_{2r}(\phi) \subseteq X_{2r}(\phi)$ and

$$\|\hat{f}\|_{2r} \leq A(r,M)\, B_{2r}^*(f) \tag{1.19}$$

for all $f \in HL_{2r}$.

However, the following theorem shows that more than (1.18) is true.

Theorem 3. Let $\phi_1, \phi_2, \cdots$ be a uniformly bounded orthonormal P-system for an $r \geq 2$ on a bounded interval. Then $HL_{2r}(\phi)$ is strictly contained in $X_{2r}(\phi)$.

Hence Theorem 2 applies to a wider class of trigonometric Fourier series in L_{2r} $(r \geq 2)$ than the Men'shov-Paley Theorem (1.15)(b).

Of course in Theorem 2 a rearrangement is required to put S^* in L_{2r}, whereas this is not necessary in (1.15)(b).

Actually, for the exponential system, the fact that HL_{2r} is strictly contained in X_{2r} $(r \geq 2)$ is a consequence of the following well-known theorem [1, vol. I, p. 215]: Let $0 < n_1 < n_2 < n_3 < \cdots$ be a sequence of integers such that $n_{k+1}/n_k > \alpha > 1$ $(k = 1, 2, \cdots)$. If $\sum_{-\infty}^{\infty} |a_k|^2 < \infty$ then

$$\sum_{1}^{\infty} \left(a_k e^{i n_k x} + a_{-k} e^{-i n_k x} \right)$$

is the Fourier series of a g which belongs to L_q for all $q < \infty$. Thus for $r > 1$ if we choose positive numbers a_k such that $\sum_{-\infty}^{\infty} a_k^2 < \infty$ but $B_{2r}^{\star}(a) = \infty$, the resulting g will belong to X_{2r} but not to HL_{2r}. For example, $a_k = (k^{1/2} \log k)^{-1}$ $(k \geq 2)$.

However, we have discovered a proof of Theorem 3 which does not use the above-mentioned result on gap series and which works for more general orthonormal systems ϕ. Furthermore, it produces examples of functions in $X_{2r}(\phi) - HL_{2r}(\phi)$ whose coefficients are all nonzero.

Interpolation of Theorems 1 and 2 between adjacent values of r. We would like to know whether

$$\int_{\Omega} \int_{0}^{2\pi} [S^*(x, \sigma, f)]^q \, dx \, d\sigma \leq A(\alpha, q) \int_{0}^{2\pi} |f|^q \tag{1.20}$$

holds for non-even q. (The notation of Theorem 2 is used here.)

We do not believe that this problem admits an easy solution. Even the apparently simpler Majorant Problem of Hardy and Littlewood [3] remains unsolved. They produced an analytic trigonometric polynomial T such that $\|T\|_3 > \|\hat{T}\|_3$, and posed the problem of whether (1.8) can be extended to read

$$\|f\|_q \le A_q \|\hat{f}\|_q \tag{1.21}$$

for non-even q.

Actually, (1.21) would follow from (1.20). For if f is a trigonometric polynomial then $|f(x)| \le S^*(x,\sigma,f)$ for all σ. Hence (1.21) would hold for trigonometric polynomials. Applying it to the Cesaro sums of f and $\hat{f}$ and using Fatou's Lemma would give (1.21) in general.

However, there is another kind of connection between (1.20) and (1.21). If $f \sim \sum_1^\infty a_\nu \phi_\nu$ let $S_m(x,\sigma,f) = \sum_1^m a_{\sigma_\nu} \phi_{\sigma_\nu}(x)$. We observe that (1.20) holds if and only if

$$\int_\Omega \int_0^{2\pi} |S_{m(x,\sigma)}(x,\sigma,f)|^q \, dx \, d\sigma \le A(\alpha,q) \int_0^{2\pi} |\hat{f}|^q \tag{1.22}$$

for all positive integer-valued functions $m(x,\sigma)$ (such that $S_{m(x,\sigma)}(x,\sigma)$ is measurable on $\Omega \times [0,2\pi]$), where $A(\alpha,q)$ does not depend on the choice of $m(x,\sigma)$. Let $m(x,\sigma)$ be a fixed function. The mapping M such that $M(f)$ is the function $S_{m(x,\sigma)}(x,\sigma,f)$ on $\Omega \times [0,2\pi]$ is a linear transformation from $L_1(H)$ $(H=[0,2\pi])$ into the measurable functions on $\Omega \times H$. Theorem 2 implies that M is bounded from X_{2r} to $L_{2r}(\Omega \times H)$. It is tempting to ask whether M can be interpolated between r and $r+1$. By this we mean that for q between $2r$ and $2r+2$ there may be Banach spaces X_q and Y_q such that $X_{2r} \supseteq X_q \supseteq X_{2r+2}$, $L_{2r}(\Omega \times H) \supseteq Y_q \supseteq L_{2r+2}(\Omega \times H)$ and M is bounded from X_q to Y_q with norm depending on q. We would expect that Y_q would be $L_q(\Omega \times H)$, and that X_q would be the set of f which have an $\hat{f}$ in $L_q(H)$. We hope that the norm on X_q would be equivalent to $\|\hat{f}\|_q$.

Observe that the Majorant Problem can be looked at in the same way; here we consider the identity transformation J as a bounded linear transformation from X_{2r} to $L_{2r}(H)$ with norm 1, and we ask whether J is bounded from the proposed space X_q to $L_q(H)$ $(2r < q < 2r+2)$ with norm ≤ 1. The example of $\|T\|_3 > \|\hat{T}\|_3$ shows that with this point of view we do not expect that the norm on X_q will be precisely $\|\hat{f}\|_q$. In fact, $\|\hat{f}\|_q$ may not even be a norm.

The remaining sections may be read in any order. They consist of proofs and discussions of Theorems 1, 2, and 3 of this section.

II. Proof of Theorem 1

The initial step in this proof is essentially the same as for the case $r = 1$, treated by Garsia. The basic tool is the Bohnenblust-Spitzer Theorem [7] which we state for convenience. Let $x_1, \cdots, x_n$ be real numbers. For a permutation $\sigma = (\sigma_1, \cdots, \sigma_n)$ of $\{1, \cdots, n\}$ let

$$S(\sigma) = \max_{1 \le m \le n} \Big(\sum_{\nu=1}^{m} x_{\sigma_\nu} \Big)^+ .$$

$(x^+ = \max(x, 0),\ x^- = (-x)^+)$. If σ is the product of the disjoint cycles $C_1, \cdots, C_p$ write

$$R(\sigma) = \sum_{k=1}^{p} \Big(\sum_{\nu \in C_k} x_\nu \Big)^+ .$$

(2.1) <u>There is a 1-1 mapping</u> V <u>of the set of all</u> $n!$ <u>permutations onto itself such that</u>

$$R(V(\sigma)) = S(\sigma) \tag{2.2}$$

<u>for all</u> σ.

Thus we can replace sums over all σ involving S by corresponding sums involving R.

Returning to Theorem 1 define the four functions

$$\begin{matrix} S_R^{\pm}(x,\sigma) \\ S_I^{\pm}(x,\sigma) \end{matrix} = \max_{1 \le m \le n} \Big({}^{Re}_{Im} \sum_{\nu=1}^{m} a_{\sigma_\nu} \phi_{\sigma_\nu}(x) \Big)^{\pm} .$$

Then

$$S^*(x,\sigma) \le S_R^+(x,\sigma) + S_R^-(x,\sigma) + S_I^+(x,\sigma) + S_I^-(x,\sigma),$$

$$[S^*(x,\sigma)]^q \le 4^{q-1}(S_R^{+q} + S_R^{-q} + S_I^{+q} + S_I^{-q}) \quad (q=2r),$$

$$\frac{1}{n!}\sum_\sigma [S^*(x,\sigma)]^q \le \frac{4^{q-1}}{n!}\sum_\sigma [S_R^{+q} + S_R^{-q} + S_I^{+q} + S_I^{-q}].$$

We use (2.1) to rewrite the sum over σ. To enable us to reverse order of summation a little later let $\sum_C$ denote a sum over <u>all</u> cycles $C = (\nu_1, \cdots, \nu_\ell)$ of indices ν_i $(1 \le \nu_i \le n)$, and define

$$\begin{aligned} X_C(\sigma) &= 1 \text{ if } C \text{ is a cycle of } \sigma \\ &= 0 \text{ otherwise.} \end{aligned}$$

With these conventions, $\frac{1}{n!}\sum_\sigma [S^*(x,\sigma)]^q$

$$\le \frac{4^{q-1}}{n!}\sum_\sigma \Big\{ \Big[\sum_C X_C(\sigma)\big(Re\sum_{\nu\in C} a_\nu\phi_\nu(x)\big)^+\Big]^q + \Big[\sum_C X_C(\sigma)\big(Re\sum_{\nu\in C} a_\nu\phi_\nu(x)\big)^-\Big]^q$$

$$+ \Big[\sum_C X_C(\sigma)\big(Im\sum_{\nu\in C} a_\nu\phi_\nu(x)\big)^+\Big]^q + \Big[\sum_C X_C(\sigma)\big(Im\sum_{\nu\in C} a_\nu\phi_\nu(x)\big)^-\Big]^q \Big\}$$

$$\le \frac{4^q}{n!}\sum_\sigma \Big(\sum_C X_C(\sigma)\Big|\sum_{\nu\in C} a_\nu\phi_\nu(x)\Big|\Big)^q.$$

Integrating, expanding the qth power, and exchanging orders of summation, we have

$$\frac{1}{n!}\sum_\sigma \int [S^*(x,\sigma)]^q dx \le \frac{4^q}{n!}\sum_{C_1\cdots C_q}\sum_\sigma X_{C_1}(\sigma)\cdots X_{C_q}(\sigma)\int \Big|\sum_{\nu\in C_1} a_\nu\phi_\nu \cdots \sum_{\nu\in C_q} a_\nu\phi_\nu\Big|. \quad (2.3)$$

We now classify the nonzero terms in the right hand side of (2.3). In order that $\sum_\sigma X_{C_1}(\sigma)\cdots X_{C_q}(\sigma)$ be nonzero for a given $(C_1,\cdots,C_q)$, any two C_i, C_j must be identical as cycles or have disjoint sets of indices, otherwise the $C_1,\cdots,C_q$ couldn't all belong to the same permutation. Each such $(C_1,\cdots,C_q)$ determines a partition P of $\{1,\cdots,q\}$ into p disjoint classes such that $C_i=C_j$ if i and j are in the same class, $C_i\cap C_j=\phi$ if i and j are in different classes. Let S_k $(1\le k\le p)$ be the set of indices of the cycle C_i where i is in the k^{th} class of P. Let ℓ_k be the number of indices in S_k. Write

$$\frac{1}{n!}\sum_\sigma\int[S^*(x,\sigma)]^q dx \le 4^q\sum_P I_P, \qquad (2.4)$$

$$I_P=\frac{1}{n!}\sum_{\ell_1\cdots\ell_p}\sum_{S_1\cdots S_p}\sum_{C_1\cdots C_q}^{P}\sum_\sigma X_{C_1}(\sigma)\cdots X_{C_q}(\sigma)\int\Big|\sum_{\nu\in C_1}a_\nu\phi_\nu\cdots\sum_{\nu\in C_q}a_\nu\phi_\nu\Big|, \qquad (2.5)$$

where $\ell_k\ge 1$ $(k=1,\cdots,p)$, $\ell_1+\cdots+\ell_p\le n$, $|S_k|=\ell_k$, and $\sum^P$ runs over all $(C_1,\cdots,C_q)$ which determine the partition P and the disjoint sets $S_1,\cdots,S_p$.

For given $(S_1,\cdots,S_p)$ the number of $(C_1,\cdots,C_q)$ which can be chosen is $\ell_1!\cdots\ell_p!/(\ell_1\cdots\ell_p)$, since there are $\ell!/\ell$ cycles corresponding to a given set. The number $\sum_\sigma X_{C_1}(\sigma)\cdots X_{C_q}(\sigma)$ of σ containing $C_1,\cdots,C_q$ is just $(n-\ell_1-\cdots-\ell_p)!$ since we are free to permute $\{1,\cdots,n\}-\bigcup_{k=1}^p S_k$. The integrals in (2.5) depend only on P and $S_1,\cdots,S_p$. Therefore

$$I_p = \sum_{\ell_1 \cdots \ell_p} \left[\ell_1 \cdots \ell_p \binom{n}{\ell_1, \cdots, \ell_p, n-\ell_1-\cdots-\ell_p} \right]^{-1} \sum_{S_1 \cdots S_p} \int \left| \sum_{\nu \in C_1} a_\nu \phi_\nu \cdots \sum_{\nu \in C_q} a_\nu \phi_\nu \right| . \quad (2.6)$$

By Hölder's inequality for q functions, the last integral is not more than

$$\left(\prod_{i=1}^{q} \int \left| \sum_{\nu \in C_i} a_\nu \phi_\nu \right|^q \right)^{1/q} .$$

The concavity of the q^{th} root then gives

$$\sum_{S_1 \cdots S_p} \left(\prod_{i=1}^{q} \int \left| \sum_{\nu \in C_i} a_\nu \phi_\nu \right|^q \right)^{1/q}$$

$$\leq \left(\sum_{S_1 \cdots S_p} 1 \right)^{1-\frac{1}{q}} \left(\sum_{S_1 \cdots S_p} \prod_{i=1}^{q} \int \left| \sum_{\nu \in C_i} a_\nu \phi_\nu \right|^q \right)^{1/q} . \quad (2.7)$$

But $\sum 1$, the number of $(S_1, \cdots, S_p)$ such that $|S_k| = \ell_k$ $(k=1, \cdots, p)$ is equal to

$$\binom{n}{\ell_1, \cdots, \ell_p, n-\ell_1-\cdots-\ell_p}$$

and so

$$I_p \leq \sum_{\ell_1 \cdots \ell_p} (\ell_1 \cdots \ell_p)^{-1} \binom{n}{\ell_1, \cdots, \ell_p, n-\ell_1-\cdots-\ell_p}^{-\frac{1}{q}} \left(\sum_{S_1 \cdots S_p} \prod_{i=1}^{q} \int \left| \sum_{\nu \in C_i} a_\nu \phi_\nu \right|^q \right)^{\frac{1}{q}} . \quad (2.8)$$

Now write

$$\prod_{i=1}^{q} \int \Big|\sum_{\nu \in C_i} a_\nu \phi_\nu\Big|^{2n} = \int \Big| \sum_{N \in C_1 \times \cdots \times C_q} a_N \phi_N(X)\Big|^{2n} dv$$

$$= \sum_{N_1 \cdots N_n M_1 \cdots M_n \in C_1 \times \cdots \times C_q} \int \prod_{j=1}^{n} a_{N_j} \overline{a}_{M_j} \phi_{N_j} \overline{\phi}_{M_j} \, dv. \quad (2.9)$$

Here $N = (\nu_1, \cdots, \nu_q)$, $X = (x_1, \cdots, x_q)$, $dv = dx_1 \cdots dx_q$, $a_N = a_{\nu_1} \cdots a_{\nu_q}$, and $\phi_N(x) = \phi_{\nu_1}(x_1) \cdots \phi_{\nu_q}(x_q)$. Then

$$\sum_{S_1 \cdots S_p} \prod_{i=1}^{q} \int \Big|\sum_{\nu \in C_i} a_\nu \phi_\nu\Big|^{q}$$

$$\leq \sum_{\text{all } N_1, \cdots, N_n, M_1, \cdots, M_n} \zeta(N_1, \cdots, N_n, M_1, \cdots, M_n) \Big| \int \prod_{j=1}^{n} a_{N_j} \overline{a}_{M_j} \phi_{N_j} \overline{\phi}_{M_j} \, dv \Big|,$$

where $\zeta(N_1, \cdots, N_n, M_1, \cdots, M_n)$ is the number of $(S_1, \cdots, S_p)$ such that $|S_k| = \ell_k$ $(k = 1, \cdots, p)$, and $N_1, \cdots, N_n, M_1, \cdots, M_n$ are all in the $C_1 \times \cdots \times C_q$ determined by the partition P and by $(S_1, \cdots, S_p)$.

The crucial step in the proof comes now — we use the fact that $\phi_1, \cdots, \phi_n$ is a P-system for n . This gives us that

$$\int \prod_{j=1}^{n} \phi_{N_j}(X) \, \phi_{M_j}(X) \, dv \geq 0,$$

this integral being a product of integrals of form (1.4). Hence

$$\sum_{S_1 \cdots S_p} \prod_{i=1}^{q} \int \Big|\sum_{\nu \in C_i} a_\nu \phi_\nu\Big|^{q}$$

$$\leq (\max \zeta) \sum_{\text{all } N_1 \cdots N_n M_1 \cdots M_n} \int \prod_{j=1}^{n} |a_{N_j} a_{M_j}| \phi_{N_j} \bar{\phi}_{M_j} \, dv$$

$$= (\max \zeta) \left(\int \left| \sum_{\nu=1}^{n} |a_\nu| \phi_\nu(x) \right|^q dx \right)^q, \qquad (2.10)$$

where ζ is maximized over all $(N_1, \cdots, N_n, M_1, \cdots, M_n)$ with $\ell_1, \cdots, \ell_p$ fixed. This maximum is easy to obtain. The requirement that $N_1, \cdots, N_n, M_1, \cdots, M_n$ all belong to $C_1 \times \cdots \times C_q$ is, when written out (letting $N_\ell = (\nu_{\ell 1}, \cdots, \nu_{\ell q})$, $M_\ell = (\mu_{\ell 1}, \cdots, \mu_{\ell q})$ $(\ell = 1, \cdots, n)$),

$$\nu_{1i}, \cdots, \nu_{ni}, \mu_{1i}, \cdots, \mu_{ni} \in C_i \quad (i = 1, \cdots, q).$$

For each k $(k = 1, \cdots, p)$ choose an i in the k^{th} class of P and let n_k be one of the $\nu_{1i}, \cdots, \mu_{ni}$. Define $N = (\nu_1, \cdots, \nu_q)$, where $\nu_i = n_k$ if i is in the k^{th} class. Then $\{N_1, \cdots, N_n, M_1, \cdots, M_n\} \subseteq C_1 \times \cdots \times C_q$ implies $N \in C_1 \times \cdots \times C_q$. Hence $\zeta(N_1, \cdots, M_n) \leq \zeta(N, \cdots, N)$, which is simply the number of $(S_1, \cdots, S_q)$ such that $n_k \in S_k$, $|S_k| = \ell_k$ $(k = 1, \cdots, p)$, and the S_k are disjoint. This number is

$$\binom{n-p}{\ell_1 - 1, \cdots, \ell_p - 1, n - \ell_1 - \cdots - \ell_p} = \max \zeta.$$

Putting this in (2.10) and referring to (2.8) gives

$$I_p \leq [n(n-1)\cdots(n-p+1)]^{-1/q} \sum_{\ell_1 \cdots \ell_p} (\ell_1 \cdots \ell_p)^{\frac{1}{q} - 1} \int |\hat{T}|^q, \qquad (2.11)$$

where the sum is over all $(\ell_1, \cdots, \ell_p)$ such that $\ell_k \geq 1$ $(k = 1, \cdots, p)$ and $\ell_1 + \cdots + \ell_p \leq n$.

To complete the proof of (1.6), the crude estimate

$$\sum_{\ell_1 \cdots \ell_p} (\ell_1 \cdots \ell_p)^{\frac{1}{q}-1} < \left(\sum_{\ell=1}^{n} \ell^{\frac{1}{q}-1}\right)^p < \left(\int_0^n \ell^{\frac{1}{q}-1} d\ell\right)^p = (q n^{1/q})^p$$

is good enough. (It turns out that

$$\int_{\ell_k \geq 0,\, \sum \ell_k \leq n} (\ell_1 \cdots \ell_p)^{\frac{1}{q}-1} d\ell_1 \cdots d\ell_p = \frac{[\Gamma(1/q)]^p}{\Gamma(1+p/q)} n^{p/q}$$

$$\sim (q n^{1/q})^p \text{ as } q \to \infty.)$$

Now

$$n^p / [n(n-1) \cdots (n-p+1)] \leq p^p / p!$$

since $n \geq p$ and the left side is a decreasing function of n. Finally

$$I_p \leq q^p (p^p/p!)^{1/q} \int |\hat{T}|^q$$

$$\leq q^{q+1} (q!)^{-1/q} \int |\hat{T}|^q$$

since $p \leq q$. This, together with (2.4), gives (1.6), and Theorem 1 is proved.

III. Discussion of Theorem 1.

In this section we will write $T = \sum_{\nu=1}^{n} a_\nu \phi_\nu$ as before, where the ϕ_ν will be specified as needed. Given the ϕ_ν we will denote the expression $(1/n!)\int [S^*(x,\sigma,T)]^q dx$ by $J_q(T)$.

Theorem 1 says that

$$J_{2r}(T) \leq A_r \int |\hat{T}|^{2r} \tag{3.1}$$

if (ϕ_ν) is a P-system for r. We would like to know whether we can replace $\int |\hat{T}|^{2r}$ by a "smaller" quantity. The obvious candidate is $\int |T|^{2r}$, but we do not know whether

$$J_q(T) \leq A_q \int |T|^q \tag{3.2}$$

holds or not for trigonometric polynomials if $q > 2$. Theorem 1 would be "best possible" and (3.2) could not hold if we could prove that

$$J_q(T) \geq A_q \int |\hat{T}|^q, \tag{3.3}$$

but we will see later that (3.3) is false for $q = 2r > 2$. However if the a_ν are real it is easy to see that

(3.4) For some σ (depending on T and $q \geq 1$),

$$\| S^*(\cdot,\sigma,T)\|_q \geq \frac{1}{2} \|\hat{T}\|_q . \tag{3.5}$$

We merely write $T = T^+ - T^-$, $\hat{T} = T^+ + T^-$, where $T^+ = \sum a_\nu^+ \phi_\nu$, $T^- = \sum a_\nu^- \phi_\nu$. We may suppose $\|T^+\|_q \geq \|T^-\|_q$. Thus $\|\hat{T}\|_q \leq \|T^+\|_q + \|T^-\|_q \leq 2\|T^+\|_q \leq 2\|S^*(\cdot,\sigma,T)\|_q$, where σ is any permutation which puts the terms of T^+ first. (There are at least $(n/2)!$ of these σ.) In the complex case, a basically similar argument (see Appendix 2) gives $\|S^*(x,\sigma,T)\|_{2r} \geq A_r \|\hat{T}\|_{2r}$ for some σ, provided that (ϕ_ν) is a P-system for r.

Returning to the problem of whether (3.1) is best possible in some sense, we can obtain some definite results by comparing $J_q(T)$ to expressions $N_q(|a|)$ which depend only on the absolute value sequence $|a| = (|a_1|, |a_2|, \cdots)$. The trivial observation $S^*(x,\sigma,T) \geq |T(x)|$ for all σ gives

$$J_q(T) \geq \int |\hat{T}|^q, \tag{3.6}$$

which in turn yields

(3.7) <u>Theorem 1 is the best result of the form</u>

$$J_q(T) \leq N_q(|a|) \tag{3.8}$$

<u>within a factor</u> A_q.

For if (3.7) holds then $N_q(|a|) \geq J_q(\hat{T}) \geq \int |\hat{T}|^q$, i.e., $\int |\hat{T}|^q$ is the smallest $N_q(|a|)$ such that (3.8) can hold.

The observation (3.6) easily gives us a result of form

$$J_q(T) \geq N_q(|a|) \tag{3.9}$$

in the opposite direction. Assume that the ϕ_ν are orthonormal on an interval of length $h < \infty$, so that for $q \geq 2$ we have $h^{1/2-1/q}\|T\|_q \geq \|T\|_2 = \|\hat{T}\|_2 = \left(\sum_1^n |a_\nu|^2\right)^{1/2}$. Then (3.6) gives

$$J_q(T) \geq h^{1-q/2}\left(\sum_{\nu=1}^n |a_\nu|^2\right)^{q/2}. \tag{3.10}$$

(3.11) <u>Let</u> $q = 2r$. <u>If</u> (ϕ_ν) <u>is a uniformly bounded</u> $(|\phi_\nu| \leq M)$ <u>orthonormal system on an interval of length</u> h <u>then (3.10) is the best inequality of form (3.9), within a factor</u> $A(r, M, h)$.

To prove this it will be sufficient to show

(3.12) <u>For any positive integer</u> r <u>and nonnegative</u> $b_1, \cdots, b_n$ <u>there is a</u> $T = \sum_1^n a_\nu \phi_\nu$ <u>with</u> $|a_\nu| = b_\nu$ $(\nu = 1, \cdots, n)$ <u>such that</u>

$$J_{2r}(T) \leq A_r \int \left(\sum_{\nu=1}^n b_\nu^2 |\phi_\nu|^2\right)^r. \tag{3.13}$$

For then under the assumptions of (3.11) we have

$J_{2r}(T) \leq A_r h M^{2r}\left(\sum |a_\nu|^2\right)^r$ for this T.

In (3.12) the ϕ_ν are arbitrary. The method of proof will be to average (3.1) over sign changes of coefficients. In the case of real functions and coefficients let $R_1(\omega), \cdots, R_n(\omega)$ $(0 \leq \omega \leq 1)$ be the first n Rademacher functions. In the complex case let $R_1(\omega), \cdots, R_n(\omega)$ be independent random variables each of which takes on the values $\exp\left[i\, 2\pi k/(r+1)\right]$ $(k = 0, 1, \cdots, r)$ with equal probability. The latter R_ν have been contrived so that

$$\int R_\nu^\alpha \bar{R}_\nu^\beta = \int R_\nu^{\alpha-\beta} = \delta_{\alpha\beta} \quad (\alpha, \beta = 0, 1, \cdots, r;\ \nu = 1, \cdots, n). \quad (3.14\,C)$$

In the real case we have

$$\begin{aligned} \int R_\nu^\alpha &= 1 \quad \alpha \text{ even} \\ &= 0 \quad \alpha \text{ odd} \end{aligned} \quad (\nu = 1, \cdots, n). \qquad (3.14\,R)$$

These properties ensure that the functions $\psi_\nu(\omega, x) = R_\nu(\omega)\,\phi_\nu(x)$ form a P-system for r on $\{(\omega, x)\}$. (The complex R_ν are needed only if the ϕ_ν are not themselves a P-system.) We can thus apply Theorem 1 to $U(\omega, x) = \sum_1^n b_\nu \psi_\nu(\omega, x) = \hat{U}(\omega, x)$. This gives

$$\frac{1}{n!} \sum_\sigma \int d\omega \int dx\, [S^*(\omega, x; \sigma; U)]^{2r} \le A_r \int d\omega \int dx\, |\hat{U}(\omega, x)|^{2r}. \quad (3.15)$$

In the real case, [1] chap. V, theorem 8.4 gives

$$\int \Big| \sum_{\nu=1}^n b_\nu \phi_\nu(x) R_\nu(\omega) \Big|^{2r} d\omega \le A_r \Big(\sum_{\nu=1}^n |b_\nu \phi_\nu(x)|^2 \Big)^r, \quad (3.16)$$

and the same is true in the complex case. For if $c_\nu = b_\nu \phi_\nu(x)$ $(\nu = 1, \cdots, n;$ x fixed) then

$$\int \left| \sum_{\nu=1}^{n} c_\nu R_\nu(\omega) \right|^{2r} d\omega$$

$$= \sum \binom{r}{\alpha_1 \cdots \alpha_n} \binom{r}{\beta_1 \cdots \beta_n} c_1^{\alpha_1} \bar{c}_1^{\beta_1} \cdots c_n^{\alpha_n} \bar{c}_n^{\beta_n}$$

$$\cdot \left(\int R_1^{\alpha_1 - \beta_1} \right) \left(\int R_2^{\alpha_2 - \beta_2} \right) \cdots \left(\int R_n^{\alpha_n - \beta_n} \right),$$

where $\alpha_i \geq 0$, $\beta_i \geq 0$ $(i = 1, \cdots, n)$, $\alpha_1 + \cdots + \alpha_n = \beta_1 + \cdots + \beta_n = r$. By (3.14c) this expression becomes

$$\sum \binom{r}{\alpha_1 \cdots \alpha_n}^2 |c_1|^{2\alpha_1} \cdots |c_n|^{2\alpha_n} \leq A_r \left(\sum_{\nu=1}^{n} |c_\nu|^2 \right)^r,$$

where

$$A_r = \max_{(\alpha_i)} \binom{r}{\alpha_1 \cdots \alpha_n}.$$

From (3.15) and (3.16) we obtain

$$\int J_{2r}(U(\omega, \cdot))\, d\omega \leq A_r \int \left(\sum_{\nu=1}^{n} b_\nu^2 |\phi_\nu(x)|^2 \right)^r dx.$$

Hence (3.13) holds with $T(x) = U(\omega, x)$ for appropriate ω.

A consequence of (3.11) is that (3.3) cannot hold for the trigonometric system. For if $q > 2$ there is no upper bound on $\|\hat{T}\|_q / \|\hat{T}\|_2$ as $n \to \infty$. (Let T be the n^{th} Féjèr kernel.) In fact it is easy to show that the last statement holds for any complete orthonormal P-system for n $(q = 2n > 2)$ on a bounded interval.

The behavior of $J_{2n}(T)$ under sign changes of coefficients as revealed by (3.1), (3.6), (3.10), and (3.12) gives us reason to believe that (3.2) may be true for systems satisfying the requirements of (3.11), say. For if we keep $\hat{T}$ fixed and vary the signs of the coefficients of T, the quantities $[J_{2n}(T)]^{1/2n}$ and $\|T\|_{2n}$ both run between the extremes $\|\hat{T}\|_{2n}$ and $\|\hat{T}\|_2$ (within constants not depending on n), and furthermore both quantities reach these two extremes (within constants) for the same sign changes.

IV. Proof of Theorem 2.

Given (n_k) and σ, set

$$S_0 = 0, \quad S_n = \sum_{\nu=1}^{n} a_\nu \phi_\nu, \quad S_n(x,\sigma) = \sum_{\nu=1}^{n} a_{\sigma_\nu} \phi_{\sigma_\nu}(x) \quad (n = 1, 2, \cdots),$$

$$S_*(x) = \sup_k |S_{n_k}(x)|,$$

$$\delta_k^*(x,\sigma) = \max_{n_k < n \le n_{k+1}} |S_n(x,\sigma) - S_{n_k}(x,\sigma)|,$$

$$\hat{\delta}_k = \sum_{\nu=n_k+1}^{n_{k+1}} |a_\nu| \phi_\nu . \qquad (k = 0, 1, \cdots)$$

Notice that $S_{n_k}(x,\sigma) = S_{n_k}(x)$ for all k, x, and σ since σ preserves the blocks $(n_k, n_{k+1}]$. Hence for $n_k + 1 \le n \le n_{k+1}$ we write

$$S_n(x,\sigma) = S_{n_k}(x) + (S_n(x,\sigma) - S_{n_k}(x,\sigma)),$$

$$|S_n(x,\sigma)|^{2r} \le 2^{2r-1}(|S_{n_k}(x)|^{2r} + |S_n(x,\sigma) - S_{n_k}(x,\sigma)|^{2r})$$

$$\le 2^{2r-1}(|S_{n_k}(x)|^{2r} + [\delta_k^*(x,\sigma)]^{2r}).$$

Hence

$$[S^*(x,\sigma)]^{2r} \le 2^{2r-1}(S_*^{2r}(x) + \sum_{k=0}^{\infty} [\delta_k^*(x,\sigma)]^{2r}). \qquad (4.1)$$

Theorem 1 applied to the $\delta_k^*(x,\sigma)$ can be stated as

$$\int_\Omega d\sigma \int_0^{2\pi} dx\, [\delta_k^*(x,\sigma)]^{2n} \leq A_n \int_0^{2\pi} |\hat{\delta}_k|^{2n}.$$

Therefore

$$\int_\Omega d\sigma \int_0^{2\pi} dx\, [S^*(x,\sigma)]^{2n} \leq 2^{2n-1} \left(\int_0^{2\pi} S_*^{2n} + \sum_{k=0}^{\infty} \int_0^{2\pi} |\hat{\delta}_k|^{2n} \right). \quad (4.2)$$

The theory of Littlewood and Paley [1, chap. XV] on Fourier series of L_p functions for $p>1$ yields

$$\int_0^{2\pi} S_*^{2n} \leq A(\alpha,n) \int_0^{2\pi} |f|^{2n}, \quad \sum_{k=0}^{\infty} \int_0^{2\pi} |\hat{\delta}_k|^{2n} \leq A(\alpha,n) \int_0^{2\pi} |\hat{f}|^{2n}$$

which with (4.2) proves (1.13).

A slightly weaker form of Theorem 2 in which the sequence (n_k) may depend on $\hat{f}$ can be proved by using the following special case of M. Riesz' theorem on partial sums of Fourier series of functions in L_p $(p>1)$ instead of the deep Littlewood-Paley theory.

(4.3) Let $f \in X_{2n}$. The partial sums S_n and $\hat{S}_n$ of the Fourier series of f and $\hat{f}$ satisfy

a) $\|S_n\|_{2n} \leq \|\hat{S}_n\|_{2n} \leq \|\hat{f}\|_{2n}$

b) $\|f - S_n\|_{2n} \leq \|\hat{f} - \hat{S}_n\|_{2n} \to 0$ as $n \to \infty$.

Part (a) follows from (1.10) and the fact that $S_n \prec \hat{S}_n \prec f$.

For part (b) we note similarly that $\hat{f}-\hat{S}_n = \widehat{f-S_n}$; thus we need only prove that $\hat{S}_n \to \hat{f}$ strongly in L_{2r}. We proceed as in M. Riesz' theorem. Given $\varepsilon > 0$ let $T(x)$ be the Cesàro mean $\sigma_N(x,\hat{f})$, where N is chosen so that $\|\hat{f}-\sigma_N(\cdot,\hat{f})\|_{2r} < \varepsilon$. Letting $S_n(T)$ and $S_n(\hat{f}-T)$ denote the partial sums of the Fourier series of T and $\hat{f}-T$, we have

$$\hat{S}_n = S_n(T) + S_n(\hat{f}-T),$$

$$\hat{S}_n - \hat{f} = S_n(T) - \hat{f} + S_n(\hat{f}-T)$$

$$= T - \hat{f} + S_n(\hat{f}-T)$$

if $n \geq N$. But $\hat{f}-T$ has nonnegative coefficients. By (a),

$$\|\hat{S}_n - \hat{f}\|_{2r} \leq \|\hat{f}-T\|_{2r} + \|\hat{f}-T\|_{2r} < 2\varepsilon \quad (n \geq N).$$

We now prove our weak form of Theorem 2.

If $n_k + 1 \leq n \leq n_{k+1}$ $(k = 1, 2, \cdots)$ write

$$S_n(x,\sigma) = f(x) + (S_{n_k}(x) - f(x)) + (S_n(x,\sigma) - S_{n_k}(x,\sigma)).$$

This decomposition leads to

$$[S^*(x,\sigma)]^{2r} \le [\delta_0^*(x,\sigma)]^{2r}$$

$$+ 3^{2r-1}\left(|f(x)|^{2r} + \sum_{k=1}^{\infty} |S_{n_k}(x) - f(x)|^{2r} + \sum_{k=1}^{\infty} [\delta_k^*(x,\sigma)]^{2r}\right).$$

Integrating over Ω and using Theorem 1 gives

$$\int_{\Omega} d\sigma \int_0^{2\pi} dx \, [S^*(x,\sigma)]^{2r} \le A_r \int_0^{2\pi} |\hat{\delta}_0|^{2r}$$

$$+ 3^{2r-1}\left(\int_0^{2\pi} |f|^{2r} + \sum_{k=1}^{\infty} \int_0^{2\pi} |S_{n_k} - f|^{2r} + A_r \sum_{k=1}^{\infty} \int_0^{2\pi} |\hat{\delta}_k|^{2r} \right). \quad (4.4)$$

By (4.3), $\int |\hat{\delta}_0|^{2r} \le \int |\hat{f}|^{2r}$ and the partial sums $\hat{S}_n$ of the Fourier series of $\hat{f}$ tend to $\hat{f}$ in L_{2r} . Hence we can make $n_1, n_2, \cdots$ increase so fast that

$$\sum_{k=1}^{\infty} \int_0^{2\pi} |\hat{S}_{n_k} - \hat{f}|^{2r} \le \int_0^{2\pi} |\hat{f}|^{2r},$$

say. (We may assume that f is not a.e. zero; all we are saying is that this sum may be made arbitrarily small.) Then also

$$\sum_{k=1}^{\infty} \int_0^{2\pi} |S_{n_k} - f|^{2r} \leq \int_0^{2\pi} |\hat{f}|^{2r}, \quad \sum_{k=1}^{\infty} \int_0^{2\pi} |\hat{\delta}_k|^{2r} \leq \int_0^{2\pi} |\hat{f}|^{2r},$$

since $\hat{f} - \hat{S}_{n_k}$ majorizes $S_{n_k} - f$ and $\hat{\delta}_k$. Hence (1.13) follows with $A(\alpha, r)$ replaced by A_r.

V. Proof of Theorem 3.

We are to show that there exists a function $g \sim \sum_1^\infty a_\nu \phi_\nu$ with positive coefficients a_ν such that $\int |g|^{2r} < \infty$ but $B_{2r}^{\star}(a) = \infty$. The proof is carried out in two stages. First we obtain an estimate on the L_{2r} norm of finite sums with nonnegative coefficients, averaged over all rearrangements of the coefficients. Next, such sums will be pieced together to form the function g . The arguments are due jointly to the author and Professor Garsia.

Rearrangements of coefficients in finite sums. Given a sum $\sum_1^n b_\nu \phi_\nu$ with $b_\nu \geq 0$ $(\nu = 1, \cdots, n)$ we will establish an upper bound on the expression

$$I = \frac{1}{n!} \sum_\sigma \int \Big| \sum_{\nu=1}^{n} b_{\sigma_\nu} \phi_\nu \Big|^{2r}, \tag{5.1}$$

where r is a positive integer. As usual, $\sum_\sigma$ sums over all permutations $\sigma = (\sigma_1, \cdots, \sigma_n)$ of $\{1, \cdots, n\}$. Expanding the $2r^{th}$ power gives

$$I = \frac{1}{n!} \int \sum_\sigma \sum_{\nu_1 \cdots \nu_r \mu_1 \cdots \mu_r} \prod_{j=1}^{r} b_{\sigma_{\nu_j}} \phi_{\nu_j} b_{\sigma_{\mu_j}} \bar{\phi}_{\mu_j} . \tag{5.2}$$

We classify the $(\nu_1, \cdots, \nu_r, \mu_1, \cdots, \mu_r)$, writing

$$I = \sum_s \sum_{\alpha, \beta} k(\alpha, \beta) \sum_N \frac{1}{n!} \sum_\sigma b_{\sigma_{n_1}}^{\gamma_1} \cdots b_{\sigma_{n_s}}^{\gamma_s}$$

$$\cdot \int \phi_{n_1}^{\alpha_1} \bar{\phi}_{n_1}^{\beta_1} \cdots \phi_{n_s}^{\alpha_s} \bar{\phi}_{n_s}^{\beta_s} , \tag{5.3}$$

where $1 \le s \le \min(2r, n)$, $\alpha = (\alpha_1, \cdots, \alpha_s)$, $\beta = (\beta_1, \cdots, \beta_s)$, $\alpha_i \ge 0$, $\beta_i \ge 0$, $\gamma_i = \alpha_i + \beta_i$ $(i = 1, \cdots, s)$, $1 \le \gamma_1 \le \gamma_2 \le \cdots \le \gamma_s$, $\alpha_1 + \cdots + \alpha_s = \beta_1 + \cdots + \beta_s = r$, $N = (n_1, \cdots, n_s)$ where the n_i are distinct, and $k(\alpha, \beta)$ is a positive combinatorial factor bounded above by an A_r.

The sum over σ in (5.3) does not depend on the exact identity of the n_i; thus

$$I = \sum_s \sum_{\alpha,\beta} k(\alpha,\beta) \frac{(n-s)!}{n!} \sum_M b_{m_1}^{\gamma_1} \cdots b_{m_s}^{\gamma_s}$$

$$\cdot \int \sum_N \phi_{n_1}^{\alpha_1} \bar{\phi}_{n_1}^{\beta_1} \cdots \phi_{n_s}^{\alpha_s} \bar{\phi}_{n_s}^{\beta_s}, \tag{5.4}$$

where $M = (m_1, \cdots, m_s)$, the m_i distinct. Now since $b_\nu \ge 0$, and the ϕ_ν form a P-system for r, we will remove the restriction that M and N consist of distinct indices. This adds only nonnegative terms to the right side of (5.4). Thus

$$I \le \sum_s \sum_{\alpha,\beta} k(\alpha,\beta) \frac{(n-s)!}{n!} \sum_{\nu=1}^{n} b_\nu^{\gamma_1} \cdots \sum_{\nu=1}^{n} b_\nu^{\gamma_s}$$

$$\cdot \int \sum_{\nu=1}^{n} \phi_\nu^{\alpha_1} \bar{\phi}_\nu^{\beta_1} \cdots \sum_{\nu=1}^{n} \phi_\nu^{\alpha_s} \bar{\phi}_\nu^{\beta_s}$$

$$= \sum_s \sum_{\alpha,\beta} k(\alpha,\beta)\, U(\alpha,\beta). \tag{5.5}$$

We will estimate the various $U(\alpha,\beta)$ (which are nonnegative). Let $|\phi_\nu| \le M \ (\nu = 1, 2, \cdots)$ and let the length of the interval be h . First observe

$$\int \Big|\sum_{\nu=1}^{n} \phi_\nu\Big|^{\ell} = \int \Big|\sum \phi_\nu\Big|^{\ell-2} \Big|\sum \phi_\nu\Big|^{2}$$

$$\le (Mn)^{\ell-2} \int \Big|\sum \phi_\nu\Big|^2 = M^{\ell-2} n^{\ell-1}, \qquad (5.6)$$

if $\ell \ge 2$. (This is a primitive form of (1.16).) Also note that

$$\frac{(n-s)!}{n!} \le A_r \frac{1}{n^s} .$$

<u>Case 1</u>: $\gamma_1 \ge 2$. We simply estimate $|\phi_\nu| \le M$ and thus

$$U(\alpha,\beta) \le \frac{A_r}{n^s} \sum b_\nu^{\gamma_1} \cdots \sum b_\nu^{\gamma_s} \, M^{\gamma_1} n \cdots M^{\gamma_s} n \, h$$

$$= A_r h M^{2r} \sum b_\nu^{\gamma_1} \cdots \sum b_\nu^{\gamma_s} = U_1 . \qquad (5.7.1)$$

<u>Case 2</u>: $\gamma_1 = \cdots = \gamma_\ell = 1, \ \ell \ge 2,$ and $\gamma_{\ell+1} \ge 2$ in case $\ell < s$. Here (using 5.6)

$$\int \sum \phi_\nu^{\alpha_1} \bar{\phi}_\nu^{\beta_1} \cdots \sum \phi_\nu^{\alpha_s} \bar{\phi}_\nu^{\beta_s}$$

$$\le M^{\gamma_{\ell+1}} n \cdots M^{\gamma_s} n \int \Big|\sum \phi_\nu^{\ell} \le M^{2r-2} n^{s-1},$$

$$U(\alpha,\beta) \le A_r M^{2r-2} \frac{1}{n} \Big(\sum b_\nu\Big)^{\ell} \sum b_\nu^{\gamma_{\ell+1}} \cdots \sum b_\nu^{\gamma_s} = U_2 . \ (5.7.2)$$

Case 3. $\gamma_1 = 1$ and $\gamma_2 \geq 2$ in case $s \geq 2$. By Schwarz' Inequality,

$$\int (\sum \phi_\nu^{\alpha_1} \bar{\phi}_\nu^{\beta_1})(\sum \phi_\nu^{\alpha_2} \bar{\phi}_\nu^{\beta_2} \cdots \sum \phi_\nu^{\alpha_s} \bar{\phi}_\nu^{\beta_s})$$

$$\leq \left[\int |\sum \phi_\nu|^2 \int |\sum \phi_\nu^{\alpha_2} \bar{\phi}_\nu^{\beta_2} \cdots \sum \phi_\nu^{\alpha_s} \bar{\phi}_\nu^{\beta_s}|^2 \right]^{1/2}$$

$$\leq [n(M^{\gamma_2} n \cdots M^{\gamma_s} n)^2 h]^{1/2} = h^{1/2} M^{2r-1} n^{s-1/2},$$

$$\mho(\alpha,\beta) \leq A_r h^{1/2} M^{2r-1} n^{-1/2} \sum b_\nu \sum b_\nu^{\gamma_2} \cdots \sum b_\nu^{\gamma_s} = \mho_3 . \quad (5.7.3)$$

Piecing together finite sums. Our function g will be obtained by modifying the series $\sum_{\nu=1}^{\infty} \nu^{-\lambda} \phi_\nu$, where $\lambda = 1 - 1/2r$, r being an integer greater than 1. If $0 = n_0 < n_1 < n_2 < \cdots$ is a sequence of integers (to be determined later) let $\sigma = (\sigma_1, \sigma_2, \cdots)$ be a permutation of $\{1, 2, \cdots\}$ which rearranges the terms in each block $\{n_k + 1, \cdots, n_{k+1}\}$ $(k = 0, 1, 2, \cdots)$ among themselves. Write

$$\Phi_k(x,\sigma) = \sum_{\nu = n_k + 1}^{n_{k+1}} \sigma_\nu^{-\lambda} \phi_\nu(x), \qquad (k = 0, 1, \cdots)$$

$$B_k = \sum_{\nu = n_k + 1}^{n_{k+1}} \nu^{-2\lambda r} \nu^{2r-2} = \sum_{n_k+1}^{n_{k+1}} \frac{1}{\nu} .$$

Then $\sum_0^\infty B_k = \sum_1^\infty \nu^{-1} = \infty$. Choose positive convergence factors $\varepsilon_0 \geq \varepsilon_1 \geq \varepsilon_2 \geq \cdots$ such that $\sum_0^\infty \varepsilon_k < \infty$. The factor

ε_k will multiply the k^{th} block of $\sum \nu^{-\lambda}\phi_\nu$. Make (n_k) increase so fast that

$$\sum_{k=1}^{\infty} \varepsilon_k^{2r} B_k = \infty . \tag{5.8}$$

Let $I_{m,n}$ be the expression I of (5.1) associated with $\sum_{m+1}^{n} \nu^{-\lambda}\phi_\nu$. (Here $n-m$ replaces n in (5.1).). We will show later that $I_{m,n} \le A(r,M,h)$ for all m and n. Since $I_{n_k, n_{k+1}}$ is the average of $\int |\Phi_k(x,\sigma)|^{2r} dx$ over all σ , there exists σ such that

$$\int |\Phi_k(x,\sigma)|^{2r} dx \le A(r,M,h)$$

for all k. Then $\sum_0^\infty \varepsilon_k \|\Phi_k(\cdot,\sigma)\|_{2r} < \infty$ and hence $\sum_0^\infty \varepsilon_k \Phi_k(x,\sigma)$, when written out, is the Fourier series of a $g \in L_{2r}$ whose coefficients are $a_\nu = \varepsilon_k \sigma_\nu^{-\lambda}$ for $n_k + 1 \le \nu \le n_{k+1}$, with decreasing rearrangement $a_\nu^* = \varepsilon_k \nu^{-\lambda}$. (This is why we took $\varepsilon_k \downarrow$.) Thus (5.8) states that $B_{2r}^*(a) = \infty$.

It remains to be proved that $I_{m,n} \le A(r,M,h)$. By (5.5) and (5.7.1, 2, 3) we need only prove that the expressions U_1 , U_2 , U_3 associated with $\sum_{m+1}^{n} \nu^{-\lambda}\phi_\nu$ are all bounded by an $A(r,M,h)$. Evidently $\sum b_\nu^\gamma \le \sum_{m+1}^n \nu^{-3/2} \le A$, an absolute constant greater than 1, if $\gamma \ge 2$ and $r \ge 2$. Since $s \le 2r$,

$$U_1 \le A_r h M^{2r} A^s \le A_r h M^{2r} A^{2r}.$$

Then to prove the assertion for U_2 we observe that

$$\frac{1}{n-m}\left(\sum b_\nu\right)^\ell = \frac{1}{n-m}\left(\sum_{m+1}^n \frac{1}{\nu^\gamma}\right)^\ell$$

$$< \frac{1}{n-m}\left(\int_m^n x^{-\gamma}dx\right)^\ell = (2r)^\ell \frac{(n^{1/2r} - m^{1/2r})^\ell}{n-m}$$

$$= (2r)^\ell n^{\ell/2r-1}\frac{(1-x)^\ell}{1-x^{2r}} \qquad \left(x = \left(\frac{m}{n}\right)^{1/2r}\right)$$

$$= (2r)^\ell n^{\ell/2r-1}\frac{(1-x)^{\ell-1}}{1+x+\cdots+x^{2r-1}} < (2r)^\ell n^{\ell/2r-1} \le (2r)^{2r}$$

since $\ell \le 2r$. Similarly for U_3 consider

$$\frac{1}{(n-m)^{1/2}} \sum b_\nu < 2r \frac{n^{1/2r} - m^{1/2r}}{(n-m)^{1/2}}$$

$$= 2r\, n^{1/2r - 1/2} \frac{1-x}{(1-x^{2r})^{1/2}} = 2r\, n^{1/2r-1/2}\left[\frac{1-x}{1+x+\cdots+x^{2r-1}}\right]^{\frac{1}{2}}$$

$$< 2r\, n^{1/2r - 1/2} \leq 2r,$$

since $2r > 2$.

If we form a probability space Ω from the σ as in Theorem 2, we can see that there are many σ which give a $g \in L_{2r}$. The statement $I_{n_k, n_{k+1}} \leq A(r, M, h)$ becomes

$$\int_\Omega d\sigma \int dx\, |\Phi_k(x,\sigma)|^{2r} \leq A(r, M, h).$$

By convexity of the $2r^{th}$ power, we have $\int_\Omega \|\Phi_k(\cdot,\sigma)\|_{2r} d\sigma \leq A(r,M,h)$ also.

Then

$$\sum_{k=0}^{\infty} \varepsilon_k \int_\Omega \|\Phi_k(\cdot,\sigma)\|_{2r}\, d\sigma < \infty.$$

Hence for a.e. σ in Ω , $\sum_0^\infty \varepsilon_k \|\Phi_k(\cdot,\sigma)\|_{2r} < \infty$ and $g_\sigma \sim \sum_0^\infty \varepsilon_k \ \Phi_k(\cdot,\sigma)$ is in L_{2r} .

A corollary of the proof of Theorem 3 is that $\|\hat{f}\|_{2r}$ is a "smaller" expression for finite sums than $B^*_{2r}(a)$:

(5.9) Let $\phi_1, \phi_2, \cdots$ be a uniformly bounded $(|\phi_\nu| \leq M \ (\nu=1,2,\cdots))$ orthonormal P-system for an $r \geq 2$ on an interval of length $h < \infty$. Then for each n there is a sum $T = \sum_1^n a_\nu \phi_\nu$ with $a_\nu > 0 \ (\nu = 1,\cdots,n)$ such that

$$(\log n) \int |T|^{2r} \leq A(r,M,h)\, B^{*2r}_{2r}(a). \qquad (5.10)$$

The relation (5.10) holds for $T(x) = \Phi_0(x,\sigma)$ if we set $n_1 = n$ and choose σ so that $\int |\Phi_0(x,\sigma)|^{2r} dx \leq A(r,M,h)$. Always $B_0 \approx \log n_1$.

Theorem 3 for the trigonometric system. In case that $\phi_1, \phi_2, \cdots$ is the exponential or cosine system, we can state Theorem 3 in a different form by using the following theorem of Hardy and Littlewood [4][1, vol. II, P. 129]:

(5.11) Let $a_1 \geq a_2 \geq a_3 \geq \cdots$ decrease to zero. The series $\sum_1^\infty a_\nu \phi_\nu$ is the Fourier series of an $h \in L_p$, $p > 1$, if and only if

$$\sum_{\nu=0}^{\infty} a_\nu^p \, \nu^{p-2} < \infty,$$

<u>and in this case the ratio</u>

$$\int_0^{2\pi} |h|^p \Big/ \sum a_\nu^p \nu^{p-2}$$

(<u>excluding</u> $h=0$ <u>a.e.</u>) <u>lies between two positive constants which depend only on</u> p .

We immediately deduce modified forms of Theorem 3 and (5.9) when $\phi_1, \phi_2, \cdots$ is the exponential or cosine system.

(5.12) <u>For each integer</u> $r \geq 2$ <u>there is a function</u> $g^* \sim \sum_1^\infty a_\nu^* \phi_\nu$ <u>with</u> $a_1^* \geq a_2^* \geq a_3^* \geq \cdots \downarrow 0$ <u>such that</u>

a) $\int_0^{2\pi} |g^*|^{2r} = \infty$

b) <u>there is a rearrangement</u> $a_1, a_2, \cdots$ <u>of</u> $a_1^*, a_2^*, \cdots$ <u>such that</u> $\sum_1^\infty a_\nu \phi_\nu$ <u>is the Fourier series of a</u> $g \in L_{2r}$.

(5.13) <u>For each</u> r <u>and</u> n <u>there is a trigonometric polynomial</u> $T = \sum_1^n a_\nu \phi_\nu$ <u>with</u> $a_\nu > 0 \ (\nu = 1, \cdots, n)$ <u>such that</u>

$$(\log n) \int_0^{2\pi} |T|^{2r} \leq A_r \int_0^{2\pi} \Big|\sum_{\nu=1}^{n} a_\nu^* \phi_\nu\Big|^{2r}. \tag{5.14}$$

We do not know whether the $\log n$ may be replaced by something larger.

Hardy and Littlewood [5] and Gabriel [6] proved that if $q \geq 2$ and $\sum_1^\infty a_\nu^* \phi_\nu \sim g^* \in L_q$ then $\sum_1^\infty a_\nu \phi_\nu \sim g \in L_q$ and

$$\int_0^{2\pi} |g|^{2r} \leq A_q \int_0^{2\pi} |g^*|^{2r}, \tag{5.15}$$

where $A_{2r}=1$ if r is an integer. But we have never seen any strengthening of this result in the direction of (5.12) or (5.13). The rearrangement of $(a_\nu^\star)$ which gives (5.14) was not explicitly constructed; we might ask whether this would be possible for the case of the cosine system. Perhaps the nondecreasing rearrangement of the coefficients gives the least value of $\|T\|_{2r}$.

Appendix 1

(A1.1) a) If (ϕ_ν) is a linearly independent P-system for r then $N(T) = \|\hat{T}\|_{2r}$ is a norm on the space of finite linear combinations of the ϕ_ν .

b) The set $X_{2r} \subseteq L_{2r}$ of $f \sim \sum_{-\infty}^{\infty} a_\nu e^{i\nu x}$ such that $\sum_{-\infty}^{\infty} |a_\nu| e^{i\nu x}$ is the Fourier series of an $\hat{f} \in L_{2r}$ is a Banach space under the norm $\|\hat{f}\|_{2r}$.

Proof. a) We need only show the triangle inequality. Recall that (1.9) is true for trigonometric polynomials with no assumptions on the $e^{i\nu x}$ other than that they form a P-system. Thus we may use (1.9) here. If $T = \sum_{\nu=1}^{n} a_\nu \phi_\nu$, $U = \sum_{\nu=1}^{n} b_\nu \phi_\nu$ then

$$\widehat{T+U} \prec \hat{T} + \hat{U},$$

$$\|\widehat{T+U}\|_{2r} \leq \|\hat{T} + \hat{U}\|_{2r}$$

$$\leq \|\hat{T}\|_{2r} + \|\hat{U}\|_{2r}. \tag{A1.2}$$

b) Applying (1.9) to two elements of X_{2r} exactly as in (A1.2), we have that X_{2r} is a linear space and $\|\hat{\ }\|_{2r}$ is a norm on it. To prove completeness, given $f_n \in X_{2r}$ $(n = 1, 2, \cdots)$ satisfying

$$\|\widehat{f_n - f_m}\|_{2r} \to 0 \quad as \quad n, m \to \infty \qquad (A1.3)$$

we will first identify the element of X_{2r} to which (f_n) will converge. Since $\widehat{f_n - f_m}$ is a majorant of both $f_n - f_m$ and $\hat{f}_n - \hat{f}_m$, (A1.3) and (1.9) imply that (f_n) and $(\hat{f}_n)$ are both Cauchy sequences in L_{2r}. Thus there are $f, g \in L_{2r}$ such that $\|f_n - f\|_{2r} \to 0$, $\|\hat{f}_n - g\|_{2r} \to 0$. This also implies convergence of the Fourier coefficients of these functions. Thus $g = \hat{f}$, $f \in X_{2r}$.

To prove $\|\widehat{f_n - f}\|_{2r} \to 0$, first observe that $\|\widehat{f_n - f}\|_2 = \|f_n - f\|_2 \to 0$ as $n \to \infty$, and that $\widehat{f_n - f_m} - \widehat{f_n - f} \prec \widehat{f_m - f}$ for all m, n. Then for each n, $\|\widehat{f_n - f_m} - \widehat{f_n - f}\|_2 \le \|\widehat{f_m - f}\|_2 \to 0$ as $m \to \infty$. Hence there is a subsequence (f_{m_k}) (depending on n) such that $\widehat{f_n - f_{m_k}} \to \widehat{f_n - f}$ a.e. as $k \to \infty$. By Fatou's Lemma, $\|\widehat{f_n - f}\|_{2r} \le \liminf_{k \to \infty} \|\widehat{f_n - f_{m_k}}\|_{2r}$, which tends to 0 as $n \to \infty$ by (A1.3).

Appendix 2

(A2.1) Let (ϕ_ν) be a complex-valued P-system for n, and $T = \sum_{\nu=1}^{n} a_\nu \phi_\nu$, a_ν complex. Then there is a σ such that

$$\int [S^*(x,\sigma,T)]^{2n} dx \geq A_n \int |\hat{T}|^{2n}. \qquad (A2.2)$$

Proof. Divide up the unit circle of the complex plane into m equal segments $E_0, \cdots, E_{m-1}$, where $E_k = \{e^{ix}: (k-\frac{1}{2})/m < x \leq (k+\frac{1}{2})/m\}$, and classify a_ν according to which segment $e^{i \arg a_\nu}$ falls into. Thus write $T = T_0 + \cdots + T_m$, where $T_k = \sum (a_\nu \phi_\nu : e^{i \arg a_\nu} \in E_k)$ $(k = 0, \cdots, m-1)$. As an approximation to $\hat{T}$ write

$$U = \sum_{k=0}^{m-1} e^{-i 2\pi k/m} T_k$$

$$= \sum_{\nu=1}^{n} |a_\nu| e^{i\theta_\nu} \phi_\nu$$

where $-\pi/m < \theta_\nu \leq \pi/m$. Then

$$\int |U|^{2n} = \sum |a_{\nu_1} \cdots a_{\nu_n} a_{\mu_1} \cdots a_{\mu_n}| e^{i(\theta_{\nu_1} + \cdots + \theta_{\nu_n} - \theta_{\mu_1} - \cdots - \theta_{\mu_n})} \int \phi_{\nu_1} \cdots \phi_{\nu_n} \bar{\phi}_{\mu_1} \cdots \bar{\phi}_{\mu_n}$$

$$= \sum |a_{\nu_1} \cdots a_{\nu_n} a_{\mu_1} \cdots a_{\mu_n}| \cos(\theta_{\nu_1} + \cdots + \theta_{\nu_n} - \theta_{\mu_1} - \cdots - \theta_{\mu_n}) \int \phi_{\nu_1} \cdots \phi_{\nu_n} \bar{\phi}_{\mu_1} \cdots \bar{\phi}_{\mu_n}. \quad (A2.3)$$

If we make $2r \cdot \pi/m = \pi/3$, $m = 6r$, then the cosine is $\geq 1/2$. The integrals in (A2.3) are all nonnegative. Hence

$$\int |U|^{2r} \geq \frac{1}{2} \sum |a_{\nu_1} \cdots a_{\nu_r} a_{\mu_1} \cdots a_{\mu_r}| \int \phi_{\nu_1} \cdots \phi_{\nu_r} \bar{\phi}_{\mu_1} \cdots \bar{\phi}_{\mu_r}$$

$$= \frac{1}{2} \int |\hat{T}|^{2r}.$$

Now we may assume with no loss of generality that the largest of the $\|T_k\|_{2r}$ is $\|T_1\|_{2r}$. Then

$$\|\hat{T}\|_{2r} \leq 2^{1/2r} \|U\|_{2r} \leq 2^{1/2r} \sum_{k=0}^{m-1} \|T_k\|_{2r}$$

$$\leq 2^{1/2r} 6r \|T_1\|_{2r} \leq 2^{1/2r} 6r \|S^*(\cdot, \sigma, T)\|_{2r}$$

for all σ which put the terms of T_1 first.

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