TIME-DEPENDENT MONOENERGETIC NEUTRON TRANSPORT IN TWO ADJACENT SEMI-INFINITE MEDIA

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ABSTRACT

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An exact solution to the monoenergetic Boltzmann equation is obtained for the case of a plane isotropic burst of neutrons introduced at the interface separating two adjacent, dissimilar, semi-infinite media. The method of solution used is to remove the time dependence by a Laplace transformation, solve the transformed equation by the normal mode expansion method, and then invert to recover the time dependence.

The general result is expressed as a sum of definite, multiple integrals, one of which contains the uncollided wave of neutrons originating at the source plane. It is possible to obtain a simplified form for the solution at the interface, and certain numerical calculations are made there.

The interface flux in two adjacent moderators is calculated and plotted as a function of time for several moderator materials. For each case it is found that the flux decay curve has an asymptotic slope given accurately by diffusion theory. Furthermore, the interface current is observed to change directions when the scattering and absorption cross sections of the two moderator materials are related in a certain manner. More specifically, the reflection process in two adjacent moderators appears to depend initially on the scattering properties and for long times on the absorption properties of the

media.

This analysis contains both the single infinite and semi-infinite medium problems as special cases. The results in these two special cases provide a check on the accuracy of the general solution since they agree with solutions of these problems obtained by separate analyses.

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I. STATEMENT OF THE PROBLEM

In this thesis the effect of a discontinuity in material properties on a neutron distribution in time is investigated. An exact solution to one specific problem is obtained, and certain conclusions are made from an investigation of the behavior exhibited by this solution.

When a pulse of neutrons is initiated in a finite homogeneous body, the long-term flux decay is found to be that of a single mode decaying exponentially in time. The exponential decay constant is found to depend to a large extent on the absorption cross section, to a lesser extent on the diffusion coefficient and buckling, and to a still lesser degree on the spectrum of the thermal neutrons.

If the assembly is composed of two adjacent homogeneous regions rather than a single region, the decaying flux will be some function of the properties of both regions. Exactly how the decaying neutron flux is affected by the discontinuity is unknown, both as to the magnitude of the effect and to its explicit dependence on the nuclear properties of the system. The answer to this question was the major objective of the research summarized in this thesis.

In attempting to analyze a time-dependent heterogeneous system to determine neutron flux behavior, the related problem of reflection at an interface also can be critically examined. In fact, an exact solution of a time-dependent heterogeneous problem provides one with an exact description of the mechanism of reflection. Hence an attempt to gain some insight into the reflection of neutrons was made the secondary objective of this research.

For several years prior to the initiation of the research described in this thesis, there had been appearing in the literature exact solutions to certain monoenergetic, single-region, time-dependent and time-independent problems in plane geometry. Several monoenergetic, two-region, time-independent problems had also been analyzed in this geometry. So it appeared that if a monoenergetic, two-region, time-dependent problem were properly posed in plane geometry, its exact solution could perhaps be obtained. Furthermore, the exact solution to a time-dependent, two-region problem would in itself be a contribution to neutron transport theory, since it would demonstrate that the analytical techniques used could be successfully applied to this type of problem.

The problem as initially conceived consisted of two thick slabs of dissimilar material somewhere in which a point burst of high energy neutrons is introduced. An accurate mathematical description of this system requires that the neutron distribution, subsequent to the burst, satisfy a Boltzmann equation. This Boltzmann equation has as independent variables space, time, and velocity (speed and direction). However, the analysis of the physical problem was restricted to a one-speed study to reduce the complexity of the Boltzmann equation, and to identify this work as closely as possible with those problems successfully treated in the literature.

Through this literature study, it also became apparent that the mathematical model of the physical problem would have to be made dependent on only one spatial dimension (slab geometry) if an exact solution were desired. Thus, the point source in the analysis was replaced by a plane source parallel to the interface separating the two slabs.

In an actual system, a point source can be made to approximate a plane source by physically relocating the point source or detector after each of a series of bursts, or by using a set of detectors judiciously spaced for each burst. Thus, the point source system can be physically integrated to obtain the plane source results.

As the investigation of the literature proceeded, it became clear that problems having finite geometries had solutions of a more complicated nature than those having geometries of infinite extent. More exactly, the coefficients in the normal mode expansion for problems of finite geometry were not obtained explicitly in the literature, whereas those coefficients in infinite geometry problems could be, and usually were, written down completely. Hence, the physical problem consisting of two dissimilar slabs was replaced by one of two adjacent, dissimilar, semi-infinite media.

Several further simplifications were made in the mathematical model to decrease certain anticipated analytical difficulties but which in no way restrict the applicability of the analysis. The plane source was located at the interface separating the two media, thereby effectively reducing the problem from a three to a two-region study. Making the source isotropic was another slight modification employed.

The final mathematical simplification was to assume isotropic scattering in the laboratory system for both materials. This is a good approximation only in scattering collisions between neutrons and very heavy nuclei. However, anisotropic scattering effects can be included in an exact solution of the two-region problem and were omitted here again simply to reduce the complexity of the analysis.

In summary, this thesis contains an analysis of the decay of a monoenergetic neutron population in two adjacent, semi-infinite media following a plane isotropic burst of neutrons at the interface. Furthermore, this analysis can be expected to yield both the single half-space and full-space solutions as special cases.

The method of solution can be briefly stated as the removal of the time dependence by a Laplace transformation, followed by the solution of the transformed equation, and concluded by an inversion to recover the time dependence. An exact solution as well as two approximate ones (diffusion and P-1) are developed and compared for certain moderator materials.

The exact solution to this problem contains the uncollided beam of neutrons which never interacts, and which travels at speed v through the two media. In addition, because of the infinite extent of the system, no discrete spatial modes appear in the result.

The detailed solution to this problem is discussed in the remainder of this thesis. First, however, a brief review of the literature is given to provide additional background to the problem.

II. REVIEW OF THE LITERATURE

This chapter is divided into four sections. Section A contains a brief discussion of the Boltzmann equation, the foundation upon which the remainder of the work in this thesis is based. Section B is a review of the development of the normal mode expansion (NME) method for solving the Boltzmann equation in slab geometry. Section C covers work done by several authors on time-dependent problems, utilizing the NME method. Section D is a brief review of the steady-state, two-media problem.

A. Discussion of the Boltzmann Equation

The mathematical development of neutron transport theory has the Boltzmann equation as its basis[1,2]. This equation relates the behavior of a system containing many neutrons to the interaction properties between the neutrons and the bulk media, and it takes into account these changes in the system by their effect on a neutron distribution function. The assumptions used in deriving the Boltzmann equation are listed below.

1. Neutrons are considered as stable particles since their 13-minute half-life is much longer than their average lifetime in an assembly.

2. Neutrons interact only through nuclear forces, which are short ranged. The atomic nuclei are assumed stationary.

3. Neutrons move at constant velocity between collisions.

4. Neutron-neutron collisions are neglected.

5. Collisions between neutrons and atomic nuclei are well defined events and take place instantaneously. As the result of a collision, a neutron may be captured, it may cause fission, or it may be scattered (elastically or inelastically).

6. At a given speed, the probability of collision per unit path length is constant.

7. The neutron density is assumed to be continuous in both position and velocity, and given a neutron distribution at any time it should be possible to determine its behavior uniquely at any subsequent time.

Using these assumptions, it is possible to write down a general transport equation for the neutron distribution. The following symbols are used:

t = time.

$\vec{r}$ = position vector of a neutron.

$\vec{v}$ = velocity vector of a neutron; $\vec{v} = v\vec{\Omega}$, $\vec{\Omega}$ = unit direction vector for the velocity.

$N(\vec{r}, v\vec{\Omega}, t)\, dV dv d\Omega$ = probable number of neutrons at time t in the volume element dV about $\vec{r}$, traveling with speed dv about v in the direction $d\Omega$ about $\vec{\Omega}$.

$\sigma_t(\vec{r}, v)$ = macroscopic cross section (probability of interaction per unit path length), which is assumed to be a function of position and energy only. Similarly σ_s, σ_c, σ_f are the macroscopic cross sections for scattering, capture, and fission interactions, respectively.

Hence, we can write $\sigma_t = \sigma_s + \sigma_c + \sigma_f$.

$f_1(\vec{r}, v'\vec{\Omega}' \rightarrow v\Omega)\, dv d\Omega$ = probability that a neutron, if it collides in dV about $\vec{r}$ with original velocity $v'\vec{\Omega}'$, will scatter into dv about v, and $d\Omega$ about $\vec{\Omega}$.

$f_2(\vec{r}, v'\vec{\Omega}' \rightarrow v\vec{\Omega})\, dv d\Omega$ = probability that a neutron, arising from a fission collision in dV about $\vec{r}$, between a neutron of velocity $v'\vec{\Omega}'$ and a nucleus, will be emitted in dv about v, and $d\Omega$ about $\vec{\Omega}$.

$\alpha(\vec{r}, v)$ = average number of secondary neutrons produced per fission collision between a neutron with speed v and a nucleus in dV about $\vec{r}$.

$S(\vec{r}, v\vec{\Omega}, t)\, dV dv d\Omega dt$ = probable number of neutrons emitted by independent sources in the volume element dV about $\vec{r}$, with speed dv about v, and in the direction $d\Omega$ about $\vec{\Omega}$ at time t to t + dt. Independent sources imply sources whose existence and strength are independent of the neutron population of the system.

Considering the flow of neutrons into and out of a differential volume of phase space then yields the following neutron transport equation:[1]

$$\frac{\partial N}{\partial t}(\vec{r}, v\vec{\Omega}, t) + v\vec{\Omega}\cdot \nabla N + v\sigma_t(\vec{r}, v)\, N =$$

$$\int dv'v' \int d\Omega' \Big[\sigma_s(\vec{r}, v')\, f_1(\vec{r}, v'\vec{\Omega}' \rightarrow v\vec{\Omega}) + \qquad (1)$$

$$\alpha(\vec{r}, v')\sigma_f(\vec{r}, v')f_2(\vec{r}, v'\vec{\Omega}' \rightarrow v\vec{\Omega})\Big]\, N(\vec{r}, v'\vec{\Omega}', t) + S(\vec{r}, v\vec{\Omega}, t).$$

In neutron transport theory one is concerned with the description of neutrons having energies from a few MEV (fission) to approximately 1/40 ev (thermal equilibrium with the surrounding medium). Equation (1) is valid over this energy range[1]. Furthermore, since neutron-neutron collisions are neglected and since the neutrons are assumed to have no major effect on the containing medium, the neutron transport equation is linear.

Boundary conditions to Equation (1) follow from the physical interpretation of $N(\vec{r}, v\vec{\Omega}, t)$. A few are listed below.

1. At the surface of a system which is nonreentrant, $N(\vec{r}, v\vec{\Omega}, t) = 0$ for all $\vec{\Omega}$ entering the system.

2. The number of neutrons coming directly (without collision) from infinity is zero.

3. At a point on an interface between two media, the number of neutrons which leave one medium with a certain velocity will enter the next medium with this velocity unchanged.

These can be simplified in most problems to:

1. $N(\vec{r}, v\vec{\Omega}, t) = 0$ for $\vec{r}$ on the surface and $\vec{\Omega}$ entering the system.

2. (a) $\lim_{\vec{r} \to \infty} N(\vec{r}, v\vec{\Omega}, t) = 0$ if the sources are located in a finite region of space.

   (b) $\lim_{\vec{r} \to \infty} N(\vec{r}, v\vec{\Omega}, t) = Ke^{+r/l}$ if a source at infinity is considered (Milne problem).

3. $N(\vec{r}, v\vec{\Omega}, t)$ is continuous at interfaces.

We shall now make several additional assumptions, consistent with our problem, in order to reduce Equation (1) to a simpler form. First, assuming that the medium is homogeneous results in the following simplifications:

$$\sigma_t(\vec{r}, v) = \sigma_t(v),$$

$$\alpha(\vec{r}, v) = \alpha(v),$$

$$f_1(\vec{r}, v'\vec{\Omega}' \rightarrow v\vec{\Omega}) = f_1(v'\vec{\Omega}' \rightarrow v\vec{\Omega}),$$

$$f_2(\vec{r}, v'\vec{\Omega}' \rightarrow v\vec{\Omega}) = f_2(v'\vec{\Omega}' \rightarrow v\vec{\Omega}).$$

For a system containing several different homogeneous media, these assumptions hold within each medium.

Next, the neutrons are assumed to be monoenergetic; thus,

$$f_1(v'\vec{\Omega}' \rightarrow v\vec{\Omega}) = f_1(\vec{\Omega}' \rightarrow \vec{\Omega})\delta(v' - v),$$

$$f_2(v'\vec{\Omega}' \rightarrow v\vec{\Omega}) = f_2(\vec{\Omega}' \rightarrow \vec{\Omega})\delta(v' - v).$$

Finally, the scattering function is assumed to be independent of angle — isotropic in the laboratory system. To maintain the proper normalization of

$$\int f_{1,2}(\vec{\Omega}' \rightarrow \vec{\Omega})d\Omega' = 1$$

we set accordingly,

$$f_1(\vec{\Omega}' \rightarrow \vec{\Omega}) = f_2(\vec{\Omega}' \rightarrow \vec{\Omega}) = \frac{1}{4\pi}.$$

Substituting these relations into Equation (1) and simplifying yields

$$\frac{\partial N(\vec{r},\vec{\Omega},t)}{\partial t} + v\vec{\Omega}\cdot\nabla N + v\sigma_t N = \frac{(\sigma_s + \alpha\sigma_f)}{4\pi} v \int_{\Omega'} N(\vec{r}, \vec{\Omega}', t)\, d\Omega' + S(\vec{r},\vec{\Omega},t). \tag{2}$$

Since the work in this thesis is restricted to problems having plane symmetry in space and azimuthal symmetry in angle, let us define $\mu \equiv \cos\Omega_x$; hence

$$\frac{\partial N(x,\mu,t)}{\partial t} + v\mu\frac{\partial N}{\partial x} + v\sigma_t N = \frac{(\sigma_s + \alpha\sigma_f)}{2} v \int_{-1}^{+1} N(x,\mu',t)\, d\mu' + S(x,\mu,t). \tag{3}$$

The last change in the basic equation is brought about through the definitions

$$c \equiv \frac{\sigma_s + \alpha\sigma_f}{\sigma_t} \quad , \qquad \sigma_t \equiv \sigma,$$

and

$$\Psi(x,\mu,t) \equiv vN(x,\mu,t).$$

Thus, Equation (3) becomes

$$\frac{1}{v}\frac{\partial\Psi(x,\mu,t)}{\partial t} + \mu\frac{\partial\Psi}{\partial x} + \sigma\Psi = \frac{c\sigma}{2}\int_{-1}^{+1}\Psi(x,\mu',t)\, d\mu' + S(x,\mu,t). \tag{4}$$

It is the solution to this equation which is sought for two adjacent, semi-infinite media in which a plane isotropic burst of neutrons is introduced at the interface. The remainder of this chapter contains a summary of several investigations which have been made of Equations (3) and (4), both in their time-dependent and time-independent forms.

B. The Normal Mode Expansion Method

If Equation (4) is reduced to the time-independent and homogeneous form, one is led to an equation which was considered by Case in 1960[3];

$$\mu \frac{\partial \Psi(y,\mu)}{\partial y} + \Psi = \frac{c}{2} \int_{-1}^{+1} \Psi(y,\mu')d\mu', \tag{5}$$

where

$$y = \sigma x.$$

In Reference (3) a new method for solving Equation (5) is presented.

Based on work done previously in the field of plasmas[4,5], Case finds that both regular and singular — in the sense that they be distributions — solutions to Equation (5) are physically admissible and, in fact, necessary if a complete solution to Equation (5) is desired.

Solutions to Equation (5) are assumed to have the form

$$\psi_\nu(y,\mu) = e^{-y/\nu}\varphi_\nu(\mu) . \tag{6}$$

Substituting Equation (6) into Equation (5) and requiring

$$\int_{-1}^{+1} \varphi_\nu(\mu)d\mu = 1,$$

yields

$$(\nu - \mu)\varphi_\nu(\mu) = \frac{c\nu}{2} . \tag{7}$$

Prior to the Case paper[3], to obtain $\varphi_\nu(\mu)$ one simply divided by the factor $(\nu - \mu)$, thereby implicitly assuming $\nu \notin [-1, 1]$. This operation results in the two solutions to Equation (7) given by

$$\varphi_{o\pm}(\mu) = \frac{c}{2} \frac{\nu_o}{\nu_o \mp \mu}, \tag{8a}$$

where

$$1 = c\nu_o \tanh^{-1} \frac{1}{\nu_o} \tag{8b}$$

(ν_o is real and greater than 1 for $c < 1$, and is pure imaginary for $c > 1$).

What Case observed was that not all of the solutions to Equation (7) are given by Equation (8a). Permitting ν to lie in the interval $[-1, 1]$, Case obtained, as another solution to Equation (7),

$$\varphi_\nu(\mu) = \frac{c}{2} P \frac{\nu}{\nu - \mu} + \lambda(\nu)\delta(\mu - \nu), \tag{9a}$$

where

$$\lambda(\nu) = 1 - c\nu \tanh^{-1} \nu, \tag{9b}$$

and where the P in Equation (9a) indicates principal value integration.

An important property of the solutions to Equation (7), as given by the functions $\varphi_{o\pm}(\mu)$ and $\varphi_\nu(\mu)$, is that these functions are found to be orthogonal over the range $-1 \leq \mu \leq +1$ with a weight function μ. That is

$$\int_{-1}^{+1} \mu \varphi_{\nu'}(\mu) \varphi_\nu(\mu) d\mu = 0 \quad , \quad \nu \neq \nu'. \tag{10}$$

Also the normalization integrals for $\nu = \nu'$ are presented in Reference (3).

A second and more important property of the φ's is that certain of them form a complete set of functions along any portion of the $[-1, 1]$ range over which μ is defined. Moreover, from the proof of completeness provided in Reference (3), a procedure is outlined for determining the coefficients in an expansion of the form (valid for $-1 \leq \mu \leq +1$)

$$f(\mu) = a_{o+}\varphi_{o+}(\mu) + a_{o-}\varphi_{o-}(\mu) + \int_{-1}^{+1} A(\nu)\varphi_{\nu}(\mu)d\nu.$$

This procedure can be used as an alternative of, and an extension to, the full-range orthogonality property.

Hence, the general solution to Equation (5) can be written as

$$\Psi(y, \mu) = a_{o+}\psi_{o+}(y, \mu) + a_{o-}\psi_{o-}(y, \mu) + \int_{-1}^{+1} A(\nu)\psi_{\nu}(y, \mu)d\nu, \qquad (11)$$

where the coefficients, in principle, can be obtained. In Reference (3), the author applies this method of solution to several time-independent problems of interest in plane geometry; the Green's function for a uniform infinite medium, and several half-space problems including the Milne problem. In addition, the solution to the time-dependent equation in an infinite medium is discussed, and this is presented in more detail in Section C.

However, except for the cases where the full-range orthogonality relations are used, simple forms for the expansion coefficients are not determined in Reference (3). The derivation of simple forms for these coefficients in less than full range problems (e.g., for $o \leq \mu \leq 1$) is included in a second paper by Case[6], published in the form of a report in 1961.

For a coherent account of how expansion coefficients in partial μ-space problems are obtained and subsequently simplified, it is necessary to look at the results of the completeness proof[3] in more detail. In addition, since most time-independent problems are similar in their sequence of analysis, the work done on only one problem in this reference will be reviewed – the Milne problem.

In the Milne problem one must solve Equation (5) in the right half plane subject to the boundary conditions;

(a) $\Psi(o,\mu) = 0 \qquad \mu \geqq 0,$

(b) $\Psi(y,\mu) \rightarrow \psi_o(y,\mu)$ as $y \rightarrow \infty,$

where

$$\psi_{o-}(y,\mu) = e^{y/\nu_o}\varphi_{o-}(\mu).$$

The solution to Equation (5) which satisfies condition (b) is

$$\Psi(y,\mu) = \psi_{o-}(y,\mu) + a_{o+}\psi_{o+}(y,\mu) + \int_0^1 A(\nu)\psi_\nu(y,\mu)d\nu. \tag{12}$$

Applying condition (a) yields

$$-\varphi_{o-}(\mu) = a_{o+}\varphi_{o+}(\mu) + \int_0^1 A(\nu)\varphi_\nu(\mu)d\nu. \tag{13}$$

Now the completeness proof[3] provides the following prescription for obtaining these coefficients.

$$A(\nu) = \frac{2}{c\nu}\left[N^+(\nu) - N^-(\nu)\right],$$

$$N_c(z) = \frac{1}{2\pi i X_1(z)} \int_0^1 \frac{\gamma(\mu')\psi'(\mu')d\mu'}{(\mu'-z)},$$

$$\int_0^1 \gamma(\mu')\psi'(\mu')d\mu' = 0 \quad , \quad X_1(z) = \frac{X_{10}(z)}{1-z} \quad , \quad X_{10}(z) = e^{\Gamma_c(z)}, \tag{14}$$

$$\Gamma_c(z) = \frac{1}{2\pi i} \int_0^1 \frac{2i\theta(\mu)d\mu}{\mu - z} \quad , \quad \theta(\mu) = \arg \Lambda^+(\mu) ,$$

$$\Lambda(z) = 1 - \frac{cz}{2} \int_{-1}^{+1} \frac{d\mu}{z-\mu} \quad , \quad \gamma(\mu) = \frac{c\mu}{2} \frac{X_1^-(\mu)}{\Lambda^-(\mu)},$$

$$\psi'(\mu) = -\varphi_{o-}(\mu) - a_{o+}\varphi_{o+}(\mu).$$

By means of these formulae, both a_{o+} and $A(\nu)$ are completely determined — if not explicitly, at least in theory. In fact, from the third relation in Equation (14), the expression for a_{o+} is given in Reference (6) as well as in Reference (3), by

$$a_{o+} = \frac{-\int_0^1 \gamma(\mu')\varphi_{o-}(\mu')d\mu'}{\int_0^1 \varphi_{o+}(\mu')\gamma(\mu')d\mu'} . \tag{15}$$

It is at this point that Case introduces several identities concerning $X_1(z)$, based on Cauchy's integral formula. He had previously[3] obtained an identity for the $X(z)$ found in the full-space completeness theorem, but the general applicability of these identities in determining simple forms for the expansion coefficients did not become known until his 1961 report[6] was published.

For the problem reviewed here, the $X(z)$ identity used to reduce Equation (15) is given as

$$X_1(z) = \int_0^1 \frac{\gamma(\mu')d\mu'}{\mu'-z} . \tag{16}$$

Hence, Case obtains

$$a_{o+} = \frac{X_1(-\nu_o)}{X_1(\nu_o)} \tag{17}$$

and a similar result for $N(z)$, thus permitting $A(\nu)$ to be written down in a simple form as well. In fact, at $x = 0$, the emerging angular flux is found to be

$$\Psi(o,\mu) = \frac{c\nu_o^{\,2}\; X_1(-\nu_o)}{X_1(\mu)\,(\nu_o^2 - \mu^2)} . \tag{18}$$

Once the identities in $X(z)$ were observed to be vital in the simplification of these problems, a number of them were obtained, in addition to those presented in Reference (6). For each problem solved by the NME method and published in the literature, relations for $X(z)$ were usually determined immediately and applied in the simplification of the solution.

In addition to the reductions brought to light due to the X(z) identities, Case also notes, in Reference (6), some possible extensions of the NME method to slightly more complicated problems, all of a time-independent nature. He examines the problem of a critical slab in some detail and discusses the generalization of the NME method to include anisotropic scattering. Finally, he considers the Milne problem for two adjacent half-spaces, which is related quite closely to the work described in this thesis, and certain aspects of which are described in more detail in Section D.

While the usefulness of the NME method was being demonstrated by its application to a variety of problems in slab geometry, and while the method itself was being generalized to include energy, time, anisotropic scattering effects, and changes in geometry, it was not until April 1964[7] that the necessity of using the completeness proof to obtain the expansion coefficients was shown to be unnecessary. The orthogonality relations first proved in the full μ-space by Case had been extended to any fraction of the μ-space by the suitable choice of a weighting function.

In Reference (7) orthogonality relations are developed and applied in the steady-state solution of both half-space and two adjacent half-space problems. Our discussion centers on the development of the two-media orthogonality relations found in Reference (7).

Although the completeness proof is no longer needed to obtain the coefficients in an expansion such as Equation (13), certain results from the completeness proof are needed to obtain the proper weight functions. Hence, the orthogonalization method of obtaining the expansion coefficients is not entirely divorced from the completeness proof.

From the two-media problem, described in Reference (7), the following formulae are obtained for a geometry where subscript 1 implies $x < 0$ and subscript 2 impliex $x > 0$ in Equation (5).

$$\left.\begin{aligned}
&\varphi_{n\pm}(\mu) = \frac{c_n\nu_{0n}}{2}\,\frac{1}{\nu_{0n}\mp\mu} \quad , \quad 1 = c_n\nu_{0n}\tanh^{-1}\frac{1}{\nu_{0n}} \, , \; n = 1,2, \\
&\varphi_\nu(\mu) = \frac{\nu c(\nu)}{2}\, P\,\frac{1}{\nu-\mu} + \ell(\nu)\delta(\nu-\mu) \quad , \quad c(\nu) = \begin{cases} c_2 & \nu > 0 \\ c_1 & \nu < 0 \end{cases} , \\
&\ell(\nu) = 1-\nu c(\nu)\tanh^{-1}\nu \quad , \quad L_k^{\pm}(\nu) = \ell(\nu) \pm \frac{i\pi\nu c(\nu)}{2} \, , \\
&X_k(z) = X_1(-z)\,X_2(z) \quad , \quad \Gamma_k(\mu) = \frac{\mu c(\mu)\,X_k^+(\mu)}{2L_k^+(\mu)} \, .
\end{aligned}\right] \qquad (19a)$$

The $X_n(z)$ are the appropriate half-space functions obtained from the completeness proof, and the angular dependence is described in terms of $\varphi_{1\pm}(\mu)$, $\varphi_{2\pm}(\mu)$, $\varphi_{1\nu}(\mu)$, $\varphi_{2\nu}(\mu)$, in the appropriate half-spaces.

Again, the basis for the method is linked directly to several $X(z)$ identities. The identities of interest for the two-media problem are given in Reference (7) as

$$\left.\begin{aligned}
&\int_{-1}^{+1} \mu^n \Gamma_k(\mu)\,\frac{d\mu}{\mu-z} = z^n X_k(z) \quad , \quad n = 0,1 \, , \\
&\int_{-1}^{+1} \mu^2 \Gamma_k(\mu)\,\frac{d\mu}{\mu-z} = z^2 X_k(z) - 1 \, .
\end{aligned}\right\} \qquad (19b)$$

Based on these identities, the three linearly independent functions, $\mu^n \Gamma_k(\mu)$, $n = 0, 1, 2$, can be combined to include any two of the four discrete eigenfunctions in the orthogonality relations. Kuscer, et al., chose $\varphi_{1-}(\mu)$ and $\varphi_{2+}(\mu)$ by using the weight function

$$(\nu_{01} + \mu)\,(\nu_{02}-\mu)\;\Gamma_k(\mu) \equiv W_k(\mu). \tag{21}$$

Some of the resulting integrals obtained were

$$\left.\begin{aligned}
&\int_{-1}^{+1} W_k(\mu)\varphi_\nu(\mu)\varphi_{\nu'}(\mu)d\mu = W_k(\nu)L_k^+(\nu)L_k^-(\nu)\delta(\nu-\nu'),\\
&\int_{-1}^{+1} \varphi_{2+}(\mu)\varphi_\nu(\mu)W_k(\mu)d\mu = \int_{-1}^{+1} \varphi_{1-}(\mu)\varphi_\nu(\mu)W_k(\mu)d\mu = 0,\\
&\int_{-1}^{+1} \varphi_{1-}(\mu)\varphi_{2+}(\mu)W_k(\mu)d\mu = 0,\\
&\int_{-1}^{+1} \varphi_{2+}^{\,2}(\mu)W_k(\mu)d\mu = -\left(\frac{c_2\nu_{02}}{2}\right)^2 (\nu_{02} + \nu_{01})\, X_k(\nu_{02}),\\
&\int_{-1}^{+1} \varphi_{1-}^{\,2}(\mu)W_k(\mu)d\mu = \left(\frac{c_1\nu_{01}}{2}\right)^2 (\nu_{02} + \nu_{01})\, X_k(-\nu_{01}).
\end{aligned}\right] \tag{22}$$

Thus, the coefficients in steady-state, two-region problems can be obtained with the orthogonality relations listed above. Also, these results contain as special cases both the full and half-space orthogonality relations, as noted in Reference (7), and they can be obtained by a suitable choice of constants.

To summarize then, Equation (5) has been looked at in some detail within the framework of the NME method, and exact solutions have been found to a variety of problems based on this equation. Simplifications to the method of solution have come through the application of identities in X(z) both to expressions for the expansion coefficients obtained from the completeness proof and to the development of certain orthogonality relations.

Extensions of the NME method to include anisotropic scattering effects have been successful and can be found in the literature (Mika[8], Zelazny[9], et al., Shure[10], et al).

The addition of energy dependence to Equation (5) has been examined in some detail for several simple models including both constant cross sections[11, 12] and energy-dependent cross sections[13]. A study of the time-dependent thermalization problem using the NME method has also been made (see for instance, Reference 14). However, attempts to extend the NME method to other geometries have met with little success[15].

Three time-dependent problems have been solved and their results published in the literature. They will be discussed in the next section.

C. Three Time-Dependent Problems

The additional dependence of the neutron flux on time increases the complexity of solutions to Equation (4) as compared to the complexity of the time-independent solutions discussed in the previous section. Two methods of time removal have been applied successfully to Equation (4), and both methods are described in this section.

1. The Single Infinite Medium

In this problem the units are chosen such that $v = \sigma = 1$, reducing Equation (4) to

$$\frac{\partial \Psi(x,\mu,t)}{\partial t} + \mu \frac{\partial \Psi}{\partial x} + \Psi = \frac{c}{2} \int_{-1}^{+1} \Psi(x,\mu',t)d\mu' + S(x,\mu,t). \qquad (21)$$

In Reference (3), Case solves Equation (21) in a uniform infinite medium with

$$S(x,\mu,t) = \delta(x-x_0)\delta(\mu-\mu_0). \qquad (22)$$

Assuming solutions of the form

$$\psi(x,\mu,t) = e^{ikx} e^{-(1+i\alpha k)t} \varphi_{\alpha,k}(\mu), \qquad (23)$$

Case states — based on his work in Reference (5) — that the $\varphi_{\alpha,k}(\mu)$ are orthogonal and complete for $\mu \epsilon [-1,1]$. Hence, the solution can be written down almost immediately as

$$\Psi(x,\mu,t) = \frac{e^{-t}}{2\pi} \int_{-\infty}^{+\infty} e^{ik(x-x_0)} dk \left[\frac{\varphi_{o,k}(\mu_o)\varphi_{o,k}(\mu)}{N_{o,k}} e^{-ik\alpha_o t} \left\{ 1(k+\frac{\pi c}{2}) - 1(k-\frac{\pi c}{2}) \right\} \right.$$

$$\left. + \int_{-1}^{+1} \frac{\varphi_{\alpha,k}(\mu_o)\varphi_{\alpha,k}(\mu) e^{-ik\alpha t} d\alpha}{N_{\alpha,k}} \right], \qquad (24)$$

where all the terms are defined explicitly in Reference (3).

This problem differs from the thesis problem in that in the problem above the material properties are uniform throughout space and the source is angularly dependent, whereas in the thesis problem the material properties are not the same on the two sides of the source plane nor is the source angularly dependent. Although the angular dependence of the source is not a major obstacle in applying this method to the thesis problem, the difference in material properties precludes the use of the type of solutions assumed here (Equation 23), without some additional modifications.

2. The Single Slab

A time-dependent solution to the neutron transport equation in a finite slab was reported by Bowden in 1963[2]. This problem is physically quite different from the thesis problem. Also, due to the finiteness of the geometry, one expects a fundamental mode and higher harmonics to be set up in a slab problem as compared to an absence of modes in the case of two semi-infinite media. However, many similarities in the method of analysis compel us to look in some detail at this work[2]. Both this problem and the one following are summarized in Reference (16).

Based on assumptions of isotropic scattering and a mono-energetic system of neutrons, Bowden gives as the governing equation within a slab, extending from $-a' \leq x \leq +a'$, the homogeneous form of Equation (3), with the accompanying conditions

$$\left.\begin{aligned} N(\pm a', \mu, t) &= 0 \quad , \quad \mu \lessgtr 0, t > 0 , \\ N(x, \mu, 0) &= f'(x, \mu). \end{aligned}\right\}$$

By the change of variables

$$\left.\begin{aligned} y &= \sigma_t x \ , \quad t' = \sigma_t vt \ , \quad f(y,\mu) = f'(x,\mu) \ , \\ u(y,\mu,t) &= N(x,\mu,t)\, e^{\sigma_t vt} \ , \quad a = \sigma_t a', \end{aligned}\right]$$

Bowden obtains from Equation (3) and the related boundary conditions listed above,

$$\left.\begin{aligned} &\frac{\partial u(y,\mu,t)}{\partial t'} + \mu \frac{\partial u}{\partial y} = \frac{c}{2} \int_{-1}^{+1} u(y,\mu',t')d\mu' \ , \\ &u(\pm a,\mu,t') = 0 \ , \quad \mu \gtrless 0, \ t' > 0, \\ &u(y,\mu,0) = f(y,\mu). \end{aligned}\right\} \tag{25}$$

From theorems proved by Lehner and Wing[17], Bowden concludes that the solution to Equation (25) can be written as

$$u(y,\mu,t') = \sum_{j=0}^{N} (f, \Psi^*_{\alpha_j})\, \Psi_{\alpha_j}(y,\mu)\, e^{\alpha_j t'} + \lim_{\omega \to \infty} \frac{1}{2\pi i} \int_{\gamma_1 - i\omega}^{\gamma_1 + i\omega} R_s(y,\mu) e^{st'} ds, \tag{26}$$

where the $\Psi_{\alpha_j}(y,\mu)$ satisfy the equation and boundary conditions

$$\left.\begin{aligned} &-\mu \frac{\partial \Psi_{\alpha_j}(y,\mu)}{\partial y} + \frac{c}{2} \int_{-1}^{+1} \Psi_{\alpha_j}(y,\mu')d\mu' = \alpha_j \Psi_{\alpha_j}(y,\mu) \ , \\ &\Psi_{\alpha_j}(\pm a,\mu) = 0 \ , \quad \mu \gtrless 0. \end{aligned}\right]$$

The $\Psi^*_{\alpha_j}(y,\mu)$ satisfy

$$\left.\begin{array}{l} \dfrac{\mu\partial\Psi^*_{\alpha_j}(y,\mu)}{\partial y} + \dfrac{c}{2}\displaystyle\int_{-1}^{+1}\Psi^*_{\alpha_j}(y,\mu')\,d\mu' = \alpha_j\,\Psi^*_{\alpha_j}(y,\mu)\ , \\ \qquad\qquad\qquad \Psi^*_{\alpha_j}(\pm a,\mu) = 0 \quad , \ \mu \gtrless 0\ . \end{array}\right]$$

$R_s(y,\mu)$ satisfies

$$\left.\begin{array}{l} s\ R_s(y,\mu) + \mu\dfrac{\partial R_s}{\partial y} - \dfrac{c}{2}\displaystyle\int_{-1}^{+1} R_s(y,\mu')d\mu' = f(y,\mu), \\ \qquad\qquad\qquad R_s(\pm a,\mu) = 0\ ,\ \mu \lessgtr 0, \end{array}\right]$$

and

$$(h,g) = \int_{-a}^{+a} dy \int_{-1}^{+1} h(y,\mu)\,\overline{g(y,\mu)}\,d\mu\ .$$

Bowden notes that although Lehner and Wing[17] suggest the form that the solution will assume (Equation 26, et seq.), they do not suggest the shape of the eigenfunctions nor the eigenvalue distribution. To learn of the solution in more detail, Bowden applies the NME method to Equation (25).

Removing the time dependence by a Laplace transformation converts Equations (25) to

$$\left.\begin{array}{l} \mu\dfrac{\partial u_s(y,\mu)}{\partial y} + su_s = \dfrac{c}{2}\displaystyle\int_{-1}^{+1} u_s(y,\mu')d\mu' + f(y,\mu)\ , \\ \qquad\qquad u_s(\pm a,\mu) = 0\ ,\ \mu \gtrless 0\ , \end{array}\right] \qquad (27)$$

where

$$u_s(y,\mu) = \int_0^\infty e^{-st'} u(y,\mu,t')dt' .$$

Bowden first solves the homogeneous part of Equation (27),

$$\mu \frac{\partial \Psi_s(y,\mu)}{\partial y} + s\Psi_s = \frac{c}{2} \int_{-1}^{+1} \Psi_s(y,\mu')d\mu' , \tag{28}$$

and then solves the inhomogeneous equation by the use of a Green's function. In the homogeneous equation, he assumes solutions of the form $\Psi_s(y,\mu) = \varphi_s(\mu,\nu)e^{-sy/\nu}$, and the ν-plane is broken up into two regions, $\nu\epsilon[-1,1]$ and $\nu\not\epsilon[-1,1]$. For $\nu\epsilon[-1,1]$, the expected continuous solution which holds for all s,

$$\left.\begin{aligned} \psi_s(y,\mu) &= \varphi_s(\mu,\nu)e^{-sy/\nu}, \\ \varphi_s(\mu,\nu) &= \frac{c\nu}{2} P\frac{1}{\nu-\mu} + \lambda_s(\nu)\delta(\nu-\mu) , \\ \lambda_s(\nu) &= s - c\nu \tanh^{-1}\nu , \end{aligned}\right\} \tag{29}$$

is obtained. Also, for $\nu\not\epsilon[-1,1]$, there appear the discrete solutions,

$$\left.\begin{aligned} \varphi_s(\mu,\nu) &= \frac{c\nu}{2}\,\frac{1}{\nu-\mu} , \\ s &= c\nu\tanh^{-1}\left(\frac{1}{\nu}\right), \end{aligned}\right\} \tag{30}$$

as anticipated.

However, because s and ν are complex, both the number of discrete solutions and the region of the s-plane where they exist are unknown. Bowden solves both these problems by looking in detail at

$$\Omega_s(\nu) \equiv s - c\nu \tanh^{-1}\left(\frac{1}{\nu}\right) . \tag{31}$$

A discrete solution will exist for that ν which makes $\Omega_s(\nu)$ vanish, provided $\nu \notin [-1, 1]$.

To determine the number of zeroes of $\Omega_s(\nu)$, for a fixed s, in the ν-plane with a branch cut from -1 to +1, Bowden applies a theorem from complex variable theory, the principle of the argument (see for instance, Copson, pg. 119)[18], which has also been applied in neutron transport theory by Zelazny[9]. The theorem states that if a function $f(\nu)$ is regular within and on a closed contour C, except for poles, none of which lie on C, and if $f(\nu)$ is non-zero on C but has zeroes within C, then the excess of the number of zeroes over the number of poles of $f(\nu)$ within C is $\frac{1}{2\pi}$ times the increase in argument of $f(\nu)$ as ν goes once around C (an n^{th} order pole is counted as n first order poles and similarly for the zeroes).

The contour chosen by Bowden in the ν-plane is shown in Figure 1, where $\rho \to 0$ and $R \to \infty$. Along C_R, Bowden notes that $\lim_{R \to \infty} \Omega_s(\nu) = s - c$. He also notes that if $s = c$, the only two solutions of Equation (31) are $\nu_0 = \pm\infty$; $s \neq c$ was assumed in the remainder of the analysis. Now, there is no change in the argument of $\Omega_s(\nu)$ along C_R, nor is there any change in the argument observed by traversing the contours $C_{\rho-}$ and $C_{\rho+}$. Thus,

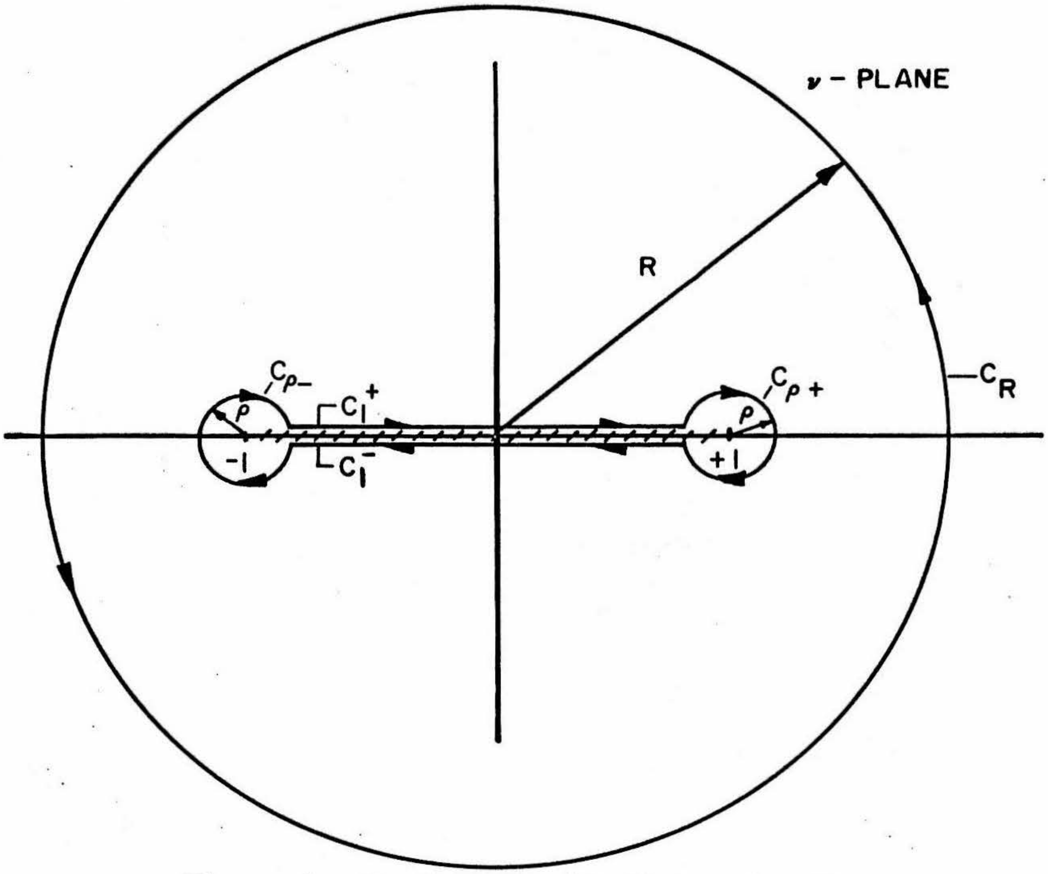

Figure 1 - Contour Used to Determine the Number of Discrete Roots

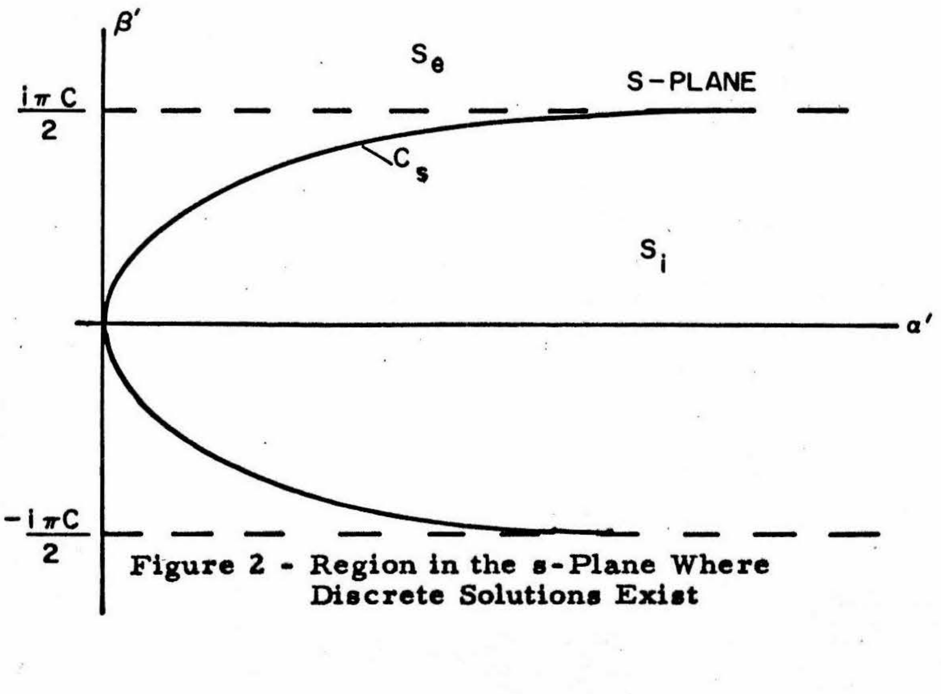

Figure 2 - Region in the s-Plane Where Discrete Solutions Exist

all changes in the argument of $\Omega_s(\nu)$ occur along C_1^+ and C_1^-.

Above and below the branch cut, $\Omega_s(\nu)$ has the functional form

$$\Omega_s^{\pm}(\mu) = s - c\mu \tanh^{-1}\mu \pm i\pi c\mu/2 . \qquad (32)$$

Bowden writes $s = \alpha' + i\beta'$ and obtains

$$\arg \Omega_s^{\pm}(\mu) = \tan^{-1}\left\{ \frac{\beta' \pm \pi c\mu/2}{\alpha' - c\mu \tanh^{-1}\mu} \right\} .$$

Investigating this expression, the author finds that the argument of $\Omega_s(\nu)$ changes by 4π – and hence indicates two zeroes since there are no poles – only within a certain region of the s-plane. Outside this region, the argument does not change by a factor of 2π indicating that no zeroes of $\Omega_s(\nu)$ occur there. By a slight modification of the contour shown in Figure 1, it is also noted that if s lies on the boundary separating these two regions, $\Omega_s(\nu)$ has two zeroes.

The region where $\Omega_s(\nu)$ vanishes, or where the two discrete solutions exist, is defined as S_i. That region where $\Omega_s(\nu)$ does not vanish is defined as S_e. These regions are shown in Figure 2. The equation for C_s (see Figure 2) is given as

$$\alpha' = \frac{2\beta'}{\pi} \tanh^{-1} \frac{2\beta'}{\pi c} .$$

By inspection of Equation (31), one can easily see that if ν_o is a zero, then $-\nu_o$ is also zero. Choosing ν_o to have a positive real part, two discrete solutions can then be written:

$$\left.\begin{aligned} \psi_{s\pm}(y,\mu) &= \varphi_{s\pm}(\mu)\, e^{\mp sy/\nu_o} , \\ \varphi_{s\pm}(\mu) &= \frac{c\nu_o}{2}\,\frac{1}{\nu_o \mp \mu} , \\ s &= c\nu_o \tanh^{-1}\frac{1}{\nu_o} , \quad \mathrm{Re}\,\nu_o > 0 . \end{aligned}\right\} \tag{33}$$

Here for $s \epsilon S_i$, the two discrete solutions defined by Equation (33) and the continuous solution defined in Equation (29) are known to exist, while for $s \epsilon S_e$, only the continuous solution, Equation (29), exists.

Bowden next proves that for $s \epsilon S_i$ the functions $\varphi_{s\pm}(\mu)$ and $\varphi_s(\mu, \nu)$ form a complete set over the space $-1 \leqq \mu \leqq +1$. For $s \epsilon S_e$, $\varphi_s(\mu, \nu)$ is itself complete over the same space. In addition, full-range orthogonalization and normalization integrals are obtained in Reference (2). Thus, the form of solution found for $s \epsilon S_i$ is different from that found for $s \epsilon S_e$ and is characterized by the presence or absence of discrete solutions.

The remainder of the slab problem will be summarized rather briefly. For $s \epsilon S_i$, Bowden then solves Equation (28) subject to the boundary conditions

$$\Psi_s(\pm a, \mu) = 0 \quad , \quad \mu \lesseqgtr 0 .$$

A similar analysis for the adjoint $\Psi_s^*(y, \mu)$ leads to the conclusion that the eigenvalues of the transformed angular flux and its adjoint are found to be identical, and that

$$\Psi_{\alpha_j}^*(y,\mu) = \overline{\Psi_{\alpha_j}(y,\mu)} ,$$

where the bar denotes complex conjugate. Next, the solution to the inhomogeneous problem, Equation (27), is solved in both S_i and S_e with the aid of a Green's function. It is observed[2], based on the work of Lehner and Wing[17], that the solutions in S_i and S_e are representations of a function which is analytic in the right half complex plane of s, save for eigenvalues of the homogeneous problem which lie on the real axis. Hence, by shifting the inversion path to the left but not past the imaginary axis, the author obtains as a final result Equation (26), where now the quantities involved are completely specified.

3. The Single Semi-Infinite Medium

In April 1964, a paper[19] was published on the solution to the time-dependent monoenergetic Boltzmann equation for a single semi-infinite medium. The source applied to the half-space was a burst of neutrons entering the surface of the medium in some preferred direction at $t = 0$. Although there are obvious differences between this single half-space problem and the thesis problem, one can expect many similarities in the analysis of the two problems, since a two adjacent half-space analysis contains a certain single half-space problem as a special case. These similarities in the analysis are discussed in this section.

Starting with the homogeneous form of Equation (4) for $x > 0$, the transformations $v\sigma t = t'$, $\sigma x = y$, yield

$$\frac{\partial \Psi(y,\mu,t')}{\partial t'} + \mu \frac{\partial \Psi}{\partial y} + \Psi = \frac{c}{2} \int_{-1}^{+1} \Psi(x,\mu',t')d\mu' . \qquad (34a)$$

Boundary conditions for the problem are given in Reference (19) as

$$\left.\begin{aligned} \Psi(0,\mu,t') &= \delta(\mu-\mu_o)\,\delta(t') \quad , \quad \mu_o > 0, \\ \lim_{y\to\infty} \Psi(y,\mu,t') &= 0 \quad , \quad t' \geqq 0 \;. \end{aligned}\right\} \qquad (34b)$$

Kuscer and Zweifel also note that when

$$x' = cy \quad , \quad t'' = ct', \text{ and } \Psi(y,\mu,t') = e^{-(1-c)t'}\chi(y,\mu,t'),$$

Equation (34a) becomes

$$\frac{\partial\chi(x',\mu,t'')}{\partial t''} + \mu\frac{\partial\chi}{\partial x'} = \frac{1}{2}\int_{-1}^{+1}\chi(x',\mu',t'')d\mu' \;, \qquad (35a)$$

and the boundary conditions are transformed into

$$\left.\begin{aligned} \chi(0,\mu,t'') &= \delta(\mu-\mu_o)\,\delta(t''), \ \mu_o > 0, \\ \lim_{x'\to\infty} \chi(x',\mu,t'') &= 0 \;,\;, \quad t'' \geqq 0 \;. \end{aligned}\right\} \qquad (35b)$$

Removing the time dependence from Equations (35a, b) by a Laplace transformation, where

$$\Psi_s(x',\mu) = \int_0^\infty e^{(1-s)t''}\chi(x',\mu,t'')dt'' \;,$$

results in

$$\left.\begin{aligned} &\mu\frac{\partial\Psi_s(x',\mu)}{\partial x'} + s\Psi_s = \frac{1}{2}\int_{-1}^{+1}\Psi_s(x',\mu')d\mu' \;, \\ &\lim_{x'\to\infty}\Psi_s(x',\mu) = 0 \quad , \quad \Psi_s(0,\mu) = \delta(\mu-\mu_o) \;, \; \mu_o > 0 \;. \end{aligned}\right\} \qquad (36)$$

Kuscer and Zweifel, when solving Equation (36) by the NME method, also note the existence of two regions in the s-plane where the allowed solutions in Equation (36) are characterized by the presence or absence of discrete terms. A study of the S_i and S_e regions for this problem would yield a picture quite similar to that shown in Figure 2. In this problem, because of the slightly different transformations from those made in the previous problem, one should set $c = 1$ (in Figure 2) to get the correct description of the boundary between the two regions S_i and S_e.

In addition to this small normalization difference, it is observed in Reference (19) that a branch point appears at $s = 1$ and a branch cut in the s-plane exists along the real s-axis for $0 \leq s \leq 1$. As explained in Reference (19), this can be thought of as a limiting case of the slab problem treated earlier[(2)], in which a finite number of poles occur along the real s-axis. We note that as the slab thickness is increased, an increasing number of poles fill the interval on the real s-axis more and more densely and in the limit become a branch cut.

A physical reason for the appearance of the branch cut in the semi-infinite medium problem as opposed to the finite number of poles for the slab is that in the finite medium one expects standing modes to be set up, whereas in the unbounded geometry no standing modes are expected.

Kuscer and Zweifel write down the solution to Equation (36) in each region of the s-plane (again the presence and absence of discrete terms is apparent), and they also note some important simplifications which occur in the transformed angular flux at $x' = 0$. To perform the inversion, it is necessary to have a function analytic in some right half plane. It is known that such a function exists since the transformed Equation (36) has a continuous solution along the real s-axis for $1 < s < \infty$. Also, it can be seen that in S_i and S_e the two representations of $\Psi_s(x', \mu)$ are analytic. What Kuscer and Zweifel show is that there are no singularities of $\Psi_s(x', \mu)$ along the border[20] between the two regions, and that the values assumed by the solution along the border are the same when the border is approached from either side. Hence, the function represented by $\Psi_s(x', \mu)$ in both regions of the s-plane is analytic in the right half s-plane except for a branch cut from 0 to 1. Thus, the inversion path can be shifted to that shown in Figure 3.

The technique sketched in the last paragraph to prove that the solutions to Equation (36) in each region of the s-plane are analytic continuations of each other will also be used, with the same objective in mind, in the thesis problem.

The final form of the solution (not shown here) contains the four integrals $I_1, \ldots, I_4$, and the solution can be reduced still further at $x' = 0$. A final point of interest can be found in their discussion of the results, where Kuscer and Zweifel note the appearance of an uncollided beam of neutrons which also can be expected in the two-media problem.

Several papers have been published which deal with the steady-state, angular flux distribution in two adjacent, semi-infinite media. The next section contains a review of this work.

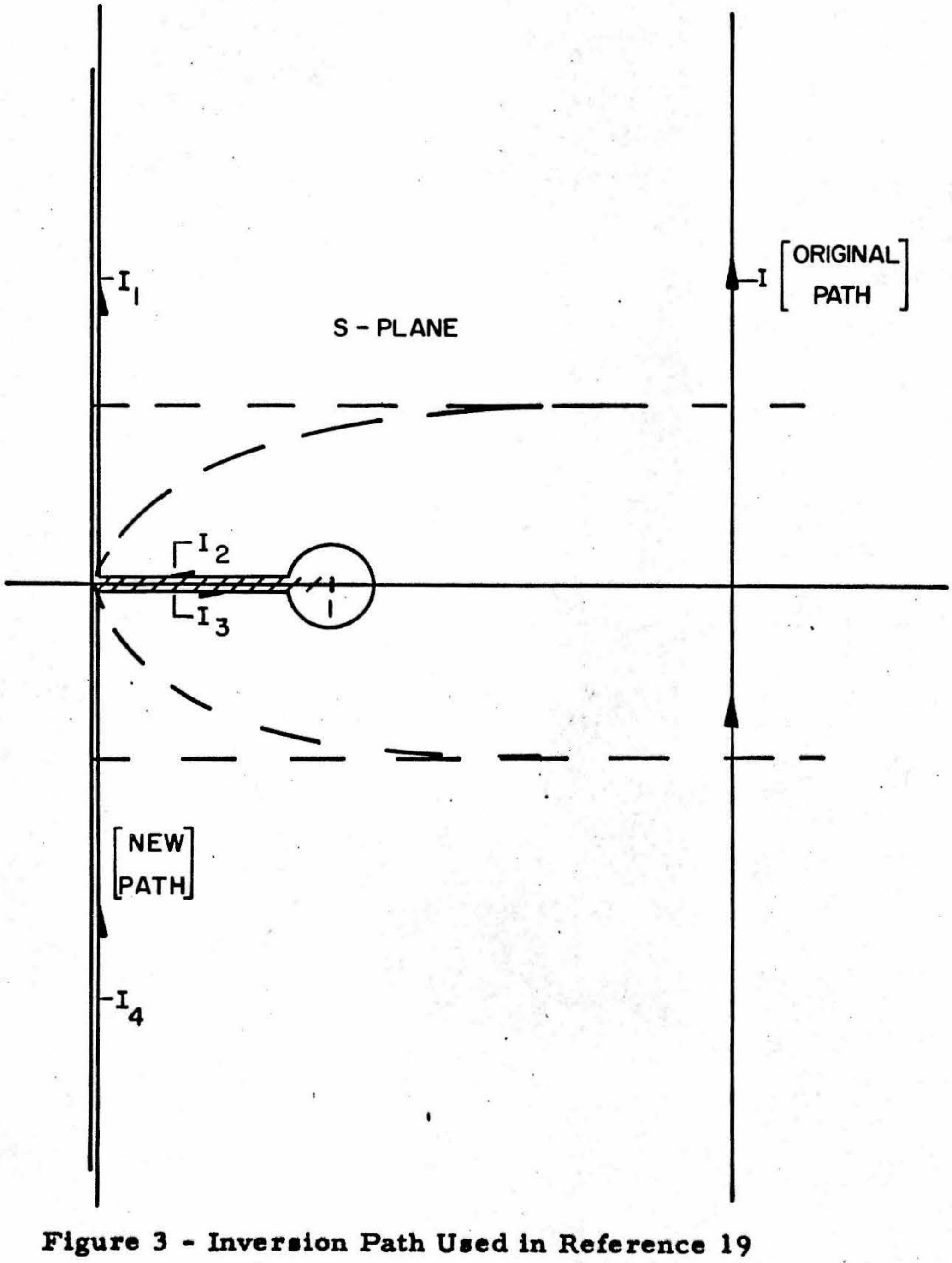

Figure 3 - Inversion Path Used in Reference 19

D. The Steady-State, Two Adjacent Semi-Infinite Media Problem

Prior to the discovery of the NME method, the solutions to problems involving half-spaces were derived with the aid of Fourier transforms in general, and by the use of the Weiner-Hopf method in particular. This method is applied to the two-media problem in References (1) and (21). Another discussion of the two-media problem, utilizing this method, can be found in a paper published in 1957 by K. M. Case[22], where the two media Green's function is obtained for an isotropic plane source.

In applying the NME method to this geometry, one starts with an equation of the form

$$\mu \frac{\partial \Psi(y,\mu)}{\partial y} + \Psi = \begin{cases} \dfrac{c_1}{2} \\ \dfrac{c_2}{2} \end{cases} \int_{-1}^{+1} \Psi(y,\mu')d\mu' \qquad \begin{matrix} y < 0 \\ y > 0 \, , \end{matrix}$$

with boundary and source conditions appropriate to the problem under consideration. Again, both discrete and continuous solutions are obtained, and they can be shown[6] to be complete in μ-space. In fact, the $X(z)$ function for this problem is found to be the product of the two separate half-space functions:

$$X(z) = X_2(z) \; X_1(-z) \; .$$

Hence the expansion coefficients can be determined from either the completeness proof or the orthogonality relations and in principle these problems are solvable.

A more complete examination of the steady-state, two-media problem than that given in Reference (6) can be found in a paper by Mendelson and Summerfield[23]. After a brief review of the completeness proof obtained previously[6] for this geometry, the authors discuss the solutions to several problems of interest - the Milne problem, a uniform source in one half-space, and the Green's function (angularly dependent).

In solving these problems, the authors obtain reductions in the complexity of certain of the solutions at the interface separating the two media through the evaluation of the integrals over the continuous eigenfunctions in ν-space. This appears to be a type of reverse Hilbert problem and its success seems to depend heavily on a "feel" for the problem under consideration. From the brief discussion of this technique provided in Reference (23), major reductions in the complexity of the solution at the interface in the thesis problem are achieved.

On the basis of this brief survey of problems treated by the NME method, it is expected that several factors appearing in these problems should also appear in the solution for the time-dependent flux in the two-media system. First, removing the time by a Laplace transformation should result in a transformed equation which can be solved by the NME method. Next, there should be several regions in the s-plane, where the forms of solution are different but which describe an analytic function in some right half s-plane, save for a portion of the real axis. This portion of the real axis should be a branch, due to the unbounded geometry of the two media.

Finally, some reduction in the complexity of the solution at the interface between the two media should occur.

III. TIME-DEPENDENT NEUTRON TRANSPORT IN TWO ADJACENT SEMI-INFINITE MEDIA

A. Summary

With the assumptions of a monoenergetic neutron spectrum, a system of two homogeneous regions, and isotropic scattering in the laboratory system, the governing equation for the angular flux in each spatial region is (see Equation 4),

$$\frac{1}{v} \frac{\partial \Psi(x,\mu,t)}{\partial t} + \mu \frac{\partial \Psi}{\partial x} + \sigma \Psi = \frac{c\sigma}{2} \int_{-1}^{+1} \Psi(x,\mu',t)d\mu' + S(x,\mu,t) . \quad (37)$$

The geometry for this problem is shown in Figure 4. An isotropic burst of neutrons of strength s [in units of neutrons/unit area x time] is assumed to occur at the interface at $t = 0$. That is

$$S(x,\mu,t) = s\delta(x)\delta(t).$$

A minor simplification to Equation (37) can be made by setting $vt = \tau$ and $vs = q$. Thus the equation, whose solution we seek, is written as

$$\frac{\partial \Psi(x,\mu,\tau)}{\partial \tau} + \mu \frac{\partial \Psi}{\partial x} + \sigma(x)\Psi = \frac{c(x)\sigma(x)}{2} \int_{-1}^{+1} \Psi(x,\mu',\tau)d\mu' + q\delta(x)\delta(\tau) . \quad (38)$$

$c(x)$ and $\sigma(x)$ imply that the cross sections may be different in each half space. $\Psi(x,\mu,\tau)$ is required also to satisfy the boundary

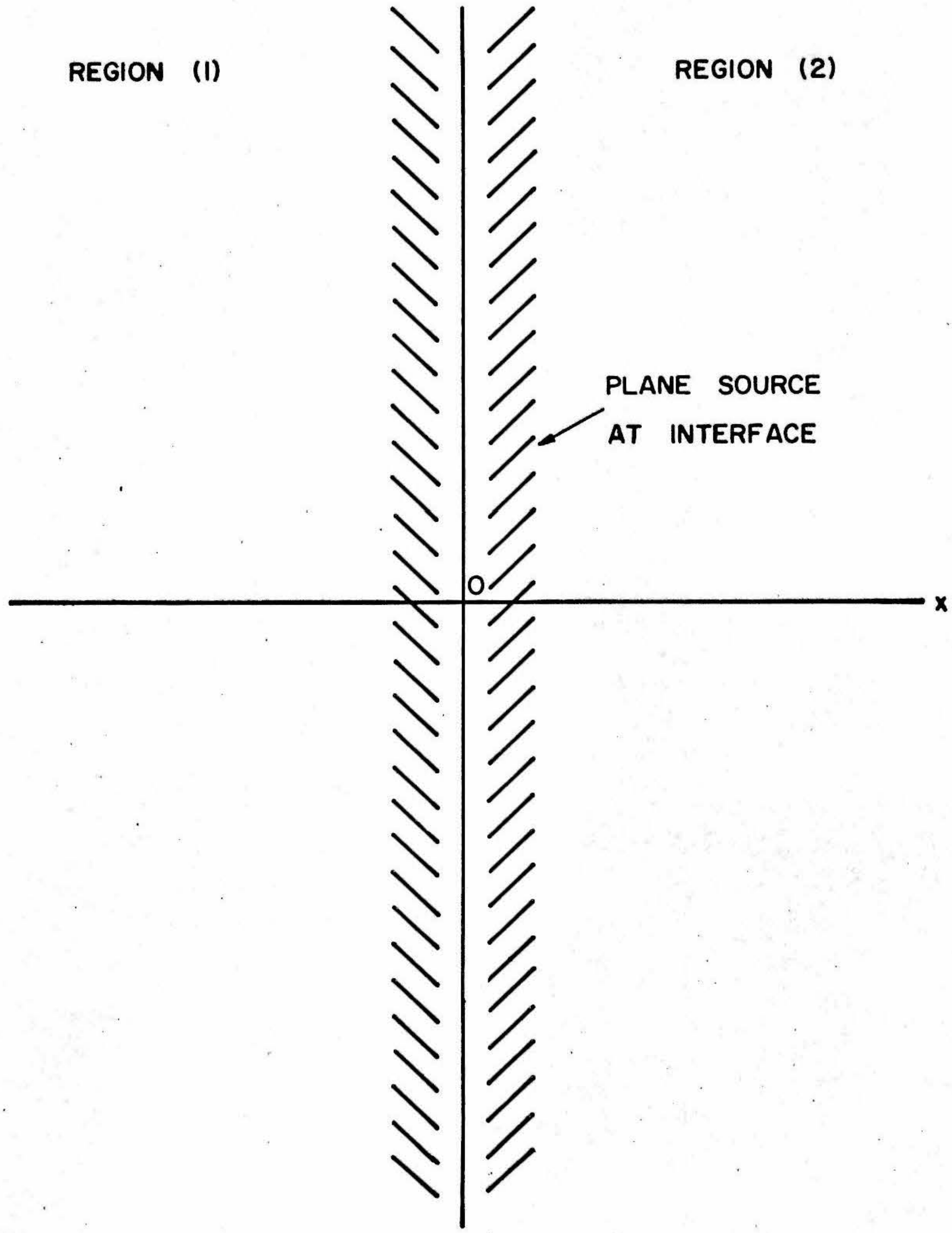

Figure 4 - Geometry for Two Media Problem

and initial conditions,

$$\lim_{x \to \infty} \Psi(x, \mu, \tau) = 0 \quad , \qquad \Psi(x, \mu, 0-) = 0 \tag{39a}$$

and from Equation (38) the jump condition,

$$\mu \left[\Psi(0+, \mu, \tau) - \Psi(0-, \mu, \tau)\right] = q\delta(\tau). \tag{39b}$$

The method of solution to be used is given in sequence below:

1. Remove the τ dependence by a Laplace transformation.

2. Solve the transformed equation in some right half s-plane by the NME method.

3. Determine the properties of the solution there.

4. Invert the solution from s to τ, deforming the path of integration in a suitable manner.

5. Simplify the solution where possible (at the interface).

Steps 1 and 2 are covered in this chapter, Steps 3 and 4 in Chapter IV, and Step 5 in Chapter V.

Once $\Psi(x, \mu, \tau)$ is determined, both the flux and current can be obtained by simply applying the usual definitions

$$\Phi(x, \tau) = \int_{-1}^{+1} \Psi(x, \mu, \tau) d\mu ,$$

and

$$J(x, \tau) = \int_{-1}^{+1} \mu\Psi(x, \mu, \tau) d\mu . \tag{40}$$

B. Time-Removal

If we define $\overline{\Psi}_s(x,\mu)$ by

$$\overline{\Psi}_s(x,\mu) = \int_0^{\infty} e^{-s\tau}\,\Psi(x,\mu,\tau)d\tau \;, \tag{41}$$

then to recover the τ dependence we use

$$\Psi(x,\mu,\tau) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{s\tau}\overline{\Psi}_s(x,\mu)ds \;. \tag{42}$$

Hence, it will be necessary to obtain $\overline{\Psi}_s(x,\mu)$ at least along the path given in the inversion integral(42) where γ is a real number greater than the real parts of the locations of all singularities and branch cuts of $\overline{\Psi}_s(x,\mu)$. It will be observed later that from a knowledge of the behavior of $\overline{\Psi}_s(x,\mu)$ in some right half s-plane we shall be able to deform the inversion path, through the use of Cauchy's theorem, and obtain a simpler form than Equation (42) for our final result.

Multiplying Equation (38) by $e^{-s\tau}$ and integrating over τ from 0 to ∞ yields

$$[s+\sigma(x)]\overline{\Psi}_s(x,\mu) + \mu\,\frac{\partial\overline{\Psi}_s}{\partial x} = \frac{c(x)\sigma(x)}{2}\int_{-1}^{+1}\overline{\Psi}_s(x,\mu')d\mu' + q\delta(x) \;. \tag{43}$$

The transformed boundary conditions (39a) become

$$\lim_{x\to\pm\infty}\overline{\Psi}_s(x,\mu) = 0 \;, \tag{44a}$$

and the initial condition (39b) transforms into the jump condition

$$\mu[\overline{\Psi}_s(0+,\mu) - \overline{\Psi}_s(0-,\mu)] = q \ . \tag{44b}$$

C. General Solution of the Transformed Equation

For $x < 0$, Equation (43) becomes

$$(s+\sigma_1)\overline{\Psi}_s(s,\mu) + \mu \frac{\partial \overline{\Psi}_s}{\partial x} = \frac{c_1\sigma_1}{2}\int_{-1}^{+1} \overline{\Psi}_s(s,\mu')d\mu' \ , \tag{45a}$$

and for $x > 0$,

$$(s+\sigma_2)\overline{\Psi}_s(s,\mu) + \mu \frac{\partial \overline{\Psi}_s}{\partial x} = \frac{c_2\sigma_2}{2}\int_{-1}^{+1} \overline{\Psi}_s(x,\mu')d\mu' \ . \tag{45b}$$

To solve Equation (45a) one assumes a solution of the form

$$\psi_{1\nu}(x,\mu,s) = \varphi_{1\nu}(\mu)\, e^{-(s+\sigma_1)x/\nu} \ . \tag{46}$$

Substituting Equation (46) into (45a) yields

$$(s+\sigma_1)[\nu-\mu]\varphi_{1\nu}(\mu) = \frac{c_1\sigma_1}{2}\,\nu\int_{-1}^{+1} \varphi_{1\nu}(\mu')d\mu' \ .$$

Assuming $\int_{-1}^{+1}\varphi_{1\nu}(\mu')d\mu' = (s+\sigma_1)$, permits us to write this last equation in the simple form

$$(\nu-\mu)\,\varphi_{1\nu}(\mu) = \frac{c_1\sigma_1\nu}{2} \ . \tag{47}$$

Since $-1 \leqq \mu \leqq +1$, when ν is in this range, one should expect a form of solution different from that which occurs when $\nu \notin [-1, 1]$.

We shall look for solutions in both these regions of the ν-plane.

$\nu \epsilon [-1, 1]$. Here the solution to Equation (47) is written in the form

$$\varphi_{1\nu}(\mu) = \frac{c_1\sigma_1}{2} P \frac{\nu}{\nu-\mu} + \lambda_1(\nu)\delta(\nu-\mu) , \tag{48}$$

where P indicates principal value integration.

This form is an extension of the work of Dirac[24] and has been applied successfully in transport theory by Case[3]. To obtain $\lambda_1(\nu)$, one uses the normalization relation; hence

$$\left.\begin{aligned} (s+\sigma_1) &= \int_{-1}^{+1} \varphi_{1\nu}(\mu')d\mu' = \lambda_1(\nu) + \frac{c_1\sigma_1\nu}{2} \ln \frac{1+\nu}{1-\nu} , \\ &\text{or} \\ \lambda_1(\nu) &= (s+\sigma_1) - \frac{c_1\sigma_1\nu}{2} \ln \frac{1+\nu}{1-\nu} . \end{aligned}\right\} \tag{49}$$

$\nu \not\epsilon [-1, 1]$. Here we write the solution to Equation (47) simply as

$$\varphi_{1\nu}(\mu) = \frac{c_1\sigma_1}{2} \frac{\nu}{\nu-\mu} . \tag{50}$$

Requiring the same normalization as in Equation (49) imposes the following relationship between s and ν:

$$s + \sigma_1 = \frac{c_1\sigma_1\nu}{2} \ln \frac{\nu+1}{\nu-1} = c_1\sigma_1\nu\tanh^{-1}\frac{1}{\nu} . \tag{51}$$

So our discrete solutions place a restriction on the values of ν, relating them to s in a manner similar to that found in Bowden's work[2]. In fact, if we let $\frac{s+\sigma_1}{\sigma_1}$ be replaced for the moment by just s, the results of Bowden can be carried over to this problem. Figure 5 shows the region of the s-plane where a solution to Equation (51) exists.

The two solutions to Equation (51) which exist for $s \epsilon S_{1i}$ are defined as ν_{01} and $-\nu_{01}$ (the second solution is the negative of the first) where ν_{01} has a real part > 0. The dependence of ν_{01} on s is sketched in Figure 5.

Since for ν_{01} real and greater than one, the value of s was chosen real, thereby fixing the branch of the logarithm in Equation (51), we note the existence of two series of values of ν_{01} corresponding to a given s when s is real and between $-\sigma_1$ and $-\sigma_1(1-c_1)$. Thus, there exists a branch in the s-plane along the real axis from $-\sigma_1$ to $-\sigma_1(1-c_1)$. This keeps Equation (51) single-valued. A similar condition was encountered in Reference (19).

Summarizing these discrete solutions to Equation (45a), we have

$$\left.\begin{aligned} \psi_{1+}(x,\mu,s) &= \varphi_{1+}(\mu)\, e^{-(s+\sigma_1)x/\nu_{01}} \quad , \\ \psi_{1-}(x,\mu,s) &= \varphi_{1-}(\mu)\, e^{(s+\sigma_1)x/\nu_{01}} \quad , \end{aligned}\right\} \tag{52}$$

where

$$\varphi_{1\pm}(\mu) = \frac{c_1\sigma_1}{2} \quad \frac{\nu_{01}}{\nu_{01}\mp\mu} \quad , \tag{53}$$

and

$$s+\sigma_1 = \frac{c_1\sigma_1\nu_{01}}{2} \ln \frac{\nu_{01}+1}{\nu_{01}-1} \quad ; \quad \text{Re}\,\nu_{01} \geqq 0 \, . \tag{54}$$

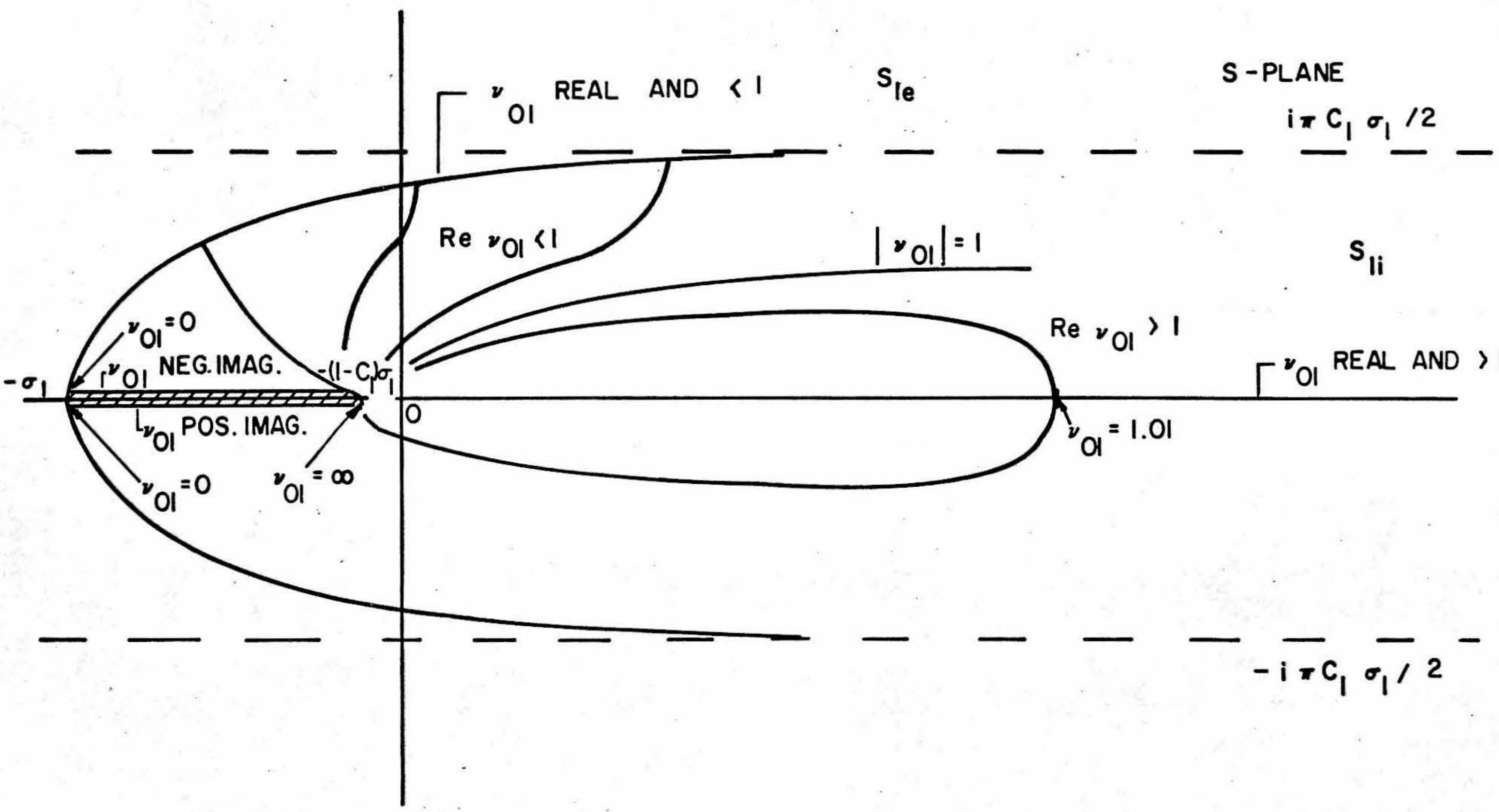

Figure 5 - Region Where Discrete Solutions Exist in the s-Plane

The continuous solution to Equation (45a) is given by Equation (46) with $\varphi_{1\nu}(\mu)$ and $\lambda_1(\nu)$ being defined by Equations (48) and (49) respectively.

A general solution to Equation (45a) can now be written in both S_{1i} and S_{1e}. For $s \varepsilon S_{1i}$,

$$\overline{\Psi}_s(s,\mu) = -a_{1+}\,\psi_{1+}(x,\mu,s) - a_{1-}\,\psi_{1-}(x,\mu,s) - \int_{-1}^{+1} A_1(\nu)\psi_{1\nu}(x,\mu,s)d\nu\ . \tag{55a}$$

For $s \varepsilon S_{1e}$,

$$\overline{\Psi}_s(x,\mu) = -\int_{-1}^{+1} A_1(\nu)\,\psi_{1\nu}(x,\mu,s)d\nu\ . \tag{55b}$$

The $A_1(\nu)$ used in (55a) and (55b) may be different since the regions of the s-plane where each is defined do not overlap. Also, the minus signs are used to simplify certain equations which appear later.

Equation (45b) can be solved in a manner identical to that used to solve Equation (45a); only the subscripts are changed. Hence, for $\nu \varepsilon [-1, 1]$,

$$\left.\begin{aligned}
\psi_{2\nu}(x,\mu,s) &= \varphi_{2\nu}(\mu)\, e^{-(s+\sigma_2)\,x/\nu}\ , \\
\varphi_{2\nu}(\mu) &= \frac{c_2\sigma_2}{2}\, P\,\frac{\nu}{\nu-\mu} + \lambda_2(\nu)\delta(\nu-\mu)\ , \\
\lambda_2(\nu) &= s + \sigma_2 - \frac{c_2\sigma_2\nu}{2}\,\ell n\,\frac{1+\nu}{1-\nu}\ .
\end{aligned}\right\} \tag{56}$$

Also for $s \epsilon S_{2i}$ – adjusting Figure 5 for $x > 0$ – the following discrete solutions occur;

$$\left.\begin{aligned} \psi_{2+}(x, \mu, s) &= \varphi_{2+}(\mu)\, e^{-(s+\sigma_2)x/\nu_{02}} , \\ \psi_{2-}(x, \mu, s) &= \varphi_{2-}(\mu)\, e^{(s+\sigma_2)x/\nu_{02}} , \end{aligned}\right\} \quad (57)$$

where

$$\varphi_{2\pm}(\mu) = \frac{c_2\sigma_2}{2} \frac{\nu_{02}}{\nu_{02} \mp \mu} ,$$

and

$$s + \sigma_2 = \frac{c_2\sigma_2\nu_{02}}{2} \ln \frac{\nu_{02} + 1}{\nu_{02} - 1} , \quad \text{Re}\ \nu_{02} \geqq 0. \quad (59)$$

Thus, the general solution to Equation (45b) is written for $s \epsilon S_{2i}$,

$$\overline{\Psi}_s(x, \mu) = a_{2+}\psi_{2+}(x, \mu, s) + a_{2-}\psi_{2-}(x, \mu, s) + \int_{-1}^{+1} A_2(\nu)\psi_{2\nu}(x, \mu, s)d\nu , \quad (60a)$$

and for $s \epsilon S_{2e}$,

$$\overline{\Psi}_s(x, \mu) = \int_{-1}^{+1} A_2(\nu)\psi_{2\nu}(x, \mu, s)d\nu . \quad (60b)$$

D. Application of the Boundary Conditions at $x = \pm \infty$

The general solutions, Equations (55a, b) and (60a, b) must be modified to insure that they vanish as $x \to \pm \infty$. First let us consider

$s \epsilon S_{1i}$, which implies $x < 0$. From an inspection of the discrete terms (Equation 52)

$$\psi_{1\pm}(x, \mu, s) = \varphi_{1\pm}(\mu)\, e^{\mp(s+\sigma_1)x/\nu_{01}} ,$$

we note that the x-dependence is exhibited only in the exponential factor. If ν_{01} is real and greater than one, $(s+\sigma_1)/\nu_{01}$ is always positive and the behavior of $\psi_{1\pm}(x, \mu, s)$ is readily seen for large x. To learn of the behavior of $\psi_{\pm}(x, \mu, s)$ for any $s \epsilon S_{1i}$, we must look more closely at

$$\frac{s + \sigma_1}{\nu_{01}} \equiv \zeta .$$

From Equation (54), this can be written as

$$\zeta = \frac{c_1 \sigma_1}{2} \ln \frac{\nu_{01} + 1}{\nu_{01} - 1} . \qquad (61)$$

Letting $\nu_{01} = \alpha_1'' + i\beta_1''$ and taking only the real part of ζ, since the imaginary part does not contribute to the magnitude of the exponential, gives

$$\left.\begin{aligned} \operatorname{Re} \zeta &= \frac{c_1 \sigma_1}{2} \ln \sqrt{1 + \frac{4\alpha''}{(\alpha''-1)^2 + \beta''^2}} , \quad \alpha''^2 + \beta''^2 > 1 , \\ \operatorname{Re} \zeta &= \frac{c_1 \sigma_1}{2} \ln \sqrt{1 + \frac{4\alpha''}{(1-\alpha'')^2 + \beta''^2}} , \quad \alpha''^2 + \beta''^2 < 1 . \end{aligned}\right]$$

So for $\alpha'' > 0$ the $\mathrm{Re}\,\zeta > 0$, and the behavior of $\psi_{1\pm}(x,\mu,s)$ for large x is the same as it was for ν_{01} real and greater than one.

Based on this result, we set $a_{1+} = 0$ since $\psi_{1+}(x,\mu,s)$ does not tend to zero as $x \to -\infty$. A similar argument holds for $x > 0$, thereby requiring $a_{2-} = 0$ for $s \epsilon S_{2i}$.

For $x < 0$, to make the continuous solution

$$\psi_{1\nu}(x,\mu,s) = \varphi_{1\nu}(\mu)\, e^{-(s+\sigma_1)x/\nu}, \quad -1 \le \nu \le +1$$

satisfy the boundary condition as $x \to -\infty$, we require

$$\mathrm{Re}\, s > -\sigma_1 \quad , \quad \text{and } A_1(\nu) = 0 \quad , \quad 0 < \nu \le 1.$$

A similar argument for $x > 0$ requires

$$\mathrm{Re}\, s > -\sigma_2 \quad , \quad \text{and } A_2(\nu) = 0 \quad , \quad -1 \le \nu < 0.$$

Thus the solutions to Equations (45a, b) which vanish at $x = \pm\infty$ are:

for $x < 0$ and $\mathrm{Re}\, s > -\sigma_1$,

$$\left.\begin{aligned} \overline{\Psi}_s(x,\mu) &= -a_1\psi_{1-}(x,\mu,s) - \int_{-1}^{0} A_1(\nu)\psi_{1\nu}(x,\mu,s)d\nu \quad , \quad s\epsilon S_{1i}, \\ \overline{\Psi}_s(x,\mu) &= -\int_{-1}^{0} A_1(\nu)\psi_{1\nu}(x,\mu,s)d\nu \quad , \quad s\epsilon S_{1e}\,; \end{aligned}\right\} \quad (62)$$

for $x > 0$ and $\operatorname{Re} s > -\sigma_2$,

$$\left.\begin{aligned} \overline{\Psi}_s(x,\mu) &= a_{2+}\psi_{2+}(x,\mu,s) + \int_0^1 A_2(\nu)\psi_{2\nu}(x,\mu,s)d\nu \ , \quad s\epsilon\, S_{2i} \ , \\ \overline{\Psi}_s(s,\mu) &= \int_0^1 A_2(\nu)\psi_{2\nu}(x,\mu,s)d\nu \ , \quad s\epsilon\, S_{2e} \ . \end{aligned}\right\} \qquad (63)$$

E. Relationship Between the Regions in the s-Plane and the Properties of the Half-Spaces

If the two half-spaces had the same properties, then the regions S_{1i} and S_{2i} would be identical, and of course so would S_{ie} and S_{2e}. However, since the choice of material properties (c and σ) is quite arbitrary, we cannot expect the regions in the s-plane to remain identical for $x > 0$ and $x < 0$. By permitting the c and σ for each half space to take on physically realizable values, and by superimposing the s-plane for $x > 0$ onto the s-plane for $x < 0$, one can show that the resultant s-plane structure falls into only two general cases.

Without loss of generality, one can require $\sigma_1 \geqq \sigma_2$. Then the two general cases which include all choices of c and σ are shown in Figures 6a and 6b, and are defined as Case I and Case II. From Figures 6a and 6b, we see that in Case I, $c_1\sigma_1 > c_2\sigma_2$, whereas in Case II, $c_1\sigma_1 < c_2\sigma_2$.

It will be convenient to discard the notation S_{1i} etc. and replace it with the (a), (b), (c), (d), designation shown in Figures 6a and 6b. The connection between these notations is given

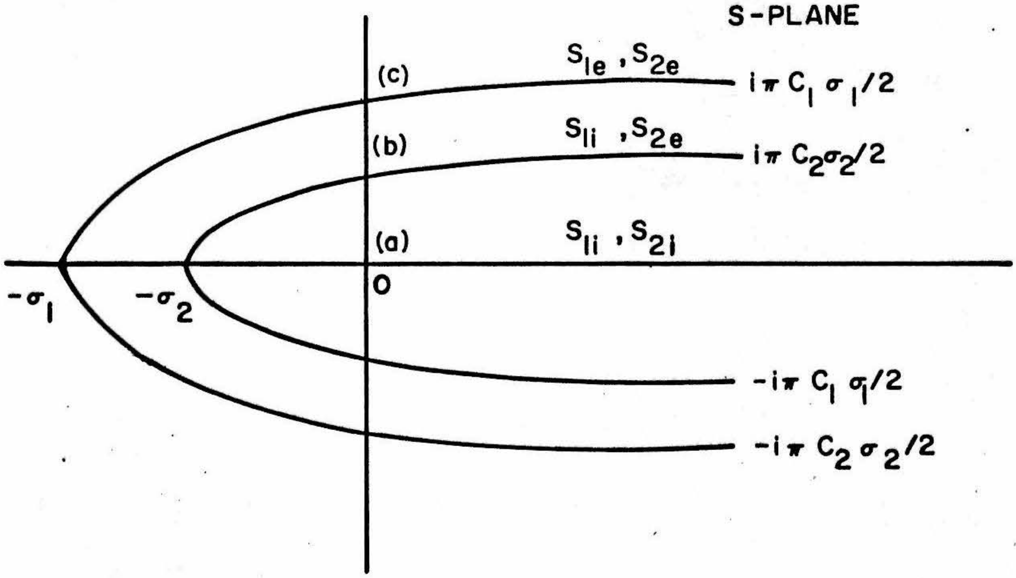

Figure 6a - Superimposed s-Plane Regions For Case I

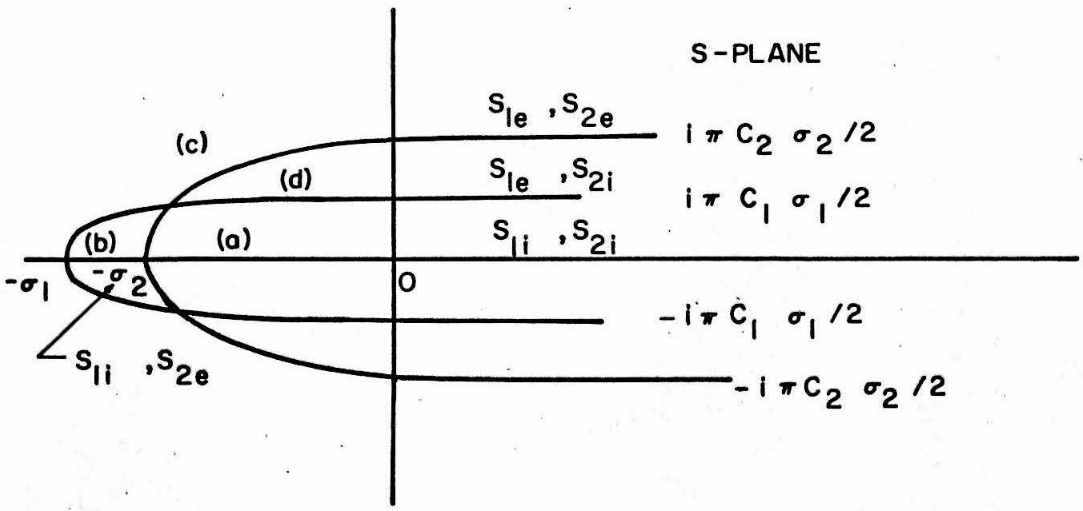

Figure 6b - Superimposed s-Plane Regions For Case II

below:

New Designation	Old Designation Case I	Old Designation Case II
(a)	$s\epsilon S_{1i}$, $s\epsilon S_{2i}$	$s\epsilon S_{1i}$, $s\epsilon S_{2i}$
(b)	$s\epsilon S_{1i}$, $s\epsilon S_{2e}$	$s\epsilon S_{1i}$, $s\epsilon S_{2e}$
(c)	$s\epsilon S_{1e}$, $s\epsilon S_{2e}$	$s\epsilon S_{1e}$, $s\epsilon S_{2e}$
(d)	-----	$s\epsilon S_{1e}$, $s\epsilon S_{2i}$

Hence, in region (a) of the s-plane, the transformed solutions (62, 63) contain discrete terms for all x in Cases I and II, whereas in region (b) the transformed solutions contain discrete terms only for $x < 0$ in Cases I and II. Regions (c) and (d) can be discussed in a similar manner.

For Cases I and II, the solutions to Equations (45a, b) are defined in all of region (a) and in at least part of regions (b) and (c). In Case II, the solutions to Equations (45 a, b) are also defined in all of region (d). Hence, the solutions must be determined explicitly in at least a portion of each of these regions. This is done through the utilization of the jump condition (44b) at the interface separating the two media.

It is found later that the distinction between Cases I and II is unnecessary in that the final form of solution becomes the same for both cases.

F. Application of the Jump Condition, Equation (44b), to the Solution

Let us write out the solutions for Cases I and II. The solutions in regions (a), (b), and (c) are the same for both Cases I and II.

In (a),

$$\left.\begin{aligned}\overline{\Psi}_s(x,\mu) &= -a_1\psi_{1-}(x,\mu,s) - \int_{-1}^{0} A_1(\nu)\psi_{1\nu}(x,\mu,s)d\nu, \quad x < 0\,;\\ \overline{\Psi}_s(x,\mu) &= a_{2+}\psi_{2+}(x,\mu,s) + \int_{0}^{1} A_2(\nu)\psi_{2\nu}(x,\mu,s)d\nu, \quad x > 0.\end{aligned}\right\} \quad (64)$$

In (b),

$$\left.\begin{aligned}\overline{\Psi}_s(x,\mu) &= -a_{1-}\psi_{1-}(x,\mu,s) - \int_{-1}^{0} A_1(\nu)\psi_{1\nu}(x,\mu,s)d\nu\,, \quad x < 0;\\ \overline{\Psi}_s(x,\mu) &= \int_{0}^{1} A_2(\nu)\psi_{2\nu}(x,\mu,s)d\nu\,, \quad x > 0\,.\end{aligned}\right\} \quad (65)$$

In (c),

$$\left.\begin{aligned}\overline{\Psi}_s(x,\mu) &= -\int_{-1}^{0} A_1(\nu)\psi_{1\nu}(x,\mu,s)d\nu\,, \quad x < 0;\\ \overline{\Psi}_s(s,\mu) &= \int_{0}^{1} A_2(\nu)\psi_{2\nu}(x,\mu,s)d\nu\,, \quad x > 0.\end{aligned}\right\} \quad (66)$$

In addition, for Case II we have

in (d),

$$\left.\begin{aligned}\overline{\Psi}_s(x,\mu) &= -\int_{-1}^{0} A_1(\nu)\,\psi_{1\nu}(x,\mu,s)d\nu\,, \quad x < 0;\\ \overline{\Psi}_s(x,\mu) &= a_{2+}\psi_{2+}(x,\mu,s) + \int_{0}^{1} A_2(\nu)\psi_{2\nu}(x,\mu,s)d\nu\,, \quad x > 0.\end{aligned}\right\} \quad (67)$$

Again we note that although the coefficients have the same designations in different regions of the s-plane, their functional form may be, and indeed will be, different.

Utilizing Equations (64) through (67), the jump condition (44b) is now written in each region of the s-plane.

In (a),
$$q = \mu\left[a_{2+}\varphi_{2+}(\mu) + a_{1-}\varphi_{1-}(\mu) + \int_{-1}^{0} A_1(\nu)\varphi_{1\nu}(\mu)d\nu + \int_{0}^{1} A_2(\nu)\varphi_{2\nu}(\mu)d\nu\right]. \tag{68}$$

In (b),
$$q = \mu\left[a_{1-}\varphi_{1-}(\mu) + \int_{-1}^{0} A_1(\nu)\varphi_{1\nu}(\mu)d\nu + \int_{0}^{1} A_2(\nu)\varphi_{2\nu}(\mu)d\nu\right]. \tag{69}$$

In (c),
$$q = \mu\left[\int_{-1}^{0} A_1(\nu)\varphi_{1\nu}(\mu)d\nu + \int_{0}^{1} A_2(\nu)\varphi_{2\nu}(\mu)d\nu\right]. \tag{70}$$

In (d),
$$q = \mu\left[a_{2+}\varphi_{2+}(\mu) + \int_{-1}^{0} A_1(\nu)\varphi_{1\nu}(\mu)d\nu + \int_{0}^{1} A_2(\nu)\varphi_{2\nu}(\mu)d\nu\right]. \tag{71}$$

Note that the jump condition is the same in each of the regions (a), (b), and (c) for Cases I and II. For Case II there exists the jump condition in region (d) given by Equation (71).

G. Completeness and Evaluation of the Coefficients

In Appendix I, it is proved that for an arbitrary function $\psi''(\mu)$, satisfying relatively weak conditions on the interval $-1 \leqq \mu \leqq +1$, the functions

$$\varphi_{2+}(\mu)\ ,\ \varphi_{1-}(\mu)\ ,\ \varphi_{1\nu}(\mu)\ ,\ -1 \leqq \nu < 0\ ,\ \varphi_{2\nu}(\mu),\ 0 < \nu \leq 1, \text{ for } s\varepsilon(a),$$

$$\varphi_{1-}(\mu)\ ,\ \varphi_{1\nu}(\mu)\ ,\ -1 \leq \nu < 0\ ,\ \varphi_{2\nu}(\mu)\ ,\ 0 < \nu \leq 1, \text{ for } s\varepsilon(b),$$

$$\varphi_{1\nu}(\mu) \ , \ -1 \le \nu < 0 \ , \ \varphi_{2\nu}(\mu) \ , \ 0 < \nu \le 1 \ , \ \text{for } s\varepsilon(c) \ ,$$

$$\varphi_{2+}(\mu) \ , \ \varphi_{1\nu}(\mu) \ , \ -1 \le \nu < 0 \ , \ \varphi_{2\nu}(\mu) \ , \ 0 < \nu \le 1 \ , \ \text{for } s\varepsilon(d),$$

when multiplied by μ, form complete sets.

So an expansion of the form

$$\psi''(\mu) = \left[a_{2+}\varphi_{2+}(\mu) + a_{1-}\varphi_{1-}(\mu) + \int_{-1}^{0} A_1(\nu)\varphi_{1\nu}(\mu)d\nu + \int_{0}^{1} A_2(\nu)\varphi_{2\nu}(\mu)d\nu \right]\mu$$

exists, for example, in region (a) of the s-plane, Similarly, the right-hand sides of Equations (69), (70), and (71) can be used to represent an arbitrary function on the interval $-1 \leqq \mu \leqq +1$. Hence, the NME method of solution has yielded a set of eigenfunctions which are complete in μ-space.

Another extremely useful fact, aside from completeness but obtained from the completeness proof as well, is that a method for obtaining the coefficients is provided. This method is utilized in Appendix II to obtain the coefficients in Equations (68) through (71). The results are summarized below, where

$$X_o(z) = \exp\left[\frac{1}{2\pi i} \left\{ \int_{-1}^{0} \ln \frac{\bar{\lambda}_1(\mu') + i\pi\mu'}{\bar{\lambda}_1(\mu') - i\pi\mu'} \frac{d\mu'}{\mu' - z} + \int_{0}^{1} \ln \frac{\bar{\lambda}_2(\mu') + i\pi\mu'}{\bar{\lambda}_2(\mu') - i\pi\mu'} \frac{d\mu'}{\mu' - z} \right\} \right],$$

$$\bar{\lambda}_1(\nu) = \frac{2}{c_1\sigma_1} \lambda_1(\nu) \quad , \quad \bar{\lambda}_2(\nu) = \frac{2}{c_2\sigma_2} \lambda_2(\nu) \ ,$$

and where $\lambda_1(\nu)$ and $\lambda_2(\nu)$ are given by Equations (49) and (56) respectively.

Region (a):

$$
\left.
\begin{aligned}
a_{1-} &= \left\{\frac{2qX_a(0)}{c_1\sigma_1(\nu_{02}+\nu_{01})}\right\} \left\{\frac{\nu_{02}}{\nu_{01}X_a(-\nu_{01})}\right\} \\
a_{2+} &= \left\{\frac{-2qX_a(0)}{c_2\sigma_2(\nu_{02}+\nu_{01})}\right\} \left\{\frac{\nu_{01}}{\nu_{01}X_a(+\nu_{02})}\right\} \\
A_1(\nu) &= \frac{-\,2qX_a(0)\,\nu_{01}\nu_{02}}{c_1\sigma_1\nu X_a^-(\nu)\,(\lambda_1(\nu)+i\pi\nu)\,(\nu-\nu_{02})\,(\nu+\nu_{01})} \qquad -1 \leqq \nu < 0 \\
A_2(\nu) &= \frac{-2qX_a(0)\nu_{01}\nu_{02}}{c_2\sigma_2\nu X_a^-(\nu)\,(\lambda_2(\nu)+i\pi\nu)\,(\nu-\nu_{02})\,(\nu+\nu_{01})} \qquad 0 < \nu \leqq +1 \\
X_a(z) &= \frac{X_0(z)}{(1-z)\,(-1-z)}
\end{aligned}
\right] \tag{72}
$$

Region (b):

$$
\left.
\begin{aligned}
a_{1-} &= \frac{2q\,X_b(0)}{c_1\sigma_1\nu_{01}X_b(-\nu_{01})} \\
A_1(\nu) &= \frac{2qX_b(0)\,\nu_{01}}{c_1\sigma_1\nu(\nu+\nu_{01})X_b^-(\nu)\,(\lambda_1(\nu)+i\pi\nu)} \qquad -1 \leqq \nu < 0 \\
A_2(\nu) &= \frac{2qX_b(0)\,\nu_{01}}{c_2\sigma_2\nu(\nu+\nu_{01})X_b^-(\nu)\,(\lambda_2(\nu)+i\pi\nu)} \qquad 0 < \nu \leqq 1 \\
X_b(z) &= \frac{X_0(z)}{(-1-z)}
\end{aligned}
\right] \tag{73}
$$

Region (c):

$$\left.\begin{aligned} A_1(\nu) &= \frac{2qX_c(0)}{c_1\sigma_1\nu X_c^-(\nu)(\lambda_1(\nu)+i\pi\nu)} \qquad -1\leqq\nu<0 \\ A_2(\nu) &= \frac{2qX_c(0)}{c_2\sigma_2\nu X_c^-(\nu)(\lambda_2(\nu)+i\pi\nu)} \qquad 0<\nu\leq 1 \\ X_c(z) &= X_0(z) \end{aligned}\right\} \quad (74)$$

Region (d):

$$\left.\begin{aligned} a_{2+} &= \frac{-2qX_d(0)}{c_2\sigma_2\nu_{02}X_d(\nu_{02})} \\ A_1(\nu) &= \frac{-2qX_d(0)\nu_{02}}{c_1\sigma_1\nu(\nu-\nu_{02})X_d^-(\nu)(\lambda_1(\nu)+i\pi\nu)} \qquad -1\leqq\nu<0 \\ A_2(\nu) &= \frac{2qX_d(0)\nu_{02}}{c_2\sigma_2\nu(\nu-\nu_{02})X_d^-(\nu)(\lambda_2(\nu)+i\pi\nu)} \qquad 0<\nu\leqq 1 \\ X_d(z) &= \frac{X_0(z)}{1-z} \end{aligned}\right\} \quad (75)$$

If one accepts the completeness of the functions $\varphi_{1-}(\mu)$, $\varphi_{2+}(\mu)$, $\varphi_{1\nu}(\mu)$, $-1\leq\nu<0$, $\varphi_{2\nu}(\mu)$, $0<\nu\leq 1$, or certain of them, then an alternative method of obtaining the coefficients is available, based on the orthogonality properties of these functions[7]. This alternative method is employed to obtain several of the coefficients in Section C of Appendix II. Of course, the final results are identical in the two approaches but the intermediate steps are quite dissimilar.

H. Final Form of the Transformed Solution

Now that the coefficients have been evaluated, one need only substitute them into the general solutions, Equations (64) through (67), to obtain the final form of the transformed solution in each region of the s-plane for arbitrary x. These solutions are written out below.

Region (a):

$$\overline{\Psi}_s(x,\mu) = \frac{-2qX_a(0)}{c_1\sigma_1}\left[\frac{\nu_{02}\varphi_{1-}(\mu)\, e^{+(s+\sigma_1)x/\nu_{01}}}{\nu_{01}(\nu_{02}+\nu_{01})X_a(-\nu_{01})}\right.$$

$$\left. -\int_{-1}^{0}\frac{\nu_{01}\nu_{02}\varphi_{1\nu}(\mu)\, e^{-(s+\sigma_1)x/\nu}\, d\nu}{\nu(\nu-\nu_{02})(\nu+\nu_{01})X_a^-(\nu)(\lambda_1(\nu)+i\pi\nu)}\right], \quad x<0. \tag{76}$$

$$\overline{\Psi}_s(x,\mu) = \frac{-2qX_a(0)}{c_2\sigma_2}\left[\frac{\nu_{01}\varphi_{2+}(\mu)\, e^{-(s+\sigma_1)x/\nu_{02}}}{\nu_{02}(\nu_{02}+\nu_{01})X_a(\nu_{02})}\right.$$

$$\left. +\int_{0}^{1}\frac{\nu_{01}\nu_{02}\varphi_{2\nu}(\mu)\, e^{-(s+\sigma_2)x/\nu}\, d\nu}{\nu(\nu-\nu_{02})(\nu+\nu_{01})X_a^-(\nu)(\lambda_2(\nu)+i\pi\nu)}\right], \quad x>0. \tag{77}$$

Region (b):

$$\overline{\Psi}_s(x,\mu) = \frac{-2qX_b(0)}{c_1\sigma_1}\left[\frac{\varphi_{1-}(\mu)\, e^{(s+\sigma_1)x/\nu_{01}}}{\nu_{01}X_b(-\nu_{01})}\right.$$

$$\left. +\int_{-1}^{0}\frac{\nu_{01}\varphi_{1\nu}(\mu)\, e^{-(s+\sigma_1)x/\nu}\, d\nu}{\nu(\nu+\nu_{01})X_b^-(\nu)(\lambda_1(\nu)+i\pi\nu)}\right], \quad x<0. \tag{78}$$

$$\overline{\Psi}_s(x,\mu) = \frac{2qX_b(0)}{c_2\sigma_2}\left[\int_0^1 \frac{\nu_{01}\varphi_{2\nu}(\mu)\, e^{-(s+\sigma_2)x/\nu}\, d\nu}{\nu(\nu+\nu_{01})X_b^-(\nu)(\lambda_2(\nu)+i\pi\nu)}\right], \quad x>0. \qquad (79)$$

Region (c):

$$\overline{\Psi}_s(x,\mu) = \frac{-2qX_c(0)}{c_1\sigma_1}\int_{-1}^0 \frac{\varphi_{1\nu}(\mu)\, e^{-(s+\sigma_1)x/\nu}\, d\nu}{\nu X_c^-(\nu)(\lambda_1(\nu)+i\pi\nu)}, \quad x<0. \qquad (80)$$

$$\overline{\Psi}_s(x,\mu) = \frac{2qX_c(0)}{c_2\sigma_2}\int_0^1 \frac{\varphi_{2\nu}(\mu)\, e^{-(s+\sigma_2)x/\nu}\, d\nu}{\nu X_c^-(\nu)(\lambda_2(\nu)+i\pi\nu)}, \quad x>0. \qquad (81)$$

Region (d):

$$\overline{\Psi}_s(x,\mu) = \frac{2qX_d(0)}{c_1\sigma_1}\int_{-1}^0 \frac{\nu_{02}\varphi_{1\nu}(\mu)\, e^{-(s+\sigma_1)x/\nu}\, d\nu}{(\nu-\nu_{02})\nu X_d^-(\nu)(\lambda_1(\nu)+i\pi\nu)}, \quad x<0. \qquad (82)$$

$$\overline{\Psi}_s(x,\mu) = \frac{-2qX_d(0)}{c_2\sigma_2}\left[\frac{\varphi_{2+}(\mu)\, e^{-(s+\sigma_2)x/\nu_{02}}}{\nu_{02}X_d(\nu_{02})} + \int_0^1 \frac{\nu_{02}\varphi_{2\nu}(\mu)\, e^{-(s+\sigma_2)x/\nu}\, d\nu}{\nu(\nu-\nu_{02})X_d^-(\nu)(\lambda_2(\nu)+i\pi\nu)}\right], \quad x>0. \qquad (83)$$

Hence the solutions to the transformed Equations (45a, b) satisfying the boundary and interface conditions (44a, b) have been found in those parts of regions (a), (b), and (c) of the s-plane in Case I, and in those parts of regions (a), (b), (c), and (d) of the s-plane in Case II, having $\operatorname{Re} s > -\sigma_1$ for $x < 0$ and $\operatorname{Re} s > -\sigma_2$ for $x > 0$.

In order to invert these solutions and recover the time dependence, we must study the behavior of the transformed solutions in detail. This investigation is done in the next chapter where the subsequent inversion is performed.

IV. BEHAVIOR OF THE TRANSFORMED SOLUTIONS IN THE s-PLANE AND THE INVERSION FROM s TO τ

A. Behavior of $\overline{\Psi}_s(x, \mu)$ in the s-Plane

In the last chapter, the solutions to the transformed Equations (45a, b), which satisfy the proper boundary and interface conditions, were obtained in certain regions of the s-plane. These regions are shown in Figures 7a, b for Case I and in Figures 8a, b for Case II.

In order to satisfy the boundary conditions for large x, it is necessary to keep Re s to the right of $-\sigma_2$ for $x > 0$, and to the right of $-\sigma_1$ for $x < 0$; hence the dashed line boundary to the regions where the solutions exist.

Let us first investigate the behavior of the solutions within each region of the s-plane for both Cases I and II. In regions (a), (b), and (d) there exist certain relationships between ν_{01} and s, and ν_{02} and s (Equations (54) and (59));

$$s + \sigma_1 = \frac{c_1\sigma_1}{2} \nu_{01} \ln \frac{\nu_{01} + 1}{\nu_{01} - 1}, \tag{84a}$$

$$s + \sigma_2 = \frac{c_2\sigma_2}{2} \nu_{02} \ln \frac{\nu_{02} + 1}{\nu_{02} - 1}, \tag{84b}$$

where Re ν_{01}, $\nu_{02} \geqq 0$. It is stated in Section C of the last chapter that when ν_{01} is real and greater than one, s is real. As a consequence of this, a branch is drawn in the s-plane from $-\sigma_1$ to $-\sigma_1(1-c_1)$ along the real axis, in order to keep Equation (84a) one to one. Similarly a branch is drawn from $-\sigma_2$ to $-\sigma_2(1-c_2)$ to

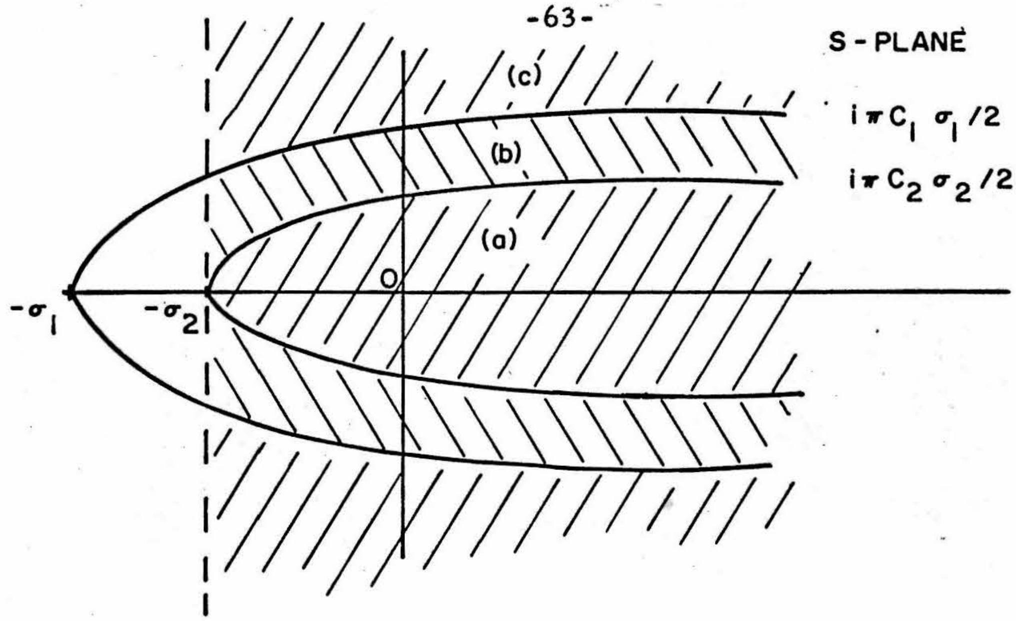

Figure 7a - Regions Where Solutions Exist in Case I for x > 0

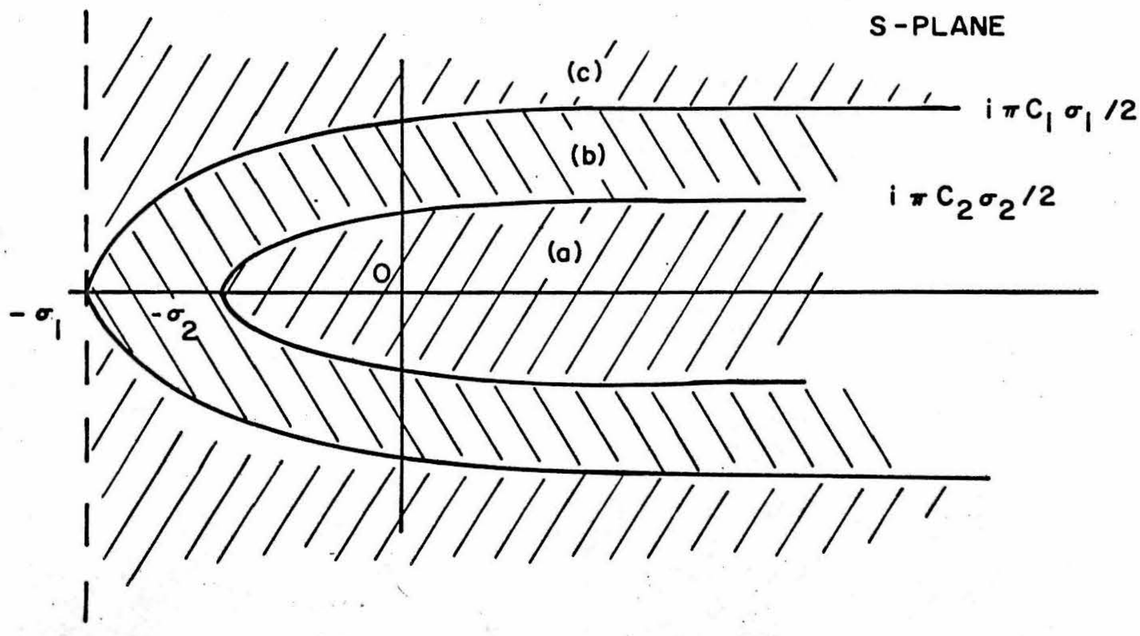

Figure 7b - Regions Where Solutions Exist in Case I for x < 0

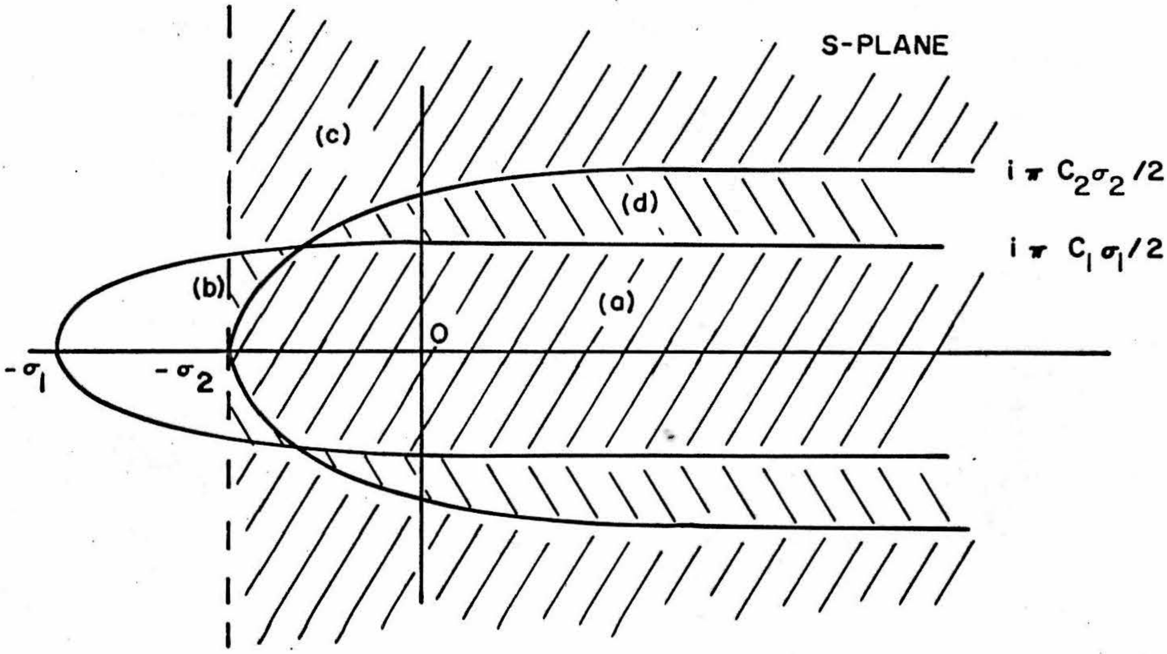

Figure 8a - Regions Where Solutions Exist in Case II for $x > 0$

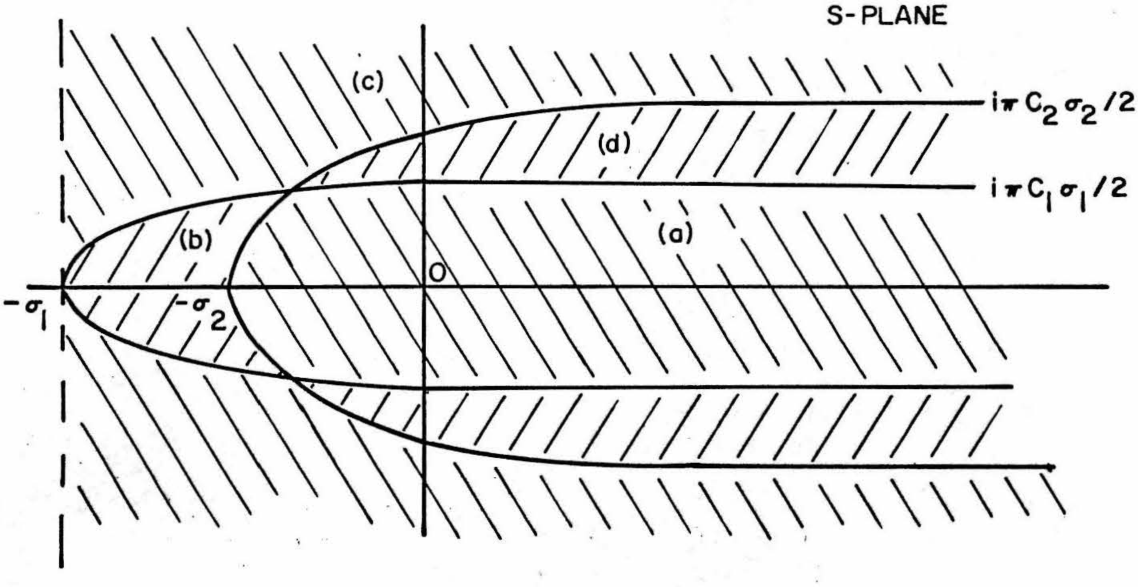

Figure 8b - Regions Where Solutions Exist in Case II for $x < 0$

keep Equation (84b) single-valued.

The branches from $-\sigma_1$ to $-\sigma_1(1-c_1)$ and from $-\sigma_2$ to $-\sigma_2(1-c_2)$ occur in regions (a) and (b) of the s-plane. It is necessary to look at the transformed solutions in these regions to see if the branches required in Equations (84a, b) are carried over into the s-plane where the solutions are described. By inspection of Equations (76) through (79) ($\overline{\Psi}_s(x, \mu)$ in regions (a) and (b)) it is easily seen, due to the discontinuous nature of ν_{01} and ν_{02} along the lengths of the real s-axis defined above, that $\overline{\Psi}_s(x, \mu)$ is itself discontinuous there. Hence, branches exist in the s-plane for the functions $\overline{\Psi}_s(x, \mu)$ from $-\sigma_1$ to $-\sigma_1(1-c_1)$ and from $-\sigma_2$ to $-\sigma_2(1-c_2)$. These two branches overlap in many problems of physical interest.

Aside from the existence of these two branches along the real s-axis, Equations (76) through (83) appear to have no other singularities or branches anywhere within the regions of the s-plane where they are defined.

In Appendix III, it is shown (partially) that the solutions are continuous along the boundaries between adjacent regions of the s-plane and that they take on identical values along these borders, when the borders are approached from either region. Hence, it can be concluded that the solutions to the transformed Equations (45a, b) of the previous chapter are analytic in the right-half planes $\text{Re}\, s > -\sigma_1$ for $x < 0$, and $\text{Re}\, s > -\sigma_2$ for $x > 0$, save for two branch cuts along the real s-axis from $-\sigma_1$ to $-\sigma_1(1-c_1)$ and from

$-\sigma_2$ to $-\sigma_2(1-c_2)$ for $x < 0$, or from $-\sigma_2$ to $-\sigma_1(1-c_1)$ and from $-\sigma_2$ to $-\sigma_2(1-c_2)$ for $x > 0$, in both Cases I and II.

When $x = 0$, the region of analyticity in the s-plane can be extended because the restraining factor $e^{-(s+\sigma)x/\nu}$ is no longer present. It is easy to show for this case that the solutions to the transformed equations (either $x = 0+$ or $x = 0-$) are analytic in the entire s-plane save for the two branches on the real s-axis from $-\sigma_1$ to $-\sigma_1(1-c_1)$ and from $-\sigma_2$ to $-\sigma_2(1-c_2)$.

Since the inversion integral of the Laplace transformation only requires a knowledge of the transformed function along a line parallel to the imaginary s-axis and to the right of all discontinuities of $\overline{\Psi}_s(x,\mu)$, we are in a position to recover the time dependence. Because of our more detailed knowledge of $\overline{\Psi}_s(x,\mu)$, we shall be able to change the inversion path and obtain a solution which more clearly depicts the properties of the time-dependent flux. The actual inversion will be carried out next.

B. Inversion Path Related to Media Properties

The original inversion path for a typical case is shown in Figure 9. Because of our knowledge of $\overline{\Psi}_s(x,\mu)$, we can shift this path to the left, picking up the contributions along the branches, until $\mathrm{Re}\, s = -\sigma_1$ or $-\sigma_2$, depending upon whether $x > 0$ or $x < 0$. The new path for $x < 0$ is sketched in Figure 9 as well for the case $\sigma_1 > \sigma_2 > a_1\sigma_1 > a_2\sigma_2$.

One can conclude from this shift in the inversion path that the distinction between Cases I and II is no longer necessary. Now let us look at the inversion in detail.

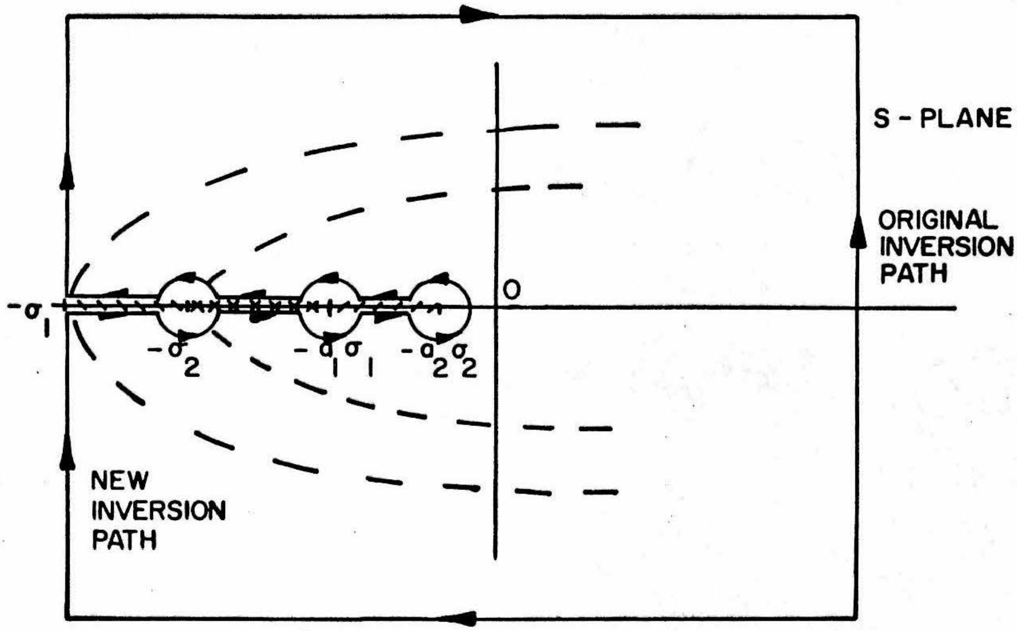

Figure 9 - Inversion Path in Case I for $x<0$ with $\sigma_1>\sigma_2>a_1\sigma_1>a_2\sigma_2$

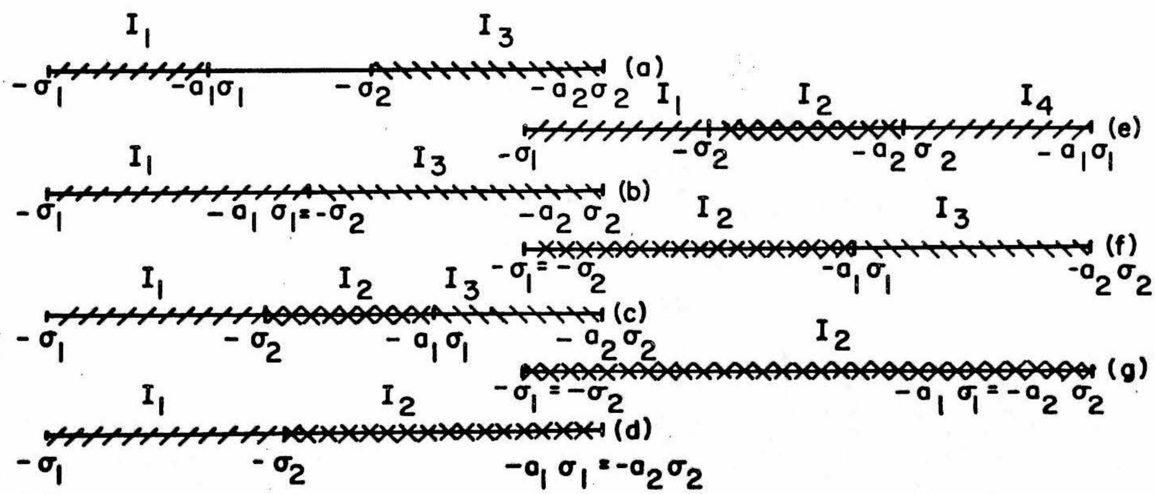

Figure 10 - Possible Locations of the Branches in the s-Plane

The two branches defined along the real s-axis from $-\sigma_1$ to $-a_1\sigma_1$ and from $-\sigma_2$ to $-a_2\sigma_2$ where $a_1 = 1-c_1$, $a_2 = 1-c_2$, are dependent on the cross sections of the half-spaces. It is necessary that $\sigma_1 \geqq \sigma_2$, but this in no way restricts the generality of the solution. A summary of the possible locations of these branches is given in Figure 10.

C. Example of the Inversion for $x < 0$

As an example of how the inversion is carried out let us look at two half-spaces having properties which correspond to branch (c) in Figure 10; that is $\sigma_1 > \sigma_2 > a_1\sigma_1 > a_2\sigma_2$. Many actual moderator combinations would fall into this case. A detailed sketch of the inversion path is given in Figure 11.

We know from Equation (42) that

$$\Psi(x, \mu, \tau) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{s\tau} \overline{\Psi}_s(x, \mu) ds \quad ,$$

or that

$$\Psi(x, \mu, \tau) \equiv \frac{1}{2\pi i} \; I_o(\overline{\Psi}_s) \quad ,$$

where $I_o(\overline{\Psi}_s)$ is shown in Figure 11. But our knowledge of $\overline{\Psi}_s(x, \mu)$ as a function of s permits us to write

$$I_o(\overline{\Psi}_s) = I_{y-}(\overline{\Psi}_s) + I_{5-} + I_{1-} + I_{\rho_{1-}} + I_{2-} + I_{\rho_{2-}} + I_{3-} + I_{\rho_3}$$
$$+ I_{3+} + I_{\rho_{2+}} + I_{2+} + I_{\rho_{1+}} + I_{1+} + I_{5+} + I_{y+}(\overline{\Psi}_s), \qquad (85)$$

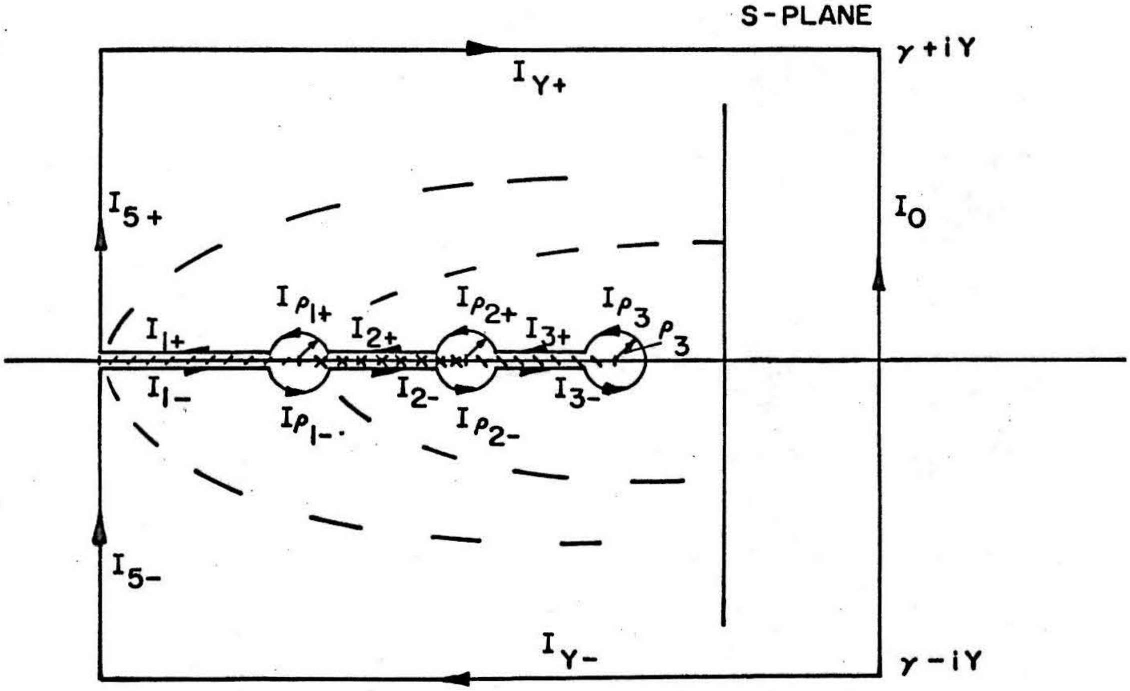

Figure 11 - Inversion Path for Branch (c) of Figure 10 for $x < 0$

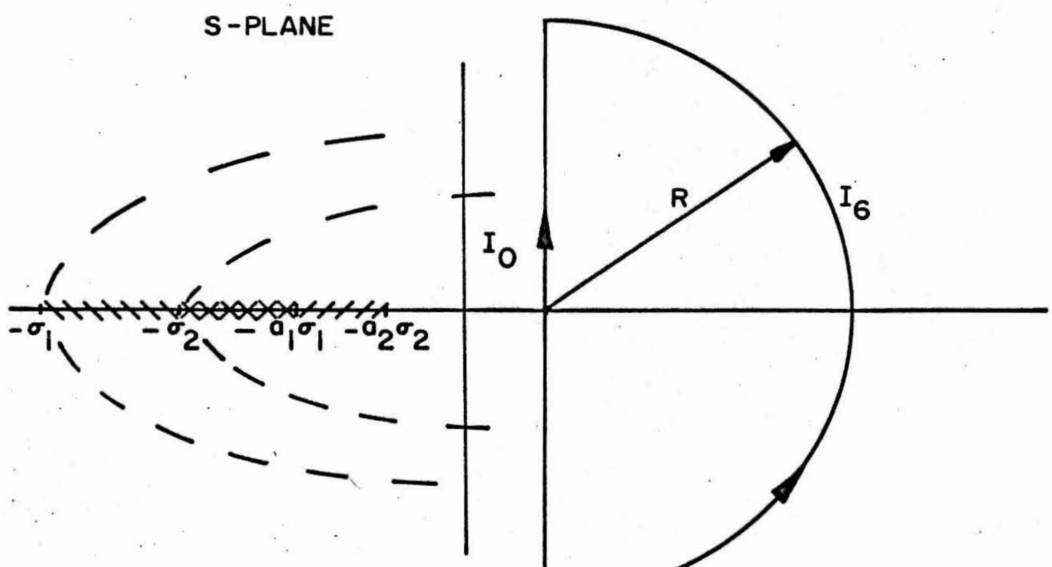

Figure 12 - Path of Inversion for $vt < |x|$

where the I's correspond to the path integrals sketched in Figure 11. It can be shown that many of these integrals vanish.

Investigating the behavior of $\overline{\Psi}_s(x,\mu)$ from Equation (80) for large s – at the extremities of I_o – yields the following asymptotic form which is defined as $\overline{\Psi}_{unc}(x,\mu,s)$;

$$\overline{\Psi}_s(x,\mu) \sim \frac{q\, e^{-(s+\sigma_1)x/\mu}\, 1(-\mu)}{-\mu} \equiv \overline{\Psi}_{unc}(x,\mu,s) \ , \qquad (86)$$

where

$$1(-\mu) = \begin{cases} 1 \ , & \mu < 0 \\ 0 \ , & \mu > 0 \ . \end{cases}$$

The next term in the asymptotic expansion of $\overline{\Psi}_s(x,\mu)$ has a $\frac{1}{s}$ behavior, etc. Thus, for large s ,

$$\overline{\Psi}_s(x,\mu) - \overline{\Psi}_{unc}(x,\mu,s) \sim 0\left(\frac{1}{s}\right).$$

Before proceeding, we observe that

$$\Psi_{unc}(x,\mu,\tau) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \overline{\Psi}_{unc}(x,\mu,s)\, e^{s\tau} ds = \frac{q\, e^{-\sigma_1 x/\mu}\, 1(-\mu)\,\delta(\tau - \frac{x}{\mu})}{-\mu} \ , \qquad (87)$$

which will be shown later to be the uncollided angular flux (hence the notation). Also, one can conclude, by an inspection of Equations (76) through (83), that

$$\Psi(x,\mu,\tau) = 0 \quad \text{when} \quad \tau < |x| \ . \qquad (88)$$

This is physically plausible and is demonstrated mathematically by setting $I_0 = I_6$ (see Figure 12) and observing that $I_6 \to 0$ as $R \to \infty$, after the δ-function behavior is factored out.

Equation (42) can now be written as

$$\Psi(x,\mu,\tau) - \Psi_{unc}(x,\mu,\tau) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{s\tau}\left[\overline{\Psi}_s(x,\mu) - \overline{\Psi}_{unc}(x,\mu,s)\right]ds \,. \tag{89}$$

Also, Equation (85) again holds for the new integrands;

$$I_0(\overline{\Psi}_s - \overline{\Psi}_{unc}) = I_{y-}(\overline{\Psi}_s - \overline{\Psi}_{unc}) + \ldots + I_{y+}(\overline{\Psi}_s - \overline{\Psi}_{unc}) \,. \tag{90}$$

But now we see that $I_{y\pm} \to 0$ as $y \to \infty$.

In addition, since $\overline{\Psi}_{unc}(x,\mu,s)$ is analytic within the region about which we are integrating, its contribution above and below the cuts cancels. Hence, defining

$$I_1(\overline{\Psi}_s(x,\mu) - \overline{\Psi}_{unc}(x,\mu,s)) \equiv I_{1-}(\overline{\Psi}_s - \overline{\Psi}_{unc}) + I_{1+}(\overline{\Psi}_s - \overline{\Psi}_{unc}) \,, \tag{91}$$

we can write

$$I_1(\overline{\Psi}_s(x,\mu) - \overline{\Psi}_{unc}(x,\mu,s)) = I_1(\overline{\Psi}_s(x,\mu)) \,. \tag{92a}$$

Similarly,

$$I_2(\overline{\Psi}_s - \overline{\Psi}_{unc}) = I_2(\overline{\Psi}_s) \,, \tag{92b}$$

$$I_3(\overline{\Psi}_s - \overline{\Psi}_{unc}) = I_3(\overline{\Psi}_s) \,. \tag{92c}$$

Considering next the integrals around the small semicircles having radii ρ_1, ρ_2, and around the circle with radius ρ_3, it can be shown that as $\rho_1, \rho_2, \rho_3 \to 0$, these integrals vanish.

So Equation (90) becomes

$$I_0(\overline{\Psi}_s - \overline{\Psi}_{unc}) = I_1(\overline{\Psi}_s) + I_2(\overline{\Psi}_s) + I_3(\overline{\Psi}_s) + I_5(\overline{\Psi}_s - \overline{\Psi}_{unc}), \tag{93}$$

where $I_5 \equiv I_{5-} + I_{5+}$. Hence the result for this example can be expressed as

$$\Psi(x,\mu,\tau) = \Psi_{unc}(x,\mu,\tau) + \frac{1}{2\pi i}\left[I_1(\overline{\Psi}_s) + I_2(\overline{\Psi}_s) + I_3(\overline{\Psi}_s) + I_5(\overline{\Psi}_s - \overline{\Psi}_{unc})\right]. \tag{94}$$

Exact expressions for the $I_1, \ldots, I_5$ have not been written down yet; they will be included in the general case considered next.

D. Inversion in the General Case ($x < 0$)

All of the cases shown in Figure 10 can be done at once by utilizing step functions appropriately. The integrals along the branches are identified by the terminology shown in Figure 10. Thus, the complete solution can be written, for $x < 0$, as

$$\Psi(x,\mu,\tau) = \Psi_{unc}(x,\mu,\tau) + \frac{1}{2\pi i}\left[I_1(\overline{\Psi}_s) + I_2(\overline{\Psi}_s) + I_3(\overline{\Psi}_s) + I_4(\overline{\Psi}_s) + I_5(\overline{\Psi}_s - \overline{\Psi}_{unc})\right], \tag{95}$$

where

$$I_1(\overline{\Psi}_s) = \int_{-\sigma_1}^{-a_1\sigma_1 1(a_1\sigma_1 - \sigma_2) - \sigma_2 1(\sigma_2 - a_1\sigma_1)} e^{s\tau} ds \left\{\overline{\Psi}_s(x,\mu)^- - \overline{\Psi}_s(x,\mu)^+\right\}, \tag{96}$$

$$I_2(\overline{\Psi}_s) = 1(\sigma_2 - a_1\sigma_1) \int_{-\sigma_2}^{-a_1\sigma_1 1(a_1\sigma_1-\sigma_2 a_2)-a_2\sigma_2 1(a_2\sigma_2-a_1\sigma_1)} e^{s\tau} ds \left\{ \overline{\Psi}_s(x,\mu)^- - \overline{\Psi}_s(x,\mu)^+ \right\} , \tag{97}$$

$$I_3(\overline{\Psi}_s) = 1(a_1\sigma_1 - a_2\sigma_2) \int_{-\sigma_2 1(a_1\sigma_1-\sigma_2) - a_1\sigma_1 1(\sigma_2 - a_1\sigma_1)}^{-a_2\sigma_2} e^{s\tau} ds \left\{ \overline{\Psi}_s(x,\mu)^- - \overline{\Psi}_s(x,\mu)^+ \right\} , \tag{98}$$

$$I_4(\overline{\Psi}_s) = 1(a_2\sigma_2 - a_1\sigma_1) \int_{-a_2\sigma_2}^{-a_1\sigma_1} e^{s\tau} ds \left\{ \overline{\Psi}_s(x,\mu)^- - \overline{\Psi}_s(x,\mu)^+ \right\} , \tag{99}$$

$$I_5(\overline{\Psi}_s - \overline{\Psi}_{unc}) = \int_{-\sigma_1 - i\infty}^{-\sigma_1 + i\infty} e^{s\tau} ds \left\{ \overline{\Psi}_s(x,\mu) - \overline{\Psi}_{unc}(x,\mu,s) \right\} , \tag{100}$$

and $\Psi_{unc}(x,\mu,\tau)$ is given by Equation (87).

In these expressions we have used $\overline{\Psi}_s(x,\mu)^-$ to denote that the function is to be considered below the branch cut, and $\overline{\Psi}_s(x,\mu)^+$ implies that s is above the branch. Also, we note that

$$1(x-a) = \begin{cases} 1 , & x > a \\ \frac{1}{2} , & x = a \\ 0 , & x < 0 . \end{cases} \tag{101}$$

Let us obtain explicit expressions now for $I_1(\overline{\Psi}_s)$ through $I_5(\overline{\Psi}_s - \overline{\Psi}_{unc})$.

<u>$I_1(\Psi_s)$:</u> This integration is contained entirely in region (b) of the s-plane. For this region $\Psi_s(x,\mu)$ is given explicitly by Equation (78). We know from Equation (84a) that

$$s+\sigma_1 = \frac{c_1\sigma_1\nu_{01}}{2}\,\ln\,\frac{\nu_{01}+1}{\nu_{01}-1} \quad ,$$

and that when s is above the cut ν_{01} is a negative imaginary number. Similarly, for s below the cut, ν_{01} is a positive imaginary number. So we can write

$$s+\sigma_1 = \frac{c_1\sigma_1}{2}\,\xi\left[\pi + \tan^{-1}\frac{2\,\xi}{\xi^2-1}\right] , \quad \begin{array}{l} 0 < \xi < \infty \\ 0 > \tan^{-1} > -\pi \end{array} , \qquad (102)$$

where

$$\nu_{01} = \begin{cases} -i\xi & \text{above cut} \\ +i\xi & \text{below cut.} \end{cases}$$

With these simplifications, it is easy to obtain the following expression for $I_1(\Psi_s)$ from Equation (96), where the notation $\varphi_{1-}^{-}(\mu)$ implies that the function is evaluated below the cut.

$$I_1(\Psi_s) = 2iq\int_{-\sigma_1}^{-a_1\sigma_1 l(a_1\sigma_1-\sigma_2)-\sigma_2 l(\sigma_2-a_1\sigma_1)} \frac{2X_b(0)e^{s\tau}ds}{c_1\sigma_1}\left\{\frac{1}{\xi}\,\mathrm{Re}\left[\frac{\varphi_{1-}^{-}(\mu)\,e^{-i(s+\sigma_1)x/\xi}}{X_b(-i\xi)}\right]\right. \qquad (103)$$

$$\left. -\int_{-1}^{0}\frac{\xi\varphi_{1\nu}(\mu)\,e^{-(s+\sigma_1)x/\nu}d\nu}{(\nu^2+\xi^2)X_b^-(\nu)(\lambda_1(\nu)+i\pi\nu)}\right\} .$$

$I_2(\overline{\Psi}_s)$: This integration is contained in region (a) of the s-plane, and $\overline{\Psi}_s(x,\mu)$ is given by Equation (76). Here ν_{01} is again pure imaginary and can be expressed in terms of ξ (see Equation (102)). Similarly, ν_{02} exhibits a discontinuity when the branch is approached. Paralleling the analysis leading to Equation (102) let us define η such that

$$\nu_{02} = \begin{cases} -i\eta & \text{above cut} \\ +i\eta & \text{below cut.} \end{cases}$$

Then from Equation (84b),

$$s+\sigma_2 = \frac{c_2\sigma_2\eta}{2}\left[\pi + \tan^{-1}\frac{2\eta}{\eta^2-1}\right], \qquad \begin{array}{l} 0 < \eta < \infty \\ 0 > \tan^{-1} > -\pi \end{array} . \tag{104}$$

After some further algebraic manipulation on Equation (97), one obtains

$$I_2(\overline{\Psi}_s) = 2iq\, l\, (\sigma_2 - a_1\sigma_1)\int_{-\sigma_2}^{-a_1\sigma_1 l(a_1\sigma_1 - a_2\sigma_2) - a_2\sigma_2 l(a_2\sigma_2 - a_1\sigma_1)} \left\{\frac{2X_a(0)e^{+s\tau}ds}{c_1\sigma_1}\right\}$$

$$\left\{\frac{\eta}{\xi(\xi+\eta)}\,\mathrm{Re}\left[\frac{\varphi_{1-}^{-}(\mu)e^{-i(s+\sigma_1)x/\xi}}{X_a(-i\xi)}\right]\right. \tag{105}$$

$$\left. -\int_{-1}^{0}\frac{\xi\eta(\eta-\xi)\varphi_{1\nu}(\mu)e^{-(s+\sigma_1)x/\nu}\,d\nu}{(\nu^2+\eta^2)(\nu^2+\xi^2)X_a^{-}(\nu)(\lambda_1(\nu)+i\pi\nu)}\right\} .$$

$\underline{I_3(\overline{\Psi}_s)}$: Here ν_{01} is real and greater than one, and we can use Equation (84a) to obtain it. However, Equation (104) is still necessary for η. With this in mind and utilizing Equation (76) for $\overline{\Psi}_s(x,\mu)$, we can write $I_3(\overline{\Psi}_s)$ from Equation (98) as

$$I_3(\overline{\Psi}_s) = 2iql(a_1\sigma_1 - a_2\sigma_2)\int_{-\sigma_1 l(a_1\sigma_1-\sigma_2)-a_1\sigma_1 l(\sigma_2-a_1\sigma_1)}^{-\sigma_2\sigma_2} \frac{2X_a(0)e^{s\tau}ds}{c_1\sigma_1}\left\{\frac{-\eta\varphi_{1-}(\mu)e^{(s+\sigma_1)x/\nu_{01}}}{(\nu_{01}^2+\eta^2)X_a(-\nu_{01})}\right.$$

$$\left. + \int \frac{\nu_{01}\eta\varphi_{1\nu}(\mu)e^{-(s+\sigma_1)x/\nu}d\nu}{(\nu+\nu_{01})(\nu^2+\eta^2)X_a^-(\nu)(\lambda_1(\nu)+i\pi\nu)}\right\}. \tag{106}$$

$\underline{I_4(\overline{\Psi}_s)}$: Here ν_{01} is imaginary and will be replaced by $i\xi$, whereas ν_{02} is real, greater than one, and obeys (Equation (84b))

$$s + \sigma_2 = \frac{c_2\sigma_2}{2}\nu_{02}\,\ln\frac{\nu_{02}+1}{\nu_{02}-1} .$$

Hence, with Equation (76) for $\overline{\Psi}_s(x,\mu)$, Equation (99) becomes

$$I_4(\overline{\Psi}_s) = 2iql(a_2\sigma_2 - a_1\sigma_1)\int_{-a_2\sigma_2}^{-a_1\sigma_1} \frac{2X_a(0)e^{s\tau}ds}{c_1\sigma_1}\left\{\frac{\nu_{02}}{\xi}\,\mathrm{Re}\left[\frac{\varphi_{1-}^{-}(\mu)e^{-i(s+\sigma_1)x/\xi}}{(\nu_{02}+i\xi)X_a(-i\xi)}\right]\right.$$

$$\left. + \int_{-1}^{0} \frac{\xi\nu_{02}\varphi_{1\nu}(\mu)e^{-(s+\sigma_1)x/\nu}d\nu}{(\nu^2+\xi^2)(\nu-\nu_{02})X_a^{-}(\nu)(\lambda_1(\nu)+i\pi\nu)}\right\} . \tag{107}$$

<u>$I_5(\overline{\Psi}_s - \overline{\psi}_{unc})$</u>: This integration occurs in region (c) of the s-plane; Equation (80) is the representation for $\overline{\Psi}_s(x,\mu)$ in this region. Writing Equation (100) explicitly yields

$$I_5(\overline{\Psi}_s - \overline{\psi}_{unc}) = q\int_{-\sigma_1-i\infty}^{-\sigma_1+i\infty} e^{s\tau}ds\left\{\frac{-2X_c(0)}{c_1\sigma_1}\cdot\int_{-1}^{0}\frac{\varphi_{1\nu}(\mu)e^{-(s+\sigma_1)x/\nu}d\nu}{\nu X_c^{-}(\nu)(\lambda_1(\nu)+i\pi\nu)}\right. \tag{108}$$

$$\left. - \frac{e^{-(s+\sigma_1)x/\mu}\,{}_1(-\mu)}{-\mu}\right\} .$$

Thus, the complete solution for $x < 0$ has been obtained.

E. Inversion in the General Case ($x > 0$)

To carry out the inversion for $x > 0$, we cannot shift the path to $\mathrm{Re}\, s = -\sigma_1$ but must be content with a shift to $\mathrm{Re}\, s = -\sigma_2$. As a result $I_1(\overline{\Psi}_s)$ will not appear in this case. For the special case discussed previously (branch (c) of Figure 10) the inversion path corresponding to Figure 11 for $x < 0$ is given in Figure 13 for $x > 0$. From this figure we can see that $I_1(\overline{\Psi}_s)$ does not appear, and also that I_5 travels through two regions in the s-plane, rather than just through region (c), as was the case previously.

Using Figure 13 as a guide, the complete solution can be written down for $x > 0$, using the notation developed for $x < 0$.

$$\psi(x,\mu,\tau) = \psi_{unc}(x,\mu,\tau) + \frac{1}{2\pi i}\left[I_2(\overline{\Psi}_s)+I_3(\overline{\Psi}_s)+I_4(\overline{\Psi}_s)+I_5(\overline{\Psi}_s-\overline{\psi}_{unc})\right]. \quad (109)$$

$$\overline{\Psi}_{unc}(x,\mu,s) = \frac{q e^{-(s+\sigma_2)x/\mu}\, 1(\mu)}{\mu}, \quad \psi_{unc}(x,\mu,\tau) = \frac{q e^{-\sigma_2 x/\mu}\, 1(\mu)\,\delta(\tau-\frac{x}{\mu})}{\mu}. \quad (110)$$

$$I_2(\overline{\Psi}_s) = 2iq\,1(\sigma_2 - a_1\sigma_1)\int_{-\sigma_2}^{-a_1\sigma_1 1(a_1\sigma_1-a_2\sigma_2)-a_2\sigma_2 1(a_2\sigma_2-a_1\sigma_1)} \frac{2X_a(0)\, e^{s\tau} ds}{c_2\sigma_2}\left\{\frac{\xi}{\eta(\eta+\xi)}\mathrm{Re}\left[\frac{\varphi_{2+}^{-}(\mu) e^{i(s+\sigma_2)x/\eta}}{X_a(i\eta)}\right]\right.$$
$$\left. + \int_0^1 \frac{\xi\eta(\eta-\xi)\varphi_{2\nu}(\mu) e^{-(s+\sigma_2)x/\nu}\, d\nu}{(\nu^2+\eta^2)(\nu^2+\xi^2)X_a^-(\nu)(\lambda_2(\nu)+i\pi\nu)}\right\}. \quad (111)$$

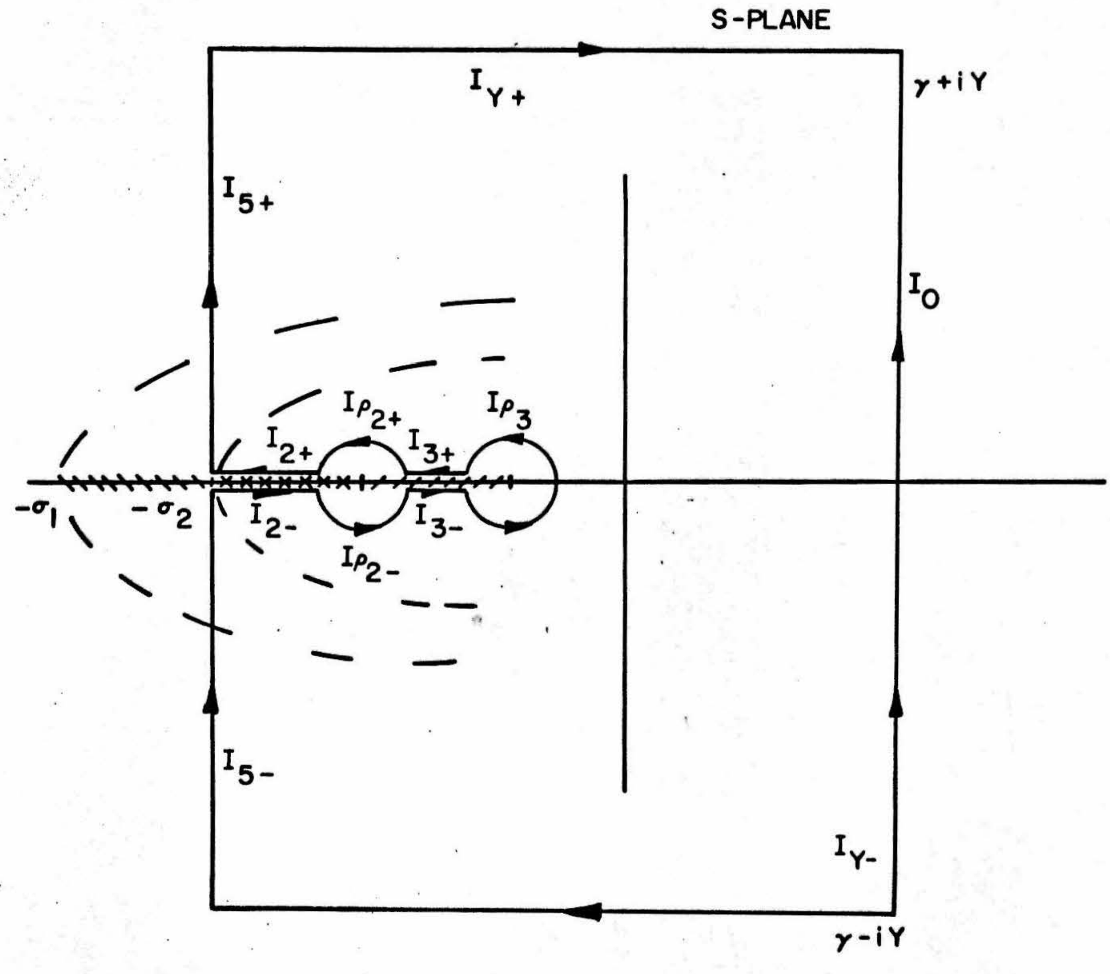

Figure 13 - Inversion Path for Branch (c) of Figure 10 for x>0

$$I_3(\overline{\Psi}_s) = 2iq(a_1\sigma_1 - a_2\sigma_2)\int_{-\sigma_2 1(a_1\sigma_1-\sigma_2)-a_1\sigma_1 1(\sigma_2-a_1\sigma_1)}^{-a_2\sigma_2} \frac{2X_a(0)e^{s\tau}ds}{c_2\sigma_2}\left\{\frac{\nu_{01}}{\eta}\ \mathrm{Re}\left[\frac{\varphi_{2+}^{-}(\mu)e^{i(s+\sigma_2)x/\eta}}{(\nu_{01}+i\eta)X_a(i\eta)}\right]\right.$$

$$\left. - \int_0^1 \frac{\eta\nu_{01}\varphi_{2\nu}(\mu)e^{-(s+\sigma_2)x/\nu}\,d\nu}{(\nu+\nu_{01})(\nu^2+\eta^2)X_a^-(\nu)(\lambda_2(\nu)+i\pi\nu)}\right\}. \tag{112}$$

$$I_4(\overline{\Psi}_s) = 2iq1(a_2\sigma_2 - a_1\sigma_1)\int_{-a_2\sigma_2}^{-a_1\sigma_1} \frac{-2X_a(0)e^{s\tau}ds}{c_2\sigma_2}\left\{\frac{\xi\varphi_{2+}(\mu)e^{-(s+\sigma_2)x/\nu_{02}}}{(\nu_{02}^2+\xi^2)X_a(\nu_{02})}\right. \tag{113}$$

$$\left. + \int_0^1 \frac{\xi\nu_{02}\varphi_{2\nu}(\mu)e^{-(s+\sigma_2)x/\nu}\,d\nu}{(\nu-\nu_{02})(\nu^2+\xi^2)X_a^-(\nu)(\lambda_2(\nu)+i\pi\nu)}\right\}.$$

$$I_5(\overline{\Psi}_s - \psi_{unc}) = \int_{-\sigma_2 - i\infty}^{-\sigma_2 - i\omega_0} qe^{s\tau} ds \left\{ \frac{2X_c(0)}{c_2\sigma_2} \int_0^1 \frac{\varphi_{2\nu}(\mu) e^{-(s+\sigma_2)x/\nu} d\nu}{\nu X_c^-(\nu)(\lambda_2(\nu) + i\pi\nu)} - \frac{e^{-(s+\sigma_2)x/\mu}\, 1(\mu)}{\mu} \right\}$$

$$+ \int_{-\sigma_2 - i\omega_0}^{-\sigma_2 - i0} qe^{s\tau} ds \left\{ \frac{2X_b(0)}{c_2\sigma_2} \int_0^1 \frac{\nu_{01}\varphi_{2\nu}(\mu) e^{-(s+\sigma_2)x/\nu} d\nu}{\nu(\nu+\nu_{01}) X_b^-(\nu)(\lambda_2(\nu) + i\pi\nu)} - \frac{e^{-(s+\sigma_2)x/\mu}\, 1(\mu)}{\mu} \right\}$$

$$+ \int_{-\sigma_2 + i0}^{-\sigma_2 + i\omega_0} qe^{s\tau} ds \left\{ \frac{2X_b(0)}{c_2\sigma_2} \int_0^1 \frac{\nu_{01}\varphi_{2\nu}(\mu) e^{-(s+\sigma_2)x/\nu} d\nu}{\nu(\nu+\nu_{01}) X_b^-(\nu)(\lambda_2(\nu) + i\pi\nu)} - \frac{e^{-(s+\sigma_2)x/\mu}\, 1(\mu)}{\mu} \right\} \tag{114}$$

$$+ \int_{-\sigma_2 + i\omega_0}^{-\sigma_2 + i\infty} qe^{s\tau} ds \left\{ \frac{2X_c(0)}{c_2\sigma_2} \int_0^1 \frac{\varphi_{2\nu}(\mu) e^{-(s+\sigma_2)x/\nu} d\nu}{\nu X_c^-(\nu)(\lambda_2(\nu) + i\pi\nu)} - \frac{e^{-(s+\sigma_2)x/\mu}\, 1(\mu)}{\mu} \right\}.$$

In I_5, ω_0 is determined by the relation

$$\omega_0 = \frac{c_1 \sigma_1 \pi \nu_{01}}{2} \quad , \qquad 0 < \nu_{01} < 1,$$

since ω_0 is on the boundary between regions (b) and (c) in the s-plane.

So we have written down the solution for $x > 0$ as well as for $x < 0$. This complete solution has been obtained as a sum of several, definite, multiple integrals. That this solution satisfies the original equation can be shown in a straightforward manner by substitution.

In Appendix IV, it is shown that the solution to Equation (37) subject to the boundary conditions that $\Psi(x, \mu, \tau)$ vanish at $x = \pm\infty$, if it exists, is unique. Since we have a solution to this equation satisfying the requisite boundary conditions, we can state that it is the unique solution.

F. <u>The Uncollided Angular Flux</u>

If one were to derive an equation to describe only the uncollided flux, it would take the form

$$\frac{\partial \Psi_{unc}(x, \mu, \tau)}{\partial \tau} + \mu \frac{\partial \Psi_{unc}}{\partial x} + \sigma(x)\Psi_{unc} = q\delta(x)\delta(\tau). \qquad (115)$$

But the quantity we have previously called the uncollided flux — that quantity whose Laplace transform does not vanish as $s \to \infty$ — can be shown to satisfy Equation (115). Thus, it is indeed the uncollided flux.

V. SIMPLIFICATION OF THE SOLUTION

A. The Angular Flux at the Interface (x = 0)

When x = 0, the solution written out in the previous chapter can be greatly simplified. Even though x need approach zero from only one side, it will be necessary later to have expressions for both x = 0+ and x = 0-.

For x = 0- (approaching the interface from region (1)), we know from Equation (95) that

$$\Psi(0-,\mu,\tau) = \Psi_{unc}(0-,\mu,\tau) + \frac{1}{2\pi i}[I_1(\overline{\Psi}_s) + \ldots + I_4(\overline{\Psi}_s) + I_5(\overline{\Psi}_s - \overline{\Psi}_{unc})]. \quad (116)$$

It can be shown for this case that $\overline{\Psi}_s(x,\mu)$ is analytic for $\text{Re}\, s \leqq -\sigma_1$ and that $I_5(\overline{\Psi}_s - \overline{\Psi}_{unc})$ behaves as $|s|^{-1}$ for large s. Hence, we can set I_5 equal to zero when x = 0-.

Letting x = 0+ (approaching the interface from region (2)) allows us to write Equation (109) as

$$\Psi(0+,\mu,\tau) = \Psi_{unc}(0+,\mu,\tau) + \frac{1}{2\pi i}[I_2(\overline{\Psi}_s) + \ldots + I_4(\overline{\Psi}_s) + I_5(\overline{\Psi}_s - \overline{\Psi}_{unc})]. \quad (117)$$

In this case we note that $\overline{\Psi}_s(x,\mu)$ is analytic for $\text{Re}\, s \leqq -\sigma_2$ save for a branch from $-\sigma_1$ to $-\sigma_2$ along the real axis, and I_5 behaves as $|s|^{-1}$ for large s. Hence we can set

$$I_5(\overline{\Psi}_s - \overline{\Psi}_{unc}) = I_1(\overline{\Psi}_s) \quad (117a)$$

for x = 0+.

Thus the solution for both x = 0+ and x = 0- has the same general form:

$$\Psi(0\pm,\mu,\tau) = \Psi_{unc}(0\pm,\mu,\tau) + \frac{1}{2\pi i}\left[I_1(\overline{\Psi}_s) + \ldots + I_4(\overline{\Psi}_s)\right] . \tag{118}$$

In Equation (118) we know, from Equations (87) and (110), that

$$\Psi_{unc}(0\pm,\mu,\tau) = \frac{q\ \delta\ (\tau)1(\pm\mu)}{\pm\ \mu} . \tag{119}$$

From Equations (103) and (117a),

$$\left.\begin{aligned}
I_1(\overline{\Psi}_s(0+,\mu)) &= 2iq\int_{-\sigma_1}^{-a_1\sigma_1 1(a_1\sigma_1-\sigma_2)-\sigma_2 1(\sigma_2-a_1\sigma_1)} \frac{2\xi X_b(0)e^{s\tau}ds}{c_2\sigma_2}\int_0^1 \frac{\varphi_{2\nu}(\mu)d\nu}{(\nu^2+\xi^2)X_b^-(\nu)(\lambda_2(\nu)+i\pi\nu)}, \\
I_1(\overline{\Psi}_s(0-,\mu) &= 2iq\int_{-\sigma_1}^{-a_1\sigma_1 1(a_1\sigma_1-\sigma_2)-\sigma_2 1(\sigma_2-a_1\sigma_1)} \frac{2X_b(0)e^{s\tau}ds}{c_1\sigma_1}\left\{ \mathrm{Re}\left\{\frac{\varphi_{1-}^-(\mu)}{X_b(-i\xi)}\right\}\right. \\
&\qquad \left. - \int_{-1}^{0} \frac{\xi\varphi_{1\nu}(\mu)d\nu}{(\nu^2+\xi^2)X_b^-(\nu)(\lambda_1(\nu)+i\pi\nu)}\right\} .
\end{aligned}\right] \tag{120}$$

From Equations (105) and (111),

$$I_2(\overline{\Psi}_s(0+,\mu)) = 2iq\,l(\sigma_2-a_1\sigma_1)\int_{-\sigma_2}^{-a_1\sigma_1 l(a_1\sigma_1-a_2\sigma_2)-a_2\sigma_2 l(a_2\sigma_2-a_1\sigma_1)} \frac{2X_a(0)e^{+s\tau}ds}{c_2\sigma_2}\left\{\frac{\xi}{\eta(\eta+\xi)}\,\mathrm{Re}\left\{\frac{\varphi_{2+}^-(\mu)}{X_a(i\eta)}\right\}\right.$$

$$\left.+\int_0^1 \frac{\xi\eta(\eta-\xi)\varphi_{2\nu}(\mu)d\nu}{(\nu^2+\eta^2)(\nu^2+\xi^2)X_a^-(\nu)(\lambda_2(\nu)+i\pi\nu)}\right\},$$

$$I_2(\overline{\Psi}_s(0-,\mu)) = 2iq\,l(\sigma_2-a_1\sigma_1)\int_{-\sigma_2}^{-a_1\sigma_1 l(a_1\sigma_1-a_2\sigma_2)-a_2\sigma_2 l(a_2\sigma_2-a_1\sigma_1)} \frac{2X_a(0)e^{s\tau}ds}{c_1\sigma_1}\left\{\frac{\eta}{\xi(\eta+\xi)}\,\mathrm{Re}\left\{\frac{\varphi_{1-}^-(\mu)}{X_a(-i\xi)}\right\}\right.$$

$$\left.-\int_{-1}^{0} \frac{\xi\eta(\eta-\xi)\varphi_{1\nu}(\mu)d\nu}{(\nu^2+\eta^2)(\nu^2+\xi^2)X_a^-(\nu)(\lambda_1(\nu)+i\pi\nu)}\right\}. \tag{121}$$

From Equations (106) and (112),

$$I_3(\overline{\Psi}_s(0+,\mu)) = 2iq\,l(a_1\sigma_1-a_2\sigma_2)\int_{-\sigma_2 l(a_1\sigma_1-\sigma_2)-a_1\sigma_1 l(\sigma_2-a_1\sigma_1)}^{-a_2\sigma_2} \frac{2X_a(0)e^{s\tau}ds}{c_2\sigma_2}\left\{\frac{\nu_{01}}{\eta}\,\mathrm{Re}\left\{\frac{\varphi_{2+}^-(\mu)}{(\nu_{01}+i\eta)X_a(i\eta)}\right\}\right.$$

$$\left.\int_0^1 \frac{\eta\nu_{01}\varphi_{2\nu}(\mu)d\nu}{(\nu+\nu_{01})(\nu^2+\eta^2)X_a^-(\nu)(\lambda_2(\nu)+i\pi\nu)}\right\},$$

$$I_3(\overline{\Psi}_s(0-,\mu)) = 2iq\,l(a_1\sigma_1-a_2\sigma_2)\int_{-\sigma_2 l(a_1\sigma_1-\sigma_2)-a_1\sigma_1 l(\sigma_2-a_1\sigma_1)}^{-a_2\sigma_2} \frac{2X_a(0)e^{s\tau}ds}{c_1\sigma_1}\left\{\frac{-\eta\varphi_{1-}(\mu)}{(\nu_{01}^2+\eta^2)X_a(-\nu_{01})}\right.$$

$$\left.+\int_{-1}^{0} \frac{\nu_{01}\eta\varphi_{1\nu}(\mu)d\nu}{(\nu+\nu_{01})(\nu^2+\eta^2)X_a^-(\nu)(\lambda_1(\nu)+i\pi\nu)}\right\}. \tag{122}$$

From Equations (107) and (113),

$$I_4(\overline{\Psi}_s(0+,\mu)) = 2iql(a_2\sigma_2-a_1\sigma_1)\int_{-a_2\sigma_2}^{-a_1\sigma_1} \frac{-2X_a(0)e^{s\tau}ds}{c_2\sigma_2}\left\{\frac{\xi\varphi_{2+}(\mu)}{(\nu_{02}^2+\xi^2)X_a(\nu_{02})}\right.$$

$$\left. + \int_0^1 \frac{\xi\nu_{02}\varphi_{2\nu}(\mu)d\nu}{(\nu-\nu_{02})(\nu^2+\xi^2)X_a^-(\nu)(\lambda_2(\nu)+i\pi\nu)}\right\},$$

(123)

$$I_4(\overline{\Psi}_s(0-,\mu)) = 2iql(a_2\sigma_2-a_1\sigma_1)\int_{-a_2\sigma_2}^{-a_1\sigma_1} \frac{2X_a(0)e^{s\tau}ds}{c_1\sigma_1}\left\{\frac{\nu_{02}}{\xi}\mathrm{Re}\left\{\frac{\varphi_{1-}^-(\mu)}{(\nu_{02}+i\xi)X_a(-i\xi)}\right\}\right.$$

$$\left. + \int_{-1}^0 \frac{\xi\nu_{02}\varphi_{1\nu}d\nu}{(\nu^2+\xi^2)(\nu-\nu_{02})X_a^-(\nu)(\lambda_1(\nu)+i\pi\nu)}\right\}.$$

It is of course necessary that the solution be the same at the interface no matter how it is approached; that is,

$$I_1(\overline{\Psi}_s(0+,\mu)) = I_1(\overline{\Psi}_s(0-,\mu)) \text{ , etc.}$$

In Appendix V, the expressions (120) through (123) are simplified through the evaluation of the integrals over ν and through some subsequent algebraic manipulations. Their simplified forms, listed as Equations (A102), (A104), (A105), (A106), are given below:

$$I_1(\overline{\Psi}_s(0,\mu)) = 2iq\int_{-\sigma_1}^{-a_1\sigma_1 l(a_1\sigma_1-\sigma_2)-\sigma_2 l(\sigma_2-a_1\sigma_1)} \frac{\sqrt{-b_1b_2}\; e^{s\tau}ds}{b_2-b_1} \begin{cases} \sqrt{\frac{b_1-2}{b_2-2}}\;\frac{1}{1-\mu}\;\frac{X_{10}(\mu)}{X_{20}(\mu)} & -1<\mu<0 \\ \sqrt{\frac{b_2-2}{b_1-2}}\;\frac{-1-\mu}{\mu^2+\xi^2}\;\frac{X_{20}(-\mu)}{X_{10}(-\mu)} & 0<\mu<1 \end{cases} \qquad (124)$$

$$I_2(\overline{\Psi}_s(0,\mu)) = 2iq\,1(\sigma_2 - a_1\sigma_1)\int_{-\sigma_2}^{-a_1\sigma_1 1(a_1\sigma_1 - a_2\sigma_2) - a_2\sigma_2 1(a_2\sigma_2 - a_1\sigma_1)} \frac{(\eta-\xi)\sqrt{b_1 b_2}\, e^{s\tau} ds}{(b_1 b_2)} \begin{cases} \sqrt{\dfrac{b_1 - 2}{b_2 - 2}}\, \dfrac{1}{\mu^2 + \eta^2}\, \dfrac{X_{10}(\mu)}{X_{20}(\mu)} & -1<\mu<0 \\[2ex] \sqrt{\dfrac{b_2 - 2}{b_1 - 2}}\, \dfrac{1}{\mu^2 + \xi^2}\, \dfrac{X_{20}(-\mu)}{X_{10}(-\mu)} & 0<\mu<1 \end{cases} \quad (125)$$

$$I_3(\overline{\Psi}_s(0,\mu)) = 2iq\,1(a_1\sigma_1 - a_2\sigma_2)\int_{-\sigma_2 1(a_1\sigma_1 - \sigma_2) - a_1\sigma_1 1(\sigma_2 - a_1\sigma_1)}^{-a_2\sigma_2} \frac{\sqrt{b_1 b_2}\, e^{s\tau} ds}{(b_2 - b_1)} \begin{cases} \sqrt{\dfrac{b_1 - 2}{2 - b_2}}\, \dfrac{\mu - \nu_{01}}{\mu^2 + \eta^2}\, \dfrac{X_{10}(\mu)}{X_{20}(\mu)} & -1<\mu<0 \\[2ex] \sqrt{\dfrac{2 - b_2}{b_1 - 2}}\, \dfrac{-1}{\mu + \nu_{01}}\, \dfrac{X_{20}(-\mu)}{X_{10}(-\mu)} & 0<\mu<1 \end{cases} \quad (126)$$

$$I_4(\overline{\Psi}_s(0,\mu)) = 2iq\,1(a_2\sigma_2 - a_1\sigma_1)\int_{-a_2\sigma_2}^{-a_1\sigma_1} \frac{\sqrt{b_1 b_2}\, e^{s\tau} ds}{(b_2 - b_1)} \begin{cases} \sqrt{\dfrac{2 - b_1}{b_2 - 2}}\, \dfrac{-1}{\mu - \nu_{02}}\, \dfrac{X_{10}(\mu)}{X_{20}(\mu)} & -1<\mu<0 \\[2ex] \sqrt{\dfrac{b_2 - 2}{2 - b_1}}\, \dfrac{\mu + \nu_{02}}{\mu^2 + \xi^2}\, \dfrac{X_{20}(-\mu)}{X_{10}(-\mu)} & 0<\mu<1 \end{cases} \quad (127)$$

where $b_n = \dfrac{2}{c_n \sigma_n}\,(s + \sigma_n)$, $n = 1,2$,

and $X_{n0}(z) = \exp\left\{\dfrac{1}{2\pi i}\displaystyle\int_0^1 \ell n\, \dfrac{\lambda_n(\mu') + i\pi\mu'}{\lambda_n(\mu') - i\pi\mu'}\, \dfrac{d\mu'}{\mu' - z}\right\}$, $n = 1, 2$.

B. The Interface Flux and Current

The flux and current can be separated into an uncollided and a collided portion for arbitrary x. To obtain the uncollided flux and current for any x, we know, from Equations (87) and (110), that

$$\Psi_{unc}(x,\mu,\tau) = \begin{cases} \frac{q}{\mu} e^{-\sigma_2 x/\mu} 1(\mu)\delta(\tau - \frac{x}{\mu}), & x > 0 \\ -\frac{q}{\mu} e^{-\sigma_1 x/\mu} 1(-\mu)\delta(\tau - \frac{x}{\mu}), & x < 0 \end{cases} \quad . \tag{128}$$

Hence from the definitions of flux and current [Equations (40)], we obtain, making use of Equation (128),

$$\Phi_{unc}(x,\tau) = \int_{-1}^{+1} \Psi_{unc}(x,\mu,\tau)d\mu = \begin{cases} \frac{q}{\tau} e^{-\sigma_2 \tau} 1(\tau - x) & x > 0 \\ \frac{q}{\tau} e^{-\sigma_1 \tau} 1(\tau + x) & x < 0 , \end{cases} \tag{129}$$

$$J_{unc}(x,\tau) = \int_{-1}^{+1} \mu \Psi_{unc}(x,\mu,\tau)d\mu = \begin{cases} \frac{qx}{\tau^2} e^{-\sigma_2 \tau} 1(\tau - x) & x > 0 \\ \frac{qx}{\tau^2} e^{-\sigma_1 \tau} 1(\tau + x) & x < 0 . \end{cases} \tag{130}$$

By inspection, there exists no uncollided current at the interface (as expected) and there is an ambiguity in the uncollided flux. This ambiguity exists because of the abrupt discontinuity in total cross section at the interface. To solve this difficulty we define

$$\Phi_{unc}(0,\tau) \equiv \frac{q}{\tau} e^{-\tau\left(\frac{\sigma_1 + \sigma_2}{2}\right)} , \quad \tau > 0 . \tag{131}$$

Both the collided interface flux and current can be written in simple forms. How these simple forms are obtained is discussed in Appendix V. The resultant expressions for the total interface flux and current are given below.

From Equations (A107),

$$\Phi(0,\tau) = \Phi_{unc}(0,\tau) + \frac{1}{2\pi i}\left[I_1(0,\tau)+I_2(0,\tau)+I_3(0,\tau)+I_4(0,\tau)\right], \qquad (132)$$

$$J(0,\tau) = \frac{1}{2\pi i}\left[J_1(0,\tau)+J_2(0,\tau)+J_3(0,\tau)+J_4(0,\tau)\right] . \qquad (133)$$

From Equation (A110),

$$I_1(0,\tau) = 2iq\int_{-\sigma_1}^{-a_1\sigma_1 1(a_1\sigma_1-\sigma_2)-\sigma_2 1(\sigma_2-a_1\sigma_1)} \frac{-b_1b_2}{(b_1-b_2)}\,\frac{e^{s\tau}ds}{\xi} . \qquad (134)$$

From Equation (A113),

$$I_2(0,\tau) = 2iq1(\sigma_2-a_1\sigma_1)\int_{-\sigma_2}^{-a_1\sigma_1 1(a_1\sigma_1-a_2\sigma_2)-a_2\sigma_2 1(a_2\sigma_2-a_1\sigma_1)} \frac{b_1b_2 \quad (\xi-\eta) \quad e^{s\tau}ds}{(b_1-b_2) \qquad \xi\eta} . \qquad (135)$$

From Equation (A114),

$$I_3(0,\tau) = 2iq1(a_1\sigma_1-a_2\sigma_2)\int_{-\sigma_2 1(a_1\sigma_1-\sigma_2)-a_1\sigma_1 1(\sigma_2-a_1\sigma_1)}^{-a_2\sigma_2} \frac{b_1b_2}{b_1-b_2}\,\frac{e^{s\tau}ds}{\eta} . \qquad (136)$$

From Equation (A115),

$$I_4(0,\tau) = 2iq1(a_2\sigma_2-a_1\sigma_1)\int_{-a_2\sigma_2}^{-a_1\sigma_1} \frac{b_1b_2}{b_2-b_1}\,\frac{e^{s\tau}ds}{\xi} . \qquad (137)$$

From Equation (A116),

$$J_1(0,\tau) = 2iq\int_{-\sigma_1}^{-a_1\sigma_1 1(a_1\sigma_1-\sigma_2)-\sigma_2 1(\sigma_2-a_1\sigma_1)} \frac{\sqrt{-b_1b_2}\,\sqrt{(2-b_2)(2-b_1)}\;e^{s\tau}ds}{(b_1-b_2)} . \qquad (138)$$

From Equation (A117),

$J_2(0,\tau) = 0$.

From Equation (A118),

$$J_3(0,\tau) = 2iql(a_1\sigma_1 - a_2\sigma_2)\int_{-\sigma_2 l(a_1\sigma_1-\sigma_2)-a_1\sigma_1 l(\sigma_2-a_1\sigma_1)}^{-a_2\sigma_2} \frac{\sqrt{b_1 b_2}\ \sqrt{(b_1-2)(2-b_2)}\ e^{s\tau}ds}{(b_2 - b_1)} \quad . \tag{139}$$

From Equation (A119),

$$J_4(0,\tau) = 2iql(a_2\sigma_2 - a_1\sigma_1)\int_{-a_2\sigma_2}^{-a_1\sigma_1} \frac{\sqrt{b_1 b_2}\ \sqrt{2-b_1)(b_2-2)}\ e^{s\tau}ds}{(b_2 - b_1)} \quad . \tag{140}$$

C. Simplifications for $x \neq 0$.

For $x \neq 0$, there seem to be no simplifications comparable to those made when $x = 0$. However, to obtain the total flux from the general results (Equations (95) and (109)), it is necessary only to integrate out the μ dependence, where we note from Section C of Chapter III,

$$\int_{-1}^{+1} \varphi_n(\mu)d\mu = s + \sigma_n \ , \quad n = 1, 2.$$

Thus when $x > 0$, for example, Equation (109) is integrated over μ from -1 to +1 to obtain

$$\Phi(x,\tau) = \Phi_{unc}(x,\tau) + \frac{1}{2\pi i}\left[I_1(x,\tau)+\ldots+I_5(x,\tau)\right] \ , \tag{141}$$

where $\Phi_{unc}(x,\tau)$ is given by Equation (129) and the $I_n(x,\tau)$ are the $I_n(\overline{\Psi}_s(x,\mu))$ with the $\varphi_n(\mu)$ replaced by $(s+\sigma_n)$.

D. Two Special Cases

The two adjacent media problem contains both the single-infinite medium problem and the single semi-infinite medium problem as special cases. Let us obtain these results now.

1. The Single Infinite Medium Problem

From an inspection of the general solution for $x < 0$, given by Equations (95) through (100), one can see that

$$I_1(\overline{\Psi}_s) = I_3(\overline{\Psi}_s) = I_4(\overline{\Psi}_s) = 0$$

for the single infinite medium. Furthermore, from Equation (105),

$$I_2(\overline{\Psi}_s(x,\mu)) = 2iq \int_{-\sigma}^{-a\sigma} \frac{2X_a(0)e^{s\tau}ds}{c\sigma} \left\{ \frac{1}{2\xi} \mathrm{Re} \left\{ \frac{\varphi_-^-(\mu)e^{-i(s+\sigma)x/\xi}}{X_a(-i\xi)} \right\} \right\}, \quad x < 0 . \tag{142}$$

The subscripts have been dropped since there is only one medium. Also, no changes are needed in $I_5(\overline{\Psi}_s - \overline{\Psi}_{unc})$ and $\Psi_{unc}(x,\mu,\tau)$ other than the dropping of subscripts. Hence, we obtain

$$\Psi(x,\mu,\tau) = \Psi_{unc}(x,\mu,\tau) + \frac{1}{2\pi i}\left[I_2(\overline{\Psi}_s)+I_5(\overline{\Psi}_s - \overline{\Psi}_{unc})\right] , \tag{143}$$

with $I_5(\overline{\Psi}_s - \overline{\Psi}_{unc})$ given by Equation (108) and $\Psi_{unc}(x,\mu,\tau)$ by Equation (87).

We also note that for $x > 0$, the general solution can be written in exactly the same form (Equation (143))where now $I_2(\overline{\Psi}_s)$ becomes, from Equation (111),

$$I_2(\overline{\Psi}_s) = 2iq \int_{-\sigma}^{-a\sigma} \frac{2X_a(0)e^{s\tau}ds}{c\sigma} \left\{ \frac{1}{2\xi} \operatorname{Re} \left\{ \frac{\varphi_+^-(\mu)e^{i(s+\sigma)x/\xi}}{X_a(i\xi)} \right\} \right\}, \quad x>0, \quad (144)$$

where the $I_5(\overline{\Psi}_s - \overline{\Psi}_{unc})$ of Equation (114) becomes (region (b) being absent)

$$I_5(\overline{\Psi}_s - \overline{\Psi}_{unc}) = q \int_{-\sigma-i\infty}^{-\sigma+i\infty} e^{s\tau} ds \left\{ \frac{2X_c(0)}{c\sigma} \int_0^1 \frac{\varphi_\nu(\mu)e^{-(s+\sigma)x/\nu}\, d\nu}{\nu X_c^-(\nu)(\lambda(\nu)+i\pi\nu)} - \frac{e^{-(s+\sigma)x/\mu}\, 1(\mu)}{\mu} \right\}, \quad (145)$$

$$x > 0,$$

and where the uncollided angular flux is given by Equation (110).

The single infinite medium problem can be solved in an alternative manner quite easily by applying the full-range orthogonality relations developed by Case[3] to obtain the expansion coefficients in the transformed solution. When this is done and the final solution compared to that given above, the two methods are found to yield identical expressions for $\Psi(x, \mu, \tau)$.

At the source plane, we can again obtain a simple form for the flux (the current is zero) either directly from Equation (143), or by a simplification of the general result for the interface flux, Equation (132). Either method yields

$$\Phi(0,\tau) = q \left\{ \frac{e^{-\sigma\tau}}{\tau} + \frac{2}{\pi c\sigma} \int_{-\sigma}^{-a\sigma} \frac{e^{s\tau}(s+\sigma)\, ds}{\eta \left\{ 1 - \frac{c\sigma\eta^2}{(s+\sigma)(1+\eta^2)} \right\}} \right\}. \quad (146)$$

2. The Single Semi-Infinite Medium Problem

(This problem was done with angular source dependence in Reference (19).)

Since we have required $\sigma_1 \geqq \sigma_2$, let us assume that the left half space contains the semi-infinite medium. The right half space can be either vacuum or a pure absorber. Hence, we set

$$c_2 = 0 , \quad \nu_{02} = 1 ,$$

and observe that

$$X_{20}(z) = 1 .$$

This choice of constants reduces the two-region solution to that of the half space.

If we now look at the general form for the solution, Equations (95) through (100), we find that these equations become, for the semi-infinite medium $(x < 0)$,

$$\Psi(x,\mu,\tau) = \Psi_{unc}(x,\mu,\tau) + \frac{1}{2\pi i}\left[I(\overline{\Psi}_s) + I_5(\overline{\Psi}_s - \overline{\Psi}_{unc})\right] , \tag{147}$$

where

$$I(\overline{\Psi}_s) = 2iq\int_{-\sigma}^{-a\sigma} \frac{2X_{10}(0)e^{s\tau}ds}{-c_1\sigma_1}\left\{\frac{1}{\xi}\,\mathrm{Re}\left\{\frac{\varphi_{-}(\mu)e^{-i(s+\sigma)x/\xi}(-1+i\xi)}{X_{10}(i\xi)}\right\} - \int_{-1}^{0}\frac{\xi\varphi_{\nu}(\mu)e^{-(s+\sigma)x/\nu}(-1-\nu)\,d\nu}{(\nu^2+\xi^2)X_{10}^{+}(-\nu)(\lambda(\nu)+i\pi\nu)}\right\} , \tag{148}$$

$$I_5(\overline{\Psi}_s - \overline{\Psi}_{unc}) = q\int_{-\sigma-i\infty}^{-\sigma+\infty} e^{s\tau}ds\left\{\frac{-2X_{10}(0)}{c_1\sigma_1}\int_{-1}^{0}\frac{\varphi_{\nu}(\mu)e^{-(s+\sigma)x/\nu}\,d\nu}{\nu X_{10}^{+}(-\nu)(\lambda(\nu)+i\pi\nu)} - \frac{e^{-(s+\sigma_1)x/\mu}\,1(-\mu)}{-\mu}\right\} \tag{149}$$

and $\Psi_{unc}(x,\mu,\tau)$ is given by Equation (87).

The single half space problem can be solved independent of the two media problem. When this is done and the results compared, they are again found to be identical.

At the boundary of the half space, certain simplifications can be made to Equations (147) through (149) which lead to

$$\Psi(0,\mu,\tau) = q\,\frac{1(\mu)\delta(\tau)}{\mu} + \frac{q}{\pi}\int_{-\sigma}^{-a\sigma} \frac{\sqrt{b_1}}{\sqrt{2-b_1}}\,\frac{-1-\mu}{\mu^2+\xi^2}\,\frac{e^{s\tau}\,ds}{X_{10}(-\mu)}\ , \tag{150}$$

$$\Phi(0,\tau) = \frac{q}{\tau}\,e^{\frac{-\sigma_1\tau}{2}} + \frac{q}{\pi}\int_{-\sigma_1}^{-a_1\sigma_1} \frac{b_1 e^{s\tau}\,ds}{\xi}\ , \tag{151}$$

$$J(0,\tau) = \frac{q}{\pi}\int_{-\sigma}^{-a\sigma} e^{s\tau}\sqrt{b_1}\ \sqrt{2-b_1}\ \ ds\ . \tag{152}$$

So when the general two region problem is reduced to the single infinite or semi-infinite region problems, the results obtained agree with those derived by alternative, but related, methods.

E. <u>Integration of $\Psi(x,\mu,\tau)$ over all τ</u>

Since $\overline{\Psi}_s(x,\mu) = \int_0^\infty e^{-s\tau}\,\Psi(x,\mu,\tau)d\tau$, we can write

$$\lim_{s\to 0}\int_0^\infty e^{s\tau}\,\Psi(x,\mu,\tau)d\tau = \int_0^\infty \Psi(x,\mu,\tau)d\tau = \overline{\Psi}_0(x,\mu),\ \text{for } c < 1\ .$$

But s = 0 is a point in region (a) of the s-plane. Thus from the transformed solutions given by Equations (76) and (77), we can write

$$\int_0^\infty \Psi(x,\mu,\tau)d\tau = \frac{-2qX_a(0)}{c_1\sigma_1}\left\{\frac{\nu_{02}\varphi_{1-}(\mu)e^{\sigma_1 x/\nu_{01}}}{\nu_{01}(\nu_{02}+\nu_{01})X_a(-\nu_{01})}\right.$$

$$\left. - \int_{-1}^{0} \frac{\nu_{01}\nu_{02}\varphi_{1\nu}(\mu)e^{-\sigma_1 x/\nu} d\nu}{\nu(\nu-\nu_{02})(\nu+\nu_{01})X_a^-(\nu)(\lambda_1(\nu)+i\pi\nu)}\right\}, \quad x<0, \tag{153}$$

$$\int_0^\infty \Psi(x,\mu,\tau)d\tau = \frac{-2qX_a(0)}{c_2\sigma_2}\left\{\frac{\nu_{01}\varphi_{2+}(\mu)e^{-\sigma_2 x/\nu_{02}}}{\nu_{02}(\nu_{02}+\nu_{01})X_a(\nu_{02})}\right.$$

$$\left. + \int_{0}^{1} \frac{\nu_{01}\nu_{02}\varphi_{2\nu}(\mu)e^{-\sigma_2 x/\nu} d\nu}{\nu(\nu-\nu_{02})(\nu+\nu_{01})X_a^-(\nu)(\lambda_2(\nu)+i\pi\nu)}\right\}, \quad x>0. \tag{154}$$

In Equations (153) and (154), for the evaluation of ν_{01}, $X_a(-\nu_{01})$, etc., we must set $s = 0$.

This work can be extended by noting that

$$\overline{\Psi}_s(x,\mu) = \sum_{n=0}^{\infty} \frac{(-1)^n s^n}{n!} \int_0^\infty \tau^n \, \Psi(x,\mu,\tau)\, d\tau \quad . \tag{155}$$

Thus expanding $\overline{\Psi}_s(x,\mu)$ in a power series about $s = 0$ will give the moments of the neutron angular distribution in time.

VI. NUMERICAL ANALYSIS AND CONCLUSIONS

The exact solution to the problem of finding the time-dependent, monoenergetic neutron distribution in two adjacent, semi-finite media has been obtained in closed form. This exact solution was written as a combination of multiple, definite integrals, one of which is improper. At the interface (source plane) it was possible to reduce this solution and write it as a sum of simple, definite integrals, all having finite limits.

Rather than approximating these solutions for short and long times, possibly by the use of expansions, an exact numerical evaluation was made with the intention of exhibiting the system properties for all time. Also, because of the simplicity in the structure of the solution at the source plane as compared to $x \neq 0$, the numerical work was done for $x = 0$.

Three different cases were investigated and are described in sequence. In all cases, the total flux and current were the quantities calculated.

A. One Infinite Medium

For a single medium with a plane isotropic burst of neutrons appearing at $x = 0$ when $t = 0$, the total flux is given by Equation (146);

$$\Phi(o, \tau) = q\left[\frac{e^{-\sigma\tau}}{\tau} + \frac{2}{\pi c \sigma}\int_{-\sigma}^{-a\sigma} \frac{e^{s\tau}(s+\sigma)\, ds}{\eta\left[1 - \dfrac{c\sigma\eta^2}{(s+\sigma)(1+\eta^2)}\right]}\right]. \tag{156}$$

Equation (104) relates η to s.

Using P-1 theory[25] (see Appendix VI for approximate solutions), it can be shown that (Equation (A151))

$$\Phi(o,\tau) = \Phi_o(o,\tau) = \sqrt{3}\, q\delta(\tau) + \frac{\sqrt{3}q}{\pi}\int_{-\sigma}^{-a\sigma} e^{s\tau}\sqrt{\frac{s+\sigma}{-s-a\sigma}}\, ds. \qquad (157)$$

From diffusion theory (Equation (A165)), we can write

$$\Phi(o,\tau) = \frac{qe^{-a\sigma\tau}}{\sqrt{\pi\tau D}} \quad . \qquad (158)$$

An infinite medium of pure carbon was considered and the flux at the source plane calculated as a function of time utilizing Equations (156), (157), and (158) — the total current at the interface is zero. The results are shown in Figure 14.

For long times ($\tau > 20$), we see that both the diffusion and P-1 theory representations of the flux become almost identical to that given by exact theory. In fact for $c \approx 1$, Equations (156) and (157) have as asymptotic forms Equation (158) for large τ.

The analysis in this thesis was made on monoenergetic systems, presumably in the thermal energy range. In many experimental situations where thermal neutron decay is being measured, high energy neutrons are introduced into the system and slowing down occurs. Since the slowing-down time in graphite is approximately 150 μ-sec, the short-time deviations shown in Figure 14 would then be obscured. However, one can expect diffusion theory to be accurate soon after the slowing down time has elapsed.

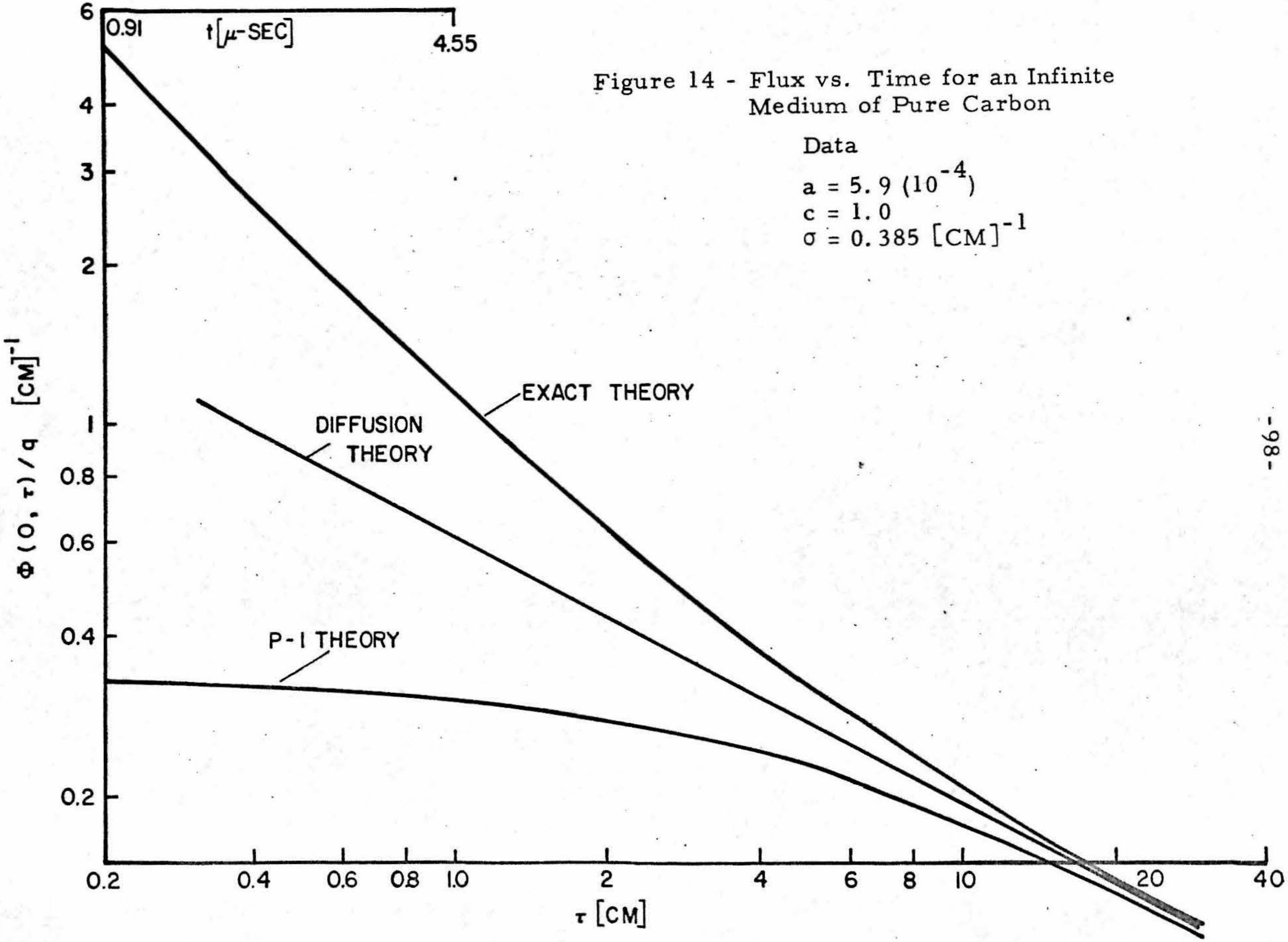

Figure 14 - Flux vs. Time for an Infinite Medium of Pure Carbon

Data

$a = 5.9\ (10^{-4})$
$c = 1.0$
$\sigma = 0.385\ [CM]^{-1}$

From Figure 14, it can be seen that the source plane flux is more accurately represented by diffusion than by P-1 theory for short times. The diffusion approximation results from making several simplifying assumptions in the P-1 equations [see Appendix VI]. However, it does not necessarily follow that these assumptions must make diffusion theory less accurate than P-1 theory—in fact at the source plane just the opposite effect is seen to occur. The second order equation which results from P-1 theory is given by Equation (A155). For $c \approx 1$, the major difference between P-1 and diffusion theory is contained in the term

$$\frac{1}{\sigma} \frac{\partial^2 \varphi(x, \tau)}{\partial \tau^2} ,$$

which is neglected in diffusion theory. If σ is very large, this term becomes negligible and one then expects the two approximate solutions to converge. This can be seen by observing that for $c \approx 1$ and large σ Equation (157) reduces to Equation (158).

Hence the mathematical reason for the deviation between P-1 and diffusion theory shown in Figure 14 is that the term containing the second derivative in time is not negligible for carbon for short times.

This second derivative in time introduces a delta function in time (which can be thought of as an approximation to the uncollided flux) and other modifications into the diffusion solution for the flux. The net effect of these modifications must be to initially remove neutrons from the source plane. Thus one can expect diffusion theory, which does not take into account this initial loss of neutrons from the source plane, to have a higher neutron flux at $x = 0$

for short times. This ordering of the two approximations is expected to hold for all moderator materials.

B. Two Adjacent Semi-Infinite Media (Actual Materials)

Three different physical systems, containing half spaces of carbon, water, and beryllium, were examined; the data used are shown below[26].

Material	$\sigma[CM]^{-1}$	c	a
C	0.385	1.0	$5.9(10^{-4})$
H_2O	3.63	0.9945	$5.37(10^{-3})$
Be	0.869	0.9988	$1.265(10^{-3})$

For the material properties noted above, we must evaluate the expressions listed below, which were obtained from Equations (131) through (140).

$$\Phi(0,\tau) = \Phi_{unc}(0,\tau) + \frac{1}{2\pi i}\left[I_1(0,\tau) + I_2(0,\tau) + I_3(0,\tau)\right]. \tag{159}$$

$$J(0,\tau) = \frac{1}{2\pi i}\left[J_1(0,\tau) + J_3(0,\tau)\right]. \tag{160}$$

$$I_1(0,\tau) = 2iq\int_{-\sigma_1}^{-\sigma_2} \frac{-b_1 b_2}{(b_1-b_2)}\, e^{s\tau}\,\frac{ds}{\xi}\,. \tag{161}$$

$$I_2(0,\tau) = 2iq\int_{-\sigma_2}^{-a_1\sigma_1} \frac{b_1 b_2}{(b_1-b_2)}\,\frac{(\xi-\eta)}{\xi\eta}\, e^{s\tau}\, ds. \tag{162}$$

$$I_3(0,\tau) = 2iq\int_{-a_1\sigma_1}^{-a_2\sigma_2} \frac{b_1 b_2}{(b_1-b_2)}\,\frac{e^{s\tau}ds}{\eta}\,. \tag{163}$$

$$\Phi_{unc}(0,\tau) = \frac{q}{\tau}\, e^{-\frac{\tau}{2}(\sigma_1 + \sigma_2)} \quad . \tag{164}$$

$$J_1(0,\tau) = 2iq \int_{-\sigma_1}^{-\sigma_2} \frac{\sqrt{-b_1 b_2}\ \sqrt{(2-b_1)(2-b_2)}\ e^{s\tau} ds}{(b_1 - b_2)} \quad . \tag{165}$$

$$J_3(0,\tau) = 2iq \int_{-a_1\sigma_1}^{-a_2\sigma_2} \frac{\sqrt{b_1 b_2}\ \sqrt{(b_1-2)(2-b_2)}\ e^{s\tau}\ ds}{(b_2 - b_1)} \quad . \tag{166}$$

Keeping in mind the convention of placing the material with the larger total cross section to the left, the flux given by Equation (159) was numerically evaluated for two adjacent half spaces of (1) carbon and beryllium, (2) water and carbon, and (3) water and beryllium. The results are plotted on log-log paper for short times (Figures 15a, 16a, 17a) and on semi-log paper for long times (Figures 15b, 16b, 17b).

For short times the two-region flux decay curves lie below their single region counterparts having the larger total, i. e., scattering, cross sections (since absorption is weak). A material with a high scattering scross section acts to retain the neutrons near their initial positions as opposed to a material with a lower scattering cross section. Hence one would expect, for example, on the basis of the scattering cross sections involved, that neutrons would remain at the source plane longer in pure water than they would in carbon and water.

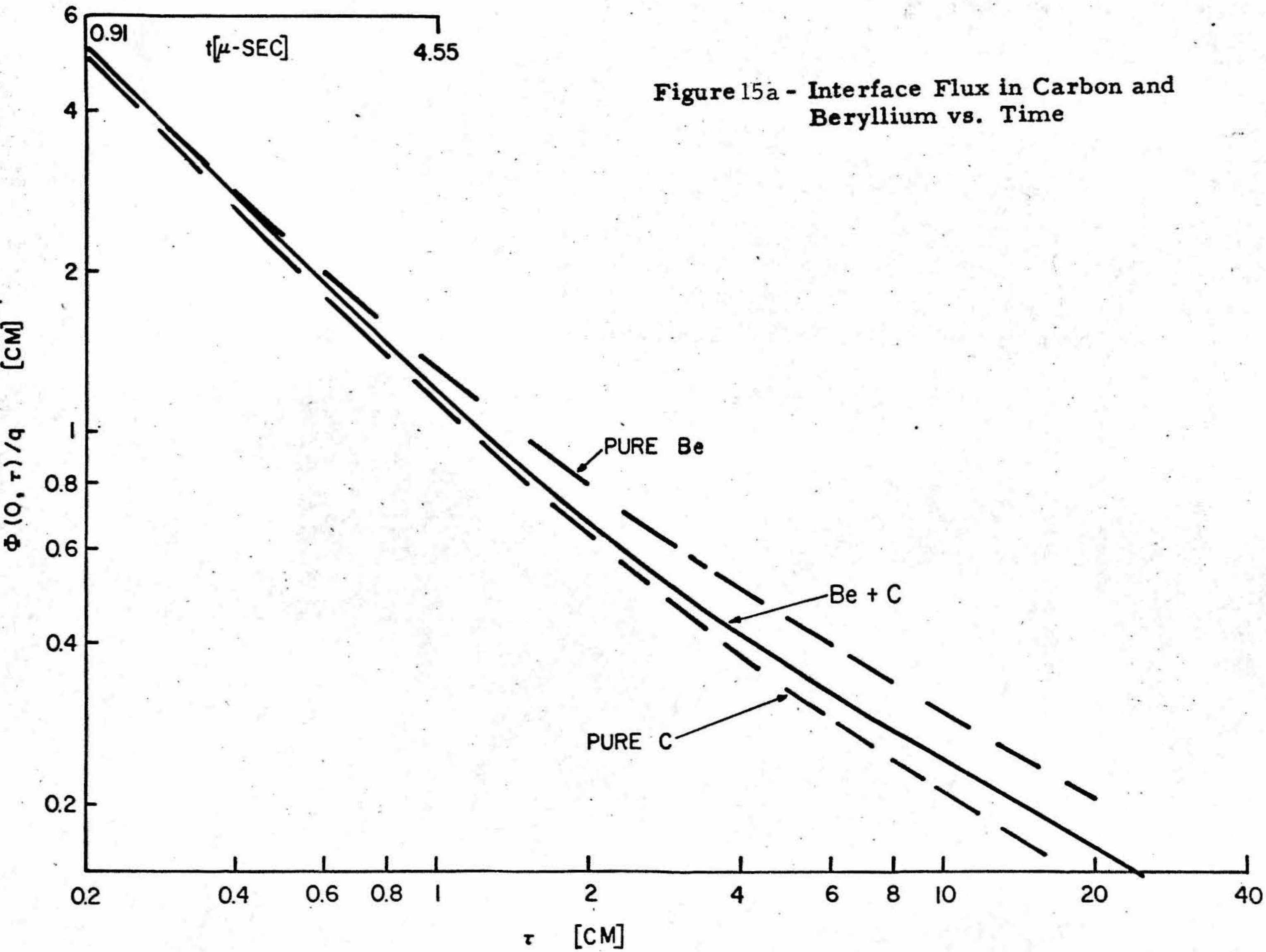

Figure 15a - Interface Flux in Carbon and Beryllium vs. Time

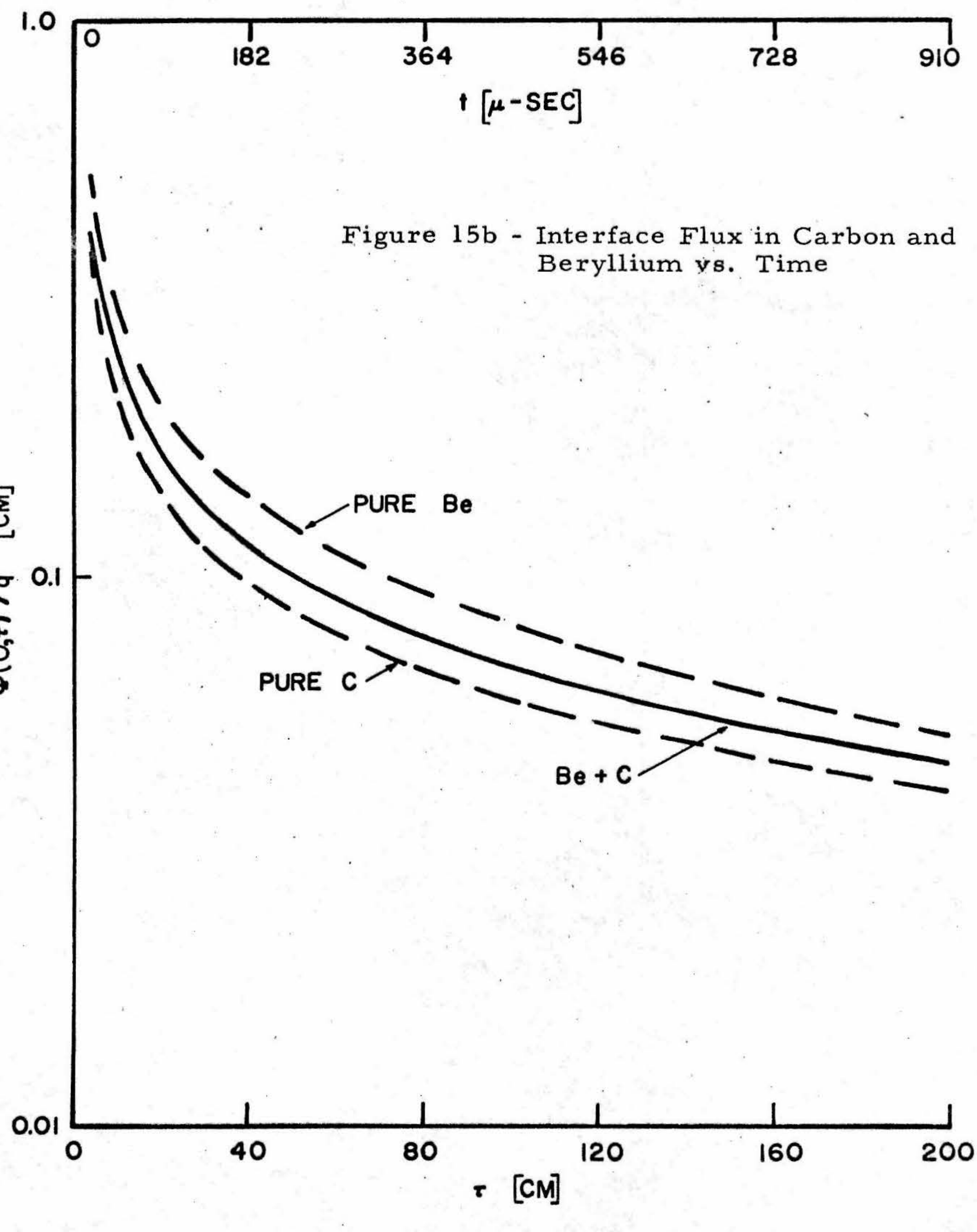

Figure 15b - Interface Flux in Carbon and Beryllium vs. Time

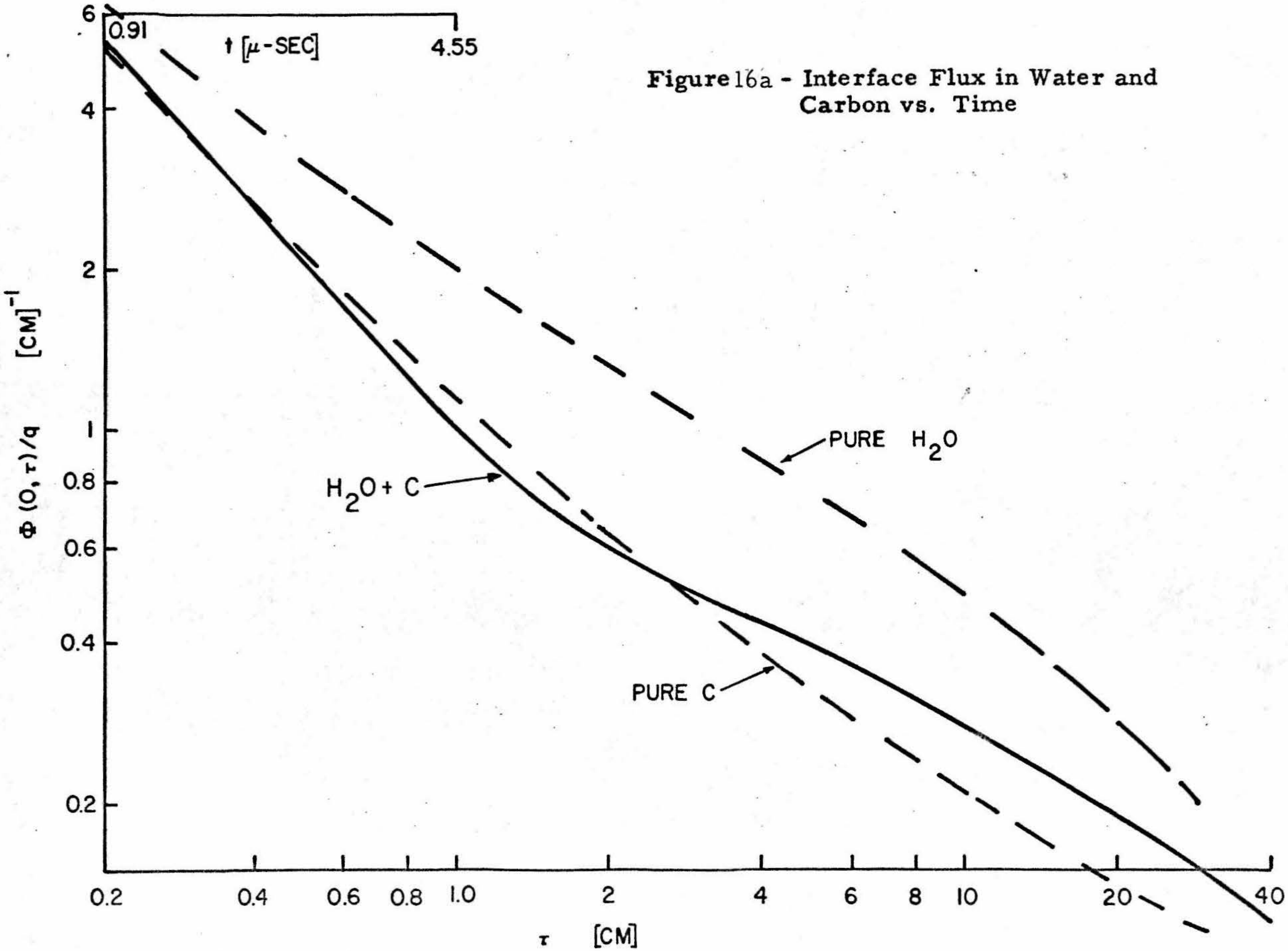

Figure 16a - Interface Flux in Water and Carbon vs. Time

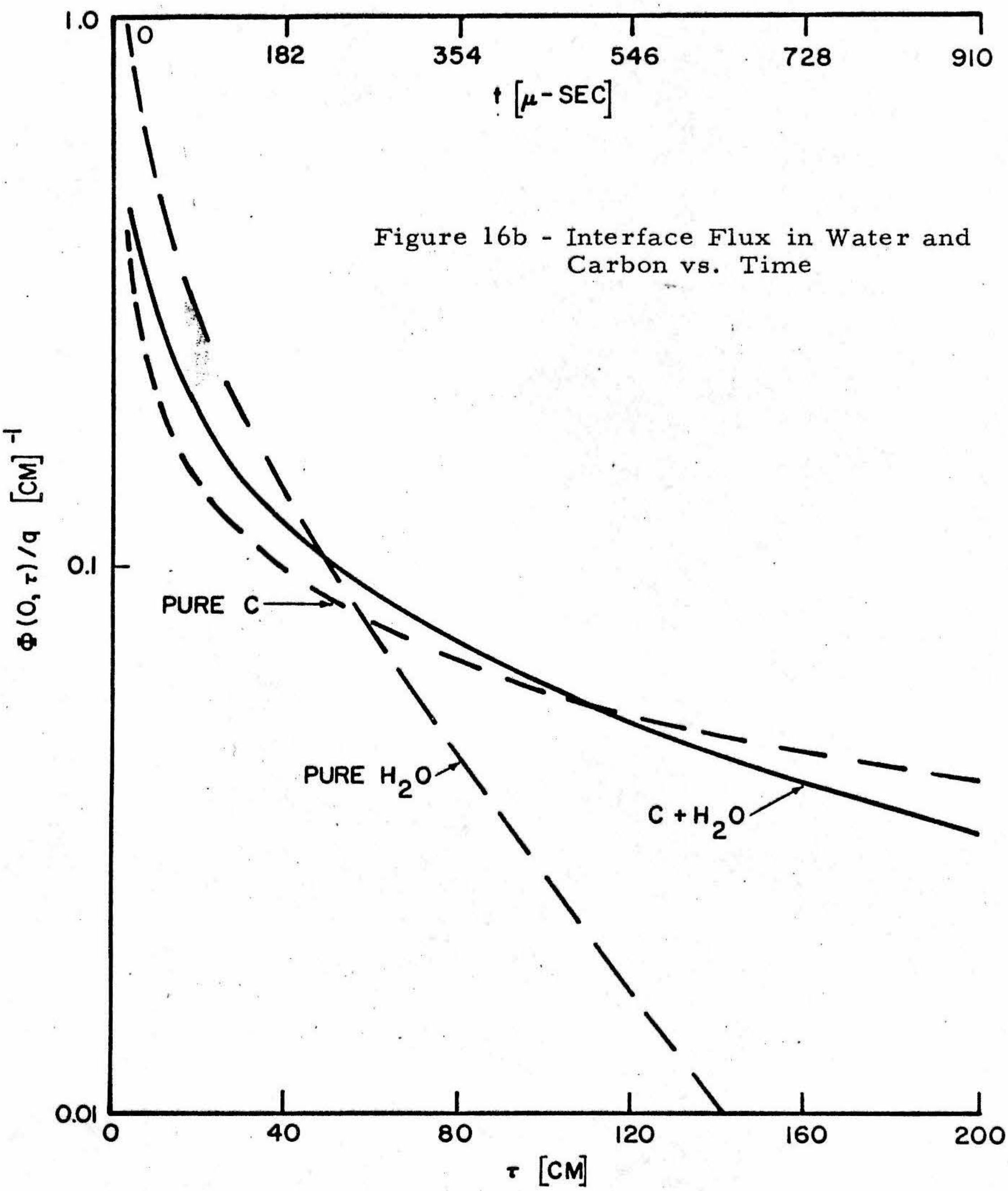

Figure 16b - Interface Flux in Water and Carbon vs. Time

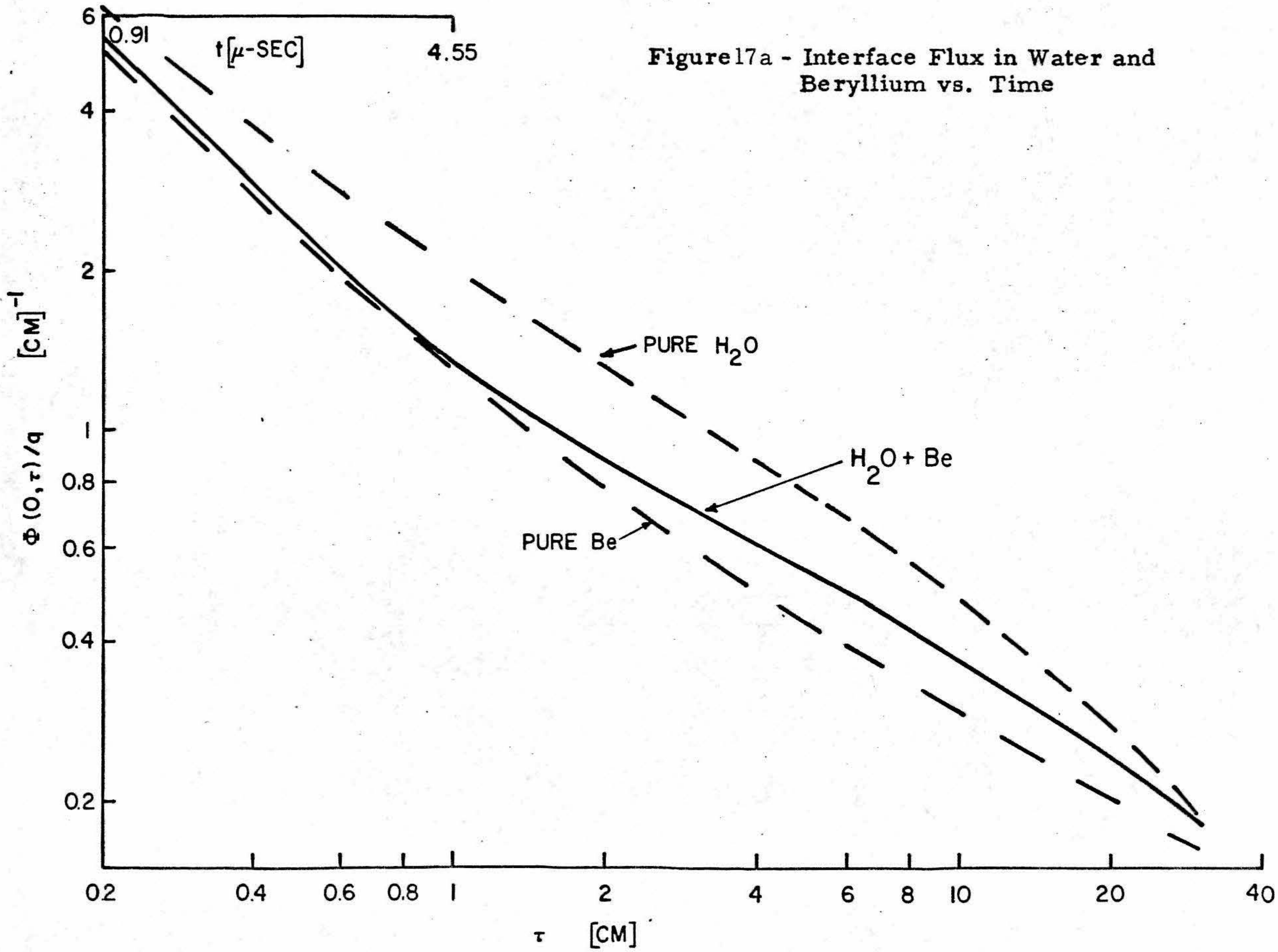

Figure 17a - Interface Flux in Water and Beryllium vs. Time

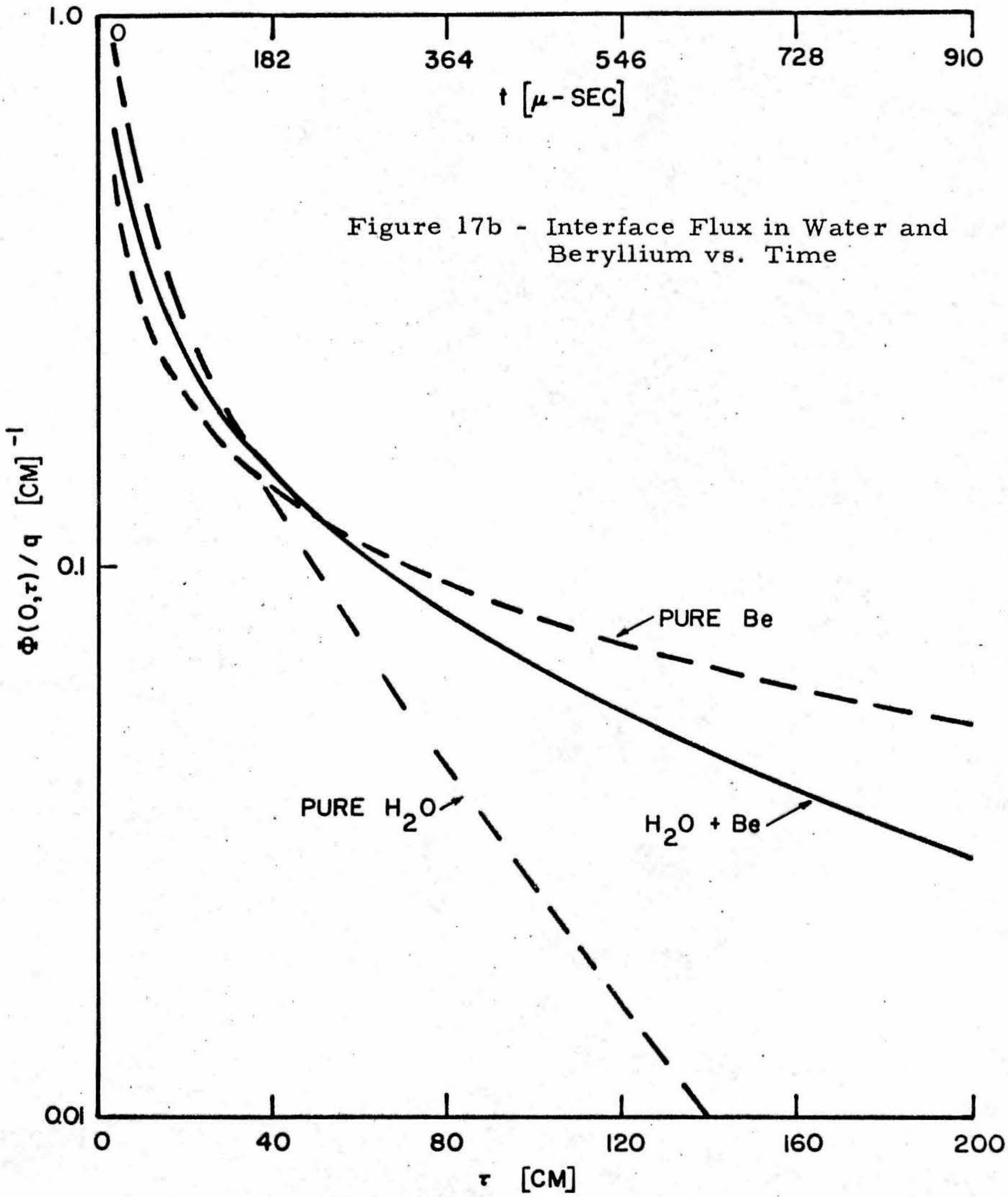

Figure 17b - Interface Flux in Water and Beryllium vs. Time

For long times one can show that the two-region flux decay is asymptotically given by

$$\frac{\varphi}{q} \sim K_e \frac{e^{-a_2 \sigma_2 \tau}}{\tau^{3/2}} ,$$

which is quite similar to the single region asymptotic form given by Equation (158). Hence on semi-log paper the two-region flux decay curve will become parallel to the weaker-absorbing, single-region decay for large τ. This asymptotic trend is clearly shown in Figures 15b, 16b and 17b.

During intermediate times the smoothly decaying flux curves indicate a rather complicated dependence on half-space properties. It was hoped that the net interface current behavior would shed some light on this dependence. Therefore, Equation (160) was numerically evaluated. The results of this calculation are exhibited in Figure 18 for the three problems studied.

Because we are investigating the behavior of the current at the source plane, we expect the current to start and remain at zero when both regions are identical. If the regions have different scattering properties, the interface current may assume a non-zero finite value immediately after the burst of neutrons is initiated.

From Figure 18, we see that for short times ($\tau < 2$) the current is positive, and the neutrons flow from the region with the larger to the region with the smaller total cross section. However, for long times ($\tau > 10$), the curves indicate that the net flow of neutrons

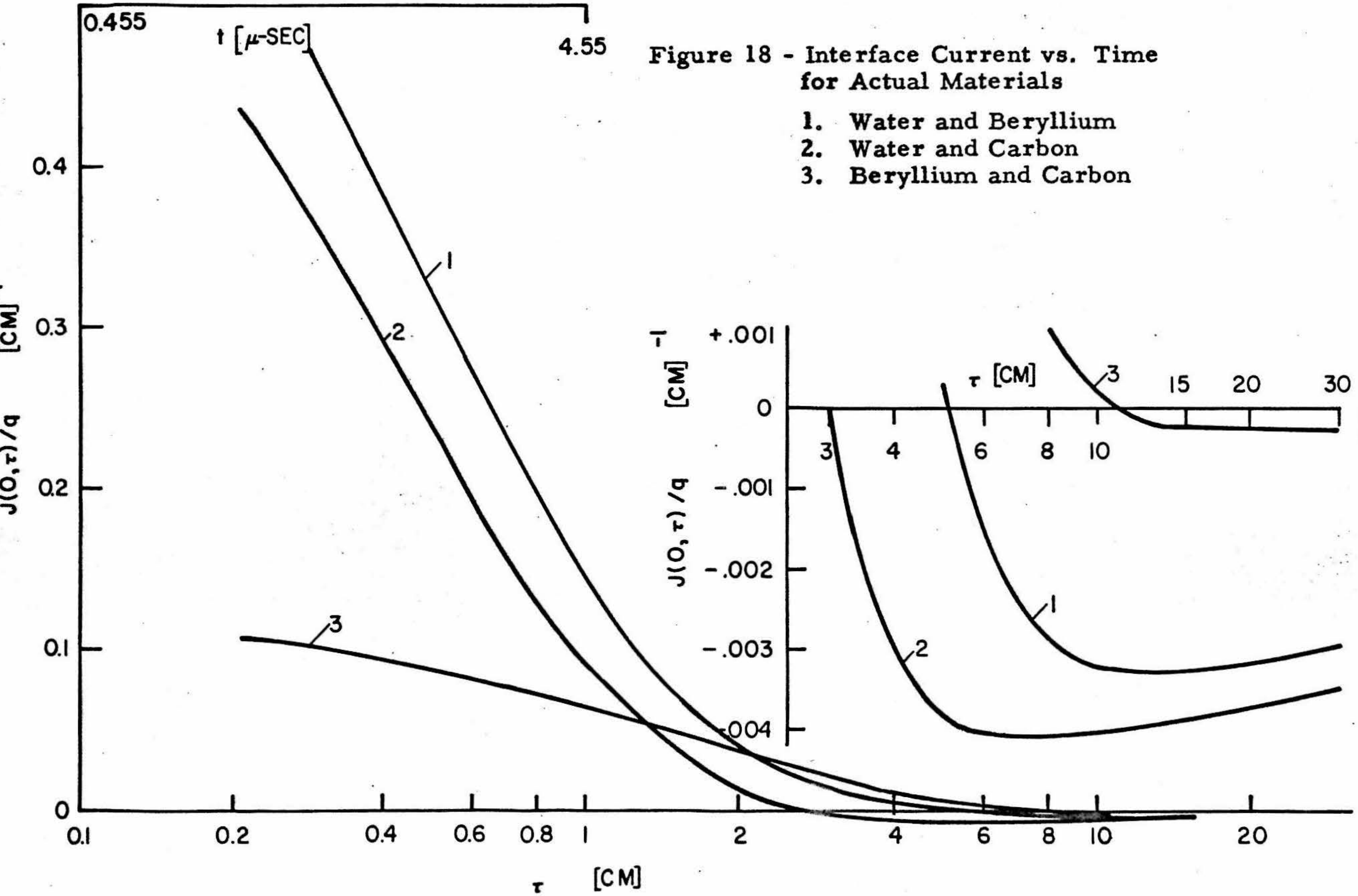

Figure 18 - Interface Current vs. Time for Actual Materials

1. Water and Beryllium
2. Water and Carbon
3. Beryllium and Carbon

has changed direction and that they now flow from the region with the lower to the region with the higher absorption cross section. An inspection of the equation describing the current (Equation (160)) indicates that the current will never again become positive.

Hence when a plane isotropic burst of neutrons is initiated at the interface separating two dissimilar semi-infinite moderators, the interface flux is found to decay smoothly in time, and the interface current is seen to change directions once, at most. To obtain additional data on the interface current behavior, another problem was studied. The results are noted in the next section.

C. Two Adjacent Semi-Infinite Media (Fictitious Cross Sections)

In this problem, the net current at the source plane was calculated for a series of values of (a), (c), and (σ) in each half-space. Starting with two pure scatterers (note that $J_3(0, \tau)$ vanishes), both having scattering cross sections of $\sigma = 0.5$ $[CM]^{-1}$, the scattering cross section in the left half plane was increased in steps of 0.1 $[CM]^{-1}$ to a final value of $\sigma_1 = 1.0$ $[CM]^{-1}$, keeping σ_2 in the right half plane at its initial value. The net interface current was calculated as a function of time at each step. The results are shown in Figure 19.

Next, keeping the right half plane as a pure scatterer of cross section $\sigma_2 = 0.5$ $[CM]^{-1}$ and the total cross section in the left half plane at $\sigma_1 = 1.0$ $[CM]^{-1}$, the absorption cross section in the left half plane was increased in steps of 0.1 $[CM]^{-1}$ from 0 to 0.5 $[CM]^{-1}$.

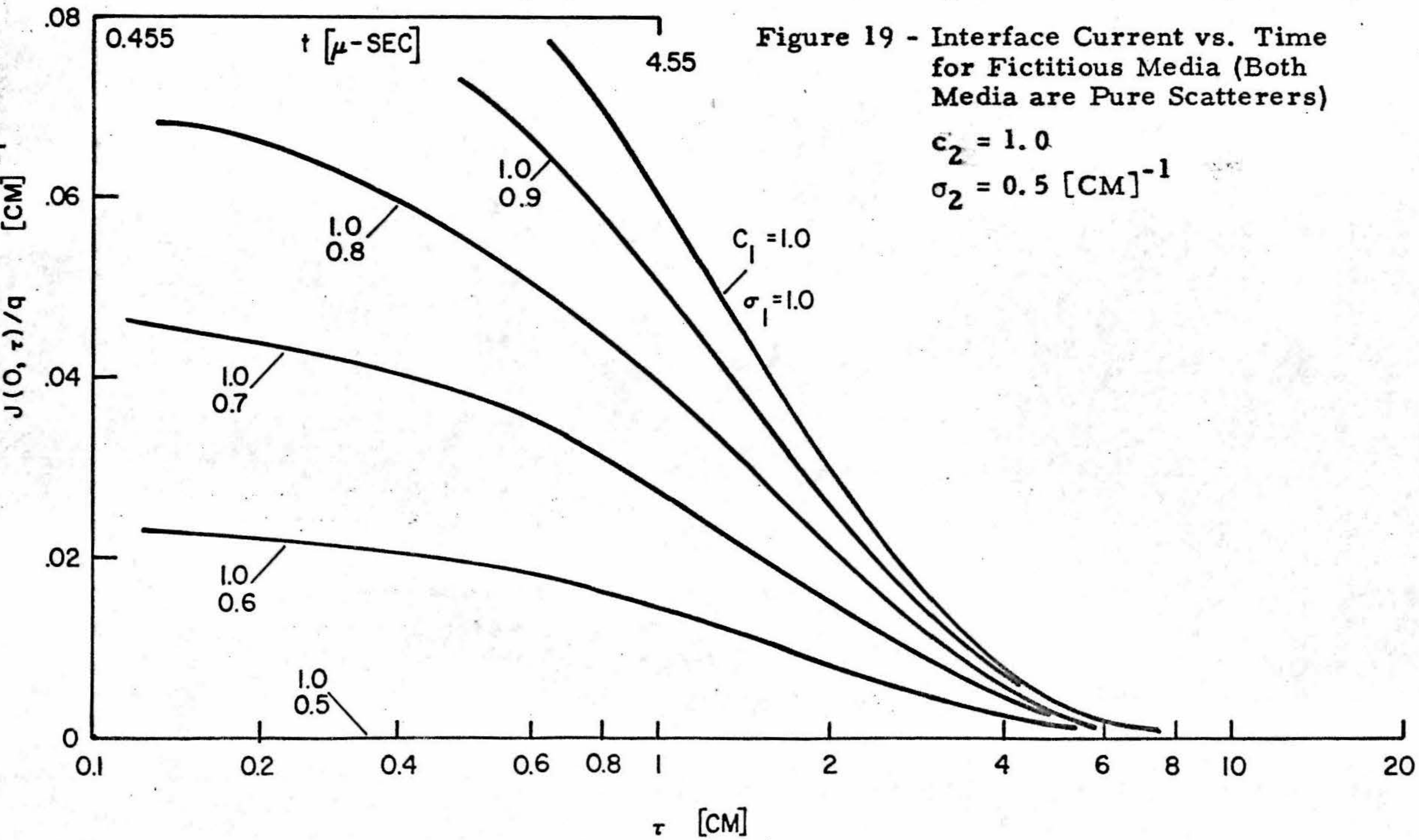

Figure 19 - Interface Current vs. Time for Fictitious Media (Both Media are Pure Scatterers)

$c_2 = 1.0$

$\sigma_2 = 0.5\ [\mathrm{CM}]^{-1}$

The interface current was again evaluated as a function of time at each step. The results are given in Figure 20.

An inspection of Figures 19 and 20, together with an examination of the interface current and flux expressions brings to light the following explanation for changes in the interface current and flux in two adjacent moderator materials.

If there are no differences in the cross sections of the two half-spaces, there will of course be no interface current and the interface flux will be smoothly decaying in time (no reverse curvature). When there is a difference in the scattering but no difference in the absorption cross section, there will be a net flow of neutrons, for all $\tau > 0$, from the half-space with the larger to the half-space with the smaller scattering cross section. Similarly, if the scattering cross section is the same on both sides but the absorption cross section is greater on the left than on the right, there will be a net flow of neutrons to the left for all $\tau > 0$.

Letting both scattering and absorption be different on each side yields two further variations in the description of the current, If $c_1\sigma_1 > c_2\sigma_2$ and $a_1\sigma_1 < a_2\sigma_2$, there will always be a net current from the left to the right. However, if $c_1\sigma_1 > c_2\sigma_2$ and $a_1\sigma_1 > a_2\sigma_2$, the current may change directions after a certain time. Initially, the neutrons will travel from the region of high to that of low scattering probability. However, after a certain time there will be a net flow of neutrons from the region of low to high absorption probability. It appears then that the scattering process is initially

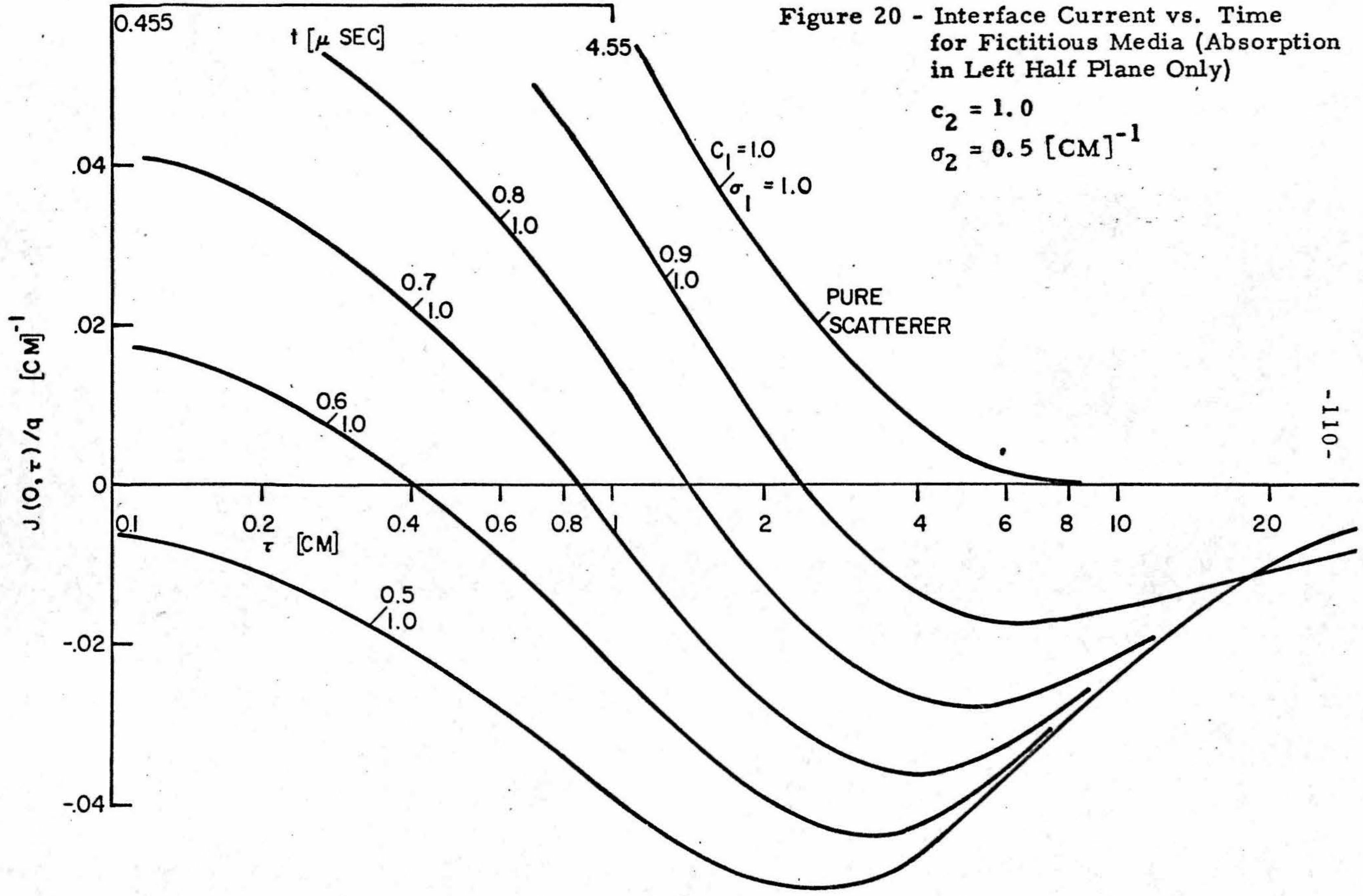

Figure 20 - Interface Current vs. Time for Fictitious Media (Absorption in Left Half Plane Only)

$c_2 = 1.0$

$\sigma_2 = 0.5\ [CM]^{-1}$

dominant and that the absorption mechanism is a long-term one. Corresponding to these currents, small bumps in the flux versus time curves will appear.

The calculations discussed in this section were based on pure moderator half-spaces. However, the only mathematical restriction on the solution is $c \leqq 1$. Recalling the definitions of c and a, we have

$$c = \frac{\sigma_s + \alpha\sigma_f}{\sigma_t} \quad , \quad a = 1 - c \; .$$

Hence fissionable materials could also be considered in the numerical work and were not included simply as a matter of convenience.

D. Summary

It has been shown in this thesis that the NME method can be applied successfully to two-region, time-dependent, monoenergetic neutron transport in one spatial dimension. Although only one problem was specifically solved here, there appear to be no restrictions in extending this method to other time-dependent, two-region problems.

The exact solution to the two adjacent, semi-infinite media problem exhibited here also describes explicitly certain physical effects which one would expect. First the existence of uncollided waves traveling through the two media was shown and the form of these waves was given explicitly, Next, the lack of a fundamental mode due to the infinite extent of the system was observed. Further, some insight into the mechanism of reflection at an interface was gained through the numerical calculations.

The magnitude of flux deviations due to differences in the properties of the two media appeared, however, to be rather small at the interface. In contrast to this, the current was seen to depend markedly on the difference in properties of the two media.

Although flux determinations away from the interface were not numerically made, the expressions, though involved, could be evaluated to determine flux as a function of time at any distance from the source plane. Also, expressions for $< t^n >$ are obtainable from the analysis, although they were not determined by the author.

Finally, the semi-infinite and infinite medium solutions were obtained as special cases of the general two-media result.

E. Possible Extensions

Although an isotropic source was assumed in this analysis, an angularly dependent source could be used with only minor changes, and with no complications occurring in the subsequent analysis. However, adding in anisotropic scattering would lengthen and complicate the analysis considerably, as would energy dependence.

However, without changing the basic equation or the physics of the source, two slightly different extensions could be made to the thesis problem, thereby obtaining additional information on time-dependent neutron fluxes. First, the geometry could be changed to that of a reflected slab. Because of the finite thickness and consequently the possibility of discrete modes being set up, one would then be able to compare the properties of these decaying modes with those found in experiments and with predictions made by less exact theories. However, the coefficients in the normal mode expansion for this problem are not as easily obtainable as in the present instance. and this could reduce the apparent utility of the method in the reflected slab problem.

Alternatively, if one were to place the source off axis, but retain the two semi-infinite media, the two-region problem solved in this thesis would become conceptually a three-region problem. However, this does not seem to be an insurmountable difficulty. Further, the solution to this problem would clearly exhibit the effect of an interface on a traveling neutron wave front and may be of some value in this regard.

As a final extension, suppose now that the source (off axis in the two semi-infinite media problem) is of an oscillatory nature ($e^{i\omega t}$). It seems possible that some type of waves could be set up in the region between the source plane and the interface. One might then be able to relate ω to the properties and geometry of the system. This perhaps would be of some use in determining the nuclear properties of materials.

APPENDIX I - COMPLETENESS PROOF

In this appendix it is proved that certain of the functions $\varphi_{1-}(\mu), \varphi_{2+}(\mu), \varphi_{1\nu}(\mu)$, and $\varphi_{2\nu}(\mu)$ are complete in the space $-1 \leqslant \mu \leqslant +1$ for a given function $\psi'(\mu)$. There also occurs, as a result of this proof, a prescription for obtaining the coefficients in the expansion which is used to represent the function $\psi'(\mu)$.

To carry out the proof presented here, the restrictions needed on $\psi'(\mu)$ are

$$\text{(a)} \quad |\psi'(\mu_2) - \psi'(\mu_1)| \leqq B\, |\mu_2 - \mu_1|^{\gamma} \quad 0 < \gamma \leqq 1 \quad ,$$

where $\quad -1 < \mu_1, \mu_2 < 1 \quad ,$ and

$$\text{(b)} \quad \psi'(\mu) \sim \frac{1}{|\mu-1|^{\gamma_+}} \; , \; \gamma_+ < 1 \quad ; \; \psi'(\mu) \sim \frac{1}{|\mu+1|^{\gamma_-}} \; , \; \gamma_- < 1 \quad ;$$

as $\mu \to +1, \; -1$ respectively.

Condition (a) is termed the Hölder condition in the literature, and conditions (a) and (b) together, the H^* conditions.

More specifically, utilizing the notation shown in Figures 6a and 6b to designate certain regions of the s-plane corresponding to Cases I and II, we shall find that, when multiplied by μ,

$$\varphi_{2+}(\mu) \; , \; \varphi_{1-}(\mu) \; , \; \varphi_{1\nu}(\mu), -1 < \nu < 0 \; , \; \varphi_{2\nu}(\mu), 0 < \nu < 1$$

are complete for $-1 \leqslant \mu \leqslant +1$ in (a),

$$\varphi_{1-}(\mu) \; , \; \varphi_{1\nu}(\mu), -1 < \nu < 0 \; , \; \varphi_{2\nu}(\mu), 0 < \nu < 1$$

are complete for $-1 \leqslant \mu \leqslant +1$ in (b),

$$\varphi_{1\nu}(\mu), -1 < \nu < 0 \ , \ \varphi_{2\nu}(\mu), 0 < \nu < 1$$

are complete for $-1 \leqslant \mu \leqslant +1$ in (c),

$$\varphi_{2+}(\mu) \ , \ \varphi_{1\nu}(\mu), -1 < \nu < 0 \ , \ \varphi_{2\nu}(\mu), 0 < \nu < 1$$

are complete for $-1 \leqslant \mu \leqslant +1$ in (d).

The procedure used here parallels closely that of Case[3, 6] which itself leans heavily on the solution to the Hilbert problem as presented in Muskhelishvili[27]. First, we attempt to show that one can express $\psi'(\mu)$ in the form

$$\psi'(\mu) = \mu \left[\int_{-1}^{0} A_1(\nu)\varphi_{1\nu}(\mu)d\nu + \int_{0}^{1} A_2(\nu)\varphi_{2\nu}(\mu)d\nu \right] \quad , \tag{A1}$$

where the $\psi'(\mu)$ satisfies the H^* conditions and the $A(\nu)$ are as yet unspecified but are required to have properties similar to those of $\psi'(\mu)$.

If the expressions for $\varphi_{1\nu}(\mu)$ and $\varphi_{2\nu}(\mu)$, Equations (48) and (56), are substituted into Equation (A1), and the following notational changes are made,

$$\left.\begin{aligned} \frac{c_1\sigma_1 A_1(\nu)}{2} &= B(\nu) \quad , & \frac{c_2\sigma_2 A_2(\nu)}{2} &= A(\nu) \quad , \\ \frac{2\lambda_1(\mu)}{c_1\sigma_1} &= \hbar_1(\mu) \quad , & \frac{2\lambda_2(\mu)}{c_2\sigma_2} &= \hbar_2(\mu) \quad , \end{aligned}\right\} \tag{A2}$$

then Equation (A1) becomes

$$\psi'(\mu) = \mu\left[\int_{-1}^{0} \frac{\nu B(\nu)d\nu}{\nu-\mu} + \int_{0}^{1} \frac{\nu A(\nu)d\nu}{\nu-\mu} + \begin{cases} B(\mu)\lambda_1(\mu) \ , & \mu < 0 \\ A(\mu)\lambda_2(\mu) \ , & \mu > 0 \end{cases}\right] , \quad \text{(A3)}$$

where the integrations are principal value, whenever necessary.

Next, a new function $N(z)$ is defined;

$$N(z) \equiv \frac{1}{2\pi i}\left[\int_{-1}^{0} \frac{\nu B(\nu)d\nu}{\nu-z} + \int_{0}^{1} \frac{\nu A(\nu)d\nu}{\nu-z}\right] . \quad \text{(A4)}$$

From the conditions on $A_1(\nu)$, $A_2(\nu)$ and thus on $A(\nu)$, $B(\nu)$, we note that

1. $N(z)$ is analytic in the z-plane cut from -1 to $+1$,
2. $\lim_{z\to\infty} N(z) = \frac{K}{z}$,
3. $N(z) \sim \frac{1}{|z-1|^{\gamma_+}}$ $\gamma_+ < 1$ as $z \to 1$, and similarly for $z \to -1$,
4. $N^{\pm}(\mu) = \frac{1}{2\pi i}\left[\int_{-1}^{0} \frac{\nu B(\nu)d\nu}{\nu-\mu} + \int_{0}^{1} \frac{\nu A(\nu)d\nu}{\nu-\mu} \pm i\pi \begin{cases} A(\mu), \mu > 0 \\ B(\mu), \mu < 0 \end{cases}\right]$.

Hence

$$N^+(\mu) - N^-(\mu) = \mu \begin{cases} A(\mu), \ \mu > 0 \\ B(\mu), \ \mu < 0 \end{cases} ,$$

and

$$N^+(\mu) + N^-(\mu) = \frac{1}{\pi i}\left[\int_{-1}^{0} \frac{\nu B(\nu)d\nu}{\nu-\mu} + \int_{0}^{1} \frac{\nu A(\nu)d\nu}{\nu-\mu}\right] . \quad \text{(A5)}$$

Substituting Equations (A5) into Equation (A3) yields, after some rearranging,

$$\frac{\lambda(\mu)+i\pi\mu}{\lambda(\mu)-i\pi\mu} N^+(\mu) - N^-(\mu) = \frac{\psi'(\mu)}{\lambda(\mu)-i\pi\mu} , \quad \text{(A6)}$$

where the subscripts on $\lambda(\mu)$ have been dropped since $\lambda(\mu) = \lambda_1(\mu)$ for $\mu \epsilon [-1,0]$, and $\lambda(\mu) = \lambda_2(\mu)$ for $\mu \epsilon [0,+1]$. So the requirement of a solution to Equation (A1) has been transformed into the requirement of a solution to Equation (A6).

Equation (A6) is designated as an inhomogeneous Hilbert problem on an open contour, and is one of a class of such problems discussed in Muskhelishvili[27]. However, the method of solution to Equation(A6)carried out here, though using some of the notation of Muskhelishvili is based on a variation of this work which was introduced by Case[3]. Following Case, let us partially define two new functions $F(z)$ and $X(z)$ by the relation

$$F(z) = X(z)N(z) - \frac{1}{2\pi i} \int_{-1}^{+1} \frac{\psi'(\mu')X^-(\mu')d\mu'}{(\lambda(\mu')-i\pi\mu')(\mu'-z)} \quad . \tag{A7}$$

We seek the properties which must be assumed by $X(z)$ in order to make $F(z)$ vanish everywhere. Furthermore, we must determine this $X(z)$ explicitly. If this can be accomplished, then completeness will have been proved.

If $X(z) \sim z^{\alpha}$ for large z, then $F(z) \sim z^{\alpha-1}$. So in order to make $F(z)$ vanish as $z \to \infty$, we require $\mathrm{Re}\,\alpha < 1$. Also if $X(z)$ is analytic in the cut plane, then $F(z)$ will be as well. As $z \to 1$, if we assume $X(z) \sim |1-z|^{\delta_+}$, then $F(z)$ will be bounded provided $\delta_+ > \gamma_+$ and similarly as $z \to -1$, $\delta_- > \gamma_-$.

Considering the jump in $F(z)$ as one crosses the cut $-1 \leqslant z \leqslant +1$, one obtains

$$F^+(\mu) - F^-(\mu) = X^+(\mu)N^+(\mu) - X^-(\mu)N^-(\mu) - \frac{\psi'(\mu)X^-(\mu)}{\lambda(\mu)-i\pi\mu} .$$

Setting this discontinuity equal to zero yields

$$\frac{X^+(\mu)}{X^-(\mu)} N^+(\mu) - N^-(\mu) = \frac{\psi'(\mu)}{\lambda(\mu)-i\pi\mu} . \quad \text{(A8)}$$

Comparing Equations (A8) and (A6) we see that

$$\frac{X^+(\mu)}{X^-(\mu)} = \frac{\lambda(\mu)+i\pi\mu}{\lambda(\mu)-i\pi\mu} . \quad \text{(A9)}$$

Hence we have constructed a function $F(z)$ which is analytic in the z-plane cut from -1 to $+1$, vanishes at infinity, and has no discontinuity across the cut. By Liouville's theorem this function must be zero. If we assume further that $X(z)$ is nonvanishing in the cut plane, Equation (A7) becomes

$$N(z) = \frac{1}{2\pi i X(z)} \int_{-1}^{+1} \frac{\psi'(\mu')X^-(\mu')d\mu'}{(\lambda(\mu)-i\pi\mu')(\mu'-z)} . \quad \text{(A10)}$$

Let us investigate Equation (A10) to see if it satisfies the conditions on $N(z)$. It is certainly analytic in the cut plane. Now $z \to 1$, we know that $N(z) \sim \frac{1}{|z-1|^{\gamma_+}}$ where $\gamma_+ < 1$. For the same condition to hold in Equation (A10) it is necessary that $\delta_+ < 1$; similarly as $z \to -1$, we require $\delta_- < 1$ to make $N(z)$ have the proper behavior. Finally, as $z \to \infty$, we find from Equation (A10),

$$N(z) \sim z^{-1-\alpha} \int_{-1}^{+1} \frac{\psi'(\mu')X^-(\mu')d\mu'}{\lambda(\mu)-i\pi\mu'} \left[1 + \frac{\mu'}{z} + \frac{\mu'^2}{z^2} + \ldots\right] . \quad \text{(A11)}$$

But we require $N(z) \sim \frac{1}{z}$ as $z \to \infty$. To reconcile these conditions we require the following, depending upon the choice of α. First, α must be an integer from 0 to $-\infty$. Then if $\alpha = 0$ no further requirement is necessary, and if $\alpha = -n$, we require

$$\int_{-1}^{+1} \frac{\psi'(\mu')X^{-}(\mu')\mu'^{\ell}d\mu'}{\lambda(\mu')-i\pi\mu'} = 0 \quad , \quad \ell = 0, 1, \ldots, n-1 \quad . \qquad \text{(A12)}$$

This last equation seems to put an additional restriction on the $\psi'(\mu)$ allowed. However, it will be seen later that this is indeed not the case, and that it is just this apparent restriction (Equation (A12)) which will permit us to determine all the coefficients.

The next step in this proof is to obtain a $X(z)$ function satisfying the conditions summarized below.

1. $X(z)$ is analytic and nonvanishing in the z-plane cut from -1 to +1.

2. $X(z) \sim z^{\alpha}$ as $z \to \infty$, where $\alpha = 0, -1, -2, \ldots$.

3. As $z \to +1$, $X(z) \sim |1-z|^{\delta_+}$ where $\delta_+ < 1$, and similarly for $z \to -1$.

4. $$\frac{X^{+}(\mu)}{X^{-}(\mu)} = \frac{\lambda(\mu)+i\pi\mu}{\lambda(\mu)-i\pi\mu} \quad , \quad -1 \leqslant \mu \leqslant +1 \quad .$$

This is a homogeneous Hilbert problem and has been handled many times in the literature[27]. Defining

$$\left.\begin{aligned} \Gamma(z) &= \frac{1}{2\pi i} \int_{-1}^{+1} \frac{\ell n G(\mu')d\mu'}{\mu'-z} \quad , \\ \text{where} & \\ G(\mu') &= \frac{\lambda(\mu')+i\pi\mu'}{\lambda(\mu')-i\pi\mu'} \quad , \end{aligned}\right\} \qquad \text{(A13)}$$

it is easily seen that

$$X_o(z) = e^{\Gamma(z)} \tag{A14}$$

satisfies conditions (1) and (4).

Let us next look at the behavior of $X_o(z)$ as $z \to \pm 1$. We shall find that this behavior depends upon the value of s chosen, and that it will differ depending only on whether s is in (a), (b), (c), or (d) of Figures 6a and 6b.

As $z \to 1$, $\Gamma(z) \sim \frac{\ln G(1)}{2\pi i} \ln(1-z)$, and as $z \to -1$, $\Gamma(z) \sim \frac{\ln G(-1)}{2\pi i} \ln(-1-z)$. Thus $X_o(z) \sim (1-z)^{\ln G(1)/2\pi i}$ as $z \to +1$, and $X_o(z) \sim (-1-z)^{-\ln G(-1)/2\pi i}$ as $z \to -1$. Writing $\ln G(\mu') = \ln|R| + i\theta(\mu')$, we note that only the imaginary part is important since $\ln G(\pm 1)$ is divided by i.

To obtain $G(\pm 1)$ we start at $\mu' = 0$, where we assume zero phase ($\ln G(o) = 0$), and travel along the path of integration to $\mu' = \pm 1$, treating the argument of $\ln G(\mu')$ as a continuous variable.

The values of $\theta(\pm 1)$ so obtained for each region of the s-plane are listed below, together with the $X_o(z)$ behavior.

Region	$\theta(+1)$	$X_o(z)$ $(z \to +1)$	$\theta(-1)$	$X_o(z)$ $(z \to -1)$
(a)	2π	$(1-z)^{+1}$	-2π	$(-1-z)^{+1}$
(b)	0	$(1-z)^{o}$	-2π	$(-1-z)^{1}$
(c)	0	$(1-z)^{o}$	0	$(-1-z)^{o}$
(d)	2π	$(1-z)^{1}$	0	$(-1-z)^{o}$.

From this behavior for $X_o(z)$ we construct a $X(z)$ which satisfies the four conditions required, by suitably multiplying $X_o(z)$ by $(1-z)^{\gamma_1}(-1-z)^{\gamma_2}$, where γ_1 and γ_2 are 0 or -1. More specifically, we find as suitable X(z)'s,

$$\begin{aligned}
&\text{in (a)}, \quad X_a(z) = [(-1-z)(1-z)]^{-1} X_o(z) \;, \quad \alpha = -2 \;, \\
&\text{in (b)}, \quad X_b(z) = (-1-z)^{-1} X_o(z) \;, \quad \alpha = -1 \;, \\
&\text{in (c)}, \quad X_c(z) = X_o(z) \;, \quad \alpha = 0 \;, \\
&\text{in (d)}, \quad X_d(z) = (1-z)^{-1} X_o(z) \;, \quad \alpha = -1 \;.
\end{aligned} \tag{A15}$$

The value of α corresponding to each region in the s-plane is also listed, and this implies that Equation (A12) must be satisfied in regions (a), (b), and (d).

Let us apply these results to the solution of the equations obtained from the jump condition at the interface, Equations (68) through (71), which are rewritten below.

$$\text{In (a)}, \quad q-\mu[a_{2+}\varphi_{2+}(\mu)+a_{1-}\varphi_{1-}(\mu)] = \mu\left[\int_{-1}^{o} A_1(\nu)\varphi_{1\nu}(\mu)d\nu + \int_{o}^{1} A_2(\nu)\varphi_{2\nu}(\mu)d\nu\right]. \tag{A16}$$

$$\text{In (b)}, \quad q-\mu[a_{1-}\varphi_{1-}(\mu)] = \mu\left[\int_{1}^{o} A_1(\nu)\varphi_{1\nu}(\mu)d\nu + \int_{o}^{1} A_2(\nu)\varphi_{2\nu}(\mu)d\nu\right]. \tag{A17}$$

$$\text{In (c)}, \quad q = \mu\left[\int_{-1}^{o} A_1(\nu)\varphi_{1\nu}(\mu)d\nu + \int_{o}^{1} A_2(\nu)\varphi_{2\nu}(\mu)d\nu\right] . \tag{A18}$$

$$\text{In (d)}, \quad q-\mu[a_{2+}\varphi_{2+}(\mu)] = \mu\left[\int_{-1}^{o} A_1(\nu)\varphi_{1\nu}(\mu)d\nu + \int_{o}^{1} A_2(\nu)\varphi_{2\nu}(\mu)d\nu\right]. \tag{A19}$$

Defining the left hand side of each of these equations by $\psi'(\mu)$, they are seen to satisfy the H^* conditions and they also contain, in cases (a), (b) and (d), unspecified quantities. Investigating Equation (A16) first (region (a) of the s-plane), we can apply the completeness theorem provided, since $\alpha = -2$,

$$\int_{-1}^{+1} \frac{\psi'(\mu')X^{-}(\mu')d\mu'\mu'^{\ell}}{\lambda(\mu') - i\pi\mu'} = 0 \quad , \quad \ell = 0, 1 \quad .$$

This set of two inhomogeneous equations completely define a_{1-} and a_{2+}. Thus Equation (A12) does not restrict the function $\psi'(\mu)$ but provides for the determination of the discrete coefficients. Once the discrete coefficients are obtained, both $A_1(\nu)$ and $A_2(\nu)$ can be obtained through the machinery used in completeness proof.

A similar sequence can be used to obtain the coefficients in the other regions of the s-plane. Thus Equation (A12) provides just enough of a requirement to determine the discrete coefficients in each region.

Hence completeness has been proved in each region of the s-plane for both Cases I and II, and a prescription for obtaining the coefficients has been determined. The actual evaluation of these coefficients is the subject of the next appendix.

APPENDIX II - OBTAINING THE COEFFICIENTS IN THE NORMAL MODE EXPANSION

The explicit form of the coefficients a_{1-}, a_{2+}, $A_1(\nu)$ and $A_2(\nu)$ in each region of the s-plane are evaluated in this appendix. They result from the completeness proof with the help of several identities in the X(z). As a check, several of these coefficients are extracted from the jump conditions, Equations (68) through (71), by the applications of certain orthogonality relations.

A. Identities in X(z)

Before applying the machinery of the completeness proof to obtain the coefficients, let us obtain several identities which are useful in the simplification of the resulting expressions for the coefficients. These expressions are all based on Cauchy's theorem, and similar expressions can be found in the literature.

Consider region (a). Here $X_a(z)$ is analytic in the plane cut from $-1 \leqslant z \leqslant +1$, and behaves asymptotically as z^{-2}. Hence

$$X_a(z) = \frac{1}{2\pi i} \int_c \frac{X_a(z')dz'}{z'-z} , \qquad (A19)$$

where c is the contour shown in Figure 21. Noting that the contribution of the large circle to the integral tends to zero as the radius of this circle tends to ∞, and that the contributions of the small circles about ±1 also vanish as their radii tend to zero, we are left with

$$X_a(z) = \frac{1}{2\pi i} \int_{-1}^{+1} \frac{d\mu'}{\mu'-z} [X_a^+(\mu') - X_a^-(\mu')] .$$

This can be further simplified to

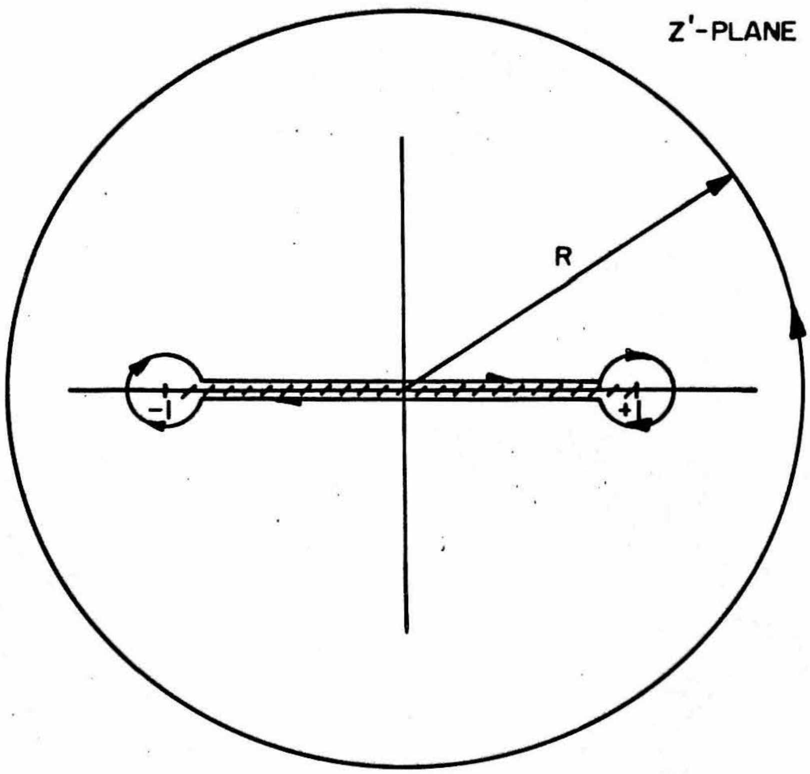

Figure 21 - Contour for Integral Equation in X(z)

$$X_a(z) = \int_{-1}^{+1} \frac{\mu' X_a^-(\mu')d\mu'}{(\lambda(\mu')-i\pi\mu')(\mu'-z)} \quad . \tag{A20}$$

From a derivation identical in every detail to this, one can also obtain

$$zX_a(z) = \int_{-1}^{+1} \frac{\mu'^2 X_a^-(\mu')d\mu'}{(\lambda(\mu')-i\pi\mu')(\mu'-z)} \quad . \tag{A21}$$

Based on a partial fraction expansion, making use of Equation (A20),it can be shown that

$$\frac{X_a(z_2)-X_a(z_1)}{z_2-z_1} = \int_{-1}^{+1} \frac{\mu' X_a^-(\mu')d\mu'}{(\lambda(\mu')-i\pi\mu')(\mu'-z_1)(\mu'-z_2)} \quad . \tag{A22}$$

Let us next consider $X_b(z)$ in region (b). Formula (A19) and Figure 21 are still valid, and the figure will so remain throughout this section. In this region $X_b(z) \sim z^{-1}$. Hence Formula (A20) remains valid here;

$$X_b(z) = \int_{-1}^{+1} \frac{\mu' X_b^-(\mu')d\mu'}{(\lambda(\mu)-i\pi\mu')(\mu'-z)} \quad . \tag{A23}$$

An expression similar to Formula (A21) is presently not needed, but Formula (A22) is identical in this region; that is,

$$\frac{X_b(z_2)-X_b(z_1)}{z_2-z_1} = \int_{-1}^{+1} \frac{\mu' X_b^-(\mu')d\mu'}{(\lambda(\mu')-i\pi\mu')(\mu'-z_1)(\mu'-z_2)} \quad . \tag{A24}$$

In region (c), the evaluation of Formula (A19) over the contour shown in Figure 21 yields, in addition to the integral from -1 to +1, a contribution over the large circle. Hence

$$X_c(z) = 1 + \int_{-1}^{+1} \frac{\mu' X_c^-(\mu')d\mu'}{(\mu'-z)(\lambda(\mu')-i\pi\mu')} \quad . \tag{A25}$$

The form of Equation (A24) is retained in this region however;

$$\frac{X_c(z_2)-X_c(z_1)}{z_2-z_1} = \int_{-1}^{+1} \frac{\mu' X_c^-(\mu')d\mu'}{(\lambda(\mu')-i\pi\mu')(\mu'-z_1)(\mu'-z_2)} \quad . \tag{A26}$$

In region (d), the Formulae for $X_d(z)$ are identical to those in region (b) and are not written out here.

B. Evaluation of the Coefficients

From the completeness proof, the following list of formulae will yield the coefficients, after some algebra, in terms of known functions.

$$N^+(\mu)-N^-(\mu) = \mu \begin{cases} \dfrac{c_2\sigma_2 A_2(\mu)}{2} & \mu > 0 \\ \dfrac{c_1\sigma_1 A_1(\mu)}{2} & \mu < 0 \end{cases} \quad . \tag{A27}$$

$$N(z) = \frac{1}{2\pi i X(z)} \int_{-1}^{+1} \frac{\psi'(\mu')X^-(\mu')d\mu'}{(\lambda(\mu')-i\pi\mu')(\mu'-z)} \quad . \tag{A28}$$

$$\int_{-1}^{+1} \frac{\psi'(\mu')X^-(\mu')\mu'^{\ell}d\mu'}{\lambda(\mu')-i\pi\mu'} = 0 \quad , \quad \ell = 0,1, \ldots, -\alpha-1 \quad , \tag{A29}$$

where $\alpha = -1,-2, \ldots$.

Region in s-plane	α	$\psi'(\mu)$
(a)	-2	$q-\mu[a_{2+}\varphi_{2+}(\mu) + a_{1-}\varphi_{1-}(\mu)]$
(b)	-1	$q-\mu\,[a_{1-}\varphi_{1-}(\mu)]$
(c)	0	q
(d)	-1	$q - \mu\,[a_{2+}\varphi_{2+}(\mu)]$

The coefficients in each region of the s-plane are evaluated separately.

1. Region (a): Let us first obtain a_{1-} and a_{2+} through the use of Equation (A29). Substituting $\psi'(\mu)$ into Equation (A29) yields the following two equations for $\ell = 0$ and $\ell = 1$;

$$\left.\begin{aligned} a_{1-}\int_{-1}^{+1}\frac{\mu' X_a^-(\mu')\varphi_{1-}(\mu')d\mu'}{\lambda(\mu')-i\pi\mu'} + a_{2+}\int_{-1}^{+1}\frac{\mu' X_a^-(\mu')\varphi_{2+}(\mu')d\mu'}{\lambda(\mu')-i\pi\mu'} \\ = q\int_{-1}^{+1}\frac{X_a^-(\mu')d\mu'}{\lambda(\mu')-i\pi\mu'} \\ a_{1-}\int_{-1}^{+1}\frac{\mu'^2 X_a^-(\mu')\varphi_{1-}(\mu')d\mu'}{\lambda(\mu')-i\pi\mu'} + a_{2+}\int_{-1}^{+1}\frac{\mu'^2 X_a^-(\mu')\varphi_{2+}(\mu')d\mu'}{\lambda(\mu')-i\pi\mu'} \\ = q\int_{-1}^{+1}\frac{\mu' X_a^-(\mu')d\mu'}{\lambda(\mu')-i\pi\mu'} \end{aligned}\right] .$$

These two simultaneous equations can be simplified first by exhibiting the functional forms of $\varphi_{1-}(\mu)$ and $\varphi_{2+}(\mu)$, and then by applying the identities given in Section A for $X_a(z)$. The result is

$$\left.\begin{aligned} a_{1-}\frac{c_1\sigma_1}{2}\nu_{01}X_a(-\nu_{01}) - a_{2+}\frac{c_2\sigma_2}{2}\nu_{02}X_a(\nu_{02}) = qX_a(o) \\ a_{1-}\frac{c_1\sigma_1}{2}\nu_{01}^2X_a(-\nu_{01}) + a_{2+}\frac{c_2\sigma_2}{2}\nu_{02}^2X_a(\nu_{02}) = 0 \end{aligned}\right\} .$$

Solving yields

$$\left.\begin{aligned} a_{1-} &= \frac{2q}{c_1\sigma_1}\frac{X_a(o)}{(\nu_{02}+\nu_{01})}\frac{\nu_{02}}{\nu_{01}X_a(-\nu_{01})} \\ a_{2+} &= -\frac{2q}{c_2\sigma_2}\frac{X_a(o)}{(\nu_{02}+\nu_{01})}\frac{\nu_{01}}{\nu_{02}X_a(+\nu_{02})} \end{aligned}\right\} . \qquad \text{(A30)}$$

Next we obtain $A_1(\mu)$ and $A_2(\mu)$. Substituting $\psi'(\mu)$ into Equation (A28) gives the following expression for $N(z)$;

$$N(z) = \frac{1}{2\pi i X_a(z)}\left[q\int_{-1}^{+1}\frac{X_a^-(\mu')d\mu'}{(\lambda(\mu')-i\pi\mu')(\mu'-z)} - a_{2+}\int_{-1}^{+1}\frac{\mu'\varphi_{2+}(\mu')X_a^-(\mu')d\mu'}{(\lambda(\mu')-i\pi\mu')(\mu'-z)}\right.$$

$$\left. -a_{1-}\int_{-1}^{+1}\frac{\mu'\varphi_{1-}(\mu')X_a^-(\mu')d\mu'}{(\lambda(\mu')-i\pi\mu')(\mu'-z)}\right] .$$

Using Identity (A22) of Section A and substituting for a_{2+} and a_{1-} from Equation (A30) reduces this to

$$N(z) = \frac{q}{2\pi i}\left[\frac{1}{z} - \frac{X_a(o)}{zX_a(z)} - \frac{X_a(o)\nu_{01}}{(\nu_{01}+\nu_{02})X_a(\nu_{02})(z-\nu_{02})}\right.$$

$$+\frac{X_a(o)\nu_{01}}{X_a(z)(\nu_{02}+\nu_{01})(z-\nu_{02})} - \frac{X_a(o)\nu_{02}}{(\nu_{02}+\nu_{01})X_a(-\nu_{01})(z+\nu_{01})}$$

$$\left.+\frac{X_a(o)\nu_{02}}{(\nu_{02}+\nu_{01})X_a(z)(z+\nu_{01})}\right] .$$

Finally, after some additional simplifications, we obtain

$$N^+(\mu)-N^-(\mu) = \frac{-qX_a(o)\nu_{01}\nu_{02}}{(\mu-\nu_{02})(\mu+\nu_{01})X_a^-(\mu)(\lambda(\mu)+i\pi\mu)} .$$

Thus the continuous coefficients can be expressed as

$$\left.\begin{aligned} A_1(\mu) &= \frac{-2qX_a(o)\nu_{01}\nu_{02}}{c_1\sigma_1\mu X_a^-(\mu)(\lambda(\mu)+i\pi\mu)(\mu-\nu_{02})(\mu+\nu_{01})} & \mu < 0 \\ A_2(\mu) &= \frac{-2qX_a(o)\nu_{01}\nu_{02}}{c_2\sigma_2\mu X_a^-(\mu)(\lambda(\mu)+i\pi\mu)(\mu-\nu_{02})(\mu+\nu_{01})} & \mu > 0 \end{aligned}\right\} . \quad (A31)$$

2. <u>Region(b):</u> Substituting the appropriate $\psi'(\mu)$ into Equation (A29) yields the following expression for the coefficient a_{1-},

$$a_{1-}\int_{-1}^{+1}\frac{\mu'\varphi_{1-}(\mu')X_b^-(\mu')d\mu'}{\lambda(\mu')-i\pi\mu'} = q\int_{-1}^{+1}\frac{X_b^-(\mu')d\mu'}{\lambda(\mu')-i\pi\mu'} .$$

Applying Identity (A23) to this equation gives

$$a_{1-} = \frac{2qX_b(o)}{c_1\sigma_1\nu_{01}X_b(-\nu_{01})} . \qquad \text{(A32)}$$

The expression for $N(z)$ in region (b) becomes, from Equation (A28),

$$N(z) = \frac{1}{2\pi i X_b(z)}\left[q\int_{-1}^{+1}\frac{X_b^-(\mu')d\mu'}{(\lambda(\mu')-i\pi\mu')(\mu'-z)} - a_{1-}\int_{-1}^{+1}\frac{\mu'\varphi_{1-}(\mu')X_b^-(\mu')d\mu'}{(\lambda(\mu')-i\pi\mu')(\mu'-z)}\right].$$

Applying Identity (A24) to this expression and replacing a_{1-} by Equation (A32) yields

$$N(z) = \frac{q}{2\pi i}\left[\frac{1}{z} - \frac{X_b(o)}{zX_b(z)} + \frac{X_b(o)}{(z+\nu_{01})X_b(z)} - \frac{X_b(o)}{X_b(-\nu_{01})(z+\nu_{01})}\right] .$$

With this expression we can obtain

$$N^+(\mu)-N^-(\mu) = \frac{qX_b(o)\nu_{01}}{(\mu+\nu_{01})X_b^-(\mu)(\lambda(\mu)+i\pi\mu)} .$$

Hence the continuous coefficients in region (b) are

$$\left.\begin{aligned} A_1(\mu) &= \frac{2qX_b(o)\nu_{01}}{c_1\sigma_1\mu(\mu+\nu_{01})X_b^-(\mu)(\lambda(\mu)+i\pi\mu)} & \mu < 0 \\ A_2(\mu) &= \frac{2qX_b(o)\nu_{01}}{c_2\sigma_2\mu(\mu+\nu_{01})X_b^-(\mu)(\lambda(\mu)+i\pi\mu)} & \mu > 0 \end{aligned}\right\} . \qquad \text{(A33)}$$

3. <u>Region (c):</u> In this region no discrete coefficient is defined, nor is there a requirement to obtain one since Equation (A29) does not

apply here ($\alpha = 0$). So we need only determine the continuous coefficients $A_1(\mu)$ and $A_2(\mu)$.

The expression for $N(z)$ in this case is

$$N(z) = \frac{q}{2\pi i X_c(z)} \int_{-1}^{+1} \frac{X_c^-(\mu')d\mu'}{(\lambda(\mu')-i\pi\mu')(\mu'-z)} .$$

Evaluating the integral through the use of Identity (A26) gives

$$N(z) = \frac{q}{2\pi i}\left[\frac{1}{z} - \frac{X_c(o)}{zX_c(z)}\right].$$

Hence

$$N^+(\mu)-N^-(\mu) = \frac{qX_c(o)}{X_c^-(\mu)(\lambda(\mu)+i\pi\mu)} ,$$

and the continuous coefficients in region (c) are

$$\left.\begin{aligned} A_1(\mu) &= \frac{2qX_c(o)}{c_1\sigma_1\mu X_c^-(\mu)(\lambda(\mu)+i\pi\mu)} & \mu < 0 \\ A_2(\mu) &= \frac{2qX_c(o)}{c_2\sigma_2\mu X_c^-(\mu)(\lambda(\mu)+i\pi\mu)} & \mu > 0 \end{aligned}\right\} . \qquad \text{(A34)}$$

4. Region (d): Here the expression for the discrete coefficient is

$$a_{2+} \int_{-1}^{+1} \frac{\mu'\varphi_{2+}(\mu')X_d^-(\mu')d\mu'}{\lambda(\mu')-i\pi\mu'} = q \int_{-1}^{+1} \frac{X_d^-(\mu')d\mu'}{\lambda(\mu')-i\pi\mu'} .$$

This yields, when the integrals are evaluated,

$$a_{2+} = \frac{-2qX_d(o)}{c_2\sigma_2\nu_{02}X_d(\nu_{02})} . \qquad \text{(A35)}$$

For $N(z)$, the following expression must be evaluated;

$$N(z) = \frac{1}{2\pi i X_d(z)}\left[q\int_{-1}^{+1} \frac{X_d^-(\mu')d\mu'}{(\lambda(\mu')-i\pi\mu')(\mu'-z)} - a_{2+}\int_{-1}^{+1} \frac{\mu'\varphi_{2+}(\mu')X_d^-(\mu')d\mu'}{(\lambda(\mu')-i\pi\mu')(\mu'-z)}\right] .$$

Once the integtals are evaluated, and the results simplified, one obtains

$$N(z) = \frac{q}{2\pi i}\left[\frac{1}{z} - \frac{X_d(o)}{zX_d(z)} + \frac{X_d(o)}{(z-\nu_{02})X_d(z)} - \frac{X_d(o)}{(z-\nu_{02})X_d(\nu_{02})}\right] .$$

Thus

$$N^+(\mu)-N^-(\mu) = \frac{-qX_d(o)\nu_{02}}{(\mu-\nu_{02})X_d^-(\mu)(\lambda(\mu)+i\pi\mu)} ,$$

and the continuous coefficients are found to be

$$\left.\begin{aligned} A_1(\mu) &= \frac{-2qX_d(o)\nu_{02}}{c_1\sigma_1\mu(\mu-\nu_{02})X_d^-(\mu)(\lambda(\mu)+i\pi\mu)} \qquad \mu < 0 \\ A_2(\mu) &= \frac{-2qX_d(o)\nu_{02}}{c_2\sigma_2\mu(\mu-\nu_{02})X_d^-(\mu)(\lambda(\mu)+i\pi\mu)} \qquad \mu > 0 \end{aligned}\right\} . \qquad (A36)$$

C. Obtaining Several Coefficients by Orthogonality Relations

In Section B of Chapter II a method(7) to obtain the expansion coefficients by orthogonality relations was discussed for the steady-state two-media problem. This method is used in the thesis problem to obtain some of the coefficients in region (a) of the s-plane, hence providing a check on the results obtained from the completeness proof. The formulae used here are quite similar to those employed in Reference (7), differing only by constant factors.

From the jump condition in region (a), Equation (68), we seek the coefficients in the equation

$$q=\mu\left[a_{2+}\varphi_{2+}(\mu)+a_{1-}\varphi_{1-}(\mu)+\int_{-1}^{o} A_1(\nu)\varphi_{1\nu}(\mu)d\nu+\int_{o}^{1} A_2(\nu)\varphi_{2\nu}(\mu)d\nu\right]. \quad (A37)$$

Utilizing Identities (A20) and (A21) as well as

$$z^2X_a(z)-1 = \int_{-1}^{+1} \frac{\mu'^3 X_a^-(\mu')d\mu'}{(\lambda(\mu')-i\pi\mu')(\mu'-z)} \quad , \quad (A38)$$

and following closely the work in Reference (7), we define a weight function

$$W(\mu) = \frac{(\nu_{02}-\mu)(\nu_{01}+\mu)X_a^-(\mu)}{\lambda(\mu)+i\pi\mu} \quad , \quad (A39)$$

such that when Equation (A37) is multiplied by $W(\mu)\varphi_n(\mu)$ and integrated over μ, only one term has a non-zero value on the right hand side of Equation (A37).

Based on this brief sketch of the method let us evaluate a_{2+}, $A_1(\nu)$, and $A_2(\nu)$. First a_{2+} will be obtained.

We multiply Equation (A37) by $W(\mu)\varphi_{2+}(\mu)$ and integrate over μ from -1 to +1. It is relatively easy to show that (see Equations (22) for comparison)

$$\left.\begin{aligned}
&\int_{-1}^{+1} W(\mu)\varphi_{2+}(\mu)d\mu = \frac{c_2\sigma_2}{2}\ \nu_{01}\nu_{02}X_a(o) \quad , \\
&\int_{-1}^{+1} \mu W(\mu)\varphi_{2+}^2(\mu)d\mu = -\left(\frac{c_2\sigma_2\nu_{02}}{2}\right)^2 X_a(\nu_{02})(\nu_{01}+\nu_{02}) \quad , \\
&\int_{-1}^{+1} \mu W(\mu)\varphi_{1-}(\mu)\varphi_{2+}(\mu)d\mu = 0 \\
&\int_{-1}^{+1} \mu W(\mu)\varphi_{2+}(\mu)d\mu\left[\int_{-1}^{o} A_1(\nu)\varphi_{1\nu}(\mu)d\nu + \int_{o}^{1} A_2(\nu)\varphi_{2\nu}(\mu)d\nu\right] = 0 \quad .
\end{aligned}\right]$$

Hence

$$a_{2+} = \frac{-2qX_a(o)\nu_{01}}{c_2\sigma_2\nu_{02}(\nu_{02}+\nu_{01})X_a(\nu_{02})} \quad , \tag{A40}$$

which is identical to the result obtained from the completeness theorem.

To obtain $A_1(\nu)$ and $A_2(\nu)$ by this method we define

$$\varphi_\nu(\mu) = \begin{cases} \varphi_{1\nu}(\mu) & -1 < \nu < 0 \;, \\ \varphi_{2\nu}(\mu) & 0 < \nu < 1 \;, \end{cases} \qquad A(\nu) = \begin{cases} A_1(\nu) & -1 < \nu < 0 \\ A_2(\nu) & 0 < \nu < 1 \end{cases} ,$$

and write Equation (A37) as

$$q = \mu\left[a_{2+}\varphi_{2+}(\mu) + a_{1-}\varphi_{1-}(\mu) + \int_{-1}^{+1} A(\nu)\varphi_\nu(\mu)d\nu\right] \quad . \tag{A41}$$

Now we multiply Equation (A41) by $W(\mu)\varphi_{\nu'}(\mu)$ and integrate over μ. Here we find

$$\int_{-1}^{+1} qW(\mu)\varphi_{\nu'}(\mu)d\mu = \frac{c(\nu')\sigma(\nu')}{2}\, q\nu_{01}\nu_{02}X_a(o) \quad ,$$

$$\int_{-1}^{+1} W(\mu)\varphi_{\nu'}(\mu)\begin{Bmatrix}\varphi_{1-}(\mu) \\ \varphi_{2+}(\mu)\end{Bmatrix} d\mu = 0$$

$$\int_{-1}^{+1} \mu W(\mu)\varphi_{\nu'}(\mu)d\mu \int_{-1}^{+1} A(\nu)\varphi_\nu(\mu)d\nu =$$

$$A(\nu')\left(\frac{c(\nu')\sigma(\nu')}{2}\right)^2 [\pi^2\nu'^2+\lambda^2(\nu')]\left[\frac{(\nu_{02}-\nu')(\nu_{01}+\nu')\nu' X_a^-(\nu')}{(\lambda(\nu')-i\pi\nu')}\right] .$$

The last integral is done with the aid of the Poincaré-Bertrand theorem[27]. Hence, we can write

$$A(\nu) = \frac{2qX_a(o)\nu_{01}\nu_{02}}{c(\nu)\sigma(\nu)[(\nu_{02}-\nu)(\nu_{01}+\nu)\nu X_a^-(\nu)(\lambda(\nu)+i\pi\nu)]} \quad , \qquad \text{(A42)}$$

where

$$c(\nu)\sigma(\nu) = \begin{cases} c_1\sigma_1 & -1 < \nu < 0 \\ c_2\sigma_2 & 0 < \nu < 1 \end{cases} .$$

This result is again identical to the $A(\nu)$ obtained from the completeness proof.

APPENDIX III – ANALYTIC CONTINUATION OF THE TRANSFORMED SOLUTIONS

This appendix contains a summary of a proof that the $\overline{\Psi}_s(x, \mu)$ obtained for Case I in regions (a), (b), (c) of the s-plane and given by Equations (76) through (81) are the sectional representations of a function which is analytic in some right half s-plane, except for singularities or branches which occur within each s-plane region. Similarly the $\overline{\Psi}_s$ obtained for Case II in regions (a), (b), (c), (d) of the s-plane and given by Equations (76) through (83) are shown to be representations of a function analytic in some right half s-plane, save for discontinuities within each s-plane region.

The basis of this proof is a theorem[28] which follows from Cauchy's Integral Formula and which can be stated in the following manner. If $f_1(z)$ is analytic in D, and $f_2(z)$ is analytic in D_2, where D_1 and D_2 are adjacent regions having a common border γ, and if the values of $f_1(z)$ and $f_2(z)$ are continuous on γ, and are identical for each value of z on γ, then there exists a function analytic in $D_1 U D_2 U \gamma$ which coincides with $f_1(z)$ in D_1 and with $f_2(z)$ in D_2.

Let us show, for Case I and $x < 0$, that the solutions in each region of the s-plane are indeed analytic continuations of each other. Figure 22 contains the information needed.

The solution within each region of the s-plane and given by Equations (76), (78), (80) is written out below.

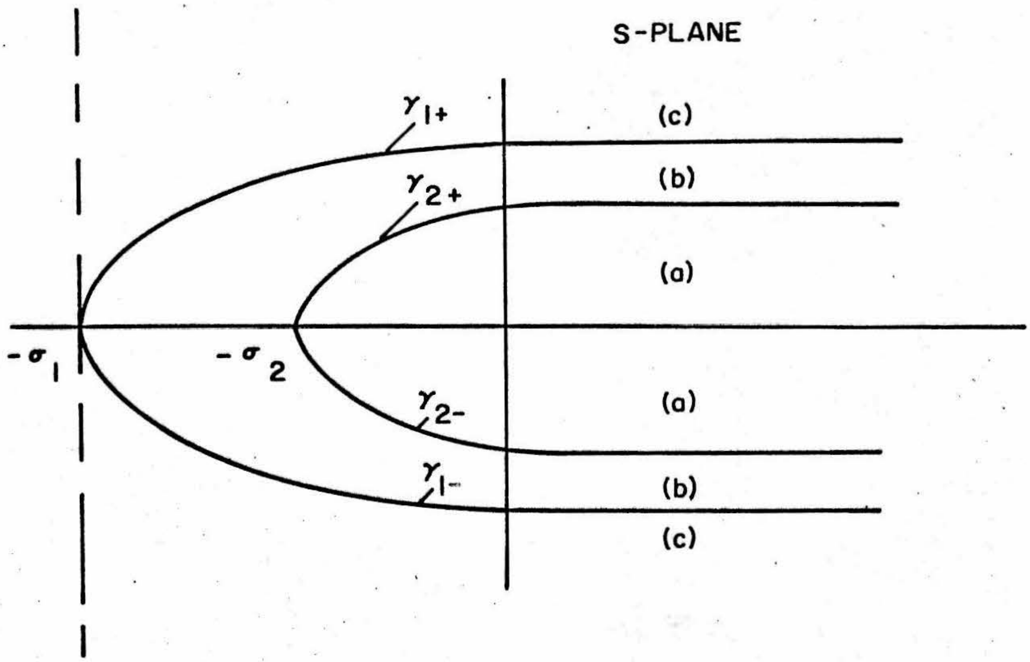

Figure 22 - Sketch of the s-Plane for Case I, $x<0$

$$\overline{\Psi}_s(x,\mu) = \frac{-2qX_a(0)}{c_1\sigma_1}\left[\frac{\nu_{02}\varphi_{1-}(\mu)e^{(s+\sigma_1)x/\nu_{01}}}{\nu_{01}(\nu_{02}+\nu_{01})X_a(-\nu_{01})}\right.$$

$$\left. -\int_{-1}^{0}\frac{\nu_{01}\nu_{02}\varphi_{1\nu}(\mu)e^{-(s+\sigma_1)x/\nu}\,d\nu}{\nu(\nu-\nu_{02})(\nu+\nu_{01})X_a^-(\nu)(\lambda_1(\nu)+i\pi\nu)}\right],\quad s\in(a). \tag{A43}$$

$$\overline{\Psi}_s(x,\mu) = \frac{-2qX_b(0)}{c_1\sigma_1}\left[\frac{\varphi_{1-}(\mu)\,e^{(s+\sigma_1)x/\nu_{01}}}{\nu_{01}X_b(-\nu_{01})}+\right.$$

$$\left.\int_{-1}^{0}\frac{\nu_{01}\varphi_{1\nu}(\mu)e^{-(s+\sigma_1)x/\nu}\,d\nu}{\nu(\nu+\nu_{01})X_b^-(\nu)(\lambda_1(\nu)+i\pi\nu)}\right],\quad s\in(b). \tag{A44}$$

$$\overline{\Psi}_s(x,\mu) = \frac{-2qX_c(0)}{c_1\sigma_1}\int_{-1}^{0}\frac{\varphi_{1\nu}(\mu)\,e^{-(s+\sigma_1)x/\nu}\,d\nu}{\nu X_c^-(\nu)(\lambda_1(\nu)+i\pi\nu)},\quad s\in(c). \tag{A45}$$

We shall look first at the border between regions (b) and (c) in the lower half s-plane and prove that Equations (A44) and (A45) both describe the same continuous function along γ_{1-}. Approaching γ_{1-} from region (c), we note that $(\lambda_1(\nu)-i\pi\nu)$ has a zero somewhere within $-1<\nu<0$, since on γ_{1-}, $0<\nu_{01}<1$. In fact, based on Equation (49), it can be shown that

$$\lambda_1(\nu)-i\pi\nu = \left\{\ell n\,\frac{1+\nu_{01}}{1-\nu_{01}}+\frac{2\nu_{01}}{1-\nu_{01}^2}-i\pi\right\}(\nu+\nu_{01})+O(\nu+\nu_{01})^2 .$$

The appearance of this zero requires that a branch point and an accompanying branch cut be considered in the integrals contained in $X(z)$. For convenience $X_0(z)$ (Equation (A14)) is written out below:

$$X_0(z) = \exp \frac{1}{2\pi i} \left\{ \int_{-1}^{0} \ell n \frac{\lambda_1(\mu')+i\pi\mu'}{\lambda_1(\mu')-i\pi\mu'} \frac{d\mu'}{\mu'-z} + \int_{0}^{1} \ell n \frac{\lambda_2(\mu')+i\pi\mu'}{\lambda_2(\mu')-i\pi\mu'} \frac{d\mu'}{\mu'-z} \right\}. \tag{A46}$$

In the plane of integration (μ'-plane), the location of the branch point as a function of s, for s near γ_{1-}, is shown in Figure 23.

The branch cuts are chosen as shown, and the phase at $\mu' = 0$ is zero as required. Hence, the paths of integration in the μ'-plane for the function $X_0(z)$ are shown in Figures 24a and 24b as $s \to \gamma_{1-}$ from regions (b) and (c) respectively.

Thus we obtain, for $s \to \gamma_{1-}$ from region (c),

$$X_0(z) = \left(\frac{\nu_{01}+z}{1+z}\right)^{1/2} e^{\frac{1}{2\pi i} \Gamma_{\nu_{01}}(z)} ,$$

where $\Gamma_{\nu_{01}}(z)$, obtained from Equation (A46), implies the principal value integral with respect to ν_{01},

$$\Gamma_{\nu_{01}}(z) = P \int_{-1}^{0} \ell n \frac{\lambda_1(\mu')+i\pi\mu'}{\lambda_1(\mu')-i\pi\mu'} \frac{d\mu'}{\mu'-z} + \int_{0}^{1} \ell n \frac{\lambda_2(\mu')+i\pi\mu'}{\lambda_2(\mu')-i\pi\mu'} \frac{d\mu'}{\mu'-z} .$$

Finally, for $z = \nu$, where $-1 < \nu < 0$, we obtain

$$X_0^+(\nu) = \left(\frac{\nu_{01}+\nu}{1+\nu}\right)^{1/2} \left(\frac{\lambda_1(\nu)+i\pi\nu}{\lambda_1(\nu)-i\pi\nu}\right)^{1/2} e^{\frac{1}{2\pi i} \Gamma_{\nu_{01}}(\nu)} , \tag{A47}$$

and the integration in the μ'-plane is now principal value with respect to ν, as well as ν_{01}.

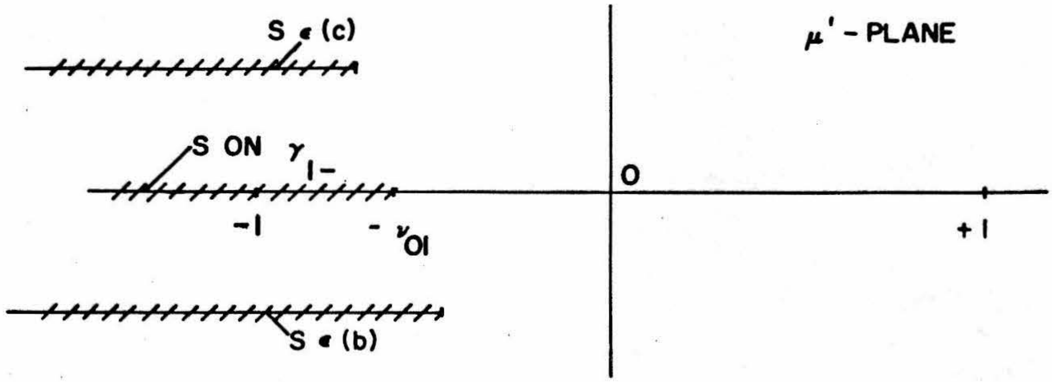

Figure 23 - Location of Branch Point and Line for s near γ_{1-}

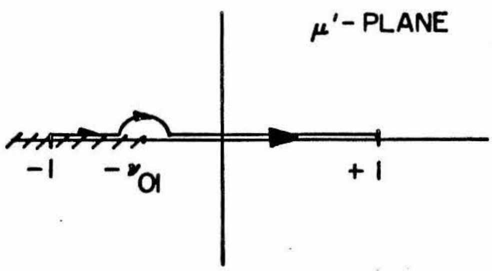

Figure 24a - Path of Integration in $X_o(z)$ for $s \rightarrow \gamma_{1-}$ from (b)

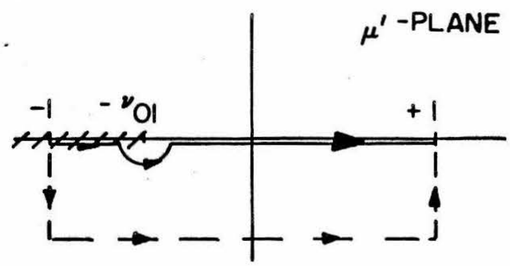

Figure 24b - Path of Integration in $X_o(z)$ for $s \rightarrow \gamma_{1-}$ from (c)

It might seem that when $z = -\nu_{01}$, $X_0(z)$ is undefined, in the sense that it is unbounded. However, this is not the case since an alternative path of integration may be chosen (shown as a dotted line in Figure 24b) along which the integrals exist. So we can conclude that $X_0(z)$ is defined for all values of z, even for s along γ_{1-}. Also $\Gamma_{\nu_{01}}(-\nu_{01})$ exists.

Utilizing Equation (A47) permits one to write $\overline{\Psi}_s(x,\mu)$ for $s \to \gamma_{1-}$ from (c), as follows:

$$\overline{\Psi}_s(x,\mu) = \frac{-2q\sqrt{\nu_{01}}}{c_1\sigma_1}\, e^{\frac{1}{2\pi i}\Gamma_{\nu_{01}}(0)} \int_{-1}^{0} \left\{ \frac{e^{-(s+\sigma_1)x/\nu}\,(1+\nu)^{1/2}\, e^{-\frac{1}{2\pi i}\Gamma_{\nu_{01}}(\nu)}}{\nu(\nu_{01}+\nu)^{1/2}(\lambda_1(\nu)+i\pi\nu)^{1/2}} \right\} \tag{A48}$$

$$\left\{ \frac{1}{(\lambda_1(\nu)-i\pi\nu)^{1/2}} \right\} \left\{ \frac{c_1\sigma_1}{2}\,\nu P\frac{1}{\nu-\mu} + \lambda_1(\nu)\delta(\nu-\mu) \right\} d\nu \;.$$

The path of integration in Equation (A48) (for that part of the integrand not containing the δ-function) is similar to that shown in Figure 24b, and is shown in Figure 25a. No branch is needed this time, and the principal value integration about μ is indicated for $\mu \neq -\nu_{01}$.

For $\mu \neq -\nu_{01}$ then, Equation (A48) can be written as

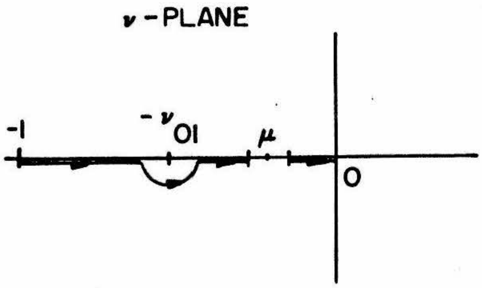

Figure 25a - Path of Integration in $\Psi_s(x, \mu)$ for $s \to \gamma_{1-}$ from (c) with $\mu \neq -\nu_{01}$

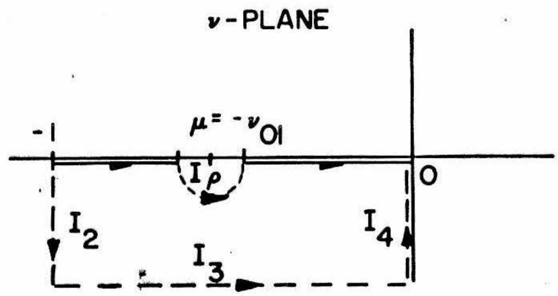

Figure 25b - Path of Integration in $\Psi_s(x, \mu)$ for $s \to \gamma_{1-}$ from (c) with $\mu = -\nu_{01}$

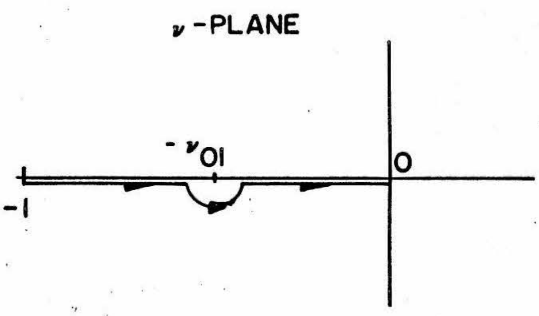

Figure 26 - Path of Integration Used in Defining the δ-Function

$$\overline{\Psi}_s(x,\mu) = \left\{ \frac{-2q\sqrt{\nu_{01}}}{c_1\sigma_1} \, e^{\frac{1}{2\pi i}\Gamma_{\nu_{01}}(0)} \right\}$$

$$\left\{ P_\mu P_{-\nu_{01}} \int_{-1}^{0} \frac{e^{-(s+\sigma_1)x/\nu} (1+\nu)^{1/2} e^{-\frac{1}{2\pi i}\Gamma_{\nu_{01}}(\nu)} c_1\sigma_1 d\nu}{(\nu+\nu_{01})^{1/2} (\lambda(\nu)^2+\pi^2\nu^2)^{1/2} 2(\nu-\mu)} \right.$$

$$+ \frac{1(-\mu)\lambda_1(\mu) e^{-(s+\sigma_1)x/\mu} (1+\mu)^{1/2} e^{-\frac{1}{2\pi i}\Gamma_{\nu_{01}}(\mu)}}{\mu(\nu_{01}+\mu)^{1/2} (\lambda_1(\mu)^2+\pi^2\mu^2)^{1/2}} \tag{A49}$$

$$\left. + \frac{i\pi e^{(s+\sigma_1)x/\nu_{01}} (1-\nu_{01})^{1/2} e^{-\frac{1}{2\pi i}\Gamma_{\nu_{01}}(-\nu_{01})} c_1\sigma_1}{(-2i\pi\nu_{01})^{1/2} 2(-\nu_{01}-\mu)\left[\ell n \frac{1+\nu_{01}}{1-\nu_{01}} + \frac{2\nu_{01}}{1-\nu_{01}^2} - i\pi\right]} \right\} .$$

When $\mu = -\nu_{01}$, the path of integration in Equation (A48) (excluding the δ-function portion) becomes slightly less complicated and is shown in Figure 25b by the solid line.

It can be shown that $\overline{\Psi}_s(x,\mu)$ is bounded when $\mu = -\nu_{01}$, or that the singularity which seems apparent at this point is actually removable, and hence nonexistent. Defining

$$I_1 \equiv P_{-\nu_{01}} \int_{-1}^{0} \frac{e^{-(s+\sigma_1)x/\nu} (1+\nu)^{1/2} e^{-\frac{1}{2\pi i}\Gamma_{\nu_{01}}(\nu)} \nu c_1\sigma_1 d\nu}{\nu(\nu+\nu_{01})^{1/2} (\lambda_1^2(\nu)+\pi^2\nu^2)^{1/2} 2(\nu+\nu_{01})} ,$$

and

$$I_{\delta_c} \equiv \int_{-1}^{0} \frac{e^{-(s+\sigma_1)x/\nu}\,(1+\nu)^{1/2}\, e^{-\frac{1}{2\pi i}\Gamma_{\nu_{01}}(\nu)}\,\lambda_1(\nu)\delta(\nu+\nu_{01})d\nu}{\nu(\nu+\nu_{01})^{1/2}\,(\lambda_1^2(\nu)+\pi^2\nu^2)^{1/2}} \;,$$

Equation (A48) becomes, for $\nu_{01} = -\mu$,

$$\overline{\Psi}_s(x,\mu) = \frac{-2q\sqrt{\nu_{01}}}{c_1\sigma_1}\, e^{\frac{1}{2\pi i}\Gamma_{\nu_{01}}(0)}\,[I_1+I_{\delta_c}] \;. \qquad \text{(A50)}$$

Due to the analyticity of the integrand in the ν-plane we can write

$$I_1 + I_\rho = I_2 + I_3 + I_4 \doteq \text{bounded functions (B.F.)} \;,$$

since I_2 through I_4 are easily seen to exist. Hence

$$I_1 = -I_\rho + \text{B.F.}$$

But it can be shown that

$$I_\rho = -\frac{2}{\rho}\left[\frac{e^{(s+\sigma_1)x/\nu_{01}}\,(1-\nu_{01})^{1/2}\, e^{-\frac{1}{2\pi i}\Gamma_{\nu_{01}}(-\nu_{01})}\, c_1\sigma_1}{(-2i\pi\nu_{01})^{1/2}\; 2\left[\ell n\,\frac{1+\nu_{01}}{1-\nu_{01}} + \frac{2\nu_{01}}{1-\nu_{01}^2} - i\pi\right]^{1/2}}\right] + \text{B.F.},$$

where $\rho = |\nu + \nu_{01}|$.

Hence

$$I_1 = \frac{2}{\rho}\left[\frac{e^{(s+\sigma_1)x/\nu_{01}}\,(1-\nu_{01})^{1/2}\, e^{-\frac{1}{2\pi\nu}\Gamma_{\nu_{01}}(-\nu_{01})}\, c_1\sigma_1}{(-2i\pi\nu_{01})^{1/2}\; 2\left[\ell n\,\frac{1+\nu_{01}}{1-\nu_{01}} + \frac{2\nu_{01}}{1-\nu_{01}^2} - i\pi\right]^{1/2}}\right] + \text{B.F.} \qquad \text{(A50a)}$$

So I_1 does have a first order pole with the behavior near $\nu = -\nu_{01}$ shown above.

We can write I_{δ_c} as

$$I_{\delta_c} \doteq \frac{e^{(s+\sigma_1)x/\nu_{01}} \, (1-\nu_{01})^{1/2} \, e^{-\frac{1}{2\pi i}\Gamma_{\nu_{01}}(-\nu_{01})} \, \lambda_1(-\nu_{01})}{-\nu_{01}(-2i\pi\nu_{01})^{1/2}\left(\ell n \frac{1+\nu_{01}}{1-\nu_{01}} + \frac{2\nu_{01}}{1-\nu_{01}^2} - i\pi\right)^{1/2}} \, K(\rho) \, , \tag{A50b}$$

where

$$K(\rho) = \int_{-1}^{0} \frac{\delta(\nu+\nu_{01})d\nu}{\nu+\nu_{01}} \quad .$$

From Figure 26, and the property of the δ-function

$$\int_{-1}^{0} \frac{F(\nu)d\nu}{\nu+\nu_{01}} = P\int_{-1}^{0} \frac{F(\nu)d\nu}{\nu+\nu_{01}} + i\pi \int_{-1}^{0} F(\nu)\delta(\nu+\nu_{01})d\nu \, ,$$

we set $F(\nu) = (\nu+v_{01})^{-1}$ and observe that the left-hand side of this expression is bounded. This yields

$$\int_{-1}^{0} \frac{\delta(\nu+\nu_{01})d\nu}{\nu+\nu_{01}} = -\frac{2}{i\pi\rho} + \text{B.F.} = K(\rho)$$

for the behavior of $K(\rho)$ near $\nu = -\nu_{01}$. Hence, one can write

$$I_{\delta_c} = +I_\rho + \text{B.F.}$$

Thus both parts of Equation (A50) (I_1 and I_{δ_c}) are singular at $\nu = -\nu_{01}$ but their sum is bounded, implying that $\overline{\Psi}_s(x,\mu)$ is itself bounded for $\mu = -\nu_{01}$ and hence for all μ when s is on γ_{1-}.

So we have obtained expressions for $\overline{\Psi}_s(x,\mu)$ as $s \to \gamma_{1-}$ from $s \in c$, which indicate that $\overline{\Psi}_s(x,\mu)$ is continuous along γ_{1-}. Let us obtain expressions similar to these for $s \to \gamma_{1-}$ from $s \in b$.

Here, based on Figure 24a, we learn that

$$X_0(z) = \left(\frac{1+z}{\nu_{01}+z}\right)^{1/2} e^{\frac{1}{2\pi i}\Gamma_{\nu_{01}}(z)} ,$$

and

$$X_0^-(\nu) = \left(\frac{1+\nu}{\nu_{01}+\nu}\right)^{1/2} \left(\frac{\lambda_1(\nu)-i\pi\nu}{\lambda_1(\nu)+i\pi\nu}\right)^{1/2} e^{\frac{1}{2\pi i}\Gamma_{\nu_{01}}(\nu)} .$$

Another obscure but important fact is that $X(-\nu_{01}) = X^-(-\nu_{01})$ along γ_{1-}. With these formulae in mind we can write for $\overline{\Psi}_s(x,\mu)$ as $s \to \gamma_{1-}$ from (b),

$$\overline{\Psi}_s(x,\mu) = \left\{\frac{-2q\, e^{+\frac{1}{2\pi i}\Gamma_{\nu_{01}}(0)} \sqrt{\nu_{01}}}{c_1\sigma_1}\right\}$$

$$\left\{\frac{e^{(s+\sigma_1)x/\nu_{01}}\, e^{-\frac{1}{2\pi i}\Gamma_{\nu_{01}}(-\nu_{01})}\,(1-\nu_{01})^{1/2} c_1\sigma_1 2\pi i}{(-2\pi i\nu_{01})^{1/2}\, 2(-\nu_{01}-\mu)\left(\ln\frac{1+\nu_{01}}{1-\nu_{01}} + \frac{2\nu_{01}}{1-\nu_{01}^2} - i\pi\right)^{1/2}}\right. \tag{A51}$$

$$\left. + \int_{-1}^{0} \frac{e^{-(s+\sigma_1)x/\nu}\,(1+\nu)^{1/2}\, e^{-\frac{1}{2\pi i}\Gamma_{\nu_{01}}(\nu)}\, d\nu}{\nu(\nu+\nu_{01})^{1/2}(\lambda_1^2(\nu)+\pi^2\nu^2)^{1/2}} \left\{\frac{c_1\sigma_1}{2}\,\nu\, P\frac{1}{\nu-\mu} + \lambda_1(\nu)\delta(\nu-\mu)\right\}\right\} ,$$

where the non-delta function integration is carried out over the paths shown in Figure 27a for $\mu \neq -\nu_{01}$ and in Figure 27b for $\mu = -\nu_{01}$.

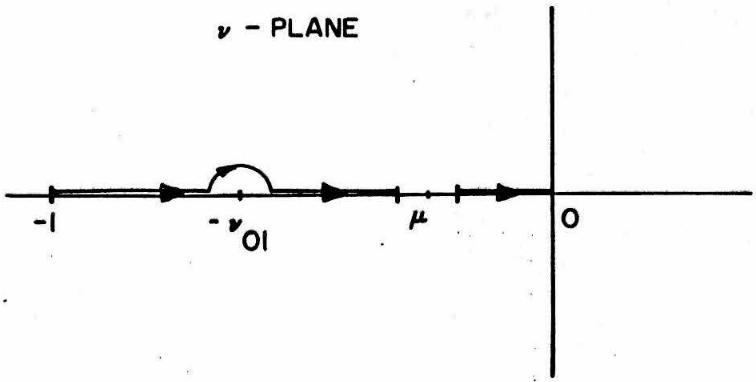

Figure 27a - Path of Integration in $\overline{\Psi}_s(x,\mu)$ for $s \to \gamma_{1-}$ from (b) with $\mu \neq -\nu_{01}$

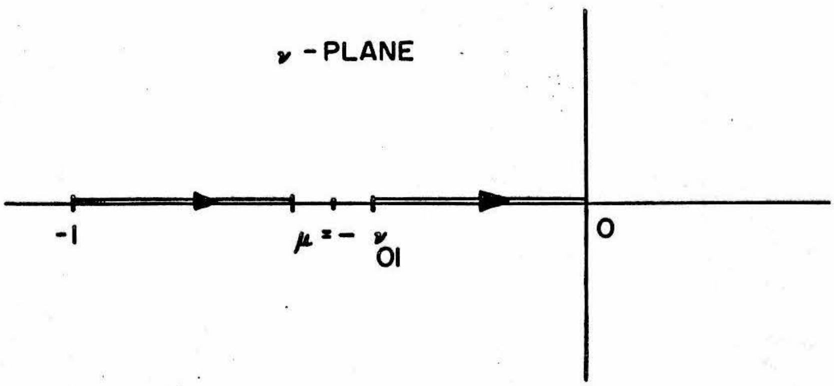

Figure 27b - Path of Integration in $\overline{\Psi}_s(x,\mu)$ for $s \to \gamma_{1-}$ from (b) with $\mu = -\nu_{01}$

If one expands Equation (A51) for $\mu \neq -\nu_{01}$, the result is identical to that given by Equation (A49). Also, if one notes that in the discrete term the factor $1/(\nu_{01}+\mu)$ can be replaced by $\int_{-1}^{0} \frac{\delta(\nu+\nu_{01})d\nu}{\nu+\nu_{01}}$ when $\mu \rightarrow -\nu_{01}$, then one finds that $\overline{\Psi}_s(x,\mu)$ is bounded here as well.

To prove that the $\overline{\Psi}_s(x,\mu)$ at the border of the two regions are identical for $\mu = -\nu_{01}$ requires an additional step. First we write the solutions at $\mu = -\nu_{01}$ when approached from both regions (b) and (c):

$$\overline{\Psi}_s(x,\mu) = H(\nu_{01}) [I_1 + I_{\delta_c}] \qquad , s \in c ,$$

$$\overline{\Psi}_s(x,\mu) = H(\nu_{01}) [I_1 + D.T. + I_{\delta_b}], s \in b ,$$

where $H(\nu_{01})$ is obtained from Equation (A50), and D.T. indicates the discrete term in Equation (A51). Thus we must prove that

$$I_{\delta_c} = D.T. + I_{\delta_b} . \tag{A52}$$

Now $I_{\delta_c} = M(\nu_{01}) i\pi \int_{-1}^{0} \frac{\delta(\nu+\nu_{01})d\nu}{\nu+\nu_{01}}$, where $M(\nu_{01})$ is given by comparison with Equation (A50b). This can be rewritten as

$$I_{\delta_c} = M(\nu_{01}) \left[P\int_{-1}^{0} \frac{d\nu}{(\nu+\nu_{01})^2} - \int_{\underset{\rightarrow\cup}{-1}}^{0} \frac{d\nu}{(\nu+\nu_{01})^2} \right] .$$

In a similar manner we can write

$$D.T. + I_{\delta_b} = M(\nu_{01}) \left[P\int_{-1}^{0} \frac{d\nu}{(\nu+\nu_{01})^2} - \int_{\underset{\rightarrow\cap}{-1}}^{0} \frac{d\nu}{(\nu+\nu_{01})^2} \right] .$$

But $\int_{\underset{\rightarrow\cup}{-1}}^{0} = \int_{\underset{\rightarrow\cap}{-1}}^{0} \cdot$, and Equation (A52) is proved.

Thus $\overline{\Psi}_s(x,\mu)$ is continuous along γ_{1-} and takes on the same set of values there when γ_{1-} is approached from either region (c) or (b). So the $\overline{\Psi}_s(x,\mu)$ in regions (c) and (b) describe a function which is analytic in a region which includes γ_{1-} and adjacent sections of regions (b) and (c).

An investigation of the solutions along γ_{2-} between regions (a) and (b) proceeds in a manner similar to that carried out along γ_{1-} except that no apparent singularity appears here since ν_{01} is always complex for s along γ_{2-}.

The methods applied in the γ_{1-} and γ_{2-} study were also applied to the boundaries separating all the various regions in both Cases I and II with the results always being the same. Hence we can state in both Cases I and II for $x > 0$ that the $\overline{\Psi}_s(x,\mu)$ represent a function which is analytic in the right half plane Re $s > -\sigma_2$, save for two branch cuts (possibly overlapping) along the real s-axis from $-\sigma_2$ to $-\sigma_2(1-c_2)$, and from $-\sigma_2$ to $-\sigma_1(1-c_1)$. For $x < 0$, the $\overline{\Psi}_s(x,\mu)$ represent a function analytic in the right half plane Re $s > -\sigma_1$ save for two branch cuts (possibly overlapping) along the real s-axis from $-\sigma_2$ to $-\sigma_2(1-c_2)$ and from $-\sigma_1$ to $-\sigma_1(1-c_1)$.

APPENDIX IV — UNIQUENESS OF SOLUTION

The uniqueness of the solution to Equation (38)

$$\frac{\partial \Psi(x,\mu,\tau)}{\partial \tau} + \mu \frac{\partial \Psi}{\partial x} + \sigma(x)\Psi = \frac{c(x)\sigma(x)}{2} \int_{-1}^{+1} \Psi(x,\mu',\tau)d\mu' + q\delta(x)\delta(\tau) , \tag{A53}$$

subject to the boundary conditions

$$\lim_{x \to \pm\infty} \Psi(x,\mu,\tau) = 0 , \tag{A54}$$

is proved in this appendix. The method used is quite standard and can be found, for instance, in Churchill[29]. Both the notation and the geometry are shown in Figure 28.

Let us assume two solutions to Equation (A53), $\Psi_1(x,\mu,\tau)$ and $\Psi_2(x,\mu,\tau)$. Then their difference

$$\Psi_d(x,\mu,\tau) = \Psi_1(x,\mu,\tau) - \Psi_2(x,\mu,\tau)$$

must satisfy the homogeneous equation

$$\frac{\partial \Psi_d(x,\mu,\tau)}{\partial \tau} + \mu \frac{\partial \Psi_d}{\partial x} + \sigma(x)\Psi_d = \frac{c(x)\sigma(x)}{2} \int_{-1}^{+1} \Psi_d(x,\mu',\tau)d\mu' , \tag{A55}$$

and the boundary conditions

$$\lim_{x \to \pm\infty} \Psi_d(x,\mu,\tau) = 0 . \tag{A56}$$

Assume further that $\Psi_d(x,\mu,0) = 0$, or that the two solutions are identical for $\tau = 0$.

Multiplying Equation (A55) by $\Psi_d(x,\mu,\tau)$ and integrating over all space and angle yields

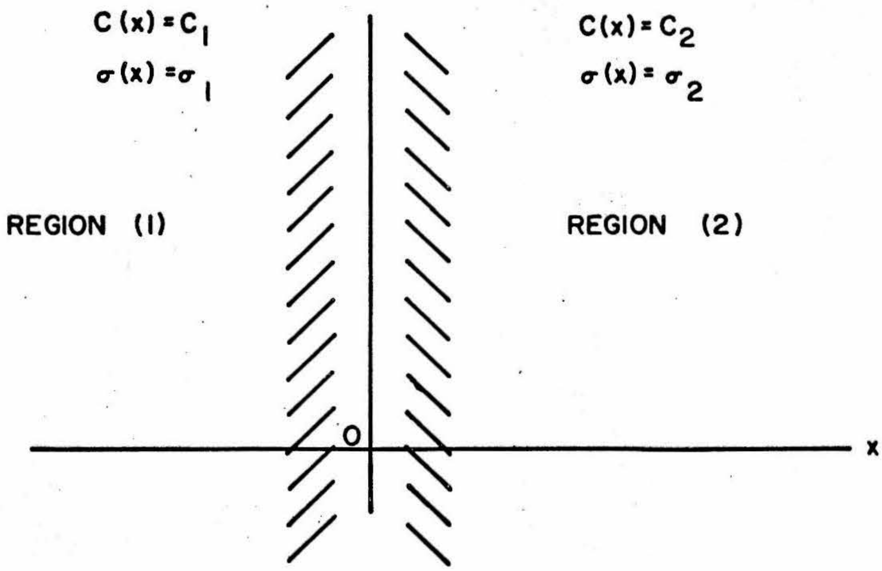

Figure 28 - Geometry and Notation Used in Uniqueness Proof

$$\frac{d}{d\tau}\left\{\int_{-\infty}^{+\infty} dx \int_{-1}^{+1} \Psi_d^2(x,\mu,\tau)d\mu\right\} + \int_{-\infty}^{+\infty} \sigma(x)dx \int_{-1}^{+1} \Psi_d^2(x,\mu,\tau)d\mu$$

$$= \int_{-\infty}^{+\infty} \frac{c(x)\sigma(x)}{2}\, dx \int_{-1}^{+1} \Psi_d(x,\mu,\tau)d\mu \int_{-1}^{+1} \Psi_d(x,\nu,\tau)\, d\nu \ . \tag{A57}$$

Let us rewrite this equation in the more suitable form

$$\frac{d}{d\tau}\left\{\int_{-\infty}^{+\infty} dx \int_{-1}^{+1} \Psi_d^2(x,\mu,\tau)d\mu\right\} = \int_{-\infty}^{0} \frac{c_1\sigma_1}{2}\, dx \left[\int_{-1}^{+1} \Psi_d(x,\mu,\tau)d\mu\right]^2$$

$$+ \int_{0}^{\infty} \frac{c_2\sigma_2}{2}\, dx \left[\int_{-1}^{+1} \Psi_d(x,\mu,\tau)\, d\mu\right]^2 \tag{A58}$$

$$- \int_{-\infty}^{0} \sigma_1 dx \int_{-1}^{+1} \Psi_d^2(x,\mu,\tau)\, d\mu - \int_{0}^{\infty} \sigma_2 dx \int_{-1}^{+1} \Psi_d^2\,(x,\mu,\tau)\, d\mu \ .$$

From the Schwartz inequality

$$\int_a^b f^2(x)dx \int_a^b g^2(x)dx \geqq \left[\int_a^b f(x)g(x)dx\right]^2 \quad ,$$

where f(x) and g(x) are real quantities, we can conclude that

$$\int_{-1}^{+1} \Psi_d^2(x,\mu,\tau)\, d\mu \geqq \frac{1}{2}\left[\int_{-1}^{+1} \Psi_d(x,\mu,\tau)d\mu\right]^2 \ .$$

Substituting this result into Equation (A58) yields

$$\frac{d}{d\tau}\left\{\int_{-\infty}^{+\infty} dx \int_{-1}^{+1} \Psi_d^2(x,\mu,\tau)d\mu\right\} \leqq \sigma_1(c_1-1)\int_{-\infty}^{0} dx \int_{-1}^{+1} \Psi_d^2(x,\mu,\tau)d\mu$$

$$+ \sigma_2(c_2-1)\int_{0}^{\infty} dx \int_{-1}^{+1} \Psi_d^2(x,\mu,\tau)d\mu \ . \quad \text{(A59)}$$

Since $\Psi_d^2(x,\mu,\tau) \geqq 0$ for all x, μ, and τ, we can write

$$\frac{d}{d\tau}\left\{\int_{-\infty}^{+\infty} dx \int_{-1}^{+1} \Psi_d^2(x,\mu,\tau)d\mu\right\} \leqq 0 \quad \text{for} \quad 0 < c_1, c_2 \leqq 1.$$

This implies (since $\Psi_d(x,\mu,0) = 0$) that $\Psi_d(x,\mu,\tau) = 0$ for $\tau \geqq 0$. Hence $\Psi_1(x,\mu,\tau) = \Psi_2(x,\mu,\tau)$, and the solution to Equation (A53) satisfying the accompanying conditions (A54), if it exists, is unique.

APPENDIX V - EVALUATION OF THE INTERFACE INTEGRALS

The simplifications found in this appendix are based mainly on the work of Mendelson and Summerfield[23]. As in Appendix II, we shall need several identities in $X(z)$, and these will be obtained first.

A. Several Identities in $X(z)$

An inspection of the interface integrals - Equation (120) through (123) - indicate that identities in $X(z)$ are needed only in regions (a) and (b) of the s-plane. We expect different identities to hold in each region.

Define

$$X_{10}(z) \equiv \exp \frac{1}{2\pi i} \int_0^1 \ln \frac{\bar{\lambda}_1(\mu')+i\pi\mu'}{\bar{\lambda}_1(\mu')-i\pi\mu'} \frac{d\mu'}{\mu'-z} \quad , \tag{A60}$$

where the subscript 1 on $X_{10}(z)$ corresponds to the subscript 1 on $\bar{\lambda}_1$. Then

$$X_a(z) = \frac{1}{(1-z)(-1-z)} X_{10}(-z)X_{20}(z) \quad , \tag{A61}$$

and

$$X_b(z) = \frac{1}{(-1-z)} X_{10}(-z)X_{20}(z) \quad . \tag{A62}$$

Let us define, in addition,

$$\Gamma_1'(z) \equiv \frac{1}{2\pi i} \int_0^1 \ln \frac{\bar{\lambda}_1(\mu')+i\pi\mu'}{\bar{\lambda}_1(\mu')-i\pi\mu'} \frac{d\mu'}{\mu'-z} \quad . \tag{A63}$$

Then

$$\Gamma_1'(-z) = \frac{1}{2\pi i} \int_{-1}^{0} \ln \frac{\lambda_1(\mu')+i\pi\mu'}{\lambda_1(\mu')-i\pi\mu'} \frac{d\mu'}{\mu'-z} \quad , \tag{A64}$$

and hence

$$\Gamma_1'(z)+\Gamma_1'(-z) = \frac{1}{2\pi i} \int_{-1}^{+1} \ln \frac{\lambda_1(\mu')+i\pi\mu'}{\lambda_1(\mu')-i\pi\mu'} \frac{d\mu'}{\mu'-z} \quad . \tag{A65}$$

Let us evaluate the integral in Equation (A65). We note that

$$\lambda_1(\mu')\pm i\pi\mu' = \frac{2}{c_1\sigma_1} (s+\sigma_1)-\mu' \ln \frac{1+\mu'}{1-\mu'} \pm i\pi\mu' \quad . \tag{A66}$$

But, if we think of μ' as $\mathrm{Re}\, z'$, Equation (A66) takes the form

$$\lambda_1(\mu')\pm i\pi\mu' = \left(\frac{2}{c_1\sigma_1} (s+\sigma_1)-z' \ln \frac{z'+1}{z'-1}\right)_{\pm} \quad . \tag{A67}$$

Hence Equation (A65) can be written in the alternative form

$$\Gamma_1'(z)+\Gamma_1'(-z) = \frac{1}{2\pi i} \int_c \ln\left(\frac{2}{c_1\sigma_1} (s+\sigma_1)-z' \ln \frac{z'+1}{z'-1}\right) \frac{dz'}{z'-z} \quad , \tag{A68}$$

where the contour c is shown in Figure 29. The integral over the small circles around $z' = \pm 1$ can be added since in the limit as their radii tend to zero their contributions also tend to zero.

Observing that

$$\lim_{z'\to\infty} \ln\left(\frac{2}{c_1\sigma_1} (s+\sigma_1)-z' \ln \frac{z'+1}{z'-1}\right) = \ln\left(\frac{2}{c_1\sigma_1} (s+\sigma_1)-2\right),$$

and forcing the integrand in Equation (A68) to vanish as $z'\to\infty$, we write Equation (A68) as

$$\Gamma_1'(z)+\Gamma_1'(-z) = \frac{1}{2\pi i} \int_c \ln \left[\frac{\frac{2}{c_1\sigma_1} (s+\sigma_1)-z' \ln \frac{z'+1}{z'-1}}{\frac{2}{c_1\sigma_1} (s+\sigma_1)-2}\right] \frac{dz'}{z'-z} \quad , \tag{A69}$$

since contributions above and below the cut cancel.

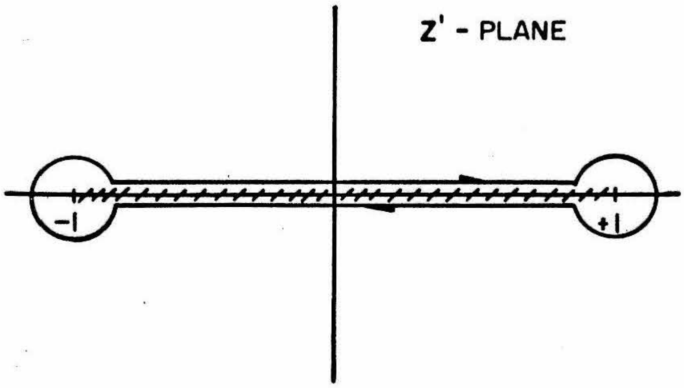

Figure 29 - Contour Used in Construction of Integral Equation in $\Gamma'(z)$

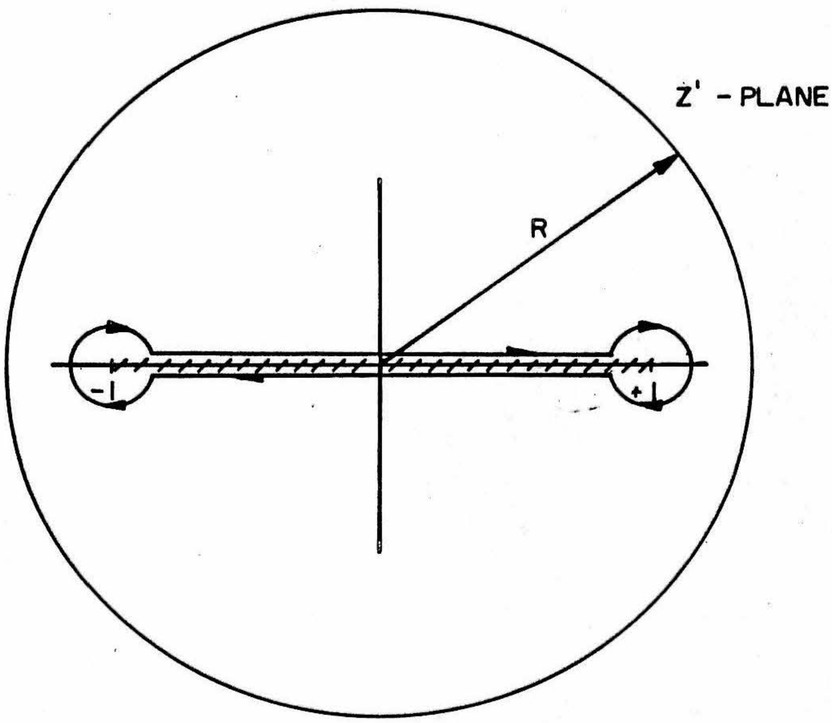

Figure 30 - Final Contour Used to Obtain $\Gamma'(z)$

In both regions (a) and (b) the argument of the logarithm has a zero of first order at $z' = \pm\nu_{01}$. To remove this we divide by the factor $\left(\frac{z'^2-\nu_{01}^2}{z'^2-1}\right)$, where the (z'^2-1) is added to keep the desired behavior at infinity. Hence

$$\Gamma_1'(z)+\Gamma_1'(-z)=\frac{1}{2\pi i}\int_{c'}\ell n\left[\frac{\left\{\frac{2}{c_1\sigma_1}(s+\sigma_1)-z'\ell n\frac{z'+1}{z'-1}\right\}(z'^2-1)}{\left\{\frac{2}{c_1\sigma_1}(s+\sigma_1)-2\right\}\left\{z'^2-\nu_{01}^2\right\}}\right]\frac{dz'}{z'-z}\ , \tag{A70}$$

where now c' is shown in Figure 30 (the integral over the large circle vanishes as $R\to\infty$).

Using the notation $b_1 = \frac{2}{c_1\sigma_1}(s+\sigma_1)$, and applying Cauchy's integral formula to Equation (A70) yields

$$\Gamma_1'(z)+\Gamma_1'(-z) = \ell n\left[\frac{(b_1-z\,\ell n\,\frac{z+1}{z-1})(z^2-1)}{(b_1-2)(z^2-\nu_{01}^2)}\right] . \tag{A71}$$

Thus

$$X_{10}(z)X_{10}(-z) = \frac{(b_1-z\,\ell n\,\frac{z+1}{z-1})(z^2-1)}{(b_1-2)(z^2-\nu_{01}^2)}\ ,\quad s\in(a)\cup(b)\ . \tag{A72}$$

In region (a), through a similar derivation, it can be shown that

$$X_{20}(z)X_{20}(-z) = \frac{(b_2-z\,\ell n\,\frac{z+1}{z-1})(z^2-1)}{(b_2-2)(z^2-\nu_{02}^2)}\ ,\quad s\in a\ . \tag{A73}$$

However, the argument of the logarithm in the equation for region (b) corresponding to Equation (A69) has no zero. Hence for this case

$$X_{20}(z)X_{20}(-z) = \frac{b_2 - z\,\ell n\frac{z+1}{z-1}}{b_2-2}\;, \qquad s \in b\;. \tag{A74}$$

B. Additional Identities Useful in the Current and Flux Integrals

From Equations (A20), (A21), and (A23), we have

$$X_a(z) = \int_{-1}^{+1} \frac{\mu' X_a^-(\mu')d\mu'}{(\lambda(\mu')-i\pi\mu')(\mu'-z)}\;, \tag{A75}$$

$$X_b(z) = \int_{-1}^{+1} \frac{\mu' X_a^-(\mu')d\mu'}{(\lambda(\mu')-i\pi\mu')(\mu'-z)}\;, \tag{A76}$$

$$zX_a(z) = \int_{-1}^{+1} \frac{\mu'^2 X_a^-(\mu')d\mu'}{(\lambda(\mu')-i\pi\mu')(\mu'-z)}\;. \tag{A77}$$

With derivations similar to that described for Equation (A25), we can obtain the following additional identities:

$$z^2X_a(z)-1 = \int_{-1}^{+1} \frac{\mu'^3 X_a^-(\mu')d\mu'}{(\lambda(\mu')-i\pi\mu')(\mu'-z)}\;, \quad \text{(also given as Equation (A38))} \tag{A78}$$

$$X_b(z)+1 = \int_{-1}^{+1} \frac{\mu'^2 X_b^-(\mu')d\mu'}{(\lambda(\mu')-i\pi\mu')(\mu'-z)}\;. \tag{A79}$$

Manipulating Equations (A75) through (A79), where use is made of Equations (A72) through (A74), provides us with the following set of identities, corresponding to Equations (A75) through (A79):

$$X_a(z) = \frac{1}{b_1-2}\int_{-1}^{0}\frac{\mu X_{20}(\mu)d\mu}{X_{10}(\mu)(\mu^2-\nu_{01}^2)(\mu-z)} + \frac{1}{b_2-2}\int_{0}^{1}\frac{\mu X_{10}(-\mu)d\mu}{X_{20}(-\mu)(\mu^2-\nu_{02}^2)(\mu-z)} \quad . \tag{A80}$$

$$X_b(z) = \frac{1}{b_1-2}\int_{-1}^{0}\frac{\mu(1-\mu)X_{20}(\mu)d\mu}{(\mu^2-\nu_{01}^2)X_{10}(\mu)(\mu-z)} + \frac{1}{b_2-2}\int_{0}^{1}\frac{\mu X_{10}(-\mu)d\mu}{(-1-\mu)X_{20}(-\mu)(\mu-z)} \quad . \tag{A81}$$

$$zX_a(z) = \frac{1}{b_1-2}\int_{-1}^{0}\frac{\mu^2 X_{20}(\mu)d\mu}{(\mu^2-\nu_{01}^2)X_{10}(\mu)(\mu-z)} + \frac{1}{b_2-2}\int_{0}^{1}\frac{\mu^2 X_{10}(-\mu)d\mu}{(\mu^2-\nu_{02}^2)X_{20}(-\mu)(\mu-z)} \quad . \tag{A82}$$

$$z^2X_a(z)-1 = \frac{1}{b_1-2}\int_{-1}^{0}\frac{\mu^3 X_{20}(\mu)d\mu}{(\mu^2-\nu_{01}^2)X_{10}(\mu)(\mu-z)} + \frac{1}{b_2-2}\int_{0}^{1}\frac{\mu^3 X_{10}(-\mu)d\mu}{(\mu^2-\nu_{02}^2)X_{20}(-\mu)(\mu-z)} \quad . \tag{A83}$$

$$zX_b(z)+1 = \frac{1}{b_1-2}\int_{-1}^{0}\frac{\mu^2(1-\mu)X_{20}(\mu)d\mu}{(\mu^2-\nu_{01}^2)X_{10}(\mu)(\mu-z)} + \frac{1}{b_2-2}\int_{0}^{1}\frac{\mu^2 X_{10}(-\mu)d\mu}{(-1-\mu)X_{20}(-\mu)(\mu-z)} \quad . \tag{A84}$$

C. Simplification of $I_n(\overline{\Psi}_s(0\pm,\mu))$

1. $I_1(\overline{\Psi}_s(0,\mu))$. An inspection of Equation (120) reveals the fact that by judiciously choosing the range of μ, we can remove the principal value signs from the integration over ν. Thus

$$I_1(\overline{\Psi}_s(0,\mu)) =$$

$$2iq \int_{-\sigma_1}^{-a_1\sigma_1 1(a_1\sigma_1-\sigma_2)-\sigma_2 1(\sigma_2-a_1\sigma_1)} X_b(0)e^{s\tau}\, ds \begin{cases} \dfrac{2\xi}{c_2\sigma_2} K_+(\mu, s) \quad , & -1 < \mu < 0, \\ \dfrac{2}{c_1\sigma_1}\left\{ \mathrm{Re}\left\{\dfrac{\varphi_{1-}^{-}(\mu)}{X_b^{-}(-i\xi)}\right\} - \xi K_-(\mu, s)\right\} , & 0 < \mu < 1 , \end{cases} \tag{A85}$$

where

$$K_+(\mu, s) = \frac{c_2\sigma_2}{2}\int_0^1 \frac{\nu d\nu}{(\nu-\mu)(\nu^2+\xi^2)X_b^-(\nu)(\lambda_2(\nu)+i\pi\nu)} \qquad -1< \mu < 0 , \tag{A86}$$

and

$$K_-(\mu, s) = \frac{c_1\sigma_1}{2}\int_{-1}^0 \frac{\nu d\nu}{(\nu-\mu)(\nu^2+\xi^2)X_b^-(\nu)(\lambda_1(\nu)+i\pi\nu)} \qquad 0 < \mu < 1 . \tag{A87}$$

We shall now perform the integrations over ν in Equations (A86) and (A87). Considering Equation (A86) first, we define

$$I_+(z) \equiv \int_0^1 \frac{\nu d\nu}{(\nu-z)(\nu^2+\xi^2)X_b^-(\nu)(\lambda_2(\nu)+i\pi\nu)} , \tag{A88}$$

where we note that for z real and equal to μ, $(-1< \mu< 0)$,

$$K_+(\mu, s) = \frac{c_2\sigma_2}{2} I_+(\mu) . \tag{A89}$$

From Equation (A88) it is seen that $I_+(z)$ is analytic in the z-plane cut along the real axis from $0 \leq z \leq 1$, and that $I_+(z)$ is of order z^{-1} as z tends to infinity. Also for $0 < \mu < 1$,

$$I_+^+(\mu) - I_+^-(\mu) = \frac{2\pi i\mu}{(\mu^2+\xi^2)X_b^-(\mu)(\lambda_2(\mu)+i\pi\mu)} \quad . \tag{A90}$$

It is easy to prove that the $I_+(z)$ satisfying the conditions stated in the last paragraph is unique, if it exists. The procedure used in this uniqueness proof is to assume two solutions, take the difference and note that this difference is analytic everywhere and vanishes at infinity; hence is zero. Thus the solution is unique, if it exists.

To obtain this solution, we start by substituting for $X_b^-(\mu)$, utilizing Equations (A72) and (A74). Hence

$$I_+^+(\mu)-I_+^-(\mu) = \frac{2\pi i\mu(-1-\mu)X_{20}(-\mu)(b_2-2)}{(\mu^2+\xi^2)X_{10}(-\mu)(b_2-z\,\ell n\,\frac{z+1}{z-1})_+\,(b_2-z\ell n\frac{z+1}{z-1})_-} \quad , \tag{A91}$$

where we note that

$$(b_2-z\;\ell n\,\frac{z+1}{z-1}\,)_+ = \lambda_2(\mu) + i\pi\mu \qquad 0<\mu<1 \; . \tag{A92}$$

Following closely the work of Mendelson and Summerfield[(23)], we consider

$$L_+(z) = \frac{X_{10}(z)}{X_{20}(z)} \quad . \tag{A93}$$

Now $L_+(z)$ is analytic in the z-plane cut from 0 to 1, and has a jump across the cut given by

$$L_+^+(\mu)-L_+^-(\mu) = \frac{(\mu^2-1)(b_2-2)(2\pi i\mu)(b_2-b_1)\;\;X_{20}(-\mu)}{(\mu^2-\nu_{01}^2)(b_1-2)(b_2-z\,\ell n\,\frac{z+1}{z-1})_+(b_2-z\,\ell n\,\frac{z+1}{z-1})_-X_{10}(-\mu)} \; . \tag{A94}$$

Hence

$$\frac{b_1-2}{b_2-b_1}\;\frac{1}{1-z}\;\frac{X_{10}(z)}{X_{20}(z)} = \frac{b_1-2}{b_2-b_1}\;\frac{1}{1-z}\;L_+(z) \tag{A95}$$

has the proper discontinuity across the branch. Also either side of Equation (A95) is analytic in the remainder of the z-plane and behaves as $\frac{1}{z}$ for $z \to \infty$.

Therefore, we can write

$$I_+(z) = \frac{b_1-2}{b_2-b_1}\;\frac{1}{1-z}\;\frac{X_{10}(z)}{X_{20}(z)}\;, \tag{A96}$$

and when this is substituted into Equation (A89),

$$K_+(\mu, s) = \frac{c_2\sigma_2}{2}\;\frac{b_1-2}{b_2-b_1}\;\frac{1}{1-\mu}\;\frac{X_{10}(\mu)}{X_{20}(\mu)}\,, \quad -1 < \mu < 0\,. \tag{A97}$$

To obtain $K_-(\mu, s)$ we start with

$$I_-(z) = \int_{-1}^{0} \frac{\nu\, d\nu}{(\nu - z)(\nu^2+\xi^2)X_b^-(\nu)(\lambda_2(\nu)+i\pi\nu)}\;. \tag{A98}$$

Proceeding as above we can show

$$L_-(z) = \frac{(-1-z)(b_2-2)}{(b_1-b_2)(z^2+\xi^2)}\;\frac{X_{20}(-z)}{X_{10}(-z)} \tag{A99}$$

has the proper discontinuity for $z \in [-1, 0]$. It also behaves as $\frac{1}{z}$ for large z, but certain poles exist at

$$z = \pm i\,\xi\,.$$

We remove these poles by adding appropriate terms to obtain

$$I_-(z) = \frac{(-1-z)(b_2-2)X_{20}(-z)}{(b_1-b_2)(z^2+\xi^2)X_{10}(-z)} - \frac{(-1+i\xi)(b_2-2)X_{20}(i\xi)}{(b_1-b_2)2i\xi(z+i\xi)X_{10}(i\xi)}$$

$$- \frac{(-1-i\xi)(b_2-2)X_{20}(-i\xi)}{(b_1-b_2)2i\xi(z-i\xi)X_{10}(-i\xi)} \quad . \qquad \text{(A100)}$$

Thus $K_-(\mu)$ has the form

$$K_-(\mu, s) = \frac{c_1\sigma_1}{2}\frac{b_2-2}{b_1-b_2}\left[\frac{(-1-\mu)X_{20}(-\mu)}{(\mu^2+\xi^2)X_{10}(-\mu)} - \frac{(-1+i\xi)X_{20}(i\xi)}{2i\xi(\mu+i\xi)X_{10}(i\xi)}\right.$$

$$\left. - \frac{(-1-i\xi)X_{20}(-i\xi)}{2i\xi(\mu-i\xi)X_{10}(-i\xi)}\right], \quad 0 < \mu < 1 \ . \qquad \text{(A101)}$$

Substituting Equations (A97) and (A101) into Equation (A85) and simplifying yields

$$I_1(\overline{\Psi}_s(0,\mu)) = 2iq\int_{-\sigma_1}^{-a_1\sigma_1 l(a_1\sigma_1-\sigma_2)-\sigma_2 l(\sigma_2-a_1\sigma_1)} \frac{\sqrt{-b_1b_2}\; e^{s\tau}\, ds}{(b_2-b_1)} \begin{cases} \sqrt{\dfrac{2-b_1}{2-b_2}}\,\dfrac{1}{1-\mu}\,\dfrac{X_{10}(\mu)}{X_{20}(\mu)}, & 1 < \mu < 0 , \\[2ex] \sqrt{\dfrac{2-b_2}{2-b_1}}\,\dfrac{-1-\mu}{\mu^2+\xi^2}\,\dfrac{X_{20}(-\mu)}{X_{10}(-\mu)}, & 0 < \mu < 1 , \end{cases} \qquad \text{(A102)}$$

where we have used

$$X_b(0) = \frac{-\sqrt{-b_1b_2}}{\xi\sqrt{(2-b_1)(2-b_2)}} \quad . \qquad \text{(A103)}$$

2. <u>The Remaining Integrals</u>

The procedure used to simplify $I_1(\overline{\Psi}_s(0,\mu))$ was applied to I_2, I_3, and I_4 $(\overline{\Psi}_s(0,\mu))$ as well. Since the method is similar to that described for I_1, it is not presented here; only the final results are shown. For Expressions (121), (122), (123) we find

$$I_2(\overline{\Psi}_s(0,\mu)) = 2iql(\sigma_2 - a_1\sigma_1)\int_{-\sigma_2}^{-a_1\sigma_1 l(a_1\sigma_1 - a_2\sigma_2) - a_2\sigma_2 l(a_2\sigma_2 - a_1\sigma_1)} \frac{(\eta-\xi)\sqrt{b_1 b_2}\, e^{s\tau}\, ds}{(b_1 - b_2)} \begin{cases} \sqrt{\dfrac{b_1-2}{b_2-2}}\, \dfrac{1}{\mu^2+\eta^2}\, \dfrac{X_{10}(\mu)}{X_{20}(\mu)}, & -1 < \mu < 0 \\[2ex] \sqrt{\dfrac{b_2-2}{b_1-2}}\, \dfrac{1}{\mu^2+\xi^2}\, \dfrac{X_{20}(-\mu)}{X_{10}(-\mu)}, & 0 < \mu < 1 , \end{cases} \tag{A104}$$

$$I_3(\overline{\Psi}_s(0,\mu)) = 2iql(a_1\sigma_1 - a_2\sigma_2)\int_{-\sigma_2 l(a_1\sigma_1 - \sigma_2) - a_1\sigma_1 l(\sigma_2 - a_1\sigma_1)}^{-a_2\sigma_2} \frac{\sqrt{b_1 b_2}\, e^{s\tau}\, ds}{(b_2 - b_1)} \begin{cases} \sqrt{\dfrac{b_1-2}{2-b_2}}\, \dfrac{\mu-\nu_{01}}{\mu^2+\eta^2}\, \dfrac{X_{10}(\mu)}{X_{20}(\mu)}, & -1 < \mu < 0 \\[2ex] \sqrt{\dfrac{2-b_2}{b_1-2}}\, \dfrac{-1}{\mu+\nu_{01}}\, \dfrac{X_{20}(-\mu)}{X_{10}(-\mu)}, & 0 < \mu < 1 , \end{cases} \tag{A105}$$

$$I_4(\overline{\Psi}_s(0,\mu)) = 2iql(a_2\sigma_2 - a_1\sigma_1)\int_{-a_2\sigma_2}^{-a_1\sigma_1} \frac{\sqrt{b_1 b_2}\, e^{s\tau}\, ds}{(b_2 - b_1)} \begin{cases} \sqrt{\dfrac{2-b_1}{b_2-2}}\, \dfrac{-1}{\mu-\nu_{02}}\, \dfrac{X_{10}(\mu)}{X_{20}(\mu)}, & -1 < \mu < 0 \\[2ex] \sqrt{\dfrac{b_2-2}{2-b_1}}\, \dfrac{\mu+\nu_{02}}{\mu^2+\xi^2}\, \dfrac{X_{20}(-\mu)}{X_{10}(-\mu)}, & 0 < \mu < 1 . \end{cases} \tag{A106}$$

D. The Flux and Current Integrals

Since we are interested in obtaining simplified expressions for

$$\Phi(0,\tau) = \int_{-1}^{+1} \Psi(0,\mu,\tau)d\mu, \text{ and } J(0,\tau) = \int_{-1}^{+1} \mu\, \Psi(0,\mu,\tau)d\mu\ , \qquad \text{(A107)}$$

it is necessary to evaluate

$$I_n(0,\tau) = \int_{-1}^{+1} I_n(\overline{\Psi}_s(0,\mu))d\mu\ , \quad n = 1,2,3,4\ , \qquad \text{(A108)}$$

and

$$J_n(0,\tau) = \int_{-1}^{+1} \mu\, I_n(\overline{\Psi}_s(0,\mu))d\mu, \quad n = 1,2,3,4\ . \qquad \text{(A109)}$$

The $I_n(0,\tau)$ integrations can be done by two alternative methods. Integrating Equation (A102) in a straightforward manner over μ, and making use of Equation (A81) yields

$$I_1(0,\tau) = 2iq \int_{-\sigma_1}^{-a_1\sigma_1 1(a_1\sigma_1-\sigma_2)-\sigma_2 1(\sigma_2-a_1\sigma_1)} \frac{-b_1 b_2}{b_1-b_2}\, \frac{e^{s\tau}ds}{\xi}\ . \qquad \text{(A110)}$$

Alternatively, we can start with $I_1(\overline{\Psi}_s(0+,\mu))$ given by Equation (120), integrate this over μ (which merely replaces $\varphi_{2\nu}(\mu)$ by $s+\sigma_2$), and then evaluate the integral over ν. This integral over ν is defined as

$$I_+' \equiv \int_0^1 \frac{d\nu(s+\sigma_2)}{(\nu^2+\xi^2)X_b^-(\nu)(\lambda_2(\nu)+i\pi\nu)}\ . \qquad \text{(A111)}$$

By inspection then

$$I_+' = (s+\sigma_2)I_+(0)\ , \qquad \text{(A112)}$$

where $I_+(0)$ is given by Equation (A96). Hence we arrive again at Equation (A110) as a final result but by an alternative method.

Applying either of these methods to the remaining $I_n(0,\tau)$ integrals [(A104), (A105), (A106)] yields

$$I_2(0,\tau) = 2iql(\sigma_2 - a_1\sigma_1)\int_{-\sigma_2}^{-a_1\sigma_1 l(a_1\sigma_1 - a_2\sigma_2) - a_2\sigma_2 l(a_2\sigma_2 - a_1\sigma_1)} \frac{b_1 b_2}{b_1 - b_2}\,\frac{\xi - \eta}{\xi\eta}\, e^{s\tau} ds \,, \tag{A113}$$

$$I_3(0,\tau) = 2iql(a_1\sigma_1 - a_2\sigma_2)\int_{-\sigma_2 l(a_1\sigma_1 - \sigma_2) - a_1\sigma_1 l(\sigma_2 - a_1\sigma_1)}^{-a_2\sigma_2} \frac{b_1 b_2}{b_1 - b_2}\,\frac{e^{s\tau} ds}{\eta} \,, \tag{A114}$$

$$I_4(0,\tau) = 2iql(a_2\sigma_2 - a_1\sigma_1)\int_{-a_2\sigma_2}^{-a_1\sigma_1} \frac{b_1 b_2}{b_2 - b_1}\,\frac{e^{s\tau} ds}{\xi} \,. \tag{A115}$$

The first of the two methods described above was used to obtain the currents in Equation (A109). The results are listed below, where we again make use of Equations (A102), (A104), (A105), (A106).

$$J_1(0,\tau) = 2iq\int_{-\sigma_1}^{-a_1\sigma_1 l(a_1\sigma_1 - \sigma_2) - \sigma_2 l(\sigma_2 - a_1\sigma_1)} \frac{\sqrt{-b_1 b_2}}{b_1 - b_2}\sqrt{(2-b_1)(2-b_2)}\, e^{s\tau} ds \,. \tag{A116}$$

$$J_2(0,\tau) = 0 \,. \tag{A117}$$

$$J_3(0,\tau) = 2iql(a_1\sigma_1 - a_2\sigma_2)\int_{-\sigma_2 l(a_1\sigma_1 - \sigma_2) - a_1\sigma_1 l(\sigma_2 - a_1\sigma_1)}^{-a_2\sigma_2} \frac{\sqrt{b_1 b_2}}{b_2 - b_1}\sqrt{(b_1-2)(2-b_2)}\, e^{s\tau} ds \,. \tag{A118}$$

$$J_4(0,\tau) = 2iql(a_2\sigma_2-a_1\sigma_1)\int_{-a_2\sigma_2}^{-a_1\sigma_1} \frac{\sqrt{b_1b_2}}{b_2-b_1}\sqrt{(2-b_1)(b_2-2)}\, e^{s\tau}ds. \quad (A119)$$

APPENDIX VI – DIFFUSION AND P-1 THEORY

In this appendix the two-region, time-dependent problem is solved in an approximate manner by the use of Legendre polynomials and by diffusion theory.

A. Legendre Polynomial Expansion Method

The basic equation, source, and boundary conditions, as given by Equations (38) and (39a,b), are written below.

$$\frac{\partial \Psi(x,\mu,\tau)}{\partial \tau} + \mu \frac{\partial \Psi}{\partial x} + \sigma(x)\Psi = \frac{c(x)\sigma(x)}{2} \int_{-1}^{+1} \Psi(x,\mu',\tau)d\mu' + q\delta(x)\delta(\tau) \quad . \tag{A120}$$

$$\lim_{x \to \pm\infty} \Psi(x,\mu,\tau) = 0 \;, \quad \text{for } \tau \geqq 0. \tag{A121}$$

$$\mu[\Psi(0+,\mu,\tau) - \Psi(0-,\mu,\tau)] = q\delta(\tau) \; . \tag{A122}$$

By multiplying Equation (A120) by $e^{-s\tau}$ and integrating over τ from zero to infinity we obtain

$$\mu \frac{\partial \overline{\Psi}_s(x,\mu)}{\partial x} + (s + \sigma(x))\overline{\Psi}_s = \frac{c(x)\sigma(x)}{2} \int_{-1}^{+1} \overline{\Psi}_s(x,\mu')d\mu' + q\delta(x) \; . \tag{A123}$$

Now the assumption is made that $\overline{\Psi}_s(x,\mu)$ can be written as follows:

$$\overline{\Psi}_s(x,\mu) = \sum_{n=0}^{\infty} \left(\frac{2n+1}{2}\right) \overline{\varphi}_n(x,s)\, P_n(\mu) \quad . \tag{A124}$$

Substituting this expression into Equation (A123) and recalling that

$$\mu P_n(\mu) = \frac{n+1}{2n+1} P_{n+1}(\mu) + \frac{n}{2n+1} P_{n-1}(\mu), \; n = 0, 1, 2, \ldots, \tag{A125}$$

provides for the following change in form of Equation (A123):

$$\sum_{n=0}^{\infty} \frac{(2n+1)}{2} P_n(\mu) \left\{ \frac{1}{2n+1} \quad \frac{n d\bar{\varphi}_n(x,s)}{dx} + (n+1) \frac{d\bar{\varphi}_{n+1}(x,s)}{dx} \right.$$

$$\left. + (s+\sigma(x)) \bar{\varphi}_n(x,s) \right\} = \frac{c(x)\sigma(x)}{2} \bar{\varphi}_0(x,s) + q\,\delta(x) \; . \tag{A126}$$

Multiplying Equation (A126) by $P_0(\mu)$, $P_1(\mu)$, . . . , and integrating from -1 to +1 over μ after each multiplication yields the following infinite set of coupled differential equations:

$$\left.\begin{array}{l} \dfrac{d\bar{\varphi}_1(x,s)}{dx} + [s+\sigma(x)]\; \bar{\varphi}_0(x,s) = c(x)\sigma(x)\bar{\varphi}_0(x,s) + 2q\delta(x) \\ \dfrac{1}{3}\; \dfrac{d\bar{\varphi}_0(x,s)}{dx} + \dfrac{2}{3}\; \dfrac{d\bar{\varphi}_2(x,s)}{dx} + [s+\sigma(x)]\; \bar{\varphi}_1(x,s) = 0 \\ \qquad\qquad\qquad \cdot \\ \qquad\qquad\qquad \cdot \\ \qquad\qquad\qquad \cdot \end{array}\right\} \tag{A127}$$

From an inspection of Equations (A127) and the original boundary conditions it can be seen that the boundary conditions (and source condition) become, for Equation (A127),

$$\left.\begin{array}{l} \lim\limits_{x \to \pm\infty} \bar{\varphi}_n(x,s) = 0 \quad , \quad n = 0, 1, 2, \ldots, \\ \bar{\varphi}_n(0-,s) = \bar{\varphi}_n(0+,s) \quad , \quad n \neq 1, \\ \bar{\varphi}_1(0+,s) - \bar{\varphi}_1(0-,s) = 2q \quad . \end{array}\right\} \tag{A128}$$

It is usually assumed that a finite number of terms in the expansion of the angular flux, Equation (A124), will accurately represent the true angular neutron distribution. With this in mind we use as our approximation, only the first two terms in the expansion (P-1 approximation). Hence we assume

$$\left.\begin{aligned} &\frac{d\overline{\varphi}_1(x,s)}{dx} + [s+\sigma(x)]\ \overline{\varphi}_0(x,s) = c(x)\sigma(x)\overline{\varphi}_0(x,s) \qquad x \neq 0, \\ &\frac{1}{3}\ \frac{d\overline{\varphi}_0(x,s)}{dx} + [s+\sigma(x)]\ \overline{\varphi}_1(x,s) = 0 \quad , \end{aligned}\right\} \qquad (A129)$$

where

$$\left.\begin{aligned} &\lim_{x \to \pm\infty} \overline{\varphi}_{0,1}(x,s) = 0 \\ &\overline{\varphi}_0(0+,s) = \overline{\varphi}_0(0-,s) \\ &\overline{\varphi}_1(0+,s) - \overline{\varphi}_1(0-,s) = 2q \quad . \end{aligned}\right\} \qquad (A130)$$

If Equation (A124) is integrated over μ, one obtains

$$\overline{\varphi}_0(x,s) = \int_{-1}^{+1} \Psi_s(x,\mu)d\mu \ , \qquad (A131)$$

implying that $\overline{\varphi}_0(x,s)$ is the transformed flux.

Let us obtain an explicit expression for this quantity through the solution of Equations (A129). The details are omitted and only the result exhibited because of the simplicity of the calculations; thus

$$\overline{\varphi}_0(x,s) = \frac{2q}{\sqrt{\frac{s+a_2\sigma_2}{3(s+\sigma_2)}} + \sqrt{\frac{s+a_1\sigma_1}{3(s+\sigma_1)}}} \begin{cases} e^{-\sqrt{3(s+\sigma_2)(s+a_2\sigma_2)}\,x}, & x > 0 \\ e^{+\sqrt{3(s+\sigma_1)(s+a_1\sigma_1)}\,x} & x < 0 \,. \end{cases} \qquad (A132)$$

To recover the time dependence we use

$$\Phi_0(x,\tau) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \overline{\varphi}_0(x,s)\, e^{s\tau}\, ds \,. \qquad \text{(A133)}$$

First let us look at the inversion in detail for $x > 0$. An examination of $\overline{\varphi}_0(x,s)$ reveals the existence of two branches along the real axis in the s-plane, from $-\sigma_1$ to $-a_1\sigma_1$ and from $-\sigma_2$ to $-a_2\sigma_2$ (identical to those found in the exact solution). The function $\overline{\varphi}_0(x,s)$ is analytic everywhere else. The behavior of $\overline{\varphi}_0(x,s)$ is found to be, for large s,

$$\overline{\varphi}_0(x,s) \sim \sqrt{3}\, q\, e^{-\sqrt{3}\, x \frac{(\sigma 2 + a 2\, \sigma 2)}{2}}\, e^{-\sqrt{3}\, xs} \equiv \overline{\varphi}_{unc}(x,s), \qquad \text{(A134)}$$

which we have suitably defined as $\overline{\varphi}_{unc}(x,s)$.

By inspection,

$$\Phi_{unc}(x,\tau) = \sqrt{3}\, q\, e^{-\sqrt{3}\, x \frac{\sigma_2 + a_2\sigma_2}{2}}\, \delta(\tau - \sqrt{3}\, x) \,. \qquad \text{(A135)}$$

Equation (A133) can now be written as

$$\Phi_0(x,\tau) = \Phi_{unc}(x,\tau) + \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{s\tau}\, [\overline{\varphi}_0(x,s) - \overline{\varphi}_{unc}(x,s)]\, ds. \qquad \text{(A136)}$$

But since $\overline{\varphi}_0(x,s) - \overline{\varphi}_{unc}(x,s) = 0(\frac{1}{s})$ for large s, the original inversion path can be changed, as shown in Figure 31. Of course this figure depicts only one special case ($\sigma_1 > \sigma_2 > a_1\sigma_1 > a_2\sigma_2$). However, with this as a guide the general solution can be written down.

Many simplifications can be noted in the general solution to this problem, some of which parallel those made in the exact solution. They are listed below.

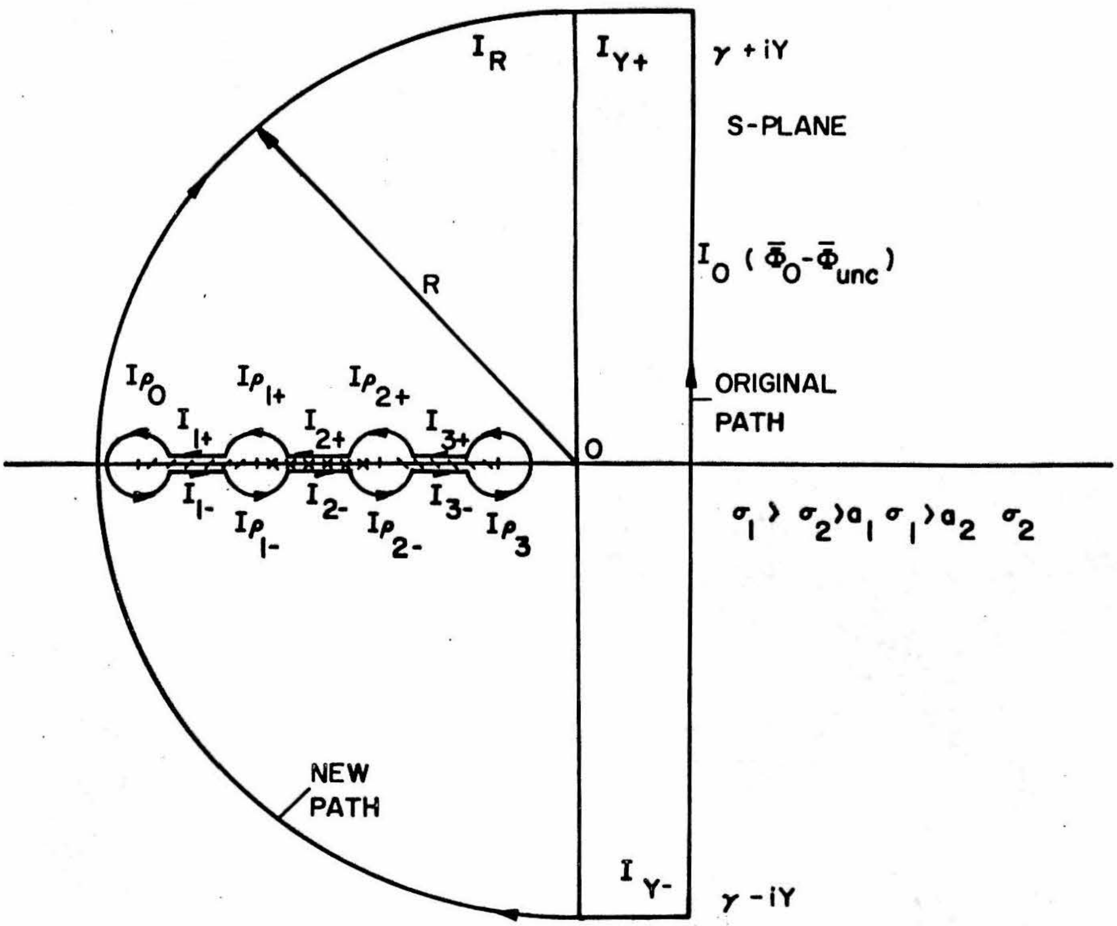

Figure 31 - Typical Path of Inversion for P-1 Solution

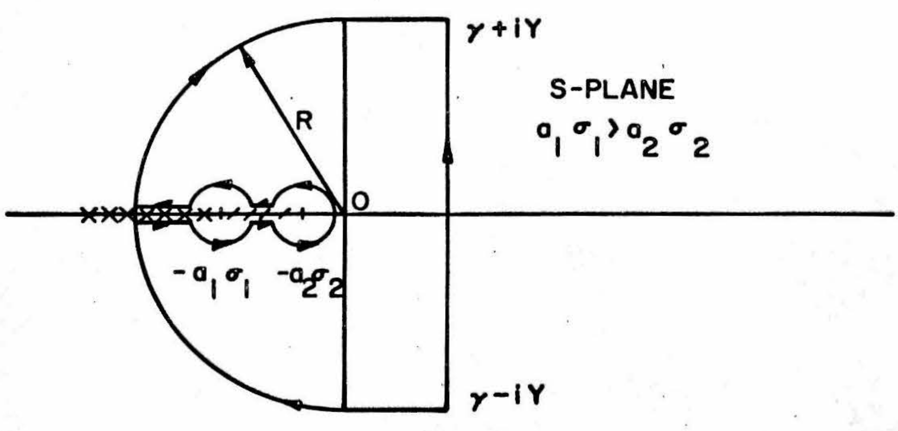

Figure 32 - Inversion Path for Diffusion Solution

1. The solution exists only for $\tau > \sqrt{3}\, x$; for $\tau < \sqrt{3}\, x$, the flux is zero.

2. The integrals I_{Y-}, I_{Y+}, and I_R vanish.

3. The integrals $I_{\rho_{0+}}, \ldots, I_{\rho_3}$ vanish.

4. $I_{1-}(\overline{\varphi}_0 - \overline{\varphi}_{unc}) + I_{1+}(\overline{\varphi}_0 - \overline{\varphi}_{unc}) = I_{1-}(\overline{\varphi}_0) - I_{1+}(\overline{\varphi}_0) \equiv I_1(\overline{\varphi}_0)$, etc., since the uncollided flux is continuous across the branches.

We can now write the general solution for the flux as

$$\Phi_0(x,\tau) = \Phi_{unc}(x,\tau) + \frac{1}{2\pi i}\left[I_1(\overline{\varphi}_0) + I_2(\overline{\varphi}_0) + I_3(\overline{\varphi}_0) + I_4(\overline{\varphi}_0)\right] 1(\tau - \sqrt{3}\,x), \; (x > 0), \tag{A137}$$

where the explicit forms for $I_1, \ldots, I_4$ are given below.

$$I_1(\overline{\varphi}_0) = 4iq\sqrt{3} \int_{-\sigma_1}^{-a_1\sigma_1 1(a_1\sigma_1 - \sigma_2) - \sigma_2 1(\sigma_2 - a_1\sigma_1)} \mathrm{Im}\left\{ \frac{e^{s\tau}\, e^{\sqrt{3}\,x\sqrt{(-s-\sigma_2)(-s-a_2\sigma_2)}}}{\sqrt{\dfrac{-s-a_2\sigma_2}{-s-\sigma_2}} - i\sqrt{\dfrac{-s-a_1\sigma_1}{s+\sigma_1}}} \right\} ds\,. \tag{A138}$$

$$I_2(\overline{\varphi}_0) = 4iq\sqrt{3}\; 1\,(\sigma_2 - a_1\sigma_1) \int_{-\sigma_2}^{-a_1\sigma_1 1(a_1\sigma_1 - a_2\sigma_2) - a_2\sigma_2 1(a_2\sigma_2 - a_1\sigma_1)} \left\{ \frac{e^{s\tau}\, ds}{\sqrt{\dfrac{-s-a_2\sigma_2}{s+\sigma_2}} + \sqrt{\dfrac{-s-a_1\sigma_1}{s+\sigma_1}}} \right\}$$

$$\left\{ \cos\left\{ \sqrt{3}\, x\, \sqrt{(s+\sigma_2)(-s-a_2\sigma_2)} \right\} \right\}. \tag{A139}$$

$$I_3(\overline{\varphi}_0)=4i\sqrt{3}\,q\,1(a_1\sigma_1-a_2\sigma_2)\int_{-\sigma_2 1(a_1\sigma_1-\sigma_2)-a_1\sigma_1 1(\sigma_2-a_1\sigma_1)}^{-a_2\sigma_2} \mathrm{Im}\left\{\frac{e^{s\tau}\, e^{i\sqrt{3}x\sqrt{(s+\sigma_2)(-s-a_2\sigma_2)}}}{\sqrt{\frac{s+a_1\sigma_1}{s+\sigma_1}}-i\sqrt{\frac{-s-a_2\sigma_2}{s+\sigma_2}}}\right\} ds. \quad (A140)$$

$$I_4(\overline{\varphi}_0)=4i\sqrt{3}\,q\,1(a_2\sigma_2-a_1\sigma_1)\int_{-a_2\sigma_2}^{-a_1\sigma_1} e^{s\tau}\, e^{-\sqrt{3}x\sqrt{(s+\sigma_2)(s+a_2\sigma_2)}}\, ds \quad \mathrm{Im}\left\{\frac{1}{\sqrt{\frac{s+a_2\sigma_2}{s+\sigma_2}}-i\sqrt{\frac{-s-a_1\sigma_1}{s+\sigma_1}}}\right\}.$$

(A141)

For $x < 0$, the details of the inversion are similar to those mentioned above. The final form of the solution is written out below.

$$\Phi_0(x,\tau) = \Phi_{unc}(x,\tau)+\frac{1}{2\pi i}[I_1(\overline{\varphi}_0)+I_2(\overline{\varphi}_0)+I_3(\overline{\varphi}_0)+I_4(\overline{\varphi}_0)]\,1(\tau+\sqrt{3}x),\ x<0.$$

(A142)

$$\Phi_{unc}(x,\tau) = \sqrt{3}\,q\,e^{\sqrt{3}x\frac{\sigma_1+a_1\sigma_1}{2}}\,\delta(\tau+\sqrt{3}\,x). \quad (A143)$$

$$I_1(\overline{\varphi}_0) = 4iq\sqrt{3}\int_{-\sigma_1}^{-a_1\sigma_1(a_1\sigma_1-\sigma_2)-\sigma_2 1(\sigma_2-a_1\sigma_1)} e^{s\tau}\, ds\ \mathrm{Im}\left\{\frac{e^{-i\sqrt{3}x\sqrt{(s+\sigma_1)(-s-a_1\sigma_1)}}}{\sqrt{\frac{-s-a_2\sigma_2}{-s-\sigma_2}}-i\sqrt{\frac{-s-a_1\sigma_1}{s+\sigma_1}}}\right\}. \quad (A144)$$

$$I_2(\bar{\varphi}_0) = 4iq\sqrt{3}\ 1(\sigma_2 - a_1\sigma_1) \int_{-\sigma_2}^{-a_1\sigma_1 1(a_1\sigma_1 - a_2\sigma_2) - a_2\sigma_2 1(a_2\sigma_2 - a_1\sigma_1)} \frac{e^{s\tau}\, ds \cos\left\{\sqrt{3}\, x\, \sqrt{(s+\sigma_1)(-s-a_1\sigma_1)}\right\}}{\sqrt{\frac{-s-a_2\sigma_2}{s+\sigma_2}} + \sqrt{\frac{-s-a_1\sigma_1}{s+\sigma_1}}} . \quad (A145)$$

$$I_3(\bar{\varphi}_0) = 4iq\sqrt{3}\, 1(a_1\sigma_1 - a_2\sigma_2) \int_{-\sigma_2 1(a_1\sigma_1 - \sigma_2) - a_1\sigma_1 1(\sigma_2 - a_1\sigma_1)}^{-a_2\sigma_2} e^{s\tau}\, ds\, e^{\sqrt{3}\, x \sqrt{(s+\sigma_1)(s+a_1\sigma_1)}}\ \mathrm{Im}\left\{ \frac{1}{\sqrt{\frac{s+a_1\sigma_1}{s+\sigma_1}} - i\sqrt{\frac{-s-a_2\sigma_2}{s+\sigma_2}}} \right\} .$$

(A146)

$$I_4(\bar{\varphi}_0) = 4iq\sqrt{3}\ 1(a_2\sigma_2 - a_1\sigma_1) \int_{-a_2\sigma_2}^{-a_1\sigma_1} e^{s\tau}\, ds\ \mathrm{Im}\left\{ \frac{e^{-i\sqrt{3}x\sqrt{(s+\sigma_1)(-s-a_1\sigma_1)}}}{\sqrt{\frac{s+a_2\sigma_2}{s+\sigma_2}} - i\sqrt{\frac{-s-a_1\sigma_1}{s+\sigma_1}}} \right\} .$$

(A147)

Further slight simplifications can be made at $x = 0$. However, it is of more interest to this thesis to obtain the results when the two adjacent media are identical so that they can be compared to the exact solution for the single infinite medium. Setting $\sigma_1 = \sigma_2 = \sigma$, and $a_1\sigma_1 = a_2\sigma_2 = a\sigma$, we obtain, for all x,

$$\Phi_0(x, \tau) = \Phi_{unc}(x, \tau) + \frac{1}{2\pi i}\ I_2^1\,(\bar{\varphi}_0)\ 1\,(\tau - \sqrt{3}\,|x|), \quad (A148)$$

where

$$\Phi_{unc}(x, \tau) = q\,\sqrt{3}\,\delta\,(\tau - \sqrt{3}\,|x|)\, e^{-\sqrt{3}\,|x|\,\frac{\sigma + a\sigma}{2}} , \quad (A149)$$

and

$$I_2^1(\overline{\varphi}_0) = 2iq\sqrt{3}\int_{-\sigma}^{-a\sigma} e^{s\tau}\sqrt{\frac{s+\sigma}{-s-a\sigma}}\,ds\,\cos\left\{\sqrt{3}\,|x|\sqrt{(s+\sigma)(-s-a\sigma)}\right\}. \quad (A150)$$

Finally, at x = 0, we obtain, for the single infinite medium,

$$\Phi_0(0,\tau) = \sqrt{3}\,q\,\delta(\tau) + \frac{\sqrt{3}q}{\pi}\int_{-\sigma}^{-a\sigma} e^{s\tau}\sqrt{\frac{s+\sigma}{-s-a\sigma}}\,ds \quad . \quad (A151)$$

B. Diffusion Theory

Consider Equation (A120) with an arbitrary source,

$$\frac{\partial\Psi(x,\mu,\tau)}{\partial\tau} + \mu\frac{\partial\Psi}{\partial x} + \sigma(x)\Psi = \frac{c(x)\sigma(x)}{2}\int_{-1}^{+1}\Psi(x,\mu',\tau)d\mu' + S(x,\tau)\,,$$

and assume (A152)

$$\Psi(x,\mu,\tau) = \sum_{n=0}^{\infty}\varphi_n(x,\tau)P_n(\mu)\left(\frac{2n+1}{2}\right).$$

When this series is terminated at n = 1, Equation (A152) can be written as two coupled equations:

$$\frac{\partial\varphi_0(x,\tau)}{\partial\tau} + \frac{\partial\varphi_1}{\partial x} + \sigma(x)\varphi_0 = c(x)\sigma(x)\varphi_0 + 2S(x,\tau), \quad (A153a)$$

$$\frac{\partial\varphi_1}{\partial\tau} + \frac{1}{3}\frac{\partial\varphi_0}{\partial x} + \sigma(x)\varphi_1 = 0\,. \quad (A153b)$$

Differentiating Equation (A153a) with respect to τ, Equation (A153b) with respect to x (recalling that $\sigma(x)$ and c(x) are constant within a region), and subtracting the results yields, after some simplifications,

$$\frac{\partial^2\varphi_0}{\partial\tau^2} - \frac{1}{3}\,\frac{\partial^2\varphi_0}{\partial x^2} + \sigma(1+a)\,\frac{\partial\varphi_0}{\partial\tau} + a\,\sigma^2\varphi_0 = 2\sigma S + 2\frac{\partial S}{\partial\tau}\,. \qquad \text{(A154)}$$

Recalling that $\tau = vt$, and that $\varphi_0(x,\tau) = vn(x,\tau)$, we can write Equation (A154) as

$$-\frac{v}{3\sigma}\,\frac{\partial^2 n(x,t)}{\partial x^2} + av\sigma n = -\frac{1}{v\sigma}\,\frac{\partial^2 n}{\partial t^2} - (1+a)\,\frac{\partial n}{\partial t} + 2S + \frac{2}{v\sigma}\,\frac{\partial S}{\partial t}\,. \qquad \text{(A155)}$$

To reduce this to the diffusion equation it is assumed that (see, for instance, Meghreblian and Holmes[26])

$$\lim_{v\to\infty} \frac{v}{\sigma} \qquad \text{is finite,}$$

$$\text{and} \quad \lim_{v\to\infty} a\sigma v \qquad \text{is finite.}$$

Hence Equation (A155) reduces to

$$-\frac{1}{3\sigma}\,\frac{\partial^2[vn(x,t)]}{\partial x^2} + a\sigma vn = -\frac{\partial n}{\partial t} + 2S\,, \qquad \text{(A156)}$$

or in a more familiar notation, with the source noted explicitly,

$$-D(x)\,\frac{\partial^2\Phi(x,\tau)}{\partial x^2} + a(x)\sigma(x)\Phi = -\frac{\partial\Phi}{\partial\tau} + 2q\,\delta(x)\delta(\tau). \qquad \text{(A157)}$$

Here $D = \frac{1}{3\sigma(x)}$, and the geometry is again that of two-adjacent half spaces.

Let us now solve Equation (A157). Removing the time dependence by a Laplace transformation yields

$$-D(x)\,\frac{d^2\bar{\varphi}(x,s)}{dx^2} + (a(x)\sigma(x)+s)\bar{\varphi}(x,s) = 2q\delta(x)\,. \qquad \text{(A158)}$$

The boundary and source conditions can be written as

$$\lim_{x \to \pm\infty} \overline{\varphi}(x, s) = 0 , \tag{A159}$$

$$D_1 \frac{d\overline{\varphi}(0-, s)}{dx} - D_2 \frac{d\overline{\varphi}(0+, s)}{dx} = 2q . \tag{A160}$$

Solving Equation (A158) subject to conditions (A159) and (A160) provides us with

$$\overline{\varphi}(x, s) = \frac{2q}{\sqrt{D_1(s+a_1\sigma_1)}\ \sqrt{D_2(s+a_2\sigma_2)}} e^{+\sqrt{\frac{s+a_1\sigma_1}{D_1}}x} \qquad x < 0, \tag{A161}$$

$$\overline{\varphi}(x, s) = \frac{2q}{\sqrt{D_1(s+a_1\sigma_1)}\ \sqrt{D_2(s+a_2\sigma_2)}} e^{-\sqrt{\frac{s+a_2\sigma_2}{D_2}}x} \qquad x > 0 . \tag{A162}$$

To invert when $x < 0$, we note that $-a_1\sigma_1$ and $-a_2\sigma_2$ are branch points, and we assume that $a_1\sigma_1 > a_2\sigma_2$. Hence, we can write, based on Figure 32,

$$\Phi(x, \tau) = \frac{2q}{\pi}\left[\int_{-\infty}^{-a_1\sigma_1} \frac{e^{s\tau} \cos\left\{\sqrt{-s-a_1\sigma_1}\ \frac{x}{\sqrt{D_1}}\right\}}{\sqrt{D_1(-s-a_1\sigma_1)} + \sqrt{D_2(-s-a_2\sigma_2)}}\, ds \right.$$

$$\left. -\mathrm{Im}\left\{\int_{-a_1\sigma_1}^{-a_2\sigma_2} \frac{e^{s\tau}\, e^{\sqrt{s+a_1\sigma_1}\ \frac{x}{\sqrt{D_1}}}\, ds}{\sqrt{D_1(s+a_1\sigma_1)} + i\sqrt{D_2(-s-a_2\sigma_2)}}\right\}\right], \quad x < 0. \tag{A163}$$

Similarly,

$$\Phi(x,\tau) = \frac{2q}{\pi}\left[\int_{-\infty}^{-a_1\sigma_1} \frac{e^{s\tau}\cos\left\{\sqrt{-s-a_2\sigma_2}\frac{x}{\sqrt{D_2}}\right\}ds}{\sqrt{D_1(-s-a_1\sigma_1)}+\sqrt{D_2(-s-a_2\sigma_2)}}\right.$$

$$\left.-\mathrm{Im}\left\{\int_{-a_1\sigma_1}^{-a_2\sigma_2} \frac{e^{s\tau}\, e^{-\frac{+i x\sqrt{-s-a_2\sigma_2}}{\sqrt{D_2}}}\, ds}{\sqrt{D_1(s+a_1\sigma_1)}+i\sqrt{D_2(-s-a_2\sigma_2)}}\right\}\right],\quad x>0\,. \tag{A164}$$

A simplification of these results occurs for the special case of $a_1\sigma_1 = a_2\sigma_2$; then

$$\Phi(x,\tau) = \frac{2q\,e^{-a\sigma\tau}}{\sqrt{\pi\tau}\,(\sqrt{D_1}+\sqrt{D_2})}\begin{cases} e^{-\frac{x^2}{4D_2\tau}}, & x>0\,,\\ e^{-\frac{x^2}{4D_1\tau}}, & x<0\end{cases}\;. \tag{A165}$$

Furthermore, we note that

$$\int_0^\infty \Phi(x,\tau)\,d\tau = \lim_{s\to 0}\overline{\varphi}(x,s)\;. \tag{A166}$$

Hence,

$$\int_0^\infty \Phi(x,\tau)d\tau = \frac{2q}{\sqrt{a_1\sigma_1 D_1}+\sqrt{a_2\sigma_2 D_2}}\begin{cases} e^{\sqrt{\frac{a_1\sigma_1}{D_1}}\,x}, & x<0\\ e^{-\sqrt{\frac{a_2\sigma_2}{D_2}}\,x}, & x>0\end{cases}\;. \tag{A167}$$

In addition, using the same technique in P-1 theory yields

$$\int_0^\infty \Phi_0(x,\tau)d\tau = \frac{2q}{\sqrt{\frac{a_2}{3}} + \sqrt{\frac{a_1}{3}}} \begin{cases} e^{-\sqrt{3a_2}\,\sigma_2 x}, & x > 0 \\ e^{+\sqrt{3a_1}\,\sigma_1 x}, & x < 0 \;. \end{cases} \tag{A168}$$

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