ON CERTAIN DISCRETE INEQUALITIES AND THEIR CONTINUOUS ANALOGUES

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Abstract

In a 1955 paper, Ky Fan, Olga Taussky, and John Todd presented discrete analogues of inequalities of Wirtinger type, and by taking limits they were able to recover the continuous inequalities. We generalize their techniques to mixed and higher derivatives and inequalities with weight functions in the integrals. We have also considered analogues of inequalities of Müller and Redheffer and have used these inequalities to derive a necessary and sufficient condition on ordered pairs of numbers so that the first number is the square norm of the k^{th} derivative of some periodic function and the second number is the square norm of the m^{th} derivative of the same periodic function.

Introduction

In 1955 [7] Ky Fan, Olga Taussky, and John Todd discovered discrete analogues of certain integral inequalities involving functions and their derivatives. They considered inequalities of the Wirtinger type: $\int_0^{2\pi} x(t)^2 dt \leq \int_0^{2\pi} x'(t)^2 dt$, where x has period 2π and $\int_0^{2\pi} x(t) dt = 0$. By taking limits they were able to derive continuous inequalities by matrix techniques and avoided the differential equations of the calculus of variations. At the suggestion of Professor Todd we have attempted to generalize the techniques of [7] to polynomials in the derivatives of x and inequalities with weight functions in the integrals. We have also considered analogues of inequalities of Müller [11] and Redheffer [13] and have used these inequalities to derive a necessary and sufficient condition on ordered pairs of numbers so that the first number is the square norm of the k^{th} derivative of some periodic function and the second number is the square norm of the m^{th} derivative of the same periodic function. This last result is the L_2 analogue of a result of Kolomogoroff [10] on the uniform norm.

I. General Technique

The following is the basic technique of Fan, Taussky, and Todd [7] and will be used to derive most of our inequalities. Let S be a real symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and let $v_1, v_2, \cdots, v_n$ be the corresponding linearly independent eigenvectors:

$$Sv_i = \lambda_i v_i \quad (i=1,2,\cdots,n) \; .$$

If P is a polynomial with real coefficients, then P(S) is a real symmetric $n \times n$ matrix and we know that P(S) has eigenvalues $P(\lambda_1), P(\lambda_2), \cdots, P(\lambda_n)$ with corresponding eigenvectors $v_1, v_2, \cdots, v_n$ so that $P(S)v_i = P(\lambda_i)v_i$ $(i=1,2,\cdots,n)$.

By a well-known property of symmetric matrices

$$\left[\min_{1\leq i\leq n}\{P(\lambda_i)\}\right] \cdot (x,x) \leq (x,P(S)x) \leq \left[\max_{1\leq i\leq n}\{P(\lambda_i)\}\right] \cdot (x,x)$$

for every n-vector x, where $(x,y) = x^T y$ for n-vectors x and y.

Now we must have

$$\inf_{\lambda_n \leq t \leq \lambda_1}\{P(t)\} \leq \min_{1\leq i\leq n}\{P(\lambda_i)\}$$

and

$$\sup_{\lambda_n \leq t \leq \lambda_1}\{P(t)\} \geq \max_{1\leq i\leq n}\{P(\lambda_i)\},$$

hence

$$\left[\inf_{\lambda_n \leq t \leq \lambda_1}\{P(t)\}\right] \cdot (x,x) \leq (x,P(S)x) \leq \left[\sup_{\lambda_n \leq t \leq \lambda_1}\{P(t)\}\right] \cdot (x,x)$$

for every n-vector x.

We note that equality is possible in the last expression if and only if $P(\lambda_i) = \sup_{\lambda_n \leq t \leq \lambda_1} \{P(t)\}$ for some i between 1 and n or $P(\lambda_i) = \inf_{\lambda_n \leq t \leq \lambda_1} \{P(t)\}$ for some i between 1 and n. If equality does occur then the set of extremal vectors must be the space spanned by the eigenvectors corresponding to λ_i .

For reasons of simplicity we will only consider vectors with real components and real symmetric matrices. In all cases the extension to complex vectors and Hermitian matrices will be apparent.

II. Periodic Boundary Conditions

Because of the absence of troublesome boundary conditions the periodic case is easiest to handle. We may periodically extend any n-vector $x = (x_1, x_2, \cdots, x_n)^T$ be setting $x_{i+rn} = x_i$ for $i = 1,2,\cdots,n$ and r any integer.

<u>Definition</u>: If x is a periodically extended n-vector then for $m = 0,1,2,\cdots$

$$x^{(m)} = (\Delta^m x_1, \Delta^m x_2, \cdots, \Delta^m x_n)^T,$$

where

$$\Delta^m x_i = \sum_{r=0}^{m} (-1)^{m-r} \binom{m}{r} x_{i-[m/2]+r} \quad (i=1,2,\cdots,n).$$

We call $x^{(m)}$ the m^{th} difference of the n-vector x.

It is clear that $x^{(r+s)} = (x^{(r)})^{(s)}$ for $r,s = 0,1,2,\cdots$.

The following lemma, which is crucial for our inequalities, was proved by Fan-Taussky-Todd [17] for $m = 0,1,2$.

<u>Lemma 1</u>. If x is a periodically extended n-vector, then $(x^{(m)}, x^{(m)}) = (x, P^m x)$ $(m=0,1,2,\cdots)$, where P is the $n \times n$ symmetric circulant

$$\begin{bmatrix} 2 & -1 & 0 & \cdots & -1 \\ -1 & 2 & -1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ -1 & 0 & \cdots & -1 & 2 \end{bmatrix}.$$

Proof:

$$Px = \begin{bmatrix} -x_n + 2x_1 - x_2 \\ \vdots \\ -x_1 + 2x_n - x_{n-1} \end{bmatrix} = -x^{(2)}, \text{ hence } Px^{(j)} = -x^{(j+2)}.$$

By partial summation

$$(x,x^{(2)}) = \sum_{k=1}^{n} x_k(x_{k-1}-2x_k+x_{k+1}) = -\sum_{k=1}^{n}(x_{k+1}-x_k)^2$$

$$= -(x^{(1)},x^{(1)}),$$

yielding

$$(x^{(j)},Px^{(j)}) = (x^{(j+1)},x^{(j+1)}).$$

Thus if m is even we have

$$(x^{(m)},x^{(m)}) = (P^{\frac{m}{2}}x,P^{\frac{m}{2}}x) = (x,P^m x),$$

while if m is odd

$$(x^{(m)},x^{(m)}) = (x^{(m-1)},Px^{(m-1)}) = (P^{\frac{m-1}{2}}x,P \cdot P^{\frac{m-1}{2}}x) = (x,P^m x).$$

Rutherford [14] has shown that P has eigenvalues $4\sin^2(\frac{k\pi}{n})$ $(k=1,2,\cdots,n)$, hence $\lambda_n = 0$, $\lambda_{n-1} = \lambda_{n-2} = 4\sin^2(\frac{\pi}{n})$, $\cdots,\lambda_1 = 4\sin^2([\frac{n}{2}]\frac{\pi}{n})$. The eigenvector corresponding to λ_n is $(1,1,\cdots,1)^T$.

At this point we have two alternatives open to us. If F is a polynomial with real coefficients then we may use the method that we have described in Section I or else we may also add the auxiliary condition $\sum_{k=1}^{n} x_i = 0$, which is equivalent to the requirement $(x,e) = 0$, where $e = (1,1,\cdots,1)^T$. Since e is the eigenvector corresponding to $\lambda_n = 0$, the auxiliary orthogonality condition implies that we need only take our maximum and minimum over $\{F(\lambda_1),F(\lambda_2),\cdots,F(\lambda_{n-1})\}$.

The following example should illustrate the differences in the two approaches.

Let $P(t) = t^m$ for $m \geq 1$. Since P(t) is increasing for non-negative t and since $0 = \lambda_n < \lambda_{n-1} \leq \cdots \leq \lambda_1$, we have

$$\min_{1\leq i\leq n}\{P(\lambda_i)\} = 0 \text{ and } \min_{1\leq i\leq n-1}\{P(\lambda_i)\} = P(\lambda_{n-1}) = 4^m \sin^{2m}(\tfrac{\pi}{n}).$$

Thus if x is a periodically extended n-vector then we have the trivial inequality $(x^{(m)},x^{(m)}) \geq 0(x,x) = 0$; however if we add the auxiliary condition $\sum_{i=1}^{n} x_i = 0$, then we obtain

$$(x^{(m)},x^{(m)}) \geq 4^m \sin^{2m}(\tfrac{\pi}{n})(x,x).$$

We note that the above inequalities are best possible and we include a discussion of the possibility of equality. For brevity this will be the only case where we discuss equality in the discrete case.

If $(x^{(m)},x^{(m)}) = 0$, then x must be in the subspace spanned by the eigenvectors corresponding to $\lambda_n = 0$. Thus $x = a\cdot(1,1,\cdots,1)^T$, where a is a real number.

If $\sum_{i=1}^{n} x_i = 0$ and $(x^{(m)},x^{(m)}) = 4^m \sin^{2m}(\frac{\pi}{n})\cdot(x,x)$, then x must be in the subspace spanned by the eigenvectors corresponding to $\lambda_{n-1} = \lambda_{n-2} = 4\sin^2(\frac{\pi}{n})$. It is known, [7], that this subspace is spanned by the vectors $u = (u_1,u_2,\cdots,u_n)^T$ and $w = (w_1,w_2,\cdots,w_n)^T$, where $u_j = \cos(\frac{2\pi j}{n})$ and $w_j = \sin(\frac{2\pi j}{n})$. Thus $x = a\cdot u + b\cdot w$, where a and b are real numbers.

Hence we have established the following extension of Theorem 10 of [7].

Theorem 1. If x is a periodically extended n-vector and if $\sum_{i=1}^{n} x_i = 0$, then

$$(x^{(m)},x^{(m)}) \geq 4^m \sin^{2m}(\tfrac{\pi}{n}) \cdot (x,x).$$

Equality holds if and only if x is the periodic extension of a vector of the form $a\cdot u + b\cdot w$.

Corollary 1.1. If $x(t) \in C^m[a,b]$, x(t) has period b-a, and $\int_a^b x(t)dt = 0$, then

$$\int_a^b \{x^{(m)}(t)\}^2 dt \geq \left(\frac{2\pi}{b-a}\right)^{2m} \int_a^b \{x(t)\}^2 dt.$$

Proof:

If we let $\Delta = \frac{b-a}{n+1}$ and $y_i = x(t_i) - \frac{1}{n}\sum_{j=1}^{n} x(t_j)$, where $t_i = a + i\Delta$, then

$$\frac{(y^{(m)},y^{(m)})}{\Delta^{2m}} \cdot \Delta \geq \frac{4^m \sin^{2m}(\frac{\pi}{n})}{\Delta^{2m}} \cdot (y,y)\Delta .$$

The result now follows when we let $n \to \infty$.

A careful inspection of the proof of Theorem 1 will show that we only used the periodic extension property of the n-vector x on $x_{1-[\frac{m}{2}]}, x_{2-[\frac{m}{2}]}, \cdots, x_0$ and $x_{n+1}, x_{n+2}, \cdots, x_{n+m-[\frac{m}{2}]}$. Hence we may weaken our requirement $x(t)$ has period $b - a$ to a condition on the end points, namely $x(a) = x(b)$, $x'(a) = x'(b), \cdots, x^{(m-1)}(a) = x^{(m-1)}(b)$.

Halperin and Pitt [8], Müller [11], Nirenberg [12], and Redheffer [13] have developed inequalities of the form:

$$\int_a^b \{x'(t)\}^2 dt \leq \alpha \int_a^b \{x''(t)\}^2 dt + H(\alpha) \int_a^b \{x(t)\}^2 dt,$$

where $x(t) \in C^2[a,b]$, $\alpha > 0$. Redheffer obtained the best possible value of $H(\alpha)$. We will now develop generalizations of the form:

$$\int_a^b \{x^{(k)}(t)\}^2 dt \leq \alpha \int_a^b \{x^{(m)}(t)\}^2 dt + H_{k,m}(\alpha) \int_a^b \{x(t)\}^2 dt,$$

$1 \leq k < m$ under the further restriction that $x(a) = x(b)$, $x'(a) = x'(b), \cdots, x^{(m-1)}(a) = x^{(m-1)}(b)$. These inequalities will be established by taking limits of the appropriate discrete inequalities. Incidentally we observe that $H_{1,2}(\alpha)$ for our periodic case is much smaller than Redheffer's general value of $\frac{1}{\alpha} + \frac{12}{(b-a)^2}$.

This inequality will also appear in Sections IV and V, where it will serve as a model to illustrate our techniques.

If we let $P(t) = \theta^k t^k - \alpha\theta^m t^m$, then we wish to maximize $P(t)$ on $[\lambda_n,\lambda_1] = [0,4\sin^2([\frac{n}{2}]\frac{\pi}{n})]$. Since $P'(t) = k\theta^k t^{k-1} - m\alpha\theta^m t^{m-1}$, if we assume that $\theta > 0$ then the maximum value of $P(t)$ on $t \geq 0$ occurs at $t = \frac{1}{\theta}(\frac{k}{m\alpha})^{1/m-k}$. Thus $\sup_{\lambda_n \leq t \leq \lambda_1}\{P(t)\} \leq P(\frac{1}{\theta}\{\frac{k}{m\alpha}\}^{1/m-k}) = (\frac{k}{m\alpha})^{k/m-k}(1-\frac{k}{m})$.

An inspection of the graph of $P(t)$ will yield the following cases:

If $\lambda_1 \geq \frac{1}{\theta}(\frac{k}{m\alpha})^{1/m-k}$ then

$$\sup_{\lambda_n \leq t \leq \lambda_1}\{P(t)\} = P(\tfrac{1}{\theta}[\tfrac{k}{m\alpha}]^{1/m-k}) = (\tfrac{k}{m\alpha})^{k/m-k}(1-\tfrac{k}{m}).$$

If $\lambda_1 \leq \frac{1}{\theta}(\frac{k}{m\alpha})^{1/m-k}$ then

$$\sup_{\lambda_n \leq t \leq \lambda_1}\{P(t)\} = P(\lambda_1) = P(4\sin^2([\tfrac{n}{2}]\tfrac{\pi}{n}))$$

$$= 4^k\theta^k \sin^{2k}([\tfrac{n}{2}]\tfrac{\pi}{n})\{1-4^{m-k}\theta^{m-k}\alpha \sin^{2m-2k}([\tfrac{n}{2}]\tfrac{\pi}{n})\}.$$

If n is even then $P(\lambda_1) = 4^k\theta^k\{1-4^{m-k}\theta^{m-k}\alpha\}$.

Hence we have the following result.

<u>Theorem 2</u>. If x is a periodically extended n-vector, α and θ are positive constants, $1 \leq k < m$ then we have the following inequality:

$$\theta^k(x^{(k)},x^{(k)}) \leq \alpha\theta^m(x^{(m)},x^{(m)}) + (\tfrac{k}{m\alpha})^{k/m-k}(1-k/m)(x,x).$$

If $4 \leq \frac{1}{\theta}(\frac{k}{m\alpha})^{1/m-k}$, then we may improve the above inequality to:

$$\theta^k(x^{(k)},x^{(k)}) - \alpha\theta^m(x^{(m)},x^{(m)}) \leq 4^k\theta^k\{1-4^{m-k}\theta^{m-k}\alpha\}\cdot(x,x).$$

Proof:

We note that $4 \leq \frac{1}{\theta}(\frac{k}{m\alpha})^{1/m-k}$ certainly implies that $4\sin^2([\frac{n}{2}]\frac{\pi}{n}) \leq \frac{1}{\theta}(\frac{k}{m\alpha})^{1/m-k}$ and P(t) is increasing on the interval $[0,\frac{1}{\theta}(\frac{k}{m\alpha})^{1/m-k}]$.

If we take $\theta = (\frac{n+1}{b-a})^2$ and let $n \to \infty$ then we obtain a continuous analogue of Theorem 2.

Corollary 2.1. If $x(t) \in C^m[a,b]$, $1 \leq k < m$, $x(a) = x(b)$, $x'(a) = x'(b),\cdots,x^{(m-1)}(a) = x^{(m-1)}(b)$, and if $\alpha > 0$ then we have the following inequality:

$$\int_a^b \{x^{(k)}(t)\}^2 dt \leq \alpha\int_a^b \{x^{(m)}(t)\}^2 dt + (\frac{k}{m\alpha})^{k/m-k}(1-k/m)\int_a^b \{x(t)\}^2 dt.$$

If we let k = 1, m = 2 then our constant has the value $\frac{1}{4\alpha}$ as compared to the Redheffer value of $\frac{1}{\alpha} + \frac{12}{(b-a)^2}$.

One may now inquire about the possibility of equality in Corollary 2.1; by our previous discussion we see that this is possible if and only if

$$\lim_{n\to\infty}[\max_{1\leq i\leq n}\{\theta^k\lambda_i^k - \alpha\theta^m\lambda_i^m\}] = (\frac{k}{m\alpha})^{k/m-k}(1-\frac{k}{m}).$$

However a brief investigation shows that the above expression is not always satisfied, hence Corollary 2.1 is in general not best possible. A little more work will yield the best inequality.

We notice that the smaller eigenvalues of P approach zero as $1/n^2$ and that the positive root of $\theta^k t^k - \alpha\theta^m t^m$ exhibits a similar behavior:

$$\lambda_n = 0,\ \lambda_{n-1} = 4\sin^2(\tfrac{\pi}{n}) \approx \frac{4\pi^2}{n^2},\ \lambda_{n-3} = 4\sin^2(\tfrac{2\pi}{n}) \approx \frac{8\pi^2}{n^2}, \ldots$$

$$\text{root} = \frac{(b-a)^2}{(n+1)^2} \cdot \left(\frac{1}{\alpha}\right)^{1/m-k}.$$

Hence if we set

$$L = \left[\frac{b-a}{2\pi} \cdot \left(\frac{1}{\sqrt{\alpha}}\right)^{1/m-k}\right],$$

where [ ] is the greatest integer function, then L is the limiting number of distinct eigenvalues of P which are greater than 0 and less than or equal to the root.

If $L = 0$, which occurs when $\alpha > \left(\frac{b-a}{2\pi}\right)^{2m-2k}$, then

$$\lim_{n\to\infty} \left[\max_{1\le i\le n}\{\theta^k\lambda_i^k - \alpha\theta^m\lambda_i^m\}\right] = 0,$$

yielding the best possible result:

<u>Corollary 2.2.</u> Let $x(t), k, m$ be as in Corollary 2.1 and let $\alpha > \left(\frac{b-a}{2\pi}\right)^{2m-2k}$, then

$$\int_a^b \{x^{(k)}(t)\}^2 dt \le \alpha \int_a^b \{x^{(m)}(t)\}^2 dt.$$

If we add the auxiliary condition $\int_a^b x(t)dt = 0$, then

$$\lim_{n\to\infty} \left[\max_{1\le i\le n-1}\{\theta^k\lambda_i^k - \alpha\theta^m\lambda_i^m\}\right] = \left(\frac{2\pi}{b-a}\right)^{2k} - \alpha\left(\frac{2\pi}{b-a}\right)^{2m} \le 0$$

and we obtain the best possible result:

<u>Corollary 2.3.</u> Let $x(t), k, m, \alpha$ be as in Corollary 2.2 and let x also satisfy $\int_a^b x(t)dt = 0$, then

$$\int_a^b \{x^{(k)}(t)\}^2 dt \le \alpha \int_a^b \{x^{(m)}(t)\}^2 dt + \{(\tfrac{2\pi}{b-a})^{2k} - \alpha(\tfrac{2\pi}{b-a})^{2m}\} \cdot \int_a^b \{x(t)\}^2 dt.$$

If $L \ge 1$ then there are limiting eigenvalues between zero and the positive root. Hence if we set

$$J = \left[\frac{b-a}{2\pi} \cdot \left(\sqrt{\frac{k}{m\alpha}}\right)^{1/m-k}\right],$$

then J is the limiting number of distinct eigenvalues of P which are greater than 0 and less than or equal to the maximum of $\theta^k t^k - \alpha\theta^m t^m$.

Thus if

$$\left[\frac{b-a}{2\pi} \cdot \left(\sqrt{\frac{k}{m\alpha}}\right)^{1/m-k}\right] = \frac{b-a}{2\pi} \cdot \left(\sqrt{\frac{k}{m\alpha}}\right)^{1/m-k}$$

then

$$\lim_{n\to\infty} \left[\max_{1\le i\le n}\{\theta^k\lambda_i^k - \alpha\theta^m\lambda_i^m\}\right] = \left(\frac{k}{m\alpha}\right)^{k/m-k}(1-k/m)$$

and Corollary 2.1 is best possible. Otherwise we may replace $(\frac{k}{m\alpha})^{k/m-k}(1-k/m)$ by $\max\{(\frac{2J\pi}{b-a})^{2k} - \alpha(\frac{2J\pi}{b-a})^{2m}, (\frac{2(J+1)\pi}{b-a})^{2k} - \alpha(\frac{2(J+1)\pi}{b-a})^{2m}\}$.

We may summarize the above results by the following best possible inequalities.

<u>Corollary 2.4</u>. If $x(t) \in C^m[a,b]$, $1 \le k < m$, $x(a) = x(b)$, $x'(a) = x'(b), \cdots, x^{(m-1)}(a) = x^{(m-1)}(b)$, and if $\alpha > 0$ then

$$\int_a^b \{x^{(k)}(t)\}^2 dt \le \alpha \int_a^b \{x^{(m)}(t)\}^2 dt + H_{k,m}(\alpha) \int_a^b \{x(t)\}^2 dt,$$ where

$H_{k,m}(\alpha) = 0$ for $\alpha > (\frac{b-a}{2\pi})^{2m-2k}$,

$H_{k,m}(\alpha) = (\frac{k}{m\alpha})^{k/m-k}(1-k/m)$ if $\frac{b-a}{2\pi} \cdot \left(\sqrt{\frac{k}{m\alpha}}\right)^{1/m-k}$ is a positive integer, and otherwise

$$H_{k,m}(\alpha) = \max\{(\frac{2J\pi}{b-a})^{2k} - \alpha(\frac{2J\pi}{b-a})^{2m}, (\frac{2(J+1)\pi}{b-a})^{2k} - \alpha(\frac{2(J+1)\pi}{b-a})^{2m}\}.$$

Corollary 2.5. Let $x(t), k, m, \alpha$ be as in Corollary 2.4 and let x also satisfy $\int_a^b x(t)dt = 0$, then

$$\int_a^b \{x^{(k)}(t)\}^2 dt \leq \alpha \int_a^b \{x^{(m)}(t)\}^2 dt + G_{k,m}(\alpha) \int_a^b \{x(t)\}^2 dt,$$

where

$$G_{k,m}(\alpha) = (\frac{2\pi}{b-a})^{2k} - \alpha(\frac{2\pi}{b-a})^{2m} \text{ for } \alpha > (\frac{b-a}{2\pi})^{2m-2k},$$

and otherwise

$$G_{k,m}(\alpha) = H_{k,m}(\alpha).$$

It is not difficult to see that $H_{k,m}(\alpha)$ and $G_{k,m}(\alpha)$ are piecewise linear functions of α, which are monotonic decreasing, and that the jumps in the derivatives of $H_{k,m}$ and $G_{k,m}$ occur for those positive numbers α_i which yield

$$(\frac{2\pi i}{b-a})^{2k} - \alpha_i(\frac{2\pi i}{b-a})^{2m} = (\frac{2\pi i+2\pi}{b-a})^{2k} - \alpha_i(\frac{2\pi i+2\pi}{b-a})^{2m},$$

where i is a non-negative integer. We have $\infty > \alpha_0 > \alpha_1 > \cdots > 0$ and $\lim_{i\to\infty} \alpha_i = 0$. Note that

$$\int_a^b \{\sin^{(k)}(\frac{2\pi i t}{b-a})\}^2 dt = \alpha \int_a^b \{\sin^{(m)}(\frac{2\pi i t}{b-a})\}^2 dt + H_{k,m}(\alpha) \int_a^b \{\sin(\frac{2\pi i t}{b-a})\}^2 dt,$$

for $\alpha_i \leq \alpha \leq \alpha_{i-1}$. This accounts for the piecewise linearity of $H_{k,m}$ and $G_{k,m}$ and will be of great importance in Section III.

At this stage it is apparent that there is an unlimited number of possible inequalities. In fact every polynomial will give discrete inequalities and polynomials in θt yield continuous analogues.

It is known, see for example [1], that if $f(t)$ is represented by a power series with real coefficients and if A is a real symmetric matrix with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ which lie strictly within the circle of convergence of the power series for $f(t)$, then $f(A)$ is real symmetric and has eigenvalues $f(\lambda_1), f(\lambda_2), \cdots, f(\lambda_n)$. This result enables us to extend our polynomial inequalities to discrete inequalities on analytic functions.

Theorem 3. If x is a periodically extended n-vector and if $f(t) = \sum_{m=0}^{\infty} a_m t^m$, where the a_m are real numbers and the series converges on a set containing $[0,4]$ in its interior - if n is odd then we need only require convergence on a set containing $[0, 4\sin^2([\frac{n}{2}]\frac{\pi}{n})]$ in its interior - then

$$[\inf_{0 \leq t \leq 4}\{f(t)\}](x,x) \leq \sum_{m=0}^{\infty} a_m (x^{(m)}, x^{(m)}) \leq [\sup_{0 \leq t \leq 4}\{f(t)\}](x,x).$$

The following inequalities are examples of the application of Theorem 3.

Corollary 3.1. If x is a periodically extended n-vector then

$$\sum_{m=1}^{\infty} \frac{1}{m!} (x^{(m)},x^{(m)}) \le (e^4-1)(x,x).$$

<u>Corollary 3.2</u>. If x is a periodically extended n-vector and $\theta > 0$ then

$$-(x,x) \le \sum_{m=0}^{\infty} (-1)^m \frac{\theta^{2m+1}}{(2m+1)!} (x^{(2m+1)},x^{(2m+1)}) \le (x,x).$$

Proof:

$$\sin t = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} t^{2m+1} \text{ and } |\sin t| \le 1 .$$

<u>Corollary 3.3</u>. If $x(t) \in C^{\infty}[b-a]$ and $x(t)$ has period $b - a$ then

$$-\int_a^b \{x(t)\}^2 dt \le \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} \int_a^b \{x^{(2m+1)}(t)\}^2 dt \le \int_a^b \{x(t)\}^2 dt.$$

III. Inequalities of Kolmogoroff Type

In analysis one frequently wants to obtain inclusion regions for the norms of functions; i.e., if M is in the inclusion region, then there exists a function f, from a certain set of functions, with norm of f equal to M. Kolmogoroff [10] established a result of this type using the uniform norm over the function set $C^m[0,\infty]$. If we set $M_k(x) = \sup_{0\leq t \leq\infty} \{|x^{(k)}(t)|\}$ $(k=0,1,2,\cdots,m)$, then Kolmogoroff [10] gave a necessary and sufficient condition "in order that to a triple of positive numbers $M_0, M_k, M_m (0 < k < m)$ there should correspond a function x(t) for which

$$M_0 = M_0(x),\ M_k = M_k(x),\ M_m = M_m(x)."$$

In this section we will establish a similar result on a triple of positive numbers relative to the square norm, where our set of functions will be $C^m[a,b] \cap \{x|x(a) = x(b), x'(a) = x'(b), \cdots, x^{(m-1)}(a) = x^{(m-1)}(b)\}$. We will also add the restriction $\int_a^b x(t)dt = 0$ in order to eliminate the constant function.

If x(t) is an element of our function class, then we set

$$A_k(x) = \int_a^b \{x^{(k)}(t)\}^2 dt \quad (k=0,1,2,\cdots,m).$$

We may assume without loss of generality that we have normalized x so that $A_0(x) = 1$. We will establish the following result:

<u>Theorem 4</u>. In order that to a pair of numbers a_k, a_m there should

correspond a function $x(t) \in C^m[a,b] \cap \{x | x(a) = x(b),$
$x'(a) = x'(b), \cdots, x^{(m-1)}(a) = x^{(m-1)}(b), \int_a^b x(t)dt = 0\}$
for which

$$A_0(x) = 1,\ A_k(x) = a_k,\ A_m(x) = a_m$$

it is necessary and sufficient that

$$a_k \geq \left(\frac{2\pi}{b-a}\right)^{2k},\ a_m \geq \left(\frac{2\pi}{b-a}\right)^{2m},\text{ and}$$

$$a_k - \alpha a_m \leq G_{k,m}(\alpha) \text{ for all } \alpha > 0.$$

Proof of Theorem 4:

The necessity of the conditions is a direct consequence of Corollaries 1.1 and 2.5.

The proof of the sufficiency will be divided into several cases and we will use the notation of Section II throughout. We will denote the normalized sin $\left(\frac{2\pi i t}{b-a}\right)$ by $S_i(t)$. We recall that on the interval $\alpha_i \leq \alpha \leq \alpha_{i-1}$ we have $A_k(S_i) - \alpha A_m(S_i) = G_{k,m}(\alpha)$.

Case 1: $a_k - \alpha a_m = G_{k,m}(\alpha)$ for at least two values of α, say $\alpha = t_1$ and $\alpha = t_2$.

We immediately have that $a_k - \alpha a_m = G_{k,m}(\alpha)$ for all $\alpha \in [t_1, t_2]$. Since we know that the S_i are maximizing functions there is an integer i with $\alpha_i \leq t_1 < t_2 \leq \alpha_{i-1}$ and $a_k - \alpha a_m = G_{k,m}(\alpha)$ $= A_k(S_i) - \alpha A_m(S_i)$ for $\alpha_i \leq \alpha \leq \alpha_{i-1}$. Thus $a_k = A_k(S_i)$, $a_m = A_m(S_i)$.

Case 2: $a_k - \alpha a_m = G_{k,m}(\alpha)$ for only one value of α, say $\alpha = t_1$.

We immediately have that $t_1 = \alpha_1$ for exactly one integer i.

Thus

$$a_k - \alpha_i a_m = G_{k,m}(\alpha_i) = A_k(S_i) - \alpha_i A_m(S_i) =$$

$$A_k(S_{i+1}) - \alpha_i A_m(S_{i+1}).$$

Also since $a_k - \alpha a_m \leq G_{k,m}(\alpha)$ for all $\alpha > 0$ we must have $A_m(S_i) < a_m < A_m(S_{i+1})$. Thus there exist b_i, b_{i+1} with $0 < b_i, b_{i+1} < 1$, $b_i^2 + b_{i+1}^2 = 1$, and $a_m = A_m(b_i S_i + b_{i+1} S_{i+1}) = b_i^2 A_m(S_i) + b_{i+1}^2 A_m(S_{i+1})$. [Note that the last equality follows from the orthogonality of S_i and all its derivatives to S_{i+1} and all its derivatives]. Now,

$$A_k(b_i S_i + b_{i+1} S_{i+1}) - \alpha_i a_m =$$
$$A_k(b_i S_i + b_{i+1} S_{i+1}) - \alpha_i A_m(b_i S_i + b_{i+1} S_{i+1}) =$$
$$b_i^2[A_k(S_i) - \alpha_i A_m(S_i)] + b_{i+1}^2[A_k(S_{i+1}) - \alpha_i A_m(S_{i+1})] =$$
$$(b_i^2 + b_{i+1}^2)[a_k - \alpha_i a_m].$$

Hence $a_k = A_k(b_i S_i + b_{i+1} S_{i+1})$.

Case 3: $a_k - \alpha a_m < G_{k,m}(\alpha)$ for all $\alpha > 0$ and $a_m \geq (\frac{4\pi}{b-a})^{2m} = A_m(S_2)$.

We know that there is exactly one integer $i \geq 2$ with $(\frac{2i\pi}{b-a})^{2m} \leq a_m < (\frac{2(i+1)\pi}{b-a})^{2m}$ or $A_m(S_i) \leq a_m < A_m(S_{i+1})$. If we repeat the construction of Case 2 we can find b_i and b_{i+1} with $a_m = A_m(b_i S_i + b_{i+1} S_{i+1})$ and since $a_k - \alpha_i a_m < G_{k,m}(\alpha_i) =$ $A_k(b_i S_i + b_{i+1} S_{i+1}) - \alpha_i A_m(b_i S_i + b_{i+1} S_{i+1}) = A_k(b_i S_i + b_{i+1} S_{i+1}) - \alpha_i a_m$

we have

$$a_k < A_k(b_iS_i+b_{i+1}S_{i+1}).$$

By our conditions on a_k and a_m there is an integer $j > i + 1$ and an $\bar{\alpha} > 0$ such that $A_k(S_j) - \bar{\alpha}A_m(S_j) = A_k(S_1) - \bar{\alpha}A_m(S_1)$ $< a_k - \bar{\alpha}a_m$. Now $A_m(S_1) < A_m(S_i) \leq a_m < A_m(S_{i+1}) < A_m(S_j)$, hence there are b_1, b_j with $a_m = A_m(b_1S_1+b_jS_j)$ and since

$$a_k - \bar{\alpha}a_m > G_{k,m}(\bar{\alpha}) = A_k(b_1S_1+b_jS_j) - \bar{\alpha}A_m(b_1S_1+b_jS_j)$$
$$= A_k(b_1S_1+b_jS_j) - \bar{\alpha}a_m$$

we have

$$a_k > A_k(b_1S_1+b_jS_j).$$

By the orthogonality of $b_1S_1 + b_jS_j$ and $b_iS_i + b_{i+1}S_{i+1}$ there are constants c and d with $c^2 + d^2 = 1$ for normalization and

$$a_m = A_m(cb_1S_1 + cb_jS_j + db_iS_i + db_{i+1}S_{i+1}),$$
$$a_k = A_k(cb_1S_1 + cb_jS_j + db_iS_i + db_{i+1}S_{i+1}).$$

Case 4: $a_k - \alpha a_m < G_{k,m}(\alpha)$ for all α and $a_m < (\frac{4\pi}{b-a})^{2m} = A_m(S_2)$.

If we repeat the process of Cases 2 and 3 we can find b_1, b_2 such that $a_m = A_m(b_1S_1+b_2S_2)$ and $a_k < A_k(b_1S_1+b_2S_2)$.

We cannot simply repeat the process of Case 3 because $b_1S_1 + b_2S_2$ will not be orthogonal to $b_1'S_1 + b_2'S_2$. However we note that the normalized multiple of cos $(\frac{2\pi t}{b-a})$, which we denote by $C_1(t)$, also satisfies $A_k(C_1) - \alpha A_m(C_1)$ for $\alpha_1 \leq \alpha \leq \alpha_0$ and is orthogonal to all the S_i. Hence we can find c_1, c_j such that

$a_m = A_m(c_1C_1+c_jS_j)$ and

$$a_k > A_k(c_1C_1+c_jS_j)$$

and we continue as in Case 3.

Thus we have proven Theorem 4.

IV. Zero Boundary Conditions

We extend the n-vector $x = (x_1, x_2, \cdots, x_n)^T$ to an n+2-vector $\overline{x}$ with zero boundary conditions by setting $x_0 = x_{n+1} = 0$ ($\overline{x}$ is called the zero extension of the n-vector x). If we set

$$(\overline{x},\overline{x}) = \sum_{i=0}^{n+1} x_i^{\,2}$$

$$(\overline{x}^{(1)},\overline{x}^{(1)}) = \sum_{i=0}^{n} (x_{i+1}-x_i)^2$$

$$(\overline{x}^{(2)},\overline{x}^{(2)}) = \sum_{i=1}^{n} (x_{i-1}-2x_i+x_{i+1})^2$$

then Fan-Taussky-Todd [7] have shown that

$$(\overline{x}^{(m)},\overline{x}^{(m)}) = (x, Z^m x) = x^T Z^m x \quad (m=0,1,2),$$

where Z is the n x n tridiagonal matrix

$$\begin{bmatrix} 2 & -1 & 0 & . & . & . & \\ -1 & 2 & -1 & 0 & & & \\ 0 & . & . & . & . & & \\ & . & . & . & . & . & \\ & & . & . & . & . & . \\ & & & . & & & \end{bmatrix}.$$

Rutherford [14] has shown that Z has eigenvalues $4\cos^2(\frac{\pi k}{2n+2})$ $(k=1,2,\cdots,n)$, hence the eigenvalues of Z lie in the interval $[\mu_n, \mu_1] = [4\sin^2(\frac{\pi}{2n+2}), 4\cos^2(\frac{\pi}{2n+2})]$. This immediately yields the following:

Theorem 5. If $\overline{x}$ is the zero extension of the n-vector x and $f(t) = at^2 + bt + c$, a,b,c real, then:

$$\min_{t \in [\mu_n,\mu_1]}\{f(t)\}(\overline{x},\overline{x}) \le a(\overline{x}^{(2)},\overline{x}^{(2)}) + b(\overline{x}^{(1)},\overline{x}^{(1)}) + c(\overline{x},\overline{x})$$

$$\le \max_{t\in [\mu_n,\mu_1]}\{f(t)\}(\overline{x},\overline{x}).$$

If we let $f(t) = \theta t - \alpha\theta^2 t^2$ then we can derive an analogue of Theorem 2 for the zero boundary condition case. A brief reference to Section II will show that we have the following three cases to consider.

(i) if $\mu_1 = 4\cos^2(\frac{\pi}{2n+2}) \le 1/2\alpha\theta$ then $\max_{t\in[\mu_1,\mu_n]}\{f(t)\} = f(\mu_1)$

$= \theta\mu_1\{1-\alpha\theta\mu_1\}$.

(ii) if $\mu_n = 4\sin^2(\frac{\pi}{2n+2}) \ge 1/2\alpha\theta$ then $\max_{t\in[\mu_1,\mu_n]}\{f(t)\} = f(\mu_n)$

$= \theta\mu_n\{1-\alpha\theta\mu_n\}$.

(iii) in all cases $f(t) \le f(1/2\alpha\theta) = 1/4\alpha$.

Corollary 5.1. If $\overline{x}$ is the zero extension of the n-vector x and $\alpha,\theta > 0$, then we have:

$$\theta(\overline{x}^{(1)},\overline{x}^{(1)}) \le \alpha\theta^2(\overline{x}^{(2)},\overline{x}^{(2)}) + 1/4\alpha(\overline{x},\overline{x}).$$

If $4\cos^2(\frac{\pi}{2n+2}) \le 1/2\alpha\theta$ then the $1/4\alpha$ may be replaced by $4\theta\cos^2(\frac{\pi}{2n+2})\{1-4\alpha\theta\cos^2(\frac{\pi}{2n+2})\}$.

If $4\sin^2(\frac{\pi}{2n+2}) \ge 1/2\alpha\theta$ then the $1/4\alpha$ may be replaced by $4\theta\sin^2(\frac{\pi}{2n+2})\{1-4\alpha\theta\sin^2(\frac{\pi}{2n+2})\}$.

Before we pass to the limit we must investigate cases (i) and (ii) when $\theta = (\frac{n+1}{b-a})^2$ and $n \to \infty$.

(i) $4\cos^2(\frac{\pi}{2n+2}) \le 1/2\alpha\theta \Rightarrow 4\cos^2(\frac{\pi}{2n+2}) \le 1/2\alpha(\frac{b-a}{n+1})^2 \Rightarrow$

$\alpha \le \frac{1}{8}(\frac{b-a}{n+1})^2 \sec^2(\frac{\pi}{2n+2})$, i.e. $\alpha = 0$.

(ii) $4\sin^2(\frac{\pi}{2n+2}) \ge 1/2\alpha\theta \Rightarrow 4\sin^2(\frac{\pi}{2n+2}) \ge 1/2\alpha(\frac{b-a}{n+1})^2 \Rightarrow$

$\alpha \ge \frac{1}{8}(\frac{b-a}{n+1})^2/\sin^2(\frac{\pi}{2}\cdot\frac{1}{n+1})$, i.e. $\alpha > \frac{(b-a)^2}{2\pi^2}$.

<u>Corollary 5.2</u>. If $x(t) \in C^2[a,b], x(a) = x(b) = 0$, $\alpha > 0$ then we have:

$$\int_a^b \{x'(t)\}^2 dt \le \alpha\int_a^b \{x''(t)\}^2 dt + 1/4\alpha \int_a^b \{x(t)\}^2 dt.$$

If we also have $\alpha > (\frac{(b-a)^2}{2\pi^2}$ then the $1/4\alpha$ may be replaced by $\frac{\pi^2}{(b-a)^2}\{1- \frac{\alpha\pi^2}{(b-a)^2}\}$.

If we take limits carefully letting $a_n = 0$, $b_n = \sqrt{n}$ and $\theta = (\frac{n+1}{\sqrt{n}})^2$ then we obtain

<u>Corollary 5.3</u>. If $x(t) \in C^2[0,\infty]$, $x(0) = 0$, $\alpha > 0$, and all integrals exist then

$$\int_0^\infty \{x'(t)\}^2 dt \le \alpha\int_0^\infty \{x''(t)\}^2 dt + 1/4\alpha \int_0^\infty \{x(t)\}^2 dt.$$

We note that it is possible to derive best possible analogues of Corollaries 5.2 and 5.3 in the same spirit as the extensions of Corollary 2.1. Also we may derive a Kolmogoroff type necessary and sufficient condition for this zero boundary case in the same manner as we did in Section III.

V. Free Boundary Conditions

If $x = (x_0, x_1, \cdots, x_{n+1})$ is an n+2-vector, then we define a new n+2-vector h by:

$$h_k = \frac{x_{n+1} - x_0}{n+1} k + x_0 \qquad (k=0,1,\cdots,n+1).$$

Notice that $h_0 = x_0$, $h_{n+1} = x_{n+1}$ and h_k is on the line between x_0 and x_{n+1}, hence the vector $\overline{y} = x - h$ is an n+2-vector with $\overline{y}_0 = \overline{y}_{n+1} = 0$ or $\overline{y}$ is the zero extension of some n-vector y. Therefore we may apply Theorem 5 to $\overline{y}$, obtaining:

$$\min_{t \in [\mu_n, \mu_1]} \{f(t)\}(\overline{y}, \overline{y}) \leq a(\overline{y}^{(2)}, \overline{y}^{(2)}) + b(\overline{y}^{(1)}, \overline{y}^{(1)}) + c(\overline{y}, \overline{y})$$

$$\leq \max_{t \in [\mu_n, \mu_1]} \{f(t)\}(\overline{y}, \overline{y}),$$

where $f(t) = at^2 + bt + c$ and $[\mu_n, \mu_1]$ is as before.

We must now try to relate $(\overline{y}^{(i)}, \overline{y}^{(i)})$ to $(x^{(i)}, x^{(i)})$. We find

$$(\overline{y}^{(2)}, \overline{y}^{(2)}) = \sum_{i=1}^{n} \{\overline{y}_{i-1} - 2\overline{y}_i + \overline{y}_{i+1}\}^2$$

$$= \sum_{i=1}^{n} \{x_{i-1} - h_{i-1} - 2x_i + 2h_i + x_{i+1} - h_{i+1}\}^2$$

$$= \sum_{i=1}^{n} \{x_{i-1} - 2x_i + x_{i+1}\}^2 = (x^{(2)}, x^{(2)}),$$

$$(\bar{y}^{(1)},\bar{y}^{(1)}) = \sum_{i=0}^{n} \{\bar{y}_i-\bar{y}_{i+1}\}^2 = \sum_{i=0}^{n} \{x_i-x_{i+1} + \frac{x_{n+1}-x_0}{n+1}\}^2$$

$$= \sum_{i=0}^{n} \{x_i-x_{i+1}\}^2 + 2\,\frac{x_{n+1}-x_0}{n+1} \sum_{i=0}^{n} \{x_i-x_{i+1}\}+\{\frac{x_{n+1}-x_0}{n+1}\}^2 \sum_{i=0}^{n} 1$$

$$= (x^{(1)},x^{(1)}) - \frac{\{x_0-x_{n+1}\}^2}{n+1}\,,$$

and

$$(\bar{y},\bar{y}) = (x-h,x-h) = (x,x) - 2(x,h) + (h,h),$$

which is not very convenient to use. Substitution of these expressions into the previous inequality yields the free boundary analogue of Corollary 5.1.

<u>Theorem 6</u>. Let x be an n+2-vector, let $\bar{y}$ be the n+2-vector obtained by setting $\bar{y}_k = x_k - \frac{x_{n+1}-x_0}{n+1}\,k - x_0$, and let $\alpha,\theta > 0$ then we have:

$$\theta(x^{(1)},x^{(1)}) \le \alpha\theta^2(x^{(2)},x^{(2)}) + \frac{1}{4\alpha}\,(\bar{y},\bar{y}) + \frac{\theta(x_{n+1}-x_0)^2}{(n+1)}\,.$$

We may replace $1/4\alpha$ by the same coefficients as in Corollary 5.1.

Taking limits yields the continuous case.

<u>Corollary 6.1</u>. If $x(t) \in C^2[a,b]$ and $\alpha > 0$ then we have:

$$\int_a^b\{x'(t)\}^2dt \le \alpha\int_a^b\{x''(t)\}^2dt + \frac{1}{4\alpha}\int_a^b\{y(t)\}^2dt + \frac{\{x(b)-x(a)\}^2}{b-a}\,,$$

where $y(t) = x(t) - x(a) - \frac{x(b)-x(a)}{b-a}\cdot(t-a)$. If $\alpha > \frac{(b-a)^2}{2\pi^2}$ then the $1/4\alpha$ may be replaced by $\frac{\pi^2}{(b-a)^2}\{1-\frac{\alpha\pi^2}{(b-a)^2}\}$.

As one can see the free boundary case is untidy because of the presence of an auxiliary function which is used to reduce the problem to the case of zero boundary conditions. However, this technique can be used to extend many of the results of [7] to the free boundary case; for example:

Theorem 7. (analogue of Theorem 9 of [7]). If $x_0, x_1, \cdots, x_{n+1}$ are any real numbers, then

$$\sum_{i=0}^{n} \{x_i - x_{i+1}\}^2 \geq 4 \sin^2\left(\frac{\pi}{2n+2}\right) \sum_{i=0}^{n+1} \{x_i - x_0 - i \frac{x_{n+1}-x_0}{n+1}\}^2 + \frac{(x_{n+1}-x_0)^2}{(n+1)} .$$

Theorem 8. (analogue of Theorem 11 of [7]). If $x_0, x_1, \cdots, x_{n+1}$ are any real numbers, then

$$\sum_{i=1}^{n} \{x_{i-1} - 2x_i + x_{i+1}\}^2 \geq 16 \sin^4\left(\frac{\pi}{2n+2}\right) \sum_{i=0}^{n+1} \{x_i - x_0 - i \frac{x_{n+1}-x_0}{n+1}\}^2 .$$

VI. Extension to General Inner Products

In I-V we have considered inequalities with the usual inner product $(f,g) = \int_a^b f(t)g(t)dt$, or with various discrete analogues of this inner product. If $w(t)$ is a positive integrable function on $[a,b]$, then $(f,g)w = \int_a^b f(t)g(t)w(t)dt$ is also an inner product on $C[a,b]$, [6]. A number of authors has investigated discrete and continuous inequalities in these general inner products, see Beesack [2]; Block [3],[4] and Coles [5].

Beesack used the properties of the Riccati equation $x' + x^2 + p = 0$, to develop inequalities between $\int_a^b \{x'(t)\}^2 dt$ and $\int_a^b \{x(t)\}^2 p(t)dt$. We will attempt to use a discrete variational technique to produce our inequalities. For reasons of simplicity we will restrict ourselves to the case of zero-boundary conditions.

The Euler equation for the functional $F[x] = \int_a^b [\{x'(t)\}^2 q(t) - c\{x(t)\}^2 p(t)]dt$ is $(qx')' + cpx = 0$. Therefore c will be the smallest eigenvalue of the Sturm-Liouville problem $(qx')' + \lambda px = 0$, $x(a) = x(b) = 0$. Our task will be to discretize this problem and produce estimates for the relevant eigenvalues.

We let $\overline{x}$ be the zero extension of the n-vector $x = (x_1, x_2, \cdots, x_n)^T$ and q_i, p_i $(i=0,1,\cdots,n+1)$ be $2n + 4$ positive numbers. An application of Theorem 7 yields

$$\sum_{i=0}^{n} \{x_{i+1}-x_i\}^2 q_i \geq \min_{0\leq j\leq n} q_j \sum_{i=0}^{n} \{x_{i+1}-x_i\}^2$$

$$\geq 4 \sin^2(\frac{\pi}{2n+2}) \cdot \min_{0 \leq j\leq n} q_j \sum_{i=1}^{n} x_i^{\,2}$$

$$\geq 4 \sin^2(\frac{\pi}{2n+2}) \cdot \frac{\min_{0\leq j \leq n} q_j}{\max_{1 \leq k\leq n} p_k} \sum_{i=1}^{n} x_i^{\,2} p_i \ .$$

A simple passage to the limit $n \to \infty$ yields

$$\int_a^b \{x'(t)\}^2 q(t)dt \geq \frac{\pi^2}{(b-a)^2} \cdot \frac{\min_{s\in[a,b]} q(s)}{\max_{r\in[a,b]} p(r)} \cdot \int_a^b \{x(t)\}^2 p(t)dt.$$

At this point we find that a great deal of work is involved in obtaining sharper inequalities. We introduce the vector y, with $y_i = \sqrt{p_i}\, x_i$. Then

$$\sum_{i=0}^{n} \{x_{i+1}-x_i\}^2 q_i = \sum_{i=0}^{n} \{\frac{y_{i+1}}{\sqrt{p_{i+1}}} - \frac{y_i}{\sqrt{p_i}}\}^2 q_i$$

$$= y_1^2 \frac{q_0+q_1}{p_1} - \frac{2y_1y_2q_1}{\sqrt{p_1p_2}} + y_n^{\,2} \frac{q_{n-1}+q_n}{p_n} + \sum_{i=2}^{n-1} \{y_i^{\,2} \frac{q_{i-1}+q_i}{p_i} - \frac{2y_iy_{i+1}q_i}{\sqrt{p_ip_{i+1}}}\}$$

$$= y^T Q y,$$

where Q is the n x n real symmetric tridiagonal matrix

$$
\begin{bmatrix}
\frac{q_0+q_1}{p_1} & \frac{-q_1}{\sqrt{p_1p_2}} & 0 & & & & & \\
\frac{-q_1}{\sqrt{p_1p_2}} & \frac{q_1+q_2}{p_2} & \frac{-q_2}{\sqrt{p_2p_3}} & 0 & & & & \\
0 & \cdot & \cdot & \cdot & \cdot & & & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & & \\
 & & 0 & \frac{-q_{i-1}}{\sqrt{p_{i-1}p_i}} & \frac{q_{i-1}+q_i}{p_i} & \frac{-q_i}{\sqrt{p_ip_{i+1}}} & 0 & \\
 & & & \cdot & \cdot & \cdot & \cdot & \\
 & & & & \cdot & \cdot & \cdot & \\
 & & & & & 0 & \frac{-q_{n-1}}{\sqrt{p_{n-1}p_n}} & \frac{q_{n-1}+q_n}{p_n}
\end{bmatrix}.
$$

Hence we have the following inequality.

<u>Theorem 9</u>. If $\bar{x}$ is the zero-extension of the n-vector x and if p_i, q_i $(i=0,1,\cdots,n+1)$ are positive numbers, then

$$\lambda\min \sum_{i=1}^{n} x_i^2 p_i \le \sum_{i=0}^{n} \{x_{i+1}-x_i\}^2 q_i \le \lambda\max \sum_{i=1}^{n} x_i^2 p_i,$$

where λ_{min} and λ_{max} are respectively the smallest and largest eigenvalues of the matrix Q.

Because of the rather complicated nature of Q it seems hopeless to attempt to directly evaluate the eigenvalues of Q. Therefore we will consider diagonal similarity transformations of Q in the hope of obtaining a more tractable matrix which will have the same eigenvalues as Q.

Let D be the n x n matrix $\text{diag}\{\alpha_1\sqrt{p_1}, \alpha_2\sqrt{p_2}, \cdots, \alpha_n\sqrt{p_n}\}$, with $\alpha_i \neq 0$ $(i=1,2,\cdots,n)$ and all α_i of the same sign. Then $D^{-1}QD$ has the same eigenvalues as Q.

$$D^{-1}QD = \begin{bmatrix} \frac{q_0+q_1}{p_1} & \frac{-\alpha_2 q_1}{\alpha_1 p_1} & 0 & & & & & \\ \frac{-\alpha_1 q_1}{\alpha_2 p_2} & \frac{q_1+q_2}{p_2} & \frac{-\alpha_3 q_2}{\alpha_2 p_2} & 0 & & & & \\ 0 & \cdot & \cdot & \cdot & & & & \\ & & \cdot & \cdot & \cdot & & & \\ & & & \frac{-\alpha_{i-1} q_{i-1}}{\alpha_i p_i} & \frac{q_{i-1}+q_i}{p_i} & \frac{-\alpha_{i+1} q_i}{\alpha_i p_i} & & \\ & & & & \cdot & \cdot & \cdot & \\ & & & & & \cdot & \cdot & \cdot \\ & & & & & 0 & \frac{-\alpha_{n-1} q_{n-1}}{\alpha_n p_n} & \frac{q_{n-1}+q_n}{p_n} \end{bmatrix}.$$

If $A = (a_{ij})$ is an $n \times n$ matrix, then the Theorem of Gerschgorin [15] tells us that the eigenvalues of A lie within the union of the discs:

$$0 \leq |a_{ii} - Z| \leq R_i(A) \quad (i=1,2,\cdots,n),$$

where

$$R_i(A) = \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|.$$

In our problem

$$R_1(D^{-1}QD) = \frac{\alpha_2 q_1}{\alpha_1 p_1}$$

$$R_i(D^{-1}QD) = \frac{\alpha_{i-1}q_{i-1}}{\alpha_i p_i} + \frac{\alpha_{i+1}q_i}{\alpha_i p_i} \qquad (i=2,3,\cdots,n-1)$$

$$R_n(D^{-1}QD) = \frac{\alpha_{n-1}q_{n-1}}{\alpha_n p_n} \quad .$$

Therefore the Theorem of Gerschgorin yields

$$\lambda\min \geq \min_{1\leq i\leq n} \left\{\frac{q_{i-1}+q_i}{p_i} - R_i(D^{-1}QD)\right\}$$

$$\lambda\max \leq \max_{1\leq i\leq n} \left\{\frac{q_{i-1}+q_i}{p_i} + R_i(D^{-1}QD)\right\}.$$

Now for $i = 2,3,\cdots,n-1$ we have

$$\frac{q_{i-1}+q_i}{p_i} - R_i(D^{-1}QD)$$

$$= \frac{1}{\alpha_i p_i}\{\alpha_i q_{i-1}+\alpha_i q_i-\alpha_{i-1}q_{i-1}-\alpha_{i+1}q_i\}$$

$$= \frac{1}{\alpha_i p_i}\{-q_{i-1}(\alpha_{i-1}-2\alpha_i+\alpha_{i+1})+\alpha_{i+1}q_{i-1}-\alpha_i q_{i-1}+\alpha_i q_i-\alpha_{i+1}q_i\}$$

$$= \frac{1}{\alpha_i p_i}\{-q_{i-1}(\alpha_{i-1}-2\alpha_i+\alpha_{i+1})-(q_i-q_{i-1})(\alpha_{i+1}-\alpha_i)\}.$$

$$= \frac{-q_{i-1}}{\alpha_i p_i}\Delta^2\alpha_i - \frac{1}{\alpha_i p_i}\Delta q_{i-1}\cdot\Delta\alpha_i.$$

We may describe the cases $i = 1$ and n in either of two ways. We may treat $i = 1$ and n as special cases or we may observe that for the special choices $\alpha_0 = \alpha_{n+1} = 0$ we have

$$\frac{q_0+q_1}{p_1} - \frac{\alpha_2 q_1}{\alpha_1 p_1} = \frac{1}{\alpha_1 p_1}\{q_0(\alpha_1-\alpha_0)-q_1(\alpha_2-\alpha_1)\}$$

$$= \frac{-q_0}{\alpha_1 p_1}\Delta^2\alpha_1 - \frac{1}{\alpha_1 p_1}\Delta q_0 \cdot \Delta\alpha_1,$$

and

$$\frac{q_{n-1}+q_n}{p_n} - \frac{\alpha_{n-1}q_{n-1}}{\alpha_n p_n} = \frac{-q_{n-1}}{\alpha_n p_n}\Delta^2\alpha_n - \frac{1}{\alpha_n p_n}\Delta q_{n-1}\Delta\alpha_n \,.$$

Hence the use of Gerschgorin Circles yields the following:

Theorem 10. If $\overline{x}$ is the zero-extension of the n-vector x; if p_i, q_i $(i=0,1,\cdots,n+1)$ are positive numbers; and if $\overline{\alpha}$ is the zero-extension of the n-vector α, where α_i $(i=1,2,\cdots,n)$ are all of the same sign and non-zero, then

$$\sum_{i=0}^{n}\{x_{i+1}-x_i\}^2 q_i \geq \min_{1\leq i\leq n}\{\frac{-q_{i-1}}{\alpha_i p_i}\Delta^2\alpha_i - \frac{1}{\alpha_i p_i}\Delta q_{i-1}\cdot\Delta\alpha_i\}\cdot\sum_{j=1}^{n}x_j^2 p_j \,.$$

If we take limits we find

Corollary 10.1. If $x(t) \in C'[a,b]$, $x(a) = x(b) = 0$ and the following conditions are satisfied:

(i) $p(t) \in C[a,b]$, $p(t) > 0$ on $a < t < b$

(ii) $q(t) \in C'[a,b]$, $q(t) > 0$ on $a < t < b$

(iii) $\alpha(t) \in C^2[a,b]$, $\alpha(t) \neq 0$ on $a < t < b$,

then

$$\int_a^b\{x'(t)\}^2 q(t)dt \geq \min_{a<s<b}\{\frac{-1}{\alpha(s)p(s)}\cdot(q\alpha')'\}\cdot\int_a^b\{x(t)\}^2 p(t)dt.$$

Note: $(\frac{n+1}{b-a})^2\{\frac{q_0+q_1}{p_1} - R_1(D^{-1}QD)\} \to \infty$ as $n \to \infty$ unless $\alpha(a) = 0$, similarly for $\alpha(b)$. Hence the boundary conditions on α are automatically taken into account.

Since, p,q, and x are independent of α, we have

$$\int_a^b \{x'(t)\}^2 q(t)dt \geq M \int_a^b \{x(t)\}^2 p(t)dt,$$

where

$$M = \max_{\alpha} \min_{a<s<b} \{\frac{-1}{\alpha(s)p(s)} \cdot (q\alpha')'\}.$$

It seems remarkable that the constant M, which we derived from the Gerschgorin Circle bound, is actually best possible. This follows from the fact that the eigenfunction u(t) belonging to the smallest eigenvalue c of the Sturm-Liouville Problem $(qx')' + \lambda px = 0$, $x(a) = x(b) = 0$ does not vanish on $a < t < b$, [9]. Hence setting $\alpha(t) = u(t)$ yields $M = c$.

An additional dividend of this technique is that we have derived computable lower bounds for the smallest eigenvalue of the Sturm-Liouville Problem.

A natural question to ask is: How close is $\min_{1\leq i\leq n}\{\frac{-q_{i-1}}{\alpha_i p_i}\Delta^2\alpha_i - \frac{1}{\alpha_i p_i}\cdot \Delta q_{i-1} \cdot \Delta\alpha_i\}$ to λmin, the smallest eigenvalue of Q, for a particular choice of α?

In order to answer this question we will use the techniques of non-negative matrices and matrices whose inverses are non-negative. Varga [16] provides a rather complete account of these techniques. Our estimates will be derived from the following result, which appears to be of independent interest.

Theorem 11. If $A = (a_{ij})$ is an irreducibly diagonally dominant $n \times n$ matrix with $a_{ij} \leq 0$ for all $i \neq j$, and $a_{ii} > 0$ $(i=1,2,\cdots,n)$, then A has an eigenvalue λ with $0 < \lambda \leq \max_{1\leq i\leq n} \{ \sum_{j=1}^{n} a_{ij} \}$.

Corollary 11.1. Equality holds in Theorem 11 if and only if

$$\max_{1\leq i\leq n}\{ \sum_{j=1}^{n} a_{ij} \} = \sum_{\ell=1}^{n} a_{k\ell} \quad (k=1,2,\cdots,n).$$

Proof of Theorem 11:

The irreducible diagonal dominance of A tells us that

$$\max_{1\leq i\leq n} \{ |a_{ii}| - \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}| \} > 0.$$

Hence

$$\max_{1\leq i\leq n} \{ \sum_{j=1}^{n} a_{ij} \} > 0.$$

By Corollary 1, P. 85, [16] we have that A is non-singular and $A^{-1} > 0$. If λ is an eigenvalue of A with minimum modulus, then $\rho(A^{-1}) = 1/|\lambda|$, where $\rho(M)$ is the spectral radius of M.

Since $A^{-1} > 0$, Lemma 2.6, P. 40 [16] tells us that A^{-1} is primitive, hence A^{-1} has exactly one eigenvalue whose modulus is equal to $\rho(A^{-1})$ and this eigenvalue is positive. Therefore

$$1/\lambda = \rho(A^{-1}) > 0 \text{ and } \lambda > 0.$$

Let $B = A + K$, where K is the $n \times n$ diagonal matrix whose i^{th} diagonal entry is

$$\max_{1 \leq j \leq n}\{\sum_{k=1}^{n} a_{jk}\} - \sum_{\ell=1}^{n} a_{i\ell}.$$

Thus K is a non-negative matrix and $A \leq A + K = B$.

If μ is an eigenvalue of minimum modulus of B, then the Theorem of Gerschgorin yields

$$|\mu| \geq \max_{1 \leq i \leq n}\{\sum_{j=1}^{n} a_{ij}\}.$$

However if we let e be the n-vector $\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$, then

$$Be = \max_{1 \leq i \leq n}\{\sum_{j=1}^{n} a_{ij}\} \cdot e.$$

Thus $\max_{1 \leq i \leq n}\{\sum_{j=1}^{n} a_{ij}\}$ is the eigenvalue of minimum modulus of B, yielding

$$\rho(B^{-1}) = [\max_{1 \leq i \leq n}\{\sum_{j=1}^{n} a_{ij}\}]^{-1}.$$

Also by Corollary 1, P. 85 [16]

$$B^{-1} > 0.$$

A and B satisfy Varga's definition of an M-matrix, hence $A \leq B$ and Exercise 9, P. 87, [16] yield $0 < B^{-1} \leq A^{-1}$.

Thus Theorem 2.1 (3), P. 30, [16] yields

$$\rho(B^{-1}) \leq \rho(A^{-1})$$

or

$$1/\max_{1\leq i\leq n}\{\sum_{j=1}^{n} a_{ij}\} \leq 1/\lambda$$

or

$$\lambda \leq \max_{1\leq i\leq n}\{\sum_{j=1}^{n} a_{ij}\}.$$

Proof of Corollary 11.1:

K = 0 if and only if

$$\max_{1\leq i\leq n}\{\sum_{j=1}^{n} a_{ij}\} = \sum_{\ell=1}^{n} a_{k\ell} \quad (k=1,2,\cdots,n).$$

Thus A = B if and only if equality holds for all k, hence $A^{-1} = B^{-1}$ if and only if equality holds for all k. Theorem 2.1(3), p. 30, [16] completes the proof of the corollary.

We are now in a position to give upper bounds for λmin, the least eigenvalue of Q, for a particular choice of α.

We will restrict ourselves to the case $\frac{q_{i-1}+q_i}{p_i} - R_i(D^{-1}QD) \geq 0$ $(i=1,2,\cdots,n)$, with strict inequality for some i. This case is attainable since the choice $\alpha_i=1(i=1,2,\cdots,n)$ yields

$$\frac{q_0+q_1}{p_1} - R_i(D^{-1}QD) = \frac{q_0}{p_1}$$

$$\frac{q_{i-1}+q_i}{p_i} - R_i(D^{-1}QD) = 0 \quad (i=2,3,\cdots,n-1)$$

$$\frac{q_{n-1}+q_n}{p_n} - R_n(D^{-1}QD) = \frac{q_n}{p_n} .$$

With this restriction $D^{-1}QD$ is strictly diagonally dominant and satisfies the other conditions of Theorem 11. Therefore we have the following inequalities.

Theorem 12. If $\lambda\min$ is the smallest eigenvalue of Q and α is chosen so that $\frac{q_{i-1}+q_i}{p_i} - R_i(D^{-1}QD) \geq 0$ $(i=1,2,\cdots,n)$ with strict inequality for some i then

$$\min_{1\leq i\leq n}\{\frac{q_{i-1}+q_i}{p_i} - R_i(D^{-1}QD)\} \leq \lambda\min \leq \max_{1\leq i\leq n}\{\frac{q_{i-1}+q_i}{p_i} - R_i(D^{-1}QD)\}.$$

Corollary 12.1. Equality holds in Theorem 12 if and only if

$$\min_{1\leq i\leq n}\{\frac{q_{i-1}+q_i}{p_i} - R_i(D^{-1}QD)\} = \max_{1\leq i\leq n}\{\frac{q_{i-1}+q_i}{p_i} - R_i(D^{-1}QD)\}.$$

If we take limits we find

Corollary 12.2. If $\alpha(t) \in C^2[a,b]$, $\alpha(t) \neq 0$ on $a < t < b$, $\alpha(a) = \alpha(b) = 0$, and the following conditions are satisfied

(i) $p(t) \in C[a,b]$, $p(t) > 0$ on $a < t < b$

(ii) $q(t) \in C'[a,b]$, $q(t) > 0$ on $a < t < b$

(iii) $(-q\alpha')'/p\alpha > 0$ on (a,b)

then

$$\min_{a<s<b}\{\frac{-1}{p(s)\alpha(s)} \cdot (q\alpha')'\} \leq c \leq \max_{a<s<b}\{\frac{-1}{p(s)\alpha(s)} \cdot (q\alpha')'\},$$

where c is the smallest eigenvalue of the Sturm-Liouville Problem $(qx')' + \lambda px = 0$, $x(a) = x(b) = 0$.

Note: We must add the condition $\alpha(a) = \alpha(b) = 0$ in order to avoid the trivial estimate $c \leq +\infty$ which arises as the

$$\lim_{n\to\infty}\left(\frac{n+1}{b-a}\right)^2\left\{\frac{q_0+q_1}{p_1} - R_1(D^{-1}QD)\right\} = \lim_{n\to\infty}\left(\frac{n+1}{b-a}\right)^2\left\{\frac{q_{n-1}+q_n}{p_n} - R_n(D^{-1}QD)\right\}$$

unless $\alpha(a) = \alpha(b) = 0$.

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