RINGS FAITHFULLY REPRESENTED ON THEIR LEFT SOCLE

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ABSTRACT

In 1964 A. W. Goldie [1] posed the problem of determining all rings with identity and minimal condition on left ideals which are faithfully represented on the right side of their left socle. Goldie showed that such a ring which is indecomposable and in which the left and right principal indecomposable ideals have, respectively, unique left and unique right composition series is a complete blocked triangular matrix ring over a skewfield. The general problem suggested above is very difficult. We obtain results under certain natural restrictions which are much weaker than the restrictive assumptions made by Goldie.

We characterize those rings in which the principal indecomposable left ideals each contain a unique minimal left ideal (Theorem (4. 2)). It is sufficient to handle indecomposable rings (Lemma (1. 4)). Such a ring is also a blocked triangular matrix ring. There exist r positive integers $K_1, \ldots, K_r$ such that the i, j$^{\underline{th}}$ block of a typical matrix is a $K_i \times K_j$ matrix with arbitrary entries in a subgroup D_{ij} of the additive group of a fixed skewfield D. Each D_{ii} is a sub-skewfield of D and $D_{ri} = D$ for all i. Conversely, every matrix ring which has this form is indecomposable, faithfully represented on the right side of its left socle, and possesses the property that every principal indecomposable left ideal contains a unique minimal left ideal.

The principal indecomposable left ideals may have unique composition series even though the ring does not have minimal condition on right ideals. We characterize this situation by defining a partial ordering ρ on $\{1, 2, \ldots, r\}$ where we set $i \rho j$ if $D_{ij} \neq 0$. Every principal indecomposable left ideal has a unique composition series if and only if the diagram of ρ is an inverted tree and every D_{ij} is a one-dimensional left vector space over D_{ii} (Theorem (5. 4)).

We show (Theorem (2. 2)) that every ring A of the type we are studying is a unique subdirect sum of less complex rings $A_1, \ldots, A_s$ of the same type. Namely, each A_i has only one isomorphism class of minimal left ideals and the minimal left ideals of different A_i are non-isomorphic as left A-modules. We give (Theorem (2. 1)) necessary and sufficient conditions for a ring which is a subdirect sum of rings A_i having these properties to be faithfully represented on the right side of its left socle. We show ((4. F), p. 42) that up to technical trivia the rings A_i are matrix rings of the form

$$\begin{bmatrix} Q_1 & & & \\ & Q_2 & O & \\ & & \ddots & \\ \multicolumn{3}{c}{\rule{3em}{0.4pt}} & Q_p \end{bmatrix}$$

. Each Q_j comes from the faithful irreducible matrix representation of a certain skewfield over a fixed skewfield D. The bottom row is filled in by arbitrary elements of D.

In Part V we construct an interesting class of rings faithfully represented on their left socle from a given partial ordering on a finite set, given skewfields, and given additive groups. This class of rings contains the ones in which every principal indecomposable left ideal has a unique minimal left ideal. We identify the uniquely determined subdirect summands mentioned above in terms of the given partial ordering (Proposition (5. 2)). We conjecture that this technique serves to construct all the rings which are a unique subdirect sum of rings each having the property that every principal indecomposable left ideal contains a unique minimal left ideal.

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I. Preliminaries

Unless otherwise specified A will be a ring with identity and minimal condition on left ideals. All left and right A-modules, when they come into consideration, are assumed to be unitary. By the radical N of A we mean the Jacobson radical. That is, $N = \{ a \in A \mid aM = 0$ for every irreducible left A-module $M \}$. If I is a left and T a two-sided ideal of A, we define the right quotient (I:T) of T in I to be $\{ a \in A \mid Ta \subseteq I \}$. (I:T) is a left ideal of A and is a two-sided ideal of A if I is two-sided.

In the above terminology, the left socle S (or S(A) if it is necessary to specify the ring A) is simply 0:N. S is a two-sided ideal and is the largest completely reducible left ideal contained in A.

The rings in which we are primarily interested, are rings A in which $0{:}S(A) = 0$. In other words, rings which act on the right side of their left socle faithfully. If this is the case, we shall say that the ring is faithfully represented on its left socle.

A commutative ring which is faithfully represented on its socle must be a direct sum of fields: $0 = NS = SN$ so $N = 0$.

The first lemma is basic and will be used repeatedly:

Lemma (1.1). Suppose $0{:}S = 0$ and that T is a two-sided ideal of A contained in S and satisfying $T \cap (0{:}T) = 0$. Then $TI = T \cap I$ for any left ideal I of A.

Proof. Since T is a two-sided and I a left ideal, $TI \subseteq T \cap I$. Now let L be a minimal left ideal contained in $T \cap I$. Since $L \subseteq T$, $L \nsubseteq 0{:}T$. So $TL \neq 0$. Since L is minimal, this implies: $L = TL \subseteq TI$. But $T \cap I \subseteq S \cap I \subseteq S$. Hence $T \cap I$ is completely reducible as a left

A-module. It follows that $T \cap I$ as a sum of minimal left ideals, each of which is contained in TI by the above argument, is contained in TI.

(1.A) Note that lemma (1.1) certainly holds for $T = S$. We shall see later (1.D) that it also holds for any T having a complement in S, i.e., having a two-sided ideal Q such that $S = T \dotplus Q$.

(1.B) At this time it is convenient to recall some facts and definitions pertaining to rings A with identity and minimal condition on left ideals. Recall that idempotents $e_1, e_2, \ldots, e_n$ of A are called mutually orthogonal if $e_i e_j = 0$ for $i \neq j$. Note: we do not consider 0 (either the element or the ideal) to be idempotent. If $e = e_1 + e_2 + \cdots + e_n$, then e is an idempotent and $Ae = Ae_1 \dotplus Ae_2 \dotplus \cdots \dotplus Ae_n$. Hence, if Ae is indecomposable, then $n = 1$ and e cannot be written as a sum of more than one orthogonal idempotent. Such an idempotent e is called primitive and the corresponding ideal Ae a (left) principal indecomposable ideal. Thus A is a finite direct sum of principal indecomposables corresponding to a decomposition of 1 as a sum of primitive idempotents. Although 1 can be written in many ways as a sum of primitive idempotents, the non-isomorphic P.I.'s (principal indecomposables) are finite in number and uniquely determined by the ring A. Moreover, if $A = \dot{\sum}_{\substack{1 \le i \le r \\ 1 \le \alpha \le K_i}} Ae_{i\alpha}$ where the $Ae_{i\alpha}$ are P.I.'s and $Ae_{i\alpha} \simeq Ae_{j\beta}$ iff $i = j$, then $Ae_{11}, Ae_{21}, \ldots, Ae_{r1}$ constitute a full set of non-isomorphic P.I.'s and the numbers $K_1, K_2, \ldots, K_r$ and r are uniquely determined by A.

Recall that if Ae is a P. I., where e is a primitive idempotent, then $Ne \subset Ae$ is the unique maximal subideal of Ae. Thus, if M is a left A-module possessing a composition series, then $eM \neq 0$ iff M has a composition factor isomorphic to Ae/Ne. Two P. I. 's Ae and Af are isomorphic iff $Ae/Ne \simeq Af/Nf$. Also, Ae is a minimal left ideal iff $Ne = 0$. On the other hand, if L is any minimal left ideal, then either $L^2 = 0$ or $L^2 = L$. In the latter case L contains (as does any non-nilpotent left ideal) an idempotent e. Obviously $L = Ae$, and since L is a fortiori indecomposable, e is a primitive idempotent. Thus, the idempotent minimal left ideals are precisely the minimal left ideals which are at the same time principal indecomposables.

(1.C). As a particular example, suppose that the ring A contains a unique P. I. Then A itself must be this P. I. Since A/N is a ring, A/N must be a skewfield by the discussion in (1. B). Recall that such a ring A is called a completely primary ring. If A contains an idempotent minimal left ideal, then this ideal must be A, A is a skewfield, and $0{:}S = 0{:}A = 0$. Conversely, if A contains no idempotent minimal left ideal, then A is not a skewfield and $S^2 \subseteq NS = 0$ so that $0{:}S \neq 0$. Thus, a completely primary ring is faithfully represented on its left socle iff it is a skewfield.

It will be important to remember that if A is any ring (with identity and minimal condition on left ideals) and e an idempotent of A, then Ae is a P. I. iff eAe is completely primary.

We will now prove a theorem which is at the same time a generalization of (1. C) and an alternate characterization of rings which are faithfully represented on their left socle:

Theorem (1. 2). A is faithfully represented on its left socle iff every minimal left ideal is module-isomorphic to an idempotent minimal left ideal of A.

Proof. Let L be a minimal left ideal. Since $eL \neq 0$ for some primitive idempotent e, L is isomorphic to Ae/Ne (see (1. B)). If $0{:}S = 0$, then $SL \neq 0$ so that $S(Ae/Ne) \neq 0$. That is, $SAe \nsubseteq Ne$. Therefore $SAe = Ae$ since Ne is the unique maximal subideal of Ae. But S is two-sided. So we have $Ae \subseteq S$. Therefore $Ne = NAe \subseteq NS = 0$. I. e., Ae is an idempotent minimal left ideal.

For the converse, suppose that $0{:}S \neq 0$. Then there is a minimal left ideal L such that $SL = 0$. If L is isomorphic to an idempotent minimal left ideal L', then $L'L \neq 0$ (because $L'L = 0$ implies $L'^2 = 0$ since L and L' are isomorphic). But $L'L \subseteq SL$, a contradiction.

As an application of theorem (1. 2) we may show that a quasi-Frobenius ring cannot be represented faithfully on its left socle unless it is semi-simple. By a quasi-Frobenius ring we mean a ring with identity and minimal condition on left ideals with the property that every left ideal is the left-annihilator of its right-annihilator and every right ideal is the right-annihilator of its left-annihilator (by the left (right) annihilator of a subset we just mean the set of all elements of the ring which act on the left (right) side of the subset as zero).

It is known ([2] - p. 401) that if A is a quasi-Frobenius ring, then every irreducible left A-module is isomorphic to a minimal left ideal of A. Let e be a primitive idempotent and suppose that A is faithfully represented on its left socle. Then, since Ae/Ne is irreducible, Ae/Ne is isomorphic to an idempotent minimal left ideal L by theorem (1. 2). But L is a P. I. (1. B), say $L = Ae'$ where e' is a primitive idempotent. Hence, $Ae/Ne \simeq Ae'$ so $Ne = 0$ (again by (1. B)). It follows that $N = 0$ and A is semi-simple.

We now restrict our attention to rings which are faithfully represented on their left socle. For these rings the socle itself has nice properties reminiscent of semi-simple rings.

Theorem (1. 3). Let $L_1, \ldots, L_r$ be a full set of non-isomorphic, idempotent, minimal left ideals of A. Let S_i be the sum of all minimal left ideals isomorphic to L_i. Then $S_i = L_iA$ and $S = S_1 \dotplus \cdots \dotplus S_r$ is the unique decomposition of S as a direct sum of two-sided, indecomposable ideals of A.

Proof. Since $L_i \subseteq S_i$ and S_i is two-sided, $L_iA \subseteq S_i$. Let L'_i be any minimal left ideal isomorphic to L_i. Since L_i is non-nilpotent, $L_iL'_i \neq 0$. Therefore, $L_iL'_i = L'_i$ so that $L'_i \subseteq L_iA$. We have shown that $S_i = L_iA$.

Now $S = S_1 \dotplus \cdots \dotplus S_r$ follows from theorem (1. 2) and the fact that S is completely reducible as a left A-module.

Let $S = T_1 \dotplus \cdots \dotplus T_s$ where the T_i are indecomposable two-sided ideals of A. For any T_i, we have by lemma (1. 1) that $T_i = S \cap T_i = ST_i = \dot{\sum_j} S_jT_i$. It follows from the indecomposability of T_i that $T_i \subseteq S_j$, some j. On the other hand, let $L_j = Ae_j$ where e_j is a primitive idempotent. We can express e_j uniquely as $e_j = \sum_k e_{jk}$ where $e_{jk} \in T_k$. Since the T_k are two-sided ideals they mutually annihilate each other from which it follows that the e_{jk}

form a set of mutually orthogonal idempotents. But e_j is a primitive idempotent. So there can only be one e_{jk}, say $e_j = e_{jk^*}$. We have $L_j = Ae_j \subseteq T_{k^*}$ so that $S_j = L_jA \subseteq T_{k^*}A \subseteq T_{k^*}$ by the first part of the proof. The desired result follows.

Corollary. If T is any two-sided ideal of A such that $0{:}T = 0$, then $T \supseteq S$ and $TS = S$.

Proof. Let L be any minimal left ideal. Since $0{:}T = 0$, we have $TL = L$. Thus, $L \subseteq T$. Hence, $S \subseteq T$. But then, since $0{:}TS = (0{:}T){:}S = 0{:}S = 0$ and $TS \subseteq S$, we must have $TS = S$.

Note that the above corollary essentially says that the left socle representation is the faithful representation of smallest degree that one may achieve by representing A (on the right) on its two-sided ideals.

(1. D). By theorem (1. 3), the promise made in (1. A) will be fulfilled if we can show that $(0{:} \sum_{i \in I} S_i) \cap \sum_{i \in I} S_i = 0$ for any subset I of $\{1, 2, \ldots, r\}$. If $(0{:} \sum_{i \in I} S_i) \cap \sum_{i \in I} S_i \neq 0$, there exists a minimal left ideal L such that $L \subseteq \sum_{i \in I} S_i$ and $(\sum_{i \in I} S_i)L = 0$. But $(\sum_{i \notin I} S_i)L = 0$ since $L \subseteq \sum_{i \in I} S_i$. Hence $L \subseteq (0{:} \sum_{i \in I} S_i) \cap (0{:} \sum_{i \notin I} S_i) = 0{:} \sum_{i=1}^{r} S_i = 0{:}S = 0$, a contradiction.

At this point it is natural to inquire into the relationship between the socle of A and the socles of the indecomposable rings which occur when A is written as a direct sum of indecomposable two-sided ideals.

It is easy to prove:

Lemma (1.4). Let $A = B_1 \dotplus B_2 \dotplus \cdots \dotplus B_n$ be the decomposition of A into indecomposable two-sided ideals B_i. Then $0{:}S(A) = 0{:}S(B_1) \dotplus 0{:}S(B_2) \dotplus \cdots \dotplus 0{:}S(B_n)$ where the quotients $0{:}S(B_i)$ are understood as being taken in B_i. In particular, A is faithfully represented on its left socle iff every B_i is faithfully represented on its left socle.

Proof. Since $B_iB_j = 0$ if $i \neq j$, we get from the decomposition $A = \dot{\sum_i} B_i$, a decomposition $S = \dot{\sum_i} S \cap B_i = \dot{\sum_i} S(B_i)$ and a decomposition $0{:}S = \dot{\sum_i} (0{:}S) \cap B_i$. Now $0 = S(0{:}S) = \sum_i S(B_i)\ ((0{:}S) \cap B_i)$ so $S(B_i)\ ((0{:}S) \cap B_i) = 0$ for every i. I. e., $(0{:}S) \cap B_i \subseteq 0{:}S(B_i)$, all i, and $0{:}S \subseteq \dot{\sum_i} 0{:}S(B_i)$ follows. But $S(0{:}S(B_i)) = S(B_i)(0{:}S(B_i)) = 0$ for all i. Hence $0{:}S(B_i) \subseteq 0{:}S$, all i, and the lemma follows.

(1. E). To get better results along the same lines as the preceding, we must say what we mean by "block theory". First of all, two principal indecomposables Ae and Af (e, f idempotents) of any ring A with identity and minimal condition on left ideals are said to be linked if there exist primitive idempotents $e_0 = e, e_1, \ldots, e_n = f$ such that Ae_{i-1} and Ae_i have a common composition factor for $1 \leq i \leq n$. Linking is then an equivalence relation on the set of all principal indecomposables of A. By a block we mean the sum of all principal indecomposables in an equivalence class. If $B_1, \ldots, B_m$ are the distinct blocks, then $A = B_1 \dotplus \cdots \dotplus B_m$ is the unique representation of A as a direct sum of two-sided indecomposable ideals.

In the light of the above considerations, it is trivially true that any ring with identity and minimal condition on left ideals containing only one isomorphism class of minimal left ideals is indecomposable. The converse is hardly true even for rings which are faithfully represented on their socle.

Example (1.1). Consider the ring of all matrices of the form $\begin{bmatrix} a & o & o \\ b & d & o \\ c & o & e \end{bmatrix}$ where a, b, c, d, e are arbitrary elements of some fixed skewfield D. This ring has an identity and minimal condition on both left and right ideals. Its left socle S consists of everything of the form $\begin{bmatrix} o & o & o \\ b & d & o \\ c & o & e \end{bmatrix}$ so that $0:S = 0$. The idempotents $\begin{bmatrix} o & o & o \\ o & 1 & o \\ o & o & o \end{bmatrix}$ and $\begin{bmatrix} o & o & o \\ o & o & o \\ o & o & 1 \end{bmatrix}$ generate (as left modules over the ring) two idempotent minimal left ideals which are non-isomorphic since they annihilate each other.

However, this ring is indecomposable. One may easily see this by either calculating its center, observing that all its minimal right ideals are isomorphic, or using block theory. If one takes the last course of action, one quickly notices that the principal indecomposables (these are just the left ideals generated by $\begin{bmatrix} 1 & o & o \\ o & o & o \\ o & o & o \end{bmatrix}$, $\begin{bmatrix} o & o & o \\ o & 1 & o \\ o & o & o \end{bmatrix}$ and $\begin{bmatrix} o & o & o \\ o & o & o \\ o & o & 1 \end{bmatrix}$) are "linked by minimal left ideals". This is effectually what we will prove next.

Going back to our standard assumption that A is faithfully represented on its left socle, we have:

Lemma (1.5). Two P.I.'s (principal indecomposables) I and J are linked iff there exists a sequence $I = I_0, I_1, \ldots, I_n = J$ of P.I.'s such that some minimal left ideal of I_{j-1} is isomorphic to some minimal left ideal of I_j for $1 \leq j \leq n$.

Proof. The sufficiency is trivial. To prove the necessity it is sufficient to prove that, given P.I.'s I and J having a common composition factor, there exists a P.I. K such that I has a minimal left ideal isomorphic to a minimal left ideal of K and K has a minimal left ideal isomorphic to a minimal left ideal of J.

Suppose then that I and J are P.I.'s having a common composition factor isomorphic to Ae/Ne where e is a primitive idempotent. This means that $eI \neq 0$ and $eJ \neq 0$. By lemma (1.1) and the hypothesis that $0:S = 0$, we have $(S \cap Ae)I = SeI \neq 0$ and $(S \cap Ae)J = SeJ \neq 0$. Since $S \cap Ae \neq 0$ is completely reducible, it follows that there exists a minimal left ideal $L \subseteq Ae$ and an element $i \in I$ such that $Li \neq 0$. Therefore L is isomorphic to the minimal left ideal Li in I. Similarly, there exists a minimal left ideal L' in Ae isomorphic to a minimal left ideal in J. Hence, $K = Ae$ does the job and we are done.

Lemma (1.6). Let $A = B_1 \dotplus \cdots \dotplus B_n$ and $S = S_1 \dotplus \cdots \dotplus S_m$ be the decompositions of A and S respectively into two-sided indecomposable ideals of A. Then $S(B_i) = \sum_{\substack{j \\ S_j \cap B_i \neq 0}} S_j$.

Proof. On one hand, we have a decomposition

$$1) \quad S = \dot{\sum_{1 \leq i \leq n}} S(B_i) \quad .$$

On the other hand, $S = \dot{\sum}_{1 \leq j \leq m} S_j$ is, according to theorem (1. 3), the unique decomposition of S into two-sided indecomposables of A, and hence must refine the decomposition in 1).

We now give an example which shows that P. I. 's may be linked in a rather complicated way in spite of lemma (1. 5).

Example (1. 2). Let A be the ring of all matrixes of the form

$$\begin{bmatrix} a_{11} & o & o & o & o & o & o \\ o & a_{22} & o & o & o & o & o \\ a_{31} & a_{32} & a_{33} & o & o & o & o \\ o & o & o & a_{44} & o & o & o \\ o & a_{52} & o & a_{54} & a_{55} & o & o \\ o & o & o & o & o & a_{66} & o \\ a_{71} & o & o & o & o & a_{76} & a_{77} \end{bmatrix}$$

where the a_{ij} are elements of a skewfield D. Let e_{ij} be the matrix units of this ring. We see that Ae_{33}, Ae_{55} and Ae_{77} are the idempotent minimal left ideals and that every minimal left ideal is isomorphic to one of them. Hence $0:S = 0$ by theorem (1. 2). In fact, we see that the socle breaks up into the direct sum of three indecomposables: $S = S_3 \dot{+} S_5 \dot{+} S_7$ where $S_i = e_{ii}A$. We may convince ourselves, keeping lemma (1. 5) in mind, that A is really indecomposable.

The way, in the previous example, in which the indecomposable constituents of the socle are generated by idempotents suggests a theorem (compare with [1] - theorem (6. 2)).

Theorem (1.7). Let $A = Ae_1 \dotplus \cdots \dotplus Ae_s \dotplus Af_{11} \dotplus \cdots \dotplus Af_{1K_1} \dotplus \cdots \dotplus Af_{t1} \dotplus \cdots \dotplus Af_{tK_t}$ be a decomposition of A into P. I. 's Ae_i, Af_{ij} where the Af_{ij} are idempotent minimal left ideals and arranged so that $Af_{ij} \simeq Af_{i'j'}$ iff $i = i'$ and where none of the Ae_i are minimal. Define idempotents $g, g_1, g_2, \ldots$ by $g_1 = f_{11} + \cdots + f_{1K_1}, \ldots, g_t = f_{t1} + \cdots + f_{tK_t}$; $g = g_1 + \cdots + g_t$ and let $S_i = Af_{i1}A$. Then $S = gA$ and $S_i = g_iA$ for $1 \leq i \leq t$.

Proof. Note that by theorem (1.3), $S = S_1 \dotplus S_2 \dotplus \cdots \dotplus S_t$ is just the decomposition of S into indecomposable two-sided ideals of A.

Now we have arranged things so that of all the given primitive idempotents $e_1, \ldots, e_s$; $f_{11}, \ldots, f_{1K_1}$;$\ldots$; $f_{i1}, \ldots, f_{iK_i}$;$\ldots$;$f_{t1}, \ldots, f_{tK_t}$, only the idempotents $f_{i1}, f_{i2}, \ldots, f_{iK_i}$ fail to annihilate Af_{i1} on the left (see 1. B). It follows that $Af_{i1} = f_{i1}Af_{i1} + f_{i2}Af_{i1} + \cdots + f_{iK_i}Af_{i1} = g_iAf_{i1}$. Hence $g_iL = L$ whenever L is a minimal left ideal isomorphic to Af_{i1}. Therefore, $g_iS_i = S_i$ by theorem (1.3). Since $Ag_i = \sum_{\alpha=1}^{K_i} Af_{i\alpha} \subseteq S_i$, $g_i \in S_i$. So $g_iA \subseteq S_i$ (S_i is two-sided). We have: $S_i = g_iS_i \subseteq g_iA \subseteq S_i$, and we are done.

II. The Subdirect Sum Representation

Recall that A is a subdirect sum of the rings A_α, $\alpha \in \Omega$, if $A \subseteq \bigoplus_{\alpha \in \Omega} A_\alpha$ and the natural projection: $A \to A_\alpha$ is onto. If $T_\alpha = \text{Ker}\,(A \to A_\alpha)$, then $\bigcap_\alpha T_\alpha = 0$ and $A_\alpha \simeq A/T_\alpha$. If T'_α, $\alpha \in \Omega$ is a collection of two-sided ideals of A such that $\bigcap_\alpha T'_\alpha = 0$, then A is a subdirect sum of the rings A/T'_α, $\alpha \in \Omega$. By the minimal condition, there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Omega$ such that $0 = \bigcap_{i=1}^{n} T_{\alpha_i}$. That is, A is a subdirect sum of a finite subset of the A_α. We may assume that this last intersection is irredundant - i. e., that $T_{\alpha_1} \cap \ldots \cap \hat{T}_{\alpha_i} \cap \ldots \cap T_{\alpha_n} \supset 0$ for every i. So in what follows, when we say that A is a subdirect sum of certain rings, we shall always mean that A is a finite subdirect sum of these rings which is irredundant in the above sense.

We start in by giving three examples.

Example (2. 1). Let A be the ring of all matrices of the form $\begin{bmatrix} a & o & o \\ o & a & o \\ b & c & d \end{bmatrix}$ where a, b, c, d are arbitrary elements of some division ring D. This ring is faithfully represented on its left socle and has only one isomorphism class of minimal left ideals. The left ideals generated by the matrix units e_{31} and e_{32} are, in fact, two-sided. They give rise as explained in the first paragraph of this section to a representation of A as a subdirect sum of rings A_1 and A_2 having the properties:

1) Both A_1 and A_2 are ring isomorphic to the ring of all matrices of the form $\begin{bmatrix} a & o \\ b & c \end{bmatrix}$, a, b, c $\in D$.

2) A_1 and A_2 are faithfully represented on their left socles and both contain only one isomorphism class of minimal left ideals.
3) The minimal left ideals of A_1 and A_2 are isomorphic as left A-modules.

Example (2. 2). A is the ring of all matrices

$$\begin{bmatrix} a & o & o & o \\ b & c & o & o \\ o & o & a & o \\ o & d & e & f \end{bmatrix}$$ where a, b, c, d, e, f $\in$ D, D a skewfield. The

product of two elements of A is given by:

$$\begin{bmatrix} a & o & o & o \\ b & c & o & o \\ o & o & a & o \\ o & d & e & f \end{bmatrix} \begin{bmatrix} a' & o & o & o \\ b' & c' & o & o \\ o & o & a' & o \\ o & d' & e' & f' \end{bmatrix} = \begin{bmatrix} aa' & o & o & o \\ ba' + cb' & cc' & o & o \\ o & o & aa' & o \\ o & dc' + fd' & ea' + fe' & ff' \end{bmatrix}.$$

A is not faithfully represented on its left socle. The kernel of the socle representation is obviously Ae_{21}. Ae_{21} and $e_{44}A$ are two-sided and $Ae_{21} \cap e_{44}A = 0$. Hence we have a representation of A as a subdirect sum of the rings A_1 and A_2 where $A_1 \simeq A/Ae_{21} \simeq$ the ring of all matrices of the form $\begin{bmatrix} c & o & o \\ o & a & o \\ d & e & f \end{bmatrix}$; a, c, d, e, f $\in$ D

and $A_2 \simeq A/e_{44}A \simeq$ the ring of all matrices of the form $\begin{bmatrix} a & o \\ b & c \end{bmatrix}$; a, b, c $\in$ D.

The rings A_1 and A_2 are both faithfully represented on their left socles and both contain only one class of minimal left ideals. Thus a subdirect sum of rings faithfully represented on their left socles need not itself be faithfully represented on its left socle. However, the minimal left ideals of A_1 and A_2 are non-isomorphic as left A-modules. One can see that A is even a unique subdirect sum of rings possessing these two properties.

Example (2. 3). The ring of example (1. 1) can be shown to be the subdirect sum of two rings each ring-isomorphic to the ring of all matrices of the form $\begin{bmatrix} a & o \\ b & c \end{bmatrix}$; a, b, c $\in$ D. Note that this ring has only one isomorphism class of minimal left ideals.

Theorem (2. 1). Suppose A is a subdirect sum of rings A_i having the properties

(i) A_i contains only one isomorphism class of minimal left ideals.

(ii) The minimal left ideals of different A_i are non-isomorphic as left A-modules.

Then A is faithfully represented on its left socle iff every minimal left ideal of each A_i is isomorphic as a left A-module to an idempotent minimal left ideal of A. This being the case, each A_i is faithfully represented on its left socle.

Proof. Set $T_i = \mathrm{Ker}(A \to A_i)$, $1 \leq i \leq q$, and let $L_1, \ldots, L_p$ be a full set of non-isomorphic minimal left ideals of A.

If L is minimal and $L \nsubseteq T_i$, some i, then $L \simeq L + T_i/T_i \simeq$ every minimal left ideal of A_i. Furthermore, every minimal left ideal of A fails to be contained in some T_i since $\bigcap_i T_i = 0$. Therefore, every L_i fails to be contained in a unique T_j by (ii). We may assume $L_1 \nsubseteq T_1, \ldots, L_p \nsubseteq T_p$. Therefore, $T_i \supseteq S$ if $i > p$. But then, $S \cap (\bigcap_{j \neq i} T_j) \subseteq T_i \cap (\bigcap_{j \neq i} T_j) = 0$ so that $\bigcap_{j \neq i} T_j = 0$ which contradicts the irredundancy. Therefore, $q = p$.

If $0{:}S = 0$, theorem (1. 2) says that each L_i is isomorphic to an idempotent minimal left ideal L_i' of A. In this case, the image $\overline{L_i'}$ of L_i' in A_i satisfies $(\overline{L_i'})^2 = \overline{L_i'}$. Applying theorem (1. 2) once more, we see that each A_i is faithfully represented on its left socle and that any minimal left ideal of an A_i is isomorphic to an idempotent minimal ideal of A.

Suppose that $0:S \neq 0$. Let S_i be the sum of all minimal left ideals of A isomorphic to L_i. Clearly $T_i \supseteq S_1 \dotplus \cdots \dotplus \hat{S}_i \dotplus \cdots \dotplus S_p$. Therefore $T_i \cap S_i = 0$ since $\bigcap_i T_i = 0$. Hence $T_i \subseteq 0:S_i$. If $T_i = 0:S_i$ for all i, then $0:S = 0:\sum_i S_i = \bigcap_i (0:S_i) = \bigcap_i T_i = 0$, a contradiction. Hence $T_i \neq 0:S_i$, for some i. Say $T_1 \subset 0:S_1$.

Now there exists a left ideal I such that $T_1 \subset I \subseteq 0:S_1$ and such that $\widetilde{I}$ is a minimal left ideal of $\widetilde{A}$ where $\widetilde{A} = A/T_1$. Since $\widetilde{I}$ is a left A-module, there exists a primitive idempotent $e_1 \in A$ such that $e_1\widetilde{I} \neq \widetilde{0}$. $\widetilde{I} \simeq Ae_1/Ne_1$ since $\widetilde{I}$ is an irreducible left A-module.

We shall assume that $Ne_1 = 0$ and show that this leads to a contradiction. If $Ne_1 = 0$, Ae_1 is a minimal left ideal of A so that $Ae_1 \subseteq S_i$, some i. Then $e_1 \in S_i$. So $e_1 I \subseteq S_i I \subseteq S_i \subseteq T_j$ for $j \neq i$. But $e_1 I \nsubseteq T_1$ as otherwise $e_1\widetilde{I} = \widetilde{0}$. Therefore $e_1 \in S_1$. So $e_1 I = 0$ since $I \subseteq 0:S_1$. That is, $e_1\widetilde{I} = \widetilde{0}$ which gives the desired contradiction.

We have shown that $A_1 \simeq A/T_1$ has a minimal left ideal which is not isomorphic to an idempotent minimal left ideal of A, which proves the theorem.

Theorem (2. 2). If A is faithfully represented on its left socle, then A is a unique subdirect sum of rings A_i having the properties
(1) A_i is faithfully represented on its left socle and has only one isomorphism class of minimal left ideals.
(2) The minimal left ideals of different A_i are non-isomorphic as left A-modules.

Proof. Let $S = S_1 \dotplus \cdots \dotplus S_r$ be the decomposition of the socle into indecomposable two-sided ideals of A. Then: $\bigcap_i (0:S_i) = 0:\sum_i S_i = 0:S = 0$. Furthermore $\bigcap_{j \neq i}(0:S_j) = 0:\sum_{j \neq i} S_j \supseteq S_i$. So the intersection $\bigcap_i (0:S_i) = 0$ is irredundant. Therefore, A is a subdirect sum of the rings $A/(0:S_i)$.

Let L be a minimal left ideal of $\bar{A}$ where $\bar{A} = A/(0:S_i)$. Then $L = \bar{I}$ where I is a left ideal of A such that $I \supset 0:S_i$. Let $Ae \subseteq S_i$ where e is an idempotent and Ae is a minimal left ideal. Using lemma (1. 1) (see also 1. A) and theorem (1. 3), we have $S_i eI = S_i AeAI = (S_i S_i)I = (S_i \cap S_i)I = S_i I \neq 0$, since $I \supset (0:S_i)$. Hence $\bar{e}L = \bar{e}\bar{I} \neq \bar{0}$. Now $\bar{A}\bar{e}$ is a P. I. of $\bar{A}$ and $\bar{N} = N + (0:S_i)/(0:S_i)$ is the radical of $\bar{A}$. Therefore $L \simeq \bar{A}\bar{e}/\bar{N}\bar{e} \simeq \bar{A}\bar{e}$. It now follows by theorem (1. 2) that the rings $A_i = A/(0:S_i)$ satisfy properties (1) and (2) of the statement of the theorem we are proving.

On the other hand, suppose that A is a subdirect sum of rings A_i satisfying (1) and (2). Set $T_i = \mathrm{Ker}\,(A \to A_i)$. We may prove as in theorem (2. 1) (only it is even easier now) that $T_i \subseteq 0:S_i$. Now S_i contains an idempotent e such that Ae is a minimal left ideal of A. $e \notin T_i$ since $e \notin 0:S_i$. Hence it follows by theorem (1. 3) that $S_i \equiv S(A/T_i) \pmod{T_i}$. Therefore $x \in 0:S_i$ implies that $S_i x \equiv 0 \pmod{T_i}$ so that $x \in T_i$ since A/T_i is faithfully represented on its left socle by hypothesis. We have that $T_i = 0:S_i$, all i, which proves the uniqueness.

Corollary. If A is faithfully represented on its left socle and contains a unique minimal two-sided ideal, then A contains only one isomorphism class of minimal left ideals.

Proof. The hypothesis just means that 0 is a meet irreducible of the lattice of two-sided ideals of A.

Of course this was obvious earlier from the point of view that if $S = \dot{\sum_i} S_i$ is the decomposition of S into two-sided indecomposables S_i, then each S_i contains a minimal two-sided ideal.

It follows from lemma (6. 3) and theorem (6. 4) of [1] that any ring satisfying property (1) of our previous theorem (2. 2) is a subdirect sum of rings which also satisfy property (1), but have the added property of containing a unique two-sided minimal ideal. We conjecture that such a subdirect sum representation is also unique. Of course, the number of subdirect summands is uniquely determined.

Theorem (2. 3). If T, Q are two-sided ideals such that $A = T \dotplus Q$, then T is a subdirect sum of a subset of the uniquely determined (by theorem 2. 2) set of rings of which A is a subdirect sum (assuming, of course, that $0{:}S = 0$).

Proof. It is clearly sufficient to prove this for a block B of A. Let $S_1, \dots, S_t$ be the indecomposable two-sided direct summands of the socle which are contained in B. Then $S(B) = S_1 \dotplus \dots \dotplus S_t$ by lemma (1. 6). Since B is faithfully represented on its left socle (lemma (1. 4)), it follows that $0{:}S(B)$ is just the sum of all blocks of A which are different from B. Therefore $B \simeq A/0{:}S(B) = \overline{A}$. By lemma (1. 1), we have $(0{:}S(B)){:}S(B) = 0{:}S(B)S(B) = 0{:}S(B)$ and $(0{:}S(B)){:}S_i = 0{:}S(B)S_i = 0{:}S_i$ for all i. Hence $\overline{0} = \overline{0{:}S(B)} = \overline{0}{:}\sum_i \overline{S}_i$ $= \bigcap_i (\overline{0}{:}\overline{S}_i) = \bigcap_i \overline{(0{:}S_i)}$. This intersection is clearly irredundant, so we have that B is a subdirect sum of the rings $\overline{A}/\overline{(0{:}S_i)} \simeq A/(0{:}S(B) + 0{:}S_i) = A/0{:}S_i$ and we are done.

From theorem (2. 2) we see that it would behoove us to study more closely rings which have only one isomorphism class of minimal left ideals.

III. Some Rings Which Have Only One Isomorphism Class of Minimal Left Ideals

It follows trivially from lemma (1. 5) that an indecomposable ring which is faithfully represented on its left socle and in which every (left) P. I. contains a unique minimal left ideal has only one isomorphism class of minimal left ideals. Goldie [1] has shown this by a different method. A particular case of this is an indecomposable ring in which every left P. I. possesses a unique composition series (and in which $0:S = 0$). Goldie [1] has characterized such rings in which every right P. I. also possesses a unique composition series. They are just the blocked-triangular matrix rings with entries in a skewfield.

For completeness, we shall give an example of an indecomposable ring in which both the left and right P. I. 's have unique composition series and which is not faithfully represented on its left socle. It is a factor ring of a ring of Goldie's characterization theorem.

Example (3. 1). Let A be the ring of all matrices of the form
$\begin{bmatrix} a & o & o \\ b & c & o \\ d & e & f \end{bmatrix}$; where a, b, c, d, e, f are arbitrary elements of a skewfield D.

By the above remarks, it is sufficient to show (except for the indecomposability) that some factor ring of A is not faithfully represented on its left socle.

Set $T = Ae_{31}$, $I = Ae_{21} + Ae_{31}$, $N = Ae_{21} + Ae_{31} + Ae_{32}$ (N is the radical of A) and $e = e_{22}$ where the e_{ij} are matrix units.

Clearly I is a left ideal and T is a two-sided ideal (in fact, T happens to be the unique minimal two-sided ideal of A).

Now $eI \nsubseteq T$ and $Ne \nsubseteq T$. Therefore $\overline{eI} \neq \overline{0}$ and $\overline{Ne} \neq \overline{0}$ where $\overline{A} = A/T$. But $\overline{I}$ is obviously minimal so that $\overline{I} \simeq \overline{Ae}/\overline{Ne}$ (since $\overline{N}$ is the radical of $\overline{A}$ and $\overline{Ae}$ is a P. I. of $\overline{A}$). By theorem (1. 2) $\overline{A}$ is not faithfully represented on its left socle. It is clear by block theory (see (1. E)) that $\overline{A}$ is indecomposable.

(3. A). Before we give more examples it is convenient to define what we mean by Loewy length. By the Loewy length $\ell(I)$ of a left ideal I of A we just mean the length of the series $I \supset NI \supset N^2 I \supset \ldots \supset 0$. This is the same as the length of the series $0 \subset (0:N) \cap I \subset (0:N^2) \cap I \subset \ldots \subset I$. We call $\ell(A)$ the exponent of the radical. Observe that $N^{\ell(I) - K} I \subseteq (0:N^K) \cap I$ for $0 \leq K \leq \ell(I)$ is trivial. In particular, if I has a unique minimal left ideal, lemma (1. 1) implies that it is $SI = N^{\ell(I) - 1} I$.

We shall now give some examples of indecomposable rings in which $0:S = 0$ and every left P. I. has a unique minimal left ideal. We shall not try to really prove any of our assertions about these examples since we shall eventually have general theorems which will easily handle all of them.

(3. B). First, since all of our examples will be matrix rings, let us establish a convention about matrices which will make our work much easier. Namely, $\begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1n} \\ R_{21} & R_{22} & \cdots & R_{2n} \\ \vdots & \vdots & & \vdots \\ R_{n1} & R_{n2} & \cdots & R_{nn} \end{bmatrix}$ where the R_{ij} are n^2

rings (some of them may be the null-ring), shall stand for the set of all n x n matrices whose i, $j^{\underline{th}}$ entry is an arbitrary element of R_{ij}. For example, $\begin{bmatrix} D & 0 \\ E & E \end{bmatrix}$ where D and E are skewfields and $E \supseteq D$ is just the ring of all matrices of the form $\begin{bmatrix} a & o \\ b & c \end{bmatrix}$ where $a \in D$ and $b, c \in E$.

Example (3.2). The ring $A = \begin{bmatrix} D & 0 & 0 & 0 \\ 0 & D & 0 & 0 \\ D & 0 & D & 0 \\ D & D & D & D \end{bmatrix}$; D a skewfield.

A has four P.I.'s given by Ae_{11}, Ae_{22}, Ae_{33}, Ae_{44} where the e_{ij} are matrix units. The radical N is given by

$$N = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ D & 0 & 0 & 0 \\ D & D & D & 0 \end{bmatrix}, \quad N^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ D & 0 & 0 & 0 \end{bmatrix} \text{ and } N^3 = 0. \text{ Hence}$$

$t(A) = 3 = t(Ae_{11})$, $t(Ae_{22}) = 2 = t(Ae_{33})$. Note that the number of non-isomorphic left P.I.'s = 4 > 3 = exponent of the radical. Note also that $e_{33}Ae_{11} \neq 0$ and $e_{12}Ae_{11} = 0$.

It is true that in this ring every left P.I. has a unique composition series. By Goldie's characterization theorem mentioned above, some right P.I. must have at least two different composition series. Indeed, it is easy to see that the right P.I. $e_{44}A$ has two non-isomorphic minimal right ideals.

Example (3.3). The ring $A = \begin{bmatrix} D & 0 & 0 & 0 \\ D & D & 0 & 0 \\ D & 0 & D & 0 \\ D & D & D & D \end{bmatrix}$. Here $t(Ae_{11}) = 3$ and $t(Ae_{22}) = t(Ae_{33}) = 2$ but $e_{33}Ae_{11} \neq 0$ and $e_{22}Ae_{11} \neq 0$. Ae_{11} fails to have a unique composition series.

Example (3. 4). The ring $\begin{bmatrix} D & 0 & 0 \\ E & E & 0 \\ F & F & F \end{bmatrix}$ where D, E and F are skewfields such that $F \supseteq E \supseteq D$. Every left P. I. of this ring has a unique composition series. This ring has minimal condition on right ideals iff F is finite dimensional as a right vector space over D.

Example (3. 5). Let K be a field and let K(x) be the field generated over K by the algebraic element x of degree $n > 1$. Set $H = Kx$. We have $K(x) \supseteq H \not\supseteq K$ and H is one-dimensional as a left vector space over K. Then every left P. I. of the ring $\begin{bmatrix} K & 0 & 0 \\ H & K & 0 \\ K(x) & K(x) & K(x) \end{bmatrix}$ has a unique composition series. This example would not work if $[H:K] > 1$.

We now prove two technical lemmas (we are again assuming $0{:}S = 0$):

Lemma (3. 1). If I is a left ideal of A containing a unique minimal left ideal and f is an A-homomorphism: $I \to A$, then f is either zero or a monomorphism.

Proof. By lemma (1. 1), the unique minimal left ideal of I must be SI. If $\text{Ker } f \neq 0$ where $f{:}I \to A$, then $Sf(I) = f(SI) \subseteq f(\text{Ker } f) = 0$ which implies $f(I) = 0$.

Lemma (3. 2). Let A be a ring in which every P. I. has a unique minimal left ideal. Suppose that Ae is a P. I. (e idempotent) and that I is any left ideal. Then

(1) $AeI \neq 0$ iff there exists a subideal J of I such that Ae is isomorphic to J. In particular, $AeI = 0$ if $\ell(Ae) > \ell(I)$.
(2) If $\ell(Ae) \leq \ell(I)$, then $eN^{\ell(I) - \ell(Ae) + 1}I = 0$.

Proof. (1) If $Aei \neq 0$, there exists $i \in I$ such that $Aei \neq 0$. Therefore $x \in Ae \to xi$ gives a non-zero A-homomorphism of Ae onto $Aei \leq I$ ($\leq$ means "subideal of"). By lemma (3. 1) this must be an isomorphism. The converse is obvious.
(2) $SeN^{\ell(I) - \ell(Ae) + 1}I = N^{\ell(Ae) - 1}eN^{\ell(I) - \ell(Ae) + 1}I \subseteq N^{\ell(I)}I = 0.$
The first equality sign is a consequence of the discussion in (3. A).

An immediate consequence of (2) is that eAe is a skewfield for any primitive idempotent e. Hence, if M is any left A-module possessing a composition series, then eM is a left vector space over eAe of dimension equal to the number of composition factors of M isomorphic to Ae/Ne (see [3] - theorem 9. 5A).

We now give a lemma which shows why the one-dimensionality was necessary in example (3. 5):

Lemma (3. 3). Suppose that every P. I. of A has a unique minimal left ideal. Let I be a left ideal and Ae a P. I.. Then
(1) Every subideal of I isomorphic to Ae is of the form Aex for some $x \in I$.
(2) AeI is an irredundant (but not necessarily direct) sum of subideals of I isomorphic to Ae, the number of irredundant summands being uniquely determined as the number of composition factors of I isomorphic to Ae/Ne.
(3) If Ae is minimal and I contains a unique minimal left ideal, then eI is a one-dimensional left vector space over eAe.

Proof. (1) follows from lemma (3. 1) and the fact that every left ideal has finite composition series length.

To prove (2), we write $eI = eAex_1 \dot{+} \cdots \dot{+} eAex_n$ where $x_i \in eI$. Then $AeI = Aex_1 + \cdots + Aex_n$ and $Aex_i \simeq Ae$ since $Aex_i \neq 0$. Now $Aex_i \subseteq \sum_{j \neq i} Aex_j$ implies $eAex_i \subseteq \sum_{j \neq i} eAex_j$ which is impossible since eI was the direct sum of the $eAex_j$. Hence the sum $Aex_1 + \cdots + Aex_n$ is irredundant.

To prove uniqueness, assume that AeI is the irredundant sum of $I_1, I_2, \ldots, I_p$ where $I_j \simeq Ae$ and $I_j \leq I$, $1 \leq j \leq p$. By (1), there exists $y_j \in I_j$ such that $I_j = Aey_j$. Therefore, $AeI = \sum_{j=1}^{p} Aey_j$ so that $eI = eAeI = \sum_{j=1}^{p} eAey_j$. If this last sum is not direct, then there exists i such that $eAey_i \subseteq \sum_{j \neq i} eAey_j$ so that $Aey_i \subseteq \sum_{j \neq i} Aey_j$. But this contradicts the irredundancy of $I_1, I_2, \ldots, I_p$. Hence we must have $p = n$.

(3) is a direct consequence of (2). In fact, from the proof in (2), we can see that (3) holds in any ring with minimal condition on left ideals and an identity.

It is trivial that any ring with identity and minimal condition on left ideals contains a P. I. of Loewy length ρ where ρ is the exponent of the radical. It is easy to see that the rings of examples (3. 2) through (3. 5) contain P. I. 's of every possible Loewy length. On the other hand, we shall give later (example (3. 6)) an example of a very simple ring which is faithfully represented on its left socle, contains only one isomorphism class of minimal left ideals, but which does not possess P. I. 's of every possible Loewy length.

Nevertheless, we can prove:

Lemma (3.4). If every P. I. of A has a unique minimal left ideal, then A contains P. I.'s of Loewy length 1, 2, ..., ρ where $\rho = \exp(N)$.

Proof. Let $A = \dot{\sum}_{i \in I} Ae_i$ be a decomposition of A into P. I.'s Ae_i (the e_i's are idempotents). Fix K, $1 \leq K \leq \rho$. Then $N^{K-1} = \dot{\sum}_{j \in J} N^{K-1} e_j$ where $J = \{i \in I \mid N^{K-1} e_i \neq 0\}$. $J \neq \Phi$ since $K - 1 < \rho$.

Suppose that $N^K e_j \neq 0$ for every $j \in J$. Then $SN^{K-1} e_j \subseteq Se_j \subseteq N^K e_j$ for all $j \in J$. Therefore, $SN^{K-1} = \sum_{j \in J} SN^{K-1} e_j \subseteq \sum_{j \in J} N^K e_j = N^K$ since, for $i \in I$, $SN^{K-1} e_i \neq 0$ iff $N^{K-1} e_i \neq 0$ iff $N^K e_i \neq 0$ iff $i \in J$. But $SN^{K-1} \subseteq N^K$ implies $SN^{K-1} N^{\rho - K} \subseteq N^K N^{\rho - K}$ which implies $SN^{\rho - 1} \subseteq 0$ so that $N^{\rho - 1} = 0$, a contradiction. Hence there exists $j \in J$ such that $N^K e_j = 0$ whence $\ell(Ae_j) = K$.

Lemma (3.4) is a generalization of lemma (8.1) in [1].

Lemma (3.5). Let $Ae_1, \ldots, Ae_r$ be a full set of non-isomorphic P. I.'s of A. Then there exists a permutation $i_1, \ldots, i_r$ of $1, \ldots, r$ such that $Ae_{i_\alpha} Ae_{i_\beta} = 0$ if $\alpha < \beta$. If $j_1, \ldots, j_r$ is any such permutation, then Ae_{j_r} must be minimal.

Proof. By lemma (3.2) - (1), $\ell(Ae_i) \geq \ell(Ae_j)$ implies $Ae_i Ae_j = 0$ if $i \neq j$. So all we have to do is order the Ae_i by letting the Ae_i of Loewy length ρ come first (in any order) followed by the ones of Loewy length $\rho - 1$ (also in any order) finally ending up with the minimal P. I.'s coming last.

To prove the last statement of the lemma, we must only observe that given any P. I. Af there exists a minimal P. I. Ae such that $AeAf \neq 0$. Namely, take $Ae \simeq Sf$ (theorem (1. 2)).

Example (3. 6). Let A be the ring of all matrices of the form $\begin{bmatrix} a & o & o \\ b & a & o \\ c & d & e \end{bmatrix}$; a, b, c, d, e $\in$ D, D a skewfield. A is faithfully represented on its left socle and has only one isomorphism class of minimal left ideals. Set $e_1 = \begin{bmatrix} 1 & o & o \\ o & 1 & o \\ o & o & o \end{bmatrix}$ and $e_2 = \begin{bmatrix} o & o & o \\ o & o & o \\ o & o & 1 \end{bmatrix}$.
e_1Ae_1 is ring-isomorphic to the ring of all matrices $\begin{bmatrix} a & o \\ b & a \end{bmatrix}$. Hence e_1Ae_1 is completely primary but not a skewfield (see 1. C). It is easy to see that $\ell(Ae_1) = 3$ and $\ell(Ae_2) = 1$ so that lemma (3. 3) fails in this case. Notice that A/S is isomorphic to e_1Ae_1 so that A/S is not faithfully represented on its left socle.

In spite of examples (3. 6) and (3. 1) we can prove:

Theorem (3. 6). Suppose A is a ring such that every P. I. has a unique minimal left ideal. Let $Ae_1, \ldots, Ae_r$ be a full set of non-isomorphic P. I.'s of A of Loewy length K where $1 \leq K \leq \rho = \exp(N)$. Set $\bar{A} = A/0{:}N^{K-1}$. Then $\overline{Ae_1}, \ldots, \overline{Ae_r}$ constitute a full set of non-isomorphic minimal left ideals of $\bar{A}$. In particular, $\bar{A}$ is faithfully represented on its left socle (by theorem 1. 2).

Proof. $N^{K-1}Ne_i = N^Ke_i = 0$ so $Ne_i \subseteq (0{:}N^{K-1}) \cap Ae_i \subseteq Ae_i$; but $(0{:}N^{K-1}) \cap Ae_i = Ae_i$ implies $Ae_i \subseteq 0{:}N^{K-1}$ so that $N^{K-1}e_i = 0$, a contradiction. Therefore $(0{:}N^{K-1}) \cap Ae_i = Ne_i$. It follows that

$\overline{Ae_i} \simeq Ae_i/Ne_i$. Hence the $\overline{Ae_i}$ are certainly minimal and distinct from one another.

Let L be a minimal left ideal of $\overline{A}$. Then $L = \overline{I}$ where I is a left ideal of A containing $0{:}N^{K-1}$. Since $\overline{I}$ is an irreducible left A-module, there exists a primitive idempotent $e \in A$ such that $e\overline{I} \neq \overline{0}$. Hence there exists $i \in I$ such that $Aei \nsubseteq 0{:}N^{K-1}$ whence $I = Aei + 0{:}N^{K-1}$. Then $N^{K-1}I = N^{K-1}Aei \neq 0$ (since $\overline{I} \neq \overline{0}$) so that $\mathfrak{t}(I) = \mathfrak{t}(Aei)$. Since $\overline{N}$ is the radical of $\overline{A}$, $\overline{NI} = \overline{0}$ which implies $N^K I = N^{K-1}NI = 0$. Therefore $\mathfrak{t}(Aei) = \mathfrak{t}(I) = K$. But since Ae contains a unique minimal left ideal, $Ae \simeq Aei$ so that, in particular, $\mathfrak{t}(Ae) = K$. Hence $Ae \simeq Ae_j$, some j such that $1 \leq j \leq r$. Since $L = \overline{I}$ is irreducible and $eL \neq \overline{0}$, we now have $L \simeq Ae/Ne \simeq Ae_j/Ne_j$ which proves the theorem.

Corollary. If Af is a P. I. of Loewy length $K + 1$ where $1 \leq K \leq \rho - 1$, then there exists a P. I. Ae of Loewy length K such that $AeAf \neq 0$.

Proof. For $K = 1$, this is a consequence of $0{:}S = 0$. Hence we assume that $K \geq 2$. Set $\overline{A} = A/0{:}N^{K-1}$. Now $\overline{Af} \neq \overline{0}$, since $\mathfrak{t}(Af) = K + 1$. Hence there exists a minimal left ideal L of $\overline{A}$ such that $L \subseteq \overline{Af}$. By the theorem, $L \simeq Ae/Ne$ where Ae is a P. I. of Loewy length K. But $\overline{Af} \simeq Af/Af \cap (0{:}N^{K-1})$ and this just means that $AeAf \neq 0$.

In the rest of this section we assume that every P. I. of A has a unique minimal left ideal (and that $0{:}S = 0$). We now give some material which will eventually lead to a sort of classification of rings in which every P. I. has a unique composition series.

(3.C). If $\tau(Ae) = K$, the corollary to theorem (3.6) provides us with left ideals $I_1, I_2, \ldots, I_K$ such that $Ae = I_K \supset I_{K-1} \supset \ldots \supset I_1 \supset 0$ with the further property that I_j is isomorphic to a P.I. of Loewy length j. If Ae has a composition series without repeated factors, then the I_j are uniquely determined according to lemma (3.3) - (2). In this case one can show that $I_j \subseteq N^{\rho-j}$.

In what follows we denote by $\ell(I)$ the composition series length of the left ideal I of A.

Lemma (3.7). The P.I. Ae of Loewy length K has a unique composition series iff it has a composition series of the form $Ae = I_K \supset I_{K-1} \supset \ldots \supset I_1 \supset 0$ where each I_j is isomorphic to a P.I. of Loewy length j.

Proof. If Ae has a unique composition series, then it is obviously given by $Ae \supset Ne \supset \cdots \supset N^{K-1}e \supset 0$. Then the series described in (3.C) does the job since it has length K.

On the other hand, let $Ae = I_K \supset I_{K-1} \supset \cdots \supset I_1 \supset 0$ be a composition series such that $I_j \underset{\Phi_j}{\simeq} Ae_j$ where $\tau(Ae_j) = j$. Then, of course, $\ell(Ae_j) = j$. Since I_j/I_{j-1} is irreducible, the restriction of Φ_j to I_{j-1} gives an isomorphism of I_{j-1} onto Ne_j. Hence $I_j/I_{j-1} \simeq Ae_j/Ne_j$. Therefore the Ae_j/Ne_j are the only composition factors of Ae and they are, furthermore, all different.

Now let I be any subideal of Ae and, say, $\ell(I) = i$. Then there must be i different Ae_j such that $Ae_jI \neq 0$. Since $Ae_jI \neq 0$ implies $Ae_j \simeq J \leq I$, these must be $Ae_1, Ae_2, \ldots, Ae_i$. Therefore, $Ae_i \simeq Ae_iI$ by lemma (3.3) - (2). Hence $Ae_iI = I$ since $Ae_iI \leq I$ and $\ell(Ae_iI) = \ell(Ae_i) = \tau(Ae_i) = i = \ell(I)$. But then $I = I_i$ since I_i is the unique subideal of Ae isomorphic to Ae_i (because Ae has a composition series without repeated factors).

Theorem (3.8). If a P.I. Ae of A has a composition series without repeated factors, then it has a unique composition series iff there exist no two non-isomorphic P.I.'s Af and Ag such that $fAe \neq 0$, $gAe \neq 0$ and $\mathfrak{t}(Af) = \mathfrak{t}(Ag)$.

Proof. Suppose that Af/Nf and Ag/Ng are composition factors of Ae where Af and Ag are P.I.'s and $\mathfrak{t}(Af) = \mathfrak{t}(Ag)$. Then $Af \simeq I \leq Ae$ and $Ag \simeq J \leq Ae$. If Ae has a unique composition series, then either $I \subseteq J$ or $J \subseteq I$, say $I \subseteq J$. But $(Af)^2 \neq 0 \Rightarrow AfI \neq 0$ $\Rightarrow AfJ \neq 0 \Rightarrow AfAg \neq 0$. Therefore, $Af \simeq Ag$ since $\mathfrak{t}(Af) = \mathfrak{t}(Ag)$, which proves the necessity.

For the sufficiency, we construct the series $Ae = I_K \supset I_{K-1} \supset \cdots \supset I_1 \supset 0$ where I_j is isomorphic to a P.I. Ae_j of Loewy length j (see 3.C). Then Ae_j/Ne_j is a composition factor of Ae for $1 \leq j \leq K$. Suppose Af/Nf is also a composition factor. Then $\mathfrak{t}(Af) = p$ for some p such that $1 \leq p \leq K$ since Af is isomorphic to a subideal of Ae. By hypothesis, $Af \simeq Ae_p$ so that $Af/Nf \simeq Ae_p/Ne_p$. Hence the Ae_j/Ne_j are the only composition factors of A. Since Ae has no repeated composition factors $Ae = I_K \supset I_{K-1} \supset \cdots \supset I_1 \supset 0$ is indeed a composition series and we are done by the previous lemma.

Corollary. If every P.I. has a composition series without repeated factors and if the exponent of the radical is equal to the number of non-isomorphic P.I.'s, then every P.I. has a unique composition series.

Proof. This follows directly from the theorem together with lemma (3. 4).

Note that the rings of examples (3. 4) and (3. 5) have the property of the corollary whereas the ring of example (3. 2) does not. However, by using theorem (3. 8) it is now trivial to prove that every P. I. of the ring of example (3. 2) has a unique composition series.

The rings which we are considering have many special right A-module properties even though they may well not have minimal condition on right ideals (example (3. 4)). For instance, we may prove a lemma for right ideals analogous to lemma (3. 1).

Lemma (3. 9). If Ae is a P. I. and I a right ideal, then $IeA \neq 0$ implies that eA is isomorphic to a subideal of I.

Proof. $IeA \neq 0$ implies that there exists $x \in I$ such that $xeA \neq 0$. Hence $\Phi: y \in eA \to xy$ is an A-homomorphism $eA \twoheadrightarrow xeA \leq I$. Since $xe \neq 0$, $0 \subset Sxe \subseteq Se$ whence $Sxe = Se$ (Se is the unique minimal left ideal of Ae). Therefore, if $xy = 0$ for $y \in eA$, then $Sxy = Sxey = Sey = Sy = 0$ so that $y = 0$. Therefore Φ must be an isomorphism.

Let us agree to call a P. I. Ae dominant if Ae is isomorphic to every P. I. which has a composition factor isomorphic to Ae/Ne (compare with the definition in [1] - p. 283). Clearly, every left ideal is isomorphic to a subideal of some dominant P. I. Also, every P. I. of Loewy length $\rho = \exp(N)$ is dominant. In fact, if $\ell(Ae) = \rho$, then $SeN = N^{\rho-1}eN \subseteq N^{\rho} = 0$ so that $eN = 0$ and eA is minimal.

Hence we can meaningfully define the right socle S' of A to be the sum of all minimal right ideals. By the left annihilator of a left ideal I we mean $(I)_{\ell} = \{a \in A \mid aI = 0\}$.

Lemma (3.10). Let e be a primitive idempotent of A. Then eA is a minimal right ideal iff Ae is dominant. Every minimal right ideal is isomorphic to an idempotent minimal right ideal and, furthermore, $(S')_{\ell} = 0$.

Proof. Suppose Ae is dominant. If eA is not minimal, then $eN \neq 0$ so that $eNf \neq 0$ for some primitive idempotent f. But $Ae \simeq Af$ since Ae is dominant and we contradict lemma (3.2) - (2).

Conversely, if eA is minimal, then $eN = 0$. Let Af be any P.I. Then $AeAf \neq 0 \Rightarrow e(Af/Nf) \neq 0 \Rightarrow Ae/Ne \simeq Af/Nf \Rightarrow Ae \simeq Af$ whence Ae is dominant.

Now let $I \neq 0$ be a right ideal. Then $Ig \neq 0$ for some primitive idempotent g. By lemma (3.9) I contains a right ideal isomorphic to gA. There exists a dominant P.I. Ah such that $gAh \neq 0$. Again by lemma (3.9), gA contains a subideal isomorphic to hA. Hence I contains a subideal isomorphic to hA; i.e., we have shown that every right ideal contains a right ideal isomorphic to an idempotent minimal right ideal. The rest of the proof is exactly like the proof of theorem (1.2) so that we omit it.

This lemma does not go through without all the special assumptions. For instance, the dual of the ring in example (3.6) is the ring of all matrices of the form $\begin{bmatrix} a & o & o \\ b & c & o \\ d & e & c \end{bmatrix}$; $a, b, c, d, e \in D$, D a skewfield. This ring is not faithfully represented on its left socle.

(3. B). With regard to dominant P. I. 's one can show that $(Ae)_{\ell} = 0$ for some dominant P. I. Ae iff all the minimal right ideals of A are isomorphic iff A contains a unique (up to isomorphism) dominant principle indecomposable. We shall explain later (4. B) what this means in terms of matrix representations. For now note that in example (3. 2) Ae_{11} is the unique P. I. of maximal Loewy length but that Ae_{22} is dominant whereas the unique (when there is one) dominant P. I. must have maximal Loewy length.

IV. Matrix Representations of Rings Which Are Faithfully Represented on Their Left Socle

(4. A). We proceed to construct a large class of examples which, in fact, turn out to be typical of indecomposable rings which are faithfully represented on their left socle and in which every P. I. (principal indecomposable) contains a unique minimal left ideal.

Let D be a skewfield and let D_{ij} be r^2 subgroups of the additive group of D with the properties:

(1) $D_{ij} = 0$ if $i < j$.

(2) $D_{ij}D_{jk} \subseteq D_{ik}$ for all i, j, k (under the natural multiplication in D).

(3) For every i, D_{ii} is a skewfield and $D_{ri} = D$.

(4) D_{ij} is finite dimensional as a left vector space over D_{ii}, for every i, j.

Note that (2) implies that every D_{ij} is a left D_{ii}-module as well as a right D_{jj}-module. Hence, by (3), the statement in (4) makes sense. Another consequence of (2) is that $D_{ik} = 0$ for $i \geq k$ implies that either $D_{ij} = 0$ or $D_{jk} = 0$.

Set $A = \left\{(a_{ij})^{1 \leq i, j \leq r} \mid a_{ij} \in D_{ij}\right\}$. (2) together with the fact that every D_{ii} is a skewfield imply that A is a ring and that the

$r \times r$ matrix $\begin{bmatrix} 1 & & & O \\ & 1 & & \\ & & \ddots & \\ O & & & 1 \end{bmatrix}$, where 1 is the identity of D,

is the identity of A. The assumption (in (3)) that $D_{ri} = D$ for every i is a key one. It will be clear that without this hypothesis it would not be true that every P. I. contains a unique minimal left ideal.

Let e_i be the matrix with 1 in the i, i position and 0 every place else. Then $A = Ae_1 \dot{+} Ae_2 \dot{+} \dots \dot{+} Ae_r$. Each e_iAe_i is obviously isomorphic to the skewfield D_{ii}. Furthermore, the set N of all matrices in A having no non-zero elements on their main diagonal is clearly the largest left ideal of A possessing the property that all of its elements are nilpotent. Hence every Ae_i will be a P. I. and N will be the (Jacobson) radical of A if we can only show that A has minimal condition on left ideals.

It is convenient at this time to introduce the simple ring $[D]_r$ of all $r \times r$ matrices with arbitrary entries in D. The rule $d \cdot (d_{ij}) = (dd_{ij})$ for $d \in D$ and $(d_{ij}) \in [D]_r$ makes $[D]_r$ into a D-module. Let e_{ij} be the matrix units of $[D]_r$. Then, since $e_i = e_{ii}$ for $1 \leq i \leq r$, we have: $e_iAe_j = D_{ij} \cdot e_{ij}$ for every i, j.

Now let $x_1, x_2, \dots, x_n$ be a basis for D_{ij} as a left vector space over D_{ii}. Then $D_{ij} = \dot{\sum}_{1 \leq k \leq n} D_{ii}x_k$ so $e_iAe_j = D_{ij} \cdot e_{ij}$

$$= \dot{\sum_{1 \leq k \leq n}} (D_{ii}x_k) \cdot e_{ij} = \dot{\sum_{1 \leq k \leq n}} (D_{ii} \cdot e_{ii})(x_k \cdot e_{ij}) = \dot{\sum_{1 \leq k \leq n}} e_iAe_i(x_k \cdot e_{ij}).$$

Hence the dimension of e_iAe_j as a left vector space over e_iAe_i is the same as the dimension of D_{ij} as a left vector space over D_{ii}. Similarly, the dimension of e_iAe_j as a right vector space over e_jAe_j is the same as the dimension of D_{ij} as a right vector space over D_{jj} (whether this dimension is finite or not).

We proceed to show that A has minimal condition on left ideals by showing that each Ae_j has a composition series. To this end, set $f_i = e_i + e_{i+1} + \dots + e_r$ for $i = 1, \dots, r$ and $f_{r+1} = 0$. Keeping in mind that f_iAe_j is just $\sum_{k=i}^{r} D_{kj} \cdot e_{kj}$, we let M_i be the left A-factor module $f_iAe_j/f_{i+1}Ae_j$ for $1 \leq i \leq r$. M_i is isomorphic to D_{ij} as an additive group. For t, s such that $t > s \geq i$,

$e_tAe_sf_iAe_j \subseteq e_tAe_j \subseteq f_{i+1}Ae_j$. Therefore, since $N = \sum_{\substack{t>s \\ 1\leq s, t\leq r}} e_tAe_s$, $NM_i = 0$. Hence M_i is a left-module over the semi-simple ring A/N. Now clearly, the action of A/N on M_i is the same as the action, by left multiplication, of D_{ii} on D_{ij}. But, since D_{ii} is a ring-homomorphic image of A/N, this just means that M_i is a direct sum of p irreducible A-modules(where p is the left dimension of D_{ij} over D_{ii}), all of them isomorphic to Ae_i/Ne_i. In particular, by (3), e_rAe_j is isomorphic as a left A-module to Ae_r, an idempotent minimal left ideal of A.

Now let I be any minimal left ideal contained in Ae_j. Since $e_iAe_k \subseteq N$ if $i \neq k$, $Ae_i \simeq Ae_k$ iff $i = k$. Hence, since I is minimal, there exists a unique e_i such that $e_iI = e_iAI \neq 0$. It follows that e_iI is a left ideal and hence that $e_iI = I$. Therefore, $I = e_iAIe_j \subseteq e_iAe_j$. But the action, by left multiplication, of e_rAe_i on e_iAe_j is the same as the action by left multiplication of $D_{ri} = D$ on D_{ij}. In particular, $e_rAe_iI \neq 0$. Therefore, $e_rAe_iI = I$ since e_rAe_i is a left ideal. We must have $i = r$. But $I = e_rAe_rIe_j \subseteq e_rAe_j$. Since e_rAe_j is minimal, $I = e_rAe_j$.

Since every minimal left ideal is isomorphic to a minimal left ideal contained in some Ae_i, theorem (1. 2) implies that A is faithfully represented on its left socle. That A is indecomposable is trivial. Furthermore, since by lemma (1. 1) $S = \dot{\sum_j} Se_j = \dot{\sum_j} (S \cap Ae_j)$ (S is the left socle of A), we have shown directly that $S = \dot{\sum_j} e_rAe_j = e_rA$. Notice that, on the other hand, $S = e_rA$ is a direct consequence of theorem (1. 7).

By the same argument as above, one can show that A has minimal condition on right ideals iff D_{ij} is finite dimensional as a right vector space over D_{jj} for all i, j, that is, iff e_rAe_i is finite dimensional as a right vector space over e_iAe_i for all i. Note that $D_{ij} = D$ for all $D_{ij} \neq 0$ iff e_rAe_i is a one-dimensional right vector space over e_iAe_i for all i. On the other hand, as in example (3.5), D_{ij} can well be a one-dimensional left vector space over D_{ii} for all $D_{ij} \neq 0$ without having $D_{ij} = D$ for every $D_{ij} \neq 0$.

(4.B). Let us investigate, momentarily, what it means for a P.I. Ae_i to be dominant. Clearly the dominance of Ae_i means that $D_{ik} = 0$ for $1 \leq k < i$. Hence, if $D_{i1} \neq 0$ for every i, Ae_1 is the unique dominant P.I. of A and all the minimal right ideals are isomorphic even though a given e_iA may contain an infinite direct sum of them.

On the other hand, if there exists a unique dominant P.I., let us assume inductively that $D_{i1} = 0$ and $D_{k1} \neq 0$ if $k < i$. But then $D_{ik} = 0$ for $k < i$ by (2) and Ae_i is dominant. This is a contradiction if $i \neq 1$, since Ae_1 is dominant. Hence the existence of a unique dominant P.I. just means that $D_{i1} \neq 0$ for all i. Note also that if $D_{i1} = 0$ and if j is minimal such that $D_{ij} \neq 0$, then e_iAe_j is a direct sum of minimal right ideals.

(4.C). We wish now to investigate what it means for A to have the following property (P):

(P) If i and i' are two distinct indices satisfying: $|\{k \mid D_{ki} \neq 0\}| = |\{k \mid D_{ki'} \neq 0\}|$, and j is another index with $j < i$, then either $D_{ij} = 0$ or $D_{i'j} = 0$.

It follows from what has been shown above that the composition series length of Ae_i is given by $\sum_{\substack{k \\ D_{ki} \neq 0}} [D_{ki}:D_{kk}]$ where $[D_{ki}:D_{kk}]$ is the left dimension of D_{ki} over D_{kk}. Hence, if $[D_{ki}:D_{kk}] = 1$

whenever $D_{ki} \neq 0$, lemma (3.7) implies that $|\{k | D_{ki} \neq 0\}| = \mathfrak{t}(Ae_i)$ (see (3.A) for the notation) iff Ae_i has a unique composition series. On the other hand, if Ae_i has a unique composition series, it is clear that $[D_{ki}:D_{kk}] = 1$ for every $D_{ki} \neq 0$.

We are now in a position to prove:

Proposition (4.1). Every P.I. of A has a unique composition series iff (P) holds and $[D_{ij}:D_{ii}] = 1$ for every i, j such that $D_{ij} \neq 0$.

Proof. We use induction on r for both the necessity and the sufficiency. The case $r = 1$ is trivial.

Let A* be the subring of A consisting of all $(r - 1) \times (r - 1)$ matrices formed by deleting the first column and the first row of every matrix in A. In other words, A* is just the ring $\{(a_{ij})^{2 \leq i, j \leq r} | a_{ij} \in D_{ij}; i, j \neq 1\}$ considered as imbedded in the natural way in A. Note that A* is a direct sum of the P.I.'s Ae_2, $Ae_3, \ldots, Ae_r$ which are equal in the same order to $A^*e_2, \ldots, A^*e_r$.

Suppose now that every P.I. of A has a unique composition series. Then the same thing clearly holds for every P.I. of A*. Hence we may assume by induction that (P) holds for $j \neq 1$. If (P) fails, then by lemma (3.7) there exist P.I.'s Ae_i and $Ae_{i'}$ such that $i \neq i'$, $\mathfrak{t}(Ae_i) = \mathfrak{t}(Ae_{i'})$, $Ae_iAe_1 \neq 0$ and $Ae_{i'}Ae_1 \neq 0$. But this is in direct violation of theorem (3.8).

Conversely, assume that (P) holds. Since (P) certainly holds in A*, we may assume by induction that Ae_2, $Ae_3, \ldots, Ae_r$ each have a unique composition series. If Ae_1 does not have a unique composition series, then there exist by theorem (3.8) i and i' such that $Ae_iAe_1 \neq 0$ and $Ae_{i'}Ae_1 \neq 0$ where $i, i' > 1$, $i \neq i'$ and $\mathfrak{t}(Ae_i) = \mathfrak{t}(Ae_{i'})$. But then it follows from lemma (3.7) that (P) is violated.

(4. D). Consider the series $0 = T_o \subset T_1 \subset T_2 \subset \ldots \subset T_r = S$ where S is the left socle and $T_i = e_rAe_1 \dot{+} e_rAe_2 \dot{+} \cdots \dot{+} e_rAe_i$. It is obvious that each T_i is a two-sided ideal. Also, T_i/T_{i-1} is isomorphic as a left A-module as well as a left e_rAe_r-module to e_rAe_i. Hence $0 \subset T_1 \subset T_2 \subset \ldots \subset T_r = S$ is a two-sided composition series for the socle and T_i/T_{i-1} is a one dimensional left vector space over e_rAe_r for $1 \leq i \leq r$.

Let $u_i \neq 0 \in e_rAe_i$. Then, since $u_i \in T_i - T_{i-1}$, u_i is a basis of T_i as a left vector space over e_rAe_r modulo T_{i-1}. Hence $u_1, u_2, \ldots, u_r$ form a basis for S as a left vector space over e_rAe_r. If we consider the socle as a left e_rAe_r and a right A representation module, it is not hard to see that the faithful matrix representation of A obtained by using the above basis is just the ring A.

(4. E). Let $K_1, K_2, \ldots, K_r$ be any r positive integers and set

$$\widetilde{A} = \left\{ (a_{ij}^{\alpha\beta})_{\substack{1 \leq i, j \leq r \\ 1 \leq \alpha \leq K_i \\ 1 \leq \beta \leq K_j}} \mid a_{ij}^{\alpha\beta} \in D_{ij} \right\}.$$

Let $e_{i\alpha}$ be the matrix $(a_{i'j}^{\alpha'\beta})$ where $a_{i'j}^{\alpha'\beta} = \begin{cases} 1 & \text{if } \alpha = \alpha' = \beta \text{ and } i = i' = j \\ 0 & \text{otherwise} \end{cases}$. Then by the construction in (0. B) (see the appendix) and by lemma (0. 7) - (3) $\widetilde{A}$ is a ring with identity and minimal condition on left ideals and

$$\widetilde{A} = \dot{\sum_{\substack{1 \leq i \leq r \\ 1 \leq \alpha \leq K_i}}} \widetilde{A}e_{i\alpha}$$

is a decomposition of $\widetilde{A}$ into P. I. 's $\widetilde{A}e_{i\alpha}$ such that $\widetilde{A}e_{i\alpha} \simeq \widetilde{A}e_{j\beta}$ iff $i = j$. Lemma (0. 6) - (2) implies that every P. I. of A contains a unique minimal left ideal. Theorem (0. 9) - (2) and (3) imply that $\widetilde{A}$ is faithfully represented on its left socle and indecomposable. So, by the corollary to theorem (0. 8), $\widetilde{A}$ is the unique indecomposable ring with identity and minimal condition on left ideals which is faithfully represented on its left socle, in which every P. I. contains a unique minimal left ideal, whose reduced ring is A, and which has the set of multiplicities $K_1, K_2, \ldots, K_r$ where K_i is the multiplicity corresponding to the primitive idempotent e_i (see (0. C)).

At this point we perhaps ought to interpret the previous results obtained for A in terms of $\widetilde{A}$. But this is so trivial using the more general lemmas of the appendix that we omit doing so.

We now prove the statement made in the first sentence of (4. A):

Theorem (4. 2). Suppose that A is an indecomposable ring with identity and minimal condition on left ideals which is faithfully represented on its left socle and possesses the property that every principle indecomposable left ideal contains a unique minimal left ideal. Then there exist positive integers $K_1, K_2, \ldots, K_r$ and a skewfield D whose additive group has r^2 subgroups D_{ij} satisfying

(1) For every i, k, j, $D_{ik}D_{kj} \subseteq D_{ij}$ (under the natural multiplication in D) and $D_{ij} = 0$ if $i < j$.

(2) For every i, D_{ii} is a sub-skewfield of D and $D_{ri} = D$.

(3) D_{ij} is finite dimensional as a left vector space over D_{ii} for every i, j.

such that A is isomorphic to the ring of $r \times r$ blocked triangular matrices in which the i, j$\underline{\text{th}}$ block of a typical matrix is a $K_i \times K_j$ matrix with arbitrary entries in D_{ij}.

Proof. Write $A = Ae_1 \dotplus Ae_2 \dotplus \cdots \dotplus Ae_r$ where the Ae_i are P. I. 's and the e_i are idempotents. Now in the language of the appendix, the reduced ring of A (see (0. C) in the appendix) has all the properties of A stated in the first sentence of the statement of the theorem plus the added property of being a direct sum of non-isomorphic P. I. 's. This is an immediate consequence of lemma (0. 7) - (3), theorem (0. 9) - (2) and (3) and lemma (0. 6) - (2). Hence, from the discussion in (4. E), we see that we may assume that $Ae_i \not\cong Ae_j$ if $i \neq j$. By lemma (3. 5) we may assume $e_iAe_j = 0$ if $i < j$. From the construction in (0. B) we may further assume that $A = \left\{(a_{ij})^{1 \leq i, j \leq r} \mid a_{ij} \in e_iAe_j\right\}$.

Again by lemma (3. 5) Ae_r is a minimal left ideal. Hence, for $1 \leq i \leq r$, there exists $u_i \in e_rAe_i$ such that $e_rAe_i = e_rAe_ru_i$ (lemma (3. 3) - (3)).

We define r^2 Z-homomorphisms (Z is the ring of integers) Φ_{ij} by $\Phi_{ij}: a_{ij} \in e_iAe_j \rightarrow a'_{ij} \in e_rAe_r$ where $a'_{ij}u_j = u_ia_{ij}$. Clearly, Φ_{ij} is a ring-homomorphism for $i = j$. Set $D_{ij} = \Phi_{ij}(e_iAe_j)$ and $D = e_rAe_r$. We have so far that every D_{ij} is a subgroup of the additive group of D, $D_{ij} = 0$ if $i < j$, and every D_{ii} is a skewfield (because every e_iAe_i is a skewfield - see the remark following lemma (3. 2).) Also, since $u_r \in e_rAe_r$, we have: $e_rAe_ru_i = e_rAe_i = u_re_rAe_i$. It follows that $D_{ri} = D$ for all i.

Since A is indecomposable, lemma (1. 5) implies that A has only one isomorphism class of minimal left ideals. Therefore, $0:e_rA = 0:Ae_rA = 0:S = 0$ so that every Φ_{ij} is a monomorphism. Hence the map $(a_{ij}) \rightarrow (a'_{ij})$ is a Z-monomorphism. We must prove that this map is a ring-monomorphism. But the equation $u_ia_{ik}b_{kj} = a'_{ik}u_kb_{kj} = a'_{ik}b'_{kj}u_j$ is precisely what we need to do this.

That is, $A \simeq \left\{(a'_{ij})^{1\leq i,j\leq r} \mid a'_{ij} \in D_{ij}\right\}$. That $D_{ik}D_{kj} \subseteq D_{ij}$ follows from the fact that the matrices (a'_{ij}) form a ring. So the proof is completed.

The matrix representations so far achieved have been by (4. D) or the proof of theorem (4. 2) just the representation of the ring on its left socle. On the other hand, it is easy to produce a ring of matrices which does not come from the representation of the ring on any two-sided ideal:

Example (4. 1). Consider the ring $\begin{bmatrix} a & o & o \\ b & c & o \\ d & o & e \end{bmatrix}$ of example (1. 1)

(a, b, c, d, e are elements of a skewfield D). If this ring came from the faithful representation of the ring on some two-sided ideal, then the socle representation would, by the corollary to theorem (1. 3), have to have degree $\leq$ 3. But

$$0 \subset \begin{bmatrix} 0 & 0 & 0 \\ D & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \subset \begin{bmatrix} 0 & 0 & 0 \\ D & D & 0 \\ 0 & 0 & 0 \end{bmatrix} \subset \begin{bmatrix} 0 & 0 & 0 \\ D & D & 0 \\ D & 0 & 0 \end{bmatrix} \subset \begin{bmatrix} 0 & 0 & 0 \\ D & D & 0 \\ D & 0 & D \end{bmatrix} = S$$

is a two-sided composition series for the socle of length 4 (see (3. B) for the notation).

(4. D) shows that the rings considered previously in this section which are a direct sum of non-isomorphic P. I. 's have the pleasant property that a two-sided composition series for the socle has the same length as the left A-module length of the socle. This is hardly true in general:

Example (4. 2). Consider the ring of all matrices of the form $\begin{bmatrix} a & b & o \\ -b & a & o \\ c & d & e \end{bmatrix}$ where a, b, c, d, and e are elements of the set R of real numbers. Clearly S is a direct sum of the isomorphic minimal left ideals $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ R & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & R & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & R \end{bmatrix}$ whence the left composition series length of the socle is 3 (in particular, we see that this ring is faithfully represented on its left socle). But neither the minimal left ideal $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ R & 0 & 0 \end{bmatrix}$ nor the minimal left ideal $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & R & 0 \end{bmatrix}$ is two-sided. Hence the two-sided ideal $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ R & R & 0 \end{bmatrix}$ is a two-sided composition factor of the socle. So we see that the length of a two-sided composition series for the socle is 2.

More generally, if D, E and F are skewfields such that $F \subseteq D$ and $E \to E'$ is an irreducible representation of E over F as matrices of degree d, then the ring $\begin{bmatrix} E' & 0 \\ \underbrace{DD\ldots D}_{d} & D \end{bmatrix}$ is faithfully represented on its left socle, the left composition series length of the socle is $d + 1$, and the two-sided composition series length of the socle is 2.

We now endeavor to see what the faithful socle representation of general rings looks like.

(4. F). Suppose $A = \sum' Ae_i$ where the Ae_i are P. I. 's and $Ae_i \not\cong Ae_j$ if $i \neq j$. Let $f_1, f_2, \dots, f_n$ be the distinct e_i having the property that Ae_i is minimal and let D_i be the skewfield $f_i A f_i$. By theorem (1. 7) we may assume that $S_i = f_i A$ where $S = S_1 \dotplus S_2 \dotplus \cdots \dotplus S_n$ is the decomposition of the socle into indecomposable two-sided ideals of A. Finally, let $0 \subset T_{11} \subset T_{12} \subset \cdots \subset T_{1\alpha_1} \subset T_{21} \subset T_{22} \subset \cdots \subset T_{2\alpha_2} \subset \cdots \subset T_{n1} \subset T_{n2} \subset \cdots \subset T_{n\alpha_n} = S$ be a two-sided composition series for the socle such that $T_{i\alpha_i} = S_1 \dotplus S_2 \dotplus \cdots \dotplus S_i$ for $1 \leq i \leq n$.

Now $T_{i,\alpha_i - 1}$ contains $f_1, f_2, \dots, f_{i-1}$. If T_{i,α_i-1} also contains f_i it follows, since T_{i,α_i-1} is two-sided, that $T_{i,\alpha_i-1} \supseteq T_{i\alpha_i}$ which is absurd. Hence, since $S_1 \dotplus \cdots \dotplus S_{i-1} \subseteq T_{i,\alpha_i-1} \subset T_{i\alpha_i}$, we have $T_{i,\alpha_i-1} \subseteq S_1 \dotplus \cdots \dotplus S_{i-1} \dotplus S_i \cap N$. But as left A-modules, $T_{i,\alpha_i}/(S_1 \dotplus \cdots \dotplus S_{i-1} \dotplus S_i \cap N) \simeq S_i/S_i \cap N \simeq Af_i$. It follows that $T_{i,\alpha_i-1} = S_1 \dotplus \cdots \dotplus S_i \cap N$ and that $T_{i\alpha_i}/T_{i,\alpha_i-1}$ is a one-dimensional left vector space over D_i. Note that $f_i \in T_{i\alpha_i}$ whereas f_i does not belong to any T_{pq} properly below $T_{i\alpha_i}$.

For $1 \leq \beta \leq \alpha_i$, $T_{i\beta}/T_{i,\beta-1}$ (we set $T_{i,\beta-1} = T_{i-1,\alpha_{i-1}}$ if $\beta = 1$) is clearly isomorphic to a left A-submodule of $S_i = f_i A$. Hence each $T_{i\beta}/T_{i,\beta-1}$ is a left vector space over D_i of dimension, say, $\gamma_{i\beta}$.

Now the series $S \supset SN \supset SN^2 \supset \cdots \supset SN^\rho = 0$ may be refined to a two-sided composition series for S. Hence $(T_{i\beta}/T_{i,\beta-1})N = 0$. So each $T_{i\beta}/T_{i,\beta-1}$ is a right $\bar{A}$-module where $\bar{A}$ is the semi-simple ring A/N. By the assumption that A is the direct sum of non-isomorphic P. I. 's Ae_k, $\bar{A}$ is actually the direct sum (as two-sided ideals) of the skewfields $\bar{e}_k \bar{A} \bar{e}_k$. Since $T_{i\beta}/T_{i,\beta-1}$ is irreducible as a left D_i and a right $\bar{A}$ representation module, there exists a unique e_j such that $(T_{i\beta}/T_{i,\beta-1})e_j \neq 0$, that is, a unique e_j such that $\overline{e_j A e_j}$ is faithfully represented on $T_{i\beta}/T_{i,\beta-1}$ on the right. A necessary condition for this to happen is that $S_i \cap Ae_j \neq 0$. But no e_k can right annihilate every two-sided composition factor of the socle (because $0{:}S = 0$) so every e_p with $S_i \cap Ae_p \neq 0$ has the property that $(T_{i\beta'}/T_{i,\beta'-1})e_p \neq 0$ for some β'.

Choose $u^1_{i\beta}, u^2_{i\beta}, \ldots, u^{\gamma_{i\beta}}_{i\beta} \in T_{i\beta}$ forming a left D_i-basis for $T_{i\beta}$ modulo $T_{i,\beta-1}$ and such that each $u^k_{i\beta} = u^k_{i\beta}e_j$ where $(T_{i\beta}/T_{i,\beta-1})e_j \neq 0$. Then, of course, the set of all $u^k_{i\beta}$ form a basis for S as the direct sum of the left vector spaces S_i. Note, in fact, that $u^1_{i\alpha_i}S = S_i \equiv \dot{\sum}_{\substack{1 \leq \beta \leq \alpha_i \\ 1 \leq k \leq \gamma_{i\beta}}} D_i u^k_{i\beta} \pmod{T_{i-1,\alpha_{i-1}}}$ and that $u^k_{i\beta}S = 0$ if $\beta \neq \alpha_i$ (for every i).

Let p_i be the left A-module length of S_i. Using the prescribed basis, we can now see that the socle representation of A produces a ring of $n \times n$ blocked matrices of the form

$$\begin{bmatrix} P_1 & & & O \\ & P_2 & & \\ & & \ddots & \\ O & & & P_n \end{bmatrix}$$

where P_i is a $p_i \times p_i$ matrix with entries in D_i having the form

$$\begin{bmatrix} Q_{i1} & & & & & O \\ & Q_{i2} & & & & \\ & & \ddots & & & \\ & ? & & Q_{i\beta} & & \\ & & & & \ddots & \\ \hline & & & & & Q_{i\alpha_i} \end{bmatrix}$$

where $Q_{i\beta}$ is a $\gamma_{i\beta} \times \gamma_{i\beta}$ matrix belonging to the faithful representation of $\overline{e}_j\overline{A}\overline{e}_j$ on $T_{i\beta}/T_{i,\beta-1}$. The bottom row is just filled in by arbitrary elements of D_i (recall that $\gamma_{i\alpha_i} = 1$).

For each fixed i, $a \in A \rightarrow P_i(a)$ is obviously the representation of A on S_i. Hence the n rings $\{P_i(a) \mid a \in A\}$ are just the uniquely determined subdirect summands of theorem (2. 2).

If $\overline{e}_j\overline{A}\overline{e}_j$ is also faithfully represented on $T_{i_1\beta_1}/T_{i_1,\beta_1-1}$, $T_{i_2\beta_2}/T_{i_2,\beta_2-1}, \ldots, T_{i_{\delta(j)}\beta_{\delta(j)}}/T_{i_{\delta(j)},\beta_{\delta(j)}-1}$ (these obviously don't in general need to be equivalent representations) where $i_1 = i$, $\beta_1 = \beta$, we have $\delta(j)$ ring isomorphisms $\Phi_1 = 1, \Phi_2, \ldots, \Phi_{\delta(j)}$ such that in every matrix, $\Phi_k(Q_{i\beta}) = Q_{i_k\beta_k}$. It can easily be true that $i_k \neq i$, some k, i. e., - this is precisely the sort of thing that happens when A is indecomposable. In fact, by what was said above plus lemma (1. 5) we can theoretically tell whether or not the ring is indecomposable just by looking at the diagonal blocks $Q_{i\beta}$. Note that if $\overline{e}_j\overline{A}\overline{e}_j$ happens to be some D_k, then $\delta(j) = \delta(k) = 1$.

If S has a two-sided composition series whose length is equal to the left A-module length of the socle, then every $\gamma_{i\beta}$ is equal to one. In this case it is not hard to see that $\delta(j)$ is just the left A-module length of $Se_j = S \cap Ae_j$. Also in this case, we have that every $\overline{e}_j\overline{A}\overline{e}_j$ is imbedded irreducibly as a sub-skewfield in at least one, but at most $\delta(j)$ different D_k.

We now give an example which shows why it is in general so hard to say anything about the off-diagonal elements.

Example (4.3). Let K be a field of characteristic 2 for which there exists an element $\theta \in K$ such $\sqrt{\theta} \notin K$ (for instance we could take $K = GF_2(X)$, $\theta = X$) and set $E = K(\sqrt{\theta})$. Let A be the finite dimensional K-algebra given by $\begin{bmatrix} 1 \otimes_K E & 0 \\ E \otimes_K E & E \otimes_K 1 \end{bmatrix}$. Note that A is faithfully represented on its left socle, the socle being given by $\begin{bmatrix} 0 & 0 \\ E \otimes_K E & E \otimes_K 1 \end{bmatrix}$ and that $E \otimes_K E$ is isomorphic as a K-algebra to $E[X]/(X - \sqrt{\theta})^2$, a ring with radical.

Since $E \otimes_K E$ is a two-dimension left vector space over $E \otimes 1$ as well as a two-dimensional right vector space over $1 \otimes E$, the socle of A has a two-sided composition series of the form $0 \subset \begin{bmatrix} 0 & 0 \\ T_1 & 0 \end{bmatrix} \subset \begin{bmatrix} 0 & 0 \\ E \otimes_K E & 0 \end{bmatrix} \subset \begin{bmatrix} 0 & 0 \\ E \otimes_K E & E \otimes 1 \end{bmatrix}$, where T_1 is an ideal in $E \otimes_K E$. This is also a left composition series. We pick a basis by taking $u_1 \neq 0 \in T_1$, $u_2 \in E \otimes_K E - T_1$ and $u_3 \neq 0 \in E \otimes 1$. This basis gives us a representation of A on its left socle as 3×3 triangular matrices with entries in $E \otimes 1$. The 2×2 triangular blocks in the upper left hand corner belong to the left $E \otimes 1$, right $1 \otimes E$ representation module $E \otimes_K E$. Hence, this 2×2 representation is not diagonalizable since $E \otimes_K E$ is not a completely reducible left $(E \otimes 1) \otimes_K (1 \otimes E) = E \otimes_K E$ module (i.e., $E \otimes_K E$ is hardly semi-simple). It follows that the socle representation of A will always contain off diagonal elements which in given matrices, are related in some way to the diagonal elements.

On the other hand, the material in (4.F) gives us many clues as to how to construct all sorts of examples. For instance:

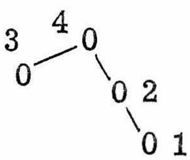
Example (4. 4). The ring of all matrices of the form

$$\begin{bmatrix} a & 0 & 0 & 0 & 0 & 0 \\ b & a & 0 & 0 & 0 & 0 \\ c & d & e & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & b & f & 0 \\ 0 & 0 & 0 & g & h & i \end{bmatrix}$$

where a, b, ... $\in$ D and D is a skewfield, is indecomposable and faithfully represented on its left socle (and has minimal condition on both left and right ideals).

V. Some Rings Defined by Partial Orderings

There are many ways of defining a ring by a partial order on the first n integers. Even the most naive approach is a natural one for certain types of examples:

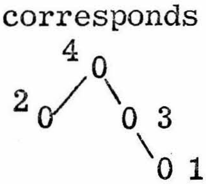
Example (5.1). Consider the partial ordering ρ given by the diagram 

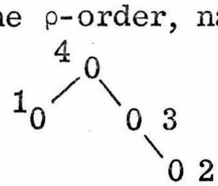
. Corresponding to it we consider the ring

$$A = \begin{bmatrix} D & 0 & 0 & 0 \\ D & D & 0 & 0 \\ 0 & 0 & D & 0 \\ D & D & D & D \end{bmatrix}$$

where D is a skewfield. That is, the ring of all 4×4 matrices obtained by putting an arbitrary element of D in the $i, j^{\underline{th}}$ position if $i \,\rho\, j$ and putting 0 in the $i, j^{\underline{th}}$ position if $i \not\rho\, j$.

By (4.A), A is an indecomposable ring with identity and minimal condition on left and right ideals which is faithfully represented on its left (and right) socle. By proposition (4.1) every left P.I. of A even has a unique composition series. We shall show later that any labeling of the above diagram merely gives us a ring isomorphic to A. In particular, there are exactly two more ways of labeling the diagram such that the natural $\geq$ order corresponds to the ρ-order, namely



and


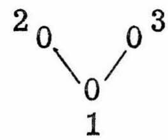
giving us rings

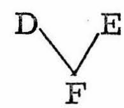
$$\begin{bmatrix} D & 0 & 0 & 0 \\ 0 & D & 0 & 0 \\ D & 0 & D & 0 \\ D & D & D & D \end{bmatrix}$$

and

$\begin{bmatrix} D & 0 & 0 & 0 \\ 0 & D & 0 & 0 \\ 0 & D & D & 0 \\ D & D & D & D \end{bmatrix}$ which are both isomorphic to A even though all three "look different". Actually, this just reflects the fact that the socle of A has three distinct two-sided composition series (and also that the permutation in lemma (3. 5) is not unique).

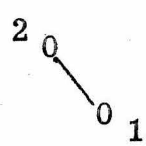
Other approaches are also possible:

Example (5. 2). Consider the diagram 

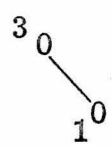
. If we take the same tack as in the previous example, we just get the by now familiar ring of example (1. 1). Instead, we take three skewfields D, E, and F whose inclusion lattice looks like 

and we construct corresponding to the diagram the ring A given by

$\begin{bmatrix} F & 0 & 0 \\ D & D & 0 \\ E & 0 & E \end{bmatrix}$. We shall see that this is an indecomposable ring with identity and minimal condition on left ideals which is also faithfully represented on its left socle.

The sub-partial order 

gives a ring.

$\begin{bmatrix} F & 0 & 0 \\ D & D & 0 \\ 0 & 0 & 0 \end{bmatrix} \simeq \begin{bmatrix} F & 0 \\ D & D \end{bmatrix}$ and 

gives a ring $\begin{bmatrix} F & 0 & 0 \\ 0 & 0 & 0 \\ E & 0 & E \end{bmatrix} \simeq \begin{bmatrix} F & 0 \\ E & E \end{bmatrix}$.

Note that by (4. A), these latter rings have the property that every P. I. contains a unique minimal left ideal. Furthermore it is easy to see that A is isomorphic to the ring of matrices of the form

$\begin{bmatrix} a & o & o & o \\ b & c & o & o \\ o & o & a & o \\ o & o & d & e \end{bmatrix}$ where $a \in F$, $b, c \in D$, $d, e \in E$ and that this ring

is actually the ring produced by representing A on its left socle. From the general left socle representation obtained in (4. F) we see that the unique subdirect summands (in the sense of theorem (2. 2)) of A are given by $\begin{bmatrix} F & 0 \\ D & D \end{bmatrix}$ and $\begin{bmatrix} F & 0 \\ E & E \end{bmatrix}$.

Now let ρ be a partial ordering of the set $\{1, 2, \ldots, r\}$ and let $m_1, m_2, \ldots, m_n$ be a full set of distinct ρ-maximal elements. Let ρ_α be the restriction of ρ to the set $\{j \mid m_\alpha \rho j\}$. Each ρ_α is a sub - partial order on $\{1, 2, \ldots, r\}$. Let $D_1, D_2, \ldots, D_n$ be skewfields and D_{ij} be r^2 Z-modules such that whenever $i\rho j$ the following properties hold:

(a) $D_{ij} \neq 0$

(b) $D_{ij} \subseteq D_\alpha \cap D_\beta$ if $(i, j) \in \rho_\alpha \cap \rho_\beta$

(c) $D_{ik}D_{kj} \subseteq D_{ij}$

(d) D_{ii} is a skewfield and D_{ij} is a finite dimensional left vector space over D_{ii}

(e) $D_{m_\alpha m_\alpha} = D_\alpha$

We notice that since $\rho = \bigcup_{\alpha=1}^{n} \rho_\alpha$, every D_{ij} such that $i\rho j$ is contained, by (b), in some D_α. We are tacitly assuming that in this case D_{ij} has the multiplicative structure of D_α. It follows from (b) that the multiplication in (c) is well defined (where, of course, $D_{ik}D_{kj}$ is just the 0 of D_{ij} if $i \not\rho k$ or $k \not\rho j$).

By the ring defined by ρ, D_α, and D_{ij} we mean the set $A_\rho = \left\{(a_{ij})^{1 \le i, j \le r} \,|\, a_{ij} \in D_{ij} \text{ if } i \rho j \text{ and } a_{ij} = 0 \text{ otherwise}\right\}$. The verification that A_ρ is a ring with identity given by

$$\begin{bmatrix} 1_1 & & O \\ & \ddots & \\ O & & 1 \end{bmatrix}, \quad \text{where the } 1 \text{ in the } i,i \text{ position is the}$$

identity of D_{ii}, is trivial.

Now if Φ is a permutation on $\{1, 2, \ldots, r\}$, it is clear that A_ρ is isomorphic to the ring $\left\{(a_{\Phi(i)\Phi(j)})^{1 \le i, j \le r} \,|\, a_{\Phi(i)\Phi(j)} \in D_{\Phi(i)\Phi(j)}\right.$ if $\Phi(i) \rho \Phi(j)$ and $\left. a_{\Phi(i)\Phi(j)} = 0 \text{ otherwise}\right\}$. Therefore, since every partial order on $\{1, 2, \ldots, r\}$ can be extended to a total order on $\{1, 2, \ldots, r\}$, we may, and do assume that $i \not\rho j$ if $i < j$. That is, we assume that the ring A_ρ is a triangular matrix ring.

Set e_i equal to the matrix with 1 in the i, i position and 0 every place else. Then, exactly as in (4. A), we show that the radical N of $A_\rho = A$ is $\sum_{i \neq j} e_i A e_j$ and that the left A-factor module $(e_i + e_{i+1} + \cdots + e_r)Ae_j/(e_{i+1} + \cdots + e_r)Ae_j$ is the direct sum of $[D_{ij}:D_{ii}]$ irreducible left A-modules all of which are isomorphic to Ae_i/Ne_i (provided that $i \rho j$).

Now if i is ρ-maximal, $Ae_iAe_j = \sum_k e_k Ae_iAe_j = e_iAe_iAe_j = e_iAe_j$ and e_iAe_j is a left ideal. Hence if i is maximal and $i \rho j$, e_iAe_j is a direct sum of $[D_{ij}:D_{ii}]$ minimal left ideals isomorphic to Ae_i since $Ne_i = 0$.

On the other hand, if I is a minimal left ideal contained in Ae_j, there exists (as in (4. A)) a unique e_k such that $e_k I = I$, whence

$I \subseteq e_k Ae_j$. We pick i such that i is maximal and $i \rho k$. Since $D_{ik} \neq 0$ (by (2)), it follows that $e_i Ae_k I \neq 0$. But $e_i Ae_k$ is an ideal as we saw above. Hence $I = e_i Ae_k I \subseteq e_i Ae_j$ and I is isomorphic to the idempotent minimal left ideal Ae_i.

We summarize:

Proposition (5. 1). The following hold:

(1) $A = A_\rho$ is a ring with identity and minimal condition on left ideals and is the direct sum of the non-isomorphic P. I.'s Ae_j. The left composition series length of an Ae_j is given by

$$\ell(Ae_j) = \sum_{\substack{i \\ i \rho j}} [D_{ij}:D_{ii}]$$ where $[D_{ij}:D_{ii}]$ is the left dimension of D_{ij} as a vector space over D_{ii}. The radical N of A is the set of all matrices of A which have no non-zero elements on their main diagonal.

(2) A is faithfully represented on its left socle S. $e_{m_\alpha} Ae_j$ is a left ideal and is the direct sum of $[D_{m_\alpha j}:D_\alpha]$ minimal left ideals. If we set $S_\alpha = \sum_{\substack{j \\ m_\alpha \rho j}} e_{m_\alpha} Ae_j$, then $S = S_1 \dotplus S_2 \dotplus \cdots \dotplus S_n$ is the unique decomposition of the socle into two-sided indecomposable ideals of A. Furthermore, $\ell(S_\alpha) = \sum_{\substack{j \\ m_\alpha \rho j}} [D_{m_\alpha j}:D_\alpha]$.

We prove:

Proposition (5. 2).

(1) If $D_{m_\alpha i}$ is a skewfield whenever $m_\alpha \rho i$, then $m_\alpha \rho i$ implies that $D_\alpha \subseteq D_{m_\alpha i}$ and also that $D_{ij} \subseteq D_{m_\alpha j}$ if $i \rho j$.

(2) A is indecomposable iff the diagram of ρ is connected.

(3) The n rings A_{ρ_α} where A_{ρ_α} is the ring determined by ρ_α, D_α, and the r^2 Z-modules D_{ij} are the uniquely determined subdirect summands of A of theorem (2. 2).

(4) If $D_{m_\alpha i} = D_\alpha$ whenever $m_\alpha \rho i$, then every P. I. of A_{ρ_α} contains a unique minimal left ideal.

Proof. (1) This follows by $D_{m_\alpha i} D_{ij} \subseteq D_{m_\alpha j}$ and (a).

(2) Two P. I. 's Ae_i and Ae_j are linked iff there exist e_{α_1}, $e_{\alpha_2}, \ldots, e_{\alpha_m}$ and $e_{\beta_1}, e_{\beta_2}, \ldots, e_{\beta_{m-1}}$ where $e_{\alpha_1} = e_i$ and $e_{\alpha_m} = e_j$ such that $e_{\beta_k} A e_{\alpha_k} \neq 0$ and $e_{\beta_k} A e_{\alpha_{k+1}} \neq 0$ for $1 \leq k \leq m-1$. So (2) follows.

(3) By proposition (5. 1) - (2), $S_\alpha = \sum_{\substack{i \\ m_\alpha \rho i}} e_{m_\alpha} A e_i$. Hence

$0{:}S_\alpha = \sum_{\substack{i, j \\ m_\alpha \not\rho i}} e_i A e_j$. Therefore $A/0{:}S_\alpha \simeq A_{\rho_\alpha}$ both as a ring and a left A-module, since ρ_α is a sub - partial ordering on $\{1, 2, \ldots, r\}$.

(4) ρ_α has only one maximal element, namely m_α. Hence, as we saw above, every minimal left ideal of $A_{\rho_\alpha} e_j$, where $m_\alpha \rho_\alpha j$, is contained in the left ideal $m_\alpha A_{\rho_\alpha} e_j$. But $[D_{m_\alpha j} : D_\alpha] = 1$ means that $m_\alpha A_{\rho_\alpha} e_j$ is minimal. The rest follows by (1) and (4. A).

From now on we assume that $D_{m_\alpha i} = D_\alpha$ whenever $m_\alpha \rho i$. We can then prove:

Proposition (5.3). (1) The socle of $A = A_\rho$ has a two-sided composition series of length equal to the length of a left composition series for the socle.

(2) Every P.I. of A contains a unique minimal left ideal iff $A = A_{\rho_1} \dot{+} A_{\rho_2} \dot{+} \cdots \dot{+} A_{\rho_n}$, that is, iff the A_{ρ_α} are the two-sided blocks of A (see (1.E)).

Proof. (1) If we fix α, every $m_\alpha Ae_j$ is minimal. We order the j such that $m_\alpha \rho j$; say $j_1 < j_2 < \cdots < j_p$. Then it is easy to show by induction that $0 \subset e_{m_\alpha} Ae_{j_1} \subset e_{m_\alpha} Ae_{j_1} + e_{m_\alpha} Ae_{j_2} \subset \cdots \subset e_{m_\alpha} Ae_{j_1} + e_{m_\alpha} Ae_{j_2} + \cdots + e_{m_\alpha} Ae_{j_p} = S_\alpha$ is a two-sided composition series for S_α.

(2) If we fix j, the different $m_\alpha Ae_j$ give all the minimal left ideals of A contained in Ae_j. Hence (proposition (5.1) - (2)), every P.I. of A has a unique minimal left ideal iff $\sum_{\alpha=1}^{n} |\{j \mid m_\alpha \rho j\}| = r$ iff $\rho_\alpha \cap \rho_\beta = \Phi$ for $\alpha \neq \beta$ and (2) follows.

By the proof of proposition (5.3) - (2), we see that $A = A_{\rho_1} \dot{+} \cdots \dot{+} A_{\rho_n}$ iff $\ell(S) = r$. Hence, by (1) of the same proposition, we must have $A = A_{\rho_1} \dot{+} \cdots \dot{+} A_{\rho_n}$ if A is in its faithful socle representation. But if $A = A_{\rho_1} \dot{+} \cdots \dot{+} A_{\rho_n}$, it is completely obvious that A is in its socle representation. On the other hand, if $\ell(S) \neq r$, that is, if $\ell(S) > r$, then A cannot come from any faithful representation on a two-sided ideal of A (see example (4. 1)) by the corollary to theorem (1. 3).

Let ρ'_α be the partial order on $\{1, 2, \ldots, \ell(S_\alpha)\}$ induced in the natural way by ρ_α. In other words, ρ'_α is defined by $i'\rho'_\alpha j'$, where $1 \leq i', j' \leq m'_\alpha = \ell(S_\alpha)$, if $i \rho_\alpha j$ where $i, j \in \{k \mid m_\alpha \rho_\alpha k\}$. Then, it is a matter of calculation to show that A is isomorphic to

$$\left\{ (a^{i'j'}_{\alpha\beta})_{\substack{1 \leq \alpha, \beta \leq n \\ 1 \leq i' \leq m'_\alpha \\ 1 \leq j' \leq m'_\beta}} \;\middle|\; a^{i'j'}_{\alpha\beta} = a_{ij} \text{ if } \alpha = \beta,\ i'\rho'_\alpha j',\ (a_{ij}) \in A \text{ and } a^{i'j'}_{\alpha\beta} = 0 \text{ otherwise} \right\}$$

and that A_{ρ_α} is isomorphic to the subring given by

$$\left\{ (a^{i'j'}_{\alpha\alpha})_{\substack{1 \leq i' \leq m'_\alpha \\ 1 \leq j' \leq m'_\alpha}} \right\}.$$

The former ring clearly gives the socle representation of A. On comparison with the general socle representation obtained in (4. F) it seems not unreasonable to conjecture that the rings A_ρ give all the rings with identity and minimal condition on left ideals, which are a direct sum of non-isomorphic P. I. 's and faithfully represented on their left socle, and which are a unique subdirect sum in the sense of theorem (2. 2) of rings which, besides all the above properties, have the added properties of being indecomposable and being such that every P. I. contains a unique minimal left ideal. However, we cannot prove this.

We now set out to determine, in a sense to be made specific, the rings which are faithfully represented on their left socle and which possess the property that every left P. I. has a unique composition series.

Set $G(j) = |\{i \mid i \rho j\}|$. Then, in terms of ρ, property (P) of (4. C) just says:

(P') If $i > j$ and $G(i) = G(i')$ for $i \neq i'$, then $i \not\rho j$ or $i' \not\rho j$.

Then if ρ has a greatest element, (P') is equivalent to:

(P'') No two non-ρ comparable elements have a lower bound.

Proof. If (P') fails, (P'') obviously fails.

On the other hand, assume that (P'') fails. Then there exist i, j, k such that $i \not\rho j$, $j \not\rho i$, but $i \rho k$ and $j \rho k$. Let m be minimal such that $m \rho i$ and $m \rho j$ (m exists since ρ has a greatest element).

If m covers i and m covers j, then $G(i) = G(j)$ for $i \neq j$, $i \rho k$, $j \rho k$ and $i > k$ so that (P') fails. Hence, there exists s such that m covers s and, say, $s \rho i$ where $s \neq i$. Since m is minimal, $s \not\rho j$. By the same argument as before, there exists t such that m covers t and $t \rho j$, $t \neq j$. But then, $G(s) = G(t)$, $s \neq t$ (by the minimal property of m), $s \rho k$, $t \rho k$ and $s > k$. Hence (P'') is violated and we are done.

Thus we have shown by propositions (4. 1) and (5. 3) - (2) that if $[D_{ij}:D_{ii}] = 1$ whenever $i \rho j$, then A_ρ is indecomposable and every P. I. of A_ρ has a unique composition series iff ρ has a greatest element and (P'') holds.

Note that P'' just says that the diagram of ρ is an inverted tree (provided, of course, that ρ has a greatest element).

We have now, with the aid of theorem (4. 2) and the appendix shown more than enough to prove:

Theorem (5.4). Let $A = \dot{\sum}_{\substack{1 \leq i \leq r \\ 1 \leq \alpha \leq K_i}} Ae_{i\alpha}$ be a decomposition of the indecomposable ring A with identity and minimal condition on left ideals into a direct sum of P. I.'s $Ae_{i\alpha}$ such that $Ae_{i\alpha} \simeq Ae_{j\beta}$ iff $i = j$. Suppose that A is faithfully represented on its left socle and that every P. I. contains a unique minimal left ideal. Let ρ be the partial ordering on $\{1, 2, \ldots, r\}$ defined by $i \rho j$ if $e_{\alpha_{i1}} A e_{\alpha_{j1}} \neq 0$. Then every P. I. of A has a unique composition series iff ρ satisfies (P'') and every P. I. of A has a composition series without repeated factors.

Appendix

Suppose that A is a simple ring. That is, a ring with identity and minimal condition on left ideals possessing no proper two-sided ideals. Such a ring A is a finite direct sum of minimal left ideals. If A contains only one minimal left ideal, then A is obviously a skewfield. One feels that Wedderburn's structure theorem should follow automatically - that is, that the most general simple ring is a complete matrix ring over a skewfield. This is actually true in a much broader sense.

In order to proceed we need some technical facts:

Lemma (0.1). If A is any ring and e and f are idempotents of A, then $\mathrm{Hom}_A(Ae, Af) = \{R_x | x \in eAf\}$ where by R_x is meant the map of Ae into Af induced by right multiplying by an element x of eAf.

Proof. If $a \in A$, $b \in Ae$, $x \in eAf$; $R_x(ab) = abx = aR_x(b) \in Af$ so $x \to R_x$ is a Z-homomorphism (Z = ring of integers): $eAf \to \mathrm{Hom}_A(Ae, Af)$. Also, $\Phi \in \mathrm{Hom}_A(Ae, Af) \to \Phi(e) = \Phi(e^2)$ $= e\Phi(e) \in eAf$ is a Z-homomorphism.

Now $x \in eAf \to R_x \to R_x(e) = ex = x$ and $\Phi \in \mathrm{Hom}_A(Ae, Af)$ $\to \Phi(e) \to R_{\Phi(e)} = \Phi$ since, if $y \in Ae$, $R_{\Phi(e)}(y) = y\Phi(e) = \Phi(ye) = \Phi(y)$.

Lemma (0.2). If $Ae \underset{\Phi}{\simeq} Af$ and I is a right ideal, then $\Phi(Ie) = If$.

Proof. There exist $a, b \in A$ such that $\Phi(e) = af$ and $\Phi^{-1}(f) = be$. Hence, $\Phi(Ie) = I\Phi(e) = Iaf \subseteq If$. But $Ie \subseteq \Phi^{-1}(If) = I\Phi^{-1}(f) = Ibe \subseteq Ie$. Therefore, $\Phi^{-1}(If) = Ie$ so $If = \Phi(Ie)$.

Lemma (0.3). If J is a left ideal and $Ae \simeq Af$, then $AeJ = AfJ$.

Proof. By lemma (0.1), there exist $u \in eAf$ and $v \in fAe$ such that $Aeu = Af$ and $Afv = Ae$. Hence $AfJ = AeuJ \subseteq AeJ$ and $AeJ = AfvJ \subseteq AfJ$.

Lemma (0.4). If $Ae \underset{\Phi}{\simeq} Ae'$ and $Af \underset{\psi}{\simeq} Af'$ then $eAf \underset{\Phi^{-1}(e')\psi}{\simeq} e'Af'$ (as Z-modules) where by $\Phi^{-1}(e')\psi$ is meant the map $x \in eAf \to \Phi^{-1}(e')\psi(x) \in e'Af'$ of eAf into $e'Af'$.

Proof. By lemma (0.1), corresponding to the sequence of natural isomorphisms $eAf \to \mathrm{Hom}_A(Ae, Af) \to \mathrm{Hom}_A(Ae', Af') \to e'Af'$, we have the sequence $x \to R_x \to \psi o R_x o \Phi^{-1} \to \psi o R_x o \Phi^{-1}(e') = \Phi^{-1}(e')\psi(x)$ since $\psi o R_x o \Phi^{-1}(e') = \psi(\Phi^{-1}(e')x) = \Phi^{-1}(e')\psi(x)$.

Lemma (0.5). Suppose that $Ae \underset{\alpha}{\simeq} Ae'$, $Af \underset{\beta}{\simeq} Af'$, and $Ag \underset{\gamma}{\simeq} Ag'$. Then the diagram

$$\begin{array}{ccccc} eAf & \times & fAg & \rightarrow & eAg \\ \downarrow{\scriptstyle \alpha^{-1}(e')\beta} & & \downarrow{\scriptstyle \beta^{-1}(f')\gamma} & & \downarrow{\scriptstyle \alpha^{-1}(e')\gamma} \\ e'Af' & \times & f'Ag' & \rightarrow & e'Ag' \end{array}$$

commutes with respect to the multiplication in A.

Proof. Let $x \in eAf$ and $y \in fAg$. Then $x \to x' = \alpha^{-1}(e')\beta(x)$, $y \to y' = \beta^{-1}(f')\gamma(y)$, and $xy \to (xy)' = \alpha^{-1}(e')\gamma(xy)$. We calculate:
$x'y' = \alpha^{-1}(e')\beta(x)\beta^{-1}(f')\gamma(y) = \alpha^{-1}(e')\beta^{-1}(\beta(x)f')\gamma(y)$
$= \alpha^{-1}(e')\beta^{-1}(\beta(x))\gamma(y) = \alpha^{-1}(e')x\gamma(y) = \alpha^{-1}(e')\gamma(xy) = (xy)'$.

(0. A). Suppose that $A = \dot{\sum}_{\substack{1\le i\le r\\ 1\le\alpha\le K_i}} Ae_{i\alpha}$ where $1 = \sum_{\substack{1\le i\le r\\ 1\le\alpha\le K_i}} e_{i\alpha}$, the $e_{i\alpha}$ form a set of mutually orthogonal idempotents, and $Ae_{i\alpha} \simeq Ae_{j\beta}$ if $i = j$ (there is always one such decomposition, namely, $A = A1$). In this section we shall use the notation $\sum_{\substack{i,\alpha\\ j,\beta\\ \vdots}}$ to mean $\sum_{\substack{1\le i,j,\ldots\le r\\ 1\le\alpha\le K_i\\ 1\le\beta\le K_j\\ \vdots}}$.

Since $A = \dot{\sum}_{i,\alpha} e_{i\alpha}A$, we have $A = \dot{\sum}_{i,\alpha} e_{i\alpha}(\dot{\sum}_{j,\beta} Ae_{j\beta}) = \dot{\sum}_{\substack{i,\alpha\\ j,\beta}} e_{i\alpha}Ae_{j\beta}$ (direct sum as Z-modules). We now construct an additive subgroup A' of the additive group of A by setting $A' = \dot{\sum}_{1\le i,j\le r} e_{i1}Ae_{j1}$. Note that $e_{i1}A'e_{j1} = e_{i1}Ae_{j1}$. It is clear that $(A')^2 \subseteq A'$, whence A' is a ring, and that A' has an identity $1'$ given by $1' = \sum_{i=1}^{r} e_{i1}$. Therefore, $A' = \dot{\sum}_{1\le i\le r} A'e_{i1}$ since the e_{i1} form a set of mutually orthogonal idempotents in A'. Let us note here that if we fix integers $n_1, n_2, \ldots, n_r$, where $1 \le n_i \le K_i$, then the ring $A'' = \sum_{\substack{i,\alpha\\ j,\beta}} e_{in_i}Ae_{jn_j}$ is ring-isomorphic to A'. This follows easily from lemmas (0. 4) and (0. 5).

We now see how we can recover the ring A from the ring A':

(0. B). For each ordered pair (i, j), $1 \le i, j \le r$, we set $A^{\alpha, \beta} = A$ for $1 \le \alpha \le K_i$ and $1 \le \beta \le K_j$. We define a Z-module $\widetilde{A}$ by setting $\widetilde{A} = \bigoplus_{\substack{i, \alpha \\ j, \beta}} e_{i1} A^{\alpha, \beta} e_{j1}$. That is, $\widetilde{A}$ is the external direct sum $\bigoplus_{\substack{1 \le i, j \le r \\ 1 \le \alpha \le K_i \\ 1 \le \beta \le K_j}} e_{i1} A^{\alpha, \beta} e_{j1}$ of the Z-modules $e_{i1} A^{\alpha, \beta} e_{j1}$. We define a multiplication in $\widetilde{A}$ by setting $\widetilde{a}\widetilde{b} = \bigoplus_{\substack{i, \alpha \\ j, \beta}} (\sum_{k, \gamma} a_{ik}^{\alpha\gamma} b_{kj}^{\gamma\beta})$ where $\widetilde{a} = \bigoplus_{\substack{i, \alpha \\ j, \beta}} a_{ij}^{\alpha\beta}$ and $\widetilde{b} = \bigoplus_{\substack{i, \alpha \\ j, \beta}} a_{ij}^{\alpha\beta}$; $a_{ij}^{\alpha\beta}, b_{ij}^{\alpha\beta} \in e_{i1} A^{\alpha, \beta} e_{j1}$. It is a routine matter to verify that $\widetilde{A}$ is a ring with identity $\widetilde{1} = \bigoplus_{\substack{i, \alpha \\ j, \beta}} f_{ij}^{\alpha\beta}$ where

$$f_{ij}^{\alpha\beta} = \begin{cases} e_{i1} & \text{if } i = j \text{ and } \alpha = \beta \\ 0 & \text{otherwise} \end{cases}.$$

We see that $\widetilde{A}$ is just the ring of all matrices $(a_{ij}^{\alpha\beta})_{\substack{1 \le i, j \le r \\ 1 \le \alpha \le K_i \\ 1 \le \beta \le K_j}}$ where $a_{ij}^{\alpha\beta} \in e_{i1} A e_{jr}$. That is, the ring of all $r \times r$ blocked matrices

$$\begin{bmatrix} B_{11} & B_{12} \cdots & B_{1r} \\ B_{21} & B_{22} \cdots & B_{2r} \\ \vdots & \vdots & \vdots \\ B_{r1} & B_{r2} \cdots & B_{rr} \end{bmatrix}$$

where B_{ij} is the $K_i \times K_j$ matrix

$$\begin{bmatrix} a_{ij}^{11} & a_{ij}^{12} & \cdots & a_{ij}^{1K_1} \\ a_{ij}^{21} & a_{ij}^{22} & \cdots & a_{ij}^{2K_2} \\ \vdots & \vdots & & \vdots \\ a_{ij}^{K_i 1} & a_{ij}^{K_i 2} & \cdots & a_{ij}^{K_i K_j} \end{bmatrix}, \quad a_{ij}^{\alpha\beta} \in e_{i1} A e_{jr}.$$

$$\text{Set } e_{i1}^{\alpha} = \bigoplus_{\substack{i',\alpha' \\ j',\beta}} g_{i'j}^{\alpha'\beta} \text{ where } g_{i'j}^{\alpha'\beta} = \left\{ \begin{array}{ll} e_{i1} & \text{if } i'=i=j \text{ and } \alpha'=\beta=\alpha \\ 0 & \text{otherwise} \end{array} \right\}.$$

Then the e_{i1}^{α} form a set of mutually orthogonal idempotents in $\widetilde{A}$ and $\widetilde{1} = \sum_{i,\alpha} e_{i1}^{\alpha}$. Hence $\widetilde{A} = \dot{\sum}_{i,\alpha} \widetilde{A} e_{i1}^{\alpha}$ (internal direct sum as left $\widetilde{A}$-modules) and $\widetilde{A} = \dot{\sum}_{\substack{i,\alpha \\ j,\beta}} e_{i1}^{\alpha} \widetilde{A} e_{j1}^{\beta}$ (internal direct sum as Z-modules). We clearly can and do identity $e_{i1}^{1} \widetilde{A} e_{j1}^{1}$ with $e_{i1} A e_{j1}$ as subrings of $\widetilde{A}$ for every i, j.

$$\text{Set } 1_i^{\alpha} = \bigoplus_{\substack{i',\alpha' \\ j,\beta}} h_{i'j}^{\alpha'\beta} \text{ where } h_{i'j}^{\alpha'\beta} = \left\{ \begin{array}{ll} e_{i1} & \text{if } \alpha'=1, \beta=\alpha, \text{ and } i'=j=i \\ 0 & \text{otherwise} \end{array} \right\}.$$

Then $x \in \widetilde{A} e_{i1}^{1} \to x 1_i^{\alpha}$ is an $\widetilde{A}$-isomorphism $\widetilde{A} e_{i1}^{1} \to \widetilde{A} e_{i1}^{\alpha} \to 0$ for $1 \leq \alpha \leq K_i$. By lemmas (0. 4) and (0. 5) there exist Z-module isomorphisms

$$\Phi_{ij}^{\alpha\beta} : e_{i\alpha} A e_{j\beta} \to e_{i1} A e_{j1} \to 0$$

$$\psi_{ij}^{\alpha\beta} : e_{i1}^{1} \widetilde{A} e_{j1}^{1} \to e_{i1}^{\alpha} \widetilde{A} e_{j1}^{\beta} \to 0$$

such that the diagrams

$$\begin{array}{ccccc}
e_{i\alpha}Ae_{k\gamma} & \times & e_{k\gamma}Ae_{j\beta} & \rightarrow & e_{i\alpha}Ae_{j\beta} \\
\Phi_{ik}^{\alpha\gamma}\downarrow & & \Phi_{kj}^{\gamma\beta}\downarrow & & \Phi_{ij}^{\alpha\beta}\downarrow \\
e_{i1}Ae_{k1} & \times & e_{k1}Ae_{j1} & \rightarrow & e_{i1}Ae_{j1}
\end{array}$$

commute with respect to multiplication in A and such that the diagrams

$$\begin{array}{ccccc}
e_{i1}^{1}\tilde{A}e_{k1}^{1} & \times & e_{k1}^{1}\tilde{A}e_{j1}^{1} & \rightarrow & e_{i1}^{1}\tilde{A}e_{j1}^{1} \\
\psi_{ik}^{\alpha\gamma}\downarrow & & \psi_{kj}^{\gamma\beta}\downarrow & & \psi_{ij}^{\alpha\beta}\downarrow \\
e_{i1}^{\alpha}\tilde{A}e_{k1}^{\gamma} & \times & e_{k1}^{\gamma}\tilde{A}e_{j1}^{\beta} & \rightarrow & e_{i1}^{\alpha}\tilde{A}e_{j1}^{\beta}
\end{array}$$

commute with respect to multiplication in $\tilde{A}$. If we put $\omega_{ij}^{\alpha\beta} = \psi_{ij}^{\alpha\beta}\,\Phi_{ij}^{\alpha\beta}$, then it is obvious that the map $a = \sum_{\substack{i,\alpha\\ j,\beta}} a_{ij}^{\alpha\beta} \in A$ (where $a_{ij}^{\alpha\beta} \in e_{i\alpha}Ae_{j\beta}$) $\rightarrow \tilde{a} = \sum_{\substack{i,\alpha\\ j,\beta}} \omega_{ij}^{\alpha\beta}(a_{ij}^{\alpha\beta})$ is a Z-isomorphism, $A \rightarrow \tilde{A} \rightarrow 0$. That this map is actually a ring isomorphism follows from the fact that the diagrams

$$\begin{array}{ccccc}
e_{i\alpha}Ae_{k\gamma} & \times & e_{k\gamma}Ae_{j\beta} & \rightarrow & e_{i\alpha}Ae_{j\beta} \\
\omega_{ik}^{\alpha\gamma}\downarrow & & \omega_{kj}^{\gamma\beta}\downarrow & & \omega_{ij}^{\alpha\beta}\downarrow \\
e_{i1}^{\alpha}\tilde{A}e_{k1}^{\gamma} & \times & e_{k1}^{\gamma}\tilde{A}e_{j1}^{\beta} & \rightarrow & e_{i1}^{\alpha}\tilde{A}e_{j1}^{\beta}
\end{array}$$

commute with respect to multiplication in A in the top row and multiplication in $\tilde{A}$ in the bottom row.

We now prove some lemmas which relate the properties of A and A':

Lemma (0.6). Let A and A' be the rings defined in (0.A). Then the following statements hold:

(1) $e_{i1}A'e_{j1} = e_{i1}Ae_{j1}$

(2) $I \to I \cap A' = I'$ gives a lattice isomorphism between the lattice of left ideals of A contained in Ae_{j1} and the lattice of left ideals of A' contained in $A'e_{j1}$. The inverse is given by $I' \to AI'$.

(3) $Ae_{i1}Ae_{j1} \cap A' = A'e_{i1}A'e_{j1}$

(4) $T \to T \cap A' = T'$ gives a lattice isomorphism between the lattice of two-sided ideals of A and the lattice of two-sided ideals of A'. The inverse is given by $T' \to AT'A$.

(5) If P is a left and Q a two-sided ideal of A, then $(P:Q) \cap A' = (P \cap A'):(Q \cap A')$ where the quotient on the right hand side of the equality is taken in A' (and the quotient in the left hand side is taken, of course, in A).

(6) If P and Q are both two-sided ideals of A, then $PQ \cap A' = (P \cap A')(Q \cap A')$.

Proof. (1) This is immediate by the definitions.

(2) Suppose $I \leq Ae_{j1}$. Then it is clear that $I \cap A' = \sum_i e_{i1}I \leq A'e_{j1}$. But then, $A(I \cap A') = \sum_i Ae_{i1}I = \sum_{i,\alpha} Ae_{i\alpha}I = AI = I$ by lemma (0.3).

On the other hand, suppose $J \leq A'e_{j1}$. Since $AJ \leq Ae_{j1}$,
$AJ \cap A' = \sum_i e_{i1}AJ = \sum_i e_{i1}(\sum_{k,\alpha} Ae_{k\alpha})J = \sum_i e_{i1}(\sum_k Ae_{k1}J) = \sum_{i,k} e_{i1}Ae_{k1}J$
$= A'J = J$.

(3) $Ae_{i1}Ae_{j1} \cap A' = \sum_k e_{k1}Ae_{i1}Ae_{j1} = \sum_k e_{k1}A'e_{i1}A'e_{j1} = A'e_{i1}A'e_{j1}$.

(4) Let T be a two-sided ideal of A. Then clearly, $T \cap A'$ $= \sum_{i,j} e_{i1}Te_{j1}$ is a two-sided ideal of A'. $A(T \cap A')A = \sum_{i,j} Ae_{i1}Te_{j1}A$ $\subseteq \sum_{i,\alpha \atop j,\beta} Ae_{i\alpha}Te_{j\beta}A = ATA = T$. Now by lemma (0.1), $Ae_{j1}A$ contains

$Ae_{j\beta}$ for $1 \leq \beta \leq K_j$. Therefore, $Te_{j1}A \supseteq Te_{j\beta}$ by lemma (0. 2) since T is a right ideal. Also, since T is a left ideal, $Ae_{i1}T \supseteq e_{i\alpha}T$, $1 \leq \alpha \leq K_i$, by lemma (0. 3). Hence $Ae_{i1}Te_{j1}A \supseteq e_{i\alpha}Te_{j1}A \supseteq e_{i\alpha}Te_{j\beta}$ for $1 \leq i \leq K_i$ and $1 \leq j \leq K_j$. Thus, it follows that $A(T \cap A')A = T$.

Conversely, assume that U is a two-sided ideal of A'. Since AUA is a two-sided ideal of A, $AUA \cap A' = \sum_{i,j} e_{i1}AUAe_{j1}$

$$= \sum_{\substack{i,j \\ i',j'}} e_{i1}Ae_{i'1}Ue_{j'1}Ae_{j1} = A'UA' = U.$$

(5) The inclusion from left to right is trivial.

Let $a' \in P \cap A' : Q \cap A'$ where $a' \in A'$. Then $Qa' = AQa'$

$$= \sum_{\substack{i,\alpha \\ j}} Ae_{i\alpha}Qe_{j1}a' \overset{\text{lemma (0. 3)}}{=} \sum_{i,j} Ae_{i1}Qe_{j1}a' = A(Q \cap A')a' \subseteq A(P \cap A')$$

$\subseteq AP \subseteq P$ which proves the inclusion from right to left.

(6) $$PQ \cap A' = \sum_{i,j} e_{i1}PQe_{j1} = \sum_{\substack{i,j \\ k,\alpha}} e_{i1}PAe_{k\alpha}Qe_{j1} \overset{\text{lemma (0.3)}}{=} \sum_{i,j,k} e_{i1}PAe_{k1}Qe_{j1}$$

$$= \sum_{i,j,k} e_{i1}Pe_{k1}Qe_{j1} = (\sum_{i,j} e_{i1}Pe_{j1})(\sum_{i',j'} e_{i'1}Qe_{j'1}) = (P \cap A')(Q \cap A').$$

Lemma (0. 7). If A and A' are the rings defined in (0. A), then

(1) A has minimal condition on left ideals iff A' has minimal condition on left ideals.

(2) If either A or A' has minimal condition on left ideals, then every $Ae_{i\alpha}$ is a P. I. of A iff Ae_{i1} is a P. I. of A iff $A'e_{i1}$ is a P. I. of A'.

(3) If A has minimal condition on left ideals, $Ae_{i1} \not\simeq Ae_{j1}$ for $i \neq j$, and Ae_{i1} is a P. I. of A for every i, then A' has minimal condition on left ideals and $A' = \dot{\sum_i} A'e_{i1}$ is a decomposition of A' into non-isomorphic P. I. 's $A'e_{i1}$. Conversely if A' has minimal condition on left ideals and $A' = \dot{\sum_i} A'e_{i1}$ is such a decomposition of A', then A has minimal condition on left ideals and $A = \dot{\sum_{i,\alpha}} Ae_{i\alpha}$ is a decomposition of A into P. I. 's $Ae_{i\alpha}$ with the property that $Ae_{i\alpha} \simeq Ae_{j\beta}$ iff $i = j$.

Proof: (1) This is immediate from (2) of the preceding lemma.
(2) First of all, it follows from the proof of lemma (0. 1) that the $e_{i\alpha}Ae_{i\alpha}$ are isomorphic rings for $1 \leq \alpha \leq K_i$. Thus by (1), the $e_{i\alpha}Ae_{i\alpha}$ are completely primary for $1 \leq \alpha \leq K_i$ iff $e_{i1}Ae_{i1}$ is completely primary iff $e_{i1}A'e_{i1}$ is completely primary (since $e_{i1}Ae_{i1} = e_{i1}A'e_{i1}$) and (2) follows.
(3) By (2), we must only show that $Ae_{i1} \simeq Ae_{j1}$ iff $A'e_{i1} \simeq A'e_{j1}$. But this follows from lemma (0. 6) - (3).

(0. C). Suppose now that A is a ring with identity and minimal condition on left ideals. Let $Ae_1, Ae_2, \ldots, Ae_r$ be a full set of non-isomorphic P. I. 's of A where the e_i form a set of mutually orthogonal idempotents. In this section we call the ring $\sum_{1 \leq i, j \leq r} e_iAe_j$ the reduced ring of A. We may make this definition since lemmas (0.4) and (0. 5) imply that any two such subrings of A are isomorphic. Since $\sum_i e_i$ is an idempotent of A, $\sum_i Ae_i = A \sum_i e_i$ is a direct summand of A as a left A-module. Hence the Ae_i may be included in a set of P. I. 's of A whose direct sum is A and

everything said above for A and A' holds for A and its reduced ring.

If we write $A = \dot{\sum}_{\substack{1 \leq i \leq r \\ 1 \leq \alpha \leq K_i}} Af_{i\alpha}$ where the $Af_{i\alpha}$ are P. I. 's and $Af_{i\alpha} \simeq Af_{j\beta}$ iff $i = j$, then the unordered set $K_1, K_2, \ldots, K_r$ is uniquely determined by the ring A. In this section we shall call $K_1, K_2, \ldots, K_r$ the set of multiplicities of A. Two rings with identity and minimal condition on left ideals possessing the same set of multiplicities are called equivalent if they have isomorphic reduced rings. Now each integer K_i is uniquely determined by the P. I. Af_{i1} and hence by the primitive idempotent f_{i1}. We shall say that K_i is the multiplicity corresponding to the primitive idempotent f_{i1}.

With this terminology we can state:

Theorem (0. 8). There is a one-to-one correspondence between the set of equivalence classes of all rings with identity and minimal condition on left ideals having a given, fixed set of multiplicities and the set of isomorphism classes of rings with identity and minimal condition on left ideals which are a direct sum of non-isomorphic P. I. 's. This correspondence is given by $A \rightarrow$ the reduced ring of A.

Proof. This follows from the definitions in (0. C) together with the construction in (0. B) and lemma (0. 7) - (3).

Corollary. Suppose that the ring A' with identity and minimal condition on left ideals is a direct sum of the non-isomorphic P. I. 's $A'e_1, A'e_2, \ldots, A'e_r$. Then there exists, up to isomorphism, a unique ring A with identity and minimal condition on left ideals whose reduced ring is A' having the set of multiplicities $K_1, K_2, \ldots, K_r$ where K_i is the multiplicity corresponding to the primitive idempotent e_i.

Proof. In view of the theorem, we must only observe that by lemma (0. 7) - (2), every e_i may be regarded as a primitive idempotent of A.

Example (0. 1). Let A be the ring of all matrices of the form

$$\begin{bmatrix} a & o & o & o \\ o & b & o & o \\ c & d & a & o \\ e & f & g & h \end{bmatrix}$$ where a, b,... are arbitrary elements of a skewfield D. A is a ring with identity and minimal condition on left ideals. Set $f_1 = e_{11} + e_{33}$, $f_2 = e_{22}$ and $f_3 = e_{44}$ where the e_{ii} are matrix units of the complete ring of 4×4 matrices with entries in D. Then A is a direct sum of the non-isomorphic P. I. 's Af_1, Af_2, Af_3.

We wish to construct the ring with identity and minimal condition on left ideals having the set of multiplicities 2, 3, 1 whose reduced ring is A and where 2 is the multiplicity corresponding to f_1, 3 corresponds to f_2, and 1 corresponds to f_3. We could proceed by employing the construction of (0. B). However, it is easier to make a "good guess", namely, let A* be the ring of all matrices of the form

$$\begin{bmatrix} a_1 & a_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_3 & a_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_1 & b_2 & b_3 & 0 & 0 & 0 \\ 0 & 0 & b_4 & b_5 & b_6 & 0 & 0 & 0 \\ 0 & 0 & b_7 & b_8 & b_9 & 0 & 0 & 0 \\ c_1 & c_2 & d_1 & d_2 & d_3 & a_1 & a_2 & 0 \\ c_3 & c_4 & d_4 & d_5 & d_6 & a_3 & a_4 & 0 \\ e_1 & e_2 & f_1 & f_2 & f_3 & g_1 & g_2 & h_1 \end{bmatrix} \quad \text{where } a_i, b_j, \ldots \in D.$$

Clearly A^* is a ring with identity. Let $g_{11} = e'_{11} + e'_{66}$, $g_{12} = e'_{22} + e'_{77}$, $g_{21} = e'_{33}$, $g_{22} = e'_{44}$, $g_{23} = e'_{55}$, $g_{31} = e'_{88}$ where the e'_{ii} are matrix units of the complete ring of 8×8 matrices with entries in D. Then $A^* = A^*g_{11} \dotplus A^*g_{12} + A^*g_{21} + A^*g_{22} + A^*g_{23} \dotplus A^*g_{31}$. But the ring $g_{11}A^*g_{11} \dotplus g_{11}A^*g_{21} \dotplus g_{11}A^*g_{31} \dotplus g_{21}A^*g_{11} \dotplus g_{21}A^*g_{21} \dotplus g_{21}A^*{}_{31} \dotplus g_{31}A^*g_{11} \dotplus g_{31}A^*g_{21} \dotplus g_{31}A^*g_{31}$

is the ring of all matrices

$$\begin{bmatrix} a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ c_1 & 0 & d_1 & 0 & 0 & a_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ e_1 & 0 & f_1 & 0 & 0 & g_1 & 0 & h_1 \end{bmatrix}$$

which is isomorphic to A. Hence, by the corollary to theorem (0. 8) and lemma (0. 7) - (3), A^* is the desired ring.

Theorem (0. 9). Suppose that A is a ring with identity and minimal condition on left ideals and that A' is the reduced ring of A. Then the following hold:

(1) If N is the radical of A, then $N' = N \cap A'$ is the radical of A' and $AN'A = N$.

(2) A is indecomposable iff A' is.

(3) If S is the socle of A, then $S' = S \cap A'$ is the socle of A' and $AS'A = S$. Furthermore, A is faithfully represented on its left socle iff A' is faithfully represented on its left socle.

(4) If $S = S_1 \dot{+} \cdots \dot{+} S_r$ is the decomposition of S into indecomposable two-sided ideals of A, then $S' = S'_1 \dot{+} \cdots \dot{+} S'_r$ where $S'_i = S_i \cap A'$ is the decomposition of the socle of A' into indecomposable two-sided ideals of A' and $AS'_iA = S_i$ (see theorem (1.3)).

(5) $\tau(A) = \tau(A')$ and if $A = \dot{\sum}_{\substack{1 \leq i \leq t \\ 1 \leq \alpha \leq K_i}} Ae_{i\alpha}$ where the $Ae_{i\alpha}$ are P.I.'s

such that $Ae_{i\alpha} \simeq Ae_{j\beta}$ iff $i = j$ and $A'f_1, \ldots, A'f_r$ are a complete set of non-isomorphic P.I.'s of A' where $Af_i \simeq Ae_{i1}$, then $\tau(Ae_{i\alpha}) = \tau(A'f_i)$ for $1 \leq i \leq t$ and $1 \leq \alpha \leq K_i$ (see 3.A).

(6) If T is a two-sided ideal of A, then $A'/T \cap A'$ is isomorphic to the reduced ring of A/T.

Proof. Let $A = \dot{\sum}_{\substack{1 \leq i \leq r \\ 1 \leq \alpha \leq K_i}} Ae_{i\alpha}$ where the $Ae_{i\alpha}$ are P.I.'s of A and

$Ae_{i\alpha} \simeq Ae_{j\beta}$ iff $i = j$. We may take (see (0.C)) $A' = \dot{\sum}_{1 \leq i, j \leq r} e_{i1}Ae_{j1}$

(direct sum as Z-modules). Then $A' = \dot{\sum}_{1 \leq i \leq r} A'e_{i1}$ is a decomposition of A' into the direct sum of non-isomorphic P.I.'s $A'e_{i1}$ of A' (lemma (0.7) - (3)).

(1) Let N be the radical of A and N' be the radical of A'. That $N \cap A' \subseteq N'$ is clear. On the other hand, since $N'e_{i1} \subset A'e_{i1}$, we have by lemma (0.6) - (2) that $AN'e_{i1} \subset Ae_{i1}$. Therefore, $AN'e_{i1} \subseteq Ne_{i1}$ so that $AN' = \sum_i AN'e_{i1} \subseteq N$. Hence $N' \subseteq N$ whence $N' = N \cap A'$. Lemma (0.6) - (4) implies that $N = AN'A$.

(2) By lemma (0.6) - (2) and (3), $Ae_{i1}Ae_{j1} = 0$ iff $A'e_{i1}A'e_{j1} = 0$. It follows that Ae_{i1} and Ae_{j1} have a common composition factor iff $A'e_{i1}$ and $A'e_{j1}$ do. Thus (2) is implied by block theory (see (1.E)).

(3) This is a simple consequence of (1) together with lemma (0.6) - (4) and (5).

(4) This is immediate from (3) plus lemma (0.6) - (4).

(5) Since $A' \cap Te_{i1} = (A' \cap T)e_{i1}$ for any two-sided ideal T of A, (5) follows from (1) together with lemma (0.6) - (2), (4) and (5).

(6) Let $\overline{A} = A/T$. Then $\dot{\sum}_{\substack{1 \le i \le r \\ 1 \le \alpha \le K_i}} \overline{A}\overline{e}_{i\alpha}$ is a decomposition of $\overline{A}$ into the direct sum of P.I.'s $\overline{A}\overline{e_{i\alpha}}$ of $\overline{A}$. Since $e_{i1} \in T$ implies $e_{i\alpha} \in T$ for $1 \le \alpha \le K_i$ (lemma (0.3)), $\overline{Ae_{i1}} = \overline{0}$ iff $\overline{Ae_{i\alpha}} = \overline{0}$ for every α such that $1 \le \alpha \le K_i$. Hence $\dot{\sum}_{1 \le i, j \le r} \overline{e_{i1}Ae_{j1}}$ is the reduced ring of $\overline{A}$. But the ring $A'/T \cap A'$ is isomorphic to the ring $(A' + T)/T = \overline{A}'$ so that (6) follows.

Index of Notations

Index of Terms

References

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