ON AN EMBEDDING PROPERTY OF GENERALIZED CARTER SUBGROUPS

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ABSTRACT

If $\mathcal{E}$ and $\mathcal{F}$ are saturated formations, we say that $\mathcal{E}$ is strongly contained in $\mathcal{F}$ if for any solvable group G with $\mathcal{E}$-subgroup, E, and $\mathcal{F}$-subgroup, F, some conjugate of E is contained in F. In this paper, we investigate the problem of finding the formations which strongly contain a fixed saturated formation $\mathcal{E}$.

Our main results are restricted to formations, $\mathcal{E}$, such that $\mathcal{E} = \{G|G/F(G) \in \mathcal{T}\}$, where $\mathcal{T}$ is a non-empty formation of solvable groups, and F(G) is the Fitting subgroup of G. If $\mathcal{T}$ consists only of the identity, then $\mathcal{E} = \mathcal{N}$, the class of nilpotent groups, and for any solvable group, G, the $\mathcal{N}$-subgroups of G are the Carter subgroups of G.

We give a characterization of strong containment which depends only on the formations $\mathcal{E}$, and $\mathcal{F}$. From this characterization, we prove:

If $\mathcal{T}$ is a non-empty formation of solvable groups, $\mathcal{E} = \{G|G/F(G) \in \mathcal{T}\}$, and $\mathcal{E}$ is strongly contained in $\mathcal{F}$, then

(1) there is a formation $\mathcal{V}$ such that $\mathcal{F} = \{G|G/F(G) \in \mathcal{V}\}$.

(2) If for each prime p, we assume that $\mathcal{T}$ does not contain the class, $\mathcal{S}_{p'}$, of all solvable p'-groups, then either $\mathcal{E} = \mathcal{F}$, or $\mathcal{F}$ contains all solvable groups.

This solves the problem for the Carter subgroups.

We prove the following result to show that the hypothesis of (2) is not redundant:

If $\mathfrak{R} = \{G \mid G/F(G) \in \mathfrak{S}_{r'}\}$, then there are infinitely many formations which strongly contain $\mathfrak{R}$.

Chapter I

Introduction

In 1961, Roger Carter [2] discovered the interesting result that a finite solvable group contains a nilpotent subgroup which is its own normalizer. These subgroups are now called Carter subgroups. In the same paper, Carter showed that any two Carter subgroups are conjugate, and that they possess certain interesting cover-avoidance properties.

Since the discovery of the Carter subgroups, several papers, devoted to the investigation of relations between the system normalizers of a finite solvable group and its Carter subgroups, have been published. In another direction, Gaschütz [5] has discovered a quite general theory from which most of the properties of Carter subgroups follow as a special case. One consequence of this theory is that there are many types of subgroups of a finite solvable group, G, which possess properties analogous to the properties of Carter subgroups; a subgroup of G which satisfies the necessary conditions is called a generalized Carter subgroup.

We shall, from now on, assume that all groups under discussion are finite and solvable. To describe our results we need

Definition 1. A formation, $\mathfrak{F}$, is a collection of groups which satisfies:

(1) If $G \in \mathfrak{F}$, and N is normal in G, then $G/N \in \mathfrak{F}$.

(2) If N_1, N_2 are normal in G, and $G/N_i \in \mathfrak{F}$, then $G/N_1 \cap N_2 \in \mathfrak{F}$.

A formation $\mathfrak{F}$ is <u>saturated</u> if it satisfies, in addition to (1) and (2),

(3) If $G/D(G) \in \mathfrak{F}$, and $D(G)$ is the Frattini subgroup of G, then $G \in \mathfrak{F}$.

The generalized Carter subgroups may be defined as follows:

<u>Definition 2</u>. Let G be a group, and $\mathfrak{F}$ a formation. A subgroup, F, of G is called an <u>$\mathfrak{F}$-subgroup</u> of G provided

(4) $F \in \mathfrak{F}$

(5) If U is any subgroup of G containing F, and if N is a normal subgroup of U such that $U/N \in \mathfrak{F}$, then $FN = U$ (i.e., F covers U/N).

One of the basic results on formations is that if $\mathfrak{F}$ is any saturated formation, and if G is any solvable group, then $\mathfrak{F}$-subgroups of G always exist, and all such groups are conjugate (Gaschütz, [5]).

In this thesis we shall investigate the behaviour of the $\mathcal{E}$-subgroups of a solvable group G with respect to the $\mathfrak{F}$-subgroups of G for two saturated formations $\mathcal{E}$, and $\mathfrak{F}$.

<u>Definition 3</u>. Let $\mathcal{E}$ and $\mathfrak{F}$ be saturated formations. $\mathcal{E}$ is <u>strongly contained in</u> $\mathfrak{F}$ (written $\mathcal{E} \ll \mathfrak{F}$) provided

(6) If G is any solvable group which has $\mathcal{E}$-subgroup, E, and $\mathfrak{F}$-subgroup, F, then some conjugate of E is contained in F.

We shall be concerned with the problem: Given a saturated formation

$\mathcal{E}$, find all formations $\mathcal{F}$ such that $\mathcal{E} \ll \mathcal{F}$. From the definition, it is clear that there are always two formations, $\mathcal{F}$, which strongly contain $\mathcal{E}$, viz. $\mathcal{E}$, and $\mathcal{S}$, the formation of all solvable groups. We comment that the relation $\ll$ forms a partial ordering of the collection of all saturated formations. It is a consequence of the definition that if $\mathcal{E} \ll \mathcal{F}$, then $\mathcal{E}$ is contained in $\mathcal{F}$.

The two main results of this thesis appear in Chapter V. The first of these shows that if $\mathcal{T}$ is a non-empty formation, and $\mathcal{E} = \{G | G/F(G) \in \mathcal{T}\}$ (F(G) is the Fitting subgroup of G), then any formation $\mathcal{F}$ which strongly contains $\mathcal{E}$ has essentially the same structure as $\mathcal{E}$ in that there is a non-empty formation $\mathcal{V}$ such that $\mathcal{F} = \{G | G/F(G) \in \mathcal{V}\}$. We then apply this result to find a large class of formations which are maximal in the partial ordering $\ll$. In particular, if $\mathcal{N}_i$ denotes the formation of groups of nilpotent length at most i, then $\mathcal{N}_i$ is maximal in the ordering $\ll$. It is easily shown that for $\mathcal{N} = \mathcal{N}_1$, the $\mathcal{N}$-subgroups of a group G are the Carter subgroups of G, so the problem is solved for the Carter subgroups.

The machinery for proving these results is developed in Chapter IV. We begin by developing a characterization of strong containment which depends only on the two formations $\mathcal{E}$ and $\mathcal{F}$. This characterization depends on the knowledge that if $\mathcal{E}$ is a saturated formation, then $\mathcal{E}$ is a locally defined formation (see Chapter II), a result proved by Lubeseder in [10]. In certain cases, we are able to strengthen our characterization of strong containment so that it gives a complete description of the minimal local definition for $\mathcal{F}$ as a necessary condition for strong containment.

In Chapter VI, we present an example which shows that Hypothesis II of our main theorem is not redundant. The formation which gives the example is $\mathcal{R} = \{G \mid G/F(G)$ is an r'-group$\}$. It is apparent from Theorem 11 that $\mathcal{R}$ is not maximal in the partial ordering $\ll$. In fact, there is an infinite number of formations which strongly contain $\mathcal{R}$.

In Chapter III, we give some preliminary results which are used throughout the paper. In particular, we give a cover-avoidance characterization of the $\mathcal{F}$-subgroups of a solvable group. In Theorem 5, we find the $\mathcal{F}$-subgroups of a certain type of semi-direct product. We also show how these subgroups are embedded in the product. The remaining results in Chapter III are known, but their proofs are included for the sake of completeness.

In Chapter II, we give notation along with the results quoted from previous papers. Some of the results in this chapter are proved because we have found no suitable reference to them in the literature.

Chapter II

Notation and Quoted Results

We shall use the following notation:

G - A finite solvable group

$|G|$ - The order of G

$H \leq G$ - H is a subgroup of G (proper if $H < G$)

$H \lhd G$ - H is normal in G

$\subseteq$ - A subset of (proper subset if $\subset$)

$D(G)$ - The Frattini subgroup of G, the intersection of all maximal subgroups of G

$F(G)$ - The Fitting subgroup of G, the maximal normal nilpotent subgroup of G

$F_i(G)$ - The inverse image in G of $F(G/F_{i-1}(G))$

$\mathrm{Aut}(G)$ - The automorphism group of G

Ξ - A field

Z_p - The field of integers mod p, p a prime

$\Xi(G)$ - The group ring of G over the field Ξ

$\simeq_{\Xi(G)}$ - Is $\Xi(G)$ - isomorphic to

π - A set of primes

π' - The complementary set of primes

$O_\pi(G)$ - The maximal normal π-subgroup of G

$O_{\pi_1,\ldots,\pi_n}(G)$ - The inverse image in G of $O_{\pi_n}(G/O_{\pi_1,\ldots,\pi_{n-1}}(G))$

$O^{\pi}(G)$ - The smallest normal subgroup N of G such that G/N is a π-group

$O^{\pi_1,\ldots,\pi_n}(G) - O^{\pi_n}(O^{\pi_1,\ldots,\pi_{n-1}}(G))$.

If $H \lhd K$, and $H \leq K \leq G$, then K/H is a section of G. K/H is a π-section if K/H is a π-group. If $L \leq G$, then L covers the section K/H provided $(L \cap K)H = K$, and avoids K/H if $K \cap L \leq H$.

If F acts on G (i.e., every element of F induces an automorphism of G), and F fixes both H and K, then K/H is an F-invariant section of G. If F acts irreducibly on K/H, then K/H is an F-composition factor of G. An F-composition series of G is a series, $G = G_0 > G_1 > \cdots > G_n = 1$, such that G_i/G_{i+1} is an F-composition factor of G for each i. The F-composition length of G is n, the length of any F-composition series of G.

If F = G, then K/H is a chief factor of G, and the series, $G = G_0 > G_1 > \cdots > G_n = 1$, is a chief series of G.

If K/H is an F-invariant section of G, then $C_F(K/H)$ denotes the kernel of the representation of F on K/H. $C_{K/H}(F)$ denotes the elements of K/H fixed by every element of F.

If F acts on G, and K/H is an F-invariant section of G which is also an elementary abelian p-group, then we may look at K/H as a vector space over Z_p, hence also as a $Z_p(F)$-module with the action

$$(Hk)f = Hk^f \qquad \text{for } f \in F,\ Hk \in K/H.$$

If M is a right $Z_p(G)$-module, we may form the semi-direct product GM, where the action of G on M by conjugation is

$$m^g = g^{-1}mg = (m)g \qquad \text{for } m \in M,\ g \in G.$$

We shall frequently switch back and forth between additive and multiplicative notation.

If M and N are right $\Xi(G)$-modules, then $M \oplus N$ will denote the direct sum of these modules, and $\mathrm{Hom}_{\Xi(G)}(M,N)$ will denote the collection of all $\Xi(G)$-linear homomorphisms from M into N.

If H is a second group, and T is a $\Xi(G)$-module, then the tensor product $M \otimes_{\Xi} T$ (sometimes written $M \otimes T$) is a $\Xi(G \times H)$-module under the canonical action

$$(m \otimes t)(g \times h) = mg \otimes th$$

A <u>principal indecomposable $\Xi(G)$-module</u> is an indecomposable component of the regular $\Xi(G)$-module. If M is a completely reducible $\Xi(G)$-module, and $M = I_1 \oplus \cdots \oplus I_s$ is a decomposition of M as a direct sum of irreducible submodules, I_k, then for each k we let M_{I_k} be the direct sum of all copies of I_k which appear in this decomposition of M. M_{I_k} is called a <u>primary component</u> of M.

The <u>dual $\Xi(G)$-module</u> to M will be denoted by $\hat{M}$. We recall that $\hat{M}$ is the collection of all Ξ-linear functionals from M into Ξ, and that $\hat{M}$ becomes a right $\Xi(G)$-module if we let G operate on $\hat{M}$ so that for g in G, and μ in $\hat{M}$, μg is the functional defined by

$$(m)\mu g = (mg^{-1})\mu \qquad \text{for } m \in M$$

If N is a $\Xi(G)$-submodule of M, we set $N^{\perp} = \{\mu \in \hat{M} \mid (N)\mu = 0\}$; we note that the following relations hold if we assume M is finitely generated and identify M with its second dual.

$$(N^{\perp})^{\perp} = N, \ (0)^{\perp} = \hat{M}, \ M^{\perp} = (0) \tag{7}$$

If N, U are $\Xi(G)$-submodules of M such that $N \subseteq U$, then $U^{\perp} \subseteq N^{\perp}$,

$$N^{\perp}/U^{\perp} \simeq {}_{\Xi(G)}(\widehat{U/N}), \text{ and} \tag{8}$$

$$C_G(U/N) = C_G((\widehat{U/N})). \tag{9}$$

Furthermore, if I_1 and I_2 are submodules of M, then

$$M = I_1 \oplus I_2 \Longleftrightarrow \hat{M} = I_1^{\perp} \oplus I_2^{\perp}. \tag{10}$$

Suppose $H \triangleleft G$, and U is a $\Xi(H)$-module. For a fixed g in G, we can define a new $\Xi(H)$-module U^g by looking at U as a Ξ-vector space and specifying a new action for H,

$$u \bullet h = u(ghg^{-1}) \qquad \text{for } h \in H$$

The module U^g is called the <u>conjugate</u> of U by g.

If $F \leq G$, and M is a $\Xi(G)$-module, then $M|_F$ denotes the $\Xi(F)$-module obtained from M by restriction to $\Xi(F)$. If U is a $\Xi(F)$-module, then $U|^G$ is the $\Xi(G)$-module induced from U. One of the most convenient ways to define $U|^G$ is presented in Green's paper [8].

Set $U|^G = U \otimes_{\Xi(F)} \Xi(G)$, then $U|^G$ can be written as a direct sum of Ξ-subspaces in the following way: Let $1 = g_1, g_2, \cdots, g_s$ be a complete set of coset representatives for F in G. Then

$$U|^G = U \otimes 1 \oplus U \otimes g_2 \oplus \cdots \oplus U \otimes g_s$$

This decomposition does not depend on the choice of coset representatives. The action of G on $U|^G$ is characterized by giving the action of G on each element $m \otimes g_i$:

(11) $$(m \otimes g_i)g = mf_g \otimes g_j \qquad \text{where } g_i g = f_g g_j,$$

and $f_g \in F$.

Using (11), one can show that U is canonically embedded as a submodule of $U|^G|F$ by the mapping

(12) $$\Delta: \; u \longrightarrow u \otimes 1 \qquad \text{for } u \in U.$$

We shall use the following results:

(13) ([1], p.8) Let $M = I_1 \oplus \cdots \oplus I_n$ be a decomposition of the $\Xi(G)$-module M into a direct sum of irreducible $\Xi(G)$-submodules. Let M_{I_k} be the primary component of M associated with I_k. If W is any $\Xi(G)$-submodule of M isomorphic to I_k, then $W \subseteq M_{I_k}$.

(14) ([4], p.75) If Ξ is a field of characteristic p, G is a group, and M is an irreducible $\Xi(G)$-module, then $O_p(G)$ lies in the kernel of the representation of G on M.

(15) Let H be a p'-subgroup of G, and let M be the $Z_p(G)$-module which gives the permutation representation of G on the cosets of H, $1_H|^G$. If $M = J_1 \oplus \cdots \oplus J_n$ is any decomposition of M into a direct sum of indecomposable $Z_p(G)$-modules, then each J_k is a principal indecomposable $Z_p(G)$-module.

Proof. If R is the regular $Z_p(H)$-module, then $R|^G$ is the regular $Z_p(G)$-module. Since H is a p'-group, R is completely reducible to a direct sum of $Z_p(H)$-irreducible submodules, one of which is the identity $Z_p(H)$-module, I. We have, therefore, a decomposition $R = I \oplus L$. Thus $R|^G = I|^G \oplus L|^G$, so because $M = I|^G$, each J_k is a direct summand of the regular $Z_p(G)$-module.

(16) ([3], p. 372) Let G be a group, and J a principal indecomposable $\Xi(G)$-module for some field Ξ. Then J has a unique maximal proper $\Xi(G)$-submodule which we shall denote by $\sqrt{J}$.

(17) (cf. [7], Theorem 1) Suppose A is a group of automorphisms of the group G, and $(|A|,|G|) = 1$. If H/K is an A-invariant section of G, then $C_G(A)$ covers $C_{H/K}(A)$.

The next three results are well-known elementary facts that prove to be quite useful. Their proofs will be included in an appendix.

(18') If $N \lhd G$, and $\pi_1,\cdots,\pi_n$ are sets of primes, then

$$O_{\pi_1,\cdots,\pi_n}(N) = N \cap O_{\pi_1,\cdots,\pi_n}(G).$$

(18) Let G be a group, then $O_{p',p}(G) = \cap\, C_G(K)$, where the intersection is taken over all p-chief factors, K, of G.

(19) Let G be a group, then $F(G) = \cap\, O_{p',p}(G)$, where the intersection is taken over all primes p.

We shall give a short summary of the theory of formations which is found in the papers of Gaschütz, and Lubeseder ([5],[6], and [10]). Let $\mathcal{S}$ be the collection of all solvable groups.

Definition 4. For each prime p, let $\mathfrak{F}_p$ be a formation. Let

$$\mathfrak{F} = \left\{ G \in \mathcal{S} \;\middle|\; \begin{array}{l} (\alpha)\ \text{If } \mathfrak{F}_p \text{ is non-empty, and } K \text{ is a p-chief factor} \\ \quad\ \text{of } G, \text{ then } G/C_G(K) \in \mathfrak{F}_p. \\ (\beta)\ \text{If } \mathfrak{F}_p \text{ is empty, then } G \text{ is a p'-group.} \end{array} \right\}$$

$\mathfrak{F}$ is a formation which is called the formation locally defined by the family $\{\mathfrak{F}_p\}$. In general, a formation, $\mathfrak{F}$, is locally defined if there exists a family, $\{\mathfrak{F}_p\}$, of formations such that $\mathfrak{F}$ is locally defined by the family $\{\mathfrak{F}_p\}$.

Because of (18) and the fact that $\mathfrak{F}_p$ is a formation, (2) is equivalent to the condition:

(α') If $\mathfrak{F}_p$ is non-empty, then $G/O_{p',p}(G) \in \mathfrak{F}_p$.

The family, $\{\mathfrak{F}_p\}$, of formations which locally defines $\mathfrak{F}$ is usually not unique. If, however, $\{\mathfrak{F}_p\}$ and $\{\mathfrak{F}'_p\}$ are two families of formations which locally define the same formation $\mathfrak{F}$, then the family $\{\mathcal{H}_p \mid \mathcal{H}_p = \mathfrak{F}_p \cap \mathfrak{F}'_p\}$ also defines $\mathfrak{F}$. Thus there is a unique minimal local definition for any locally defined formation $\mathcal{F}$. For example, the minimal local definition of the formation of all p-groups, $\mathcal{H}$, is

$$\mathcal{H}_p = \{1\}$$

$$\mathcal{H}_q = \text{empty if } q \neq p$$

(20) ([5], p. 302; [6], p. 198) If $\mathcal{F}$ is locally defined, then $\mathcal{F}$ is saturated.

(21) ([10]) If $\mathcal{F}$ is saturated, then $\mathcal{F}$ is locally defined.

From now on, we shall use the terms saturated and locally defined interchangeably.

Definition 5. If $\mathcal{F}$ is a formation, and G is a group, then

$G_{\mathcal{F}} = \bigcap_{N \in \mathfrak{X}} N$ where $\mathfrak{X} = \{N \mid N \lhd G, \text{ and } G/N \in \mathcal{F}\}$.

Using this definition, a subgroup, F, of G is an $\mathcal{F}$-subgroup of G if, and only if,

(22) $F \in \mathcal{F}$, and

(23) if $F \leq U \leq G$, then F covers $U/U_{\mathcal{F}}$.

(24) ([5], p. 301) If the formation, $\mathcal{F}$, is saturated, then every solvable group G has an $\mathcal{F}$-subgroup. All the $\mathcal{F}$-subgroups of G are conjugate.

(25) ([5], P. 301) Let $\mathcal{F}$ be a formation, and G a group. Let F be an element of $\mathcal{F}$ such that $F \leq G$. Then

(a) If F is an $\mathcal{F}$-subgroup of G, and $F \leq U \leq G$, then F is an $\mathcal{F}$-subgroup of U.

(b) If $N \lhd G$, and F is an $\mathcal{F}$-subgroup of G, then FN/N is an $\mathcal{F}$-subgroup of G/N.

(c) If $N \lhd G$, F_1/N is an $\mathcal{F}$-subgroup of G/N, and F is an $\mathcal{F}$-subgroup of F_1, then F is an $\mathcal{F}$-subgroup of G.

Chapter III

Preliminary Results

The first theorem of this chapter is a result similar to Clifford's theorem.

Theorem 1. Let H be a normal p'-subgroup of G, Ξ a field of characteristic p, and M an indecomposable $\Xi(G)$-module. Then $M|_H$ is a completely reducible $\Xi(H)$-module whose irreducible components form a single class, C, of conjugate $\Xi(H)$-modules under conjugation by elements of G. Let L, J be two $\Xi(G)$-submodules of M such that $L \subset J$. Then the distinct $\Xi(H)$-irreducible components of $(J/L)|_H$ are precisely the elements of C.

Proof. The complete reducibility of $M|_H$ follows directly from the theorem of complete reducibility. Let $M|_H = I_1 \oplus \cdots \oplus I_s$ be a decomposition of M as a direct sum of irreducible $\Xi(H)$-submodules, and for each $k = 1,\cdots,s$, let M_k be the primary component of $M|_H$ associated with I_k. Consider the module $I_k g$ for some g in G. The action of H on this module is the same as the action of H on the conjugate module I_k^g because of the formula:

$$(i_k g)h = i_k(ghg^{-1})g \ .$$

Now $I_k g$ is isomorphic to some I_j, hence by (13), $I_k g \subseteq M_j$. Clearly $I_j g^{-1}$ is isomorphic to I_k, so $I_j g^{-1} \subseteq M_k$. Therefore

$M_k^g \simeq M_k g = M_j$, so G permutes the primary components of $M|_H$ by conjugation.

Let $C_1, \cdots, C_r$ be the conjugacy classes of primary components of $M|_H$ under the action of G. Set $T_i = \sum_{M_k \in C_i} \oplus M_k$. Because $M|_H$ is the direct sum of its primary components, $M|_H = T_1 \oplus \cdots \oplus T_r$. But each T_i is a $\Xi(G)$-submodule of M, so we contradict the indecomposability of M unless $r = 1$. Let C be the conjugacy class of irreducible components of $M|_H$.

Obviously, all of the irreducible $\Xi(H)$-submodules of $(J/L)|_H$ lie in C. Since G is transitive on the elements of C, and J/L is a $\Xi(G)$-module (non-zero), every element of C must appear as a component of $(J/L)|_H$.

Lemma 1. Let G be solvable, and M be a $Z_p(G)$-module. M is faithful if, and only if, $M|_{F(G)}$ is faithful.

Proof. The lemma follows a fortiori from the statement that if $1 < N \lhd G$, then $1 < N \cap F(G)$. Now $1 < N$, and N is solvable, so $1 < F(N) = N \cap F(G)$, so the lemma holds.

The second theorem in this chapter is the "Frobenius reciprocity theorem" for modules. It is one of the main tools used in this paper. For ordinary representations, the usual reciprocity theorem may be derived from this isomorphism by noting that if μ, and η are the characters of two $\Xi(G)$-modules, M and N, then the scalar product

(μ,η) is the Ξ-dimension of $\mathrm{Hom}_{\Xi(G)}(M,N)$.

Theorem 2. Let G be a group, and $H \leq G$. Let M be a $\Xi(H)$-module, and N a $\Xi(G)$-module. Then $\mathrm{Hom}_{\Xi(G)}(M|^G,N)$, and $\mathrm{Hom}_{\Xi(H)}(M,N|_H)$ are isomorphic as vector spaces over Ξ.

Proof. We define a map Θ, from $\mathrm{Hom}_{\Xi(G)}(M|^G,N)$ into $\mathrm{Hom}_{\Xi(H)}(M,N|_H)$ by

$$\Theta: \eta \longrightarrow \Delta \circ \eta|_{M\otimes 1} = \Theta(\eta), \qquad \text{for } \eta \in \mathrm{Hom}_{\Xi(G)}(M|^G,N),$$

where Δ is the canonical map defined by (12). Θ is clearly well defined. If $m \in M$, and $h \in H$, then

$$[mh]\Theta(\eta) = [mh\otimes 1]\eta|_{M\otimes 1} = [(m\otimes 1)h]\eta|_{M\otimes 1}$$
$$= \{[m \otimes 1]\eta|_{M\otimes 1}\}h = \{[m]\Theta(\eta)\}h\ .$$

In this computation, we have used (11) and the fact that $\eta|_{M\otimes 1}$ is an element of $\mathrm{Hom}_{\Xi(H)}(M\otimes 1,N)$. Both Δ and $\eta|_{M\otimes 1}$ are Ξ-linear maps, so $\Theta(\eta)$ is a $\Xi(H)$-homomorphism from M into $N|_H$. Since restriction is a Ξ-linear operation, it follows that Θ is a Ξ-linear map from $\mathrm{Hom}_{\Xi(G)}(M|^G,N)$ into $\mathrm{Hom}_{\Xi(H)}(M,N|_H)$.

Suppose $\Theta(\eta) = 0$. Then $\eta|_{M\otimes 1} = 0$, so using (11) we have

$$(m\otimes g_\ell)\eta = ((m\otimes 1)g_\ell)\eta = \{(m\otimes 1)\eta\}g_\ell = 0\,g_\ell = 0,$$

since η is a $\Xi(G)$-homomorphism. Therefore Θ is a 1-1 map.

Let μ be an arbitrary element of $\mathrm{Hom}_{\Xi(H)}(M,N|_H)$. We set

$(\sum_{\ell=1}^{s}(m_\ell \otimes g_\ell))\mu^* = \sum_{\ell=1}^{s}(m_\ell)\mu g_\ell$, where $g_1 = 1, g_2, \cdots, g_s$ are a set of coset representatives for H in G, and $[G:H] = s$. Since N is a $\Xi(G)$-module, μ^* is a well defined map from $M|^G$ into N. If $g \in G$, and $v = \sum_{\ell=1}^{s} m_\ell \otimes g_\ell$ lies in $M|^G$, then

$$
\begin{aligned}
(vg)\mu^* &= \left\{\sum_{\ell=1}^{s}(m_\ell \otimes g_\ell)g\right\}\mu^* = \left\{\sum_{\ell=1}^{s} m_\ell h_g \otimes g_{\gamma(\ell)}\right\}\mu^* \\
&= \sum_{\ell=1}^{s}(m_\ell h_g)\mu g_{\gamma(\ell)} \quad = \sum_{\ell=1}^{s}(m_\ell)\mu h_g g_{\gamma(\ell)} \\
&= \left\{\sum_{\ell=1}^{s}(m_\ell)\mu g_\ell\right\}g \quad = (v)\mu^* g ,
\end{aligned}
$$

where γ is the permutation of the cosets Hg_i induced by multiplication on the right by g. We have used the fact that μ is a $\Xi(H)$-homomorphism, as well as equation (11). Because μ is Ξ-linear, μ^* is Ξ-linear, hence μ^* lies in $\mathrm{Hom}_{\Xi(G)}(M|^G, N)$.

If $m \in M$, then

$$[m]\Theta(\mu^*) = (m \otimes 1)\mu^*|_{M \otimes 1} = (m)\mu 1 = (m)\mu,$$

therefore $\Theta(\mu^*) = \mu$. This shows that Θ maps $\mathrm{Hom}_{\Xi(G)}(M|^G, N)$ onto $\mathrm{Hom}_{\Xi(H)}(M, N|_H)$, and completes the proof.

We shall begin a discussion of the properties of $\mathcal{F}$-subgroups of solvable groups. It is useful to know the behavior of $G_{\mathcal{F}}$ (cf. Def. 5) under homomorphisms, so we prove

Lemma 2. Let $\mathcal{F}$ be a formation, G a group, and H a normal subgroup of G. Then

$$(G/H)_{\mathcal{F}} = G_{\mathcal{F}}H/H$$

Proof. Let F be the inverse image, in G, of $(G/H)_{\mathcal{F}}$. Then G/F is isomorphic to $(G/H)/(G/H)_{\mathcal{F}}$, hence G/F lies in $\mathcal{F}$. Therefore, $G_{\mathcal{F}}H \leq F$.

Since $G_{\mathcal{F}} \leq G_{\mathcal{F}}H$, $G/G_{\mathcal{F}}H$ is an element of $\mathcal{F}$. Furthermore, $G/G_{\mathcal{F}}H$ is isomorphic to $(G/H)/(G_{\mathcal{F}}H/H)$. Therefore $F/H \leq G_{\mathcal{F}}H/H$, so we also have $F \leq G_{\mathcal{F}}H$. This shows that $F = G_{\mathcal{F}}H$, and completes the proof.

The next two theorems give cover-avoidance properties of $\mathcal{F}$-subgroups. They generalize a remark made by Carter in [2].

Theorem 3. Let $\mathcal{F}$ be a formation locally defined by the family $\{\mathcal{F}_p\}$, G be a group, F a subgroup of G which lies in $\mathcal{F}$, and K an F-composition factor of G. Then

(a) F either covers, or avoids K;

(b) if F covers K, and $p \mid |K|$, then $F/C_F(K) \in \mathcal{F}_p$;

(c) if F is an $\mathcal{F}$-subgroup of G, and $p \mid |K|$, then

$$\text{(C)} \quad F/C_F(K) \in \mathcal{F}_p \Longrightarrow F \text{ covers } K.$$

Proof. Let $K = L/M$ be the F-composition factor in question. Statement (a) follows from the fact that F acts irreducibly on K, and $(L \cap F)M/M$ is an F-invariant subgroup of K.

If F covers K, then $L/M = (L \cap F)M/M$. If we look at F as a set of operators on K, then K is operator isomorphic to $L \cap F/M \cap F$, a p-chief factor of F. Therefore the kernel of the representation of F on $L \cap F/M \cap F$ is $C_F(K)$. Because F lies in $\mathcal{F}$, it follows that $F/C_F(K) \in \mathcal{F}_p$. This proves (b).

Now suppose F is an $\mathcal{F}$-subgroup of G, and K is a p-section of G such that $F/C_F(K)$ lies in $\mathcal{F}_p$. To show F covers K, it suffices to show that F covers the larger section FL/M. But by (25a), F is an $\mathcal{F}$-subgroup of FL, hence it is sufficient to show $\overline{F} = FL/M$ is an element of $\mathcal{F}$ since F, by definition, covers any such factor of FL.

If $q \neq p$, then K, as a normal q'-subgroup of $\overline{F}$, is contained in $O_{q'}(\overline{F})$. Therefore $O_{q',q}(F)L/M \leq O_{q',q}(\overline{F})$, so $\overline{F}/O_{q',q}(\overline{F})$ is isomorphic to a quotient group of $FL/O_{q',q}(F)L$. $FL/O_{q',q}(F)L$ is isomorphic to $F/O_{q',q}(F)(F \cap L)$, a factor of $F/O_{q',q}(F)$. Since $F \in \mathcal{F}$, $F/O_{q',q}(F) \in \mathcal{F}_q$, hence $\overline{F}/O_{q',q}(\overline{F})$ is also an element of $\mathcal{F}_q$.

Let $U = F_{\mathcal{F}_p}$. Since $F \in \mathcal{F}$, $F/O_{p',p}(F) \in \mathcal{F}_p$, hence $U \leq O_{p',p}(F)$. Since $F/C_F(K) \in \mathcal{F}_p$, $U \leq C_F(K)$, hence K is contained in the center of UL/M. Therefore UL/M has a normal p-complement. By (18'), UL/M is contained in $O_{p',p}(\overline{F})$, thus $\overline{F}/O_{p',p}(\overline{F})$ is a factor group of $FL/UL \simeq F/U(F \cap L)$. Therefore, $\overline{F}/O_{p',p}(\overline{F}) \in \mathcal{F}_p$. Because $\mathcal{F}$ is locally defined, $\overline{F}$ lies in $\mathcal{F}$.

Our next theorem will show that property (C) characterizes the $\mathcal{F}$-subgroups of a solvable group G. In order to obtain as weak an hypothesis as possible, we prove two lemmas. Property (C) actually applies only to specific F-composition factors of G, so when we say that property (C) holds for an F-composition series, $G = G_0 > G_1 > \cdots > G_n = 1$, of G, we mean that F satisfies this property for all factors, G_i/G_{i+1}, of the series for which the hypothesis of property (C) holds.

Lemma 3. Suppose $\mathcal{F}$ is a formation locally defined by $\{\mathcal{F}_p\}$, $F \in \mathcal{F}$, and $F \leq G$. Let A/B be an F-invariant section of G such that $A > C > B$ defines a fixed F-composition series of A/B. If property (C) holds for this series, then property (C) holds for every F-composition series of A/B.

Proof. We may assume that a second F-composition series of A/B exists and is defined by $A > D > B$ where $D \neq C$. Then we must have $CD > C > C \cap D$, so the fact that C and D are F-invariant shows that $A = CD$, and $B = C \cap D$. Therefore

$$(26) \qquad A/B \simeq C/B \times D/B, \; A/C \simeq D/B, \; A/D \simeq C/B,$$

where the decomposition is an operator decomposition, and the isomorphisms are operator isomorphisms.

Suppose the decomposition (26) is unique. If $F/C_F(A/D)$ lies in $\mathcal{F}_p$, it follows from (26) that $F/C_F(C/B)$ lies in $\mathcal{F}_p$. Since property

(C) holds for the series $A > C > B$, F covers C/B. Therefore $(F \cap A)D \geq (F \cap C)D \geq CD = A$, so F covers A/D. Suppose $F/C_F(D/B)$ lies in $\mathcal{F}_q$, then $F/C_F(A/C)$ lies in $\mathcal{F}_q$. By property (C), F covers A/C. Because of the uniqueness of the decomposition, and the fact that $F \cap A$ is not contained in C, either $A = (F \cap A)B$, or $D = (F \cap A)B$. In the former case, F covers all of A/B, hence F covers the subgroup D/B. In the latter case, $F \cap A = F \cap D$ since $F \cap A \leq D$, so $D = (F \cap D)B$. Therefore, in either case, F covers D/B.

The decomposition (26) is unique if $(|A/C|,|C/B|) = 1$, so we may assume A/B is an elementary abelian p-group for some prime p. This means that we can look at A/B as a $Z_p(F)$-module. If A/C and C/B are distinct $Z_p(F)$-modules, then it follows from (13) that the decomposition is unique. Thus we may assume that $D/B \simeq_{Z_p(F)} C/B$. In this case, it follows from property (C) for $A > C > B$ that F either covers, or avoids A/B. Therefore Lemma 3 holds in all cases.

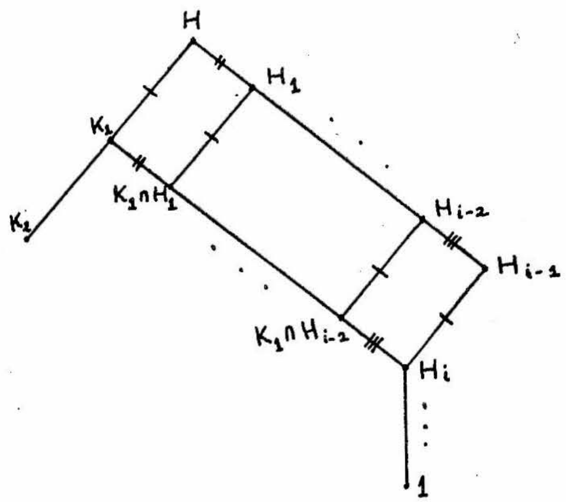
Lemma 4. Assume $F \in \mathcal{F}$, $H \leq G$, and $F \leq N_G(H)$. If property (C) holds for a fixed F-composition series of H, then it holds for every F-composition series of H.

Proof. Let $H = H_0 > H_1 > \cdots > H_n = 1$ be the fixed F-composition series of H for which property (C) holds. We use induction on n. The lemma is true if $n = 1$.

Let $H = K_0 > K_1 > \cdots > K_n = 1$ be a second F-composition series for H. If $K_1 = H_1$, property (C) holds for the series

$K_0 > K_1 > \cdots > K_n = 1$ by induction.

Suppose $K_1 \neq H_1$, and let i be the smallest integer such that $K_1 \cap H_i = H_i$. Because $H_i \leq K_1 \cap H_{i-1} < H_{i-1}$, we have $H_i = K_1 \cap H_{i-1}$, so that we have the following lattice diagram:



Since H_1 is part of an F-composition series for H, $F \leq N_G(H_1)$. Because of the isomorphisms indicated in the diagram,
$H_1 > K_1 \cap H_1 > \cdots > K_1 \cap H_{i-1} = H_i > H_{i+1} > \cdots > H_n = 1$ is an F-composition series for H_1 which has length n-1. By induction, property (C) holds for this series. Therefore, property (C) holds for the F-composition series of $H/H_1 \cap K_1$ defined by the series $H > H_1 > H_1 \cap K_1$. By Lemma 3, property (C) holds for the F-composition series $H > K_1 > K_1 \cap H_1 > \cdots > K_1 \cap H_{i-1} = H_i > \cdots > H_n = 1$ of H. In particular, property (C) holds, by induction, for any F-composition series of K_1. Therefore property (C) holds for the series $K_0 > K_1 > \cdots > K_n = 1$.

<u>Theorem 4.</u> Let $\mathcal{F}$ be a formation locally defined by $\{\mathcal{F}_p\}$. Let G be a group, $F \leq G$, and $F \in \mathcal{F}$. If property (C) holds for a fixed F-composition series $G = G_0 > G_1 > \cdots > G_n = 1$ of G, then F is an $\mathcal{F}$-subgroup of G.

<u>Proof.</u> We use induction on $|G|$. By Lemma 4, we may assume that the series $G = G_0 > G_1 > \cdots > G_n = 1$ is a refinement of the chief series $G = H_0 > H_1 > \cdots > H_{m-1} > H_m = 1$. Then $H_{m-1} = G_k$ for some k. H_{m-1} is a minimal normal subgroup of G, so we set $\overline{G}_i = G_i/G_k$ for $i = 0,1,\cdots,k$, $\overline{F} = FG_k/G_k$, and $\overline{G} = \overline{G}_0$. Our first step is to show that $\overline{F}$ is an $\mathcal{F}$-subgroup of $\overline{G}$.

If $m = 1$, then $\overline{G} = \overline{F} = \overline{1}$, so the result is trivial. If $m > 1$, then $H_{m-1} < G$, and $\overline{G} = \overline{G}_0 > \overline{G}_1 > \cdots > \overline{G}_k = \overline{1}$ is an $\overline{F}$-composition series of $\overline{G}$. By induction, to show that $\overline{F}$ is an $\mathcal{F}$-subgroup of $\overline{G}$, it is sufficient to verify property (C) for the series $\overline{G}_0 > \overline{G}_1 > \cdots > \overline{G}_k = \overline{1}$.

For each i, set $K_i = G_i/G_{i+1}$, and $\overline{K}_i = \overline{G}_i/\overline{G}_{i+1}$. Since $G_k \leq G_{i+1}$ for $i < k$, G_k centralizes the section K_i for $i < k$. Therefore, $C_{\overline{F}}(\overline{K}_i) = C_F(K_i)G_k/G_k$ for all $i < k$. Thus,

$$\overline{F}/C_{\overline{F}}(\overline{K}_i) \simeq FG_k/C_F(K_i)G_k \simeq F/C_F(K_i)(F \cap G_k).$$

But $F \cap G_k \leq C_F(K_i)$, so we have

$$\overline{F}/C_{\overline{F}}(\overline{K}_i) \simeq F/C_F(K_i) \qquad \text{for } i < k. \tag{27}$$

Suppose $\overline{K}_i$ is a p-section of $\overline{G}$ such that $\overline{F}/C_{\overline{F}}(\overline{K}_i) \in \mathfrak{F}_p$. By (27), $F/C_F(K_i) \in \mathfrak{F}_p$, so F covers K_i . Therefore, $(FG_k \cap G_i)G_{i+1} = (F \cap G_i)G_kG_{i+1} = (F \cap G_i)G_{i+1} = G_i$. By taking homomorphic images, and noting that $FG_k \cap G_i/G_k = \overline{F} \cap \overline{G}_i$, we get $(\overline{F} \cap \overline{G}_i)\overline{G}_{i+1} = \overline{G}_i$. Thus $\overline{F}$ covers $\overline{K}_i$. Therefore property (C) holds for the $\overline{F}$-composition series $\overline{G} = \overline{G}_0 > \cdots > \overline{G}_k = \overline{1}$ of $\overline{G}$.

Now that we know $\overline{F}$ is an $\mathfrak{F}$-subgroup of $\overline{G}$, it follows from (25c) that we can complete our proof by showing that F is an $\mathfrak{F}$-subgroup of FG_k .

Suppose $FG_k < G$. We consider the series $FG_k = D_0 \geq D_1 \geq \cdots \geq D_n = 1$, where $D_i = FG_k \cap G_i$ for each i. Suppose $D_i > D_{i+1}$ for some i. Then

$$D_i/D_{i+1} \simeq (FG_k \cap G_i)G_{i+1}/G_{i+1} > 1$$

This is an operator isomorphism, hence because F is irreducible on K_i, we have

$$(28) \qquad D_i/D_{i+1} \simeq G_i/G_{i+1}$$

Therefore the distinct terms of the series, $D_0 \geq D_1 \geq \cdots \geq D_n = 1$, form an F-composition series for FG_k which passes through G_k. Since F covers FG_k/G_k, and since $D_i = G_i$ for $i \geq k$, property (C) holds for this composition series. By induction, F is an $\mathfrak{F}$-subgroup of FG_k.

Suppose $G = FG_k$. Since G_k is a minimal normal subgroup of G, F acts irreducibly on G_k. Therefore F either covers, or avoids G_k.

If F covers G_k, then $F = G$, so F is an $\mathcal{F}$-subgroup of FG_k. Suppose F avoids G_k, and $G_k(=G_{n-1})$ is a p-group. Then $FG_k/C_{FG_k}(G_k) \simeq F/C_F(G_k)$ cannot lie in $\mathcal{F}_p$ since property (C) holds for G_k. Therefore $(FG_k)_{\mathcal{F}} \geq G_k$. Since FG_k/G_k lies in $\mathcal{F}$, $(FG_k)_{\mathcal{F}} = G_k$.

If $F \leq U \leq FG_k$, then $U = F$, or $U = FG_k$. The above remarks show that F covers $U/U_{\mathcal{F}}$ in both cases. Therefore F is an $\mathcal{F}$-subgroup of FG_k, and the proof is complete.

As one application of Theorem 4, we prove

<u>Theorem 5</u>. Let $\mathcal{F}$ be a formation locally defined by $\{\mathcal{F}_p\}$, $H \in \mathcal{F}$, and let I be a finitely generated $Z_p(H)$-module. Let $G = HI$ be the semi-direct product of I by H where the action of H on I by conjugation is the canonical action given in Chapter II. Then,

(a) $F = HC_I(O_{p'}(H_{\mathcal{F}_p}))$ is an $\mathcal{F}$-subgroup of G.

(b) As a $Z_p(H)$-module, $I = C_I(O_{p'}(H_{\mathcal{F}_p})) \oplus G$.

<u>Proof</u>. Let $W = C_I(O_{p'}(H_{\mathcal{F}_p}))$. Our first task is to show HW lies in $\mathcal{F}$. Suppose $q \neq p$, then W is a q'-group, so $O_{q',q}(F) = O_{q',q}(H)W$. Therefore,

$$F/O_{q',q}(F) \simeq H/O_{q',q}(H).$$

Since H lies in $\mathcal{F}$, $F/O_{q',q}(F) \in \mathcal{F}_q$.

Let $U = H_{\mathcal{F}_p}$. Then $O_{p'}(U)$ centralizes W. Since $H/O_{p',p}(H)$ lies in $\mathcal{F}_p$, $U \leq O_{p',p}(U)$. Therefore $O_{p',p}(UW) = UW$. UW is normal in F, so by (18'), $UW \leq O_{p',p}(F)$. Therefore $F/O_{p',p}(F)$ is isomorphic to a quotient group of H/U. Since $H/U \in \mathcal{F}_p$, $F/O_{p',p}(F) \in \mathcal{F}_p$. Therefore, F lies in $\mathcal{F}$.

Now let $G = G_0 > G_1 > \cdots > G_n = 1$ be an F-composition series for G such that $G_\ell = I$ for some ℓ. In order to check property (C), we need only consider $K_i = G_i/G_{i+1}$ for $i \geq \ell$, since F covers G/I. W centralizes every K_i, so we have

$$F/C_F(K_i) \simeq H/C_H(K_i).$$

If $i \geq \ell$, and $F/C_F(K_i) \in \mathcal{F}_p$, then $C_H(K_i) \geq U$. In particular, $O_{p'}(U)$ centralizes K_i. I is a p-group, so by the covering lemma, (17), W covers K_i. Therefore F covers K_i, so property (C) holds for the series $G_0 > G_1 > \cdots > G_n = 1$. By Theorem 4, F is an $\mathcal{F}$-subgroup of G.

We shall prove the next assertion by induction on the length of a $Z_p(H)$-composition series of I. Since $H \simeq G/I$, $G_{\mathcal{F}} \subseteq I$. Since F covers $G/G_{\mathcal{F}}$, it follows that, as a $Z_p(H)$-module, $I = W + G_{\mathcal{F}}$, so the major difficulty is to show that the sum is direct. If I is irreducible, one of the two summands must be the zero subspace, so the theorem is true if $n = 1$.

Suppose $I = I_0 \supset I_1 \supset \cdots \supset I_n = (0)$ is a $Z_p(H)$-composition series for I, and $n > 1$. I_{n-1} is a minimal normal subgroup of G, and HI_{n-1}/I_{n-1} acts on the quotient module $\bar{I} = I/I_{n-1}$ in the same way that H acts on $\bar{I}$. Therefore $\bar{G} = G/I_{n-1}$ is isomorphic to the semi-direct product $H\bar{I}$ where the action of H by conjugation is the module action. By induction,

$$(29) \qquad \bar{I} = C_{\bar{I}}(O_{p'}(U)) \oplus (\bar{G})_{\mathcal{J}}.$$

<u>CASE I</u>. $(\bar{0}) \subset C_{\bar{I}}(O_{p'}(U)) \subseteq \bar{I}$.

Set $T/I_{n-1} = (\bar{G})_{\mathcal{J}}$. By the covering lemma, (17), $C_{\bar{I}}(O_{p'}(U)) = WI_{n-1}/I_{n-1}$. By (29), $T \cap W$ is a normal subgroup of G contained in I_{n-1}.

If $T \cap W = (0)$, then $I = T \oplus W$, and reverting to multiplicative notation, we have $F \cap T = 1$. Therefore, T is clearly the smallest normal subgroup of G whose factor group is covered by F. Since F is an $\mathcal{J}$-subgroup of G, it follows that $T = G_{\mathcal{J}}$.

Suppose $T \cap W = I_{n-1}$. The hypothesis of CASE I states that $T \subset I$, so the theorem is true, by induction, for the semi-direct product $G_1 = HT$. Therefore, $T = C_T(O_{p'}(U)) \oplus T_1$, where $T_1 = (G_1)_{\mathcal{J}}$. Now $I_{n-1} = C_T(O_{p'}(U))$, and because I is abelian, $T_1 \lhd G$. Therefore, as a $Z_p(H)$-module,

$$I = W \oplus T_1.$$

As before, this forces $T_1 = G_{\mathcal{J}}$ since $G/T_1 \simeq F$ is the largest possible factor of G covered by F. Therefore, the theorem holds in CASE I.

<u>CASE II</u>. $(\bar{0}) = C_{\bar{I}}(O_{p'}(U))$.

By assumption, $W \subseteq I_{n-1}$. $O_{p'}(U)$ is normal in H, so W is normal in G. If $W = (0)$, then H is an $\mathcal{J}$-subgroup of G, and $I = G_{\mathcal{J}}$. We may, therefore, assume that $W = I_{n-1}$.

Let $\hat{I}$ be the dual of I. By (8)

$$\hat{I}_{n-1} \simeq \hat{I}/I_{n-1}^{\perp}\ ,\ \text{and}\ I_{n-1}^{\perp} \simeq (\widehat{I/I_{n-1}})$$

Since $O_{p'}(U)$ centralizes I_{n-1}, it follows from (9), that $O_{p'}(U)$ centralizes $\hat{I}_{n-1}$. By the covering lemma, (17), $C_{\hat{I}}(O_{p'}(U))$ covers $\hat{I}/I_{n-1}^{\perp}$.

Now $O_{p'}(U) \lhd H$, so $C_{I_{n-1}^{\perp}}(O_{p'}(U))$ is a $Z_p(H)$-submodule of $I_{n-1}^{\perp}$. By (8),

$$I/[C_{I_{n-1}^{\perp}}(O_{p'}(U))]^{\perp} \simeq \widehat{[C_{I_{n-1}^{\perp}}(O_{p'}(U))]}.$$

Since $O_{p'}(U)$ centralizes $C_{I_{n-1}^{\perp}}(O_{p'}(U))$, it follows from (9), that $O_{p'}(U)$ centralizes $I/[C_{I_{n-1}^{\perp}}(O_{p'}(U))]^{\perp}$. Thus using the covering lemma again, we know that W covers the factor $I/[C_{I_{n-1}^{\perp}}(O_{p'}(U))]^{\perp}$.

But $C_{I_{n-1}^{\perp}}(O_{p'}(U)) \subseteq I_{n-1}^{\perp}$, so $W = I_{n-1} \subseteq [C_{I_{n-1}^{\perp}}(O_{p'}(U))]^{\perp}$, hence we must have

$$I = [C_{I_{n-1}^{\perp}}(O_{p'}(U))]^{\perp}.$$

Therefore $C_{I_{n-1}^{\perp}}(O_{p'}(u)) = (0)$, so $C_{\hat{I}}(O_{p'}(U))$ avoids $I_{n-1}^{\perp}$. These statements show that

$$\hat{I} = C_{\hat{I}}(O_{p'}(U)) \oplus I_{n-1}^{\perp} .$$

But by (10), this splitting induces a splitting in I,

$$I = [C_{\hat{I}}(O_{p'}(U))]^{\perp} \oplus I_{n-1}$$

As before, it follows from the fact that $F = HI_{n-1}$ is an $\mathcal{F}$-subgroup of G, that $G_{\mathcal{F}} = [C_{\hat{I}}(O_{p'}(U))]^{\perp}$. This completes the proof in all cases.

Remark. This result cannot be extended to the case where I is a p-group of class 2 because of the following example. Let I be the quaternion group. I has an automorphism, h, of order 3 such that h acts fixed-point free on I/D(I), and centralizes D(I). Let H be the cyclic group of order 3 generated by h, and let G = HI. A Carter subgroup of G is H x D(I), but D(I) has no complement in I, so no splitting is possible.

Chapter IV

Strong Containment

In this chapter, we shall characterize strong containment. In certain cases, we can make our characterization more precise by giving generating sets for certain of the formations $\mathcal{F}_p$ in the minimal local definition of $\mathcal{F}$. The results of this chapter form the basis for our results in Chapter V.

<u>Lemma 5</u>. Let $\mathcal{E}$ and $\mathcal{F}$ be two non-empty saturated formations; let $\mathcal{E}$ be locally defined by $\{\mathcal{E}_p\}$. Let G be a group of minimal order satisfying

(P) An $\mathcal{E}$-subgroup of G is not contained in any $\mathcal{F}$-subgroup of G.

If F is an $\mathcal{F}$-subgroup of G, and E is an $\mathcal{E}$-subgroup of F, then

(a) $G_{\mathcal{F}} = M$ is a minimal normal subgroup of G; G is the semi-direct product of M by F; F acts faithfully and irreducibly on M.

(b) If M is a p-group, then $E^* = EC_M(O_{p'}(E_{\mathcal{E}_p}))$ is an $\mathcal{E}$-subgroup of G, and $1 < C_M(O_{p'}(E_{\mathcal{E}_p})) \leq M$.

<u>Proof</u>. If $G \in \mathcal{F}$, then G = F contains every $\mathcal{E}$-subgroup of G, hence G does not satisfy (P). Therefore $G \notin \mathcal{F}$; in particular, G is not the identity. Let $M \neq 1$ be a minimal normal subgroup of G. By (25b), FM/M is an $\mathcal{F}$-subgroup of G/M. Because of the minimality of $|G|$ with respect to the property (P), some $\mathcal{E}$-subgroup of G/M is

contained in FM/M. Since all $\mathcal{E}$-subgroups of G/M are conjugate, we can find an $\mathcal{E}$-subgroup, $\tilde{E}$, of G such that $\tilde{E}M \leq FM$. $\tilde{E}$, as an $\mathcal{E}$-subgroup of G, is also an $\mathcal{E}$-subgroup of FM. Because G satisfies (P), no conjugate of $\tilde{E}$ under FM can be contained in F. Therefore $FM \leq G$, and satisfies (P), so $G = FM$.

G/M lies in $\mathcal{F}$, but G does not lie in $\mathcal{F}$, so $G_{\mathcal{F}} = M$. Since $F \cap M$ is a normal subgroup of G, properly contained in M, $F \cap M = 1$, so G is the semi-direct product of M by F. Since M was arbitrary to begin with, and we showed $M = G_{\mathcal{F}}$, M is the unique minimal normal subgroup of G. Therefore F acts faithfully and irreducibly on M. This proves (a).

G/M is isomorphic to F, so EM/M is an $\mathcal{E}$-subgroup of G/M. By (25c), an $\mathcal{E}$-subgroup of EM is also an $\mathcal{E}$-subgroup of G. By Theorem 5, $E^* = EC_M(O_{p'}(E_{\mathcal{E}_p}))$ is an $\mathcal{E}$-subgroup of EM. Since E^* is not contained in F, $1 < C_M(O_{p'}(E_{\mathcal{E}_p})) \leq M$.

Before stating the characterization, we introduce some notation.

<u>Definition 6</u>. If $\mathcal{E}$ and $\mathcal{F}$ are two saturated formations, and $\mathcal{E}$ is locally defined by $\{\mathcal{E}_p\}$, set

(a) $\pi(\mathcal{E}) = \{p \mid \mathcal{E}_p \text{ is non-empty}\}$. $\pi(\mathcal{E})$ is called the <u>characteristic</u> of $\mathcal{E}$.

(b) If $p \in \pi(\mathcal{E})$, set

$$\Phi_p = \left\{ H \in \mathcal{F} \;\middle|\; \begin{array}{l} H \text{ has a faithful irreducible } Z_p(H)\text{-module, } M\text{, such} \\ \text{that if } E \text{ is an } \mathcal{E}\text{-subgroup of } H\text{, then} \\ 1 < C_M(O_{p'}(E_{\mathcal{E}_p})) \leq M. \end{array} \right\}$$

(c) If $p \in \pi(\mathcal{E})$, set

$$\theta_p = \left\{ H \in \Phi_p \;\middle|\; \begin{array}{l} \text{There is at least one faithful irreducible } Z_p(H)\text{-module,} \\ M\text{, such that } 1 < C_M(O_{p'}(E_{\mathcal{E}_p})) < M. \end{array} \right\}$$

<u>Theorem 6</u>. Suppose $\mathcal{E}$ and $\mathcal{F}$ are two saturated formations locally defined by $\{\mathcal{E}_p\}$ and $\{\mathcal{F}_p\}$ respectively. Then

$$\mathcal{E} \ll \mathcal{F} \iff \Phi_p \subseteq \mathcal{F}_p \quad \text{for each } p \in \pi(\mathcal{E}).$$

<u>Proof</u>. Suppose $\Phi_p \subseteq \mathcal{F}_p$ for each $p \in \pi(\mathcal{E})$, and $\mathcal{E}$ is not strongly contained in $\mathcal{F}$. Then the class of groups satisfying property (P) is non-empty, so we choose G to be an element of minimal order in this class. By Lemma 5, if $G_{\mathcal{F}}$ is a p-group, then p divides the order of an $\mathcal{E}$-subgroup of G, hence $p \in \pi(\mathcal{E})$. By Lemma 5, if F is an $\mathcal{F}$-subgroup of G, then F lies in Φ_p. Therefore, F is an element of $\mathcal{F}_p$.

Since $G_{\mathcal{F}}$ is the unique minimal normal subgroup of G, $G_{\mathcal{F}} = O_{p',p}(G)$. Therefore $F \simeq G/G_{\mathcal{F}} = G/O_{p',p}(G)$ lies in $\mathcal{F}_p$. If $q \neq p$, then $G_{\mathcal{F}} \leq O_{q'}(G)$, so $O_{q',q}(G) = G_{\mathcal{F}}O_{q',q}(F)$. Therefore,

$$G/O_{q',q}(G) \simeq F/O_{q',q}(F)$$

Since $F \in \mathcal{F}$, it follows that $G/O_{q',q}(G) \in \mathcal{F}_q$. By ($\alpha'$), G lies in

$\mathfrak{F}$, a contradiction to the fact that $G_{\mathfrak{F}} > 1$. Therefore $\mathcal{E} << \mathfrak{F}$.

Suppose $\mathcal{E} << \mathfrak{F}$, $p \in \pi(\mathcal{E})$, and $F \in \Phi_p$. Let M be the faithful irreducible $Z_p(F)$-module mentioned in the definition of Φ_p. Set $G = FM$, where the action of F on M by conjugation is the module action. By Theorem 5, an $\mathfrak{F}$-subgroup of G is $F^* = FC_M(O_{p'}(F_{\mathfrak{F}_p}))$, hence $G = F^*M$. Let E be an $\mathcal{E}$-subgroup of F. Since EM/M is an $\mathcal{E}$-subgroup of G/M, it follows from (25c) and Theorem 5, that $E^* = EC_M(O_{p'}(E_{\mathcal{E}_p}))$ is an $\mathcal{E}$-subgroup of G. E^* does not avoid M, and because $\mathcal{E} << \mathfrak{F}$, E^* is contained in some $\mathfrak{F}$-subgroup of G, hence F^* does not avoid M. Since $O_{p'}(F_{\mathfrak{F}_p}) \lhd F$, $C_M(O_{p'}(F_{\mathfrak{F}_p}))$ is normal in G, hence $F^* = G$.

Since G is an element of $\mathfrak{F}$,and F acts faithfully on the p-chief factor, M, of G, we have $F \simeq G/C_G(M) \in \mathfrak{F}_p$. Therefore $\Phi_p \subseteq \mathfrak{F}_p$.

Because of this characterization, if $\mathcal{E} << \mathfrak{F}$, and $p \in \pi(\mathcal{E})$, then $\Phi_p \subseteq \mathfrak{F}_p$ for any $\mathfrak{F}_p$ which lies in some local definition. This leads naturally to the question: Suppose $\{\mathfrak{F}_p\}$ is the unique minimal local definition for $\mathfrak{F}$. If $p \in \pi(\mathcal{E})$, is $\mathfrak{F}_p$ the smallest formation generated by the set Φ_p? The answer to this question is yes, provided Θ_p is non-empty for at least two primes. We have not been able to relax the hypothesis on the Θ_p's. In the next few lemmas, we shall investigate properties of the Φ_p's and Θ_p's as a prelude to giving a partial answer to this question.

Lemma 6. Let $\mathcal{E}$ and $\mathcal{F}$ be non-empty saturated formations with local definitions $\{\mathcal{E}_p\}$ and $\{\mathcal{F}_p\}$ respectively. Suppose $\mathcal{E} \ll \mathcal{F}$, and G is an element of $\mathcal{F}$ with $\mathcal{E}$-subgroup, E.

(a) Suppose $G \in \Phi_q$. G lies in θ_q if, and only if, $O_{q'}(E_{\mathcal{E}_q}) > 1$.

(b) If $q \in \pi(\mathcal{E})$, $O_q(G) = 1$, and the permutation representation, $1_{O_{q'}(E_{\mathcal{E}_q})}|^G$, is faithful, then G lies in $\langle \Phi_q \rangle$, the smallest formation generated by the set Φ_q.

(c) Let M be a faithful irreducible $Z_p(H)$-module. If G = HM, and $1_{O_{q'}(E_{\mathcal{E}_q})}|^G$ is faithful for some q in $\pi(\mathcal{E}) - \{p\}$, then $G \in \Phi_q$.

(d) For each r,s in $\pi(\mathcal{E})$, $\theta_r \subseteq \langle \Phi_s \rangle$.

Proof. Suppose $G \in \Phi_q$. Then G has a faithful irreducible $Z_q(G)$-module, I, such that $1 < C_I(O_{q'}(E_{\mathcal{E}_q})) \leq I$. If J is any such module, $J = C_J(O_{q'}(E_{\mathcal{E}_q}))$ if, and only if, $O_{q'}(E_{\mathcal{E}_q}) = 1$ since J is faithful. This proves (a).

Let G be a group satisfying the hypothesis of (b). Let T be the $Z_q(G)$-module which gives the representation $1_{O_{q'}(E_{\mathcal{E}_q})}|^G$. By (15), we can write T as a direct sum of principal indecomposable $Z_q(G)$-modules,

$$T = T_1 \oplus \cdots \oplus T_s.$$

For each $k = 1,\cdots,s$, it follows from (16) that T_k has a unique maximal proper $Z_q(G)$-submodule, $\sqrt{T_k}$. We set $J_k = T_k/\sqrt{T_k}$.

Since $O_q(G) = 1$, $F(G)$ is a normal q'-subgroup of G, hence by Theorem 1, the distinct irreducible components of $J_k|_{F(G)}$ are exactly the same as the distinct irreducible components of $T_k|_{F(G)}$. Therefore, since T is faithful, it follows that if $J = J_1 \oplus \cdots \oplus J_s$, then $J|_{F(G)}$ is faithful. By Lemma 1, J is a faithful $Z_q(G)$-module.

We now apply the Frobenius reciprocity theorem for modules. For each $k = 1,\cdots,s$,

$$(0) \subset \mathrm{Hom}_{Z_q(G)}(T,J_k) \simeq \mathrm{Hom}_{Z_q(O_{q'}(E_{\ell_q}))}(1,J_k|_{O_{q'}(E_{\ell_q})})$$

Therefore $1 < C_{J_k}(O_{q'}(E_{\ell_q})) \leq J_k$, for each k.

Set $G_k = G/C_G(J_k)$. Then $E_k = EC_G(J_k)/C_G(J_k)$ is an ℓ-subgroup of G_k. By Lemma 2, $(E_k)_{\ell_q} = E_{\ell_q}C_G(J_k)/C_G(J_k)$. Therefore $O_{q'}((E_k)_{\ell_q}) = O_{q'}(E_{\ell_q})C_G(J_k)/C_G(J_k)$, so

$$1 < C_{J_k}(O_{q'}(E_{\ell_q})) = C_{J_k}(O_{q'}((E_k)_{\ell_q})) \leq J_k.$$

Since G lies in $\mathcal{F}$, $q \in \pi(\ell)$, and J_k is a faithful irreducible $Z_q(G_k)$-module, G_k lies in Φ_q for each k. Therefore $G = G/\bigcap_k C_G(J_k)$ lies in $\langle \Phi_q \rangle$, the smallest formation generated by the set Φ_q. This proves (b).

The proof of (c) is essentially the same as the proof of (b). Let $G = HM$ be the group mentioned in the hypothesis of (c). Let T be the $Z_q(G)$-module which gives the faithful representation $1_{O_{q'}(E_{\ell_q})}|^G$.

Once again, T has a decomposition into a direct sum, $T = T_1 \oplus \cdots \oplus T_s$, of principal indecomposable $Z_q(G)$-modules. Because of faithfulness it follows that M is non-trivial on some T_k, say T_1. Since M is minimal normal in G, M must act faithfully on T_1. Since M is a q'-group, it follows from Theorem 1, that if $J = T_1/\sqrt{T_1}$, then $J|_M$ is faithful. But M is the unique minimal normal subgroup of G, and by (14), $O_p(H) = 1$, so $F(G) = M$. By Lemma 1, J is a faithful irreducible $Z_q(G)$-module. By Frobenius reciprocity,

$$1 < C_J(O_{q'}(E_{\mathcal{E}_q})) \leq J.$$

Therefore, the fact that $q \in \pi(\mathcal{E})$ says that G is, by definition, an element of Φ_q. This proves (c).

Part (d) is the only statement in Lemma 6 which requires the assumption $\mathcal{E} \ll \mathcal{F}$. Suppose $H \in \theta_r$, E is an $\mathcal{E}$-subgroup of H, and M is a faithful irreducible $Z_r(H)$-module such that $1 < C_M(O_{r'}(E_{\mathcal{E}_r})) < M$. Set $G = HM$. By Theorem 5, $F = HC_M(O_{r'}(H_{\mathcal{F}_r}))$ is an $\mathcal{F}$-subgroup of G, and since $E \simeq EM/M$ is an $\mathcal{E}$-subgroup of G/M, $E^* = EC_M(O_{r'}(E_{\mathcal{E}_r}))$ is an $\mathcal{E}$-subgroup of G. Since $\mathcal{E} \ll \mathcal{F}$, F cannot avoid M, hence $F = G$ is an element of $\mathcal{F}$.

Let N be the intersection of all the conjugates of $O_{s'}(E^*_{\mathcal{E}_s})$ in G. Then $N \lhd G$, and $N \cap M \leq E^* \cap M = C_M(O_{r'}(E_{\mathcal{E}_r})) < M$. Therefore $N \cap M = 1$. This shows that $1_{O_{s'}(E^*_{\mathcal{E}_s})}|^G|_M$ is faithful. Because

$M = F(G)$, it follows from Lemma 1 that $1_{O_s,(E^*_{\ell_s})}|^G$ is faithful. By part (c), $G \in \Phi_s$ for $s \in \pi(\ell) - \{r\}$. Therefore $H \simeq G/M$ is an element of $\langle \Phi_s \rangle$, for $s \in \pi(\ell) - \{r\}$. Since $\theta_r \subseteq \Phi_r$, it follows that $\theta_r \subseteq \langle \Phi_s \rangle$ for each s in $\pi(\ell)$. This proves (d).

The next lemma has an elegant proof. This proof was told to me by Dr. E. C. Dade, and it shortens this part of the original proof of the next theorem considerably.

Lemma 7. Let A, B be two groups and assume $Z(A) = 1$. If M is a faithful $Z_p(A)$-module, and T is a faithful $Z_p(B)$-module, then $M \otimes T$ is a faithful $Z_p(A \times B)$-module.

Proof. If V is a vector space over Z_p, we let $GL(V)$ denote the general linear group on V. Then $A \times B \leq GL(M) \times GL(T) = C$, so we examine the kernel, K, of the representation of C on $M \otimes T$. Let $m_1, \cdots, m_r$ be a Z_p-basis for M, and $t_1, \cdots, t_s$ be a Z_p-basis for T. Then $\{m_i \otimes t_j \mid 1 \leq i \leq r, 1 \leq j \leq s\}$ is a Z_p-basis for $M \otimes T$. Suppose $f \times g \in K$, and

$$m_i f = \sum \varphi_{ik} m_k, \qquad \text{for each } i$$

$$t_j g = \sum \gamma_{j\ell} t_\ell, \qquad \text{for each } j.$$

Then $m_i \otimes t_j = m_i \otimes t_j(f \times g) = (\sum \varphi_{ik} m_k) \otimes (\sum \gamma_{j\ell} t_\ell)$. So

$m_i \otimes t_j = \sum_{k,\ell} \varphi_{ik}\gamma_{j\ell}\, m_k \otimes t_\ell$, hence we have

$$\varphi_{ik}\gamma_{j\ell} = \begin{cases} 0 & \text{if } (i,j) \neq (k,\ell) \\ 1 & \text{if } (i,j) = (k,\ell) \end{cases}$$

Therefore $m_i f = \varphi m_i$ for each i, and $t_j g = \varphi^{-1} t_j$ for each j.

Suppose $a \times b \in (A \times B) \cap K$. Since $Z(A) = 1$, and our results show that $K \leq Z(GL(M)) \times Z(GL(T))$, we must have $a = 1$. But this means that the constant, φ, is the identity, so $b = 1$. Therefore $A \times B$ acts faithfully on $M \otimes T$.

<u>Lemma 8.</u> Suppose $A \in \Phi_p - \theta_p$, $B \in \theta_p$, and either $Z(A)$, or $Z(B)$ is the identity. Then $A \times B \in \langle \theta_p \rangle$, the smallest formation generated by the set θ_p.

<u>Proof</u>. Let E be an $\mathcal{E}$-subgroup of A, and $\tilde{E}$ an $\mathcal{E}$-subgroup of B. Since $A \in \Phi_p - \theta_p$, it follows from Lemma 6a that $O_{p'}(E_{\mathcal{E}_p}) = 1$. But E lies in $\mathcal{E}$, so $E/O_{p',p}(E)$ is an element of $\mathcal{E}_p$, consequently $E_{\mathcal{E}_p} \leq O_{p',p}(E)$. Therefore $E_{\mathcal{E}_p}$ is a p-group. Since $B \in \theta_p$, $O_{p'}(\tilde{E}_{\mathcal{E}_p}) > 1$.

Now $E \times \tilde{E}$ is an $\mathcal{E}$-subgroup of $A \times B$. We wish to examine $O_{p'}((E \times \tilde{E})_{\mathcal{E}_p})$. Since $(E \times \tilde{E})/(E_{\mathcal{E}_p} \times \tilde{E}_{\mathcal{E}_p}) \in \mathcal{E}_p$, $(E \times \tilde{E})_{\mathcal{E}_p}$ is a normal subgroup of $E \times \tilde{E}$ contained in $E_{\mathcal{E}_p} \times \tilde{E}_{\mathcal{E}_p}$. We define a subgroup, W, of $\tilde{E}_{\mathcal{E}_p}$ by: $W = \{e \in \tilde{E}_{\mathcal{E}_p} \mid \exists t \in (E \times \tilde{E})_{\mathcal{E}_p} \ni : t = d \times e, \text{ and } d \in E_{\mathcal{E}_p}\}$.

In other words, W is just the collection of all elements of $\widetilde{E}_{\mathcal{E}_p}$ which appear as components of elements of $(E \times \widetilde{E})_{\mathcal{E}_p}$. W is clearly a subgroup of $\widetilde{E}_{\mathcal{E}_p}$, and since $(E \times \widetilde{E})_{\mathcal{E}_p}$ is normal in $E \times \widetilde{E}$, W is normal in $\widetilde{E}$. By construction, $(E \times \widetilde{E})_{\mathcal{E}_p} \le E \times W$, hence $\widetilde{E}/W \simeq (E \times \widetilde{E})/(E \times W)$ lies in $\mathcal{E}_p$. Therefore $W = \widetilde{E}_{\mathcal{E}_p}$.

Now if e is any element of $O_{p'}(\widetilde{E}_{\mathcal{E}_p})$, then there is an element d in $E_{\mathcal{E}_p}$, such that $t = d \times e$ lies in $(E \times \widetilde{E})_{\mathcal{E}_p}$. But $E_{\mathcal{E}_p}$ is a p-group, hence by taking an appropriate power of t, we see that e lies in $(E \times \widetilde{E})_{\mathcal{E}_p}$. Therefore,

(30) $O_{p'}(\widetilde{E}_{\mathcal{E}_p}) \le O_{p'}((E \times \widetilde{E})_{\mathcal{E}_p}) \le O_{p'}(E_{\mathcal{E}_p} \times \widetilde{E}_{\mathcal{E}_p}) = O_{p'}(\widetilde{E}_{\mathcal{E}_p})$.

By assumption, A has a faithful irreducible $Z_p(A)$-module, M, and B has a faithful irreducible $Z_p(B)$-module, T, such that $1 < C_T(O_{p'}(\widetilde{E}_{\mathcal{E}_p})) < T$. By Lemma 7, $M \otimes T$ is a faithful $Z_p(A \times B)$-module.

Since $M \otimes T|_B$ is isomorphic to a multiple of T, if we let U be any $Z_p(A \times B)$-composition factor of $M \otimes T$, then $U|_B$ is also a multiple of T, so because of (30) we have

(31) $\quad 1 < C_U(O_{p'}(\widetilde{E}_{\mathcal{E}_p})) = C_U(O_{p'}((E \times \widetilde{E})_{\mathcal{E}_p})) < U.$

for each U.

Let $G = (A \times B)/C_{A\times B}(U)$, then $\bar{E} = (E \times \tilde{E})C_{A\times B}(U)/C_{A\times B}(U)$ is an $\mathcal{E}$-subgroup of G. By Lemma 2, $\bar{E}_{\mathcal{E}_p} = (E \times \tilde{E})_{\mathcal{E}_p} C_{A\times B}(U)/C_{A\times B}(U)$, and by (31), $O_{p'}((E \times \tilde{E})_{\mathcal{E}_p})$ does not centralize U, hence $O_{p'}(\bar{E}_{\mathcal{E}_p}) > 1$. Since $1 < C_U(O_{p'}((E \times \tilde{E})_{\mathcal{E}_p}))$, and G is an element of $\mathcal{J}$, it follows from Lemma 6a that G lies in θ_p.

Let V be the direct sum of all $Z_p(A \times B)$-composition factors occurring in a composition series of $M \otimes T$. By (14), $F(A \times B) = F(A) \times F(B)$ is a p'-group, so the fact that $M \otimes T$ is faithful implies $V|_{F(A \times B)}$ is faithful. By Lemma 1, V is a faithful completely reducible $Z_p(A \times B)$-module. Therefore $(A \times B)/\bigcap_U C_{A\times B}(U)$ lies in $\langle \theta_p \rangle$, where the intersection is taken over the irreducible components of V. Since V is faithful this intersection is the identity, hence $A \times B \in \langle \theta_p \rangle$.

<u>Corollary 1</u>. If $\mathcal{E} \ll \mathcal{J}$, and there is an element B in θ_p such that $Z(B) = 1$, then $\langle \Phi_p \rangle \subseteq \langle \Phi_q \rangle$ for each q in $\pi(\mathcal{E})$.

<u>Proof</u>. By Lemma 8, if $A \in \Phi_p - \theta_p$, then $A \times B \in \langle \theta_p \rangle$. Therefore $A \in \langle \theta_p \rangle$, so $\langle \Phi_p \rangle = \langle \theta_p \rangle$. By Lemma 6d, if $q \in \pi(\mathcal{E})$, then $\theta_p \subseteq \langle \Phi_q \rangle$, hence $\langle \Phi_p \rangle \subseteq \langle \Phi_q \rangle$.

<u>Theorem 7</u>. Suppose $\mathcal{E} \ll \mathcal{J}$, and θ_p, θ_r are non-empty for two primes p,r in the characteristic of $\mathcal{E}$. Let $\{\mathcal{J}_q\}$ be the unique minimal local definition of $\mathcal{J}$. Then

$$\mathcal{F}_q = \langle \Phi_q \rangle \qquad \text{for each } q \text{ in } \pi(\mathcal{E}).$$

<u>Proof</u>. We define a new formation, $\tilde{\mathcal{F}}$, locally by setting

$$(32) \qquad \begin{aligned} \langle \Phi_q \rangle &= \tilde{\mathcal{F}}_q && \text{for } q \in \pi(\mathcal{E}) \\ \mathcal{F}_q &= \tilde{\mathcal{F}}_q && \text{for } q \in \pi(\mathcal{E})' \end{aligned}$$

Since $\mathcal{E} \ll \mathcal{F}$, $\tilde{\mathcal{F}}_q \subseteq \mathcal{F}_q$ for each q, by Theorem 6. Therefore $\tilde{\mathcal{F}} \subseteq \mathcal{F}$.

Let $\tilde{\Phi}_s$ be the set specified in Definition 6 for the formation $\tilde{\mathcal{F}}$. Since $\tilde{\mathcal{F}} \subseteq \mathcal{F}$, $\tilde{\Phi}_s \subseteq \Phi_s \subseteq \tilde{\mathcal{F}}_s$ for each s in $\pi(\mathcal{E})$. Therefore $\mathcal{E} \ll \tilde{\mathcal{F}}$, by Theorem 6.

Suppose $\tilde{\mathcal{F}} \subset \mathcal{F}$. If G is an element of minimal order in $\mathcal{F} - \tilde{\mathcal{F}}$, then G is a semi-direct product, $G = \tilde{F}M$, where $\tilde{F}$ is an $\tilde{\mathcal{F}}$-subgroup of G, and acts faithfully and irreducibly on the elementary abelian t-group M. Since G lies in $\mathcal{F} - \tilde{\mathcal{F}}$, $\tilde{F} \simeq G/C_G(M)$ lies in $\mathcal{F}_t - \tilde{\mathcal{F}}_t$. For t in $\pi(\mathcal{E})'$ this contradicts the definition of $\tilde{\mathcal{F}}_t$, hence $t \in \pi(\mathcal{E})$.

Since $\mathcal{E} \ll \tilde{\mathcal{F}}$, $\tilde{F}$, as an $\tilde{\mathcal{F}}$-subgroup of G, must contain some $\mathcal{E}$-subgroup, E, of G. Thus for any prime q, the permutation representation $1_{O_{q'}(E_{\mathcal{E}_q})}|^G$ is faithful. By Lemma 6c, G lies in Φ_q for each q in $\pi(\mathcal{E}) - \{t\}$.

By Lemma 6d, $\theta_q \subseteq \tilde{\mathcal{F}}_s$ for each q,s in $\pi(\mathcal{E})$, hence if $G \in \theta_q$ for some q in $\pi(\mathcal{E}) - \{t\}$, $G \in \tilde{\mathcal{F}}_t$. Suppose, therefore, that $G \in \Phi_q - \theta_q$ for each q in $\pi(\mathcal{E}) - \{t\}$. One of the primes p,r is unequal to t, say p. Then G is an element of $\Phi_p - \theta_p$ such that $Z(G) = 1$. Since

θ_p is non-empty there is a group H in θ_p, so by Lemma 8, $G \times H \in \langle \theta_p \rangle$. By Lemma 6d, $G \times H \in \tilde{\mathcal{F}}_t$, hence in each case $\bar{F}$, as a factor group of G, must lie in $\tilde{\mathcal{F}}_t$, a contradiction.

Therefore $\tilde{\mathcal{F}} = \mathcal{F}$. Since $\{\tilde{\mathcal{F}}_q\}$ forms a local definition for $\tilde{\mathcal{F}}$, we have $\Phi_q \subseteq \mathcal{F}_q \subseteq \tilde{\mathcal{F}}_q$ for $q \in \pi(\mathcal{E})$, so the proof of Theorem 7 is complete.

Because we could not relax the hypothesis on the θ_p's, we thought it appropriate to include

<u>Theorem 8</u>. Suppose $\mathcal{E} \ll \mathcal{F}$, and $p \in \pi(\mathcal{E})$. θ_p is empty if, and only if, for each element F of $\mathcal{F}$, an $\mathcal{E}$-subgroup, E, of F either covers, or avoids each p-chief factor of F.

<u>Proof</u>. Suppose an $\mathcal{E}$-subgroup of F either covers or avoids each p-chief factor of F for every F in $\mathcal{F}$. Let $F \in \Phi_p$, and let E be an $\mathcal{E}$-subgroup of F. Let M be a faithful irreducible $Z_p(F)$-module such that $C_M(O_{p'}(E_{\mathcal{E}_p})) > 1$. By Theorem 5, and the fact that $\mathcal{E} \ll \mathcal{F}$, $F^* = FC_M(O_{p'}(F_{\mathcal{F}_p}))$ is an $\mathcal{F}$-subgroup of FM, acts irreducibly on M, and does not avoid M. Therefore $F^* = FM$; M is a p-chief factor of $G = FM$ which is not avoided by the $\mathcal{E}$-subgroup, $E^* = EC_M(O_{p'}(E_{\mathcal{E}_p}))$, of G. Therefore $M = C_M(O_{p'}(E_{\mathcal{E}_p}))$, so θ_p is empty.

Suppose θ_p is empty, $F \in \mathcal{F}$, and E is an $\mathcal{E}$-subgroup of F which does not avoid the p-chief factor $K = L/N$ of F. Let $\bar{F} = F/C_F(K)$. Our first assertion is that the semi-direct product $\bar{F}K$ lies in

$\mathcal{F}$ (the action of $\overline{F}$ on K is the action induced by the action of F on K). By Theorem 5, $\overline{F}^* = \overline{F}C_K(O_{p'}(\overline{F}_{\mathcal{F}_p}))$ is an $\mathcal{F}$-subgroup of $\overline{F}K$. Therefore $\overline{F}^*$ acts irreducibly on K, and $\overline{F}^*/C_{\overline{F}^*}(K) \simeq \overline{F}$. Since $F \in \mathcal{F}$, $\overline{F} \in \mathcal{F}_p$ by definition. By Theorem 3, $\overline{F}^*$ covers K, hence $\overline{F}K$ lies in $\mathcal{F}$.

$\overline{E} = EC_F(K)/C_F(K)$ is an $\mathcal{E}$-subgroup of $\overline{F}$. By Lemma 2, $\overline{E}_{\mathcal{E}_p} = E_{\mathcal{E}_p}C_F(K)/C_F(K)$. Because $O_{p',p}(E_{\mathcal{E}_p}) = E_{\mathcal{E}_p}$, it follows that $O_{p'}(\overline{E}_{\mathcal{E}_p}) = O_{p'}(E_{\mathcal{E}_p})C_F(K)/C_F(K)$. Therefore $C_K(O_{p'}(E_{\mathcal{E}_p})) = C_K(O_{p'}(\overline{E}_{\mathcal{E}_p}))$. $O_{p'}(E_{\mathcal{E}_p})$ centralizes every p-section of E, hence $O_{p'}(E_{\mathcal{E}_p})$ centralizes $(L \cap E)N/N$, a non-identity subgroup of K. Therefore,

$$1 < C_K(O_{p'}(\overline{E}_{\mathcal{E}_p})) \leq K$$

Thus $\overline{F}$ lies in Φ_p. θ_p is empty, so it follows from Lemma 6 that $\overline{E}_{\mathcal{E}_p}$ is a p-group. If U is any $\overline{E}$-composition factor of K, then $\overline{E}_{\mathcal{E}_p}$ centralizes U since $\overline{E}_{\mathcal{E}_p} \leq O_p(\overline{E})$. Upon taking inverse images in E, we see that $C_E(U) \geq E_{\mathcal{E}_p}$, so that $E/C_E(U)$ lies in $\mathcal{E}_p$. By Theorem 3, E covers U, hence E also covers all of K.

Chapter V

Structure Theorems

Throughout this chapter we shall make the following assumptions:

Hypothesis I. $\mathcal{L}$ and $\mathcal{F}$ are saturated formations such that

(a) $\mathcal{N} \subseteq \mathcal{L} \ll \mathcal{F}$;

(b) there is a non-empty formation, $\mathcal{T}$, such that $\mathcal{L} = \{G \in \mathcal{S} \mid G/F(G) \in \mathcal{T}\}$.

Our first theorem says that the structure of $\mathcal{F}$ is essentially the same as the structure of $\mathcal{L}$ in that we can show the existence of a formation, $\mathcal{V}$, such that $\mathcal{F} = \{G \in \mathcal{S} \mid G/F(G) \in \mathcal{V}\}$.

We first prove two lemmas.

Lemma 9. Let $\mathcal{T}$ be a non-empty formation. Let $\mathcal{G}$ be the formation locally defined by setting $\mathcal{G}_p = \mathcal{T}$ for each p. Let $\mathcal{L} = \{G \in \mathcal{S} \mid G/F(G) \in \mathcal{T}\}$. Then $\mathcal{L} = \mathcal{G}$.

Proof. Suppose $G \in \mathcal{L}$. Because $O_{p',p}(G) \geq F(G)$, $G/F(G) \in \mathcal{T}$ implies that, for each p, $G/O_{p',p}(G)$ lies in $\mathcal{T}$. By (α'), G lies in $\mathcal{G}$.

Suppose $G \in \mathcal{G}$. Then, for each p, $G/O_{p',p}(G) \in \mathcal{T}$. Since $\mathcal{T}$ is a formation, and $F(G) = \bigcap_p O_{p',p}(G)$, $G/F(G)$ lies in $\mathcal{T}$. Therefore $G \in \mathcal{L}$, so $\mathcal{L} = \mathcal{G}$.

Lemma 10. Suppose $\mathcal{L}$, $\mathcal{F}$ satisfy Hypothesis I.

(a) If G is a group with $\mathcal{L}$-subgroup, E, and $E \in \mathcal{T}$, then E = G.

(b) If $\{\mathcal{F}_q\}$ is any local definition for $\mathcal{F}$, and G is an element of $\mathcal{T}$ such that $O_q(G) = 1$, then G lies in $\mathcal{F}_q$.

<u>Proof</u>. We prove (a) by induction on the nilpotent length of G. If G is nilpotent, there is nothing to prove. Since E lies in $\mathcal{T}$, $EF(G)$ lies in $\mathcal{E}$. Since E is an $\mathcal{E}$-subgroup of G, E covers $U/U_{\mathcal{E}}$ for any subgroup, U, of G which contains E. Therefore $E \geq F(G)$. Set $\overline{G} = G/F(G)$, then $\overline{E} = E/F(G)$ is an $\mathcal{E}$-subgroup of $\overline{G}$. By induction, $\overline{E} = \overline{G}$, hence $E = G$.

Let $\{\mathcal{F}_q\}$ be any local definition for $\mathcal{F}$. Suppose $G \in \mathcal{T}$, and $O_p(G) = 1$. Let M be the regular $Z_p(G)$-module, and form $G_1 = GM$. Since G lies in $\mathcal{T}$, G_1 lies in $\mathcal{E}$. It is a simple consequence of strong containment that $\mathcal{E} \subseteq \mathcal{F}$, hence $G_1 \in \mathcal{F}$. Since $O_p(G) = 1$, and G acts faithfully on M, $M = O_{p',p}(G_1)$. Therefore $G_1/M \in \mathcal{F}_p$. Since $G \simeq G_1/M$, G lies in $\mathcal{F}_p$. This proves (b).

<u>Theorem 9</u>. Suppose $\mathcal{E}$ and $\mathcal{F}$ satisfy Hypothesis I. Then there is a formation, $\mathcal{V}$, containing $\mathcal{T}$, such that $\mathcal{F} = \{G \in \mathcal{S} \mid G/F(G) \in \mathcal{V}\}$.

<u>Proof</u>. If $\mathcal{E} = \mathcal{F}$, the formation $\mathcal{T}$ satisfies the requirements of the theorem. Assume $\mathcal{E} \subset \mathcal{F}$. By Lemma 9, we know that the family, $\{\mathcal{E}_p \mid \mathcal{E}_p = \mathcal{T}$ for each p$\}$, of formations is a local definition for $\mathcal{E}$. We shall use this family for the local definition of $\mathcal{E}$ throughout the remainder of the proof. Let $\{\mathcal{F}_q\}$ be the unique minimal local definition of $\mathcal{F}$. A second application of Lemma 9 says that we need

only show $\mathcal{F}_r = \mathcal{F}_s$ for each pair of primes, r,s. In view of Theorem 7 and Corollary 1, we begin by examining the set θ_s for various primes s. Since $\mathcal{N} \subseteq \mathcal{E}$, $\pi(\mathcal{E})$ contains all primes, so θ_s and Φ_s are defined for each s.

Let G be an element of minimal order in $\mathcal{F} - \mathcal{E}$. By minimality, if N is any normal non-identity subgroup of G, then G/N lies in $\mathcal{E}$. Therefore $G_{\mathcal{E}}$ is the unique minimal normal non-identity subgroup of G. If E is an $\mathcal{E}$-subgroup of G, then $EG_{\mathcal{E}} = G$, and $E \cap G_{\mathcal{E}} = 1$. Furthermore, E acts faithfully and irreducibly on $G_{\mathcal{E}}$. We set $M = G_{\mathcal{E}}$, and note that M is an elementary abelian p-group for some prime p.

Since G is not in $\mathcal{E}$, E cannot lie in $\mathcal{T}$. Therefore $F(E) \geq E_{\mathcal{T}} > 1$. But it follows from (14) that F(E) is a p'-group, so for some prime $r \neq p$, $E_{\mathcal{T}}$ has a non-identity normal Sylow r-subgroup, R. If $s \neq r$, then

$$0_{s'}(E_{\mathcal{E}_s}) = 0_{s'}(E_{\mathcal{T}}) \geq R > 1.$$

Because M is the unique minimal normal subgroup of G, and $E \cap M = 1$, $1_{0_{s'}(E_{\mathcal{E}_s})}|^G$ is faithful for each s. By Lemma 6, G lies in θ_s for each prime $s \neq r,p$. Since E is faithful and irreducible on M, $Z(G) = 1$.

We now choose a prime $s \neq r,p$. Then $G \in \theta_s$, so there exists a faithful irreducible $Z_s(G)$-module, J, such that $1 < C_J(0_{s'}(E_{\mathcal{E}_s})) < J$.

Set $G_1 = GJ$. Since E is isomorphic to an $\mathcal{E}$-subgroup of G_1/J, it follows from (25c) and Theorem 5 that $E^* = EC_J(O_{s'}(E_{\mathcal{E}_s}))$ is an $\mathcal{E}$-subgroup of G_1. An $\mathcal{F}$-subgroup of G_1 covers G_1/J since $G \in \mathcal{F}$; it cannot avoid J because $\mathcal{E} << \mathcal{F}$. Therefore G_1 lies in $\mathcal{F}$. Because E is a quotient group of E^* which does not lie in $\mathcal{T}$, E^* cannot lie in $\mathcal{T}$, hence

$$1 < (E^*)_{\mathcal{T}} = (E^*)_{\mathcal{E}_p} \leq E_{\mathcal{T}} C_J(O_{s'}(E_{\mathcal{E}_s})).$$

$(E^*)_{\mathcal{T}}$ is a p'-group because $E_{\mathcal{T}}$ is a subgroup of the p'-group, $F(E)$, and $s \neq p$. $1_{(E^*)_{\mathcal{T}}}|^{G_1}$ is faithful since J is the unique minimal normal subgroup of G_1, and $(E^*)_{\mathcal{T}} \cap J \leq C_J(O_{s'}(E_{\mathcal{E}_s})) < J$. It follows from parts (a) and (c) of Lemma 6 that $G_1 \in \theta_p$. By construction, $Z(G_1) = 1$, hence we have established

(33) If $s \neq r$, then there is a group, X, in θ_s such that $Z(X) = 1$.

We now apply the results of Chapter IV. $\pi(\mathcal{E})$ contains all primes, so it follows from Theorem 7, and (33) that $\mathcal{F}_s = \langle \Phi_s \rangle$ for each prime s. By Corollary 1, we have

(34)
$$\mathcal{F}_s = \mathcal{F}_q \quad \text{for } s,q \text{ in } r'$$
$$\mathcal{F}_s \subseteq \mathcal{F}_r \quad \text{for each } s.$$

For $s \neq r$, we set $\mathcal{F}_s = \mathcal{V}$. The final step in the proof will be to show $\Phi_r \subseteq \mathcal{V}$.

By part (d) of Lemma 6, $\theta_r \subseteq \mathcal{F}_s$ for each s, so $\theta_r \subseteq \mathcal{V}$. Suppose

$H \in \Phi_r - \theta_r$, and E is an $\mathcal{E}$-subgroup of H. Then $O_{r'}(E_{\mathcal{E}_r}) = 1$, so $E_{\mathcal{E}_r} = E_{\mathcal{T}}$ is an r-group.

Suppose $E_{\mathcal{T}} = 1$, then E lies in $\mathcal{T}$. By Lemma 10, E = H, and if s is any prime not dividing $r|H|$, $O_s(H) = 1$, so $H \in \mathcal{F}_s = \mathcal{V}$.

Suppose $E_{\mathcal{T}} > 1$. By (14), $O_r(H) = 1$, so $1_{E_{\mathcal{T}}}|^H$ is faithful. Since $H \in \Phi_r$, H also lies in $\mathcal{F}$. Thus if s is a prime which does not divide $|H|$, it follows from part (b) of Lemma 6 that $H \in \langle \Phi_s \rangle$. Therefore

$$\Phi_r \subseteq \mathcal{V} \subseteq \mathcal{F}_r = \langle \Phi_r \rangle .$$

Since $\mathcal{V} = \mathcal{F}_s$ for each s, Lemma 9 says that $\mathcal{F} = \{G \in \mathcal{S} \mid G/F(G) \in \mathcal{V}\}$. The fact that $\mathcal{V}$ contains $\mathcal{T}$ is a consequence of part (b) of Lemma 10.

We are interested in finding formations which are maximal in the partial ordering $\ll$. Since $\mathcal{E} \ll \mathcal{F}$ implies $\mathcal{E} \subseteq \mathcal{F}$, we shall assume $\mathcal{E} \subset \mathcal{F}$, as well as Hypothesis I. Since $\mathcal{E} = \{G \in \mathcal{S} \mid G/F(G) \in \mathcal{T}\}$, we shall fix our local definition for $\mathcal{E}$ by setting $\mathcal{E}_p = \mathcal{T}$ for each p. We shall assume that $\{\mathcal{F}_p\}$ is the minimal local definition for $\mathcal{F}$. By the proof of Theorem 9, there is a formation, $\mathcal{V}$, containing $\mathcal{T}$, such that $\mathcal{F}_p = \mathcal{V}$ for each p. Since $\mathcal{E} \subset \mathcal{F}$, we must have $\mathcal{T} \subset \mathcal{V}$.

Before stating our main theorem, we prove several lemmas. The proof of Lemma 12 contains the essential constructions used in the proof of our main theorem.

Lemma 11. Let G be a group, and $1 < H \leq G$. Assume that the permutation representation, $1_H|^G$, is faithful. If M is the $Z_p(G)$-module which gives $1_H|^G$, set $U = \bigcap_{g\in G} C_M(H)g$. Then U is a $Z_p(G)$-submodule of M, and M/U is a faithful $Z_p(G)$-module.

<u>Proof</u>. We can choose the cosets of H in G to be a Z_p-basis for M, i.e., $M = Z_p\cdot H \oplus Z_p\cdot Hg_2 \oplus \cdots \oplus Z_p\cdot Hg_s$, where $s = [G:H]$, and the operation of G on M is by right multiplication.

For each g in G, $C_M(H)g = C_M(H^g)$, hence $U = \bigcap_{g\in G} C_M(H^g) = C_M(\bigcup_{g\in G} H^g)$. In other words, if N is the normal closure of H in G, then $U = C_M(N)$. Since $N < G$, U is a $Z_p(G)$-submodule of M.

For $i = 1,\cdots,m$, let $\mathcal{O}_i$ denote one of the orbits of the cosets $H, Hg_2,\cdots,Hg_s$ under action by N. Since $N < G$, G permutes these orbits transitively, thus all orbits have the same number of elements. The number of elements in any orbit is $[N:H]$, and since $1_H|^G$ is faithful, H cannot be normal in N, so $[N:H] \geq 3$. For each i, we set $\mathcal{O}_i = \{Hg_{i1},\cdots,Hg_{ir}\}$ where $r = [N:H]$.

Let $u_i = \sum_{k=1}^{r} Hg_{ik}$, for $i = 1,\cdots,m$. Our first assertion is that $u_1,\cdots,u_m$ form a Z_p-basis for $C_M(N)$. The elements $u_1,\cdots,u_m$ are clearly linearly independent, and since $\mathcal{O}_i$ is an orbit of N, the u_i's satisfy $u_in = u_i$ for each n in N. Therefore $u_1,\cdots,u_m$ span a subspace of $C_M(N)$.

Suppose $w = \sum_{i=1}^{m} \sum_{k=1}^{r} \alpha_{ik} Hg_{ik}$ lies in $C_M(N)$. Then

$$\sum_{i=1}^{m} \sum_{k=1}^{r} \{\alpha_{ik} Hg_{ik} - \alpha_{ik} Hg_{ik} n\} = 0$$

for each n in N. Now N acts as a transitive permutation group on each orbit, hence for each i, and each k,j such that $1 \leq i \leq m$, and $1 \leq k,j \leq r$, there is an element n_{ikj} of N such that

$$Hg_{ik} n_{ikj} = Hg_{ij}.$$

If we use this relation in the sum above, as well as the fact that n_{ijk} permutes the cosets of $\mathcal{O}_i$, it follows that, for each i,

$$\alpha_{i1} = \alpha_{i2} = \cdots = \alpha_{ir} = \alpha_i$$

Thus $W = \sum_{i=1}^{m} \alpha_i u_i$, so the u_i span $C_M(N)$. Therefore, a Z_p-basis for M/U is the collection of cosets:

$$(35) \quad \begin{array}{ll} U + Hg_{11} = U + H, & U + Hg_{12}, \cdots, U + Hg_{1,r-1} \\ U + Hg_{21} & , U + Hg_{21}, \cdots, U + Hg_{2,r-1} \\ \vdots & \qquad\qquad\qquad \vdots \\ U + Hg_{m1} & , U + Hg_{m2}, \cdots, U + Hg_{m,r-1} \end{array}$$

Suppose x lies in the kernel of the representation of G on M/U. Then, for $1 \leq i \leq m$, and $1 \leq j \leq r - 1$, we have

$$Hg_{ij}x - Hg_{ij} = \sum_{k=1}^{m} \alpha_k u_k$$

where the α_k are suitably chosen elements of Z_p. Since $Hg_{ij}x$ is a coset, and each u_k is a sum of at least three distinct cosets, we must have each $\alpha_k = 0$. Since x permutes the orbits of N, it follows from the fact that x fixes Hg_{i1} that x fixes each orbit, $\mathcal{O}_i$. Therefore

$$Hg_{ir}x = (u_i - \sum_{k=1}^{r-1} Hg_{ik})x = Hg_{ir},$$

so x lies in the kernel of the representation of G on the cosets of H. Therefore x = 1, hence M/U is faithful.

Lemma 12. Suppose $\mathcal{E}$, $\mathcal{F}$ satisfy Hypothesis I, $\mathcal{E} \subset \mathcal{F}$, and there is an element, H, in $\mathcal{V} \cap \mathcal{E} - \mathcal{T}$ such that $O_p(H) = 1$. Then $\mathcal{V} \supseteq \{G \in \mathcal{F} \mid F(G) \text{ is a p-group}\}$.

Proof. Let G be an element of $\mathcal{F}$ such that F(G) is a p-group. Let E be an $\mathcal{E}$-subgroup of G, and assume $O_{p'}(E_{\mathcal{T}}) > 1$. Since F(G) is a p-group, $O_{p'}(G) = 1$, hence $1_{O_{p'}(E_{\mathcal{T}})}|^G$ is faithful. Let M be the $Z_p(G)$-module which gives the permutation representation on the cosets of $O_{p'}(E\ )$; let $U = \bigcap_{g \in G} C_M(O_{p'}(E_{\mathcal{T}})^g)$. By Lemma 11, the $Z_p(G)$-module, $M_1 = M/U$, is faithful.

Set $X = GM_1$, then $F(X) = F(G)M_1$, so $X/F(X) \simeq G/F(G)$. Since G lies in $\mathcal{F}$, $X/F(X) \in \mathcal{V}$, hence X lies in $\mathcal{F}$. By (25c) and Theorem 5, $E^* = EC_{M_1}(O_{p'}(E_{\mathcal{T}}))$ is an $\mathcal{E}$-subgroup of X. But $C_M(O_{p'}(E_{\mathcal{T}})) \supseteq U$, hence $C_{M_1}(O_{p'}(E\ ))=C_M(O_{p'}(E_{\mathcal{T}}))/U$. Let T be the intersection of all conjugates of E^* in X. Since $E^* \cap M_1 = C_{M_1}(O_{p'}(E_{\mathcal{T}}))$, it follows that

$$T \cap M_1 = \bigcap_{g \in X} C_{M_1}(O_{p'}(E_{\mathcal{T}})^g) = 1.$$

But if $K \lhd X$, and $K \cap M_1 = 1$, then $K \leq C_X(M_1)$. $C_X(M_1) = C_G(M_1)M_1$, so the fact that G is faithful on M_1 says that M_1 is self-centralizing in X, consequently, $K = 1$. Therefore $T = 1$, so $1_{E^*}|^X$ is faithful. It is clear, now, that $1_{E^*_{\mathcal{T}}}|^X$ is also faithful, so if t is any prime which does not divide the order of X, then $X \in \langle \Phi_t \rangle = \mathcal{F}_t = \mathcal{V}$, by Lemma 6. Therefore G, as a factor group of X, also lies in $\mathcal{V}$.

We may now assume $O_{p'}(E_{\mathcal{T}}) = 1$, so $E_{\mathcal{T}}$ is a p-group. It is time to use H. If $R = I_1 \oplus \cdots \oplus I_t$ is a decomposition of the regular $Z_p(H)$-module into its principal indecomposable constituents, we set $K_\ell = I_\ell / \sqrt{I_\ell}$ for $\ell = 1,\cdots,t$. Since R is faithful, and $F(H)$ is a p'-group, it follows from Theorem 1 and Lemma 1 that $R^* = K_1 \oplus \cdots \oplus K_t$ is faithful. Since H does not lie in $\mathcal{T}$, it follows that for some ℓ, $B = H/C_H(K_\ell)$ does not lie in $\mathcal{T}$. Let $K = K_\ell$, then B is an element of $\mathcal{V} \cap \mathcal{E} - \mathcal{T}$ which has K as a faithful irreducible $Z_p(B)$-module.

Let U be the regular $Z_p(G)$-module, and set $W = (B \times G)(K \otimes U)$, where the action of $B \times G$ on $K \otimes U$ by conjugation is the canonical action given in Chapter II. To show $G \in \mathcal{V}$, it is sufficient to show $W \in \mathcal{V}$, since G is a factor group of W.

By (14), F(B) is a p'-group, and by assumption, F(G) is a p-group. Therefore if N is the kernel of the representation of $F(B \times G)$ on $K \otimes U$, then $N = N \cap F(B) \times N \cap F(G)$. Since B and G act faithfully on $K \otimes U$, $K \otimes U|_{F(B \times G)}$ is faithful. By Lemma 1, $K \otimes U$ is a faithful $Z_p(B \times G)$-module. Therefore $O_{p'}(W) = 1$, so $F(W) = F(G)(K \otimes U)$. Since $W/F(W) \simeq B \times G/F(G)$, $W/F(W) \in \mathcal{V}$, so W lies in $\mathcal{F}$.

An $\mathcal{E}$-subgroup of $B \times G$ is $B \times E$, so by (25c) and Theorem 5, $E^* = (B \times E)C_{K \otimes U}(O_{p'}((B \times E)_{\mathcal{T}})$ is an $\mathcal{E}$-subgroup of W. Since $B \in \mathcal{V} \cap \mathcal{E} - \mathcal{T}$, $1 < B_{\mathcal{T}} \leq F(B)$, so $B_{\mathcal{T}}$ is a p'-group. By assumption $E_{\mathcal{T}}$ is a p-group. Let V be the collection of elements of $B_{\mathcal{T}}$ which appear as components of elements of $(B \times E)_{\mathcal{T}}$. Since $(B \times E)_{\mathcal{T}} \lhd B \times E$, $V \lhd B$, and it is clear that $(B \times E)_{\mathcal{T}} \leq V \times E$. Therefore $B/V \simeq (B \times E)/(V \times E)$ lies in $\mathcal{T}$, hence $V = B_{\mathcal{T}}$. If $v \in V$, then there is an element u in E such that $v \times u \in (B \times E)_{\mathcal{T}}$. Since $B_{\mathcal{T}}$ is a p'-group, and $E_{\mathcal{T}}$ is a p-group, v is equal to a power of $v \times u$. Therefore

$$B_{\mathcal{T}} = O_{p'}((B \times E)_{\mathcal{T}}). \tag{36}$$

Now $K \otimes U|_{B_\tau}$ is a multiple of $K|_{B_\tau}$, so it follows from Theorem 1 that $c_{K \otimes U}(B_\tau) = 1$. By (36), $B \times E$ is an $\mathcal{E}$-subgroup of W.

Let t be a prime which does not divide $|W|$. The fact that $1_{B\times E}|^W$ is faithful implies that $1_{(B\times E)_\tau}|^W$ is faithful, so by part (b) of Lemma 6, W is an element of $\langle \Phi_t \rangle = \mathcal{F}_t = \mathcal{V}$. Therefore G lies in $\mathcal{V}$ in every case, so Lemma 12 has been proven.

Because of the preceeding lemma, we give

<u>Definition 7</u>. Set $\eta = \{p \mid \mathcal{V} \cap \mathcal{E} - \mathcal{T}$ contains a group, H, with $O_p(H) = 1\}$. We call a prime, p, <u>special</u> if $p \in \eta'$.

<u>Lemma 13</u>. If $\mathcal{E}$, $\mathcal{F}$ satisfy Hypothesis I, and $\mathcal{E} \subset \mathcal{F}$, then there is at most one special prime.

<u>Proof</u>. Let G be an element of minimal order in $\mathcal{F} - \mathcal{E}$. Then G is the semi-direct product, EM, where E is an $\mathcal{E}$-subgroup of G, and M is the unique minimal normal subgroup of G.

Since E acts faithfully and irreducibly on M, $M = F(G)$. By Lemma 10, E is not an element of $\mathcal{T}$, and since $G \in \mathcal{F}$, $G/F(G)$ lies in $\mathcal{V}$, hence $E \in \mathcal{V} \cap \mathcal{E} - \mathcal{T}$.

Since $O_r(E) \cap O_s(E) = 1$ for two distinct primes r,s, $E/O_t(E)$ can lie in $\mathcal{T}$ for at most one prime t. If $s \neq t$, then $E/O_s(E) \in \mathcal{V} \cap \mathcal{E} - \mathcal{T}$, hence $\eta' \subseteq \{t\}$.

Remark. If general, we cannot control the choice of G enough to be certain that there are no special primes. This is the basis for the example in Chapter VI, and the reason behind

Hypothesis II. Let $G = EM$ be a fixed element of minimal order in $\mathcal{F} - \mathcal{L}$. If r is any prime such that $E/O_r(E)$ lies in $\mathcal{T}$, we assume that $\mathcal{S}_{r'}$ is not contained in $\mathcal{T}$. (Such a prime does not necessarily exist.)

Theorem 10. Suppose $\mathcal{L}$ and $\mathcal{F}$ satisfy Hypotheses I and II. If $\mathcal{L} \subset \mathcal{F}$, then $\mathcal{F} = \mathcal{S}$, the collection of all solvable groups.

Proof. Our first step is to show that $\mathcal{U}$ contains the collection, $\mathcal{S}_\eta$, of all solvable η-groups. By Lemma 12, the fact that $\mathcal{N} \subseteq \mathcal{L} \subseteq \mathcal{F}$ shows that $\mathcal{U}$ contains the collection of all nilpotent η-groups. Proceeding by induction, we assume that $\mathcal{U}$ contains the collection, $\mathcal{N}^i_\eta$, of all solvable η-groups of nilpotent length at most i. But $\mathcal{N}^{i+1}_\eta = \{G \in \mathcal{S} \mid G/F(G) \in \mathcal{N}^i_\eta\}$, thus $\mathcal{F}$ contains all solvable η-groups of nilpotent length at most i + 1.

Let $G \in \mathcal{N}^{i+1}_\eta$, and $F(G) = P_1 \times \cdots \times P_s$, where P_i is the Sylow p_i-subgroup of $F(G)$. Set $N_i = \prod_{k \neq i} P_k$, and let R_i be the regular $Z_{p_i}(G/N_i)$-module for $i = 1, \cdots, s$. We allow G to act on $R = R_1 \times \cdots \times R_s$ by conjugation according to the rule

(37) $(r_1, r_2, \cdots, r_s)^g = ([r_1]N_1 g, [r_2]N_2 g, \cdots, [r_s]N_s g)$.

Then we form the semi-direct product $X = GR$. By construction, N_i centralizes the p_i-group, R_i, hence the group, $F(G)R$, is nilpotent. Since $F(X)/R$ is a normal nilpotent subgroup of X/R, and $X/R \simeq G$, $F(X) \leq F(G)R$. Therefore $F(X) = F(G)R$, so $X/F(X) \simeq G/F(G)$. Since $G/F(G) \in \mathcal{V}$, it follows that X lies in $\mathcal{F}$.

For each i, set $\overline{X}_i = X/N_i(\prod_{k\neq i} R_k)$, $\overline{R}_i = N_iR/N_i(\prod_{k\neq i} R_k)$, and $\overline{G}_i = G(\prod_{k\neq i} R_k)/N_i(\prod_{k\neq i} R_k)$. By modularity

$$G(\prod_{k\neq i} R_k) \cap N_iR = (G \cap N_iR)(\prod_{k\neq i} R_k) = ((G \cap R)N_i)(\prod_{k\neq i} R_k) = N_i(\prod_{k\neq i} R_k).$$

Thus $\overline{X}_i$ is the semi-direct product of $\overline{R}_i$ by $\overline{G}_i$, hence

$$C_{\overline{X}_i}(\overline{R}_i) = C_{\overline{G}_i}(\overline{R}_i)\overline{R}_i$$

Because $\overline{G}_i$ acts faithfully on $\overline{R}_i$, it follows that $\overline{R}_i$ is a self-centralizing normal p_i-subgroup of $\overline{X}_i$. Therefore $O_{p_i'}(\overline{X}_i) = 1$, so $F(\overline{X}_i)$ is a p_i-group. But p_i lies in η, so by Lemma 12, $\overline{X}_i$ is an element of $\mathcal{V}$ for each i. Since $\bigcap_i\{N_i(\prod_{k\neq i} R_k)\} = 1$, X lies in $\mathcal{V}$ also. Therefore $G \in \mathcal{V}$. By induction, $\mathcal{S}_\eta \subseteq \mathcal{V}$.

By Lemma 13, if EM is the minimal element of $\mathcal{F}-\mathcal{E}$ mentioned in Hypothesis II, then there is at most one prime, r_0, such that $E/O_{r_0}(E)$ lies in $\mathcal{T}$, thus η contains r_0'. Therefore,

$$\mathcal{S}_{r_0'} \subseteq \mathcal{S}_\eta \subseteq \mathcal{V} \subseteq \mathcal{F}.$$

Suppose $\mathcal{E}$ does not contain $\mathcal{S}_{r_0'}$, and let $G_1 = E_1M_1$ be an element of minimal order in $\mathcal{S}_{r_0'} - \mathcal{E}$. By Lemma 10, E_1 is an element of $\mathcal{V} \cap \mathcal{E} - \mathcal{T}$, and since $E_1 \in \mathcal{S}_{r_0'}$, $0_{r_0}(E_1) = 1$. Therefore η contains all primes. 1, not a comma

Suppose $\mathcal{E}$ contains $\mathcal{S}_{r_0'}$. By assumption $\mathcal{T}$ does not contain $\mathcal{S}_{r_0'}$, so we can choose H in $\mathcal{S}_{r_0'} - \mathcal{T}$. Since $\mathcal{S}_{r_0'} \subseteq \mathcal{V}$, H is an element of $\mathcal{V} \cap \mathcal{E} - \mathcal{T}$ with $0_{r_0}(H) = 1$. Therefore η contains all primes in every case, so we have

$$\mathcal{S} = \mathcal{S}_{\eta} \subseteq \mathcal{V} \subseteq \mathcal{F} \subseteq \mathcal{S},$$

which completes the proof of Theorem 10.

<u>Corollary 2</u>. Let $\mathcal{N}_i$ be the collection of groups of nilpotent length at most i. If $\mathcal{N}_i \ll \mathcal{F}$, then $\mathcal{F} = \mathcal{N}_i$, or $\mathcal{F} = \mathcal{S}$.

<u>Proof</u>. If we set $\mathcal{N}_0 = \{1\}$, then for $i \geq 1$, $\mathcal{N}_i = \{G \in \mathcal{S} \mid G/F(G) \in \mathcal{N}_{i-1}\}$. For each prime, p, $\mathcal{S}_{p'}$ is not contained in $\mathcal{N}_{i-1}$, hence the hypothesis of Theorem 10 is satisfied. The result follows from Theorem 10.

Chapter VI

An Example

Let r be a prime. Throughout this chapter, we shall set $\mathcal{R} = \{G \in \mathcal{S} \mid G/F(G) \in \mathcal{S}_{r'}\}$. For each p, we set $\mathcal{R}_p = \mathcal{S}_{r'}$; $\{\mathcal{R}_p\}$ forms a local definition for $\mathcal{R}$ by Lemma 9. In this chapter, we shall characterize the formations which strongly contain $\mathcal{R}$. The formation $\mathcal{R}$ provides an example which shows that Hypothesis II is not redundant.

<u>Lemma 14</u>. Let G be a group with Sylow r-subgroup R. Then $N_G(R)$ is an $\mathcal{R}$-subgroup of G.

<u>Proof</u>. Because R is a Sylow r-subgroup of G, $N_G(R)/R$ is an r'-group. Therefore $N_G(R)$ lies in $\mathcal{R}$.

Suppose $N_G(R) \leq U \leq G$. We wish to show $N_G(R)$ covers $U/U_{\mathcal{R}}$. Clearly $U_{\mathcal{R}} = O^{r',r}(U)$. Since $R \leq O^{r'}(U)$, it follows that $O^{r'}(U)$ acts transitively on the Sylow r-subgroups of U, hence $U = O^{r'}(U)N_G(R)$. Since R covers every r-section of U, it follows that $N_G(R)$ covers $O^{r'}(U)/O^{r',r}(U)$. Therefore $N_G(R)$ covers $U/O^{r',r}(U)$. By definition, $N_G(R)$ is an $\mathcal{R}$-subgroup of G.

Suppose $\mathcal{F} \gg \mathcal{R}$, and $\mathcal{F} \supset \mathcal{R}$. If $\{\mathcal{F}_q\}$ is the minimal local definition of $\mathcal{F}$, it follows from Theorem 9 that $\mathcal{F}_q = \mathcal{F}_s$ for each q,s. We set $\mathcal{V} = \mathcal{F}_q$. If H lies in $\mathcal{V} \cap \mathcal{R}$, then H has a normal Sylow r-subgroup, so $H/O_r(H)$ lies in $\mathcal{S}_{r'}$. Therefore, Hypothesis II is violated for the prime r. It follows from Lemma 13 that r is the unique special prime associated with $\mathcal{J}$ and $\mathcal{R}$. The next theorem gives a class of formations which strongly contain $\mathcal{R}$.

<u>Theorem 11</u>. Let $\mathcal{T}$ be a non-empty formation. Let $\mathcal{V} = \{G \in \mathcal{S} \mid G/O_{r'}(G) \in \mathcal{T}\}$, then $\mathcal{V}$ is a formation. If $\mathcal{J} = \{G \in \mathcal{S} \mid G/F(G) \in \mathcal{V}\}$, then $\mathcal{J} \gg \mathcal{R}$.

<u>Proof</u>. Suppose $G \in \mathcal{V}$, and $N \triangleleft G$. Then $O_{r'}(G)N/N \leq O_{r'}(G/N)$. Since $G/O_{r'}(G) \in \mathcal{T}$, we also have $(G/N)/O_{r'}(G/N) \in \mathcal{T}$. Therefore, G/N lies in $\mathcal{V}$.

Suppose N_1, N_2 are two normal subgroups of G such that $G/N_i \in \mathcal{V}$ for $i = 1,2$. For each i, let $M_i/N_i = O_{r'}(G/N_i)$, then $G/M_i \in \mathcal{T}$ for each i, hence $G/M_1 \cap M_2 \in \mathcal{T}$. For each i, $(M_1 \cap M_2)N_i/N_i$ is an r'-group, hence $M_1 \cap M_2/N_i \cap M_1 \cap M_2$ is an r'-group for each i. Therefore $M_1 \cap M_2/N_1 \cap N_2$ is an r'-group, so it follows that the factor group of $G/N_1 \cap N_2$ by $O_{r'}(G/N_1 \cap N_2)$ lies in $\mathcal{T}$. Therefore $G/N_1 \cap N_2 \in \mathcal{V}$, thus $\mathcal{V}$ is a formation.

To show $\mathcal{R} \ll \mathcal{J}$, it is sufficient to show that $\Phi_p \subseteq \mathcal{V}$ for each prime p. Suppose $G \in \Phi_r$, then G has a faithful irreducible $Z_r(G)$-module. By (14), this means that $O_r(G) = 1$. Since $G \in \mathcal{J}$, $G/F(G) \in \mathcal{V}$. Since $O_r(G) = 1$, F(G) is an r'-group, so $G/O_{r'}(G) \in \mathcal{T}$.

Therefore G lies in $\mathcal{V}$.

Suppose $G \in \Phi_p$ for $p \neq r$. An $\mathcal{R}$-subgroup of G is $N_G(R)$ where R is a Sylow r-subgroup of G. Since $p \neq r$, $O_{p'}((N_G(R))_{\mathcal{S}_{r'}}) = R$. Therefore G has a faithful irreducible $Z_p(G)$-module, J, such that $1 < C_J(R) \leq J$. By Theorem 1, either $C_J(O_r(G)) = J$, or it is the identity. The latter possibility cannot occur because $1 < C_J(R) \leq C_J(O_r(G))$. Therefore the fact that J is faithful says that $O_r(G) = 1$, so F(G) is an r'-group. $G \in \mathcal{F}$, so the same argument as in the preceeding paragraph shows that $G/O_{r'}(G) \in \mathcal{T}$. Therefore $G \in \mathcal{V}$. By Theorem 6, $\mathcal{R} \ll \mathcal{F}$.

Since our choice of $\mathcal{T}$ is arbitrary, it follows that we can choose an infinite number of distinct formations which strongly contain $\mathcal{R}$. Our next theorem is the converse to Theorem 11.

<u>Theorem 12</u>. Suppose $\mathcal{F} \gg \mathcal{R}$, and $\{\mathcal{F}_q\}$ is the minimal local definition for $\mathcal{F}$. Then there is a non-empty formation $\mathcal{T}$ such that $\mathcal{F}_q = \{G \in \mathcal{S} \mid G/O_{r'}(G) \in \mathcal{T}\}$.

<u>Proof</u>. Suppose $\mathcal{F} \supset \mathcal{R}$. By Theorem 9, there is a formation $\mathcal{V}$ such that $\mathcal{F}_q = \mathcal{V}$ for each q. Our first step is to show that $\mathcal{V}$ is the smallest formation generated by $\{H \in \mathcal{F} \mid O_r(H) = 1\}$. Let $\hat{\mathcal{V}}$ be the smallest formation generated by this set.

Suppose $H \in \mathcal{F}$, and $O_r(H) = 1$. Let $K = I_1 \oplus \cdots \oplus I_s$ be the decomposition of the regular $Z_r(H)$-module, K, into principal indecomposable submodules. By Theorem 1, Lemma 1, and the fact that $F(H)$ is an r'-group, it follows that H acts faithfully on $J_1 \oplus \cdots \oplus J_s$, where $J_k = I_k/\sqrt{I_k}$ for each k. For each k, set $H_k = H/C_H(J_k)$. Then J_k is a faithful irreducible $Z_r(H_k)$-module. If R_k is a Sylow r-subgroup of H_k, then $N_{H_k}(R_k)$ is an $\mathcal{R}$-subgroup of H_k, and $(N_{H_k}(R_k))_{\mathcal{S}_{r'}} = (N_{H_k}(R_k))_{\mathcal{R}_r} = R_k$. Therefore $O_{r'}((N_{H_k}(R_k))_{\mathcal{R}_r}) = 1$. By definition, $H_k \in \Phi_r$ for each k, so $\mathcal{R} \ll \mathcal{F}$ implies $H_k \in \mathcal{F}_r = \mathcal{V}$. Therefore H lies in $\mathcal{V}$. Since the generators of $\tilde{\mathcal{V}}$ all lie in $\mathcal{V}$, $\tilde{\mathcal{V}}$ is contained in $\mathcal{V}$. We know that $\mathcal{V}$ is the smallest formation generated by Φ_r, from the proof of Theorem 9. Thus if we show $\Phi_r \subseteq \tilde{\mathcal{V}}$, we have shown $\mathcal{V} \subseteq \tilde{\mathcal{V}}$. If $G \in \Phi_r$, then G has a faithful irreducible $Z_r(G)$-module, and $G \in \mathcal{F}$. By (14), $O_r(G) = 1$ so by definition, $G \in \tilde{\mathcal{V}}$. Therefore $\mathcal{V} = \tilde{\mathcal{V}}$.

Let $\mathcal{T}$ be the smallest formation generated by the set $\{H/O_{r'}(H) \mid H \in \mathcal{V}\}$. Set $\mathcal{V}_1 = \{G \in \mathcal{S} \mid G/O_{r'}(G) \in \mathcal{T}\}$. We want to show $\mathcal{V} = \mathcal{V}_1$. By construction, $\mathcal{V} \subseteq \mathcal{V}_1$.

Since the generators of $\mathcal{T}$ are elements of $\mathcal{V}$, we must have $\mathcal{T} \subseteq \mathcal{V}$. Therefore, if $G \in \mathcal{V}_1$, then $G/O_{r'}(G)$ lies in $\mathcal{V}$. To show G lies in $\mathcal{V}$, we use induction on the nilpotent length of $O_{r'}(G)$. If $O_{r'}(G)$ is nilpotent, then $G/O_{r'}(G) \in \mathcal{V}$ forces $G/F(G) \in \mathcal{V}$. Therefore G is an element of $\mathcal{F}$. By our first paragraph, $G/O_r(G)$ lies in $\mathcal{V}$, so G also lies in $\mathcal{V}$ since $O_r(G) \cap O_{r'}(G) = 1$.

We note that $O_{r'}(G/F(O_{r'}(G))) = O_{r'}(G)/F(O_{r'}(G))$, hence by induction, if $G \in \mathcal{V}_1$, then $G/F(O_{r'}(G)) \in \mathcal{V}$. Therefore G lies in $\mathcal{F}$. By our first paragraph $G/O_r(G)$ lies in $\mathcal{V}$, so $G/O_r(G) \cap F(O_{r'}(G)) = G$ also lies in $\mathcal{V}$. Therefore $\mathcal{V} = \mathcal{V}_1$. This completes the proof in the case when $\mathcal{R} \subset \mathcal{F}$.

If $\mathcal{R} = \mathcal{F}$, we let $\mathcal{T}$ be the formation consisting only of the identity. We must then show that $\{\mathcal{R}_q\}$ is the minimal local definition for $\mathcal{R}$.

Let $\{\tilde{\mathcal{R}}_q\}$ be the minimal local definition for $\mathcal{R}$. Suppose p is an arbitrary prime, $G \in \mathcal{S}_{r'} = \mathcal{R}_p$, and t is a prime which does not divide $rp|G|$. Let K be the regular $Z_t(G)$-module. Set $G_1 = GK$. Let K_1 be the regular $Z_p(G_1)$-module. Set $G_2 = G_1K_1$. Since G acts faithfully on K, and G_1 acts faithfully on K_1, $O_{p',p}(G_2) = K_1$. Depending on the choice of p, G_2 is either an r'-group, or has K_1 as a normal Sylow r-subgroup. Therefore $G_2 \in \mathcal{R}$, hence $G_2/O_{p',p}(G_2) = G_2/K_1 \in \tilde{\mathcal{R}}_p$. Since G is a factor group of G_1, it follows that $G \in \tilde{\mathcal{R}}_p$. Therefore $\mathcal{S}_{r'} \subseteq \tilde{\mathcal{R}}_p$. This completes the proof.

Appendix

In this appendix, we shall give proofs to (18'), (18), and (19).

(18') If $N \lhd G$, and $\pi_1, \cdots, \pi_n$ are sets of primes, then

$$O_{\pi_1,\cdots,\pi_n}(N) = N \cap O_{\pi_1,\cdots,\pi_n}(G).$$

<u>Proof</u>. Use induction on N. If $n = 1$, and $\pi = \pi_1$, then $O_\pi(G)$ is the unique maximal normal π-subgroup of G, hence $O_\pi(G)$ is characteristic in G. Therefore, $O_\pi(N)$, as a characteristic subgroup of the normal subgroup N, is normal in G. Thus $O_\pi(N) \leq O_\pi(G) \cap N$. But $N \cap O_\pi(G)$ is a normal π-subgroup of N, hence is contained in $O_\pi(N)$.

We may assume that $T = O_{\pi_1,\cdots,\pi_{n-1}}(N) = O_{\pi_1,\cdots,\pi_{n-1}}(G) \cap N$. Then $O_{\pi_1,\cdots,\pi_n}(N)O_{\pi_1,\cdots,\pi_{n-1}}(G)/O_{\pi_1,\cdots,\pi_{n-1}}(G)$ is a normal π_n-subgroup of $\overline{G} = G/O_{\pi_1,\cdots,\pi_{n-1}}(G)$, and must be contained in $O_{\pi_n}(\overline{G})$. Therefore $O_{\pi_1,\cdots,\pi_n}(N) \leq O_{\pi_1,\cdots,\pi_n}(G) \cap N$. But $O_{\pi_1,\cdots,\pi_n}(G) \cap N/T$ is a normal π_n-subgroup of N/T, hence $O_{\pi_1,\cdots,\pi_n}(G) \cap N \leq O_{\pi_1,\cdots,\pi_n}(N)$. This proves (18').

(18) Let G be a group. Then $O_{p',p}(G) = \cap\, C_G(K)$, where the intersection is taken over all p-chief factors, K, of G.

<u>Proof</u>. By looking at a chief series of G which passes through $O_{p',p}(G)$, it is clear that $O_{p',p}(G)$ centralizes every p-chief factor of G. Let $P = O_{p',p}(G)/O_{p'}(G)$, and $E = P/D(P)$. Then G is represented

on E, and the kernel of this representation is $O_{p',p}(G)$ (cf. [9], p. 7). If g is any p'-element of $\cap C_G(K)$, then g centralizes E, since it centralizes every p-chief factor of G. Therefore g lies in $O_{p'}(G)$. This shows that $\cap C_G(K)$ has a normal p-complement, so by (18'), $\cap C_G(K) \leq O_{p',p}(G)$.

(19) If G is a group, then $F(G) = \bigcap_p O_{p',p}(G)$, where the intersection is taken over all primes p.

Proof. Since F(G) is nilpotent, it has a normal p-complement for every p. By (18'), $F(G) \leq \bigcap_p O_{p',p}(G)$. Therefore it is sufficient to show that $\bigcap_p O_{p',p}(G)$ is nilpotent. But look at any chief series of G passing through $\bigcap_p O_{p',p}(G)$. By (18), $\bigcap_p O_{p',p}(G)$ centralizes every factor in this series, hence the part of this chief series covered by $\bigcap_p O_{p',p}(G)$ forms a central series for $\bigcap_p O_{p',p}(G)$, so $\bigcap_p O_{p',p}(G)$ is nilpotent. This proves (19).

References

1. Artin, E., Nesbitt, C. J., and R. M. Thrall, RINGS WITH MINIMUM CONDITION. University of Michigan Press, Ann Arbor, 1944.

2. Carter, R. W., "Nilpotent self-normalizing subgroups of soluble groups." Math. Zeit., 75(1961), pp. 136-139.

3. Curtis, W., and I. Reiner, REPRESENTATION THEORY OF FINITE GROUPS AND ASSOCIATIVE ALGEBRAS. Interscience, New York, 1962.

4. Fong, P., and W. Gaschütz, "A note on the modular representations of solvable groups." J. für Math., 208(1961), pp. 73-78.

5. Gaschütz, W., "Zur Theorie der endlichen auflösbaren Gruppen." Math. Zeit., 80 (1963), pp. 300-305.

6. Gaschütz, W., and U. Lubeseder, "Kennzeichnung gesättigter Formationen." Math. Zeit., 82(1963), pp. 198-199.

7. Glauberman, G. "Fixed points in groups with operator groups." Math. Zeit., 84(1964), pp. 120-125.

8. Green, J. A., "On the indecomposable representations of a finite group." Math. Zeit., 70(1959), pp. 430-445.

9. Hall, P., and G. Higman, "On the p-length of p-soluble groups and reduction theorems for Burnside's problem." Proc. London Math. Soc. (3)6(1956), pp. 1-42.

10. Lubeseder, U., Unpublished Thesis. University of Kiel, Kiel Germany.