ON AN EMBEDDING PROPERTY OF GENERALIZED
CARTER SUBGROUPS

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ABSTRACT

If £ and $\mathcal{F}$ are saturated formations, we say that £ is strongly contained in $\mathcal{F}$ if for any solvable group $G$ with £-subgroup, $E$, and $\mathcal{F}$-subgroup, $F$, some conjugate of $E$ is contained in $F$. In this paper, we investigate the problem of finding the formations which strongly contain a fixed saturated formation £.

Our main results are restricted to formations, £, such that £ = \{G| G/F(G) ∈ $\mathcal{F}$\}, where $\mathcal{T}$ is a non-empty formation of solvable groups, and F(G) is the Fitting subgroup of G. If $\mathcal{F}$ consists only of the identity, then £ = $\mathcal{N}$, the class of nilpotent groups, and for any solvable group, G, the $\mathcal{N}$-subgroups of G are the Carter subgroups of G.

We give a characterization of strong containment which depends only on the formations £, and $\mathcal{F}$. From this characterization, we prove:

If $\mathcal{T}$ is a non-empty formation of solvable groups, £ = \{G| G/F(G) ∈ $\mathcal{F}$\}, and £ is strongly contained in $\mathcal{F}$, then
(1) there is a formation $\mathcal{U}$ such that $\mathcal{F}$ = \{G| G/F(G) ∈ $\mathcal{U}$\}.
(2) If for each prime $p$, we assume that $\mathcal{T}$ does not contain the class, $\mathcal{S}_p$, of all solvable $p'$-groups, then either £ = $\mathcal{F}$, or $\mathcal{F}$ contains all solvable groups.

This solves the problem for the Carter subgroups.
We prove the following result to show that the hypothesis of (2) is not redundant:

If \( \mathcal{K} = \{ G | G/F(G) \in \mathcal{G} \} \), then there are infinitely many formations which strongly contain \( \mathcal{K} \).
Chapter I
Introduction

In 1961, Roger Carter [2] discovered the interesting result that a finite solvable group contains a nilpotent subgroup which is its own normalizer. These subgroups are now called Carter subgroups. In the same paper, Carter showed that any two Carter subgroups are conjugate, and that they possess certain interesting cover-avoidance properties.

Since the discovery of the Carter subgroups, several papers, devoted to the investigation of relations between the system normalizers of a finite solvable group and its Carter subgroups, have been published. In another direction, Gaschütz [5] has discovered a quite general theory from which most of the properties of Carter subgroups follow as a special case. One consequence of this theory is that there are many types of subgroups of a finite solvable group, G, which possess properties analogous to the properties of Carter subgroups; a subgroup of G which satisfies the necessary conditions is called a generalized Carter subgroup.

We shall, from now on, assume that all groups under discussion are finite and solvable. To describe our results we need

**Definition 1.** A formation, ℱ, is a collection of groups which satisfies:
(1) If $G \in \mathcal{F}$, and $N$ is normal in $G$, then $G/N \in \mathcal{F}$.

(2) If $N_1, N_2$ are normal in $G$, and $G/N_1 \in \mathcal{F}$, then $G/N_1 \cap N_2 \in \mathcal{F}$.

A formation $\mathcal{F}$ is saturated if it satisfies, in addition to (1) and (2),

(3) If $G/D(G) \in \mathcal{F}$, and $D(G)$ is the Frattini subgroup of $G$, then $G \in \mathcal{F}$.

The generalized Carter subgroups may be defined as follows:

**Definition 2.** Let $G$ be a group, and $\mathcal{F}$ a formation. A subgroup, $F$, of $G$ is called an $\mathcal{F}$-subgroup of $G$ provided

(4) $F \in \mathcal{F}$

(5) If $U$ is any subgroup of $G$ containing $F$, and if $N$ is a normal subgroup of $U$ such that $U/N \in \mathcal{F}$, then $FN = U$ (i.e., $F$ covers $U/N$).

One of the basic results on formations is that if $\mathcal{F}$ is any saturated formation, and if $G$ is any solvable group, then $\mathcal{F}$-subgroups of $G$ always exist, and all such groups are conjugate (Gaschütz, [5]).

In this thesis we shall investigate the behaviour of the $\mathcal{E}$-subgroups of a solvable group $G$ with respect to the $\mathcal{F}$-subgroups of $G$ for two saturated formations $\mathcal{E}$, and $\mathcal{F}$.

**Definition 3.** Let $\mathcal{E}$ and $\mathcal{F}$ be saturated formations. $\mathcal{E}$ is strongly contained in $\mathcal{F}$ (written $\mathcal{E} \ll \mathcal{F}$) provided

(6) If $G$ is any solvable group which has $\mathcal{E}$-subgroup, $E$, and $\mathcal{F}$-subgroup, $F$, then some conjugate of $E$ is contained in $F$.

We shall be concerned with the problem: Given a saturated formation
\( \mathcal{E} \), find all formations \( \mathcal{F} \) such that \( \mathcal{E} \ll \mathcal{F} \). From the definition, it is clear that there are always two formations, \( \mathcal{F} \), which strongly contain \( \mathcal{E} \), viz. \( \mathcal{E} \), and \( \mathcal{S} \), the formation of all solvable groups. We comment that the relation \( \ll \) forms a partial ordering of the collection of all saturated formations. It is a consequence of the definition that if \( \mathcal{E} \ll \mathcal{F} \), then \( \mathcal{E} \) is contained in \( \mathcal{F} \).

The two main results of this thesis appear in Chapter V. The first of these shows that if \( \mathcal{F} \) is a non-empty formation, and 
\[
\mathcal{E} = \{ G \mid G/F(G) \in \mathcal{F} \} \quad (F(G) \text{ is the Fitting subgroup of } G),
\]
then any formation \( \mathcal{F} \) which strongly contains \( \mathcal{E} \) has essentially the same structure as \( \mathcal{E} \) in that there is a non-empty formation \( \mathcal{V} \) such that 
\[
\mathcal{F} = \{ G \mid G/F(G) \in \mathcal{V} \}. \]
We then apply this result to find a large class of formations which are maximal in the partial ordering \( \ll \). In particular, if \( \mathcal{N}_i \) denotes the formation of groups of nilpotent length at most \( i \), then \( \mathcal{N}_i \) is maximal in the ordering \( \ll \). It is easily shown that for \( \mathcal{N} = \mathcal{N}_i \), the \( \mathcal{N} \)-subgroups of a group \( G \) are the Carter subgroups of \( G \), so the problem is solved for the Carter subgroups.

The machinery for proving these results is developed in Chapter IV. We begin by developing a characterization of strong containment which depends only on the two formations \( \mathcal{E} \) and \( \mathcal{F} \). This characterization depends on the knowledge that if \( \mathcal{E} \) is a saturated formation, then \( \mathcal{E} \) is a locally defined formation (see Chapter II), a result proved by Lubeseder in [10]. In certain cases, we are able
to strengthen our characterization of strong containment so that it gives a complete description of the minimal local definition for \( \mathcal{F} \) as a necessary condition for strong containment.

In Chapter VI, we present an example which shows that Hypothesis II of our main theorem is not redundant. The formation which gives the example is \( \mathcal{R} = \{ G \mid G/F(G) \text{ is an } r'-\text{group} \} \). It is apparent from Theorem 11 that \( \mathcal{R} \) is not maximal in the partial ordering \( \ll \). In fact, there is an infinite number of formations which strongly contain \( \mathcal{R} \).

In Chapter III, we give some preliminary results which are used throughout the paper. In particular, we give a cover-avoidance characterization of the \( \mathcal{F} \)-subgroups of a solvable group. In Theorem 5, we find the \( \mathcal{F} \)-subgroups of a certain type of semi-direct product. We also show how these subgroups are embedded in the product. The remaining results in Chapter III are known, but their proofs are included for the sake of completeness.

In Chapter II, we give notation along with the results quoted from previous papers. Some of the results in this chapter are proved because we have found no suitable reference to them in the literature.
Chapter II
Notation and Quoted Results

We shall use the following notation:

- $G$ - A finite solvable group
- $|G|$ - The order of $G$
- $H \leq G$ - $H$ is a subgroup of $G$ (proper if $H < G$)
- $H \triangleleft G$ - $H$ is normal in $G$
- $\subseteq$ - A subset of (proper subset if $\subset$)
- $D(G)$ - The Frattini subgroup of $G$, the intersection of all maximal subgroups of $G$
- $F(G)$ - The Fitting subgroup of $G$, the maximal normal nilpotent subgroup of $G$
- $F_1(G)$ - The inverse image in $G$ of $F(G/F_1(G))$
- $\text{Aut}(G)$ - The automorphism group of $G$
- $\mathbb{E}$ - A field
- $\mathbb{Z}_p$ - The field of integers mod $p$, $p$ a prime
- $\mathbb{E}(G)$ - The group ring of $G$ over the field $\mathbb{E}$
- $\cong$ - $\mathbb{E}(G)$ - isomorphic to $\mathbb{E}(G)$
- $\pi$ - A set of primes
- $\pi'$ - The complementary set of primes
- $O_{\pi}(G)$ - The maximal normal $\pi$-subgroup of $G$
The inverse image in $G$ of $0_{\prod_{1}^{\infty} G} / 0_{\prod_{1}^{\infty} (G)}$ is a $\pi$-group.

The smallest normal subgroup $N$ of $G$ such that $G/N$ is a $\pi$-group is

$$0_{\prod_{1}^{\infty} n} (G) = 0_{\prod_{1}^{\infty} n}(0_{\prod_{1}^{\infty} (G)}).$$

If $H \not< K$, and $H \leq K \leq G$, then $K/H$ is a section of $G$. $K/H$ is a $\pi$-section if $K/H$ is a $\pi$-group. If $L \leq G$, then $L$ covers the section $K/H$ provided $(L \cap K)H = K$, and avoids $K/H$ if $K \cap L \not< H$.

If $F$ acts on $G$, i.e., every element of $F$ induces an automorphism of $G$, and $F$ fixes both $H$ and $K$, then $K/H$ is an $F$-invariant section of $G$. If $F$ acts irreducibly on $K/H$, then $K/H$ is an $F$-composition factor of $G$. An $F$-composition series of $G$ is a series

$$G = G_0 > G_1 > \ldots > G_n = 1,$$

such that $G_i/G_{i+1}$ is an $F$-composition factor of $G$ for each $i$. The $F$-composition length of $G$ is $n$, the length of any $F$-composition series of $G$.

If $F = G$, then $K/H$ is a chief factor of $G$, and the series

$$G = G_0 > G_1 > \ldots > G_n = 1,$$

is a chief series of $G$.

If $K/H$ is an $F$-invariant section of $G$, then $C_F(K/H)$ denotes the kernel of the representation of $F$ on $K/H$. $C_{K/H}(F)$ denotes the elements of $K/H$ fixed by every element of $F$.

If $F$ acts on $G$, and $K/H$ is an $F$-invariant section of $G$ which is also an elementary abelian $p$-group, then we may look at $K/H$ as a vector space over $\mathbb{Z}_p$, hence also as a $\mathbb{Z}_p(F)$-module with the action...
\[(Hk)f = Hk^f \quad \text{for} \ f \in F, \ Hk \in K/H.\]

If \(M\) is a right \(\mathbb{Z}_p(G)\)-module, we may form the semi-direct product \(GM\), where the action of \(G\) on \(M\) by conjugation is
\[m^g = g^{-1}mg = (m)g \quad \text{for} \ m \in M, \ g \in G.\]

We shall frequently switch back and forth between additive and multiplicative notation.

If \(M\) and \(N\) are right \(\mathbb{E}(G)\)-modules, then \(M \oplus N\) will denote the direct sum of these modules, and \(\text{Hom}_{\mathbb{E}(G)}(M,N)\) will denote the collection of all \(\mathbb{E}(G)\)-linear homomorphisms from \(M\) into \(N\).

If \(H\) is a second group, and \(T\) is a \(\mathbb{E}(G)\)-module, then the tensor product \(M \otimes_{\mathbb{E}(G)} T\) (sometimes written \(M \otimes T\)) is a \(\mathbb{E}(G \times H)\)-module under the canonical action
\[(m \otimes t)(g \times h) = mg \otimes th\]

A principal indecomposable \(\mathbb{E}(G)\)-module is an indecomposable component of the regular \(\mathbb{E}(G)\)-module. If \(M\) is a completely reducible \(\mathbb{E}(G)\)-module, and \(M = I_1 \oplus \cdots \oplus I_s\) is a decomposition of \(M\) as a direct sum of irreducible submodules, \(I_k\), then for each \(k\) we let \(M_{I_k} \), be the direct sum of all copies of \(I_k\) which appear in this decomposition of \(M\). \(M_{I_k}\) is called a primary component of \(M\).

The dual \(\mathbb{E}(G)\)-module to \(M\) will be denoted by \(\hat{M}\). We recall that \(\hat{M}\) is the collection of all \(\mathbb{E}\)-linear functionals from \(M\) into \(\mathbb{E}\), and that \(\hat{M}\) becomes a right \(\mathbb{E}(G)\)-module if we let \(G\) operate on \(\hat{M}\) so that
for \( g \) in \( G \), and \( \mu \) in \( \hat{M} \), \( \mu g \) is the functional defined by

\[
(m)\mu g = (mg^{-1})\mu \quad \text{for } m \in M
\]

If \( N \) is a \( \Xi(G) \)-submodule of \( M \), we set

\[
\downarrow N = \{ \mu \in \hat{M} \mid (N)\mu = 0 \};
\]

we note that the following relations hold if we assume \( M \) is finitely generated and identify \( M \) with its second dual.

\[
(7) \quad \downarrow (\downarrow N) = N, \quad (0)\downarrow = \hat{M}, \quad M\downarrow = (0)
\]

If \( N, U \) are \( \Xi(G) \)-submodules of \( M \) such that \( N \subseteq U \), then \( U\downarrow \subseteq N\downarrow \),

\[
(8) \quad \downarrow N\downarrow /\downarrow U\downarrow \cong \Xi(G)(\overline{U}/\overline{N}), \quad \text{and}
\]

\[
(9) \quad CG(U/N) = CG((U/N)).
\]

Furthermore, if \( I_1 \) and \( I_2 \) are submodules of \( M \), then

\[
(10) \quad M = I_1 \oplus I_2 \iff \hat{M} = I_1\downarrow \oplus I_2\downarrow.
\]

Suppose \( H < G \), and \( U \) is a \( \Xi(H) \)-module. For a fixed \( g \) in \( G \), we can define a new \( \Xi(H) \)-module \( U^G \) by looking at \( U \) as a \( \Xi \)-vector space and specifying a new action for \( H \),

\[
u \cdot h = u(ghg^{-1}) \quad \text{for } h \in H
\]

The module \( U^G \) is called the conjugate of \( U \) by \( g \).

If \( F \leq G \), and \( M \) is a \( \Xi(G) \)-module, then \( M|_F \) denotes the \( \Xi(F) \)-module obtained from \( M \) by restriction to \( \Xi(F) \). If \( U \) is a \( \Xi(F) \)-module, then \( U^G \) is the \( \Xi(G) \)-module induced from \( U \). One of the most convenient ways to define \( U^G \) is presented in Green's paper [8].
Set $U^G = U \otimes \Xi(F)$, then $U^G$ can be written as a direct sum of $\Xi$-subspaces in the following way: Let $1 = g_1, g_2, \ldots, g_s$ be a complete set of coset representatives for $F$ in $G$. Then

$$U^G = U \otimes 1 \oplus U \otimes g_2 \oplus \cdots \oplus U \otimes g_s$$

This decomposition does not depend on the choice of coset representatives. The action of $G$ on $U^G$ is characterized by giving the action of $G$ on each element $m \otimes g_i$:

$$(11) \quad (m \otimes g_i)g = mf_g \otimes g_j \quad \text{where } g_ig_j = f_ig_j,$$

and $f_g \in F$.

Using (11), one can show that $U$ is canonically embedded as a submodule of $U^G/F$ by the mapping

$$\Lambda: u \mapsto u \otimes 1 \quad \text{for } u \in U.$$ (12)

We shall use the following results:

(13) ([1], p.8) Let $M = I_1 \oplus \cdots \oplus I_n$ be a decomposition of the $\Xi(G)$-module $M$ into a direct sum of irreducible $\Xi(G)$-submodules. Let $M_{I_k}$ be the primary component of $M$ associated with $I_k$. If $W$ is any $\Xi(G)$-submodule of $M$ isomorphic to $I_k$, then $W \subseteq M_{I_k}$.

(14) ([4], p.75) If $\Xi$ is a field of characteristic $p$, $G$ is a group, and $M$ is an irreducible $\Xi(G)$-module, then $O_p(G)$ lies in the kernel of the representation of $G$ on $M$. 


Let $H$ be a $p'$-subgroup of $G$, and let $M$ be the $Z_p(G)$-module which gives the permutation representation of $G$ on the cosets of $H$, $1_H^G$. If $M = J_1 \oplus \cdots \oplus J_n$ is any decomposition of $M$ into a direct sum of indecomposable $Z_p(G)$-modules, then each $J_k$ is a principal indecomposable $Z_p(G)$-module.

\textbf{Proof.} If $R$ is the regular $Z_p(H)$-module, then $R^G$ is the regular $Z_p(G)$-module. Since $H$ is a $p'$-group, $R$ is completely reducible to a direct sum of $Z_p(H)$-irreducible submodules, one of which is the identity $Z_p(H)$-module, $I$. We have, therefore, a decomposition $R = I \oplus L$. Thus $R^G = I^G \oplus L^G$, so because $M = I^G$, each $J_k$ is a direct summand of the regular $Z_p(G)$-module.

(16) ([3], p. 372) Let $G$ be a group, and $J$ a principal indecomposable $E(G)$-module for some field $E$. Then $J$ has a unique maximal proper $E(G)$-submodule which we shall denote by $\sqrt{J}$.

(17) (cf. [7], Theorem 1) Suppose $A$ is a group of automorphisms of the group $G$, and $(|A|, |G|) = 1$. If $H/K$ is an $A$-invariant section of $G$, then $C_G(A)$ covers $C_{H/K}(A)$.

The next three results are well-known elementary facts that prove to be quite useful. Their proofs will be included in an appendix.

(18') If $N < G$, and $\pi_1, \ldots, \pi_n$ are sets of primes, then

$$0_{\pi_1, \ldots, \pi_n}(N) = N \cap 0_{\pi_1, \ldots, \pi_n}(G).$$
Let $G$ be a group, then $0_{p'}(G) = \cap C_G(K)$, where the intersection is taken over all $p$-chief factors, $K$, of $G$.

Let $G$ be a group, then $F(G) = \cap 0_{p'}(G)$, where the intersection is taken over all primes $p$.

We shall give a short summary of the theory of formations which is found in the papers of Gaschütz, and Lubeseder ([5],[6], and [10]). Let $\mathcal{A}$ be the collection of all solvable groups.

**Definition 4.** For each prime $p$, let $\mathcal{F}_p$ be a formation. Let

$$\mathcal{F} = \left\{ G \in \mathcal{A} \middle| \begin{array}{l}
(a) \text{ If } \mathcal{F}_p \text{ is non-empty, and } K \text{ is a } p\text{-chief factor of } G, \text{ then } G/C_G(K) \in \mathcal{F}_p. \\
(b) \text{ If } \mathcal{F}_p \text{ is empty, then } G \text{ is a } p'\text{-group.}
\end{array} \right\}$$

$\mathcal{F}$ is a formation which is called the formation locally defined by the family $\{\mathcal{F}_p\}$. In general, a formation $\mathcal{F}$ is locally defined if there exists a family, $\{\mathcal{F}_p\}$, of formations such that $\mathcal{F}$ is locally defined by the family $\{\mathcal{F}_p\}$.

Because of (18) and the fact that $\mathcal{F}_p$ is a formation, (2) is equivalent to the condition:

$$(\alpha') \text{ If } \mathcal{F}_p \text{ is non-empty, then } G/0_{p'},(G) \in \mathcal{F}_p.$$ 

The family, $\{\mathcal{F}_p\}$, of formations which locally defines $\mathcal{F}$ is usually not unique. If, however, $\{\mathcal{F}_p\}$ and $\{\mathcal{F}'_p\}$ are two families of formations which locally define the same formation $\mathcal{F}$, then the family $\{\mathcal{H}_p | \mathcal{H}_p = \mathcal{F}_p \cap \mathcal{F}_p'\}$ also defines $\mathcal{F}$. Thus there is a unique
minimal local definition for any locally defined formation \( \mathcal{F} \). For example, the minimal local definition of the formation of all \( p \)-groups, \( \mathcal{H}_p \), is

\[
\mathcal{H}_p = \{1\} \\
\mathcal{H}_q = \text{empty if } q \neq p
\]

(20) ([5], p. 302; [6], p. 198) If \( \mathcal{F} \) is locally defined, then \( \mathcal{F} \) is saturated.

(21) ([10]) If \( \mathcal{F} \) is saturated, then \( \mathcal{F} \) is locally defined.

From now on, we shall use the terms saturated and locally defined interchangeably.

**Definition 5.** If \( \mathcal{F} \) is a formation, and \( G \) is a group, then

\[
G_{\mathcal{F}} = \bigcap_{N \in \mathcal{X}} N \text{ where } \mathcal{X} = \{N | N < G, \text{ and } G/N \in \mathcal{F}\}.
\]

Using this definition, a subgroup, \( F \), of \( G \) is an \( \mathcal{F} \)-subgroup of \( G \) if, and only if,

(22) \( F \in \mathcal{F} \), and

(23) if \( F \leq U \leq G \), then \( F \) covers \( U/U_{\mathcal{F}} \).

(24) ([5], p. 301) If the formation, \( \mathcal{F} \), is saturated, then every solvable group \( G \) has an \( \mathcal{F} \)-subgroup. All the \( \mathcal{F} \)-subgroups of \( G \) are conjugate.

(25) ([5], p. 301) Let \( \mathcal{F} \) be a formation, and \( G \) a group. Let \( F \) be an element of \( \mathcal{F} \) such that \( F \leq G \). Then

(a) If \( F \) is an \( \mathcal{F} \)-subgroup of \( G \), and \( F \leq U \leq G \), then \( F \) is an \( \mathcal{F} \)-subgroup of \( U \).
(b) If \( N \triangleleft G \), and \( F \) is an \( \mathcal{H} \)-subgroup of \( G \), then \( FN/N \) is an \( \mathcal{H} \)-subgroup of \( G/N \).

(c) If \( N \triangleleft G \), \( F_1/N \) is an \( \mathcal{H} \)-subgroup of \( G/N \), and \( F \) is an \( \mathcal{H} \)-subgroup of \( F_1 \), then \( F \) is an \( \mathcal{H} \)-subgroup of \( G \).
Chapter III
Preliminary Results

The first theorem of this chapter is a result similar to Clifford's theorem.

**Theorem 1.** Let $H$ be a normal $p'$-subgroup of $G$, $E$ a field of characteristic $p$, and $M$ an indecomposable $E(G)$-module. Then $M|_H$ is a completely reducible $E(H)$-module whose irreducible components form a single class, $C$, of conjugate $E(H)$-modules under conjugation by elements of $G$. Let $L$, $J$ be two $E(G)$-submodules of $M$ such that $L \subseteq J$. Then the distinct $E(H)$-irreducible components of $(J/L)|_H$ are precisely the elements of $C$.

**Proof.** The complete reducibility of $M|_H$ follows directly from the theorem of complete reducibility. Let $M|_H = I_1 \oplus \cdots \oplus I_s$ be a decomposition of $M$ as a direct sum of irreducible $E(H)$-submodules, and for each $k = 1, \cdots, s$, let $M_k$ be the primary component of $M|_H$ associated with $I_k$. Consider the module $I_k g$ for some $g$ in $G$. The action of $H$ on this module is the same as the action of $g$ on the conjugate module $I_k^g$ because of the formula:

$$(i_k g) h = i_k (ghg^{-1}) g.$$ 

Now $I_k g$ is isomorphic to some $I_j$, hence by (13), $I_k g \subseteq M_j$. Clearly $I_j g^{-1}$ is isomorphic to $I_k$, so $I_j g^{-1} \subseteq M_k$. Therefore
\( M_k^g \cong M_k g = M_j \), so \( G \) permutes the primary components of \( M|_H \) by conjugation.

Let \( C_1, \ldots, C_r \) be the conjugacy classes of primary components of \( M|_H \) under the action of \( G \). Set \( T_i = \sum_{M_k \in C_i} \otimes M_k \). Because \( M|_H \) is the direct sum of its primary components, \( M|_H = T_1 \oplus \cdots \oplus T_r \). But each \( T_i \) is a \( \Xi(G) \)-submodule of \( M \), so we contradict the indecomposability of \( M \) unless \( r = 1 \). Let \( C \) be the conjugacy class of irreducible components of \( M|_H \).

Obviously, all of the irreducible \( \Xi(H) \)-submodules of \( (J/L)|_H \) lie in \( C \). Since \( G \) is transitive on the elements of \( C \), and \( J/L \) is a \( \Xi(G) \)-module (non-zero), every element of \( C \) must appear as a component of \( (J/L)|_H \).

**Lemma 1.** Let \( G \) be solvable, and \( M \) be a \( Z_p(G) \)-module. \( M \) is faithful if, and only if, \( M|_{F(G)} \) is faithful.

**Proof.** The lemma follows a fortiori from the statement that if \( 1 < N < G \), then \( 1 < N \cap F(G) \). Now \( 1 < N \), and \( N \) is solvable, so \( 1 < F(N) = N \cap F(G) \), so the lemma holds.

The second theorem in this chapter is the "Frobenius reciprocity theorem" for modules. It is one of the main tools used in this paper. For ordinary representations, the usual reciprocity theorem may be derived from this isomorphism by noting that if \( \mu \), and \( \tau \) are the characters of two \( \Xi(G) \)-modules, \( M \) and \( N \), then the scalar product...
(μ, η) is the \( \Xi \)-dimension of \( \text{Hom} \, \Xi(G)(M, N) \).

**Theorem 2.** Let \( G \) be a group, and \( H \leq G \). Let \( M \) be a \( \Xi(H) \)-module, and \( N \) a \( \Xi(G) \)-module. Then \( \text{Hom} \, \Xi(G)(M|_H^G, N) \) and \( \text{Hom} \, \Xi(H)(M, N|_H) \) are isomorphic as vector spaces over \( \Xi \).

**Proof.** We define a map \( \Theta \), from \( \text{Hom} \, \Xi(G)(M|_H^G, N) \) into \( \text{Hom} \, \Xi(H)(M, N|_H) \) by

\[
\Theta: \eta \mapsto \Lambda \circ \eta|_M \otimes 1 = \Theta(\eta), \quad \text{for } \eta \in \text{Hom} \, \Xi(G)(M|_H^G, N),
\]

where \( \Lambda \) is the canonical map defined by (12). \( \Theta \) is clearly well defined. If \( m \in M \), and \( h \in H \), then

\[
[mh] \Theta(\eta) = [mh \otimes 1] \eta|_M \otimes 1 = [(m \otimes 1)h] \eta|_M \otimes 1 = \{[(m \otimes 1)\eta]|_M \otimes 1\}h = \{[m] \Theta(\eta)\}h.
\]

In this computation, we have used (11) and the fact that \( \eta|_M \otimes 1 \) is an element of \( \text{Hom} \, \Xi(H)(M \otimes 1, N) \). Both \( \Lambda \) and \( \eta|_M \otimes 1 \) are \( \Xi \)-linear maps, so \( \Theta(\eta) \) is a \( \Xi(H) \)-homomorphism from \( M \) into \( N|_H \). Since restriction is a \( \Xi \)-linear operation, it follows that \( \Theta \) is a \( \Xi \)-linear map from \( \text{Hom} \, \Xi(G)(M|_H^G, N) \) into \( \text{Hom} \, \Xi(H)(M, N|_H) \).

Suppose \( \Theta(\eta) = 0 \). Then \( \eta|_M \otimes 1 = 0 \), so using (11) we have

\[
(m \otimes g_\xi) \eta = ((m \otimes 1)g_\xi) \eta = \{(m \otimes 1)\eta\}g_\xi = 0 \cdot g_\xi = 0,
\]

since \( \eta \) is a \( \Xi(G) \)-homomorphism. Therefore \( \Theta \) is a 1-1 map.

Let \( \mu \) be an arbitrary element of \( \text{Hom} \, \Xi(H)(M, N|_H) \). We set
\[
\left( \sum_{\ell=1}^{s} (m_\ell \otimes g_\ell) \right)^* = \sum_{\ell=1}^{s} (m_\ell \mu g_\ell), \quad \text{where } g_1 = 1, g_2, \ldots, g_s \text{ are a set of coset representatives for } H \text{ in } G, \quad \text{and } \left[ G : H \right] = s. \quad \text{Since } N \text{ is a } \Xi(G)\text{-module, } \mu^* \text{ is a well defined map from } M^G \text{ into } N. \text{ If } g \in G, \text{ and } v = \sum_{\ell=1}^{s} m_\ell \otimes g_\ell \text{ lies in } M^G, \text{ then }
\]
\[
(vg)^* = \left( \sum_{\ell=1}^{s} (m_\ell \otimes g_\ell) g \right)^* = \left( \sum_{\ell=1}^{s} m_\ell h_g \otimes g_\gamma(\ell) \right)^* = \sum_{\ell=1}^{s} (m_\ell) h_g g_\gamma(\ell) = (v) \mu^* g,
\]
where \( \gamma \) is the permutation of the cosets \( Hg_1 \) induced by multiplication on the right by \( g \). We have used the fact that \( \mu \) is a \( \Xi(H) \)-homomorphism, as well as equation (11). Because \( \mu \) is \( \Xi \)-linear, \( \mu^* \) is \( \Xi \)-linear, hence \( \mu^* \) lies in \( \text{Hom } \Xi(G)(M^G, N) \).

If \( m \in M \), then
\[
[m] \otimes (\mu^*) = (m \otimes 1) \mu^*|_{M \otimes 1} = (m) \mu 1 = (m) \mu,
\]
therefore \( \Theta(\mu^*) = \mu \). This shows that \( \Theta \) maps \( \text{Hom } \Xi(G)(M^G, N) \) onto \( \text{Hom } \Xi(H)(M, N|_H) \), and completes the proof.

We shall begin a discussion of the properties of \( \mathcal{F} \)-subgroups of solvable groups. It is useful to know the behavior of \( G_{\mathcal{F}} \) (cf. Def. 5) under homomorphisms, so we prove
Lemma 2. Let $\mathcal{F}$ be a formation, $G$ a group, and $H$ a normal subgroup of $G$. Then

$$(G/H)_{\mathcal{F}} = G_{\mathcal{F}}H/H$$

Proof. Let $F$ be the inverse image, in $G$, of $(G/H)_{\mathcal{F}}$. Then $G/F$ is isomorphic to $(G/H)/(G/H)_{\mathcal{F}}$, hence $G/F$ lies in $\mathcal{F}$. Therefore, $G_{\mathcal{F}}H \leq F$.

Since $G_{\mathcal{F}} \leq G_{\mathcal{F}}H$, $G/G_{\mathcal{F}}H$ is an element of $\mathcal{F}$. Furthermore, $G/G_{\mathcal{F}}H$ is isomorphic to $(G/H)/(G_{\mathcal{F}}H/H)$. Therefore $F/H \leq G_{\mathcal{F}}H/H$, so we also have $F \leq G_{\mathcal{F}}H$. This shows that $F = G_{\mathcal{F}}H$, and completes the proof.

The next two theorems give cover-avoidance properties of $\mathcal{F}$-subgroups. They generalize a remark made by Carter in [2].

Theorem 3. Let $\mathcal{F}$ be a formation locally defined by the family $\{\mathcal{F}_p\}$, $G$ be a group, $F$ a subgroup of $G$ which lies in $\mathcal{F}$, and $K$ an $F$-composition factor of $G$. Then

(a) $F$ either covers, or avoids $K$;
(b) if $F$ covers $K$, and $p \mid |K|$, then $F/C_F(K) \in \mathcal{F}_p$;
(c) if $F$ is an $\mathcal{F}$-subgroup of $G$, and $p \mid |K|$, then

$$(C) \frac{F}{C_F(K)} \in \mathcal{F}_p \implies F \text{ covers } K.$$  

Proof. Let $K = L/M$ be the $F$-composition factor in question.

Statement (a) follows from the fact that $F$ acts irreducibly on $K$, and $(L \cap F)M/M$ is an $F$-invariant subgroup of $K$. 

If $F$ covers $K$, then $L/M = (L \cap F)M/M$. If we look at $F$ as a set of operators on $K$, then $K$ is operator isomorphic to $L \cap F/M \cap F$, a $p$-chief factor of $F$. Therefore the kernel of the representation of $F$ on $L \cap F/M \cap F$ is $C_F(K)$. Because $F$ lies in $\mathcal{I}$, it follows that $F/C_F(K) \in \mathcal{J}_p$. This proves (b).

Now suppose $F$ is an $\mathcal{I}$-subgroup of $G$, and $K$ is a $p$-section of $G$ such that $F/C_F(K)$ lies in $\mathcal{J}_p$. To show $F$ covers $K$, it suffices to show that $F$ covers the larger section $FL/M$. But by (25a), $F$ is an $\mathcal{I}$-subgroup of $FL$, hence it is sufficient to show $\overline{F} = FL/M$ is an element of $\mathcal{I}$ since $F$, by definition, covers any such factor of $FL$.

If $q \neq p$, then $K$, as a normal $q'$-subgroup of $F$, is contained in $0_{q'}(F)$. Therefore $0_{q',q'(F)L/M} \leq 0_{q',q'(F)}$, so $\overline{F}/0_{q',q'(F)}$ is isomorphic to a quotient group of $FL/0_{q',q'(F)L}$. $FL/0_{q',q'(F)L}$ is isomorphic to $F/0_{q',q'(F)(F \cap L)}$, a factor of $F/0_{q',q'(F)}$. Since $F \in \mathcal{I}$, $F/0_{q',q'(F)} \in \mathcal{I}_q$, hence $\overline{F}/0_{q',q'(F)}$ is also an element of $\mathcal{I}_q$.

Let $U = F_{\mathcal{I}_q}$. Since $F \in \mathcal{I}$, $F/0_{p',p'(F)} \in \mathcal{J}_p$, hence $U \leq 0_{p',p'(F)}$. Since $F/C_F(K) \in \mathcal{J}_p$, $U \leq C_F(K)$, hence $K$ is contained in the center of $UL/M$. Therefore $UL/M$ has a normal $p$-complement. By (18'), $UL/M$ is contained in $0_{p',p'(F)}$, thus $\overline{F}/0_{p',p'(F)}$ is a factor group of $FL/UL \cong F/U(F \cap L)$. Therefore, $\overline{F}/0_{p',p'(F)} \in \mathcal{J}_p$. Because $\mathcal{J}$ is locally defined, $F$ lies in $\mathcal{I}$.
Our next theorem will show that property (C) characterizes the \( \mathcal{F} \)-subgroups of a solvable group \( G \). In order to obtain as weak an hypothesis as possible, we prove two lemmas. Property (C) actually applies only to specific \( F \)-composition factors of \( G \), so when we say that property (C) holds for an \( F \)-composition series, \( G = G_0 > G_1 > \cdots > G_n = 1 \), of \( G \), we mean that \( F \) satisfies this property for all factors, \( G_i / G_{i+1} \), of the series for which the hypothesis of property (C) holds.

**Lemma 3.** Suppose \( \mathcal{F} \) is a formation locally defined by \( \{ \mathcal{I}_p \} \), \( F \in \mathcal{F} \), and \( F \leq G \). Let \( A/B \) be an \( F \)-invariant section of \( G \) such that \( A > C > B \) defines a fixed \( F \)-composition series of \( A/B \). If property (C) holds for this series, then property (C) holds for every \( F \)-composition series of \( A/B \).

**Proof.** We may assume that a second \( F \)-composition series of \( A/B \) exists and is defined by \( A > D > B \) where \( D \neq C \). Then we must have \( CD > C > C \cap D \), so the fact that \( C \) and \( D \) are \( F \)-invariant shows that \( A = CD \), and \( B = C \cap D \). Therefore

\[
A/B \cong C/B \times D/B, \quad A/C \cong D/B, \quad A/D \cong C/B,
\]

where the decomposition is an operator decomposition, and the isomorphisms are operator isomorphisms.

Suppose the decomposition (26) is unique. If \( F/C_p(A/D) \) lies in \( \mathcal{I}_p \), it follows from (26) that \( F/C_p(C/B) \) lies in \( \mathcal{I}_p \). Since property
(C) holds for the series $A > C > B$, $F$ covers $C/B$. Therefore 
\[(F \cap A)D \supset (F \cap C)D = CD = A, \] so $F$ covers $A/D$. Suppose $F/C_F(D/B)$ lies in $\mathcal{J}_q$, then $F/C_F(A/C)$ lies in $\mathcal{J}_q$. By property (C), $F$ covers $A/C$. Because of the uniqueness of the decomposition, and the fact that $F \cap A$ is not contained in $C$, either $A = (F \cap A)B$, or $D = (F \cap A)B$. In the former case, $F$ covers all of $A/B$, hence $F$ covers the subgroup $D/B$. In the latter case, $F \cap A = F \cap D$ since $F \cap A \leq D$, so $D = (F \cap D)B$. Therefore, in either case, $F$ covers $D/B$.

The decomposition (26) is unique if $(\pi(A/C), |C/B|) = 1$, so we may assume $A/B$ is an elementary abelian $p$-group for some prime $p$. This means that we can look at $A/B$ as a $\mathbb{Z}_p(F)$-module. If $A/C$ and $C/B$ are distinct $\mathbb{Z}_p(F)$-modules, then it follows from (13) that the decomposition is unique. Thus we may assume that $D/B \cong \mathbb{Z}_p(F)/C/B$.

In this case, it follows from property (C) for $A > C > B$ that $F$ either covers, or avoids $A/B$. Therefore Lemma 3 holds in all cases.

**Lemma 4.** Assume $F \in \mathcal{J}$, $H \leq G$, and $F \leq N_G(H)$. If property (C) holds for a fixed $F$-composition series of $H$, then it holds for every $F$-composition series of $H$.

**Proof.** Let $H = H_0 > H_1 > \cdots > H_n = 1$ be the fixed $F$-composition series of $H$ for which property (C) holds. We use induction on $n$. The lemma is true if $n = 1$.

Let $H = K_0 > K_1 > \cdots > K_n = 1$ be a second $F$-composition series for $H$. If $K_1 = H_1$, property (C) holds for the series
$K_0 > K_1 > \cdots > K_n = 1$ by induction.

Suppose $K_1 \neq H_1$, and let $i$ be the smallest integer such that $K_i \cap H_i = H_i$. Because $H_i \leq K_i \cap H_{i-1} < H_{i-1}$, we have $H_i = K_i \cap H_{i-1}$, so that we have the following lattice diagram:

Since $H_1$ is part of an $F$-composition series for $H$, $F \leq N_G(H_1)$. Because of the isomorphisms indicated in the diagram,

$H_1 > K_1 \cap H_1 > \cdots > K_1 \cap H_{i-1} = H_i > H_{i+1} > \cdots > H_n = 1$ is an $F$-composition series for $H_1$ which has length $n-1$. By induction, property (C) holds for this series. Therefore, property (C) holds for the $F$-composition series of $H/H_1 \cap K_1$ defined by the series $H > H_1 > H_1 \cap K_1$. By Lemma 3, property (C) holds for the $F$-composition series $H > K_1 > K_1 \cap H_1 > \cdots > K_1 \cap H_{i-1} = H_i > \cdots > H_n = 1$ of $H$. In particular, property (C) holds, by induction, for any $F$-composition series of $K_1$. Therefore property (C) holds for the series $K_0 > K_1 > \cdots > K_n = 1$. 
Theorem 4. Let $\mathcal{F}$ be a formation locally defined by $\{\mathcal{F}_p\}$. Let $G$ be a group, $F \leq G$, and $F \in \mathcal{F}$. If property (C) holds for a fixed $F$-composition series $G = G_0 > G_1 > \cdots > G_n = 1$ of $G$, then $F$ is an $\mathcal{F}$-subgroup of $G$.

Proof. We use induction on $|G|$. By Lemma 4, we may assume that the series $G = G_0 > G_1 > \cdots > G_n = 1$ is a refinement of the chief series $G = H_0 > H_1 > \cdots > H_{m-1} > H_m = 1$. Then $H_{m-1} = G_k$ for some $k$. $H_{m-1}$ is a minimal normal subgroup of $G$, so we set $\overline{G}_i = G_i/G_k$ for $i = 0, 1, \cdots, k$, $\overline{F} = FG_k/G_k$, and $\overline{G} = \overline{G}_0$. Our first step is to show that $\overline{F}$ is an $\mathcal{F}$-subgroup of $\overline{G}$.

If $m = 1$, then $\overline{G} = \overline{F} = \overline{I}$, so the result is trivial. If $m > 1$, then $H_{m-1} < G$, and $\overline{G} = \overline{G}_0 > \overline{G}_1 > \cdots > \overline{G}_k = \overline{I}$ is an $\overline{F}$-composition series of $\overline{G}$. By induction, to show that $\overline{F}$ is an $\mathcal{F}$-subgroup of $\overline{G}$, it is sufficient to verify property (C) for the series $\overline{G}_0 > \overline{G}_1 > \cdots > \overline{G}_k = \overline{I}$.

For each $i$, set $K_i = G_i/G_{i+1}$, and $\overline{K}_i = \overline{G}_i/\overline{G}_{i+1}$. Since $G_k \leq G_{i+1}$ for $i < k$, $G_k$ centralizes the section $K_i$ for $i < k$. Therefore, $C_F(\overline{K}_i) = C_F(K_i)G_k/G_k$ for all $i < k$. Thus,

$$\overline{F}/C_F(\overline{K}_i) \cong FG_k/C_F(K_i)G_k \cong F/C_F(K_i)(F \cap G_k).$$

But $F \cap G_k \leq C_F(K_i)$, so we have

$$\overline{F}/C_F(\overline{K}_i) \cong F/C_F(K_i) \quad \text{for } i < k.$$
Suppose $K_1$ is a $p$-section of $G$ such that $F/\mathcal{C}_F(K_1) \in \mathcal{J}_p$. By (27), $F/\mathcal{C}_F(K_1) \in \mathcal{J}_p$, so $F$ covers $K_1$. Therefore,
\[(FG_k \cap G_1)G_{i+1} = (F \cap G_1)G_{i+1} = (F \cap G_1)G_{i+1} = G_i.\]
By taking homomorphic images, and noting that $FG_k \cap G/G_k = F \cap G_k$, we get
\[(F \cap G_k)G_{i+1} = G_i.\]
Thus $F$ covers $K_1$. Therefore property (C) holds for the $F$-composition series $G = G_0 > \cdots > G_k = 1$ of $G$.

Now that we know $F$ is an $\mathcal{J}$-subgroup of $G$, it follows from (25a) that we can complete our proof by showing that $F$ is an $\mathcal{J}$-subgroup of $FG_k$.

Suppose $FG_k < G$. We consider the series $FG_k = D_0 \geq D_1 \geq \cdots \geq D_n = 1$, where $D_i = FG_k \cap G_i$ for each $i$. Suppose $D_i > D_{i+1}$ for some $i$. Then
\[D_i/D_{i+1} \cong (FG_k \cap G_i)G_{i+1}/G_{i+1} > 1\]
This is an operator isomorphism, hence because $F$ is irreducible on $K_1$, we have
\[(28) \quad D_i/D_{i+1} \cong G_i/G_{i+1}\]
Therefore the distinct terms of the series, $D_0 \geq D_1 \geq \cdots \geq D_n = 1$, form an $F$-composition series for $FG_k$ which passes through $G_k$. Since $F$ covers $FG_k/G_k$, and since $D_i = G_i$ for $i \geq k$, property (C) holds for this composition series. By induction, $F$ is an $\mathcal{J}$-subgroup of $FG_k$.

Suppose $G = FG_k$. Since $G_k$ is a minimal normal subgroup of $G$, $F$ acts irreducibly on $G_k$. Therefore $F$ either covers, or avoids $G_k$. 

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If $F$ covers $G_k$, then $F = G$, so $F$ is an $\mathcal{F}$-subgroup of $FG_k$. Suppose $F$ avoids $G_k$, and $G_k (= G_{n-1})$ is a $p$-group. Then $FG_k/C_{FG_k}(G_k) \simeq F/C_F(G_k)$ cannot lie in $\mathcal{F}_p$ since property (C) holds for $G_k$. Therefore $(FG_k)_p \geq G_k$. Since $FG_k/G_k$ lies in $\mathcal{F}$, $(FG_k)_p = G_k$.

If $F \leq U \leq FG_k$, then $U = F$, or $U = FG_k$. The above remarks show that $F$ covers $U/U_\mathfrak{q}$ in both cases. Therefore $F$ is an $\mathcal{F}$-subgroup of $FG_k$, and the proof is complete.

As one application of Theorem 4, we prove

**Theorem 5.** Let $\mathcal{F}$ be a formation locally defined by $\{\mathcal{F}_p\}$, $H \in \mathcal{F}$, and let $I$ be a finitely generated $\mathbb{Z}_p(H)$-module. Let $G = HI$ be the semi-direct product of $I$ by $H$ where the action of $H$ on $I$ by conjugation is the canonical action given in Chapter II. Then,

(a) $F = HC_\mathcal{F} \bigl(0_p, (H_\mathfrak{q}) \bigr)$ is an $\mathcal{F}$-subgroup of $G$.

(b) As a $\mathbb{Z}_p(H)$-module, $I = C_\mathcal{F} \bigl(0_p, (H_\mathfrak{q}) \bigr) \oplus G$.

**Proof.** Let $W = C_\mathcal{F} \bigl(0_p, (H_\mathfrak{q}) \bigr)$. Our first task is to show $HW$ lies in $\mathcal{F}$. Suppose $q \neq p$, then $W$ is a $q'$-group, so $0_{q', q}(F) = 0_{q', q}(H)W$. Therefore,

$$F/0_{q', q}(F) \simeq H/0_{q', q}(H).$$

Since $H$ lies in $\mathcal{F}$, $F/0_{q', q}(F) \in \mathcal{F}_{q'}$.

Let $U = H_\mathfrak{q}$. Then $0_p, (U)$ centralizes $W$. Since $H/0_{p', p}(H)$ lies in $\mathcal{F}_p$, $U \leq 0_{p', p}(U)$. Therefore $0_{p', p}(UW) = UW$. $UW$ is normal in $F$, so by (18'), $UW \leq 0_{p', p}(F)$. Therefore $F/0_{p', p}(F)$ is isomorphic to
a quotient group of $H/U$. Since $H/U \in \mathcal{F}_p$, $F/O_p, p(F) \in \mathcal{F}_p$. Therefore, $F$ lies in $\mathcal{F}$.

Now let $G = G_0 > G_1 > \cdots > G_n = 1$ be an $F$-composition series for $G$ such that $G_\ell = I$ for some $\ell$. In order to check property (C), we need only consider $K_i = G_i/G_{i+1}$ for $i \geq \ell$, since $F$ covers $G/I$. $W$ centralizes every $K_i$, so we have

$$F/C_p(K_i) \cong H/C_H(K_i).$$

If $i \geq \ell$, and $F/C_p(K_i) \in \mathcal{F}_p$, then $C(H)(K_i) \geq U$. In particular, $0_p(U)$ centralizes $K_i$. $I$ is a $p$-group, so by the covering lemma, $W$ covers $K_i$. Therefore $F$ covers $K_i$, so property (C) holds for the series $G_0 > G_1 > \cdots > G_n = 1$. By Theorem 4, $F$ is an $\mathcal{F}$-subgroup of $G$.

We shall prove the next assertion by induction on the length of a $Z_p(H)$-composition series of $I$. Since $H = G/I$, $G/H \subseteq I$. Since $F$ covers $G/\mathcal{F}$, it follows that, as a $Z_p(H)$-module, $I = W + G/\mathcal{F}$, so the major difficulty is to show that the sum is direct. If $I$ is irreducible, one of the two summands must be the zero subspace, so the theorem is true if $n = 1$.

Suppose $I = I_0 \supset I_1 \supset \cdots \supset I_n = (0)$ is a $Z_p(H)$-composition series for $I$, and $n > 1$. $I_{n-1}$ is a minimal normal subgroup of $G$, and $H/\mathcal{n-1}/I_{n-1}$ acts on the quotient module $\overline{I} = I/I_{n-1}$ in the same way that $H$ acts on $\overline{I}$. Therefore $\overline{G} = G/I_{n-1}$ is isomorphic to the
semi-direct product $H \overline{\times} H$ where the action of $H$ by conjugation is the module action. By induction,

\[(29) \quad \overline{I} = C_H(0_p, (U)) \oplus (\overline{G})_{\overline{f}}.\]

\textbf{CASE I.} $(\overline{G}) \subseteq C_H(0_p, (U)) \subseteq \overline{I}$.

Set $T/I_{n-1} = (\overline{G})_{\overline{f}}$. By the covering lemma, (17),

$C_T(0_p, (U)) = WI_{n-1}/I_{n-1}$. By (29), $T \cap W$ is a normal subgroup of $G$ contained in $I_{n-1}$.

If $T \cap W = (0)$, then $I = T \oplus W$, and reverting to multiplicative notation, we have $F \cap T = 1$. Therefore, $T$ is clearly the smallest normal subgroup of $G$ whose factor group is covered by $F$. Since $F$ is an $\overline{f}$-subgroup of $G$, it follows that $T = G_{\overline{f}}$.

Suppose $T \cap W = I_{n-1}$. The hypothesis of CASE I states that $T \subseteq I$, so the theorem is true, by induction, for the semi-direct product $G_1 = HT$. Therefore, $T = C_T(0_p, (U)) \oplus T_1$, where $T_1 = (G_1)_{\overline{f}}$.

Now $I_{n-1} = C_T(0_p, (U))$, and because $I$ is abelian, $T_1 \triangleleft G$. Therefore, as a $Z_p(H)$-module,

$I = W \oplus T_1$.

As before, this forces $T_1 = G_{\overline{f}}$ since $G/T_1 \cong F$ is the largest possible factor of $G$ covered by $F$. Therefore, the theorem holds in CASE I.

\textbf{CASE II.} $(\overline{G}) = C_H(0_p, (U))$.

By assumption, $W \subseteq I_{n-1}$. $0_p, (U)$ is normal in $H$, so $W$ is normal
in G. If \( W = (0) \), then \( H \) is an \( \mathfrak{p} \)-subgroup of \( G \), and \( I = G_{\mathfrak{p}} \). We may, therefore, assume that \( W = I_{n-1} \).

Let \( \hat{I} \) be the dual of \( I \). By (8)

\[
\hat{I}_{n-1} \cong I/I_{n-1}, \text{ and } I_{n-1} \cong (I/I_{n-1})
\]

Since \( O_{p^*}(U) \) centralizes \( I_{n-1} \), it follows from (9), that \( O_{p^*}(U) \) centralizes \( \hat{I}_{n-1} \). By the covering lemma, (17), \( C_{\bar{I}}(O_{p^*}(U)) \) covers \( \hat{I}/I_{n-1} \).

Now \( O_{p^*}(U) \triangleleft H \), so \( C_{\bar{I}}(O_{p^*}(U)) \) is a \( Z_{p^*}(H) \)-submodule of \( I_{n-1} \). By (8),

\[
I/[C_{\bar{I}}(O_{p^*}(U))]_{n-1} \cong [C_{\bar{I}}(O_{p^*}(U))].
\]

Since \( O_{p^*}(U) \) centralizes \( C_{\bar{I}}(O_{p^*}(U)) \), it follows from (9), that \( O_{p^*}(U) \) centralizes \( I/[C_{\bar{I}}(O_{p^*}(U))]_{n-1} \). Thus using the covering lemma again, we know that \( W \) covers the factor \( I/[C_{\bar{I}}(O_{p^*}(U))]_{n-1} \).

But \( C_{\bar{I}}(O_{p^*}(U)) \subseteq I_{n-1} \), so \( W = I_{n-1} \subseteq [C_{\bar{I}}(O_{p^*}(U))]_{n-1} \), hence we must have

\[
I = [C_{\bar{I}}(O_{p^*}(U))].
\]

Therefore \( C_{\bar{I}}(O_{p^*}(U)) = (0) \), so \( C_{\bar{I}}(O_{p^*}(U)) \) avoids \( I_{n-1} \). These statements show that
\[ \hat{I} = C_{I}(0_{p},(U)) \oplus I_{n-1}. \]

But by (10), this splitting induces a splitting in \( I \),
\[ I = [C_{I}(0_{p},(U))]^{\perp} \oplus I_{n-1}. \]

As before, it follows from the fact that \( F = H_{n-1} \) is an \( I \)-subgroup of \( G \), that \( G^{I} = [C_{I}(0_{p},(U))]^{\perp} \). This completes the proof in all cases.

Remark. This result cannot be extended to the case where \( I \) is a \( p \)-group of class 2 because of the following example. Let \( I \) be the quaternion group. \( I \) has an automorphism, \( h \), of order 3 such that \( h \) acts fixed-point free on \( I/D(I) \), and centralizes \( D(I) \). Let \( H \) be the cyclic group of order 3 generated by \( h \), and let \( G = H \cdot I \). A Carter subgroup of \( G \) is \( H \times D(I) \), but \( D(I) \) has no complement in \( I \), so no splitting is possible.
Chapter IV
Strong Containment

In this chapter, we shall characterize strong containment. In certain cases, we can make our characterization more precise by giving generating sets for certain of the formations $\mathcal{F}_p$ in the minimal local definition of $\mathcal{F}$. The results of this chapter form the basis for our results in Chapter V.

Lemma 5. Let $\mathcal{E}$ and $\mathcal{F}$ be two non-empty saturated formations; let $\mathcal{E}$ be locally defined by $[\mathcal{E}_p]$. Let $G$ be a group of minimal order satisfying

$$(P) \quad \text{An } \mathcal{E}\text{-subgroup of } G \text{ is not contained in any } \mathcal{F}\text{-subgroup of } G.$$ 

If $F$ is an $\mathcal{F}$-subgroup of $G$, and $E$ is an $\mathcal{E}$-subgroup of $F$, then

(a) $G/F = M$ is a minimal normal subgroup of $G$; $G$ is the semi-direct product of $M$ by $F$; $F$ acts faithfully and irreducibly on $M$.

(b) If $M$ is a $p$-group, then $E^* = E_C(0_p)(E_{p'})$ is an $\mathcal{E}$-subgroup of $G$, and $1 < C_M(0_p)(E_{p'}) \leq M$.

Proof. If $G \in \mathcal{F}$, then $G = F$ contains every $\mathcal{E}$-subgroup of $G$, hence $G$ does not satisfy $(P)$. Therefore $G \not\in \mathcal{F}$; in particular, $G$ is not the identity. Let $M \neq 1$ be a minimal normal subgroup of $G$. By (25b), $FM/M$ is an $\mathcal{F}$-subgroup of $G/M$. Because of the minimality of $|G|$ with respect to the property $(P)$, some $\mathcal{E}$-subgroup of $G/M$ is
contained in $FM/M$. Since all $\ell$-subgroups of $G/M$ are conjugate, we can find an $\ell$-subgroup, $E$, of $G$ such that $EM \leq FM$. $E$, as an $\ell$-subgroup of $G$, is also an $\ell$-subgroup of $FM$. Because $G$ satisfies (P), no conjugate of $E$ under $FM$ can be contained in $F$. Therefore $FM \leq G$, and satisfies (P), so $G = FM$.

$G/M$ lies in $\mathcal{F}$, but $G$ does not lie in $\mathcal{F}$, so $G \not\in \mathcal{F}$. Since $F \cap M$ is a normal subgroup of $G$, properly contained in $M$, $F \cap M = 1$, so $G$ is the semi-direct product of $M$ by $F$. Since $M$ was arbitrary to begin with, and we showed $M = G \not\in \mathcal{F}$, $M$ is the unique minimal normal subgroup of $G$. Therefore $F$ acts faithfully and irreducibly on $M$. This proves (a).

$G/M$ is isomorphic to $F$, so $EM/M$ is an $\ell$-subgroup of $G/M$. By (25c), an $\ell$-subgroup of $EM$ is also an $\ell$-subgroup of $G$. By Theorem 5, $E^* = EM(0_p(E_p^{E_p}))$ is an $\ell$-subgroup of $EM$. Since $E^*$ is not contained in $F$, $1 < CM(0_p(E_p^{E_p})) \leq M$.

Before stating the characterization, we introduce some notation.

**Definition 6.** If $\ell$ and $\mathcal{F}$ are two saturated formations, and $\ell$ is locally defined by $\{\ell_p\}$, set

(a) $\pi(\ell) = \{p | \ell_p \text{ is non-empty}\}$. $\pi(\ell)$ is called the characteristic of $\ell$.

(b) If $p \in \pi(\ell)$, set
\[ t_p = \begin{cases} 
H \in T & \text{H has a faithful irreducible } \mathbb{Z}_p(H)-\text{module, } M, \text{ such that if } E \text{ is an } \ell\text{-subgroup of } H, \text{ then} \\
1 < C_M(0_p, (E_p^\ell)) \leq M. 
\end{cases} \]

(c) If \( p \in \pi(\ell) \), set

\[ \theta_p = \begin{cases} 
H \in \hat{\mathcal{F}} & \text{There is at least one faithful irreducible } \mathbb{Z}_p(H)-\text{module, } M, \text{ such that } 1 < C_M(0_p, (E_p^\ell)) < M. 
\end{cases} \]

**Theorem 6.** Suppose \( \ell \) and \( \mathcal{F} \) are two saturated formations locally defined by \( \{ \ell_p \} \) and \( \{ \mathcal{F}_p \} \) respectively. Then

\[ \ell \ll \mathcal{F} \iff \theta_p \subseteq \mathcal{F}_p \text{ for each } p \in \pi(\ell). \]

**Proof.** Suppose \( \theta_p \subseteq \mathcal{F}_p \) for each \( p \in \pi(\ell) \), and \( \ell \) is not strongly contained in \( \mathcal{F} \). Then the class of groups satisfying property (P) is non-empty, so we choose \( G \) to be an element of minimal order in this class. By Lemma 5, if \( G_{\mathcal{F}_q} \) is a p-group, then \( p \) divides the order of an \( \ell\)-subgroup of \( G \), hence \( p \in \pi(\ell) \). By Lemma 5, if \( F \) is an \( \mathcal{F}_p \)-subgroup of \( G \), then \( F \) lies in \( \theta_p \). Therefore, \( F \) is an element of \( \mathcal{F}_p \).

Since \( G_{\mathcal{F}_q} \) is the unique minimal normal subgroup of \( G \),

\[ G_{\mathcal{F}_q} = 0_p,_{p'}(G). \]

Therefore \( F \cong G/G_{\mathcal{F}_q} = G/0_p,_{p'}(G) \) lies in \( \mathcal{F}_p \). If \( q \neq p \), then \( G_{\mathcal{F}_q} \leq 0_{q',p}(G) \), so \( 0_{q',p}(G) = G/0_{q',p}(F) \). Therefore,

\[ G/0_{q',p}(G) \cong F/0_{q',p}(F) \]

Since \( F \in \mathcal{F} \), it follows that \( G/0_{q',p}(G) \in \mathcal{F}_q \). By (a'), \( G \) lies in
a contradiction to the fact that \( G_{\ell} > 1 \). Therefore \( \ell \ll \mathcal{I} \).

Suppose \( \ell \ll \mathcal{I} \), \( p \in \pi(\mathcal{E}) \), and \( F \in \mathcal{I}_p \). Let \( M \) be the faithful irreducible \( \mathbb{Z}_p(\mathbb{F}) \)-module mentioned in the definition of \( \mathcal{I}_p \). Set \( G = FM \), where the action of \( \mathbb{F} \) on \( M \) by conjugation is the module action. By Theorem 5, an \( \mathcal{I} \)-subgroup of \( G \) is \( F^* = FC_M(0_p, (F_{\mathcal{I}_p})) \), hence \( G = F^*M \). Let \( E \) be an \( \ell \)-subgroup of \( F \). Since \( EM/M \) is an \( \ell \)-subgroup of \( G/M \), it follows from (25c) and Theorem 5, that \( E^* = EC_M(0_p, (E_p)) \) is an \( \ell \)-subgroup of \( G \). \( E^* \) does not avoid \( M \), and because \( \ell \ll \mathcal{I} \), \( E^* \) is contained in some \( \mathcal{I} \)-subgroup of \( G \), hence \( F^* \) does not avoid \( M \). Since \( 0_p, (F_{\mathcal{I}_p}) \subset F \), \( C_p(0_p, (F_{\mathcal{I}_p})) \) is normal in \( G \), hence \( F^* = G \).

Since \( G \) is an element of \( \mathcal{I} \), and \( F \) acts faithfully on the \( p \)-chief factor, \( M \), of \( G \), we have \( F \cong G/C_p(M) \in \mathcal{I}_p \). Therefore \( \mathcal{I}_p \subseteq \mathcal{I}_p \).

Because of this characterization, if \( \ell \ll \mathcal{I} \), and \( p \in \pi(\mathcal{E}) \), then \( \mathcal{I}_p \subseteq \mathcal{I}_p \) for any \( \mathcal{I}_p \) which lies in some local definition. This leads naturally to the question: Suppose \( \{ \mathcal{I}_p \} \) is the unique minimal local definition for \( \mathcal{I} \). If \( p \in \pi(\mathcal{E}) \), is \( \mathcal{I}_p \) the smallest formation generated by the set \( \mathcal{I}_p \)? The answer to this question is yes, provided \( \mathcal{I}_p \) is non-empty for at least two primes. We have not been able to relax the hypothesis on the \( \mathcal{I}_p \)'s. In the next few lemmas, we shall investigate properties of the \( \mathcal{I}_p \)'s and \( \mathcal{I}_p \)'s as a prelude to giving a partial answer to this question.
Lemma 6. Let $\mathcal{E}$ and $\mathcal{F}$ be non-empty saturated formations with local definitions $\{\ell_r\}$ and $\{\ell_p\}$ respectively. Suppose $\mathcal{E} \ll \mathcal{F}$, and $G$ is an element of $\mathcal{F}$ with $\ell$-subgroup, $E$.

(a) Suppose $G \in \ell_q$. $G$ lies in $\theta_q$ if, and only if, $0_q(\ell_q E_q) > 1$.

(b) If $q \in \pi(\ell)$, $0_q(G) = 1$, and the permutation representation, $1_{0_q(\ell_q E_q)}^G$, is faithful, then $G$ lies in $\langle \ell_q \rangle$, the smallest formation generated by the set $\ell_q$.

(c) Let $M$ be a faithful irreducible $Z_q(G)$-module. If $G = HM$, and $1_{0_q(\ell_q E_q)}^G$ is faithful for some $q$ in $\pi(\ell) - \{p\}$, then $G \in \ell_q$.

(d) For each $r, s$ in $\pi(\ell)$, $\theta_r \subseteq \langle \ell_s \rangle$.

Proof. Suppose $G \in \ell_q$. Then $G$ has a faithful irreducible $Z_q(G)$-module, $I$, such that $1 < C_I(0_q(\ell_q E_q)) \leq I$. If $J$ is any such module, $J = C_J(0_q(\ell_q E_q))$ if, and only if, $0_q(\ell_q E_q) = 1$ since $J$ is faithful.

This proves (a).

Let $G$ be a group satisfying the hypothesis of (b). Let $T$ be the $Z_q(G)$-module which gives the representation $1_{0_q(\ell_q E_q)}^G$. By (15), we can write $T$ as a direct sum of principal indecomposable $Z_q(G)$-modules,

$$T = T_1 \oplus \cdots \oplus T_s.$$

For each $k = 1, \cdots, s$, it follows from (16) that $T_k$ has a unique maximal proper $Z_q(G)$-submodule, $\sqrt{T_k}$. We set $J_k = T_k/\sqrt{T_k}$. 
Since \( 0_q(G) = 1 \), \( F(G) \) is a normal \( q' \)-subgroup of \( G \), hence by Theorem 1, the distinct irreducible components of \( J_k F(G) \) are exactly the same as the distinct irreducible components of \( T_k F(G) \). Therefore, since \( T \) is faithful, it follows that if \( J = J_1 \oplus \cdots \oplus J_s \), then \( J F(G) \) is faithful. By Lemma 1, \( J \) is a faithful \( Z_q(G) \)-module.

We now apply the Frobenius reciprocity theorem for modules. For each \( k = 1, \ldots, s \),

\[
(0) \subseteq \text{Hom}_{Z_q(G)}(T, J_k) \cong \text{Hom}_{Z_q(G)}(O_q(\mathfrak{e}_q), (1 \epsilon_k 1_q) J_k F(G) \epsilon_q) = \text{Hom}_{Z_q(G)}(O_q(\mathfrak{e}_q)) (1 \epsilon_k 1_q) \epsilon_q
\]

Therefore \( 1 < C_{J_k} (O_q(\mathfrak{e}_q)) \leq J_k \) for each \( k \).

Set \( G_k = G/C_{G}(J_k) \). Then \( E_k = E_{G}(J_k)/C_{G}(J_k) \) is an \( \mathfrak{e} \)-subgroup of \( G_k \). By Lemma 2, \( (E_k) \mathfrak{e}_q = E_{G}(J_k)/C_{G}(J_k) \). Therefore \( O_q((E_k) \mathfrak{e}_q) = O_q((E_k) \mathfrak{e}_q) C_{G}(J_k)/C_{G}(J_k) \), so

\[
1 < C_{J_k} (O_q((E_k) \mathfrak{e}_q)) = C_{J_k} (O_q((E_k) \mathfrak{e}_q)) \leq J_k.
\]

Since \( G \) lies in \( \mathfrak{e}_q \), \( q \in \pi(\mathfrak{e}) \), and \( J_k \) is a faithful irreducible \( Z_q(G_k) \)-module, \( G_k \) lies in \( \mathfrak{s}_q \) for each \( k \). Therefore \( G = G/\cap_{k} C_{G}(J_k) \) lies in \( \langle \mathfrak{s}_q \rangle \), the smallest formation generated by the set \( \mathfrak{s}_q \). This proves (b).

The proof of (c) is essentially the same as the proof of (b).

Let \( G = \text{HM} \) be the group mentioned in the hypothesis of (c). Let \( T \) be the \( Z_q(G) \)-module which gives the faithful representation \( 1_{O_q((E_k) \mathfrak{e}_q)} |^G \).
Once again, \( T \) has a decomposition into a direct sum, \( T = T_1 \oplus \cdots \oplus T_s' \), of principal indecomposable \( Z_q(G) \)-modules. Because of faithfulness it follows that \( M \) is non-trivial on some \( T_k' \), say \( T_1' \). Since \( M \) is minimal normal in \( G \), \( M \) must act faithfully on \( T_1' \). Since \( M \) is a \( q' \)-group, it follows from Theorem 1, that if \( J = T_1'/\sqrt{T_1} \), then \( J|_M \) is faithful. But \( M \) is the unique minimal normal subgroup of \( G \), and by (14), \( 0_p(H) = 1 \), so \( F(G) = M \). By Lemma 1, \( J \) is a faithful irreducible \( Z_q(G) \)-module. By Frobenius reciprocity,

\[
i < C_j(O_{q'}, (E_{q'})) \leq J.
\]

Therefore, the fact that \( q \in \pi(\ell) \) says that \( G \) is, by definition, an element of \( \hat{q} \). This proves (c).

Part (d) is the only statement in Lemma 6 which requires the assumption \( \ell \ll \mathcal{I} \). Suppose \( H \in \Theta \), \( E \) is an \( \ell \)-subgroup of \( H \), and \( M \) is a faithful irreducible \( Z_r(H) \)-module such that \( 1 < C_M(O_{r}, (E_{r})) < M \).

Set \( G = HM \). By Theorem 5, \( F = HC_M(O_{r}, (H_{r})) \) is an \( \mathcal{I} \)-subgroup of \( G \), and since \( E = EM/M \) is an \( \ell \)-subgroup of \( G/M \), \( E^* = EC_M(O_{r}, (E_{r})) \) is an \( \ell \)-subgroup of \( G \). Since \( \ell \ll \mathcal{I} \), \( F \) cannot avoid \( M \), hence \( F = G \) is an element of \( \mathcal{I} \).

Let \( N \) be the intersection of all the conjugates of \( O_{s'}, (E^*_{\ell}) \) in \( G \). Then \( N \triangleleft G \), and \( N \cap M = E^* \cap M = C_M(O_{r}, (E_{r})) < M \). Therefore \( N \cap M = 1 \). This shows that \( 1_{O_{s'}, (E^*_{\ell})}|^G_M \) is faithful. Because
\[ M = F(G) \text{, it follows from Lemma 1 that } 1_{s}^{0}(E_{s}^{*})_{G}^{G} \text{ is faithful.} \]

By part (c), \( G \in \mathcal{S}_{s} \) for \( s \in \pi(\mathcal{L}) - \{r\} \). Therefore \( H \cong G/M \) is an element of \( \langle \mathcal{S}_{s} \rangle \), for \( s \in \pi(\mathcal{L}) - \{r\} \). Since \( \mathcal{S}_{s} \leq \mathcal{S}_{r} \), it follows that \( \mathcal{S}_{r} \leq \langle \mathcal{S}_{s} \rangle \) for each \( s \) in \( \pi(\mathcal{L}) \). This proves (d).

The next lemma has an elegant proof. This proof was told to me by Dr. E. C. Dade, and it shortens this part of the original proof of the next theorem considerably.

**Lemma 7.** Let \( A, B \) be two groups and assume \( Z(A) = 1 \). If \( M \) is a faithful \( Z_{p}(A) \)-module, and \( T \) is a faithful \( Z_{p}(B) \)-module, then \( M \otimes T \) is a faithful \( Z_{p}(A \times B) \)-module.

**Proof.** If \( V \) is a vector space over \( Z_{p} \), we let \( GL(V) \) denote the general linear group on \( V \). Then \( A \times B \leq GL(M) \times GL(T) = C \), so we examine the kernel, \( K \), of the representation of \( C \) on \( M \otimes T \). Let \( m_{1}, \ldots, m_{r} \) be a \( Z_{p} \)-basis for \( M \), and \( t_{1}, \ldots, t_{s} \) be a \( Z_{p} \)-basis for \( T \). Then \( \{ m_{i} \otimes t_{j} | 1 \leq i \leq r, 1 \leq j \leq s \} \) is a \( Z_{p} \)-basis for \( M \otimes T \). Suppose \( f \times g \in K \), and

\[
m_{i}f = \sum \phi_{ik} m_{k}, \quad \text{for each } i
\]

\[
t_{j}g = \sum \gamma_{jk} t_{k}, \quad \text{for each } j
\]

Then \( m_{i} \otimes t_{j} = m_{i} \otimes t_{j}(f \times g) = \left( \sum \phi_{ik} m_{k} \right) \otimes \left( \sum \gamma_{jk} t_{k} \right) \). So
m_i \otimes t_j = \sum_{k,l} \varphi_{ik}^j \lambda_{kl} m_k \otimes t_l; \text{ hence we have}

\begin{align*}
\varphi_{ik}^j & = 0 \text{ if } (i,j) \neq (k,l) \\
\varphi_{ik}^j & = 1 \text{ if } (i,j) = (k,l)
\end{align*}

Therefore \( m_i f = \varphi_m m_i \) for each \( i \), and \( t_j g = \varphi^{-1} t_j \) for each \( j \).

Suppose \( a \times b \in (A \times B) \cap K \). Since \( Z(A) = 1 \), and our results show that \( K \leq Z(GL(M)) \times Z(GL(T)) \), we must have \( a = 1 \). But this means that the constant, \( \varphi \), is the identity, so \( b = 1 \). Therefore \( A \times B \) acts faithfully on \( M \otimes T \).

**Lemma 8.** Suppose \( A \in \varnothing \ p - \varrho p \), \( B \in \varrho p \), and either \( Z(A) \), or \( Z(B) \) is the identity. Then \( A \times B \in \langle \varrho p \rangle \), the smallest formation generated by the set \( \varrho p \).

**Proof.** Let \( E \) be an \( \ell \)-subgroup of \( A \), and \( \mathcal{E} \) an \( \ell \)-subgroup of \( B \). Since \( A \in \varnothing \ p - \varrho p \), it follows from Lemma 6a that \( 0, (E_{\ell})_{\varrho p} = 1 \). But \( E \) lies in \( \ell \), so \( E/0, (E) \) is an element of \( \ell \), consequently \( E_{\ell} \leq 0, (E) \). Therefore \( E_{\ell} \) is a \( p \)-group. Since \( B \in \varrho p \), \( 0, (\mathcal{E}_{\ell})_{\varrho p} > 1 \).

Now \( E \times \mathcal{E} \) is an \( \ell \)-subgroup of \( A \times B \). We wish to examine \( 0, ((E \times \mathcal{E})_{\ell})_{\varrho p} \). Since \( (E \times \mathcal{E})_{\ell} \) is a normal subgroup of \( E \times \mathcal{E} \) contained in \( E_{\ell} \times \mathcal{E}_{\ell} \). We define a subgroup, \( W \), of \( E_{\ell} \) by:

\[ W = \{ e \in E_{\ell} \mid \exists t \in (E \times \mathcal{E})_{\ell} \ : \ t = d \times e, \text{ and } d \in E_{\ell} \} \]
In other words, $W$ is just the collection of all elements of $\mathcal{E}_p$ which appear as components of elements of $(E \times \mathcal{E})_p$. $W$ is clearly a subgroup of $\mathcal{E}_p$, and since $(E \times \mathcal{E})_p$ is normal in $E \times \mathcal{E}$, $W$ is normal in $\mathcal{E}$. By construction, $(E \times \mathcal{E})_p \leq E \times W$, hence $\mathcal{E}/W \simeq (E \times \mathcal{E})/(E \times W)$ lies in $\mathcal{E}_p$. Therefore $W = \mathcal{E}_p$.

Now if $e$ is any element of $0_p, (\mathcal{E}_p)$, then there is an element $d$ in $\mathcal{E}_p$, such that $t = d \times e$ lies in $(E \times \mathcal{E})_p$. But $\mathcal{E}_p$ is a $p$-group, hence by taking an appropriate power of $t$, we see that $e$ lies in $(E \times \mathcal{E})_p$. Therefore,

\[(30) \ 0_p, (\mathcal{E}_p) \leq 0_p, ((E \times \mathcal{E})_p) \leq 0_p, (E \times \mathcal{E}_p) = 0_p, (\mathcal{E}_p) .\]

By assumption, $A$ has a faithful irreducible $\mathbb{Z}_p(A)$-module, $M$, and $B$ has a faithful irreducible $\mathbb{Z}_p(B)$-module, $T$, such that $1 < c_T(0_p, (\mathcal{E}_p)) < T$. By Lemma 7, $M \otimes T$ is a faithful $\mathbb{Z}_p(A \times B)$-module.

Since $M \otimes T|_B$ is isomorphic to a multiple of $T$, if we let $U$ be any $\mathbb{Z}_p(A \times B)$-composition factor of $M \otimes T$, then $U|_B$ is also a multiple of $T$, so because of $(30)$ we have

\[(31) \ 1 < c_U(0_p, (\mathcal{E}_p)) = c_U(0_p, ((E \times \mathcal{E})_p)) < U .\]

for each $U$. 
Let $G = (A \times B)/C_{AB}(U)$, then $E = (E \times E)C_{AB}(U)/C_{AB}(U)$ is an $\ell$-subgroup of $G$. By Lemma 2, $E_{\ell_p} = (E \times E)_{\ell_p}C_{AB}(U)/C_{AB}(U)$, and by (31), $0_{\ell_p}((E \times E)_{\ell_p})$ does not centralize $U$, hence $0_{\ell_p}(E_{\ell_p}) > 1$. Since $1 < C_{U}(0_{\ell_p}((E \times E)_{\ell_p}))$, and $G$ is an element of $\mathcal{F'}$, it follows from Lemma 6a that $G$ lies in $\theta_{\ell_p}$.

Let $V$ be the direct sum of all $Z_{p}(A \times B)$-composition factors occurring in a composition series of $M \otimes T$. By (14), $F(A \times B) = F(A) \times F(B)$ is a $p'$-group, so the fact that $M \otimes T$ is faithful implies $V|_{F(A \times B)}$ is faithful. By Lemma 1, $V$ is a faithful completely reducible $Z_{p}(A \times B)$-module. Therefore $(A \times B)/C_{AB}(U)$ lies in $\langle \theta_{\ell_p} \rangle$, where the intersection is taken over the irreducible components of $V$.

Since $V$ is faithful this intersection is the identity, hence $A \times B \in \langle \theta_{\ell_p} \rangle$.

**Corollary 1.** If $\ell \ll \mathcal{F}$, and there is an element $B$ in $\theta_{\ell_p}$ such that $Z(B) = 1$, then $\langle \theta_{\ell_p} \rangle \subseteq \langle \theta_{\ell_q} \rangle$ for each $q$ in $\pi(\ell)$.

**Proof.** By Lemma 8, if $A \in \theta_{\ell_p} - \theta_{\ell_p}$, then $A \times B \in \langle \theta_{\ell_p} \rangle$. Therefore $A \in \langle \theta_{\ell_p} \rangle$, so $\langle \theta_{\ell_p} \rangle = \langle \theta_{\ell_p} \rangle$. By Lemma 6d, if $q \in \pi(\ell)$, then $\theta_{\ell_p} \subseteq \langle \theta_{\ell_q} \rangle$, hence $\langle \theta_{\ell_p} \rangle \subseteq \langle \theta_{\ell_q} \rangle$.

**Theorem 7.** Suppose $\ell \ll \mathcal{F}$, and $\theta_{\ell_p}, \theta_{\ell_r}$ are non-empty for two primes $\ell_p, \ell_r$ in the characteristic of $\ell$. Let $\{ \mathcal{F}_q \}$ be the unique minimal local definition of $\mathcal{F}$. Then
\[ \mathcal{F}_q = \langle \delta_q \rangle \quad \text{for each } q \in \pi(\mathcal{L}). \]

**Proof.** We define a new formation, \( \tilde{\mathcal{F}} \), locally by setting
\[ \langle \delta_q \rangle = \tilde{\mathcal{F}}_q \quad \text{for } q \in \pi(\mathcal{L}) \]
(32)
\[ \mathcal{F}_q = \tilde{\mathcal{F}}_q \quad \text{for } q \in \pi(\mathcal{L})' \]

Since \( \ell \ll F \), \( \tilde{\mathcal{F}}_q \subseteq F_q \) for each \( q \), by Theorem 6. Therefore \( \tilde{\mathcal{F}} \subseteq F \).

Let \( \mathcal{S}_s \) be the set specified in Definition 6 for the formation \( \mathcal{F} \). Since \( \tilde{\mathcal{F}} \subseteq F \), \( \mathcal{S}_s \subseteq \mathcal{S} \subseteq \tilde{\mathcal{F}}_s \) for each \( s \in \pi(\mathcal{L}) \). Therefore \( \ell \ll \mathcal{F} \), by Theorem 6.

Suppose \( \tilde{\mathcal{F}} \subset F \). If \( G \) is an element of minimal order in \( F - \mathcal{F} \), then \( G \) is a semi-direct product, \( G = \mathcal{F}M \), where \( \mathcal{F} \) is an \( \mathcal{F} \)-subgroup of \( G \), and acts faithfully and irreducibly on the elementary abelian \( t \)-group \( M \). Since \( G \) lies in \( \mathcal{F} - \mathcal{F} \), \( \mathcal{F} \simeq G/C_G(M) \) lies in \( \mathcal{F}_t - \mathcal{F}_t \).

For \( t \) in \( \pi(\mathcal{L})' \), this contradicts the definition of \( \mathcal{F}_t \), hence \( t \in \pi(\mathcal{L}) \).

Since \( \ell \ll \mathcal{F} \), \( F \), as an \( \mathcal{F} \)-subgroup of \( G \), must contain some \( \ell \)-subgroup, \( E \), of \( G \). Thus for any prime \( q \), the permutation representation \( _q^1 \mathcal{O}_q^{(E/M)} \) is faithful. By Lemma 6c, \( G \) lies in \( \mathcal{S}_q \) for each \( q \) in \( \pi(\mathcal{L}) - \{t\} \).

By Lemma 6d, \( \theta_q \subseteq \mathcal{S}_q \) for each \( q, s \) in \( \pi(\mathcal{L}) \), hence if \( G \in \theta_q \) for some \( q \) in \( \pi(\mathcal{L}) - \{t\} \), \( G \in \mathcal{F}_t \). Suppose, therefore, that \( G \in \mathcal{S}_q - \theta_q \) for each \( q \) in \( \pi(\mathcal{L}) - \{t\} \). One of the primes \( p, r \) is unequal to \( t \), say \( p \). Then \( G \) is an element of \( \mathcal{S}_p - \theta_p \) such that \( Z(G) = 1 \). Since
is non-empty there is a group \( H \) in \( \theta_p \), so by Lemma 8, \( G \times H \in \langle \theta_p \rangle \).

By Lemma 6d, \( G \times H \in \widetilde{\mathcal{F}}_t \), hence in each case \( \mathcal{F}_p \), as a factor group of \( G \), must lie in \( \mathcal{F}_t \), a contradiction.

Therefore \( \widetilde{\mathcal{F}}_t = \mathcal{F}_t \). Since \( \{\mathcal{F}_q\} \) forms a local definition for \( \mathcal{F}_p \), we have \( \mathcal{F}_q \subseteq \mathcal{F}_q \subseteq \mathcal{F}_q \) for \( q \in \pi(\ell) \), so the proof of Theorem 7 is complete.

Because we could not relax the hypothesis on the \( \theta_p \)'s, we thought it appropriate to include

**Theorem 8.** Suppose \( \ell \ll \mathcal{F}_p \), and \( p \in \pi(\ell) \). \( \theta_p \) is empty if, and only if, for each element \( F \) of \( \mathcal{F}_p \), an \( \ell \)-subgroup, \( E \), of \( F \) either covers, or avoids each \( p \)-chief factor of \( F \).

**Proof.** Suppose an \( \ell \)-subgroup of \( F \) either covers or avoids each \( p \)-chief factor of \( F \) for every \( F \) in \( \mathcal{F}_p \). Let \( F \in \mathcal{F}_p \), and let \( E \) be an \( \ell \)-subgroup of \( F \). Let \( M \) be a faithful irreducible \( \mathbb{Z}_p(F) \)-module such that \( C_M(0_p, (E_p^\ell)) > 1 \). By Theorem 5, and the fact that \( \ell \ll \mathcal{F}_p \), \( F^{*} = \mathcal{F}_M(0_p, (F_{p_{\mathcal{F}_p}})) \) is an \( \mathcal{F} \)-subgroup of \( \mathcal{F}_M \), acts irreducibly on \( M \), and does not avoid \( M \). Therefore \( F^{*} = \mathcal{F}_M \); \( M \) is a \( p \)-chief factor of \( G = \mathcal{F}_M \) which is not avoided by the \( \ell \)-subgroup, \( E^{*} = \mathcal{F}_M(0_p, (E_p^\ell)) \), of \( G \).

Therefore \( M = C_M(0_p, (E_p^\ell)) \), so \( \theta_p \) is empty.

Suppose \( \theta_p \) is empty, \( F \in \mathcal{F}_p \), and \( E \) is an \( \ell \)-subgroup of \( F \) which does not avoid the \( p \)-chief factor \( K = L/N \) of \( F \). Let \( \overline{F} = F/C_p(K) \). Our first assertion is that the semi-direct product \( \overline{F}K \) lies in
(the action of \( F \) on \( K \) is the action induced by the action of \( F \) on \( K \)). By Theorem 5, \( F^* = \overline{FC}_K(0_p,(\overline{F}_p)) \) is an \( \mathcal{J} \)-subgroup of \( \overline{FK} \). Therefore \( F^* \) acts irreducibly on \( K \), and \( F^*/C_{F^*}(K) \cong \overline{F} \). Since \( F \in \mathcal{J} \), \( \overline{F} \in \mathcal{J}_p \) by definition. By Theorem 3, \( F^* \) covers \( K \), hence \( \overline{FK} \) lies in \( \mathcal{J} \).

\[ E = E_{CF}(K)/C_F(K) \] is an \( \mathcal{E} \)-subgroup of \( \overline{F} \). By Lemma 2,

\[ \overline{E}_p = E_{p} C_F(K)/C_F(K). \]

Because \( 0_p, (E_p) = E_p \), it follows that

\[ 0_p, (\overline{E}_p) = 0_p, (E_p) C_F(K)/C_F(K). \]

Therefore \( C_K(0_p,(E_p)) = C_K(0_p,(\overline{E}_p)). \)

\[ 0_p, (E_p) \] centralizes every \( p \)-section of \( E \), hence \( 0_p, (E_p) \) centralizes \( (L \cap E)N/N \), a non-identity subgroup of \( K \). Therefore,

\[ 1 < C_K(0_p,(\overline{E}_p)) \subseteq K \]

Thus \( \overline{F} \) lies in \( \mathcal{E}_p \). \( \mathcal{E}_p \) is empty, so it follows from Lemma 6 that

\( \overline{E}_p \) is a \( p \)-group. If \( U \) is any \( \overline{E} \)-composition factor of \( K \), then

\( \overline{E}_p \) centralizes \( U \) since \( \overline{E}_p \leq 0_p(E) \). Upon taking inverse images in \( E \), we see that \( C_E(U) \geq \overline{E}_p \), so that \( E/C_E(U) \) lies in \( \mathcal{E}_p \). By Theorem 3, \( E \) covers \( U \), hence \( E \) also covers all of \( K \).
Chapter V

Structure Theorems

Throughout this chapter we shall make the following assumptions:

Hypothesis I. \( \mathcal{L} \) and \( \mathcal{F} \) are saturated formations such that
(a) \( \mathcal{N} \subseteq \mathcal{L} \ll \mathcal{F} \);
(b) there is a non-empty formation, \( \mathcal{T} \), such that \( \mathcal{L} = \{ G \in \mathcal{L} | G/F(G) \in \mathcal{T} \} \).

Our first theorem says that the structure of \( \mathcal{F} \) is essentially the same as the structure of \( \mathcal{L} \) in that we can show the existence of a formation, \( \mathcal{U} \), such that \( \mathcal{F} = \{ G \in \mathcal{L} | G/F(G) \in \mathcal{U} \} \).

We first prove two lemmas.

Lemma 9. Let \( \mathcal{T} \) be a non-empty formation. Let \( \mathcal{Y} \) be the formation locally defined by setting \( \mathcal{Y}_p = \mathcal{T} \) for each \( p \). Let \( \mathcal{L} = \{ G \in \mathcal{L} | G/F(G) \in \mathcal{Y} \} \). Then \( \mathcal{L} = \mathcal{Y} \).

Proof. Suppose \( G \in \mathcal{L} \). Because \( 0_{p',p}(G) \supseteq F(G) \), \( G/F(G) \in \mathcal{T} \) implies that, for each \( p \), \( G/0_{p',p}(G) \) lies in \( \mathcal{T} \). By \( \mathcal{G}(\alpha') \), \( G \) lies in \( \mathcal{Y} \).

Suppose \( G \in \mathcal{Y} \). Then, for each \( p \), \( G/0_{p',p}(G) \in \mathcal{T} \). Since \( \mathcal{T} \) is a formation, and \( F(G) = \cap_{p} 0_{p',p}(G) \), \( G/F(G) \) lies in \( \mathcal{T} \). Therefore \( G \in \mathcal{L} \), so \( \mathcal{L} = \mathcal{Y} \).

Lemma 10. Suppose \( \mathcal{L}, \mathcal{F} \) satisfy Hypothesis I.
(a) If \( G \) is a group with \( \mathcal{L} \)-subgroup, \( E \), and \( E \in \mathcal{T} \), then \( E = G \).
(b) If \( \{ f_q \} \) is any local definition for \( \mathcal{F} \), and \( G \) is an element of \( \mathcal{T} \) such that \( 0_q(G) = 1 \), then \( G \) lies in \( \mathcal{F}_q \).

**Proof.** We prove (a) by induction on the nilpotent length of \( G \). If \( G \) is nilpotent, there is nothing to prove. Since \( E \) lies in \( \mathcal{T} \), \( EF(G) \) lies in \( \mathcal{E} \). Since \( E \) is an \( \ell \)-subgroup of \( G \), \( E \) covers \( U/U_0 \) for any subgroup, \( U \), of \( G \) which contains \( E \). Therefore \( E \geq F(G) \). Set \( \overline{G} = G/F(G) \), then \( \overline{E} = E/F(G) \) is an \( \ell \)-subgroup of \( \overline{G} \). By induction, \( \overline{E} = \overline{G} \), hence \( E = G \).

Let \( \{ f_q \} \) be any local definition for \( \mathcal{F} \). Suppose \( G \in \mathcal{T} \), and \( 0_p(G) = 1 \). Let \( M \) be the regular \( Z_p(G) \)-module, and form \( G_1 = GM \). Since \( G \) lies in \( \mathcal{T} \), \( G_1 \) lies in \( \mathcal{E} \). It is a simple consequence of strong containment that \( \ell \subseteq \mathcal{I} \), hence \( G_1 \in \mathcal{F} \). Since \( 0_p(G) = 1 \), and \( G \) acts faithfully on \( M \), \( M = 0_p, p(G_1) \). Therefore \( G_1/M \in \mathcal{F}_p \). Since \( G \sim G_1/M \), \( G \) lies in \( \mathcal{F}_p \). This proves (b).

**Theorem 9.** Suppose \( \ell \) and \( \mathcal{F} \) satisfy Hypothesis I. Then there is a formation, \( \mathcal{V} \), containing \( \mathcal{T} \), such that \( \mathcal{F} = \{ G \in \mathcal{I} \mid G/F(G) \in \mathcal{V} \} \).

**Proof.** If \( \ell = \mathcal{I} \), the formation \( \mathcal{T} \) satisfies the requirements of the theorem. Assume \( \ell \subseteq \mathcal{I} \). By Lemma 9, we know that the family, \( \{ \ell_p \mid \ell_p = \mathcal{T} \text{ for each } p \} \), of formations is a local definition for \( \ell \).

We shall use this family for the local definition of \( \ell \) throughout the remainder of the proof. Let \( \{ f_q \} \) be the unique minimal local definition of \( \mathcal{F} \). A second application of Lemma 9 says that we need
only show \( \mathcal{J}_r = \mathcal{J}_s \) for each pair of primes, \( r, s \). In view of Theorem 7 and Corollary 1, we begin by examining the set \( \theta_s \) for various primes \( s \). Since \( \mathcal{N} \subseteq \mathcal{E} \), \( \pi(\mathcal{E}) \) contains all primes, so \( \theta_s \) and \( \mathcal{E}_s \) are defined for each \( s \).

Let \( G \) be an element of minimal order in \( \mathcal{J} - \mathcal{E} \). By minimality, if \( N \) is any normal non-identity subgroup of \( G \), then \( G/N \) lies in \( \mathcal{E} \). Therefore \( G\mathcal{E} \) is the unique minimal normal non-identity subgroup of \( G \). If \( E \) is an \( \mathcal{E} \)-subgroup of \( G \), then \( E_G = G \), and \( E \cap G\mathcal{E} = 1 \). Furthermore, \( E \) acts faithfully and irreducibly on \( G\mathcal{E} \). We set \( M = G\mathcal{E}\), and note that \( M \) is an elementary abelian \( p \)-group for some prime \( p \).

Since \( G \) is not in \( \mathcal{E} \), \( E \) cannot lie in \( \mathcal{J} \). Therefore \( F(E) \geq E_{r'} \). But it follows from (14) that \( F(E) \) is a \( p' \)-group, so for some prime \( r \neq p \), \( E_{r'} \) has a non-identity normal Sylow \( r \)-subgroup, \( R \). If \( s \neq r \), then

\[
0_s, (E_{r'} \mathcal{E}_s) = 0_s, (E_{r'} \mathcal{E}_s) \geq R > 1.
\]

Because \( M \) is the unique minimal normal subgroup of \( G \), and \( E \cap M = 1 \),

\[
1_{0_s, (E\mathcal{E}_s)}^G \text{ is faithful for each } s.
\]

By Lemma 6, \( G \) lies in \( \theta_s \) for each prime \( s \neq r, p \). Since \( E \) is faithful and irreducible on \( M \),

\( Z(G) = 1 \).

We now choose a prime \( s \neq r, p \). Then \( G \in \theta_s \), so there exists a faithful irreducible \( Z_s(G) \)-module, \( J \), such that

\[
1 < \text{c}_j(0_s, (E\mathcal{E}_s)) < J.
\]
Set $G_1 = GJ$. Since $E$ is isomorphic to an $\ell$-subgroup of $G_1/J$, it follows from (25c) and Theorem 5 that $E^* = E_{CJ}(0_s,(E_{\ell s}))$ is an $\ell$-subgroup of $G_1$. An $\ell$-subgroup of $G_1$ covers $G_1/J$ since $G \in \mathcal{F}_i$; it cannot avoid $J$ because $\mathcal{E} \ll \mathcal{F}$. Therefore $G_1$ lies in $\mathcal{F}$. Because $E$ is a quotient group of $E^*$ which does not lie in $\mathcal{F}$, $E^*$ cannot lie in $\mathcal{F}$, hence

$$1 < (E^*)^{\mathcal{F}_i} = (E^*)^{\mathcal{F}_p} \leq E_{CJ}(0_s,(E_{\ell s})).$$

$(E^*)^{\mathcal{F}_i}$ is a $p'$-group because $E_{\mathcal{F}_i}$ is a subgroup of the $p'$-group, $F(E)$, and $s \neq p$. $1(\mathcal{F}_i)^{G_1}$ is faithful since $J$ is the unique minimal normal subgroup of $G_1$, and $(E^*)^{\mathcal{F}_i} \cap J \leq C_J(0_s,(E_{\ell s})) < J$. It follows from parts (a) and (c) of Lemma 6 that $G_1 \in \mathcal{P}_p$. By construction, $Z(G_1) = 1$, hence we have established

(33) If $s \neq r$, then there is a group, $X$, in $\mathcal{E}_s$ such that $Z(X) = 1$.

We now apply the results of Chapter IV. $\pi(\ell)$ contains all primes, so it follows from Theorem 7, and (33) that $\mathcal{F}_s = \langle \bar{s}_s \rangle$ for each prime $s$. By Corollary 1, we have

$$\mathcal{F}_s = \mathcal{F}_q \quad \text{for } s, q \text{ in } r'$$

(34)

$$\mathcal{F}_s \subseteq \mathcal{F}_r \quad \text{for each } s.$$ 

For $s \neq r$, we set $\mathcal{F}_s = \mathcal{U}$. The final step in the proof will be to show $\bar{s}_r \subseteq \mathcal{U}$.

By part (d) of Lemma 6, $\mathcal{P}_r \subseteq \mathcal{F}_s$ for each $s$, so $\mathcal{P}_r \subseteq \mathcal{U}$. Suppose
\( H \in \hat{\mathcal{F}}_r = \theta_r \), and \( E \) is an \( \ell \)-subgroup of \( H \). Then \( 0_r(E) = 1 \), so \( E \) is an \( r \)-group.

Suppose \( E \in \mathcal{T} \), then \( E \) lies in \( \mathcal{T} \). By Lemma 10, \( E = H \), and if \( s \) is any prime not dividing \( r \mid H \mid \), \( 0_s(H) = 1 \), so \( H \in \mathcal{T}_s = \mathcal{U} \).

Suppose \( E \in \mathcal{T} \). By (14), \( 0_r(H) = 1 \), so \( 1 \subseteq H \) is faithful. Since \( H \in \hat{\mathcal{F}}_r \), \( H \) also lies in \( \mathcal{T} \). Thus if \( s \) is a prime which does not divide \( |H| \), it follows from part (b) of Lemma 6 that \( H \in \langle \hat{\mathcal{I}}_s \rangle \).

Therefore
\[
\hat{\mathcal{F}}_r \subseteq \mathcal{U} \subseteq \mathcal{T}_r = \langle \hat{\mathcal{F}}_r \rangle.
\]

Since \( \mathcal{U} = \mathcal{T}_s \) for each \( s \), Lemma 9 says that \( \mathcal{T} = \{ G \in \mathcal{L} \mid G/F(G) \in \mathcal{U} \} \).

The fact that \( \mathcal{U} \) contains \( \mathcal{T} \) is a consequence of part (b) of Lemma 10.

We are interested in finding formations which are maximal in the partial ordering \( \ll \). Since \( \mathcal{L} \ll \mathcal{T} \) implies \( \mathcal{L} \subseteq \mathcal{T} \), we shall assume \( \mathcal{L} \subseteq \mathcal{T} \), as well as Hypothesis I. Since \( \mathcal{L} = \{ G \in \mathcal{L} \mid G/F(G) \in \mathcal{T} \} \), we shall fix our local definition for \( \mathcal{L} \) by setting \( \mathcal{L}_p = \mathcal{T} \) for each \( p \). We shall assume that \( \{ \mathcal{T}_p \} \) is the minimal local definition for \( \mathcal{T} \).

By the proof of Theorem 9, there is a formation, \( \mathcal{U} \), containing \( \mathcal{T} \), such that \( \mathcal{T}_p = \mathcal{U} \) for each \( p \). Since \( \mathcal{L} \subseteq \mathcal{T} \), we must have \( \mathcal{T} \subseteq \mathcal{U} \).

Before stating our main theorem, we prove several lemmas. The proof of Lemma 12 contains the essential constructions used in the proof of our main theorem.

**Lemma 11.** Let \( G \) be a group, and \( 1 < H \leq G \). Assume that the permu-
tation representation, \(1_H^G\), is faithful. If \(M\) is the \(Z_p(G)\)-module which gives \(1_H^G\), set \(U = \cap_{g \in G} C_M(H)g\). Then \(U\) is a \(Z_p(G)\)-submodule of \(M\), and \(M/U\) is a faithful \(Z_p(G)\)-module.

Proof. We can choose the cosets of \(H\) in \(G\) to be a \(Z_p\)-basis for \(M\), i.e., \(M = Z_p \cdot H \oplus Z_p \cdot Hg_2 \oplus \cdots \oplus Z_p \cdot Hg_s\), where \(s = [G:H]\), and the operation of \(G\) on \(M\) is by right multiplication.

For each \(g\) in \(G\), \(C_M(H)g = C_M(H^g)\), hence \(U = \cap_{g \in G} C_M(H^g) = C_M(\cup H^g)\). In other words, if \(N\) is the normal closure of \(H\) in \(G\), then \(U = C_M(N)\). Since \(N < G\), \(U\) is a \(Z_p(G)\)-submodule of \(M\).

For \(i = 1, \ldots, m\), let \(\hat{O}_i\) denote one of the orbits of the cosets \(H, Hg_2, \ldots, Hg_s\) under action by \(N\). Since \(N < G\), \(G\) permutes these orbits transitively, thus all orbits have the same number of elements. The number of elements in any orbit is \([N:H]\), and since \(1_H^G\) is faithful, \(H\) cannot be normal in \(N\), so \([N:H] \geq 3\). For each \(i\), we set \(\hat{O}_i = \{Hg_{i1}, \ldots, Hg_{ir}\}\) where \(r = [N:H]\).

Let \(u_i = \sum_{k=1}^{r} Hg_{ik}\), for \(i = 1, \ldots, m\). Our first assertion is that \(u_1, \ldots, u_m\) form a \(Z_p\)-basis for \(C_M(N)\). The elements \(u_1, \ldots, u_m\) are clearly linearly independent, and since \(\hat{O}_i\) is an orbit of \(N\), the \(u_i\)'s satisfy \(u_in = u_i\) for each \(n\) in \(N\). Therefore \(u_1, \ldots, u_m\) span a subspace of \(C_M(N)\).
Suppose \( w = \sum_{i=1}^{m} \sum_{k=1}^{r} \alpha_{ik} h_{igik} \) lies in \( C_\mathcal{M}(N) \). Then

\[
\sum_{i=1}^{m} \sum_{k=1}^{r} \{ \alpha_{ik} h_{igik} - \alpha_{ik} h_{i,gik} \} = 0
\]

for each \( n \) in \( N \). Now \( N \) acts as a transitive permutation group on each orbit, hence for each \( i \), and each \( k, j \) such that \( 1 \leq i \leq m \), and \( 1 \leq k, j \leq r \), there is an element \( n_{ikj} \) of \( N \) such that

\[
h_{igik} n_{ikj} = h_{ij}.
\]

If we use this relation in the sum above, as well as the fact that \( n_{ijk} \) permutes the cosets of \( C_1 \), it follows that, for each \( i \),

\[
\alpha_{1i} = \alpha_{2i} = \cdots = \alpha_{ri} = \alpha_i
\]

Thus \( W = \sum_{i=1}^{m} \alpha_i u_i \), so the \( u_i \) span \( C_\mathcal{M}(N) \). Therefore, a \( \mathbb{Z}_p \) basis for \( M/U \) is the collection of cosets:

\[
\begin{align*}
U + h_{11} &= U + H, \quad U + h_{12}, \cdots, U + h_{1,r-1} \\
U + h_{21} &= U + h_{21}, \cdots, U + h_{2,r-1} \\
&\vdots \\
U + h_{m1} &= U + h_{m2}, \cdots, U + h_{m,r-1}
\end{align*}
\]

(35)

Suppose \( x \) lies in the kernel of the representation of \( G \) on \( M/U \). Then, for \( 1 \leq i \leq m \), and \( 1 \leq j \leq r - 1 \), we have
where the $\alpha_k$ are suitably chosen elements of $\mathbb{Z}_p$. Since $H_{g_{ij}}x$ is a coset, and each $u_k$ is a sum of at least three distinct cosets, we must have each $\alpha_k = 0$. Since $x$ permutes the orbits of $N$, it follows from the fact that $x$ fixes $H_{g_{11}}$ that $x$ fixes each orbit, $\mathcal{O}_i$. Therefore

$$H_{g_{ir}}x = (u_i - \sum_{k=1}^{r-1} H_{g_{ik}})x = H_{g_{ir}},$$

so $x$ lies in the kernel of the representation of $G$ on the cosets of $H$. Therefore $x = 1$, hence $M/U$ is faithful.

**Lemma 12.** Suppose $\mathcal{E}, \mathcal{F}$ satisfy Hypothesis I, $\mathcal{E} \subset \mathcal{F}$, and there is an element, $H$, in $V \cap \mathcal{E} - \mathcal{T}$ such that $0_p(H) = 1$. Then $V \supset \{ G \in \mathcal{F} | F(G) is a p-group \}$.

**Proof.** Let $G$ be an element of $\mathcal{F}$ such that $F(G)$ is a $p$-group. Let $E$ be an $\mathcal{E}$-subgroup of $G$, and assume $0_p'(E_T) > 1$. Since $F(G)$ is a $p$-group, $0_p'(G) = 1$, hence $1_{0_p'(E_T)}^G$ is faithful. Let $M$ be the $\mathbb{Z}_p(G)$-module which gives the permutation representation on the cosets of $0_p'(E_T)$; let $U = \cap_{g \in G} M(0_p'(E_T)^g)$. By Lemma 11, the $\mathbb{Z}_p(G)$-module, $M_1 = M/U$, is faithful.
Set \( X = GM_1 \), then \( F(X) = F(G)M_1 \), so \( X/F(X) \cong G/F(G) \). Since \( G \) lies in \( \mathcal{F} \), \( X/F(X) \in \mathcal{U} \), hence \( X \) lies in \( \mathcal{F} \). By (25c) and Theorem 5, \( E^* = EC_{M_1}(0_{p^*}(E_\gamma)) \) is an \( \ell \)-subgroup of \( X \). But \( C_{M}(0_{p^*}(E_\gamma)) \supseteq U \), hence \( C_{M}(0_{p^*}(E_\gamma))/U \). Let \( T \) be the intersection of all conjugates of \( E^* \) in \( X \). Since \( E^* \cap M_1 = C_{M_1}(0_{p^*}(E_\gamma)) \), it follows that \( T \cap M_1 = \bigcap_{g \in X} C_{M_1}(0_{p^*}(E_\gamma)^g) = 1. \)

But if \( K < X \), and \( K \cap M_1 = 1 \), then \( K \leq C_X(M_1) \). \( C_X(M_1) = C_G(M_1)M_1 \), so the fact that \( G \) is faithful on \( M_1 \) says that \( M_1 \) is self-centralizing in \( X \), consequently, \( K = 1 \). Therefore \( T = 1 \), so \( 1_{E^*}X \) is faithful. It is clear, now, that \( 1_{E^*}X \) is also faithful, so if \( t \) is any prime which does not divide the order of \( X \), then \( X \in \langle \tilde{s}_t \rangle = \mathcal{F}_t = \mathcal{U} \), by Lemma 6. Therefore \( G \), as a factor group of \( X \), also lies in \( \mathcal{U} \).

We may now assume \( 0_{p^*}(E_\gamma) = 1 \), so \( E_\gamma \) is a \( p \)-group. It is time to use \( H \). If \( R = I_1 \oplus \cdots \oplus I_t \) is a decomposition of the regular \( \mathbb{Z}_p(H) \)-module into its principal indecomposable constituents, we set \( K_\lambda = I_\lambda / \sqrt{I_\lambda} \) for \( \lambda = 1, \cdots, t \). Since \( R \) is faithful, and \( F(H) \) is a \( p^* \)-group, it follows from Theorem 1 and Lemma 1 that \( R^* = K_1 \oplus \cdots \oplus K_t \) is faithful. Since \( H \) does not lie in \( \mathcal{V} \), it follows that for some \( \lambda, B = H/C_{H}(K_\lambda) \) does not lie in \( \mathcal{V} \). Let \( K = K_\lambda \), then \( B \) is an element of \( \mathcal{V} \cap \mathcal{U} - \mathcal{V} \) which has \( K \) as a faithful irreducible \( \mathbb{Z}_p(B) \)-module.
Let $U$ be the regular $Z_p(G)$-module, and set $W = (B \times G) (K \otimes U)$, where the action of $B \times G$ on $K \otimes U$ by conjugation is the canonical action given in Chapter II. To show $G \in \mathcal{U}$, it is sufficient to show $\bar{W} \in \mathcal{U}$, since $G$ is a factor group of $W$.

By (14), $F(B)$ is a $p'$-group, and by assumption, $F(G)$ is a $p$-group. Therefore if $N$ is the kernel of the representation of $F(B \times G)$ on $K \otimes U$, then $N = N \cap F(B) \times N \cap F(G)$. Since $B$ and $G$ act faithfully on $K \otimes U$, $K \otimes U|_{F(B \times G)}$ is faithful. By Lemma 1, $K \otimes U$ is a faithful $Z_p(B \times G)$-module. Therefore $0_{p'}(W) = 1$, so $F(W) = F(G)(K \otimes U)$. Since $W/F(W) \cong B \times G/F(G)$, $W/F(W) \in \mathcal{U}$, so $W$ lies in $\mathcal{U}$.

An $\mathcal{U}$-subgroup of $B \times G$ is $B \times E$, so by (25c) and Theorem 5, $E^* = (B \times E)_{C \otimes U} (0_{p'}((B \times E)_{\gamma}))$ is an $\mathcal{U}$-subgroup of $W$. Since $B \in \mathcal{U} \cap \mathcal{U} - \gamma$, $1 < B_\gamma \leq F(B)$, so $B_\gamma$ is a $p'$-group. By assumption $E_\gamma$ is a $p$-group. Let $V$ be the collection of elements of $B_\gamma$ which appear as components of elements of $(B \times E)_\gamma$. Since $(B \times E)_\gamma \subset B \times E$, $V \subset B$, and it is clear that $(B \times E)_\gamma \leq V \times E$. Therefore $B/V \cong (B \times E)/(V \times E)$ lies in $\gamma$, hence $V = B_\gamma$. If $v \in V$, then there is an element $u$ in $E$ such that $v \times u \in (B \times E)_\gamma$. Since $B_\gamma$ is a $p'$-group, and $E_\gamma$ is a $p$-group, $v$ is equal to a power of $v \times u$. Therefore

$$E_\gamma = 0_{p'}((B \times E)_\gamma).$$
Now $K \otimes U|_{B_{\mathcal{E}}}$ is a multiple of $K|_{B_{\mathcal{E}}}$, so it follows from Theorem 1 that $C_K \otimes U(B_{\mathcal{E}}) = 1$. By (36), $B \times E$ is an $\mathcal{E}$-subgroup of $W$.

Let $t$ be a prime which does not divide $|W|$. The fact that $1_{B \times E}|^W$ is faithful implies that $1_{(B \times E)^{\mathcal{E}}}|^W$ is faithful, so by part (b) of Lemma 6, $W$ is an element of $\langle \delta_t \rangle = \mathcal{T}_t = W$. Therefore $G$ lies in $\mathcal{U}$ in every case, so Lemma 12 has been proven.

Because of the preceding lemma, we give

**Definition 7.** Set $\mathcal{N} = \{p \mid V \cap \mathcal{L} \not\subset \mathcal{N} \text{ contains a group, } H, \text{ with } O_p(H) = 1\}$. We call a prime, $p$, special if $p \in \mathcal{N}$.

**Lemma 13.** If $\mathcal{E}$, $\mathcal{T}$ satisfy Hypothesis I, and $\mathcal{E} \subset \mathcal{T}$, then there is at most one special prime.

**Proof.** Let $G$ be an element of minimal order in $\mathcal{T} - \mathcal{E}$. Then $G$ is the semi-direct product, $EM$, where $E$ is an $\mathcal{E}$-subgroup of $G$, and $M$ is the unique minimal normal subgroup of $G$.

Since $E$ acts faithfully and irreducibly on $M$, $M = F(G)$. By Lemma 10, $E$ is not an element of $\mathcal{T}$, and since $G \in \mathcal{T}$, $G/F(G)$ lies in $\mathcal{U}$, hence $E \in \mathcal{V} \cap \mathcal{L} - \mathcal{T}$.

Since $O_r(E) \cap O_s(E) = 1$ for two distinct primes $r, s$, $E/0_t(E)$ can lie in $\mathcal{T}$ for at most one prime $t$. If $s \neq t$, then $E/0_s(E) \in \mathcal{V} \cap \mathcal{L} - \mathcal{T}$, hence $\mathcal{N} \not\subseteq \{t\}$. 

Remark. If general, we cannot control the choice of \( G \) enough to be certain that there are no special primes. This is the basis for the example in Chapter VI, and the reason behind

Hypothesis II. Let \( G = EM \) be a fixed element of minimal order in \( \gamma - \ell \). If \( \ell \) is any prime such that \( E/\mathfrak{e}(E) \) lies in \( \mathcal{T} \), we assume that \( \mathfrak{e}_\mathfrak{e} \) is not contained in \( \mathcal{T} \). (Such a prime does not necessarily exist.)

Theorem 10. Suppose \( \ell \) and \( \mathcal{I} \) satisfy Hypotheses I and II. If \( \ell \in \mathcal{I} \), then \( \mathcal{I} = \mathcal{G} \), the collection of all solvable groups.

Proof. Our first step is to show that \( \mathcal{U} \) contains the collection, \( \mathcal{G}_\eta \), of all solvable \( \eta \)-groups. By Lemma 12, the fact that \( \mathcal{U} \subseteq \ell \subseteq \mathcal{I} \) shows that \( \mathcal{U} \) contains the collection of all nilpotent \( \eta \)-groups.

Proceeding by induction, we assume that \( \mathcal{U} \) contains the collection, \( \mathcal{N}^i_\eta \), of all solvable \( \eta \)-groups of nilpotent length at most \( i \). But \( \mathcal{N}^{i+1}_\eta = \{ G \in \mathcal{G} \mid G/F(G) \in \mathcal{N}^i_\eta \} \), thus \( \mathcal{I} \) contains all solvable \( \eta \)-groups of nilpotent length at most \( i + 1 \).

Let \( G \in \mathcal{N}^{i+1}_\eta \), and \( F(G) = P_1 \times \cdots \times P_s \), where \( P_i \) is the Sylow \( p_i \)-subgroup of \( F(G) \). Set \( N_i = \prod_{k \neq i} P_k \), and let \( R_i \) be the regular \( \mathbb{Z}_{P_i} (G/N_i) \)-module for \( i = 1, \cdots, s \). We allow \( G \) to act on \( R = R_1 \times \cdots \times R_s \) by conjugation according to the rule

\[
(37) \ (r_1, r_2, \ldots, r_s)^G = ([r_1]_{N_1} g, [r_2]_{N_2} g, \ldots, [r_s]_{N_s} g).
\]
Then we form the semi-direct product $X = GR$. By construction, $N_i$ centralizes the $p_i$-group, $R_i$, hence the group, $F(G)R$, is nilpotent. Since $F(X)/R$ is a normal nilpotent subgroup of $X/R$, and $X/R \cong G$, $F(X) \leq F(G)R$. Therefore $F(X) = F(G)R$, so $X/F(X) \cong G/F(G)$. Since $G/F(G) \in \mathcal{U}$, it follows that $X$ lies in $\mathcal{F}$.

For each $i$, set $\overline{x_i} = X/N_i(\prod_{k \neq i} R_k)$, $\overline{R_i} = N_i R/N_i(\prod_{k \neq i} R_k)$, and $\overline{G_i} = G(\prod_{k \neq i} R_k)/N_i(\prod_{k \neq i} R_k)$. By modularity

$$G(\prod_{k \neq i} R_k) \cap N_i R = (G \cap N_i R)(\prod_{k \neq i} R_k) = ((G \cap R)N_i)(\prod_{k \neq i} R_k) = N_i(\prod_{k \neq i} R_k).$$

Thus $\overline{x_i}$ is the semi-direct product of $\overline{R_i}$ by $\overline{G_i}$, hence

$$C_{\overline{x_i}}(\overline{R_i}) = C_{\overline{G_i}}(\overline{R_i})\overline{R_i}.$$

Because $\overline{G_i}$ acts faithfully on $\overline{R_i}$, it follows that $\overline{R_i}$ is a self-centralizing normal $p_i$-subgroup of $\overline{x_i}$. Therefore $0, (\overline{x_i}) = 1$, so $F(\overline{x_i})$ is a $p_i$-group. But $p_i$ lies in $\eta$, so by Lemma 12, $\overline{x_i}$ is an element of $\mathcal{U}$ for each $i$. Since $\cap_i (N_i(\prod_{k \neq i} R_k)) = 1$, $X$ lies in $\mathcal{U}$ also. Therefore $G \in \mathcal{U}$. By induction, $\ell_\eta \subseteq \mathcal{U}$.

By Lemma 13, if $EM$ is the minimal element of $\mathcal{F}'$ mentioned in Hypothesis II, then there is at most one prime, $r_0$, such that $E/0_{r_0}(E)$ lies in $\mathcal{F}'$, thus $\eta$ contains $r_0$. Therefore,

$$\ell_{r_0} \subseteq \ell_\eta \subseteq \mathcal{U} \subseteq \mathcal{F}.$$
Suppose \( \mathcal{E} \) does not contain \( \mathbf{s}_{r_0}^i \), and let \( G_1 = E_1M_1 \) be an element of minimal order in \( \mathbf{s}_{r_0}^i - \mathcal{E} \). By Lemma 10, \( E_1 \) is an element of \( \bigcup \mathcal{E} - \mathcal{T} \), and since \( E_1 \in \mathbf{s}_{r_0}^i \), \( \sigma_{r_0}(E_1) = 1 \). Therefore \( \mathcal{H} \) contains all primes.

Suppose \( \mathcal{E} \) contains \( \mathbf{s}_{r_0}^i \). By assumption \( \mathcal{T} \) does not contain \( \mathbf{s}_{r_0}^i \), so we can choose \( H \) in \( \mathbf{s}_{r_0}^i - \mathcal{T} \). Since \( \mathbf{s}_{r_0}^i \subseteq \mathcal{V} \), \( H \) is an element of \( \bigcup \mathcal{E} - \mathcal{T} \) with \( \sigma_{r_0}(H) = 1 \). Therefore \( \mathcal{H} \) contains all primes in every case, so we have

\[
\mathbf{s} = \mathbf{s}_{\mathcal{H}} \subseteq \mathcal{V} \subseteq \mathcal{F} \subseteq \mathbf{s},
\]

which completes the proof of Theorem 10.

**Corollary 2.** Let \( \mathcal{N}_i \) be the collection of groups of nilpotent length at most \( i \). If \( \mathcal{N}_i \ll \mathcal{F} \), then \( \mathcal{F} = \mathcal{N}_i \), or \( \mathcal{F} = \mathbf{s} \).

**Proof.** If we set \( \mathcal{N}_0 = \{1\} \), then for \( i \geq 1 \), \( \mathcal{N}_i = \{ G \in \mathbf{s} | G/F(G) \in \mathcal{N}_{i-1} \} \).

For each prime, \( p \), \( \mathbf{s}_p \) is not contained in \( \mathcal{N}_{i-1} \), hence the hypothesis of Theorem 10 is satisfied. The result follows from Theorem 10.
Let r be a prime. Throughout this chapter, we shall set
\( \mathcal{L} = \{ G \in \mathcal{S} | G/F(G) \in \mathcal{L}_r \} \). For each p, we set \( \mathcal{L}_p = \mathcal{L}_{rp} \); \( \{ \mathcal{L}_p \} \) forms a local definition for \( \mathcal{L} \) by Lemma 9. In this chapter, we shall characterize the formations which strongly contain \( \mathcal{L} \). The formation \( \mathcal{L} \) provides an example which shows that Hypothesis II is not redundant.

**Lemma 14.** Let G be a group with Sylow r-subgroup R. Then \( N_G(R) \) is an \( \mathcal{L} \)-subgroup of G.

**Proof.** Because R is a Sylow r-subgroup of G, \( N_G(R)/R \) is an \( r' \)-group. Therefore \( N_G(R) \) lies in \( \mathcal{L} \).

Suppose \( N_G(R) \leq U \leq G \). We wish to show \( N_G(R) \) covers \( U/U_R \).

Clearly \( U_{\mathcal{L}} = 0^{r'}(U) \). Since \( R \leq 0^{r'}(U) \), it follows that \( 0^{r'}(U) \) acts transitively on the Sylow r-subgroups of U, hence \( U = 0^{r'}(U)N_G(R) \).

Since R covers every r-section of U, it follows that \( N_G(R) \) covers \( 0^{r'}(U)/0^{r'}F(U) \). Therefore \( N_G(R) \) covers \( U/0^{r'}F(U) \). By definition, \( N_G(R) \) is an \( \mathcal{L} \)-subgroup of G.

Suppose \( \mathcal{I} \gg \mathcal{L} \), and \( \mathcal{I} \supseteq \mathcal{L} \). If \( \{ T_q \} \) is the minimal local definition of \( \mathcal{I} \), it follows from Theorem 9 that \( T_q = T_s \) for each q,s.

We set \( \mathcal{V} = T_q \). If H lies in \( \mathcal{V} \cap \mathcal{L} \), then H has a normal Sylow
r-subgroup, so \( H/O_r(H) \) lies in \( \mathfrak{R} \). Therefore, Hypothesis II is violated for the prime \( r \). It follows from Lemma 13 that \( r \) is the unique special prime associated with \( \mathfrak{T} \) and \( \mathfrak{R} \). The next theorem gives a class of formations which strongly contain \( \mathfrak{R} \).

**Theorem 11.** Let \( \mathfrak{T} \) be a non-empty formation. Let
\[
\mathfrak{U} = \{ G \in \mathfrak{L} | G/O_r, (G) \in \mathfrak{T} \},
\]
then \( \mathfrak{U} \) is a formation. If
\[
\mathfrak{T} = \{ G \in \mathfrak{L} | G/F(G) \in \mathfrak{U} \},
\]
then \( \mathfrak{T} \gg \mathfrak{R} \).

**Proof.** Suppose \( G \in \mathfrak{U} \), and \( N < G \). Then \( O_r, (G)N/N \leq O_r, (G/N) \).
Since \( G/O_r, (G) \in \mathfrak{T} \), we also have \( (G/N)/O_r, (G/N) \in \mathfrak{T} \). Therefore, \( G/N \) lies in \( \mathfrak{U} \).

Suppose \( N_1, N_2 \) are two normal subgroups of \( G \) such that \( G/N_1 \in \mathfrak{U} \) for \( i = 1,2 \). For each \( i \), let \( M_1/N_1 = O_r, (G/N_1) \), then \( G/M_1 \in \mathfrak{T} \) for each \( i \), hence \( G/M_1 \cap M_2 \in \mathfrak{T} \). For each \( i \), \( (M_1 \cap M_2)N_1/N_1 \) is an \( r^i \)-group, hence \( M_1 \cap M_2/N_1 \cap M_1 \cap M_2 \) is an \( r^i \)-group for each \( i \). Therefore \( M_1 \cap M_2/N_1 \cap N_2 \) is an \( r^i \)-group, so it follows that the factor group of \( G/N_1 \cap N_2 \) by \( 0_r, (G/N_1 \cap N_2) \) lies in \( \mathfrak{T} \). Therefore \( G/N_1 \cap N_2 \in \mathfrak{U} \), thus \( \mathfrak{U} \) is a formation.

To show \( \mathfrak{R} \ll \mathfrak{T} \), it is sufficient to show that \( \mathfrak{R} \subseteq \mathfrak{U} \) for each prime \( p \). Suppose \( G \in \mathfrak{S} \), then \( G \) has a faithful irreducible \( Z_r(G) \)-module. By (14), this means that \( 0_r, (G) = 1 \). Since \( G \in \mathfrak{T} \), \( G/F(G) \in \mathfrak{V} \). Since \( 0_r, (G) = 1, F(G) \) is an \( r^i \)-group, so \( G/O_r, (G) \in \mathfrak{T} \).
Therefore G lies in $\mathcal{V}$.

Suppose $G \in \Phi_p$ for $p \neq r$. An $p$-subgroup of $G$ is $N_G(R)$ where $R$ is a Sylow $r$-subgroup of $G$. Since $p \neq r$, $0_p ((N_G(R))_{p,r}) = R$. Therefore G has a faithful irreducible $Z_p(G)$-module, $J$, such that $1 < C_J(R) \leq J$. By Theorem 1, either $C_J(0_r(G)) = J$, or it is the identity. The latter possibility cannot occur because $1 < C_J(R) \leq C_J(0_r(G))$. Therefore the fact that $J$ is faithful says that $0_r(G) = 1$, so $F(G)$ is an $r^*$-group. $G \in \mathcal{Q}$, so the same argument as in the preceding paragraph shows that $G/0_r(G) \in \mathcal{F}$. Therefore $G \in \mathcal{V}$. By Theorem 6, $\mathcal{R} \ll \mathcal{F}$.

Since our choice of $\mathcal{F}$ is arbitrary, it follows that we can choose an infinite number of distinct formations which strongly contain $\mathcal{R}$. Our next theorem is the converse to Theorem 11.

**Theorem 12.** Suppose $\mathcal{L} \gg \mathcal{R}$, and $\{\mathcal{L}_q\}$ is the minimal local definition for $\mathcal{L}$. Then there is a non-empty formation $\mathcal{T}$ such that $\mathcal{L}_q = \{ G \in \mathcal{L} | G/0_r(G) \in \mathcal{T} \}$.

**Proof.** Suppose $\mathcal{L} \gg \mathcal{R}$. By Theorem 9, there is a formation $\mathcal{V}$ such that $\mathcal{L}_q = \mathcal{V}$ for each $q$. Our first step is to show that $\mathcal{V}$ is the smallest formation generated by $\{H \in \mathcal{L} | 0_r(H) = 1\}$. Let $\mathcal{U}$ be the smallest formation generated by this set.

Suppose $H \in \mathcal{L}$, and $0_r(H) = 1$. Let $K = I_1 \oplus \cdots \oplus I_s$ be the decomposition of the regular $Z_r(H)$-module, $K$, into principal
indecomposable submodules. By Theorem 1, Lemma 1, and the fact that $F(H)$ is an $r'$-group, it follows that $H$ acts faithfully on $J_1 \oplus \cdots \oplus J_s$, where $J_k = I_k / \overline{I}_k$ for each $k$. For each $k$, set $H_k = H/C_H(J_k)$. Then $J_k$ is a faithful irreducible $Z_r(H_k)$-module. If $R_k$ is a Sylow $r$-subgroup of $H_k$, then $N_{H_k}(R_k)$ is a $p$-subgroup of $H_k$, and $(N_{H_k}(R_k))_{p'} = (N_{H_k}(R_k))_p = R_k$. Therefore $O_r((N_{H_k}(R_k))_p) = 1$. By definition, $H_k \in \mathfrak{p}_r$ for each $k$, so $\langle p \rangle \leq \mathfrak{p}$ implies $H_k \in \mathfrak{p}_r = \mathfrak{p}$. Therefore $H$ lies in $\mathfrak{U}$. Since the generators of $\mathfrak{U}$ all lie in $\mathfrak{U}$, $\mathfrak{U}$ is contained in $\mathfrak{U}$. We know that $U$ is the smallest formation generated by $\mathfrak{p}_r$, from the proof of Theorem 9. Thus if we show $\mathfrak{p}_r \subseteq \mathfrak{U}$, we have shown $\mathfrak{U} \subseteq \mathfrak{U}$. If $G \in \mathfrak{p}_r$, then $G$ has a faithful irreducible $Z_r(G)$-module, and $G \in \mathfrak{U}$. By (14), $O_r(G) = 1$ so by definition, $G \in \mathfrak{U}$. Therefore $\mathfrak{U} = \mathfrak{U}$.

Let $\mathfrak{T}$ be the smallest formation generated by the set \{ $H/O_r(H) | H \in \mathfrak{U}$ \}. Set $U_1 = \{ G \in \mathfrak{U} | G/O_r(G) \in \mathfrak{T} \}$. We want to show $\mathfrak{U} = U_1$. By construction, $\mathfrak{U} \subseteq U_1$.

Since the generators of $\mathfrak{T}$ are elements of $\mathfrak{U}$, we must have $\mathfrak{T} \subseteq \mathfrak{U}$. Therefore, if $G \in U_1$, then $G/O_r(G)$ lies in $\mathfrak{U}$. To show $G$ lies in $\mathfrak{U}$, we use induction on the nilpotent length of $O_r(G)$. If $O_r(G)$ is nilpotent, then $G/O_r(G) \in \mathfrak{U}$ forces $G/F(G) \in \mathfrak{U}$. Therefore $G$ is an element of $\mathfrak{T}$. By our first paragraph, $G/O_r(G)$ lies in $\mathfrak{U}$, so $G$ also lies in $\mathfrak{U}$ since $O_r(G) \cap O_r'(G) = 1$. 


We note that \( 0_{r^*}(G/F(0_{r^*}(G))) = 0_{r^*}(G)/F(0_{r^*}(G)) \), hence by induction, if \( G \in \mathcal{U}_1 \), then \( G/F(0_{r^*}(G)) \in \mathcal{U} \). Therefore \( G \) lies in \( \mathcal{F} \). By our first paragraph \( G/O_r(G) \) lies in \( \mathcal{U} \), so \( G/O_r(G) \cap F(0_{r^*}(G)) = G \) also lies in \( \mathcal{U} \). Therefore \( \mathcal{U} = \mathcal{U}_1 \). This completes the proof in the case when \( \mathcal{R} \subset \mathcal{F} \).

If \( \mathcal{R} = \mathcal{F} \), we let \( \mathcal{T} \) be the formation consisting only of the identity. We must then show that \( \{ \mathcal{T} \} \) is the minimal local definition for \( \mathcal{R} \).

Let \( \{ \mathcal{R}_q \} \) be the minimal local definition for \( \mathcal{R} \). Suppose \( p \) is an arbitrary prime, \( G \in \mathcal{R}_p = \mathcal{R}_p \), and \( t \) is a prime which does not divide \( rp|G| \). Let \( K \) be the regular \( Z(G) \)-module. Set \( G_1 = GK \).

Let \( K_1 \) be the regular \( Z_p(G_1) \)-module. Set \( G_2 = G_1 K_1 \). Since \( G \) acts faithfully on \( K \), and \( G_1 \) acts faithfully on \( K_1 \), \( 0_{p^*,p}(G_2) = K_1 \).* Depending on the choice of \( p \), \( G_2 \) is either an \( r \)-group, or has \( K_1 \) as a normal Sylow \( r \)-subgroup. Therefore \( G_2 \in \mathcal{R} \), hence \( G_2/0_{p^*,p}(G_2) = G_2/K_1 \in \mathcal{R}_p \). Since \( G \) is a factor group of \( G_1 \), it follows that \( G \in \mathcal{R}_p \). Therefore \( \mathcal{R}_p \leq \mathcal{R}_p \). This completes the proof.
Appendix

In this appendix, we shall give proofs to (18'), (18), and (19).

(18') If \( N \triangleleft G \), and \( \pi_1, \ldots, \pi_n \) are sets of primes, then

\[
0_{\pi_1, \ldots, \pi_n}(N) = N \cap 0_{\pi_1, \ldots, \pi_n}(G).
\]

Proof. Use induction on \( N \). If \( n = 1 \), and \( \pi = \pi_1 \), then \( 0_{\pi}(G) \) is the unique maximal normal \( \pi \)-subgroup of \( G \), hence \( 0_{\pi}(G) \) is characteristic in \( G \). Therefore, \( 0_{\pi}(N) \), as a characteristic subgroup of the normal subgroup \( N \), is normal in \( G \). Thus \( 0_{\pi}(N) \leq 0_{\pi}(G) \cap N \). But \( N \cap 0_{\pi}(G) \) is a normal \( \pi \)-subgroup of \( N \), hence is contained in \( 0_{\pi}(N) \).

We may assume that \( T = 0_{\pi_1, \ldots, \pi_n}(N) = 0_{\pi_1, \ldots, \pi_n}(G) \cap N \).

Then \( 0_{\pi_1, \ldots, \pi_n}(N)0_{\pi_1, \ldots, \pi_n}(G)/0_{\pi_1, \ldots, \pi_n}(G) \) is a normal \( \pi_n \)-subgroup of \( \overline{G} = G/0_{\pi_1, \ldots, \pi_n}(G) \), and must be contained in \( 0_{\pi_n}(\overline{G}) \). Therefore \( 0_{\pi_1, \ldots, \pi_n}(N) \leq 0_{\pi_1, \ldots, \pi_n}(G) \cap N \). But \( 0_{\pi_1, \ldots, \pi_n}(G) \cap N/T \) is a normal \( \pi_n \)-subgroup of \( N/T \), hence

\[
0_{\pi_1, \ldots, \pi_n}(G) \cap N \leq 0_{\pi_1, \ldots, \pi_n}(N). \quad \text{This proves (18').}
\]

(18) Let \( G \) be a group. Then \( 0_{p_1, \ldots, p_n}(G) = \cap C_G(K) \), where the intersection is taken over all \( p \)-chief factors, \( K \), of \( G \).

Proof. By looking at a chief series of \( G \) which passes through \( 0_{p_1, \ldots, p_n}(G) \), it is clear that \( 0_{p_1, \ldots, p_n}(G) \) centralizes every \( p \)-chief factor of \( G \). Let \( P = 0_{p_1, \ldots, p_n}(G)/0_{p_n}(G) \), and \( E = P/D(P) \). Then \( G \) is represented
on E, and the kernel of this representation is $0_{p',p}(G)$ (cf. [9], p. 7). If g is any $p'$-element of $\cap C_G(K)$, then g centralizes E, since it centralizes every $p$-chief factor of G. Therefore g lies in $0_{p',p}(G)$. This shows that $\cap C_G(K)$ has a normal $p$-complement, so by (18'), $\cap C_G(K) \leq 0_{p',p}(G)$.

(19) If G is a group, then $F(G) = \cap_{p} 0_{p',p}(G)$, where the intersection is taken over all primes p.

Proof. Since $F(G)$ is nilpotent, it has a normal $p$-complement for every p. By (18'), $F(G) \leq \cap_{p} 0_{p',p}(G)$. Therefore it is sufficient to show that $\cap_{p} 0_{p',p}(G)$ is nilpotent. But look at any chief series of G passing through $\cap_{p} 0_{p',p}(G)$. By (18), $\cap_{p} 0_{p',p}(G)$ centralizes every factor in this series, hence the part of this chief series covered by $\cap_{p} 0_{p',p}(G)$ forms a central series for $\cap_{p} 0_{p',p}(G)$, so $\cap_{p} 0_{p',p}(G)$ is nilpotent. This proves (19).
References


