ON THE SET OF EIGENVALUES OF A CLASS OF EQUIMODULAR MATRICES

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ABSTRACT

The structure of the set $\mathcal{Q}(A)$ of all eigenvalues of all complex matrices (elementwise) equimodular with a given n x n non-negative matrix A is studied. The problem was suggested by O. Taussky and some aspects have been studied by R.S. Varga and B.W. Levinger.

If every matrix equimodular with A is non-singular, then A is called regular. A new proof of the P. Camion-A.J. Hoffman characterization of regular matrices is given.

The set $\mathcal{Q}(A)$ consists of $m \leqslant n$ closed annuli centered at the origin. Each gap, γ, in this set can be associated with a class of regular matrices and with a (unique) permutation, $\pi(\gamma)$. The association depends on both the combinatorial structure of A and the size of the a_{ii}. Let A be associated with the set of r permutations, $\pi_1, \pi_2, \ldots, \pi_r$, where each gap in $\mathcal{Q}(A)$ is associated with one of the π_k. Then $r \leqslant n$, even when the complement of $\mathcal{Q}(A)$ has n+1 components. Further, if $\pi(\gamma)$ is the identity, the real boundary points of γ are eigenvalues of real matrices equimodular with A. In particular, if A is essentially diagonally dominant, every real boundary point of $\mathcal{Q}(A)$ is an eigenvalue of a real matrix equimodular with A.

Several conjectures based on these results are made which if verified would constitute an extension of the Perron-Frobenius Theorem, and an algebraic method is introduced which unites the study of regular matrices with that of $\mathcal{Q}(A)$.

I. INTRODUCTION.

We shall call two n x n complex matrices $B = (b_{ij})$ and $C = (c_{ij})$ equimodular if $|b_{ij}| = |c_{ij}|$, $i,j = 1,2,\ldots,n$. Clearly the relation "B equimodular with C" establishes an equivalence on the set of all n x n complex matrices, and we shall denote by $\mathcal{S}(B)$ the class of all n x n complex matrices equimodular with a given matrix B. We shall denote by $\mathcal{R}(B)$ the set of all eigenvalues of all matrices in $\mathcal{S}(B)$. When we discuss certain general features of $\mathcal{R}(B)$ which do not depend on B, it will often be convenient to refer to this set as the eigenvalue set of B or even as the eigenvalue set of a general matrix. It is the purpose of this thesis to investigate such eigenvalue sets.

In this introductory section, we shall review what is already known about eigenvalue sets and conduct a heuristic survey of problems toward whose solution we may profitably direct the course of our investigations. We shall also introduce definitions and notational conventions which recur throughout our work, and we shall state without proof several well-known results which we shall need in later sections.

Not a great deal is known about the eigenvalue set of a general matrix. O. Taussky [2] asked for a characterization of this set, and perhaps the best response to this request is found in a paper by Varga and Levinger [3], in which the authors characterize the eigenvalue set in terms of minimal Gerschgorin sets [4,5]. This characterization is particularly elegant for the eigenvalue set of an essentially diagonally dominant matrix, but becomes more complex in the general case. In Section IV, we shall once again have cause to refer to this paper, for one of the main results of that section is quite similar to a conjecture of Varga and Levinger, although the approach we use is completely different from the method of minimal Gerschgorin sets.

Aside from the paper of Varga and Levinger, the most important guides we have for the investigation of the eigenvalue sets are the many results in the literature which deal with bounds on the set of eigenvalues of a given matrix. Perhaps the most important of these results are the Gerschgorin Disk Theorem and the Perron-Frobenius Theorem. These two results are stated in their entirety at the close of this section, but roughly speaking, the most important evidence they provide is that the eigenvalue set of a diagonally dominant matrix does not contain the origin and that if A is a non-negative matrix, then the largest non-negative boundary point of $\mathcal{R}(A)$ is an eigenvalue of A .

At this point, we can say very little about the eigenvalue set of a general matrix. However, we can prove that every such set has a certain basic form.

Theorem 0. Let B be an $n \times n$ complex matrix. Then $\mathcal{R}(B)$ is symmetric about the origin of the complex plane. Furthermore, $\mathcal{R}(B)$ is a closed set which has at most n components, and every component of $\mathcal{R}(B)$ contains the same number of eigenvalues (counting multiplicities) of each matrix in $\mathcal{S}(B)$.

Proof: First of all, if α is a positive number in $\mathcal{R}(B)$, then α is an eigenvalue of some matrix C in $\mathcal{S}(B)$. If θ is any real number such that $0 \leq \theta < 2\pi$, then $\alpha e^{i\theta}$ is an eigenvalue of $e^{i\theta}C$, and since $e^{i\theta}C$ is in $\mathcal{S}(B)$, it follows that $\mathcal{R}(B)$ contains every complex number on the circle $|z| = \alpha$.

Now, let R be any matrix in $\mathcal{S}(B)$ other than B . It is well-known -- e.g. [7] -- that the eigenvalues of an n x n complex matrix regarded as a function of n^2 complex variables are continuous functions of those variables. Since the domain of these functions -- i.e. $\mathcal{S}(B)$ -- is a compact, connected set in complex E^{n^2} space, it follows that $\mathcal{R}(B)$ is closed and that every component of $\mathcal{R}(B)$ must contain exactly the same number of eigenvalues of R (including multiplicities) as it does those of B . Since B can have at most n distinct eigenvalues and since R was chosen arbitrarily from $\mathcal{S}(B)$, this completes the proof of the theorem.

Except for the basic form described in Theorem 0 , none of the features of the eigenvalue set of a general matrix is immediately discernible. In fact, matrices which are very "similar" can have completely different types of eigenvalue sets. For example, if I is the identity matrix, then $\mathcal{R}(I)$ is the unit circle. Let k be a positive number, and let $D_k = \text{diag}\ (1/k, 1/k^2, \ldots, 1/k^n)$. Then for each such D_k , $\mathcal{R}(D_k + I)$ consists of the union of n distinct circles. Thus, we see that two matrices which are "close" in the topological sense can have eigenvalue sets which do not even have the same number of components. It is just as obvious that two algebraically similar matrices may have entirely different types of eigenvalue sets. For example, let J be the matrix all of whose entries are equal to one. Then J is orthogonally similar to the diagonal matrix $D = \text{diag}(0, 0, \ldots, n)$. We shall see later that $\mathcal{R}(J)$ is the disk $|z| \leq n$, but $\mathcal{R}(D)$ is clearly the union of the origin and the circle $|z| = n$. Therefore, it appears that if we are to find a general technique for analyzing the eigenvalue set of a matrix, then very little can be gained by regarding a matrix as a linear transformation or as a point in complex E^{n^2} space.

Mindful of these preliminary observations, we shall begin our investigations of $\mathcal{R}(A)$ with a view toward answering the following questions :

1. Is every real boundary point of $\mathcal{R}(A)$ an eigenvalue of some real matrix in $\mathcal{S}(A)$?
2. If A and B are non-negative matrices such that $B \geqslant A$, under what conditions is it true that $\mathcal{R}(B) \supseteq \mathcal{R}(A)$?
3. Is there any essential difference between the case when $\mathcal{R}(A)$ contains the origin and the case when it does not ?
4. What causes the gaps between components in $\mathcal{R}(A)$? Is it possible to predict whether a gap will appear in $\mathcal{R}(A)$ without having precise knowledge of $\mathcal{R}(A)$ itself ?
5. Under what conditions is one of the annuli which comprise $\mathcal{R}(A)$ actually a circle ?

In addition to the above questions, there are several other important issues with which we shall deal in our investigations. However, it is difficult to motivate or even describe these problems until we have introduced a certain amount of basic terminology. Therefore, each section will begin with a paragraph which describes and motivates the main ideas and results contained in the section, and in Section VI, at the end of our investigations, we shall summarize the main results of our work.

It is convenient at this point to introduce a few basic definitions, notational conventions, and background results which will be used throughout our work.

First of all, a complex matrix B will be called regular if every matrix in $\mathcal{S}(B)$ is non-singular. Equivalently, B is regular if $\mathcal{R}(B)$ does not contain the origin.

Italic capital letters such as A,B,C, ... will be used to denote n x n complex matrices. The letter A will be reserved for non-negative matrices. Script letters with subindices -- e.g. $\mathcal{A}_{ij}$, $\mathcal{B}_{ij}$, $\mathcal{C}_{ij}$, $\mathcal{T}_{ij}$, $\mathcal{V}_{ij}$, -- will be used to denote the determinant of the n-1 x n-1 submatrix formed by deleting the i^{th} row and j^{th} column from the n x n complex matrix denoted by the corresponding italic capital letter -- e.g. A,B,C,T,V . Although script letters are also used in the symbols $\mathcal{S}(A)$, $\mathcal{R}(A)$ as well as in the symbols for certain other sets which will be introduced later, there should be no difficulty in deducing from the context in which a script letter appears whether it denotes a set or a determinant. If $B = (b_{ij})$ is a complex matrix, the symbol $|B|$ will denote the non-negative matrix whose $(i,j)^{th}$ entry is $|b_{ij}|$. This symbol should not be confused with the symbol sometimes used in the literature to denote the determinant of B . We shall always denote the determinant of B by (det B) .

The symbol " | " will be used to indicate the end of the proof of a theorem or a lemma.

A set of n matrix positions which includes exactly one position from every row and every column will be called a <u>generalized diagonal.</u> For example, three of the $3!$ generalized diagonals of a 3×3 matrix are $\{(1,1), (2,2), (3,3)\}$; $\{(1,1), (2,3), (3,2)\}$; and $\{(1,2), (2,3), (3,1)\}$. A matrix whose only non-zero elements lie along a generalized diagonal will be called a <u>generalized permutation matrix.</u> Where we speak of such a matrix, we shall tacitly assume it to be non-singular -- i.e. we assume it has n non-zero terms.

A matrix A will be called <u>reducible</u> if there exists a permutation matrix Q such that

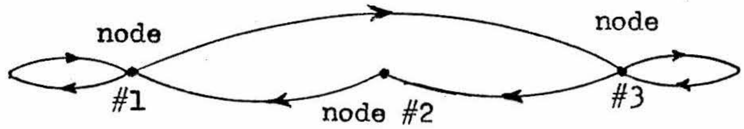
$$QAQ' = \left(\begin{array}{c|c} A_1 & A_2 \\ \hline 0 & A_3 \end{array}\right),$$

where A_1 and A_3 are square submatrices of A and 0 is a block of zeroes. If A is not reducible, it will be called irreducible.

Let $A = (a_{ij})$ be an $n \times n$ non-negative matrix. Then by the <u>graph of A</u> we mean the directed graph on n nodes which has an edge directed from node $\#k$ to node $\#\ell$ if and only if $a_{k\ell} \neq 0$. For example, consider the following matrix :

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 4 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

Then the following is the graph of A :



A sequence of edges in the graph of A which begins at node #r and ends at node #s will be called a path from node #r to node #s . The graph of A is said to be strongly connected if there exists at least one path from each node in the graph to every other node. Equivalently, the graph of A is strongly connected if there exists a closed path in the graph which passes through each and every node.

The symbol $\sum_{i \neq j}'$ will denote the deleted summation $\sum_{\substack{i=1 \\ i \neq j}}^{n}$. A complex n x n matrix $B = (b_{ij})$ will be called diagonally dominant if we have the following inequalities for $k = 1,2,\ldots, n$:

$$|b_{kk}| > \sum_{j \neq k}' |b_{kj}| \quad .$$

A complex matrix B will be called essentially diagonally dominant if there exists a positive diagonal matrix D such that BD is diagonally dominant.

Let $B = (b_{ij})$ be an n x n complex matrix. Then the disk in the complex plane described by the inequality

$$|z-b_{kk}| \leqslant \sum_{j \neq k}' |b_{kj}|$$

will be called the k^{th} Gerschgorin disk of B . We say that this disk is isolated if its intersection with the union of the other (n-1)

Gerschgorin disks of B is void. Thus, it follows that an isolated Gerschgorin disk contains one and only one diagonal element of B .

Finally, if B is an n x n complex matrix, then by the spectral radius of B we shall mean the maximum of the moduli of the eigenvalues of B . Thus, if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of B , the spectral radius of B is equal to $\max_{1 \leq k \leq n} |\lambda_k|$.

We conclude this introductory section by stating without proof five basic results which are used throughout our work.

i) Let f be a continuous, real-valued function whose domain is a connected set in a finite-dimensional metric space. Then if the range of f includes both positive and negative numbers, it must also include zero. If the domain of f is also compact, then its range is a closed segment of the real line.

ii) Gerschgorin Disk Theorem. Let B be a complex n x n matrix. Then all the eigenvalues of B are contained in the union of the n Gerschgorin disks of B . Furthermore, if the k^{th} Gerschgorin disk of B is isolated, then this disk contains exactly one of the eigenvalues of B .

It follows from ii) that a diagonally dominant matrix B must be regular, since the union of the Gerschgorin disks of any matrix in $\mathcal{S}(B)$ cannot contain the origin.

iii) Perron-Frobenius Theorem. Let A be a non-negative, irreducible matrix. Then A has a simple, positive eigenvalue α which is equal to the spectral radius of A and which is associated with a positive eigenvector. Furthermore, if B is any complex matrix such that $A \geqslant |B|$, then α is greater than the spectral radius of B with the sole exception of the case when there exists a unitary diagonal matrix D and a real number θ such that $A = (e^{i\theta})DBD^{-1}$, in which case B has a simple eigenvalue of modulus α .

Clearly, this result implies that the largest positive boundary point of $\mathcal{R}(A)$ is an eigenvalue of A , since every matrix B in $\mathcal{S}(A)$ is such that $A \geqslant |B|$.

iv) Let B be a complex matrix. Then the matrix power series $I + B^2 + B^3 + \ldots$ converges to $(I-B)^{-1}$ if and only if the spectral radius of B is less than one.

v) A matrix is irreducible if and only if its graph is strongly connected.

II. REGULAR MATRICES

The main purpose of this section is to introduce and develop a technique which will prove to be the crux of all our main results. Using this technique, we shall establish a natural means of associating each regular matrix with a permutation matrix, and as a by-product of this result, we give a new proof of a theorem of Camion and Hoffman which completely characterizes regular matrices. Finally, we obtain a property of non-negative regular matrices which will be used to a great extent in Section IV .

We begin this section with a lemma which expresses analytically the obvious fact that m line segments can be laid out in the form of a closed m-gon if and only if the length of no single one of them exceeds the sum of the lengths of the rest.

Lemma 1.1. *Let* $x_1 \geq x_2 \geq \cdots \geq x_m$ *be a set of* m *non-negative numbers.* Then it is possible to find real numbers $\theta_1, \theta_2, \ldots, \theta_m$ *such that* $\sum_{j=1}^{m} x_j e^{i\theta_j} = 0$ if and only if $x_1 \leq \sum_{j=2}^{m} x_j$.

Proof: Suppose for real numbers $\theta_1, \theta_2, \ldots, \theta_m$ we have $\sum_{j=1}^{m} x_j e^{i\theta_j} = 0$. This can be written as $-x_1 e^{i\theta_1} = \sum_{j=2}^{m} x_j e^{i\theta_j}$ and it follows that $x_1 = \left| \sum_{j=2}^{m} x_j e^{i\theta_j} \right| \leq \sum_{j=2}^{m} x_j$.

To prove the converse, we use induction on m . For $m = 2$, if $x_1 \leq x_2$, we must have $x_1 - x_2 = 0$ since $x_1 \geq x_2$ by assumption.

In the general case, suppose we have the inequality $x_1 \leqslant \sum_{j=2}^{m} x_j$. Then if we have $x_1 + x_m \leqslant \sum_{j=2}^{m-1} x_j$ we may apply the inductive hypothesis to the m-1 numbers $x_1 + x_m, x_2, \ldots, x_m$ to obtain the desired result. However, suppose $x_1 + x_m > \sum_{j=2}^{m-1} x_j$ so that $x_1 > (\sum_{j=2}^{m-1} x_j) - x_m \geqslant 0$. Let $z(\theta) = (\sum_{j=2}^{m-1} x_j) - x_m$. Then $z(\theta)$ varies continuously between $\sum_{j=2}^{m} x_j$ and $(\sum_{j=2}^{m-1} x_j) - x_m$ as θ varies between π and 0 . Since we have $(\sum_{j=1}^{m-1} x_j) - x_m < x_1 \leqslant \sum_{j=1}^{m} x_j$ it follows that $|z(\theta_1)| = x_1$ for some θ_1 with $0 \leqslant \theta_1 \leqslant \pi$. Therefore, there exists a number θ_2 with $0 \leqslant \theta_2 \leqslant 2\pi$ such that

$$z(\theta_1) = x_1 \, e^{i\theta_2} = (\sum_{j=1}^{m-1} x_j) - x_m \, e^{i\theta_1} .$$

This can be re-written as

$$x_1 \, e^{i\theta_2} + \sum_{j=1}^{m-1} x_j \, e^{i\pi} + x_m \, e^{i\theta_1} = 0 ,$$

and the proof of the lemma is complete. ∎

It may appear that we can gain little insight into the nature of eigenvalue sets of general matrices by concentrating on regular matrices, for "most" matrices are not regular. However, as we shall show in Section III, if A is a complex matrix and λ is a positive number not in $\mathcal{R}(A)$, then the translated matrix $A - \lambda I$ must be regular. This indicates that a characterization of regular matrices is of basic importance in discussing many of the features of the eigenvalue set of a general complex matrix. The following lemma is the first step in the derivation of such a characterization.

<u>Lemma 1.2. An $n \times n$ non-negative matrix A is regular if and only if for each (row) index</u> k and every matrix B in $\mathcal{S}(A)$, we have $\sum_{j=1}^{n} |b_{kj}|\,|\mathcal{B}_{kj}|\, e^{i\theta_j} \neq 0$ for any n real numbers $\theta_1, \theta_2, \ldots, \theta_n$.

Proof: Suppose A is not regular. Then there exists a singular matrix $B = (b_{ij})$ in $\mathcal{S}(A)$. Let k be a (row) index. Then since B is singular, we must have

$$\sum_{j=1}^{n} (-1)^{|k-j|}\, b_{kj}\, \mathcal{B}_{kj} = \det B = 0 \ .$$

Clearly this implies the existence of real numbers $\theta_1, \theta_2, \ldots, \theta_n$ such that $\sum_{j=1}^{n} |b_{kj}|\,|\mathcal{B}_{kj}|\, e^{i\theta_j} = 0$.

Conversely, suppose A is regular. Let k be a (row) index and let B be an arbitrary element of $\mathcal{S}(A)$. Suppose there exist real numbers $\theta_1, \theta_2, \ldots, \theta_n$ such that $\sum_{j=1}^{n} |b_{kj}|\,|\mathcal{B}_{kj}|\, e^{i\theta_j} = 0$.
For $j = 1,2,\ldots,n$, let γ_j be the argument of the complex number $b_{kj}\,\mathcal{B}_{kj}$. Then we have

$$\sum_{j=1}^{n} b_{kj}\, \mathcal{B}_{kj}\, e^{i(\theta_j - \gamma_j)} = 0 \ .$$

Clearly, this implies the existence of a singular matrix C which differs from B only in the signs of the elements of the k^{th} row. Therefore, this matrix C is in $\mathcal{S}(A)$, and this contradicts the fact that A is regular. Hence, it is impossible to find numbers $\theta_1, \theta_2, \ldots, \theta_n$ such that $\sum_{j=1}^{n} |b_{kj}|\,|\mathcal{B}_{kj}|\, e^{i\theta_j} = 0$. Since k was an arbitrary (row) index and since B was chosen arbitrarily from $\mathcal{S}(A)$ this completes the proof of the lemma. ▌

Lemma 1.2 indicates that we can obtain valuable information about regular matrices by investigating the set of complex numbers

$$\left\{ b_{k1}\,\mathcal{B}_{k1} ,\; b_{k2}\,\mathcal{B}_{k2} ,\; \ldots ,\; b_{kn}\,\mathcal{B}_{kn} \right\}$$

for a fixed row index k. We shall now combine the results of Lemma 1.1 and Lemma 1.2 to demonstrate an important property of this set, but in anticipation of this and later results, it is convenient to first introduce a function which proves to be a valuable tool in our investigations.

<u>Definition 1</u>. Let $B = (b_{ij})$ be a complex $n \times n$ matrix. Let r be an arbitrary (row) index and s, an arbitrary (column) index. Then $F_{rs}(B)$ will denote the following function:

$$F_{rs}(B) = |b_{rs}|\,|\mathcal{B}_{rs}| - \sum_{j \neq s}{}' |b_{rj}|\;|\mathcal{B}_{rj}|$$

Clearly, for fixed r and s, $F_{rs}(B)$ is a continuous, real-valued function of the elements of B if B is regarded as a point in complex E^{n^2} space. One reason why this function is so useful is contained in the following lemma.

<u>Lemma 1.3. Let B be a complex matrix. If there exist two indices r and s such that $F_{rs}(B) = 0$, then B is not regular. If there exist two indices k and ℓ such that $F_{k\ell}(C) > 0$ for every matrix C in $\mathcal{S}(B)$, then B is regular.</u>

Proof: The first statement is an obvious consequence of Lemma 1.2. Now let k and ℓ be two indices such that $F_{k\ell}(C) > 0$ for every matrix in $\mathcal{S}(B)$. Suppose B is not regular. Then there exists a matrix C in $\mathcal{S}(B)$ which is singular, and we must have

$$\sum_{j=1}^{n}(-1)^{|k-j|}\, c_{kj}\,\mathcal{C}_{kj} = \det C = 0 .$$

From this it follows that

$$|c_{k\ell}|\,|\mathcal{C}_{k\ell}| \leq \sum_{j\neq\ell}{}' |c_{kj}|\,|\mathcal{C}_{kj}| \quad ,$$

and we must have $F_{k\ell}(C) \leq 0$. However, this contradicts the assumption that $F_{k\ell}(C) > 0$, and we conclude that B is regular, as desired. |

Now we prove a converse to the second part of Lemma 1.3 .

Lemma 1.4. Let A be a regular matrix. Then for each (row) index k , there exists one and only one (column) index $\ell = \ell(k)$ such that $F_{k\ell}(B) > 0$ for all matrices B in $\mathcal{S}(A)$.

Proof: Let k be a (row) index which will remain fixed throughout the proof. Since A is regular, we conclude from Lemma 1.2 that it is impossible to find real numbers $\theta_1,\theta_2,\ldots,\theta_n$ such that

$$\sum_{j=1}^{n} |b_{kj}|\,|\mathcal{B}_{kj}|\, e^{i\theta_j} = 0 \ .$$

Choose ℓ so that $|b_{k\ell}|\,|\mathcal{B}_{k\ell}| = \max_{1\leq j\leq n} |b_{kj}|\,|\mathcal{B}_{kj}|$. Then it follows from Lemma 1.1 that we must have

$$|b_{k\ell}|\,|\mathcal{B}_{k\ell}| > \sum_{j\neq K}{}' |b_{kj}|\,|\mathcal{B}_{kj}| \ .$$

Moreover, ℓ is unique in the sense that for $m \neq \ell$ we must have

$$|b_{km}|\,|\mathcal{B}_{km}| < \sum_{j\neq m}{}' |b_{kj}|\,|\mathcal{B}_{kj}| \ .$$

Thus, we conclude that $F_{k\ell}(B) > 0$, but we still must show that this index ℓ depends only on k and not on the choice of B from $\mathcal{S}(A)$. Assume that there exists a matrix C in $\mathcal{S}(A)$ such that

$F_{k\ell}(C) \leqslant 0$. Since $F_{k\ell}$ is a continuous, real-valued function on $\mathcal{S}(A)$, and since $\mathcal{S}(A)$ is a connected set in complex E^{n^2} space which contains both B and C , it follows that $F_{k\ell}(T) = 0$ for some matrix T in $\mathcal{S}(A)$. However, this contradicts the fact that A is regular, and we conclude that $F_{k\ell}(C) > 0$ for every matrix C in $\mathcal{S}(A)$. Therefore, the (column) index ℓ depends only on the (row) index k and not on the choice of B from $\mathcal{S}(A)$, as desired. |

As a consequence of Lemma 1.4, we can now define the important concept of a pivotal position for a class of regular matrices.

Definition 2. Let A be a regular matrix and suppose that (k,ℓ) is an ordered pair of indices such that $F_{k\ell}(A) > 0$. Then we say that (k,ℓ) is a pivotal position for matrices in $\mathcal{S}(A)$, and if $B = (b_{ij})$ is any matrix in $\mathcal{S}(A)$, the element $b_{k\ell}$ will be called the pivotal element in the k^{th} row of B .

It follows from Lemma 1.4 that every row of a regular matrix contains exactly one pivotal element and that the position of this element is an invariant of the class $\mathcal{S}(A)$. Moreover, we shall show in Lemma 1.5 that the set of pivotal positions of the class $\mathcal{S}(A)$ is actually an invariant of a much larger class of regular matrices, which we shall now define.

Definition 3. Let $B = (b_{ij})$ be a regular n x n matrix. We shall denote by $\mathcal{N}(B)$ the set of all complex matrices $C = (c_{ij})$ whose elements satisfy the following inequalities for $k = 1,2,\ldots,n$:

$$|c_{k\ell}| \geq |b_{k\ell}| \quad \text{where } b_{k\ell} \text{ is the pivotal element in the } k^{th} \text{ row of } B \tag{2.1}$$
$$|c_{ks}| \leq |b_{ks}| \quad \text{for } s = 1,2,\ldots,n \quad s \neq \ell$$

i.e. such a matrix C is obtained from B by increasing (in modulus) the pivotal elements in B and by decreasing (in modulus) the others. For example, consider the matrix

$$B = \begin{pmatrix} 3 & 2 & 3 \\ 2 & 8 & 3 \\ 1 & 6 & 15 \end{pmatrix}$$

We shall see that B is regular and that the pivotal positions of B are (1,1); (2,2); and (3,3) . Therefore, a typical member of $\mathcal{N}(B)$ is the following complex matrix:

$$C = \begin{pmatrix} 5 & 0 & 1 \\ 2 & 9i & 2 \\ 0 & 2+2i & 15 \end{pmatrix}$$

<u>Lemma 1.5. Let A be a regular n x n matrix. Every matrix in $\mathcal{N}(A)$ is regular and has the same pivotal positions as matrices in $\mathcal{S}(A)$</u> .

Proof: To simplify notation, we shall assume that A is the non-negative matrix in $\mathcal{S}(A)$. We shall first prove the lemma for a subset of $\mathcal{N}(A)$ and then proceed recursively to the completely general case. Let k be a fixed (row) index, and let $\mathcal{N}_k(A)$ be the set of all complex matrices $B = (b_{ij})$ whose elements satisfy the following inequalities for $k = 1,2,\ldots,n$:

$$|b_{k\ell}| \geq a_{k\ell} \quad \text{where } a_{k\ell} \text{ is the pivotal element in the } k^{th} \text{ row of } A$$

$$|b_{ks}| \leq a_{ks} \quad s = 1,2,\ldots,n \quad s \neq \ell \tag{2.2}$$

$$|b_{ij}| = a_{ij} \quad i,j = 1,2,\ldots,n \quad i \neq k$$

In other words, $\mathcal{N}_k(A)$ is the subset of $\mathcal{N}(A)$ which contains those matrices which differ from certain matrices in $\mathcal{S}(A)$ only in the elements of the k^{th} row. Let $B = (b_{ij})$ be an arbitrary matrix in $\mathcal{N}_k(A)$, and let $C = (c_{ij})$ be the corresponding matrix in $\mathcal{S}(A)$ -- i.e. B and C are identical except for the elements of the k^{th} row ; for $j = 1,2,\ldots,n$, the modulus of the complex number c_{kj} satisfies the inequalities (2.2) , and its argument is equal to the argument of b_{kj} . Then the inequalities (2.2) , together with the fact that $F_{k\ell}(C) > 0$, imply that $F_{k\ell}(B) > 0$, since the minors of the elements of the k^{th} row of B are identical with the minors of the corresponding elements of the k^{th} row of C -- i.e. we have $\mathcal{B}_{kj} = \mathcal{C}_{kj}$, $j = 1,2,\ldots,n$. Therefore, since B was chosen arbitrarily from $\mathcal{N}_k(A)$, it follows from Lemma 1.3 that every matrix in $\mathcal{N}_k(A)$ is regular and has (k,ℓ) as a pivotal position.

Now suppose (r,s) is a pivotal position for matrices in $\mathcal{S}(A)$ but not for matrices in $\mathcal{S}(B)$. Then we must have $F_{rs}(A) > 0$ and $F_{rs}(B) \leq 0$. Since $\mathcal{N}_k(A)$ is a connected subset of complex E^{n^2} space which contains both A and B , it follows that there exists a matrix T in $\mathcal{N}_k(A)$ such that $F_{rs}(T) = 0$. However, this implies that T is not regular, and this contradicts the already established fact that every matrix in $\mathcal{N}_k(A)$ is regular. Hence, we conclude that $F_{rs}(B) > 0$, and (r,s) must be a pivotal position for all matrices in $\mathcal{N}_k(A)$. Therefore, every matrix in $\mathcal{N}_k(A)$ is regular and has the same pivotal positions as matrices in $\mathcal{S}(A)$.

Now, let $U = (u_{ij})$ be an arbitrary matrix in $\mathcal{N}(A)$. We define the following sequence of matrices $\{V_s\}$:

$$V_s = (v_{ij}^{(s)}) \ , \quad \text{where} \quad v_{rj}^{(s)} = u_{rj} \quad r \leqslant s \quad j = 1,2,\ldots,n$$

$$\text{and} \quad v_{ij}^{(s)} = a_{ij} \quad i > s \quad j = 1,2,\ldots,n$$

Since U is in $\mathcal{N}(A)$, we know that V_1 is in $\mathcal{N}_1(A)$, and by what we have already proved, it follows that V_1 is regular and has the same pivotal positions as matrices in $\mathcal{S}(A)$. Then V_2 is in $\mathcal{N}_2(V_1)$, and V_2 is regular and has the same pivotal positions as matrices in $\mathcal{S}(V_1)$ and $\mathcal{S}(A)$. Proceeding in like fashion, we conclude that V_n is regular and has the same pivotal positions as matrices in $\mathcal{S}(V_{n-1}), \mathcal{S}(V_{n-2}), \ldots, \mathcal{S}(V_1)$ and $\mathcal{S}(A)$, and since $V_n = U$ by construction, this completes the proof of the lemma. ❙

Now we are in a position to prove our first main result, which says that if A is regular, then the pivotal positions of the class $\mathcal{S}(A)$ lie along a generalized diagonal. In anticipation of future applications, this result will be stated in a slightly different, although equivalent form.

Theorem 1. Let A be a regular matrix. Then there exists a permutation matrix P such that PA is a regular matrix all of whose pivotal positions lie along the main diagonal.

Proof: First we shall show that the pivotal positions of the class $\mathcal{S}(A)$ lie along a generalized diagonal. For this purpose, we consider the matrix $B = (b_{ij})$ whose elements satisfy the following conditions for $k = 1,2,\ldots,n$:

$$b_{k\ell} = a_{k\ell} \quad \text{if } a_{k\ell} \text{ is the pivotal element in the } k^{th} \text{ row of } A$$
$$b_{ks} = 0 \quad s = 1,2,\ldots,n \quad s \neq \ell$$

Clearly B is in $\mathcal{N}(A)$. Therefore, by Lemma 1.5, B must be regular and have the same pivotal positions as matrices in $\mathcal{S}(A)$. However, each row of B has exactly one non-zero element, and such a matrix can be non-singular if and only if it is actually a generalized permutation matrix. Therefore, the pivotal positions of the class $\mathcal{S}(B)$ and hence of the class $\mathcal{S}(A)$ lie along a generalized diagonal. Let $Q = (q_{ij})$ be that permutation matrix whose elements satisfy the following conditions for $k = 1,2,\ldots,n$:

$$q_{k\ell} = 1 \text{ if } b_{k\ell} \neq 0$$
$$q_{ks} = 0 \text{ if } b_{ks} = 0$$

Let $P = Q^{-1}$. It is clear that PA is a regular matrix, and since pre-multiplication by a permutation matrix merely permutes the rows of a matrix, we conclude that if (r,s) is a pivotal position in A , then (s,s) is a pivotal position in PA. Therefore, all the pivotal positions of PA lie along the main diagonal, as desired. |

Corollary 1.1. *If A and P are as in Theorem 1, then AP is also a regular matrix whose pivotal elements all lie along the main diagonal.*

Proof: It is clear that AP is regular, and the location of the pivotal elements of AP follows from the fact that BP is a diagonal matrix, where B is the matrix defined in the proof of Theorem 1. |

Since the pivotal positions of a regular matrix A lie along a generalized diagonal, it is clear that if P_1 and P_2 are permutation matrices such that P_1A and P_2A both have all their pivotal positions along the main diagonal, then we must have $P_1 = P_2$. This fact allows us to make the following definition.

Definition 4. Let A be a regular matrix, and let P be that (unique) permutation matrix such that PA has all its pivotal positions along the main diagonal. Then we say that A is *associated with* P. If A is associated with I, the identity matrix, we say that A is a regular matrix in *normal form*.

The results of Theorem 1 and Lemma 1.5 indicate that a regular matrix in normal form behaves in many ways like a diagonal matrix. Indeed, we shall show that a regular matrix in normal form is actually essentially diagonally dominant, and this fact when combined with Theorem 1 gives the characterization of regular matrices which constitutes Theorem 2. The following lemma is used primarily in Theorem 2 to provide a connection between Theorem 1 and the concept of diagonal dominance, but it is also a rather interesting result in itself.

Lemma 2.1. *Let* A *be an* $n \times n$ *non-negative regular matrix in normal form, and let* $B = (b_{ij})$ *be any matrix in* $\mathcal{N}(A)$ *such that* $b_{kk} > 0$ *for* $k = 1,2,\ldots,n$. *Then the real parts of all the eigenvalues of* B *are positive.*

Proof: Let λ be an eigenvalue of B , and suppose that the real part of λ is non-positive. Write λ as $\lambda = \operatorname{Re}\lambda + i \operatorname{Im}\lambda$. Then for $k = 1,2,\ldots,n$, we have $b_{kk} - \operatorname{Re}\lambda = b_{kk} + |\operatorname{Re}\lambda| \geqslant a_{kk}$ since $b_{kk} \geqslant a_{kk}$ and $\operatorname{Re}\lambda \leqslant 0$. It follows that $B - \lambda I$ is in $\mathcal{N}(A)$, since B is in $\mathcal{N}(A)$ and we have for $k = 1,2,\ldots,n$:

$$|b_{kk} - \lambda| = \left[(b_{kk} + |\operatorname{Re}\lambda|)^2 + (\operatorname{Im}\lambda)^2 \right]^{\frac{1}{2}} \geqslant a_{kk}$$

Therefore, it follows from Lemma 1.5 that $B - \lambda I$ is regular, and we conclude that $B - \lambda I$ cannot be singular. This contradicts the assumption that λ is an eigenvalue of B , and we conclude that the real part of λ must be positive, as desired. ▌

In a recent paper, Camion and Hoffman [1] proved that a matrix A is regular if and only if there exists a permutation matrix P and a positive diagonal matrix D such that PAD is diagonally dominant. In Theorem 2, we obtain the same characterization as a consequence of Theorem 1 and Lemma 2.1, although in order to demonstrate the existence of a suitable positive diagonal matrix D , we are forced to use a technique which has little in common with the rest of the work in this section.

<u>Theorem 2. Let A be an $n \times n$ non-negative matrix. Then A is regular if and only if there exists a permutation matrix P and a positive diagonal matrix D such that PAD is diagonally dominant.</u>

Proof: Suppose PAD is diagonally dominant. Let B be any matrix in $\mathcal{S}(A)$. Then PBD is diagonally dominant and must therefore be non-singular. Hence, B is non-singular, and since B was chosen arbitrarily from $\mathcal{S}(A)$, we conclude that A is regular.

Conversely, suppose A is regular. Let P be the permutation matrix associated with A , and let $B = PA$. We must show that B is essentially diagonally dominant. Let $C = (c_{ij})$ be the matrix whose elements satisfy the following conditions:

$$c_{ij} = -b_{ij} \quad i,j = 1,2,\ldots,n \quad i \neq j$$
$$c_{kk} = b_{kk} \quad k = 1,2,\ldots,n$$

We shall first show that C^{-1} is a non-negative matrix, and then we shall use this fact to show that B is essentially diagonally dominant.

Choose $\sigma > 0$ so small that the matrix $S = I - \sigma C$ is non-negative. Let $p(S)$ be the largest non-negative real boundary point of $\mathcal{R}(S)$. Then it follows from the Perron-Frobenius Theorem that $p(S)$ is actually an eigenvalue of S , and we must have the following:

$$0 = \det(S - p(S)I) = \det[(I-\sigma C) - p(S)I] = \det([1-p(S)]I - \sigma C) .$$

Consequently, $[1-p(S)]\sigma^{-1}$ is a real eigenvalue of C, and it follows from Lemma 2.1 that this eigenvalue must be positive. Since σ is positive, we conclude that $1 > p(S) \geqslant 0$, and this inequality insures the convergence of the matrix series $I + S + S^2 + \ldots$ to the matrix $(I-S)^{-1} = (\sigma C)^{-1}$. Since S and all its powers are non-negative matrices, we conclude that $(\sigma C)^{-1} = \sigma^{-1}C^{-1}$ is also non-negative, and since $\sigma > 0$, it follows that C^{-1} is a non-negative matrix.

Let e_n denote the transpose of the row vector $(1,1,\ldots,1)$, and consider the vector $x = C^{-1}e_n$. Since C^{-1} is non-negative, we know that $x > 0$. Let D be the positive diagonal matrix such that $x = De_n$. Then the vector CDe_n is positive, and this means we have, for $i = 1,2,\ldots n$:

$$d_i c_{ii} - \sum_{j \neq i}' c_{ij} d_j > 0$$

Clearly, this is equivalent to the statement that C is essentially diagonally dominant, and this completes the proof of the theorem. |

<u>Corollary 2.1. Let A be a non-negative regular matrix, and let P and D be as in Theorem 2. If B is any matrix in $\mathcal{N}(A)$, then PBD is also diagonally dominant.</u>

Proof: This is a simple consequence of the inequalities (2.1), which arise in the definition of $\mathcal{N}(A)$. |

<u>Corollary 2.2. A diagonally dominant matrix is a regular matrix in normal form.</u> |

With Theorem 2, it is easy to see why the matrix

$$B = \begin{pmatrix} 3 & 2 & 3 \\ 2 & 8 & 3 \\ 1 & 6 & 15 \end{pmatrix}$$

which appeared on page 16 is regular. Let $D = \text{diag}\,(1, 1/2, 1/3)$. Then we have

$$BD = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 1 \\ 1 & 3 & 5 \end{pmatrix}$$

Since BD is diagonally dominant, it follows from Corollary 2.2 that B is a regular matrix in normal form.

While the characterization of regular matrices given in Theorem 2 provides an excellent means of constructing examples of various types of regular matrices, it would be desirable for certain purposes to have a more tractable test for non-regularity. For example, it is obvious that the matrix

$$J = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

is not regular, but this is not an immediate consequence of Theorem 2, and using Theorem 2 alone, it is quite difficult to see why the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

is not regular. However, in Theorem 3 we obtain a *necessary* condition for the regularity of a matrix which is usually much easier to apply than Theorem 2.

Theorem 3. Let A be an n x n non-negative, regular matrix. Then the product of the pivotal elements of A is greater than the product of the elements of A which lie along any other generalized diagonal.

Proof: Because of Theorem 1, it suffices to prove the theorem for regular matrices in normal form. Therefore, we assume that the pivotal elements of A are all diagonal elements.

Let $(1,\ell_1)$, $(2,\ell_2),\ldots,$ (n,ℓ_n) be any generalized diagonal which contains at least one off-diagonal position, and let $B = (b_{ij})$ be that non-negative matrix whose elements satisfy the following conditions, for $k = 1,2,\ldots,n$:

$$b_{kk} = a_{kk}$$
$$b_{k\;k} = a_{k\ell_k}$$
$$b_{kj} = 0 \qquad\qquad j = 1,2,\ldots,n \quad j \neq k,\ell_k$$

Thus, B has non-zero elements along only the main diagonal and the generalized diagonal under consideration. Furthermore, B is in $\mathcal{N}(A)$ and is therefore a regular matrix in normal form, by Lemma 1.5 . If the permutation $k \leftrightarrow \ell_k$ is not cyclic of order n , then B is reducible, and the theorem follows by induction, since each diagonal block of a block triangular regular matrix must be regular. Hence, we may assume that $k \leftrightarrow \ell_k$ is cyclic of order n . Suppose we have

$$\prod_{k=1}^{n} b_{k\ell_k} = \prod_{k=1}^{n} a_{k\ell_k} \geqslant \prod_{k=1}^{n} a_{kk} = \prod_{k=1}^{n} b_{kk} .$$

Then since the generalized diagonal under consideration contains at least one off-diagonal position, it follows that by decreasing only the off-diagonal elements of B, we can construct a non-negative matrix $C = (c_{ij})$ which is in $\mathcal{N}(B)$ and for which we have

$$\prod_{k=1}^{n} c_{k\ell_k} = \prod_{k=1}^{n} c_{kk} = \prod_{k=1}^{n} b_{kk} = \prod_{k=1}^{n} a_{kk}$$

Since $k \leftrightarrow \ell_k$ is cyclic of order n, there exists a permutation matrix P such that

$$PCP' = \begin{pmatrix} c'_{11} & c'_{1\ell_1} & & & 0 \\ & c'_{22} & c'_{2\ell_2} & & \\ & 0 & \ddots & \ddots & \\ & & & \ddots & c'_{n-1,\ell_{n-1}} \\ c'_{n\ell_n} & & & & c'_{nn} \end{pmatrix}$$

However, the matrix C cannot be regular, for we have

$$\det(PCP') = \prod_{k=1}^{n} c'_{kk} \pm \prod_{k=1}^{n} c'_{k\ell_k} = \prod_{k=1}^{n} c_{kk} \pm \prod_{k=1}^{n} c_{kk} \quad .$$

This contradicts the fact that all matrices in $\mathcal{N}(B)$ are regular, and we conclude that the theorem holds in all cases. |

We conclude from Theorem 3 that if the product of the elements along no single generalized diagonal of A exceeds the product of the elements along every other generalized diagonal, then A cannot be regular. For example, consider the following matrix:

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} .$$

No generalized diagonal product exceeds 2, but two different such products equal 2. Thus, no single generalized diagonal product of A dominates the rest, and A cannot be regular. Unfortunately, as one might suspect, even if this condition is satisfied, it is in general not enough to guarantee that a matrix be regular. For example, the matrix

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

is equimodular with the singular matrix

$$B = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

even though the product of the diagonal elements of A is greater than the product of the elements along any other generalized diagonal. Despite its limitations, Theorem 3 will nevertheless prove to be the key to all our investigation in Section IV.

We conclude this section with several results which are interesting in themselves but are isolated from the main body of our investigations.

Theorem 4. *Let* A *be an* $n \times n$ *non-negative regular matrix in normal form. Let* P *be an* $n \times n$ *permutation matrix and let* $D = \mathrm{diag}\,(d_1, d_2, \ldots, d_n)$ *be a positive diagonal matrix. Then* PAP', DA, AD, and DAD^{-1} *are all regular matrices in normal form.*

Proof: It is clear that each of these four product matrices is regular. We shall use Theorem 3 to prove they are all in normal form.

Let $(1,\ell_1)$, $(2,\ell_2)$, ..., (n,ℓ_n) be any generalized diagonal. Then the elements $a_{1\ell_1}, a_{2\ell_2}, \ldots, a_{n\ell_n}$ also lie on a generalized diagonal of PAP'. In particular, the product of the diagonal elements of A equals that of PAP', and every other product of elements of PAP' which lie along a generalized diagonal is equal to a product of elements of A which lie along a generalized diagonal. Therefore, since A is in normal form, it follows from Theorem 3 that no such generalized diagonal product of elements of PAP' is as great as the product of the diagonal elements of PAP', and PAP' must be in normal form.

Now let us consider the matrices $DA = (d_i \, a_{ij})$ and $AD = (d_j \, a_{ij})$. Let $(1,\ell_1)$, $(2,\ell_2)$, ..., (n,ℓ_n) be any generalized diagonal which contains at least one off-diagonal position. Then since A is in normal form, we have

$$\prod_{k=1}^{n} d_k a_{k\ell_k} = \left(\prod_{k=1}^{n} d_k\right)\left(\prod_{k=1}^{n} a_{k\ell_k}\right) < \left(\prod_{k=1}^{n} d_k\right)\left(\prod_{k=1}^{n} a_{kk}\right) = \prod_{k=1}^{n} d_k a_{kk}$$

Since $(1,\ell_1)$, $(2,\ell_2)$, ..., (n,ℓ_n) was an arbitrary generalized diagonal, it follows from Theorem 3 that DA is in normal form. Similarly, we can show that

$$\prod_{k=1}^{n} d_{\ell_k} a_{k\ell_k} < \prod_{k=1}^{n} d_{\ell_k} a_{\ell_k \ell_k} \quad ,$$

and this allows us to conclude that AD is in normal form. Finally, since D^{-1} is also a positive diagonal matrix, it follows from what we have already proved that $(DA)D^{-1} = DAD^{-1}$ is in normal form. ▌

Theorem 5. *Let A be a non-negative regular $n \times n$ matrix in normal form, and let B be a principal submatrix of A -- i.e. B is a submatrix of A all of whose diagonal elements are also diagonal elements of A. Then B is a regular matrix in normal form.*

Proof: Let k be an integer such that $1 \leq k \leq n$, and let B be a $k \times k$ principal submatrix of A. Then there exists a permutation matrix P such that

$$PAP' = \left(\begin{array}{c|c} B & A_1 \\ \hline A_2 & A_3 \end{array}\right),$$

where A_3 is an $(n-k) \times (n-k)$ principal submatrix of A, and A_1 and A_2 are $k \times (n-k)$ and $(n-k) \times k$ submatrices of A, respectively. Therefore, since A is in normal form, it follows from Theorem 4 that PAP' is in normal form, and we see that it suffices to prove the theorem for the case when $B = (b_{ij})$ is the submatrix of A whose elements satisfy $b_{ij} = a_{ij}$, for $i,j = 1,2,\ldots,k$.

Now, let C be the $n \times n$ non-negative block diagonal matrix which has the following form:

$$C = \left(\begin{array}{c|c} B & O_1 \\ \hline O_2 & D \end{array}\right)$$

where O_1 , and O_2 are $k \times (n-k)$ and $(n-k) \times k$ blocks of zeroes, respectively, and $D = \text{diag}(a_{k+1,k+1}, a_{k+2,k+2}, \ldots, a_{nn})$. Then C is in $\mathcal{N}(A)$, and it follows from Lemma 1.5 that C is a regular matrix in normal form. Suppose B is not regular, and let U be a singular matrix in $\mathcal{S}(B)$. Then the singular matrix

$$T = \left(\begin{array}{c|c} U & O_1 \\ \hline O_2 & D \end{array}\right)$$

is in $\mathcal{S}(C)$, and this contradicts the fact that C is regular. Therefore, B must be regular. Because of the block diagonal form of C , it follows from Theorem 3 that the product of the diagonal elements of B must be greater than the product of the elements of B which lie along any other generalized diagonal, and B must be in normal form. Since k was an arbitrary integer such that $1 \leq k \leq n$ and since B was an arbitrary $k \times k$ principal submatrix of A , this completes the proof of the theorem. |

<u>Theorem 6. Let</u> A <u>be an</u> $n \times n$ <u>non-negative regular matrix, and let</u> P <u>be the permutation matrix associated with</u> A . <u>Let</u> B <u>be any real matrix in</u> $\mathcal{S}(A)$ <u>such that all the pivotal elements of</u> B <u>are positive. Then we have</u> det $B > 0$ <u>if</u> det $P = +1$ <u>and</u> det $B < 0$ <u>if</u> det $P = -1$.

Proof: Let $C = PB$. Since C is a regular matrix in normal form such that $c_{ii} > 0$ for $i = 1,2,\ldots,n$, it follows from Lemma 2.1 that all the real eigenvalues of C are positive, and since C is real, its complex eigenvalues occur in conjugate pairs. Hence, the product of the eigenvalues of C is positive, and since $B = P'C$, we must have $\det B = (\det P')(\det C) > 0$ if $\det P' > 0$, and $\det B < 0$ if $\det P < 0$. Since P is a permutation matrix, we have $\det P = \det P' = \pm 1$, and we conclude that $\det B$ has the same sign as $\det P$. ▎

Corollary 6.1. *Let A be an $n \times n$ non-negative regular matrix in normal form, and let $B = (b_{ij})$ be any real matrix in $\mathscr{S}(A)$ such that $b_{kk} > 0$ for $k = 1,2,\ldots,n$. Then $\det B > 0$.* ▎

Corollary 6.2. *Let $B = (b_{ij})$ be a real symmetric regular matrix in normal form. If $b_{kk} > 0$ for $k = 1,2,\ldots,n$, then B is positive definite.*

Proof: It follows from Theorem 5 that the principal submatrices of B of all orders are regular matrices in normal form. Since the diagonal elements of B are all positive, it follows from Corollary 6.1 that the determinant of each such principal submatrix must be positive. Therefore, B is positive definite, as desired. ▎

III. NORMAL GAPS IN THE EIGENVALUE SET OF A GENERAL COMPLEX MATRIX

In Section II, we established a natural means of associating each regular matrix with a permutation matrix, and we shall use this result to show that each gap between components of the eigenvalue set of a general complex matrix may be associated with a permutation matrix in a meaningful fashion. In this section we shall be mainly interested in gaps which are associated with the identity matrix; such gaps will be called normal gaps. We shall prove that the unbounded gap of every eigenvalue set is normal, and we shall show that certain interior gaps in the eigenvalue sets of special types of matrices are normal. We also prove that if $A = (a_{ij})$ is a non-negative regular matrix in normal form and $B = (b_{ij})$ is a matrix in $\mathcal{N}(A)$ such that $|b_{kk}| = a_{kk}$ for $k = 1,2,\ldots,n$, then the eigenvalue set of B is contained in that of A.

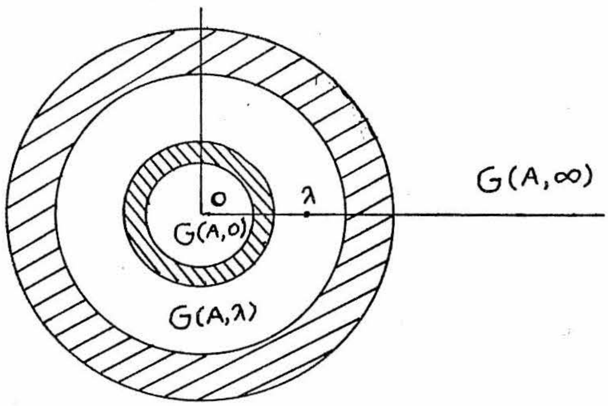
For several of the following results, it will be convenient to have a simple means of referring to the gaps in the eigenvalue set of a matrix.

<u>Definition 5</u>. Let B be an $n \times n$ complex matrix, and let λ be any non-negative number which is not in $\mathcal{R}(B)$. Then we shall denote by $G(B,\lambda)$ the entire gap in $\mathcal{R}(B)$ which contains λ. The unbounded gap in $\mathcal{R}(B)$ will always be denoted by $G(B,\infty)$. If B is regular, the gap which contains the origin will always be denoted

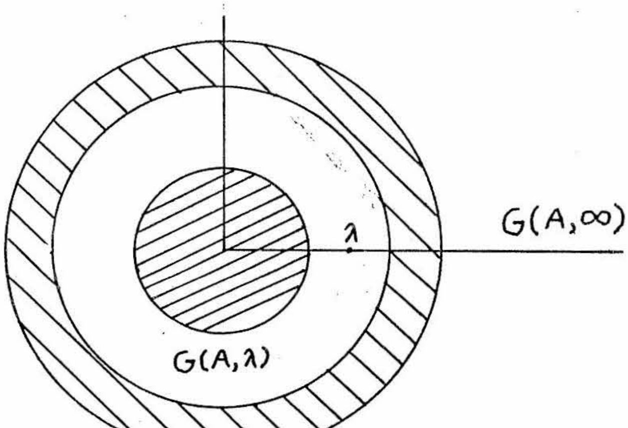
by G(B,0) . If $\lambda_1 \geqslant 0$ is in one gap of $\mathcal{R}$(B) and $\lambda_2 > \lambda_1$ is in another, then we say that G(B, λ_1) is the more interior of the two gaps and that G(B, λ_2) is the more exterior.

The following diagram illustrates how Definition 5 is used in both the regular and non-regular cases:

Regular Case



Non-Regular Case



Our first goal in this section is to show that each gap in the eigenvalue set of a general complex matrix may be associated with a permutation matrix. To obtain this result, we consider the following set of matrices, which is closely related to a certain gap in the eigenvalue set of a given matrix.

Definition 6. Let B be a complex $n \times n$ matrix, and let λ be a non-negative number not in $\mathcal{R}(B)$. Then we denote by $\mathcal{H}(B, \lambda)$ the set of all matrices of the form $C - \sigma I$, where C is a matrix in $\mathcal{S}(B)$ and σ is a non-negative number in the gap $G(B, \lambda)$.

We shall show in Theorem 7 that all the matrices in $\mathcal{H}(B, \lambda)$ are regular and are associated with the same permutation matrix in the sense of Definition 4, but first we must show that a non-regular matrix B is equimodular with a singular matrix which has the same diagonal elements as B.

Lemma 7.1. Let $B = (b_{ij})$ be a complex $n \times n$ matrix which is not regular. Then there exists a singular matrix $C = (c_{ij})$ in $\mathcal{S}(B)$ such that $c_{kk} = b_{kk}$, for $k = 1, 2, \ldots, n$.

Proof: Since B is not regular, there exists a singular matrix $U = (u_{ij})$ in $\mathcal{S}(B)$. For $k = 1, 2, \ldots, n$, let θ_k and α_k be the arguments of the complex numbers u_{kk} and b_{kk}, respectively, and consider the unitary diagonal matrix :

$$D = \operatorname{diag}\left(e^{i(\alpha_1 - \theta_1)}, \ldots, e^{i(\alpha_n - \theta_n)} \right).$$

Then the matrix $C = UD$ is singular, and we have for $k = 1,2,\ldots, n$:

$$c_{kk} = |u_{kk}| e^{i[\theta_k + (\alpha_k - \theta_k)]} = |b_{kk}| e^{i\alpha_k} = b_{kk} .$$

Since D is unitary, it follows that C is in $\mathcal{S}(B)$, and this completes the proof of the lemma. ▎

Theorem 7. Let B be a complex $n \times n$ matrix, and let λ be a positive number not in $\mathcal{R}(B)$. Then all the matrices in $\mathcal{H}(B, \lambda)$ are regular and are associated with the same permutation matrix.

Proof: Let $C - \sigma I$ be an arbitrary matrix in $\mathcal{H}(B, \lambda)$, where C is in $\mathcal{S}(B)$ and σ is a non-negative number in the gap $G(B, \lambda)$. Suppose $C - \sigma I$ is not regular. Then there exists a singular matrix $U = (u_{ij})$ in $\mathcal{S}(C - \sigma I)$, and because of Lemma 7.1 , we may assume that $u_{kk} = c_{kk} - \sigma I$, $k = 1,2,\ldots,n$. Let $V = U + \sigma I$. Then V is in $\mathcal{S}(B)$, and since U is singular, it follows that σ is an eigenvalue of V . However, this contradicts the fact that σ is not in $\mathcal{R}(B)$, and we conclude that $C - \sigma I$ must be regular. Since $C - \sigma I$ was chosen arbitrarily from $\mathcal{H}(B, \lambda)$, it follows that every matrix in $\mathcal{H}(B, \lambda)$ is regular.

Now we shall show that every matrix in $\mathcal{H}(B, \lambda)$ has the same pivotal positions as $B - \lambda I$. Once again let $C - \sigma I$ be an arbitrary matrix in $\mathcal{H}(B, \lambda)$. Let k be an arbitrary index and let (k, ℓ) be the pivotal position in the k^{th} row of $B - \lambda I$. Suppose we have

$$F_{k\ell}(B - \lambda I) > 0 \quad \text{and} \quad F_{k\ell}(C - \sigma I) \leq 0 .$$

Since $\mathcal{H}(B, \lambda)$ is a connected set in complex E^{n^2} space which contains both $B-\lambda I$ and $C-\sigma I$, it follows that there exists a matrix T in $\mathcal{H}(B, \lambda)$ such that $F_{k\ell}(T) = 0$. However, this contradicts the fact that every matrix in $\mathcal{H}(B, \lambda)$ is regular, and we conclude that $F_{k\ell}(C-\sigma I) > 0$. Since $C-\sigma I$ was chosen arbitrarily from $\mathcal{H}(B, \lambda)$, and since k was an arbitrary (row) index, it follows that every matrix in $\mathcal{H}(B, \lambda)$ has the same pivotal positions as $B-\lambda I$ and hence is associated with the same permutation matrix as $B-\lambda I$, and this completes the proof of the theorem. ▮

The value of Theorem 7 lies in the fact that if we know $\lambda \geqslant 0$ is not in the eigenvalue set of a given matrix B, and if we also know that one matrix in $\mathcal{H}(B, \lambda)$ is associated with a certain permutation matrix P, then every matrix in $\mathcal{H}(B, \lambda)$ must be associated with P. Therefore, we make the following definition:

<u>Definition 7</u>. Let B be an $n \times n$ complex matrix, and let λ be a non-negative number not in $\mathcal{R}(B)$. Then the pivotal positions shared by all the matrices in $\mathcal{H}(B, \lambda)$ will be called the pivotal positions of the gap $G(B, \lambda)$. If P is the permutation matrix associated with every matrix in $\mathcal{H}(B, \lambda)$, then we say that the gap $G(B, \lambda)$ is associated with P. A gap which is associated with the identity matrix will be called a <u>normal gap</u>.

It will be seen that normal gaps play an important part in our investigations. One reason for the importance of such gaps is found in the following result, which states that the unbounded gap of any eigenvalue set is normal.

Theorem 8. *Let B be an n x n complex matrix. Then $G(B,\infty)$ is a normal gap.*

Proof: Let C be the matrix in $\mathcal{S}(B)$ which has all non-negative off-diagonal elements and all non-positive diagonal elements, and let $\alpha = \max_{1 \le k \le n} \sum_{j \ne k}' c_{kj}$. If λ is any positive number greater than both α and the largest non-negative boundary point of $\mathcal{R}(B)$, then $C-\lambda I$ is diagonally dominant, and it follows from Corollary 2.2 that $C-\lambda I$ is a regular matrix in normal form. Since λ is in $G(B,\infty)$, we conclude that $G(B,\infty)$ is a normal gap. |

Theorem 8 guarantees that at least one gap in every eigenvalue set is normal. In Theorem 9, we show that if a certain gap in an eigenvalue set is normal, then every gap exterior to this normal gap is also normal. In particular, every gap in the eigenvalue set of a regular matrix in normal form is normal.

Theorem 9. *Let B be an n x n complex matrix, and let λ be a non-negative number in a normal gap in $\mathcal{R}(B)$. Then if $\rho > \lambda$ is not in $\mathcal{R}(B)$, the gap $G(B,\rho)$ must be normal.*

Proof: Let $C = (c_{ij})$ be that matrix in $\mathcal{S}(B)$ which has all non-negative off-diagonal elements and all non-positive diagonal elements. Since $G(B,\lambda)$ is a normal gap, it follows from Theorem 7 that $C-\lambda I$ is a regular matrix in normal form. Furthermore, $C-\rho I$

is in $\mathcal{N}(C-\lambda I)$ since we have for $k = 1,2,\ldots,n$:

$$|c_{kk} - \rho| = |b_{kk}| + \rho > |b_{kk}| + \lambda = |c_{kk} - \lambda| \, .$$

Therefore, it follows from Lemma 1.5 that $C-\rho I$ is a regular matrix in normal form, and we conclude from Theorem 7 that the gap $G(B,\rho)$ is normal. ▌

Corollary 9.1. *Let B be a regular matrix in normal form. Then every gap in $\mathcal{Q}(B)$ is normal.*

Proof: Since the gap $G(B,0)$ is normal, the corollary follows from Theorem 8. ▌

The following result is perhaps the most interesting in this section, for it shows, in particular, that if $A = (a_{ij})$ is an $n \times n$ non-negative essentially diagonally dominant matrix and $B = (b_{ij})$ is a non-negative matrix such that $A \geqslant B$ and $a_{kk} = b_{kk}$ for $k = 1,2,\ldots,n$, then the eigenvalue set of B is contained in that of A .

Theorem 10. *Let $A = (a_{ij})$ be an $n \times n$ non-negative matrix such that every gap in $\mathcal{Q}(A)$ is normal, and let $B = (b_{ij})$ be a complex matrix whose elements satisfy the following conditions for* $k = 1,2,\ldots,n$:

$$|b_{kk}| = a_{kk}$$

$$|b_{kj}| \leqslant a_{kj} \quad j = 1,2,\ldots,n \quad j \neq k \tag{3.1}$$

Then we have $\mathcal{Q}(A) \supseteq \mathcal{Q}(B)$.

Proof: Let λ be any non-negative number which is not in $\mathcal{R}(A)$, and let $C = (c_{ij})$ be an arbitrary matrix in $\mathcal{S}(B)$. Because of inequalities (3.1) the elements of C satisfy the following conditions, for $k = 1,2,\dots,n$:

$$|c_{kj}| \leqslant a_{kj} \qquad j = 1,2,\dots,n \qquad j \neq k$$

$$|a_{kk} - \lambda| \leqslant |c_{kk} - \lambda|$$

Since every gap in $\mathcal{R}(A)$ is normal, it follows that $A-\lambda I$ is a regular matrix in normal form. Therefore, $C-\lambda I$ is in $\mathcal{N}(A-\lambda I)$ and we conclude from Lemma 1.5 that $C-\lambda I$ is regular. Hence, λ cannot be an eigenvalue of C, and since C was chosen arbitrarily from $\mathcal{S}(B)$, λ cannot be in $\mathcal{R}(B)$. Since λ was any non-negative number not in $\mathcal{R}(A)$, we conclude that every number which is not in $\mathcal{R}(A)$ also cannot be in $\mathcal{R}(B)$, and this means that $\mathcal{R}(A) \supseteq \mathcal{R}(B)$.

<u>Corollary 10.1. Let</u> $B = (b_{ij})$ <u>be an</u> $n \times n$ <u>regular matrix in normal form, and let</u> C <u>be a</u> $k \times k$ <u>principal submatrix of</u> B. <u>Then</u> $\mathcal{R}(B) \supseteq \mathcal{R}(C)$.

Proof: Let λ be a positive number not in $\mathcal{R}(B)$, and let T be any matrix in $\mathcal{S}(C)$. To simplify notation, suppose B can be partitioned as follows:

$$B = \left(\begin{array}{c|c} C & B_2 \\ \hline B_3 & B_4 \end{array}\right) \tag{3.2}$$

Let S be the following n x n partitioned matrix:

$$S = \left(\begin{array}{c|c} T & B_2 \\ \hline B_3 & B_4 \end{array}\right)$$

Then S is in $\mathcal{S}(B)$, and it follows from Theorem 10 that $S-\lambda I$ is a regular matrix in normal form. Hence, it follows from Theorem 5 that $T-\lambda I$ is a regular matrix in normal form, and since T was chosen arbitrarily from $\mathcal{S}(C)$, and since λ was any positive number not in $\mathcal{R}(B)$, we conclude that $\mathcal{R}(B) \supseteq \mathcal{R}(C)$. If B is not in form (3.2) , then we proceed exactly as in Theorem 5. |

Corollary 10.2. Let $B = (b_{ij})$ be an n x n regular matrix in normal form. Then each diagonal element of B is contained in $\mathcal{R}(B)$. |

Actually, using a continuity argument similar to that used in Theorem 0 , we can show that the eigenvalue set of any matrix in $\mathcal{N}(B)$ whose diagonal elements equal those of B must have at least as many components as $\mathcal{R}(B)$. This fact can be used to show that each component of $\mathcal{R}(B)$ must contain at least one diagonal element of B . Clearly, this result is not necessarily true unless B is a regular matrix in normal form.

If a non-negative matrix A has an isolated Gerschgorin disk, then it follows from the Gerschgorin Disk Theorem that the annulus formed by rotating this disk about the origin must contain at least one component of $\mathcal{R}(A)$. In Theorem 11, we obtain another fact about the eigenvalue set of a matrix with an isolated Gerschgorin disk.

<u>Theorem 11. Let $A = (a_{ij})$ be a non-negative $n \times n$ matrix. If the k^{th} Gerschgorin disk of A is isolated and at least one of the diagonal elements of A is less than a_{kk}, then every gap in $\mathcal{R}(A)$ which contains numbers greater than a_{kk} must be normal.</u>

Proof: Let $\lambda > a_{kk}$ be in a gap in $\mathcal{R}(A)$. Let $B = (b_{ij})$ be that matrix in $\mathcal{S}(A)$ which has all non-negative off-diagonal elements and all non-positive diagonal elements. We shall show that $B-\lambda I$ is diagonally dominant.

Let r be any index such that $a_{rr} > a_{kk}$ and s, such that $a_{ss} < a_{kk}$. Since the k^{th} Gerschgorin disk of A is isolated, we have

$$a_{rr} - \sum_{j \neq r}{}' a_{rj} > a_{kk} + \sum_{j \neq k}{}' a_{kj}$$

and

$$a_{kk} - \sum_{j \neq k}{}' a_{kj} > a_{ss} + \sum_{j \neq s}{}' a_{sj} \qquad (3.3)$$

Therefore, we must have

$$|b_{rr} - \lambda| = a_{rr} + \lambda > \sum_{j \neq r}{}' a_{rj} = \sum_{j \neq r}{}' b_{rj}$$

and

$$|b_{ss} - \lambda| = a_{ss} + \lambda > a_{ss} + a_{kk} > \sum_{j \neq s}{}' a_{sj} = \sum_{j \neq s}{}' b_{sj}$$

Finally, since at least one diagonal element of A is less than a_{kk}, it follows from inequality (3.3) that

$$|b_{kk} - \lambda| = a_{kk} + \lambda > \sum_{j \neq k}{}' a_{kj} + \lambda > \sum_{j \neq k}{}' a_{kj} \qquad (3.4)$$

Therefore, $B-\lambda I$ is diagonally dominant, and it follows from Corollary 2.2 that $B-\lambda I$ is in normal form. Hence, we conclude from Theorem 7 that the gap $G(B,\lambda)$ must be normal, and since λ was any positive number not in $\mathcal{R}(A)$ and greater than a_{kk}, this completes the proof of the theorem. ∎

In Theorem 11, the condition that at least one diagonal element of A be less than a_{kk} may be removed if we require that λ be in a gap in $\mathcal{R}(A)$ which includes numbers greater than the larger of the two numbers a_{kk} and $\sum_{j \neq k}' a_{kj}$, for this is enough to guarantee that inequality (3.4) will hold. On the other hand, if a_{kk} is the smallest diagonal element of A, then it is possible for numbers greater than a_{kk} to lie in a gap in $\mathcal{R}(A)$ which is not normal. For example, consider the following 2 x 2 matrix:

$$A = \begin{pmatrix} 5 & 1 \\ 1 & 0 \end{pmatrix} .$$

Clearly A is a regular matrix associated with the permutation matrix

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ,$$

but both Gerschgorin disks of A are isolated, and one component of $\mathcal{R}(A)$ must be contained in the disk $|z| \leqslant 1$. Therefore, there exist numbers greater than $a_{22} = 0$ which lie in the non-normal gap in $\mathcal{R}(A)$ which is associated with P. However, we can obtain a result similar to Theorem 11 even if a_{kk} is the smallest diagonal element of A.

<u>Theorem 12. Let</u> $A = (a_{ij})$ <u>be a non-negative</u> $n \times n$ <u>matrix, and let the</u> k^{th} <u>Gerschgorin disk of</u> A <u>be isolated, where</u> a_{kk} <u>is in the smallest diagonal element of</u> A . <u>If</u> a_{kk} <u>is in a component of</u> $\mathcal{R}(A)$, <u>then every gap in</u> $\mathcal{R}(A)$ <u>which contains numbers greater than</u> a_{kk} <u>is normal. If</u> a_{kk} <u>lies in a gap in</u> $\mathcal{R}(A)$, <u>then every gap in</u> $\mathcal{R}(A)$ <u>which is exterior to the gap</u> $G(A, a_{kk})$ <u>is normal.</u>

Proof: Since the k^{th} Gerschgorin disk of A is isolated, it follows that elements of only one component of $\mathcal{R}(A)$ can be contained in the annulus (or disk) formed by rotating the disk

$$|z - a_{kk}| \leq \sum_{j \neq k}' a_{kj}$$

about the origin.

Therefore, if a_{kk} lies in a component of $\mathcal{R}(A)$, it follows that any gap in $\mathcal{R}(A)$ which contains numbers greater than a_{kk} must also contain numbers greater than $a_{kk} + \sum_{j \neq k}' a_{kj}$, for the greatest positive boundary point of the k^{th} Gerschgorin disk of A must lie in the more exterior of the gaps which bound the component of $\mathcal{R}(A)$ containing a_{kk} . If λ is greater than $a_{kk} + \sum_{j \neq k}' a_{kj}$, then inequality (3.4) holds, and the proof that $G(A, \lambda)$ is normal follows exactly as in Theorem 11.

Similarly, if a_{kk} is in a gap in $\mathcal{R}(A)$, then the "next" gap contains $a_{kk} + \sum_{j \neq k}' a_{kj}$, and it follows as before that this gap and every other gap exterior to $G(A, a_{kk})$ must be normal. |

We conclude this section with a result which concerns the eigenvalue set of a positive definite matrix. Although this result is interesting, it is included mainly to illustrate the method by which the technique which we developed in Section II may be used to obtain results which are basically combinatorial in nature.

<u>Theorem 13. Let B be an n x n positive definite real symmetric matrix. Then every gap in $\mathcal{R}(B)$ is normal.</u>

Proof: We shall use induction on n . Let $n = 2$. Since $B = (b_{ij})$ is positive definite, we must have $b_{11} > 0$; $b_{22} > 0$; and $b_{11} b_{22} > b_{12} b_{21} = (b_{12})^2$. Therefore, B must actually be a 2 x 2 regular matrix in normal form, and it follows from Theorem 9 that every gap in $\mathcal{R}(B)$ is normal.

Now, for a general n x n matrix B , let λ be a positive number which is not in $\mathcal{R}(B)$, and let A be the non-negative matrix in $\mathcal{S}(B)$. It follows from Theorem 7 that $-A-\lambda I$ is a regular matrix. Let $S = A + \lambda I$. We shall prove that S is in normal form.

Consider the non-negative matrix $C = (c_{ij})$ whose elements satisfy the following conditions:

$$c_{kk} = s_{kk}$$

$$c_{k\ell} = s_{k\ell} \quad \text{if } s_{k\ell} \text{ is the pivotal element in the } k^{th} \text{ row of } S$$

$$c_{kj} = 0 \quad \text{otherwise}$$

Then C is in $\mathcal{N}(S)$ and it follows from Lemma 1.5 that C is regular with the same pivotal positions as S . We shall now show that C is reducible.

A matrix is irreducible if and only if its graph is strongly connected. However, since S is symmetric, C must be symmetric, and furthermore, C has at most n non-zero off-diagonal elements. If C has less than n non-zero off-diagonal elements, it is clearly reducible. On the other hand, if C has exactly n non-zero off-diagonal elements, it follows from the symmetry of C that n must be even, and the graph of C consists of n/2 closed paths of length two. Thus, for $n > 2$, C must be reducible.

Since C is reducible, there exists a permutation matrix P such that

$$PCP' = \left(\begin{array}{c|c} C_1 & C_2 \\ \hline 0 & C_4 \end{array}\right) ,$$

where for some integer m with $1 \leq m < n$, 0 is an $(n-m) \times m$ block of zeroes; C_2 is an $m \times (n-m)$ submatrix of C; and C_1 and C_4 are $m \times m$ and $(n-m) \times (n-m)$ submatrices of C, respectively.

Now consider the matrix $U = PBP'$. Partition U conformally with PCP' -- i.e. regard U as the partitioned matrix

$$U = \left(\begin{array}{c|c} U_1 & U_2 \\ \hline U_3 & U_4 \end{array}\right) ,$$

where U_1, U_2, U_3, U_4 have the same dimensions as C_1, C_2, C_3, C_4, respectively. Since B is positive definite, it follows that U is positive definite, and the principal submatrices U_1 and U_4 must also be positive definite. Moreover, since $\mathcal{R}(B) = \mathcal{R}(U)$, and since λ is not in $\mathcal{R}(B)$, it follows from Corollary 10.1 that λ is not in either $\mathcal{R}(U_1)$ or $\mathcal{R}(U_4)$. Therefore, by the inductive hypothesis, λ must be in a normal gap in both $\mathcal{R}(U_1)$ and $\mathcal{R}(U_4)$, and $U_1 + \lambda I$ and $U_4 + \lambda I$ must be regular matrices in normal form. Let V be the following block diagonal n x n matrix:

$$V = \left(\begin{array}{c|c} U_1 & 0 \\ \hline 0 & U_4 \end{array}\right)$$

Since the matrix $V + \lambda I$ is in $\mathcal{N}(PSP')$, it follows that $V + \lambda I$ has the same pivotal positions as PSP'. However, since both $U_1 + \lambda I$ and $U_4 + \lambda I$ are in normal form, it follows that $V + \lambda I$ is in normal form, and this in turn implies that PSP' is in normal form. Therefore, we conclude that S is in normal form, and it follows from Theorem 7 that the gap $G(B, \lambda)$ is normal. Since λ was an arbitrary positive number not in $\mathcal{R}(B)$, we conclude that every gap in $\mathcal{R}(B)$ is normal, as desired. |

IV. THE ANALOGUE OF A CONJECTURE OF VARGA AND LEVINGER.

In their characterization of the eigenvalue set, Varga and Levinger proved that at most $(n+1)$ permutations are needed to characterize the eigenvalue set of a general $n \times n$ matrix A in terms of the minimal Gerschgorin sets of matrices related to A by these permutations. Furthermore, they conjectured that actually only n such permutations are needed to characterize $\Omega(A)$. Although we do not actually obtain an analytic characterization of $\Omega(A)$, we can still show that $\Omega(A)$ may be associated with at most $(n+1)$ permutations in a meaningful fashion, for it follows from Theorem 0 that there exist at most $(n+1)$ gaps in $\Omega(A)$, and each of these gaps is associated with a permutation matrix in the sense of Definition 7. Therefore, motivated by the analagous conjecture of Varga and Levinger, we make the following conjecture:

<u>Conjecture 4.1</u>. Let A be a non-negative $n \times n$ matrix. Then there can exist at most n different permutation matrices associated with gaps in $\Omega(A)$.

In this section, we shall verify this conjecture, and in addition, we shall develop a new means of classifying gaps which is more refined than the one provided by Definition 7. It will be seen that this new gap classification system actually yields a certain amount of information about the components of $\Omega(A)$.

In constructing a proof of Conjecture 4.1 , we shall depend mainly on a comparison technique which utilizes the results of Theorem 3 and Theorem 7. The following example illustrates this technique:

Example 4.1.

Let $A = (a_{ij})$ be a general 4×4 non-negative matrix. Suppose that a certain gap in $\mathcal{R}(A)$ -- say $G(A, \lambda)$ -- is associated with the permutation matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Then the non-negative matrix $A - \lambda I$ must be a regular matrix whose pivotal positions are (1,1); (2,3); (3,4); and (4,2) , and it follows from Theorem 3 that, among others, the following inequalities hold:

$$\begin{aligned} |a_{11} - \lambda| a_{23}\, a_{34}\, a_{42} &> a_{12}\, a_{23}\, a_{34}\, a_{41} \\ |a_{11} - \lambda| a_{23}\, a_{34}\, a_{42} &> |a_{22} - \lambda| a_{13}\, a_{34}\, a_{41} \\ |a_{11} - \lambda| a_{23}\, a_{34}\, a_{42} &> |a_{11} - \lambda||a_{22} - \lambda| a_{34}\, a_{43} \end{aligned} \tag{4.1}$$

Consider the following matrix:

$$B = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & -a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & -a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & -a_{44} \end{pmatrix}$$

Then B is in $\mathcal{S}(A)$, and we conclude from Theorem 7 that $B-\lambda I$ is a regular matrix associated with P . Therefore, in addition to inequalities (4.1) , it follows from Theorem 3 that, among others, we have the following inequalities:

$$|a_{11} - \lambda| a_{23} a_{34} a_{42} > |-a_{22} - \lambda| a_{13} a_{34} a_{41}$$

$$|a_{11} - \lambda| a_{23} a_{34} a_{42} > |a_{11} - \lambda||-a_{22} - \lambda| a_{34} a_{43} \qquad (4.2)$$

$$|a_{11} - \lambda| a_{23} a_{34} a_{42} > |a_{11} - \lambda||-a_{22} - \lambda||-a_{33} - \lambda||-a_{44} - \lambda|$$

Inequalities such as (4.1) and (4.2) occur for each and every gap in $\mathcal{R}(A)$. Thus, if there exists a second gap in $\mathcal{R}(A)$ which is not associated with P , the "difference" between this gap and $G(A,\lambda)$ must be due to a fundamental change in the above inequalities.

However, suppose ρ is in a second gap in $\mathcal{R}(A)$ whose pivotal positions are (1,1); (2,4); (3,2); and (4,3) . Then we have the following pair of inequalities:

$$|a_{11} - \lambda| a_{23} a_{34} a_{42} > |a_{11} - \lambda| a_{24} a_{32} a_{43}$$

and

$$|a_{11} - \rho| a_{24} a_{32} a_{43} > |a_{11} - \rho| a_{23} a_{34} a_{42}$$

These two inequalities are clearly incompatible. We shall now show that this dilemma results from the fact that two gaps can be associated with different permutation matrices only if they do not have the same diagonal pivotal positions.

<u>Theorem 14. Let A be an n x n non-negative matrix and let $P = (p_{ij})$ and $Q = (q_{ij})$ be permutation matrices which are associated with two different gaps in $\mathcal{G}(A)$. If $p_{kk} = q_{kk}$ for $k = 1,2,\ldots,n$, then $P = Q$.</u>

Proof: Let λ_1 and λ_2 be any positive numbers in the gaps which are associated with P and Q, respectively, and assume $\lambda_1 > \lambda_2$. Let $A_1 = -A - \lambda_1 I$ and let $A_2 = -A - \lambda_2 I$. It follows from Theorem 7 that PA_1 and QA_2 are regular matrices in normal form, and it is clear that A_1 and A_2 have the same diagonal pivotal positions, since $p_{kk} = q_{kk}$ for $k = 1,2,\ldots,n$.

Let $C = (c_{ij})$ be the non-negative matrix whose elements satisfy the following conditions, for $k = 1,2,\ldots,n$:

$c_{kk} = a_{kk} + \lambda_1$	if $p_{kk} = q_{kk} = 1$
$c_{kk} = 0$	if $p_{kk} = q_{kk} = 0$
$c_{ks} = a_{ks}$	if (k,s) is a pivotal position in the k^{th} row of either A_1 or A_2
$c_{kj} = 0$	otherwise

It is clear that C is in $\mathcal{N}(A_1)$. Since $\lambda_1 > \lambda_2$ and since A_2 has the same diagonal pivotal positions as A_1, it follows that C is in $\mathcal{N}(A_2)$ also. Therefore, both PC and QC are regular matrices in normal form, and we conclude from Theorem 1 that $P = Q$. |

It is not hard to see that we could prove Conjecture 4.1 at this point if we could only show that any two gaps with the same <u>number</u> of diagonal pivotal positions must be associated with the same permutation matrix, for no $n \times n$ matrix can have exactly $(n-1)$ diagonal pivotal positions. However, this is not the case, as the following example illustrates:

<u>Example 4.2</u>. Consider the following 3×3 matrix:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 5 & 1 \end{pmatrix}$$

Then, using Theorem 2, it is easy to see that A is a regular matrix associated with

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

Furthermore, the number 1 lies in a gap in $\mathcal{R}(A)$ which is associated with the permutation matrix

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Therefore, both $G(A,0)$ and $G(A,1)$ have one diagonal pivotal position, but P and Q are not identical.

One feature of the preceding example merits further investigation. Namely, in the gap $G(A,0)$, the $(3,3)$ position is pivotal, and we have $a_{33} - 0 = 1 > 0$; however, in the gap $G(A,1)$, where $(1,1)$ is the diagonal pivotal position, we have $a_{11} - 1 = -1 < 0$.

Furthermore, if A is a general $n \times n$ non-negative matrix, and if (k,k) is a pivotal position for a certain gap $G(A,\lambda)$, then it follows from Theorem 7 that a_{kk} cannot lie in the gap $G(A,\lambda)$, for (k,k) cannot be a pivotal position of $A - a_{kk}I$. Therefore, we conclude that every positive number in $G(A,\lambda)$ must either be strictly greater than, or strictly less than a_{kk} . This observation enables us to make the following definition:

Definition 8. Let A be an $n \times n$ non-negative matrix, and let λ be a non-negative number not in $\mathcal{R}(A)$. Let (k,k) be a pivotal position for the gap $G(A,\lambda)$. Then the (k,k) position will be called subordinate if every non-negative number in $G(A,\lambda)$ is strictly greater than a_{kk} . If every non-negative number in $G(A,\lambda)$ is strictly less than a_{kk} , then (k,k) will be called non-subordinate. Furthermore, the gap $G(A,\lambda)$ is said to be of type $\langle r,s \rangle$ if it has r diagonal pivotal positions, exactly s of which are subordinate.

For example, if A is the matrix in Example 4.2 , the gap $G(A,0)$ is of type $\langle 1,0 \rangle$, since A has no negative diagonal pivotal elements, and the gap $G(A,1)$ is of type $\langle 1,1 \rangle$, since the $(1,1)$ position is pivotal in this gap and we have $1 > a_{11} = 0$.

Thus, at least in the preceding example, we see that this new method of classifying gaps enables us to distinguish between two gaps which have the same number of diagonal pivotal positions but are associated with different permutation matrices. Actually, this phenomenon is quite general, for we shall show in Theorem 15 that if A is a non-negative matrix such that two gaps in $\mathcal{R}(A)$ have the same number of subordinate diagonal pivotal positions, then these gaps must be associated with the same permutation matrix. However, we shall first prove a lemma which will enable us to use an inductive technique in the proof of Theorem 15.

Lemma 15.1. Let A *be a non-negative* $n \times n$ *matrix, and let there be a gap in* $\mathcal{R}(A)$ *of type* $\langle r,s \rangle$ *which has* (k,k) *as a pivotal position. Let* A_k *denote the* $(n-1) \times (n-1)$ *submatrix formed from* A *by deleting the* k^{th} *row and* k^{th} *column. Then if* (k,k) *is non-subordinate, there exists a gap in* $\mathcal{R}(A_k)$ *of type* $\langle r-1,s \rangle$ *, and if* (k,k) *is subordinate, there exists a gap in* $\mathcal{R}(A_k)$ *of type* $\langle r-1,s-1 \rangle$.

Proof: First of all, we note that it is impossible to have $r = s$ in the first case or $s = 0$ in the second case because of the way the lemma is worded.

We shall assume that $k = n$. It will be seen that this involves no real loss of generality but merely facilitates the construction used in the proof.

Let λ be any positive number in the given gap in $\mathcal{R}(A)$. We shall show that λ is in a gap in $\mathcal{R}(A_n)$ of type $\langle r-1,s \rangle$ if (n,n) is non-subordinate, and of type $\langle r-1,s-1 \rangle$ if (n,n) is subordinate.

Let $B = (b_{ij})$ be an arbitrary matrix in $\mathcal{S}(A_n)$, and let C be the following n x n block diagonal partitioned matrix:

$$C = \left(\begin{array}{c|c} B & O_1 \\ \hline O_2 & a_{nn} \end{array} \right),$$

where O_1 and O_2 are 1 x (n-1) and (n-1) x 1 blocks of zeroes, respectively. Since (n,n) is a pivotal position for the gap $G(A,\lambda)$, it follows that $A-\lambda I$ is regular and that $C-\lambda I$ is in $\mathcal{N}(A-\lambda I)$. Therefore, $C-\lambda I$ is non-singular, and λ cannot be an eigenvalue of B. Since B was chosen arbitrarily from $\mathcal{S}(A_n)$, we conclude that λ is in a gap in $\mathcal{R}(A_n)$.

Furthermore, $C-\lambda I$ has the same pivotal positions as $A-\lambda I$, and from this we conclude that except for the (n,n) position, which is excluded, the diagonal pivotal position of the gap $G(A_n,\lambda)$ are either subordinate or non-subordinate according to their status in $G(A,\lambda)$. If (n,n) is non-subordinate in $G(A,\lambda)$, then the gap $G(A_n,\lambda)$ has the same number of subordinate positions as the gap $G(A,\lambda)$, but it has one less diagonal pivotal position than $G(A,\lambda)$, since (n,n) is not included. Hence, in this case, the gap $G(A_n,\lambda)$ is of type $\langle r-1,s\rangle$. Similarly, if (n,n) is subordinate, then $G(A_n,\lambda)$ contains one less subordinate position -- namely, (n,n) -- and one less diagonal pivotal position than the gap $G(A,\lambda)$. Hence, in this case, $G(A_n,\lambda)$ is of type $\langle r-1,s-1\rangle$. |

Now we are in a position to prove Theorem 15.

<u>Theorem 15. Let A be an n x n non-negative matrix. Then any two gaps in $\mathcal{Q}(A)$ which have the same number of subordinate diagonal pivotal positions must be associated with the same permutation matrix.</u>

Proof: We shall proceed by induction on n .

For $n = 1$, the theorem is obviously true. Now assume the theorem holds for all matrices of order not greater than $n-1$.

Suppose there exist two gaps in $\mathcal{Q}(A)$ which have the same number of subordinate diagonal pivotal positions. Let $\lambda_1 > 0$ be in the more interior of these gaps and let $\lambda_2 > \lambda_1$ be in the other. Suppose that $G(A, \lambda_1)$ is of type $\langle r,s \rangle$ and that $G(A, \lambda_2)$ is of type $\langle t,s \rangle$. Naturally, we must have $0 \leqslant s \leqslant r,t \leqslant n$, but otherwise, s,r and t are completely arbitrary. Before proceeding to the main part of the proof, we shall make a few simplifying observations and assumptions.

First, suppose there exists an index ℓ_1 such that (ℓ_1, ℓ_1) is non-subordinate for both gaps. It follows from Lemma 15.1 that λ_1 and λ_2 are in gaps in $\mathcal{Q}(A_{\ell_1})$ of types $\langle r-1, s \rangle$ and $\langle t-1, s \rangle$, respectively. By the inductive hypothesis, we conclude that $G(A_{\ell_1}, \lambda_1)$ and $G(A_{\ell_1}, \lambda_2)$ are both associated with the same permutation matrix, say Q . Then it follows that $G(A, \lambda_1)$ and $G(A, \lambda_2)$ are both associated with the permutation matrix $P = (p_{ij})$, where $p_{\ell_1,\ell_1} = 1$ and $P_{\ell_1} = Q$. A similar argument shows that the theorem holds if a diagonal pivotal position is subordinate for both gaps. Furthermore, if (k,k) is subordinate for $G(A, \lambda_1)$, then it is subordinate for $G(A, \lambda_2)$, since $\lambda_1 > a_{kk}$ implies $\lambda_2 > a_{kk}$.

Therefore, it suffices to consider the case in which the only diagonal positions which are pivotal for both gaps are non-subordinate for the gap $G(A, \lambda_1)$ and subordinate for $G(A, \lambda_2)$. Suppose there are precisely p such diagonal positions, where $0 \leqslant p \leqslant s$. Clearly we must also have $r + t - p \leqslant n$.

In the proof that follows, we shall need to be able to index the diagonal pivotal positions of both gaps, and in order to avoid confusing the proof with a complicated, albeit completely general indexing scheme, we shall make a few assumptions concerning the location of these diagonal pivotal positions. Since the calculations used in the proof involve only arithmetic inequalities among products of elements along certain generalized diagonals, it will be seen that this assumption involves no real loss of generality. Accordingly, we will assume that the p common diagonal pivotal positions for the two gaps are $(1,1)$; $(2,2)$; ...; (p,p); that the $(r-s)-p$ other non-subordinate and s subordinate diagonal pivotal positions for the gap $G(A,\lambda_1)$ are $(p+1,p+1)$; ...: $(r-s,r-s)$, and $(r-s+1,r-s+1)$; ...; (r,r), respectively; and finally that the $(s-p)$ other subordinate and $(t-s)$ non-subordinate diagonal pivotal positions for $G(A,\lambda_2)$ are $(r+1,r+1)$; ...; $(r+s-p,r+s-p)$ and $(r+s-p+1,r+s-p+1)$; ...; and $(r+t-p,r+t-p)$, respectively.

Moreover, let α denote the product of the off-diagonal pivotal elements of the gap $G(A,\lambda_1)$, and let β denote the product of the off-diagonal pivotal elements of the gap $G(A,\lambda_2)$. If $G(A,\lambda_1)$ is normal, then let α equal one, and similarly, if $G(A,\lambda_2)$ is normal, we take β equal one. Clearly α and β are invariants of the two gaps and depend in no way on the choice of λ_1 and λ_2. Finally, we shall need the following two matrices $B = (b_{ij})$ and $C = (c_{ij})$ whose elements satisfy the following conditions:

$$b_{ij} = a_{ij} \quad i,j = 1,2, \ldots, n \quad i \neq j$$
$$b_{ii} = a_{ii} \quad i = 1,2, \ldots, r$$
$$b_{ii} = -a_{ii} \quad i = r+1, r+2, \ldots, n$$

and

$$c_{ij} = a_{ij} \quad i,j = 1,2, \ldots, n \quad i \neq j$$
$$c_{ii} = a_{ii} \quad i = 1,2, \ldots, p;\ r+1, \ldots, r+t-p$$
$$c_{ii} = -a_{ii} \quad i = p+1, p+2, \ldots, r;\ r+t-p+1, \ldots, n$$

Clearly both B and C are in $\mathcal{S}(A)$. Furthermore, B is such that the product of the pivotal elements of the matrix $|B-\lambda_1 I|$ is no greater than the product of the pivotal elements of any other matrix of the form $|R-\lambda_1 I|$, where R is in $\mathcal{S}(A)$. The matrix C has an analagous connection with the gap $G(A, \lambda_2)$.

Now suppose the two gaps are not associated with the same permutation matrix. If we apply Theorem 3 to the regular matrix $|B-\lambda_1 I|$, we obtain the following inequality, since the terms on the right lie on a generalized diagonal of $|B-\lambda_1 I|$:

$$\left(\prod_{i=1}^{r} |a_{ii} - \lambda_1|\right)\alpha > \left(\prod_{i=1}^{p} |a_{ii} - \lambda_1|\right)\left(\prod_{i=r+1}^{r+t-p} |-a_{ii} - \lambda_1|\right)\beta \qquad (4.3)$$

In the gap $G(A, \lambda_1)$, the positions $(1,1); (2,2); \ldots; (r-s, r-s)$ are non-subordinate, and the positions $(r-s+1, r-s+1); \ldots; (r,r)$ are subordinate. Therefore, we conclude from Definition 8 that we have the following:

$$|a_{ii} - \lambda| = a_{ii} - \lambda \qquad i = 1,2, \ldots, r-s$$

and

$$|a_{ii} - \lambda| = \lambda - a_{ii} \qquad i = r-s+1, \ldots, r$$

Thus, we may rewrite inequality (4.3) as follows:

$$\left[\prod_{i=1}^{p}(a_{ii}-\lambda_1)\right]\left[\prod_{i=p+1}^{r-s}(a_{ii}-\lambda_1)\right]\left[\prod_{i=r-s+1}^{r}(\lambda_1-a_{ii})\right]\alpha >$$

$$\left[\prod_{i=1}^{p}(a_{ii}-\lambda_1)\right]\left[\prod_{i=r+1}^{r+(s-p)}(a_{ii}+\lambda_1)\right]\left[\prod_{i=r+s-p+1}^{r+t-p}(a_{ii}+\lambda_1)\right]\beta$$

As it stands, this inequality gives us little information, but we may weaken it to obtain the following useful inequality:

$$\lambda_1^{s}\left(\prod_{i=p+1}^{r-s}a_{ii}\right)\alpha > \lambda_1^{(s-p)}\left(\prod_{i=r+s-p+1}^{r+t-p}a_{ii}\right)\beta \qquad (4.4)$$

If we apply similar arguments to the matrix $|C-\lambda_2 I|$, we obtain the following inequality:

$$\left(\prod_{i=1}^{p}|a_{ii}-\lambda_2|\right)\left(\prod_{i=r+1}^{r+t-p}|a_{ii}-\lambda_2|\right)\beta >$$

$$\left(\prod_{i=1}^{p}|a_{ii}-\lambda_2|\right)\left(\prod_{i=p+1}^{r}|-a_{ii}-\lambda_2|\right)\alpha \qquad (4.5)$$

Following exactly the same procedure as before, we obtain the weaker inequality:

$$\lambda_2^{(s-p)}\left(\prod_{i=r+s-p+1}^{r+t-p}a_{ii}\right)\beta > \lambda_2^{s}\left(\prod_{i=p+1}^{r-s}a_{ii}\right)\alpha \qquad (4.6)$$

Combining inequalities (4.4) and (4.6) , we obtain the inequality

$$\lambda_1^p \left(\prod_{i=p+1}^{r-s} a_{ii}\right)\alpha > \lambda_2^p \left(\prod_{i=p+1}^{r-s} a_{ii}\right)\alpha \qquad (4.7)$$

Since all the quantities in inequality (4.7) are positive numbers, we conclude that $\lambda_1^p > \lambda_2^p$. However, $\lambda_2 > \lambda_1$ by assumption, and since this contradiction is a logical result of the existence of inequalities (4.3) and (4.5) , we conclude that at least one of these two inequalities cannot hold. This means that the gaps must have exactly the same pivotal position so that the two sides of (4.3) and (4.5) are identical, and this is equivalent to the statement that the two gaps correspond to the same permutation matrix. |

With the results of Theorem 1, we are now in a position to prove Conjecture 4.1.

Theorem 16. Let A be a regular n x n non-negative matrix for which there exist (n+1) gaps in $\mathcal{R}(A)$. Then at least two of these gaps are associated with the same permutation matrix.

Proof: Suppose that no two gaps in $\mathcal{R}(A)$ are associated with the same permutation matrix. It follows from Theorem 15 that there exists one and only one gap with exactly s subordinate diagonal pivotal positions for each s = 0, 1,2,...,n . However, if s = (n-1) for a certain gap, then that gap must have either (n-1) or n diagonal pivotal positions, and since no n x n permutation matrix has (n-1) non-zero diagonal elements, it follows that any gap with (n-1) or n subordinate diagonal pivotal positions must be associated with the identity. Thus, in any case at least two gaps in $\mathcal{R}(A)$ must be associated with the same permutation matrix. |

Corollary 16.1. *Let A be an n x n non-negative matrix. Then there can exist at most n different permutation matrices associated with gaps in* $\mathcal{R}(A)$.

Proof: It is clear that (n+1) different permutation matrices may be needed only if there are (n+1) gaps in $\mathcal{R}(A)$, and in this case, it follows from Theorem 16 that at least two of the gaps in $\mathcal{R}(A)$ must be associated with the same permutation matrix. |

The results of Theorem 15 and 16 seem to indicate that gaps may be distinguished from one another by the number of subordinate diagonal pivotal positions which they determine. However, there is still the possibility that two gaps may have the same number of subordinate positions if they are associated with the same permutation matrix, and we would like to remove this qualification if possible. We shall prove in Theorem 17 that two gaps which are associated with the same permutation matrix cannot have the same number of subordinate diagonal pivotal positions. It will be seen that the proof of this result for a general non-negative matrix A depends almost entirely on the fact that the result holds for a non-negative generalized permutation matrix. Before we proceed to the proof of Theorem 17, let us analyze the eigenvalue set of a specific generalized permutation matrix.

Example 4.2.

Consider the following matrix:

$$A = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & 0 & a_{34} \\ 0 & 0 & a_{43} & 0 \end{pmatrix}$$

Suppose we have $a_{11} > (a_{34}\ a_{43})^{\frac{1}{2}} > a_{22}$. Then $\mathcal{R}(A)$ consists of the three circles $|z| = a_{11}$; $|z| = a_{22}$; and $|z| = (a_{34}\ a_{43})^{\frac{1}{2}}$. Clearly the gap $G(A,0)$ is associated with

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and is of type $\langle 2,0 \rangle$. Let λ_1 , satisfy $a_{22} < \lambda_1 < (a_{34}\ a_{43})^{\frac{1}{2}}$. Then the gap $G(A, \lambda_1)$ is associated with

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and is of type $\langle 2,1 \rangle$. Let λ_2 satisfy $a_{11} > \lambda_2 > (a_{34}\ a_{43})^{\frac{1}{2}}$. Then $\lambda_2^{\ 2} > a_{34}\ a_{43}$, and the gap $G(A, \lambda_2)$ is associated with the identity and is of type $\langle 4,3 \rangle$. The gap $G(A,\infty)$ is clearly associated with the identity and is of type $\langle 4,4 \rangle$.

The reasoning used in this example may be applied to any generalized permutation matrix A to show that if two gaps in $\mathcal{R}(A)$ are associated with the same permutation matrix, then these gaps cannot have the same number of subordinate diagonal pivotal positions.

However, before we prove this result, we shall need to define the concept of a cyclic matrix and to show that every generalized permutation matrix can be combinatorially decomposed into a direct sum of cyclic submatrices.

Definition 9. For $k \geqslant 2$, let A be a $k \times k$ non-negative generalized permutation matrix. Then A will be called *cyclic of order k* if the permutation related to A is cyclic of length k. Alternatively, A is cyclic of order k if A^k is a positive diagonal matrix but any smaller power of A has positive off-diagonal elements. If A is cyclic of order k, the positive k^{th} root of the product of the positive elements of A will be called the *radius* of A, and we shall denote this number by $(\text{rad } A)$.

If A is cyclic of order k, the significance of the radius of A lies in the fact that $\mathcal{R}(A)$ consists entirely of the circle $|z| = (\text{rad } A)$.

Lemma 17.1. Let A be an n x n non-negative generalized permutation matrix. Then there exists a permutation matrix Q such that

$$QAQ' = \begin{pmatrix} A_0 & & & & O \\ & A_1 & & & \\ & & \cdot & & \\ & & & \cdot & \\ O & & & & A_m \end{pmatrix} \qquad (4.8)$$

where A_o is a positive diagonal submatrix of A and $A_1, A_2, \ldots, A_m$ are all cyclic submatrices of A of various orders.

Proof: This result is well known and is an immediate consequence of the fact that a permutation may be written as the composition of disjoint orbital cycles. The cycles of length one correspond to non-zero diagonal elements of A, and the cycles of length k correspond to $k \times k$ submatrices of A which are cyclic of order k. |

Therefore, if A is a generalized permutation matrix, it follows from Lemma 17.1 that $\mathcal{R}(A) = \bigcup_{i=0}^{m} \mathcal{R}(A_i)$, and it is easily seen that $\mathcal{R}(A)$ consists of a set of circles. These circles fall into two fundamentally different categories.

Definition 10. Let A be an $n \times n$ non-negative generalized permutation matrix. If $a_{kk} \neq 0$, then the circle $|z| = a_{kk}$ will be called a *primary component* of $\mathcal{R}(A)$. If A_ℓ is a cyclic submatrix of the standard decomposition (4.8) of A given in Lemma 17.1, then the circle $|z| = (\text{rad } A_\ell)$ will be called a *cyclic component* of $\mathcal{R}(A)$.

Lemma 17.2. *Let A be an $n \times n$ non-negative generalized permutation matrix, and let two gaps in $\mathcal{R}(A)$ be associated with the same permutation matrix. Then these gaps can be separated by only primary components in $\mathcal{R}(A)$. Furthermore, the two gaps cannot have the same number of subordinate diagonal pivotal positions.*

Proof: We shall assume that A is in the block diagonal form (4.8) of Lemma 17.1. Let λ_1 be a positive number in the more interior of the gaps in question, and let $\lambda_2 > \lambda_1$ be in the other. Suppose the gaps are separated by a cyclic component of $\mathcal{G}(A)$. Then there is a cyclic submatrix of A , say A_ℓ , such that

$$\lambda_2 > (\text{rad } A_\ell) > \lambda_1$$

Since $G(A, \lambda_1)$ and $G(A, \lambda_2)$ are both associated with the same permutation matrix, it follows from Theorem 7 that $-A-\lambda_1 I$ and $-A-\lambda_2 I$ are regular and have the same pivotal positions. From the block diagonal form of A , it follows that $-A_\ell - \lambda_1 I$ and $-A_\ell - \lambda_2 I$ are also regular and have the same pivotal positions -- namely, the pivotal positions of $G(A, \lambda_1)$ and $G(A, \lambda_2)$ which lie in the submatrix A_ℓ . Since A_ℓ is cyclic, it follows that $-A_\ell - \lambda_1 I$ and $-A_\ell - \lambda_2 I$ have non-zero elements along only the main diagonal and one other generalized diagonal, which includes no diagonal positions. Since A_ℓ is cyclic, each diagonal element of $-A_\ell - \lambda_1 I$ is equal to $-\lambda_1$, and each diagonal element of $-A_\ell - \lambda_2 I$ is equal to $-\lambda_2$. Suppose that A is cyclic of order k . Then the product of the diagonal elements of $-A_\ell - \lambda_1 I$ is equal to $(-\lambda_1)^k$, and the product of the elements along the only other generalized diagonal of $-A_\ell - \lambda_1 I$ is equal to $\pm(\text{rad } A_\ell)^k$. Since we have

$$(\text{rad } A_\ell)^k > \lambda_1^k \quad ,$$

it follows from Theorem 3 that the pivotal elements of $-A_\ell - \lambda_1 I$ are the off-diagonal elements of $-A_\ell - \lambda_1 I$. On the other hand, since

$$\lambda_2^k > (\text{rad } A_\ell)^k \quad ,$$

we conclude that the pivotal elements of $-A_\ell - \lambda_2 I$ are the diagonal elements of $-A_\ell - \lambda_2 I$, and this contradicts the statement that $-A_\ell - \lambda_1 I$ and $-A_\ell - \lambda_2 I$ have the same pivotal positions.

Hence, we conclude that no cyclic component of $\mathcal{Q}(A)$ can separate $G(A, \lambda_1)$ and $G(A, \lambda_2)$, and since these two gaps were assumed to be distinct, they must be separated by a primary component of $\mathcal{Q}(A)$.

Thus, there exists a diagonal element of A, say a_{rr}, which satisfies

$$\lambda_2 > a_{rr} > \lambda_1 .$$

Since a_{rr} is the only non-zero element in the r^{th} row and r^{th} column of A, it follows that (r,r) must be a pivotal position for both gaps, and it is clearly subordinate for $G(A, \lambda_2)$ and non-subordinate for $G(A, \lambda_1)$. Hence, the two gaps cannot have the same number of subordinate diagonal pivotal positions, and in fact, if the gaps are separated by exactly s primary components of $\mathcal{Q}(A)$, then the gap $G(A, \lambda_2)$ has exactly s more subordinate positions than $G(A, \lambda_1)$. |

This result, coupled with Theorem 15, allows us to conclude that no two gaps in the eigenvalue set of a generalized permutation matrix can have the same number of subordinate diagonal pivotal positions. In order to prove that this result can be extended to the general case, we shall now establish a connection between the gaps in the eigenvalue set of a general matrix A and gaps in the eigenvalue sets of certain generalized permutation matrices which are closely related to A .

Definition 11. Let A be an $n \times n$ non-negative matrix, and let $P = (p_{ij})$ be a permutation matrix. Then we shall denote by $\mathcal{J}(P,A)$ the set of all $n \times n$ complex matrices $C = (c_{ij})$ whose elements satisfy the following conditions, for $k = 1,2,\ldots,n$:

$$|c_{k\ell}| = a_{k\ell} \qquad \text{if } p_{\ell k} = 1$$
$$|c_{kj}| \leqslant a_{kj} \qquad \text{if } p_{jk} = 0$$

Lemma 17.3. *Let A be an $n \times n$ non-negative matrix, and let $P = (p_{ij})$ be associated with a certain gap in $\mathcal{R}(A)$. Let C be any matrix in $\mathcal{J}(P,A)$. Then the gap in $\mathcal{R}(A)$ which is associated with P is contained in a gap in $\mathcal{R}(C)$ which is also associated with P .*

Proof: Let λ be any positive number in the given gap in $\mathcal{R}(A)$. Let $R = (r_{ij})$ be any matrix in $\mathcal{S}(C)$. Let $B = (b_{ij})$ be the matrix whose entries satisfy the following conditions, for $k = 1,2,\ldots,n$:

$$b_{kk} = a_{kk} \qquad \text{if } p_{kk} = 1$$
$$b_{kk} = -a_{kk} \qquad \text{if } p_{kk} = 0$$
$$b_{kj} = a_{kj} \qquad j = 1,2,\ldots,n \qquad j \neq k$$

Then B belongs to $\mathcal{S}(A)$, and it follows from Theorem 7 that $B-\lambda I$ is a regular matrix associated with P . Furthermore, $R-\lambda I$ belongs to $\mathcal{N}(B-\lambda I)$, since we have the following inequalities for $k = 1,2,\ldots,n$:

$$a_{kk}+\lambda = |b_{kk}-\lambda| \geqslant |r_{kk}-\lambda| \qquad \text{if } p_{kk} = 0$$
$$|a_{kk}-\lambda| = |b_{kk}-\lambda| \leqslant |r_{kk}-\lambda| \qquad \text{if } p_{kk} = 1$$

Therefore, $R-\lambda I$ is a regular matrix associated with P . Since R was chosen arbitrarily from $\mathcal{S}(C)$ and λ, from $G(A,\lambda)$, it follows that $G(A,\lambda)$ is contained in a gap in $\mathcal{R}(C)$, and we conclude from Theorem 7 that this gap, $G(C,\lambda)$, must be associated with P , since $R-\lambda I$ belongs to $\mathcal{H}(C,\lambda)$. |

<u>Theorem 17. Let</u> A <u>be an</u> $n \times n$ <u>non-negative matrix. Then no two gaps in</u> $\mathcal{G}(A)$ <u>can have the same number of subordinate diagonal pivotal positions.</u>

Proof: Because of Theorem 15, it suffices to prove that any two gaps in $\mathcal{G}(A)$ which are associated with the same permutation matrix cannot have the same number of subordinate diagonal pivotal positions.

Suppose that two gaps in $\mathcal{G}(A)$ are associated with the same permutation matrix P . Let $\lambda_1 > 0$ be in the more interior of these two gaps, and let $\lambda_2 > \lambda_1$ be in the other. Since $G(A,\lambda_1)$ and $G(A,\lambda_2)$ are disjoint gaps, at least one component of $\mathcal{G}(A)$ must lie between the circles $z = \lambda_1$ and $z = \lambda_2$. Let $B = (b_{ij})$ be the $n \times n$ non-negative generalized permutation matrix in $\mathcal{F}(P,A)$-- i.e. for $k = 1,2,\ldots,n$, we have

$$b_{k\ell} = a_{k\ell} \qquad \text{if } p_{\ell k} = 1$$
$$b_{kj} = 0 \qquad \text{if } p_{jk} = 0$$

Let α be an eigenvalue of A contained in a component of $\mathcal{G}(A)$ which separates the gaps, $G(A,\lambda_1)$ and $G(A,\lambda_2)$. Since $\mathcal{F}(P,A)$ is a closed, connected subset of complex E^{n^2} space which contains both A and B , there exists a "path" in $\mathcal{F}(P,A)$ which connects A to B, and the eigenvalues of the matrices on this path form continuous paths in the complex plane. One of these paths contains α and at least one eigenvalue of B. Moreover, since Lemma 17.3 informs us that no matrix in $\mathcal{F}(P,A)$ can have a complex number with modulus equal to λ_1 or λ_2

as an eigenvalue, we conclude that there must exist at least one component of $\mathcal{R}(B)$ between the circles $|z| = \lambda_1$ and $|z| = \lambda_2$. Furthermore, it follows from Lemma 17.3 that λ_1 and λ_2 both lie in gaps in $\mathcal{R}(B)$ which are associated with P, and we conclude from Lemma 17.2 that $G(B, \lambda_1)$ and $G(B, \lambda_2)$ cannot have the same number of subordinate diagonal pivotal positions. Consequently, $G(A, \lambda_1)$ and $G(A, \lambda_2)$ also cannot have the same number of subordinate positions, and the proof of the theorem is complete. |

We can also use Lemma 17.3 to obtain a certain amount of information about the components of $\mathcal{R}(A)$, but this information is useful only if we know which permutation matrices are associated with the gaps in $\mathcal{R}(A)$.

Theorem 18. Let A be an n x n non-negative matrix, and let a component of $\mathcal{R}(A)$ be bounded by gaps which are associated with the permutation matrices P and Q. Then at least one eigenvalue of every matrix in the set $\mathcal{F}(P,A) \cap \mathcal{F}(Q,A)$ must be contained in the given component of $\mathcal{R}(A)$.

Proof: Let B be any matrix in $\mathcal{F}(P,A) \cap \mathcal{F}(Q,A)$. Let $G(A, \lambda)$ be the gap in $\mathcal{R}(A)$ which bounds the given component of $\mathcal{R}(A)$ and which is associated with P, and let $G(A,\rho)$ be the gap which is associated with Q. It follows from Lemma 17.3 that the gap $G(A, \lambda)$ is contained in the gap $G(B, \lambda)$ in $\mathcal{R}(B)$ and that $G(B, \lambda)$ is associated with P. Similarly, the gap $G(A,\rho)$ is contained in the gap $G(B,\rho)$, which is associated with Q. Therefore, since $\mathcal{F}(P,A) \cap \mathcal{F}(Q,A)$ is a closed, connected subset of complex E^{n^2} space which contains both A and B , it follows that $G(B,\lambda)$ and $G(B,\rho)$ must be separated by at least one component of $\mathcal{R}(B)$. Therefore, $G(A,\lambda)$ and $G(A,\rho)$ must also be separated by at least one component of $\mathcal{R}(B)$, and this means that at least one eigenvalue of B must be contained in the given component of $\mathcal{R}(A)$. |

We conclude this section with two results which illustrate how Theorem 17 may be used to determine the number and nature of the components of certain eigenvalue sets.

Lemma 19.1 Let A be an n x n non-negative matrix. If A is not regular, then each gap in $\mathcal{R}(A)$ must have at least one diagonal pivotal position. If A is regular, then only the gap $G(A,0)$ can have no diagonal pivotal positions.

Proof: Suppose A is not regular. Let λ be a positive number in a gap in $\mathcal{R}(A)$ which has no diagonal pivotal positions. Then A is in $\mathcal{N}(-A-\lambda I)$, and this contradicts the fact that A is not regular.

Similarly, if A is regular, then if $G(A,0)$ has no diagonal pivotal positions, it follows from Theorem 17 that every other gap in $\mathcal{R}(A)$ must have at least one diagonal pivotal position. If $G(A,0)$ has at least one diagonal pivotal position, we conclude from a simple application of Theorem 3 that it is impossible for any gap in $\mathcal{R}(A)$ to have no diagonal pivotal positions. |

Theorem 19. Let $A = (a_{ij})$ *be an* $n \times n$ *non-negative matrix such that* $a_{kk} = 0$ *for* $k = 1,2,\ldots,n$ *. Then* $\mathcal{R}(A)$ *can have at most* $(n-1)$ *components.*

Proof: Since all the diagonal elements of A are equal to zero, it follows that all the diagonal pivotal positions of each gap in $\mathcal{R}(A)$ are subordinate.

If A is regular, then $G(A,0)$ has no diagonal pivotal positions, and it follows from Theorem 17 that no two gaps in $\mathcal{R}(A)$ can have the same number of subordinate diagonal pivotal positions. Therefore, in the worst possible case, there can exist one and only one gap of type $\langle s,s \rangle$, for $s = 0,1,2,\ldots,(n-2),n$, since no gap can have exactly $(n-1)$ diagonal pivotal positions. Hence, if A is regular, there can be at most n gaps in $\mathcal{R}(A)$, and this is equivalent to the statement that $\mathcal{R}(A)$ has at most $(n-1)$ components.

If A is not regular, then it follows from Lemma 19.1 that each gap in $\mathcal{R}(A)$ must have at least one diagonal pivotal position. Hence, there can only be gaps of type $\langle s,s \rangle$ for $s = 1,2,\ldots,(n-2),n$, and it follows that $\mathcal{R}(A)$ can have at most $(n-1)$ gaps. Therefore, since A is not regular, $\mathcal{R}(A)$ can have no more than $(n-1)$ components. |

Lemma 20.1. Let A *be an* $n \times n$ *non-negative matrix. Then* $G(A,\infty)$ *is of type* $\langle n,n \rangle$.

Proof: Since $G(A,\infty)$ is associated with the identity, it has n diagonal pivotal positions, all of which must be subordinate, since $G(A,\infty)$ is unbounded. Hence, $G(A,\infty)$ is of type $\langle n,n \rangle$. |

<u>Theorem 20. Let J_n be the $n \times n$ matrix all of whose entries are equal to one. Then $\mathcal{R}(J_n)$ is the disk $|z| \leq n$.</u>

Proof: Let λ be a positive number which is not in $\mathcal{R}(J_n)$. Then we conclude from Theorem 7 and the symmetry of J_n that $-J_n - \lambda I$ must be in normal form. Furthermore, $J_n - \lambda I$ must also be a regular matrix in normal form, and it follows from Theorem 3 that $\lambda > 1$, since the product of the elements of $J_n - \lambda I$ which lie along any generalized diagonal that includes no diagonal positions is equal to one. Since all the diagonal elements of J_n are equal to one, it follows that all the diagonal pivotal positions of $G(A, \lambda)$ are subordinate, and $G(A, \lambda)$ must be of type $\langle n,n \rangle$. We conclude from Lemma 20.1 and Theorem 17 that the only gap in $\mathcal{R}(A)$ is $G(A, \infty)$.

Furthermore, it is clear that n is an eigenvalue of J_n, and if σ is any number greater than n, then $-J_n - \sigma I$ is diagonally dominant and hence, a regular matrix in normal form. Consequently, n must be the only positive boundary point of $\mathcal{R}(J_n)$, and we conclude that $\mathcal{R}(J_n)$ is the disk $|z| \leq n$. |

V. BOUNDARY PROPERTIES OF NORMAL GAPS.

It follows from the Perron-Frobenius Theorem that the largest non-negative boundary point of the eigenvalue set of a non-negative matrix A is actually an eigenvalue of A . Since it is possible for $\mathcal{R}(A)$ to have as many as 2n non-negative boundary points, it is unrealistic to expect that every non-negative boundary point of $\mathcal{R}(A)$ will be an eigenvalue of A . However, we shall prove in this section that the positive boundary points of a normal gap are actually eigenvalues of certain <u>real</u> matrices equimodular with A . In particular, if A is a regular matrix in normal form, then every real boundary point of $\mathcal{R}(A)$ is an eigenvalue of a real matrix equimodular with A . We shall also show that if A is an irreducible regular matrix in normal form, then the diagonal elements of A are all interior points of $\mathcal{R}(A)$, and it will be seen that this in turn implies that no component of $\mathcal{R}(A)$ can be a circle.

In order to prove that the real boundary points of a normal gap in $\mathcal{R}(A)$ are actually eigenvalues of real matrices in $\mathcal{S}(A)$, we proceed along lines which are basically analagous to the method used in the proof of Theorem 1. However, certain continuity arguments must be altered to take into account the fact that the set of all real matrices equimodular with A is not a connected subset of real E^{n^2} space. We begin with a definition.

Definition 12. Let $B = (b_{ij})$ be an $n \times n$ real matrix. Then by $\mathcal{S}_o(B)$ we shall denote the set of all real matrices $C = (c_{ij})$ in $\mathcal{S}(B)$ whose diagonal elements satisfy $c_{ii} = b_{ii}$, for $i = 1,2,\ldots,n$.

We shall call B semi-regular if every matrix in $\mathcal{S}_o(B)$ is non-singular.

We say that B is in normal form if for every matrix C in $\mathcal{S}_o(B)$ we have $F_{kk}(C) > 0$, for $k = 1,2,\ldots,n$.

Lemma 21.1. Let B be an $n \times n$ real matrix. If there exists an index k such that $F_{kk}(C) > 0$ for every matrix C in $\mathcal{S}_o(B)$, then B is semi-regular.

Proof: If C is a singular matrix, then $F_{kk}(C) \leqslant 0$. Hence, $\mathcal{S}_o(B)$ contains no singular matrices, and it follows that $\mathcal{S}_o(B)$ is semi-regular. |

Lemma 21.2. Let B be an $n \times n$ real matrix. If there exists a matrix C in $\mathcal{S}_o(B)$ and an index k such that $F_{kk}(C) = 0$, then B is not semi-regular.

Proof: Let C be a matrix in $\mathcal{S}_o(B)$ such that $F_{kk}(C) = 0$. Then we have

$$|c_{kk}|\,|\mathcal{C}_{kk}| = \sum_{j \neq k}{}' |c_{kj}|\,|\mathcal{C}_{kj}| = \pm c_{kk}\mathcal{C}_{kk}$$

As in Lemma 1.3 we can use this equality to construct a singular matrix $T = (t_{ij})$ which is in $\mathcal{S}_o(B)$ and which differs from C only in the off-diagonal elements of the k^{th} row. Since $\mathcal{S}_o(B)$ contains a singular matrix, it follows that B cannot be semi-regular. |

Definition 13. Let $B = (b_{ij})$ be an $n \times n$ real matrix. Then we shall denote by $\mathcal{N}_o(B)$ the set of all real matrices $C = (c_{ij})$ whose elements satisfy the following conditions for $k = 1,2,\ldots,n$:

$$c_{kk} = b_{kk}$$
$$|c_{kj}| \leqslant |b_{kj}| \qquad j = 1,2,\ldots,n \qquad j \neq k$$

In the proof of Theorem 21, we shall show that a boundary point of a normal gap in the eigenvalue set of a non-negative matrix A must actually be an eigenvalue of a matrix in $\mathcal{N}_o(A)$. We shall now prove two lemmas which will enable us to use this fact to prove that such a boundary point must be an eigenvalue of a matrix in $\mathcal{S}_o(A)$.

Lemma 21.3. Let B be an $n \times n$ semi-regular matrix in normal form. Then every matrix in $\mathcal{N}_o(B)$ is also semi-regular and in normal form.

Proof: The proof of this lemma is entirely analagous to the proof of Lemma 1.5. Therefore, we shall demonstrate the validity of the lemma for those matrices in $\mathcal{N}_o(B)$ which differ from B only in the elements of a single row. The general result is obtained by using a recursive argument analagous to that employed in Lemma 1.5.

Accordingly, for an index k , which will remain fixed throughout the proof, we denote by $\mathcal{M}_k(B)$ the set of all real matrices $U = (u_{ij})$ whose elements satisfy the following inequalities:

$$u_{kk} = b_{kk}$$
$$|u_{kj}| \leqslant |b_{kj}| \qquad j = 1,2,\ldots,n \qquad j \neq k \tag{5.1}$$
$$u_{ij} = b_{ij} \qquad i,j = 1,2,\ldots,n \qquad i \neq k$$

Let C be any element of $\mathcal{M}_k(B)$. We shall show that C is a semi-regular matrix in normal form. Let $T = (t_{ij})$ be any matrix in $\mathcal{S}_o(C)$ and let $V = (v_{ij})$ be the corresponding matrix in $\mathcal{S}_o(B)$. By this we mean that V is in $\mathcal{S}_o(B)$; that v_{kj} has the same sign as t_{kj} for $j = 1,2,\ldots,n$; and that $v_{ij} = t_{ij}$ for $i,j = 1,2,\ldots,n$ and $i \neq k$.

Then $F_{kk}(V) > 0$, since B is in normal form. Since we have $\mathcal{V}_{kj} = \mathcal{T}_{kj}$ for $j = 1,2,\ldots,n$, the fact that T is in $\mathcal{S}_o(C)$ together with inequalities (5.1) allow us to conclude that $F_{kk}(T) > 0$, also. Since T was chosen arbitrarily from $\mathcal{S}_o(C)$, it follows from Lemma 21.1 that C is semi-regular, and since C was chosen arbitrarily from $\mathcal{M}_k(B)$ we conclude that every matrix in $\mathcal{M}_k(B)$ is semi-regular.

Now, suppose for some index $m \neq k$, we have $F_{mm}(T) \leq 0$. Let $\mathcal{M}_k(V)$ denote the set of all real matrices $R = (r_{ij})$ whose elements satisfy the following conditions:

$$\begin{aligned} r_{kk} &= v_{kk} \\ |r_{kj}| &\leq |v_{kj}| \qquad j = 1,2,\ldots,n \qquad j \neq k \\ r_{ij} &= v_{ij} \qquad i,j = 1,2,\ldots,n \qquad i \neq k \end{aligned}$$

Since B is in normal form, it follows that $F_{mm}(V) > 0$, and since $\mathcal{M}_k(V)$ is a connected subset of real E^{n^2} space which contains both V and T , we conclude that there exists a matrix R in $\mathcal{M}_k(V)$ such that $F_{mm}(R) = 0$. However, it follows from Lemma 21.2 that R cannot be semi-regular, and since R is equimodular with a matrix in $\mathcal{M}_k(B)$, this contradicts the already established fact that every matrix in $\mathcal{M}_k(B)$ is semi-regular. Hence, we conclude that $F_{mm}(T) > 0$ for $m = 1,2,\ldots,n$, and since T was chosen arbitrarily from $\mathcal{S}_o(C)$, it follows that C is in normal form. |

<u>Lemma 21.4. Let A be an $n \times n$ non-negative matrix, and let σ be a non-negative boundary point of a normal gap in $\mathcal{R}(A)$. If $A-\sigma I$ is semi-regular, then it is in normal form.</u>

Proof: We shall assume that σ is a non-negative right boundary point of a normal gap in $\mathcal{R}(A)$. By this, we mean that points slightly smaller than σ are not in $\mathcal{R}(A)$. The proof for non-negative left boundary points is entirely analagous to the one given here.

Let λ be a positive number in the gap for which σ is a boundary point, and let B be any matrix in $\mathcal{S}_o(A-\sigma I)$. Then $B = C-\sigma I$, where C is a matrix in $\mathcal{S}_o(A)$. Let ρ be any number in the open interval (λ, σ). Then ρ is in $G(A, \lambda)$, and it follows from Theorem 7 that $A-\rho I$ is a regular matrix in normal form, since $G(A, \lambda)$ is a normal gap. Hence, we must have $F_{kk}(A-\rho I) > 0$, for $k = 1,2,\ldots,n$. Since for each fixed k, $F_{kk}(A-\rho I)$ is a continuous function of the elements of its argument, $A-\rho I$, it follows that it is a continuous function of ρ, and we conclude that $F_{kk}(A-\sigma I) \geq 0$, for $k = 1,2,\ldots,n$. Furthermore, since $A-\sigma I$ is assumed to be semi-regular, it follows from Lemma 21.2 that we must have strict inequality for each index k. Therefore, $A-\sigma I$ is in normal form, as desired. |

Theorem 21. Let A be an n x n non-negative matrix, and let σ be a non-negative boundary point of a normal gap in $\mathcal{R}(A)$. Then σ is an eigenvalue of a matrix in $\mathcal{S}_o(A)$.

Proof: First we shall show that the matrix $A-\sigma I$ cannot be regular. Suppose that $A-\sigma I$ is regular. Then $A-\sigma I$ is certainly semi-regular, and it follows from Lemma 21.4 and Theorem 7 that $A-\sigma I$ is a regular matrix in normal form. If B is any matrix in $\mathcal{S}(A)$, then $B-\sigma I$ is in $\mathcal{N}(A-\sigma I)$. Hence, no matrix in $\mathcal{S}(A)$ can have σ as an eigenvalue, and this contradicts the fact that σ is in $\mathcal{R}(A)$. Consequently, we conclude that $A-\sigma I$ is not regular.

Since $A-\sigma I$ is not regular, it follows that σ is an eigenvalue of a matrix $C = (c_{ij})$ in $\mathcal{S}(A)$ whose diagonal elements are all non-negative. Let y be the eigenvector associated with σ , where $y = (|y_1|e^{i\theta_1} , |y_2| e^{i\theta_2} , \ldots , |y_n| e^{i\theta_n})$, say . Let $D = \text{diag} (d_1, d_2, \ldots, d_n)$ be the unitary diagonal matrix whose diagonal elements satisfy the following conditions:

$$d_k = e^{i\theta_k} \quad \text{if} \quad y_k \neq 0$$

$$d_k = 1 \quad \text{if} \quad y_k = 0 .$$

Then $x = D^{-1}y$ is a real vector, and since $Cy = \sigma y$, we have $D^{-1}CDx = \sigma x$. Let $T = D^{-1} C D$. Then T is in $\mathcal{S}(A)$; the diagonal elements of T satisfy $t_{kk} = a_{kk}$, for $k = 1,2,\ldots,n$; and T has σ as an eigenvalue associated with the real eigenvector x . Let $T = T_1 + iT_2$, where T_1 and T_2 are real matrices.

Then we have

$$\sigma x = Tx = T_1 x + iT_2 x \quad ,$$

and since σx is real, we must have $T_1 x = \sigma x$ and $T_2 x = 0$.

Therefore, the matrix $T_1 - \sigma I$ is not semi-regular. Furthermore since T has the same diagonal elements as A , we conclude that T_1 is in $\mathcal{N}_0(A)$, and $T_1 - \sigma I$ is in $\mathcal{N}_0(A - \sigma I)$. Hence, it follows from Lemma 21.3 that $A - \sigma I$ cannot be a semi-regular matrix in normal form, and we conclude from Lemma 21.4 that $A - \sigma I$ cannot even be semi-regular. Therefore, there must be a singular matrix in $\mathcal{S}_0(A - \sigma I)$, and this is equivalent to the statement that σ is an eigenvalue of a matrix in $\mathcal{S}_0(A)$. |

<u>Corollary 21.1</u> <u>If</u> B <u>is a regular matrix in normal form, then every real boundary point of</u> $\mathcal{R}(B)$ <u>is an eigenvalue of a real matrix equimodular with</u> B . |

<u>Corollary 21.2.</u> <u>Let</u> B <u>be a positive definite, real symmetric matrix. Then every real boundary point of</u> $\mathcal{R}(B)$ <u>is an eigenvalue of a real matrix equimodular with</u> B . |

Using Theorem 21, we can also obtain a result which is similar to the Perron-Frobenius Theorem, but considerably weaker.

<u>Theorem 22.</u> <u>Let</u> A <u>be an</u> $n \times n$ <u>non-negative matrix. Then the largest non-negative boundary point of</u> $\mathcal{R}(A)$ <u>is an eigenvalue of a matrix in</u> $\mathcal{S}_0(A)$. <u>Furthermore, if</u> B <u>is a non-negative matrix such that</u> $B \geqslant A$, <u>then the largest non-negative boundary point of</u> $\mathcal{R}(B)$

<u>is at least as large as that of</u> $\mathcal{R}(A)$.

Proof: Since $G(A,\infty)$ is a normal gap, it follows from Theorem 21 that the non-negative boundary point of $G(A,\infty)$ is an eigenvalue of a matrix in $\mathcal{S}_0(A)$.

Now, let λ be any positive number in $G(B,\infty)$. Since $G(B,\infty)$ is a gap of type $\langle n,n \rangle$ and since $B \geqslant A$, it follows that all the diagonal elements of both A and B are less than λ . Let $C = (c_{ij})$ be any matrix in $\mathcal{S}(A)$. Then we have the following inequalities, for $k = 1,2,\ldots,n$:

$$|c_{kk}-\lambda| \geqslant |a_{kk}-\lambda| = \lambda-a_{kk} \geqslant \lambda-b_{kk} = |b_{kk}-\lambda|$$

Therefore, $C-\lambda I$ is in $\mathcal{N}(B-\lambda I)$, and $C-\lambda I$ must be a regular matrix in normal form. Therefore, λ is in a gap in $\mathcal{R}(A)$, and since λ was chosen arbitrarily from $G(B,\infty)$, it follows that $G(A,\infty)$ is contained in $G(B,\infty)$, as desired. ▎

Unfortunately, the method used to prove Theorem 21 is non-constructive, and even though we can say that the largest non-negative boundary point of the eigenvalue set of a non-negative matrix A is an eigenvalue of a matrix in $\mathcal{S}_0(A)$, we cannot prove that this point is an eigenvalue of A . Furthermore, if A is irreducible, we cannot prove that $B \geqslant A$ implies that the spectral radius of B is greater than that of A , and although it is possible to show that the eigenvector which is associated with the largest non-negative boundary point of $\mathcal{R}(A)$ is non-negative, we cannot prove that if A is irreducible, then this eigenvector is positive.

In fact, the concept of irreducibility rarely fits into our investigations, for if A is a reducible matrix, then there exists an irreducible matrix B such that the boundary points of $\mathcal{R}(B)$ are arbitrarily close to those of $\mathcal{R}(A)$ and the gaps in $\mathcal{R}(B)$ are of the same type as the corresponding gaps in $\mathcal{R}(A)$. Thus, the eigenvalue sets of A and B are practically indistinguishable by the methods we employ. However, our final result shows that this is not altogether true, for we are able to prove that the diagonal elements of an irreducible regular matrix in normal form are actually interior points of its eigenvalue set.

Theorem 23. *Let* A *be an* n x n *non-negative, irreducible, regular matrix in normal form. Then every diagonal element of* A *is an interior point of* $\mathcal{R}(A)$.

Proof: It follows from Corollary 10.2 that every diagonal element of A is in $\mathcal{R}(A)$. We shall prove that no diagonal element of A can be a boundary point of $\mathcal{R}(A)$. To simplify the proof, we shall show that, in particular, a_{11} cannot be a boundary point of $\mathcal{R}(A)$. We shall use induction on n.

For $n = 2$, the fact that A is irreducible means that $a_{12} a_{21} \neq 0$. Hence, $A - a_{11}I$ must be regular, and it follows from Theorem 21 that a_{11} cannot be a boundary point of $\mathcal{R}(A)$.

Now, in the general case, since A is irreducible, the graph of A must be strongly connected. Hence, there must be a closed path in the graph of A which includes node #1. By paring off superfluous circuits of this closed path, it is possible to obtain a closed subpath which includes node #1 and which passes through no node more than once. This means there exists a sequence of, say, $m \leqslant n$ non-zero off-diagonal terms in A which lie along a generalized diagonal of an $m \times m$ principal submatrix of A that contains a_{11}. If $m < n$, it follows that there exists an irreducible principal submatrix of A which contains a_{11}. In this case, we conclude from the inductive hypothesis that a_{11} is an interior point of the eigenvalue set of this principal submatrix, and the theorem follows from Corollary 10.1 .

If $m = n$, then the n non-zero off-diagonal elements must lie along a generalized diagonal which is cyclic of order n. Hence, there is no loss of generality in assuming that these elements are $a_{12}; a_{23}; \ldots; a_{n-1,n}; a_{n1}$. Let $C = (c_{ij})$ be the matrix whose elements satisfy the following conditions:

$$\begin{aligned}
c_{kk} &= a_{kk} && k = 1,2,\ldots, n\\
c_{k,k+1} &= a_{k,k+1} && k = 1,2,\ldots, n-1\\
c_{n1} &= a_{n1} &&\\
c_{ij} &= 0 && \text{otherwise}
\end{aligned}$$

Then C is in $\mathcal{N}(A)$, and it follows from Corollary 10.2 that a_{11} is in $\mathcal{R}(C)$. Let B be an arbitrary matrix in $\mathcal{S}_o(C-a_{11}I)$. Then we have

$$\det B = \pm a_{n1} \prod_{k=1}^{n-1} a_{k,k+1} .$$

It follows that $C-a_{11}I$ is semi-regular, and we conclude from Theorem 21 that a_{11} cannot be a boundary point of $\mathcal{R}(C)$. Therefore, a_{11} must be an interior point of $\mathcal{R}(C)$, and it follows from Theorem 10 that a_{11} must also be an interior point of $\mathcal{R}(A)$. |

Corollary 23.1. *Let A be an n x n non-negative, irreducible, regular matrix in normal form. Then no component of $\mathcal{R}(A)$ can be a circle.*

Proof: Let $D = \text{diag}(a_{11}, a_{22}, \ldots, a_{nn})$. Since every gap in $\mathcal{R}(A)$ is normal, it follows from Theorem 18 that an eigenvalue of D is contained in every component of $\mathcal{R}(A)$. In particular, this means that a circular component of $\mathcal{R}(A)$ would have to contain a diagonal element of A , and this is impossible since every diagonal element of A is an interior point of $\mathcal{R}(A)$. Hence, no component of $\mathcal{R}(A)$ can be a circle. |

VI. SUMMARY.

In this section, we shall summarize the results of the preceding sections and make a few conjectures about possible future developments.

Now that we have completed our investigations, we can say the following about the structure of the eigenvalue set of a general $n \times n$ non-negative matrix A :

1. The set $\mathcal{R}(A)$ consists of $k \leqslant n$ closed annuli, each of which is centered at the origin. One of these annuli is a disk if $\mathcal{S}(A)$ contains a singular matrix. (Theorem 0) .

2. Each gap in $\mathcal{R}(A)$ may be regarded as a class of regular matrices, each of which is associated with the same permutation matrix. Thus, each gap in $\mathcal{R}(A)$ may be associated with a permutation in a natural fashion. (Theorem 7) .

3. The unbounded gap, $G(A, \infty)$ is always associated with the identity permutation. (Theorem 8) .

4. There exist $m \leqslant n$ permutations, $\varphi_1, \varphi_2, \ldots, \varphi_m$, such that each gap in $\mathcal{R}(A)$ is associated with one of $\varphi_1, \varphi_2, \ldots, \varphi_m$. This is true even if $\mathcal{R}(A)$ has $(n+1)$ gaps. Hence, we say that $\mathcal{R}(A)$ is associated with the permutations $\varphi_1, \varphi_2, \ldots, \varphi_m$. (Theorem 16) .

5. Let P_k denote the permutation matrix related to φ_k, for $k = 1,2, \ldots, n$. Then $\mathcal{R}(A)$ can be characterized as the complement of the set of all complex numbers z such that for each matrix B in $\mathcal{S}(A)$, there exists a permutation, say φ_k, and a positive diagonal matrix D such that the matrix $P_k(B - |z| I)D$ is diagonally dominant. (Theorem 2 and Theorem 7).

It is clearly not feasible to compute $\mathcal{R}(A)$ from the above characterization. Instead, we use tools such as Theorem 3 in conjunction with Theorem 7 to determine key points in the gaps in $\mathcal{R}(A)$, and once we have determined the number of gaps and the permutation matrix with which each is associated, we can use Theorem 18 to construct points in the components of $\mathcal{R}(A)$. To illustrate this technique, we shall analyze the eigenvalue set of a specific matrix.

Example 6.1.

Consider the following matrix:

$$A = \begin{pmatrix} 16 & 0 & 1/16 & 0 \\ 0 & 0 & 1 & 0 \\ 1/32 & 0 & 0 & 1 \\ 0 & 1 & 9 & 2 \end{pmatrix}$$

Because of the structure of A, it follows from Theorem 1 that any gap in $\mathcal{R}(A)$ must be associated with one of the following permutation matrices :

$$P_1 = \begin{pmatrix} 1&0&0&0 \\ 0&0&0&1 \\ 0&1&0&0 \\ 0&0&1&0 \end{pmatrix} \quad P_2 = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \end{pmatrix}$$

$$P_3 = \begin{pmatrix} 0&0&1&0 \\ 0&1&0&0 \\ 1&0&0&0 \\ 0&0&0&1 \end{pmatrix} \quad I = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}$$

Since 1/16 and 1/32 are small relative to the other numbers in A , we eliminate P_3 from consideration and we shall assume that any gap in $\mathcal{R}(A)$ is associated with P_1, P_2, or I .

First of all, we observe that A must be regular since the submatrix formed from A by deleting the 1st row and 1st column is regular, and the minor of the (1,3) position is zero for all matrices in $\mathcal{S}(A)$. Since no product of elements of A which lie along generalized diagonal exceeds 16 , it follows from Theorem 3 that the gap G(A,0) is associated with P_1 . Since G(A,0) is of type $\langle 1,0 \rangle$, any other gap in $\mathcal{R}(A)$ must have at least one subordinate diagonal pivotal position. If $\mathcal{R}(A)$ has a gap of type $\langle 1,1 \rangle$, then this gap must have (1,1) as a pivotal position, since any gap which includes numbers greater than 16 is clearly normal. Since (1,1) is the sole diagonal pivotal position of only P_1 , it follows that any gap of type $\langle 1,1 \rangle$ must be associated with P_1 . If a gap is associated with P_2 , then every gap in $\mathcal{R}(A)$ which is more exterior than this gap must be normal, since any such gap must have at least two subordinate diagonal pivotal positions.

Now we shall determine whether each diagonal element of A is in $\mathcal{R}(A)$ or in a gap in $\mathcal{R}(A)$. Clearly, the zero diagonal elements of A are in the gap $G(A,0)$. The number 2 cannot be in a gap in which $(2,2)$ is pivotal. Hence, if 2 is not in $\mathcal{R}(A)$, it must be in a gap associated with P_1 or P_2. We observe that the product of the elements of $A-2I$ which lie along the generalized diagonal $(1,1)$; $(2,2)$; $(3,4)$; $(4,3)$ is greater than the product of elements along $(1,1)$; $(2,3)$; $(3,4)$; $(4,2)$. Hence, if 2 is in a gap, it follows from Theorem 3 and Theorem 7 that this gap must be associated with P_2. Let $D = \text{diag}\,(1, 9/16, 1, 33/16)$. Then if B is any matrix in $\mathcal{S}(A)$, the matrix $P_2(B-2I)D$ is diagonally dominant. Hence, it follows from Theorem 2 and Theorem 7 that 2 is in a gap in $\mathcal{R}(A)$ which is associated with P_2, and this gap must be of type $\langle 2,1\rangle$. On the other hand, the number 16 cannot be in a gap in $\mathcal{R}(A)$ since $(1,1)$ must be a pivotal position of every gap in $\mathcal{R}(A)$. Moreover, if B is any matrix in $\mathcal{S}(A)$, then it is clear that $B-14I$ is diagonally dominant. Hence, it follows that 14 is in a normal gap in $\mathcal{R}(A)$ of type $\langle 4,3\rangle$. Thus, we conclude that there are four gaps in $\mathcal{R}(A)$:

$G(A,0)$, which is associated with P_1 and is of type $\langle 1,0\rangle$;
$G(A,2)$, which is associated with P_2 and is of type $\langle 2,1\rangle$;
$G(A,14)$, which is associated with I and is of type $\langle 4,3\rangle$;
$G(A,\infty)$, which is associated with I and is of type $\langle 4,4\rangle$.

Now we shall determine a few key points in $\mathcal{R}(A)$. First of all, it follows from Theorem 18 that the component of $\mathcal{R}(A)$ which is bounded by $G(A,0)$ and $G(A,2)$ contains at least one component of $\mathcal{R}(C_1)$, where C_1 is the following matrix :

$$C_1 = \begin{pmatrix} 16 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 9 & 0 \end{pmatrix}$$

Thus, this first component contains only one eigenvalue of every matrix in $\mathcal{S}(A)$, since this property holds for matrices in $\mathcal{S}(C_1)$, and the smallest boundary point of $\mathcal{R}(A)$ is no greater than $1/8$. Similarly, the largest positive boundary point of this component is somewhat less than $1/2$. The second component of $\mathcal{R}(A)$ is bounded by $G(A,2)$ and $G(A,14)$, and contains at least one component of $\mathcal{R}(C_2)$, where

$$C_2 = \begin{pmatrix} 16 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -9 & 2 \end{pmatrix}$$

The characteristic polynomial of C_2 is $(x-16)(x^3-2x^2+9x-1)$. This polynomial has a pair of complex roots whose modulus is approximately 3 . Therefore, the second component of $\mathcal{R}(A)$ contains two eigenvalues of every matrix in $\mathcal{S}(A)$, and contains numbers at least as small as 3 . In addition to a component of $\mathcal{R}(C_2)$, this second component of $\mathcal{R}(A)$ must contain a component of $\mathcal{R}(C_3)$, where

$$C_3 = \begin{pmatrix} 16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 9 & 2 \end{pmatrix}$$

Since the only two eigenvalues of C_3 which can be in the second component of (A) are $1+\sqrt{10}$ and $1-\sqrt{10}$, and since 2 is in $G(A,2)$, we conclude that the smallest positive boundary point of $\mathcal{R}(A)$ must be slightly larger than 2. Further examination shows that the largest positive boundary point of this gap is slightly less than 5. Finally, the last component of $\mathcal{R}(A)$ contains 16, and its two positive boundary points are both approximately equal to 16. Hence, $\mathcal{R}(A)$ is closely approximated by the union of the following three annuli:

$$1/8 \leq |z| \leq 1/2 \quad ; \quad 2 \leq |z| \leq 5 \quad ; \quad 31/2 \leq |z| \leq 33/2$$

The results we have obtained answer only part of the questions raised in the introductory section. Actually, we believe that much more can be said about $\mathcal{R}(A)$. In fact, we conjecture that the Perron-Frobenius Theorem is a special case of a more general property of the set $\mathcal{R}(A)$ which may be stated as follows:

Conjecture 6.1. Let A *be an irreducible, non-negative,* $n \times n$ *matrix, and let* $\sigma > 0$ *be a boundary point of* $\mathcal{R}(A)$. *Then* σ *is an eigenvalue of a certain real matrix* B *in* $\mathcal{S}(A)$, *and a matrix* C *equimodular with* A *has an eigenvalue of modulus* σ *if and only if* $C = e^{i\theta}DBD^{-1}$, *where* D *is a unitary diagonal matrix.*

We also conjecture that it is possible to describe the matrix B of Conjecture 6.1 explicitly, in terms of the pivotal positions of the gap for which it is a boundary point.

Conjecture 6.2. *Let $G(A,\lambda)$ be a gap in $\mathcal{R}(A)$. Then the matrix B of Conjecture 6.1 may be described as follows, for* $k = 1,2,\ldots,n$:

a) *If (k,k) is a subordinate pivotal position in $G(A,\lambda)$, then*

$$b_{kj} = a_{kj} \qquad j = 1,2,\ldots,n$$

b) *If (k,k) is a non-subordinate pivotal position in $G(A,\lambda)$,*

$$b_{kk} = a_{kk}$$

$$b_{kj} = -a_{kj} \qquad j = 1,2,\ldots,n \quad j \neq k$$

c) *If (k,ℓ) with $\ell \neq k$ is the pivotal position in the k^{th} row in $G(A,\lambda)$, then*

$$b_{k\ell} = a_{k\ell}$$

$$b_{kj} = -a_{kj} \qquad j = 1,2,\ldots,n \quad j \neq \ell$$

Moreover, we further conjecture that Theorem 17 may be extended in the following sense:

Conjecture 6.3. *Let $G(A,\lambda)$ and $G(A,\mu)$ be gaps in (A) and suppose that $\mu > \lambda > 0$. Then $G(A,\mu)$ has more subordinate diagonal pivotal positions than $G(A,\lambda)$.*

It appears that a more systematic approach to the study of $\mathcal{R}(A)$ is possible if we regard this study as a special case of a more general problem. Before we can be more specific, we shall require the following definition:

Definition: Let A and B be non-negative, n x n matrices. Then the pair (A,B) is said to be *bi-regular* if $\det(R+S) \neq 0$ for all matrices R in $\mathcal{S}(A)$ and S in $\mathcal{S}(B)$.

The motivation for this definition is provided by Theorem 7, for if λ is not in $\mathcal{R}(A)$, then the pair $(A, \lambda I)$ is bi-regular. By regarding the set $\mathcal{R}(A)$ in terms of bi-regularity, it is possible to construct a unified theory in which the intrinsic properties of $\mathcal{R}(A)$ are made evident. Specifically, it can be shown that the open set of all n x n regular matrices has $n!$ components, each of which contains a permutation matrix. A similar result holds for the set of all bi-regular matrices of a given order. In particular, it can be shown that each component of this set contains a bi-regular "splitting" of the form (P,Q), where $P+Q$ is a permutation matrix. If $B = 0$, then (A,B) is bi-regular if and only if A is regular, and in this case, the corresponding splitting has $Q = 0$, and P is a permutation matrix. This is equivalent to Theorem 1. If $B = \lambda I$, where λ is not in $\mathcal{R}(A)$, then the corresponding splitting is closely related to the subordinate and non-subordinate diagonal pivotal positions of the gap $G(A, \lambda)$. This technique not only promises to yield interesting results concerning $\mathcal{R}(A)$, but also suggests that the study of what could be called multi-regular sets of matrices might be profitable.

Bibliography

[1] Paul Camion and A. J. Hoffman, "On the Nonsingularity of Complex Matrices," to appear.

[2] Olga Taussky, "On the Variation of the Characteristic Roots of a Finite Matrix Under Various Changes of its Elements," Recent Advances in Matrix Theory, ed. H. Schneider, University of Wisconsin Press, 1964: 125-138.

[3] B. W. Levinger and R. S. Varga, "On a Problem of O. Taussky," to appear.

[4] Richard S. Varga, "Minimal Gerschgorin Sets," to appear.

[5] B. W. Levinger and R. S. Varga, "Minimal Gerschgorin Sets II," to appear.

[6] John Todd, "On Smallest Gerschgorin Disks for Eigenvalues," Numer. Math. 7, 1965: 171-175.

[7] A. M. Ostrowski, "Solution of Equations and Systems of Equations," Academic Press, New York: 192-194.

[8] Richard S. Varga, "Matrix Iterative Analysis," Prentice-Hall, Inc., 1962.