THE VACUUM REGGE TRAJECTORY
IN CONVENTIONAL FIELD THEORY

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ABSTRACT

The effect on the scattering amplitude of the existence of a pole in the angular momentum plane near $J = 1$ in the channel with the quantum numbers of the vacuum is calculated. This is then compared with a fourth order calculation of the scattering of neutral vector mesons from a fermion pair field in the limit of large momentum transfer. The presence of the third double spectral function in the perturbation amplitude complicates the identification of pole trajectory parameters, and the limitations of previous methods of treating this are discussed. A gauge invariant scheme for extracting the contribution of the vacuum trajectory is presented which gives agreement with unitarity predictions, but further calculations must be done to determine the position and slope of the trajectory at $s = 0$. The residual portion of the amplitude is compared with the Gribov singularity.

TABLE OF CONTENTS

I. INTRODUCTION

This investigation is one step in a grand plan[(1)-(5)] designed to delineate the role of Regge poles in conventional field theory. In a sense, the Regge pole hypothesis is a theory about theories. It tries to explain how various field theories might possibly resolve what appear to be paradoxes. How, for example, a field theory might yield sensible conclusions about higher spin particles, and yet be unrenormalizable if these particles are inserted into the Lagrangian as "elementary"; or how the exchange of these higher spin particles is compatible with the high energy behavior of scattering processes in the cross channel when even the lowest order terms in perturbation theory seem to be too large. Another incentive for investigating the relation between Regge poles and field theory is the belief that such a relation will reveal properties of field theory which are independent of the results of a perturbation expansion. One application of this kind of information would be to the problem of whether or not field theory in its present form is correct for the strong interactions. Suppose, for example, that many of the experimental consequences of the hypothesis of Regge poles[(6)] for the strong interactions were in fact verified. Then any theory which was to explain the strong interactions would have to contain the features which the strong interactions exhibited--in this case Regge behavior. Thus the knowledge that field theory did not contain Regge behavior would tell us at least that field theory could be excluded as a source of further information, without our having to develop a non-perturbative calculating scheme.

A third motivation for this study is the inverse of the previous one, namely, that if a connection between field theory and Regge poles theory could be established, it would be possible to use field theory as a guide to develop further understanding of Regge poles. This point of view becomes particularly important if the Regge theory turns out to be a valuable tool in understanding experimental results. This hope was dimmed in the recent past by first the speculation[7] and later the confirmation[8] of the existence of cuts as well as poles in the angular momentum plane. However, recent research seems to indicate[9] that at least some of the simple experimental consequences of Regge poles are retained even when cuts are present, although the energies at which these phenomena become observable are higher. More research is seriously needed on this point.

In the sense in which the Regge pole hypothesis makes predictions about field theories, Feynman diagrams become the laboratory in which these predictions may be tested. Some of these "experiments" have already been performed. In the field theory where neutral vector mesons interact with spin one-half nucleons, i.e., massive quantum electrodynamics, the nucleon has been shown to lie on a Regge trajectory[3] while the meson does not[10]. In the same field theory, the experiment begun in Reference (5) to determine the existence of a vacuum Regge trajectory is continued here after the development of more refined techniques. Considering this and other field theories with particles of different spins, a set of "empirical" criteria emerges for the development of a Regge trajectory in a given theory. Among these are the factorizability of the Regge pole residues and the existence of a nonsense channel, which will be discussed later.

Before we begin the actual experiment, it would be well to consider what is meant by a vacuum Regge trajectory, and what properties it has that would justify searching for it. The remarks in this section are very general, and are not intended as a thorough introduction to the subject such as may be found in the book by Frautschi[(11)] to which the reader is referred for further details. Let s and t respectively be the conventional energy and momentum transfer variables for the relativistic scattering of two spinless particles. Then, as we shall see later, if the scattering is dominated by a Reggeized intermediate state we expect the invariant scattering matrix element to behave like

$$M(s,t) = C(s)t^{\alpha(s)} \tag{1.1}$$

at large t, where $\alpha(s)$ represents the Regge trajectory and $C(s)$ is made up of factors that appear in (3.15). Then if we may continue this function to a range of the parameters s and t which represents scattering in the crossed channel, we would find for the cross section in the high energy (t) limit

$$\frac{d\sigma}{ds} = \frac{|C(s)|^2}{16\pi} t^{2\alpha(s)-2} . \tag{1.2}$$

By use of the optical theorem we would also find for the total cross section in the high energy limit

$$\sigma_{tot} = \mathrm{Im}\, C(0)t^{\alpha(0)-1} \tag{1.3}$$

where again we are in the channel where t represents the energy, and $s = 0$ gives the forward scattering direction. In order for the trajectory to match the inference from the data that the total cross sections for strong interactions become constant at sufficiently high energy[12], it must have

$$\alpha(0) = 1 \; . \tag{1.4}$$

Expanding the trajectory about $s = 0$, we see from (1.2) that

$$\frac{d\sigma}{ds} = \frac{|C(s)|^2}{16\pi} t^{2s\alpha'(0)} \; . \tag{1.5}$$

Since s becomes more negative with increasing scattering angle it is clear that we must have

$$a'(0) > 0 \tag{1.6}$$

in order to match the observed peak in the forward direction as t becomes larger and larger. This choice of sign insures also that the width of the peak for large energies becomes inversely proportional to the logarithm of the energy. These properties and their relation to the experiments are the motivation for a theoretical interest in the Regge trajectory. We expect also that the vacuum trajectory does not correspond to a physical particle, and hence cannot give a real pole when it crosses integer values. Otherwise it shares with ordinary Regge poles the other properties which emerge from a more detailed examination of the inner workings of the theory; some of these will be mentioned later.

We follow the general method of Reference (5) in evaluating the trajectory. We calculate the scattering amplitude in two ways, once by the Feynman rules, and again by inserting our conjectures for the vacuum singularity of the partial wave amplitudes into the partial wave expansion. A comparison of these amplitudes permits the identification of the parameters introduced by the conjectures, unless the forms of these amplitudes are so dissimilar that a simple identification scheme does not apply. The problem is then to revise the scheme so that an interpretation is possible.

Some details of the field theory used in the main body of the text are included in Appendix A along with the notational conventions. Chapter II describes the preparation of the states affected by the vacuum trajectory and derives the partial wave expansion for the relevant processes. Chapter III reviews the Reggeization procedure for particles with spin, and makes the predictions of the results expected in lowest order perturbation theory. The results of the calculations of Reference (5) and their agreement with the predictions are summarized. In Chapter IV, we calculate the fourth order $\gamma\gamma$ scattering amplitude, discuss its properties and derive the prescription for evaluating the trajectory. Trajectory parameters using alternate prescriptions are discussed. Chapter V gives a summary of the results and suggests further areas of inquiry.

II. BACKGROUND FORMALISM

1. Preparation of the States

Our discussion of the scattering amplitudes in this problem follows closely that of Reference (5). We start with the helicity states of Jacob and Wick[13] and prepare symmetrized parity eigenstates of the $\gamma\gamma$ and $N\text{-}\overline{N}$ systems.

Applying the parity operator P to a two particle helicity state yields

$$P|JM;\lambda_a\lambda_b> = \eta_a\eta_b(-1)^{J-s_a-s_b}|JM;-\lambda_a-\lambda_b> \tag{2.1}$$

where the subscripts a and b refer to the particles of the same name, while λ, η and s represent the helicity, intrinsic parity and spin respectively. In our problem the γ's have $s = 1$, $\eta = -1$, and the $N\text{-}\overline{N}$ system has $s = \frac{1}{2}$, $\eta_a\eta_b = -1$ so that for both these systems $\eta_a\eta_b(-1)^{-s_a-s_b} = +1$.

Indicating by P_{12} the operator which interchanges particles 1 and 2, we have,

$$P_{12}|JM;\lambda_1\lambda_2> = (-1)^{J-s_1-s_2}|JM;\lambda_2\lambda_1> . \tag{2.2}$$

The charge conjugation operator C yields +1 when applied to a 2γ state and $-P_{12}$ when applied to an $N\text{-}\overline{N}$ state. This last minus sign is a consequence of the anticommutation of the fermion creation operators.

If the two vector mesons which make up the $\gamma\gamma$ state were distinguishable, say by a mass difference, then we would have nine states for the two particle system. This total number of states remains the same when we introduce the symmetrized states for the indistinguishable particles of the $\gamma\gamma$ system by setting

$$|JM;\lambda_1\lambda_2>_{\mathfrak{s}} \equiv \frac{1}{\sqrt{2}}(1+P_{12})\,|JM;\lambda_1\lambda_2> = \frac{1}{\sqrt{2}}\left\{|JM;\lambda_1\lambda_2) + (-1)^J\,|JM;\lambda_2\lambda_1)\right\}. \quad (2.3)$$

This prescription limits the states with $\lambda_1 = \lambda_2$ to even J values and also changes their normalization relative to the others. Since the raw $|JM;\lambda_1\lambda_2>$ states are normalized, a state such as $|JM;00>_{\mathfrak{s}}$ has length $\sqrt{2}$. We follow Reference (5) in permitting different state normalization for the sake of a uniformity in the formalism. Table I lists the symmetrized states, their allowed J values and normalization. As will become obvious later when we discuss the factorization of the Regge pole residues, the relative state normalization does not affect our answer.

Table I

$\gamma\gamma$ System. Symmetrized States

State	Allowed J Value	Norm	Parity Reflection
$\lvert JM;11>_{\mathfrak{s}}$	Even	$\sqrt{2}$	$+\ \lvert JM;-1-1>_{\mathfrak{s}}$
$\lvert JM;10>_{\mathfrak{s}}$	Even Odd	1	$\pm\ \lvert JM;0-1>_{\mathfrak{s}}$
$\lvert JM;1-1>_{\mathfrak{s}}$	Even Odd	1	$\begin{matrix}+\\+\end{matrix}\ \lvert JM;1-1>_{\mathfrak{s}}$
$\lvert JM;00>_{\mathfrak{s}}$	Even	$\sqrt{2}$	$+\ \lvert JM;00>_{\mathfrak{s}}$
$\lvert JM;0-1>_{\mathfrak{s}}$	Even Odd	1	$\pm\ \lvert JM;10>_{\mathfrak{s}}$
$\lvert JM;-1-1>_{\mathfrak{s}}$	Even	$\sqrt{2}$	$+\ \lvert JM;-1-1>_{\mathfrak{s}}$

Now we define the parity eigenstates by introducing

$$|JM;\lambda_1\lambda_2>_\pm = \frac{1}{\sqrt{2}}\left[1 \pm (-1)^J P\right] |JM;\lambda_1\lambda_2>_{s}$$

$$= \frac{1}{\sqrt{2}}\left[|JM;\lambda_1\lambda_2>_{s} \pm |JM;-\lambda_1-\lambda_2>_{s}\right] \qquad (2.4)$$

for the $\gamma\gamma$ system, and

$$|JM;\lambda_1\lambda_2>_\pm = \frac{1}{\sqrt{2}}\left[1 \pm (-1)^J P\right] |JM;\lambda_1\lambda_2>$$

$$= \frac{1}{\sqrt{2}}\left[|JM;\lambda_1\lambda_2> \pm |JM;-\lambda_1-\lambda_2>\right] \qquad (2.5)$$

for the N-$\overline{N}$ system. For example, in terms of raw states

$$|JM;10>_\pm = \frac{1}{2}\left\{|JM;10> + |JM;01> \pm |JM;0-1> \pm |JM;-10>\right\} \qquad (2.6)$$

while

$$|JM;00>_\pm = 2|JM;00> \qquad \text{for even } J \text{ only.} \qquad (2.7)$$

Again, this procedure introduces various factors of $\sqrt{2}$ in the state normalizations. The complete list of states and their properties is listed in Tables II and III.

Table II

$\gamma\gamma$ System. Symmetrized Parity Eigenstates

State	Allowed J Value	Norm	Parity
$\|JM;11\rangle_+$	Even	$\sqrt{2}$	+
$\|JM;11\rangle_-$	Even	$\sqrt{2}$	-
$\|JM;10\rangle_+$	Even Odd	1 1	+ -
$\|JM;10\rangle_-$	Even Odd	1 1	- +
$\|JM;1-1\rangle_+$	Even	$\sqrt{2}$	+
$\|JM;1-1\rangle_-$	Odd	$\sqrt{2}$	+
$\|JM;00\rangle_+$	Even	2	+

The N-$\overline{N}$ states prepared according to Eq. (2.5) are automatically eigenstates of C and with unit norm.

Table III

N-$\overline{N}$ System. Parity Eigenstates

State	Allowed J Value	Parity	C
$\|JM;\frac{1}{2}\frac{1}{2}\rangle_+$	Even Odd	+ -	+ -
$\|JM;\frac{1}{2}\frac{1}{2}\rangle_-$	Even Odd	- +	+ -
$\|JM;\frac{1}{2}-\frac{1}{2}\rangle_+$	Even Odd	+ -	+ -
$\|JM;\frac{1}{2}-\frac{1}{2}\rangle_-$	Even Odd	- +	- +

All the permissible processes for these systems must conserve T, (time reversal) C and P separately. In this problem we are interested only in those processes whose intermediate states have the quantum numbers of the vacuum and which are physical for even values of J. (In another case, where we might be exploring the possible Regge behavior of the vector meson itself, we would look for states which were physical at odd J and had $C = -1$.) Otherwise, we would get the appearance of a particle-like pole from the conjectured singularity near $J = 1$. With these restrictions, the only states which contribute are $|JM;11\rangle_+$, $|JM;10\rangle_+$, $|JM;1\text{-}1\rangle_+$, $|JM;00\rangle_+$, and $|JM;\tfrac{1}{2}\tfrac{1}{2}\rangle_+$, $|JM;\tfrac{1}{2}\text{-}\tfrac{1}{2}\rangle_+$ all for even J. Thus there are 10 $\gamma+\gamma \rightarrow \gamma+\gamma$ processes, 3 $N\text{-}\overline{N} \rightarrow N+\overline{N}$ processes and 8 $N+\overline{N} \rightarrow \gamma+\gamma$ processes to consider, including time reversal symmetry.

2. Partial Wave Expansions

Many of the symbols which were written by hand in a draft of Reference (5) were slurred over when the paper was set in type. In this section we repeat the arguments of that paper in great detail to serve as an erratum, and to expand the treatment of our particular case.

The first task is to see what modifications occur in the partial wave expansion as a result of the symmetrization of states. Then we incorporate these results into the formalism that treats parity conserving amplitudes. Finally, we give the expansions for the processes of the last section.

Our starting point is the partial wave expansion of Jacob and Wick

$$f_{\lambda_c\lambda_d;\lambda_a\lambda_b}(\theta) = \sqrt{\frac{k_f}{k_i}} \sum_J (2J+1)\langle JM;\lambda_c\lambda_d|F|JM;\lambda_a\lambda_b\rangle d^J_{\lambda\mu}(\theta) \qquad (2.8)$$

where $\lambda = \lambda_a - \lambda_b$, $\mu = \lambda_c - \lambda_d$, k_i and k_f are the magnitudes of the center of mass momenta for the initial and final states respectively, $d^J_{\lambda\mu}(\theta)$ are the usual Wigner matrices, and where $f_{\lambda_c\lambda_d;\lambda_a\lambda_b}$, the scattering amplitude for the process $a + b \rightarrow c + d$, gives the differential cross section

$$\frac{d\sigma}{d\Omega} = |f_{\lambda_c\lambda_d;\lambda_a\lambda_b}(\theta)|^2 . \qquad (2.9)$$

The matrix F is related to the usual S matrix by

$$S_{fi} = \delta_{fi} + 2i\sqrt{k_f k_i}\, F_{fi} \,. \tag{2.10}$$

Equation (2.3) gives us the effect of the state symmetrization on states of definite J. To determine the effect of symmetrization on the scattering amplitude we must back up to a point in the development before the decomposition into states of definite J has been carried out. The step which logically precedes Eq. (2.3),

$$|\vec{k};\lambda_1\lambda_2> = \frac{1}{\sqrt{2}}\left\{|\vec{k};\lambda_1\lambda_2> + (-1)^{2s-\lambda_1+\lambda_2} e^{i\pi J_y}|\vec{k};\lambda_2\lambda_1>\right\}, \tag{2.11}$$

gives the effect of P_{12} on a two γ state with center of mass momentum $\vec{k}$. Considering for a moment the process $N + \overline{N} \rightarrow \gamma + \gamma$ we express the matrix element of F between an initial state and a symmetrized final state in terms of a matrix element of $e^{i\pi J_y}F$ between unsymmetrized states. By inserting complete sets of $|JM;\lambda_1\lambda_2>$ states we arrive at a suitable decomposition into angular momentum pieces. Using the properties of $d^J_{\lambda\mu}(\theta)$ we derive the symmetrized scattering amplitude:

$$f^{S}_{m;\ell}(\theta) = \frac{1}{\sqrt{2}}\left\{f_{m;\ell}(\theta) + (-1)^{-\mu} f_{\tilde{m};\ell}(\theta-\pi)\right\} \quad \text{for } N+\overline{N}\rightarrow\gamma+\gamma \tag{2.12}$$

writing m for the ordered pair of helicities $\lambda_c\lambda_d$, $\tilde{m}$ for $\lambda_d\lambda_c$, and ℓ for $\lambda_a\lambda_b$.

The analytic properties of the functions $d^J_{\lambda\mu}(\theta)$ are greatly clarified by introducing the functions

$$e^J_{\lambda\mu}(z) \equiv d^J_{\lambda\mu}(\theta)\chi_{\lambda\mu}(\theta) \tag{2.13}$$

where $z = \cos\theta$ and where

$$\chi_{\lambda\mu}(\theta) \equiv (\sqrt{2}\sin\frac{\theta}{2})^{-|\lambda-\mu|}(\sqrt{2}\cos\frac{\theta}{2})^{-|\lambda+\mu|} . \tag{2.14}$$

Further, introduce new scattering amplitudes

$$\mathfrak{F}_{m;\ell} \equiv \chi_{\lambda\mu}(\theta)\, f_{m;\ell}(\theta) \tag{2.15}$$

so that the partial wave expansions of the $\mathfrak{F}$'s become, from Eq. (2.8)

$$\mathfrak{F}_{m;\ell}(z) = \sqrt{\frac{k_f}{k_i}}\sum_J (2J+1)F^J_{m;\ell}\, e^J_{\lambda\mu}(z) \tag{2.16}$$

$$\mathfrak{F}_{\tilde{m};\ell}(-z) = \sqrt{\frac{k_f}{k_i}}\sum_J (2J+1)\, F^J_{\tilde{m};\ell}\, e^J_{\lambda\mu}(-z) \tag{2.17}$$

where we have written

$$F^J_{\lambda_c\lambda_d;\lambda_a\lambda_b} \equiv \langle JM;\lambda_c\lambda_d|F|JM;\lambda_a\lambda_b\rangle . \tag{2.18}$$

Now since

$$\chi_{\lambda-\mu}(\theta-\pi) = (-1)^{\lambda+\mu}\chi_{\lambda\mu}(\theta) , \tag{2.19}$$

$$\mathcal{F}^{\mathcal{S}}_{m;\ell} \equiv \chi_{\lambda\mu}(\theta) f^{\mathcal{S}}_{m;\ell}(\theta) \qquad \text{for } N + \overline{N} \rightarrow \gamma + \gamma \tag{2.20}$$

$$= \frac{1}{\sqrt{2}} \left\{ \mathcal{F}_{m;\ell}(z) + (-1)^{\lambda} \mathcal{F}_{\tilde{m};\ell}(-z) \right\} \tag{2.21}$$

$$= \frac{1}{\sqrt{2}} \sqrt{\frac{k_f}{k_i}} \sum_J (2J+1) \left\{ \frac{e^J_{\lambda\mu}(z) + (-1)^{\lambda} e^J_{\lambda-\mu}(-z)}{\sqrt{2}} \cdot \frac{F^J_{m;\ell} + F^J_{\tilde{m};\ell}}{\sqrt{2}} \right.$$

$$\left. + \frac{e^J_{\lambda\mu}(z) - (-1)^{\lambda} e^J_{\lambda\mu}(-z)}{\sqrt{2}} \cdot \frac{F^J_{m;\ell} - F^J_{\tilde{m};\ell}}{\sqrt{2}} \right\} \tag{2.22}$$

Defining the physical symmetrized matrix element for this process

$$F^{J\mathcal{S}}_{m;\ell} \equiv {}_{\mathcal{S}}\langle JM; \lambda_c \lambda_d | F | JM; \lambda_a \lambda_b \rangle , \tag{2.23}$$

Eq. (2.3) leads us to

$$F^{J\mathcal{S}}_{m;\ell} = \frac{1}{\sqrt{2}} \left[F^J_{m;\ell} + (-1)^J F^J_{\tilde{m};\ell} \right] \qquad \text{for } N + \overline{N} \rightarrow \gamma + \gamma. \tag{2.24}$$

Recall from Appendix A of Reference (3)

$$e^J_{\lambda-\mu}(-z) = (-1)^{J+\lambda} e^J_{\lambda\mu}(z) . \tag{2.25}$$

So the combination

$$e^J_{\lambda\mu}(z) \pm (-1)^\lambda e^J_{\lambda-\mu}(-z) = e^J_{\lambda\mu}(z)\left[1 \pm (-1)^{J+\lambda+\lambda}\right]$$

$$= e^J_{\lambda\mu}(z)\left[1 \pm (-1)^J\right] \qquad (2.26)$$

since 2λ is even. (This same result is derived in Reference (5) with no restriction on λ or μ.) Thus

$$\mathcal{F}^{\mathcal{S}}_{m;\ell}(z) = \sqrt{\frac{k_f}{k_i}} \sum_J (2J+1) F^{J\mathcal{S}}_{m;\ell}\, e^J_{\lambda\mu}(z) , \qquad (2.27)$$

and only the symmetrized matrix elements appear.

The partial wave matrix elements in this paper will always be defined between the appropriate set of states. Thus the product of the irregularities in the state normalizations will give the corresponding irregularities in the partial wave matrix elements.

For the process $\gamma + \gamma \rightarrow \gamma + \gamma$ both the initial and final states must be symmetrized. The derivation of the partial wave expansion is the same as the previous case except for minor modifications. Equation (2.12) acquires a factor $\sqrt{2}$ on the RHS, Eq. (2.21) becomes

$$\mathcal{F}^{\mathcal{S}}_{m;\ell}(z) = \mathcal{F}_{m;\ell}(z) + (-1)^\lambda \mathcal{F}_{\tilde{m};\ell}(-z) \qquad \gamma + \gamma \rightarrow \gamma + \gamma \qquad (2.28)$$

and since we must take

$$F^{J\mathcal{S}}_{m;\ell} \equiv {}_{\mathcal{S}}\langle JM;\lambda_c\lambda_d | F | JM;\lambda_a\lambda_b\rangle_{\mathcal{S}} , \qquad (2.29)$$

$$F^{J\mathcal{S}}_{m;\ell} = \left[F^{J}_{m;\ell} + (-1)^J F^{J}_{\tilde{m};\ell}\right] \quad \text{for } \gamma+\gamma \to \gamma+\gamma \, . \qquad (2.30)$$

These two changes work in opposite directions so that the formula Eq. (2.27) remains unaltered; it is only necessary to remember that the definitions of $\mathfrak{F}^{\mathcal{S}}_{m;\ell}(z)$ and $F^{J\mathcal{S}}_{m;\ell}$ depend on the case in question.

The last step is the construction of the parity conserving amplitudes to match the parity eigenstates. From the choice of states Eq. (2.4) and the requirement

$$F^{J\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b} \equiv {}_{\pm}\langle JM;\lambda_c\lambda_d|F|JM;\lambda_a\lambda_b\rangle_{\pm} \qquad (2.31)$$

we get

$$F^{J\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b} = F^{J\mathcal{S}}_{\lambda_c\lambda_d;\lambda_a\lambda_b} \pm F^{J\mathcal{S}}_{-\lambda_c-\lambda_d;\lambda_a\lambda_b} \, . \qquad (2.32)$$

As in Reference (3), we introduce the parity conserving scattering amplitude

$$f^{\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(z) \equiv \mathfrak{F}^{\mathcal{S}}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(z) \pm (-1)^{\lambda+\lambda_m} \mathfrak{F}^{\mathcal{S}}_{-\lambda_c-\lambda_d;\lambda_a\lambda_b} \qquad (2.33)$$

where $\lambda_m \equiv \mathrm{Max}[|\lambda|, |\mu|]$. Equations (2.32) and (2.33) retain the same form for ordinary parity eigenstates as for these symmetrized parity eigenstates. By introducing the functions

$$e^{J\pm}_{\lambda\mu}(z) \equiv \frac{1}{2}\left\{e^{J}_{\lambda\mu}(z) \pm (-1)^{\lambda+\lambda_m} e^{J}_{\lambda-\mu}(z)\right\} \qquad (2.34)$$

we obtain finally, for all cases

$$f^{\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(z) = \sqrt{\frac{k_f}{k_i}} \sum_J (2J+1)\left\{e^{J+}_{\lambda\mu}(z)\, F^{J\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b} + e^{J-}_{\lambda\mu}(z)\, F^{J\mp}_{\lambda_c\lambda_d;\lambda_a\lambda_b}\right\}. \qquad (2.35)$$

For any state of definite P, C and helicities, the sum in (2.35) is automatically restricted to even or odd J by the definition of the state. Which one to choose in any given case may be determined from Tables II and III.

From the inversion formula of the original Jacob and Wick partial wave expansion,

$$F^{J}_{\lambda_c\lambda_d;\lambda_a\lambda_b} = \frac{1}{2}\sqrt{\frac{k_i}{k_f}} \int_{-1}^{1} dz\, f_{\lambda_c\lambda_d;\lambda_a\lambda_b}(\theta)\, d^{J}_{\lambda\mu}(\theta) \qquad (2.36)$$

we find

$$F^{J\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b} = \frac{1}{2}\sqrt{\frac{k_i}{k_f}} \int_{-1}^{1} dz \left\{f^{\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(z)\, c^{J+}_{\lambda\mu}(z) + f^{\mp}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(z)\, c^{J-}_{\lambda\mu}(z)\right\} \qquad (2.37)$$

where

$$c^{J\pm}_{\lambda\mu} \equiv \frac{1}{2}\left\{\frac{d^{J}_{\lambda\mu}(\theta)}{\chi_{\lambda\mu}(\theta)} \pm (-1)^{\lambda+\lambda_m} \frac{d^{J}_{\lambda-\mu}(\theta)}{\chi_{\lambda-\mu}(\theta)}\right\}. \qquad (2.38)$$

General information about the $e^{J\pm}_{\lambda\mu}(z)$ and $c^{J\pm}_{\lambda\mu}(z)$ is contained in Reference (3), Appendix A. The few functions we use in this paper are listed in our Appendix B.

With the reader's indulgence, we point out this last aspect of normalization and factors of $\sqrt{2}$. The symmetrized scattering amplitude defined by (2.3) and (2.12) is equal to the physical scattering amplitude (the one whose square gives the differential cross section for the process) only when the states which define it are themselves normalized. This is not the case if one of the symmetrized states is, for example, the state $|JM;00>_g$. On the other hand, the amplitude calculated from the Feynman rules always represents the physical scattering amplitude so that we need the appropriate factors from Table I to relate them.

The parity conserving amplitudes $f^{\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}$ do not represent physical entities; they are constructs of physical quantities which bring the partial wave expansions into a convenient form. The additional factors which this construction produces in cases where the original states were already parity eigenstates must be included in the state normalizations. Thus the factors to use in unitarity relations are the ones given in Table II. These factors need not be considered in any other portion of the paper since they may be absorbed into the factorized Regge pole residues, as will be seen in the next chapter.

III. THE BASIC PROGRAM

1. Reggeization

In the past few years, several clear and unhurried descriptions of the Regge phenomenon and its application to relativistic scattering have been published, in addition to the original pioneering articles. Among these we cite the particularly useful works of Frautschi[11] and Oehme[14], which have good bibliographies. We therefore do not feel it necessary to give more than an outline of the Reggeization procedure, except to emphasize points with special application to our case.

The basic idea is to treat the angular momentum variable J, which appears as a discrete variable in the partial wave expansions of the scattering amplitude, as a continuous variable in the full complex plane. Then, just as singularities in the energy variable have physical significance, so do the singularities in the angular momentum variable; they may represent stable particles or resonances. Furthermore, the asymptotic form of the scattering amplitude for fixed s as $z \to \infty$ is determined by the singularity in the J-plane with the largest value of Re J. Because of the crossing relations in relativistic scattering this asymptotic form is related to the scattering amplitude for high energy and fixed momentum transfer in the crossed channel.

The "Regge pole hypothesis" is the conjecture that the J singularities of the scattering amplitude are very simple, that is, they are simple poles in the region $\mathrm{Re}\, J \geq -\frac{1}{2}$, $\mathrm{Im}\, J \geq 0$, which depend on s. Though other types of singularities have been discovered, the hope remains that some of the physical consequences of the hypothesis of simple singularities will be unaltered.

The largest share of the trick of Reggeizing is to find a unique function $F(J)$ of the continuous variable J, which will be equal to the partial wave function F^J at the positive integers. We must rule out the possibility of adding to this function some multiple of say, $\sin \pi J$ which would leave it unchanged at the integers; this might be done by specifying some bound on the function at large $\operatorname{Im} J$. All the mathematical requirements are contained in the hypotheses of Carlson's Theorem[(15)]. This theorem provides that $F(J)$ is a unique interpolation if it is suitably bound. The ordinary partial wave projection formula (2.36) is the natural expression to use for such an interpolation, but the functions $d^J_{\lambda\mu}(\theta)$, being composed of sines, cosines and Legendre polynomials, are not sufficiently bounded at large $\operatorname{Im} J$ to satisfy the hypotheses of Carlson's theorem. We must look elsewhere for a workable formula.

Considering for a moment the scattering of distinguishable spin zero particles of mass λ, the center of mass momentum k satisfies

$$z = \cos \theta = 1 + \frac{t}{2k^2} \tag{3.1}$$

$$4k^2 = s - 4\lambda^2 . \tag{3.2}$$

The scattering amplitude for fixed s obeys the dispersion relation

$$8\pi\sqrt{s}\, f(s,t) = \sum_{n=1}^{N-1} C_n(s)t^n + \frac{t^N}{\pi}\int_{t_o}^{\infty} dt' \frac{A_t(s,t')}{t'^N(t'-t)} + \frac{u^N}{\pi}\int_{u_o}^{\infty} du' \frac{A_u(s,u')}{u'^N(u'-u)}, \tag{3.3}$$

where A_t and A_u are the absorptive parts of $f(s,t)$ in the t and u channels respectively, and N is sufficiently large to insure the convergence of the integrals and the possible divergence of f as $t \to \infty$.

Since $d^J_{oo}(\theta) = P_J(\cos\theta)$, the partial wave amplitudes from (2.36) become, for $J > N$

$$F^J(s) = \frac{1}{8\pi^2\sqrt{s}}\int_{z_o}^{\infty} dz\, Q_J(z)\left\{A_t(s,2k^2(z-1)) + (-1)^J A_u(s,2k^2(z-1))\right\} \tag{3.4}$$

where the $Q_J(z)$ are the Legendre functions of the second kind. Strangely enough, although the $Q_J(z)$ cause no trouble, $F^J(s)$ violates the bound specified in Carlson's theorem because of the factor $(-1)^J$ in the integrand so that no interpolation formula valid for all J can be found except in special cases. However, by separating the physical values we must match into two sets, one for even J and the other for odd J, we can find two functions which interpolate uniquely. We call these functions F^e and F^o.

$$F^e(J) \equiv \frac{1}{8\pi^2\sqrt{s}}\int_{z_o}^{\infty} dz\, Q_J(z)\left\{A_t(s,2k^2(z-1)) + A_u(s,2k^2(z-1))\right\} \tag{3.5}$$

$$F^{o}(J) \equiv \frac{1}{8\pi^2\sqrt{s}} \int_{z_o}^{\infty} dz\, Q_J(z) \left\{ A_t(s, 2k^2(z-1)) - A_u(s, 2k^2(z-1)) \right\} \qquad (3.6)$$

$$F^{e}(J) = F^{J}, \; J \text{ even integer} ; \quad F^{o}(J) = F^{J}, \; J \text{ odd integer}. \qquad (3.7)$$

These are the standard Froissart-Gribov continuations; $F^{\substack{e\\o}}(J)$ is analytic for $\mathrm{Re}\, J > N$. The separation of the scattering amplitude into parity conserving processes in Chapter II is a way of giving physical significance to $F^{e}(J)$ and $F^{o}(J)$ separately, since for each process the sum in (2.35) runs over either even or odd integers.

We now extend the definitions (3.5), (3.6) to the left of $\mathrm{Re}\, J = N$ by analytic continuation. This procedure would be rigorous if we had more knowledge of the analytic properties of A_t and A_u, but for this project we must assume it to be merely heuristic. Further, we assume, unless contrary evidence appears subsequently, that A_t and A_u are such as to cancel the poles which would appear in $F(J)$ because of the poles in Q_J at the negative integers.

To determine the effect on $f(s,t)$ of poles in $F^{e}(J)$, introduce the functions

$$\mathcal{P}_J(z) \equiv \frac{\Gamma(J+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(J+1)} (2z)^J F\left(-\frac{J}{2}, \frac{1-J}{2}, \frac{1-2J}{2} ; \frac{1}{z^2}\right) \qquad (3.8)$$

which have the property[16]

$$\mathcal{P}_J(z) = P_J(z) \qquad \text{at } J = 0, 1, 2, \ldots \qquad (3.9)$$

$$\mathcal{P}_J(z) = 0 \qquad \text{at } J = -1, -2, -3, \ldots \qquad (3.10)$$

The interesting part of the partial wave sum may be written,

$$f(s,t) = \sum_{J=0,2,\dots} (2J+1)P_J(z)F^J \tag{3.11}$$

$$= \sum_{J=0,1,2,\dots} (2J+1)\tfrac{1}{2}\left[P_J(z)F^e(J) + P_J(-z)F^e(J)\right] \tag{3.12}$$

$$= \sum_{J=-\infty}^{\infty} (2J+1)\tfrac{1}{2}\left[\mathcal{P}_J(z)F^e(J) + \mathcal{P}_J(-z)F^e(J)\right] . \tag{3.13}$$

Now we use the Sommerfeld-Watson technique to convert this sum to a contour integral

$$f(s,t) = \frac{1}{4i}\int_C \frac{dJ(2J+1)}{\sin \pi J}\mathcal{P}_J(-z)F^e(J) + \frac{1}{4i}\int_C dJ\,\frac{(2J+1)}{\sin \pi J}\mathcal{P}_J(z)F^e(J) \tag{3.14}$$

where the contour C encircles the real axis as in the limit $R \to \infty$, $\epsilon \to 0$ of Fig. 1. Besides the poles from $\sin \pi J$, the poles at the half-integers from $\mathcal{P}_J(z)$ contribute to each integral, but these contributions cancel in pairs J, $J-1$ of half-integers, except for the one at $J = \frac{1}{2}$ which is cancelled by the factor $2J+1$ in the numerator.

The function $\mathcal{P}_J(z)$ as a function of z has a cut running from $+1$ to $-\infty$, i.e., a left-hand cut, while $\mathcal{P}_J(-z)$ has only a right-hand cut. Thus we have duplicated the form of a dispersion relation for $f(s,t)$ in z (or t). To the extent that the weight functions in such a dispersion relation are uniquely determined, we shall have information about $F^e(J)$.

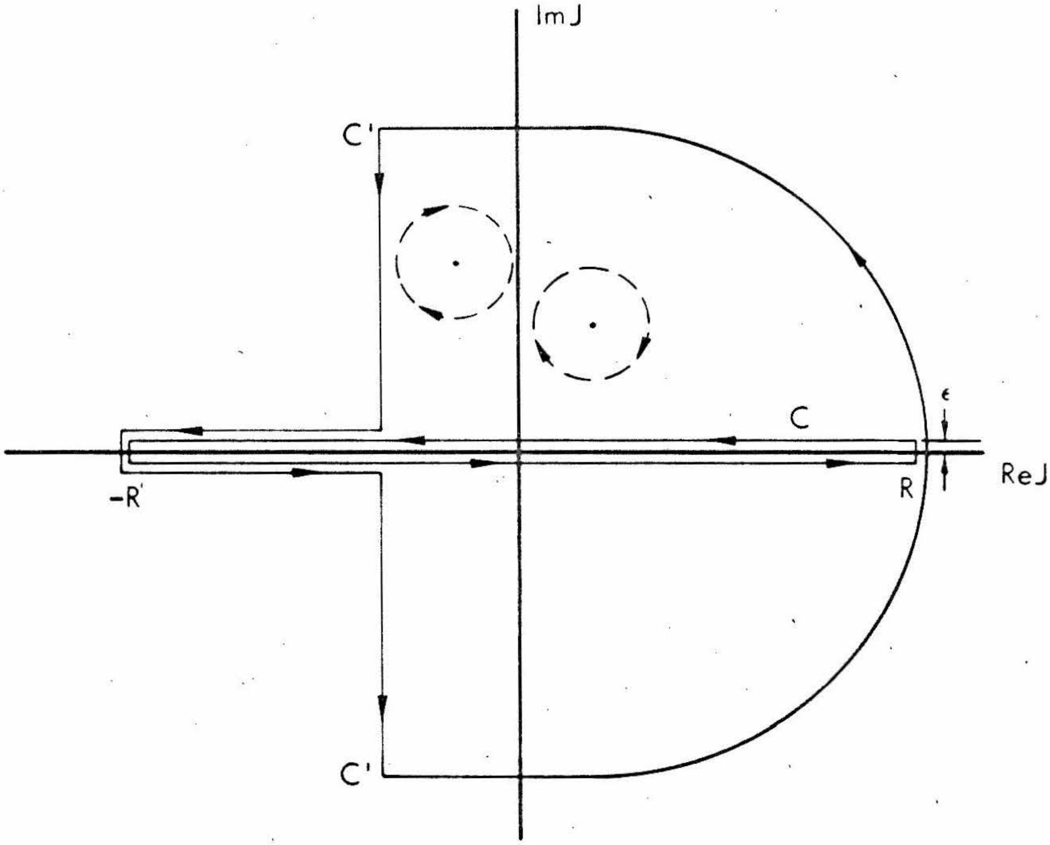

Fig. 1 — Integration contours

We now open up the contour to the form C' as in Fig. 1, picking up the contribution of the poles of $F^e(J)$.

$$f(s,t) = \frac{1}{4i}\int_{C'} dJ\,\frac{(2J+1)}{\sin \pi J}\left[P_J(-z) + P_J(z)\right] F^e(J)$$

$$- \sum_i \frac{(2\alpha_i(s)+1)\pi\beta_i(s)}{2 \sin \pi\alpha_i(s)}\left[P_{\alpha_i}(-z) + P_{\alpha_i}(z)\right] \qquad (3.15)$$

where the i^{th} pole of $F^e(J)$ occurs at $\alpha_i(s)$ with a residue of $\beta_i(s)$. Since $P_\alpha(z)$ goes asymptotically for large z as z^α, in this limit the contribution of the Regge pole with the largest value of Re J dominates all the others and the integral over the contour C'. We conjecture that this leading pole term is in the neighborhood of $J = 1$. The first terms in the expansion of $f(s,t)$ about the point $J = 1$ are evaluated at $J = 1$ and will cancel if we are not careful to keep the pieces separate. Then we would have to carry the expansion to higher terms and the perturbation theory expansion to higher orders in order to get non-vanishing terms to compare.

The situation for particles with spin is very similar except for a few complications. Equations (3.5) and (3.6) become

$$F^{\binom{e}{o}\pm}(J) \equiv \frac{1}{8\pi^2\sqrt{s}} \int_{z_o}^{\infty} dz\, (A_t^{\pm} C^{J+} + A_t^{\mp} C^{J-})$$

$$+ (\pm)(-1)^{\lambda_m} \frac{1}{8\pi^2\sqrt{s}} \int_{z_o}^{\infty} dz\, (A_u^{\pm} C^{J+} - A_u^{\mp} C^{J-}) , \qquad (3.16)$$

where the $A_t^{\pm}$, $A_u^{\pm}$ are the weight functions of $f^{\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(s,t)$ in a dispersion relation for fixed s and where the $C^{J\pm}_{\lambda\mu}(z)$ are obtained from the $c^{J\pm}_{\lambda\mu}(z)$ by replacing the $P_J(z)$ in their definitions by $Q_J(z)$. Again assuming $F^{e\pm}(J)$ is finite for negative J and considering only the contributions from $F^{e\pm}$ we have

$$f^{\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(z) = \sqrt{\frac{k_f}{k_i}} \sum_{\text{Even } J \geq \lambda_m} (2J+1) \Big\{ F^{J\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b} e^{J+}_{\lambda\mu}(z)$$

$$+ F^{J\mp}_{\lambda_c\lambda_d;\lambda_a\lambda_b} e^{J-}_{\lambda\mu}(z) \Big\} \qquad (3.17)$$

$$= \sqrt{\frac{k_f}{k_i}} \sum_{J=0,1,2,\ldots} (2J+1) \Big\{ F^{e\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(J)^{\frac{1}{2}} \left[E^{J+}_{\lambda\mu}(z) + (-1)^{\lambda_m} E^{J+}_{\lambda\mu}(-z) \right]$$

$$+ F^{e\mp}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(J)^{\frac{1}{2}} \left[E^{J-}_{\lambda\mu}(z) - (-1)^{\lambda_m} E^{J-}_{\lambda\mu}(-z) \right] \Big\} ,$$

$$(3.18)$$

where the $E^{J\pm}_{\lambda\mu}(z)$ are obtained from the $e^{J\pm}_{\lambda\mu}(z)$ by replacing the $P_J(z)$ in their definitions by $\mathcal{P}_J(z)$. After this, the extension of the sum to include the negative integers is made more difficult because of the existence of certain "nonsense"(17) values of J. These are the values of J for a particular set of helicities for which $J < \lambda_m$. Physically this is equivalent to a vector of length J which is shorter than its projection. It is not surprising that the formalism treats these nonsense values different from the others. The $E^{J\mp}_{\lambda\mu}$ are non-vanishing for $J = -1, -2, \ldots, -\lambda_m$ but provide terms which exactly cancel the contributions from $E^{J\pm}_{\lambda\mu}$ for $J = 0, 1, \ldots, \lambda_m - 1$, as shown in Appendix B of Reference (3). Thus the sum in (3.18) can be extended to include the negative integers, and the Sommerfeld-Watson transformation together with a shift of contours gives us the contribution from the Regge poles.

2. The Vacuum Trajectory

Now we conjecture that the leading Regge pole for $F^e(J)$ occurs at a value

$$J = \alpha(s) = 1 + \Delta(s) \tag{3.19}$$

where Δ vanishes as the coupling constant $g \to 0$. We calculate $\Delta(s)$ by comparing the perturbation theory calculation for the relevant processes with the deductions from the hypothesis of a Regge pole at $\alpha(s)$. There are two expansions involved here and their relation is sometimes delicate. Since we have the leading Regge pole, we want the high z limit of perturbation theory. But since we are comparing the Regge predictions with a particular order of perturbation theory, we want only that term which involves the correct power of the coupling constant, even though higher powers of the coupling constant may be associated with larger z dependence. In any given order of Δ, the contributions from the $E^{J+}(z)$ dominate those of the $E^{J-}(z)$, so that we neglect the latter in what follows. Thus the Regge pole contributions are:

$$- \frac{(2\alpha+1)\pi\beta}{2 \sin \pi\alpha} \sqrt{\frac{k_f}{k_i}} \left[(-1)^{\lambda_m} E^{\alpha+}_{\lambda\mu}(-z) + E^{\alpha+}_{\lambda\mu}(z) \right]$$

$$\xrightarrow[\Delta \to 0]{} \sqrt{\frac{k_f}{k_i}} \; \frac{3\beta}{2\Delta} \left[(-1)^{\lambda_m} E^{\alpha+}_{\lambda\mu}(-z) + E^{\alpha+}_{\lambda\mu}(z) \right] . \tag{3.20}$$

Further, we know that in the s-plane, the residues of single-particle poles and of resonance poles are factorizable as a simple consequence of the unitarity condition, i.e., they may be written as the product of two factors--one from the initial state and the other from the final state. Since the $F^{\pm}(J)$ obey similar unitarity conditions in the complex J-plane[14] we expect that the residues of a Regge trajectory factor in the same way[18],[19]. Denoting these factored residues by $\xi_{\lambda_1\lambda_2}$ where the subscripts give the helicities of the corresponding state, we see that extra factors in the F's due to the state normalizations may be absorbed into the corresponding $\xi_{\lambda_1\lambda_2}$. However, we must be careful when dealing with unitarity relations which are non-homogeneous.

We see from the inversion formula (2.37) and the form of the $c^{J\pm}_{\lambda\mu}(z)$, Appendix B that some of the partial wave amplitudes have square root factors multiplying them which should really have been taken into account in the derivation of (3.20) since otherwise they would introduce troublesome branch points. For our purposes, which center about $J = 1$, we may consider these analyticity problems to be taken care of for all such factors except $J - 1$, which we keep track of separately. These special factors appear only in channels with a nonsense state, i.e., $|JM;1\text{-}1\rangle_{\pm}$. Physically, the explanation is that in this channel, as J decreases, the formalism must have some way of eliminating the nonsense J contribution. Keeping track of this factor is important in our assigning the correct order of perturbation theory to each contribution.

Combining all our hypotheses about $F^{e}(J)$ we have the following:

$$
\begin{aligned}
F^{e+}_{ij} &\sim \frac{\xi_i \xi_j}{J-\alpha} \\
F^{e+}_{si} &\sim \frac{\xi_s \xi_i}{J-\alpha} \\
F^{e+}_{ni} &\sim \sqrt{J-1}\,\frac{\xi_n \xi_i}{J-\alpha} \\
F^{e+}_{nn} &\sim J-1\,\frac{\xi_n^2}{J-\alpha} \\
F^{e+}_{ns} &\sim \sqrt{J-1}\,\frac{\xi_n \xi_s}{J-\alpha} \\
F^{e+}_{rs} &\sim \frac{\xi_r \xi_s}{J-\alpha}
\end{aligned}
\tag{3.21}
$$

where the subscripts i, j each represent any of the states $|\frac{1}{2}\,\frac{1}{2}>_+$, $|\frac{1}{2}\,-\frac{1}{2}>_+$; r, s each represent any of the states $|1\,0>_+$, $|1\,1>_+$, $|0\,0>_+$; and n represents the nonsense state $|1\,-1>_+$, and where $\sim$ indicates that we are looking at the behavior near $J = \alpha$; we have omitted a piece which is regular at that point. Of all the possible amplitudes for $\gamma + \gamma \to \gamma + \gamma$, only one of them will give us information about $\Delta(s)$ in a fourth order perturbation calculation. We substitute these conjectures into the expression for the pole contributions to the amplitude, (3.20), choosing the appropriate $E^{J+}_{\lambda\mu}$ from Appendix B for the partial wave expansions (2.35). In evaluating the small Δ, large z limit of the $E^{\alpha+}_{\lambda\mu}(z)$, the following formulas are useful:

$$
\begin{aligned}
&\wp_\alpha(z) \to z\left\{1 + \Delta\left[\log z + (1 - \log 2)\right] + \mathcal{O}(\Delta^2)\right\} \\
&\wp'_\alpha(z) \to 1 + \Delta\left[\log z + (2 - \log 2)\right] + \mathcal{O}(\Delta^2) \\
&\wp''_\alpha(z) \to \frac{\Delta}{z}\left\{1 + \Delta\left[\log z + (2 - \log 2)\right] + \mathcal{O}(\Delta^2)\right\} \\
&\wp'''_\alpha(z) \to -\frac{\Delta}{z^2}\left\{1 + \Delta\left[\log z + (1 - \log 2)\right] + \mathcal{O}(\Delta^2)\right\} \\
&\wp^{iv}_\alpha(z) \to \frac{2\Delta}{z^3}\left\{1 + \Delta\left[\log z + (\tfrac{1}{2} - \log 2)\right] + \mathcal{O}(\Delta^2)\right\} ,
\end{aligned}
\tag{3.22}
$$

with similar formulas for $z \to -z$.

The final form of the predictions of the Regge pole hypothesis are:

for $N + \overline{N} \to N + \overline{N}$

$$
\begin{aligned}
&f^+_{\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}} \to \frac{3}{2}\,\xi^2_{\frac{1}{2}\frac{1}{2}}\left[- z \log(-z) + z \log z\right] \\
&f^+_{\frac{1}{2}-\frac{1}{2};\frac{1}{2}\frac{1}{2}} \to \frac{3}{2\sqrt{2}}\,\xi_{\frac{1}{2}-\frac{1}{2};\frac{1}{2}\frac{1}{2}}\left[- \log(-z) + \log z\right] \\
&f^+_{\frac{1}{2}-\frac{1}{2};\frac{1}{2}-\frac{1}{2}} \to \frac{3}{4}\,\xi^2_{\frac{1}{2}-\frac{1}{2}}\left[- \log(-z) + \log z\right] ;
\end{aligned}
\tag{3.23}
$$

for $\overline{N} + \overline{N} \to \gamma + \gamma$, letting p be the center of mass momentum of the $N\overline{N}$ system, k, the center of mass momentum of the $\gamma\gamma$ system,

$$
\begin{aligned}
f^{+}_{11;\frac{1}{2}\frac{1}{2}} &\to \frac{3}{2\Delta}\sqrt{\frac{k}{p}}\,\xi_{11}\xi_{\frac{1}{2}\frac{1}{2}}[-z+z] \\
f^{+}_{11;\frac{1}{2}-\frac{1}{2}} &\to \frac{3}{2\sqrt{2}\,\Delta}\sqrt{\frac{k}{p}}\,\xi_{11}\xi_{\frac{1}{2}-\frac{1}{2}}[1-1] \\
f^{+}_{10;\frac{1}{2}\frac{1}{2}} &\to \frac{3}{2\sqrt{2}\,\Delta}\sqrt{\frac{k}{p}}\,\xi_{10}\xi_{\frac{1}{2}\frac{1}{2}}[-1+1] \\
f^{+}_{10;\frac{1}{2}-\frac{1}{2}} &\to \frac{3}{4\Delta}\sqrt{\frac{k}{p}}\,\xi_{10}\xi_{\frac{1}{2}-\frac{1}{2}}[-1+1] \\
f^{+}_{1-1;\frac{1}{2}\frac{1}{2}} &\to \frac{3}{2\sqrt{6}}\sqrt{\frac{k}{p}}\,\xi_{1-1}\xi_{\frac{1}{2}\frac{1}{2}}\left[-\frac{1}{z}+\frac{1}{z}\right] \\
f^{+}_{1-1;\frac{1}{2}-\frac{1}{2}} &\to \frac{\sqrt{3}}{4}\sqrt{\frac{k}{p}}\,\xi_{1-1}\xi_{\frac{1}{2}-\frac{1}{2}}\left[-\frac{1}{z}+\frac{1}{z}\right] \\
f^{+}_{00;\frac{1}{2}\frac{1}{2}} &\to \frac{3}{2\Delta}\sqrt{\frac{k}{p}}\,\xi_{00}\xi_{\frac{1}{2}\frac{1}{2}}[-z+z] \\
f^{+}_{00;\frac{1}{2}-\frac{1}{2}} &\to \frac{3}{2\sqrt{2}\,\Delta}\sqrt{\frac{k}{p}}\,\xi_{00}\xi_{\frac{1}{2}-\frac{1}{2}}[1-1]\ .
\end{aligned}
\tag{3.24}
$$

For $\gamma + \gamma \to \gamma + \gamma$,

$$
f^{+}_{1-1;1-1} \to \frac{\Delta}{4}\xi^{2}_{1-1}\left[-\frac{1}{z}+\frac{1}{z}\right]
$$

$$
f^{+}_{1-1;00} \to \frac{3}{2\sqrt{6}}\,\xi_{1-1}\xi_{00}\left[-\frac{1}{z}+\frac{1}{z}\right]
$$

$$
\begin{aligned}
f^{+}_{00;00} &\rightarrow \frac{3}{2\Delta}\,\xi^{2}_{00}\,[\,-z + z] \\
f^{+}_{1-1;11} &\rightarrow \frac{3}{2\sqrt{6}}\,\xi_{1-1}\xi_{11}\,[\,-\tfrac{1}{z} + \tfrac{1}{z}\,] \\
f^{+}_{1-1;10} &\rightarrow \frac{\sqrt{3}}{4}\,\xi_{1-1}\xi_{10}[\,-\tfrac{1}{z} + \tfrac{1}{z}\,] \\
f^{+}_{10;10} &\rightarrow \frac{3}{2\Delta}\,\xi^{2}_{10}[\,-1 + 1] \\
f^{+}_{10;11} &\rightarrow \frac{3}{2\sqrt{2}\,\Delta}\,\xi_{10}\xi_{11}[\,-1 + 1] \\
f^{+}_{10;00} &\rightarrow \frac{3}{2\sqrt{2}\,\Delta}\,\xi_{10}\xi_{00}[\,-1 + 1] \\
f^{+}_{11;11} &\rightarrow \frac{3}{2\Delta}\,\xi^{2}_{11}\,[\,-z + z] \\
f^{+}_{11;00} &\rightarrow \frac{3}{2\Delta}\,\xi_{11}\xi_{00}[\,-z + z]\ .
\end{aligned}
\tag{3.25}
$$

In each case we have written first the term which was derived from the function with the right-hand cut in z. In the case of $N + \overline{N} \rightarrow N + \overline{N}$, the contribution of the lowest order term was dropped because it gave a contribution lower than first order in the coupling constant. In the above predictions, the terms which behave as 1 or z as $z \rightarrow \infty$ in this lowest order correspond merely to the polynomial subtraction constants in a dispersion relation in z of the form (3.3). The decomposition of the amplitude into two such terms is arbitrary; only their sum, zero, has significance. The terms which behave as z^{-1} or log z, however, must come from the integrals over the cuts in a dispersion relation. These have analytic structure, and are the asymptotic forms of terms which can be uniquely determined if their sum is given. The success of our program depends upon the identification of these terms. Of all the above processes involving a 2γ state only those which contain the nonsense state are uniquely determined in this low order. Out of this group, only the "nonsense $\rightarrow$ nonsense" transition involves $\Delta(s)$.

The $N + \overline{N} \rightarrow N + \overline{N}$ predictions are compared with the fourth order perturbation calculations (the second order process, single γ exchange, is too small at high z), the $N + \overline{N} \rightarrow \gamma + \gamma$ with the second order calculations, and the $\gamma + \gamma \rightarrow \gamma + \gamma$ with the fourth order calculations. Thus we can read off the size of the various quantitites:

$$\begin{aligned} \xi_i &\sim \mathcal{O}(g^2) \\ \xi_s &\sim \mathcal{O}(g^4) \\ \xi_n &\sim \mathcal{O}(g^0) \\ \Delta &\sim \mathcal{O}(g^4) \end{aligned} \qquad (3.26)$$

where the subscript i represents the two nucleon states, s the three meson states which are sensible at $J = 1$, and n the $1 - 1$ nonsense state.

The equations (3.23) - (3.25) are to be regarded not only as a scheme for calculating the residues, but also as a prediction of the asymptotic behavior of the amplitude in the appropriate order of perturbation theory.

The trajectory can be evaluated from the identities

$$\Delta(s) = \frac{(\Delta\xi_n^{\,2})\,(\xi_i\xi_j)}{(\xi_n\xi_i)(\xi_n\xi_j)}\ , \tag{3.27}$$

where the groupings on the right-hand side represent the combinations determined by comparison with perturbation theory.

3. Consistency and Unitarity

We quote from Reference (5) the results from second and fourth order calculations. For $N + \overline{N} \rightarrow \gamma + \gamma$,

$$f^{+}_{1-1;\frac{1}{2}\frac{1}{2}} \rightarrow g^2 \frac{m}{4\pi k\sqrt{s}} \sqrt{\frac{k}{p}} \left[-\frac{1}{z} + \frac{1}{z} \right] \tag{3.28}$$

$$f^{+}_{1-1;\frac{1}{2}-\frac{1}{2}} \rightarrow g^2 \frac{1}{8\pi k} \sqrt{\frac{k}{p}} \left[-\frac{1}{z} + \frac{1}{z} \right] \tag{3.29}$$

where the first (second) term comes from the diagram which yields the right (left) hand cut. Using (3.24) we get

$$\xi_{1-1}\xi_{\frac{1}{2}\frac{1}{2}} = g^2 \frac{1}{\sqrt{6}\,\pi} \cdot \frac{m}{k\sqrt{s}} \tag{3.30}$$

$$\xi_{1-1}\xi_{\frac{1}{2}-\frac{1}{2}} = g^2 \frac{1}{2\sqrt{3}\,\pi} \frac{1}{k} \tag{3.31}$$

The other six processes agree in their asymptotic form with the predictions of (3.24), but for them the identification of the corresponding coefficients would be purely formal. It turns out that these other processes are also consistent with

$$\frac{\xi_{\frac{1}{2}\frac{1}{2}}}{\xi_{\frac{1}{2}-\frac{1}{2}}} = \frac{\sqrt{2}\, m}{\sqrt{s}} \tag{3.32}$$

although this may be only a coincidence. From the right-hand cut of the fourth order $N + \overline{N} \rightarrow N + \overline{N}$ calculation,

$$f^{+R}_{\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}} \rightarrow - g^4 \frac{2m^2}{\pi\sqrt{s}} I_o(s)(z \log(-z)) \tag{3.33}$$

$$f^{+R}_{\frac{1}{2}-\frac{1}{2};\frac{1}{2}\frac{1}{2}} \rightarrow - g^4 \frac{m}{\pi} I_o(s) \log(-z) \tag{3.34}$$

$$f^{+R}_{\frac{1}{2}-\frac{1}{2};\frac{1}{2}-\frac{1}{2}} \rightarrow - g^4 \frac{\sqrt{s}}{2\pi} I_o(s) \log(-z) \tag{3.35}$$

where

$$I_o(s) = \frac{1}{16\pi^2} \int_{4\lambda^2}^{\infty} \frac{ds'}{\sqrt{(s')(\frac{s'}{4} - \lambda^2)}} \; \frac{1}{s'-s-i\epsilon} \tag{3.36}$$

so that, noting again the correctness of the prediction of the asymptotic form, we have

$$\xi^2_{\frac{1}{2}\frac{1}{2}} = g^4 \frac{4m^2}{3\pi} \frac{I_o(s)}{\sqrt{s}} \tag{3.37}$$

$$\xi_{\frac{1}{2}-\frac{1}{2}} \xi_{\frac{1}{2}\frac{1}{2}} = g^4 \frac{2\sqrt{2}\, m}{3\pi} I_o(s) \tag{3.38}$$

$$\xi^2_{\frac{1}{2}-\frac{1}{2}} = g^4 \frac{2\sqrt{s}}{3\pi} I_o(s) \tag{3.39}$$

which agrees with (3.32).

As in Reference (3) and Reference (5), we extract additional information from the partial wave unitarity relations continued into the complex J plane. The possibility of doing this uniquely for both

$F^e(J)$ and $F^o(J)$ follows from Carlson's theorem. We have

$$\operatorname{Im} F^{\substack{e\\o}}_{ba}(J) = \sum_m \frac{k_m}{N_m^2} F^{\substack{e\\o}}_{ma}(J) \left[F^{\substack{e\\o}}_{mb}(J)\right]^* \tag{3.40}$$

where k_m is the center of mass momentum of the intermediate state, N_m is the normalization factor for the intermediate state (taken from Table III) and the sum is over all the intermediate states. This formula is valid in the range from the lowest threshold of the intermediate states to the first inelastic threshold. To emphasize that the "Imaginary part" above refers to the discontinuity across the cut in s, we should write

$$\operatorname{Im} F^{\substack{e\\o}}_{ba}(J) = \frac{F^{\substack{e\\o}}_{ba}(J) - \left[F^{\substack{e\\o}}_{ba}(J^*)\right]^*}{2i} \tag{3.41}$$

with a similar understanding for the complex conjugation sign on the RHS of (3.40). Applying this to the process $N + \overline{N} \rightarrow N + \overline{N}$, we consider intermediate states of the $N\overline{N}$ and $\gamma\gamma$ systems. Since (3.40) must hold in each order of the coupling constant, let us consider the fourth order terms, and then look at the high z limit. In this order, the only contributions come from the $\gamma\gamma$ nonsense state as we see from (3.26), since all the other states give lower order contributions. In the neighborhood of $J = \alpha$, by means of (3.21), we can write

$$I_m \frac{\xi_i \xi_j}{J-\alpha} = \frac{k}{2}\sqrt{J-1}\ \frac{\xi_n \xi_j}{J-\alpha} \cdot \sqrt{J-1}\ \frac{(\xi_n \xi_i)^*}{J-\alpha} \tag{3.42}$$

and taking the residues of both sides,

$$\mathrm{Im}(\xi_i \xi_j) = \frac{k}{2}(\xi_n \xi_j)(\xi_n \xi_i)^*, \qquad s \geq 4\lambda^2 . \tag{3.43}$$

This expression serves as a check on the second and fourth order work already done. Noting from (3.36) that $\mathrm{Im}\ I_o(s) = (16\pi k\sqrt{s})^{-1}$ and taking the case $i = j = \frac{1}{2}\frac{1}{2}$, we confirm that

$$\mathrm{Im}\ \frac{g^4 4m^2 I_o(s)}{3\pi\sqrt{s}} = \frac{k}{2}\left(\frac{g^2 m}{\sqrt{6}\,\pi k\sqrt{s}}\right)^2 \tag{3.44}$$

where the RHS is obtained from (3.30).

Applying this formalism to the nonsense-nonsense transition, where the fourth order contribution comes only from the two $N\bar{N}$ intermediate states, we find

$$\begin{aligned}\mathrm{Im}\left[(J-1)\frac{\xi_n^2}{J-\alpha}\right] &= p\sum_i \sqrt{J-1}\,\frac{\xi_n\xi_i}{J-\alpha}\cdot\sqrt{J-1}\,\frac{(\xi_n\xi_i)^*}{J-\alpha} \\ &= p\sum_i |\xi_n\xi_i|^2 \\ &= g^4\frac{p}{12\pi^2k^2s}(2m^2+s), \qquad s \geq 4m^2. \end{aligned} \tag{3.45}$$

IV. CALCULATION OF THE TRAJECTORY

1. The Full $\gamma\gamma$ Amplitude

The fourth order calculation of the process $\gamma + \gamma \to \gamma + \gamma$ involves two parts: the calculation of $f^{+}_{1-1;1-1}(z)$ and the separation of the answer into right and left hand cuts in the momentum transfer t. The first part of the calculation is quite lengthy, but straightforward once certain points dealing with invariant amplitudes have been clarified. The second part presents not only difficulties of calculation but fundamental ambiguities of interpretation which have not yet been resolved. These will be discussed at length after we obtain $f^{+}_{1-1;1-1}(z)$.

From (2.33),

$$f^{+}_{1-1;1-1} = \frac{f^{\mathcal{S}}_{1-1;1-1}}{(1+z)^2} + \frac{f^{\mathcal{S}}_{1-1;1-1}}{(1-z)^2} \; . \tag{4.1}$$

Here $f^{\mathcal{S}}$ is determined from the invariant matrix element M by

$$f^{\mathcal{S}} = \frac{1}{8\pi\sqrt{s}} \, M \tag{4.2}$$

and M is given directly by the Feynman rules.[20] We consider first the general case of fourth order scattering of neutral vector bosons coupled by a conserved-current interation to a fermion field. We use the method of Karplus and Neuman[21], but as we shall see, their result must be modified so that its analytic properties are recognizable. Instead of deriving their result as an intermediate step and proceding onwards, we present here a more direct way of obtaining the result we want. The physical process is completely described by specifying the four boson four-momenta $k^{(1)}$, $k^{(2)}$, $k^{(3)}$, $k^{(4)}$ and their corresponding polarization four-vectors $e^{(1)}$, $e^{(2)}$, $e^{(3)}$, $e^{(4)}$. Taking the momenta as ingoing, energy-momentum conservation is

$$\sum_i k_\mu^{(i)} = 0 . \tag{4.3}$$

Gauge invariance requires

$$\epsilon_\mu^{(i)} k_\mu^{(i)} = 0 \qquad \text{(i not summed)} \tag{4.4}$$

for each particle, and the $e^{(i)}$ are normalized as in (A13). We write the invariant matrix element as

$$M = -\frac{g^4}{8\pi^2} \epsilon_\mu^{(1)} \epsilon_\nu^{(2)} \epsilon_\lambda^{(3)*} \epsilon_\sigma^{(4)*} G_{\mu\nu\lambda\sigma}(k^{(1)},k^{(2)},k^{(3)},k^{(4)}) , \tag{4.5}$$

where particles 3 and 4 have been chosen as outgoing and the factors have been chosen for convenience in what follows. Because of the conserved current interaction $G_{\mu\nu\lambda\sigma}$ is divergenceless with respect to each index. In momentum space

$$\begin{aligned} k_\mu^{(1)} G_{\mu\nu\lambda\sigma}(k^{(1)},k^{(2)},k^{(3)},k^{(4)}) &= 0 \\ k_\nu^{(2)} G_{\mu\nu\lambda\sigma}(k^{(1)},k^{(2)},k^{(3)},k^{(4)}) &= 0 \\ k_\lambda^{(3)} G_{\mu\nu\lambda\sigma}(k^{(1)},k^{(2)},k^{(3)},k^{(4)}) &= 0 \\ k_\sigma^{(4)} G_{\mu\nu\lambda\sigma}(k^{(1)},k^{(2)},k^{(3)},k^{(4)}) &= 0 \quad . \end{aligned} \tag{4.6}$$

In fourth order, six diagrams contribute to $G_{\mu\nu\lambda\sigma}$. Three of them are shown in Fig. 2. The others are the same except that the internal fermion line circulates clockwise.

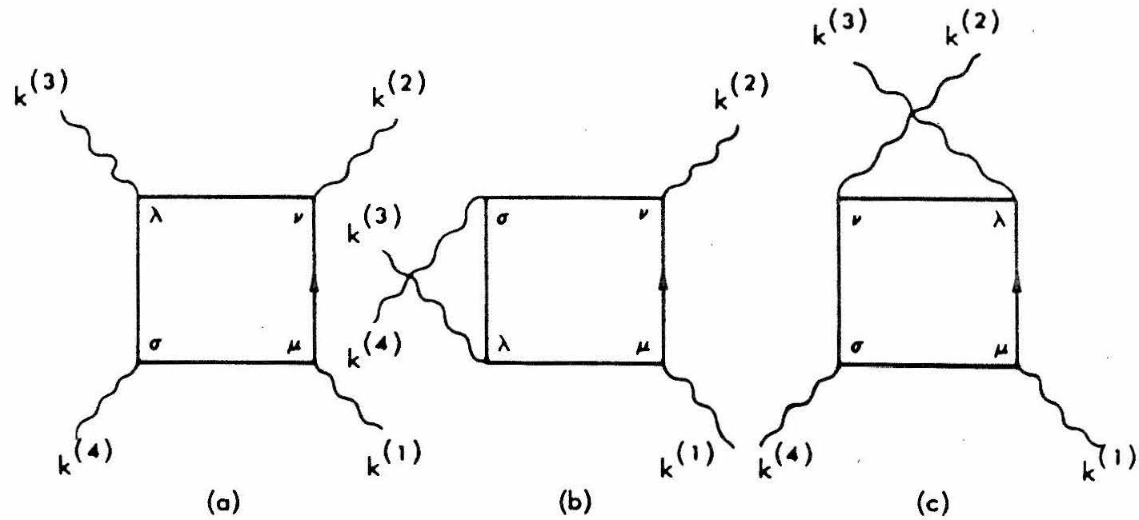

Fig.2—Diagrams for fourth order $\gamma\gamma$ scattering

In addition to being Lorentz invariant, the fourth rank tensor $G_{\mu\nu\lambda\sigma}(k^{(1)},k^{(2)},k^{(3)},k^{(4)})$ is notationally invariant; it describes the same physical process under the 24 simultaneous permutations of its arguments and corresponding indices. The contribution of the Fig. 2(a) to $G_{\mu\nu\lambda\sigma}(k^{(1)},k^{(2)},k^{(3)},k^{(4)})$ is, by the Feynman rules,

$T_{\mu\nu\lambda\sigma} =$

$$\frac{1}{i\pi^2}\int d^4p\, \frac{Tr\{\gamma_\mu(\not{p}+m)\gamma_\nu(\not{p}+\not{k}^{(2)}+m)\gamma_\lambda(\not{p}+\not{k}^{(2)}+\not{k}^{(3)}+m)\gamma_\sigma(\not{p}-\not{k}^{(1)}+m)\}}{(p^2-m^2)[(p+k^{(2)})^2-m^2][(p+k^{(2)}+k^{(3)})^2-m^2][(p-k^{(1)})^2-m^2]} \tag{4.7}$$

where the factor in front represents the product of $-\frac{8\pi^2}{g^4}$ for the factor in (4.5), $\frac{1}{(2\pi)^4 i}$ for the energy momentum delta function, (-1) for the chosed fermion loop, 2 to represent the diagram with the fermion direction reversed and g^4 for the coupling constant. We leave it understood that the poles in this and similar integrals are to be treated as if the masses had a small negative imaginary part. The symmetries of $T_{\mu\nu\lambda\sigma}$ are fewer than those of $G_{\mu\nu\lambda\sigma}$ but include the 8 simultaneous permutations of its arguments and tensor indices which leave the trace invariant--4 cyclic permutations and the complete reversal 1234 → 4321. We may obtain the contribution of the three diagrams to $G_{\mu\nu\lambda\sigma}$ from $T_{\mu\nu\lambda\sigma}$ alone by proper permutations,

$$G_{\mu\nu\lambda\sigma}(k^{(1)},k^{(2)},k^{(3)},k^{(4)}) = T_{\mu\nu\lambda\sigma}(k^{(1)},k^{(2)},k^{(3)},k^{(4)}) + T_{\mu\nu\sigma\lambda}(k^{(1)},k^{(2)},k^{(4)},k^{(3)}) + T_{\mu\lambda\nu\sigma}(k^{(1)},k^{(3)},k^{(2)},k^{(4)}) \tag{4.8}$$

so we see explicitly how $G_{\mu\nu\lambda\sigma}$ is invariant under all $3 \times 8 = 24$ permutations.

The expression (4.7) looks as if it contained a logarithmically divergent term, but Karplus and Neuman have demonstrated by the

Pauli-Villars regulator method that $G_{\mu\nu\lambda\sigma}$ is completely finite. They then demonstrate explicitly that their regularized expression for $G_{\mu\nu\lambda\sigma}$ satisfies (4.6). The same answer could be arrived at by using (4.6) to enforce a finite answer, or by using Jauch and Rohrlich's method of symmetrical integration.[22]

To simplify the writing we introduce the following condensed notation. Replace the vector momenta symbols $k^{(1)}$, $k^{(2)}$, $k^{(3)}$, $k^{(4)}$ simply by 1, 2, 3, 4 respectively when there is no possible ambiguity. For example, $G_{\mu\nu\lambda\sigma}(k^{(1)},k^{(2)},k^{(3)},k^{(4)})$ becomes $G_{\mu\nu\lambda\sigma}(1234)$ (no commas); the tensor $k_\mu^{(2)}k_\nu^{(1)}k_\lambda^{(4)}k_\sigma^{(3)}$ becomes $2_\mu 1_\nu 4_\lambda 3_\sigma$; the scalar $k_\mu^{(i)}k_\mu^{(j)}$ becomes $(i \cdot j)$; and the second of equations (4.6) becomes $2_\nu G_{\mu\nu\lambda\sigma}(1234) = 0$.

We join Karplus and Neuman in writing the tensor $G_{\mu\nu\lambda\sigma}$ in terms of the available tensors.

$$
\begin{aligned}
\epsilon_{\mu\nu\lambda\sigma}(1234) &= \sum_{i,j,k,\ell} A^{ijk\ell}(1234) i_\mu j_\nu k_\lambda \ell_\sigma + \sum_{i,j} B_1^{ij}(1234) i_\mu j_\nu g_{\lambda\sigma} \\
&\quad + \sum_{i,k} B_2^{ik}(1234) i_\mu k_\lambda g_{\nu\sigma} + \ldots + \sum_{k,\ell} B_6^{k\ell}(1234) k_\lambda \ell_\sigma g_{\mu\nu} \\
&\quad + C_1(1234) g_{\mu\nu} g_{\lambda\sigma} + C_2(1234) g_{\mu\lambda} g_{\nu\sigma} + C_3(1234) g_{\mu\sigma} g_{\nu\lambda} ,
\end{aligned} \tag{4.9}
$$

with

$$
\begin{aligned}
i &= 2, 3, 4 ; \qquad j = 1, 3, 4 \\
k &= 1, 2, 4 ; \qquad \ell = 1, 2, 3 ,
\end{aligned} \tag{4.10}
$$

where the 81 A's, the 54 B's and the 3 C's are scalar invariants which are functions of the scalar products of the four-momenta. One might imagine evaluating them by an explicit calculation of the traces and integrals in the regularized version of (4.7) and its permutations. (This would involve handling some 11,340 terms.) Although the completely antisymmetric tensor $\epsilon_{\mu\nu\lambda\sigma}$ is also available, terms containing an odd number of these tensors would give a pseudo-scalar after dotting them into the polarizations. Terms with an even number of them are expressible as combinations of the tensors already appearing in (4.9). Because of the conservation of momentum, there are only three independent four-vectors. We have eliminated $k^{(1)}$ when dealing with the tensor index μ, $k^{(2)}$ for ν, etc., as indicated by the range of the sums in (4.10). The reason for this convention is that it is invariant under simultaneous permutation of the $k^{(i)}$ and their corresponding tensor indices. Since both the value of the left hand side and the form of the right hand side of (4.9) are invariant under such permutations, certain relations among the A's must be satisfied. There are so many in fact that if the functional form of a representative set of six of them is known, the other 75 may be determined by simple permutation of the arguments. Detailed information is given in Appendix D. Similar relations hold among the B's.

A difficulty not mentioned by Karplus and Neuman is that an expression of the form of the right hand side of (4.9) is not unique. This is because any two of the tensors can be expressed identically as combinations of the remaining ones so that there is an infinite number of sets of A's, B's and C's which will yield the same function. This point is discussed more fully in Appendix C. At first glance, the simplest way to avoid this difficulty is to omit two tensors, say $C_2(1234)g_{\mu\lambda}g_{\nu\sigma}$ and $C_3(1234)g_{\mu\sigma}g_{\nu\lambda}$ from the right hand side. The coefficients of the remaining tensors are then uniquely determined, but operating on $C_1(1234)$ with certain permutations of momenta and indices would yield these unwanted tensors again. If we replaced these tensors by their values in terms of the others as given by the two identities, we would introduce additional A's. Thus under this operation some A's would go into unmanageable mixtures of A's and C's. The way out of this difficulty is to fix uniquely all the coefficients of (4.9) by regarding them as coming from some particular calculation of the traces and integrals. Then a simultaneous permutation of momenta and indices could be regarded as coming from a line by line substitution into this particular calculation. With this understanding, the results of the previous paragraph are correct.

Karplus and Neuman's big trick is to use current conservation Eq. (4.6) to derive further relations among these quantities. It is easy to show that in an expression which satisfies (4.6), if all the A's are zero, the B's and C's must vanish also. Thus, knowledge of the A's determines the B's and C's uniquely and for the case of $G_{\mu\nu\lambda\sigma}$, it is necessary to evaluate only the A's. Extending the terminology of Karplus and Neuman we call the A terms "heads," the B terms "shoulders" and the C terms "tails." Following the usual procedure, we introduce Feynman parameters to combine the denominators of Eq. (4.7), then shift the origin of the p integration to eliminate the terms in the denominator linear in p. We get

$$T_{\mu\nu\lambda\sigma}(1234) =$$

$$\frac{6}{i\pi^2}\int d\tau \int d^4p \; \frac{\mathrm{Tr}\,\{\gamma_\mu(\not{p}+\not{\chi}^{(1)}+m)\gamma_\nu(\not{p}+\not{\chi}^{(2)}+m)\gamma_\lambda(\not{p}+\not{\chi}^{(3)}+m)\gamma_\sigma(\not{p}+\not{\chi}^{(4)}+m)\,\}}{\text{Denominator}}$$

$$\begin{aligned}\text{Denominator} \equiv \{p^2 - m^2 &+ k^{(1)2}\, y_4 y_1 + k^{(2)2}\, y_1 y_2 + k^{(3)2}\, y_2 y_3 + k^{(4)2}\, y_3 y_4 \\ &- (k^{(1)}+k^{(2)})\cdot(k^{(3)}+k^{(4)})\, y_2 y_4 - (k^{(1)}+k^{(4)})\cdot(k^{(2)}+k^{(3)})\, y_1 y_3\}^4\end{aligned} \tag{4.11}$$

where

$$\begin{aligned}
\lambda^{(1)} &= -k^{(1)} y_4 + k^{(2)}(y_2+y_3) + k^{(3)} y_3 \\
\lambda^{(2)} &= -k^{(2)} y_1 + k^{(3)}(y_3+y_4) + k^{(4)} y_4 \\
\lambda^{(3)} &= k^{(1)} y_1 - k^{(3)} y_2 + k^{(4)}(y_4+y_1) \\
\lambda^{(4)} &= k^{(1)}(y_1+y_2) + k^{(2)} y_2 - k^{(4)} y_3
\end{aligned} \tag{4.12}$$

and

$$\int d\tau \equiv \int_0^1\int_0^1\int_0^1\int_0^1 \delta(1-y_1-y_2-y_3-y_4)\,dy_1\, dy_2\, dy_3\, dy_4 \, . \tag{4.13}$$

But since we need only the head terms, i.e., only those where all the tensor indices appear on the $k^{(i)}$, we may disregard all the p and m terms in the numerator, making the integration over p very simple. The "divergent" contributions never appear.

$$T_{\mu\nu\lambda\sigma}(1234) \underset{\substack{\text{heads}\\ \text{only}}}{=} \int d\tau \; \frac{\mathrm{Tr}\{\gamma_\mu \chi^{(1)} \gamma_\nu \chi^{(2)} \gamma_\lambda \chi^{(3)} \gamma_\sigma \chi^{(4)}\}}{D(1234)} \tag{4.14}$$

where

$$D(1234) \equiv [\, m^2 - k^{(1)2} y_4 y_1 - k^{(2)2} y_1 y_2 - k^{(3)2} y_2 y_3 - k^{(4)2} y_3 y_4$$
$$+ (k^{(1)} + k^{(2)}) \cdot (k^{(3)} + k^{(4)}) y_2 y_4 + (k^{(1)} + k^{(4)}) \cdot (k^{(2)} + k^{(3)}) y_1 y_3]^2$$
$$\tag{4.15}$$

Further, in doing the trace over the remaining terms, i.e., the $\lambda^{(i)}$, we get 24 terms instead of the usual 105 because all terms involving $g_{\mu\nu}$ contribute only to the shoulders and tails. We may collect the terms belonging to each head tensor, being careful to express the $\lambda^{(i)}$ in terms of the three momenta appropriate to the index carried by the $\lambda^{(i)}$ in accordance with the convention of (4.10). We call $A_1^{ijk\ell}(1234)$ the contribution of the heads to $T_{\mu\nu\lambda\sigma}(1234)$ so that

$$T_{\mu\nu\lambda\sigma} \underset{\substack{\text{heads}\\ \text{only}}}{=} \sum_{\substack{i=2,3,4;\,k=1,2,4\\ j=1,3,4;\,\ell=1,2,3}} A_1^{ijk\ell}(1234) i_\mu j_\nu k_\lambda \ell_\sigma \, . \tag{4.16}$$

The A's are calculated from the A_1's by using the symmetry operations of (4.8) and are listed in Appendix D. Because $T_{\mu\nu\lambda\sigma}$ is less symmetric than $G_{\mu\nu\lambda\sigma}$ as measured by the number of simultaneous permutations of arguments and indices which leave it invariant, the A_1's are less symmetric than the A's; a representative set of 15 of them are required to determine all the others and consequently the A's. Karplus and Neuman have done all the bookkeeping, and list values of such a set of 15 A_1's. A typical one is

$$A_1^{2111}(1234) = 8\int d\tau \, \frac{y_1(y_2+y_3+y_4)(y_1+y_2-y_3-y_4)(y_1+y_2+y_3-y_4)}{D(1234)} , \tag{4.17}$$

the others differing by the polynomial in the numerator. A complete list is given in Appendix D.

In collecting these formulas for the A_1's one discovers certain relations among them which in turn generate relations among the A's which are not obtainable by notational invariance alone.

From here on we part company with the work of Karplus and Neuman, who go on to introduce a set of tensors which satisfy (4.6) identically and express the answer in terms of these tensors and the A's. For a while considerations of the number of independent amplitudes available to describe the scattering process cast doubt on the validity of their method and some looseness in their arguments increased our suspicions, but these uncertainties have been resolved. See Appendix C for a fuller discussion. Though their final answer is correct, it contains terms with products of momenta in the denominator which yield factors of $\frac{1}{t-2}$ or $\frac{1}{t+s-2}$ in addition to the t dependence of the integrals (4.17). These factors make both the determination of the analytic properties of each term and the calculation of the asymptotic form very difficult.

Following a suggestion of J. Sullivan we cast the answer in a form whose analytic properties are more obvious by explicitly solving for the shoulders and tails in terms of the heads by use of (4.6).

For example, after contracting (4.9) with 1_μ, we have a third rank tensor which must vanish for any choice of the $k^{(i)}$. It is easy to show that this implies that the coefficient of each tensor form vanishes (unlike the case of second or fourth rank tensors). Thus the coefficient of, say $1_\nu 4_\lambda 3_\sigma$ is

$$(1\cdot 2)A^{2143}(1234) + (1\cdot 3)A^{3143}(1234) + (1\cdot 4)A^{4143}(1234) + B_6^{43}(1234) = 0 \tag{4.18}$$

and this gives $B_6^{43}(1234)$ in terms of the heads. Similarly, all the B's and C's may be expressed in terms of the A's. However, we could also obtain an equation for $B_6^{43}(1234)$ by considering the coefficient of $2_\mu 4_\lambda 3_\sigma$ after contracting with 2_ν:

$$(2\cdot 1)A^{2143}(1234) + (2\cdot 3)A^{2343}(1234) + (2\cdot 4)A^{2443}(1234) + B_6^{43}(1234) = 0\,. \tag{4.19}$$

In this way we obtain additional relations among the A's, which are not obvious from expressions such as (4.17). Two such expressions, which we write in the form

$$a(1234) = (3\cdot 4)A^{2123}(1234) + (2\cdot 4)A^{2122}(1234) + (1\cdot 4)A^{2121}(1234) = 0 \tag{4.20}$$

$$b(1234) = (1\cdot 4)A^{2311}(1234) + (3\cdot 4)A^{2311}(3134) + (2\cdot 4)A^{2311}(2314) = 0 \tag{4.21}$$

are derived in an elaborate way in Reference (21). They felt it necessary to check their arguments by an explicit calculation for a special case. The first of these identities will play a role in our later discussion of the analytic properties of the amplitude.

The final answer could be written in the form (4.9) with the coefficients of the 138 tensor amplitudes all expressed in terms of the A's. We choose to regroup the terms and list the tensor expression which multiplies each of the 81 A's. This is the form given in Appendix E. This expression agrees term by term with the expression given by Karplus and Neuman if sufficient use is made of (4.20), (4.21) and their 24 permutations. The point of all this work is that now no scalar products of momenta occur in the denominator, or in dispersion theoretic language, we have avoided a great many kinematical singularities.

The previous theory is perfectly general and applies even when the four particle box is an internal part of a diagram. Now specialize to the case where the $k^{(i)}$ represent the external momenta of identical particles of mass $\lambda < m$. Take λ to be the unit of mass and, in accordance with (4.5), choose 1 and 2 to represent the initial particles, 3 and 4 to represent the final ones. Introduce s, t and u as usual by

$$s = (k^{(1)}+k^{(2)})^2 \; ; \quad t = (k^{(1)}+k^{(4)})^2 \; ; \quad u = (k^{(1)}+k^{(3)})^2 \tag{4.22}$$

so that

$$(1\cdot 2) = (3\cdot 4) = \frac{1}{2}(s-2) \; ; \qquad (1\cdot 4) = (2\cdot 3) = \frac{1}{2}(t-2) \; ;$$
$$(1\cdot 3) = (2\cdot 4) = \frac{1}{2}(u-2) = \frac{1}{2}(2-s-t) \; . \tag{4.23}$$

Equation (4.15) becomes

$$D(1234) = [\, m^2 - (y_1y_2+y_2y_3+y_3y_4+y_4y_1) - sy_2y_4 - ty_1y_3]^2 \equiv D(s,t) \; . \tag{4.24}$$

This simple form for $D(1234)$ means that the 24 <u>a priori</u> permutations of the arguments can be reduced to only 6 different ones, and by a change of integration variables to only 3: $D(s,t)$, $D(s,u)$ and $D(t,u)$. Appendix D lists all the possibilities. Since the A's depend on their arguments only through these denominators, additional equalities among them are obtained which are given in Appendix D.

Further specialize to the actual cases at hand. In the center of mass system take:

$$\begin{array}{llll} k^{(1)}: & (\omega,0,0,k) & k^{(2)}: & (\omega,0,0,-k) \\ k^{(3)}: & (-\omega,k\sin\theta,\,0,\,k\cos\theta) & k^{(4)}: & (-\omega,-k\sin\theta,0,-k\cos\theta) \end{array} \tag{4.25}$$

with

$$\omega^2 - k^2 = 1 \; ; \qquad 4\omega^2 = s \tag{4.26}$$

Choose the helicities to be

$$\begin{array}{ll} \epsilon^{(1)}: \ \frac{1}{\sqrt{2}}(0,1,i,0) & \epsilon^{(2)}: \ \frac{1}{\sqrt{2}}(0,-1,-i,0) \end{array}$$

$$\epsilon^{(3)*}: \begin{cases} \frac{1}{\sqrt{2}}(0,-\cos\theta,i,\sin\theta) & \text{for } f^{\mathcal{S}}_{1-1;1-1} \\ \frac{1}{\sqrt{2}}(0,-\cos\theta,-i,\sin\theta) & \text{for } f^{\mathcal{S}}_{-11;1-1} \end{cases} \tag{4.27}$$

$$\epsilon^{(4)*}: \begin{cases} \frac{1}{\sqrt{2}}(0,\cos\theta,-i,-\sin\theta) & \text{for } f^{\mathcal{S}}_{1-1;1-1} \\ \frac{1}{\sqrt{2}}(0,\cos\theta,i,-\sin\theta) & \text{for } f^{\mathcal{S}}_{-11;1-1} \; . \end{cases}$$

The calculation of M is now straightforward. The tensors associated with the A's yield terms $\pm k^4 \sin^4\theta$, those associated with the B's yield $(\pm k^2 \sin^2\theta)(\pm 1 + \cos\theta)$ and those associated with the C's yield $(\pm 1 + \cos\theta)^2$. Combining the helicity cases according to (4.1), and making liberal use of the symmetries, we get the coefficients of each head. Expressing these coefficients in terms of s and t we find that the only place the t appears explicitly is in the terms

$$\frac{st}{8}\left[A^{4411} - A^{4422} - A^{3422} - A^{4311}\right] . \tag{4.28}$$

(The argument (1234) common to each of the A's is suppressed.) Two permutations of the identity $a(1234)$

$$\begin{aligned} &(2\cdot 3)A^{4311} + (2\cdot 1)A^{4111} + (2\cdot 4)A^{4411} = 0 \\ &(1\cdot 3)A^{3422} + (1\cdot 2)A^{2422} + (1\cdot 4)A^{4422} = 0 \end{aligned} \tag{4.29}$$

furnish the relations

$$\begin{aligned} &t(A^{4411} - A^{4311}) = -2A^{4311} + (s-2)\left[A^{4111} - A^{4411}\right] \\ &t(A^{3422} - A^{4422}) = -2A^{4422} + (s-2)\left[A^{2422} - A^{3422}\right] \end{aligned} \tag{4.30}$$

so that t may be entirely eliminated. The complete result is then

$$
\begin{aligned}
-\frac{8\pi^2}{g^4} M^{+}_{1-1;1-1} = {} & \frac{1}{16}(s^2-4s+8)[A^{4321}+A^{3412}+2A^{4312}] \\
& + \frac{1}{8}[A^{2341}+A^{2413}] - \frac{1}{2}(s-2)[A^{4411}+A^{4422}] \\
& + \frac{1}{8}s(s-2)[A^{4111}+A^{2422}-A^{4121}-A^{3112}] \\
& - \frac{1}{8}(s-4)^2[A^{3312}+A^{4311}]
\end{aligned}
\tag{4.31}
$$

in which the only t dependence appears in the integrals for the A's.

2. The Program and Its Difficulties

A few words about the analytic properties of the amplitude will be useful. Every A is the sum of three terms, each of which represents a contribution from one of the three basic diagrams and is obtained by applying permutations to A_1. The effect of these permutations on the scalar products of the momenta is to interchange t and u going from diagram 2(a) to 2(b), and s and u going from diagram 2(a) to 2(c). This gives a characteristic denominator to the contribution of each diagram so that M takes the form

$$M^{+}_{1-1;1-1} = \int d\tau \, \frac{N_I(s, y_i)}{D(s,t)} + \int d\tau \, \frac{N_{II}(s, y_i)}{D(s,u)} + \int d\tau \, \frac{N_{III}(s, y_i)}{D(u,t)} \tag{4.32}$$

where N_I, N_{II}, and N_{III} are the polynomials in s, y_1, y_2, y_3, y_4 obtained from (4.31) by combining integrands with the same denominator.

An elementary study of (4.24) shows that $D(s,t)$ cannot vanish anywhere in the regions of integration provided

$$\begin{aligned} s &< 4m^2 \\ t &< 4m^2 \end{aligned} \tag{4.33}$$

so that these conditions become the boundaries of the region of analyticity for the first term on the right hand side of (4.32), neglecting the s dependence of N_I. This simple examination is confirmed by the more sophisticated investigations of Karplus, Sommerfield and Wichman(23) and Tarski(24) for the general mass case. The condition on s is simply that we remain below the threshold for producing real fermion pairs. For $s < 4m^2$, we reach a branch point as we increase t, so Fig. 2(a) gives a right hand cut in t beginning at $4m^2$. In Fig. 3 we have a picture in barycentric coordinates of the region of analyticity for each diagram. The central triangle DEF

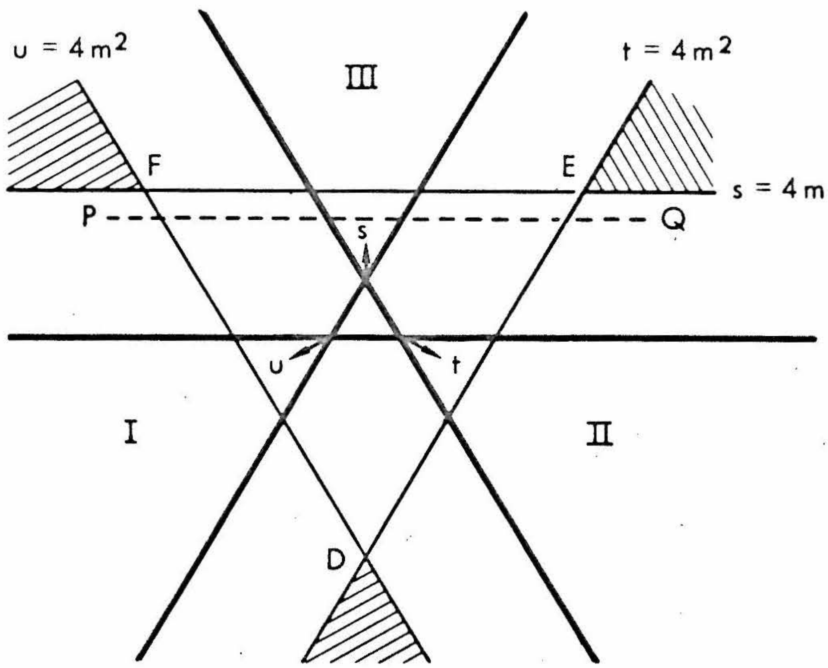

Fig.3 — Regions of analyticity

is a common analytic region for all three diagrams. If we hold s fixed at some value $s_o < 4m^2$ represented by the line PQ, we see that Fig. 2(b) yields a right hand cut beginning at $t = 4m^2$, Fig. 2(b) yields a left hand cut beginning at $t = 4 - s_o - 4m^2$, while Fig. 2(c) yields both a right hand cut beginning at $t = 4m^2$ and a left hand cut beginning at $t = 4 - s_o - 4m^2$.

The three terms in (4.32) correspond to the decomposition in the Mandelstam representation into three double-spectral functions. Because these integrals go to 0 at infinite t, if we wrote the RHS of (4.32) as a dispersion integral in t, there would be no subtraction. Thus we can unambiguously identify the contribution from the right or left hand cuts separately, even if their sum should vanish. This separability of the cuts is essential to the program.

To look at only the right hand cut, we take the first integral in (4.32) and part of the third. The question is, how do we separate the right and left hand cut contributions in this third integral? Rewriting the denominator

$$D(u,t) = [\, m^2 - (y_1y_2 + y_2y_3 + y_3y_4 + y_4y_1) - (4-s)y_2y_4 - t(y_1y_3 - y_2y_4)]^2 \quad (4.34)$$

we note that if the sign of $(y_1y_3 - y_2y_4)$ were positive (negative) $D(u,t)$ could be made to vanish at some point in the region of integration for all positive (negative) t. This strongly suggests that if we were to limit the region of integration so that $y_1y_3 - y_2y_4$ were always positive, we would get the contribution of the right hand t cut. The correctness of this intuitive result has been verified by the following procedure. Perform the y_4 integration by using the delta-function. Calculate the discontinuity across the cut in t by deforming the contour of the y variable and picking up the contributions of the two second order poles. For $4 < s < 4m^2$ contributions to the discontinuity are obtained only under severe restrictions: either 1. When $t \geq m^2$ and the remaining integration variables y_1 and y_3 are confined to a region bounded by two conic sections; or 2. When $t \leq 4 - s - 4m^2$ and y_1 and y_3 are confined to a second region bounded by two conic sections. If we repeat this procedure using the condition that $(y_1 y_3 - y_2 y_4) > 0$, the first region of the y_1, y_3 variables is unaffected while the second region becomes excluded and we get only the right hand cut in t. The other choice of sign selects only the left hand cut. (This dispersion form of the original integrals contains such complications in the specification of the boundaries of integration that it is practically useless for further computation.)

Having solved the problem of extracting the contribution of the right hand cut we examine its asymptotic form. Anticipating the results of the next section we find that the largest terms from the first diagram are of order t^{-1} as $t \rightarrow \infty$, and we can easily determine their coefficient. We have demonstrated that the right hand cut portion of the third diagram also contributes terms of this order and no larger. Their coefficient has not been determined. What seems to be a messy calculation could probably be done if there were good reason. However there is good reason not to do it.

The Mandelstam representation gives the most convenient basis for a discussion of the analytic properties of M. It is exact for this fourth order process and neglecting subtraction it is related term by term to the decomposition into Feynman diagrams.

$$M = \frac{1}{\pi^2}\int\int \frac{\rho_{st}(s',t')}{(s'-s)(t'-t)} ds'\,dt' + \frac{1}{\pi^2}\int\int \frac{\rho_{su}(s',u')}{(s'-s)(u'-u)} ds'\,du' + \frac{1}{\pi^2}\int\int \frac{\rho_{ut}(u',t')}{(u'-u)(t'-t)} du'\,dt' \qquad (4.35)$$

where, for example, the double spectral function $\rho_{ut}(u',t')$ is non-vanishing only in a region below a curve $u_o(t)$ or $t_o(u)$ in the shaded portion of Fig. 3 situated entirely below the s axis. Breaking up the contribution of the third double spectral function into right and left hand cuts in t,

$$\iint \frac{\rho_{ut}(u',t')}{(t'-t)(u'-u)} du'\,dt' = \iint \frac{\rho_{ut}(u',t')}{u'-u+t'-t} \left(\frac{1}{t'-t} + \frac{1}{u'-u}\right) du'\,dt'$$

$$= \int_{4m^2}^{\infty} dt' \int_{u_o(t')}^{\infty} du' \frac{\rho_{ut}(u',t')}{u'+s+t'-4} \left(\frac{1}{t'-t}\right)$$

$$+ \int_{4m^2}^{\infty} du' \int_{t_o(u')}^{\infty} dt' \frac{\rho_{ut}(u',t')}{u'+t'+s-4} \left(\frac{1}{u'-u}\right) . \qquad (4.36)$$

Replacing the earliest integration in both terms by an integration over $s' = 4 - u' - t'$, the RHS becomes

$$-\int_{4m^2}^{\infty} dt' \int_{-\infty}^{4-u_o(t')-t'} ds' \frac{\rho_{ut}(4-s'-t',t')}{(s'-s)(t'-t)}$$

$$-\int_{4m^2}^{\infty} du' \int_{-\infty}^{4-u'-t_o(u')} ds' \frac{\rho_{ut}(u',4-u'-s')}{(s'-s)(u'-u)} . \qquad (4.37)$$

We see explicitly how the right hand t-cut contains a discontinuity for negative s. If we now took the limit as $t \to \infty$, the coefficient of t^{-1} would contain this same negative s-cut, and $\alpha(s)$ would acquire a left hand cut in s. From the first double spectral function we get only terms which lead to a right hand s-cut in $\alpha(s)$.

The crux of the matter is that for a normal Regge pole, $\alpha(s)$ cannot have any left hand cuts in s! Oehme[14] gives a demonstration of this which we sketch briefly. Considering the spinless case for simplicity, the continued partial wave amplitude (3.5) is given by an integral over z from some small value z_o to ∞. If we break up the integral into two parts, one in which the range of integration is $z_o \leq z \leq a$, and the other in which the range of integration goes from a to ∞, the first integral is analytic in the whole J plane except for poles at the negative integers. Thus all other singularities of $F^e(J)$ must come from the second integral which we call $D(J)$. In this integral we may take a as large as we please. Making the same assumptions about continuing $D(J)$ to the left of $\mathrm{Re}\, J = N$ as in Chapter III, the Regge poles are given by

$$D^{-1}(\alpha(s)) = 0 . \tag{4.38}$$

Then we expect singularities of $\alpha(s)$ only where $D^{-1}(J)$ is singular or where

$$\left(\frac{d}{dJ} \frac{1}{D(J)} \right) \Bigg|_{J=\alpha(s)} = 0, \tag{4.39}$$

and this latter can happen where two pole trajectories cross each other at $J = \alpha(s)$. In the first case, we may expect that the left hand cuts in $A_t(s,t)$ or $A_u(s,u)$ will produce corresponding singularities in $\alpha(s)$. But this is not the case since the left hand cut from (4.37) begins at $s = 4 - u_o(t') - t'$ which recedes to $-\infty$ as we take a, and hence t', larger and larger. (In this limit $u_o(t') \rightarrow 4m^2$.) Thus $\alpha(s)$ cannot have a left hand cut unless there is another singularity present.

Since the full amplitude in fact yields a trajectory with a left hand cut, there must be some other singularity also present. That is, in addition to the vacuum trajectory we are seeking, there is an unidentified foreign object (UFO) also near $J = 1$. The original program of evaluating the Regge pole parameters was not sophisticated enough to include this possibility. We must now try to make sense out of this phenomenon. Since we no longer have a fixed prescription of what to do, let us examine the possibilities and their consequences.

If we decide that the theory is in such poor shape that the presence of a left hand cut in s is not to be excluded a priori as unreasonable, what are the other consequences of proceding with the original program and including the last diagram?

First, let us describe a somewhat lengthy calculation which gives the value of $\alpha(0)$ for the complete scattering process, including the third diagram. Consider the dispersion relation (3.3) for fixed s in our unsubtracted case. The limit as $t \rightarrow \infty$, of the portion which gives the right hand cut is

$$- \frac{1}{\pi t} \int_{t_o}^{\infty} A_t(s, t')\, dt' \qquad (4.40)$$

if that integral converges. If we could calculate A_t, the absorptive part of the scattering in the t-channel, its integral would give the coefficient of t^{-1} (or, within a factor, z^{-1}) which is just what we need to evaluate the trajectory according to (3.25) and (3.27). We did this for $s = 0$ by noting that this case corresponds to the case of forward scattering in the cross-channel where t is the energy squared. In going to the cross-channel, one must perform a Lorentz transformation to convert to the new center of mass system, and this same transformation must of course be applied to the helicities, taking care not to mix up the complex conjugation of the final state polarization vectors indicated in (4.5). The absorptive part is then obtained by making a cut in the t (energy) channel and using elastic unitarity (or the Cutkosky rules) to put the intermediate N and $\overline{N}$ particles on the mass shell. Summing over the spins and integrating over the angles which are the only free parameters of the intermediate state, and adding together the proper helicity combinations according to (4.1), we obtain for the imaginary part of the invariant Feynman amplitude,

$$\mathrm{Im}_t M^{R+}_{1-1;1-1}\Big|_{s=0} = \frac{g^4}{64\pi}\sqrt{\frac{x-1}{x}}\;\frac{1}{x^2(x-1)^4}\left\{\frac{-32x^5-64x^4-16x^3+55x^2+9x-4\frac{1}{2}}{4x-3}\right.$$

$$\left. + \frac{\log(4x-3)}{4(x-1)(2x-1)}\left[64x^6-128x^5+248x^4-40x^3-211x^2+151x-31\tfrac{1}{2}\right]\right\}, \tag{4.41}$$

where $t = 4m^2x$, and where to make the calculation simpler we have taken $m = \lambda$. At threshold, $x = 1$, it seems as if this expression diverges as $(x-1)^{-7/2}$, but the apparently random numbers which appear in both terms in the numerators combine magically to cancel the first five orders of a Taylor expansion about that point, so that it really goes to zero as $(x-1)^{3/2}$. Further details about this calcu lation are contained in Appendix F. Integrating this expression over the whole range of the cut by machine calculation we get

$$M^{R+}_{1-1;1-1}\Big|_{s=0} \approx -0.025\, \frac{g^4 m^2}{t} \tag{4.42}$$

which leads to a trajectory which does not pass through 1 at $s = 0$. This of course is not a necessary requirement of a vacuum trajectory, but it would be an encouraging feature.

A fatal drawback of the trajectory calculated using the third diagram is the magnitude of its imaginary part. Recall from (3.45) that the imaginary part should be fixed by unitarity. As will be seen in the next subsection, the first diagram alone yields the correct imaginary part, so that any further additions from the left hand s-cut will spoil the agreement.

To save the bulk of the formalism we must argue that to include the third diagram would be to include the UFO as well as the vacuum trajectory. Can we separate their contributions to the full amplitude by omitting the third diagram entirely or by omitting a portion of it? We look at these possibilities in the next section.

3. Possible Resolutions

a. Omit the Third Diagram

The simplest solution to the difficulty is to omit the third diagram altogether. This is the solution adopted by Sullivan[(25)] in his search for the vacuum Regge trajectory in the theory with vector mesons and scalar nucleons. Before discussing the merits of this choice, let us actually calculate $\alpha(s)$ in this case.

Now, instead of using the expressions for the A's in terms of the A_1's in Appendix D, we disregard the A_1 which has a denominator of the form $D(u,t)$ or $D(t,u)$. In fact, since we want only the contribution of the right hand cut we look only at the one A_1 in each A which has a denominator of the form $D(s,t)$ or $D(t,s)$. As is obvious from (4.24), the change of variables $y_1 \longleftrightarrow y_2$, $y_3 \longleftrightarrow y_4$ in the integrals converts the denominator $D(t,s)$ into $D(s,t)$. Using the symmetries of the A's and combining all the terms in (4.31) over the common denominator $D(s,t)$, we find for the contribution of the right hand t-cut an expression of the form

$$M^{R+}_{1-1;1-1} =$$

$$\int_0^1\int_0^1\int_0^1\int_0^1 \frac{N(s,y_1,y_2,y_3,y_4)\,\delta(1-y_1-y_2-y_3-y_4)}{[m^2-(y_1y_2+y_2y_3+y_3y_4+y_4y_1)-sy_2y_4-y_1y_3t]^2}\, dy_1\, dy_2\, dy_3\, dy_4 \,.$$

The asymptotic form for large t is obtained by the following well-known procedure[(26)]. Break up the integral into two parts so that in the first part the y_1 and y_3 integrations go only up to ϵ_1

and ϵ_3 respectively, where ϵ_1 and ϵ_3 are small but finite positive numbers. The second part contains the remainder of the region of integration. As t becomes large, the second part surely goes to t^{-2}; we could expand in a power series to prove this. The only hope of a larger asymptotic form comes from the first part which can not be expanded in a power series. Since N is a polynomial, we may set these variables equal to zero everywhere in the integrand but in the coefficient of t. After an elementary integration we get

$$M^{R+}_{1-1;1-1} \rightarrow \frac{\log(-t)}{-t} \int\int \frac{N(s,0,y_2,0,y_4)\,\delta(1-y_2-y_4)}{[m^2 - y_2 y_4 s]} dy_2\, dy_4 \quad (4.44)$$

which is independent of ϵ_1 and ϵ_3, and bigger than t^{-2}, unless the numerator vanishes. Indeed, calculation shows that this latter is the case for our problem. The largest asymptotic form is then obtained by taking only y_1 small, performing the integration, and adding to this the contribution for y_3 only taken small. Because the denominators so obtained have a symmetric form, the sum of these contributions may be combined, giving

$$M^{R+}_{1-1;1-1} \rightarrow$$

$$\frac{1}{-t}\int_0^1\int_0^1\int_0^1 \frac{[N_I(s,0,y_2,y,y_4)+N_I(s,y,y_2,0,y_4)]\;\delta(1-y-y_2-y_4)}{y[m^2-y(y_2+y_4)-sy_2y_4]} dy\,dy_2\,dy_4 \,. \quad (4.45)$$

If this vanishes also we may still have terms as large as $t^{-2}\log(-t)$.

In our case (4.45) applies, and we see that the prediction of (3.25) about the behavior of the asymptotic form is verified. In particular, we get, after some manipulation,

$$M^{R+}_{1-1;1-1} \rightarrow \frac{g^4}{t\pi^2}\int_0^1\int_0^1\int_0^1 \frac{sy_2(1-y_2)(-sy_2+1-2y)\delta(1-y-y_2-y_4)}{m^2-y(y_2+y_4)-sy_2y_4}\, dy\, dy_2\, dy_4 \,. \tag{4.46}$$

Combining (3.25), (3.27), (3.30) and (3.37), we find that

$$\Delta\xi_n^2 = -\frac{g^4}{4\pi^3 k^2\sqrt{s}}\, T(s) \,, \tag{4.47}$$

and the trajectory is given by

$$\Delta(s) = -\frac{2g^4}{\pi^2}\, I_o(s)T(s) \tag{4.48}$$

where $T(s)$ is the integral in (4.45), and $I_o(s)$ is the expression (3.36).

Since $I_o(0)$ is finite and $T(0) = 0$, we have

$$\alpha(0) = 1 + \Delta(0) = 1. \tag{4.49}$$

This is very fortunate. Calculating the slope of the trajectory at $s = 0$,

$$\left.\frac{d\alpha}{ds}\right|_{s=0} = -\frac{2g^4}{\pi^2}\, I_o(0)T'(0) = -\frac{g^4}{8\pi^4}\, T'(0) \tag{4.50}$$

where

$$T'(0) = \int_0^1 dy \int_0^{1-y} dy_2 \; \frac{y_2(1-y_2)(1-2y)}{m^2 - y(1-y)} \; . \tag{4.51}$$

Unfortunately, this integral is always positive for $m \geq 1$, so that the slope at the origin is negative. Sullivan[25] finds a similar discomfiture in the case with scalar nucleons.

Now we look at the discontinuity in s of $T(s)$. From an analysis of the denominator, or by noting that it is the same denominator one would get for a three particle vertex if all the particles were scalars[25], we expect a cut in s running from $4m^2$ to ∞. Calculating $\operatorname{Im} T(s)$ by deforming the contour of the y integration, and picking up the contribution of the one simple pole in the region of integration, we get

$$\operatorname{Im} T(s) = - \frac{\pi p}{3\sqrt{s}} (2m^2 + s) \; , \qquad s \geq 4m^2 \; . \tag{4.52}$$

Then we have from (4.47)

$$\operatorname{Im} (\Delta \xi_n^2) = \frac{g^4 p}{12\pi^2 k^2 s} (2m^2 + s), \quad s \geq 4m^2 \tag{4.53}$$

which agrees exactly with the prediction (3.45) of partial wave unitarity ! As marvelous as such agreement seems (especially after scores of pages of calculation) there is an important theoretical objection to discarding the third diagram which we discuss in a moment.

First, one more feature of this trajectory is worthy of mention

-- the imaginary part of the trajectory itself. It follows from very general considerations in potential theory that as s increases above threshold on the upper edge of the branch cut, $\alpha(s)$ acquires a positive imaginary part. This is related to the general interpretation that singularities of the scattering amplitude from the non-physical sheet of the s-plane correspond to resonances. This interpretation is a valuable property of trajectories which we would like to retain in relativistic scattering. We see from (4.48) that the trajectory has two thresholds, one introduced by the factor $I_0(s)$ at $4\lambda^2$, and the other by the factor $T(s)$ at $4m^2$. In the region $4\lambda^2 \leq s \leq 4m^2$ the imaginary part of α has the sign of $-T(s)\,\mathrm{Im}\,I_0(s)$. A simple calculation shows that $T(4)$ is negative so that $\mathrm{Im}\,\alpha$ at the beginning of the interval is greater than zero. More detailed investigation shows that this remains true at least until $s = 4m^2$. For $s > 4m^2$ there is an additional contribution with the sign of $-\mathrm{Re}\,I_0(s)\,\mathrm{Im}\,T(s)$. In this region $\mathrm{Re}\,I_0(s)$ is negative so that the additional contribution has a net negative sign. As s becomes larger and larger, however, we see from an application of the analog of (4.45) that $\mathrm{Re}\,T(s)$ becomes positive again. Thus for some large s the sign of the imaginary part changes sign and becomes negative. We note further that both the real and imaginary parts of $\alpha(s)$ diverge as $\log s$ for large s.

The objection to omitting the third diagram has two aspects which are closely related. The first may be seen by studying the steps (4.28) - (4.31) which lead from the full amplitude just as it is obtained, to the form of the full amplitude after all powers of t have been eliminated from the coefficients of the A's. First of all, the coefficient of t in (4.28) does not vanish. This may be verified by evaluating it at a particular point for values of the parameters where the integrals become degenerate. We ask what is the mechanism by which this t is eliminated? We may write the identity $a(1234) = 0$ (and all its permutations) in the form

$$\int \frac{y_2(1-y_2)(1-2y_1)(ty_1-sy_2+1-2y_1-2y_3)}{D(s,t)}\,d\tau$$
$$+\int \frac{(y_1+y_2)(1-y_1-y_2)(uy_1-sy_2+1-2y_1-2y_3)}{D(s,u)}\,d\tau$$
$$+\int \frac{y_2(1-y_2)(1-2y_1)(ty_1-uy_2+1-2y_1-2y_3)}{D(u,t)}\,d\tau = 0 \qquad (4.54)$$

by collecting together all the A_1's which make it up and combining the ones with similar denominators. In this equation the integrals do not vanish separately, as may be seen by taking the asymptotic form of any one of them. Thus the relation used to eliminate t does so by redistributing the contributions to the various diagrams. Furthermore, the procedure by which the original form for the full amplitude was derived was only one of many similar procedures, which all owe their equivalence to the identity $a(1234) = 0$. We cannot therefore say for any form of the amplitude that it was derived without using this identity, and hence that it possesses the "original" distribution of the contributions to each diagram. Thus, simply removing the third diagram from some particular form of the full amplitude is not an invariant procedure. To put it more simply, the equation (4.54) would be false if we crossed out the term corresponding to the third diagram.

The other closely related aspect of this same phenomenon is that of gauge invariance, which is satisfied by the full amplitude because of the current conservation equations (4.6). We should like the value we get for $\alpha(s)$ to be independent of the gauge used for the $\gamma\gamma$ calculation, which would mean in this way of carrying out the program that the tensor coefficient of the third diagram alone should satisfy (4.6). The original form of the full amplitude derived by Karplus and Neuman had this property. Since our form could have been obtained from theirs by adding multiples of a(1234) and b(1234), it would be surprising if this property were preserved. Indeed, it is not.[27]

With all this arbitrariness in the relative amounts of the third diagram in different forms of the amplitude it seems odd that we should have stumbled on to just the one form which gives the correct imaginary part (4.52). The reason for this is the key to the approach in the next subsection.

b. Omit Part of the Third Diagram

Suppose we write each integral in (4.54) in dispersion form. Because of the asymptotic behavior of these integrals we now require one subtraction each in s, t, and u. Since we are treating each integral separately it is not necessary that the points at which we make the subtractions, s_o, t_o, and u_o, satisfy $s_o + t_o + u_o = 4$. We choose for convenience to make them at $s_o = t_o = u_o = 0$. Calling the first term of (4.48) $a_I(s,t)$, etc. we have

$$a_I(0,0) + \frac{s}{\pi}\int \frac{\mathrm{Im}_s a_I(s',0)}{s'(s'-s)} ds' + \frac{t}{\pi}\int \frac{\mathrm{Im}_t a_I(0,t')}{t'(t'-t)} dt' + \frac{st}{\pi^2}\iint \frac{\mathrm{Im}_s \mathrm{Im}_t a_I(s',t')}{s't'(s'-s)(t'-t)} ds'st'$$

$$+ a_{II}(0,0) + \frac{s}{\pi}\int \frac{\mathrm{Im}_s a_{II}(s',0)}{s'(s'-s)} ds' + \frac{u}{\pi}\int \frac{\mathrm{Im}_u a_{II}(0,u')}{u'(u'-u)} du'$$

$$+ \frac{su}{\pi^2}\iint \frac{\mathrm{Im}_s \mathrm{Im}_u a_{II}(s',u')}{s'u'(s'-s)(u'-u)} ds'du'$$

$$+ a_{III}(0,0) + \frac{u}{\pi}\int \frac{\mathrm{Im}_u a_{III}(u',0)}{u'(u'-u)} du' + \frac{t}{\pi}\int \frac{\mathrm{Im}_t a_{III}(0,t')}{t'(t'-t)} dt'$$

$$+ \frac{ut}{\pi}\iint \frac{\mathrm{Im}_u \mathrm{Im}_t a_{III}(u',t')}{u't'(u'-u)(t'-t)} du'\, dt' = 0 \qquad (4.55)$$

where the single spectral functions are integrated from $4m^2$ to ∞, while the double spectral functions are integrated over the regions situated as described at (4.35). The maximal size of these regions is given by the denominators $D(s,t)$, etc., for they determine the singularities of these particular functions. The numerators, which are polynomials, only serve to make the regions smaller by cancelling out some of these singularities. We see from Fig. 3 that even at their maximal sizes, these regions do not overlap for any choice of masses. Thus the only way for (4.55) to be satisfied is if the double spectral terms all vanish separately. Indeed, the numerators in (4.54) contain just the factor which in the Landau-Eden analysis is set equal to zero to determine the condition for a pinching singularity in the variable y_1, so that the numerators cancel a possible branch point. Then we see how the rest of the cancellations occur. The pieces $\mathrm{Im}_t a_I(0,t')$ and $\mathrm{Im}_t a_{III}(0,t)$ with the right hand t-cut cancel; the pieces $\mathrm{Im}_u a_{II}(0,u')$ and $\mathrm{Im}_u a_{III}(u',0)$ with the left hand t-cut cancel, etc. We have calculated the constants and find that their sum vanishes though the separate terms do not.

The puzzles of the last subsection now become clear. First of all the identity (4.20) contains no double spectral functions so that adding multiples of it to the full amplitude does not change the imaginary part of the t-cut. If discarding the third diagram gives agreement with partial wave unitarity in one form of the full amplitude, it will for all forms. Secondly, we see that a(1234) adds cancelling amounts of right hand t-cut to the first and third diagrams; if we omit the third diagram, our estimate of the amount of this cut is spurious.

To correct this, we might think of adding, to what we have, the contribution of the third diagram to the right hand single spectral functions in t. However, since these single spectral functions in t arise from the necessity of making subtractions in the u variable, they depend on the point at which we make the subtraction. However, there is a unique prescription which accomplishes the same result, and that is to include only that portion of the third diagram which contributes to the real part of the right hand t-cut. This prescription is unique because all the imaginary part of the t-cut comes from the double spectral function, which can be uniquely determined. We can make this more specific by referring to (4.36). Ignoring subtractions, the right hand t-cut contribution to M is

$$A_t^R(s,t) = \frac{1}{\pi}\int_{u_o(t)}^{\infty} \frac{\rho_{ut}(u',t)}{u'+s+t-4} du' \qquad (4.56)$$

whose imaginary part from (4.37) is $-\rho_{ut}(4-s-t,t)$. What we want is

$$\text{Re } A_t^R(s,t) = \frac{1}{\pi}\int_{u_o(t)}^{\infty} \frac{\rho_{ut}(u',t)}{u'+s+t-4} du' + i\rho_{ut}(4-s-t,t) \,. \qquad (4.57)$$

In the case with arbitrary amounts of subtractions in both u and t, it is still true that the amount we must subtract from the contribution of the third diagram to the right hand t-cut is

$$i \,\text{Im } A_t^R(s,t) = -\, i\rho_{ut}(4-s-t,t) \,. \qquad (4.58)$$

It is now simple to show that this prescription gives a gauge invariant result. These manipulations are easy enough when the integrals are in the dispersion form, but when they are in the form of integrals over Feynman parameters the story is quite different. We have been as yet unable to calculate explicitly these additional contributions of the third diagram. We do not know whether the additional contributions will vanish at $s = 0$ so that the trajectory will go through 1 at this point. Nor do we know if the additional contributions will change the sign of the slope of the trajectory given by the contributions of the first diagram. We can, however, discuss the meaning of this new program for determing the trajectory.

We have arrived at this program by considering the properties of the terms which we assign to the vacuum trajectory. Let us consider the properties of the terms we are discarding and get information about the UFO. The imaginary part of the discontinuity in t which exists for negative s is responsible for another phenomenon in Regge pole theory--the Gribov singularity[28]. Looking first at the equal mass spinless case we give an heuristic summary of Oehme's presentation[14]. Calling $A^e(s,t)$ and $A^o(s,t)$ the combinations of the weight functions which appear in (3.5) for $F^e(J)$ and (3.6) for $F^o(J)$, respectively, we write

$$F^{\overset{e}{o}}(J) = \frac{1}{8\pi^2\sqrt{s}} \int_{z_o}^{\infty} dz\, Q_J(z) A^{\overset{e}{o}}(s, 2k^2(z-1)) \,. \tag{4.59}$$

We may express $A^{\overset{e}{o}}$ in terms of the double spectral functions as in (4.35)

$$A^{\overset{e}{o}}(s,v) = \frac{1}{\pi}\int_{u_o(v)}^{\infty} ds' \frac{\rho_{st}(s',v) \pm \rho_{su}(s',v)}{s'-s}$$

$$+ \frac{1}{\pi}\int_{u_o(v)}^{\infty} du' \frac{\rho_{ut}(u',v) \pm \rho_{ut}(v,u')}{u'+s+v-4} \,. \tag{4.60}$$

To figure out what effect the discontinuities in $A^{\overset{e}{o}}$ will have on $F^{\overset{e}{o}}(J)$ we use

$$\mathrm{Im}[\, A^{\overset{e}{o}}(s,v)Q_J(z)] = \mathrm{Im}\; Q_J(z)\cdot \mathrm{Re}\, A^{\overset{e}{o}}(s,v) + \mathrm{Re}\, Q_J(z)\cdot \mathrm{Im}\, A^{\overset{e}{o}}(s,v) \; . \tag{4.61}$$

Combining these equations and paying much attention to which side of the cut one is on, Oehme finds

$$\begin{aligned} \mathrm{Im}[\, 8\pi\sqrt{s}\, F^{\overset{e}{o}}(J) \Big|_{s\leq 0} &= \frac{1}{2}\int_4^{4-s} dv\, \frac{1}{2k^2}\, P_J\left(-1-\frac{v}{2k^2}\right) A^{\overset{e}{o}}(s,v) \\ &+ \frac{1}{\pi}\int dv\, \frac{1}{2k^2}\, Q_J\left(-1-\frac{v}{2k^2}\right)\Big[\rho_{ut}(4-s-v,v)\pm\rho_{ut}(v,4-s-v)\Big], \end{aligned} \tag{4.62}$$

where the limits on the second integral are determined by the boundaries of the double spectral function. The function P_J appears in the first integral because it is proportional to the discontinuity of the Q_J. This first integral is perfectly regular as a function of J. The second integral, which contributes only for values of s sufficiently negative, has poles at the negative integers because of the poles of Q_J there. If we had included subtractions we would have gotten a similar result with the restriction that $\mathrm{Re}\, J > N$, the number of subtractions required, because that is the range of validity of the representation (3.4). Continuing this analytically to the left, Oehme evaluates the residues at the negative integral poles $-n$ to be

$$\mathrm{Im}\left[8\pi\sqrt{s}\,F^{\substack{e\\o}}(J)\right]\Big|_{s<0} = -\frac{1}{J+n}\,\frac{1}{\pi}\int dz\; P_{J-1}(z)\left\{\rho_{tu}(x,z)\pm\rho_{tu}(x,-z)\right\} \quad (4.63)$$

where the limits of integration lie within the range -1 to +1 when s is sufficiently negative to be on the left hand s-cut of $F^{\substack{e\\o}}(J)$. Because of the reflection symmetry of $P_J(z)$ we see that the residue of $F^e(F^o)$ vanishes for n an even (odd) integer. The residue can not vanish for all the integers, however, because the completeness of the P_J would then imply that ρ_{ut} vanishes identically. Thus we get poles at least at some negative odd integers in F^e because of the existence of these third double spectral functions. The principal difference for the case with spin is that the function $C^{J\pm}_{\lambda\mu}$ which replace the Q_J involve a whole range of Q's extending from $Q_{J-\lambda_m}$ to $Q_{J+\lambda_m}$. In our case, for example, where $\lambda_m = 2$, the presence of Q_{J-2} in $C^{J+}_{2\,2}$ means that we reach the pole at Q_{-1} when J is as large as +1.

The UFO must have something to do with this Gribov singularity since they have four characteristics in common: they occur a) for $J = 1$, b) in the amplitude F^e for which this value is unphysical, c) as a result of the third double spectral function, d) in the region of the cut at negative s. However relating the UFO to the Gribov singularity is only putting our black box into another black box with a more popular name, because the Gribov singularity is not well understood. An example of the mystery which surrounds it is the following. The existence of a fixed pole in $F^e(J)$ means that if we write a dispersion relation in s for $F^e(J)$ and consider the contribution from the left hand cut, we would obtain a fixed pole at $J = -1$, for example, at all values of s. But this violates the partial wave unitarity condition (3.40) which must be valid for $4 \leq s \leq 4m^2$. Thus the pole must either be cancelled by other contributions to $F^e(J)$ or else there must be some reason why we can not continue our functions to this point.

Gribov and Pomeranchuk[29] conjecture a mechanism to cancel the fixed pole at $J = -1$ in the scalar case. They imagine that as we continue $F^e(J)$ from $\mathrm{Re}\, J > N$ down toward the pole at $J = -1$, more and more Regge poles $s = s_i(J)$ appear on the physical sheet of the s-plane coming out of the branch point at $s = 4$. Then, just looking at the negative cut and these poles,

$$8\pi\sqrt{s}\, F^e(J) = \frac{1}{\pi}\int_{-\infty}^{-s_o} \frac{\mathrm{Im}_s\, 8\pi\sqrt{s}\, F^e(J)}{s'-s}\, ds' + \sum_i \frac{\beta_i(J)}{s_i(J)-s}\,, \qquad (4.64)$$

where the upper limit $-s_o$ represents the beginning of the left hand s-cut. Then they imagine that in the neighborhood of $J = -1$, these Regge poles become as dense as you please, so that in the limit they simulate a cut at that point which exactly cancels the contribution of the pole to the left hand cut:

$$\lim_{J\to -1} \sum_i \frac{(J+1)\beta_i(J)}{s_i(J)-s} = \frac{1}{\pi}\int_{-\infty}^{-s_o} \frac{R(s')}{s'-s}\, ds' \qquad (4.65)$$

where R is the residue of the pole at $J = 1$ given in (4.63). The difference between this and our case with spin is that the speculation begins when J becomes less than $N + \lambda_m$. For $F^e_{1-1;1-1}$ we have $N = 0$, $\lambda_m = 2$ and the poles must start to emerge as we enter the nonsense region.

Sullivan discarded the entire third diagram on the grounds that it was the Gribov singularity. We see now that this was discarding too much, and that the minimum amount we may discard and claim we are free from the Gribov singularity is given by the prescription of this section.

There are several interesting questions which remain to be answered. The UFO arises so naturally from the structure of the diagrams that one wonders if it is in fact connected to the ordinary vacuum trajectory in some simple way. One could imagine that it is the crossed version of what in some other channel would look very simple. There are some indications, in fact, from the scalar case, that the chain of crossed diagrams may give a moving singularity in the same way as the ladder diagrams give the usual Regge trajectory.[30] If this should turn out to be the case we would have an alternate explanation of how the unitary relations are maintained.

VI. CONCLUDING REMARKS

Let us briefly summarize the results of the investigations of the preceding pages, make some comments to underline their significance, and suggest some areas of future inquiry. We have confirmed the results of the fourth order calculation by Karplus and Neuman of the scattering of neutral vector particles as a result of a conserved current interaction with a fermion pair field. We have also given a very simple derivation of two identities which appear in their work, and made a distinction between those symmetries which are valid for the general process and those which are true only in the fourth order case. As a result, the previous objections to the number of gauge invariant tensors used in their calculation have been resolved. An equivalent solution of the same general problem which is free of kinematical singularities has been obtained and evaluated for the particular case of the nonsense-nonsense transition. We have further cast the particular result into a form which may be written as a dispersion integral in t with no subtractions, and a scheme for separating the right and left hand t cuts in the contribution to the full amplitude of the third diagram (Fig. 2c) has been validated. Although the fourth order contributions naturally behave as $t^{-1} \log t$ for large t, the asymptotic form of the right hand t-cut in our case was shown to be t^{-1} from each of the diagrams, confirming the prediction of the vacuum trajectory hypothesis.

A number of assumptions concerning the analytic properties of the partial wave amplitudes had been made in the Reggeization procedure[3] which led to the conclusion that the vacuum pole dominated the asymptotic behavior of all the relevant processes. From the appearance of a left hand s-cut in the asymptotic form of the right hand t-cut, we conclude that another singularity, the UFO, is present and hence that not all of these assumptions can be valid. In reformulating these assumptions a posteriori we must make room for the observed phenomenon, although it is not clear how this should be done. We are in the position of having to espouse a doctrine of "maximum analyticity,"[31] together with all its connotations of vagueness and expediency.

A separate calculation of the imaginary part of the forward scattering amplitude in the cross channel reveals that a trajectory based on both right and left hand s-cuts does not pass through 1 at $s = 0$. The trajectory calculated by omitting the third diagram does have $\alpha(0) = 1$, and further, fulfills the condition on its imaginary part predicted by unitarity, but has the unphysical property that $\alpha'(0)$ is negative. However, we show that simply omitting the third diagram is an ambiguous, gauge dependent procedure, although in our helicity case, first converting the amplitude into a form which has no subtractions in t somewhat mitigates this objection.

We conclude that the correct prescription is to retain only that part of the third diagram which contributes to the real part of the right hand t-cut. This prescription is gauge invariant, and depends neither on the subtractions in a dispersion relation nor onto which of the many forms of the full $\gamma\gamma$ amplitude it is applied. Further, the trajectory it generates will always satisfy the unitarity requirement on its imaginary part. Other trajectory parameters must await a difficult calculation before their properties are determined. For the scattering of scalar particles the difficulty of extracting even the asymptotic form of integrals arising from the third diagram is documented in the literature. The situation is worse when the particles have spin. Moreover, in our case the prescription calls not only for a separation of the right and left t-cuts, but a further extraction of the s-cut before the asymptotic form may be taken. While attempting this project we can hope that the additional contributions to $T(s)$, Eq. (4.47), will be sufficiently negative to make the sum negative for $0 \leq s \leq 4\lambda^2$; in this way $\alpha'(0)$ will become physically sensible. The same prescription should be applied to the recent calculation of Sullivan[25] of the vector meson scalar nucleon case in order to cure the same ills.

The discarded portion of the amplitude contains the UFO; further investigation shows that the same portion should contain the Gribov singularity. The identification of the UFO with the Gribov singularity is strongly suggested, especially since both appear in the same channel at the same point. This singularity is not well-understood even in the spinless case where it was first found. More work is needed on this question and particularly on the role of the non-planar diagrams in producing it.

The larger question remains, however, of how this singularity should be regarded in the search for the vacuum Regge trajectory. Is it to be regarded, as we have done for the most part here, as an object which interferes with the "pure" vacuum trajectory and where the largest problem is how to separate out its contribution? Or is it to be regarded as a necessary part of the same mechanism which produces what we have chosen to idealize as the "pure" trajectory? In the latter case, it would fulfill some role, such as providing for the crossing symmetry of the trajectory, which has yet to be discovered. Looking at the situation from a purely practical point of view, both singularities contribute to the asymptotic form of the amplitude, and hence it would be a combination of both which would be measured in any experiment. Perhaps it would be best to say, at this point in our understanding, that there seems indeed to be a vacuum trajectory, but in addition, there are other phenomena in the angular-momentum plane which mask the simple properties which the vacuum trajectory was supposed to display.

Appendix A

FIELD THEORY AND CONVENTIONS

The following brief remarks characterize the field theory with which we are working. Further details may be found in Reference (20) The diagonal metric is $g_{\mu\nu}$, where

$$g_{00} = - g_{11} = - g_{22} = - g_{33} = 1 \tag{A1}$$

and we use the summation convention for repeated indices so that, if $p_\mu = (E, \vec{p})$ is the four-momentum of a nucleon of mass m, we have

$$p_\mu p_\mu = E^2 - \vec{p}^{\,2} = m^2 . \tag{A2}$$

The free Lagrangian for the nucleon in units where $\hbar = c = 1$ is

$$\mathcal{L}_N = - \frac{1}{2}\overline{\psi}(-i\gamma_\mu \partial_\mu + m)\psi - \frac{1}{2}(i\partial_\mu \overline{\psi}\gamma_\mu + m\overline{\psi})\psi \tag{A3}$$

so that using the notation $\not{a} = a_\mu \gamma_\mu$, the Euler-Lagrange equations become

$$i\not{\partial}\psi - m\psi = (\not{p} - m)\psi = 0 \tag{A4}$$

$$-i\overline{\psi}\not{\partial} - m\overline{\psi} = \overline{\psi}(\not{p} + m) = 0 . \tag{A5}$$

The only properties of the spinors which we use is the choice of normalization

$$\overline{u}(p)u(p) = 2m \tag{A6}$$

$$\overline{v}(p)v(p) = -2m \tag{A7}$$

where $u(p)$ are positive energy spinors and $v(p)$ are negative energy ones.

For the neutral vector meson, henceforth called γ, the free Lagrangian is

$$\mathcal{L}_\gamma = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) - \frac{1}{2}\lambda^2 A_\mu A_\mu \tag{A8}$$

where λ is the particle mass. The free-field equations are then

$$\Box A_\mu - \lambda^2 A_\mu = 0 \tag{A9}$$

subject to the condition

$$\partial_\mu A_\mu = 0 \tag{A10}$$

which we regard as an operator identity provided $\lambda \neq 0$. This condition means that there are only three independent fields $A_\mu(x)$ rather than the nominal four. In the decomposition of the $A_\mu(x)$ into plane waves, there are, for each momentum, only three independent solutions to the field equation; we take these to be labeled by their polarization vectors $\epsilon^i_\mu(\vec{k})$. To insure that the subsidiary condition is fulfilled, we take

$$\epsilon^i_\mu(\vec{k})k_\mu = 0 \qquad i = -1, 0, 1 \tag{A11}$$

for each of the three polarization states. These states also satisfy the completeness relation

$$\sum_{i=-1}^{1} \epsilon_{\mu}^{i}(\vec{k})\epsilon_{\nu}^{i*}(\vec{k}) = -g_{\mu\nu} + \frac{k_{\mu}k_{\nu}}{\lambda^2} . \tag{A12}$$

We normalize these states by

$$\epsilon_{\mu}^{i}(\vec{k})\,\epsilon_{\mu}^{j*}(\vec{k}) = -\delta_{ij} \tag{A13}$$

and further choose them to be helicity eigenstates. If the γ momentum is in the z-direction so that $k_{\mu} = (\omega, 0, 0, k)$ the set

$$\begin{aligned} &\text{helicity } +1 \qquad \epsilon_{\mu}^{1} = \frac{1}{\sqrt{2}}(0,1,i,0) \\ &\text{helicity } \;\; 0 \qquad \epsilon_{\mu}^{0} = \frac{1}{\lambda}(k,0,0,\omega) \\ &\text{helicity } -1 \qquad \epsilon_{\mu}^{-1} = \frac{1}{\sqrt{2}}(0,1,-i,0) \end{aligned} \tag{A14}$$

satisfies all the requirements. The violation of all the accepted phase conventions is not serious; the problems we treat have always an even number of helicities of each type. The helicities for momenta in different directions are obtained by rotation.

To obtain a conserved current coupling we make the replacements

$$\begin{aligned} \partial_{\mu}\psi &\rightarrow (\partial_{\mu} + igA_{\mu})\psi \\ \partial_{\mu}\overline{\psi} &\rightarrow (\partial_{\mu} - igA_{\mu})\overline{\psi} \end{aligned} \tag{A15}$$

in the nucleon Lagrangian (A3), and add the free γ Lagrangian (A8) to obtain for the total

$$\mathcal{L} = \mathcal{L}_N + \mathcal{L}_\gamma + \mathcal{L}_I \tag{A16}$$

where

$$\mathcal{L}_I = - g\bar{\psi}\gamma_\mu \psi A_\mu \ . \tag{A17}$$

The coefficient of A_μ in (A17) has the same form as the four-current for the free Dirac field, except of course that the ψ and $\bar{\psi}$ here obey not the sourceless equations (A4) and (A5) but the Euler-Lagrange equations derived from the complete Lagrangian. However it is still true that the current in the presence of the γ field is conserved.

If we were to replace g, the coupling constant by e, the charge, then the above formalism would become the same as that for conventional quantum electrodynamics, except for the free γ equation. Thus, the entire machinery of quantum electrodynamics may be taken over, except for the use of a slightly different propagator for the internal γ lines. Our work in fourth order does not involve any internal γ lines.

In discussing the scattering of two particles, $1 + 2 \rightarrow 3 + 4$, we find it convenient to introduce the invariant scattering matrix element M, related to the S matrix by

$$S = 1 + \frac{i(2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)}{4\sqrt{E_1 E_2 E_3 E_4}} M \tag{A18}$$

where the E_i represent the energies of the particles and the p_i represent their four-momenta. If we introduce the Mandelstam variables s and t as in (4.12) then $M(s,t)$ has two important properties which the scattering amplitude does not have. It is $M(s,t)$ which can be taken to represent the scattering in all channels as the ranges of s and t are varied. This can be demonstrated explicitly for the general fourth order $\gamma\gamma$ scattering amplitude in Appendix E. It is also $M(s,t)$ rather than the scattering amplitude which obeys the Mandelstam representation (4.35) (at least in fourth order).

Carrying out the integrations over the final particle momenta we find for the scattering amplitude

$$f = \frac{M}{8\pi\sqrt{s}} \tag{A19}$$

and f is related to the cross section by (2.9).

Appendix B

THE FUNCTIONS $e^{J\pm}_{\lambda\mu}(z)$ AND $c^{J\pm}_{\lambda\mu}(z)$

The relations below are taken from Appendix A of Reference (3). All of the properties of the e and c functions are derivable from the properties of the Wigner functions $d^J_{\lambda\mu}(z)$ available in Reference (13).

As in Chapter II we take

$$\lambda_m = \mathrm{Max}\,[\,|\lambda|, |\mu|\,] \;. \tag{B1}$$

Then the following formulas hold both for the e's and the c's:

$$e^{J\pm}_{-\mu,-\lambda}(z) = e^{J\pm}_{\lambda\mu}(z) \tag{B2}$$

$$e^{J\pm}_{\mu\lambda}(z) = (-1)^{\lambda-\mu} e^{J\pm}_{\lambda\mu}(z) \tag{B3}$$

$$e^{J\pm}_{\lambda,-\mu}(z) = \pm\,(-1)^{\lambda+\lambda_m} e^{J\pm}_{\lambda\mu}(z) \tag{B4}$$

$$e^{J\pm}_{\lambda\mu}(-z) = \pm(-1)^{J-\lambda_m} e^{J\pm}_{\lambda\mu}(z) \tag{B5}$$

$$e^{J}_{\lambda\mu}(z) = e^{J+}_{\lambda\mu}(z) + e^{J-}_{\lambda\mu}(z) \;. \tag{B6}$$

Although general recursion formulas for the e's and c's exist, we give only those functions used in the text. We have dropped the arguments z of all the functions and denote differentiation with respect to z by primes.

$$e_{00}^{J+} = P_J \; ; \qquad e_{00}^{J-} = 0$$

$$-e_{10}^{J+} = e_{01}^{J+} = \frac{P_J'}{\sqrt{J(J+1)}} \; ; \qquad e_{10}^{J-} = e_{01}^{J-} = 0$$

$$e_{11}^{J+} = \frac{P_J' + zP_J''}{J(J+1)} \; ; \qquad e_{11}^{J-} = \frac{-P_J''}{J(J+1)}$$

$$e_{02}^{J+} = \frac{P_J''}{\sqrt{(J-1)J(J+1)(J+2)}} \; ; \qquad e_{02}^{J-} = 0 \tag{B7}$$

$$e_{12}^{J+} = \frac{2P_J'' + zP_J'''}{J(J+1)\sqrt{(J-1)(J+2)}} \; ; \qquad e_{12}^{J-} = \frac{-P_J'''}{J(J+1)\sqrt{(J-1)(J+2)}}$$

$$e_{22}^{J+} = \frac{2P_J'' + 4zP_J''' + (z^2+1)P_J^{iv}}{(J-1)J(J+1)(J+2)} \; ; \qquad e_{22}^{J-} = \frac{-4P_J''' - 2zP_J^{iv}}{(J-1)J(J+1)(J+2)} \, .$$

$$c_{00}^{J+} = P_J \; ; \qquad c_{00}^{J-} = 0$$

$$-c_{10}^{J+} = c_{01}^{J+} = \frac{\sqrt{J(J+1)}}{2J+1}\left(P_{J-1} - P_{J+1}\right); \; c_{10}^{J-} = c_{01}^{J-} = 0$$

$$c_{11}^{J+} = \frac{(J+1)P_{J-1} + JP_{J+1}}{2J+1} \; ; \qquad c_{11}^{J-} = P_J$$

$$c_{02}^{J+} = \frac{\sqrt{(J-1)J(J+1)(J+2)}}{(2J-1)(2J+1)(2J+3)}\left[(2J+3)P_{J-2} - 2(2J+1)P_J + (2J-1)P_{J+2}\right];$$

$$c_{02}^{J-} = 0 \tag{B8}$$

$$c_{12}^{J+} = \frac{\sqrt{(J-1)(J+2)}}{(2J-1)(2J+1)(2J+3)}\left[(J+1)(2J+3)P_{J-2} - 3(2J+1)P_J - J(2J-1)P_{J+2}\right];$$

$$c_{12}^{J-} = \frac{\sqrt{(J-1)(J+2)}}{2J+1}\left[P_{J-1} - P_{J+1}\right]$$

$$c_{22}^{J+} = \frac{(J+1)(J+2)(2J+3)P_{J-2} + 6(J-1)(J+2)(2J+1)P_J + (J-1)J(2J-1)P_{J+2}}{(2J-1)(2J+1)(2J+3)};$$

$$c_{22}^{J-} = \frac{2(J+2)P_{J-1} + 2(J-1)P_{J+1}}{2J+1} .$$

Appendix C

SOME REMARKS CONCERNING INDEPENDENT INVARIANTS

In the $\gamma\gamma$ scattering problem considered in the text, all the information is contained in the fourth rank tensor $G_{\mu\nu\lambda\sigma}(1234)$ defined in (4.5). Ignoring for the moment the fact that $G_{\mu\nu\lambda\sigma}$ obeys the current conservation equations (4.6) we discuss the decomposition of a general tensor into independent tensor forms. The only available pieces from which these independent tensor forms may be constructed are the dynamical variables $k_\mu^{(i)}$, and the invariant tensors $g_{\mu\nu}$ and $\epsilon_{\mu\nu\lambda\sigma}$. Since the four $k_\mu^{(i)}$ are related by momentum conservation (4.3) we save time by considering only three of them. Further, we ignore the possibility that these three momenta in a particular degenerate case may not be independent. In general, if $b^j(1234)$, $j = 1,2,\ldots,N$ are a set of scalar functions of the momenta and $H^j_{\mu\nu\lambda\sigma\ldots}$ are a set of tensor functions of the momenta and the invariant tensors, we call the $H^j_{\mu\nu\lambda\sigma\ldots}$ independent tensor forms if

$$\sum_{j=1}^{N} b^j(1234)H^j_{\mu\nu\lambda\sigma\ldots} = 0 \quad \text{implies} \quad b^j(1234) \equiv 0 \,. \tag{C1}$$

For the case of first rank tensors we choose arbitrarily 1_μ, 2_μ and 3_μ to be the dynamical variables. Further defining

$$R_\mu \equiv \epsilon_{\mu\nu\lambda\sigma} 1_\nu 2_\lambda 3_\sigma \tag{C2}$$

we note that because of the antisymmetry of the indices in $\epsilon_{\mu\nu\lambda\sigma}$,

R_μ is orthogonal to the other three. Moreover

$$R_\mu = \epsilon_{\mu\nu\lambda\sigma} 2_\nu 1_\lambda 4_\sigma = \epsilon_{\mu\nu\lambda\sigma} 1_\nu 3_\lambda 4_\sigma = \epsilon_{\mu\nu\lambda\sigma} 4_\nu 3_\lambda 2_\sigma \tag{C3}$$

so that we may eliminate any vector we choose in accordance with (4.10). The vectors 2_μ, 3_μ, 4_μ, together with the pseudovector R_μ span the space of four-vectors; ignoring reflection properties, the four independent components of any four-vector can be obtained by a suitable linear combination of the four primitive four-vectors.

An arbitrary second rank tensor which contains sixteen independent components may be decomposed into a linear combination of the sixteen independent tensor forms obtained from the outer products of the four primitive vectors. Each tensor index contributes a factor of four in counting the possibilities. This decomposition is true even for the metric tensor. Indeed we have the identity

$$\epsilon_{\mu\alpha\beta\gamma}\epsilon_{\nu\rho\sigma\tau} \equiv - \begin{vmatrix} g_{\mu\nu} & g_{\mu\rho} & g_{\mu\sigma} & g_{\mu\tau} \\ g_{\alpha\nu} & g_{\alpha\rho} & g_{\alpha\sigma} & g_{\alpha\tau} \\ g_{\beta\nu} & g_{\beta\rho} & g_{\beta\sigma} & g_{\beta\tau} \\ g_{\gamma\nu} & g_{\gamma\rho} & g_{\gamma\sigma} & g_{\gamma\tau} \end{vmatrix} \tag{C4}$$

which contains the special case

$$R_\mu R_\nu = 1_\mu 1_\nu [2^2 3^2 - (2\cdot 3)^2] + 2_\mu 2_\nu [3^2 1^2 - (3\cdot 1)^2] + 3_\mu 3_\nu [1^2 2^2 - (1\cdot 2)^2]$$
$$+ (1_\mu 2_\nu + 2_\mu 1_\nu)[(2\cdot 3)(3\cdot 1) - (1\cdot 2)3^2] +$$
$$+ (2_\mu 3_\nu + 3_\mu 2_\nu)[(3\cdot 1)(1\cdot 2) - (2\cdot 3)1^2]$$
$$+ (3_\mu 1_\nu + 1_\mu 3_\nu)[(1\cdot 2)(2\cdot 3) - (3\cdot 1)2^2]$$
$$+ g_{\mu\nu}[-1^2 2^2 3^2 + (1\cdot 2)^2 3^2 + (2\cdot 3)^2 1^2 + (3\cdot 1)^2 2^2 - 2(1\cdot 2)(2\cdot 3)(3\cdot 1)] , \tag{C5}$$

so that $g_{\mu\nu}$ may be expressed in terms of the 10 tensor forms which are invariant under reflection.

Fourth rank tensors contain 256 independent components, and we may classify the 256 independent tensor forms according to whether they are even or odd under reflection. The ones containing an odd number of indices borne by the pseudovector R have odd symmetry. Table CI lists the possibilities.

Table CI

PARITY CLASSIFICATION OF 4th RANK TENSORS

Number of R's	Number of Tensors: Even	Number of Tensors: Odd	
0	81	--	
1	--	108	
2	54	--	
3	--	12	
4	1	--	
Total	136	120	256

In the decomposition of $G_{\mu\nu\lambda\sigma}(1234)$ in (4.9) only even tensors have been included, but still there are 81 A's, 54 B's and 3 C's for a total of 138. The two extra tensor forms in the decomposition (4.9) correspond to the existence of two identities among these tensor forms. These identities may be obtained from the observations

$$(R_\mu R_\nu)(R_\lambda R_\sigma) = (R_\mu R_\lambda)(R_\nu R_\sigma) = (R_\mu R_\sigma)(R_\nu R_\lambda) \tag{C6}$$

by substituting for the indicated pairings the identity (C4), first choosing the momenta appearing in each R according to the convention (4.10) by using the freedom generated by (C3). Thus any two of the tensor forms appearing in (4.8) may be eliminated to give a decomposition with uniquely determined coefficients.

Turning to the case of a fourth rank tensor which satisfies the current conservation equations (4.6) we observe that for each index we have not a choice of four four-vectors in the construction of independent tensor forms, but only three. When considering, for example the first index μ, we must choose only vectors orthogonal to 1_μ in order that the contraction with 1_μ vanishes. Since R_μ is already orthogonal to 1_μ it remains only to construct from the vectors 2_μ, 3_μ, and 4_μ two suitable linear combinations which are orthogonal to 1_μ. Call these constructions q_μ and r_μ so that we now have available for the first index the vectors q_μ, r_μ and R_μ. Further imagine performing the analogous construction q_ν and r_ν for the second index ν, in such a way that the operation $1 \rightarrow 2$ induces $q_\mu \rightarrow q_\nu$ and $r_\mu \rightarrow r_\nu$. Of course such an operation does not affect R_μ so that

$R_\mu \rightarrow R_\nu$ trivially. Making similar definitions for each of the four tensor indices we can construct 3^4 or 81 different independent conserving fourth rank tensors. According to the number of R's present they will have even or odd reflection symmetry as in table C II.

Table CII

PARITY CLASSIFICATION OF CURRENT CONSERVING TENSORS

Number of R's		Number of Tensors Even	Odd	
0		16	--	
1		--	32	
2		24	--	
3		--	8	
4		1	--	
	Total	41	40	81

The totals in Table CII have a simple interpretation if amplitudes are counted in the helicity scheme. Suppose we have four distinguishable γ's so that we cannot invoke either time reversal symmetry or identical particle symmetry. Then we have nine initial states and nine final states for a total of 81 transitions. If parity eigenstates are introduced as in (2.4) applied to the raw states then initially there are five states with parity $(-1)^J$ and four with parity $-(-1)^J$. The odd one arises because only one parity eigenstate may be formed from $|JM;00\rangle$. The same distribution holds for the final states. In a vector current interaction we have five states into five states plus four states into four states for a total of 41 transitions. With a pseudovector current interaction we would have five states into four states plus four states into five states for a total of 40 transitions.

Suppose now we consider the elastic scattering of two distinguishable γ's with a vector current interaction. Time reversal symmetry now gives us relations between the transitions so that from the 5 by 5 matrix we have only 15 independent transitions, and in the 4 by 4, only 10 for a total of 25 transitions.

We can check this by looking at the explicit form of the constructed amplitudes. If we write the full scattering amplitude as a linear combination of the independent tensors, the fact that the amplitude is invariant under certain simultaneous permutations of the momentum arguments and corresponding tensor indices means that under the same permutation each of the tensors must go into a linear combination of the others. The square transformation matrix which accomplishes this must be symmetric so that it can be diagonalized by a suitable choice of combinations of the independent tensors. For this choice of tensors each one must be invariant under the relevent permutation operation. Now write symbolically the tensors we have constructed as if they were this diagonalizing set of tensors. Omitting for brevity the subscripts $\mu, \nu, \lambda, \sigma$ in that order, the invariance under time reversal implies the relation abcd = cdab for each of the tensors, where a, b, c, and d stand for any of q, r, or R. This is because the position of each of these vectors in the tensor determines which polarization vector it gets dotted into and for any interchange of momenta the q's go into q's, the r's into r's and the R's into R's. The following combinations exhaust the 41 available independent parity conserving tensors.

qqqq, (qqqr + qrqq), (rqqq + qqrq), (qqrr + rrqq),
(qrrq + rqqr), qrqr, rqrq, (qrrr + rrqr),
(rqrr + rrrq), rrrr (= 10)

(RRqq + qqRR), RqRq, qRqR, (RqqR + qRRq),
(RRrq + rqRR), (RrRq + RqRr), (RrqR + qRRr),
(rRRq + RqrR), (rRqR + qRrR), (RRqr + qrRR),
RrRr, rRrR, (RRrr + rrRR), (RrrR + rRRr) (= 14)

RRRR (= 1)

This gives a total of 25. Similarly, it is simple to see that the axial vector part gives 20 amplitudes as it should; the presence of an odd number of R's insures that each of the 40 original tensors must be paired.

Appendix D

PROPERTIES OF THE FUNCTIONS $A^{ijk\ell}(1234)$ AND $A_1^{ijk\ell}(1234)$

The largest source of relations among the 81 functions $A^{ijk\ell}(1234)$ is the notational invariance of the $G_{\mu\nu\lambda\sigma}(1234)$ in (4.9). The effect of the simultaneous permutation of the arguments and indices of $G_{\mu\nu\lambda\sigma}(1234)$ is to produce equality of the heads under similar operations. Thus, starting with $A^{2143}(1234)$ and interchanging the second and third arguments we generate the relation $A^{2143}(1234) = A^{3412}(1324)$, where we have been careful in the superscript not only to interchange the numbers 2 and 3, but also the numbers appearing in the second and third positions. Note that the fact that the second and third arguments of the functions are in this case actually called 2 and 3 is irrelevant; for the purpose of specifying which permutation we consider, the arguments could just as well be called (abcd). If we apply all 24 permutations of the argument to the head $A^{2143}(1234)$ we generate relations among only three of the heads,

$$A^{2143}(1234) = A^{3412}(1324) = A^{4321}(2341) , \tag{D1}$$

since most of the permutations produce no change in the superscripts. Thus the 81 heads group themselves into families all the members of which can be generated if the functional form of any one of them is known. Karplus and Neuman[21] give a complete list which contains three typographical errors: $A^{4411}(3124)$ of their Eq. (19b) should read $A^{4411}(2431)$, $A^{3422}(4132)$ of (19c) should read $A^{3442}(4132)$, and $A^{3342}(4321)$ of (19d) should read $A^{3342}(4312)$. Instead of reproducing the list we give a typical member of each of the six groups and indicate the size of the family.

Group	Typical Head	Number in Group
I	A^{2143}	3
II	A^{2341}	6
III	A^{2111}	12
IV	A^{2121}	12
V	A^{2311}	24
VI	A^{2123}	24

A similar table constructed for the $A_1^{ijk\ell}(1234)$ of (4.16) would require 15 entries, since only the eight permutations which leave the trace unaltered may be used to generate equalities among them.

Another source of equalities among the heads in the general case is current conservation (4.6). It is simple to prove that in order for the third rank tensor obtained by any of the contractions (4.6) to vanish identically, the coefficient of each tensor form in (4.9) must vanish separately. By contracting with 4_σ and setting the coefficient of $2_\mu 1_\nu 2_\lambda$ equal to zero we obtain

$$(4\cdot 1)A^{2121}(1234)+(4\cdot 2)A^{2122}(1234)+(4.3)A^{2123}(1234)=a(1234)=0 \tag{D2}$$

which is (4.20). By contracting (4.9) with 4_σ and setting the coefficient of $2_\mu 3_\nu 1_\lambda$ equal to zero, we get

$$(4\cdot 1)A^{2311}+(4\cdot 2)A^{2312}(1234)+(4\cdot 3)A^{2313}(1234) = 0 \tag{D3}$$

from which, using the symmetries from group V of the table we deduce

$$(4\cdot 1)A^{2311}(1234)+(4\cdot 2)A^{2311}(2314)+(4\cdot 3)A^{2311}(3124) = b(1234) = 0 \tag{D4}$$

which is (4.21).

Applying the procedure indicated in (4.8), the six typical heads may be expressed in terms of a basic set of 15 A_1's as follows.

$$A^{2143}(1234) = A_1^{2143}(1234) + A_1^{2143}(1243) + A_1^{3412}(1324) \tag{D5}$$

$$A^{2341}(1234) = A_1^{2341}(1234) + A_1^{2413}(1243) + A_1^{2413}(4132) \tag{D6}$$

$$A^{2111}(1234) = A_1^{2111}(1234) + A_1^{2111}(1243) + A_1^{3111}(1324) \tag{D7}$$

$$A^{2121}(1234) = A_1^{2121}(1234) + A_1^{2112}(1243) + A_1^{3311}(1324) \tag{D8}$$

$$A^{2311}(1234) = A_1^{2311}(1234) + A_1^{2411}(1243) + A_1^{3121}(1324) \tag{D9}$$

$$A^{2123}(1234) = A_1^{2113}(2134) + A_1^{2141}(2143) + A_1^{3112}(2314). \tag{D10}$$

Here we have followed Karplus and Neuman in making occasional use of the permutation symmetries for the sake of convenience.

Note that the symmetries obtained up to now are perfectly general and apply even to the four γ vertex in any order of perturbation theory. In the general fourth order case the necessary A_1's are of the form

$$A_1^{ijk\ell}(1234) = \int \frac{N^{ijk\ell}(y_1, y_2, y_3, y_4)}{D(1234)} d\tau \tag{D11}$$

in the notation of (4.13) and (4.15). The numerators of the integrals are given by Karplus and Neuman:

$$N^{2143} = 32 y_1 y_3 (1-y_1)(1-y_3)$$

$$N^{3412} = 32(y_1+y_2)(y_2+y_3)(y_3+y_4)(y_4+y_1)$$

$$N^{2341} = 4\{[(1-y_4)(1-y_3)+y_3y_4][(1-y_1)(1-y_2)+y_1y_2] + [(1-y_4)y_3+(1-y_3)y_4][(1-y_2)y_1+(1-y_1)y_2]\}$$

$$N^{2413} = -4\{[(y_2+y_3)(y_3+y_4)+(y_1+y_2)(y_1+y_4)][y_1y_3+(1-y_1)(1-y_3)] + [(y_1+y_4)(y_3+y_4)+(y_1+y_2)(y_2+y_3)][y_1(1-y_3)+y_3(1-y_1)]\}$$

$$N^{2111} = 8y_1(y_2+y_3+y_4)(y_1+y_2-y_3-y_4)(y_1+y_2+y_3-y_4)$$

$$N^{3111} = -8(y_1+y_2)(y_3+y_4)(y_2+y_3+y_4-y_1)(y_1+y_2+y_3-y_4)$$

$$N^{2121} = -8y_1(y_2+y_3+y_4)(y_1+y_3+y_4-y_2)(y_1+y_2+y_3-y_4)$$

$$N^{2112} = 8y_1(y_2+y_3+y_4)(y_1+y_2-y_3-y_4)(y_2+y_3-y_1-y_4) \tag{D12}$$

$$N^{3311} = 8(y_1+y_2)(y_3+y_4)(y_1+y_3+y_4-y_2)(y_1+y_2+y_3-y_4)$$

$$N^{2311} = - N^{3121} = 4(y_1+y_2+y_3-y_4)\{(y_1+y_2)[(y_1y_2+(1-y_1)(1-y_2)] + (y_3+y_4)[y_1(1-y_2)+y_2(1-y_1)]\}$$

$$N^{2411} = 4(y_1+y_2-y_3-y_4)\{(y_1+y_4)[y_1y_4+(1-y_1)(1-y_4)] + (y_2+y_3)[y_4(1-y_1)+y_1(1-y_4)]\}$$

$$N^{2113} = -8y_1(y_2+y_3+y_4)(y_1+y_2-y_3-y_4)(y_1+y_2+y_4-y_3)$$

$$N^{2141} = 8y_1(y_2+y_3+y_4)(y_1+y_2+y_3-y_4)(y_1+y_2+y_4-y_3)$$

$$N^{3112} = -8(y_1+y_2)(y_3+y_4)(y_2+y_3-y_1-y_4)(y_2+y_3+y_4-y_1) \quad .$$

Two additional relations among the A's may now be verified:

$$A^{2311}(1234) = - A^{3121}(1234) \tag{D13}$$

$$A^{2341}(1234) = A^{4123}(1234) \quad . \tag{D14}$$

If we specialize further to our case of equal masses for all the γ's, then the functions D(1234) simplify as in (4.24) with the symmetries:

$$
\begin{aligned}
D(1234) &= D(2143) = D(3412) = D(4321) = D(s,t) \\
D(1432) &= D(2341) = D(3214) = D(4123) = D(t,s) \\
D(1243) &= D(2134) = D(3421) = D(4312) = D(s,u) \\
D(1342) &= D(2431) = D(3124) = D(4213) = D(u,s) \\
D(1324) &= D(2413) = D(3142) = D(4231) = D(u,t) \\
D(1423) &= D(2314) = D(3241) = D(4132) = D(t,u) \, .
\end{aligned} \tag{D15}
$$

Here, as in (4.24)

$$
D(s,t) = [\, m^2 - (y_1y_2 - y_2y_3 + y_3y_4 + y_4y_1) - sy_2y_4 - ty_1y_3]^2 \tag{4.24}
$$

Note that an integral with denominator $D(t,s)$ can be converted to one with denominator $D(s,t)$ by making the interchanges $y_1 \longleftrightarrow y_2$ and $y_3 \longleftrightarrow y_4$. in the integrand, and similarly for the two other pairs of denominator functions.

The relations (D15) also imply further equalities among the A's. Summarizing all the equalities which exist among the heads with the same argument, we have for the equal mass case

Group II $\quad A^{2341} = A^{4123}$; $A^{2413} = A^{3142}$; $A^{4312} = A^{3421}$;

Group III $\quad A^{2111} = A^{2122} = A^{3343} = A^{4443}$; $A^{4111} = A^{2322} = A^{3323} = A^{4441}$;
$A^{2422} = A^{3111} = A^{4442} = A^{3313}$;

Group IV $\quad A^{2121} = A^{4343}$; $A^{4411} = A^{3322}$; $A^{3311} = A^{4422}$; $A^{4141} = A^{2323}$;
$A^{2112} = A^{3443}$; $A^{3113} = A^{2442}$; (D16)

Group V $A^{2311} = A^{4122} = A^{3341} = A^{4423} = -A^{3121} = -A^{2421} = -A^{4342} = -A^{4313}$;

$A^{3123} = A^{2342} = A^{4113} = A^{2441} = -A^{2313} = -A^{3141} = -A^{2423} = -A^{4142}$;

$A^{3122} = A^{2411} = A^{4413} = A^{3342} = -A^{2312} = -A^{4112} = -A^{3441} = -A^{3423}$;

Group VI $A^{2123} = A^{2141} = A^{4143} = A^{2343}$; $A^{4121} = A^{2321} = A^{4323} = A^{4341}$;

$A^{3312} = A^{4412} = A^{3411} = A^{3422}$; $A^{4311} = A^{4322} = A^{3321} = A^{4421}$;

$A^{2142} = A^{2113} = A^{2443} = A^{3143}$; $A^{3112} = A^{2412} = A^{3413} = A^{3442}$.

The same relations hold for the A_1's.

Appendix E

THE FULL $\gamma + \gamma \rightarrow \gamma + \gamma$ AMPLITUDE

We give the complete form of the $\gamma + \gamma \rightarrow \gamma + \gamma$ amplitude despite its length. The expression below is extremely general, applying to the generalized four γ interaction even when it is an internal part of a more complicated diagram, i.e., the particles need not have the same mass, nor need they lie on the mass shell. Further, the expression below is valid in any order of perturbation theory. The only quantities which vary from order to order are the formulas for the 15 A_1's which make up the A's as in (D5)-(D10). Unlike the form given by Karplus and Neuman[21] in their equation (46) this expression is free from kinematical singularities, though identically equal to theirs. Terms are grouped as in Appendix D. Within each group the terms related by equations (D16) in the equal mass case have been placed together.

The expression below is presented as a table -- for each of the 81 A's the corresponding tensor coefficient is placed beside it. The sum of all these products is the $G_{\mu\nu\lambda\sigma}(1234)$ of (4.5) and (4.9).

Group I

A^{2143}	$2_\mu 1_\nu 4_\lambda 3_\sigma - (1\cdot 2) g_{\mu\nu} 4_\lambda 3_\sigma - (3\cdot 4) 2_\mu 1_\nu g_{\lambda\sigma} + (3\cdot 4)(1\cdot 2) g_{\mu\nu} g_{\lambda\sigma}$
A^{4321}	$4_\mu 3_\nu 2_\lambda 1_\sigma - (1\cdot 4) 3_\nu 2_\lambda g_{\mu\sigma} - (2\cdot 3) 4_\mu 1_\sigma g_{\nu\lambda} + (2\cdot 3)(1\cdot 4) g_{\mu\sigma} g_{\nu\lambda}$
A^{3412}	$3_\mu 4_\nu 1_\lambda 2_\sigma - (1\cdot 3) 4_\nu 2_\sigma g_{\mu\lambda} - (2\cdot 4) 3_\mu 1_\lambda g_{\nu\sigma} + (2\cdot 4)(1\cdot 3) g_{\mu\lambda} g_{\nu\sigma}$

Group II

$$A^{2341} \quad 2_\mu 3_\nu 4_\lambda 1_\sigma - (1\cdot 2)3_\nu 4_\lambda g_{\mu\sigma} + (2\cdot 3)1_\nu 4_\lambda g_{\mu\sigma} - (2\cdot 3)4_\lambda 1_\sigma g_{\mu\nu} - (3\cdot 4)2_\mu 1_\sigma g_{\nu\lambda}$$
$$+ (3\cdot 4)2_\lambda 1_\sigma g_{\mu\nu} + (3\cdot 4)(1\cdot 2) g_{\mu\sigma} g_{\nu\lambda} - (1\cdot 4)2_\mu 3_\nu g_{\lambda\sigma} .$$

$$A^{4123} \quad 4_\mu 1_\nu 2_\lambda 3_\sigma + (1\cdot 4)2_\mu 3_\sigma g_{\nu\lambda} - (1\cdot 4)2_\lambda 3_\sigma g_{\mu\nu} + (1\cdot 2)4_\mu 3_\nu g_{\lambda\sigma} - (1\cdot 2)4_\mu 3_\sigma g_{\nu\lambda}$$
$$- (2\cdot 3)4_\mu 1_\nu g_{\lambda\sigma} + (2\cdot 3)(1\cdot 4) g_{\mu\nu} g_{\lambda\sigma} - (3\cdot 4)1_\nu 2_\lambda g_{\mu\sigma}$$

$$A^{2413} \quad 2_\mu 4_\nu 1_\lambda 3_\sigma - (1\cdot 2)4_\nu 3_\sigma g_{\mu\lambda} + (1\cdot 2)3_\mu 4_\nu g_{\lambda\sigma} + (2\cdot 4)1_\nu 3_\sigma g_{\mu\lambda} - (2\cdot 4)1_\lambda 3_\sigma g_{\mu\nu}$$
$$- (1\cdot 3)2_\mu 4_\nu g_{\lambda\sigma} - (3\cdot 4)2_\mu 1_\lambda g_{\nu\sigma} + (3\cdot 4)1_\lambda 2_\sigma g_{\mu\nu} + (1\cdot 2)(3\cdot 4) g_{\mu\lambda} g_{\nu\sigma}$$

$$A^{3142} \quad 3_\mu 1_\nu 4_\lambda 2_\sigma + (1\cdot 3)2_\mu 4_\lambda g_{\nu\sigma} - (1\cdot 3)4_\lambda 2_\sigma g_{\mu\nu} - (1\cdot 2)3_\mu 4_\lambda g_{\nu\sigma} - (3\cdot 4)1_\nu 2_\sigma g_{\mu\lambda}$$
$$- (4\cdot 2)3_\mu 1_\nu g_{\lambda\sigma} + (1\cdot 3)(4\cdot 2) g_{\mu\nu} g_{\lambda\sigma}$$

$$A^{4312} \quad 4_\mu 3_\nu 1_\lambda 2_\sigma - (1\cdot 4)3_\nu 2_\sigma g_{\mu\lambda} - (2\cdot 3)4_\mu 1_\lambda g_{\nu\sigma} + (1\cdot 4)(2\cdot 3) g_{\mu\lambda} g_{\nu\sigma}$$
$$+ (1\cdot 3)4_\mu 2_\lambda g_{\nu\sigma} - (1\cdot 3)4_\mu 2_\sigma g_{\nu\lambda} - (4\cdot 2)3_\nu 1_\lambda g_{\mu\sigma} + (4\cdot 2)3_\nu 1_\sigma g_{\mu\lambda}$$

$$A^{3421} \quad 3_\mu 4_\nu 2_\lambda 1_\sigma - (1\cdot 3)4_\nu 2_\lambda g_{\mu\sigma} - (2\cdot 4)3_\mu 1_\sigma g_{\nu\lambda} + (1\cdot 3)(2\cdot 4) g_{\mu\sigma} g_{\nu\lambda}$$
$$+ (3\cdot 2)4_\nu 1_\lambda g_{\mu\sigma} - (3\cdot 2)4_\nu 1_\sigma g_{\mu\lambda} - (1\cdot 4)3_\mu 2_\lambda g_{\nu\sigma} + (1\cdot 4)3_\mu 2_\sigma g_{\nu\lambda}$$

Group III

$$A^{2111} \quad 2_\mu 1_\nu 1_\lambda 1_\sigma - (1\cdot 2)1_\lambda 1_\sigma g_{\mu\nu}$$

$$A^{2122} \quad 2_\mu 1_\nu 2_\lambda 2_\sigma - (1\cdot 2)2_\lambda 2_\sigma g_{\mu\nu}$$

$$A^{3343} \quad 3_\mu 3_\nu 4_\lambda 3_\sigma - (1\cdot 2)3_\mu 3_\nu g_{\lambda\sigma}$$

$$A^{4443} \quad 4_\mu 4_\nu 4_\lambda 3_\sigma - (3\cdot 4)4_\mu 4_\nu g_{\lambda\sigma}$$

A^{4111} $\quad 4_\mu 1_\nu 1_\lambda 1_\sigma - (1\cdot 4) 1_\nu 1_\lambda g_{\mu\sigma}$

A^{2322} $\quad 2_\mu 3_\nu 2_\lambda 2_\sigma - (2\cdot 3) 2_\mu 2_\sigma g_{\nu\lambda}$

A^{3323} $\quad 3_\mu 3_\nu 2_\lambda 3_\sigma - (2\cdot 3) 3_\mu 3_\sigma g_{\nu\lambda}$

A^{4441} $\quad 4_\mu 4_\nu 4_\lambda 1_\sigma - (1\cdot 4) 4_\nu 4_\lambda g_{\mu\sigma}$

A^{2422} $\quad 2_\mu 4_\nu 2_\lambda 2_\sigma - (2\cdot 4) 2_\mu 2_\lambda g_{\nu\sigma}$

A^{3111} $\quad 3_\mu 1_\nu 1_\lambda 1_\sigma - (1\cdot 3) 1_\nu 1_\sigma g_{\mu\lambda}$

A^{4442} $\quad 4_\mu 4_\nu 4_\lambda 2_\sigma - (2\cdot 4) 4_\mu 4_\lambda g_{\nu\sigma}$

A^{3313} $\quad 3_\mu 3_\nu 1_\lambda 3_\sigma - (1\cdot 3) 3_\nu 3_\sigma g_{\mu\lambda}$

<u>Group IV</u>

A^{2121} $\quad 2_\mu 1_\nu 2_\lambda 1_\sigma - (1\cdot 2) 2_\lambda 1_\sigma g_{\mu\nu}$

A^{4343} $\quad 4_\mu 3_\nu 4_\lambda 3_\sigma - (3\cdot 4) 4_\mu 3_\nu g_{\lambda\sigma}$

A^{4411} $\quad 4_\mu 4_\nu 1_\lambda 1_\sigma - (1\cdot 4) 4_\nu 1_\lambda g_{\mu\sigma}$

A^{3322} $\quad 3_\mu 3_\nu 2_\lambda 2_\sigma - (2\cdot 3) 3_\mu 2_\sigma g_{\nu\lambda}$

A^{3311} $\quad 3_\mu 3_\nu 1_\lambda 1_\sigma - (1\cdot 3) 3_\nu 1_\sigma g_{\mu\lambda}$

A^{4422} $\quad 4_\mu 4_\nu 2_\lambda 2_\sigma - (2\cdot 4) 4_\mu 2_\lambda g_{\nu\sigma}$

A^{4141} $\quad 4_\mu 1_\nu 4_\lambda 1_\sigma - (1\cdot 4) 1_\nu 4_\lambda g_{\mu\sigma}$

A^{2323} $\quad 2_\mu 3_\nu 2_\lambda 3_\sigma - (2\cdot 3) 2_\mu 3_\sigma g_{\nu\lambda}$

A^{2112} $2_\mu 1_\nu 1_\lambda 2_\sigma - (2\cdot 1) 1_\lambda 2_\sigma g_{\mu\nu}$

A^{3443} $3_\mu 4_\nu 4_\lambda 3_\sigma - (3\cdot 4) 3_\mu 4_\nu g_{\lambda\sigma}$

A^{3113} $3_\mu 1_\nu 1_\lambda 3_\sigma - (1\cdot 3) 1_\nu 3_\sigma g_{\mu\lambda}$

A^{2442} $2_\mu 4_\nu 4_\lambda 2_\sigma - (2\cdot 4) 2_\mu 4_\lambda g_{\nu\sigma}$

<u>Group V</u>

A^{2311} $2_\mu 3_\nu 1_\lambda 1_\sigma - (1\cdot 2) 3_\nu 1_\sigma g_{\mu\lambda} - (2\cdot 3) 1_\lambda 1_\sigma g_{\mu\nu} + (3\cdot 1) 2_\lambda 1_\sigma g_{\mu\nu}$

$- (3\cdot 1) 2_\mu 1_\sigma g_{\nu\lambda} + (3\cdot 1)(1\cdot 2) g_{\mu\sigma} g_{\nu\lambda}$

A^{3121} $3_\mu 1_\nu 2_\lambda 1_\sigma - (2\cdot 1) 3_\mu 1_\sigma g_{\nu\lambda} + (1\cdot 2)(1\cdot 3) g_{\mu\sigma} g_{\nu\lambda} - (3\cdot 2) 1_\nu 1_\sigma g_{\mu\lambda}$

A^{4122} $4_\mu 1_\nu 2_\lambda 2_\sigma - (1\cdot 4) 2_\lambda 2_\sigma g_{\mu\nu} - (2\cdot 1) 4_\mu 2_\lambda g_{\nu\sigma} - (4\cdot 2) 1_\nu 2_\lambda g_{\mu\sigma}$

A^{2421} $2_\mu 4_\nu 2_\lambda 1_\sigma - (1\cdot 2) 4_\nu 2_\lambda g_{\mu\sigma} - (2\cdot 4) 2_\lambda 1_\sigma g_{\mu\sigma} - (4\cdot 1) 2_\mu 2_\lambda g_{\nu\sigma}$

A^{3341} $3_\mu 3_\nu 4_\lambda 1_\sigma - (1\cdot 3) 3_\nu 4_\lambda g_{\mu\sigma} - (4\cdot 1) 3_\mu 3_\nu g_{\lambda\sigma}$

A^{4313} $4_\mu 3_\nu 1_\lambda 3_\sigma - (1\cdot 4) 3_\nu 3_\sigma g_{\mu\lambda} - (3\cdot 1) 4_\mu 3_\nu g_{\lambda\sigma} - (4\cdot 3) 3_\nu 1_\lambda g_{\mu\sigma}$

$+ (4\cdot 3) 3_\nu 1_\sigma g_{\mu\lambda}$

A^{4423} $4_\mu 4_\nu 2_\lambda 3_\sigma + (2\cdot 4) 4_\mu 3_\nu g_{\lambda\sigma} - (2\cdot 4) 4_\mu 3_\sigma g_{\nu\lambda} - (3\cdot 2) 4_\mu 4_\nu g_{\lambda\sigma}$

A^{4342} $4_\mu 3_\nu 4_\lambda 2_\sigma - (2\cdot 3) 4_\mu 4_\lambda g_{\nu\sigma} + (3\cdot 4) 4_\mu 2_\lambda g_{\nu\sigma} - (3\cdot 4) 4_\mu 2_\sigma g_{\nu\lambda}$

A^{3123} $3_\mu 1_\nu 2_\lambda 3_\sigma + (1\cdot 3) 2_\mu 3_\sigma g_{\nu\lambda} - (1\cdot 3) 2_\lambda 3_\sigma g_{\mu\nu} - (2\cdot 1) 3_\mu 3_\sigma g_{\nu\lambda}$

A^{2313} $2_\mu 3_\nu 1_\lambda 3_\sigma - (1\cdot 2) 3_\nu 3_\sigma g_{\mu\lambda} + (2\cdot 3) 1_\nu 3_\sigma g_{\mu\lambda} - (2\cdot 3) 1_\lambda 3_\sigma g_{\mu\nu}$

$$A^{2342} \quad 2_\mu 3_\nu 4_\lambda 2_\sigma - (2\cdot 3) 2_\mu 4_\lambda g_{\nu\sigma} - (3\cdot 4) 2_\mu 2_\sigma g_{\nu\lambda} - (4\cdot 2) 2_\mu 3_\nu g_{\lambda\sigma}$$

$$A^{2423} \quad 2_\mu 4_\nu 2_\lambda 3_\sigma - (2\cdot 4) 2_\mu 3_\sigma g_{\nu\lambda} - (3\cdot 2) 2_\mu 4_\nu g_{\lambda\sigma} - (4\cdot 3) 2_\mu 2_\lambda g_{\nu\sigma}$$

$$A^{4113} \quad 4_\mu 1_\nu 1_\lambda 3_\sigma - (1\cdot 4) 1_\nu 3_\sigma g_{\mu\lambda} - (3\cdot 1) 4_\mu 1_\nu g_{\lambda\sigma} + (3\cdot 1)(4\cdot 1) g_{\mu\nu} g_{\lambda\sigma}$$
$$- (4\cdot 3) 1_\nu 1_\lambda g_{\mu\sigma}$$

$$A^{3141} \quad 3_\mu 1_\nu 4_\lambda 1_\sigma - (1\cdot 3) 1_\nu 4_\lambda g_{\mu\sigma} - (3\cdot 4) 1_\nu 1_\sigma g_{\mu\lambda} - (4\cdot 1) 3_\mu 1_\nu g_{\lambda\sigma}$$
$$+ (4\cdot 1)(1\cdot 3) g_{\mu\nu} g_{\lambda\sigma}$$

$$A^{2441} \quad 2_\mu 4_\nu 4_\lambda 1_\sigma - (1\cdot 2) 4_\nu 4_\lambda g_{\mu\sigma} + (2\cdot 4) 1_\nu 4_\lambda g_{\mu\sigma} - (2\cdot 4) 4_\lambda 1_\sigma g_{\mu\nu}$$

$$A^{4142} \quad 4_\mu 1_\nu 4_\lambda 2_\sigma + (1\cdot 4) 2_\mu 4_\lambda g_{\nu\sigma} - (1\cdot 4) 4_\lambda 2_\sigma g_{\mu\nu} - (2\cdot 1) 4_\mu 4_\lambda g_{\nu\sigma}$$

$$A^{3122} \quad 3_\mu 1_\nu 2_\lambda 2_\sigma - (1\cdot 3) 2_\lambda 2_\sigma g_{\mu\nu} - (2\cdot 1) 3_\mu 2_\sigma g_{\nu\lambda} - (3\cdot 2) 1_\nu 2_\sigma g_{\mu\lambda}$$

$$A^{2312} \quad 2_\mu 3_\nu 1_\lambda 2_\sigma - (1\cdot 2) 3_\nu 2_\sigma g_{\mu\lambda} - (2\cdot 3) 1_\lambda 2_\sigma g_{\mu\nu} - (3\cdot 1) 2_\mu 2_\sigma g_{\nu\lambda}$$

$$A^{2411} \quad 2_\mu 4_\nu 1_\lambda 1_\sigma - (1\cdot 2) 4_\nu 1_\lambda g_{\mu\sigma} - (2\cdot 4) 1_\lambda 1_\sigma g_{\mu\nu} + (4\cdot 1) 1_\lambda 2_\sigma g_{\mu\nu}$$
$$+ (1\cdot 4)(1\cdot 2) g_{\mu\lambda} g_{\nu\sigma} - (1\cdot 4) 2_\mu 1_\lambda g_{\nu\sigma}$$

$$A^{4112} \quad 4_\mu 1_\nu 1_\lambda 2_\sigma - (2\cdot 1) 4_\mu 1_\lambda g_{\nu\sigma} + (2\cdot 1)(1\cdot 4) g_{\mu\lambda} g_{\nu\sigma} - (4\cdot 2) 1_\nu 1_\lambda g_{\mu\sigma}$$

$$A^{4413} \quad 4_\mu 4_\nu 1_\lambda 3_\sigma + (1\cdot 4) 3_\mu 4_\nu g_{\lambda\sigma} - (1\cdot 4) 4_\nu 3_\sigma g_{\mu\lambda} - (3\cdot 1) 4_\mu 4_\nu g_{\lambda\sigma}$$

$$A^{3441} \quad 3_\mu 4_\nu 4_\lambda 1_\sigma - (1\cdot 3) 4_\nu 4_\lambda g_{\mu\sigma} + (3\cdot 4) 4_\nu 1_\lambda g_{\mu\sigma} - (3\cdot 4) 4_\nu 1_\sigma g_{\mu\lambda}$$

$$A^{3342} \quad 3_\mu 3_\nu 4_\lambda 2_\sigma - (2\cdot 3) 3_\mu 4_\lambda g_{\nu\sigma} - (4\cdot 2) 3_\mu 3_\nu g_{\lambda\sigma}$$

$$A^{3423} \quad 3_\mu 4_\nu 2_\lambda 3_\sigma - (2\cdot 4) 3_\mu 3_\sigma g_{\nu\lambda} - (3\cdot 2) 3_\mu 4_\nu g_{\lambda\sigma} - (4\cdot 3) 3_\mu 2_\lambda g_{\nu\sigma}$$
$$+ (4\cdot 3) 3_\mu 2_\sigma g_{\nu\lambda}$$

Group VI

A^{2123} $2_\mu 1_\nu 2_\lambda 3_\sigma - (1\cdot 2)2_\lambda 3_\sigma g_{\mu\nu} - (3\cdot 2)2_\mu 1_\nu g_{\lambda\sigma} + (3\cdot 2)(1\cdot 2)g_{\mu\nu}g_{\lambda\sigma}$

A^{2141} $2_\mu 1_\nu 4_\lambda 1_\sigma - (2\cdot 1)4_\lambda 1_\sigma g_{\mu\nu}$

A^{4143} $4_\mu 1_\nu 4_\lambda 3_\sigma - (1\cdot 4)4_\lambda 3_\sigma g_{\mu\nu} - (3\cdot 4)4_\mu 1_\nu g_{\lambda\sigma} + (3\cdot 4)(1\cdot 4)g_{\mu\nu}g_{\lambda\sigma}$

A^{2343} $2_\mu 3_\nu 4_\lambda 3_\sigma - (4\cdot 3)2_\mu 3_\nu g_{\lambda\sigma}$

A^{4121} $4_\mu 1_\nu 2_\lambda 1_\sigma - (2\cdot 1)4_\mu 1_\sigma g_{\nu\lambda} + (2\cdot 1)(1\cdot 4)g_{\mu\sigma}g_{\nu\lambda} - (4\cdot 1)1_\nu 2_\lambda g_{\mu\sigma}$

A^{2321} $2_\mu 3_\nu 2_\lambda 1_\sigma - (1\cdot 2)3_\nu 2_\lambda g_{\mu\sigma} - (3\cdot 2)2_\mu 1_\sigma g_{\nu\lambda} + (3\cdot 2)(1\cdot 2)g_{\mu\sigma}g_{\nu\lambda}$

A^{4323} $4_\mu 3_\nu 2_\lambda 3_\sigma - (2\cdot 3)4_\mu 3_\sigma g_{\nu\lambda}$

A^{4341} $4_\mu 3_\nu 4_\lambda 1_\sigma - (1\cdot 4)3_\nu 4_\lambda g_{\mu\sigma}$

A^{3312} $3_\mu 3_\nu 1_\lambda 2_\sigma - (1\cdot 3)3_\nu 2_\sigma g_{\mu\lambda} - (2\cdot 3)3_\mu 1_\lambda g_{\nu\sigma} + (2\cdot 3)(1\cdot 3)g_{\mu\lambda}g_{\nu\sigma}$

A^{4412} $4_\mu 4_\nu 1_\lambda 2_\sigma - (4\cdot 1)4_\nu 2_\sigma g_{\mu\lambda} - (2\cdot 4)4_\mu 1_\lambda g_{\nu\sigma} + (2\cdot 4)(1\cdot 4)g_{\mu\lambda}g_{\nu\sigma}$

A^{3411} $3_\mu 4_\nu 1_\lambda 1_\sigma - (3\cdot 1)4_\nu 1_\sigma g_{\mu\lambda}$

A^{3422} $3_\mu 4_\nu 2_\lambda 2_\sigma - (4\cdot 2)3_\mu 2_\lambda g_{\nu\sigma}$

A^{4311} $4_\mu 3_\nu 1_\lambda 1_\sigma - (4\cdot 1)3_\nu 1_\lambda g_{\mu\sigma}$

A^{4322} $4_\mu 3_\nu 2_\lambda 2_\sigma - (3\cdot 2)4_\mu 2_\sigma g_{\nu\lambda}$

A^{3321} $3_\mu 3_\nu 2_\lambda 1_\sigma - (1\cdot 3)3_\nu 2_\lambda g_{\mu\sigma} - (2\cdot 3)3_\mu 1_\sigma g_{\nu\lambda} + (2\cdot 3)(1\cdot 3)g_{\mu\sigma}g_{\nu\lambda}$

A^{4421} $4_\mu 4_\nu 2_\lambda 1_\sigma - (1\cdot 4)4_\nu 2_\lambda g_{\mu\sigma} - (2\cdot 4)4_\mu 1_\sigma g_{\nu\lambda} + (2\cdot 4)(1\cdot 4)g_{\mu\sigma}g_{\nu\lambda}$

A^{2142} $\quad 2_\mu 1_\nu 4_\lambda 2_\sigma - (1\cdot 2) 4_\lambda 2_\sigma g_{\mu\nu}$

A^{2113} $\quad 2_\mu 1_\nu 1_\lambda 3_\sigma - (2\cdot 1) 1_\lambda 3_\sigma g_{\mu\nu} - (3\cdot 1) 2_\mu 1_\nu g_{\lambda\sigma} + (3\cdot 1)(1\cdot 2) g_{\mu\nu} g_{\lambda\sigma}$

A^{2443} $\quad 2_\mu 4_\nu 4_\lambda 3_\sigma - (3\cdot 4) 2_\mu 4_\nu g_{\lambda\sigma}$

A^{3143} $\quad 3_\mu 1_\nu 4_\lambda 3_\sigma - (1\cdot 3) 4_\lambda 3_\sigma g_{\mu\nu} - (4\cdot 3) 3_\mu 1_\nu g_{\lambda\sigma} + (4\cdot 3)(1\cdot 3) g_{\mu\nu} g_{\lambda\sigma}$

A^{3112} $\quad 3_\mu 1_\nu 1_\lambda 2_\sigma - (2\cdot 1) 3_\mu 1_\lambda g_{\nu\sigma} + (2\cdot 1)(1\cdot 3) g_{\mu\lambda} g_{\nu\sigma} - (3\cdot 1) 1_\nu 2_\sigma g_{\mu\lambda}$

A^{2412} $\quad 2_\mu 4_\nu 1_\lambda 2_\sigma - (1\cdot 2) 4_\nu 2_\sigma g_{\mu\lambda} + (4\cdot 2)(1\cdot 2) g_{\mu\lambda} g_{\nu\sigma} - (4\cdot 2) 2_\mu 1_\lambda g_{\nu\sigma}$

A^{3413} $\quad 3_\mu 4_\nu 1_\lambda 3_\sigma - (1\cdot 3) 4_\nu 3_\sigma g_{\mu\lambda}$

A^{3442} $\quad 3_\mu 4_\nu 4_\lambda 2_\sigma - (2\cdot 4) 3_\mu 4_\lambda g_{\nu\sigma}$

Appendix F

DETAILS OF CROSS CHANNEL CALCULATION

This is a description of the calculation of the absorptive part of the $\gamma\gamma$ scattering amplitude in the t-channel for $s = 0$. First we set out the general expression for this result and then show how the particular values of the particle momenta and helicities are converted to the cross channel.

In Fig. 2(a), particles 1 and 2 were taken as the ingoing particles. In the t-channel, we take 1 and 4 as the ingoing particles and evaluate the absorptive part by means of the unitarity relation for the invariant amplitude M. We make a slice across the two vertical fermion lines in Fig. 2(a), and put the internal lines on the mass shell. Then the upper and lower portions represent respectively the matrix elements for pair annihilation and creation. Of course we must make similar cuts in diagram Fig. 2(c) and the two other diagrams which have the direction of the fermion line reversed. For the case of a two particle intermediate state, the unitarity relation reads

$$\mathrm{Im} < k_2 k_3 | M | k_1 k_4 > = \frac{1}{8\pi^2} \sum_i \int d\rho < k_2 k_3 M^\dagger \, p_1 p_2 >< p_1 p_2 M k_1 k_4 > \tag{F1}$$

with

$$\int d\rho = \int\int d^4 p_1 d^4 p_2 \, \delta(p_1^2 - m^2)\delta(p_2^2 - m^2)\theta(p_{1_o})\theta(p_{2_o})\delta^4(p_1 + p_2 - k_1 - k_4) \tag{F2}$$

Here the sum over i represents the sum over the intermediate spin states, the function θ is $+1$ for positive argument and 0 for negative argument, and p_1 and p_2 are the four-momenta of the intermediate fermion antiparticle and particle, respectively. For ease in writing we will use subscripts rather than superscripts to denote the t-channel momenta and helicities.

Then we have for the pair creation matrix element

$$\langle p_1 p_2 | M | k_1 k_4 \rangle = g^2 \frac{\bar{u}(p_2)\not{e}_1(-\not{p}_1+\not{k}_4+m)\not{e}_4 v(p_1)}{(-p_1+k_4)^2-m^2}$$

$$+ g^2 \frac{\bar{u}(p_2)\not{e}_4(\not{k}_1-\not{p}_1+m)\not{e}_1 v(p_1)}{(k_1-p_1)^2-m^2} \quad . \tag{F3}$$

The annilihation matrix element is

$$\langle k_1 k_3 | M | p_1 p_2 \rangle = g^2 \frac{\bar{v}(p_1)\not{e}_3(-\not{p}_1+\not{k}_3+m)\not{e}_2 u(p_2)}{(-p_1+k_3)^2-m^2}$$

$$+ g^2 \frac{\bar{v}(p_1)\not{e}_2(-\not{p}_1+\not{k}_2+m)\not{e}_3 u(p_2)}{(-p_1+k_2)^2-m^2} \quad , \tag{F4}$$

where we have deliberately avoided the complex conjugation of the final state polarization vectors for reasons to be explained later. Making use of the spinor equations

$$(\not{p} + m)v(p) = 0 \quad , \qquad (\not{p} - m)u(p) = 0 \quad , \tag{F5}$$

we eliminate the m's in the numerator with a set of equations like

$$\bar{u}(p_2)\not{e}_1(-\not{p}_1+\not{k}_4+m)\not{e}_4 v(p_1) = \bar{u}(p_2)\not{e}_1[\not{k}_4\not{e}_4 - 2p_1\cdot e_4]\,v(p) \tag{F6}$$

$$= \bar{u}(p_2)[2e_1\cdot p_2 - \not{e}_1\not{k}_1]\,\not{e}_4 v(p_1) \tag{F7}$$

where the freedom to switch from $-\not{p}_1 + \not{k}_4$ to $\not{p}_1 - \not{k}_1$ is assured by momentum conservation. Combining these relations

$$\mathrm{Im}\langle k_2 k_3 | M | k_1 k_4 \rangle =$$

$$\frac{g^4}{8\pi^2} \sum_i \int d\rho \; \frac{\bar{u}(p_2)\not{e}_1[\not{k}_4\not{e}_4 - 2p_1\cdot e_4]\,v(p_1)}{\lambda^2 - 2p_1\cdot k_4} \; \frac{\bar{v}(p_1)[\not{e}_3\not{k}_3 - 2p_1\cdot e_3]\,\not{e}_2 u(p_2)}{\lambda^2 - 2p_1\cdot k_3}$$

$$+ \frac{\bar{u}(p_2)\not{e}_1[\not{k}_4\not{e}_4 - 2p_1\cdot e_4]\,v(p_1)}{\lambda^2 - 2p_1\cdot k_4} \; \frac{\bar{v}(p_1)[\not{e}_2\not{k}_2 - 2p_1\cdot e_2]\,\not{e}_3 u(p_2)}{\lambda^2 - 2p_1\cdot k_2}$$

$$+ \frac{\bar{u}(p_1)[-\not{e}_4\not{k}_4 + 2e_4\cdot p_1]\,\not{e}_1 v(p_2)}{\lambda^2 - 2p_1\cdot k_4} \; \frac{\bar{v}(p_2)\not{e}_2[-\not{k}_3\not{e}_3 + 2p_1\cdot e_3]\,u(p_1)}{\lambda^2 - 2p_1\cdot k_3}$$

$$+ \frac{\bar{u}(p_1)[-\not{e}_4\not{k}_4 + 2p_1\cdot e_4]\,\not{e}_1 v(p_2)}{\lambda^2 - 2p_1\cdot k_4} \; \frac{\bar{v}(p_2)\not{e}_3[-\not{k}_2\not{e}_2 + 2p_1\cdot e_2]\,u(p_1)}{\lambda^2 - 2p_1\cdot k_2} \tag{F8}$$

where in the last two terms we have first expressed all the four-vectors so that they involve p_2 and then made use of the symmetry of the integration (F2) to interchange p_1 and p_2. To perform the sum over the spin states we use the projection operators for the positive and negative energy spinors and take the trace, recalling our convention for spinor normalization (A6), (A7). The first and third terms have the same denominator, and upon rewriting the numerators we see that they are also equal. Similarly for the second and fourth terms. Thus we obtain

$\mathrm{Im}\langle k_2 k_3 \,|M|\, k_1 k_4 \rangle =$

$$\frac{g^4}{4\pi^2}\int d\rho \; \frac{\mathrm{Tr}\,\{(\not{p}_2+m)\not{e}_1[\not{k}_4\not{e}_4-2p_1\cdot e_4](\not{p}_1-m)[\not{e}_3\not{k}_3-2p_1\cdot e_3]\not{e}_2\}}{(\lambda^2-2p_1\cdot k_4)(\lambda^2-2p_1\cdot k_3)}$$

$$+\;\frac{\mathrm{Tr}\,\{(\not{p}_2+m)\not{e}_1(\not{k}_4\not{e}_4-2p_1\cdot e_4)(\not{p}_1-m)(\not{e}_2\not{k}_2-2p_1\cdot e_2)\not{e}_3\}}{(\lambda^2-2p_1\cdot k_4)(\lambda^2-2p_1\cdot k_2)} \quad . \qquad \text{(F9)}$$

The selection of the appropriate values of the momenta and helicities in passing to the crossed channel is somewhat tricky. Looking at (4.5) we see that in the s-channel, M is evaluated by complex conjugating the helicities corresponding to particles 3 and 4 before dotting them into $G_{\mu\nu\lambda\sigma}$. To evaluate M in the crossed channel we must keep this same set of complex conjugations even though in the new channel these particles are no longer the outgoing ones. We have been able to show explicitly, by use of the original Karplus and Neuman form of the amplitude, that

$$G_{\mu\nu\lambda\sigma}(s,t) = G_{\mu\sigma\lambda\nu}(t,s) \quad . \qquad \text{(F10)}$$

The switching of the roles of the particles 2 and 4 accomplishes the reversal of the roles of s and t as mentioned above and as may be seen from the set of equations (D15).

If we look at (4.25) which gives the momentum values for the s-channel problem we see that since the value $s = 0$ is below threshold, we would have to deal with imaginary components in the momenta. This difficulty can be circumvented by first making a Lorentz transformation to a frame where $k^{(1)}$ and $k^{(4)}$ will be in the center of mass system. (For clarity, first change the sign of $k^{(3)}$ and $k^{(4)}$ so that they represent the physical momenta.) The same Lorentz transformation must of course be applied to the helicities $e^{(1)}$, $e^{(2)}$, $e^{(3)*}$, $e^{(4)*}$, for each case. Then there is no difficulty encountered in setting the s-channel ω equal to zero to get the case $s = 0$. Now that we have eliminated the energy component in the s-channel we are free to use the same letter ω to denote the energy component of the γ momenta in the t-channel. With λ as the unit of mass, we obtain the set of momenta

$$\begin{array}{llll} k_1: & (\omega, 0, 0, k) & k_4: & (\omega, 0, 0, -k) \\ k_2: & (\omega, 0, 0, k) & k_3: & (\omega, 0, 0, -k) \end{array} \qquad \text{(F11)}$$

with

$$\omega^2 - k^2 = 1; \qquad 4\omega^2 = t\,. \qquad \text{(F12)}$$

The helicities are

$$\begin{array}{ll} e_1: & \frac{1}{\sqrt{2}}(k, 0, i, \omega) \\ e_4: & \begin{cases} \frac{1}{\sqrt{2}}(k, 0, -i, -\omega) & \text{for } f^{\mathcal{S}}_{1-1;1-1} \\ \frac{1}{\sqrt{2}}(k, 0, i, -\omega) & \text{for } f^{\mathcal{S}}_{-11;1-1} \end{cases} \\ e_2: & \frac{1}{\sqrt{2}}(k, 0, i, \omega) \\ e_3: & \begin{cases} \frac{1}{\sqrt{2}}(k, 0, -i, -\omega) & \text{for } f^{\mathcal{S}}_{1-1;1-1} \\ \frac{1}{\sqrt{2}}(k, 0, i, -\omega) & \text{for } f^{\mathcal{S}}_{-11;1-1}\,. \end{cases} \end{array} \qquad \text{(F13)}$$

Now that the momenta have been chosen, the operator (F2) may be evaluated more in detail. Setting $a \equiv k_1 + k_4 = (2\omega, 0, 0, 0)$ we have

$$\int d\rho = \frac{\sqrt{\omega^2 - m^2}}{8\omega} \int d\Omega \tag{F14}$$

with the further prescription

$$\begin{aligned} &\text{set} \quad p_2 = -p_1 \pm a \\ &\text{set} \quad p: (\omega, \vec{P}) \\ &\text{set} \quad |\vec{P}| = \sqrt{\omega^2 - m^2} \end{aligned} \tag{F15}$$

and where Ω represents the angles of $\vec{P}$.

The calculation now becomes very tedious even after setting $m = \lambda$, which we do merely for convenience. The evaluation of the traces, the combining of integrals of the same form, and the combination of the helicity cases according to (4.1) are so lengthy that it was felt wise to check the whole affair by doing the calculation again, but in a different gauge. The gauge invariance of the whole expression may be demonstrated explicitly from (F3) and (F4). We replace the helicities (F13) by a set which has a constant multiple of the appropriate momentum vector subtracted from it. Then, since the amplitude is linear in each helicity vector, we may multiply any two of them by (-1). The alternate set of helicities become

$$
\begin{aligned}
e_1 &: \quad \frac{1}{\sqrt{2}\,\omega}(0,0,i\omega,1) \\
e_4 &: \quad \begin{cases} \frac{1}{\sqrt{2}\,\omega}(0,0,i\omega,1) & \text{for } f^{\mathcal{S}}_{1-1;1-1} \\ \frac{1}{\sqrt{2}\,\omega}(0,0,i\omega,-1) & \text{for } f^{\mathcal{S}}_{-11;1-1} \end{cases} \\
e_2 &: \quad \frac{1}{\sqrt{2}\,\omega}(0,0,i\omega,1) \\
e_3 &: \quad \begin{cases} \frac{1}{\sqrt{2}\,\omega}(0,0,i\omega,1) & \text{for } f^{\mathcal{S}}_{1-1;1-1} \\ \frac{1}{\sqrt{2}\,\omega}(0,0,i\omega,-1) & \text{for } f^{\mathcal{S}}_{-11;1-1}\,. \end{cases}
\end{aligned}
\tag{F16}
$$

In this gauge the entire calculation has a different appearance, and the result of the two calculations can only be compared at the last step. As in a random walk, the magnitude of the coefficients in the full answer (4.41) suggests the square root of the number of steps taken; here however we have two such walks ending in the same place. In addition to the threshold properties mentioned in the text, the answer has another interesting property. The integrand, which represents the imaginary part of the amplitude with respect to t, behaves asymptotically as $t^{-2}\log t$. This agrees with what it should to correspond to a real part which behaves as t^{-1}.

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