A STUDY OF NONLINEAR PHENOMENA IN THE PROPAGATION OF ELECTROMAGNETIC WAVES IN A WEAKLY IONIZED GAS

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ABSTRACT

This thesis is a study of nonlinear phenomena in the propagation of electromagnetic waves in a weakly ionized gas externally biased with a magnetostatic field. The present study is restricted to the nonlinear phenomena arising from the interaction of electromagnetic waves in the ionized gas. The important effects of nonlinearity are wave-form distortion and generation of mixed frequencies. The wave-form distortion leads to cross modulation of one wave by a second amplitude-modulated wave.

The nonlinear effects are assumed to be small so that a perturbation method can be used. Boltzmann's kinetic equation with an appropriate expression for the collision term is solved by expanding the electron distribution function into spherical harmonics in velocity space. In turn, the electron convection current density and the conductivity tensors of the nonlinear ionized gas are found from the distribution function. Finally, the expression for the current density and Maxwell's equations are employed to investigate the effects of nonlinearity on the propagation of electromagnetic waves in the ionized gas, and also on the reflection of waves from an ionized gas of semi-infinite extent.

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I. INTRODUCTION

This thesis presents a study of nonlinear phenomena in the propagation of electromagnetic waves in a weakly ionized gas externally biased with a magnetostatic field. The present study is restricted to nonlinear phenomena arising from the interaction of electromagnetic waves in the ionized gas. The propagation of electromagnetic waves through the ionized gas disturbs the otherwise uniform electron distribution function. The resulting spatial dependence affects the conductivity tensors of the gas thus leading to nonlinear phenomena such as interactions among various electromagnetic waves in the gas.

1.1) Literature Survey

The study of nonlinear phenomena in the propagation of electromagnetic waves in a weakly ionized gas was initiated by Tellegen's paper (1) in 1933. He reported interference by cross-modulation of broadcast signals in the ionosphere by the strong Luxembourg station signal. Cross-modulation, which is the consequence of nonlinear interaction between two electromagnetic waves in the ionosphere when the disturbing wave is amplitude-modulated, is now known as the Luxembourg effect. In explaining Tellegen's observations, Bailey and Martyn (2) considered the heating effect of a passing electromagnetic wave on the collision frequency which in turn affects the propagation of another wave in the disturbed medium. With known numerical values for the physical parameters of the ionosphere and the broadcast signals, their theory predicts a detectable cross-modulation. They made use of the geometrical optics approximation and utilized the oversimplified concepts of the effective collision frequency of the free electrons.

Instead of the approach to the problems using mean free path or relaxation time technique as in (2), most investigators prefer to use the method of statistical mechanics and to solve Boltzmann's kinetic equation for the electron distribution function in an ionized gas in the presence of disturbing electromagnetic waves. From the knowledge of the electron distribution function they deduce the non-linear conductivity tensor of the disturbed gas. Chapman and Cowling (3) derived Boltzmann's kinetic equation of motion for an ionized gas in the presence of an electric field as well as a magnetic field, and also the collision term for elastic and inelastic collisions. Margenau (4) developed methods for calculating the electron distribution function for a gas discharge in the presence of an A.C. field, taking into account all types of collisions. He expanded the function into spherical harmonics in velocity space, assuming that the distribution function did not deviate much from being isotropic, and retained only the first two terms in the expansion. His expression for the isotropic part of the distribution function reduces to the Druyvesteyn's form in reference (3) for a strong electric field and to the Maxwellian form for a weak electric field.

Meanwhile several investigators studied in detail the collision term in Boltzmann's kinetic equation for collisions under different conditions. Allis (5) obtained an expression for the collision term in Boltzmann's kinetic equation by considering that a

Coulomb force exists between the two participating particles, while Desloge and Matthysse (6) arrived at the same expression for a hard collision where a short rage force exists between the two particles. In the present study, the elastic electron-molecule collisions are the dominant ones in a weakly ionized gas, and therefore the latter approach is more appropriate.

In the laboratory, Goldstein, Anderson and Clark (7) observed the nonlinear effect of cross-modulation between microwave signals propagating through a gaseous discharge plasma. Using the geometrical optics approximation, Rumi (8,9) presented an extensive analysis of the radio-wave interaction mechanism in the ionosphere. He observed that cross-modulation in the ionosphere can be produced not only by changing the electron collision frequency, but also by changing the electron density.

Later, Fain (10) and Sodha (11) determined the electron distribution function for a plasma in the presence of electromagnetic waves and a magnetostatic biasing field. The plasma was assumed to be weakly ionized so that the elastic electron-molecule collisions were dominant. Sodha also calculated the electron convection current density in the nonlinear plasma disturbed by the passing electromagnetic waves. Ginzburg (12) discussed the nonlinearity of plasmas due to the change in collision frequency, non-uniformity of the plasma and the presence of a magnetostatic field. He and Gurevick (13,14) gave a detailed analysis for the nonlinear phenomena in a plasma located in an electromagnetic field and the cross-modulation of two electromagnetic waves.

Sodha and Palumbo (15,16) derived the time-invariant change in the complex conductivity for a plasma due to the heating effect on the electrons by the passing electromagnetic waves. They then proceeded to investigate the nonlinear propagation of an amplitude-modulated electromagnetic wave in a plasma and also the nonlinear interaction of a number of electromagnetic waves. The waves were assumed to propagate in the same direction and in the absence of any external magnetostatic field. In his recent paper, Papa (17) derived the nonlinear complex conductivity tensor of a magneto-active plasma as a function of the polarization of the radio-wave. He then found the transmission and the reflection coefficients for the radio-frequency propagation through an inhomogeneous, magneto-active and nonlinear plasma. There the direction of propagation was taken to be parallel to the external magnetostatic biasing field. These authors restricted the electromagnetic waves to TEM waves. This restriction limits the usefulness of their theories when they are applied to the propagation of radio-waves in the ionosphere where the earth's magnetic field cannot be neglected and electromagnetic waves usually propagate at an angle to the direction of the earth's magnetic field.

It will be found in the present study, from the inherently nonlinear Boltzmann kinetic equation, that there are two nonlinear effects of different nature, namely the heating effect on the electrons by the passing waves and the effect of spatial dispersion due to the longitudinal components of the electric field of the waves. The heating effect on electrons gives rise to wave form distortion in the passing waves, while the effect of spatial dispersion leads to wave generation of harmonic or mixed frequencies. In the previous papers (16,17), the waves are restricted to TEM waves whose longitudinal components of the electric field along the propagation direction are zero so that the effect of spatial dispersion is absent and there is no wave generation at harmonic or mixed frequencies.

1.2) Outline of Study

In the study of nonlinear phenomena in the propagation of electromagnetic waves, many investigators have used a geometrical optics approximation which is valid only for TEM waves. In the present study, field theory will be used to treat the practical case of broadcast signals in the ionosphere with no restriction on the type of electromagnetic waves. Consequently both the heating effect on electrons and the effect of spatial dispersion will appear in the study.

To simplify the problems, nonlinearity in the disturbed gas will be treated as perturbation. The inherently nonlinear Boltzmann kinetic equation and Maxwell's equations are solved for the two dependent variables, namely, the electron distribution function and the electric field of the waves.

This study is divided into seven chapters. The first chapter defines the nonlinear problems and gives a literature survey. In Chapter II, we use an elementary kinetic theory to study not only the propagation of an electromagnetic wave in a linear, magneto-active, ionized gas, but also the reflection of a wave from such a gas of semi-infinite extent. It is necessary to include this chapter since the solutions found here will be the zeroth order solutions of the non-linear problems in later chapters. In Chapter III, the electron distribution function is expanded in velocity space into spherical harmonics which are further expanded in Fourier series. Eight terms in the expansions, instead of the usual first three terms, are retained and determined by solving Boltzmann's kinetic equation by a perturbation method. The additional five terms take into account the effect of spatial dispersion, while the usual first three terms take care of the heating effect on the electrons. In Chapter IV, the electron convection current density as well as the nonlinear conductivity tensors are derived from the knowledge of the electron distribution function. These four preliminary chapters lead to the study of nonlinear phenomena in the propagation of electromagnetic waves in Chapter V. The effects of nonlinear phenomena are wave form distortion and wave generation of harmonic or mixed frequencies. If the disturbing wave is amplitude modulated, the other wave in the same ionized gas will be cross modulated. In Chapter VI the coefficients of reflection from a weakly ionized gas are derived. Similar effects of nonlinear phenomena are also present in the reflected waves. The conclusions of this study are given in Chapter VII.

The rationalized MKS system of units is used throughout the study.

II. LINEAR, IONIZED GAS

As a preliminary investigation prior to the study of nonlinear phenomena, this chapter is devoted to the study of the propagation of an electromagnetic wave in a linear, ionized gas and the reflection of the wave from a gas of semi-infinite extent. This preliminary investigation is important as it provides the zeroth order solutions for the nonlinear problems to be studied in the following chapters. Here an elementary kinetic equation is used to describe the motion of electrons in the ionized gas which is externally biased with a magnetostatic field.

2.1) Elementary Kinetic Equation of Motion

In order to derive an expression for the conductivity tensor, the motion of the particles in the ionized gas under the influence of an external magnetostatic field and the electric field of an electromagnetic wave must be examined. In an ionized gas there are three types of particles, namely electrons, ions and neutral molecules. The neutral molecules contribute nothing to the convection current density; and because of their very large mass (in comparison with that of the electrons) the ions contribute a negligible fraction of the convection current density. Hence only the motion of the electrons need be studied. The gas is assumed to be weakly ionized and the predominant type of collision which affects the motion of the electrons is the electron-molecule collision. Since the average kinetic energy of the electrons is well below the ionization energy of the molecules, the collision is elastic. When the collision term is included in the kinetic equation, the ionized gas is found to be lossy.

From Newton's law of motion, the elementary kinetic equation of motion of an electron is

$$\frac{d\underline{v}}{dt} = -\frac{e}{m}(\underline{E} + \underline{v} \times \underline{B}_o) - \nu\,\underline{v} \tag{2.1}$$

where the first two terms on the right represent the Lorentz force on the negatively charged electron due to the electric field $\underline{E}$ of the wave and to the external magnetostatic field $\underline{B}_o$ respectively. The last term takes into account the elastic electron-molecule collision. ν is the collision frequency. The effect of the magnetic field of the wave on the electron motion is neglected.

Under an assumed time dependence $\exp(-i\omega t)$ for the electric field of a monochromatic electromagnetic wave, the steady state solution of Eq. 2.1 is given by

$$\left[(\nu - i\omega) - \underline{\omega}_g \times\right] \underline{v} = -\frac{e}{m}\underline{E}$$

where the electron gyrofrequency is

$$\underline{\omega}_g = \frac{e}{m}\underline{B}_o \quad .$$

The velocity of the electron, in terms of the electric field of the electromagnetic wave, is found to be

$$\underline{v} = -\frac{e}{m}\underline{\underline{L}} \cdot \underline{E} \tag{2.2}$$

where the dyadic operator is

$$\underline{\underline{L}}(\omega) = \frac{(\nu - i\omega)^2\ \underline{\underline{u}} + (\nu - i\omega)\ \underline{\omega}_g \times \underline{\underline{u}} + \underline{\omega}_g\underline{\omega}_g}{(\nu - i\omega)\left[(\nu - i\omega)^2 + \omega_g^2\right]}$$

and $\underline{\underline{u}}$ = the unit dyadic.

2.2) Conductivity and Dielectric Tensors

From the solution for the electron velocity $\underline{v}$ in terms of the electric field $\underline{E}$ of the electromagnetic wave, we are in the position to derive the conductivity and the dielectric tensors of the ionized gas. By definition, the electron convection current density $\underline{J}_e$ is equal to $-en\underline{v}$ where n is the electron density of the ionized gas. From expression 2.2 it follows that the electron convection current density is

$$\underline{J}_e = \epsilon_o \omega_o^2 \underline{\underline{L}} \cdot \underline{E}$$

where the electron plasma frequency is

$$\omega_o = \sqrt{\frac{ne^2}{\epsilon_o m}} \quad .$$

But the convection current density $\underline{J}_e$ is also equal to $\underline{\underline{\sigma}}_L \cdot \underline{E}$. Hence the conductivity tensor is found to be

$$\underline{\underline{\sigma}}_L(\omega) = \epsilon_o \omega_o^2 \underline{\underline{L}} \quad . \tag{2.3}$$

The corresponding dielectric tensor is given by

$$\underline{\underline{\epsilon}}_L(\omega) = \epsilon_o(\underline{\underline{u}} - \frac{\underline{\underline{\sigma}}_L}{i\omega\,\epsilon_o}) \quad . \tag{2.4}$$

The subscript L is used to indicate that $\underline{\underline{\sigma}}_L$ and $\underline{\underline{\epsilon}}_L$ are respectively the conductivity and the dielectric tensors of a linear ionized gas. Both $\underline{\underline{\sigma}}_L$ and $\underline{\underline{\epsilon}}_L$ are found to be functions of the angular frequency ω of the wave.

Since a tensor is covariant under transformation of coordinates, a cartesian coordinate system can be used without any loss of generality. For mathematical simplicity we shall choose the direction of wave propagation and that of the external magnetostatic field as those shown in Fig. 1. Accordingly, the electron gyrofrequency can be written in the cartesian system as

$$\underline{\omega}_g = \underline{e}_y \, \omega_g \sin \Theta + \underline{e}_z \, \omega_g \cos \Theta$$

and the components of the dielectric tensor become

$$\epsilon_{xx} = \epsilon_o \left[1 - \frac{X}{1 - Y^2} \right]$$

$$\epsilon_{yy} = \epsilon_o \left[1 - \frac{X(1 - Y^2 \sin^2 \Theta)}{1 - Y^2} \right]$$

$$\epsilon_{zz} = \epsilon_o \left[1 - \frac{X(1 - Y^2 \cos^2 \Theta)}{1 - Y^2} \right]$$

$$\epsilon_{xy} = -\epsilon_{yx} = \epsilon_o \frac{iXY \cos \Theta}{1 - Y^2}$$

$$\epsilon_{xz} = -\epsilon_{zx} = -\epsilon_o \frac{iXY \sin \Theta}{1 - Y^2}$$

and

$$\epsilon_{yz} = \epsilon_{zy} = \epsilon_o \frac{XY^2 \sin \Theta \cos \Theta}{1 - Y^2} \tag{2.5}$$

where the dimensionless quantities X and Y are given by

$$X = \frac{\omega_o^2}{i\omega (\nu - i\omega)}$$

$$Y = \frac{-i\omega_g}{\nu - i\omega} .$$

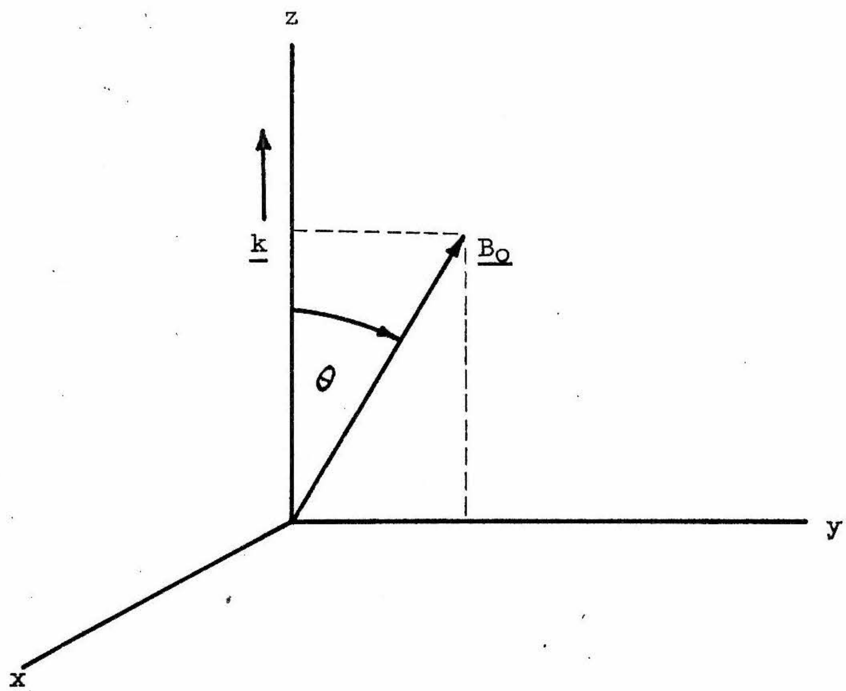

Fig. 1. Direction of wave propagation and direction of external magnetostatic field

2.3) Wave Propagation in an Ionized Gas

In this section we shall examine the properties of wave propagation using the dielectric tensor whose components are given by Eq. 2.5. The electric field of a plane monochromatic wave traveling along the z-axis has the following form

$$\underline{E}(z,t) = \underline{E}_o e^{i(kz-\omega t)}$$

where $\underline{E}_o$ represents its amplitude and is a constant vector, and k is the propagation constant to be found. A similar expression can be written for the magnetic field $\underline{H}(z,t)$.

Eliminate the magnetic field $\underline{H}(z,t)$ from Maxwell's equations,

$$\nabla \times \underline{E} = i\omega \mu_o \underline{H}$$

$$\nabla \times \underline{H} = -i\omega \underline{\underline{\epsilon}}_L \cdot \underline{E} \quad . \tag{2.6}$$

We obtain the vector wave equation for $\underline{E}$

$$\nabla \times \nabla \times \underline{E} = \omega^2 \mu_o \underline{\underline{\epsilon}}_L \cdot \underline{E} \quad . \tag{2.7}$$

For the plane wave in the ionized gas whose dielectric tensor is given in Eq. 2.5, the vector wave equation separates into a set of three equations, namely

$$\begin{aligned}
(k^2 - \omega^2\mu_o\epsilon_{xx})E_x \quad - \omega^2\mu_o\epsilon_{xy}E_y \quad - \omega^2\mu_o\epsilon_{xz}E_z &= 0 \\
- \omega^2\mu_o\epsilon_{yx} E_x + (k^2 - \omega^2\mu_o\epsilon_{yy})E_y \quad - \omega^2\mu_o\epsilon_{yz}E_z &= 0 \\
- \omega^2\mu_o\epsilon_{zx}E_x \quad - \omega^2\mu_o\epsilon_{zy}E_y \quad - \omega^2\mu_o\epsilon_{zz}E_z &= 0
\end{aligned} \tag{2.8}$$

where E_x, E_y and E_z are the three cartesian components of the electric field $\underline{E}(z,t)$. Note that the above three simultaneous equations contain three variables, namely E_x, E_y and E_z, and are homogeneous. In addition to the trivial solution $E_x = E_y = E_z = 0$, there are nontrivial solutions for the vector wave equation 2.7 only when

$$\begin{vmatrix} k^2 - \omega^2\mu_o\epsilon_{xx}, & -\omega^2\mu_o\epsilon_{xy}, & -\omega^2\mu_o\epsilon_{xz} \\ -\omega^2\mu_o\epsilon_{yx}, & k^2 - \omega^2\mu_o\epsilon_{yy}, & -\omega^2\mu_o\epsilon_{yz} \\ -\omega^2\mu_o\epsilon_{zx}, & -\omega^2\mu_o\epsilon_{zy}, & -\omega^2\mu_o\epsilon_{zz} \end{vmatrix} = 0 .$$

After some simplification, the two solutions of the above equation for k are

$$k_o = \omega\sqrt{\mu_o\epsilon_o}\left[1 - \frac{X}{1 - \frac{1}{2}\frac{Y^2\sin^2\theta}{1 - X} + \sqrt{\frac{1}{4}\frac{Y^4\sin^4\theta}{(1-X)^2} + Y^2\cos^2\theta}}\right]^{1/2} \tag{2.9}$$

and

$$k_e = \omega\sqrt{\mu_o\epsilon_o}\left[1 - \frac{X}{1 - \frac{1}{2}\frac{Y^2\sin^2\theta}{1 - X} - \sqrt{\frac{1}{4}\frac{Y^4\sin^4\theta}{(1-X)^2} + Y^2\cos^2\theta}}\right]^{1/2} . \tag{2.10}$$

The two solutions are the same as those in reference (18) except for the definitions for X and Y which are complex here to take into account the loss due to the elastic electron-molecule collisions. The two solutions for the propagation constants indicate that the anisotropic ionized gas can support two electromagnetic waves traveling with two different propagation constants respectively in an arbitrary direction with respect to the external magnetostatic field.

The following properties of wave propagation have been found in reference (18). As a function of X the propagation constant k_o resembles the propagation constant of a wave in an isotropic gas more closely than k_e does. The wave traveling with propagation constant k_o is referred to as the ordinary wave and the wave traveling with propagation constant k_e is referred to as the extraordinary wave. Waves that propagate at an arbitrary angle to the external magnetostatic field are TM waves. Both ordinary and extraordinary waves degenerate to TEM waves when they propagate parallel to the external magnetostatic field. Only the ordinary wave degenerates to a TEM wave when it propagates perpendicularly to the external magnetostatic field. In the latter case the electric field of the ordinary wave is parallel to the external magnetostatic field.

The nontrivial solution of the vector wave equation 2.7 for the ordinary wave, with the time dependence $\exp(-i\omega t)$ omitted, is found to be

$$\left.\begin{aligned} E_{xo} &= -\, iR_oA_o \\ E_{yo} &= A_o \\ E_{zo} &= \frac{Y\left(\dfrac{k_o^2}{\omega^2\mu_o\epsilon_o} - 1\right)\sin\theta}{1 - X}R_oA_o \end{aligned}\right\} e^{ik_oz} \qquad (2.11a)$$

and the corresponding magnetic field, found from Maxwell's equations, is

$$\left.\begin{aligned} H_{xo} &= -\frac{k_o}{\omega\,\mu_o}\,A_o \\ H_{yo} &= -\frac{ik_oR_o}{\omega\,\mu_o}\,A_o \\ H_{zo} &= 0 \end{aligned}\right\} \quad e^{ik_oz} \qquad (2.11b)$$

where

$$R_o = \frac{Y\,\sin^2\theta + \sqrt{Y^2\sin^4\theta + 4(1-X)^2\cos^2\theta}}{2(1-X)\,\cos\theta} \;.$$

A_o denotes an undetermined coefficient and the subscript o stands for the ordinary wave. The nontrivial solution for the extraordinary wave has field components similar to those of the ordinary wave. The field components are obtained by simply replacing the subscript o's by e's in equation 2.11, and replacing R_o by

$$R_e = \frac{Y\,\sin^2\theta - \sqrt{Y^2\sin^4\theta + 4(1-X)^2\cos^2\theta}}{2(1-X)\,\cos\theta} \;. \qquad (2.12)$$

In the next section we shall use these two nontrivial solutions for the ordinary and the extraordinary waves to find the coefficients of reflection from an ionized gas of semi-infinite extent.

2.4) Reflection from an Ionized Gas of Semi-Infinite Extent

When an electromagnetic wave propagates through the interface between two media, reflected and transmitted waves are set up. It is interesting to study these waves and the reflection coefficients when an electromagnetic wave from free space is incident on an anisotropic ionized gas of semi-infinite extent.

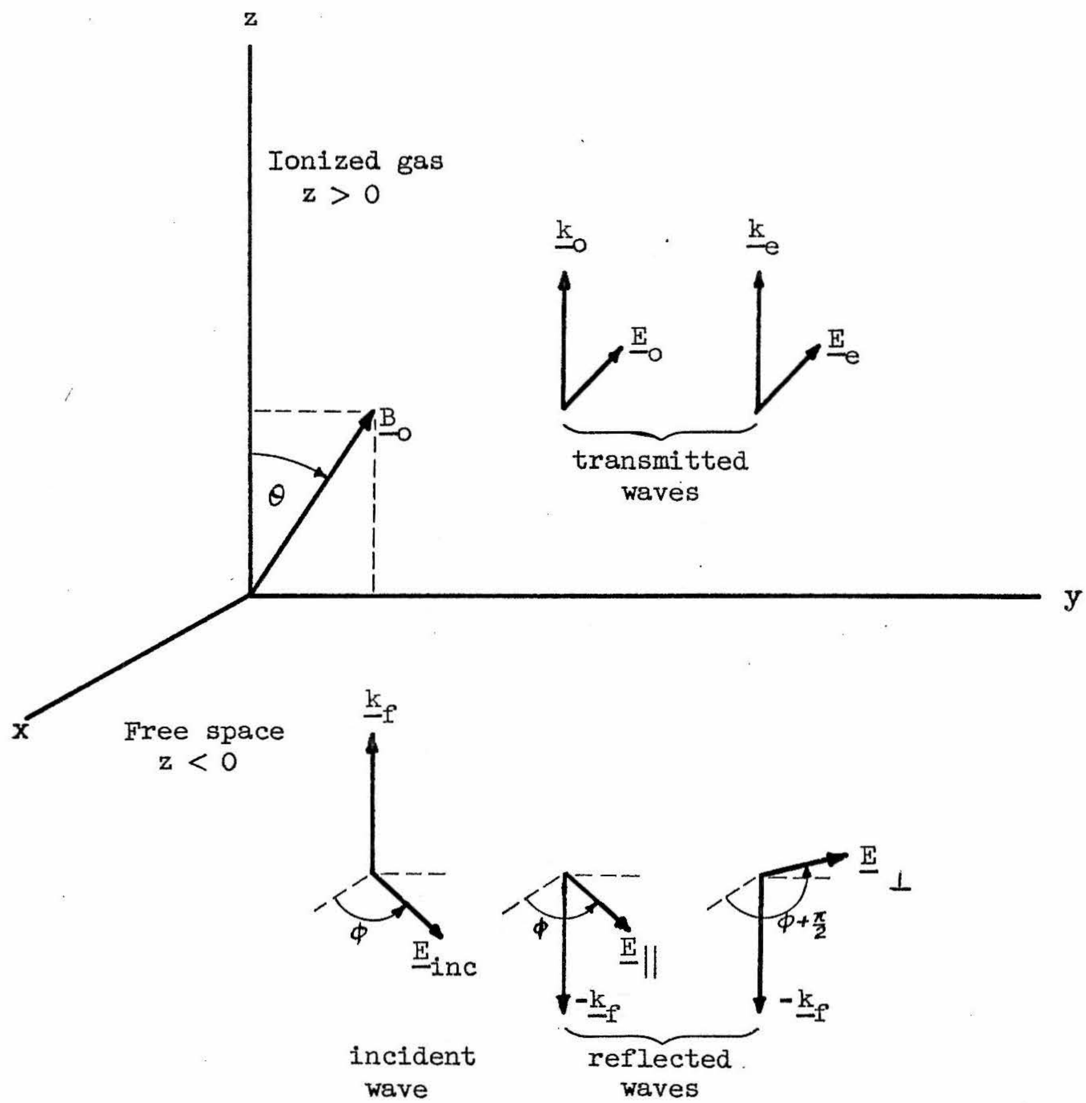

Fig. 2. Reflected and transmitted waves

For simplicity, consider a plane TEM wave from free space incident normally on the interface $z = 0$ separating free space from that occupied by an ionized gas which is externally biased with an oblique magnetostatic field. Choose a cartesian coordinate system such that the ionized gas occupies the half-space $z > 0$ as shown in Fig. 2. The external magnetostatic field is parallel to the y-z plane and makes an angle θ with the z-axis. The electric field vector of the normally incident TEM wave is assumed to make an angle ϕ with the x-axis, and the direction of propagation of the wave is parallel to the z-axis.

To find the reflected and the transmitted waves we must solve the vector wave equation for the electric fields in both half-spaces. The solutions for the electric fields of the ordinary and the extraordinary waves in the ionized gas $z > 0$ have been found in Section 2.3. Now the solutions for the electric fields of the waves in free space will be found.

In free space the vector wave equation takes the form, with the time dependence $\exp(-i\omega t)$ omitted,

$$\nabla \times \nabla \times \underline{E} = \omega^2 \mu_o \epsilon_o \underline{E} \tag{2.13}$$

Its solutions are

$$\left.\begin{aligned} E_x &= A \cos \phi \\ E_y &= A \sin \phi \\ E_z &= 0 \end{aligned}\right\} \quad e^{\pm i k_f z}$$

where k_f $(=\omega\sqrt{\mu_o \epsilon_o})$ is the free space propagation constant and A is an undetermined coefficient. From Maxwell's equation

$$\nabla \times \underline{E} = -\mu_o \frac{\partial \underline{H}}{\partial t}$$

the corresponding magnetic field components are found to be

$$H_x = \mp \frac{k_f}{\omega \mu_o} E_y$$

$$H_y = \pm \frac{k_f}{\omega \mu_o} E_x$$

$$H_z = 0 \quad .$$

The field components of the incident TEM wave of unit amplitude take the following form:

$$\left.\begin{aligned} E_x &= \cos \phi \\ E_y &= \sin \phi \\ E_z &= 0 \\ H_x &= -\frac{k_f}{\omega \mu_o} \sin \phi \\ H_y &= \frac{k_f}{\omega \mu_o} \cos \phi \\ H_z &= 0 \end{aligned}\right\} e^{ik_f z} \quad . \tag{2.14}$$

Generally the electric field vector of the reflected TEM wave is different from the electric field vector of the incident wave both in magnitude and in direction. It is possible to decompose the reflected wave into two components, one of which has the electric field parallel to that of the incident wave, while the other has the electric field perpendicular to that of the incident wave. Define the two coefficients of the reflection for these two components with respect to the incident wave as

$$\left.\begin{aligned} R_{||} &= \left|\frac{E_{||}}{E_{inc}}\right| \\ \text{and} \qquad R_{\perp} &= \left|\frac{E_{\perp}}{E_{inc}}\right| \end{aligned}\right\} \quad \text{at} \quad z = 0$$

then the field components of the two reflected waves are

$$\left.\begin{aligned} E_x &= \cos \phi \\ E_y &= \sin \phi \\ E_z &= 0 \\ H_x &= \frac{k_f}{\omega \mu_o} \sin \phi \\ H_y &= -\frac{k_f}{\omega \mu_o} \cos \phi \\ H_z &= 0 \end{aligned}\right\} R_{||} e^{-ik_f z} \qquad (2.15)$$

and

$$\left.\begin{aligned} E_x &= -\sin \phi \\ E_y &= \cos \phi \\ E_z &= 0 \\ H_x &= \frac{k_f}{\omega \mu_o} \cos \phi \\ H_y &= \frac{k_f}{\omega \mu_o} \sin \phi \\ H_z &= 0 \end{aligned}\right\} R_{\perp} e^{-ik_f z} . \qquad (2.16)$$

In the space $z > 0$, which is occupied by the ionized gas, there are two transmitted waves; namely the ordinary and the extraordinary waves. Their field components have been found and are given by equation 2.11 and equation 2.12 with two undetermined coefficients A_o and A_e .

We shall use the boundary conditions at the interface to solve for the four undetermined coefficients $R_{||}$, $R_{\perp}$, A_o and A_e . At $z = 0$ the boundary conditions require that the tangential component of the electric field be continuous and that, since there is no surface charge, the tangential component of the magnetic field be continuous. The boundary conditions lead to the following set of four equations with four unknown coefficients:

$$R_{||} \cos \phi - R_{\perp} \sin \phi + iR_o A_o + iR_e A_e = - \cos \phi$$

$$R_{||} \sin \phi + R_{\perp} \cos \phi - A_o - A_e = - \sin \phi$$

$$k_f R_{||} \sin \phi + k_f R_{\perp} \cos \phi + k_o A_o + k_e A_e = k_f \sin \phi$$

$$k_f R_{||} \cos \phi - k_f R_{\perp} \sin \phi - ik_o R_o A_o - ik_e R_e A_e = k_f \cos \phi \; . \qquad (2.17)$$

The first two equations are for the x- and the y-components of the electric field respectively, and the last two equations are for the x- and the y-components of the magnetic field respectively.

In matrix notation equation 2.17 takes the form

$$[M]\begin{bmatrix} R_{||} \\ R_{\perp} \\ A_o \\ A_e \end{bmatrix} = \begin{bmatrix} -\cos\phi \\ -\sin\phi \\ k_f\sin\phi \\ k_f\cos\phi \end{bmatrix}$$

where the matrix $[M]$ is

$$[M] = \begin{bmatrix} \cos\phi, & -\sin\phi, & iR_o, & iR_e \\ \sin\phi, & \cos\phi, & -1, & -1 \\ k_f\sin\phi, & k_f\cos\phi, & k_o, & k_e \\ k_f\cos\phi, & -k_f\sin\phi, & -ik_oR_o, & -ik_eR_e \end{bmatrix} .$$

The solutions for $R_{||}$, $R_{\perp}$, A_o and A_e can be obtained by inverting the matrix $[M]$. The two coefficients of reflection are then found to be

$$R_{||} = -\frac{(k_ok_e - k_f^2)(R_o - R_e) + (k_o - k_e)\ k_f(R_o + R_e)\ \cos 2\phi}{(k_o + k_f)(k_e + k_f)(R_o - R_e)}$$

and

$$R_{\perp} = -\frac{i(k_o - k_e)\ k_f\ [2 + i(R_o + R_e)\ \sin 2\phi]}{(k_o + k_f)(k_e + k_f)(R_o - R_e)} . \tag{2.18}$$

The amplitudes of the y-components of the transmitted ordinary wave and the transmitted extraordinary wave are found to be

$$A_o = \frac{2ik_f(\cos\phi + iR_e\sin\phi)}{(k_o + k_f)(R_o - R_e)}$$

$$A_e = -\frac{2ik_f(\cos\phi + iR_o\sin\phi)}{(k_e + k_f)(R_o - R_e)} . \tag{2.19}$$

Note that all notations used here have been defined in Sections 2.3 and 2.4.

We have now reviewed wave propagation in a linear ionized gas and the reflection of a wave from a gas of semi-infinite extent as an introduction to the treatment of the nonlinear problems considered in the following chapters.

III. ELECTRON DISTRIBUTION FUNCTION

In this chapter we shall study nonlinear phenomena in the propagation of electromagnetic waves in a weakly ionized gas. Since the nonlinear problems have great importance in the propagation of broadcast signals through the ionosphere and in reflection of signals from the ionosphere, the weakly ionized gas will be assumed to have physical properties similar to those of the ionosphere, and the electromagnetic waves will be assumed to have physical parameters similar to those of ordinary broadcast signals. Approximations derived from the above assumptions will be used extensively to simplify the mathematics and to show the effects of nonlinear phenomena explicitly.

The study starts with the use of kinetic theory to find the distribution function of free electrons in the weakly ionized gas. From the electron distribution function, the electron convection current density will be derived. Since the mass ratio of an electron to an ion is extremely small, an electron can respond to the external accelerating fields much faster than an ion. Hence the ion convection current density may be neglected in comparison with that of the electrons in the derivation of the conductivity tensors of the ionized gas. From the nonlinear conductivity tensors, the effects of nonlinear phenomena on wave propagation will be found.

3.1) Boltzmann's Kinetic Equation

In the exact kinetic theory Boltzmann's kinetic equation is used to solve for the distribution functions of charged particles. Since the free electrons contribute most to the convection current density in a weakly ionized gas, the charged particles considered here will be the free electrons whose distribution function f can be derived from Boltzmann's kinetic equation

$$\frac{\partial f}{\partial t} + \sum_i \frac{\partial x_i}{\partial t} \frac{\partial f}{\partial x_i} + \sum_i \frac{\partial v_i}{\partial t} \frac{\partial f}{\partial v_i} = \left(\frac{\delta f}{\delta t}\right)_{coll} \qquad (3.1)$$

where x_i is one of the orthogonal components of the spatial vector and v_i is one of the orthogonal components of the velocity vector of the free electrons. The collision term $\left(\frac{\delta f}{\delta t}\right)_{coll}$ on the right of equation 3.1 is the time rate of change of the distribution function due to collisions of the electrons with other particles in the ionized gas. In a weakly ionized gas most of the particles are neutral molecules, there being relatively few electrons and ions. Electron-molecule collisions effect the most change in electron distribution function. The few electron-ion and electron-electron collisions may be neglected in the kinetic equation. If the external accelerating electric field is not too strong, the average kinetic energy of the free electrons is below the ionization energy of the molecules, and the electron-molecule collisions are elastic. The collision term due to elastic electron-molecule collisions will be written in the form derived by Chapman and Cowling (3) as

$$\left(\frac{\delta f}{\delta t}\right)_{coll} = \int \left[f(v') \, F(V') - f(v) \, F(V)\right] \, uq \, d\Omega \, d^3V \qquad (3.2)$$

where f is the electron distribution function and F is the molecular distribution function. The electron travels with an initial velocity v and a velocity v' after collision, while the molecule has velocities V and V' before and after a collision respectively. Let u be the magnitude of the relative velocity before collision, i.e., $u = |\underline{v} - \underline{V}|$. q is the collision cross-section for momentum transfer between the two particles during collision. $q\, d\Omega$ is the fraction of the incoming flux of electrons which is scattered into solid angle $d\Omega$ by a single scattering center. This form, equation 3.2, for the collision term is valid for binary, elastic, hard collisions where a short-range force exists between the two colliding particles.

Now the weakly ionized gas is assumed to be externally biased with a magnetostatic field $\underline{B}_o$, and is disturbed by a number of electromagnetic waves with an electric field $\underline{E}$ and a magnetic field $\underline{B}$. Under such fields the Lorentz force on a negatively charged electron is $-e(\underline{E} + \underline{v} \times \underline{B} + \underline{v} \times \underline{B}_o)$ where e is the magnitude of charge on an electron and $\underline{v}$ is the velocity of the electron. Then the acceleration of an electron with mass m is $-\frac{e}{m}(\underline{E} + \underline{v} \times \underline{B} + \underline{v} \times \underline{B}_o)$.

It is shown in the following that the acceleration of the electron due to the magnetic field $\underline{B}$ of the wave can be neglected in comparison with that due to the electric field $\underline{E}$. From Maxwell's equations the ratio of acceleration due to $\underline{B}$ to that due to $\underline{E}$ is

$$\frac{|\underline{v} \times \underline{B}|}{|\underline{E}|} \sim \frac{v}{c}$$

where c is the light velocity in vacuum. From the electron distribution function derived later, it is found that only .04% of the electrons has velocity greater than $3\sqrt{\frac{2kT}{m}}$ where k is the

Boltzmann's constant and T is the electron temperature. This .04% of the electrons is found to contribute only 0.3% of the electron convection current density. Since only one out of 10^9 electrons has velocity greater than $5\sqrt{\frac{2kT}{m}}$, electrons having velocity greater than $5\sqrt{\frac{2kT}{m}}$ will be neglected. In the various layers of the ionosphere the electron temperature ranges from 200°C to 2000°C and the corresponding value of $\sqrt{\frac{2kT}{m}}$ ranges from 7.8×10^4 to 2.5×10^5 m sec^{-1}. The average electron velocity $\sqrt{\frac{2kT}{m}}$ is three orders of magnitude less than the light velocity in vacuum, and so the term $\underline{v} \times \underline{B}$ can be neglected.

In terms of the accelerating fields, Boltzmann's kinetic equation of motion takes the form

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \nabla_r f - \frac{e}{m}(\underline{E} + \underline{v} \times \underline{B}_o) \cdot \nabla_v f = \left(\frac{\delta f}{\delta t}\right)_{coll} \tag{3.3}$$

where ∇_r stands for the spatial gradient operator and ∇_v the velocity gradient operator.

The solution of equation 3.3 for the distribution function f is difficult to obtain, in general, unless we make an assumption that the electric field of the electromagnetic waves is not too strong and hence the electron distribution function f deviates only slightly from being isotropic. In order to show the conditions under which such an assumption is valid, the distribution function f in an isotropic ionized gas will be examined first. Then we shall derive the solution of equation 3.3 for the distribution function f in an anisotropic gas.

3.2) Distribution Function in an Isotropic Ionized Gas

In the absence of any external accelerating field, the electrons, the ions and the molecules in the ionized gas are in thermal equilibrium with random thermal velocity. As soon as electromagnetic waves are present in the gas, their electric field will give a directed velocity to the electrons. Under the assumption that the electric field is not too strong, the random thermal velocity of the electrons will be much greater than their directed velocity. Consequently, the distribution function will deviate slightly from being isotropic and it is possible to separate the distribution function into its principal part, which is isotropic in velocity space, and its directional parts which are functions of the angle between $\underline{v}$ and the electric field $\underline{E}$. Let the direction of the external electric field $\underline{E}$ be parallel to the z-axis in the cartesian coordinate system and let $\underline{v}$ make an angle Θ_v with the z-axis. Since $\underline{v}$ is symmetrical about $\underline{E}$, the distribution function can be expanded in the zeroth order spherical harmonics, namely in Legendre polynomials

$$f = \sum_{\ell} f_{\ell}(\underline{r}, v, t)\, P_{\ell}(\cos \Theta_v) \quad . \tag{3.4}$$

Note that f_{ℓ}'s are explicit functions of v, but not functions of components of $\underline{v}$.

In an isotropic ionized gas with the external electric field $\underline{E}$ lying along the z-axis, Boltzmann's kinetic equation has the form

$$\frac{\partial f}{\partial t} + v_z \frac{\partial f}{\partial z} - \frac{e}{m} E \frac{\partial f}{\partial v_z} = \left(\frac{\delta f}{\delta t}\right)_{coll} \quad . \tag{3.5}$$

Substitute the expansion of f, given in equation 3.4, into equation 3.5. Multiply both sides by $P_m(\cos\Theta_v)$ and integrate over the solid angle $d\Omega_v$ in velocity space. With the aid of the orthogonal properties of P_ℓ's , namely

$$\int P_m P_\ell \, d\Omega_\ell = \frac{4\pi}{2\ell+1} \delta_{\ell m}$$

where

$$\delta_{\ell m} = 1 \quad \text{if} \quad \ell = m$$
$$= 0 \quad \text{if} \quad \ell \neq m$$

and the differentiation

$$\frac{\partial}{\partial v_z}\left\{f_\ell P_\ell(\cos\Theta_v)\right\} = P_\ell(\cos\Theta_v)\cos\Theta_v \frac{\partial f_\ell}{\partial v} + \frac{f_\ell \sin^2\Theta_v}{v} \frac{\partial P_\ell}{\partial(\cos\Theta_v)}$$

where $v_z = v\cos\Theta_v$

and $v^2 = v_x^2 + v_y^2 + v_z^2$,

a set of equations which links various spherical harmonics of f is obtained, namely

$$\frac{\partial f_o}{\partial t} + \frac{v}{3}\frac{\partial f_1}{\partial z} - \frac{eE}{3mv^2}\frac{\partial}{\partial v}(v^2 f_1) = \left(\frac{\delta f_o}{\delta t}\right)_{coll}$$

$$\frac{\partial f_1}{\partial t} + v\left(\frac{\partial f_o}{\partial z} + \frac{2}{5}\frac{\partial f_2}{\partial z}\right) - \frac{eE}{m}\left[\frac{\partial f_o}{\partial v} + \frac{2}{5v^3}\frac{\partial}{\partial v}(v^3 f_2)\right] = \left(\frac{\delta f_1}{\delta t}\right)_{coll}$$

$$\frac{\partial f_2}{\partial t} + v\left(\frac{2}{3}\frac{\partial f_1}{\partial z} + \frac{3}{7}\frac{\partial f_3}{\partial z}\right) - \frac{eE}{m}\left[\frac{2}{3}v\frac{\partial}{\partial v}\left(\frac{f_1}{v}\right) + \frac{3}{7v^4}\frac{\partial}{\partial v}(v^4 f_3)\right] = \left(\frac{\delta f_2}{\delta t}\right)_{coll}$$

-- (3.6)

where $$\left(\frac{\delta f_\ell}{\delta t}\right)_{coll} = \frac{2\ell+1}{4\pi}\int \left(\frac{\delta f}{\delta t}\right)_{coll} P_\ell(\cos \Theta_v)\, d\Omega_v \quad . \qquad (3.7)$$

To solve the set of equations 3.6 it is necessary first to determine $\left(\frac{\delta f_\ell}{\delta t}\right)_{coll}$ from equation 3.7. Consider the case where $\ell \neq 0$. Substitute the expression 3.2 for $\left(\frac{\delta f}{\delta t}\right)_{coll}$ in equation 3.7 and expand $f(v)$ and $f(v')$ into spherical harmonics in velocity space. Then equation 3.7 takes the form

$$\left(\frac{\delta f_\ell}{\delta t}\right)_{coll} = \frac{2\ell+1}{4\pi}\int d\Omega_v\, P_\ell(\cos \Theta_v) \iint uq\, d\Omega\, d^3V$$

$$\times \left\{F(V')\sum_m f_m(v')P_m(\cos \Theta_{v'}) - F(V)\sum_m f_m(v)\, P_m(\cos \Theta_v)\right\}$$

where Θ_v is the angle between $\underline{v}$ and the z-axis and $\Theta_{v'}$ is the angle between $\underline{v}'$ and the z-axis. Choose the new spherical coordinate system (Θ,ϕ) whose polar axis coincides with $\underline{v}$ so that the direction of $\underline{v}'$ is $(\Theta,0)$, where Θ is the angle between the initial velocity $\underline{v}$ and the final velocity $\underline{v}'$ of the electron, and the direction of the original z-axis is (Θ_v,ϕ) . From geometry it is found that

$$\cos \Theta_{v'} = \cos \Theta_v \cos \Theta + \sin \Theta_v \sin \Theta \cos \phi \; .$$

The Legendre polynomial $P_m(\cos \Theta_{v'})$ can now be expanded in the new spherical coordinate system. The expansion appears in many mathematical physics texts as

$$P_m(\cos\Theta_{v'}) = P_m(\cos\Theta_v)\, P_m(\cos\Theta) + 2\sum_{n=1}^{m} \frac{(m-n)!}{(m+n)!} P_m^n(\cos\Theta_v)\, P_m^n(\cos\Theta)\, \cos n\phi \;.$$

Because of the orthogonal properties of $P_m^n(\mu)$, namely

$$\int_{-1}^{1} P_m^n(\mu)\, P_{m'}^{n'}(\mu)\, d\mu = \frac{2}{2m+1}\frac{(m+n)!}{(m-n)!}\,\delta_{nn'}\,\delta_{mm'} \;,$$

the integration of terms containing $\cos n\phi$ over the solid angle $d\Omega_v$ yields zero value and equation 3.7 becomes for $\ell \neq 0$

$$\left(\frac{\delta f_\ell}{\delta t}\right)_{coll} = \iint uq\, d\Omega\, d^3V \left\{F(V')\, f_\ell(v')\, P_\ell(\cos\Theta) - F(V)\, f_\ell(v)\right\} \;.$$

Because the mass ratio of an electron to a molecule is extremely small, there is little energy transfer during an elastic electron-molecule collision. Hence we can use the approximations $v \sim v' \sim u$ and $V \sim V'$ to obtain

$$\left(\frac{\delta f_\ell}{\delta t}\right)_{coll} = -\nu_\ell f_\ell \qquad (3.8)$$

for $\ell \neq 0$. The collision frequencies ν_ℓ are defined by

$$\nu_\ell = Nv \int q \left[1 - P_\ell(\cos\Theta)\right] d\Omega$$

and the molecule density N in the gas is

$$N = \int F(V)\, d^3V \;.$$

The isotropic part $(\frac{\delta f_o}{\delta t})_{coll}$ of the collision term has been derived by Desloge and Matthysse (6). They assumed that the electron-molecule collision was between two hard spheres having a short-range force between them. Because of the heavy mass M of the molecule, the molecular distribution function was assumed to be undisturbed by the external fields and to have a Maxwellian distribution, namely $F = (\frac{M}{2\pi kT})^{3/2} Ne^{-MV^2/2kT}$ where T was the molecular temperature and k was Boltzmann's constant. Using the laws of conservation of energy and of momentum during an elastic collision, they obtained the expression

$$(\frac{\delta f_o}{\delta t})_{coll} = \frac{1}{v^2} \frac{\partial}{\partial v} \left\{ v^2 \frac{\nu_1 kT}{M} \frac{\partial f_o}{\partial v} + v^3 \frac{\nu_1 m}{M} f_o \right\} . \tag{3.9}$$

Note that the isotropic part $(\frac{\delta f_o}{\delta t})_{coll}$ is due to energy transfer and the other parts $(\frac{\delta f_\ell}{\delta t})_{coll}$ with $\ell \neq 0$ are due to momentum transfer during collision.

Returning to equation 3.6, we are now in the position to examine conditions under which the set of equations 3.6 can be terminated at the first two equations. From the second equation of 3.6 these conditions are found to be

$$\frac{\partial f_o}{\partial z} \gg \frac{\partial f_2}{\partial z} \tag{3.10}$$

and

$$\frac{\partial f_o}{\partial v} \gg \frac{1}{v^3} \frac{\partial}{\partial v} (v^3 f_2) . \tag{3.11}$$

Assume there is only one electromagnetic wave with carrier frequency ω

in the ionized gas and neglect the dispersion term $\underline{v} \cdot \nabla_r f$ in Boltzmann's kinetic equation 3.5. Then from the second and the third equations of 3.6 we get the following approximate solutions for f_1 and f_2

$$|f_1| \sim \frac{1}{(\nu_1^2 + \omega^2)^{1/2}} \frac{e|E|}{m} \frac{\partial f_o}{\partial v}$$

and

$$|f_2| \sim \frac{2e^2E^2}{3m^2(\nu_1^2 + \omega^2)} v \left| \frac{\partial}{\partial v} \left(\frac{1}{v} \frac{\partial f_o}{\partial v}\right) \right|$$

where the approximation $\nu_1 \sim \nu_2$ is used. The collision frequencies ν_ℓ are defined by equation 3.8. The approximate solution of f_o will be found later to be $C_o e^{-mv^2/2kT}$. With the above approximate solutions for f_o, f_1 and f_2, the inequalities 3.10 and 3.11 become

$$\frac{e^2E^2}{mkT(\nu_1^2 + \omega^2)} \left(\frac{mv^2}{kT}\right) \ll 1$$

and

$$\frac{e^2E^2}{mkT(\nu_1^2 + \omega^2)} \left|-3 + \frac{mv^2}{kT}\right| \ll 1 .$$

Since we find that only one out of 10^9 electrons has velocity greater than $5\sqrt{\frac{2kT}{m}}$, for most electrons the conditions 3.10 and 3.11 are reduced further to

$$\frac{e^2E^2}{mkT(\nu_1^2 + \omega^2)} \ll 1 .$$

The inequality is well satisfied for broadcast signals in various layers of the ionosphere as shown in Table I. There the electric

TABLE I. Value of $\frac{e^2E^2}{mkT(\nu_1^2 + \omega^2)}$

Ionosphere	ω (sec^{-1})	$\frac{mkT(\nu_1^2 + \omega^2)}{e^2}$ $(v/m)^2$	$\frac{e^2E^2}{mkT(\nu_1^2 + \omega^2)}$ 10 kw	50 kw	100 kw	500 kw	1000 kw	5000 kw
D-layer(daytime)	10^5	1.5x10	$1.7x10^{-5}$	$8.5x10^{-5}$	$1.7x10^{-4}$	$8.5x10^{-4}$	$1.7x10^{-3}$	$8.5x10^{-3}$
$\nu_1 = 10^7$	10^6	1.5x10	$1.7x10^{-5}$	$8.5x10^{-5}$	$1.7x10^{-4}$	$8.5x10^{-4}$	$1.7x10^{-3}$	$8.5x10^{-3}$
$T = 300^oC$	10^7	2.9x10	$8.5x10^{-6}$	$4.2x10^{-5}$	$8.5x10^{-5}$	$4.2x10^{-4}$	$8.5x10^{-4}$	$4.2x10^{-3}$
z = 60 km	10^8	$1.5x10^3$	$1.7x10^{-7}$	$8.5x10^{-7}$	$1.7x10^{-6}$	$8.5x10^{-6}$	$1.7x10^{-5}$	$8.5x10^{-5}$
Lower part of E-layer (night)	10^5	$4.9x10^{-2}$	$2.3x10^{-3}$	$1.1x10^{-2}$	$2.3x10^{-2}$	$1.1x10^{-1}$	$2.3x10^{-1}$	1.1
$\nu_1 = 7x10^5$	10^6	$1.5x10^{-1}$	$7.6x10^{-4}$	$3.8x10^{-3}$	$7.6x10^{-3}$	$3.8x10^{-2}$	$7.6x10^{-2}$	$3.8x10^{-1}$
$T = 200^oC$	10^7	9.8	$1.1x10^{-5}$	$5.5x10^{-5}$	$1.1x10^{-4}$	$5.5x10^{-4}$	$1.1x10^{-3}$	$5.5x10^{-3}$
z = 90 km	10^8	$9.8x10^2$	$1.1x10^{-7}$	$5.5x10^{-7}$	$1.1x10^{-6}$	$5.5x10^{-6}$	$1.1x10^{-5}$	$5.5x10^{-5}$
F-layer	10^5	$9.8x10^{-3}$	$1.0x10^{-3}$	$5.0x10^{-3}$	$1.0x10^{-2}$	$5.0x10^{-2}$	$1.0x10^{-1}$	$5.0x10^{-1}$
$\nu_1 = 10^3$	10^6	$9.8x10^{-1}$	$1.0x10^{-5}$	$5.0x10^{-5}$	$1.0x10^{-4}$	$5.0x10^{-4}$	$1.0x10^{-3}$	$5.0x10^{-3}$
$T = 2000^oC$	10^7	9.8x10	$1.0x10^{-7}$	$5.0x10^{-7}$	$1.0x10^{-6}$	$5.0x10^{-6}$	$1.0x10^{-5}$	$5.0x10^{-5}$
z = 300 km	10^8	$9.8x10^3$	$1.0x10^{-9}$	$5.0x10^{-9}$	$1.0x10^{-8}$	$5.0x10^{-8}$	$1.0x10^{-7}$	$5.0x10^{-7}$

field $\underline{E}$ of the signal at a distance z km from the station having an output power W kw is found from the relation

$$E = \frac{300\sqrt{W}}{z} \cdot \frac{mv}{m} \quad .$$

From Table I it is clear that $\frac{e^2E^2}{mkT(\nu_1^2 + \omega^2)} \ll 1$ always holds even for broadcast stations having an output power as high as one megawatt and a carrier angular frequency ω of $10^5 sec^{-1}$ or higher. Hence it can be concluded that the electric field of the signal from even a powerful station can be considered small enough so that the second and higher spherical harmonics in equation 3.4 can be neglected in comparison with the zeroth and the first harmonics, namely, f_o and f_1.

We shall not find the electron distribution function of an isotropic ionized gas as the function can be obtained from the electron distribution function of an anisotropic ionized gas which will be derived in the next section.

3.3) Distribution Function in an Anisotropic Ionized Gas

In the previous section it was shown that the electron distribution function deviates slightly from being isotropic in velocity space so that the function can be approximated by the zeroth and the first spherical harmonics in velocity space. Under the same assumption that the electric field of the electromagnetic waves is not too strong, the electron distribution function can be expanded in spherical harmonics in velocity space and only the first two terms in the expansion are retained, namely

$$f = f_o + \frac{\underline{v} \cdot \underline{f}_1}{v} \quad . \tag{3.12}$$

Note that $\underline{f}_1$ appears as a vector.

To find f_o and $\underline{f}_1$ we must decompose Boltzmann's kinetic equation 3.3 into a set of equations linking f_o and $\underline{f}_1$. Insert the spherical harmonics expansion 3.12 into Boltzmann's kinetic equation 3.3, multiply both sides by $P_n^m(\cos\Theta_v)$ and integrate over the solid angle $d\Omega_v$. Here Θ_v is the angle between $\underline{v}$ and the z-axis and $d\Omega_v$ is the solid angle in velocity space. Using the orthogonal properties of the associated Legendre function $P_n^m(\mu)$, we obtain the following two equations

$$\frac{\partial f_o}{\partial t} + \frac{v}{3} \nabla_r \cdot \underline{f}_1 - \frac{e}{3mv^2} \frac{\partial}{\partial v} (v^2 \underline{E} \cdot \underline{f}_1) = (\frac{\delta f_o}{\delta t})_{coll} \tag{3.13}$$

and

$$\frac{\partial \underline{f}_1}{\partial t} + v \nabla_r f_o - \frac{e\underline{E}}{m} \frac{\partial f_o}{\partial v} - \frac{e}{m} \underline{B}_o \times \underline{f}_1 = (\frac{\delta \underline{f}_1}{\delta t})_{coll} \quad . \tag{3.14}$$

The last vector equation is obtained through its component equations. In deriving equations 3.13 and 3.14 we have assumed that f_o and $\underline{f}_1$ are functions of v, and have used the following results

$$\underline{E} \cdot \nabla_v f = \frac{\underline{v} \cdot \underline{E}}{v} \frac{\partial f_o}{\partial v} + \frac{1}{v} \underline{E} \cdot \underline{f}_1 + \underline{v} \cdot \underline{E} \left\{ \frac{\underline{v} \cdot \partial \underline{f}_1/\partial v}{v^2} - \frac{\underline{v} \cdot \underline{f}_1}{v^3} \right\}$$

and

$$(\underline{v} \times \underline{B}_o) \cdot \nabla_v f = \frac{\underline{v} \cdot (\underline{B}_o \times \underline{f}_1)}{v} \quad .$$

The components of the collision term on the right of equations 3.13 and 3.14 are still given by equations 3.9 and 3.8.

Consider equations 3.13 and 3.14. To the first approximation, by neglecting the term $v\nabla_r f_o$ in equation 3.14, $\underline{f}_1$ and $\underline{E}$ are found to be proportional. In the study of nonlinear problems, we cannot use a phasor to represent an electric field such as $\underline{E}(\underline{r},t) = \underline{E}(\underline{r})e^{-i\omega t}$ in Chapter II. The real part of the phasor will be used to represent the actual physical electric field. Since the sum of a clockwise phasor and a counterclockwise phasor is proportional to the real part of either a clockwise phasor or a counterclockwise phasor, the electric field $\underline{E}(\underline{r},t)$ of m electromagnetic waves can be explicitly written as

$$\underline{E}(r,t) = \sum_{p=1}^{m} \left\{ \underline{E}^p_{-1}(\underline{r})\, e^{-i\omega_p t} + \underline{E}^p_1(\underline{r})\, e^{i\omega_p t} \right\} \qquad (3.15)$$

where $\underline{E}^p_{-1}(\underline{r})\, e^{-i\omega_p t}$ is the clockwise phasor of the electric field of the p^{th} wave with a carrier angular frequency ω_p and $\underline{E}^p_1(\underline{r})\, e^{i\omega_p t}$ is the counterclockwise phasor. It is clear that $\underline{E}^p_{-1}$ and $\underline{E}^p_1$ are complex conjugates of each other. To the first approximation the anisotropic part $\underline{f}_1$ of the electron distribution function has a similar form,

$$\underline{f}_1(\underline{r},t) = \sum_{p=1}^{m} \left\{ \underline{f}^p_{1,-1}(\underline{r})\, e^{-i\omega_p t} + \underline{f}^p_{1,1}(\underline{r})\, e^{i\omega_p t} \right\} .$$

When these two forms of $\underline{E}(\underline{r},t)$ and $\underline{f}_1(\underline{r},t)$ are used in equation 3.13, the isotropic part f_o of the distribution function is found to have time-dependent components as well as a time-invariant component. From equation 3.14, because of the term $\underline{E}\frac{\partial f_o}{\partial v}$, $\underline{f}_1$ will have terms representing higher harmonics of the carrier frequencies. Both f_o and $\underline{f}_1$ can now be expanded into Fourier series as

$$f_o(\underline{r},v,t) = \cdots + \sum_{p=1}^{m} f^p_{0,-1}(\underline{r},v)\, e^{-i\omega_p t} + f_{0,0}(\underline{r},v) + \sum_{p=1}^{m} f^p_{0,1}(\underline{r},v)\, e^{i\omega_p t} + \cdots \tag{3.16}$$

and

$$\underline{f}_1(\underline{r},v,t) = \cdots + \sum_{p,q=1}^{m} \underline{f}^{p,q}_{1,-2}(\underline{r},v)\, e^{-i(\omega_p+\omega_q)t} + \sum_{p=1}^{m} \underline{f}^p_{1,-1}(\underline{r},v) e^{-i\omega_p t} + \sum_{p,q=1}^{m} \underline{f}^{p,q}_{1,0}(\underline{r},v)\, e^{-i(\omega_p-\omega_q)t} + \sum_{p=1}^{m} \underline{f}^p_{1,1}(\underline{r},v) e^{i\omega_p t} + \sum_{p,q=1}^{m} \underline{f}^{p,q}_{1,2}(\underline{r},v)\, e^{i(\omega_p+\omega_q)t} + \cdots\cdots \tag{3.17}$$

where higher order terms in the expansions have been omitted under the assumption that the electric field $\underline{E}$ of the waves is not too strong. Here p or q stands for the p^{th} or the q^{th} wave of m electromagnetic waves. The first subscript i of $f^{p,q}_{i,j}$ indicates the order of the spherical harmonics in velocity space, and the second subscript j is the order of harmonics in the Fourier series. The corresponding pairs in equations 3.16 and 3.17 must be complex conjugates, namely

$$f^p_{0,-1} = (f^p_{0,1})^*$$

$$\underline{f}^{p,q}_{1,-2} = (\underline{f}^{p,q}_{1,2})^*$$

$$\underline{f}^{p}_{-1,-1} = (\underline{f}^{p}_{-1,1})^*$$

and

$$\underline{f}^{p,q}_{-1,0} = (\underline{f}^{q,p}_{-1,0})^* \quad .$$

It is still difficult to solve equations 3.13 and 3.14 simultaneously for the harmonic components of f_o and $\underline{f}_1$ unless the relative magnitudes of the terms in equations 3.13, 3,14, 3.16 and 3.17 are known. Then the harmonic components can be found by means of perturbation method. Under the assumption that the electric field of the electromagnetic waves is not too strong, the terms containing $\underline{E}$ in equations 3.13 and 3.14 are small. Because the velocity of most electrons is much smaller than the velocity of light in vacuum, the term $\frac{v}{3}\nabla_r \cdot \underline{f}_1$ in equation 3.13 and the term $v\nabla_r f_o$ in equation 3.14 are also small. Since the distribution function is found to deviate slightly from being isotropic, $f^{p}_{0,-1}$, $f^{p}_{0,1}$, $\underline{f}^{p}_{-1,-1}$ and $\underline{f}^{p}_{-1,1}$ in equations 3.16 and 3.17 are of first order of smallness in comparison with $f_{0,0}$; and $\underline{f}^{p,q}_{-1,-2}$, $\underline{f}^{p,q}_{-1,0}$ and $\underline{f}^{p,q}_{-1,2}$ are of second order of smallness in comparison with $f_{0,0}$. The relative orders of magnitude of the various harmonic components of the electron distribution function can be identified by the number of superscript p's and q's.

Now we are in the position to solve equations 3.13 and 3.14 for the harmonic components of f_o and $\underline{f}_1$. From equation 3.14 the components of $\underline{f}_1$ can be found in terms of $\underline{E}$ and the components of f_o. With the solutions of the components of $\underline{f}_1$ substituted in equation 3.13, the components of f_o can be found explicitly in terms of $\underline{E}$ and the physical parameters of the ionized gas. We shall follow this procedure of solving equations 3.13 and 3.14.

Substitute the expansions 3.16 and 3.17 in equation 3.14 and equate coefficients of the corresponding terms of the same exponentials in frequency. This is possible because of their orthogonal properties, i.e., 1, $e^{-i\omega_p t}$, $e^{i\omega_p t}$, $e^{-i(\omega_p+\omega_q)t}$, etc. are orthogonal functions. Neglect terms of higher order of smallness; then, from terms of $e^{-(\omega_p+\omega_q)t}$, we obtain

$$\underline{f}^{p,q}_{-1,-2} = \frac{e}{m} \frac{\partial f^q_{0,-1}}{\partial v} \underline{\underline{L}}(\omega_p + \omega_q) \cdot \underline{E}^p_{-1} \quad . \tag{3.18}$$

Similarly, from terms of $e^{-i\omega_p t}$

$$\underline{f}^{p}_{-1,-1} = \frac{e}{m} \frac{\partial f_{0,0}}{\partial v} \underline{\underline{L}}(\omega_p) \cdot \underline{E}^p_{-1} \tag{3.19}$$

and from terms of $e^{-i(\omega_p - \omega_q)t}$

$$\underline{f}^{p,q}_{-1,0} = \frac{e}{m} \frac{\partial f^q_{0,1}}{\partial v} \underline{\underline{L}}(\omega_p - \omega_q) \cdot \underline{E}^p_{-1} \tag{3.20}$$

where the dyadic operator is given by

$$\underline{\underline{L}}(\omega) = \frac{(\nu_1 - i\omega)^2 \, \underline{\underline{u}} + (\nu_1 - i\omega) \, \underline{\omega}_g \times \underline{\underline{u}} + \underline{\omega}_g \, \underline{\omega}_g}{(\nu_1 - i\omega) \, [(\nu_1 - i\omega)^2 + \omega_g^2]} \quad .$$

$\underline{f}^p_{-1,1}$ is found from terms of $e^{i\omega_p t}$ to be the complex conjugate of $\underline{f}^p_{-1,-1}$; and $\underline{f}^{p,q}_{-1,2}$ is found from terms of $e^{i(\omega_p+\omega_q)t}$ to be the complex conjugate of $\underline{f}^{p,q}_{-1,-2}$.

To find the component $f_{0,0}$ of f_o, substitute the expansions 3.16 and 3.17 in equation 3.13 and equate coefficients of terms independent of time. Neglect terms of higher order smallness. It is found that

$$-\frac{e}{3mv^2}\frac{\partial}{\partial v}\left\{v^2\sum_{p=1}^{m}\left(\underline{E}^p_{-1}\cdot\underline{f}^p_{-1,1}+\underline{E}^p_{1}\cdot\underline{f}^p_{-1,-1}\right)\right\}$$

$$=\frac{1}{v^2}\frac{\partial}{\partial v}\left\{v^2\,\frac{\nu_1 kT}{M}\,\frac{\partial f_{0,0}}{\partial v}+v^3\,\frac{\nu_1 m}{M}\,f_{0,0}\right\} \qquad (3.21)$$

where the terms on the right are $\left(\frac{\delta f_{0,0}}{\delta t}\right)_{coll}$ and are given in equation 3.9. Use the solution of $\underline{f}^p_{-1,-1}$ given by equation 3.19 and the solution of its complex conjugate $\underline{f}^p_{-1,1}$. The solution of the differential equation 3.21 is

$$f_{0,0} = C_o e^{-\frac{mv^2}{2kT_{eff}}} \qquad (3.22)$$

where $T_{eff} = T\left\{1+\frac{2e^2M}{3m^2kT}\sum_{p=1}^{m}\right.\times$

$$\left.\frac{(\nu_1^2+\omega_p^2)(\nu_1^2+\omega_p^2+\omega_g^2)\underline{E}^p_{-1}\cdot\underline{E}^p_{1}+(\nu_1^2-3\omega_p^2+\omega_g^2)(\underline{\omega}_g\cdot\underline{E}^p_{-1})(\underline{\omega}_g\cdot\underline{E}^p_{1})}{(\nu_1^2+\omega_p^2)(\nu_1^2+\omega_p^2+\omega_g^2+2\omega_p\omega_g)(\nu_1^2+\omega_p^2+\omega_g^2-2\omega_p\omega_g)}\right\}$$

and C_o is the integration constant to be determined from the electron density. Note that T is the temperature of the neutral molecules whose distribution function has been assumed to be $\left(\frac{M}{2\pi kT}\right)^{3/2} Ne^{-Mv^2/2kT}$. Comparing $f_{0,0}$ of the electron distribution function with the molecular distribution function, we conclude that the electrons are heated up by the electromagnetic waves from T to T_{eff}.

To determine C_o we need the relation between the electron density $n(\underline{r})$ and its distribution function f. By definition

$$n(\underline{r}) = \int f \, d^3v$$

and the time-average electron density is found to be

$$n(\underline{r}) = 4\pi^2 \int_0^\infty v^2 \, f_{0,0} \, dv \; .$$

For convenience we shall relate the integration constant C_o to the electron density n at a spatial point far from the influence of the electromagnetic waves. From the above equation it is found that

$$C_o = n\left(\frac{m}{2\pi kT}\right)^{3/2} .$$

To find the other components $f_{0,\pm 1}$ of the isotropic part f_o, substitute the expressions 3.16 and 3.17 in equation 3.13 and equate coefficients of terms of the same exponentials in frequency. Neglect terms of higher order of smallness. Then, from terms of $e^{-i\omega_p t}$, it is found that

$$\left(\frac{\delta f^p_{0,-1}}{\delta t}\right)_{coll} + i\omega_p f^p_{0,-1} = \frac{v}{3} \nabla_r \cdot \underline{f}^p_{1,-1} \qquad (3.23)$$

where $\left(\frac{\delta f^p_{0,-1}}{\delta t}\right)_{coll} = \frac{1}{v^2}\frac{\partial}{\partial v}\left\{v^2 \frac{\nu_1 kT}{M} \frac{\partial f^p_{0,-1}}{\partial v} + v^3 \frac{\nu_1 m}{M} f^p_{0,-1}\right\}$

as given by equation 3.9. Under the condition $\left(\frac{\delta f^p_{0,-1}}{\delta t}\right)_{coll} \ll \omega_p f^p_{0,-1}$ the solution of equation 3.23 is found to be

$$f^p_{0,-1} = \frac{ev}{3i\omega_p m} \frac{\partial f_{0,0}}{\partial v} \nabla_r \cdot \left[\underline{\underline{L}}(\omega_p) \cdot \underline{E}^p_{-1}\right] \qquad (3.24)$$

where the solution of $\underline{f}^p_{1,-1}$ from equation 3.19 is used.

A similar expression can be found for $f^p_{0,1}$ which is the complex conjugate of $f^p_{0,-1}$.

To justify the condition

$$\left(\frac{\delta f^p_{0,-1}}{\delta t}\right)_{coll} \ll \omega_p f^p_{0,-1}$$

use the approximate solution of $f_{0,0}$, namely $C_o e^{-\frac{mv^2}{2kT}}$. From the relation between the electron density and its distribution function we find that only .04% of the electrons have velocity greater than $3\sqrt{\frac{2kT}{m}}$ m.sec^{-1} and only 2% of the electrons have velocity less than $\frac{1}{3}\sqrt{\frac{2kT}{m}}$ m.sec^{-1}. Hence most electrons have velocity around $\sqrt{\frac{2kT}{m}}$ m.sec^{-1}. Take the value $\sqrt{\frac{2kT}{m}}$ for the electron velocity v , and use the approximate solution of $f_{0,0}$ and the solution of $f^p_{0,-1}$ in equation 3.24 to evaluate

$$\left(\frac{\delta f^p_{0,-1}}{\delta t}\right)_{coll} .$$

It is found that the ratio

$$\frac{\left(\frac{\delta f^p_{0,-1}}{\delta t}\right)_{coll}}{\omega_p \; f^p_{0,-1}} \sim \frac{\nu_1 m}{\omega_p M}$$

In the ionosphere $\nu_1 = 10^3$ to 10^7 sec^{-1}

$m = 9.1 \times 10^{-31}$ kg for electron

$M = 4.7 \times 10^{-26}$ kg for nitrogen molecule

and the angular frequency of broadcast signal

$$\omega = 10^6 \text{ sec}^{-1} \text{ and up.}$$

The ratio is found to be 10^{-4} or smaller, and hence the inequality

$$\left(\frac{\delta f^p_{0,-1}}{\delta t} \right)_{coll} \ll \omega_p f^p_{0,-1} \quad \text{is justified.}$$

We shall justify the statement in Chapter I that the effect of nonlinear phenomena due to spatial dispersion disappears if the electromagnetic waves are TEM waves. The additional terms which take into account the effect of spatial dispersion are $f^p_{0,\pm 1}$, $\underline{f}^{p,q}_{-1,\pm 2}$ and $\underline{f}^{p,q}_{-1,0}$. In an ionized gas which supports TEM waves, the electron gyrofrequency $\underline{\omega}_g$ is found in Chapter II to be either zero, or parallel or perpendicular to the propagation direction of the TEM waves, and by definition of TEM waves the electric field of the waves is always perpendicular to the propagation direction. In the case where the electron gyrofrequency $\underline{\omega}_g$ is perpendicular to the propagation direction, the electric field of the TEM waves is also parallel to the electron gyrofrequency. From the properties of the TEM waves, $f^p_{0,-1}$ is found from equation 3.24 to be zero and the other four terms are also found to be zero. The effect of spatial dispersion disappears.

From the components of f_o and $\underline{f}_1$ already found, the electron distribution function f can now be obtained explicitly in terms of the electric field $\underline{E}$ and the electron velocity v . In the next chapter we shall use the electron distribution function f to derive conductivity tensors of a nonlinear ionized gas.

IV. CONDUCTIVITY TENSORS OF A NONLINEAR IONIZED GAS

To find the conductivity tensors of a nonlinear ionized gas the convection current density in the weakly ionized gas due to the disturbing electromagnetic waves will be determined first from the distribution functions of the charged particles. Then from the relation between the phasors of the current density and the corresponding phasors of the electric field, the conductivity tensors of the ionized gas as seen by the corresponding phasors of the electric field can be obtained.

Since the mass ratio of an electron to an ion is extremely small, an electron can respond to the external accelerating fields much faster than an ion can. Hence the ion convection current density can be neglected in comparison with that of electrons in the derivation of the conductivity tensors of the ionized gas. The electron density in the velocity range from v to $v + dv$ is $f\ d^3v\ m^{-3}$, and in one second there are $\underline{v}\ f\ d^3v$ electrons passing through a unit area in space in the direction of $\underline{v}$. The total convection current density due to the electrons of all velocity is given by

$$\underline{J}_e = -e\int \underline{v}\ f\ d^3v \qquad (4.1)$$

where the negative sign indicates the negative charge of the electrons. Substitute the spherical harmonic expansion in velocity space for f , and the electron convection current density is

$$\underline{J}_e = -e\int \left(\underline{v}\ f_o + \underline{v}\ \frac{\underline{v}\cdot\underline{f}_1}{v}\right) d^3v \ .$$

In cartesian coordinate systems its x-component is

$$J_{ex} = - e \iiint_{-\infty}^{\infty} v_x f_o + \frac{v_x}{v}(v_x f_{1x} + v_y f_{1y} + v_z f_{1z})\, dv_x dv_y dv_z$$

and similar expressions hold for the other two components J_{ey} and J_{ez} . Since f_o and $\underline{f}_1$ are functions of v only, and not the components of $\underline{v}$, and since they are isotropic in velocity space, $v_x f_o$, $v_x v_y f_{1y}$ and $v_x v_z f_{1z}$ are odd functions in v_x . The integration of the odd functions over dv_x from $-\infty$ to ∞ gives zero value. Because f_{1x} depends on $v(=\sqrt{v_x^2 + v_y^2 + v_z^2})$ and not on the components v_x, v_y or v_z , it is obvious that

$$\iiint_{-\infty}^{\infty} \frac{v_x^2}{v} f_{1x}\, dv_x dv_y dv_z = \iiint_{-\infty}^{\infty} \frac{v_y^2}{v} f_{1x}\, dv_x dv_y dv_z$$

$$= \iiint_{-\infty}^{\infty} \frac{v_z^2}{v} f_{1x}\, dv_x dv_y dv_z \quad .$$

Hence

$$J_{ex} = - e \iiint_{-\infty}^{\infty} \frac{v_x^2}{v} f_{1x}\, dv_x dv_y dv_z$$

$$= - \frac{e}{3} \iiint_{-\infty}^{\infty} v f_{1x}\, dv_x dv_y dv_z \quad .$$

In vector notation

$$\underline{J}_e = - \frac{e}{3} \iiint v\, \underline{f}_1\, d^3v$$

$$= - \frac{e}{3} \int_0^{\infty} v^3 \underline{f}_1 dv \int_0^{2\pi} d\phi_v \int_0^{\pi} \sin \theta_v \, d\theta_v$$

where ϕ_v and θ_v are the angles in velocity space in the spherical coordinate system. The integration over $d\phi_v$ and $d\theta_v$ is equal to 4π . Consequently

$$\underline{J}_e = - \frac{4\pi e}{3} \int_0^\infty v^3 \underline{f}_1 \, dv \quad . \tag{4.2}$$

In Section 3.3 the components of $\underline{f}_1$ of the electron distribution function in an anisotropic ionized gas were found, and now we can find $\underline{f}_1$ explicitly in terms of the electric fields of m electromagnetic waves and the physical parameters of the ionized gas. Since $\underline{f}_1$ is a function of the collision frequency ν_1 , which in turn may depend on the electron velocity v , we should know the exact dependence of ν_1 on v before the electron convection current density $\underline{J}_e$ in equation 4.2 can be evaluated. For the sake of simplicity ν_1 is assumed to be independent of v . Then the electron convection current density $\underline{J}_e$ for m disturbing electromagnetic waves can be easily evaluated from equation 4.2. Define the conductivity tensors $\underline{\underline{\sigma}}_-^p$ and $\underline{\underline{\sigma}}_+^p$ by the equation

$$\underline{J}_e(t) = \sum_{p=1}^{m} \left[\underline{\underline{\sigma}}_-^p \cdot \underline{E}_{-1}^p(\underline{r}) \, e^{-i\omega_p t} + \underline{\underline{\sigma}}_+^p \cdot \underline{E}_1^p(\underline{r}) \, e^{i\omega_p t} \right] \tag{4.3}$$

where $\underline{\underline{\sigma}}_-^p$ is the conductivity tensor of the ionized gas as seen by the clockwise phasor $\underline{E}_{-1}^p(\underline{r})$ of the electric field of the p^{th} wave and $\underline{\underline{\sigma}}_+^p$ is that seen by the counterclockwise phasor $\underline{E}_1^p(\underline{r})$. From the solution of $\underline{J}_e$ obtained from equation 4.2, the conductivity tensors $\underline{\underline{\sigma}}_-^p$ and $\underline{\underline{\sigma}}_+^p$ in the above equation are found to be

$$\underline{\underline{\sigma}}_-^p = \Bigg\{ 1 + \frac{e^2 M}{m^2\, kT} \sum_{q=1}^{m}$$

$$\times \frac{(\nu_1^2 + \omega_q^2)(\nu_1^2 + \omega_q^2 + \omega_g^2)\underline{E}_{-1}^q \cdot \underline{E}_1^q + (\nu_1^2 - 3\omega_q^2 + \omega_g^2)(\underline{\omega}_g \cdot \underline{E}_{-1}^q)(\underline{\omega}_g \cdot \underline{E}_1^q)}{(\nu_1^2 + \omega_q^2)(\nu_1^2 + \omega_q^2 + \omega_g^2 + 2\omega_q\omega_g)\,(\nu_1^2 + \omega_q^2 + \omega_g^2 - 2\omega_q\,\omega_g)} \Bigg\} \underline{\underline{\sigma}}_L\,(\omega_p)$$

$$+ \sum_{q=1}^{m} \frac{e}{i\omega_q m}\, \nabla \cdot \left[\underline{\underline{L}}(\omega_q) \cdot \underline{E}_{-1}^q \right] e^{-i\omega_q t}\, \underline{\underline{\sigma}}_L\,(\omega_p + \omega_q)$$

$$- \sum_{q=1}^{m} \frac{e}{i\omega_q m}\, \nabla \cdot \left[\underline{\underline{L}}(-\omega_q) \cdot \underline{E}_1^q \right] e^{i\omega_q t}\, \underline{\underline{\sigma}}_L(\omega_p - \omega_q) \qquad (4.4)$$

where the dyadic operator $\underline{\underline{L}}(\omega)$ has been defined in Section 3.3 and $\underline{\underline{\sigma}}_+^p = (\underline{\underline{\sigma}}_-^p)^*$.

In the above derivation for $\underline{\underline{\sigma}}_-^p$ and $\underline{\underline{\sigma}}_+^p$, the approximation is used

$$\left(\frac{T_{eff}}{T}\right)^{3/2} \sim 1 + \frac{3}{2}\,\frac{T_{eff} - T}{T}$$

where T_{eff} is given by equation 3.22. $T_{eff} \sim T$ under the assumption that the electric field of the electromagnetic waves is not too strong.

In the expression 4.4 for $\underline{\underline{\sigma}}_-^p$ the first term $\underline{\underline{\sigma}}_L(\omega_p)$ is the conductivity tensor of a linear ionized gas as given by equation 2.3 for the clockwise phasor of the electric field of an electromagnetic wave. Note that $\underline{\underline{\sigma}}_L(\omega_p)$ is the zeroth order term in the expression for $\underline{\underline{\sigma}}_-^p$. The second term on the right of equation 4.4 is due to the heating effect on the electrons in the ionized gas by the passing electromagnetic waves. The electron temperature is thereby changed from T to T_{eff} . The last two terms on the right of equation 4.4 are due to the dispersion term $\underline{v} \cdot \nabla_r f$ in Boltzmann's kinetic equation 3.3. These last two terms disappear when the electromagnetic waves in the ionized gas are TEM waves because the electron gyrofrequency $\underline{\omega}_g$ and the electric field phasors $\underline{E}^q_{\pm 1}(\underline{r})$ have a special relation with the propagation direction as mentioned in Section 3.3.

V. NONLINEAR PHENOMENA IN A WEAKLY IONIZED GAS

In the previous chapter we found the conductivity tensors of a nonlinear ionized gas. In addition to the constant terms $\underline{\underline{\sigma}}_L(\omega_p)$ and $\underline{\underline{\sigma}}_L^*(\omega_p)$, the conductivity tensor $\underline{\underline{\sigma}}_-^p$ from equation (4.4) and its complex conjugate $\underline{\underline{\sigma}}_+^p$ contain terms which depend on the electric field of the disturbing electromagnetic waves in the ionized gas. The dependence on the electric field will lead to nonlinear phenomena. Here we shall study nonlinear phenomena in two cases: (1) the effects of nonlinear phenomena on the electric field due to self-interaction, and (2) the effects of nonlinear phenomena on the electric field of a weak wave due to mutual interaction with a strong wave.

We shall solve Maxwell's equations with the electron convection current density $\underline{J}_e$, given by equation 4.3, for the electric field of the waves by means of a perturbation method.

5.1) Nonlinear Effects Due to Self-Interaction

In the ionized gas two of Maxwell's equations describing the fields of an electromagnetic wave are

$$\nabla \times \underline{E} = -\mu_o \frac{\partial \underline{H}}{\partial t}$$

$$\nabla \times \underline{H} = \underline{J}_e + \epsilon_o \frac{\partial \underline{E}}{\partial t} \quad . \qquad (2.4)$$

Eliminate $\underline{H}$ from the above pair of equations to give the vector wave equation

$$\nabla \times \nabla \times \underline{E} = -\mu_o \frac{\partial \underline{J}_e}{\partial t} - \mu_o \epsilon_o \frac{\partial^2 \underline{E}}{\partial t^2} \quad . \qquad (5.1)$$

In the study of self-interaction only one electromagnetic wave is assumed to be present in the ionized gas. Similar to the representation of the electric field of the wave by the actual physical electric field as in Chapter III, the electric field vector can be written as a sum of two phasors as

$$\underline{E}(\underline{r},t) = \underline{E}_-(\underline{r},t) + \underline{E}_+(\underline{r},t) \tag{5.2}$$

where the negative sign stands for a clockwise phasor and the positive sign stands for a counterclockwise phasor. With this representation 5.2 for the electric field $\underline{E}(\underline{r},t)$, the electron convection current density $\underline{J}_e$ in equation 5.1 can be written, from equation 4.3, as

$$\underline{J}_e(\underline{r},t) = \underline{\underline{\sigma}}_- \cdot \underline{E}_- + \underline{\underline{\sigma}}_+ \cdot \underline{E}_+ \tag{5.3}$$

where the conductivity tensor $\underline{\underline{\sigma}}_-$ is given by equation 4.4 for one wave in the ionized gas, and $\underline{\underline{\sigma}}_+$ is the complex conjugate of $\underline{\underline{\sigma}}_-$. Since the conductivity tensors $\underline{\underline{\sigma}}_\pm$ are functions of the electric field, we can solve the vector wave equation 5.1 using a perturbation method by expanding each phasor of the electric field into a series, namely for the clockwise phasor

$$\begin{aligned} \underline{E}_-(\underline{r},t) &= \left\{ \underline{E}_{-1}(\underline{r}) + \underline{\eta}_{-1}(\underline{r}) + \cdots \right\} e^{-i\omega t} \\ &\quad + \underline{E}_{-2}(\underline{r})\, e^{-i2\omega t} + \cdots \end{aligned} \tag{5.4}$$

where $\underline{\eta}_{-1}$ and $\underline{E}_{-2}$ are correction terms to take into account the terms in $\underline{\underline{\sigma}}_-$ which are dependent on the electric field. $\underline{\eta}_{-1}$ is the correction term for the wave form distortion due to the heating effect on the electrons, and corresponds to the second term on the right of equation 4.4. $\underline{\eta}_{-1}$ will be found to be proportional to the third power of the amplitude of the electric field of the wave. $\underline{E}_{-2}$ is the correction term for the harmonic wave generation due to the dispersive effect and corresponds to the third term on the right of equation 4.4. It arises indirectly through the dispersion term $\underline{v} \cdot \nabla_r f$ in Boltzmann's kinetic equation 3.3 and will be found to be proportional to the square of the amplitude of the electric field of the wave. Higher order terms in the expansion 5.4 have been omitted under the assumption that the electric field is not too strong. Since the contribution of the fourth term on the right of equation 4.4 to the clockwise phasor of the electric field is of higher order, there is no term in the expansion 5.4 of $\underline{E}_-(\underline{r},t)$ to take into account the fourth term of the conductivity tensor $\underline{\underline{\sigma}}_-$ in equation 4.4. An expansion similar to equation 5.4 is used for the counterclockwise phasor $\underline{E}_+(\underline{r},t)$.

To solve the vector wave equation 5.1 for the electric field $\underline{E}(\underline{r},t)$, substitute the expansion 5.2 for $\underline{E}(\underline{r},t)$ and the expression 5.3 for $\underline{J}_e(\underline{r},t)$ into equation 5.1. Because of the orthogonal properties of $e^{\pm i\omega t}$ and $e^{\pm i2\omega t}$, equation 5.1 can be separated into two equations containing $\underline{E}_-(\underline{r},t)$ and $\underline{E}_+(\underline{r},t)$ respectively, namely

$$\nabla \times \nabla \times \underline{E}_- = -\mu_o \frac{\partial}{\partial t}(\underline{\underline{\sigma}}_- \cdot \underline{E}_-) - \mu_o \epsilon_o \frac{\partial^2 \underline{E}_-}{\partial t^2} \tag{5.5}$$

and

$$\nabla \times \nabla \times \underline{E}_+ = -\mu_o \frac{\partial}{\partial t}(\underline{\underline{\sigma}}_+ \cdot \underline{E}_+) - \mu_o \epsilon_o \frac{\partial^2 \underline{E}_+}{\partial t^2} \quad . \tag{5.6}$$

Since equation 5.5 and equation 5.6 are similar, and $\underline{\underline{\sigma}}_-$ is the complex conjugate of $\underline{\underline{\sigma}}_+$, we find that $\underline{E}_-$ is the complex conjugate of

$\underline{E}_+$. Hence we shall consider only the clockwise phasor $\underline{E}_-$ in this and the next chapters.

To find $\underline{E}_-(\underline{r},t)$ substitute the expansion 5.2 for $\underline{E}_-(\underline{r},t)$ and the expression 4.4 for $\underline{\underline{\sigma}}_-$ into equation 5.5. Equate the coefficients of terms of the same time dependence, namely $e^{-i\omega t}$ and $e^{-i2\omega t}$. Equate further the coefficients of terms of the same order of magnitude. Note that the term $\underline{\underline{\sigma}}_L$ in equation 4.4 for $\underline{\underline{\sigma}}_-$ is one order larger than the rest in equation 4.4, and the term $\underline{E}_{-1}$ in equation 5.4 for $\underline{E}_-$ is one order larger than the rest in equation 5.4. From equation 5.5 the following set of equations is found

$$\nabla \times \nabla \times \underline{E}_{-1} - \omega^2 \mu_o \underline{\underline{\epsilon}}_L(\omega) \cdot \underline{E}_{-1} = 0 \tag{5.7}$$

$$\nabla \times \nabla \times \underline{\eta}_{-1} - \omega^2 \mu_o \underline{\underline{\epsilon}}_L(\omega) \cdot \underline{\eta}_{-1} = \underline{\underline{Q}}(\underline{E}_{-1} \cdot \underline{E}_1) \cdot \underline{E}_{-1} \tag{5.8}$$

and

$$\nabla \times \nabla \times \underline{E}_{-2} - 4\omega^2 \mu_o \underline{\underline{\epsilon}}_L(2\omega) \cdot \underline{E}_{-2} = \underline{\underline{R}}(\underline{E}_{-1}) \cdot \underline{E}_{-1} \tag{5.9}$$

where $\underline{\underline{\epsilon}}_L$ is the dielectric tensor of a linear ionized gas for an electromagnetic wave with carrier angular frequency ω . The dyadics on the right of equation 5.8 and equation 5.9 are given by

$$\underline{\underline{Q}}(\underline{E}_{-1} \cdot \underline{E}_1) = i\omega \mu_o \cdot \frac{e^2 M}{m^2 kT}$$

$$\times \frac{(\nu_1^2+\omega^2)(\nu_1^2+\omega^2+\omega_g^2)\, \underline{E}_{-1} \cdot \underline{E}_1 + (\nu_1^2 - 3\omega^2 + \omega_g^2)(\underline{\omega}_g \cdot \underline{E}_{-1})(\underline{\omega}_g \cdot \underline{E}_1)}{(\nu_1^2+\omega^2)(\nu_1^2+\omega^2+\omega_g^2+2\omega\omega_g)(\nu_1^2+\omega^2+\omega_g^2-2\omega\omega_g)}\, \underline{\underline{\sigma}}_L(\omega) ,$$

$$\underline{\underline{R}}(\underline{E}_{-1}) = \frac{2\mu_o e}{m} \nabla \cdot \left[\underline{\underline{L}}(\omega) \cdot \underline{E}_{-1} \right] \underline{\underline{\sigma}}_L(2\omega)$$

and

$$\underline{\underline{L}}(\omega) = \frac{(\nu_1 - i\omega)^2 \underline{\underline{u}} + (\nu_1 - i\omega)\, \underline{\underline{\omega}}_g \times \underline{\underline{u}} + \underline{\omega}_g \underline{\omega}_g}{(\nu_1 - i\omega)\, [(\nu_1 - i\omega)^2 + \omega_g^2]}$$

where $\underline{\underline{\sigma}}_L(\omega)$ is the conductivity tensor of a linear ionized gas for an electromagnetic wave with carrier angular frequency ω .

Equation 5.7 is identical with equation 2.7 as is expected since it is the equation for the zeroth order solution for the clockwise phasor $\underline{E}_-(\underline{r},t)$ of the electric field of the electromagnetic wave. Without any loss of generality, we take the propagation direction along the z-axis and the external magnetostatic field in the y-z plane at an angle Θ with the z-axis as shown in Fig. 1. Then the solutions of equation 5.7 are given by equation 2.11 for an ordinary wave, and by equation 2.12 for an extraordinary wave. The spatial dependence of $\underline{E}_{-1}(\underline{r})$ is $\exp(ikz)$ and that of the counterclockwise phasor $\underline{E}_1(\underline{r})$ which is the complex conjugate of $\underline{E}_{-1}(\underline{r})$ will be found to be $\exp(-ik^*z)$.

To solve equation 5.8 for the correction term $\underline{\eta}_{-1}(\underline{r})$ note that the spatial dependence of the forcing term $\underline{\underline{Q}}(\underline{E}_{-1} \cdot \underline{E}_1) \cdot \underline{E}_{-1}$ on the right of equation 5.8 is $\exp i(2k - k^*)z$. Hence the particular solution of equation 5.8 for $\underline{\eta}_{-1}(\underline{r})$ will have the spatial dependence of $\exp i(2k - k^*)z$. If the abbreviated dyadic notations

$$\underline{\underline{\mathcal{L}}}_{-1} = (2k - k^*)^2 \,(\underline{e}_x\underline{e}_x + \underline{e}_y\underline{e}_y) - \omega^2\mu_o\, \underline{\underline{\epsilon}}_L(\omega)$$

and

$$\underline{\underline{\Lambda}}_{-1} = (\underline{\underline{\mathcal{L}}}_{-1})^{-1} .$$

are used, then the particular solution of equation 5.8 is

$$\underline{\eta}_{-1}(\underline{r}) = \underline{\underline{\Lambda}}_{-1} \cdot \left[\underline{\underline{Q}}(\underline{E}_{-1} \cdot \underline{E}_1) \cdot \underline{E}_{-1} \right] . \qquad (5.10)$$

Similarly, the spatial dependence of the forcing term $\underline{\underline{R}}(\underline{E}_{-1}) \cdot \underline{E}_{-1}$ on the right of equation 5.9 is exp(i2kz). Hence the particular solution of equation 5.9 for $\underline{E}_{-2}(\underline{r})$ will have the spatial dependence of exp(i2kz). If the abbreviated dyadic notations

$$\underline{\underline{\mathcal{L}}}_{-2} = 4k^2(\underline{e}_x\underline{e}_x + \underline{e}_y\underline{e}_y) - 4\omega^2\mu_o \underline{\underline{\epsilon}}_L(2\omega)$$

and

$$\underline{\underline{\Lambda}}_{-2} = (\underline{\underline{\mathcal{L}}}_{-2})^{-1}$$

are used, then the particular solution of equation 5.9 is

$$\underline{E}_{-2}(\underline{r}) = \underline{\underline{\Lambda}}_{-2} \cdot \left[\underline{\underline{R}}(\underline{E}_{-1}) \cdot \underline{E}_{-1} \right] . \qquad (5.11)$$

In equations 5.10 and 5.11 the propagation constant k is to be taken as k_o if the wave is an ordinary wave and as k_e if the wave is an extraordinary wave.

We have solved the vector wave equation 5.1 for the electric field of an electromagnetic wave in a nonlinear ionized gas. From equation 5.4 the effects of nonlinear phenomena on the electric field of the wave are shown to be wave form distortion from the correction term $\underline{\eta}_{-1}(\underline{r})e^{-i\omega t}$ and harmonic wave generation of second harmonic frequency from the correction term $\underline{E}_{-2}(\underline{r})e^{-i2\omega t}$. When the complex propagation constant k is written explicitly as $(\beta + i\alpha)$ where β and α are the real positive phase constant and the real positive attenuation constant, respectively, then

$$ik = -\alpha + i\beta$$

$$i(2k - k^*) = -3\alpha + i\beta$$

and

$$i2k = -2\alpha + i2\beta \quad .$$

From equation 5.10 the wave form distortion is found to be proportional to the third power of the amplitude of the electric field of the wave and the attenuation constant 3α of the wave form distortion is three times that of the electric field. From equation 5.11 the harmonic wave is found to be proportional to the square of the amplitude of the electric field of the wave and the attenuation constant 2α of the harmonic wave is twice that of the electric field. Higher order correction terms in the electron distribution function f will certainly lead to further corrections to both the wave form distortion and the harmonic generation expressions. These higher order correction terms are found to be proportional to the third or higher power of the amplitude of the electric field of the wave.

In Figure 3 an antenna is located in a nonlinear ionized gas in the presence of an external magnetostatic field $\underline{B}_o$. The antenna is excited by a monochromatic source of angular frequency ω . In the far zone the electric field due to the source has just been found in terms of that in a linear ionized gas. For convenience the clockwise phasor of the electric field in the far zone is rewritten here:

$$\underline{E}_-(\underline{r},t) = \left\{\underline{E}_{-1}(\underline{r}) + \underline{\eta}_{-1}(\underline{r}) + \cdots\right\} e^{-i\omega t} + \underline{E}_{-2}(\underline{r}) e^{-i2\omega t} + \cdots$$

where $\underline{E}_{-1}(\underline{r}) e^{-i\omega t}$ = clockwise phasor of the electric field in a linear ionized gas as given in Section 2.3

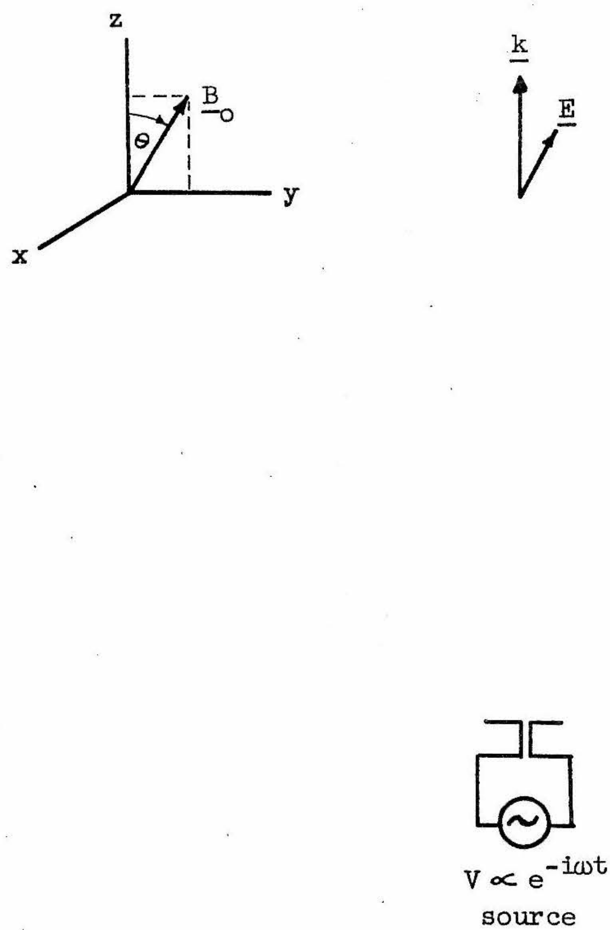

Fig. 3. Wave propagation in a nonlinear ionized gas due to self-interaction

$$\underline{\eta}_{-1}(\underline{r}) = \underline{\underline{\Lambda}}_{-1} \cdot \left[\underline{\underline{Q}}(\underline{E}_{-1} \cdot \underline{E}_1) \cdot \underline{E}_{-1} \right]$$

and
$$\underline{E}_{-2}(\underline{r}) = \underline{\underline{\Lambda}}_{-2} \cdot \left[\underline{\underline{R}}(\underline{E}_{-1}) \cdot \underline{E}_{-1} \right] .$$

The dependence of the correction terms $\underline{\eta}_{-1}(\underline{r})$ and $\underline{E}_{-2}(\underline{r})$ on the clockwise phasor $\underline{E}_{-1}(\underline{r})$ and its complex conjugate $\underline{E}_1(\underline{r})$ is explicitly indicated.

If the wave is originally amplitude-modulated with a modulation angular frequency Ω and a modulation index μ , and if $\Omega \ll \omega$, then a quasi-stationary approximation can be used to solve the vector wave equation. The solution of the electric field of the wave can be found from the previous solution by replacing each $\underline{E}_{-1}$ or $\underline{E}_1$ by $\underline{E}_{-1}(1 + \mu \cos \Omega t)$ or $\underline{E}_1(1 + \mu \cos \Omega t)$ respectively. Use the trigonomentrical equalities

$$(1+\mu \cos \Omega t)^3 = \left[(1 + \frac{3}{2} \mu^2) + \frac{3}{4} \mu (4 + \mu^2) \cos \Omega t + \frac{3}{2} \mu^2 \cos 2\Omega t + \frac{1}{4} \mu^3 \cos 3\Omega t \right]$$

and

$$(1+\mu \cos \Omega t)^2 = \left[(1 + \frac{1}{2} \mu^2) + 2\mu \cos \Omega t + \frac{1}{2} \mu^2 \cos 2\Omega t \right] .$$

The effects of nonlinear phenomena on an amplitude-modulated wave are: (1) from equation 5.10, that the wave is found to be modulated with modulation angular frequencies Ω, 2Ω and 3Ω ; and (2) from equation 5.11, that the harmonic wave is found to be modulated with modulation angular frequencies Ω and 2Ω .

If the modulation angular frequency Ω is not small in comparison with the carrier angular frequency ω, the quasi-stationary approximation cannot be used. The clockwise phasor of the electric field of the amplitude-modulated wave can be written as

$$\underline{E}_{-1}(1+\mu \cos \Omega t)e^{-i\omega t} = \frac{\mu}{2}\underline{E}_{-1}e^{-i(\omega-\Omega)t} + \underline{E}_{-1}e^{-i\omega t} + \frac{\mu}{2}\underline{E}_{-1}e^{-i(\omega+\Omega)t} .$$

Consequently the amplitude-modulated wave can be regarded as three unmodulated waves having carrier angular frequencies $(\omega-\Omega)$, ω and $(\omega+\Omega)$ respectively. The case for many waves present simultaneously in the ionized gas will be discussed in the next section.

If the wave is a TEM wave, $\underline{\underline{R}}$ is found to be zero because the electron gyrofrequency $\underline{\omega}_g$ and the electric field phasor $\underline{E}_{-1}(\underline{r})$ have a special relation with the propagation direction as mentioned in Section 3.3. From equation 5.11 we find that the harmonic wave of second harmonic angular frequency disappears. The effect of spatial dispersion, corresponding to the term $\underline{v} \cdot \nabla_r f$ in Boltzmann's kinetic equation 3.3, is always absent for TEM waves in the ionized gas.

5.2) Nonlinear Effects Due to Mutual Interaction

In the study of mutual interaction one weak and one strong electromagnetic wave are assumed to be present simultaneously in the ionized gas. We are interested in the nonlinear effects due to mutual interaction on the electric field of the weak wave. The nonlinear effects due to self-interaction on the electric field of the weak wave can be neglected in comparison with that due to mutual interaction. The nonlinear effects due to self-interaction can be easily taken into account by including the results obtained in Section 5.1.

Superscripts (1) and (2) are used to distinguish parameters for the weak wave (1) and the strong wave (2), respectively. Following the arguments given in Section 5.1, we shall solve equation 5.5 for the clockwise phasor $\underline{E}_-^{(1)}$ of the electric field of the weak wave only, since the counterclockwise phasor $\underline{E}_+^{(1)}$ is just the complex conjugate of $\underline{E}_-^{(1)}$. The clockwise phasor $\underline{E}_-^{(1)}$ can be expanded into a Fourier series, namely

$$\underline{E}_-^{(1)}(\underline{r},t) = \left\{ \underline{E}_{-1}^{(1)}(\underline{r}) + \underline{\eta}_{-1}^{1,2}(\underline{r}) + \cdots \right\} e^{-i\omega_1 t} + \underline{E}_{-2}^{1,2}(\underline{r})\, e^{-i(\omega_1+\omega_2)t} + \underline{E}_0^{1,2}(\underline{r})\, e^{-i(\omega_1-\omega_2)t} + \cdots\cdots \quad . \tag{5.12}$$

The superscripts and subscripts of $\underline{E}$'s and $\underline{\eta}$'s have the same meaning as those of the components of $\underline{f}_1$ in equation 3.17. $\underline{\eta}_{-1}^{1,2}$, $\underline{E}_{-2}^{1,2}$ and $\underline{E}_0^{1,2}$ are correction terms to take into account the terms in $\underline{\underline{\sigma}}_-^{(1)}$ which are dependent on the electric field of the strong wave (2), and those dependent on the weak wave (1) have been neglected. $\underline{\eta}_{-1}^{1,2}$ is the correction term for the wave form distortion due to the heating effect on the electrons by the strong wave. $\underline{\eta}_{-1}^{1,2}$ will be found to be proportional to the square of the amplitude of the electric field of the strong wave. $\underline{E}_{-2}^{1,2}$ and $\underline{E}_0^{1,2}$ are the correction terms for the harmonic wave generation due to the spatial dispersive effect. They arise indirectly through the dispersion term $\underline{v} \cdot \nabla_r f$ in Boltzmann's kinetic equation 3.3 and will be found to be linearly proportional to the amplitude of the electric field of the strong wave.

Higher order terms in the expansion 5.12 have been omitted under the assumption that the electric field of both waves are not too strong.

To solve equation 5.5 for the clockwise phasor $\underline{E}_-^{(1)}$ of the electric field of the weak wave, substitute the expansion 5.12 for $\underline{E}_-^{(1)}(\underline{r},t)$ and the expression 4.4 for $\underline{\underline{\sigma}}_-^{(1)}$ into equation 5.5. Equate the coefficients of terms of the same time-dependence, namely $e^{-i\omega_1 t}$, $e^{-i(\omega_1+\omega_2)t}$, and $e^{-i(\omega_1-\omega_2)t}$. Equate further the coefficients of terms of the same order of magnitude. Note that the term $\underline{\underline{\sigma}}_L(\omega_1)$ in equation 4.4 for $\underline{\underline{\sigma}}_-^{(1)}$ is one order larger than the rest in equation 4.4. Terms containing $\underline{E}_{\pm 1}^{(1)}$ in $\underline{\underline{\sigma}}_-^{(1)}$ are neglected in comparison with those containing $\underline{E}_{\pm 1}^{(2)}$. The term $\underline{E}_{-1}^{(1)}$ in equation 5.12 for $\underline{E}_-^{(1)}$ is one order larger than the rest in equation 5.12. From equation 5.5 the following set of equations is found.

$$\nabla \times \nabla \times \underline{E}_{-1}^{(1)} - \omega_1^2 \mu_o \underline{\underline{\epsilon}}_L(\omega_1) \cdot \underline{E}_{-1}^{(1)} = 0 \quad (5.13)$$

$$\nabla \times \nabla \times \underline{\eta}_{-1}^{1,2} - \omega_1^2 \mu_o \underline{\underline{\epsilon}}_L(\omega_1) \cdot \underline{\eta}_{-1}^{1,2} = \underline{\underline{Q}}(\underline{E}_{-1}^{(2)} \cdot \underline{E}_1^{(2)}) \cdot \underline{E}_{-1}^{(1)} \quad (5.14)$$

$$\nabla \times \nabla \times \underline{E}_{-2}^{1,2} - (\omega_1+\omega_2)^2 \mu_o \underline{\underline{\epsilon}}_L(\omega_1+\omega_2) \cdot \underline{E}_{-2}^{1,2} = \underline{\underline{R}}(\underline{E}_{-1}^{(2)}) \cdot \underline{E}_{-1}^{(1)} \quad (5.15)$$

and

$$\nabla \times \nabla \times \underline{E}_{o}^{1,2} - (\omega_1-\omega_2)^2 \mu_o \underline{\underline{\epsilon}}_L(\omega_1-\omega_2) \cdot \underline{E}_{o}^{1,2} = \underline{\underline{S}}(\underline{E}_{1}^{(2)}) \cdot \underline{E}_{-1}^{(1)} \quad (5.16)$$

where $\underline{\underline{\epsilon}}_L(\omega)$ is the dielectric tensor of a linear ionized gas for an electromagnetic wave with a carrier angular frequency ω. The dyadicson the right of equation 5.14, equation 5.15 and equation 5.16 are given by

$$\underline{\underline{Q}}(\underline{E}_{-1}^{(2)} \cdot \underline{E}_{1}^{(2)}) = i\omega_1\mu_o \cdot \frac{e^2 M}{m^2 kT}$$

$$\times \frac{(\nu_1^2+\omega_2^2)(\nu_1^2+\omega_2^2+\omega_g^2)\ \underline{E}_{-1}^{(2)} \cdot \underline{E}_{1}^{(2)} + (\nu_1^2 - 3\omega_2^2+\omega_g^2)(\underline{\omega}_g \cdot \underline{E}_{-1}^{(2)})(\underline{\omega}_g \cdot \underline{E}_{1}^{(2)})}{(\nu_1^2+\omega_2^2)(\nu_1^2+\omega_2^2+\omega_g^2+2\omega_2\omega_g)(\nu_1^2+\omega_2^2+\omega_g^2-2\omega_2\omega_g)}\ \underline{\underline{\sigma}}_L(\omega_1)$$

$$\underline{\underline{R}}(\underline{E}_{-1}^{(2)}) = \frac{(\omega_1+\omega_2)\mu_o e}{\omega_2 m} \nabla \cdot \left[\underline{\underline{L}}(\omega_2) \cdot \underline{E}_{-1}^{(2)}\right] \underline{\underline{\sigma}}_L(\omega_1+\omega_2)$$

$$\underline{\underline{S}}(\underline{E}_{1}^{(2)}) = -\frac{(\omega_1-\omega_2)\mu_o e}{\omega_2 m} \nabla \cdot \left[\underline{\underline{L}}(-\omega_2) \cdot \underline{E}_{1}^{(2)}\right] \underline{\underline{\sigma}}_L(\omega_1-\omega_2)$$

and

$$\underline{\underline{L}}(\omega) = \frac{(\nu_1 - i\omega)^2\ \underline{\underline{u}} + (\nu_1 - i\omega)\ \underline{\omega}_g \times \underline{\underline{u}} + \underline{\omega}_g\,\underline{\omega}_g}{(\nu_1 - i\omega)\left[(\nu - i\omega)^2 + \omega_g^2\right]}$$

where $\underline{\underline{\sigma}}_L(\omega)$ is the conductivity tensor of a linear ionized gas for an electromagnetic wave with a carrier angular frequency ω .

Equation 5.13 is identical with equation 2.7, as is expected, since it is the equation for the zeroth order solution for the clockwise phasor $\underline{E}_{-}^{(1)}(\underline{r},t)$ of the electric field of the electromagnetic wave (1). Take the propagation direction of both waves to be along the z-axis and the external magnetostatic field in the y-z plane to make an angle Θ with the z-axis as shown in Fig. 1. Then the solutions of equation 5.13 are given by equation 2.11 for an ordinary wave, and by equation 2.12 for an extraordinary wave. The spatial dependence of $\underline{E}_{-1}^{(1)}(\underline{r})$ is $\exp(ik_1 z)$ and that of the counterclockwise phasor $\underline{E}_{1}^{(1)}(\underline{r})$ which is the complex conjugate of $\underline{E}_{-1}^{(1)}(\underline{r})$ will be found to be $\exp(-ik_1^* z)$. Similarly, the spatial dependence of the clockwise phasor

$\underline{E}_{-1}^{(2)}(\underline{r})$ of the electric field of the strong wave will be found to be $\exp(ik_2 z)$ and that of the counterclockwise phasor $\underline{E}_1^{(2)}(\underline{r})$ to be $\exp(ik_2^* z)$. The propagation constant k_1 or k_2 can be k_o if the wave is an ordinary wave and k_e if the wave is an extraordinary wave. k_o and k_e are functions of ω_1 or ω_2, respectively, for the weak wave (1) or the strong wave (2).

To solve equations 5.14, 5.15 and 5.16 for the correction terms $\underline{\eta}_{-1}^{1,2}(\underline{r})$, $\underline{E}_{-2}^{1,2}(\underline{r})$ and $\underline{E}_o^{1,2}(\underline{r})$, note that their particular solutions will have the same spatial dependence as the corresponding forcing terms as in Section 5.1. On the right of the three equations the forcing terms $\underline{\underline{Q}}(\underline{E}_{-1}^{(2)} \cdot \underline{E}_1^{(2)}) \cdot \underline{E}_{-1}^{(1)}$, $\underline{\underline{R}}(\underline{E}_{-1}^{(2)}) \cdot \underline{E}_{-1}^{(1)}$ and $\underline{\underline{S}}(\underline{E}_1^{(2)}) \cdot \underline{E}_{-1}^{(1)}$ are found to have the spatial dependence of $\exp i(k_1 + k_2 - k_2^*)z$, $\exp i(k_1 + k_2)z$ and $\exp i(k_1 - k_2^*)z$ respectively. Hence the particular solutions of equations 5.14, 5.15 and 5.16 have the spatial dependence of $\exp i(k_1 + k_2 - k_2^*)z$, $\exp i(k_1 + k_2)z$ and $\exp i(k_1 - k_2^*)z$ respectively. Their particular solutions are

$$\underline{\eta}_{-1}^{1,2}(\underline{r}) = \underline{\underline{\Lambda}}_{-1}^{1} \cdot \left[\underline{\underline{Q}}(\underline{E}_{-1}^{(2)} \cdot \underline{E}_1^{(2)}) \cdot \underline{E}_{-1}^{(1)} \right] \tag{5.17}$$

$$\underline{E}_{-2}^{1,2}(\underline{r}) = \underline{\underline{\Lambda}}_{-2}^{1,2} \cdot \left[\underline{\underline{R}}(\underline{E}_{-1}^{(2)}) \cdot \underline{E}_{-1}^{(1)} \right] \tag{5.18}$$

and

$$\underline{E}_{o}^{1,2}(\underline{r}) = \underline{\underline{\Lambda}}_{o}^{1,2} \cdot \left[\underline{\underline{S}}(\underline{E}_{1}^{(2)}) \cdot \underline{E}_{-1}^{(1)} \right] \tag{5.19}$$

where the abbreviated dyadic notations are

$$\underline{\underline{\Lambda}}_{-1}^{1} = \left[(k_1 + k_2 - k_2^*)^2 \, (\underline{e}_x\underline{e}_x + \underline{e}_y\underline{e}_y) - \omega_1^2 \, \mu_o \, \underline{\underline{\epsilon}}_L(\omega_1) \right]^{-1}$$

$$\underline{\underline{\Lambda}}_{-2}^{1,2} = \left[(k_1+k_2)^2\,(\underline{e}_x\underline{e}_x+\underline{e}_y\underline{e}_y) - (\omega_1+\omega_2)^2\mu_o\,\underline{\underline{\epsilon}}_L(\omega_1+\omega_2)\right]^{-1}$$

and

$$\underline{\underline{\Lambda}}_{o}^{1,2} = \left[(k_1-k_2^*)^2\,(\underline{e}_x\underline{e}_x+\underline{e}_y\underline{e}_y) - (\omega_1-\omega_2)^2\mu_o\,\underline{\underline{\epsilon}}_L(\omega_1-\omega_2)\right]^{-1} .$$

We have solved the vector wave equation 5.1 for the electric field of a weak electromagnetic wave in a nonlinear ionized gas in the presence of a strong wave. From equation 5.12 the effects of nonlinear phenomena due to mutual interaction on the electric field of the weak wave are shown to be wave form distortion from the correction term $\underline{\eta}_{-1}^{1,2}(\underline{r})\,e^{-i\omega_1 t}$ and harmonic generation of mixed frequencies from the correction terms $\underline{E}_{-2}^{1,2}(\underline{r})\,e^{-i(\omega_1+\omega_2)t}$ and $\underline{E}_{o}^{1,2}(\underline{r})e^{-i(\omega_1-\omega_2)t}$. When the complex propagation constant k is written explicitly as $(\beta + i\alpha)$ where β and α are the real positive phase constant and the real positive attenuation constant, respectively, then

$$ik_1 = -\alpha_1 + i\beta_1$$

$$i(k_1+k_2-k_2^*) = -(\alpha_1+2\alpha_2) + i\beta_1$$

$$i(k_1+k_2) = -(\alpha_1+\alpha_2) + i(\beta_1+\beta_2)$$

and

$$i(k_1-k_2^*) = -(\alpha_1+\alpha_2) + i(\beta_1-\beta_2) .$$

From equation 5.17 the wave form distortion is found to be proportional to the square of the amplitude of the electric field of the strong wave, and the attenuation constant $(\alpha_1+2\alpha_2)$ of the wave form distortion is greater than the attenuation constant α_1 of the electric field of the weak wave. From equation 5.18 and equation 5.19, the harmonic waves of mixed frequencies $(\omega_1 \pm \omega_2)$ are found to be linearly proportional to the amplitude of the electric field of the strong wave. The attenuation constant $(\alpha_1 + \alpha_2)$ of the harmonic waves is greater than the attenuation constant α_1 of the electric field of the weak wave. The phase velocities of the harmonic waves of mixed frequencies $(\omega_1 \pm \omega_2)$ are found to be different from that of the weak wave. Higher correction terms in the electron distribution function f will certainly lead to further corrections to both the wave form distortion and the harmonic generation expressions.

In Fig. 4 two antennas are located in a nonlinear ionized gas in the presence of an external magnetostatic field $\underline{B}_o$. One of the antennas is excited by a weak source of angular frequency ω_1, while the other is excited by a strong source of angular frequency ω_2. The electric fields from the two sources interact mutually in the nonlinear ionized gas. In the far zone the electric field from the weak source has just been found. The clockwise phasor of the far-zone field of the weak source is rewritten here for convenience:

$$\underline{E}_-^{(1)}(\underline{r},t) = \left\{ \underline{E}_{-1}^{(1)}(\underline{r}) + \underline{\eta}_{-1}^{1,2}(\underline{r}) + \cdots \right\} e^{-i\omega_1 t} + \underline{E}_{-2}^{1,2}(\underline{r})\, e^{-i(\omega_1+\omega_2)t} + \underline{E}_{o}^{1,2}(\underline{r})\, e^{-i(\omega_1-\omega_2)t} + \cdots$$

where

$\underline{E}_{-1}^{(1)}(\underline{r})\, e^{-i\omega_1 t}$ = clockwise phasor of the electric field of the weak source in a linear ionized gas as given in Section 2.3

$$\underline{\eta}_{-1}^{1,2}(\underline{r}) = \underline{\underline{\Lambda}}_{-1}^{1} \cdot \left[\underline{\underline{Q}}(\underline{E}_{-1}^{(2)} \cdot \underline{E}_{1}^{(2)}) \cdot \underline{E}_{-1}^{(1)} \right]$$

$$\underline{E}_{-2}^{1,2}(\underline{r}) = \underline{\underline{\Lambda}}_{-2}^{1,2} \cdot \left[\underline{\underline{R}}(\underline{E}_{-1}^{(2)}) \cdot \underline{E}_{-1}^{(1)} \right]$$

and

$$\underline{E}_{0}^{1,2}(\underline{r}) = \underline{\underline{\Lambda}}_{o}^{1,2} \cdot \left[\underline{\underline{S}}(\underline{E}_{1}^{(2)}) \cdot \underline{E}_{-1}^{(1)} \right] .$$

The clockwise phasor of the electric field from the strong source has been found in the previous section. The dependence of the correction terms $\underline{\eta}_{-1}^{1,2}(\underline{r})$, $\underline{E}_{-2}^{1,2}(\underline{r})$ and $\underline{E}_{o}^{1,2}(\underline{r})$ on the clockwise phasors $\underline{E}_{-1}^{(1)}(\underline{r})$ of the weak wave, $\underline{E}_{-1}^{(2)}(\underline{r})$ of the strong wave and its complex conjugate $\underline{E}_{1}^{(2)}(\underline{r})$ is explicitly indicated.

If the strong wave (2) is amplitude modulated with a modulation angular frequency Ω and a modulation index μ and if $\Omega \ll |\omega_1 - \omega_2|$, then a quasi-stationary approximation can be used to solve the vector wave equation. The solution of the electric field of the weak wave can be found from the previous solution by replacing each $\underline{E}_{-1}^{(2)}$ or $\underline{E}_{1}^{(2)}$ by $\underline{E}_{-1}^{(2)}(1+\mu \cos \Omega t)$ or $\underline{E}_{1}^{(2)}(1+\mu \cos \Omega t)$ respectively. Use the trigonometrical equality

$$(1+\mu \cos \Omega t)^2 = \left[(1+\tfrac{1}{2}\mu^2) + 2\mu \cos \Omega t + \tfrac{1}{2}\mu^2 \cos 2\Omega t \right] .$$

The effects of nonlinear phenomena on the weak wave are: (a) from equation 5.17 that the weak wave is cross-modulated with modulation angular frequencies Ω and 2Ω , and (b) from equation 5.18 and equation 5.19 that the harmonic waves are modulated with a modulation angular frequency Ω .

If the modulation angular frequency Ω of the strong wave is not small in comparison with the carrier frequencies ω_1 and ω_2 , the quasi-stationary approximation cannot be used. The clockwise phasor of

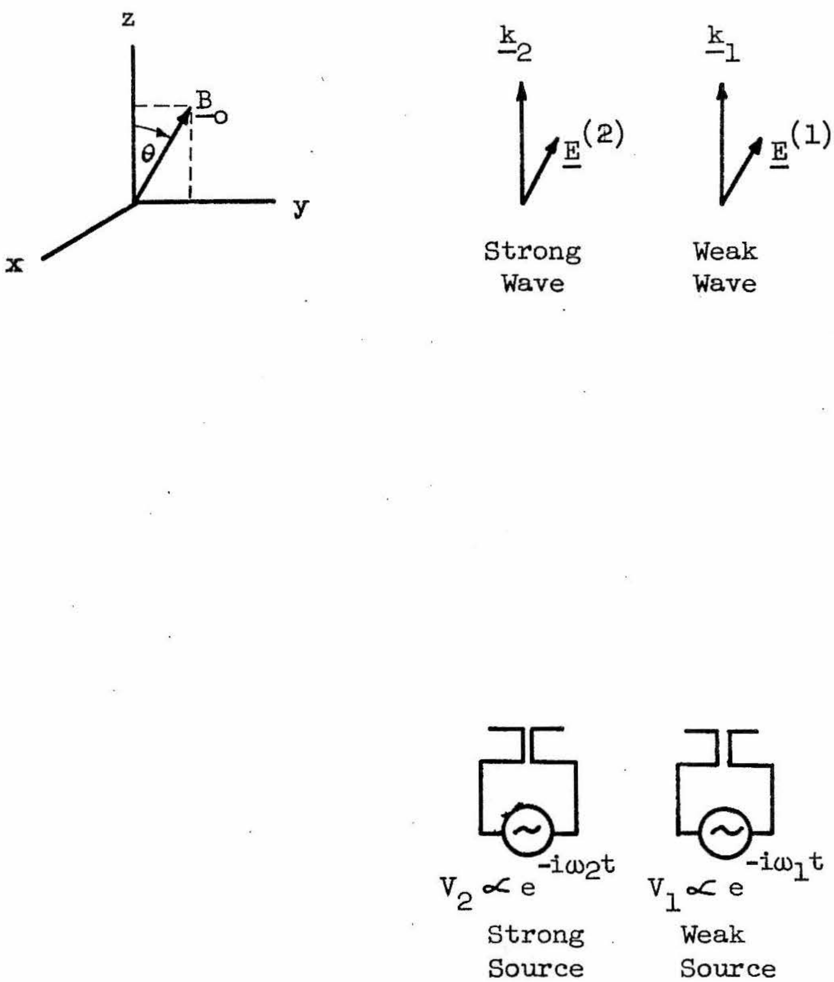

Fig. 4. Wave propagation in a nonlinear ionized gas due to mutual interaction

the electric field of the amplitude-modulated wave can be written as

$$\underline{E}_{-1}^{(2)}(1+\mu\cos\Omega t)\,e^{-i\omega_2 t} = \frac{\mu}{2}\underline{E}_{-1}^{(2)}\,e^{-i(\omega_2-\Omega)t} + \underline{E}_{-1}^{(2)}\,e^{-i\omega_2 t} + \frac{\mu}{2}\underline{E}_{-1}^{(2)}\,e^{-i(\omega_2+\Omega)t} .$$

Consequently the amplitude-modulated strong wave can be regarded as three unmodulated strong waves having carrier angular frequencies $(\omega_2-\Omega)$, ω_2 and $(\omega_2+\Omega)$ respectively. The effects of nonlinear phenomena on the weak wave due to the mutual interaction with these three waves can be found readily from the result of this section.

If there are many waves, say m, present simultaneously in the ionized gas, it is easy to extend the present result of the effects of nonlinear phenomena on the electric field of one of the waves due to both self-interaction and mutual interaction. For a wave, say wave 1, its clockwise phasor $\underline{E}_{-}^{(1)}$ of the electric field can be expanded into a Fourier series, namely

$$\underline{E}_{-}^{(1)}(\underline{r},t) = \left\{\underline{E}_{-1}^{(1)}(\underline{r}) + \sum_{p=1}^{m}\underline{\eta}_{-1}^{1,p}(\underline{r}) + \cdots\right\} e^{-i\omega_1 t} + \sum_{p=1}^{m}\underline{E}_{-2}^{1,p}(\underline{r})\,e^{-i(\omega_1+\omega_p)t} + \sum_{p=1}^{m}\underline{E}_{o}^{1,p}(\underline{r})\,e^{-i(\omega_1-\omega_p)t} + \cdots\cdots$$

where higher order correction terms have been omitted. In the above summations the terms where $p = 1$ arise from the effects of nonlinear phenomena due to self-interaction and the terms where $p = 2,3,\cdots,m$ arise from the effects of nonlinear phenomena due to mutual interaction between wave 1 and wave 2, wave 3, $\cdots$, wave m , respectively. The solutions for $\underline{\eta}_{-1}^{1,p}$, $\underline{E}_{-2}^{1,p}$ and $\underline{E}_{o}^{1,p}$ are given by equations 5.17, 5.18 and 5.19, respectively, where $p = 2$. As expected, $\underline{\eta}_{-1}^{1,1}$ and $\underline{\underline{E}}_{-2}^{1,1}$ coincide with $\underline{\eta}_{-1}$ and $\underline{E}_{-2}$, respectively, given by equations 5.10 and 5.11 in the case of self-interaction. Note that $\underline{E}_{0}^{1,1}$ is found to be zero and is omitted in equation 5.4.

If the m waves have different propagation directions, the correction terms $\underline{\eta}_{-1}^{1,p}$, $\underline{E}_{-2}^{1,p}$ and $\underline{E}_{o}^{1,p}$ are still similar to those given by equations 5.17, 5.18 and 5.19, respectively, except that the definitions of the dyadics $\underline{\underline{\Lambda}}$ will be more complicated than those given previously for waves having the same propagation direction along the z-axis. The same effects of nonlinear phenomena, namely wave form distortion and harmonic wave generation, still exist.

If the pth wave is a TEM wave, the dyadics $\underline{\underline{R}}(\underline{E}_{-1}^{p})$ and $\underline{\underline{S}}(\underline{E}_{1}^{p})$ are found to be zero because the electron gyrofrequency $\underline{\omega}_g$ and the electric field phasors $\underline{E}_{\pm 1}^{p}(\underline{r})$ of the pth wave have a special relation with its propagation direction as mentioned in Section 3.3. From equations 5.18 and 5.19, where $p = 2$, we find that the spatial dispersive effect of harmonic wave generation on wave 1 due to mutual interaction with the pth wave disappears if the pth wave is a TEM wave.

We have found the electric field of a wave in a nonlinear ionized gas where other waves are simultaneously present, and we shall use the solutions of the electric field of the waves in the ionized gas to find the reflection of the waves from the nonlinear ionized gas in the next chapter.

VI. REFLECTION FROM A NONLINEAR IONIZED GAS

In Chapter IV it was found that the conductivity tensors of a nonlinear ionized gas depended on the electric field of the disturbing electromagnetic waves. The reflected waves from such a nonlinear ionized gas will no longer be linearly proportional to the incident wave. In order to simplify the mathematics we shall examine reflected waves of normally incident waves only. We study nonlinear phenomena in the reflection from a nonlinear ionized gas of semi-infinite extent in two cases: (1) reflection of a wave from a nonlinear ionized gas, and (2) reflection of a weak wave from a nonlinear ionized gas in the presence of a disturbing strong wave.

6.1) Reflection of a Wave

When a plane electromagnetic wave from free space falls on the interface between free space and a linear ionized gas as in Section 2.4, there are two reflected waves in free space, one of which has its electric field parallel to that of the incident wave, while the other has its electric field perpendicular to that of the incident wave. In the linear ionized gas on the other side of the interface, there are two transmitted waves, namely the ordinary and the extraordinary waves. Because the conductivity tensors of the nonlinear ionized gas depend on the electric field of the waves present, the ordinary and the extraordinary waves will interact with each other in the nonlinear ionized gas. Their electric fields will be distorted and harmonic waves will be generated which have been found in Section 5.2. The above nonlinear effects modify the coefficients of reflection from the ionized gas.

Consider a plane electromagnetic wave which is normally incident on the interface separating the free space from that occupied by the nonlinear ionized gas. The gas is still assumed to be externally biased with an oblique magnetostatic field. Choose the cartesian coordinate system as shown in Fig. 2 for the incident wave and for the external magnetostatic field. The nonlinear ionized gas is assumed to occupy the half-space $z > 0$.

As mentioned in Section 5.1, the clockwise phasor of an electric field of a wave is always the complex conjugate of the corresponding counterclockwise phasor. Hence we shall solve only for the clockwise phasor of an electric field, since the counterclockwise phasor can be found readily from the clockwise phasor.

In the nonlinear ionized gas the effects of nonlinear phenomena will distort the wave forms of the two transmitted waves, namely the ordinary and the extraordinary waves, and will generate harmonic waves at the second harmonic frequency. To preserve the notations used in Chapter V, superscripts (1) and (2) are used to distinguish the ordinary wave (1) from the extraordinary wave (2). The clockwise phasor of the electric field of either wave can be expanded, according to Section 5.2, as

$$\underline{E}_{-}^{q}(\underline{r},t) = \left\{ \underline{E}_{-1}^{q}(\underline{r}) + \sum_{p=1}^{2} \underline{\eta}_{-1}^{q,p}(\underline{r}) + \cdots \right\} e^{-i\omega t} + \sum_{p=1}^{2} \underline{E}_{-2}^{q,p}(\underline{r})\, e^{-i2\omega t} + \cdots \tag{6.1}$$

where $q = 1$ or 2.

The terms $\sum_{p=1}^{2} \underline{E}_o^{q,p}(\underline{r})$ are absent because both transmitted waves have the common carrier angular frequency ω . From the vector wave equation 5.5 for the clockwise phasor of the electric field, the following set of equations is found for the clockwise phasor of the electric field of the q^th^ wave, according to Section 5.2

$$\nabla \times \nabla \times \underline{E}_{-1}^{q} - \omega^2 \mu_o \underline{\underline{\epsilon}}_L(\omega) \cdot \underline{E}_{-1}^{q} = 0 \tag{6.2}$$

$$\nabla \times \nabla \times \underline{\eta}_{-1}^{q,p} - \omega^2 \mu_o \underline{\underline{\epsilon}}_L(\omega) \cdot \underline{\eta}_{-1}^{q,p} = \underline{\underline{Q}}(\underline{E}_{-1}^{p} \cdot \underline{E}_{1}^{p}) \cdot \underline{E}_{-1}^{q} \tag{6.3}$$

and

$$\nabla \times \nabla \times \underline{E}_{-2}^{q,p} - 4\omega^2 \mu_o \underline{\underline{\epsilon}}_L(2\omega) \cdot \underline{E}_{-2}^{q,p} = \underline{\underline{R}}(\underline{E}_{-1}^{p}) \cdot \underline{E}_{-1}^{q} \tag{6.4}$$

where $\underline{\underline{\epsilon}}_L(\omega)$ is the dielectric tensor of a linear ionized gas for an electromagnetic wave with an angular frequency ω . The dyadics on the right of equations 6.3 and 6.4 are given by

$$\underline{\underline{Q}}(\underline{E}_{-1}^{p} \cdot \underline{E}_{1}^{p}) = i\omega \mu_o \cdot \frac{e^2 M}{m^2 kT}$$

$$\times \frac{(\nu_1^2 + \omega^2)(\nu_1^2 + \omega^2 + \omega_g^2) \underline{E}_{-1}^{p} \cdot \underline{E}_{1}^{p} + (\nu_1^2 - 3\omega^2 + \omega_g^2)(\underline{\omega}_g \cdot \underline{E}_{-1}^{p})(\underline{\omega}_g \cdot \underline{E}_{1}^{p})}{(\nu_1^2 + \omega^2)(\nu_1^2 + \omega^2 + \omega_g^2 + 2\omega\,\omega_g)(\nu_1^2 + \omega^2 + \omega_g^2 - 2\omega\,\omega_g)} \underline{\underline{\sigma}}_L(\omega)$$

$$\underline{\underline{R}}(\underline{E}_{-1}^{p}) = \frac{2\mu_o e}{m} \nabla \cdot \left[\underline{\underline{L}}(\omega) \cdot \underline{E}_{-1}^{p} \right] \underline{\underline{\sigma}}_L(2\omega)$$

$$\underline{\underline{L}}(\omega) = \frac{(\nu_1 - i\omega)^2 \underline{\underline{u}} + (\nu_1 - i\omega)\, \underline{\omega}_g \times \underline{\underline{u}} + \underline{\omega}_g \underline{\omega}_g}{(\nu_1 - i\omega)\left[(\nu_1 - i\omega)^2 + \omega_g^2\right]}$$

where $\underline{\underline{\sigma}}_L(\omega)$ is the conductivity tensor of a linear ionized gas for an electromagnetic wave with an angular frequency ω .

The solutions of equation 6.2 have been found in Section 2.3. For the ordinary wave (1), the solution is given by equation 2.11 with an undetermined coefficient $A_o^{(0)}$; and for the extraordinary wave (2), the solution is given by equation 2.12 with an undetermined coefficient $A_e^{(0)}$. The superscript (0) is used to distinguish the undetermined coefficients of equation 6.2 from those of equation 6.3 and equation 6.4, where superscripts (1) and (2) are used, respectively. The two other solutions of equation 6.2 are for waves traveling along the negative z-direction and they have the spatial dependence $\exp(-ikz) = \exp(\alpha - i\beta)z$ which increases with increasing z. At $z = \infty$, the radiation condition requires that all waves be finite and be zero in a lossy medium. Hence these two solutions of equation 6.2 will be omitted.

The particular solution of equation 6.3 is given by an equation similar to equation 5.17. Although equation 6.3 represents a set of four equations with q or p denoting (1) or (2), the complementary solutions to these four equations are similar to the solutions of equation 6.2. These complementary solutions are given by equations 2.11 and 2.12 with two undetermined coefficients for each equation. We shall combine the four ordinary waves into one ordinary wave with an undetermined coefficient $A_o^{(1)}$, and the four extraordinary waves into one extraordinary wave with an undetermined coefficient $A_e^{(1)}$.

Similarly, the particular solution of equation 6.4 is given by an equation similar to equation 5.18. $A_o^{(2)}$ and $A_e^{(2)}$ are the undetermined coefficients in equations 2.11 and 2.12, respectively, for the resultant ordinary and the resultant extraordinary waves of the complementary solutions of equation 6.4. Note that the carrier angular frequency in equation 6.4 is 2ω which should be used in equation 2.11 and equation 2.12.

In free space on the other side of the interface we shall choose a normally incident wave whose electric field makes an angle ϕ with the x-axis as shown in Fig. 2, and whose clockwise phasor of the electric field has an amplitude A_i . The clockwise phasors of the electric field and the magnetic field in cartesian coordinate system are

$$\left.\begin{aligned} E_x &= \cos\phi \\ E_y &= \sin\phi \\ E_z &= 0 \\ H_x &= -\frac{k_f}{\omega\,\mu_o}\sin\phi \\ H_y &= \frac{k_f}{\omega\,\mu_o}\cos\phi \\ H_z &= 0 \end{aligned}\right\} A_i\, e^{i(k_f z - \omega t)} \qquad (6.5)$$

where $k_f = \omega\sqrt{\mu_o \epsilon_o}$ is the propagation constant in free space.

In order to solve the boundary problems we shall look for reflected waves such that the incident wave, the reflected waves, and the transmitted waves satisfy the boundary conditions at $z = 0$. When wave form distortion and harmoinc wave generation in the transmitted waves due to the effects of nonlinear phenomena are taken into account, the clockwise phasors of the fields of the two *reflected* waves are

$$\left.\begin{aligned} E_x &= \cos\phi \\ E_y &= \sin\phi \\ E_z &= 0 \\ H_x &= \frac{k_f}{\omega\mu_o}\sin\phi \\ H_y &= -\frac{k_f}{\omega\mu_o}\cos\phi \\ H_z &= 0 \end{aligned}\right\} A_i\left[R_{||}^{(0)} + R_{||}^{(1)} + R_{||}^{(2)}\ e^{-i\omega t}\right] e^{-i(k_f z + \omega t)} \qquad (6.6)$$

and

$$\left.\begin{aligned} E_x &= -\sin\phi \\ E_y &= \cos\phi \\ E_z &= 0 \\ H_x &= \frac{k_f}{\omega\mu_o}\cos\phi \\ H_y &= \frac{k_f}{\omega\mu_o}\sin\phi \\ H_z &= 0 \end{aligned}\right\} A_i\left[R_{\perp}^{(0)} + R_{\perp}^{(1)} + R_{\perp}^{(2)} e^{-i\omega t}\right] e^{-i(k_f z + \omega t)}\ . \qquad (6.7)$$

The two coefficients of reflection from a nonlinear ionized gas are

$$R_{||} = R_{||}^{(0)} + R_{||}^{(1)} + R_{||}^{(2)}\ e^{-i\omega t}$$

and

$$R_{\perp} = R_{\perp}^{(0)} + R_{\perp}^{(1)} + R_{\perp}^{(2)}\ e^{-i\omega t}\ . \qquad (6.8)$$

$R_{||}^{(0)}$ and $R_{\perp}^{(0)}$ are the same as the coefficients of reflection from a linear ionized gas and have been found in Section 2.4. $R_{||}^{(1)}$ and $R_{\perp}^{(1)}$ are correction terms due to wave form distortion in the transmitted waves, and will be found to be proportional to the square of the amplitude of the incident wave. $R_{||}^{(2)}$ and $R_{\perp}^{(2)}$ are correction terms due to harmonic wave generation in the transmitted waves, and will be found to be linearly proportional to the amplitude of the incident wave.

At the interface $z = 0$ the boundary conditions are such that the tangential component of the electric field is continuous and that the tangential component of the magnetic field is continuous. We match the x- and the y-components of both the electric and the magnetic fields of the incident wave and the reflected waves with those of the transmitted waves. Equate the coefficients of terms of the same time dependence of $e^{-i\omega t}$ and $e^{-i2\omega t}$. Equate further the coefficients of terms of the same order of magnitude. We obtain the following equations in matrix notation

$$\begin{bmatrix} M_{-1} \end{bmatrix} \begin{bmatrix} A_i R_{||}^{(0)} \\ A_i R_{\perp}^{(0)} \\ A_o^{(0)} \\ A_e^{(0)} \end{bmatrix} = A_i \begin{bmatrix} -\cos\phi \\ -\sin\phi \\ k_f \sin\phi \\ k_f \cos\phi \end{bmatrix} , \qquad (6.9)$$

$$\begin{bmatrix} M_{-1} \end{bmatrix} \begin{bmatrix} A_i R_{||}^{(1)} \\ A_i R_{\perp}^{(1)} \\ A_o^{(1)} \\ A_e^{(1)} \end{bmatrix} = \sum_{q,p=1}^{2} \begin{bmatrix} \left\{ \underline{\eta}_{-1}^{q,p}(0) \right\}_x \\ \left\{ \underline{\eta}_{-1}^{q,p}(0) \right\}_y \\ (k_q + k_p - k_p^*) \left\{ \underline{\eta}_{-1}^{q,p}(0) \right\}_y \\ -(k_q + k_p - k_p^*) \left\{ \underline{\eta}_{-1}^{q,p}(0) \right\}_x \end{bmatrix} \qquad (6.10)$$

and

$$\begin{bmatrix} M_{-2} \end{bmatrix} \begin{bmatrix} A_i R_{||}^{(2)} \\ A_i R_{\perp}^{(2)} \\ A_o^{(2)} \\ A_e^{(2)} \end{bmatrix} = \sum_{q,p=1}^{2} \begin{bmatrix} \left\{ \underline{E}_{-2}^{q,p}(0) \right\}_x \\ \left\{ \underline{E}_{-2}^{q,p}(0) \right\}_y \\ (k_q + k_p) \left\{ \underline{E}_{-2}^{q,p}(0) \right\}_y \\ -(k_q + k_p) \left\{ \underline{E}_{-2}^{q,p}(0) \right\}_x \end{bmatrix} \qquad (6.11)$$

where the matrix is

$$\begin{bmatrix} M \end{bmatrix} = \begin{bmatrix} \cos \phi , & - \sin \phi , & iR_o , & iR_e \\ \sin \phi , & \cos \phi , & -1 , & -1 \\ k_f \sin \phi , & k_f \cos \phi , & k_o , & k_e \\ k_f \cos \phi , & -k_f \sin \phi , & -ik_o R_o , & -ik_e R_e \end{bmatrix}$$

In the matrix $\begin{bmatrix} M_{-1} \end{bmatrix}$, k_f, k_o, k_e, R_o and R_e are functions of ω and have been found in Sections 2.3 and 2.4. In the matrix $\begin{bmatrix} M_{-2} \end{bmatrix}$ they are functions of 2ω . $\left\{ \underline{\eta}_{-1}^{q,p}(0) \right\}_x$ and $\left\{ \underline{\eta}_{-1}^{q,p}(0) \right\}_y$ are the x- and the y-components of $\underline{\eta}_{-1}^{q,p}(z)$ evaluated at $z = 0$ respectively. k_q is the propagation constant of the transmitted waves in the ionized gas, and

$$k_q = \begin{cases} k_o & \text{if } q = 1 \\ k_e & \text{if } q = 2 \end{cases}$$

$\left\{\underline{E}^{q,p}_{-2}(0)\right\}_x$ and $\left\{\underline{E}^{q,p}_{-2}(0)\right\}_y$ are the x- and the y-components of $\underline{E}^{q,p}_{-2}(z)$ evaluated at $z = 0$.

When $A_i = 1$ equation 6.9 is identical with equation 2.17, as is expected, since it is the equation for the zeroth order solution for the reflected and the transmitted waves. $R^{(0)}_{||}$, $R^{(0)}_{\perp}$, $A^{(0)}_o$ and $A^{(0)}_e$ are found to be

$$R^{(0)}_{||} = - \frac{(k_o k_e - k_f^2)(R_o - R_e) + (k_o - k_e)k_f\ (R_o + R_e)\cos 2\phi}{(k_o + k_f)(k_e + k_f)\ (R_o - R_e)}$$

$$R^{(0)}_{\perp} = - \frac{i(k_o - k_e)k_f\left[2 + i(R_o + R_e)\ \sin 2\phi\right]}{(k_o + k_f)(k_e + k_f)(R_o - R_e)}$$

$$A^{(0)}_o = \frac{2ik_f(\cos\phi + iR_e \sin\phi)}{(k_o + k_f)\ (R_o - R_e)}\ A_i$$

and

$$A^{(0)}_e = - \frac{2ik_f(\cos\phi + iR_o \sin\phi)}{(k_e + k_f)\ (R_o - R_e)}\ A_i\ .$$

From equation 6.2, equation 2.11 and equation 2.12 $\underline{E}^q_{-1}$ is found to be proportional to $A^{(0)}_o$ if $q = 1$ and $A^{(0)}_e$ if $q = 2$. From the above expressions for $A^{(0)}_o$ and $A^{(0)}_e$, we find that $\underline{E}^q_{-1}$ is, in turn, linearly proportional to A_i .

$R^{(1)}_{||}$, $R^{(1)}_{\perp}$, $A^{(1)}_o$ and $A^{(1)}_e$ can be found from equation 6.10 by inverting the matrix $\left[M_{-1}\right]$. Because of the long and complicated solutions for $R^{(1)}_{||}$, $R^{(1)}_{\perp}$, $A^{(1)}_o$ and $A^{(1)}_e$, we shall not give their solutions here. From an equation similar to equation 5.17 the correction term $\underline{\eta}_{-1}^{q,p}$ is found to be proportional to $(\underline{E}_{-1}^{p} \cdot \underline{E}_{1}^{p})\underline{E}_{-1}^{q}$ and hence to A_i^3. From equation 6.10 $R_{||}^{(1)}$ and $R_{\perp}^{(1)}$ are found to be proportional to A_i^2, and $A_o^{(1)}$ and $A_e^{(1)}$ proportional to A_i^3.

Similarly, $R_{||}^{(2)}$, $R_{\perp}^{(2)}$, $A_o^{(2)}$ and $A_e^{(2)}$ can be found from equation 6.11 by inverting the matrix $\left[M_{-2}\right]$. However, their long and complicated solutions will not be given here. The correction term $\underline{E}_{-2}^{q,p}$ on the right of equation 6.11 is found to be proportional to A_i^2 and leads to the result that $R_{||}^{(2)}$ and $R_{\perp}^{(2)}$ are proportional to A_i, and $A_o^{(2)}$ and $A_e^{(2)}$ are proportional to A_i^2.

The results just obtained can be applied to the case shown in Fig. 5. An antenna is located in free space and is directed toward a nonlinear ionized gas of semi-infinite extent. The gas is assumed to be in the far zone of the antenna. When the plane incident wave from the antenna falls on the gas, there will be transmitted waves in the gas and reflected waves in free space. For convenience the clockwise phasors of the electric fields of the incident and the reflected waves are rewritten here:

$$\text{clockwise phasor of } \underline{E}_{inc} = \left\{ \underline{e}_x \cos\phi + \underline{e}_y \sin\phi \right\} A_i e^{i(k_f z - \omega t)}$$

$$\text{clockwise phasor of } \underline{E}_{||} = \left\{ \underline{e}_x \cos\phi + \underline{e}_y \sin\phi \right\} A_i R_{||} e^{-i(k_f z + \omega t)}$$

$$\text{clockwise phasor of } E_{\perp} = \left\{ -\underline{e}_x \sin\phi + \underline{e}_y \cos\phi \right\} A_i R_{\perp} e^{-i(k_f z + \omega t)}$$

where

$$R_{||} = R_{||}^{(0)} + R_{||}^{(1)}(A_i^2) + R_{||}^{(2)}(A_i)\, e^{-i\omega t}$$

and

$$R_{\perp} = R_{\perp}^{(0)} + R_{\perp}^{(1)}(A_i^2) + R_{\perp}^{(2)}(A_i)\, e^{-i\omega t} .$$

Note the dependence of the correction terms in $R_{||}$ and $R_{\perp}$ on the

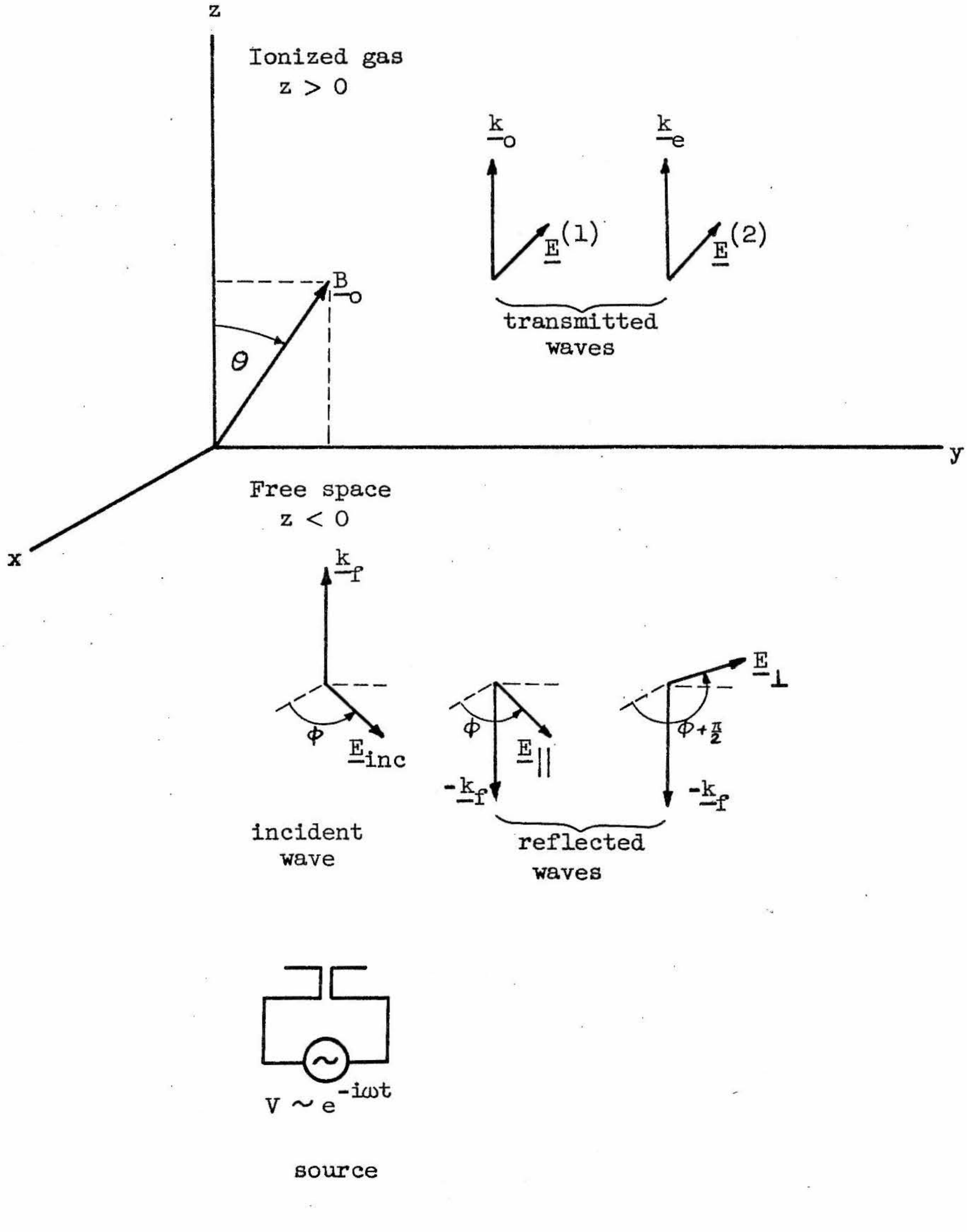

Fig. 5. Reflection of a wave from a nonlinear ionized gas

amplitude A_i of the clockwise phasor of the electric field of the incident wave.

From the coefficients of reflection given by equation 6.8 we conclude that in free space the amplitudes of the reflected waves are distorted and waves of second harmonic frequency are generated which are all due to the effects of nonlinearity in the ionized gas. If the incident wave is amplitude modulated with a modulation angular frequency Ω and a modulation index μ, the quasi-stationary solutions for $A_i R_{||}^{(1)}$ and $A_i R_{\perp}^{(1)}$ are found to be proportional to $A_i^3(1 + \mu \cos \Omega t)^3$ and those for $A_i R_{||}^{(2)}$ and $A_i R_{\perp}^{(2)}$ proportional to $A_i^2(1 + \mu \cos \Omega t)^2$. The reflected waves are now modulated with modulation angular frequencies Ω, 2Ω and 3Ω, and the harmonic waves are modulated with modulation angular frequencies Ω and 2Ω. If the direction of the external magnetostatic field is parallel to the propagation direction of the incident wave, $R_{||}^{(2)} = 0 = R_{\perp}^{(2)}$ and there will be no harmonic waves.

6.2) Reflection of a Weak Wave in the Presence of a Disturbing Strong Wave

In this study of reflection from a nonlinear ionized gas, a weak and a strong electromagnetic wave are assumed to be incident simultaneously on the interface between free space and the nonlinear ionized gas. The weak incident wave gives rise to two weak transmitted waves in the ionized gas and, similarly, the strong incident wave gives rise to two strong transmitted waves. We are interested in the effects of nonlinear phenomena on the reflected waves of the weak incident wave due to mutual interaction between the weak and the strong transmitted waves, neglecting that between the two weak transmitted waves. The electric fields of the weak transmitted waves are found in Section 5.2 to be distorted and harmonic waves are generated. The above effects of nonlinear phenomena, in turn, modify the coefficients of reflection for the weak incident wave.

For simplicity consider two plane electromagnetic waves which are normally incident on the interface separating the free space from that occupied by the nonlinear ionized gas. Choose the cartesian coordinate system as shown in Fig. 2 for the weak incident wave and for the external magnetostatic field. The strong incident wave is assumed to propagate along the z-axis and its electric field is assumed to make an arbitrary angle with the x-axis. The nonlinear gas is assumed to occupy the half-space $z > 0$.

As mentioned in Section 5.1, we shall solve for only the clockwise phasor of the electric field of a wave, since the counter-clockwise phasor is the complex conjugate of the clockwise phasor and can be found radily.

To preserve the notations used in Chapter V superscripts (1), (2), (3) and (4) are used to denote the weak ordinary wave, the weak extraordinary wave, the strong ordinary wave, and the strong extra-ordinary wave, respectively. Because of the effects of nonlinear phenomena in the ionized gas, the clockwise phasor of the electric field of either weak transmitted wave can be expanded, according to Section 5.2, as

$$\underline{E}^{q}(\underline{r},t) = \left\{ \underline{E}^{q}_{-1}(\underline{r}) + \sum_{p=3}^{4} \underline{\eta}^{q,p}_{-1}(\underline{r}) + \cdots \right\} e^{-i\omega_1 t}$$

$$+ \sum_{p=3}^{4} \underline{E}^{q,p}_{-2}(\underline{r})\, e^{-i(\omega_1+\omega_3)} + \sum_{p=3}^{4} \underline{E}^{q,p}_{o}(\underline{r})\, e^{-i(\omega_1-\omega_3)t} \qquad (6.12)$$

where $q = 1$ or 2, and the angular frequency of the weak transmitted waves (1) and (2) is ω_1, and that of the strong transmitted waves (3) and (4) is ω_3. In equation 6.12 the terms with $p = 1$ or 2 in the summations are absent because these terms are due to interaction between the two weak transmitted waves and have been neglected. From the vector wave equation 5.5 for the clockwise phasor of an electric field, the following set of equations is found for the clockwise phasor of the electric field of the qth weak transmitted wave, according to Section 5.2

$$\nabla \times \nabla \times \underline{E}^{q}_{-1} - \omega_1^2 \mu_o \underline{\underline{\epsilon}}_L(\omega_1) \cdot \underline{E}^{q}_{-1} = 0 \qquad (6.13)$$

$$\nabla \times \nabla \times \underline{\eta}^{q,p}_{-1} - \omega_1^2 \mu_o \underline{\underline{\epsilon}}_L(\omega_1) \cdot \underline{\eta}^{q,p}_{-1} = \underline{\underline{Q}}(\underline{E}^{p}_{-1} \cdot \underline{E}^{p}_{1}) \cdot \underline{E}^{q}_{1} \qquad (6.14)$$

$$\nabla \times \nabla \times \underline{E}^{q,p}_{-2} - (\omega_1+\omega_3)^2 \mu_o \underline{\underline{\epsilon}}_L(\omega_1+\omega_3) \cdot \underline{E}^{q,p}_{-2} = \underline{\underline{R}}(\underline{E}^{p}_{-1}) \cdot \underline{E}^{q}_{-1} \qquad (6.15)$$

and

$$\nabla \times \nabla \times \underline{E}^{q,p}_{o} - (\omega_1-\omega_3)^2 \mu_o \underline{\underline{\epsilon}}_L(\omega_1-\omega_3) \cdot \underline{E}^{q,p}_{o} = \underline{\underline{S}}(\underline{E}^{p}_{1}) \cdot \underline{E}^{q}_{-1} \qquad (6.16)$$

where $q = 1$ or 2 stands for the weak ordinary or the weak extraordinary wave, respectively;

and $p = 3$ or 4 stands for the strong ordinary or the strong extraordinary wave, respectively.

$\underline{\underline{\epsilon}}_L(\omega)$ is the dielectric tensor of a linear ionized gas. The dyadics on the right of the above equations are given by

$$\underline{\underline{Q}}(\underline{E}^p_{-1} \cdot \underline{E}^p_1) = i\omega_1\mu_o \cdot \frac{e^2 M}{m^2 kT}$$

$$\times \frac{(\nu_1^2+\omega_3^2)(\nu_1^2+\omega_3^2+\omega_g^2)\ \underline{E}^p_{-1}\cdot \underline{E}^p_1 + (\nu_1^2 - 3\omega_3^2+\omega_g^2)(\underline{\omega}_g\cdot \underline{E}^p_{-1})(\underline{\omega}_g \cdot \underline{E}^p_1)}{(\nu_1^2+\omega_3^2)(\nu_1^2+\omega_3^2+\omega_g^2+2\omega_3\ \omega_g^2)(\nu_1^2+\omega_3^2+\omega_g^2-2\omega_3\ \omega_g)}\ \underline{\underline{\sigma}}_L(\omega_1)$$

$$\underline{\underline{R}}(\underline{E}^p_{-1}) = \frac{(\omega_1+\omega_3)\ \mu_o e}{\omega_3 m}\ \nabla\cdot\left[\underline{\underline{L}}(\omega_3)\cdot \underline{E}^p_{-1}\right]\ \underline{\underline{\sigma}}_L(\omega_1+\omega_3)$$

$$\underline{\underline{S}}(\underline{E}^p_1) = -\frac{(\omega_1-\omega_3)\ \mu_o e}{\omega_3 m}\ \nabla\cdot\left[\underline{\underline{L}}(-\omega_3)\cdot \underline{E}^p_1\right]\underline{\underline{\sigma}}_L(\omega_1-\omega_3)$$

where $\underline{\underline{\sigma}}_L(\omega)$ is the conductivity tensor of a linear ionized gas, and

$$\underline{\underline{L}}(\omega) = \frac{(\nu_1 - i\omega)^2\ \underline{\underline{u}} + (\nu_1 - i\omega)\ \underline{\omega}_g \times \underline{\underline{u}} + \underline{\omega}_g\ \underline{\omega}_g}{(\nu_1 - i\omega)\left[(\nu_1 - i\omega)^2 + \omega_g^2\right]} .$$

This set of equations 6.13 to 6.16 is similar to the set of equations 6.2 to 6.4 in Section 6.1. The complementary solutions are given by equation 2.11 for the ordinary waves and by 2.12 for the extraordinary waves. $A_o^{(0)}$ and $A_e^{(0)}$ are the two undetermined coefficients for the solutions of equation 6.13. $A_o^{(1)}$ and $A_e^{(1)}$ are the two undetermined coefficients for the complementary solutions of equation 6.14 whose particular solution is given by an equation similar to equation 5.17. $A_o^{(2)}$ and $A_e^{(2)}$ are the two undetermined coefficients for the complementary solutions of equation 6.15 where the angular frequency is $(\omega_1+\omega_3)$. The particular solution of equation 6.15 is given by an equation similar to equation 5.18. $A_o^{(3)}$ and

$A_e^{(3)}$ are the two undetermined coefficients for the complementary solutions of equation 6.16 where the angular frequency is $(\omega_1 - \omega_3)$. The particular solution of equation 6.16 is given by an equation similar to equation 5.19.

In free space, on the other side of the interface, we shall choose a weak, normally incident wave whose electric field makes an angle ϕ with the x-axis as shown in Fig. 2, and whose clockwise phasor of the electric field has an amplitude A_1 . The clockwise phasors of the fields in cartesian coordinate system are given by equation 6.5.

In order to solve the boundary problems we shall look for reflected waves such that the incident wave, the reflected waves and the transmitted waves satisfy the boundary conditions at $z = 0$. When wave form distortion and harmonic wave generation in the two weak transmitted waves due to the effects of nonlinear phenomena in the nonlinear ionized gas are taken into account, the clockwise phasors of the fields of the two reflected waves are given by equations 6.6 and 6.7 with the coefficients of reflection

$$R_{||} = R_{||}^{(0)} + R_{||}^{(1)} + R_{||}^{(2)} e^{-i\omega_3 t} + R_{||}^{(3)} e^{i\omega_3 t}$$

and

$$R_{\perp} = R_{\perp}^{(0)} + R_{\perp}^{(1)} + R_{\perp}^{(2)} e^{-i\omega_3 t} + R_{\perp}^{(3)} e^{i\omega_3 t} . \tag{6.17}$$

$R_{||}^{(0)}$ and $R_{\perp}^{(0)}$ are the same as the coefficients of reflection from a linear ionized gas, and have been found in Section 2.4. $R_{||}^{(1)}$ and $R_{\perp}^{(1)}$ are correction terms due to wave form distortion in the weak transmitted waves, and will be found to be proportional to the square of the amplitude of the strong incident wave. $R_{||}^{(2)}$, $R_{\perp}^{(2)}$, $R_{||}^{(3)}$ and $R_{\perp}^{(3)}$ are correction terms due to harmonic wave generation in the weak transmitted waves, and will be found to be linearly proportional to the amplitude of the strong incident wave.

At the interface $z = 0$ the boundary conditions are that the tangential components of the electric and the magnetic fields are continuous. We match the x- and y-components of both the electric field and the magnetic field of the weak incident wave and the weak reflected waves with those of the weak transmitted waves. Equate the coefficients of terms of the same time dependence of $e^{-i\omega_1 t}$, $e^{-i(\omega_1+\omega_3)t}$ and $e^{-i(\omega_1-\omega_3)t}$. Equate further the coefficients of terms of the same order of magnitude. We obtain the following set of equations in matrix notation

$$\left[M_{-1}^{1}\right] \begin{bmatrix} A_1 R_{||}^{(0)} \\ A_1 R_{\perp}^{(0)} \\ A_o^{(0)} \\ A_e^{(0)} \end{bmatrix} = A_1 \begin{bmatrix} -\cos\phi \\ -\sin\phi \\ k_f \sin\phi \\ k_f \cos\phi \end{bmatrix} \tag{6.18}$$

$$\left[M_{-1}^{1}\right] \begin{bmatrix} A_1 R_{||}^{(1)} \\ A_1 R_{\perp}^{(1)} \\ A_o^{(1)} \\ A_e^{(1)} \end{bmatrix} = \sum_{q=1}^{2} \sum_{p=3}^{4} \begin{bmatrix} \left\{ \underline{\eta}_{-1}^{q,p}(0) \right\}_x \\ \left\{ \underline{\eta}_{-1}^{q,p}(0) \right\}_y \\ (k_q + k_p - k_p^*) \left\{ \underline{\eta}_{-1}^{q,p}(0) \right\}_y \\ -(k_q + k_p - k_p^*) \left\{ \underline{\eta}_{-1}^{q,p}(0) \right\}_x \end{bmatrix} \tag{6.19}$$

$$[M_{-2}^{1,3}] \begin{bmatrix} A_1 R_{\|}^{(2)} \\ A_1 R_{\perp}^{(2)} \\ A_o^{(2)} \\ A_e^{(2)} \end{bmatrix} = \sum_{q=1}^{2} \sum_{p=3}^{4} \begin{bmatrix} \{\underline{E}_{-2}^{q,p}(0)\}_x \\ \{\underline{E}_{-2}^{q,p}(0)\}_y \\ (k_q + k_p)\{\underline{E}_{-2}^{q,p}(0)\}_y \\ -(k_q + k_p)\{\underline{E}_{-2}^{q,p}(0)\}_x \end{bmatrix} \tag{6.20}$$

and

$$[M_o^{1,3}] \begin{bmatrix} A_1 R_{\|}^{(3)} \\ A_1 R_{\perp}^{(3)} \\ A_o^{(3)} \\ A_e^{(3)} \end{bmatrix} = \sum_{q=1}^{2} \sum_{p=3}^{4} \begin{bmatrix} \{\underline{E}_{o}^{q,p}(0)\}_x \\ \{\underline{E}_{o}^{q,p}(0)\}_y \\ (k_q - k_p^*)\{\underline{E}_{o}^{q,p}(0)\}_y \\ -(k_q - k_p^*)\{\underline{E}_{o}^{q,p}(0)\}_x \end{bmatrix} \tag{6.21}$$

where

$$[M] = \begin{bmatrix} \cos\phi, & -\sin\phi, & iR_o, & iR_e \\ \sin\phi, & \cos\phi, & -1, & -1 \\ k_f \sin\phi, & k_f \cos\phi, & k_o, & k_e \\ k_f \cos\phi, & -k_f \sin\phi, & -ik_o R_o, & -ik_e R_e \end{bmatrix}. \tag{6.22}$$

k_f, k_o, k_e, R_o and R_e are found in Section 2.4 to be functions of the angular frequency ω which is different for the three matrices.

For $[M_{-1}^{1}]$, $\omega = \omega_1$; for $[M_{-2}^{1,3}]$, $\omega = \omega_1 + \omega_3$;

and for $[M_o^{1,3}]$, $\omega = \omega_1 - \omega_3$.

Also $\left.\begin{aligned} k_1 &= k_o(\omega_1) \\ k_2 &= k_e(\omega_1) \end{aligned}\right\}$ for the weak transmitted waves

and $\left.\begin{aligned} k_3 &= k_o(\omega_3) \\ k_4 &= k_e(\omega_3) \end{aligned}\right\}$ for the strong transmitted waves.

If $A_1 = 1$ equation 6.18 is identical with equation 2.17 as is expected, since it is the equation for the zeroth order solution for the weak reflected waves and the weak transmitted waves. $R_{||}^{(0)}$ and $R_{\perp}^{(o)}$ are given by equation 2.18 and $A_o^{(0)}$ and $A_e^{(0)}$ are given by equation 2.19.

Since the correction terms $\underline{\eta}_{-1}^{q,p}$ are found from equations similar to equation 5.17 to be proportional to the square of the amplitude A_3 of the strong incident wave, $R_{||}^{(1)}$, $R_{\perp}^{(1)}$, $A_o^{(1)}$ and $A_e^{(1)}$ in equation 6.19 are found to be proportional to the square of the amplitude A_3 of the strong incident wave.

Since the correction terms $\underline{E}_{-2}^{q,p}$ from equations similar to equation 5.18 and the correction terms $\underline{E}_{o}^{q,p}$ from equations similar to equation 5.19 are found to be linearly proportional to the amplitude A_3 of the strong incident wave, $R_{||}^{(2)}$, $R_{\perp}^{(2)}$, $A_o^{(2)}$ and $A_e^{(2)}$ in equation 6.20 and $R_{||}^{(3)}$, $R_{\perp}^{(3)}$, $A_o^{(3)}$ and $A_e^{(3)}$ in equation 6.21 are found to be linearly proportional to the amplitude A_3 of the strong incident wave.

The results just obtained can be applied to the case shown in Fig. 6. Two antennas are located in free space and are directed toward a nonlinear ionized gas of semi-infinite extent. The gas is assumed to be in the far zone of both antennas. One antenna is excited by a weak source (1), and thereby produces a weak incident wave at the far zone. The clockwise phasor of the electric field of the weak incident wave has an amplitude A_1 . The other antenna is excited by a strong source (3), and the clockwise phasor of the electric field of the strong incident wave has an amplitude A_3 . The

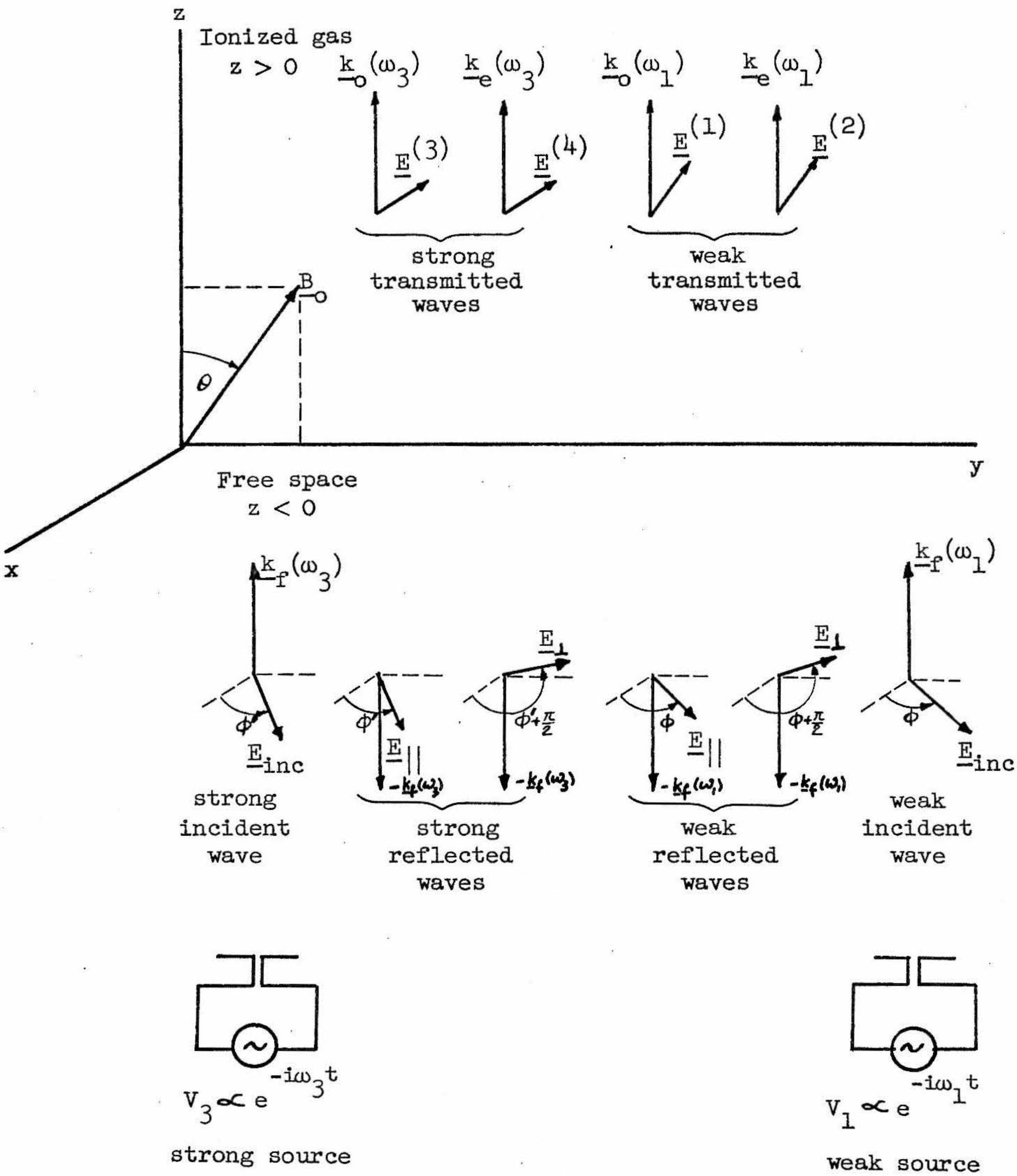

Fig. 6. Reflection of a weak wave from a nonlinear ionized gas in the presence of a strong wave.

strong transmitted waves interact mutually with the weak transmitted waves in the nonlinear ionized gas. The weak reflected waves are found to depend on the strong incident wave. For convenience, the clockwise phasors of the electric fields of the strong and the weak incident waves and the weak reflected waves are rewritten here:

$$\text{clockwise phasor of strong } \underline{E}_{inc} = \left[\underline{e}_x \cos \phi' + \underline{e}_y \sin \phi'\right] A_3 e^{i\left[k_f(\omega_3)z - \omega_3 t\right]}$$

$$\text{clockwise phasor of weak } \underline{E}_{inc} = \left[\underline{e}_x \cos \phi + \underline{e}_y \sin \phi\right] A_1 e^{i\left[k_f(\omega_1)z - \omega_1 t\right]}$$

$$\text{clockwise phasor of weak } \underline{E}_{||} = \left[\underline{e}_x \cos \phi + \underline{e}_y \sin \phi\right] A_1 R_{||}\, e^{-i\left[k_f(\omega_1)z + \omega_1 t\right]}$$

and

$$\text{clockwise phasor of weak } \underline{E}_{\perp} = \left[-\underline{e}_x \sin \phi + \underline{e}_y \cos \phi\right] A_1 R_{\perp}\, e^{-i\left[k_f(\omega_1)z + \omega_1 t\right]}$$

$$\text{where } R_{||} = R_{||}^{(0)} + R_{||}^{(1)}(A_3^2) + R_{||}^{(2)}(A_3) e^{-i\omega_3 t} + R_{||}^{(3)}(A_3)\, e^{i\omega_3 t}$$

$$\text{and } R_{\perp} = R_{\perp}^{(0)} + R_{\perp}^{(1)}(A_3^2) + R_{\perp}^{(2)}(A_3)\, e^{-i\omega_3 t} + R_{\perp}^{(3)}(A_3)\, e^{i\omega_3 t} \quad .$$

Note the dependence of the correction terms in $R_{||}$ and $R_{\perp}$ on the amplitude A_3 of the clockwise phasor of the electric field of the strong incident wave. The clockwise phasors of the electric fields of the strong reflected waves have been given in the previous section.

From the coefficients of reflection given by equation 6.17 we conclude that in free space the amplitudes of the weak reflected waves are distorted and waves of mixed frequencies are generated which are all due to the effects of nonlinearity in the ionized gas. The weak reflected waves depend on the amplitude A_1 of the weak incident wave as well as the amplitude of the strong one as indicated previously.

If the strong incident wave is amplitude modulated with a modulation angular frequency Ω and a modulation index μ, the quasi-stationary solutions for $R_{\parallel}^{(1)}$ and $R_{\perp}^{(1)}$ are found to be proportional to $A_3^2(1+\mu\cos\Omega t)^2$ and those for $R_{\parallel}^{(2)}$, $R_{\perp}^{(2)}$, $R_{\parallel}^{(3)}$ and $R_{\perp}^{(3)}$ proportional to $A_3(1+\mu\cos\Omega t)$. The weak reflected waves are cross modulated with modulation angular frequencies Ω and 2Ω, and the harmonic waves of mixed frequencies are cross modulated with modulation angular frequency Ω.

If the direction of the external magnetostatic field is parallel to the propagation direction of the strong incident wave, the strong transmitted waves will be TEM waves. The correction terms $\underline{E}_{-2}^{q,p}$ and $\underline{E}_{o}^{q,p}$ with $p = 3$ or 4 will disappear and there will be no harmonic waves of mixed frequencies in the ionized gas or free space.

If the effects of nonlinear phenomena due to the interaction between the two weak transmitted waves are not neglected, the summation signs in equations 6.19, 6.20 and 6.21 will be $\sum_{p=1}^{4}$ instead of $\sum_{p=3}^{4}$. The additional correction terms for the coefficients of reflection $R_{\parallel}$ and $R_{\perp}$ will be the same as the correction terms found in Section 5.1. If there are many incident waves present simultaneously, we can just extend the summation signs in equations 6.19, 6.20 and 6.21 to cover all transmitted waves.

If the incident waves propagate along different directions, then the matrices $[M]$'s , the dyadics $\underline{\underline{\Lambda}}$'s , the undetermined coefficients A_o , A_e , etc. will be more complicated. We still find the same effects of nonlinear phenomena, namely wave form distortion and harmonic wave generation in both the transmitted and the reflected waves.

VII. CONCLUSION

In the study of nonlinear phenomena in the propagation of electromagnetic waves in a weakly ionized gas and the reflection from the weakly ionized gas, we use statistical mechanics and field theory to solve the nonlinear problems. Boltzmann's kinetic equation is used to find the electron distribution function in a weakly ionized gas in the presence of electromagnetic waves and an external magnetostatic field. From the electron distribution function, the electron convection current density is found. Then Maxwell's equations are used to find the fields of the electromagnetic waves. A perturbation method has been used because it is justified in many practical cases. Geometrical optics approximation has been avoided because it can only be applied to TEM waves in the ionized gas. In this study we find that the amplitude of an electromagnetic wave is distorted and harmonic waves are generated. If the disturbing electromagnetic wave is amplitude modulated, the other wave is found to be cross modulated.

As pointed out in Chapter I, investigations of TEM waves in an ionized gas have little application in the propagation of broadcast signals in the ionosphere because the signals are frequently TM waves. In the present study we have analyzed the effects of nonlinear phenomena on TM waves in a weakly ionized gas, and can apply a similar analysis to the propagation of broadcast signals in the ionosphere by treating the ionosphere as stratified layers of ionized gas and by using the method in Chapter VI of matching the boundary conditions at the interfaces.

Cross modulation which is the consequence of wave form distortion had been reported in reference (1). It will be interesting to perform experiments to detect harmonic waves generated using an ionized gas and electromagnetic waves of appropriate physical parameters to maximize such a nonlinear effect.

Similar analyses can be used to study the effects of nonlinear phenomena in gas, liquid and solids such as gaseous discharge, ionic solution, semiconductors, etc.

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