RANGE OF VALIDITY OF THE METHOD OF AVERAGING

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ABSTRACT

Sufficient conditions are derived for the validity of approximate periodic solutions of a class of second order ordinary nonlinear differential equations. An approximate solution is defined to be valid if an exact solution exists in a neighborhood of the approximation.

Two classes of validity criteria are developed. Existence is obtained using the contraction mapping principle in one case, and the Schauder-Leray fixed point theorem in the other. Both classes of validity criteria make use of symmetry properties of periodic functions, and both classes yield an upper bound on a norm of the difference between the approximate and exact solution. This bound is used in a procedure which establishes sufficient stability conditions for the approximated solution.

Application to a system with piecewise linear restoring force (bilinear system) reveals that the approximate solution obtained by the method of averaging is valid away from regions where the response exhibits vertical tangents. A narrow instability region is obtained near one-half the natural frequency of the equivalent linear system. Sufficient conditions for the validity of resonant solutions are also derived, and two term harmonic balance approximate solutions which exhibit ultraharmonic and subharmonic resonances are studied.

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INTRODUCTION

Attempts to devise mathematical models for physical phenomena more often than not yield some form of nonlinear equation. Since explicit solutions are extremely rare, one usually seeks an approximate solution within a specific class. For example, Oscillation Theory is particularly interested in approximating periodic solutions of second order nonlinear differential equations.

A number of closely related approximating techniques, widely used in applications, namely the method of averaging, harmonic balance and equivalent linearization claim validity near resonance, or asymptotically as some small parameter goes to zero. These criteria are unsatisfactory in the sense that:

1) they do not guarantee the existence of an exact solution in a neighborhood of the approximation, and
2) even if existence is assumed, no quantitative measure is available for the actual error in the approximation, and hence a stability analysis based on the approximate solution is questionable.

In this work, quantitative validity criteria are derived which remove the objections cited above. Fixed point theorems from functional analysis and topology are the principal tools used.

Previous work on the validity of approximate solutions includes that of Cesari, Bass, Urabe, Holtzman and McLaughlin. Cesari(1), and later Urabe(2), derived sufficient conditions for the existence of an exact solution in a neighborhood of the n'th Galerkin (harmonic balance) approximation, and a bound on the difference between the exact and approximate solution. Urabe also showed that existence of an isolated periodic solution implies the existence of a Galerkin approximation of sufficiently high order. The validity of equivalent linearization was studied by Bass[3] for the autonomous case, and by Holtzman[4], [5] for the nonautonomous case. McLaughlin[6] used the implicit function theorem to obtain sufficient validity conditions for the Poincaré expansion.

CHAPTER I

MATHEMATICAL PRELIMINARIES

A number of ordinary differential equations frequently encountered in applications are of the form

$$\ddot{x} + f(x, \dot{x}) = g(t) \tag{1.1}$$

where f and g are piecewise continuous with at most a finite number of finite discontinuities. In this chapter, the problem of finding periodic solutions of (1.1) is reduced to a boundary value problem on an interval equal to or less than the period. Relationships between certain boundary value problems and Fredholm integral equations are derived, and several fixed point theorems are introduced.

For certain forms of f and g in (1.1) it is helpful to seek periodic solutions with appropriate symmetry properties.

1.1. Symmetry Properties of Periodic Functions.

Definition 1.1: A function u(t) is in class P_T ($u \in P_T$) if u(t) is continuously differentiable and

$$u(t) = u(t+T) \tag{1.2}$$

for all t, T a constant.

Definition 1.2: A function u(t) is in class H_T ($u \in H_T$) if $u \in P_T$ and

$$u(T/2 + t) = -u(t) \tag{1.3}$$

<u>Definition 1.3:</u> A function u(t) is in class Q_T ($u \in Q_T$) if $u \in H_T$ and

$$u(T/2 - t) = -u(t) \tag{1.4}$$

Note: Certain H_T functions may be put into class Q_T by a translation in t.

Obviously, $P_T \supset H_T \supset Q_T$. Figure 1.1a shows a function in class H_T and Figure 1.1b shows a function in class Q_T. Functions in these symmetry classes have properties given by the following Lemmas.

<u>Lemma 1.1:</u> If $u \in P_T$, then necessarily

$$\begin{aligned} u(0) &= u(T) \\ \dot{u}(0) &= \dot{u}(T) \end{aligned} \tag{1.5}$$

<u>Proof:</u> Let t=0 in (1.2), then differentiate and do the same.

<u>Lemma 1.2:</u> If $u \in H_T$, then necessarily

$$\begin{aligned} u(0) &= -u(T/2) \\ \dot{u}(0) &= -\dot{u}(T/2) \end{aligned} \tag{1.6}$$

<u>Proof:</u> Let t=0 in (1.3), then differentiate and do the same.

<u>Lemma 1.3:</u> If $u \in Q_T$ then necessarily

$$\begin{aligned} u(T/4) &= 0 \\ \dot{u}(0) &= 0 \end{aligned} \tag{1.7}$$

<u>Proof:</u> Let t = T/4 in (1.4), then

$$u(T/4) = -u(T/4)$$

$$\therefore \quad u(T/4) = 0$$

Differentiating (1.3) and (1.4) with respect to t and letting t=0 yields,

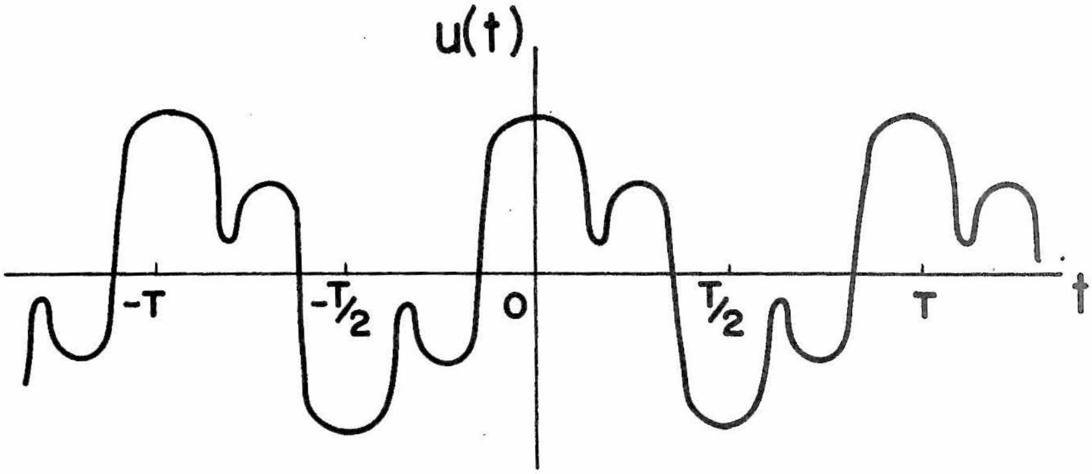

Fig. 1.1a. A CLASS H_T FUNCTION

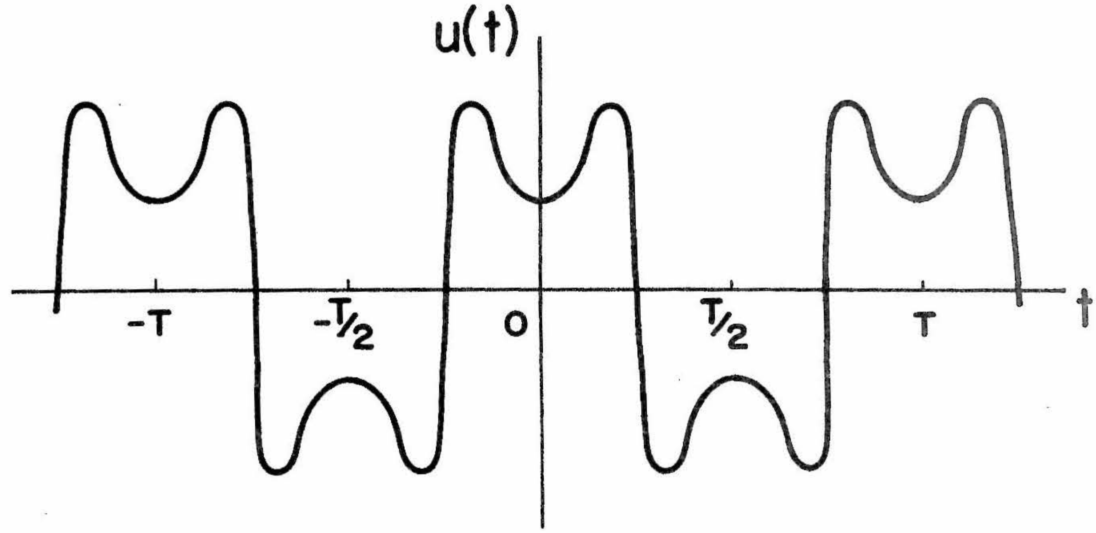

Fig. 1.1b A CLASS Q_T FUNCTION

$$\mathring{u}(T/2) = -\mathring{u}(0)$$

$$-\mathring{u}(T/2) = -\mathring{u}(0)$$

$\therefore$ $\mathring{u}(0) = -\mathring{u}(0)$ implies $\mathring{u}(0) = 0$.

1.2 Periodic Solutions via Boundary Value Problems.

The following Lemmas show that in certain cases, periodic solutions may be constructed by extending the solutions of particular boundary value problems (B. V. P. 's) to the infinite interval. Equations (1. 5) - (1. 7) serve as boundary conditions. In the sequel, it is assumed that f and g are continuous. This is a matter of convenience since many of the results apply to the case where f and g are piecewise continuous with at most a finite number of finite discontinuities.

Lemma 1.4: If

1) $g(t) \in P_T$ (1.8)

2) u(t) is a solution of the B. V. P.

$$\ddot{u} + f(u, \dot{u}) = g(t) \quad 0 < t < T$$
$$u(0) = u(T) \qquad \mathring{u}(0) = \mathring{u}(T) \tag{1.9}$$

then u(t) is extendible to a class P_T solution of (1.1). That is, the function x(t) defined by

$$x(t+nT) = u(t) \tag{1.10}$$

for $0 \leq t \leq T$, n an integer, is a class P_T solution of (1.1).

Proof: Since the differential equation remains the same on subsequent intervals $(n-1)T < t < nT$, n an integer, x(t) satisfies (1.1) for all t, and hence it is a class P_T solution.

Lemma 1.5: If

$$1)\quad g(t) \in H_T \tag{1.11}$$

$$2)\quad f(x, \dot{x}) = -f(-x, -\dot{x}) \tag{1.12}$$

3) u(t) is a solution of the B. V. P.

$$\begin{aligned} &\ddot{u} + f(u, \dot{u}) = g(t) \quad 0<t<T/2 \\ &u(0) = -u(T/2) \quad \dot{u}(0) = -\dot{u}(T/2) \end{aligned} \tag{1.13}$$

then u(t) is extendible to a class H_T solution of (1. 1). That is, x(t) defined by

$$\left.\begin{aligned} &x(t) = u(t) \quad 0 \le t \le T/2 \\ &x(t+T/2) = -u(t) \quad 0 \le t \le T/2 \\ &x(t+nT) = x(t) \quad n \text{ an integer}, \quad 0 \le t \le T \end{aligned}\right\} \tag{1.14}$$

is a class H_T solution of (1. 1).

Proof: The second of equations (1. 14) implies

$$\left.\begin{aligned} &-x(T/2+t) = u(t) \\ &-\dot{x}(T/2+t) = \dot{u}(t) \quad 0<t<T/2 \\ &-\ddot{x}(T/2+t) = \ddot{u}(t) \end{aligned}\right\} \tag{1.15}$$

Using this in the differential equation yields

$$-\ddot{x}(T/2+t) + f\Big(-x(T/2+t), -\dot{x}(T/2+t)\Big) = g(t) \quad 0<t<T/2 \tag{1.16}$$

or, using (1. 11) and (1. 12)

$$\ddot{x}(T/2+t) + f\Big(x(T/2+t), \dot{x}(T/2+t)\Big) = g(T/2+t) \quad 0<t<T/2 \tag{1.17}$$

and hence x(t) is a solution for 0<t<T satisfying

$$x(T) = x(0)$$

$$\dot{x}(T) = \dot{x}(0)$$

where (1.14) and the boundary conditions on u(t) were used. Extension to the whole real line is as in Lemma 1.4.

Lemma 1.6: If

$$1)\quad g(t)\in Q_T \tag{1.18}$$

$$2)\quad f(x,\dot{x}) = -f(-x,-\dot{x}) \tag{1.19}$$

$$3)\quad f(x,\dot{x}) = f(x,-\dot{x}) \tag{1.20}$$

4) u(t) is a solution of the B.V.P.

$$\left.\begin{aligned} &\ddot{u}+f(u,\dot{u}) = g(t) \quad 0<t<T/4 \\ &\dot{u}(0)=0 \quad u(T/4)=0 \end{aligned}\right. \tag{1.21}$$

then u(t) is extendible to a class Q_T solution of (1.1). That is, the function x(t) defined by

$$\left.\begin{aligned} &x(t) = u(t) \quad 0\le t\le T/4 \\ &x(T/2-t) = -u(t) \quad 0\le t\le T/4 \\ &x(T/2+t) = -x(t) \quad 0\le t\le T/2 \\ &x(t+nT) = x(t) \ \ n \text{ an integer}, \quad 0\le t\le T \end{aligned}\right\} \tag{1.22}$$

is a class Q_T solution of (1.1).

Proof: The second of (1.22) implies

$$\left.\begin{aligned} &x(T/2-t) = -u(t) \\ &\dot{x}(T/2-t) = \dot{u}(t) \\ &\ddot{x}(T/2-t) = -\ddot{u}(t) \end{aligned}\right\} \ 0<t<T/4 \tag{1.23}$$

Substituting into the differential equation and using (1.18) - (1.20) yields

$$\ddot{x}(T/2-t)+f\big(x(T/2-t),\dot{x}(T/2-t)\big) = g(T/2-t) \quad 0<t<T/4$$

and hence x(t) is a solution of (1.1) for $0<t<T/2$ satisfying

$$x(T/2) = -x(0)$$
$$\dot{x}(T/2) = -\dot{x}(0) = 0$$

where the boundary conditions and (1.22) were used. Extension to the whole real line is as in Lemma (1.5).

When the conditions on f and g are satisfied in Lemmas 1.4 - 1.6, the differential equation is said to allow solutions in the pertinent symmetry class. For some equations, a translation in t may be necessary to allow solutions in Q_T.

1.3. Boundary Value Problems and Fredholm Integral Equations.

The following results concerning the relationship of certain B.V.P.'s to Fredholm integral equations will be useful in what follows.

Consider the equation

$$\ddot{u} + a(t)\dot{u} + b(t)u = c(u, \dot{u}, t) \quad (1.24)$$

or, in vector-matrix form

$$\dot{\overline{u}} = A(t)\overline{u} + c(\overline{u}, t) \quad (1.25)$$

where

$$A(t) = \begin{bmatrix} 0 & , & 1 \\ -b(t), & & -a(t) \end{bmatrix}, \quad \overline{c}(\overline{u}, t) = \begin{bmatrix} 0 \\ c(u, \dot{u}, t) \end{bmatrix}, \quad u = \begin{bmatrix} u \\ \dot{u} \end{bmatrix} \quad (1.26)$$

In the following Lemmas a, b, and c are assumed to be piece-wise continuous with at most a finite number of finite discontinuities. Although the first Lemma may be deduced from developments in Hahn(7), page 359, or Halanay(8), page 225, the proof is given here in the context of this work.

<u>Lemma 1.7:</u> If $\bar{u}$, $\bar{c}(\bar{u}, t)$ and $A(t)$ are as given by (1.26) with

1) $U(t)$ the principal matrix solution (see Struble[9], page 83) of

$$\dot{\bar{u}} = A(t)\bar{u} \tag{1.27}$$

2) $[I-U(T)]^{-1}$ exists. Here I is the 2 ×2 identity matrix and $[\ ]^{-1}$ is the matrix inverse.

3)

$$H(t, s) = \begin{bmatrix} h_{11}, h_{12} \\ h_{21}, h_{22} \end{bmatrix} = \begin{cases} U(t)[I-U(T)]^{-1}U^{-1}(s) & 0 \le s < t \le T \\ U(t)[I-U(T)]^{-1}U(T)U^{-1}(s) & 0 \le t \le s \le T \end{cases} \tag{1.28}$$

4) $\bar{u}$ governed by

$$\dot{\bar{u}} = A(t)\bar{u} + \bar{c}(\bar{u}, t) \quad 0<t<T \tag{1.29}$$

$$\bar{u}(0) = \bar{u}(T) \tag{1.30}$$

then

$$\bar{u} = \int_0^T c(u, \dot{u}, s)\bar{h}(t, s)ds \tag{1.31}$$

$$\bar{h}(t, s) = \begin{bmatrix} h_{12} \\ h_{22} \end{bmatrix} \tag{1.32}$$

<u>Proof:</u> Use is made of the following well known result concerning initial value problems (see Struble[9], page 92).

Given

$$\dot{\bar{u}} = A(t)\bar{u} + \bar{f}(t) \tag{1.33}$$

$$\bar{u}(0) = \bar{u}_0 \tag{1.34}$$

then

$$\overline{u}(t) = U(t)\overline{u}_0 + U(t)\int_0^t U^{-1}(s)\overline{f}(s)ds \tag{1.35}$$

Using this result with $\overline{f}(t)$ replaced by $\overline{c}(\overline{u}, t)$, yields the following alternate form for (1. 29)

$$\overline{u}(t) = U(t)\overline{u}(0) + U(t)\int_0^t U^{-1}(s)\overline{c}(\overline{u}, s)ds \tag{1.36}$$

where now $\overline{u}(0)$ is determined from the condition (1. 30). That is,

$$\overline{u}(0) = \overline{u}(T) = U(T)\overline{u}(0) + U(T)\int_0^T U^{-1}(s)\overline{c}(\overline{u}, s)ds \tag{1.37}$$

$$\overline{u}(0) = [I-U(T)]^{-1}U(T)\int_0^T U^{-1}(s)\overline{c}(\overline{u}, s)ds \tag{1.38}$$

and hence

$$\overline{u}(t) = U(t)[I-U(T)]^{-1}U(T)\int_0^T U^{-1}(s)\overline{c}(\overline{u}, s)ds + U(t)\int_0^t U^{-1}(s)\overline{c}(\overline{u}, s)ds \tag{1.39}$$

or,

$$u(t) = \int_0^T H(t, s)\overline{c}(\overline{u}, s)ds \tag{1.40}$$

where

$$H(t, s) = \begin{cases} U(t)\left\{[I-U(t)]^{-1}U(T) + I\right\}U^{-1}(s) & 0 \le s \le t \le T \\ U(t)[I-U(T)]^{-1}U(T)U^{-1}(s) & 0 \le t \le s \le T \end{cases} \tag{1.41}$$

Now using $\bar{c}(\bar{u}, t) = \begin{bmatrix} 0 \\ c(u, \dot{u}, t) \end{bmatrix}$ gives

$$\bar{u} = \int_0^T c(u, \dot{u}, s)\bar{h}(t, s)ds \tag{1.42}$$

It remains only to show that H(t, s) as given in (1.41) is the same as in (1.29), or equivalently

$$[I-U(T)]^{-1}U(T)+I = [I-U(T)]^{-1}U(T) + [I-U(T)]^{-1}[I-U(T)]$$

$$[I-U(T)]^{-1}U(T)+I = [I-U(T)]^{-1}$$

Since the proof of the following Lemma is very similar to that of Lemma 1.7, no proof is given.

Lemma 1.8: If $\bar{u}$, $\bar{c}(\bar{u}, t)$ and A(t) are as given in (1.27) with

1) U(t) the principal matrix solution of

$$\dot{\bar{u}} = A(t)\bar{u} \tag{1.44}$$

2) $[I+U(T)]^{-1}$ exists

3)

$$H(t, s) = \begin{bmatrix} h_{11}, h_{12} \\ h_{21}, h_{22} \end{bmatrix} = \begin{cases} -U(t)[I+U(T)]^{-1}U^{-1}(s) & 0 \le s \le t \le T \\ -U(t)[I+U(T)]^{-1}U(T)U^{-1}(s) & 0 \le t \le s \le T \end{cases} \tag{1.45}$$

4) $\bar{u}$ governed by

$$\dot{\bar{u}} = A(t)\bar{u} + \bar{c}(\bar{u}, t) \quad 0 < t < T/2 \tag{1.46}$$

$$\bar{u}(0) = -\bar{u}(T/2) \tag{1.47}$$

then

$$\overline{u} = \int_0^{T/2} c(u, \dot{u}, s)\overline{h}(t, s)ds \tag{1.48}$$

$$\overline{h}(t, s) = \begin{bmatrix} h_{12} \\ h_{22} \end{bmatrix} \tag{1.49}$$

Lemma 1.9: If $\overline{u}$, $\overline{c}(\overline{u}, t)$ and $A(t)$ are as given in (1.27) with

1)

$$U(t) = \begin{bmatrix} u_1(t) & v_1(t) \\ u_2(t) & v_2(t) \end{bmatrix} \tag{1.50}$$

the principal matrix solution of

$$\dot{\overline{u}} = A(t)\overline{u} \tag{1.51}$$

2) $u_1(T/4) \neq 0.$

3)

$$\overline{h}(t, s) = \begin{bmatrix} h_1(t, s) \\ h_2(t, s) \end{bmatrix} \tag{1.52}$$

where

$$h_1(t, s) = \begin{cases} \dfrac{u_1(s)[u_1(T/4)v_1(t) - u_1(t)v_1(T/4)]}{d(T/4, s)} & 0 \le s \le t \le T/4 \\[2ex] \dfrac{u_1(t)[u_1(T/4)v_1(s) - v_1(T/4)u_1(s)]}{d(T/4, s)} & 0 \le t \le s \le T/4 \end{cases} \tag{1.53}$$

$$h_2(t, s) \quad \begin{cases} \dfrac{u_1(s)[u_1(T/4)v_2(t) - u_2(t)v_1(T/4)]}{d(T/4, s)} & 0 \le s \le t \le T/4 \\[2ex] \dfrac{u_2(t)[u_1(T/4)v_1(s) - v_1(T/4)u_1(s)]}{d(T/4, s)} & 0 \le t \le s \le T/4 \end{cases} \tag{1.54}$$

$$d(T/4, s) = u_1(T/4)[u_1(s)v_2(s) - v_1(s)u_2(s)]$$

4) $\overline{u}$ governed by

$$\dot{\overline{u}} = A(t)\overline{u} + \overline{c}(\overline{u}, t) \quad 0<t<T/4 \tag{1.55}$$

$$\mathring{u}(0) = 0 \quad u(T/4) = 0 \tag{1.56}$$

then

$$\overline{u}(t) = \int_0^{T/4} c(u, \mathring{u}, s)\overline{h}(t, s)ds \tag{1.57}$$

<u>Proof:</u> The initial value problem with $\mathring{u}(0)=0$ gives

$$\overline{u} = U(t)\begin{bmatrix} u_0 \\ 0 \end{bmatrix} + \int_0^t U(t)U^{-1}(s)\overline{c}(\overline{u}, s)ds \tag{1.58}$$

The unknown u_0 must satisfy

$$u(T/4) = 0 = u_1(T/4)u_0 + \int_0^{T/4} \frac{[u_1(s)v_1(T/4) - v_1(s)u_1(T/4)]}{[u_1(s)v_2(s) - u_2(s)v_1(s)]} c(u, \mathring{u}, s)ds \tag{1.59}$$

where the first component of the vector equation (1.58) was used. Solving for u_0 yields

$$u_0 = \int_0^{T/4} \frac{[v_1(s)u_1(T/4) - u_1(s)v_1(T/4)]}{d(T/4, s)} c(u, \mathring{u}, s)ds \tag{1.60}$$

Substituting this expression for u_0 into (1.58) and simplifying yields (1.57), the desired result.

The Fredhom integral equations (1.31), (1.48), and (1.57) are special cases of a nonlinear operator equation

$$x = Tx \tag{1.61}$$

Functional analysis and topology provide useful results concerning the solutions of such equations. References in this area include Kolmogorov and Fomin(10), Liusternik and Sobolev(11), and Saaty and Bram(12).

1.4. Fixed Point Theorems.

The various fixed point theorems are extremely useful in establishing sufficient conditions for the existence of solutions of (1.61). As noted by Saaty(13), these theorems fall into two classes, the topological fixed point theorems which merely establish existence, and the algebraic fixed point theorems which also yield uniqueness and a means of constructing the solution. Both types are useful in establishing the validity of approximate periodic solutions.

In this work, consideration is restricted to nonlinear operator equations on a complete normed linear space, i.e., a Banach space.

The following topological fixed point theorem, proved on page 45 of Saaty and Bram(12), is an extension of Brouwer's fixed point theorem to a Banach space.

Theorem 1.1 (Schauder-Leray):

If B is a closed convex subset of a Banach space X and T is a continuous operator mapping B into itself such that the image of B is compact, then there exists at least one fixed point $x^* \in B$ satisfying

$$x^* = Tx^* \tag{1.62}$$

Corollary:

If in Theorem (1.1)

1) $B = B_r = \{x \mid \|x\| \leq r\}$, a ball of radius r centered at the zero element. Here $\|\cdot\|$ represents a norm.

2) there exists a real number $\gamma \geq 0$ and a $\beta=\beta(r)$ satisfying $0<\beta<1$, such that

$$\|Tx\| \leq \gamma + \beta\|x\| \tag{1.63}$$

for all $x \in B_r$.

then there exists at least one $x^* \in B_r$ satisfying (1.62) and

$$\|x^*\| \leq \frac{\gamma}{1-\beta} \tag{1.64}$$

Proof: By Theorem (1.1) there exists an $x^* \in B_r$ satisfying (1.62). Taking the norm of (1.62) and using (1.63) yields

$$\|x^*\| \leq \gamma + \beta\|x^*\| \tag{1.65}$$

or, since $0<\beta<1$

$$\|x^*\| \leq \frac{\gamma}{1-\beta} \tag{1.66}$$

A constructive fixed point theorem is provided by the principle of contraction mappings.

Definition 1.4: An operator T mapping a normed space S into itself is a contraction mapping on S if there exists an α satisfying $0<\alpha<1$ such that for all $x, y \in S$

$$\|Tx-Ty\| \leq \alpha\|x-y\| \tag{1.67}$$

A fundamental theorem concerning contraction mapping is

Theorem 1.2:

Every contraction mapping T on a Banach space X has a unique fixed point $x^* \in X$ satsifying $x^*=Tx^*$. For any $x_0 \in X$, the sequence of iterates $x_n=Tx_{n-1}=T^n x_0$ converges to x^*, and

$$\|x_n - x^*\| \leq \frac{\alpha^n}{1-\alpha}\|Tx_0 - x_0\| \tag{1.68}$$

Proof: This theorem is the special case of Theorem 1.3, for $r \to \infty$.

This theorem requires the mapping to contract on the whole space X. Unfortunately, most operators do not satisfy this condition, so it is necessary to extend the theorem to particular subsets of X. Intuitively, one would expect a mapping to contract in the neighborhood of a point x_0 if x_0 is "close enough" to a fixed point. That is, if $\|x^*-x_0\|$ is small enough. The following theorem, given without proof on page 42 of Baily, Shampine and Waltman[14], extends the contraction mapping principle to the case where the mapping contracts on a ball and also shows how to find the radius of the ball.

Theorem 1.3 (Contraction Mapping on a Ball):

If for an operator T, there exists a number r such that

1) T maps the ball $B_r = \{w \mid \|w-x_0\| \leq r\}$ of a Banach space X into X.
2) there exists an $\alpha = \alpha(r)$ satisfying $0<\alpha<1$ and

$$\|Tx-Ty\| \leq \alpha\|x-y\| \tag{1.69}$$

for all $x, y \in B_r$.

3)

$$r \geq \frac{\|Tx_0 - x_0\|}{1-\alpha} \qquad (1.70)$$

then T has a unique fixed point $x^* \in B_r$ satisfying $x^* = Tx^*$, and x^* is the limit of the sequence $x_n = Tx_{n-1} = T^n(w_0)$ where w_0 is any point in B_r.

Proof: Since X is a Banach space, every Cauchy sequence in X converges to a limit point in X. The following argument shows that x_n defined by

$$x_n = Tx_{n-1} = T^n x_0 \qquad (1.71)$$

is a Cauchy sequence, and hence has a limit point in X.

For any n and k

$$\|x_{n+k} - x_n\| = \|x_{n+k} - x_{n+k-1} + x_{n+k-1} - x_{n+k-2} + \ldots + x_{n+1} - x_n\|$$

Using the triangle inequality

$$\|x_{n+k} - x_n\| \leq \|x_{n+k} - x_{n+k-1}\| + \|x_{n+k-1} - x_{n+k-2}\| + \ldots + \|x_{n+1} - x_n\|$$

Now for any i

$$x_{i+1} - x_i = Tx_i - Tx_{i-1}$$

$$\|x_{i+1} - x_i\| \leq \alpha\|x_i - x_{i-1}\| \leq \alpha^2\|x_{i-1} - x_{i-2}\| \leq \ldots \leq \alpha^i\|x_1 - x_0\|$$

Hence

$$\|x_{n+k} - x_n\| \leq \left(\alpha^{n+k-1} + \alpha^{n+k-2} + \ldots + \alpha^n\right)\|x_1 - x_0\|$$

$$\leq \alpha^n\left(\alpha^{k-1} + \alpha^{k-2} + \ldots + 1\right)\|x_1 - x_0\|$$

$$\leq \alpha^n\|x_1 - x_0\| \sum_{k=0}^{\infty} \alpha^k = \frac{\alpha^n}{1-\alpha}\|Tx_0 - x_0\| \qquad (1.72)$$

Since the right hand side does not depend upon k, x_n is a Cauchy sequence and hence it converges to a limit point $x^* \in B_r$ provided that all of the iterates remain in B_r. Letting n=0 in (1.72) yields

$$\|x_k - x_0\| \le \frac{\|Tx_0 - x_0\|}{1-\alpha}$$

Hence $x_k \in B_r$ for all k if

$$r \ge \frac{\|Tx_0 - x_0\|}{1-\alpha} \tag{1.73}$$

Now for $k \to \infty$ in (1.72)

$$\|x^* - x_n\| \le \frac{\alpha^n}{1-\alpha}\|Tx_0 - x_0\| \tag{1.74}$$

which provides a bound on the error for the n^{th} iterate and shows that $\|x^* - x_n\| \to 0$ as $n \to \infty$. Hence, by Hypothesis 2) $\|Tx^* - Tx_n\| \to 0$ as $n \to \infty$ so T is continuous and hence

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n$$

$$x^* = T \lim_{n\to\infty} x_n$$

$$x^* = Tx^*$$

Mathematical induction will now be used to show that the iteration scheme may begin at any point $w_0 \in B_r$. Assume $w_k \in B_r$ for all $k \le n-1$. Then

$$\|w_n - x_0\| = \|Tw_{n-1} - Tx_0 + Tx_0 - x_0\|$$

$$\le \|Tw_{n-1} - Tx_0\| + \|Tx_0 - x_0\|$$

using (1.69) and (1.70)

$$\le \alpha\|w_{n-1} - x_0\| + r(1-\alpha)$$

$$\|w_n - x_0\| \le \alpha r + r(1-\alpha) = r$$

$\therefore$ $w_n \in B_r$ for all n.

Using arguments similar to those used for the sequence x_n, it is easy to show that

$$\|w^* - w_n\| \le \frac{2r\alpha^n}{1-\alpha}$$

and hence w_n is a Cauchy sequence converging to $w^* \in B_r$, where $w^* = Tw^*$.

Now suppose there are two fixed points $x^* \in B_r$ and $w^* \in B_r$. Then

$$\|w^* - x^*\| \le \alpha \|w^* - x^*\|$$

with $0<\alpha<1$, so $\|w^* - x^*\| = 0$ implies $w^* = x^*$.

1.5. Comments on Uniqueness.

Since the conditions satisfied by r in Theorem 1.3 are inequalities, there may be a range of suitable values of r. Of course, there may be no values at all in which case contraction is not assured on any ball centered at x_0. Assume for the moment that r_{min} is the minimum value which satisfies the conditions of Theorem 1.3, and that r_{max} is the maximum value. Let $B_{min} = \{w \mid \|w - x_0\| \le r_{min}\}$ and $B_{max} = \{w \mid \|w - x_0\| \le r_{max}\}$. Obviously, there is a unique fixed point $x^* \in B_{min}$ and this is the only fixed point inside B_{max}. If there were another fixed point $y^* \in B_{max}$ then uniqueness would be violated for the contraction mapping on B_{max}. Hence, any other fixed point y^* must satisfy $\|y^* - x_0\| > r_{max}$. In the case $r_{max} \to \infty$, the fixed point $x^* \in B_{min}$ is the only solution in the whole Banach space.

CHAPTER II

VALIDITY AND STABILITY CRITERIA

The various approximating techniques are discussed by Minorsky[15], Hayashi[16], and Bogoliubov and Mitropolsky[17]. This work considers only harmonic balance, with the understanding that equivalent linearization and the method of averaging are essentially one term harmonic balance approximations.

2.1 Validity Criteria.

Consider an equation of the form (1.1) with g(t) harmonic. That is,

$$\ddot{x} + f(x, \dot{x}) = P\cos(\omega t + \varphi) \tag{2.1}$$

One term harmonic balance yields the approximate solution

$$x_0(t) = A\cos\omega t \tag{2.2}$$

with amplitude A and phase φ satisfying

$$\left.\begin{aligned} -\omega^2 A + a_1 &= P\cos\varphi \\ b_1 &= -P\sin\varphi \end{aligned}\right\} \tag{2.3}$$

where a_1 and b_1 are the coefficients of $\cos\omega t$ and $\sin\omega t$ respectively, in the Fourier expansion of $f(x_0, \dot{x}_0)$. In vector-matrix form (2.1) is

$$\frac{d\bar{x}}{dt} = A\bar{x} - \bar{f}(\bar{x}) + \bar{g}(t) \tag{2.4}$$

where

$$\overline{x} = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} \; ; \; A = \begin{bmatrix} 0 \, , & 1 \\ 0 \, , & 0 \end{bmatrix} \; ; \; \overline{f(\overline{x})} = \begin{bmatrix} 0 \\ f(x, \dot{x}) \end{bmatrix} \; ; \; \overline{g}(t) = \begin{bmatrix} 0 \\ P \cos(\omega t + \varphi) \end{bmatrix} \tag{2.5}$$

with approximate solution

$$\overline{x}_0(t) = \begin{bmatrix} A \cos \omega t \\ -A\omega \sin \omega t \end{bmatrix} \tag{2.6}$$

In this chapter, validity criteria are established for the one term approximate solution. Extension to multi-term approximations with more general periodic forcing functions is straightforward. Examples of validity criteria for two term approximations are given in Chapter III.

Suppose that (2. 1) allows solutions in a given symmetry class, and that $x_0(t)$ is in this class. To establish the validity of the approximate solution $x_0(t)$, the existence of exact solutions in the given symmetry class in a neighborhood of $x_0(t)$ is sought.

Consider the difference between the exact and approximate solution

$$\eta(t) = x(t) - x_0(t) \tag{2.7}$$

called the <u>approximation error.</u> The function $\eta(t)$ satisfies the following equation obtained by substituting (2. 7) into (2. 1) and using (2. 3)

$$\ddot{\eta} + f(x_0 + \eta, \dot{x}_0 + \dot{\eta}) = a_1 \cos \omega t + b_1 \sin \omega t \tag{2.8}$$

Alternatively, (2. 8) is written as

$$\ddot{\eta} + a(t)\dot{\eta} + b(t)\eta = a_1 \cos \omega t + b_1 \sin \omega t - f(x_0 + \eta, \dot{x}_0 + \dot{\eta}) + a(t)\dot{\eta} + b(t)\eta \tag{2.9}$$

where a(t) and b(t) are suitably chosen piecewise continuous functions with at most a finite number of finite discontinuities. In vector-matrix form (2.9) is

$$\dot{\overline{\eta}} = A(t)\overline{\eta} + \overline{d}(\overline{\eta}, t)$$

where A(t) is given by (1.26) and

$$\overline{\eta} = \begin{bmatrix} \eta \\ \dot{\eta} \end{bmatrix} ; \quad \overline{d}(\overline{\eta}, t) = \begin{bmatrix} 0 \\ a_1 \cos \omega t + b_1 \sin \omega t + a(t)\dot{\eta} + b(t)\eta \\ -f(x_0 + \eta, \dot{x}_0 + \dot{\eta}) \end{bmatrix}$$

Now using Lemma 1.7, 1.8, or 1.9, depending upon the symmetry class desired, (note that $x_0(t)$ must be in the class and the equation must allow solutions in the class) gives

$$\overline{\eta} = \int_0^I \overline{h}(t, s)\left[a_1 \cos \omega s + b_1 \sin \omega s - f(x_0 + \eta, \dot{x}_0 + \dot{\eta}) + a(s)\dot{\eta} + b(s)\eta\right] ds \qquad (2.10)$$

where $\overline{h}(t, s)$ is the appropriate kernel and I is the interval of the B.V.P. An alternate form of (2.10) is

$$\overline{\eta} = \int_0^I \overline{h}(t, s)\left[a_1 \cos \omega s + b_1 \sin \omega s - f(x_0, \dot{x}_0)\right] ds$$
$$+ \int_0^I \overline{h}(t, s)\left[f(x_0, \dot{x}_0) + a(s)\dot{\eta} + b(s)\eta - f(x_0 + \eta, \dot{x}_0 + \dot{\eta})\right] ds \qquad (2.11)$$

Since the first integral on the right hand side, denoted by

$$\overline{e}(t) = \begin{bmatrix} e(t) \\ \dot{e}(t) \end{bmatrix} \qquad (2.12)$$

does not contain $\overline{\eta}$, it can be evaluated for any given f. Hence $\overline{\eta}$ is governed by the following Fredholm integral equation

$$\overline{\eta} = \overline{e} + \int_0^I \overline{h}(t, s)\left[f(x_0, \dot{x}_0) + a(s)\dot{\eta} + b(s)\eta - f(x_0 + \eta, \dot{x}_0 + \dot{\eta})\right] ds \qquad (2.13)$$

which is just a nonlinear operator equation of the type (1.62). For purposes of notation (2.13) is written as

$$\overline{\eta} = \overline{e} + H\overline{\eta} = N\overline{\eta} \tag{2.14}$$

Solutions are sought in the Banach space V of continuous two dimensional real vector valued functions on the interval I with norm

$$\|\overline{v}(t)\| = \sup_{0 \le t \le I} \sqrt{v_1^2(t) + v_2^2(t)} \tag{2.15}$$

where

$$\overline{v}(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} \tag{2.16}$$

There will also be occasion to use the Banach space C of continuous real valued functions on I with norm

$$\|y(t)\| = \sup_{0 \le t \le I} |y(t)| \tag{2.17}$$

Consider (2.14). The operator N maps the ball $B_r = \{\overline{w} \mid \|\overline{w}\| \le r\}$ of V into V, so application of Theorem 1.3 to the operator N on V yields sufficient conditions for the existence of a solution $\overline{\eta} \in B_r$. Obviously, this implies the existence of an exact solution $\overline{x}(t)$ of (2.4) within r of $\overline{x}_0(t)$. This results in

<u>Validity Criteria I:</u> Given an equation of the form (2.1), let $\overline{x}_0(t)$, $\overline{e}(t)$ and N be as previously defined.

If a subset B_r of V can be found such that

1) there exists an $\alpha = \alpha(r)$ satisfying $0 < \alpha < 1$ and

$$\|N\overline{x} - N\overline{y}\| \le \alpha \|\overline{x} - \overline{y}\| \tag{2.18}$$

for all $\overline{x}, \overline{y} \in B_r$.

2)

$$r \geq \frac{\|\bar{e}\|}{1-\alpha} \tag{2.19}$$

then there exists an exact solution $\bar{x}(t)$ of (2. 1) of the form

$$\bar{x}(t) = \bar{x}_0(t) + \bar{\eta}(t) \tag{2.20}$$

where $\bar{\eta}(t) \in B_r$ is the limit of the sequence

$$\bar{\eta}_{i+1} = N\bar{\eta}_i \quad \bar{\eta}_0 = \bar{0} \tag{2.21}$$

and

$$\|\bar{\eta} - \bar{\eta}_i\| \leq \frac{\alpha^i \|\bar{e}\|}{1-\alpha} \tag{2.22}$$

In particular,

$$\|\bar{x}(t) - \bar{x}_0(t)\| \leq r_{min} \tag{2.23}$$

where r_{min} is the smallest value of r satisfying (2. 19).

When the sufficient conditions for validity are satisfied, an algorithm (2. 21) for computing the exact solution to any desired accuracy has been derived. In particular, using (2. 22)

$$\|\bar{x} - (\bar{x}_0 + \bar{\eta}_m)\| \leq \frac{\alpha^m \|\bar{e}\|}{1-\alpha} \tag{2.24}$$

and

$$\bar{x}(t) = \bar{x}_0(t) + \lim_{m \to \infty} \bar{\eta}_m \tag{2.25}$$

where $\bar{\eta}_m$ is given by (2. 21).

Also, since $\bar{\eta}_1 = N\bar{0} = \bar{e}$, the readily available <u>improved first approximation</u>, i. e., $\bar{x} \approx \bar{x}_0 + \bar{e}$ satisfies

$$\|\bar{x} - (\bar{x}_0 + \bar{e})\| \leq \frac{\alpha \|\bar{e}\|}{1-\alpha} \leq \alpha r_{min} \tag{2.26}$$

For the special case of $f(x, \dot{x})$ linear in $\dot{x}$, that is

$$f(x, \dot{x}) = C\dot{x} + k(x) \tag{2.27}$$

it is possible to derive sharper criteria. In (2.13) let $a(t) = C$ to obtain

$$\overline{\eta} = \overline{e} + \int_0^I \overline{h}(t, s)\left[k(x_0) + b(s)\eta - k(x_0+\eta)\right]ds \tag{2.28}$$

or, in component-operator form

$$\begin{aligned} \eta &= e + H_1\eta = N_1\eta \\ \dot{\eta} &= \dot{e} + H_2\eta = N_2\eta \end{aligned} \tag{2.29}$$

Since the right hand sides do not depend upon $\dot{\eta}$, it is possible to apply Theorem 1.3 to the first of (2.29). This leads to

Validity Criteria I-A: Given an equation of the form (2.1) with $f(x, \dot{x})$ as in (2.27). Let $x_0(t)$, N_1, N_2, and $e(t)$ be as previously defined.

If a subset $B_r = \{w \mid \|w\| \le r\}$ of C can be found such that

1) there exists an $\alpha = \alpha(r)$ satisfying $0 < \alpha < 1$ and

$$\|N_1 x - N_1 y\| \le \alpha \|x - y\| \tag{2.30}$$

for all $x, y \in B_r$.

2)

$$r \ge \frac{\|e\|}{1-\alpha} \tag{2.31}$$

then there exists an exact solution of (2.1) of the form

$$x(t) = x_0(t) + \eta(t) \tag{2.32}$$

where $\eta \in B_r$ is the limit of the sequence

$$\eta_{i+1} = N_1 \eta_i \qquad \eta_0 = 0 \tag{2.33}$$

and

$$\|\eta-\eta_i\| \le \frac{\alpha^i\|e\|}{1-\alpha} \tag{2.34}$$

In particular

$$\|x-x_0\| \le r_{min} \tag{2.35}$$

where r_{min} is the smallest value of r satisfying (2.31).

Also, if the validity criteria are satisfied, and there exists a $\delta=\delta(r)$ such that

$$\|N_2x-N_2y\| \le \delta\|x-y\| \tag{2.36}$$

for all $x, y \in B_r$, then

$$\|\dot{\eta}-\dot{\eta}_i\| \le \frac{\delta\alpha^{i-1}\|e\|}{1-\alpha} \qquad (i\ge 1) \tag{2.37}$$

and if there exists an $\epsilon=\epsilon(r)$ such that

$$\|H_2x\| \le \epsilon\|x\| \tag{2.38}$$

for all $x\in B_r$, then

$$\|\dot{\eta}\| \le \|\dot{e}\| + \frac{\epsilon\|e\|}{1-\alpha} \tag{2.39}$$

Application of Theorem (1.3) is as follows: Since N_1 maps B_r into C condition 1 is satisfied. Equations (2.30) and (2.31) are the same as conditions 2 and 3 respectively, and results up to (2.33) are direct consequences of the conclusions of the theorem. Equation (2.34) follows from (1.74). Taking the norm of the second of (2.29) and using (2.37) gives

$$\|\dot{\eta}\| \le \|\dot{e}\| + \|H_2\dot{\eta}\| \le \|\dot{e}\| + \epsilon\,\|\eta\| \tag{2.40}$$

or, using (2.34) with i=0,

$$\|\mathring{\eta}\| \leq \|\delta\| + \frac{\epsilon\|e\|}{1-\alpha} \tag{2.41}$$

which is the same as (2. 39).

Then

$$\mathring{\eta} - \mathring{\eta}_i = N_2\eta - N_2\eta_{i-1} \qquad i \geq 1 \tag{2.42}$$

and using (2. 34) and (2. 36) yields

$$\|\mathring{\eta} - \mathring{\eta}_i\| \leq \delta\|\eta - \eta_{i-1}\| \leq \frac{\delta\alpha^{i-1}\|e\|}{1-\alpha} \qquad i \geq 1 \tag{2.43}$$

When Validity Criteria I or I-A are applicable, they yield, in addition to a bound on the approximation error,

1) an algorithm, 2.21 or 2.32, for computing the solution to any desired accuracy.

2) uniqueness of the approximated solution in the given symmetry class within the ball B_{max} of radius r_{max} where r_{max} is the largest value of r satisfying the validity criteria.

Slightly sharper validity criteria which yield only existence of an exact solution near the approximation and a bound on the approximation error are obtained using Theorem 1.1 and its corollary. Application to the special case (2. 27) yields the following:

<u>Validity Criteria II-A:</u> Given an equation of the type (2. 1) where $f(x, \dot{x})$ has the form (2. 27), let x_0, N_1, N_2, H_1, H_2 and e(t) be as previously defined.

If a subset $B_r = \{w \mid \|w\| \leq r\}$ of C can be found such that

1) there exists a real number $\gamma \geq 0$ and a $\beta = \beta(r)$ satisfying $0 < \beta < 1$ and

$$\|N_1 x\| < \beta \|x\| + \gamma \tag{2.44}$$

for all $x \in B_r$.

2) $\gamma + \beta r \leq r$ (2.45)

3) there exists real numbers $\delta \geq 0$ and an $\epsilon = \epsilon(r)$ such that

$$\|N_2 x\| \leq \delta + \epsilon \|x\| \tag{2.46}$$

for all $x \in B_r$.

then there exists an exact solution of (2.1) of the form

$$x(t) = x_0(t) + \eta(t) \tag{2.47}$$

where

$$\|\eta(t)\| = \|x(t) - x_0(t)\| \leq \frac{\gamma}{1-\beta} \tag{2.48}$$

and

$$\|\dot{\eta}(t)\| = \|\dot{x}(t) - \dot{x}_0(t)\| \leq \delta + \frac{\epsilon \gamma}{1-\beta} \tag{2.49}$$

Application of Theorem 1.1 is as follows: The ball B_r is a closed convex subset of C. The continuity of $h_1(t, s)$ assures that N_1 is a continuous mapping, while conditions 1 and 2 assure that B_r is mapped into itself. Hence the images of all $x \in B_r$ under the mapping N_1 are uniformly bounded in norm by r. In addition, condition 3 assures that the images of all $x \in B_r$ have uniformly bounded first derivatives. Using the mean value theorem

$$\|x(t_2) - x(t_1)\| \leq \dot{x}(\zeta) \|t_2 - t_1\| \tag{2.50}$$

where $t_1 \le \zeta \le t_2$, and $\dot{x}(\zeta)$ is uniformly bounded. Hence all functions in the image of B_r are uniformly bounded and equicontinuous, so by Arzela's Theorem (see page 54 of Kolmogoroff and Fomin[10]) the image of B_r is compact. All of the hypotheses of Theorem (1. 1) are fulfilled and hence there is a least one $\eta(t) \in B_r$ satisfying (2. 28). Application of the corollary yields (2. 48). Then (2. 46) and (2. 48) yield (2. 49).

Successful application of the validity criteria depends upon the choice of a(s) and b(s) in (2. 13) or b(s) in (2. 28). These quantities are called the basis for the integral equation. Except for certain exceptional values which yield an unbounded kernel, the choice of basis is unrestricted. Ideally, it would be chosen so as to minimize the bound on the approximation error. However, such a choice is rather involved computationally so several alternative selections are discussed in Chapter III.

Furthermore, in application to specific problems, success depends upon determining a suitable Lipschitz constant $\alpha(r)$, alternatively called a contraction constant, for Criteria I and I-A, and upon determining suitable $\beta(r)$ and γ for Criteria II-A. Once these are obtained, the information concerning validity is contained in (2. 19), (2. 31) and (2. 45) for Criteria I, I-A and II-A, respectively. Henceforth, each of these inequalities is referred to as a validity inequality.

2.2 Comparison of Criteria I-A and II-A.

Consider the equation of a simple undamped oscillator with an odd polynomial restoring force

$$\ddot{x} + K\left(x + \epsilon_1 x^3 + \epsilon_2 x^5\right) = P \cos \omega t \qquad K, \epsilon_1, \epsilon_2 > 0 \tag{2.51}$$

The one term harmonic balance approximate solution is

$$x_0(t) = A \cos \omega t \tag{2.52}$$

where $A>0$ satisfies

$$A(K-\omega^2) + \frac{3}{4}\epsilon_1 K A^3 + \frac{5}{8} K \epsilon_2 A^5 = P \tag{2.53}$$

Solutions which are out of phase with the forcing function correspond to $P<0$.

Since (2.51) does not contain a derivative term, the approximation error satisfies an integral equation of the form (2.28) with

$$k(x) = K\left(x + \epsilon_1 x^3 + \epsilon_2 x^5\right) \tag{2.54}$$

Now $x_0 \in Q_T$ and (2.51) allows solutions in Q_T, so $h_1(t, s)$ in (2.28) is given by (1.53). For convenience, take $b(s)=K_*$, an as yet unspecified constant, in which case

$$h_1(t, s) = \begin{cases} \dfrac{-\sin\sqrt{K_*}\left(\frac{\pi}{2\omega} - t\right)\cos\sqrt{K_*}\, s}{\sqrt{K_*}\cos\frac{\sqrt{K_*}\,\pi}{2\omega}} & 0 \le s \le t \le \frac{\pi}{2\omega} \\[2ex] \dfrac{-\sin\sqrt{K_*}\left(\frac{\pi}{2\omega} - s\right)\cos\sqrt{K_*}\, t}{\sqrt{K_*}\cos\frac{\sqrt{K_*}\,\pi}{2\omega}} & 0 \le t \le s \le \frac{\pi}{2\omega} \end{cases} \tag{2.55}$$

A straightforward integration gives

$$e(t) = \frac{K\left\{\epsilon_1 A^3 + \frac{5\epsilon_2 A^5}{4}\right\}}{4\left(K_* - 9\omega^2\right)} \cos 3\omega t + \frac{\epsilon_2 K A^5}{16\left(K_* - 25\omega^2\right)} \cos 5\omega t \tag{2.56}$$

and from (2.28) and (2.29)

$$N_1 y = e(t) + \int_0^{\pi/2\omega} h_1(t,s)\left[k(x_0) + K_* y - k(x_0+y)\right] ds \tag{2.57}$$

For Criteria I-A an $\alpha = \alpha(r)$ is needed satisfying

$$\|N_1 x - N_1 y\| \leq \alpha \|x-y\| \tag{2.58}$$

for all $x, y \in B_r$. Using (2.57) and (2.54), and after some manipulation

$$\begin{aligned} N_1 x - N_1 y = \int_0^{\pi/2\omega} h_1(t,s)\Big[&\left(K_* - K - 3\epsilon_1 K x_0^2 - 5\epsilon_2 K x_0^4\right) - (x+y)\left(3\epsilon_1 K x_0 + 10\epsilon_2 K x_0^3\right) \\ &- (x^2+xy+y^2)\left(K\epsilon_1 + 10K\epsilon_2 x_0^2\right) - (x^3+xy^2+x^2y+y^3)5K\epsilon_2 x_0 \\ &- K\epsilon_2(x^4+x^3y+x^2y^2+xy^3+y^4)\Big](x-y)ds \end{aligned} \tag{2.59}$$

Now choose K_* so that the upper bound on the norm of

$$\left(K_* - K - 3\epsilon_1 K x_0^2 - 5\epsilon_2 K x_0^4\right) \tag{2.60}$$

is as small as possible, that is

$$K_* = K + \frac{3}{2}\epsilon_1 K A^2 + \frac{5}{2}\epsilon_2 K A^4 \tag{2.61}$$

which gives

$$\|K_* - K - 3\epsilon_1 K x_0^2 - 5\epsilon_2 K x_0^4\| \leq \frac{3}{2}\epsilon_1 K A^2 + \frac{5}{2}\epsilon_2 K A^4 \tag{2.62}$$

Taking the norm of (2.59) and using (2.52), (2.55), (2.62) and $x, y \in B_r$ yields

$$\|N_1x-N_1y\| \le \frac{\|x-y\|\pi K}{2\omega\sqrt{K_*}\left|\cos\frac{\sqrt{K_*}\pi}{2\omega}\right|}\left[\frac{3}{2}\epsilon_1A^2+\frac{5}{2}\epsilon_2A^4+2r\left(3\epsilon_1A+10\epsilon_2A^3\right)\right.$$

$$\left.+3r^2\left(\epsilon_1+10\epsilon_2A^2\right)+20\epsilon_2Ar^3+5\epsilon_2r^4\right] \quad (2.63)$$

$$\|N_1x-N_1y\| = \alpha(r)\|x-y\| \quad (2.64)$$

By Validity Criteria I-A, x_0 is valid provided there are values of r satisfying

$$r \ge \frac{\|e\|}{1-\alpha(r)} \quad (2.65)$$

which can be written

$$r \ge a_0+a_1r+2a_2r^2+3a_3r^3+4a_4r^4+5a_5r^5 = p(r) \quad (2.66)$$

where

$$\left.\begin{array}{ll} a_0 = \|e\| & a_3 = K\left(\epsilon_1+10\epsilon_2A^2\right)/B \\ a_1 = KA^2\left(3\epsilon_1+5\epsilon_2A^2\right)/2B & a_4 = 5K\epsilon_2A/B \\ a_2 = KA\left(3\epsilon_1+10\epsilon_2A^2\right)/B & a_5 = K\epsilon_2/B \\ \multicolumn{2}{c}{B = 2\omega\sqrt{K_*}\left|\cos\frac{\sqrt{K_*}\pi}{2\omega}\right|/\pi} \end{array}\right\} \quad (2.67)$$

Now for Criteria II-A, γ and $\beta=\beta(r)$ are needed satisfying

$$\|N_1y\| \le \gamma+\beta y \quad (2.68)$$

for all $y \in B_r$. Using K_* given by (2.61) and the same type of bounding procedure gives

$$\beta(r) = a_1+a_2r+a_3r^2+a_4r^3+a_5r^4 \quad (2.69)$$

$$\gamma = \|e\| \quad (2.70)$$

By Validity Criteria II-A, x_0 is known to be valid provided there are values of r satisfying

$$r \geq a_0+a_1r+a_2r^2+a_3r^3+a_4r^4+a_5r^5 = q(r) \tag{2.71}$$

where the a_i, i=0, 5, are given by (2.67). Figures 2.1a, b show graphs of the right and left hand sides of (2.66) and (2.71) for different values of the parameters. Obviously p(r) always lies above q(r) so $r_1 < r_2$ (see Figure 2.1a) and Criteria II-A yields a smaller bound on the approximation error. In fact, the situation depicted in Figure 2.1b may develop, wherein II-A yields a bound and I-A is not applicable. On the other hand, when I-A applies, it yields an algorithm for computing the exact solution and uniqueness of the approximated solution in the given symmetry class within the ball of radius r_3 (see Figure 2.1a).

2.3 Validity Boundaries.

A question of particular interest is: Given system parameters K, ϵ_1, ϵ_2 for what excitations P, ω do the validity criteria guarantee a valid approximate solution.

Consider the situation shown in Figure 2.1c. There is only one value of r which satisfies $r \geq p(r)$, and for a slight change in the excitation there may be no solutions. Values of the excitation for which such a situation occurs will clearly lie on boundaries between valid and "questionable" regions. Where the validity criteria are not applicable, there may still be exact solutions near the approximations, but their existence cannot be guaranteed by the methods used here.

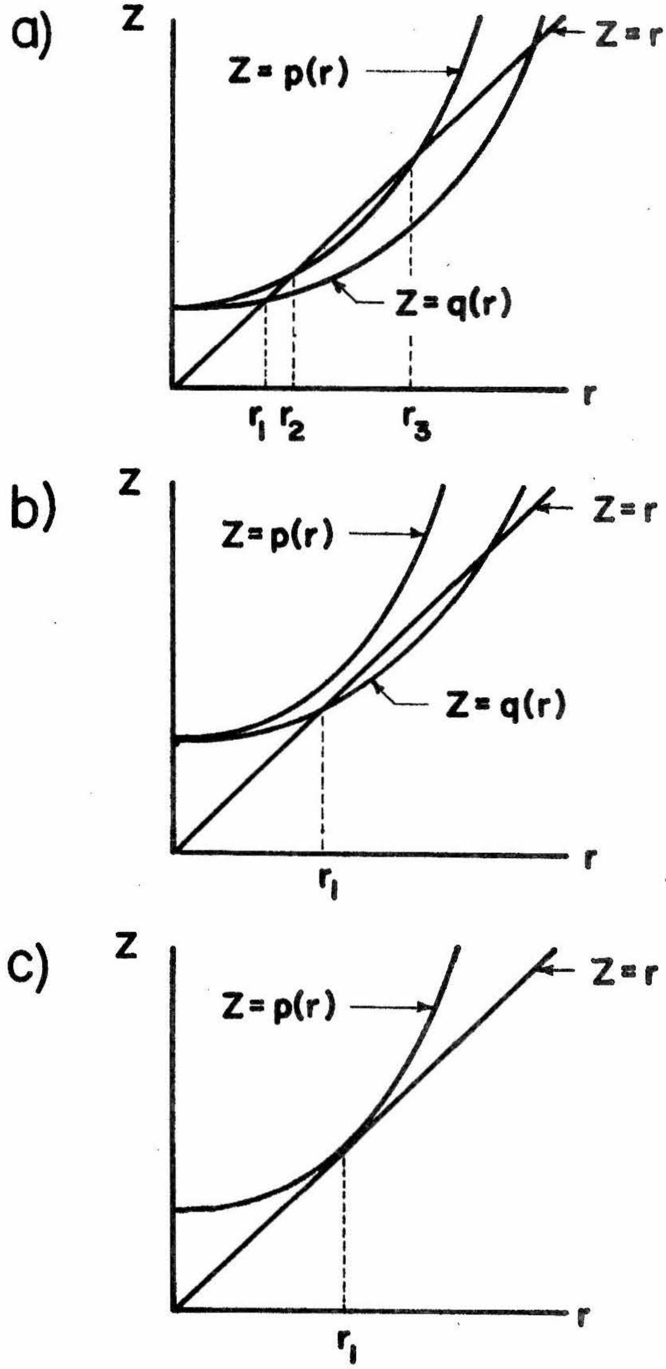

Fig. 2.1 PLOTS OF VALIDITY INEQUALITIES

It is convenient to calculate the validity boundaries as functions of A and ω rather than P and ω. Recall that A, P and ω are related by (2.53). The development is presented for Criteria I-A. Validity boundaries for II-A are determined in an analogous manner.

On a validity boundary

$$r = p(r) \tag{2.72}$$

$$1 = p'(r) \tag{2.73}$$

Given A, values of ω satisfying the above two equations are found as follows: Since $p'(r)$ is monotone increasing when $r>0$, for $p'(0)<1$ there is always a unique positive root of (2.73). Solving for given ω yields $r=r(\omega)$. The transcendental equation

$$r(\omega) = p(r(\omega)) \tag{2.74}$$

obtained from (2.72) is then solved yielding values of ω on validity boundaries. Figure (2.2) shows validity boundaries for Criteria I-A and II-A applied to (2.51) with K=1.0, $\epsilon_1=0.1$ and $\epsilon_2=0.02$. Also shown are response curves for several values of P. Indications are that the approximate solution is very good on the lower response curve for $\omega>1$ and also for certain ranges of frequency when $\omega<1$ and P is small. Of course, the validity criteria are only sufficient conditions so the approximate solution may still be quite good even though its validity cannot be demonstrated by these methods.

In addition to validity, there is also the question of whether the approximate solution is an approximation to a stable solution. When the approximation cannot be shown valid, the stability question cannot

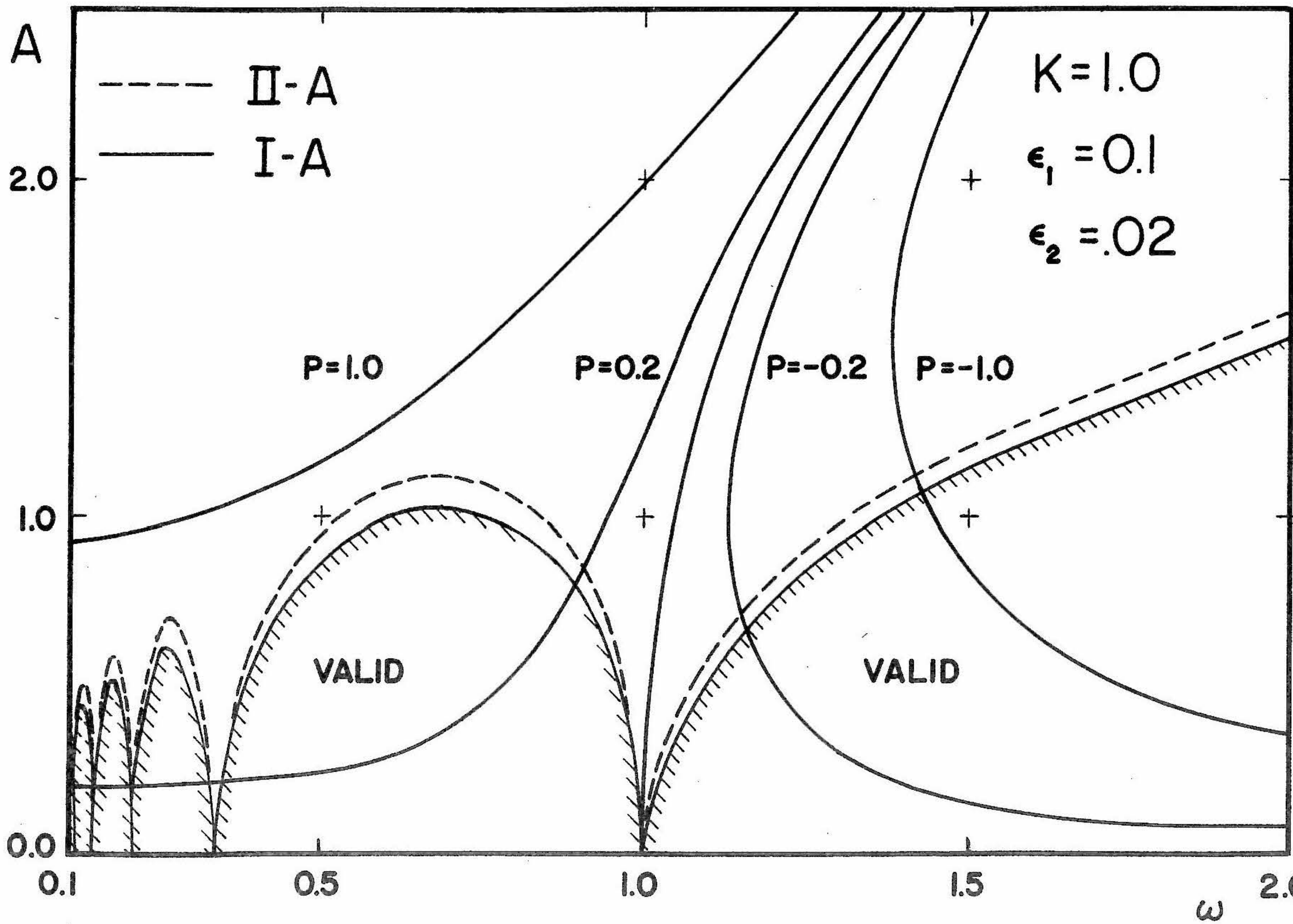

Fig. 2.2 VALIDITY BOUNDARIES - POLYNOMIAL RESTORING FORCE

be answered rigorously, and even when it is valid, a stability analysis based on the approximate solution is not exact. A precise analysis must account for the fact that the exact solution is known only to within the bound obtained for the approximation error.

2.4 Sufficient Stability Criteria.

Consider again, an equation of the form (1.1). The stability question for a periodic solution $\bar{x}(t)$ is answered by the equation of first variation (see Struble[9] page 149)

$$\ddot{y} + \frac{\partial f}{\partial \dot{x}}(x, \dot{x})\dot{y} + \frac{\partial f}{\partial x}(x, \dot{x})y = 0 \tag{2.75}$$

If this equation has any unbounded solutions then $\bar{x}(t)$ is unstable, if all solutions are bounded, then $\bar{x}(t)$ is stable, and if all solutions tend to zero as $t \to \infty$, then $\bar{x}(t)$ is asymptotically stable. Equation (2.75) is a Hill's equation of the form

$$\ddot{y} + q_1(t)\dot{y} + q_2(t)y = 0 \tag{2.76}$$

where $q_1(t)$ and $q_2(t)$ are periodic with common period P. Floquet theory says that solutions of (2.76) are of the form

$$y(t) = \varphi_1(t)e^{i\lambda_1 t} + \varphi_2(t)e^{i\lambda_2 t} \tag{2.77}$$

where φ_1 and φ_2 are periodic. The boundedness of solutions is hence determined by λ_1 and λ_2 called the characteristic exponents. Hahn[7], page 305, shows that (2.76) is transformed to

$$\ddot{z} + p(t)z = 0 \tag{2.78}$$

$$p(t) = q_2(t) - \frac{1}{4}q_1^2(t) - \frac{1}{2}\dot{q}_1(t) \tag{2.79}$$

by the transformation

$$y = ze^{-\frac{1}{2}\int_0^t q_1(s)ds} \tag{2.80}$$

Since (2.78) is also a Hill's equation it has solutions of the form (2.77). Hahn also shows that the <u>characteristic multipliers</u> μ_1 and μ_2 given by

$$\mu_1 = e^{i\nu_1 P} \qquad \mu_2 = e^{i\nu_2 P} \tag{2.81}$$

where ν_1 and ν_2 are the characteristic exponents of (2.78) satisfy

$$\mu^2 - a\mu + 1 = 0 \tag{2.82}$$

where

$$a = \left(u_1(P) + v_2(P)\right) \tag{2.83}$$

and

$$U(t) = \begin{bmatrix} u_1(t) & v_1(t) \\ u_2(t) & v_2(t) \end{bmatrix} \tag{2.84}$$

is the principal matrix solution of (2.78). Solving (2.82) gives

$$\mu_{1,2} = \frac{a}{2} \pm \left[\frac{a^2}{4} - 1\right]^{1/2} \tag{2.85}$$

from which it is easy to see that $\mu_1\mu_2 = 1$ and hence from (2.81)

$$e^{i\nu_1 P} = e^{-i\nu_2 P} \tag{2.86}$$

Thus $\nu_1 = -\nu_2 \bmod \frac{2\pi}{P}$. Letting $\nu = \nu_1 = -\nu_2 \bmod \frac{2\pi}{P}$ and using (2.81) and (2.86) gives

$$\mathrm{Im}\{\nu\} = \frac{1}{P}\ln\frac{1}{|\mu_1|} = \frac{1}{P}\ln|\mu_2| \tag{2.87}$$

$$\mathrm{Re}\{\nu\} = \frac{1}{P}(\mathrm{Arg}\,\mu_1 + 2n\pi) = \frac{1}{P}(-\mathrm{Arg}\,\mu_2 + 2m\pi) \tag{2.88}$$

for some integers m and n. Now if $a^2 \geq 4$, then μ_1 and μ_2 are both real and positive, so $\operatorname{Arg}\mu_1 = \operatorname{Arg}\mu_2 = 0$, while if $a^2 < 4$, then $|\mu_1| = |\mu_2| = 1$ so $\operatorname{Im}\{\nu\} = 0$. This leads to

$$\nu = \begin{cases} \frac{i}{P}\ln\frac{1}{|\mu_1|} & \text{for } a^2 \geq 4 \\ \frac{1}{P}(\operatorname{Arg}\mu_1 + 2n\pi) & \text{for } a^2 < 4 \end{cases} \tag{2.89}$$

now let

$$q_1(s) = \hat{q}_1 + \tilde{q}_1(s) \tag{2.90}$$

where

$$\hat{q}_1 = \frac{1}{P}\int_0^P q_1(s)ds \tag{2.91}$$

Then the characteristic exponents of (2.76) are

$$\lambda_{1,2} = \pm\nu + i\frac{\hat{q}_1 P}{2} \tag{2.92}$$

If $q_1(t) = \frac{\partial f}{\partial \dot{x}}(x, \dot{x})$ and $q_2(t) = \frac{\partial f}{\partial x}(x, \dot{x})$, then the stability of $\bar{x}(t)$ is related to $\lambda_{1,2}$ as given in Table 2.1. All except the last entry are consequences of (2.77) and the remarks following (2.75). The last case in the table involves the question of coexistence of P or 2P periodic solutions of Hill's equation (see Magnus and Winkler(18) page 5).

The above analysis provides an answer to the stability question for $\bar{x}(t)$ in terms of solutions of the equation of first variation over a single period P. Solving a Hill's equation to determine $u_1(P)$ and $v_2(P)$ is not a simple matter, even when q_1 and q_2 are known exactly. When using the equation of first variation, $\bar{x}$ and hence the periodic coefficients are known only approximately, thus adding further

Characteristic Exponents λ_1 and λ_2

$\operatorname{Im}\lambda_1$ and $\operatorname{Im}\lambda_2$	Relation of λ_1 and λ_2	Stability of $\bar{x}(t)$
$\operatorname{Im}\lambda_1<0$ and $\operatorname{Im}\lambda_2<0$		unstable
$\operatorname{Im}\lambda_1>0$ and $\operatorname{Im}\lambda_2>0$		asymptotically stable
$\operatorname{Im}\lambda_1>0$ and $\operatorname{Im}\lambda_2=0$ or $\operatorname{Im}\lambda_1=0$ and $\operatorname{Im}\lambda_2>0$		stable
$\operatorname{Im}\lambda_1=\operatorname{Im}\lambda_2=0$	$\lambda_1\neq\lambda_2 \bmod \frac{2\pi}{P}$	stable
$\operatorname{Im}\lambda_1=\operatorname{Im}\lambda_2=0$	$\lambda_1=\lambda_2 \bmod \frac{2\pi}{P}$ (occurs only when $\hat{q}_1=0$)	unstable except when $v_1(P)=u_2(P)=0$

Table 2.1. Stability from the Characteristic Exponents.

complications. A widely accepted simplification consists of analyzing the stability of the approximate solution (see, for instance, Hayashi's book[16]). Such an analysis is never precise and may sometimes lead to unjustified results as shown by an example in Chapter III.

When the validity criteria are applicable, there is a possibility of obtaining sufficient stability conditions. The equation of first variation may be written as

$$\ddot{y} + \frac{\partial f}{\partial \dot{x}}(x_0+\eta, \dot{x}_0+\dot{\eta})\dot{y} + \frac{\partial f}{\partial x}(x_0+\eta, \dot{x}_0+\dot{\eta})y = 0 \qquad (2.93)$$

which is a Hill's equation of the form

$$\ddot{y} + \left(q_1(t)+\zeta_1(t)\right)\dot{y} + \left(q_2(t)+\zeta_2(t)\right)y = 0 \qquad (2.94)$$

where $q_1(t)$ and $q_2(t)$ are known periodic functions and $\zeta_1(t)$ and $\zeta_2(t)$ are also periodic with the same period and satisfy

$$\|\zeta_1(t)\| \le \delta_1 \qquad \|\zeta_2(t)\| \le \delta_2 \qquad (2.95)$$

In some cases, it is possible to determine a range of values for $u_1(P)$ and $v_2(P)$. That is, $u_1(P)$ and $v_2(P)$ lie within certain intervals. The notation

$$[u_1(P)] = [\mathrm{MAX}(u_1(P)), \mathrm{MIN}(u_1(P))] \qquad (2.96)$$

$$[v_2(P)] = [\mathrm{MAX}(v_2(P)), \mathrm{MIN}(v_2(P))] \qquad (2.97)$$

is convenient. Using these interval values, together with (2.83), (2.85) and (2.92) gives $[\mathrm{Im}\,\lambda_1]$ and $[\mathrm{Im}\,\lambda_2]$, from which sufficient conditions for stability and instability are deduced. For example, if both intervals contain only positive values, then $x_0(t)$ approximates an asymptotically stable solution, while if either interval contains only negative values, then $x_0(t)$ approximates an unstable solution. An example in which sufficient stability conditions are computed is given in Chapter III.

CHAPTER III

EXAMPLE: A BILINEAR SYSTEM

Consider the equation

$$\ddot{x} + F(x) = P \cos \omega t \tag{3.1}$$

where

$$F(x) = \begin{cases} K-1+x & \text{for } x>1 \\ Kx & \text{for } |x| \leq 1 \\ 1-K+x & \text{for } x<-1 \end{cases} \tag{3.2}$$

and $K>0$. This restoring force is shown graphically in Figure 3.1.

The change of variables

$$x = \frac{u}{a} \quad ; \quad P = \frac{R}{K_2 a} \quad ; \quad t = \sqrt{K_2}\,\tau \quad ; \quad K = \frac{K_1}{K_2} \quad ; \quad \omega = \nu\sqrt{K_2} \tag{3.3}$$

takes the above to the more general form

$$\ddot{u} + F(u) = R \cos \nu\tau \tag{3.4}$$

where

$$F(u) = \begin{cases} K_1 a + K_2(u-a) & \text{for } u>a \\ K_1 u & \text{for } |u| \leq a \\ -K_1 a + K_2(u+a) & \text{for } u<-a \end{cases} \tag{3.5}$$

Since (3.1) is piecewise linear some exact solutions are obtained by "piecing together" solutions for the time segments on which the equation is linear. Generation of these simple solutions, i.e., solutions which cross $|x|=1$ only once in a quarter period, is discussed in Appendix A. Subsequently, these exact solutions are used to study the sharpness of bounds on the approximation error.

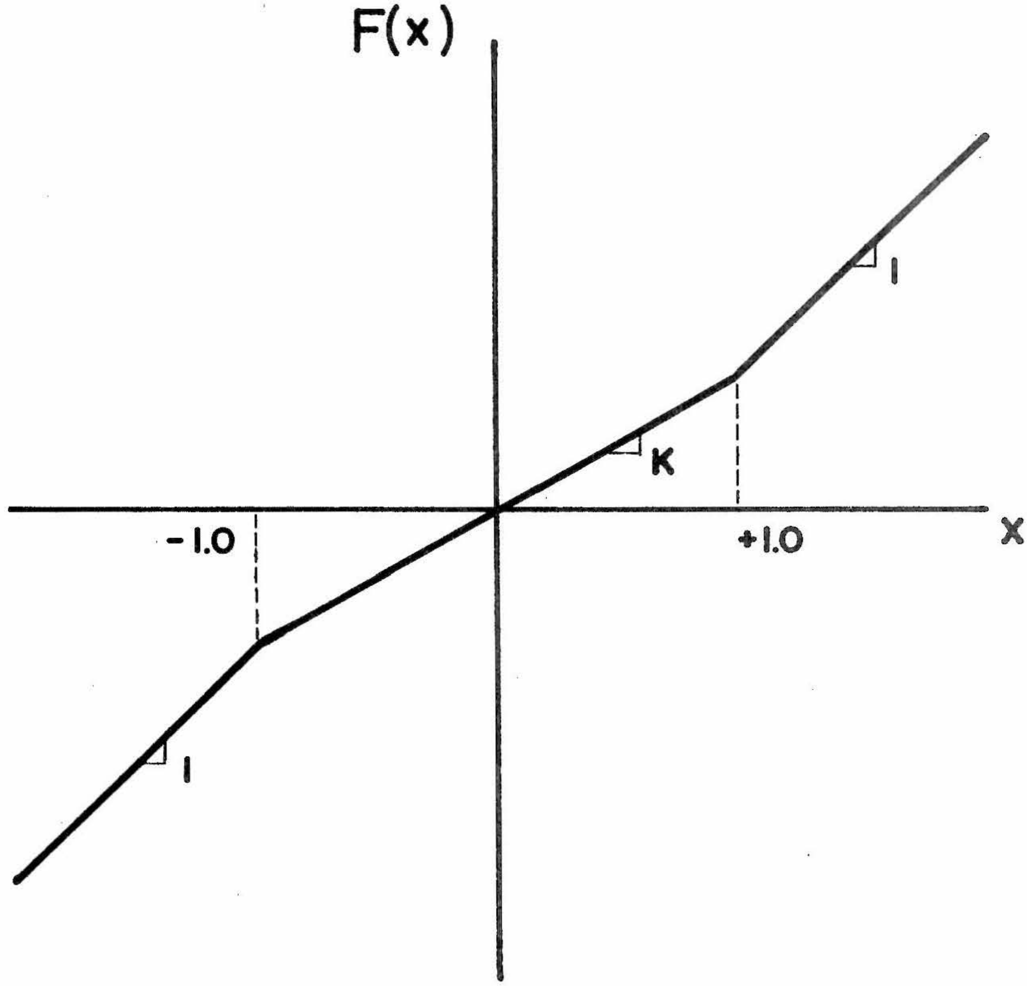

Fig. 3.1 BILINEAR RESTORING FORCE

3.1. Validity Criteria.

The first order harmonic balance approximate solution of (3.1) is

$$x_0(t) = A \cos \omega t \tag{3.6}$$

where A>1 satisfies

$$K_{EQV} - \omega^2 = \frac{P}{A} \tag{3.7}$$

$$K_{EQV} = K + \frac{2(1-K)}{\pi}\left\{\cos^{-1}\frac{1}{A} - \frac{1}{A}\left[1-\frac{1}{A^2}\right]^{1/2}\right\} \tag{3.8}$$

Solutions which are out of phase with the forcing function occur for $P<0$.

Since $x_0(t)\in Q_T$ and (3.1) allows solutions in Q_T for $T=\frac{2\pi}{\omega}$, Validity Criteria I-A is used to establish the existence of an exact Q_T solution near $x_0(t)$. In accordance with the development in Chapter II, the approximation error $\eta(t)$ satisfies

$$\eta(t) = e(t) + \int_0^{\pi/2\omega} h_1(t,s)\left[F(x_0)+b(s)\eta - F(x_0+\eta)\right]ds \tag{3.9}$$

where

$$e(t) = \int_0^{\pi/2\omega} h_1(t,s)\left[K_{EQV}x_0 - F(x_0)\right]ds \tag{3.10}$$

Take $b(s)=K_*$, an as yet unspecified constant. Then $h_1(t,s)$ is given by (2.55) and

$$\left.\begin{aligned}
e(t) &= \frac{1-K}{K_*} + \frac{(K_{EQV}-1)A\cos\omega t}{K_*-\omega^2} + \frac{\omega(1-K)\cos\sqrt{K_*}\,t}{\sqrt{K_*}(K_*-\omega^2)\cos\frac{\sqrt{K_*}\,\pi}{2\omega}} \\
&\quad\times\left\{\frac{\omega}{\sqrt{K_*}}\cos\left[\frac{\sqrt{K_*}}{\omega}\sin^{-1}\frac{1}{A}\right]-\left[A^2-1\right]^{\frac{1}{2}}\sin\left[\frac{\sqrt{K_*}}{\omega}\sin^{-1}\frac{1}{A}\right]\right\} \quad 0\le t\le\frac{1}{\omega}\cos^{-1}\frac{1}{A} \\
e(t) &= \frac{(K_{EQV}-K)}{K_*-\omega^2}A\cos\omega t + \frac{\omega(1-K)\sin\sqrt{K_*}\left(\frac{\pi}{2\omega}-t\right)}{\sqrt{K_*}(K_*-\omega^2)\cos\frac{\sqrt{K_*}\,\pi}{2\omega}} \\
&\quad\times\left\{\frac{\omega}{\sqrt{K_*}}\sin\left[\frac{\sqrt{K_*}}{\omega}\sin^{-1}\frac{1}{A}\right]-\left[A^2-1\right]^{\frac{1}{2}}\cos\left[\frac{\sqrt{K_*}}{\omega}\sin^{-1}\frac{1}{A}\right]\right\} \quad \frac{1}{\omega}\cos^{-1}\frac{1}{A}\le t\le\frac{\pi}{2\omega}
\end{aligned}\right\} \tag{3.11}$$

Using the notation of Chapter II,

$$N_1 y = e(t) + \int_0^{\pi/2\omega} h_1(t,s)\left[F(x_0)+K_* y-F(x_0+y)\right]ds \tag{3.12}$$

In Appendix B a contraction constant for this operator is derived. That is, $\alpha=\alpha(r)$ given by (B-11) is such that

$$\|N_1 y-N_1 x\| \le \alpha(r)\|y-x\| \tag{3.13}$$

for all $x, y \in B_r$. Hence, the validity inequality is

$$\frac{\|e\|}{1-\alpha(r)} \le r \tag{3.14}$$

For the present, take $K_*=K_{AVG}$, the time average value of $F'(x_0)$

$$K_{AVG} = K + \frac{2}{\pi}(1-K)\cos^{-1}\frac{1}{A} \tag{3.15}$$

Later, various other choices of the basis are discussed.

As in Chapter II, validity boundaries in the A, ω plane consist of points satisfying

$$\|e\| = r - r\alpha(r) \tag{3.16}$$

$$1 = \alpha(r) + r\alpha'(r) \tag{3.17}$$

From (B-11)

$$\alpha'(r) = \frac{[|1-K_*| - |K-K_*|]}{\omega\sqrt{K_*}\left|\cos\frac{\sqrt{K_*}\pi}{2\omega}\right|}\begin{cases} 0 & \frac{1+\sigma r}{A} < 0 \text{ or } > 1 \\ -\frac{\sigma}{[A^2-(1+\sigma r)^2]^{1/2}} & 0 \le \frac{1+\sigma r}{A} \le 1 \end{cases} \tag{3.18}$$

Multiplying (3. 17) by r and using (3. 16) gives

$$r(1-\alpha) = r^2 \frac{\partial\alpha}{\partial r} = \|e\| \tag{3.19}$$

Using (3. 18), squaring and rearranging gives

$$r^4\{|1-K_*| - |K_*-K|\}^2 + \|e\|\omega^2 K_* \cos^2\frac{\sqrt{K_*}\pi}{2\omega}\left[\{1+\sigma r\}^2 - A^2\right] = 0 \tag{3.20}$$

For given A and ω, the unique positive root of the quartic equation (3. 20) is obtained yielding $r=r(\omega)$. Roots ω of the transcendental equation

$$\|e\| = r(\omega) - r(\omega)\alpha(r(\omega)) \tag{3.21}$$

obtained from (3. 16) then correspond to points on validity boundaries. Figure 3. 2 shows validity boundaries for the case K=0.8. In the regions marked I, II, and III the sufficient validity conditions are not satisfied. The heavier solid lines are response curves determined from the approximate solution. In some parts of the shaded regions the approximate solution may still be quite good in the sense that an exact solution exists in its vicinity. However, this to be expected since the validity conditions are merely sufficient and not necessary. Furthermore, in each region there is apparently some type of behavior which

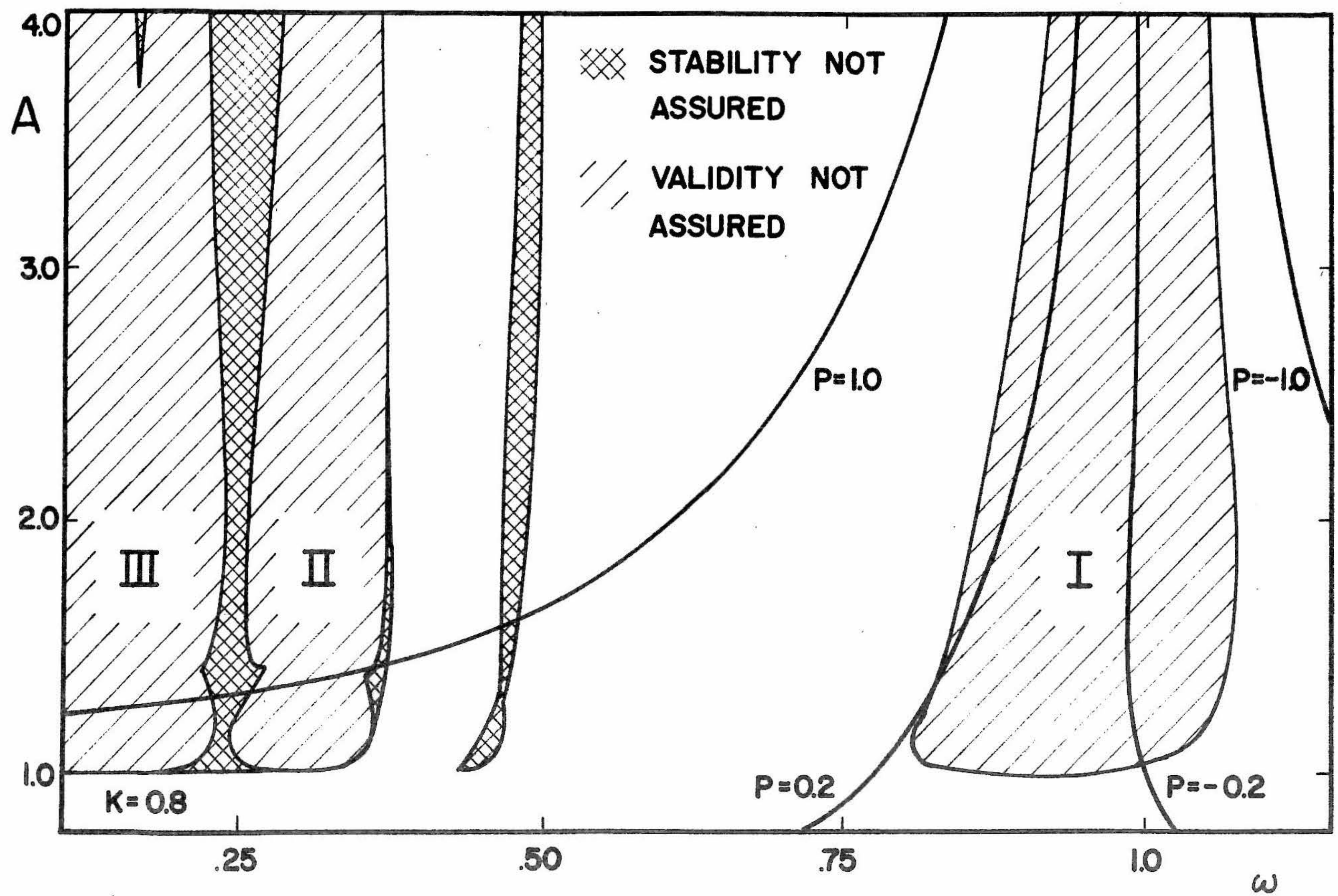

Fig. 3.2 VALIDITY AND STABILITY BOUNDARIES — BILINEAR SYSTEM

can lead to large errors in the approximate solution. In region I, some of the response curves have vertical tangents. If the location of a vertical tangent is not exact, then near the vertical tangent there may be no exact solution near the approximation. In regions II and III the system apparently has ultraharmonic resonances. The resonant response in region II is predicted by a two term harmonic balance approximation developed later in this chapter.

3.2. Bounds for the Derivative of $\eta(t)$.

In regions of validity, an upper bound on the derivative of the approximation error is obtained. The following is used

$$\eta = x_0 + \eta \tag{3.22}$$

where

$$\|\eta\| \leq r_{min} \tag{3.23}$$

and r_{min} is the smallest value of r satisfying (3.14). In the notation of Chapter II

$$H_2 y = \int_0^{\pi/2\omega} h_2(t,s)\left[F(A\cos\omega s) + K_* y - F(A\cos\omega s + y)\right]ds \tag{3.24}$$

$$h_2(t,s) = \begin{cases} \dfrac{\cos\sqrt{K_*}\left(\frac{\pi}{2\omega} - t\right)\cos\sqrt{K_*}\,s}{\cos\frac{\sqrt{K_*}\,\pi}{2\omega}} & 0 \leq s \leq t \leq \frac{\pi}{2\omega} \\ \dfrac{\sin\sqrt{K_*}\left(\frac{\pi}{2\omega} - s\right)\sin\sqrt{K_*}\,t}{\cos\frac{\sqrt{K_*}\,\pi}{2\omega}} & 0 \leq t \leq s \leq \frac{\pi}{2\omega} \end{cases} \tag{3.25}$$

Using developments in Appendix B,

$$\|H_2 y\| \leq \sqrt{K_*}\,\alpha(r_{min})\|y\| \tag{3.26}$$

for all $y \in B_{min}$. Hence from (2.38), (2.39) and (3.14)

$$\|\dot{\eta}\| \leq \|\dot{e}\| + \sqrt{K_*}\, r_{min}\,\alpha(r_{min}) \tag{3.27}$$

3.3. Stability Criteria.

Suppose that $x_0(t)$ is a valid approximate solution. If

$$\|\dot{\eta}\| \leq \dot{x}_0\left(\frac{1}{\omega}\cos^{-1}\frac{1+r_{min}}{A}\right) \tag{3.28}$$

$$= \omega\left[A^2 - (1+r_{min})^2\right]^{1/2} \tag{3.29}$$

or, from (3.27)

$$\|\dot{e}\| + \sqrt{K_*}\, r_{min}\,\alpha(r_{min}) \leq \omega\left[A^2 - (1+r_{min})^2\right]^{1/2} \tag{3.30}$$

then $\dot{x}(t)$ does not change sign when $\|x\|$ is within r_{min} of one. In this case, $x(t)$ is simple and the stability analysis of Appendix C applies. The following analysis presumes that (3.30) is satisfied.

Using (3.23) the following bounds on t_1, where $x(t_1)=1$, are obtained

$$t_- = \frac{1}{\omega}\cos^{-1}\frac{1+r_{min}}{A} \leq t_1 \leq \frac{1}{\omega}\cos^{-1}\frac{1-r_{min}}{A} = t_+ \tag{3.31}$$

In interval notation

$$[t_1] = [t_+, t_-] \tag{3.32}$$

For any given solution $x(t)=x_0(t)+\eta(t)$, $\left[u_1\left(\frac{\pi}{\omega}\right)\right]$ is computed as follows:

$$\left.\begin{aligned} MAX\left(u_1\left(\frac{\pi}{\omega}\right)\right) &= \sup_{t_- \le t_1 \le t_+} u_1\left(\frac{\pi}{\omega}\right) \\ MIN\left(u_1\left(\frac{\pi}{\omega}\right)\right) &= \inf_{t_- \le t_1 \le t_+} u_1\left(\frac{\pi}{\omega}\right) \end{aligned}\right\} \qquad (3.33)$$

where $u_1\left(\frac{\pi}{\omega}\right)$ is given by (C-6). Then according to the results given in Appendix C, x(t) is stable if

$$\left[u_1\left(\frac{\pi}{\omega}\right)\right] \subset [1,-1] \qquad (3.34)$$

and unstable if

$$\left[u_1\left(\frac{\pi}{\omega}\right)\right] \cap [1,-1] = \theta \quad , \quad \text{the null set} \qquad (3.35)$$

When neither (3.34) nor (3.35) is true, then no definitive statement is available concerning the stability of x(t).

3.4. Sufficient Stability and Instability Boundaries.

From (3.34) and (3.35), it is apparent that boundaries between stable and unstable regions occur when

$$u_1\left(\frac{\pi}{\omega}\right) = \pm 1 \qquad (3.36)$$

The stability boundaries in the A, ω plane based upon the approximate solution are calculated as follows: For given t_1 and K (3.36) with $u_1\left(\frac{\pi}{\omega}\right)$ given by C-6, is solved for ω. A is obtained from

$$A = \frac{1}{\cos \omega t_1} \qquad (3.37)$$

and P from (3.7). In valid regions where (3.30) holds, the approximate boundaries are used as initial guesses in the following scheme to determine sufficient stability and instability boundaries. For given

A and K, values of ω are determined satisfying

$$\begin{aligned} \sup_{t_- \le t_1 \le t_+} u_1\left(\frac{\pi}{\omega}\right) &= \pm 1 \quad \text{(for sufficient stability boundaries)} \\ \inf_{t_- \le t_1 \le t_+} u_1\left(\frac{\pi}{\omega}\right) &= \pm 1 \quad \text{(for sufficient instability boundaries)} \end{aligned} \tag{3.38}$$

where

$$u_1\left(\frac{\pi}{\omega}\right) = \cos\sqrt{K}\left(\frac{\pi}{\omega} - 2t_1\right)\cos 2t_1 - \frac{\left(\frac{1}{\sqrt{K}} + \sqrt{K}\right)}{2}\sin\sqrt{K}\left(\frac{\pi}{\omega} - 2t_1\right)\sin 2t_1$$

$$t_+ = \frac{1}{\omega}\cos^{-1}\frac{1 - r_{min}}{A}$$

$$t_- = \frac{1}{\omega}\cos^{-1}\frac{1 + r_{min}}{A}$$

$$r_{min} = \frac{\|e\|}{1 - \alpha(r_{min})}$$

P is then determined from (3.7).

The cross hatched areas in Figure 3.2 designate regions where the sufficient stability conditions are not satisfied. That is, either (3.30) does not hold or $\left[u_1\left(\frac{\pi}{\omega}\right)\right] \not\subset [1, -1]$. Near $\omega = \frac{1}{2}$ there are indeed some unstable solutions as may be seen in the blowup of this unstable region in Figure 3.3, where the stability boundaries based upon the approximate solution are shown together with the sufficient stability and instability boundaries. In this region, equation (3.14) is satisfied for all $r \ge r_{min}$. That is, the situation is as in Figure 3.4. Hence, the unstable solution is unique within the class $Q_{2\pi/\omega}$, so any stable solution cannot be in this class. In other words, the unstable solution

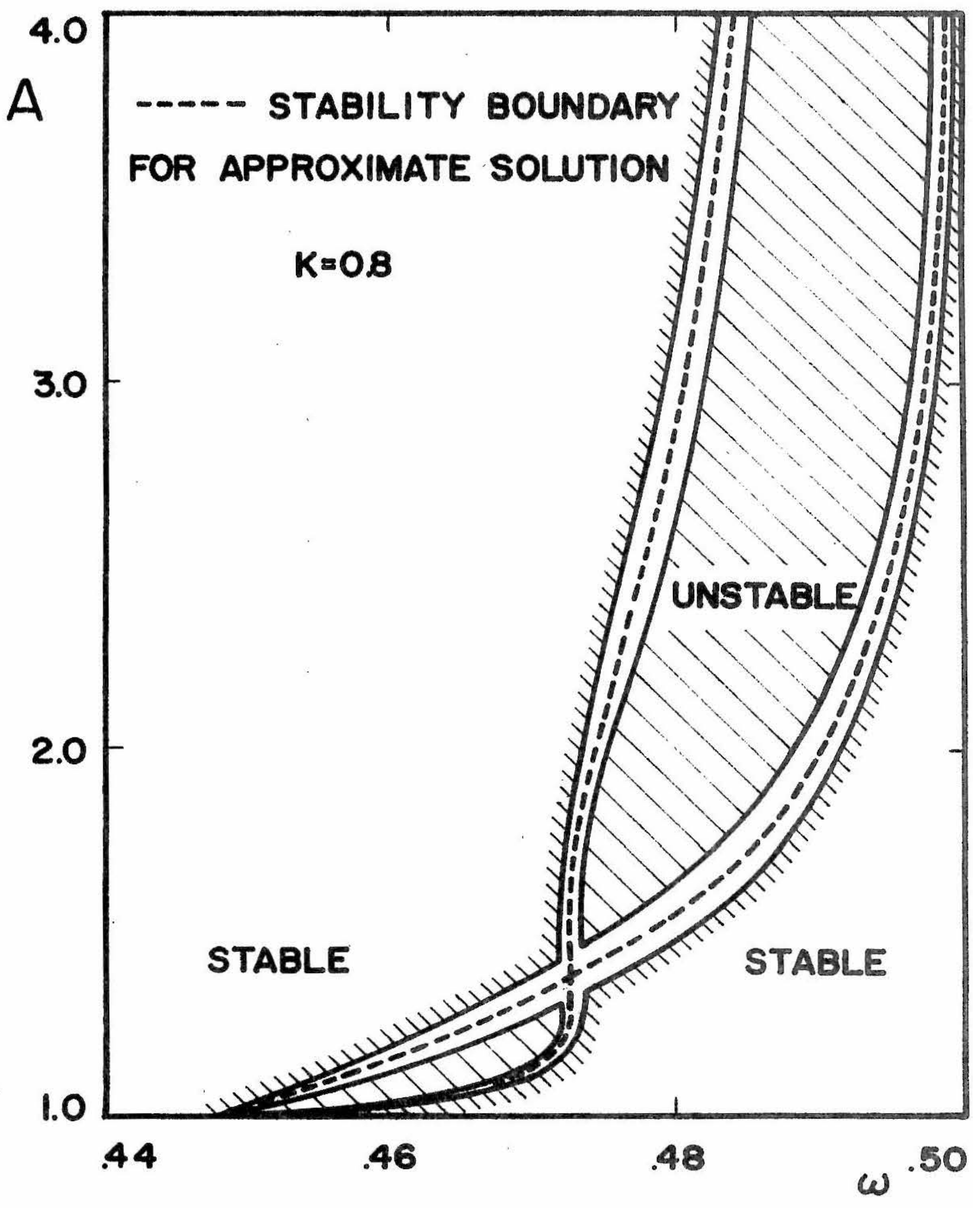

Fig. 3.3 UNSTABLE REGION NEAR $\omega = \frac{1}{2}$

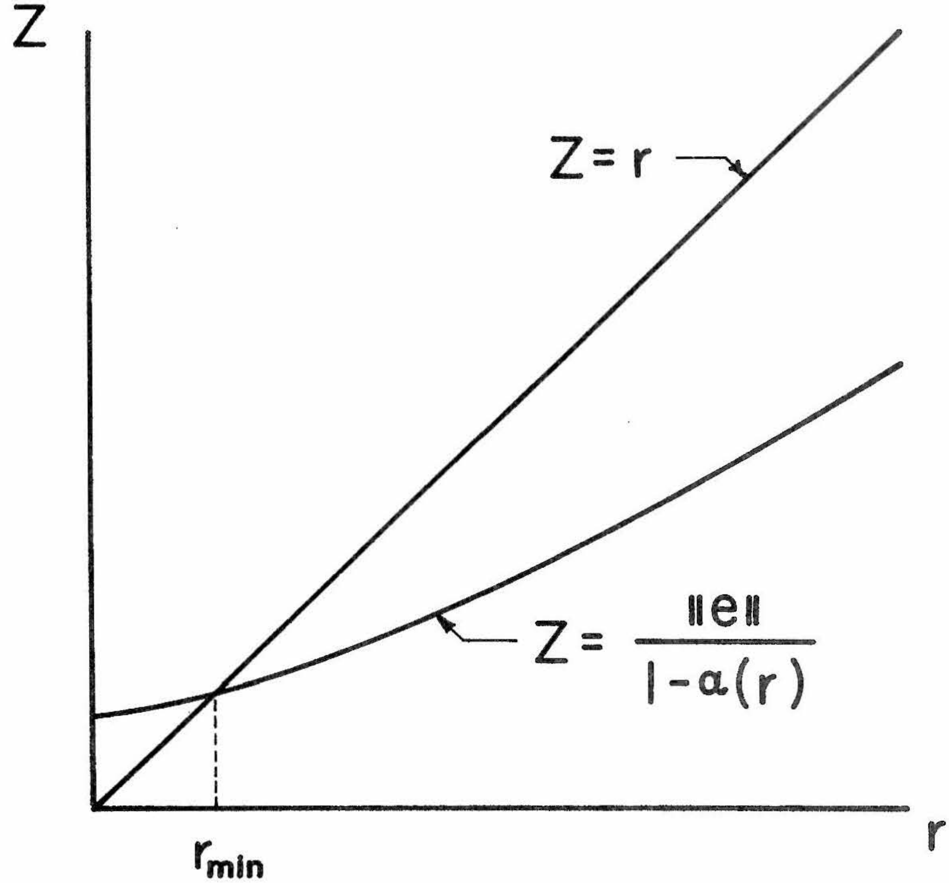

Fig. 3.4 VALIDITY INEQUALITY FOR GLOBAL UNIQUENESS

is globally unique within $Q_{2\pi/\omega}$. The same result would hold if a solution in $H_{2\pi/\omega}$ were sought, so any stable periodic solution of period $2\pi/\omega$ is in $P_{2\pi/\omega}$. This result is consistent with the work of Loud[19] who observed that equations of the form (3.1) exhibit branching phenomena in unstable regions near $\omega = \frac{1}{2n}$, n an integer.

The desirability of applying the validity criteria to class Q_T and H_T solutions may be appreciated if one observes the following: Had the validity criteria been applied in class P_T rather than Q_T, there would have been additional regions near $\omega = \frac{1}{2n}$, n an integer, where the validity criteria could not be applied. These regions would have included all of the instability region in Figure 3.3. The validity criteria in Q_T may be applied in this unstable region because the branching is to a class of solutions outside of Q_T. There are also apparently branching processes in regions I and II of Figure 3.2. However, these branching processes occur entirely within Q_T.

3.5. <u>Choice of Basis for the Integral Equation.</u>

In the integral equation for the approximation error, the linear basis, i.e., b(s) in (3.9), is chosen to facilitate application of the validity criteria. In this section, various choices of the basis for the bilinear system (3.1) with approximate solution (3.6) are compared. The symmetry class is $Q_{2\pi/\omega}$ in all cases.

The following choice of b(s) removes all linear terms from the operator H,

$$b(s) = \begin{cases} 1 & \text{for } 0 \le s \le \frac{1}{\omega}\cos^{-1}\frac{1}{A} \\ K & \text{for } \frac{1}{\omega}\cos^{-1}\frac{1}{A} \le s \le \frac{\pi}{2\omega} \end{cases} \quad (3.39)$$

In this case,

$$h_1(t,s) = \begin{cases} \left(\tilde{A}\cos t + \tilde{B}\sin t\right)\cos s/\tilde{E} & 0 \le s \le t \le \frac{1}{\omega}\cos^{-1}\frac{1}{A} \\ \left(\tilde{A}\cos s + \tilde{B}\sin s\right)\cos t/\tilde{E} & 0 \le t \le s \le \frac{1}{\omega}\cos^{-1}\frac{1}{A} \\ \sin\sqrt{K}\left(\frac{\pi}{2\omega} - s\right)\cos t/\left(\sqrt{K}\,\tilde{E}\right) & 0 \le t \le \frac{1}{\omega}\cos^{-1}\frac{1}{A} \le s \le \frac{\pi}{2\omega} \\ \sin\sqrt{K}\left(\frac{\pi}{2\omega} - t\right)\cos s/\left(\sqrt{K}\,\tilde{E}\right) & 0 \le s \le \frac{1}{\omega}\cos^{-1}\frac{1}{A} \le t \le \frac{\pi}{2\omega} \\ \sin\sqrt{K}\left(\frac{\pi}{2\omega} - t\right)\left(\tilde{C}\cos\sqrt{K}s + \tilde{D}\sin\sqrt{K}s\right)/\left(\sqrt{K}\,\tilde{E}\right) & \\ \qquad \frac{1}{\omega}\cos^{-1}\frac{1}{A} \le t \le s \le \frac{\pi}{2\omega} & \\ \sin\sqrt{K}\left(\frac{\pi}{2\omega} - s\right)\left(\tilde{C}\cos\sqrt{K}t + \tilde{D}\sin\sqrt{K}t\right)/\left(\sqrt{K}\,\tilde{E}\right) & \\ \qquad \frac{1}{\omega}\cos^{-1}\frac{1}{A} \le t \le s \le \frac{\pi}{2\omega} & \end{cases} \quad (3.40)$$

where

$$\left.\begin{aligned} \tilde{A} &= \cos\left[\frac{\sqrt{K}}{\omega}\sin^{-1}\frac{1}{A}\right]\sin\left[\frac{\cos^{-1}\frac{1}{A}}{\omega}\right] + \frac{1}{\sqrt{K}}\sin\left[\frac{\sqrt{K}}{\omega}\sin^{-1}\frac{1}{A}\right]\cos\left[\frac{\cos^{-1}\frac{1}{A}}{\omega}\right] \\ \tilde{B} &= -\cos\left[\frac{\sqrt{K}}{\omega}\sin^{-1}\frac{1}{A}\right]\cos\left[\frac{\cos^{-1}\frac{1}{A}}{\omega}\right] + \frac{1}{\sqrt{K}}\sin\left[\frac{\sqrt{K}}{\omega}\sin^{-1}\frac{1}{A}\right]\cos\left[\frac{\cos^{-1}\frac{1}{A}}{\omega}\right] \\ \tilde{C} &= \cos\left[\frac{\sqrt{K}}{\omega}\cos^{-1}\frac{1}{A}\right]\cos\left[\frac{\cos^{-1}\frac{1}{A}}{\omega}\right] + \frac{1}{\sqrt{K}}\sin\left[\frac{\sqrt{K}}{\omega}\cos^{-1}\frac{1}{A}\right]\sin\left[\frac{\cos^{-1}\frac{1}{A}}{\omega}\right] \end{aligned}\right\} \quad (3.41)$$

$$\widetilde{D} = \sin\left[\frac{\sqrt{K}}{\omega}\cos^{-1}\frac{1}{A}\right]\cos\left[\frac{\cos^{-1}\frac{1}{A}}{\omega}\right] - \frac{1}{\sqrt{K}}\cos\left[\frac{\sqrt{K}}{\omega}\cos^{-1}\frac{1}{A}\right]\sin\left[\frac{\cos^{-1}\frac{1}{A}}{\omega}\right]$$

$$\widetilde{E} = \cos\left[\frac{\sqrt{K}}{\omega}\sin^{-1}\frac{1}{A}\right]\cos\left[\frac{\cos^{-1}\frac{1}{A}}{\omega}\right] - \frac{1}{\sqrt{K}}\sin\left[\frac{\sqrt{K}}{\omega}\sin^{-1}\frac{1}{A}\right]\sin\left[\frac{\cos^{-1}\frac{1}{A}}{\omega}\right] \qquad (3.41)\ \text{cont.}$$

and

$$e(t) = (1-K) + \frac{\left(K_{EQV}-1\right)}{1-\omega^2}A\cos\omega t - \frac{(1-K)}{\widetilde{E}}\left\{\cos\left[\frac{\sqrt{K}}{\omega}\sin^{-1}\frac{1}{A}\right]\left[1-\frac{\left(K_{EQV}-\omega^2\right)}{(1-\omega^2)(K-\omega^2)}\right] + \frac{1}{\sqrt{K}}\left[A^2-1\right]^{\frac{1}{2}}\sin\left[\frac{\sqrt{K}}{\omega}\sin^{-1}\frac{1}{A}\right]\frac{\left(K_{EQV}-\omega^2\right)}{(1-\omega^2)(K-\omega^2)}\right\}\cos t$$

$$\text{for } 0 \le t \le \frac{1}{\omega}\cos^{-1}\frac{1}{A}$$

$$e(t) = \frac{\left(K_{EQV}-K\right)}{(1-\omega^2)}A\cos\omega t - \frac{(1-K)}{\sqrt{K}\widetilde{E}}\left\{\cos\left[\frac{\cos^{-1}\frac{1}{A}}{\omega}\right]\omega\left[A^2-1\right]^{\frac{1}{2}}\frac{\left(K_{EQV}-\omega^2\right)}{(1-\omega^2)(K-\omega^2)} + \sin\left[\frac{\cos^{-1}\frac{1}{A}}{\omega}\right]\left[1-\frac{\left(K_{EQV}-\omega^2\right)}{(1-\omega^2)(K-\omega^2)}\right]\right\}\sin\sqrt{K}\left(\frac{\pi}{2\omega}-t\right)$$

$$\text{for } \frac{1}{\omega}\cos^{-1}\frac{1}{A} \le t \le \frac{\pi}{2\omega} \qquad (3.42)$$

As in Appendix B for the constant coefficient case, one finds the following suitable contraction constant

$$\alpha(r) = \frac{|1-K|\left(1+\frac{1}{\sqrt{K}}\right)}{|\widetilde{E}|\,\omega}\text{MAX}\left[1, \frac{1}{\sqrt{K}}\right]\zeta(r) \qquad (3.43)$$

where

$$\zeta(r) = \begin{cases} \cos^{-1}\frac{1-r}{A} - \cos^{-1}\frac{1+r}{A} & \text{for } 0<r\leq \text{MIN}[1, A-1] \\ \frac{\pi}{2} - \cos^{-1}\frac{1+r}{A} & \text{for } 1\leq r\leq A-1 \\ \cos^{-1}\frac{1-r}{A} & \text{for } A-1\leq r\leq 1 \\ \frac{\pi}{2} & \text{for } r\geq \text{MAX}[1, A-1] \end{cases} \tag{3.44}$$

From this point, application of Validity Criteria I-A is the same as the constant basis case. Figure 3.5 shows a comparison of the percent error e_p where

$$e_p = \frac{r_{min}}{A} \times 100 \tag{3.45}$$

for various choices of the basis including the time dependent (T. D.) basis discussed above, and the following choices of constant coefficient bases: $K_* = 1$, $K_* = K_{AVG}$ defined by (3.15), and $K_* = K_{EQV}$ defined by (3.8). Also shown is $\|\eta\|$ computed as the sup of the difference between the approximate and exact solution. Where the curve is broken, just to the right of ω=0.3, there are no simple exact Q_T solutions. The parameters are P=1.0 and K=0.8, the same as those for a response curve shown in Figure 3.2.

For this case, the T. D. basis gives very good results away from the validity boundary for region II of Figure 3.2. In fact e_p for this basis is so close to the exact value that no separate curve is drawn except near the previously mentioned validity boundary. Also observe that the constant coefficient basis $K_*=K_{AVG}$ gives the widest range of validity. Hence, the time dependent basis may not always

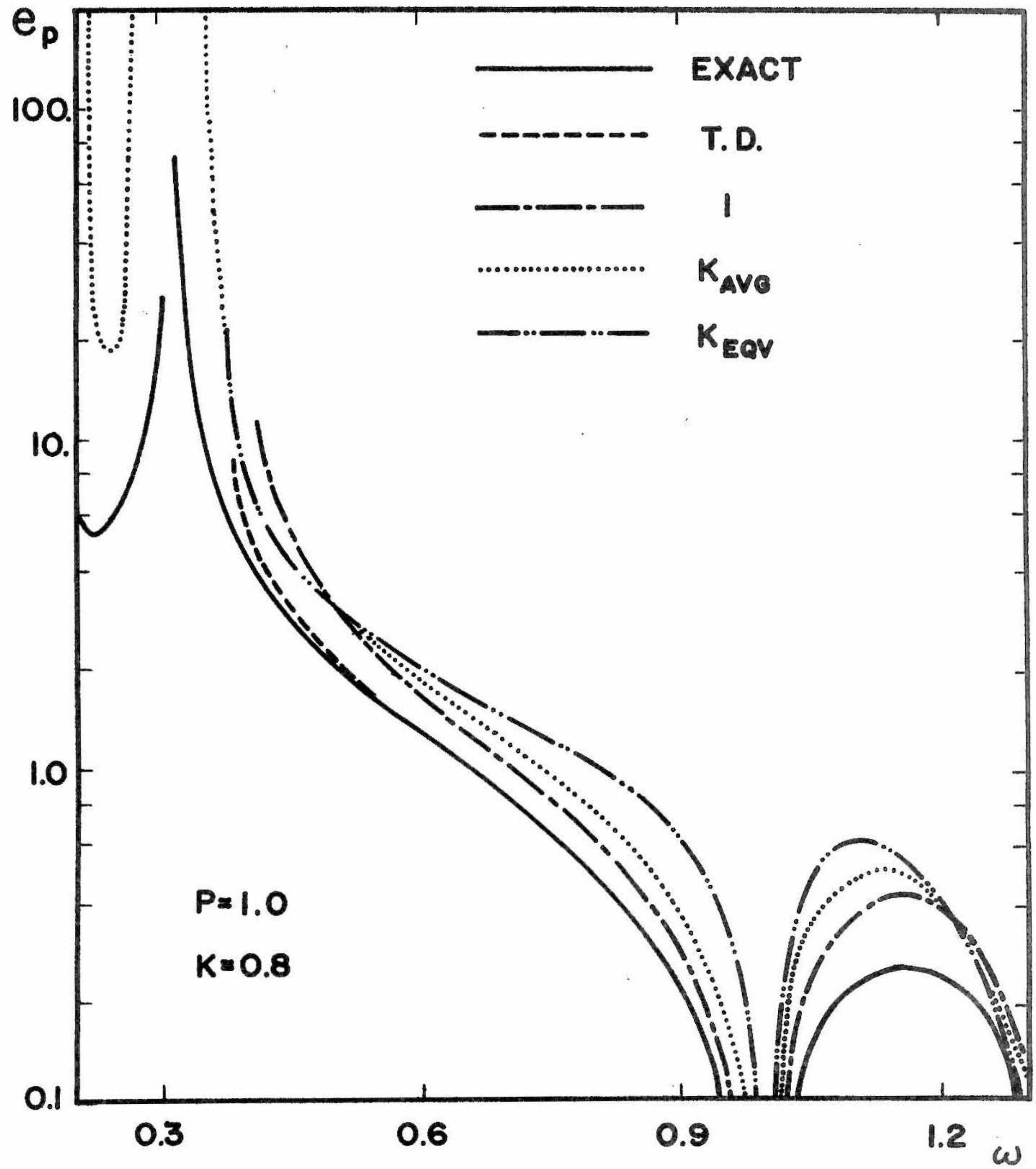

Fig. 3.5 COMPARISON OF BASES

be the best choice. The percent error for all basis choices has the character of the exact curve and compares favorably with the actual value at least for $\omega > \frac{1}{2}$. Although the comparison is plotted for only one set of values of the parameters, the same general behavior was also observed for a number of other cases.

An analytical comparison of the bases is made by considering the resonant case.

3.6. Validity Criteria at Resonance.

The following result holds for the T. D. basis and for all of the constant bases considered below.

$$\lim_{A \to \infty} \|e(t)\| \leq |1-K| \tag{3.46}$$

Also, in the limit as $A \to \infty$, equation (3.14) goes to

$$\frac{8\,|1-K|\left(1+\frac{1}{\sqrt{K}}\right)}{\pi\left|P+\frac{4(1-K)}{\pi}\right|}\,\mathrm{MAX}\left[1,\frac{1}{\sqrt{K}}\right]r^2+|1-K|\leq r \tag{3.47}$$

Solving (3.47) in the case of equality gives

$$r=\frac{\pi\left|P+\frac{4(1-K)}{\pi}\right|}{16\,|1-K|\left(1+\frac{1}{\sqrt{K}}\right)\mathrm{MAX}\left[1,\frac{1}{\sqrt{K}}\right]}\left[1\pm\left\{1-\frac{32(1-K)^2\left(1+\frac{1}{\sqrt{K}}\right)\mathrm{MAX}\ 1,\frac{1}{\sqrt{K}}}{\pi\left|P+\frac{4(1-K)}{\pi}\right|}\right\}^{1/2}\right] \tag{3.48}$$

Hence, a sufficient condition for application of the validity criteria at resonance is

$$\frac{32(1-K)^2\left(1+\frac{1}{\sqrt{K}}\right)\text{MAX}\left[1,\frac{1}{\sqrt{K}}\right]}{\pi\left|P+\frac{4(1-K)}{\pi}\right|}<1 \tag{3.49}$$

in which case the relative error r_{min}/A goes to zero as $A \to \infty$. Similar results for other choices of the basis are summarized in Table 3.1. Even at resonance, no one choice is clearly superior for all values of P and K.

Much work remains to be done in the area of optimum basis choice. However, the results given by Figure 3.5 and Table 3.1 indicate that the choice $K_* = K_{AVG}$, used in this work, compares favorably with other easily computed bases.

3.7. Two Term Harmonic Balance Approximate Solutions.

The bilinear system (3.1) allows Q_T solutions for $T = 2\pi/\Omega$, $\Omega = \omega/(2n-1)$, n a positive integer. For the case n=1, following Caughey[20], assume an approximate solution

$$x_0(t) = A\cos\omega t + U\cos 3\omega t \tag{3.50}$$

As with the one term approximation, there is no loss of generality in taking $A>0$.

In order to apply harmonic balance, it is necessary to determine a_1 and a_3 in the Fourier expansion

$$F(x_0) = a_1\cos\omega t + a_3\cos 3\omega t + \text{higher harmonics} \tag{3.51}$$

Expressions for a_1 and a_3 are given in Appendix D. Substituting (3.50) into (3.1), using (3.51) and balancing first and third harmonics yields

Basis	$\lim_{A\to\infty} \alpha(r)$	Sufficient Condition for Validity
T. D.	$\dfrac{8r\|1-K\|\left(1+\frac{1}{\sqrt{K}}\right)\mathrm{MAX}\left[1, \frac{1}{\sqrt{K}}\right]}{\pi\left\|P+\frac{4(1-K)}{\pi}\right\|}$	$\dfrac{32(1-K)^2\left(1+\frac{1}{\sqrt{K}}\right)\mathrm{MAX}\left[1, \frac{1}{\sqrt{K}}\right]}{\pi\left\|P+\frac{4(1-K)}{\pi}\right\|} < 1$
K_{EQV}	$\dfrac{4\|1-K\|(3+r)}{\pi\|P\|}$	$\dfrac{4\|1-K\|}{\sqrt{\pi\|P\|}\left(1-\frac{12\|1-K\|}{\pi\|P\|}\right)} < 1$
K_{AVG}	$\dfrac{\|1-K\|(4+2r)}{\left\|\frac{\pi P}{2}+(1-K)\right\|}$	$\dfrac{4\|1-K\|}{\sqrt{2\left\|\frac{\pi P}{2}+(1-K)\right\|}\left(1-\frac{4\|1-K\|}{\left\|\frac{\pi P}{2}+(1-K)\right\|}\right)} < 1$
1	$\dfrac{4\|1-K\|(1+r)}{\pi\left\|P+\frac{4(1-K)}{\pi}\right\|}$	$\dfrac{4\|1-K\|}{\sqrt{\pi\left\|P+\frac{4(1-K)}{\pi}\right\|}\left(1-\frac{4\|1-K\|}{\pi\left\|P+\frac{4(1-K)}{\pi}\right\|}\right)} < 1$

Table 3.1. Validity at Resonance

$$-A\omega^2 + a_1 = P$$

$$-9\omega^2 U + a_3 = 0 \qquad (3.52)$$

Just as with the one term approximate solution, sufficient conditions for existence of an exact Q_T solution near x_0 are obtained by applying a validity criteria which is essentially Criteria I-A for two term approximate solutions. The work involved in the application is computational rather than conceptual, so the details will not be given here.

3.8. Ultraharmonics.

An interesting development occurs in region II of Figure 3.2. Assuming a value of U and solving (3.52) for A and ω yields system responses shown in Figure 3.6. Note that the parameters used, i.e., $P=1.0$ and $K=0.8$ correspond to a response curve for the one term approximation shown in Figure 3.2. R is the ratio of U to A. The two term approximation predicts an ultraharmonic resonance (i.e., large amplitude response near $\omega=\omega_n/m$, m an integer ≥ 2, where ω_n is the harmonic resonant frequency) in this region.

Application of the validity criteria with $K_*=1$ shows that x_0 is valid on the lower ultraharmonic response curve in the frequency range shown in Figure 3.6. Note that there is an improvement over the one term approximation, i.e., compare with Figure 3.2. The presence of a vertical tangent in the ultraharmonic response apparently precludes application of the validity criteria over a larger frequency range.

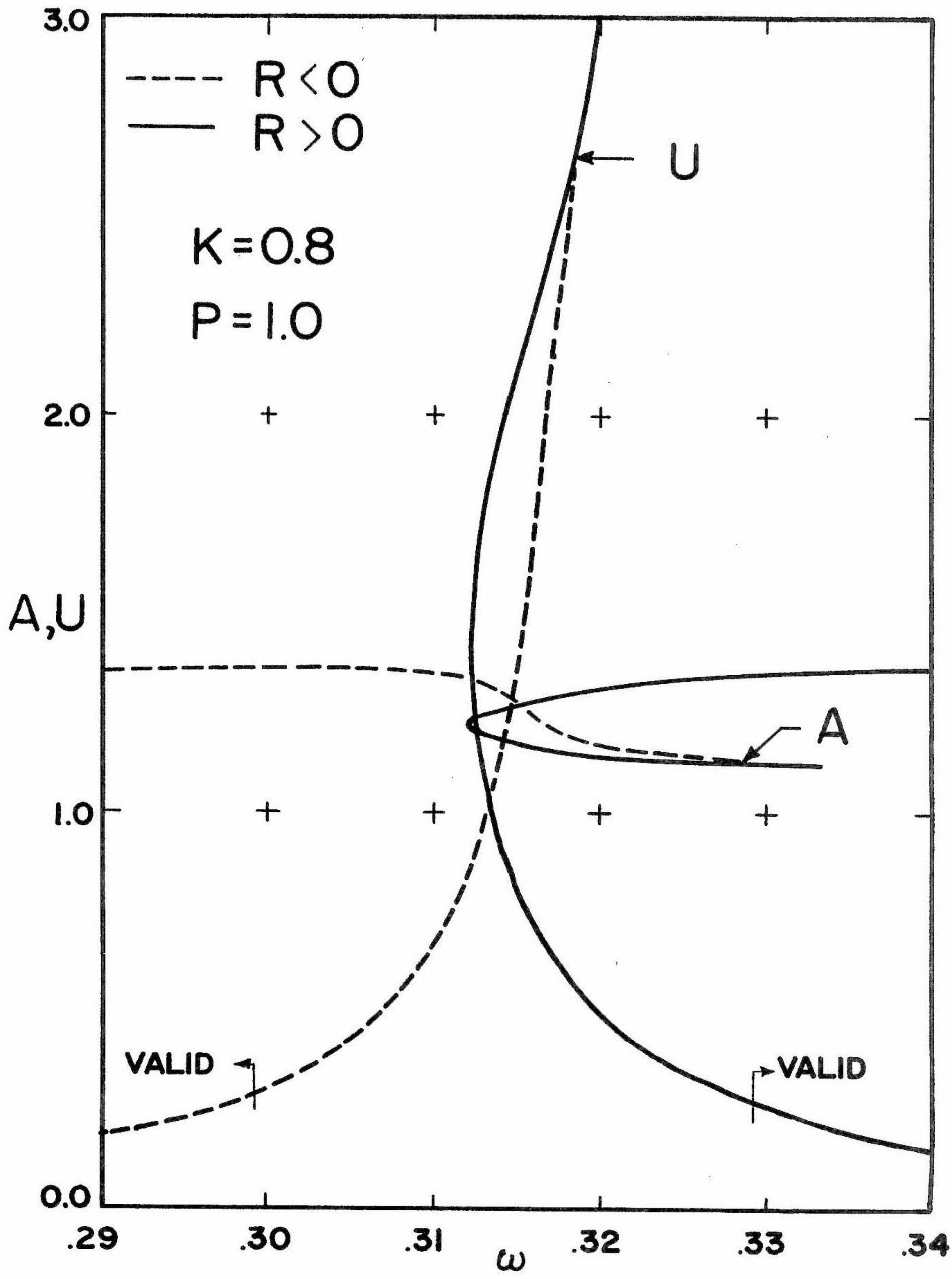

Fig. 3.6 ULTRAHARMONIC RESPONSE

With $K_{*}=1$,

$$\lim_{U \to \infty} \alpha(r) = 1 + r \tag{3.53}$$

Also, there is no constant coefficient basis which yields a smaller limiting value of $\alpha(r)$. Hence, for this particular system the existence of ultraharmonic resonant responses has not been established.

However, the two term approximation has a larger range of validity near $\omega = 1/3$. This indicates that the third harmonic is important in this region. Also, Figure 3.5 indicates that the approximation error for the one term approximate solution is large near $\omega = 1/3$. Hence, a stability analysis in this region based upon the one term approximation is not justified.

3.9. Subharmonics.

In a manner analogous to the development above, one obtains two term approximate solutions which exhibit subharmonic resonance, i.e., large amplitude response near $\omega = m\omega_n$, m an integer ≥ 2 and ω_n the harmonic resonant frequency. Assume an approximate $Q_{6\pi/\omega}$ solution

$$x_0(t) = S \cos \frac{\omega}{3} t + A \cos \omega t \tag{3.54}$$

where now $S > 0$. Harmonic balance yields

$$-\frac{S}{9}\omega^2 + b_1 = 0 \quad ; \quad -\omega^2 A + b_3 = P \tag{3.55}$$

where b_1 and b_3 are the Fourier coefficients in

$$F(x_0) = b_1 \cos \frac{\omega}{3} t + b_3 \cos \omega t + \text{higher harmonics} \tag{3.56}$$

Expressions for b_1 and b_3 are given in Appendix D.

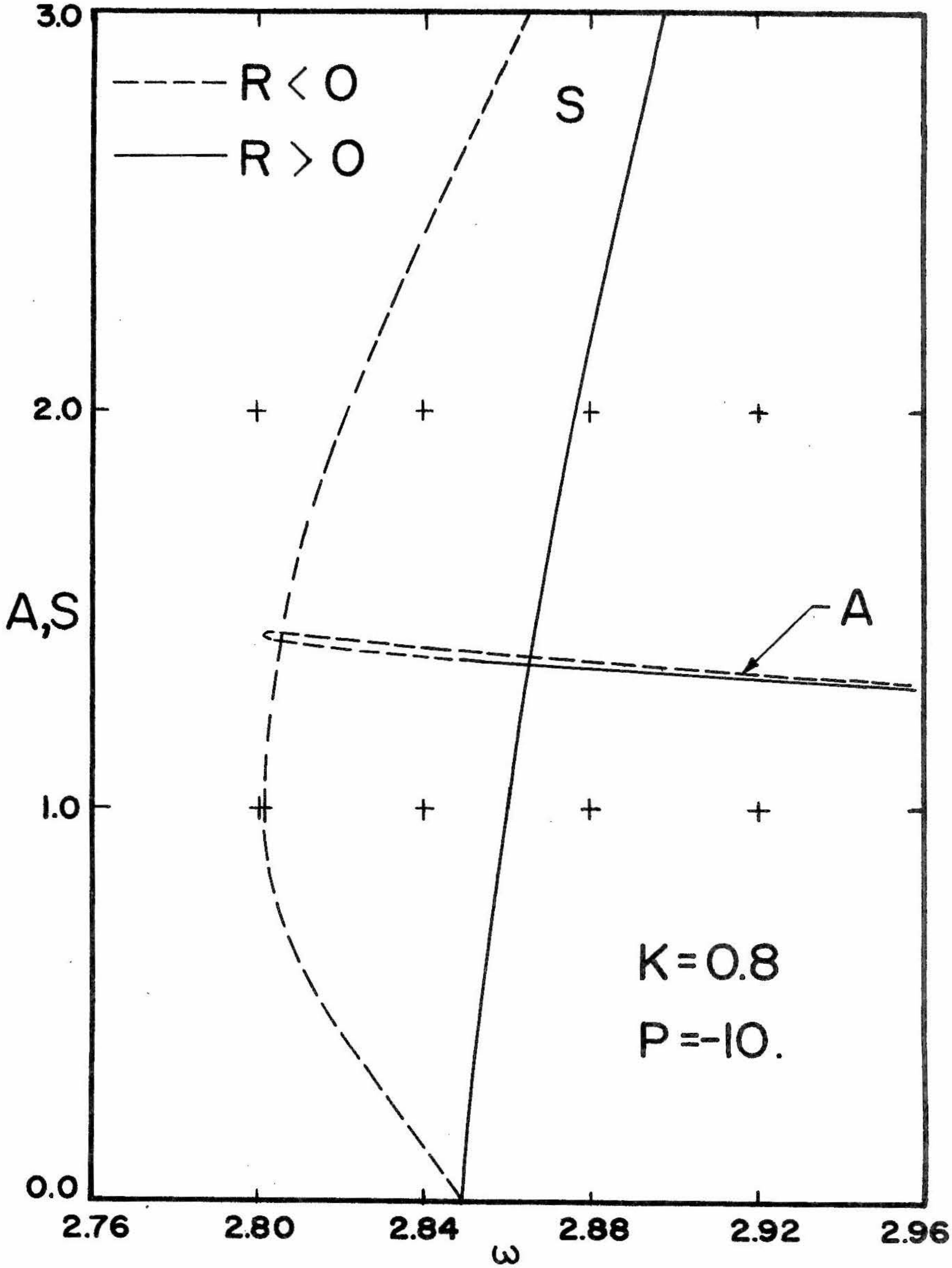

Fig. 3.7 SUBHARMONIC RESPONSE

For given S, equations (3.64) are solved for A and ω. Figure 3.7 shows the resulting subharmonic system response. The presence of vertical tangents apparently precludes application of the validity criteria for all responses shown in Figure 3.7. Also, there is no choice of constant coefficient basis which yields validity at subharmonic resonance. Hence it does not seem possible to verify the existence of subharmonic responses using this analysis; at least not with a constant coefficient basis.

CHAPTER IV

CONCLUSION

4.1 Summary

Sufficient validity criteria are derived for approximate periodic solutions of a class of second order nonlinear ordinary differential equations. An approximate solution is defined to be valid if an exact solution with the same symmetry properties exists in a neighborhood of the approximation. Although the validity criteria are stated for harmonic balance approximations, they are easily extended to include other approximate periodic solutions.

One class of validity criteria, namely I and I-A, require an integral operator associated with the approximation error to contract on a ball in an appropriate Banach space. That is, the operator must satisfy a Lipschitz condition with Lipschitz constant less than unity, and in addition, an inequality involving the Lipschitz constant and the radius of the ball must be satisfied. These criteria yield an algorithm for constructing the exact solution and some uniqueness results in addition to existence.

The second class of validity criteria, namely II-A, is based upon continuity of the same integral operator. This criteria requires that the mapping determined by the integral operator, continuously map a ball in an appropriate Banach space into a compact subset of itself.

Application of Criteria I-A and II-A to the equation of an undamped oscillator with an odd polynomial restoring force reveals that Criteria II-A yields a smaller bound on the approximation error and a larger region of validity. Criteria II-A is also somewhat easier to apply since it does not require determination of a Lipschitz constant. This is to be expected, however, since the results are not as strong as those of Criteria I-A with respect to uniqueness and the constructive algorithm.

In terms of types of hypotheses and conclusions, the existence theorems for B. V. P. 's upon which the two classes of validity criteria are based, may be compared to the existence theorems for initial value problems. The existence theorem for I and I-A is analogous to the Cauchy-Lipschitz theorem, while the existence theorem for II-A is analogous to the Cauchy-Peano theorem.

In addition to existence, both types of validity criteria yield an upper bound on a norm of the approximation error. This bound is used in a procedure for establishing sufficient stability and instability conditions for the approximated solution. A system with symmetric bilinear restoring force provides an example wherein both sufficient validity and sufficient stability and instability conditions are obtained. When the validity criteria are satisfied and the bound on the error is small, the stability analysis based upon the approximate solution is apparently quite accurate as shown in Figure 3. 3. On the other hand, when the approximation cannot be shown valid or when the bound on the approximation error is large, then a stability analysis based on the approximate solution is not justified. For example, in region II of Figure 3.2 there may be a considerable difference between the exact and approximate solutions as shown in Figure 3.5.

The example problems indicate inapplicability of the validity criteria in regions where a bifurcation (branching phenomenon) occurs within the symmetry class used. Thus the most restrictive allowable symmetry class yielded the widest range of validity.

4.2 Previous Work

The use of fixed point theorems to establish validity of approximate periodic solutions is not new, having been used in one form or another by Urabe[2] and Holtzman[4], [5] among others. However the methods proposed in these works require the pertinent integral operator to be differentiable in some sense. Thus the bilinear system studied in Chapter III cannot be handled by either of these approaches. In this work, the operator need only satisfy a much weaker Lipschitz or continuity condition. Further improvements over the previously mentioned methods result from the use of symmetry properties and selection of an arbitrary basis for the integral equation. The constructive algorithm was also apparently overlooked in earlier work.

A recent book of Falb and DeJong[21] also contains application of contraction mapping fixed point theorems to problems in oscillation theory. The approach is similar to Holtzman's work but makes use of the arbitrary basis idea, a procedure which the authors refer to as modified contraction mapping.

4.3 Future Work

Although this work considers only periodic solutions of non-linear ordinary differential equations, the basic idea of applying fixed point theorems to prove existence for the approximation error should be applicable to other types of problems, for example, transient oscillations in damped vibratory systems and also oscillations in multi-degree of freedom and continuous dynamical systems. Also, much work remains to be done in the area of optimum basis choice.

Sufficient stability and instability boundaries are determined in this work for one special case, namely, the one term harmonic balance approximate solution of the bilinear system (3.1). More general problems might be handled using the procedure given in Chapter II together with results of Caughey and Dickerson[22] in the damped case and Borg[23] in the undamped case.

APPENDIX A

SIMPLE SOLUTIONS FOR THE BILINEAR SYSTEM

Definition A1: A class Q_T solution of the bilinear system (3. 1), call it $x(t)$, is simple if $|x(t)|$ crosses +1 at one and only one value of t in $0 \leq t \leq T/4$.

Equation (3. 1) allows a Q_T solution for $T=(4n+2)\pi/\omega$, n an integer. Let T take a value from this set. For simple Q_T solutions, there is no loss of generality in assuming $x(0)>1$, in which case the governing equation is

$$\left.\begin{aligned} &\ddot{x} + x = P\cos\omega t + 1 - K \qquad 0<t<t_1 \\ &\dot{x}(0) = 0 \qquad x(t_1) = 1 \\ &\ddot{x} + Kx = P\cos\omega t \qquad t_1<t<T/4 \\ &x(t_1) = 1 \qquad x(T/4) = 0 \end{aligned}\right\} \qquad \text{(A-1)}$$

Solving for $0<t<t_1$

$$x(t) = 1-K+[A-1+K]\cos t + \frac{P}{1-\omega^2}[\cos\omega t - \cos t] \qquad 0\leq t\leq t_1 \qquad \text{(A-2)}$$

where $A>1$ is the assumed value of $x(0)$. Using the condition $x(t_1)=1$ yields

$$A = 1-K+\frac{1}{\cos t_1}\left\{K+\frac{P}{1-\omega^2}[\cos t_1 - \cos\omega t_1]\right\} \qquad \text{(A-3)}$$

Denoting $\dot{x}(t_1)$ by v and using (A-2) and (A-3)

$$v = -K\frac{\sin t_1}{\cos t_1} + \frac{P}{1-\omega^2}\left\{\frac{\cos \omega t_1 \sin t_1}{\cos t_1} - \omega \sin \omega t_1\right\} \qquad (A-4)$$

Then for $t_1 \le t \le \frac{T}{4}$

$$x(t) = \cos\sqrt{K}\,(t-t_1) + \frac{v}{\sqrt{K}}\sin\sqrt{K}\,(t-t_1)$$
$$+ \frac{P}{K-\omega^2}\left\{\cos \omega t - \cos\sqrt{K}\,(t-t_1)\cos \omega t_1\right.$$
$$\left.+ \frac{\omega}{\sqrt{K}}\sin\sqrt{K}\,(t-t_1)\sin \omega t_1\right\} \qquad (A-5)$$

The remaining condition is $x(T/4) = 0$. Using (A-4), (A-5) and simplifying gives the following equation for t_1

$$0 = \cos\sqrt{K}\left(\frac{T}{4}-t_1\right)\left[1-\frac{P\cos \omega t_1}{K-\omega^2}\right]$$
$$+ \frac{\sin\sqrt{K}\left(\frac{T}{4}-t_1\right)}{\sqrt{K}}\left\{\frac{\sin t_1}{\cos t_1}\left[\frac{P\cos \omega t_1}{1-\omega^2} - K\right] + \frac{\omega(1-K)P}{(K-\omega^2)(1-\omega^2)}\sin \omega t_1\right\} \qquad (A-6)$$

Simple Q_T solutions are then constructed as follows: For given P, K, and ω

1. All roots t_1 in $0 \le t_1 \le \frac{T}{4}$ of (A-6) are obtained.
2. Values of v and A corresponding to each root are computed using (A-4) and (A-3), respectively.
3. The function corresponding to each root and given by (A-2) and (A-5) is checked to be sure that it is a simple solution.

The last step is important, since there may be roots of (A-6) which do not yield solutions. As an example, Figure A. 1 shows a case where

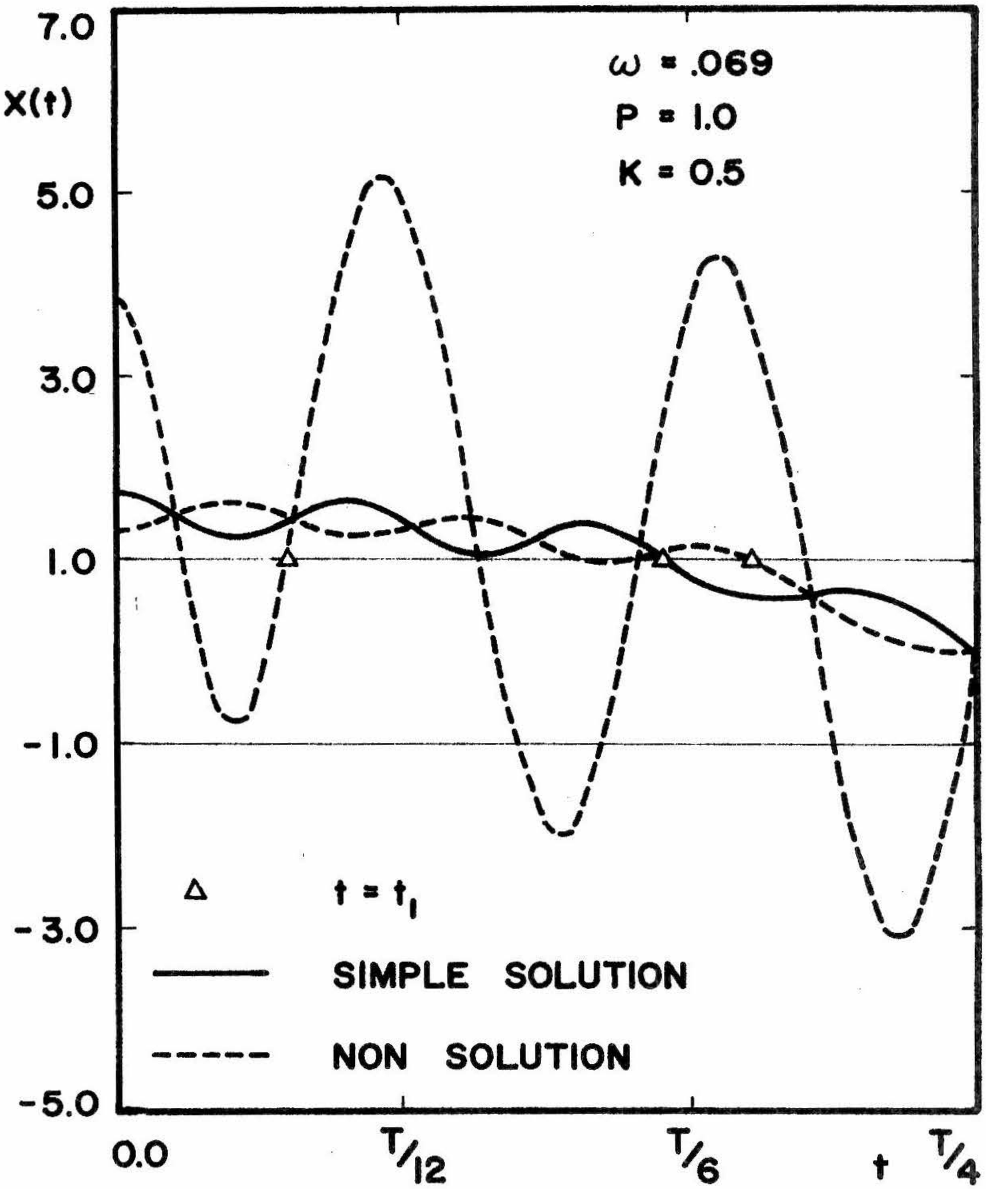

Fig. A.1 SIMPLE AND NON SOLUTIONS OF THE BILINEAR SYSTEM

(A-6) has three roots, only one of which corresponds to a solution. The other functions cross $|x| = 1$ more than once in $0 \leq t \leq \frac{T}{4}$ and hence they are not solutions.

APPENDIX B

A CONTRACTION CONSTANT

Consider the restoring force F(u) for the bilinear system shown graphically in Figure B-1. Inspection of the graph reveals that:

1) For z+x and z+y both on segment 3 or both on segment 1

$$F(z+x)-F(z+y)=x-y \tag{B-1}$$

2) For z+x and z+y both on segment 2

$$F(z+x)-F(z+y)=K(x-y) \tag{B-2}$$

3) For z+x and z+y on different segments

$$\left.\begin{aligned} K_-(x-y) &\le F(z+x)-F(z+y) \le K_+(x-y) \qquad x>y \\ K_-(x-y) &\ge F(z+x)-F(z+y) \ge K_+(x-y) \qquad x<y \end{aligned}\right\} \tag{B-3}$$

where $K_+ = MAX[1,K]$ and $K_- = MIN[1,K]$.

Hence

$$\left.\begin{aligned} \|K_*(y-x)+F(z+x)-F(z+y)\| &= |1-K_*|\ \|x-y\| \begin{cases}\text{for } z+x \text{ and } z+y \text{ both}\\ \text{on segment 3 or}\\ \text{both on segment 1}\end{cases} \\ \|K_*(y-x)+F(z+x)-F(z+y)\| &\le MAX\Big[|1-K_*|,|K_*-K|\Big]\|x-y\| \text{ for } z+x \\ &\qquad \text{and } z+y \text{ on different segments} \\ \|K_*(y-x)+F(z+x)-F(z+y)\| &= |K-K_*|\ \|x-y\| \begin{cases}\text{for } z+x \text{ and } z+y\\ \text{both on}\\ \text{segment 2}\end{cases} \end{aligned}\right\} \tag{B-4}$$

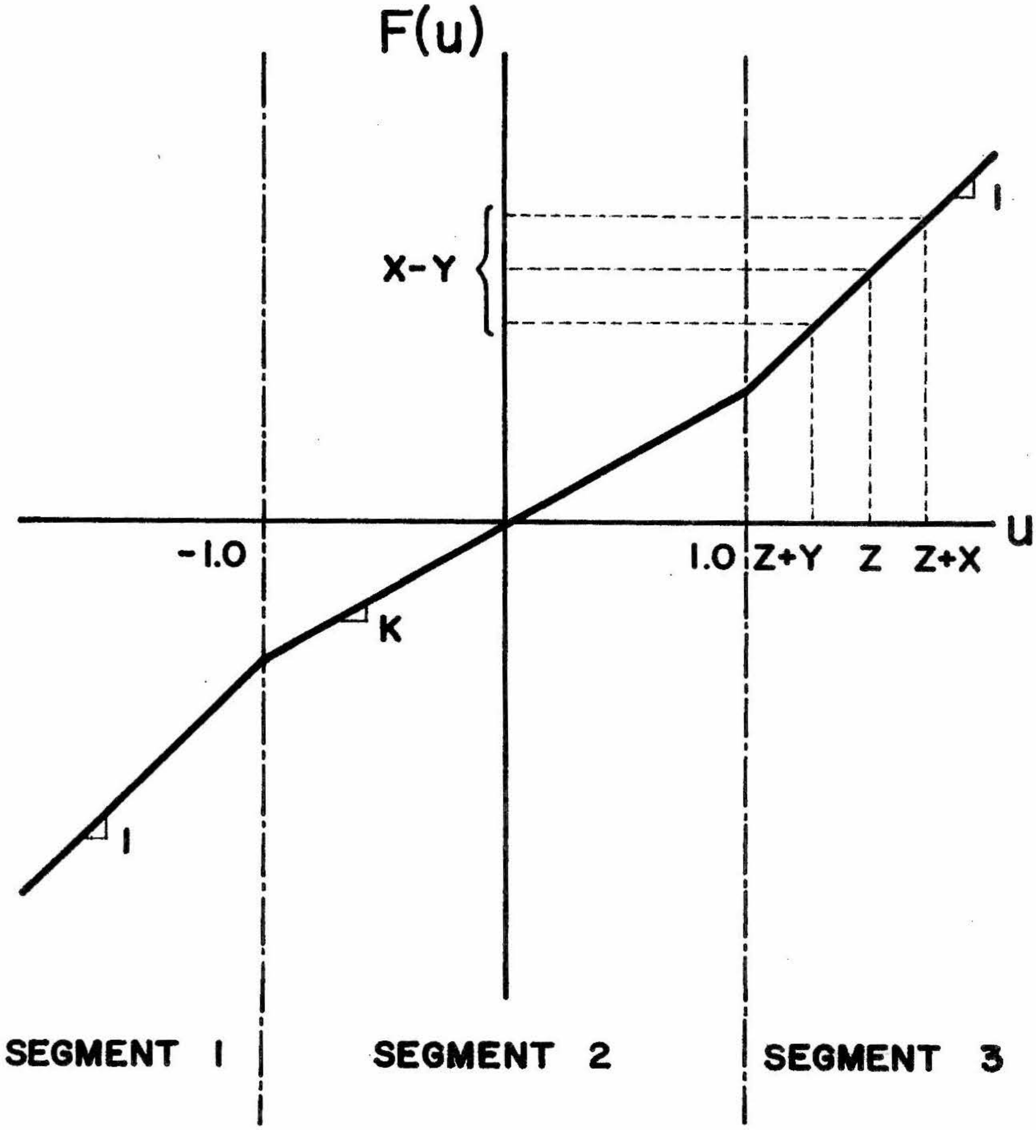

Fig. B.1 BILINEAR RESTORING FORCE

The purpose of this Appendix is to determine a suitable $\alpha = \alpha(r)$ satisfying

$$\|N_1 y - N_1 x\| \le \alpha(r)\|y-x\| \qquad (B\text{-}5)$$

for all $x, y \in B_r$, where N_1 is given by (3. 12). Using an elementary inequality for integrals

$$\|N_1 y - N_1 x\| \le \sup_{\substack{0\le t\le \frac{\pi}{2\omega} \\ 0\le s\le \frac{\pi}{2\omega}}} |h_1(t,s)| \int_0^{\pi/2\omega} \|F(x_0+x) - F(x_0+y) + K_*(y-x)\| ds \qquad (B\text{-}6)$$

From (2. 55)

$$\sup_{\substack{0\le t\le \frac{\pi}{2\omega} \\ 0\le s\le \frac{\pi}{2\omega}}} |h_1(t,s)| \le \frac{1}{\sqrt{K_*}\,\left|\cos \frac{\sqrt{K_*}\pi}{2\omega}\right|} \qquad (B\text{-}7)$$

Using $x, y \in B_r$, i. e., $\|x\| \le r$ and $\|y\| \le r$, and $x_0 = A\cos\omega s$, the following is deduced from (B-4)

1) For $r < A-1$ and $0 \le s \le \frac{1}{\omega}\cos^{-1}\frac{1+r}{A}$

$$\|F(x_0+x) - F(x_0+y) + K_*(y-x)\| = |1-K_*|\ \|y-x\| \qquad (B\text{-}8)$$

2) For $r < 1$ and $\frac{1}{\omega}\cos^{-1}\frac{1-r}{A} \le s \le \frac{\pi}{2\omega}$

$$\|F(x_0+x) - F(x_0+y) + K_*(y-x)\| = |K-K_*|\ \|y-x\| \qquad (B\text{-}9)$$

3) Otherwise, that is, $L \le s \le U$ where

$$L = \begin{cases} \frac{1}{\omega}\cos^{-1}\frac{1+r}{A} & \text{for } r < A-1 \\ 0 & \text{for } r \ge A-1 \end{cases}$$

$$U = \begin{cases} \frac{1}{\omega}\cos^{-1}\frac{1-r}{A} & \text{for } r<1 \\ 0 & \text{for } r\geq 1 \end{cases}$$

one obtains

$$\|F(x_0+x)-F(x_0+y)+K_*(y-x)\| \leq MAX\left[\,|1-K_*|,\ |K-K_*|\,\right]\|y-x\| \tag{B-10}$$

Combining with (B-6) and (B-7) yields

$$\alpha(r) = \frac{\pi}{2\omega\sqrt{K_*}\left|\cos\frac{\sqrt{K_*}\,\pi}{2\omega}\right|}\left\{|K-K_*|+\frac{2}{\pi}\left[\,|1-K_*|-|K-K_*|\,\right]\varphi(r)\right\} \tag{B-11}$$

where

$$\varphi(r) = \begin{cases} 0 & \frac{1+\sigma r}{A}>1 \\ \cos^{-1}\frac{1+\sigma r}{A} & 0\leq\frac{1+\sigma r}{A}\leq 1 \\ \frac{\pi}{2} & \frac{1+\sigma r}{A}<0 \end{cases} \tag{B-12}$$

$$\sigma = \begin{cases} +1 & \text{when } |K-K_*| > |1-K_*| \\ -1 & \text{when } |K-K_*| \leq |1-K_*| \end{cases} \tag{B-13}$$

APPENDIX C

STABILITY OF A SIMPLE Q_T SOLUTION OF THE BILINEAR SYSTEM

The equation of first variation for a simple Q_T solution x(t) of (3.1) with $T = 2\pi/\Omega$ is

$$\ddot{y} + f(t)y = 0 \tag{C-1}$$

where

$$f(t) = \begin{cases} 1 & \text{for } 0 \le t \le t_1 \\ K & \text{for } t_1 < t \le \frac{\pi}{\Omega} - t_1 \\ 1 & \text{for } \frac{\pi}{\Omega} - t_1 < t \le \frac{\pi}{\Omega} \end{cases} \qquad f(t) = f\left(t + \frac{\pi}{\Omega}\right) \tag{C-2}$$

and t_1 is the only value of t in $0 \le t \le \pi/2\Omega$ for which $|x| = 1$. Equation (3.1) allows Q_T solutions for $\Omega = \omega/(2n-1)$, n a positive integer.

Since f(t) is an even function, the stability criteria given in Chapter II have a simpler form. In the notation of Chapter II

$$u_1\left(\frac{\pi}{\Omega}\right) = v_2\left(\frac{\pi}{\Omega}\right) \tag{C-3}$$

(see Magnus and Winkler[18] page 8) and $\hat{q}_1 = 0$. Hence, the characteristic exponents for C-1 are

$$\lambda_{1,2} = \pm \begin{cases} \frac{i\Omega}{\pi} \ln \frac{1}{|\mu_1|} & \text{for } \left|u_1\left(\frac{\pi}{\Omega}\right)\right| \ge 1 \\ \frac{\Omega}{\pi}(\operatorname{Arg}\mu_1 + 2n\pi) & \text{for } \left|u_1\left(\frac{\pi}{\Omega}\right)\right| < 1 \end{cases} \tag{C-4}$$

where

$$\mu_1 = u_1\left(\frac{\pi}{\Omega}\right) + \left[u_1^2\left(\frac{\pi}{\Omega}\right) - 1\right]^{1/2} \tag{C-5}$$

Using the results summarized in Table 2. 1, (C-4) and (C-5) it is apparent that x(t) is stable when $\left|u_1\left(\frac{\pi}{\Omega}\right)\right| < 1$, unstable when $\left|u_1\left(\frac{\pi}{\Omega}\right)\right| > 1$ and unstable when $\left|u_1\left(\frac{\pi}{\Omega}\right)\right| = 1$, unless $v_1\left(\frac{\pi}{\Omega}\right)\ u_2\left(\frac{\pi}{\Omega}\right)$. Since $u_1(t)$ satisfies (C-1) and the intial conditions $u_1(0) = 1$, $\dot{u}_1(0) = 0$ it is readily obtained yielding

$$u_1\left(\frac{\pi}{\Omega}\right) = \cos\sqrt{K}\left(\frac{\pi}{\Omega} - 2t_1\right)\cos 2t_1 - \frac{\left(\frac{1}{\sqrt{K}} + \sqrt{K}\right)}{2}\sin\sqrt{K}\left(\frac{\pi}{\Omega} - 2t_1\right)\sin 2t_1 \tag{C-6}$$

APPENDIX D

FOURIER COEFFICIENTS FOR TWO TERM HARMONIC BALANCE

Consider first

$$x_0 = A \cos \omega t + U \cos 3\omega t \qquad \text{(D-1)}$$

The precise form of $F(x_0)$ where F is given by (3.2) depends upon the sign of U and upon the number of level crossings at $|x_0|=1$ per quarter cycle (CPQC), i.e., in $0 \le t \le \pi/2\omega$. There are four possible cases shown graphically in Figure D-1. The number and locations of the level crossings are determined by the roots of

$$A \cos \theta + U \cos 3\theta = 1 \qquad \text{(D-2)}$$

in $0 \le \theta \le \pi$. Equivalently (D-2) is written

$$\cos^3 \theta + \frac{(A-3U)}{4U} \cos \theta = \frac{1}{4U} \qquad \text{(D-3)}$$

The roots θ_i are called the crossing angles and they are related to the crossing times t_i where $|x_0(t_i)|=1$ by $\theta_i = \omega t_i$. If there is only one real root θ_i, then there is one CPQC; two real roots gives 2 CPQC, etc.

Let Z_i be roots of the cubic equation (D-3) satisfied by $\cos \theta_i$, that is,

$$Z_i^3 + \frac{(A-3U)}{4U} Z_i - \frac{1}{4U} = 0 \qquad \text{(D-4)}$$

Only roots Z_i in $-1 \le Z_i \le +1$ are considered since roots outside of this range correspond to complex values of θ_i. In cases where there is more than one root in this interval, let Z_1 be the largest root, Z_2 the

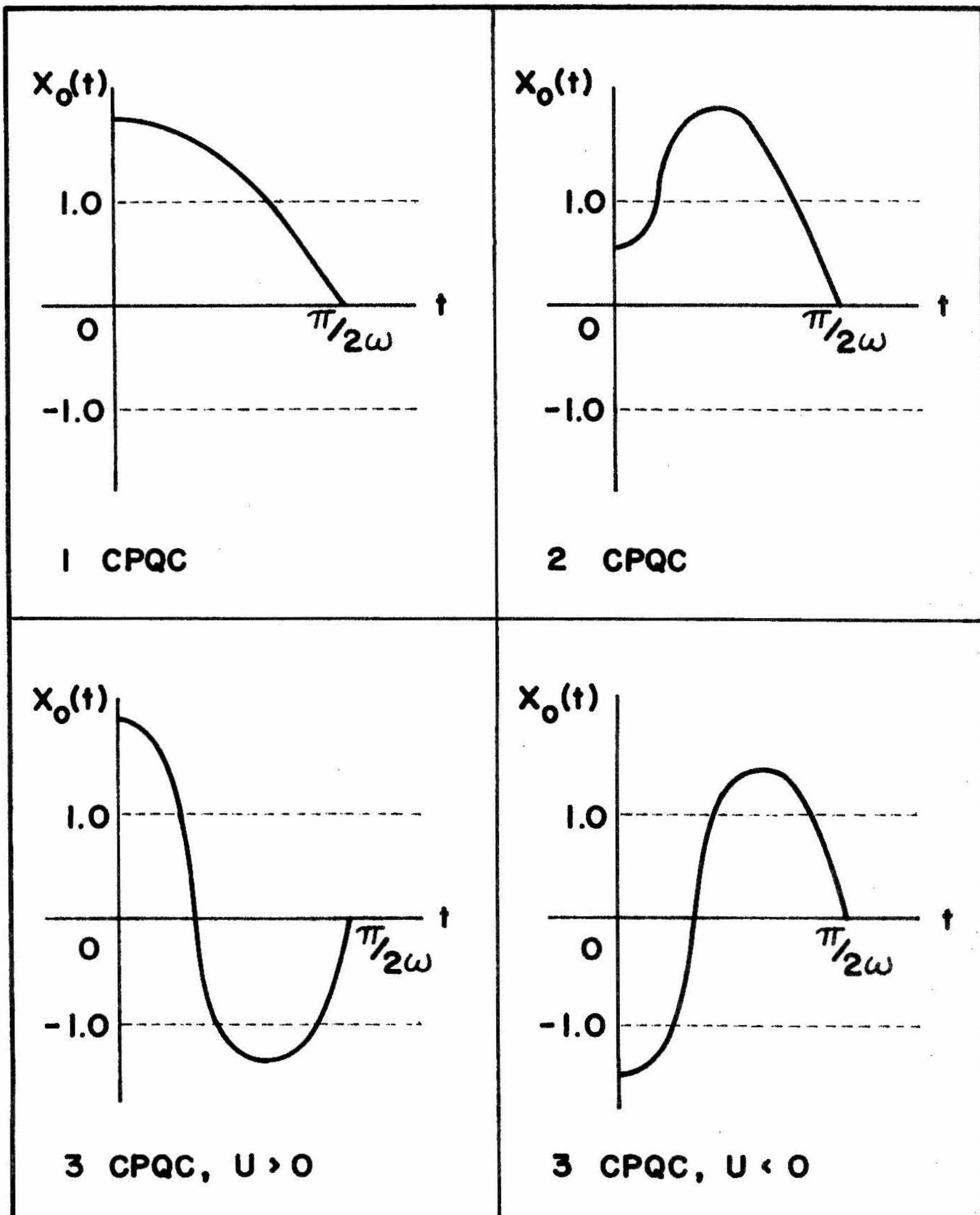

Fig. D.1 POSSIBLE LEVEL CROSSINGS FOR $X_0(t)$

next largest, and Z_3, if it exists, the smallest. Then the Z_i and θ_i are related by

$$
\left.\begin{array}{lll}
\text{one root} & : & \theta_1 = \cos^{-1} Z_1 \\
\text{two roots} & : & \theta_1 = \cos^{-1} Z_1 \\
 & & \theta_2 = \cos^{-1} Z_2 \, . \\
\text{three roots, } U>0 & : & \theta_1 = \cos^{-1} Z_1 \\
 & & \theta_2 = \pi - \cos^{-1} Z_3 \\
 & & \theta_3 = \pi - \cos^{-1} Z_2 \\
\text{three roots, } U<0 & : & \theta_1 = \pi - \cos^{-1} Z_3 \\
 & & \theta_2 = \cos^{-1} Z_1 \\
 & & \theta_3 = \cos^{-1} Z_2
\end{array}\right\} \qquad \text{(D-5)}
$$

Now expanding $F(x_0)$ in a Fourier series for each of the four cases gives

$$F(x_0) = a_1 \cos \omega t + a_3 \cos 3\omega t + \text{higher harmonics} \qquad \text{(D-6)}$$

where

$$
\left.\begin{aligned}
a_1 &= \frac{1}{\pi}\left\{\tilde{f}_1(1-K) + A\left[K+(1-K)\tilde{f}_2\right] + U(1-K)\tilde{f}_3\right\} \\
a_3 &= \frac{1}{\pi}\left\{\tilde{g}_1(1-K) + A(1-K)\tilde{g}_2 + U\left[K+(1-K)\tilde{g}_3\right]\right\}
\end{aligned}\right. \qquad \text{(D-7)}
$$

and $\tilde{f}_i$, $\tilde{g}_i$ $(i=1, 2, 3)$ are different in each case. Define functional forms

$$
\left.\begin{array}{ll}
a(x) = -4 \sin x & b(x) = -\frac{4}{3} \sin 3x
\end{array}\right\} \qquad \text{(D-8)}
$$

$$c(x) = 2x + \sin 2x \qquad d(x) = 2x + \frac{1}{3}\sin 6x$$

$$e(x) = \sin 2x + \frac{1}{2}\sin 4x \tag{D-8 cont.}$$

Then the $\tilde{f}_i$ and $\tilde{g}_i$ are:

<u>One CPQC</u>

$$\tilde{f}_1 = a(\theta_1) \qquad \tilde{g}_1 = b(\theta_1)$$
$$\tilde{f}_2 = c(\theta_1) \qquad \tilde{g}_3 = d(\theta_1) \tag{D-9}$$
$$\tilde{f}_3 = \tilde{g}_2 = e(\theta_1)$$

<u>Two CPQC</u>

$$\tilde{f}_1 = a(\theta_2)-a(\theta_1) \qquad \tilde{g}_1 = b(\theta_2)-b(\theta_1)$$
$$\tilde{f}_2 = c(\theta_2)-c(\theta_1) \qquad \tilde{g}_3 = d(\theta_2)-d(\theta_1) \tag{D-10}$$
$$\tilde{f}_3 = \tilde{g}_2 = e(\theta_2)-e(\theta_1)$$

<u>Three CPQC (U>0)</u>

$$\tilde{f}_1 = a(\theta_1)+a(\theta_2)-a(\theta_3) \qquad \tilde{g}_1 = b(\theta_1)+b(\theta_2)-b(\theta_3)$$
$$\tilde{f}_2 = c(\theta_1)+c(\theta_3)-c(\theta_2) \qquad \tilde{g}_3 = d(\theta_1)+d(\theta_2)-d(\theta_3) \tag{D-11}$$
$$\tilde{f}_3 = \tilde{g}_2 = e(\theta_1)+e(\theta_2)-e(\theta_3)$$

<u>Three CPQC (U<0)</u>

Same as (U>0), but change sign of $\tilde{f}_1$ and $\tilde{g}_1$.

For an approximate solution

$$x_0 = S\cos\frac{\omega}{3}t + A\cos\omega t \tag{D-12}$$

where $S>0$, replace A by S, U by A and ω by $\omega/3$ in all of the preceding equations. Then b_1 and b_3, instead of a_1 and a_3, are obtained from (D-6), where

$$F(x_0) = b_1 \cos\frac{\omega}{3}t + b_3 \cos \omega t + \text{higher harmonics} \qquad \text{(D-13)}$$

and x_0 is given by (D-12).

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