RANGE OF VALIDITY OF THE
METHOD OF AVERAGING

Thesis by
Daniel Adam Prelewicz

In Partial Fulfillment of the Requirements
For the Degree of
Doctor of Philosophy

California Institute of Technology
Pasadena, California
1970
(Submitted May 5, 1970)
ACKNOWLEDGMENTS

The author sincerely thanks his research advisor, Professor T. K. Caughey, for valuable guidance, suggestions and helpful discussions. Additional thanks are owed to Professor W. D. Iwan and to fellow graduate student Mr. E. J. Patula for bringing several of the references to the author's attention.

Financial support of the author's education by the Department of Health, Education and Welfare (NDEA Fellowship), the National Science Foundation (NSF Traineeship), the California Institute of Technology (teaching assistantships) and the State of California (state scholarship) is gratefully acknowledged.

The author also expresses his appreciation to his wife, Gerri, for her encouragement.

Thanks are due to Miss Julie Wright for her expert typing of the manuscript.
ABSTRACT

Sufficient conditions are derived for the validity of approximate periodic solutions of a class of second order ordinary nonlinear differential equations. An approximate solution is defined to be valid if an exact solution exists in a neighborhood of the approximation.

Two classes of validity criteria are developed. Existence is obtained using the contraction mapping principle in one case, and the Schauder-Leray fixed point theorem in the other. Both classes of validity criteria make use of symmetry properties of periodic functions, and both classes yield an upper bound on a norm of the difference between the approximate and exact solution. This bound is used in a procedure which establishes sufficient stability conditions for the approximated solution.

Application to a system with piecewise linear restoring force (bilinear system) reveals that the approximate solution obtained by the method of averaging is valid away from regions where the response exhibits vertical tangents. A narrow instability region is obtained near one-half the natural frequency of the equivalent linear system. Sufficient conditions for the validity of resonant solutions are also derived, and two term harmonic balance approximate solutions which exhibit ultraharmonic and subharmonic resonances are studied.
TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Part</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACKNOWLEDGMENTS</td>
<td></td>
<td>ii</td>
</tr>
<tr>
<td>ABSTRACT</td>
<td></td>
<td>iii</td>
</tr>
<tr>
<td>INTRODUCTION</td>
<td></td>
<td>vi</td>
</tr>
<tr>
<td>CHAPTER I.</td>
<td>MATHEMATICAL PRELIMINARIES.</td>
<td>1</td>
</tr>
<tr>
<td>1.1</td>
<td>Symmetry Properties of Periodic Functions</td>
<td>1</td>
</tr>
<tr>
<td>1.2</td>
<td>Periodic Solutions Via Boundary Value Problems</td>
<td>4</td>
</tr>
<tr>
<td>1.3</td>
<td>Boundary Value Problems and Fredholm Integral Equations</td>
<td>7</td>
</tr>
<tr>
<td>1.4</td>
<td>Fixed Point Theorems</td>
<td>13</td>
</tr>
<tr>
<td>1.5</td>
<td>Comments on Uniqueness</td>
<td>18</td>
</tr>
<tr>
<td>CHAPTER II.</td>
<td>VALIDITY AND STABILITY CRITERIA.</td>
<td>19</td>
</tr>
<tr>
<td>2.1</td>
<td>Validity Criteria</td>
<td>19</td>
</tr>
<tr>
<td>2.2</td>
<td>Comparison of Criteria I-A and II-A</td>
<td>29</td>
</tr>
<tr>
<td>2.3</td>
<td>Validity Boundaries</td>
<td>32</td>
</tr>
<tr>
<td>2.4</td>
<td>Sufficient Stability Criteria</td>
<td>36</td>
</tr>
<tr>
<td>CHAPTER III.</td>
<td>EXAMPLE: A BILINEAR SYSTEM.</td>
<td>42</td>
</tr>
<tr>
<td>3.1</td>
<td>Validity Criteria</td>
<td>44</td>
</tr>
<tr>
<td>3.2</td>
<td>Bounds for the Derivative of $\eta(t)$</td>
<td>48</td>
</tr>
<tr>
<td>3.3</td>
<td>Stability Criteria</td>
<td>49</td>
</tr>
<tr>
<td>Part</td>
<td>Title</td>
<td>Page</td>
</tr>
<tr>
<td>------</td>
<td>----------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>3.4</td>
<td>Sufficient Stability and Instability Boundaries</td>
<td>50</td>
</tr>
<tr>
<td>3.5</td>
<td>Choice of Basis for the Integral Equation</td>
<td>54</td>
</tr>
<tr>
<td>3.6</td>
<td>Validity Criteria at Resonance</td>
<td>59</td>
</tr>
<tr>
<td>3.7</td>
<td>Two Term Harmonic Balance Approximate Solutions</td>
<td>60</td>
</tr>
<tr>
<td>3.8</td>
<td>Ultraharmonics</td>
<td>62</td>
</tr>
<tr>
<td>3.9</td>
<td>Subharmonics</td>
<td>64</td>
</tr>
<tr>
<td></td>
<td><strong>CHAPTER IV. CONCLUSION.</strong></td>
<td>67</td>
</tr>
<tr>
<td>4.1</td>
<td>Summary</td>
<td>67</td>
</tr>
<tr>
<td>4.2</td>
<td>Previous Work</td>
<td>69</td>
</tr>
<tr>
<td>4.3</td>
<td>Future Work</td>
<td>70</td>
</tr>
<tr>
<td></td>
<td><strong>APPENDIX A. SIMPLE SOLUTIONS FOR THE BILINEAR SYSTEM.</strong></td>
<td>71</td>
</tr>
<tr>
<td></td>
<td><strong>APPENDIX B. A CONTRACTION CONSTANT.</strong></td>
<td>75</td>
</tr>
<tr>
<td></td>
<td><strong>APPENDIX C. STABILITY OF A SIMPLE $Q_T$ SOLUTION OF THE BILINEAR SYSTEM.</strong></td>
<td>79</td>
</tr>
<tr>
<td></td>
<td><strong>APPENDIX D. FOURIER COEFFICIENTS FOR TWO TERM HARMONIC BALANCE.</strong></td>
<td>81</td>
</tr>
<tr>
<td></td>
<td><strong>REFERENCES.</strong></td>
<td>86</td>
</tr>
</tbody>
</table>
Attempts to devise mathematical models for physical phenomena more often than not yield some form of nonlinear equation. Since explicit solutions are extremely rare, one usually seeks an approximate solution within a specific class. For example, Oscillation Theory is particularly interested in approximating periodic solutions of second order nonlinear differential equations.

A number of closely related approximating techniques, widely used in applications, namely the method of averaging, harmonic balance and equivalent linearization claim validity near resonance, or asymptotically as some small parameter goes to zero. These criteria are unsatisfactory in the sense that:

1) they do not guarantee the existence of an exact solution in a neighborhood of the approximation, and
2) even if existence is assumed, no quantitative measure is available for the actual error in the approximation, and hence a stability analysis based on the approximate solution is questionable.

In this work, quantitative validity criteria are derived which remove the objections cited above. Fixed point theorems from functional analysis and topology are the principal tools used.

Previous work on the validity of approximate solutions includes that of Cesari, Bass, Urabe, Holtzman and McLaughlin. Cesari\(^1\), and later Urabe\(^2\), derived sufficient conditions for the existence of
an exact solution in a neighborhood of the n'th Galerkin (harmonic balance) approximation, and a bound on the difference between the exact and approximate solution. Urabe also showed that existence of an isolated periodic solution implies the existence of a Galerkin approximation of sufficiently high order. The validity of equivalent linearization was studied by Bass (3) for the autonomous case, and by Holtzeman (4), (5) for the nonautonomous case. McLaughlin (6) used the implicit function theorem to obtain sufficient validity conditions for the Poincaré expansion.
CHAPTER I

MATHEMATICAL PRELIMINARIES

A number of ordinary differential equations frequently encountered in applications are of the form

\[
\ddot{x} + f(x, \dot{x}) = g(t) \tag{1.1}
\]

where \( f \) and \( g \) are piecewise continuous with at most a finite number of finite discontinuities. In this chapter, the problem of finding periodic solutions of (1.1) is reduced to a boundary value problem on an interval equal to or less than the period. Relationships between certain boundary value problems and Fredholm integral equations are derived, and several fixed point theorems are introduced.

For certain forms of \( f \) and \( g \) in (1.1) it is helpful to seek periodic solutions with appropriate symmetry properties.

1.1. Symmetry Properties of Periodic Functions.

**Definition 1.1:** A function \( u(t) \) is in class \( \mathbb{P}_T \) (\( u \in \mathbb{P}_T \)) if \( u(t) \) is continuously differentiable and

\[
u(t) = u(t + T) \tag{1.2}
\]

for all \( t, \ T \) a constant.

**Definition 1.2:** A function \( u(t) \) is in class \( \mathbb{H}_T \) (\( u \in \mathbb{H}_T \)) if \( u \in \mathbb{P}_T \) and

\[
u(T/2 + t) = -u(t) \tag{1.3}
\]
Definition 1.3: A function \( u(t) \) is in class \( Q_T \) (\( u \in Q_T \)) if \( u \in H_T \) and
\[
 u(T/2 - t) = -u(t) \tag{1.4}
\]

Note: Certain \( H_T \) functions may be put into class \( Q_T \) by a translation in \( t \).

Obviously, \( P_T \rightarrow H_T \rightarrow Q_T \). Figure 1.1a shows a function in class \( H_T \) and Figure 1.1b shows a function in class \( Q_T \). Functions in these symmetry classes have properties given by the following Lemmas.

**Lemma 1.1:** If \( u \in P_T \), then necessarily
\[
 u(0) = u(T) \\
 \dot{u}(0) = \dot{u}(T) \tag{1.5}
\]

**Proof:** Let \( t = 0 \) in (1.2), then differentiate and do the same.

**Lemma 1.2:** If \( u \in H_T \), then necessarily
\[
 u(0) = -u(T/2) \\
 \dot{u}(0) = -\dot{u}(T/2) \tag{1.6}
\]

**Proof:** Let \( t = 0 \) in (1.3), then differentiate and do the same.

**Lemma 1.3:** If \( u \in Q_T \) then necessarily
\[
 u(T/4) = 0 \\
 \dot{u}(0) = 0 \tag{1.7}
\]

**Proof:** Let \( t = T/4 \) in (1.4), then
\[
 u(T/4) = -u(T/4) \\
 \therefore u(T/4) = 0
\]

Differentiating (1.3) and (1.4) with respect to \( t \) and letting \( t = 0 \) yields,
Fig. 1.1a. A CLASS $H_T$ FUNCTION

Fig. 1.1b. A CLASS $Q_T$ FUNCTION
\[ \dot{u}(T/2) = -\dot{u}(0) \]
\[ -\dot{u}(T/2) = -\dot{u}(0) \]
\[ \therefore \dot{u}(0) = -\dot{u}(0) \implies \dot{u}(0) = 0. \]

1.2 Periodic Solutions via Boundary Value Problems.

The following Lemmas show that in certain cases, periodic solutions may be constructed by extending the solutions of particular boundary value problems (B.V.P.'s) to the infinite interval. Equations (1.5) - (1.7) serve as boundary conditions. In the sequel, it is assumed that \( f \) and \( g \) are continuous. This is a matter of convenience since many of the results apply to the case where \( f \) and \( g \) are piecewise continuous with at most a finite number of finite discontinuities.

Lemma 1.4: If

1) \( g(t) \in P_T \) \hfill \( (1.8) \)

2) \( u(t) \) is a solution of the B.V.P.
\[ \dot{u} + f(u, \dot{u}) = g(t) \quad 0 < t < T \]
\[ u(0) = u(T) \quad \dot{u}(0) = \dot{u}(T) \hfill \( (1.9) \) \]

then \( u(t) \) is extendible to a class \( P_T \) solution of (1.1). That is, the function \( x(t) \) defined by
\[ x(t+nT) = u(t) \hfill \( (1.10) \) \]
for \( 0 \leq t \leq T \), \( n \) an integer, is a class \( P_T \) solution of (1.1).

Proof: Since the differential equation remains the same on subsequent intervals \( (n-1)T < t < nT \), \( n \) an integer, \( x(t) \) satisfies (1.1) for all \( t \), and hence it is a class \( P_T \) solution.
Lemma 1.5: If

1) \( g(t) \in H_T \) \hspace{1cm} (1.11)

2) \( f(x, \dot{x}) = -f(-x, -\dot{x}) \) \hspace{1cm} (1.12)

3) \( u(t) \) is a solution of the B. V. P.

\[ \ddot{u} + f(u, \dot{u}) = g(t) \quad 0 < t < T/2 \]

\[ u(0) = -u(T/2) \quad \dot{u}(0) = -\dot{u}(T/2) \] \hspace{1cm} (1.13)

then \( u(t) \) is extendible to a class \( H_T \) solution of (1.1). That is, \( x(t) \)

defined by

\[ \begin{align*}
    x(t) &= u(t) \quad 0 \leq t \leq T/2 \\
    x(t+T/2) &= -u(t) \quad 0 \leq t \leq T/2 \\
    x(t+nT) &= x(t) \quad n \text{ an integer, } 0 \leq t \leq T
\end{align*} \] \hspace{1cm} (1.14)

is a class \( H_T \) solution of (1.1).

Proof: The second of equations (1.14) implies

\[ \begin{align*}
    -x(T/2+t) &= u(t) \\
    -\dot{x}(T/2+t) &= \dot{u}(t) \quad 0 < t < T/2 \\
    -\ddot{x}(T/2+t) &= \ddot{u}(t)
\end{align*} \] \hspace{1cm} (1.15)

Using this in the differential equation yields

\[ -\ddot{x}(T/2+t) + f(-x(T/2+t), -\dot{x}(T/2+t)) = g(t) \quad 0 < t < T/2 \] \hspace{1cm} (1.16)

or, using (1.11) and (1.12)

\[ \ddot{x}(T/2+t) + f(x(T/2+t), \dot{x}(T/2+t)) = g(T/2+t) \quad 0 < t < T/2 \] \hspace{1cm} (1.17)

and hence \( x(t) \) is a solution for \( 0 < t < T \) satisfying

\[ x(T) = x(0) \]

\[ \dot{x}(T) = \dot{x}(0) \]
where (1.14) and the boundary conditions on u(t) were used. Extension to the whole real line is as in Lemma 1.4.

Lemma 1.6: If

1) \( g(t) \in \Omega_T \)  

2) \( f(x, \dot{x}) = -f(-x, -\dot{x}) \)  

3) \( f(x, \dot{x}) = f(x, -\dot{x}) \)  

4) \( u(t) \) is a solution of the B. V. P.

\[
\ddot{u} + f(u, \dot{u}) = g(t) \quad 0 < t < T/4 \\
\dot{u}(0) = 0 \quad u(T/4) = 0
\]  

(1.18) \hspace{1cm} (1.19) \hspace{1cm} (1.20) \hspace{1cm} (1.21)

then \( u(t) \) is extendible to a class \( \Omega_T \) solution of (1.1). That is, the function \( x(t) \) defined by

\[
x(t) = u(t) \quad 0 \leq t \leq T/4 \\
x(T/2-t) = -u(t) \quad 0 \leq t \leq T/4 \\
x(T/2+t) = -x(t) \quad 0 \leq t \leq T/2 \\
x(t+nT) = x(t) \text{ n an integer, } \quad 0 \leq t \leq T
\]  

(1.22)

is a class \( \Omega_T \) solution of (1.1).

Proof: The second of (1.22) implies

\[
x(T/2-t) = -u(t) \\
\dot{x}(T/2-t) = \dot{u}(t) \quad 0 < t < T/4 \\
\ddot{x}(T/2-t) = -\ddot{u}(t)
\]  

(1.23)

Substituting into the differential equation and using (1.18) - (1.20) yields

\[
\ddot{x}(T/2-t) + f(x(T/2-t), \dot{x}(T/2-t)) = g(T/2-t) \quad 0 < t < T/4
\]

and hence \( x(t) \) is a solution of (1.1) for \( 0 < t < T/2 \) satisfying
\[ x(T/2) = -x(0) \]
\[ x(T/2) = -\dot{x}(0) = 0 \]

where the boundary conditions and (1.22) were used. Extension to the whole real line is as in Lemma (1.5).

When the conditions on \( f \) and \( g \) are satisfied in Lemmas 1.4 - 1.6, the differential equation is said to allow solutions in the pertinent symmetry class. For some equations, a translation in \( t \) may be necessary to allow solutions in \( Q_T \).

1.3. Boundary Value Problems and Fredholm Integral Equations.

The following results concerning the relationship of certain B.V.P.'s to Fredholm integral equations will be useful in what follows.

Consider the equation

\[ \ddot{u} + a(t)\dot{u} + b(t)u = c(u, \dot{u}, t) \]  \hspace{1cm} (1.24)

or, in vector-matrix form

\[ \ddot{u} = A(t)\ddot{u} + c(\ddot{u}, t) \]  \hspace{1cm} (1.25)

where

\[ A(t) = \begin{bmatrix} 0 & 1 \\ -b(t) & -a(t) \end{bmatrix}, \quad \bar{c}(\ddot{u}, t) = \begin{bmatrix} 0 \\ c(u, \dot{u}, t) \end{bmatrix}, \quad u = \begin{bmatrix} u \\ \dot{u} \end{bmatrix} \]  \hspace{1cm} (1.26)

In the following Lemmas \( a, b, \) and \( c \) are assumed to be piecewise continuous with at most a finite number of finite discontinuities. Although the first Lemma may be deduced from developments in Hahn\(^{7}\), page 359, or Halanay\(^{8}\), page 225, the proof is given here in the context of this work.
Lemma 1.7: If \( \overline{u}, \overline{c}(u, t) \) and \( A(t) \) are as given by (1,26) with

1) \( U(t) \) the principal matrix solution (see Struble\(^9\), page 83) of
\[
\dot{\overline{u}} = A(t)\overline{u}
\]  
(1.27)

2) \( [I-U(T)]^{-1} \) exists. Here \( I \) is the \( 2 \times 2 \) identity
matrix and \( [\cdot]^{-1} \) is the matrix inverse.

3)
\[
U(t, s) = \begin{bmatrix}
h_{11}&h_{12} \\
h_{21}&h_{22}
\end{bmatrix} = \begin{cases}
U(t)[I-U(T)]^{-1}U^{-1}(s) & 0 \leq s \leq t \leq T \\
U(t)[I-U(T)]^{-1}U(T)U^{-1}(s) & 0 \leq t \leq s \leq T
\end{cases}
\]  
(1.28)

4) \( \overline{u} \) governed by
\[
\dot{\overline{u}} = A(t)\overline{u} + \overline{c}(\overline{u}, t) \quad 0 < t < T
\]  
(1.29)
\[
\overline{u}(0) = \overline{u}(T)
\]  
(1.30)

then
\[
\overline{u} = \int_0^T \overline{c}(\overline{u}, \dot{\overline{u}}, s)\overline{h}(t, s)ds
\]  
(1.31)
\[
\overline{h}(t, s) = \begin{bmatrix}
h_{12} \\
h_{22}
\end{bmatrix}
\]  
(1.32)

Proof: Use is made of the following well known result concerning initial value problems (see Struble\(^9\), page 92).

Given
\[
\dot{\overline{u}} = A(t)\overline{u} + \overline{f}(t)
\]  
(1.33)
\[
\overline{u}(0) = \overline{u}_0
\]  
(1.34)
then

\[\bar{u}(t) = U(t)\bar{u}_0 + U(t) \int_0^t U^{-1}(s)\bar{f}(s)ds\]  

(1.35)

Using this result with \(\bar{f}(t)\) replaced by \(\bar{c}(\bar{u}, t)\), yields the following alternate form for (1.29)

\[\bar{u}(t) = U(t)\bar{u}(0) + U(t) \int_0^t U^{-1}(s)\bar{c}(\bar{u}, s)ds\]  

(1.36)

where now \(\bar{u}(0)\) is determined from the condition (1.30). That is,

\[\bar{u}(0) - \bar{u}(T) = U(T)\bar{u}(0) + U(T) \int_0^T U^{-1}(s)\bar{c}(\bar{u}, s)ds\]  

(1.37)

\[\bar{u}(0) = [I - U(T)]^{-1}U(T) \int_0^T U^{-1}(s)\bar{c}(\bar{u}, s)ds\]  

(1.38)

and hence

\[\bar{u}(t) = U(t) [I - U(T)]^{-1}U(T) \int_0^t U^{-1}(s)\bar{c}(\bar{u}, s)ds + U(t) \int_0^t U^{-1}(s)\bar{c}(\bar{u}, s)ds\]  

(1.39)

or,

\[\bar{u}(t) = \int_0^T H(t, s)\bar{c}(\bar{u}, s)ds\]  

(1.40)

where

\[H(t, s) = \begin{cases} U(t) \left[ [I - U(t)]^{-1}U(T) + I \right] U^{-1}(s) & 0 \leq s \leq t \leq T \\ U(t) [I - U(T)]^{-1}U(T)U^{-1}(s) & 0 \leq t \leq s \leq T \end{cases}\]  

(1.41)
Now using \( \overline{c}(\overline{u}, t) = \begin{bmatrix} 0 \\ c(u, \dot{u}, t) \end{bmatrix} \) gives
\[
\overline{u} = \int_{0}^{T} c(u, \dot{u}, s) \overline{h}(t, s) ds
\]
(1.42)

It remains only to show that \( H(t, s) \) as given in (1.41) is the same as in (1.29), or equivalently

\[
[I-U(T)]^{-1} U(T) + I = [I-U(T)]^{-1} U(T) + [I-U(T)]^{-1} [I-U(T)]
\]

\[
[I-U(T)]^{-1} U(T) + I = [I-U(T)]^{-1}
\]

Since the proof of the following Lemma is very similar to that of Lemma 1.7, no proof is given.

**Lemma 1.8:** If \( \overline{u}, \overline{c}(\overline{u}, t) \) and \( A(t) \) are as given in (1.27) with

1) \( U(t) \) the principal matrix solution of
\[
\dot{\overline{u}} = A(t) \overline{u}
\]
(1.44)

2) \( [I+U(T)]^{-1} \) exists

3)
\[
H(t, s) = \begin{bmatrix} h_{11}, h_{12} \\ h_{21}, h_{22} \end{bmatrix} = \begin{cases} -U(t) [I+U(T)]^{-1} U^{-1}(s) & 0 \leq s \leq t \leq T \\ -U(t) [I+U(T)]^{-1} U(T) U^{-1}(s) & 0 \leq t \leq s \leq T \end{cases}
\]
(1.45)

4) \( \overline{u} \) governed by
\[
\dot{\overline{u}} = A(t) \overline{u} + \overline{c}(\overline{u}, t) \quad 0 < t < T/2
\]
(1.46)
\[
\overline{u}(0) = -\overline{u}(T/2)
\]
(1.47)
then
\[
\overline{u} = \frac{1}{T/2} \int_{0}^{T/2} c(u, \dot{u}, s) \overline{h}(t, s) \, ds
\]
\[\text{(1.48)}\]
\[
\overline{h}(t, s) = \begin{bmatrix} h_{12} \\ h_{22} \end{bmatrix}
\]
\[\text{(1.49)}\]

**Lemma 1.9:** If \( \overline{u} \), \( c(u, t) \) and \( A(t) \) are as given in (1.27) with

1) \[
U(t) = \begin{bmatrix} u_1(t) & v_1(t) \\ u_2(t) & v_2(t) \end{bmatrix}
\]
\[\text{(1.50)}\]

the principal matrix solution of
\[
\dot{\overline{u}} = A(t) \overline{u}
\]
\[\text{(1.51)}\]

2) \( u_1(T/4) \neq 0 \).

3) \[
\overline{h}(t, s) = \begin{bmatrix} h_1(t, s) \\ h_2(t, s) \end{bmatrix}
\]
\[\text{(1.52)}\]

where
\[
h_1(t, s) = \begin{cases}
\frac{u_1(s)[u_1(T/4)v_1(t)-u_1(t)v_1(T/4)]}{d(T/4, s)} & 0 \leq s \leq t \leq T/4 \\
\frac{u_1(t)[u_1(T/4)v_1(s)-v_1(T/4)u_1(s)]}{d(T/4, s)} & 0 \leq t \leq s \leq T/4
\end{cases}
\]
\[\text{(1.53)}\]

\[
h_2(t, s) = \begin{cases}
\frac{u_1(s)[u_1(T/4)v_2(t)-u_2(t)v_1(T/4)]}{d(T/4, s)} & 0 \leq s \leq t \leq T/4 \\
\frac{u_2(t)[u_1(T/4)v_1(s)-v_1(T/4)u_1(s)]}{d(T/4, s)} & 0 \leq t \leq s \leq T/4
\end{cases}
\]
\[\text{(1.54)}\]
\[ d(T/4, s) = u_1(T/4)[u_1(s)v_2(s) - v_1(s)u_2(s)] \]

4) \( \overline{u} \) governed by
\[
\frac{\dot{\overline{u}}}{\overline{u}} = A(t)\overline{u} + \overline{c}(\overline{u}, t) \quad 0 < t < T/4
\]
\[ \dot{\overline{u}}(0) = 0 \quad u(T/4) = 0 \]  \hspace{1cm} (1.55)

then
\[
\overline{u}(t) = \int_0^{T/4} c(u, \dot{u}, s) h(t, s) ds
\]  \hspace{1cm} (1.57)

**Proof:** The initial value problem with \( \dot{\overline{u}}(0) = 0 \) gives
\[
\overline{u} = U(t)[u_0] + \int_0^t U(t)U^{-1}(s)c(u, u, s)ds
\]  \hspace{1cm} (1.58)

The unknown \( u_0 \) must satisfy
\[
u(T/4) = 0 = u_1(T/4)u_0 + \int_0^{T/4} \left[ \frac{u_1(s)v_1(T/4) - v_1(s)u_1(T/4)}{[u_1(s)v_2(s) - v_2(s)u_1(s)]} \right] c(u, \dot{u}, s)ds
\]  \hspace{1cm} (1.59)

where the first component of the vector equation (1.58) was used.

Solving for \( u_0 \) yields
\[
u_0 = \int_0^{T/4} \left[ \frac{v_1(s)u_1(T/4) - u_1(s)v_1(T/4)}{d(T/4, s)} \right] c(u, \dot{u}, s)ds
\]  \hspace{1cm} (1.60)

Substituting this expression for \( u_0 \) into (1.58) and simplifying yields
(1.57), the desired result.

The Fredholm integral equations (1.31), (1.48), and (1.57) are special cases of a nonlinear operator equation
Functional analysis and topology provide useful results concerning the solutions of such equations. References in this area include Kolmogorov and Fomin\(^ {10} \), Liusternik and Sobolev\(^ {11} \), and Saaty and Bram\(^ {12} \).

1.4. Fixed Point Theorems.

The various fixed point theorems are extremely useful in establishing sufficient conditions for the existence of solutions of (1.61). As noted by Saaty\(^ {13} \), these theorems fall into two classes, the topological fixed point theorems which merely establish existence, and the algebraic fixed point theorems which also yield uniqueness and a means of constructing the solution. Both types are useful in establishing the validity of approximate periodic solutions.

In this work, consideration is restricted to nonlinear operator equations on a complete normed linear space, i.e., a Banach space.

The following topological fixed point theorem, proved on page 45 of Saaty and Bram\(^ {12} \), is an extension of Brouwer's fixed point theorem to a Banach space.

Theorem 1.1 (Schauder-Leray):

If \( B \) is a closed convex subset of a Banach space \( X \) and \( T \) is a continuous operator mapping \( B \) into itself such that the image of \( B \) is compact, then there exists at least one fixed point \( x^* \in B \) satisfying

\[
x^* = Tx^* \tag{1.62}
\]
Corollary:

If in Theorem (1.1)

1) \( B = B_r = \{ x | \| x \| \leq r \} \), a ball of radius \( r \) centered at the zero element. Here \( \| \cdot \| \) represents a norm.

2) there exists a real number \( \gamma \geq 0 \) and a \( \beta = \beta(r) \) satisfying \( 0 < \beta < 1 \), such that

\[
\| T x \| \leq \gamma + \beta \| x \| \quad (1.63)
\]

for all \( x \in B_r \).

then there exists at least one \( x^* \in B_r \) satisfying (1.62) and

\[
\| x^* \| \leq \frac{\gamma}{1-\beta} \quad (1.64)
\]

Proof: By Theorem (1.1) there exists an \( x^* \in B_r \) satisfying (1.62).

Taking the norm of (1.62) and using (1.63) yields

\[
\| x^* \| \leq \gamma + \beta \| x^* \| \quad (1.65)
\]

or, since \( 0 < \beta < 1 \)

\[
\| x^* \| \leq \frac{\gamma}{1-\beta} \quad (1.66)
\]

A constructive fixed point theorem is provided by the principle of contraction mappings.

Definition 1.4: An operator \( T \) mapping a normed space \( S \) into itself is a contraction mapping on \( S \) if there exists an \( \alpha \) satisfying \( 0 < \alpha < 1 \) such that for all \( x, y \in S \)

\[
\| T x - T y \| \leq \alpha \| x - y \| \quad (1.67)
\]

A fundamental theorem concerning contraction mapping is
Theorem 1.2:

Every contraction mapping $T$ on a Banach space $X$ has a unique fixed point $x^* \in X$ satisfying $x^* = Tx^*$. For any $x_0 \in X$, the sequence of iterates $x_n = Tx_{n-1} = T^n x_0$ converges to $x^*$, and

$$\|x_n - x^*\| \leq \frac{\alpha^n}{1-\alpha} \|T x_0 - x_0\| \tag{1.68}$$

Proof: This theorem is the special case of Theorem 1.3, for $r \to \infty$.

This theorem requires the mapping to contract on the whole space $X$. Unfortunately, most operators do not satisfy this condition, so it is necessary to extend the theorem to particular subsets of $X$.

Intuitively, one would expect a mapping to contract in the neighborhood of a point $x_0$ if $x_0$ is "close enough" to a fixed point. That is, if $\|x_0 - x\|$ is small enough. The following theorem, given without proof on page 42 of Baily, Shampine and Waltman (14), extends the contraction mapping principle to the case where the mapping contracts on a ball and also shows how to find the radius of the ball.

Theorem 1.3 (Contraction Mapping on a Ball):

If for an operator $T$, there exists a number $r$ such that

1) $T$ maps the ball $B_r = \{w \mid \|w-x_0\| \leq r\}$ of a Banach space $X$ into $X$.

2) there exists an $\alpha = \alpha(r)$ satisfying $0 < \alpha < 1$ and

$$\|Tx - Ty\| \leq \alpha \|x-y\| \tag{1.69}$$

for all $x, y \in B_r$. 
then \( T \) has a unique fixed point \( x^* \in B_r \) satisfying \( x^* = Tx^* \), and \( x^* \) is the
limit of the sequence \( x_n = T x_{n-1} = T^n(w_0) \) where \( w_0 \) is any point in \( B_r \).

**Proof:** Since \( X \) is a Banach space, every Cauchy sequence in \( X \) converges to a limit point in \( X \). The following argument shows that \( x_n \)
defined by
\[
x_n = T x_{n-1} = T^n x_0
\]
is a Cauchy sequence, and hence has a limit point in \( X \).

For any \( n \) and \( k \)
\[
||x_{n+k} - x_n|| = ||x_{n+k} - x_{n+k-1} + x_{n+k-1} - x_{n+k-2} + \ldots + x_n - x_{n-1}||
\]
Using the triangle inequality
\[
||x_{n+k} - x_n|| < ||x_{n+k} - x_{n+k-1}|| + ||x_{n+k-1} - x_{n+k-2}|| + \ldots + ||x_{n+1} - x_n||
\]
Now for any \( i \)
\[
x_{i+1} - x_i = T x_i - T x_{i-1}
\]
\[
||x_{i+1} - x_i|| \leq \alpha ||x_i - x_{i-1}|| \leq \alpha^2 ||x_{i-1} - x_{i-2}|| \leq \ldots \leq \alpha^i ||x_1 - x_0||
\]
Hence
\[
||x_{n+k} - x_n|| \leq \left( \alpha^{n+k-1} + \alpha^{n+k-2} + \ldots + \alpha^n \right) ||x_1 - x_0||
\]
\[
\leq \alpha^n \left( \alpha^{k-1} + \alpha^{k-2} + \ldots + 1 \right) ||x_1 - x_0||
\]
\[
\leq \alpha^n ||x_1 - x_0|| \sum_{k=0}^{\infty} \alpha^k = \frac{\alpha^n}{1-\alpha} ||T x_0 - x_0||
\]
(1.72)
Since the right hand side does not depend upon k, \( x_n \) is a Cauchy sequence and hence it converges to a limit point \( x^* \in B_r \) provided that all of the iterates remain in \( B_r \). Letting \( n=0 \) in (1.72) yields

\[
\|x_k - x_0\| \leq \frac{\|Tx_0 - x_0\|}{1-\alpha}
\]

Hence \( x_k \in B_r \) for all \( k \) if

\[
r \geq \frac{\|Tx_0 - x_0\|}{1-\alpha}
\]

(1.73)

Now for \( k \to \infty \) in (1.72)

\[
\|x^* - x_n\| \leq \frac{\alpha}{1-\alpha} \|Tx_0 - x_0\|
\]

(1.74)

which provides a bound on the error for the \( n^{th} \) iterate and shows that \( \|x^* - x_n\| \to 0 \) as \( n \to \infty \). Hence, by Hypothesis 2) \( \|T^k x - T x_n\| \to 0 \) as \( n \to \infty \), so \( T \) is continuous and hence

\[
\lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} Tx_n
\]

\[
x^* = T \lim_{n \to \infty} x_n
\]

\[
x^* = Tx^*
\]

Mathematical induction will now be used to show that the iteration scheme may begin at any point \( w_0 \in B_r \). Assume \( w_k \in B_r \) for all \( k \leq n-1 \). Then

\[
\|w_n - x_0\| = \|Tw_{n-1} - Tx_0 + Tx_0 - x_0\|
\]

\[
< \|Tw_{n-1} - Tx_0\| + \|Tx_0 - x_0\|
\]

using (1.69) and (1.70)

\[
\leq \alpha \|w_{n-1} - x_0\| + r(1-\alpha)
\]
\[ \|w_n - x_0\| \leq \alpha r + r(1 - \alpha) = r \]

\[
\therefore w_n \in B_r \text{ for all } n. 
\]

Using arguments similar to those used for the sequence \( x_n \), it is easy to show that

\[ \|w^* - w_n\| \leq \frac{2r\alpha^n}{1 - \alpha} \]

and hence \( w_n \) is a Cauchy sequence converging to \( w^* \in B_r \), where \( w^* = Tw^* \).

Now suppose there are two fixed points \( x^* \in B_r \) and \( w^* \in B_r \). Then

\[ \|w^* - x^*\| \leq \alpha\|w^* - x^*\| \]

with \( 0 < \alpha < 1 \), so \( \|w^* - x^*\| = 0 \) implies \( w^* = x^* \).

1.5. Comments on Uniqueness.

Since the conditions satisfied by \( r \) in Theorem 1.3 are inequalities, there may be a range of suitable values of \( r \). Of course, there may be no values at all in which case contraction is not assured on any ball centered at \( x_0 \). Assume for the moment that \( r_{\min} \) is the minimum value which satisfies the conditions of Theorem 1.3, and that \( r_{\max} \) is the maximum value. Let \( B_{\min} = \{ w : \|w - x_0\| \leq r_{\min} \} \) and \( B_{\max} = \{ w : \|w - x_0\| \leq r_{\max} \} \). Obviously, there is a unique fixed point \( x^* \in B_{\min} \) and this is the only fixed point inside \( B_{\max} \). If there were another fixed point \( y^* \in B_{\max} \), then uniqueness would be violated for the contraction mapping on \( B_{\max} \). Hence, any other fixed point \( y^* \) must satisfy \( \|y^* - x_0\| > r_{\max} \). In the case \( r_{\max} \to \infty \), the fixed point \( x^* \in B_{\min} \) is the only solution in the whole Banach space.
CHAPTER II

VALIDITY AND STABILITY CRITERIA

The various approximating techniques are discussed by Minorsky (15), Hayashi (16), and Bogoliubov and Mitropolsky (17). This work considers only harmonic balance, with the understanding that equivalent linearization and the method of averaging are essentially one term harmonic balance approximations.

2.1 Validity Criteria.

Consider an equation of the form (1.1) with $g(t)$ harmonic. That is,

$$\ddot{x} + f(x, \dot{x}) = P \cos (\omega t + \varphi)$$

One term harmonic balance yields the approximate solution

$$x_0(t) = A \cos \omega t$$

with amplitude $A$ and phase $\varphi$ satisfying

$$-\omega^2 A + a_1 = P \cos \varphi$$

$$b_1 = -P \sin \varphi$$

where $a_1$ and $b_1$ are the coefficients of $\cos \omega t$ and $\sin \omega t$ respectively, in the Fourier expansion of $f(x_0, \dot{x}_0)$. In vector-matrix form (2.1) is

$$\frac{d\vec{x}}{dt} = A\vec{x} - \overline{f(\vec{x})} + \overline{g(t)}$$

where
with approximate solution

\[ \bar{x}_0(t) = \begin{bmatrix} A \cos \omega t \\ -A \omega \sin \omega t \end{bmatrix} \]

In this chapter, validity criteria are established for the one term approximate solution. Extension to multi-term approximations with more general periodic forcing functions is straightforward. Examples of validity criteria for two term approximations are given in Chapter III.

Suppose that (2.1) allows solutions in a given symmetry class, and that \( x_0(t) \) is in this class. To establish the validity of the approximate solution \( x_0(t) \), the existence of exact solutions in the given symmetry class in a neighborhood of \( x_0(t) \) is sought.

Consider the difference between the exact and approximate solution

\[ \eta(t) = x(t) - x_0(t) \] (2.7)
called the approximation error. The function \( \eta(t) \) satisfies the following equation obtained by substituting (2.7) into (2.1) and using (2.3)

\[ \ddot{\eta} + f(x_0 + \eta, \dot{x}_0 + \dot{\eta}) = a_1 \cos \omega t + b_1 \sin \omega t \] (2.8)

Alternatively, (2.8) is written as

\[ \ddot{\eta} + a(t)\eta + b(t)\eta = a_1 \cos \omega t + b_1 \sin \omega t - f(x_0 + \eta, \dot{x}_0 + \dot{\eta}) + a(t)\eta + b(t)\eta \] (2.9)

where \( a(t) \) and \( b(t) \) are suitably chosen piecewise continuous functions with at most a finite number of finite discontinuities. In vector-matrix
form (2.9) is
\[ \dot{\eta} = A(t)\eta + d(\eta, t) \]
where \( A(t) \) is given by (1.26) and
\[
\begin{align*}
\eta &= [\eta] ; \quad d(\eta, t) = \begin{bmatrix}
0 \\
a_1 \cos wt + b_1 \sin wt + a(t)\eta + b(t)\eta \\
-f(x_0 + \eta, \dot{x}_0 + \dot{\eta})
\end{bmatrix}
\end{align*}
\]

Now using Lemma 1.7, 1.8, or 1.9, depending upon the symmetry class desired, (note that \( x_0(t) \) must be in the class and the equation must allow solutions in the class) gives
\[ I = \int_{0}^{\infty} h(t, s)\left[a_1 \cos ws + b_1 \sin ws - f(x_0 + \eta, \dot{x}_0 + \dot{\eta}) + a(s)\eta + b(s)\eta \right]ds \]  
(2.10)

where \( h(t, s) \) is the appropriate kernel and I is the interval of the B.V.P.

An alternate form of (2.10) is
\[ I = \int_{0}^{\infty} h(t, s)\left[a_1 \cos ws + b_1 \sin ws - f(x_0, \dot{x}_0) \right]ds \]
\[ + \int_{0}^{\infty} h(t, s)\left[f(x_0, \dot{x}_0) + a(s)\eta + b(s)\eta - f(x_0 + \eta, \dot{x}_0 + \dot{\eta}) \right]ds \]  
(2.11)

Since the first integral on the right hand side, denoted by
\[ \bar{e}(t) = \begin{bmatrix} e(t) \\ \dot{e}(t) \end{bmatrix} \]  
(2.12)

does not contain \( \eta \), it can be evaluated for any given \( f \). Hence \( \bar{\eta} \) is governed by the following Fredholm integral equation
\[ \bar{\eta} = \bar{e} + \int_{0}^{\infty} h(t, s)\left[f(x_0, \dot{x}_0) + a(s)\eta + b(s)\eta - f(x_0 + \eta, \dot{x}_0 + \dot{\eta}) \right]ds \]  
(2.13)
which is just a nonlinear operator equation of the type (1.62). For purposes of notation (2.13) is written as

\[ \bar{\eta} = \bar{e} + H \bar{\eta} = N \bar{\eta} \]  

(2.14)

Solutions are sought in the Banach space V of continuous two dimensional real vector valued functions on the interval I with norm

\[ \| \bar{v}(t) \| = \sup_{0 \leq t \leq 1} \sqrt{v_1^2(t) + v_2^2(t)} \]  

(2.15)

where

\[ \bar{v}(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} \]  

(2.16)

There will also be occasion to use the Banach space C of continuous real valued functions on I with norm

\[ \| y(t) \| = \sup_{0 \leq t \leq 1} |y(t)| \]  

(2.17)

Consider (2.14). The operator N maps the ball \( B_r = \{ \bar{w} : \| \bar{w} \| \leq r \} \) of V into V, so application of Theorem 1.3 to the operator N on V yields sufficient conditions for the existence of a solution \( \bar{\eta} \in B_r \). Obviously, this implies the existence of an exact solution \( \bar{x}(t) \) of (2.4) within r of \( \bar{x}_0(t) \). This results in

**Validity Criteria I**: Given an equation of the form (2.1), let \( \bar{x}_0(t) \), \( \bar{e}(t) \) and N be as previously defined.

If a subset \( B_r \) of V can be found such that

1) there exists an \( \alpha = \alpha(r) \) satisfying \( 0 < \alpha < 1 \) and

\[ \| N \bar{x} - N \bar{y} \| \leq \alpha \| \bar{x} - \bar{y} \| \]  

(2.18)

for all \( \bar{x}, \bar{y} \in B_r \).
then there exists an exact solution \( \bar{x}(t) \) of (2.1) of the form
\[
\bar{x}(t) = x_0(t) + \bar{\eta}(t)
\] (2.20)
where \( \bar{\eta}(t) \in B_{r} \) is the limit of the sequence
\[
\bar{\eta}_{i+1} = N_{i} \bar{\eta}_{i} \quad \bar{\eta}_{0} = 0
\] (2.21)
and
\[
\|\bar{\eta}_{i+1} - \bar{\eta}_{i}\| \leq \alpha_{i} \|e\| \left\frac{1}{1-\alpha}\right
\] (2.22)
In particular,
\[
\|\bar{x}(t) - \bar{x}_{0}(t)\| \leq r_{\min}
\] (2.23)
where \( r_{\min} \) is the smallest value of \( r \) satisfying (2.19).

When the sufficient conditions for validity are satisfied, an algorithm (2.21) for computing the exact solution to any desired accuracy has been derived. In particular, using (2.22)
\[
\|\bar{x} - (x_{0} + \bar{\eta}_{m})\| \leq \alpha_{m} \|e\| \left\frac{1}{1-\alpha}\right
\] (2.24)
and
\[
\bar{x}(t) = x_{0}(t) + \lim_{m \rightarrow \infty} \bar{\eta}_{m}
\] (2.25)
where \( \bar{\eta}_{m} \) is given by (2.21).

Also, since \( \bar{\eta}_{1} = N_{0} - e \), the readily available improved first approximation, i.e., \( \bar{x} \approx x_{0} + e \) satisfies
\[
\|\bar{x} - (x_{0} + e)\| \leq \alpha \|e\| \left\frac{1}{1-\alpha}\right \leq \alpha r_{\min}
\] (2.26)
For the special case of \( f(x, \dot{x}) \) linear in \( \dot{x} \), that is
\[
f(x, \dot{x}) = C\dot{x} + k(x)
\]
(2.27)
it is possible to derive sharper criteria. In (2.13) let \( a(t) = C \) to obtain
\[
\bar{\eta} = \bar{e} + \int_{0}^{1} \bar{h}(t, s) [k(x_0) + b(s)\eta - k(x_0 + \eta)] \, ds
\]
(2.28)
or, in component-operator form
\[
\eta = e + H_1 \eta = N_1 \eta
\]
\[
\dot{\eta} = \dot{e} + H_2 \eta = N_2 \eta
\]
(2.29)
Since the right hand sides do not depend upon \( \eta \), it is possible to apply Theorem 1.3 to the first of (2.29). This leads to

Validity Criteria I-A: Given an equation of the form (2.1) with \( f(x, \dot{x}) \) as in (2.27). Let \( x_0(t), N_1, N_2, \) and \( e(t) \) be as previously defined.

If a subset \( B_r = \{ w \mid \| w \| \leq r \} \) of \( C \) can be found such that

1) there exists an \( \alpha = \alpha(r) \) satisfying \( 0 < \alpha < 1 \) and
\[
\| N_1 x - N_1 y \| \leq \alpha \| x - y \|
\]
(2.30)
for all \( x, y \in B_r \).

2)
\[
r \geq \| e \| \frac{1}{1-\alpha}
\]
(2.31)
then there exists an exact solution of (2.1) of the form
\[
x(t) = x_0(t) + \eta(t)
\]
(2.32)
where \( \eta \in B_r \) is the limit of the sequence
\[
\eta_{i+1} = N_1 \eta_i \quad \eta_0 = 0
\]
(2.33)
and
In particular
\[ \| x - x_0 \| \leq r_{\text{min}} \] (2.35)

where \( r_{\text{min}} \) is the smallest value of \( r \) satisfying \( (2.31) \).

Also, if the validity criteria are satisfied, and there exists a \( \delta = \delta(r) \) such that
\[ \| N_2 x - N_2 y \| \leq \delta \| x - y \| \] (2.36)

for all \( x, y \in B_r \), then
\[ \| \eta_i - \eta_1 \| \leq \frac{\delta \| e \|}{1 - \alpha} \tag{2.37} \]

and if there exists an \( \epsilon = \epsilon(r) \) such that
\[ \| H_2 x \| \leq \epsilon \| x \| \] (2.38)

for all \( x \in B_r \), then
\[ \| \eta_i \| \leq \| \epsilon \| + \frac{\epsilon \| e \|}{1 - \alpha} \] (2.39)

Application of Theorem (1.3) is as follows: Since \( N_1 \) maps \( B_r \) into \( C \), condition 1 is satisfied. Equations \( (2.30) \) and \( (2.31) \) are the same as conditions 2 and 3 respectively, and results up to \( (2.33) \) are direct consequences of the conclusions of the theorem. Equation \( (2.34) \) follows from \( (1.74) \). Taking the norm of the second of \( (2.29) \) and using \( (2.37) \) gives
\[ \| \eta_i \| \leq \| \epsilon \| + \| H_2 \eta_i \| \leq \| \epsilon \| + \epsilon \| \eta_i \| \] (2.40)

or, using \( (2.34) \) with \( i=0 \),
\[ \| \hat{\eta} \| \leq \| \delta \| + \frac{\varepsilon \| e \|}{1 - \alpha} \]  

(2.41)

which is the same as (2.39).

Then

\[ \hat{\eta}_i - \hat{\eta}_{i-1} = N_2 \eta_i - N_2 \eta_{i-1} \quad i \geq 1 \]  

(2.42)

and using (2.34) and (2.36) yields

\[ \| \hat{\eta}_i - \hat{\eta}_{i-1} \| \leq \delta \| \eta_i - \eta_{i-1} \| \leq \frac{\alpha^{i-1} \| e \|}{1 - \alpha} \quad i \geq 1 \]  

(2.43)

When Validity Criteria I or I-A are applicable, they yield, in addition to a bound on the approximation error,

1) an algorithm, \( \hat{A} \) or \( \hat{A} \), for computing the solution to any desired accuracy.

2) uniqueness of the approximated solution in the given symmetry class within the ball \( B_{max} \) of radius \( r_{max} \) where \( r_{max} \) is the largest value of \( r \) satisfying the validity criteria.

Slightly sharper validity criteria which yield only existence of an exact solution near the approximation and a bound on the approximation error are obtained using Theorem 1.1 and its corollary. Application to the special case (2.27) yields the following:

Validity Criteria II-A: Given an equation of the type (2.1) where \( f(x, \hat{x}) \) has the form (2.27), let \( x_0, N_1, N_2, H_1, H_2 \) and \( e(t) \) be as previously defined.

If a subset \( B_r = \{ w \mid \| w \| \leq r \} \) of \( C \) can be found such that
1) there exists a real number $\gamma \geq 0$ and a $\beta = \beta(r)$ satisfying $0 < \beta < 1$ and
\[
\left\|N_1 x\right\| < \beta \left\|x\right\| + \gamma
\]
for all $x \in B_r$.

2) $\gamma + \beta r < r$ \hspace{1cm} (2.45)

3) there exists real numbers $\delta \geq 0$ and an $\epsilon = \epsilon(r)$ such that
\[
\left\|N_2 x\right\| \leq \delta + \epsilon \left\|x\right\|
\]
for all $x \in B_r$.

then there exists an exact solution of (2.1) of the form
\[
x(t) = x_0(t) + \eta(t)
\]
where
\[
\left\|\eta(t)\right\| = \left\|x(t) - x_0(t)\right\| \leq \frac{\gamma}{1 - \beta}
\]
and
\[
\left\|\eta^*(t)\right\| = \left\|\dot{x}(t) - \dot{x}_0(t)\right\| \leq \delta + \frac{\epsilon \gamma}{1 - \beta}
\]

Application of Theorem 1.1 is as follows: The ball $B_r$ is a closed convex subset of $C$. The continuity of $h_1(t, s)$ assures that $N_1$ is a continuous mapping, while conditions 1 and 2 assure that $B_r$ is mapped into itself. Hence the images of all $x \in B_r$ under the mapping $N_1$ are uniformly bounded in norm by $r$. In addition, condition 3 assures that the images of all $x \in B_r$ have uniformly bounded first derivatives. Using the mean value theorem
\[
\left\|x(t_2) - x(t_1)\right\| \leq \dot{\epsilon}(\zeta) \left|t_2 - t_1\right|
\]
(2.50)
where \( t_1 < \zeta < t_2 \), and \( \hat{x}(\zeta) \) is uniformly bounded. Hence all functions in the image of \( B_r \) are uniformly bounded and equicontinuous, so by Arzela's Theorem (see page 54 of Kolmogoroff and Fomin\(^{(10)}\)) the image of \( B_r \) is compact. All of the hypotheses of Theorem (1.1) are fulfilled and hence there is a least one \( \eta(t) \in B_r \) satisfying (2.28). Application of the corollary yields (2.48). Then (2.46) and (2.48) yield (2.49).

Successful application of the validity criteria depends upon the choice of \( a(s) \) and \( b(s) \) in (2.13) or \( b(s) \) in (2.28). These quantities are called the basis for the integral equation. Except for certain exceptional values which yield an unbounded kernel, the choice of basis is unrestricted. Ideally, it would be chosen so as to minimize the bound on the approximation error. However, such a choice is rather involved computationally so several alternative selections are discussed in Chapter III.

Furthermore, in application to specific problems, success depends upon determining a suitable Lipschitz constant \( \alpha(r) \), alternatively called a contraction constant, for Criteria I and I-A, and upon determining suitable \( \beta(r) \) and \( \gamma \) for Criteria II-A. Once these are obtained, the information concerning validity is contained in (2.19), (2.31) and (2.45) for Criteria I, I-A and II-A, respectively. Henceforth, each of these inequalities is referred to as a validity inequality.
2.2 Comparison of Criteria I-A and II-A.

Consider the equation of a simple undamped oscillator with an odd polynomial restoring force

\[ x + K \left( x + \epsilon_1 x^3 + \epsilon_2 x^5 \right) = P \cos \omega t \quad K, \epsilon_1, \epsilon_2 > 0 \]  

(2.51)

The one term harmonic balance approximate solution is

\[ x_0(t) = A \cos \omega t \]  

(2.52)

where \( A > 0 \) satisfies

\[ A(K - \omega^2) + \frac{3}{4} \epsilon_1 K A^3 + \frac{5}{8} K \epsilon_2 A^5 = P \]  

(2.53)

Solutions which are out of phase with the forcing function correspond to \( P < 0 \).

Since (2.51) does not contain a derivative term, the approximation error satisfies an integral equation of the form (2.28) with

\[ k(x) = K \left( x + \epsilon_1 x^3 + \epsilon_2 x^5 \right) \]  

(2.54)

Now \( x_0 \in \Omega_T \) and (2.51) allows solutions in \( \Omega_T \), so \( h_1(t, s) \) in (2.28) is given by (1.53). For convenience, take \( b(s) = K_* \), an as yet unspecified constant, in which case

\[ h_1(t, s) = \begin{cases} 
-\sin \sqrt{K_*} \left( \frac{\pi}{2w} - t \right) \cos \sqrt{K_*} s & 0 \leq s \leq \frac{\pi}{2w} \\
\sqrt{K_*} \cos \sqrt{K_*} \frac{\pi}{2w} & \frac{\pi}{2w} < s \leq \frac{\pi}{2w} \\
-\sin \sqrt{K_*} \left( \frac{\pi}{2w} - s \right) \cos \sqrt{K_*} t & 0 \leq t \leq \frac{\pi}{2w} \\
\sqrt{K_*} \cos \sqrt{K_*} \frac{\pi}{2w} & \frac{\pi}{2w} < t \leq \frac{\pi}{2w} 
\end{cases} \]  

(2.55)
A straightforward integration gives

\[
e(t) = \frac{K\left(\varepsilon_1 A^3 + \frac{5\varepsilon_2 A^5}{4}\right)}{4\left(K_x - 9\omega^2\right)} \cos 3\omega t + \frac{\varepsilon_2 KA^5}{16\left(K_x - 25\omega^2\right)} \cos 5\omega t \quad (2.56)
\]

and from (2.28) and (2.29)

\[
\frac{\pi}{2\omega} N_1 y = e(t) + \int_0^{\pi/2\omega} h_1(t, s) k(x_0 + K\varepsilon_1 y - K\varepsilon_2 y) ds \quad (2.57)
\]

For Criteria I-A an \(\alpha = \alpha(r)\) is needed satisfying

\[
\|N_1 x - N_1 y\| \leq \alpha\|x - y\| \quad (2.58)
\]

for all \(x, y \in B_r\). Using (2.57) and (2.54), and after some manipulation

\[
\frac{\pi}{2\omega} N_1 x - N_1 y = \int_0^{\pi/2\omega} h_1(t, s) \left[\left(K_x - 3\varepsilon_1 Kx_0^2 - 5\varepsilon_2 Kx_0^4\right) - (x+y)\left(3\varepsilon_1 Kx_0 + 10\varepsilon_2 Kx_0^3\right) - (x^2 + xy + y^2)\left(5\varepsilon_2 Kx_0^3\right)\right] ds \quad (2.59)
\]

Now choose \(K_x\) so that the upper bound on the norm of

\[
\left(K_x - 3\varepsilon_1 Kx_0^2 - 5\varepsilon_2 Kx_0^4\right) \quad (2.60)
\]

is as small as possible, that is

\[
K_x = K + \frac{3}{2} \varepsilon_1 K^2 + \frac{5}{2} \varepsilon_2 K^4 \quad (2.61)
\]

which gives

\[
\|K_x - 3\varepsilon_1 Kx_0^2 - 5\varepsilon_2 Kx_0^4\| \leq \frac{3}{2} \varepsilon_1 K^2 + \frac{5}{2} \varepsilon_2 K^4 \quad (2.62)
\]

Taking the norm of (2.59) and using (2.52), (2.55), (2.62) and \(x, y \in B_r\) yields
By Validity Criteria I-A, \( x_0 \) is valid provided there are values of \( r \) satisfying

\[
\| N_1 x - N_1 y \| \leq \frac{\| x - y \| \pi K}{2 \omega \sqrt{K_*} | \cos \frac{\sqrt{K_* \pi}}{2 \omega} |} \left[ \frac{3}{2} \varepsilon_1 A^2 + \frac{5}{2} \varepsilon_2 A^4 + 2 r \left( 3 \varepsilon_1 A + 10 \varepsilon_2 A^3 \right) + 3 r^2 \left( \varepsilon_1 + 10 \varepsilon_2 A^2 \right) + 20 \varepsilon_2 A r^3 + 5 \varepsilon_2 r^4 \right] \quad (2.63)
\]

\[
\| N_1 x - N_1 y \| = \alpha(r) \| x - y \| \quad (2.64)
\]

where

\[
B = 2 \omega \sqrt{K_*} | \cos \frac{\sqrt{K_* \pi}}{2 \omega} | / \pi
\]

Now for Criteria II-A, \( \gamma \) and \( \beta=\beta(r) \) are needed satisfying

\[
\| N_1 y \| \leq \gamma + \beta y \quad (2.68)
\]

for all \( y \in B_r \). Using \( K_* \) given by (2.61) and the same type of bounding procedure gives

\[
\beta(r) = a_1 r + a_2 r^2 + a_3 r^3 + a_4 r^4 + a_5 r^5 = p(r) \quad (2.66)
\]

where

\[
\begin{align*}
    a_0 &= \| e \| \\
    a_1 &= K A \left( 3 \varepsilon_1 + 5 \varepsilon_2 A^2 \right) / B \\
    a_2 &= K A \left( 3 \varepsilon_1 + 10 \varepsilon_2 A^2 \right) / B \\
    a_3 &= K \left( \varepsilon_1 + 10 \varepsilon_2 A^2 \right) / B \\
    a_4 &= 5 K \varepsilon_2 A / B \\
    a_5 &= K \varepsilon_2 / B
\end{align*}
\]

(2.67)

\[
\gamma = \| e \| \quad (2.70)
\]
By Validity Criteria II-A, \( x_0 \) is known to be valid provided there are values of \( r \) satisfying

\[
r \geq a_0 + a_1 r + a_2 r^2 + a_3 r^3 + a_4 r^4 + a_5 r^5 = q(r)
\]

(2.71)

where the \( a_i \), \( i = 0, 5 \), are given by (2.67). Figures 2.1a, b show graphs of the right and left hand sides of (2.66) and (2.71) for different values of the parameters. Obviously \( p(r) \) always lies above \( q(r) \) so \( r_1 < r_2 \)
(see Figure 2.1a) and Criteria II-A yields a smaller bound on the approximation error. In fact, the situation depicted in Figure 2.1b may develop, wherein II-A yields a bound and I-A is not applicable.

On the other hand, when I-A applies, it yields an algorithm for computing the exact solution and uniqueness of the approximated solution in the given symmetry class within the ball of radius \( r_3 \) (see Figure 2.1a).

2.3 Validity Boundaries.

A question of particular interest is: Given system parameters \( K, \xi_1, \xi_2 \) for what excitations \( P, \omega \) do the validity criteria guarantee a valid approximate solution.

Consider the situation shown in Figure 2.1c. There is only one value of \( r \) which satisfies \( r \geq p(r) \), and for a slight change in the excitation there may be no solutions. Values of the excitation for which such a situation occurs will clearly lie on boundaries between valid and "questionable" regions. Where the validity criteria are not applicable, there may still be exact solutions near the approximations, but their existence cannot be guaranteed by the methods used here.
Fig. 2.1  PLOTS OF VALIDITY INEQUALITIES
It is convenient to calculate the validity boundaries as functions of $A$ and $w$ rather than $P$ and $w$. Recall that $A$, $P$ and $w$ are related by (2.53). The development is presented for Criteria I-A. Validity boundaries for II-A are determined in an analogous manner.

On a validity boundary

$$r = p(r)$$

$$1 = p'(r)$$

Given $A$, values of $w$ satisfying the above two equations are found as follows: Since $p'(r)$ is monotone increasing when $r > 0$, for $p'(0) < 1$ there is always a unique positive root of (2.73). Solving for given $w$ yields $r = r(w)$. The transcendental equation

$$r(w) = p(r(w))$$

obtained from (2.72) is then solved yielding values of $w$ on validity boundaries. Figure (2.2) shows validity boundaries for Criteria I-A and II-A applied to (2.51) with $K=1.0$, $\xi_1 = 0.1$ and $\xi_2 = 0.02$. Also shown are response curves for several values of $P$. Indications are that the approximate solution is very good on the lower response curve for $w > 1$ and also for certain ranges of frequency when $w < 1$ and $P$ is small. Of course, the validity criteria are only sufficient conditions so the approximate solution may still be quite good even though its validity cannot be demonstrated by these methods.

In addition to validity, there is also the question of whether the approximate solution is an approximation to a stable solution. When the approximation cannot be shown valid, the stability question cannot
Fig. 2.2 VALIDITY BOUNDARIES - POLYNOMIAL RESTORING FORCE
be answered rigorously, and even when it is valid, a stability analysis based on the approximate solution is not exact. A precise analysis must account for the fact that the exact solution is known only to within the bound obtained for the approximation error.

2.4 Sufficient Stability Criteria.

Consider again, an equation of the form (1.1). The stability question for a periodic solution \( \bar{x}(t) \) is answered by the equation of first variation (see Struble\(^9\) page 149)

\[
\dot{y} + \frac{\partial f}{\partial \dot{x}}(x, \dot{x})\dot{y} + \frac{\partial f}{\partial x}(x, \dot{x})y = 0
\]  

(2.75)

If this equation has any unbounded solutions then \( \bar{x}(t) \) is unstable, if all solutions are bounded, then \( \bar{x}(t) \) is stable, and if all solutions tend to zero as \( t \to \infty \), then \( \bar{x}(t) \) is asymptotically stable. Equation (2.75) is a Hill's equation of the form

\[
\dot{y} + q_1(t)\dot{y} + q_2(t)y = 0
\]  

(2.76)

where \( q_1(t) \) and \( q_2(t) \) are periodic with common period \( P \). Floquet theory says that solutions of (2.76) are of the form

\[
y(t) = \varphi_1(t)e^{i\lambda_1 t} + \varphi_2(t)e^{i\lambda_2 t}
\]  

(2.77)

where \( \varphi_1 \) and \( \varphi_2 \) are periodic. The boundedness of solutions is hence determined by \( \lambda_1 \) and \( \lambda_2 \) called the characteristic exponents. Hahn\(^7\), page 305, shows that (2.76) is transformed to

\[
z + p(t)z = 0
\]  

(2.78)

\[
p(t) = q_2(t) - \frac{1}{4}q_1^2(t) - \frac{1}{2}q_1(t)
\]  

(2.79)
by the transformation

\[ y = z e^{-\frac{1}{2} \int_0^t q_1(s) ds} \quad (2.80) \]

Since (2.78) is also a Hill's equation it has solutions of the form (2.77). Hahn also shows that the characteristic multipliers \( \mu_1 \) and \( \mu_2 \) given by

\[ \mu_1 = e^{i \nu_1 P}, \quad \mu_2 = e^{i \nu_2 P} \quad (2.81) \]

where \( \nu_1 \) and \( \nu_2 \) are the characteristic exponents of (2.78) satisfy

\[ u^2 - a \mu + 1 = 0 \quad (2.82) \]

where

\[ a = \left( u_1(P) + u_2(P) \right) \quad (2.83) \]

and

\[ U(t) = \begin{bmatrix} u_1(t) & \nu_1(t) \\ u_2(t) & \nu_2(t) \end{bmatrix} \quad (2.84) \]

is the principal matrix solution of (2.78). Solving (2.82) gives

\[ \mu_{1,2} = a \pm \sqrt{\frac{a^2}{4} - 1} \quad (2.85) \]

from which it is easy to see that \( \mu_1 \mu_2 = 1 \) and hence from (2.81)

\[ e^{i \nu_1 P} = e^{-i \nu_2 P} \quad (2.86) \]

Thus \( \nu_1 = -\nu_2 \mod \frac{2\pi}{P} \). Letting \( \nu = \nu_1 = -\nu_2 \mod \frac{2\pi}{P} \) and using (2.81) and (2.86) gives

\[ \text{Im}[\nu] = \frac{1}{P} \tan \frac{1}{|\mu_1|} = \frac{1}{P} \tan |\mu_2| \quad (2.87) \]

\[ \text{Re}[\nu] = \frac{1}{P} (\text{Arg} \mu_1 + 2m\pi) = \frac{1}{P} (\text{Arg} \mu_2 + 2m\pi) \quad (2.88) \]

for some integers \( m \) and \( n \). Now if \( a^2 \geq 4 \), then \( \mu_1 \) and \( \mu_2 \) are both real.
and positive, so \( \arg \mu_1 = \arg \mu_2 = 0 \), while if \( a^2 < 4 \), then \( |\mu_1| = |\mu_2| = 1 \) so \( \text{Im}\{v\} = 0 \). This leads to

\[
\nu = \begin{cases} 
\frac{i}{P} \ln \frac{1}{|\mu_1|} & \text{for } a^2 \geq 4 \\
\frac{1}{P} (\arg \mu_1 + 2n\pi) & \text{for } a^2 < 4
\end{cases} \tag{2.89}
\]

now let

\[
q_1(s) = \hat{q}_1 + \tilde{q}_1(s) \tag{2.90}
\]

where

\[
\hat{q}_1 = \frac{1}{P} \int_0^P q_1(s) ds \tag{2.91}
\]

Then the characteristic exponents of (2.76) are

\[
\lambda_{1,2} = \pm \nu + \frac{\hat{q}_1 P}{2} \tag{2.92}
\]

If \( q_1(t) = \frac{\partial f}{\partial x}(x, \dot{x}) \) and \( q_2(t) = \frac{\partial f}{\partial \dot{x}}(x, \dot{x}) \), then the stability of \( \bar{x}(t) \) is related to \( \lambda_{1,2} \) as given in Table 2.1. All except the last entry are consequences of (2.77) and the remarks following (2.75). The last case in the table involves the question of coexistence of \( P \) or \( 2P \) periodic solutions of Hill's equation (see Magnus and Winkler (13) page 5).

The above analysis provides an answer to the stability question for \( \bar{x}(t) \) in terms of solutions of the equation of first variation over a single period \( P \). Solving a Hill's equation to determine \( u_1(P) \) and \( v_2(P) \) is not a simple matter, even when \( q_1 \) and \( q_2 \) are known exactly. When using the equation of first variation, \( \bar{x} \) and hence the periodic coefficients are known only approximately, thus adding further
Characteristic Exponents $\lambda_1$ and $\lambda_2$

<table>
<thead>
<tr>
<th>$\text{Im} \lambda_1$ and $\text{Im} \lambda_2$</th>
<th>Relation of $\lambda_1$ and $\lambda_2$</th>
<th>Stability of $\mathbf{x}(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{Im} \lambda_1 &lt; 0$ and $\text{Im} \lambda_2 &lt; 0$</td>
<td>unstable</td>
<td></td>
</tr>
<tr>
<td>$\text{Im} \lambda_1 &gt; 0$ and $\text{Im} \lambda_2 &gt; 0$</td>
<td>asymptotically stable</td>
<td></td>
</tr>
<tr>
<td>$\text{Im} \lambda_1 &gt; 0$ and $\text{Im} \lambda_2 = 0$ or $\text{Im} \lambda_1 = 0$ and $\text{Im} \lambda_2 &gt; 0$</td>
<td>stable</td>
<td></td>
</tr>
<tr>
<td>$\text{Im} \lambda_1 = \text{Im} \lambda_2 = 0$</td>
<td>$\lambda_1 \neq \lambda_2 \mod \frac{2\pi}{P}$</td>
<td>stable</td>
</tr>
<tr>
<td>$\text{Im} \lambda_1 = \text{Im} \lambda_2 = 0$</td>
<td>$\lambda_1 = \lambda_2 \mod \frac{2\pi}{P}$ (occurs only when $q_1 = 0$)</td>
<td>unstable except when $v_1(P) = u_2(P) = 0$</td>
</tr>
</tbody>
</table>

Table 2.1. Stability from the Characteristic Exponents.
complications. A widely accepted simplification consists of analyzing the stability of the approximate solution (see, for instance, Hayashi's book\(^{(16)}\)). Such an analysis is never precise and may sometimes lead to unjustified results as shown by an example in Chapter III.

When the validity criteria are applicable, there is a possibility of obtaining sufficient stability conditions. The equation of first variation may be written as

\[
\dot{y} + \frac{\partial f}{\partial x}(x_0 + \eta, x_0 + \eta)\ddot{y} + \frac{\partial f}{\partial x}(x_0 + \eta, x_0 + \eta)y = 0 \tag{2.93}
\]

which is a Hill's equation of the form

\[
\dot{y} + (q_1(t) + \zeta_1(t))\ddot{y} + (q_2(t) + \zeta_2(t))y = 0 \tag{2.94}
\]

where \(q_1(t)\) and \(q_2(t)\) are known periodic functions and \(\zeta_1(t)\) and \(\zeta_2(t)\) are also periodic with the same period and satisfy

\[
\|\zeta_1(t)\| \leq \delta_1 \quad \|\zeta_2(t)\| \leq \delta_2 \tag{2.95}
\]

In some cases, it is possible to determine a range of values for \(u_1(P)\) and \(v_2(P)\). That is, \(u_1(P)\) and \(v_2(P)\) lie within certain intervals. The notation

\[
[u_1(P)] = [\text{MAX}(u_1(P)), \text{MIN}(u_1(P))] \tag{2.96}
\]

\[
[v_2(P)] = [\text{MAX}(v_2(P)), \text{MIN}(v_2(P))] \tag{2.97}
\]

is convenient. Using these interval values, together with (2.83), (2.85) and (2.92) gives \([\text{Im} \lambda_1]\) and \([\text{Im} \lambda_2]\), from which sufficient conditions for stability and instability are deduced. For example, if both intervals contain only positive values, then \(x_0(t)\) approximates an
asymptotically stable solution, while if either interval contains only negative values, then \( x_0(t) \) approximates an unstable solution. An example in which sufficient stability conditions are computed is given in Chapter III.
EXAMPLE: A BILINEAR SYSTEM

Consider the equation

\[ \ddot{x} + F(x) = P \cos \omega t \]  
(3.1)

where

\[ F(x) = \begin{cases} 
K - 1 + x & \text{for } x > 1 \\
Kx & \text{for } |x| \leq 1 \\
1 - K + x & \text{for } x < -1 
\end{cases} \]  
(3.2)

and \( K > 0 \). This restoring force is shown graphically in Figure 3.1.

The change of variables

\[ x = \frac{u}{a} ; \quad P = \frac{R}{K_2 a} ; \quad t = \sqrt{\frac{K}{K_2}} \tau ; \quad K = \frac{K_1}{K_2} ; \quad \omega = \sqrt{\frac{K}{K_2}} \]  
(3.3)

takes the above to the more general form

\[ \ddot{u} + F(u) = R \cos \omega \tau \]  
(3.4)

where

\[ F(u) = \begin{cases} 
K_1 a + K_2 (u - a) & \text{for } u > a \\
K_1 u & \text{for } |u| \leq a \\
-K_1 a + K_2 (u + a) & \text{for } u < -a 
\end{cases} \]  
(3.5)

Since (3.1) is piecewise linear some exact solutions are obtained by "piecing together" solutions for the time segments on which the equation is linear. Generation of these simple solutions, i.e., solutions which cross \(|x| = 1\) only once in a quarter period, is discussed in Appendix A. Subsequently, these exact solutions are used to study the sharpness of bounds on the approximation error.
Fig. 3.1 BILINEAR RESTORING FORCE
3.1. **Validity Criteria.**

The first order harmonic balance approximate solution of (3.1) is

\[ x_0(t) = A \cos \omega t \]  

(3.6)

where \( A > 1 \) satisfies

\[ K_{EQV} - \omega^2 = \frac{P}{A} \]

(3.7)

\[ K_{EQV} = K + \frac{2(1-K)}{\pi} \left\{ \cos^{-1} \frac{1}{A} - \frac{1}{A} \left[ 1 - \frac{1}{A^2} \right]^{1/2} \right\} \]

(3.8)

Solutions which are out of phase with the forcing function occur for \( P < 0 \).

Since \( x_0(t) \in \Omega_T \) and (3.1) allows solutions in \( \Omega_T \) for \( T = \frac{2\pi}{\omega} \), Validity Criteria I-A is used to establish the existence of an exact \( \Omega_T \) solution near \( x_0(t) \). In accordance with the development in Chapter II, the approximation error \( \eta(t) \) satisfies

\[ \eta(t) = e(t) + \int_0^{\pi/2\omega} h_1(t, s) \left[ F(x_0) + b(s) \eta - F(x_0 + \eta) \right] ds \]  

(3.9)

where

\[ e(t) = \int_0^{\pi/2\omega} h_1(t, s) \left[ K_{EQV} x_0 - F(x_0) \right] ds \]  

(3.10)

Take \( b(s) = K_\star \), an as yet unspecified constant. Then \( h_1(t, s) \) is given by (2.55) and
\[
e(t) = \frac{1-K}{K^*_x} + \frac{(K_EQ-V-1)A \cos \omega t}{K^*_x - \omega^2} + \frac{\omega (1-K) \cos \sqrt{K^*_x t}}{\sqrt{K^*_x(K^*_x - \omega^2)} \cos \frac{\sqrt{K^*_x \pi}}{2\omega}}
\]
\[
\times \left\{ \frac{w \cos \left[ \sqrt{\frac{K^*_x}{\omega}} \sin^{-1} \left( \frac{1}{A} \right) \right]}{\sqrt{\frac{K^*_x}{\omega}}} - \left[ A^2 - 1 \right]^2 \sin \left[ \sqrt{\frac{K^*_x}{\omega}} \sin^{-1} \left( \frac{1}{A} \right) \right] \right\}
\]
\[
0 \leq t \leq \frac{1}{\omega} \cos^{-1} \left( \frac{1}{A} \right)
\]
\[
(3.11)
\]
\[
e(t) = \frac{(K_EQ-V-K)}{K^*_x - \omega^2} A \cos \omega t + \frac{\omega (1-K) \sin \sqrt{K^*_x \left( \frac{\pi}{2\omega} - t \right)}}{\sqrt{K^*_x(K^*_x - \omega^2)} \cos \frac{\sqrt{K^*_x \pi}}{2\omega}}
\]
\[
\times \left\{ \frac{w \sin \left[ \sqrt{\frac{K^*_x}{\omega}} \sin^{-1} \left( \frac{1}{A} \right) \right]}{\sqrt{\frac{K^*_x}{\omega}}} - \left[ A^2 - 1 \right]^2 \cos \left[ \sqrt{\frac{K^*_x}{\omega}} \sin^{-1} \left( \frac{1}{A} \right) \right] \right\} \frac{1}{\omega} \cos^{-1} \left( \frac{1}{A} \right) \leq \frac{\pi}{2\omega}
\]
\[
(3.12)
\]
Using the notation of Chapter II,
\[
N_1 y = e(t) + \int_0^{\pi/2\omega} h(t, s) \left[ F(x_0) + K^*_x y - F(x_0 + y) \right] ds
\]
(3.13)
In Appendix B a contraction constant for this operator is derived. That is, \( \alpha = \alpha(r) \) given by (B-11) is such that
\[
\| N_1 y - N_1 x \| \leq \alpha(r) \| y - x \|
\]
for all \( x, y \in B_r \). Hence, the validity inequality is
\[
\| e \| \leq r \alpha(1-r) \leq \alpha(r) \alpha(1-r)
\]
(3.14)
For the present, take \( K^*_x = K^{AVG} \), the time average value of \( F'(x_0) \)
\[
K^{AVG} = K + \frac{2}{\pi} (1-K) \cos^{-1} \left( \frac{1}{A} \right)
\]
(3.15)
Later, various other choices of the basis are discussed.
As in Chapter II, validity boundaries in the \( A, \omega \) plane consist of points satisfying
\[ \|e\| = r - r_0(r) \]  \hspace{1cm} (3.16)  

\[ 1 = \alpha(r) + r\alpha'(r) \]  \hspace{1cm} (3.17)

From (B-11)

\[ \alpha'(r) = \frac{\left[ |1-K_0| - |K-K_0| \right]}{w\sqrt{K}} \cos \left( \frac{K_0 \pi}{2w} \right) \begin{cases} 0 & \frac{1+\sigma r}{A} < 0 \text{ or } > 1 \\ -\frac{\sigma}{A^2 - (1+\sigma r)^2} & 0 \leq \frac{1+\sigma r}{A} \leq 1 \end{cases} \]  \hspace{1cm} (3.18)

Multiplying (3.17) by \( r \) and using (3.16) gives

\[ r(1-\alpha) = r^2 \frac{\partial \alpha}{\partial r} = \|e\| \]  \hspace{1cm} (3.19)

Using (3.18), squaring and rearranging gives

\[ r^4 \left[ |1-K_0| - |K-K_0| \right]^2 + \|e\|^2 w^2 K_0 \cos^2 \left( \frac{K_0 \pi}{2w} \right) \left[ (1+\sigma r)^2 - A^2 \right] = 0 \]  \hspace{1cm} (3.20)

For given \( A \) and \( w \), the unique positive root of the quartic equation (3.20) is obtained yielding \( r = r(w) \). Roots \( w \) of the transcendental equation

\[ \|e\| = r(w) - r(w)\alpha(r(w)) \]  \hspace{1cm} (3.21)

obtained from (3.16) then correspond to points on validity boundaries. Figure 3.2 shows validity boundaries for the case \( K = 0.8 \). In the regions marked I, II, and III the sufficient validity conditions are not satisfied. The heavier solid lines are response curves determined from the approximate solution. In some parts of the shaded regions the approximate solution may still be quite good in the sense that an exact solution exists in its vicinity. However, this to be expected since the validity conditions are merely sufficient and not necessary. Furthermore, in each region there is apparently some type of behavior which
Fig. 3.2 VALIDITY AND STABILITY BOUNDARIES - BILINEAR SYSTEM
can lead to large errors in the approximate solution. In region I, some of the response curves have vertical tangents. If the location of a vertical tangent is not exact, then near the vertical tangent there may be no exact solution near the approximation. In regions II and III the system apparently has ultraharmonic resonances. The resonant response in region II is predicted by a two term harmonic balance approximation developed later in this chapter.

3.2. Bounds for the Derivative of $\eta(t)$.

In regions of validity, an upper bound on the derivative of the approximation error is obtained. The following is used

$$\eta = x_0 + \eta$$  \hspace{1cm} (3.22)

where

$$\|\eta\| \leq r_{\text{min}}$$  \hspace{1cm} (3.23)

and $r_{\text{min}}$ is the smallest value of $r$ satisfying (3.14). In the notation of Chapter II

$$H_2 y = \int_0^{\pi/2w} h_2(t, s)[F(A \cos ws) + K_\pi y - F(A \cos ws + y)] ds$$  \hspace{1cm} (3.24)

Using developments in Appendix B,
for all \( y \in B_{\min} \). Hence from (2.38), (2.39) and (3.14)

\[
\| \eta \| \leq \| \xi \| + \sqrt{K_x} r_{\min} \alpha(r_{\min})
\]  

(3.27)

### 3.3 Stability Criteria.

Suppose that \( x_0(t) \) is a valid approximate solution. If

\[
\| \xi \| \leq x_0 \left( \frac{1}{w} \cos^{-1} \left( \frac{1+r_{\min}}{A} \right) \right)
\]

(3.28)

\[
= w \left[ A^2 - (1+r_{\min})^2 \right]^{1/2}
\]

(3.29)

or, from (3.27)

\[
\| \xi \| + \sqrt{K_x} r_{\min} \alpha(r_{\min}) \leq w \left[ A^2 - (1+r_{\min})^2 \right]^{1/2}
\]

(3.30)

then \( \dot{x}(t) \) does not change sign when \( \| x \| \) is within \( r_{\min} \) of one. In this case, \( x(t) \) is simple and the stability analysis of Appendix C applies.

The following analysis presumes that (3.30) is satisfied.

Using (3.23) the following bounds on \( t_1 \), where \( x(t_1) = 1 \), are obtained

\[
t^- = \frac{1}{w} \cos^{-1} \left( \frac{1+r_{\min}}{A} \right) \leq t_1 \leq \frac{1}{w} \cos^{-1} \left( \frac{1-r_{\min}}{A} \right) = t^+
\]

(3.31)

In interval notation

\[
[t^-] = [t^+, t^-]
\]

(3.32)

For any given solution \( x(t) = x_0(t) + \eta(t) \), \( \left[ u_1 \left( \frac{\pi}{w} \right) \right] \) is computed as follows:
where \( u_1(\frac{\pi}{w}) \) is given by (C-6). Then according to the results given in Appendix C, \( x(t) \) is stable if

\[
\left[ u_1\left(\frac{\pi}{w}\right) \right] \subset [1, -1]
\]

and unstable if

\[
\left[ u_1\left(\frac{\pi}{w}\right) \right] \cap [1, -1] = \emptyset, \text{ the null set}
\]

When neither (3.34) nor (3.35) is true, then no definitive statement is available concerning the stability of \( x(t) \).

3.4. Sufficient Stability and Instability Boundaries.

From (3.34) and (3.35), it is apparent that boundaries between stable and unstable regions occur when

\[
u_1\left(\frac{\pi}{w}\right) = \pm 1
\]

The stability boundaries in the \( A, w \) plane based upon the approximate solution are calculated as follows: For given \( t_1 \) and \( K \) (3.36) with \( u_1(\frac{\pi}{w}) \) given by C-6, is solved for \( w \). \( A \) is obtained from

\[
A = \frac{1}{\cos \omega t_1}
\]

and \( P \) from (3.7). In valid regions where (3.30) holds, the approximate boundaries are used as initial guesses in the following scheme to determine sufficient stability and instability boundaries. For given
A and K, values of \( w \) are determined satisfying

\[
\begin{align*}
\sup_{t_- \leq t_1 \leq t_+} u_1(\frac{\pi}{w}) = \pm 1 & \quad \text{ (for sufficient stability boundaries)} \\
\inf_{t_- \leq t_1 \leq t_+} u_1(\frac{\pi}{w}) = \pm 1 & \quad \text{ (for sufficient instability boundaries)}
\end{align*}
\]

where

\[
\begin{align*}
u_1(\frac{\pi}{w}) &= \cos \sqrt{K} (\frac{\pi}{w} - 2t_1) \cos 2t_1 - \frac{K^{1/2}}{2} \sin \sqrt{K} (\frac{\pi}{w} - 2t_1) \sin 2t_1 \\
t_+ &= \frac{1}{w} \cos^{-1} \left( \frac{1-r_{\min}}{A} \right) \\
t_- &= \frac{1}{w} \cos^{-1} \left( \frac{1+r_{\min}}{A} \right) \\
r_{\min} &= \frac{\| e \|}{1-\alpha(r_{\min})}
\end{align*}
\]

\( P \) is then determined from (3.7).

The cross hatched areas in Figure 3.2 designate regions where the sufficient stability conditions are not satisfied. That is, either (3.30) does not hold or \([u_1(\frac{\pi}{w})] \notin [1, -1]\). Near \( w = \frac{1}{2} \) there are indeed some unstable solutions as may be seen in the blowup of this unstable region in Figure 3.3, where the stability boundaries based upon the approximate solution are shown together with the sufficient stability and instability boundaries. In this region, equation (3.14) is satisfied for all \( r \geq r_{\min} \). That is, the situation is as in Figure 3.4. Hence, the unstable solution is unique within the class \( Q_{2\pi/w} \), so any stable solution cannot be in this class. In other words, the unstable solution
Fig. 3.3 UNSTABLE REGION NEAR $\omega = \frac{1}{2}$
Fig. 3.4 VALIDITY INEQUALITY FOR GLOBAL UNIQUENESS
is globally unique within $Q_{2\pi/w}$. The same result would hold if a solution in $P_{2\pi/w}$ were sought, so any stable periodic solution of period $2\pi/w$ is in $P_{2\pi/w}$. This result is consistent with the work of Loud\textsuperscript{(19)} who observed that equations of the form (3.1) exhibit branching phenomena in unstable regions near $\omega = \frac{1}{2n}$, $n$ an integer.

The desirability of applying the validity criteria to class $Q_T$ and $H_T$ solutions may be appreciated if one observes the following: Had the validity criteria been applied in class $P_T$ rather than $Q_T$, there would have been additional regions near $\omega = \frac{1}{2n}$, $n$ an integer, where the validity criteria could not be applied. These regions would have included all of the instability region in Figure 3.3. The validity criteria in $Q_T$ may be applied in this unstable region because the branching is to a class of solutions outside of $Q_T$. There are also apparently branching processes in regions I and II of Figure 3.2. However, these branching processes occur entirely within $Q_T$.

3.5. Choice of Basis for the Integral Equation.

In the integral equation for the approximation error, the linear basis, i.e., $b(s)$ in (3.9), is chosen to facilitate application of the validity criteria. In this section, various choices of the basis for the bilinear system (3.1) with approximate solution (3.6) are compared. The symmetry class is $Q_{2\pi/w}$ in all cases.

The following choice of $b(s)$ removes all linear terms from the operator $H$, 

\[ b(s) = \begin{cases} 
1 & \text{for } 0 \leq s \leq \frac{1}{w} \cos^{-1} \frac{1}{A} \\
K & \text{for } \frac{1}{w} \cos^{-1} \frac{1}{A} \leq s \leq \frac{\pi}{2w} 
\end{cases} \] (3.39)

In this case,

\[ h_1(t, s) = \begin{cases} 
(A \cos t + B \sin t) \cos s / \hat{E} & 0 \leq s \leq \frac{1}{w} \cos^{-1} \frac{1}{A} \\
(A \cos s + \tilde{B} \sin s) \cos t / \hat{E} & 0 \leq t \leq \frac{1}{w} \cos^{-1} \frac{1}{A} \\
\sin \sqrt{\hat{K}} \left( \frac{\pi}{2w} - s \right) \cos t / \left( \sqrt{\hat{K}} \hat{E} \right) & 0 \leq s \leq \frac{1}{w} \cos^{-1} \frac{1}{A} \leq s \leq \frac{\pi}{2w} \\
\sin \sqrt{\hat{K}} \left( \frac{\pi}{2w} - t \right) (C \cos \sqrt{\hat{K}} s + \tilde{D} \sin \sqrt{\hat{K}} s) / \left( \sqrt{\hat{K}} \hat{E} \right) & 0 \leq s \leq \frac{1}{w} \cos^{-1} \frac{1}{A} \leq t \leq \frac{\pi}{2w} \\
\sin \sqrt{\hat{K}} \left( \frac{\pi}{2w} - s \right) (\tilde{C} \cos \sqrt{\hat{K}} t + \tilde{D} \sin \sqrt{\hat{K}} t) / \left( \sqrt{\hat{K}} \hat{E} \right) & \frac{1}{w} \cos^{-1} \frac{1}{A} \leq t \leq \frac{\pi}{2w} 
\end{cases} \] (3.40)

where

\[ \tilde{A} = \cos \left[ \sqrt{\hat{K}} \sin^{-1} \frac{1}{A} \right] \sin \left[ \frac{\cos^{-1} \frac{1}{A}}{w} \right] + \frac{1}{\sqrt{\hat{K}}} \sin \left[ \sqrt{\hat{K}} \sin^{-1} \frac{1}{A} \right] \cos \left[ \frac{\cos^{-1} \frac{1}{A}}{w} \right] \]

\[ \tilde{B} = -\cos \left[ \sqrt{\hat{K}} \sin^{-1} \frac{1}{A} \right] \cos \left[ \frac{\cos^{-1} \frac{1}{A}}{w} \right] + \frac{1}{\sqrt{\hat{K}}} \sin \left[ \sqrt{\hat{K}} \sin^{-1} \frac{1}{A} \right] \cos \left[ \frac{\cos^{-1} \frac{1}{A}}{w} \right] \]

\[ \tilde{C} = \cos \left[ \sqrt{\hat{K}} \cos^{-1} \frac{1}{A} \right] \cos \left[ \frac{\cos^{-1} \frac{1}{A}}{w} \right] + \frac{1}{\sqrt{\hat{K}}} \sin \left[ \sqrt{\hat{K}} \cos^{-1} \frac{1}{A} \right] \sin \left[ \frac{\cos^{-1} \frac{1}{A}}{w} \right] \] (3.41)
\[ \mathcal{D} = \sin\left(\frac{\sqrt{K}}{w} \cos^{-1} \frac{1}{A}\right) \cos\left(\frac{\cos^{-1} \frac{1}{A}}{w}\right) - \frac{1}{\sqrt{K}} \sin\left(\frac{\sqrt{K}}{w} \cos^{-1} \frac{1}{A}\right) \sin\left(\frac{\cos^{-1} \frac{1}{A}}{w}\right) \]

\[ \tilde{E} = \cos\left(\frac{\sqrt{K}}{w} \sin^{-1} \frac{1}{A}\right) \cos\left(\frac{\cos^{-1} \frac{1}{A}}{w}\right) - \frac{1}{\sqrt{K}} \sin\left(\frac{\sqrt{K}}{w} \sin^{-1} \frac{1}{A}\right) \sin\left(\frac{\cos^{-1} \frac{1}{A}}{w}\right) \]

and

\[ e(t) = (1-K) + \frac{\left(\frac{K_{\text{EQV}}}{1-w^2}\right) - (1-K) \tilde{E}}{1-w^2} \left[ \cos\left(\frac{\sqrt{K}}{w} \sin^{-1} \frac{1}{A}\right) \right] \cos t \]

\[ + \sin\left(\frac{\sqrt{K}}{w} \sin^{-1} \frac{1}{A}\right) \left[ 1 - \frac{\left(\frac{K_{\text{EQV}}}{1-w^2}\right)^2}{(1-w^2)(K-w^2)} \right] \sin \sqrt{K} \left( \frac{\pi}{2w} - t \right) \]

for \( 0 \leq t \leq \frac{1}{w} \cos^{-1} \frac{1}{A} \) \hspace{1cm} (3.41)\]

\[ e(t) = \frac{\left(\frac{K_{\text{EQV}}}{1-w^2}\right) - (1-K) \tilde{E}}{(1-w^2)} \left[ \cos\left(\frac{\sqrt{K}}{w} \sin^{-1} \frac{1}{A}\right) \right] \cos t \]

\[ + \sin\left(\frac{\sqrt{K}}{w} \sin^{-1} \frac{1}{A}\right) \left[ 1 - \frac{\left(\frac{K_{\text{EQV}}}{1-w^2}\right)^2}{(1-w^2)(K-w^2)} \right] \sin \sqrt{K} \left( \frac{\pi}{2w} - t \right) \]

for \( \frac{1}{w} \cos^{-1} \frac{1}{A} \leq t \leq \frac{\pi}{2w} \) \hspace{1cm} (3.42)

As in Appendix B for the constant coefficient case, one finds the following suitable contraction constant

\[ \alpha(r) = \frac{|1-K| \left(1+\frac{1}{\sqrt{K}}\right)}{|\tilde{E}|w} \max\left[1, \frac{1}{\sqrt{K}}\right] \zeta(r) \]

\[ (3.43) \]

where
\[
\zeta(r) = \begin{cases} 
\cos^{-1} \frac{1-r}{A} - \cos^{-1} \frac{1+r}{A} & \text{for } 0 < r \leq \text{MIN}[1, A-1] \\
\frac{\pi}{2} - \cos^{-1} \frac{1+r}{A} & \text{for } 1 < r \leq A-1 \\
\cos^{-1} \frac{1-r}{A} & \text{for } A-1 < r \leq 1 \\
\frac{\pi}{2} & \text{for } r \geq \text{MAX}[1, A-1]
\end{cases}
\] (3.44)

From this point, application of Validity Criteria I-A is the same as the constant basis case. Figure 3.5 shows a comparison of the percent error \(e_p\) where

\[
e_p = \frac{r_{\text{min}}}{A} \times 100
\] (3.45)

for various choices of the basis including the time dependent (T.D.) basis discussed above, and the following choices of constant coefficient bases: \(K_* = 1\), \(K_* = K_{AVG}\) defined by (3.15), and \(K_* = K_{E Q V}\) defined by (3.8). Also shown is ||\(\eta||\) computed as the sup of the difference between the approximate and exact solution. Where the curve is broken, just to the right of \(w=0.3\), there are no simple exact \(Q_T\) solutions. The parameters are \(P=1.0\) and \(K=0.8\), the same as those for a response curve shown in Figure 3.2.

For this case, the T.D. basis gives very good results away from the validity boundary for region II of Figure 3.2. In fact \(e_p\) for this basis is so close to the exact value that no separate curve is drawn except near the previously mentioned validity boundary. Also observe that the constant coefficient basis \(K_* = K_{AVG}\) gives the widest range of validity. Hence, the time dependent basis may not always
Fig. 3.5 COMPARISON OF BASES

- Exact
- T.D.
- I
- $K_{AVG}$
- $K_{EQV}$

$P = 1.0$
$K = 0.8$
be the best choice. The percent error for all basis choices has the character of the exact curve and compares favorably with the actual value at least for $\omega > \frac{1}{2}$. Although the comparison is plotted for only one set of values of the parameters, the same general behavior was also observed for a number of other cases.

An analytical comparison of the bases is made by considering the resonant case.

3.6. Validity Criteria at Resonance.

The following result holds for the T.D. basis and for all of the constant bases considered below.

$$\lim_{A \to \infty} \|e(t)\| \leq |1-K|$$

(3.46)

Also, in the limit as $A \to \infty$, equation (3.14) goes to

$$8 |1-K| \left(1 + \frac{1}{\sqrt{\lambda}}\right) \frac{\pi}{\pi} \frac{4(1-K)}{\pi} \text{MAX} \left[1, \frac{1}{\sqrt{\lambda}}\right] r^2 + |1-K| \leq r$$

(3.47)

Solving (3.47) in the case of equality gives

$$r = \frac{\pi}{16 |1-K| \left(1 + \frac{1}{\sqrt{\lambda}}\right) \text{MAX} \left[1, \frac{1}{\sqrt{\lambda}}\right]} \left[1 \pm \left\{ \frac{32(1-K)^2 \left(1 + \frac{1}{\sqrt{\lambda}}\right) \text{MAX} 1, \frac{1}{\sqrt{\lambda}}}{\pi \frac{4(1-K)}{\pi}} \right\}^{1/2} \right]$$

(3.48)

Hence, a sufficient condition for application of the validity criteria at resonance is
in which case the relative error \( r_{\text{min}} / A \) goes to zero as \( A \to \infty \). Similar results for other choices of the basis are summarized in Table 3.1. Even at resonance, no one choice is clearly superior for all values of \( P \) and \( K \).

Much work remains to be done in the area of optimum basis choice. However, the results given by Figure 3.5 and Table 3.1 indicate that the choice \( K_{\text{opt}} = K_{\text{AVG}} \), used in this work, compares favorably with other easily computed bases.

3.7. Two Term Harmonic Balance Approximate Solutions.

The bilinear system (3.1) allows \( Q_T \) solutions for \( T = 2\pi / \Omega \), \( \Omega = \omega / (2n - 1) \), \( n \) a positive integer. For the case \( n = 1 \), following Caughey (20), assume an approximate solution

\[
x_0(t) = A \cos \omega t + U \cos 3\omega t
\]

(3.50)

As with the one term approximation, there is no loss of generality in taking \( A > 0 \).

In order to apply harmonic balance, it is necessary to determine \( a_1 \) and \( a_3 \) in the Fourier expansion

\[
F(x_0) = a_1 \cos \omega t + a_3 \cos 3\omega t + \text{higher harmonics}
\]

(3.51)

Expressions for \( a_1 \) and \( a_3 \) are given in Appendix D. Substituting (3.50) into (3.1), using (3.51) and balancing first and third harmonics yields
\[
\lim_{r \to \infty} a(r) = \frac{8r|1-K|\left(1+\frac{1}{\sqrt{K}}\right)\text{MAX}\left[1, \frac{1}{\sqrt{K}}\right]}{\pi |P+\frac{4(1-K)}{\pi}|} < 1
\]

<table>
<thead>
<tr>
<th>Basis</th>
<th>Sufficient Condition for Validity</th>
</tr>
</thead>
</table>
| T.D.  | \[
\frac{32(1-K)^2\left(1+\frac{1}{\sqrt{K}}\right)\text{MAX}\left[1, \frac{1}{\sqrt{K}}\right]}{\pi |P+\frac{4(1-K)}{\pi}|} < 1
\] |
| \(K_{EQV}\) | \[
\frac{4|1-K|(3+r)}{\pi |P|} < 1
\] |
| \(K_{AVG}\) | \[
\frac{4|1-K|}{\sqrt{\pi |P|} \left(1-\frac{12|1-K|}{\pi |P|}\right)} < 1
\] |
| 1     | \[
\frac{4|1-K|(1+r)}{\pi |P+\frac{4(1-K)}{\pi}|} < 1
\] |

Table 3.1. Validity at Resonance
Just as with the one term approximate solution, sufficient conditions for existence of an exact $Q_T$ solution near $x_0$ are obtained by applying a validity criteria which is essentially Criteria I-A for two term approximate solutions. The work involved in the application is computational rather than conceptual, so the details will not be given here.

3.8. Ultraharmonics.

An interesting development occurs in region II of Figure 3.2. Assuming a value of $U$ and solving (3.52) for $A$ and $w$ yields system responses shown in Figure 3.6. Note that the parameters used, i.e., $P=1.0$ and $K=0.8$ correspond to a response curve for the one term approximation shown in Figure 3.2. $R$ is the ratio of $U$ to $A$. The two term approximation predicts an ultraharmonic resonance (i.e., large amplitude response near $w=w_n/m$, $m$ an integer $\geq 2$, where $w_n$ is the harmonic resonant frequency) in this region.

Application of the validity criteria with $K_w=1$ shows that $x_0$ is valid on the lower ultraharmonic response curve in the frequency range shown in Figure 3.6. Note that there is an improvement over the one term approximation, i.e., compare with Figure 3.2. The presence of a vertical tangent in the ultraharmonic response apparently precludes application of the validity criteria over a larger frequency range.
Fig. 3.6 ULTRAHARMONIC RESPONSE

- $R < 0$
- $R > 0$

$K = 0.8$
$P = 1.0$
With $K_\omega=1$,

$$\lim_{U \to \infty} \alpha(r) = 1+r$$

(3.53)

Also, there is no constant coefficient basis which yields a smaller limiting value of $\alpha(r)$. Hence, for this particular system the existence of ultraharmonic resonant responses has not been established.

However, the two term approximation has a larger range of validity near $\omega=1/3$. This indicates that the third harmonic is important in this region. Also, Figure 3.5 indicates that the approximation error for the one term approximate solution is large near $\omega=1/3$. Hence, a stability analysis in this region based upon the one term approximation is not justified.

3.9. Subharmonics.

In a manner analogous to the development above, one obtains two term approximate solutions which exhibit subharmonic resonance, i.e., large amplitude response near $\omega=m\omega_n$, $m$ an integer $\geq 2$ and $\omega_n$ the harmonic resonant frequency. Assume an approximate $Q_{6\pi/\omega}$ solution

$$x_0(t) = S \cos \frac{\omega t}{3} + A \cos \omega t$$

(3.54)

where now $S > 0$. Harmonic balance yields

$$-\frac{S}{9} \omega^2 + b_1 = 0 \quad ; \quad -\omega^2 A + b_3 = P$$

(3.55)

where $b_1$ and $b_3$ are the Fourier coefficients in

$$F(x_0) = b_1 \cos \frac{\omega}{3} t + b_3 \cos \omega t + \text{higher harmonics}$$

(3.56)

Expressions for $b_1$ and $b_3$ are given in Appendix D.
Fig. 3.7 SUBHARMONIC RESPONSE
For given $S$, equations (3.64) are solved for $A$ and $\omega$. Figure 3.7 shows the resulting subharmonic system response. The presence of vertical tangents apparently precludes application of the validity criteria for all responses shown in Figure 3.7. Also, there is no choice of constant coefficient basis which yields validity at subharmonic resonance. Hence it does not seem possible to verify the existence of subharmonic responses using this analysis; at least not with a constant coefficient basis.
4.1 Summary

Sufficient validity criteria are derived for approximate periodic solutions of a class of second order nonlinear ordinary differential equations. An approximate solution is defined to be valid if an exact solution with the same symmetry properties exists in a neighborhood of the approximation. Although the validity criteria are stated for harmonic balance approximations, they are easily extended to include other approximate periodic solutions.

One class of validity criteria, namely I and I-A, require an integral operator associated with the approximation error to contract on a ball in an appropriate Banach space. That is, the operator must satisfy a Lipschitz condition with Lipschitz constant less than unity, and in addition, an inequality involving the Lipschitz constant and the radius of the ball must be satisfied. These criteria yield an algorithm for constructing the exact solution and some uniqueness results in addition to existence.

The second class of validity criteria, namely II-A, is based upon continuity of the same integral operator. This criteria requires that the mapping determined by the integral operator, continuously map a ball in an appropriate Banach space into a compact subset of itself.

Application of Criteria I-A and II-A to the equation of an
undamped oscillator with an odd polynomial restoring force reveals that Criteria II-A yields a smaller bound on the approximation error and a larger region of validity. Criteria II-A is also somewhat easier to apply since it does not require determination of a Lipschitz constant. This is to be expected, however, since the results are not as strong as those of Criteria I-A with respect to uniqueness and the constructive algorithm.

In terms of types of hypotheses and conclusions, the existence theorems for B.V.P.'s upon which the two classes of validity criteria are based, may be compared to the existence theorems for initial value problems. The existence theorem for I and I-A is analogous to the Cauchy-Lipschitz theorem, while the existence theorem for II-A is analogous to the Cauchy-Peano theorem.

In addition to existence, both types of validity criteria yield an upper bound on a norm of the approximation error. This bound is used in a procedure for establishing sufficient stability and instability conditions for the approximated solution. A system with symmetric bilinear restoring force provides an example wherein both sufficient validity and sufficient stability and instability conditions are obtained. When the validity criteria are satisfied and the bound on the error is small, the stability analysis based upon the approximate solution is apparently quite accurate as shown in Figure 3.3. On the other hand, when the approximation cannot be shown valid or when the bound on the approximation error is large, then a stability analysis based on the approximate solution is not justified. For example, in region II
of Figure 3.2 there may be a considerable difference between the 
exact and approximate solutions as shown in Figure 3.5.

The example problems indicate inapplicability of the validity 
criteria in regions where a bifurcation (branching phenomenon) occurs 
within the symmetry class used. Thus the most restrictive allowable 
symmetry class yielded the widest range of validity.

4.2 Previous Work

The use of fixed point theorems to establish validity of approxi-
mate periodic solutions is not new, having been used in one form or 
another by Urabe\(^2\) and Holtzman\(^4\),\(^5\) among others. However the 
methods proposed in these works require the pertinent integral opera-
tor to be differentiable in some sense. Thus the bilinear system 
studied in Chapter III cannot be handled by either of these approaches. 
In this work, the operator need only satisfy a much weaker Lipschitz 
or continuity condition. Further improvements over the previously 
mentioned methods result from the use of symmetry properties and 
selection of an arbitrary basis for the integral equation. The con-
structive algorithm was also apparently overlooked in earlier work.

A recent book of Falb and DeJong\(^21\) also contains application 
of contraction mapping fixed point theorems to problems in oscillation 
theory. The approach is similar to Holtzman's work but makes use 
of the arbitrary basis idea, a procedure which the authors refer to 
as modified contraction mapping.
4.3 **Future Work**

Although this work considers only periodic solutions of non-linear ordinary differential equations, the basic idea of applying fixed point theorems to prove existence for the approximation error should be applicable to other types of problems, for example, transient oscillations in damped vibratory systems and also oscillations in multi-degree of freedom and continuous dynamical systems. Also, much work remains to be done in the area of optimum basis choice.

Sufficient stability and instability boundaries are determined in this work for one special case, namely, the one term harmonic balance approximate solution of the bilinear system (3.1). More general problems might be handled using the procedure given in Chapter II together with results of Caughey and Dickerson\(^{(22)}\) in the damped case and Borg\(^{(23)}\) in the undamped case.
APPENDIX A

SIMPLE SOLUTIONS FOR THE BILINEAR SYSTEM

Definition A1: A class \( Q_T \) solution of the bilinear system (3.1), call it \( x(t) \), is simple if \( |x(t)| \) crosses +1 at one and only one value of \( t \) in \( 0 \leq t \leq T/4 \).

Equation (3.1) allows a \( Q_T \) solution for \( T = (4n+2)\pi/\omega \), \( n \) an integer. Let \( T \) take a value from this set. For simple \( Q_T \) solutions, there is no loss of generality in assuming \( x(0) > 1 \), in which case the governing equation is

\[
\begin{align*}
\ddot{x} + x &= P \cos \omega t + 1 - K \quad 0 < t < t_1 \\
\ddot{x}(0) &= 0 \quad x(t_1) = 1 \\
\ddot{x} + Kx &= P \cos \omega t \quad t_1 < t < T/4 \\
x(t_1) &= 1 \quad x(T/4) = 0
\end{align*}
\]  

(A-1)

Solving for \( 0 < t < t_1 \)

\[
x(t) = 1 - K + \left[ A - 1 + K \right] \cos t + \frac{P}{1 - \omega^2} \left[ \cos \omega t - \cos t \right] \quad 0 \leq t \leq t_1
\]  

(A-2)

where \( A > 1 \) is the assumed value of \( x(0) \). Using the condition \( x(t_1) = 1 \) yields

\[
A = 1 - K + \frac{1}{\cos t_1} \left[ K + \frac{P}{1 - \omega^2} \left[ \cos t_1 - \cos \omega t_1 \right] \right]
\]  

(A-3)

Denoting \( \dot{x}(t_1) \) by \( v \) and using (A-2) and (A-3)
\[ v = -K \frac{\sin t_1}{\cos t_1} + \frac{P}{1-\omega^2} \left( \cos wt_1 \frac{\sin t_1}{\cos t_1} - \omega \sin wt_1 \right) \]  \hspace{1cm} (A-4) 

Then for \( t_1 \leq t \leq \frac{T}{4} \)

\[ x(t) = \cos \sqrt{K} (t-t_1) + \frac{v}{\sqrt{K}} \sin \sqrt{K} (t-t_1) \]

\[ + \frac{P}{K-\omega^2} \left( \cos wt - \cos \sqrt{K} (t-t_1) \cos wt_1 \right. \]

\[ + \left. \frac{\omega}{\sqrt{K}} \sin \sqrt{K} (t-t_1) \sin wt_1 \right) \]  \hspace{1cm} (A-5) 

The remaining condition is \( x(T/4) = 0 \). Using (A-4), (A-5) and simplifying gives the following equation for \( t_1 \)

\[ 0 = \cos \sqrt{K} \left( \frac{T}{4} - t_1 \right) \left[ 1 - \frac{P \cos wt_1}{K-\omega^2} \right] \]

\[ + \frac{\sin \sqrt{K} \left( \frac{T}{4} - t_1 \right)}{\sqrt{K}} \frac{\sin t_1}{\cos t_1} \left[ \frac{P \cos wt_1}{1-\omega^2} - K \right] + \frac{w(1-K)P}{(K-\omega^2)(1-\omega^2)} \sin wt_1 \]  \hspace{1cm} (A-6) 

Simple \( Q_T \) solutions are then constructed as follows: For given

P, \( K \), and \( \omega \)

1. All roots \( t_1 \) in \( 0 \leq t_1 \leq \frac{T}{4} \) of (A-6) are obtained.

2. Values of \( v \) and \( A \) corresponding to each root are computed using (A-4) and (A-3), respectively.

3. The function corresponding to each root and given by (A-2) and (A-5) is checked to be sure that it is a simple solution.

The last step is important, since there may be roots of (A-6) which do not yield solutions. As an example, Figure A.1 shows a case where
Fig. A.1 SIMPLE AND NON SOLUTIONS OF THE BILINEAR SYSTEM
(A-6) has three roots, only one of which corresponds to a solution. The other functions cross \( |x| = 1 \) more than once in \( 0 \leq t \leq \frac{T}{4} \) and hence they are not solutions.
APPENDIX B

A CONTRACTION CONSTANT

Consider the restoring force $F(u)$ for the bilinear system shown graphically in Figure B-1. Inspection of the graph reveals that:

1) For $z+x$ and $z+y$ both on segment 3 or both on segment 1
\[ F(z+x) - F(z+y) = x - y \]  
\( (B-1) \)

2) For $z+x$ and $z+y$ both on segment 2
\[ F(z+x) - F(z+y) = K(x-y) \]  
\( (B-2) \)

3) For $z+x$ and $z+y$ on different segments
\[ K_-(x-y) \leq F(z+x) - F(z+y) \leq K_+(x-y) \]
\[ x > y \]  
\( (B-3) \)

\[ K_-(x-y) \geq F(z+x) - F(z+y) \geq K_+(x-y) \]  
\[ x < y \]

where $K_+ = \text{MAX}[1,K]$ and $K_- = \text{MIN}[1,K]$.

Hence

\[ \|K_K(y-x) + F(z+x) - F(z+y)\| = \|1 - K_K\| \|x-y\| \begin{cases} \text{for } z+x \text{ and } z+y \text{ both on segment 3 or both on segment 1} \\ \text{on segment 3 or both on segment 1} \end{cases} \]  
\( (B-4) \)

\[ \|K_K(y-x) + F(z+x) - F(z+y)\| \leq \text{MAX} \left[ |1 - K_K|, |K_K - K| \right] \|x-y\| \begin{cases} \text{for } z+x \text{ and } z+y \text{ on different segments} \\ \text{and } z+y \text{ on different segments} \end{cases} \]  
\( (B-4) \)

\[ \|K_K(y-x) + F(z+x) - F(z+y)\| = |K - K_K| \|x-y\| \begin{cases} \text{for } z+x \text{ and } z+y \text{ both on segment} \\ \text{both on segment 2} \end{cases} \]  
\( (B-4) \)
Fig. B.1 BILINEAR RESTORING FORCE
The purpose of this Appendix is to determine a suitable $\alpha = \alpha(r)$ satisfying

$$
\|N_1 y - N_1 x\| \leq \alpha(r)\|y - x\|
$$

for all $x, y \in B_r$, where $N_1$ is given by (3.12). Using an elementary inequality for integrals

$$
\|N_1 y - N_1 x\| \leq \sup_{0 \leq t \leq \frac{\pi}{2w}} |h_1(t, s)| \int_0^{\frac{\pi}{2w}} \|F(x_0 + x) - F(x_0 + y) + K_*(y - x)\| ds
$$

(B-6)

From (2.55)

$$
\sup_{0 \leq t \leq \frac{\pi}{2w}} |h_1(t, s)| \leq \frac{1}{\sqrt{K_*} |\cos\frac{\sqrt{K_*}s}{2w}|}
$$

(B-7)

Using $x, y \in B_r$, i.e., $\|x\| \leq r$ and $\|y\| \leq r$, and $x_0 = A\cos ws$, the following is deduced from (B-4)

1) For $r < A - 1$ and $0 \leq s \leq \frac{1}{w} \cos^{-1} \frac{1+r}{A}$

$$
\|F(x_0 + x) - F(x_0 + y) + K_*(y - x)\| = |1 - K_*| \|y - x\|
$$

(B-8)

2) For $r < 1$ and $\frac{1}{w} \cos^{-1} \frac{1-r}{A} \leq s \leq \frac{\pi}{2w}$

$$
\|F(x_0 + x) - F(x_0 + y) + K_*(y - x)\| = |K - K_*| \|y - x\|
$$

(B-9)

3) Otherwise, that is, $L \leq s \leq U$ where

$$
L = \begin{cases} 
\frac{1}{w} \cos^{-1} \frac{1+r}{A} & \text{for } r < A - 1 \\
0 & \text{for } r \geq A - 1
\end{cases}
$$
\[ U = \begin{cases} \frac{1}{\omega} \cos^{-1} \frac{1-r}{A} & \text{for } r < 1 \\ 0 & \text{for } r \geq 1 \end{cases} \]

one obtains

\[ \| F(x_0 + x) - F(x_0 + y) + K_\times (y - x) \| \leq \text{MAX} \left[ |1 - K_\times|, |K - K_\times| \right] \| y - x \| \quad (B-10) \]

Combining with (B-6) and (B-7) yields

\[ \alpha(r) = \frac{\pi}{2w \sqrt{K_\times} \cos \frac{K_\times \pi}{2\omega}} \left\{ |K - K_\times| + \frac{2}{\pi} \left[ |1 - K_\times| - |K - K_\times| \right] \varphi(r) \right\} \quad (B-11) \]

where

\[ \varphi(r) = \begin{cases} 0 & \frac{1 + \sigma r}{A} > 1 \\ \cos^{-1} \frac{1 + \sigma r}{A} & 0 \leq \frac{1 + \sigma r}{A} \leq 1 \\ \frac{\pi}{2} & \frac{1 + \sigma r}{A} < 0 \end{cases} \quad (B-12) \]

\[ \sigma = \begin{cases} +1 & \text{when } |K - K_\times| > |1 - K_\times| \\ -1 & \text{when } |K - K_\times| \leq |1 - K_\times| \end{cases} \quad (B-13) \]
APPENDIX C

STABILITY OF A SIMPLE $Q_T$ SOLUTION
OF THE BILINEAR SYSTEM

The equation of first variation for a simple $Q_T$ solution $x(t)$ of
(3.1) with $T = 2\pi/\Omega$ is
$$\ddot{y} + f(t)y = 0$$
(C-1)

where
$$f(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq t_1 \\ K \text{ for } t_1 < t \leq \frac{\pi}{\Omega} - t_1 \\ f(t) = f(t + \frac{\pi}{\Omega}) & \text{for } \frac{\pi}{\Omega} - t_1 < t \leq \frac{\pi}{\Omega} \\ 1 & \text{for } \frac{\pi}{\Omega} - t_1 < t \leq \frac{\pi}{\Omega} \end{cases}$$
(C-2)

and $t_1$ is the only value of $t$ in $0 \leq t \leq \pi/2\Omega$ for which $|x|=1$. Equation (3.1)
allows $Q_T$ solutions for $\Omega = \omega/(2n-1)$, $n$ a positive integer.

Since $f(t)$ is an even function, the stability criteria given in
Chapter II have a simpler form. In the notation of Chapter II
$$u_1\left(\frac{\pi}{\Omega}\right) = v_2\left(\frac{\pi}{\Omega}\right)$$
(C-3)

(see Magnus and Winkler (18) page 8) and $q_1 = 0$. Hence, the characteristic exponents for C-1 are
$$\lambda_{1,2} = \pm \left\{ \frac{i\Omega}{\pi} \ln \left| u_1 \right| \text{ for } \left| u_1\left(\frac{\pi}{\Omega}\right)\right| \geq 1 \right\}$$
$$\left\{ \frac{\Omega}{\pi} (\text{Arg} u_1 + 2\pi n) \text{ for } \left| u_1\left(\frac{\pi}{\Omega}\right)\right| < 1 \right\}$$
(C-4)

where
\[ \mu_1 = u_1(\frac{\pi}{\Omega}) + \left[ u_1^2(\frac{\pi}{\Omega}) - 1 \right]^{1/2} \quad (C-5) \]

Using the results summarized in Table 2, (C-4) and (C-5) it is apparent that \( x(t) \) is stable when \( |u_1(\frac{\pi}{\Omega})| < 1 \), unstable when \( |u_1(\frac{\pi}{\Omega})| > 1 \) and unstable when \( |u_1(\frac{\pi}{\Omega})| = 1 \), unless \( v_1(\frac{\pi}{\Omega}) = u_2(\frac{\pi}{\Omega}) \). Since \( u_1(t) \) satisfies (C-1) and the initial conditions \( u_1(0) = 1 \), \( \dot{u}_1(0) = 0 \) it is readily obtained yielding

\[ u_1(\frac{\pi}{\Omega}) = \cos \sqrt{K} \left( \frac{\pi}{\Omega} - 2t_1 \right) \cos 2t_1 - \frac{\left( \frac{1}{\sqrt{K}} + \sqrt{K} \right)}{2} \sin \sqrt{K} \left( \frac{\pi}{\Omega} - 2t_1 \right) \sin 2t_1 \quad (C-6) \]
APPENDIX D

FOURIER COEFFICIENTS FOR
TWO TERM HARMONIC BALANCE

Consider first

\[ x_0 = A \cos \omega t + U \cos 3\omega t \]  \hspace{1cm} (D-1)

The precise form of \( F(x_0) \) where \( F \) is given by (3.2) depends upon the sign of \( U \) and upon the number of level crossings at \( |x_0| = 1 \) per quarter cycle (CPQC), i.e., in \( 0 \leq t \leq \pi/2\omega \). There are four possible cases shown graphically in Figure D-1. The number and locations of the level crossings are determined by the roots of

\[ A \cos \theta + U \cos 3\theta = 1 \]  \hspace{1cm} (D-2)

in \( 0 \leq \theta \leq \pi \). Equivalently (D-2) is written

\[ \cos^3 \theta + \frac{(A - 3U)}{4U} \cos \theta = \frac{1}{4U} \]  \hspace{1cm} (D-3)

The roots \( \theta_i \) are called the crossing angles and they are related to the crossing times \( t_i \) where \( |x_0(t_i)| = 1 \) by \( \theta_i = \omega t_i \). If there is only one real root \( \theta_1 \), then there is one CPQC; two real roots gives 2 CPQC, etc.

Let \( Z_1 \) be roots of the cubic equation (D-3) satisfied by \( \cos \theta_1 \), that is,

\[ Z_1^3 + \frac{(A - 3U)}{4U} Z_1 - \frac{1}{4U} = 0 \]  \hspace{1cm} (D-4)

Only roots \( Z_1 \) in \(-1 \leq Z_1 \leq +1\) are considered since roots outside of this range correspond to complex values of \( \theta_1 \). In cases where there is more than one root in this interval, let \( Z_1 \) be the largest root, \( Z_2 \) the
Fig. D.1 POSSIBLE LEVEL CROSSINGS FOR $X_0(t)$
next largest, and \( Z_3 \), if it exists, the smallest. Then the \( Z_i \) and \( \theta_i \) are related by

- **one root**
  \[ \theta_1 = \cos^{-1} Z_1 \]

- **two roots**
  \[ \begin{align*}
  \theta_1 &= \cos^{-1} Z_1 \\
  \theta_2 &= \cos^{-1} Z_2
  \end{align*} \]

- **three roots, \( U > 0 \)**
  \[ \begin{align*}
  \theta_1 &= \cos^{-1} Z_1 \\
  \theta_2 &= \pi - \cos^{-1} Z_3 \\
  \theta_3 &= \pi - \cos^{-1} Z_2
  \end{align*} \] (D-5)

- **three roots, \( U < 0 \)**
  \[ \begin{align*}
  \theta_1 &= \pi - \cos^{-1} Z_3 \\
  \theta_2 &= \cos^{-1} Z_1 \\
  \theta_3 &= \cos^{-1} Z_2
  \end{align*} \]

Now expanding \( F(x_0) \) in a Fourier series for each of the four cases gives

\[ F(x_0) = a_1 \cos \omega t + a_3 \cos 3\omega t + \text{higher harmonics} \] (D-6)

where

\[ a_1 = \frac{1}{\pi} \left\{ \tilde{f}_1 (1-K) + A \left[ K + (1-K) \tilde{g}_2 \right] + U(1-K) \tilde{f}_3 \right\} \] (D-7)

\[ a_3 = \frac{1}{\pi} \left\{ \tilde{g}_1 (1-K) + A (1-K) \tilde{g}_2 + U \left[ K + (1-K) \tilde{g}_3 \right] \right\} \]

and \( \tilde{f}_1, \tilde{g}_i \) \( (i = 1, 2, 3) \) are different in each case. Define functional forms

\[ \begin{align*}
  a(x) &= -4 \sin x \\
  b(x) &= -\frac{4}{3} \sin 3x
  \end{align*} \] (D-8)
\[ c(x) = 2x + \sin 2x \quad d(x) = 2x + \frac{1}{3} \sin 6x \]  
\[ e(x) = \sin 2x + \frac{1}{2} \sin 4x \]  
\[ (D-8) \text{ cont.} \]

Then the \( \tilde{f}_1 \) and \( \tilde{g}_1 \) are:

**One CPQC**

\[ \tilde{f}_1 = a(\theta_1) \quad \tilde{g}_1 = b(\theta_1) \]
\[ \tilde{f}_2 = c(\theta_1) \quad \tilde{g}_3 = d(\theta_1) \]  
\[ \tilde{f}_3 = \tilde{g}_2 = e(\theta_1) \]  
\[ (D-9) \]

**Two CPQC**

\[ \tilde{f}_1 = a(\theta_2) - a(\theta_1) \quad \tilde{g}_1 = b(\theta_2) - b(\theta_1) \]
\[ \tilde{f}_2 = c(\theta_2) - c(\theta_1) \quad \tilde{g}_3 = d(\theta_2) - d(\theta_1) \]  
\[ \tilde{f}_3 = \tilde{g}_2 = e(\theta_2) - e(\theta_1) \]  
\[ (D-10) \]

**Three CPQC (U>0)**

\[ \tilde{f}_1 = a(\theta_1) + a(\theta_2) - a(\theta_3) \quad \tilde{g}_1 = b(\theta_1) + b(\theta_2) - b(\theta_3) \]
\[ \tilde{f}_2 = c(\theta_1) + c(\theta_3) - c(\theta_2) \quad \tilde{g}_3 = d(\theta_1) + d(\theta_2) - d(\theta_3) \]  
\[ \tilde{f}_3 = \tilde{g}_2 = e(\theta_1) + e(\theta_2) - e(\theta_3) \]  
\[ (D-11) \]

**Three CPQC (U<0)**

Same as (U>0), but change sign of \( \tilde{f}_1 \) and \( \tilde{g}_1 \).

For an approximate solution

\[ x_0 = S \cos \frac{\omega}{3} t + A \cos \omega t \]  
\[ (D-12) \]
where $S > 0$, replace $A$ by $S$, $U$ by $A$ and $\omega$ by $\omega/3$ in all of the preceding equations. Then $b_1$ and $b_3$, instead of $a_1$ and $a_3$, are obtained from (D-6), where

$$F(x_0) = b_1 \cos \frac{\omega}{3} t + b_3 \cos \omega t + \text{higher harmonics} \quad \text{(D-13)}$$

and $x_0$ is given by (D-12).
REFERENCES


