ELECTROMAGNETIC WAVE PROPAGATION AND SCATTERING

IN A

RANDOMLY-INHOMOGENEOUS DIELECTRIC SPHERE

Thesis by

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In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1970

(Submitted April 23, 1970)

ACKNOWLEDGMENT

The author wishes to express his gratitude to Professor C. H. Papas for his guidance and encouragement during the course of this work.

The author is also very grateful to the National Science Foundation for their financial support.

Finally, the author wishes to express his appreciation to Mrs. R. Stratton for her efforts in typing the text.

ABSTRACT

Electromagnetic wave propagation and scattering in a sphere composed of an inhomogeneous medium having random variations in its permittivity are studied by utilizing the Born approximation in solving the vector wave equation. The variations in the permittivity are taken to be isotropic and homogeneous, and are spatially characterized by a Gaussian correlation function. Temporal variations in the medium are not considered.

Two particular problems are considered: i) finding the far-zone electric field when an electric or magnetic dipole is situated at the center of the sphere, and ii) finding the electric field at the sphere's center when a linearly polarized plane wave is incident upon it. Expressions are obtained for the mean-square magnitudes of the scattered field components; it is found that the mean of the product of any two transverse components vanishes. The cases where the wavelength is much shorter than correlation distance of the medium and where it is much longer than it are both considered.

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I. INTRODUCTION

The study of electromagnetic wave propagation in a turbulent medium has been assuming increasing importance in recent years. The advent of the laser and the employment of laser beams in various types of communication systems is perhaps the major factor which has given increased impetus to this area of research. The effects of atmospheric turbulence, which could usually be neglected at longer wavelengths, became important in analyzing the performance and limitations of a system operating at optical wavelengths.

The problem of detecting clear air turbulance, a subject of considerable importance to the airline industry, has also spurred research in electromagnetic wave propagation in a turbulent medium. Here the interest largely centers on the effects that turbulence has on waves of radar frequencies and on the factors that influence these effects[1].

Early work in the area of wave propagation in a turbulent medium was related to the subject of over-the-horizon propagation of radio waves and to the field of astronomy. In 1950, considering the problem of tropospheric turbulence, Booker and Gordon [2] derived an expression for the scattering cross-section per unit volume of a medium whose random variations in permittivity were characterized by an exponential correlation function. That same year another article by Booker, Ratcliffe, and Shinn [3] considered the problem of fluctuations of a wave reflected by the ionosphere by analyzing the diffraction from a plane screen which had random variations in its transmission characteristics. Chandrasekhar [4], in a 1952 article, developed a theory on the scintillation of stars by using geometrical optics to analyze the propagation of light through a medium with random variations in the index of refraction.

Much of the present day work being done on wave propagation in a turbulent medium is based on methods employed by Tatarski [5]. Tatarski, utilizing Kolmogorov's work on the physics of turbulence, applied Rytov's method (which Tatarski called the method of smooth perturbations) and spectral expansions to obtain structure functions for the phase and the logarithm of the amplitude for the case of a plane wave incident on a turbulent medium with a plane boundary. Other methods currently being used to attack the problem are reviewed in a recent article by Strohbehn [6].

All of the methods now employed in analyzing wave propagation in a turbulent medium begin with a scalar wave equation for the polarized component of the scattered field. In deriving the scalar wave equation it is assumed that the change in polarization of the scattered field is negligible. Generally, no attempt is made to calculate the depolarized components of the field. The method used in this work is novel in that solution proceeds directly from the vector wave equation. The solutions for the polarized and depolarized components of the scattered field are placed on an equal footing; no assumptions are made regarding their relative magnitudes.

The subject which will be considered is that of electromagnetic propagation and scattering by a sphere composed of a medium possessing random variations in its permittivity. These variations are assumed to arise from turbulent motions of the medium; however, the velocity of the turbulence is assumed to be sufficiently low so that temporal variations in the permittivity may be neglected. The variations in the permittivity are taken to be homogeneous and isotropic, and they will be characterized spatially by a Gaussian correlation function. The mean permittivity of the sphere will be assumed to be equal to that of the surrounding medium.

The subject will be dealt with by utilizing the Born approximation in solving the vector wave equation. The problem which will be treated first is that of finding the far-zone radiation pattern for the cases where electric and magnetic dipoles are situated at the center of the sphere. In particular, expressions will be obtained for the ensemble averages of the squares of the magnitudes of the components of the scattered electric field. Next, the converse problem of finding the electric field at the center of the sphere when a linearly polarized plane wave is incident upon it will be considered. In both of these problems, the cases where the wavelength is much greater than the correlation distance of the medium and where it is much less than this distance will both be treated.

The text begins by considering the wave equation and finding the form of its solution when the Born approximation is used. Next, a few pertinent aspects of random variables and correlation functions are noted. Then the problem of finding the far-zone scattered electric field for dipoles situated at the center of the sphere is considered. Finally, the problem of finding the scattered electric field generated at the sphere's center by an incident plane wave is treated.

II. SOLUTION OF THE WAVE EQUATION

This work will be concerned with the problem of calculating the electric field $\overline{E}$ in a number of different situations. Therefore, the basic equation for $\overline{E}$, the wave equation, and a number of forms of its solution will now be examined.

One begins by considering Maxwell's equations in a linear, isotropic, but not necessarily homogeneous medium. Only monochromatic waves will be of interest in this work; therefore, a time dependence of the form $e^{-i\omega t}$ may be assumed. Under these conditions Maxwell's equations have the following form:

$$\nabla \times \overline{E} = i\omega\overline{B} \tag{2.1}$$

$$\nabla \times \overline{H} = \overline{J} - i\omega\overline{D} \tag{2.2}$$

$$\nabla \cdot \overline{B} = 0 \tag{2.3}$$

$$\nabla \cdot \overline{D} = \rho \; . \tag{2.4}$$

In addition, one has the constitutive relations

$$\overline{D} = \varepsilon\overline{E} \tag{2.5}$$

$$\overline{B} = \mu\overline{H} \; . \tag{2.6}$$

2.1 Derivation of the Wave Equation

The wave equation for $\overline{E}$ will now be derived under the assumption that μ is a constant. Substitution of equation (2.6) into equation (2.1), and equation (2.5) into (2.2) yields, respectively

$$\nabla \times \overline{E} = i\omega\mu\overline{H} \tag{2.7}$$

$$\nabla \times \overline{H} = \overline{J} - i\omega\varepsilon\overline{E} . \tag{2.8}$$

By taking the curl of equation (2.7) and utilizing equation (2.8), one obtains

$$\nabla \times \nabla \times \overline{E} = i\omega\mu\overline{J} + \omega^2\mu\varepsilon\overline{E} .$$

Letting $k^2 = \omega^2\mu\varepsilon$ and rearranging terms, one obtains the usual form of the wave equation

$$\nabla \times \nabla \times \overline{E} - k^2\,\overline{E} = i\omega\mu\overline{J} . \tag{2.9}$$

2.2 Solution of the Wave Equation in a Homogeneous, Unbounded Medium

The solution for $\overline{E}$ can be obtained through the use of the field potentials [7]. In an unbounded region where μ and ε are constants, the result is

$$\overline{E}(\overline{r}) = i\omega\mu\left\{\int_V \frac{e^{ik|\overline{r}-\overline{r}'|}}{4\pi|\overline{r}-\overline{r}'|}\,\overline{J}(\overline{r}')d\overline{r}' + \frac{1}{k^2}\nabla\left[\nabla\cdot\int_V \frac{e^{ik|\overline{r}-\overline{r}'|}}{4\pi|\overline{r}-\overline{r}'|}\,\overline{J}(\overline{r}')d\overline{r}'\right]\right\} \tag{2.10}$$

where V is the volume containing the source $\overline{J}$.

If the observation point $\overline{r}$ in equation (2.10) lies outside of the volume V , then the ∇ operators may be brought inside the integral. Thus,

$$\overline{E}(\overline{r}) = i\omega\mu\left\{\int_V \frac{e^{ik|\overline{r}-\overline{r}'|}}{4\pi|\overline{r}-\overline{r}'|}\,\overline{J}(\overline{r}')d\overline{r}' + \int_V \frac{1}{k^2}\nabla\left[\nabla\cdot\left(\frac{e^{ik|\overline{r}-\overline{r}'|}}{4\pi|\overline{r}-\overline{r}'|}\,\overline{J}(\overline{r}')\right)\right]d\overline{r}'\right\} \tag{2.11}$$

when $\bar{r}$ is an exterior point of V .

One can define a dyadic operator $\nabla\nabla$ which has the property

$$\nabla\nabla \cdot (\psi\bar{C}) = \nabla[\nabla \cdot (\psi\bar{C})] = (\nabla\nabla\,\psi) \cdot \bar{C}$$

where $\bar{C}$ is a constant vector and ψ is a scalar function. In rectangular coordinates the operator is given by

$$\nabla\nabla = \sum_{i=1}^{3} \sum_{j=1}^{3} \bar{e}_i \, \bar{e}_j \frac{\partial^2}{\partial x_i \partial x_j}$$

where $x_1 = x$, $x_2 = y$, $x_3 = z$, and where $\bar{e}_1, \bar{e}_2, \bar{e}_3$ are unit vectors in the x,y,z directions, respectively. Use of the $\nabla\nabla$ operator in equation (2.11) yields

$$\bar{E}(\bar{r}) = i\omega\mu\left\{\int_V \frac{e^{ik|\bar{r}-\bar{r}'|}}{4\pi|\bar{r}-\bar{r}'|} \bar{J}(\bar{r}')d\bar{r}' + \int_V \left[\frac{1}{k^2}\nabla\nabla \frac{e^{ik|\bar{r}-\bar{r}'|}}{4\pi|\bar{r}-\bar{r}'|}\right] \cdot \bar{J}(\bar{r}')d\bar{r}'\right\} . \tag{2.12}$$

Let $\bar{\bar{I}}$ represent the identity dyadic, $\bar{C} \cdot \bar{\bar{I}} = \bar{\bar{I}} \cdot \bar{C} = \bar{C}$. In particular, $\bar{\bar{I}} = \sum_{i=1}^{3} \bar{e}_i \, \bar{e}_i$. Then (2.12) can be put into the form

$$\bar{E}(\bar{r}) = i\omega\mu \int_V \left[(\bar{\bar{I}} + \frac{1}{k^2}\nabla\nabla) \frac{e^{ik|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|}\right] \cdot \bar{J}(\bar{r}') \, d\bar{r}' . \tag{2.13}$$

Let

$$\bar{\bar{\Gamma}}(\bar{r},\bar{r}') = (\bar{\bar{I}} + \frac{1}{k^2}\nabla\nabla) \frac{e^{ik|\bar{r}-\bar{r}'|}}{4\pi|\bar{r}-\bar{r}'|} . \tag{2.14}$$

$\bar{\bar{\Gamma}}$ is known as the free space dyadic Green's function. Using this notation in (2.13), one obtains a compact expression for $\bar{E}$:

$$\overline{E}(\overline{r}) = i\omega\mu \int_V \overline{\overline{\Gamma}}(\overline{r},\overline{r}') \cdot \overline{J}(\overline{r}')\, d\overline{r}' . \qquad (2.15)$$

Recall that in obtaining equation (2.15) it was assumed that the observation point $\overline{r}$ lay outside of V. However, since there will be occasion to determine $\overline{E}$ when it lies inside V, it would be convenient to have a similar expression for this case. Using the results of Van Bladel [8], if $\overline{\overline{\Gamma}}$ is replaced by

$$\overline{\overline{\Gamma}}'(\overline{r},\overline{r}') = \overline{\overline{\Gamma}}(\overline{r},\overline{r}') - \frac{1}{3k^2}\,\delta(\overline{r}-\overline{r}')\,\overline{\overline{I}} ,$$

then

$$\overline{E}(\overline{r}) = i\omega\mu \int_V \overline{\overline{\Gamma}}'(\overline{r},\overline{r}') \cdot \overline{J}(\overline{r}')\, d\overline{r}' . \qquad (2.16)$$

This is valid regardless of whether or not $\overline{r}$ lies within V. However, when $\overline{r}$ lies within V, the expression $\int_V \overline{\overline{\Gamma}}(\overline{r},\overline{r}') \cdot J(\overline{r}') d\overline{r}'$ is assumed to mean the "principal value" [9] of the integral, i.e., the value obtained in the limit when one deletes from V a spherically shaped volume, centered at $\overline{r}$, as the radius of this volume approaches zero.

Another expression for $\overline{E}$ which will prove useful is one derived by Stratton and Chu [10]. The fields in the interior of any closed surface S are expressed in terms of the sources located within V, the volume bounded by S, and the values of the fields on S. The expression is

$$\overline{E}(\overline{r}) = \int_V [i\omega\mu\ \overline{J}(\overline{r}')\ \frac{e^{ik|\overline{r}-\overline{r}'|}}{4\pi|\overline{r}-\overline{r}'|} + \frac{1}{\varepsilon}\ \rho(\overline{r}')\nabla'\ \frac{e^{ik|\overline{r}-\overline{r}'|}}{4\pi|\overline{r}'-\overline{r}'|}]\ d\overline{r}'$$

$$- \oint_S [i\omega\mu\overline{n}' \times H(\overline{r}')\ \frac{e^{ik|\overline{r}-\overline{r}'|}}{4\pi|\overline{r}-\overline{r}'|} + (\overline{n}' \times \overline{H}(\overline{r}')) \times \nabla'\ \frac{e^{ik|\overline{r}-\overline{r}'|}}{4\pi|\overline{r}-\overline{r}'|}$$

$$+ (\overline{n}' \cdot \overline{E}(\overline{r}))\nabla'(\frac{e^{ik|\overline{r}-\overline{r}'|}}{4\pi|\overline{r}-\overline{r}'|})]\ dS' \qquad (2.17)$$

where $\overline{n}'$ is the normal to S pointing out of V.

If all the sources can be contained within a sphere of finite radius, then equation (2.17) can also be used to find the field in the region exterior to S. In this case volume V is the exterior region, and $\overline{n}'$ is directed into the interior of the surface.

If the sources are zero in the neighborhood of the point $\overline{r}$, then equation (2.17) can be put into an alternate form that will prove to be more useful. Let

$$\overline{C} = \int_V \rho(\overline{r}')\ \nabla' G(\overline{r},\overline{r}')\ d\overline{r}' \qquad (2.18)$$

where $G(\overline{r},\overline{r}') = e^{ik|\overline{r}-\overline{r}'|}/4\pi|\overline{r}-\overline{r}'|$. Due to the fact that $\nabla' G(\overline{r},\overline{r}') = -\nabla G(\overline{r},\overline{r}')$, one has

$$\overline{C} = -\int \rho(\overline{r}')\ \nabla G(\overline{r},\overline{r}')\ d\overline{r}'.$$

Since ∇ operates only on the $\overline{r}$ variables, not on the $\overline{r}'$ variables, this can be written as

$$\overline{C} = -\int_V \nabla\ [\rho(\overline{r}')\ G(\overline{r},\overline{r}')]\ d\overline{r}'.$$

It is permissible to change the order of integration and differentiation here. Hence,

$$\overline{C} = -\nabla \int_V \rho(\overline{r}') \; G(\overline{r},\overline{r}') \; d\overline{r}' .$$

Now by taking the divergence of equation (2.2), one obtains the equation of continuity

$$\nabla' \cdot \overline{J}(\overline{r}') = i\omega\rho(\overline{r}') .$$

Substitution of this into the expression for $\overline{C}$ yields

$$\overline{C} = \frac{-1}{i\omega} \nabla \int_V \nabla' \cdot \overline{J}(\overline{r}') \; G(\overline{r},\overline{r}') \; d\overline{r}' .$$

Applying the vector identity

$$\nabla' \cdot [\overline{J}(\overline{r}') \; G(\overline{r},\overline{r}')] = \nabla' \cdot \overline{J}(\overline{r}') \; G(\overline{r},\overline{r}') + \overline{J}(\overline{r}') \cdot \nabla' G(\overline{r},\overline{r}') ,$$

one finds that

$$\overline{C} = \frac{-1}{i\omega} \nabla \int_V \{\nabla' \cdot [\overline{J}(\overline{r}') \; G(\overline{r},\overline{r}')] - \overline{J}(\overline{r}') \cdot \nabla' G(\overline{r},\overline{r}')\} \; d\overline{r} .$$

Using the divergence theorem, the volume integral over the first part of the integrand may be transformed into a surface integral. Thus

$$\overline{C} = \frac{-1}{i\omega} \nabla \oint_S \overline{n}' \cdot \overline{J}(\overline{r}') \; G(\overline{r},\overline{r}') \; dS' + \frac{1}{i\omega} \nabla \int_V \overline{J}(\overline{r}') \cdot \nabla' G(\overline{r},\overline{r}') \; d\overline{r}' .$$

But since $\nabla' G = -\nabla G$ and $\nabla \cdot [\overline{J}(\overline{r}') \; G(\overline{r},\overline{r}')] = \overline{J}(\overline{r}') \cdot \nabla G(\overline{r},\overline{r}')$, one has

$$\overline{C} = \frac{-1}{i\omega} \nabla \oint_S \overline{n}' \cdot \overline{J}(\overline{r}') \, G(\overline{r},\overline{r}') \, dS' - \frac{1}{i\omega} \nabla \int_V \nabla \cdot [\overline{J}(\overline{r}') \, G(\overline{r},\overline{r}')] d\overline{r}' .$$

Bringing the ∇ operators inside the integrals, one finds that

$$\overline{C} = \frac{-1}{i\omega} \oint_S \overline{n}' \cdot \overline{J}(\overline{r}') \, \nabla G(\overline{r},\overline{r}') dS' - \frac{1}{i\omega} \int_V \nabla\{\nabla \cdot [J(\overline{r}') \, G(\overline{r},\overline{r}')]\} d\overline{r}' .$$

But since $\nabla\{\nabla \cdot [\overline{J}(\overline{r}') \, G(\overline{r},\overline{r}')]\} = [\nabla\nabla \, G(\overline{r},\overline{r}')] \cdot \overline{J}(\overline{r}')$, it has thus been shown that

$$\int_V \rho(\overline{r}') \, \nabla G(\overline{r},\overline{r}') \, d\overline{r}' = \frac{1}{i\omega} \oint_S \overline{n}' \cdot \overline{J}(\overline{r}') \, \nabla' G(\overline{r},\overline{r}') \, dS'$$
$$- \frac{1}{i\omega} \int_V [\nabla\nabla \, G(\overline{r},\overline{r}')] \cdot \overline{J}(\overline{r}') \, d\overline{r}' . \qquad (2.19)$$

Substitution of equation (2.19) into equation (2.17) yields

$$\overline{E}(\overline{r}) = \int_V [i\omega\mu \, \overline{J}(\overline{r}') \, G(\overline{r},\overline{r}') - \frac{1}{i\omega\varepsilon} [\nabla\nabla \, G(\overline{r},\overline{r}')] \cdot \overline{J}(\overline{r}')] \, d\overline{r}'$$
$$- \oint_S [i\omega\mu \, \overline{n}' \times \overline{H}(\overline{r}') \, G(\overline{r},\overline{r}') + (\overline{n}' \times E(\overline{r}')) \times \nabla' G(\overline{r},\overline{r}') +$$
$$\overline{n}' \cdot \overline{E}(\overline{r}') \, \nabla' G(\overline{r},\overline{r}') - \frac{1}{i\omega\varepsilon} \overline{n}' \cdot \overline{J}(\overline{r}') \, \nabla' G(\overline{r},\overline{r}')] \, dS' .$$

Recalling that $\overline{\overline{I}}$ represents the identity dyadic and that $\omega^2\mu\varepsilon = k^2$, this can be expressed as

$$\bar{E}(\bar{r}) = i\omega\mu \int_V [(\bar{\bar{I}} + \frac{1}{k^2}\nabla\nabla)G] \cdot \bar{J}\, d\bar{r}'$$

$$- \oint_S [i\omega\mu\bar{n}' \times \bar{H}G + (\bar{n}' \times \bar{E}) \times \nabla' G + (\bar{n}' \cdot \bar{E})\nabla' G - \frac{1}{i\omega\varepsilon}(\bar{n}' \cdot \bar{J})\nabla' G]\, dS' .$$

But since

$$\bar{\bar{\Gamma}}(\bar{r},\bar{r}') = (\bar{\bar{I}} + \frac{1}{k^2}\nabla\nabla)\, G(\bar{r},\bar{r}') ,$$

the expression for $\bar{E}$ becomes

$$\bar{E}(\bar{r}) = i\omega\mu \int_V \bar{\bar{\Gamma}} \cdot \bar{J}\, d\bar{r}' - \oint_S [i\omega\mu\bar{n}' \times \bar{H}\, G$$

$$+ (\bar{n}' \times \bar{E}) \times \nabla' G + (\bar{n}' \cdot \bar{E})\nabla' G - \frac{1}{i\omega\varepsilon}(\bar{n}' \cdot \bar{J})\nabla' G]\, dS' . \quad (2.20)$$

If all of the sources are contained in a sphere of finite radius, and if S is taken to be the sphere at infinity, then the surface integral can be shown to vanish and equation (2.20) reduces to equation (2.15).

The far zone is the region where $r >> r'$ and $kr'^2/r << 1$ for every point $\bar{r}'$ at which $\bar{J}(\bar{r}') \neq 0$. In this zone, equations (2.16), (2.20) can be simplified. Since part of this work will consider the nature of certain far-zone fields, this simplification will now be derived.

Consider

$$\bar{\bar{\Gamma}}(\bar{r},\bar{r}') = (\bar{\bar{I}} + \frac{1}{k^2}\nabla\nabla)\, \frac{e^{ik|\bar{r}-\bar{r}'|}}{4\pi|\bar{r}-\bar{r}'|} .$$

By virtue of the form of the arguments in $\bar{\bar{\Gamma}}$, ∇ may be replaced by

$-\nabla'$. Thus,

$$\overline{\overline{\Gamma}}(\overline{r},\overline{r}') = (\overline{\overline{I}} + \frac{1}{k^2}\nabla'\nabla')\frac{e^{ik|\overline{r}-\overline{r}'|}}{4\pi|\overline{r}-\overline{r}'|} .$$

Now

$$|\overline{r}-\overline{r}'| = \sqrt{r^2 - 2\overline{r}\cdot\overline{r}' + r'^2} = r\sqrt{1 - \frac{2}{r^2}\overline{r}\cdot\overline{r}' + \frac{r'^2}{r^2}} .$$

Assuming $r >> r'$, one can make the approximation

$$|\overline{r} - \overline{r}'| \sim r[1 - \frac{1}{2}(\frac{2}{r^2}\overline{r}\cdot\overline{r}' - \frac{r'^2}{r^2})] = r - \overline{e}_r\cdot\overline{r}' + \frac{r'^2}{2r}$$

where $\overline{e}_r$ is a unit vector in the direction of $\overline{r}$. Then since $kr'^2/r << 1$, one obtains

$$\frac{e^{ik|\overline{r}-\overline{r}'|}}{4\pi|\overline{r}-\overline{r}'|} \sim \frac{e^{ikr}}{4\pi r} e^{-ik\overline{e}_r\cdot\overline{r}'} .$$

Thus

$$\nabla'\frac{e^{ik|\overline{r}-\overline{r}'|}}{4\pi|\overline{r}-\overline{r}'|} \sim \frac{e^{ikr}}{4\pi r}\nabla' e^{-ik\overline{e}_r\cdot\overline{r}'} = \frac{ike^{ikr}}{4\pi r} e^{-ik\overline{e}_r\cdot\overline{r}'}\,\overline{e}_r .$$

Hence,

$$\nabla'\nabla'\frac{e^{ik|\overline{r}-\overline{r}'|}}{4\pi|\overline{r}-\overline{r}'|} \sim \frac{-ike^{ikr}}{4\pi r}\overline{e}_r\nabla' e^{-ik\overline{e}_r\cdot\overline{r}'} = -\frac{k^2e^{ikr}}{4\pi r} e^{-ik\overline{e}_r\cdot r'}\,\overline{e}_r\,\overline{e}_r .$$

Substitution of this into the expression for $\overline{\overline{\Gamma}}(\overline{r},\overline{r}')$ yields

$$\overline{\overline{\Gamma}}(\overline{r},\overline{r}') \sim \frac{e^{ikr}}{4\pi r} e^{-ik\overline{e}_r\cdot\overline{r}'}[\overline{\overline{I}} - \overline{e}_r\,\overline{e}_r] .$$

Now it is easily verified that any vector $\overline{C}$ satisfies the relation

$(\overline{\overline{I}} - \overline{e}_r \overline{e}_r) \cdot \overline{C} = -\overline{e}_r \times (\overline{e}_r \times \overline{C})$. Thus, in the far zone, $\overline{\overline{\Gamma}}(\overline{r},\overline{r}') \cdot \overline{J}(\overline{r}') \sim -\frac{e^{ikr}}{4\pi r} e^{-ik\overline{e}_r \cdot \overline{r}'} \overline{e}_r \times (\overline{e}_r \times \overline{J}(\overline{r}'))$. Hence, equations (2.15) and (2.20) become

$$\overline{E}(\overline{r}) = -\frac{i\omega\mu}{4\pi r} e^{ikr} \int_V \overline{e}_r \times (\overline{e}_r \times \overline{J}(\overline{r}')) e^{-ik\overline{e}_r \cdot \overline{r}'} d\overline{r}' \tag{2.21}$$

$$\begin{aligned}\overline{E}(\overline{r}) = {} & \frac{-i\omega\mu}{4\pi r} e^{ikr} \int_V \overline{e}_r \times (\overline{e}_r \times \overline{J}(\overline{r}')) e^{-ik\overline{e}_r \cdot \overline{r}'} d\overline{r}' \\ & - \frac{e^{ikr}}{4\pi r} \oint_S [i\omega\mu\overline{n}' \times \overline{H}(\overline{r}') - ik(\overline{n}' \times \overline{E}(\overline{r}')) \times \overline{e}_r \\ & - ik(\overline{n}' \cdot \overline{E}(r')) \overline{e}_r + \frac{k}{\omega\varepsilon} (\overline{n}' \cdot \overline{J}(\overline{r}'))\overline{e}_r] e^{-ik\overline{e}_r \cdot \overline{r}'} dS'.\end{aligned} \tag{2.22}$$

2.3 Solution of the Wave Equation in a Slightly Inhomogeneous Medium

The wave equation (2.9) was derived under the assumption that μ was a constant. In Section 2.2 the solution of the wave equation in an unbounded medium was found under the additional assumption that ε was also a constant. In this section an approximate solution of the wave equation will be obtained for the case where ε is slightly inhomogeneous. This solution is the tool which will be used to treat the problems dealt with in this work.

Now the wave equation is

$$\nabla \times \nabla \times \overline{E} - \omega^2\mu\varepsilon(\overline{r})\, \overline{E} = i\omega\mu\overline{J}$$

where $\varepsilon(\overline{r})$ has been written to emphasize the fact that the permittivity in the region is inhomogeneous. In particular, let

$$\varepsilon(\bar{r}) = \varepsilon_1[1 + \nu\, \tilde{\varepsilon}(\bar{r})] \tag{2.23}$$

where ε_1 and ν are constants, $|\nu| << 1$, and where $|\varepsilon(\bar{r})| \leq 1$ for all $\bar{r}$. Using a perturbation technique, it is assumed that

$$\bar{E} = \bar{E}_o + \nu\bar{E}_1 + \nu^2\bar{E}_2 + \cdots \quad . \tag{2.24}$$

Substituting the expressions for $\bar{E}$, $\varepsilon(\bar{r})$ into the wave equation and letting $k^2 = \omega^2\mu\varepsilon_1$, one obtains

$$\nabla\times\nabla\times(\bar{E}_o + \bar{E}_1 + {}^2\bar{E}_2 + \cdots) - k^2(1+\nu\tilde{\varepsilon})(\bar{E}_o + \nu\bar{E}_1 + \nu^2\bar{E}_2 + \cdots) = i\omega\mu\bar{J}$$

or

$$\nabla\times\nabla\times(\bar{E}_o + \nu\bar{E}_1 + \nu^2\bar{E}_2 + \cdots) - k^2(\bar{E}_o + \nu\bar{E}_1 + \nu^2 E_2 + \cdots)$$
$$= i\omega\mu\,\bar{J} + k^2\,\tilde{\varepsilon}(\nu\bar{E}_o + \nu^2\bar{E}_1 + \cdots) \ .$$

Equating terms with like powers in ν yields

$$\nabla\times\nabla\times\bar{E}_o - k^2\bar{E}_o = i\omega\mu\bar{J} \tag{2.25}$$

$$\nabla\times\nabla\times\bar{E}_1 - k^2\bar{E}_1 = k^2\tilde{\varepsilon}\,\bar{E}_o \tag{2.26}$$

$$\nabla\times\nabla\times\bar{E}_2 - k^2\bar{E}_2 = k^2\tilde{\varepsilon}\,\bar{E}_1$$
$$\vdots$$

From equation (2.25) it can be seen that $\bar{E}_o$ is the solution for a homogeneous medium, i.e., the solution when $\nu = 0$. Next, from equation (2.26), it can be seen that $\overline{E}_1$ is the solution of the wave equation when the quantity $k^2 \tilde{\varepsilon} \overline{E}_o / i\omega\mu$ acts as the source, and so on.

Now if ν is sufficiently small, one can use the approximation

$$\overline{E} \sim \overline{E}_o + \nu \overline{E}_1 . \tag{2.27}$$

This is known as the Born approximation and it will be the device used to treat problems in this work.

$\overline{E}_1$ can now be obtained through the use of the expressions derived earlier. One merely replaces $\overline{E}$ by $\overline{E}_1$ and $i\omega\mu\overline{J}$ by $k^2 \tilde{\varepsilon} \overline{E}_o$. In the interior of the inhomogeneous medium, equation (2.16) is used:

$$\overline{E}_1(\overline{r}) = k^2 \int_V \overline{\overline{\Gamma}}'(\overline{r},\overline{r}') \; \overline{E}_o(\overline{r}') \; \tilde{\varepsilon}(\overline{r}') d\overline{r}' .$$

Recalling that $\overline{\overline{\Gamma}}'(\overline{r},\overline{r}') = \overline{\overline{\Gamma}}(\overline{r},\overline{r}') - \frac{1}{3k^2} \delta(\overline{r}-\overline{r}') \; \overline{\overline{I}}$, one obtains

$$\overline{E}_1(\overline{r}) = k^2 \int_V \overline{\overline{\Gamma}}(\overline{r},\overline{r}') \cdot \overline{E}_o(\overline{r}') \; \tilde{\varepsilon}(\overline{r}') d\overline{r}' - \frac{1}{3} \tilde{\varepsilon}(\overline{r}) \; \overline{E}_o(\overline{r}) . \tag{2.28}$$

If the inhomogeneity is finite in extent, and if the observation point is sufficiently distant from the region containing the inhomogeneity, then one can use the far-zone expressions for obtaining $\overline{E}_1$. Application of equations (2.21) and (2.22) yields

$$\overline{E}_1(\overline{r}) = \frac{-k^2}{4\pi r} e^{ikr} \int_V \overline{e}_r \times (\overline{e}_r \times \overline{E}_o(\overline{r}')) \tilde{\varepsilon}(\overline{r}') \; e^{-ik\overline{e}_r \cdot \overline{r}'} \; d\overline{r}' \tag{2.29}$$

$$\overline{E}_1(\overline{r}) = \frac{-k^2}{4\pi r} e^{ikr} \int_V \overline{e}_r \times (\overline{e}_r \times \overline{E}_o(\overline{r}')) \tilde{\varepsilon}(\overline{r}') e^{-ik\overline{e}_r \cdot \overline{r}'} d\overline{r}' -$$

$$\frac{e^{ikr}}{4\pi r} \oint_S [i\omega\mu \overline{n}' \times \overline{H}_1(\overline{r}') - ik(\overline{n}' \times \overline{H}_1(\overline{r}')) \times \overline{e}_r - ik(\overline{n}' \cdot \overline{E}_1(\overline{r}'))\overline{e}_r$$

$$-ik(\overline{n}' \cdot \overline{E}_o(\overline{r}') \tilde{\varepsilon}(\overline{r}'))\overline{e}_r] e^{-ik\overline{e}_r \cdot \overline{r}'} dS'. \qquad (2.30)$$

The equations of this part will serve as a starting point for dealing with the problems to be considered in this work. Before work on these problems can begin, however, some elementary aspects of random variables should be reviewed. This is done in the next part.

III. TREATMENT OF RANDOM FUNCTIONS

In treating the problem of wave propagation in a turbulent medium, one must deal with quantities that are random functions. Therefore, it will be helpful to examine briefly the method of handling these quantities.

For the purposes of this work a random variable will be defined in a non-rigorous fashion as a variable whose values must be characterized statistically and cannot be given by a deterministic rule. Statistical characterization here denotes finding averages of various functions of the random variable or finding with what probabilities the variable will assume various values or lie within various ranges of values.

Now consider a variable w which is a function of position, i.e., $w = w(\bar{r})$, and suppose that for each fixed value of $\bar{r}$, w is a random variable. Then w is a random function and in particular it constitutes a random field. In general, the statistics of an arbitrary random field will vary from point to point.

One means of characterizing the random field w is through the use of the correlation function which is defined by $\langle w(\bar{r}_1)\, w(\bar{r}_2)\rangle$. In this expression $\langle\ \rangle$ denotes ensemble average and $\bar{r}_1$, $\bar{r}_2$ are two arbitrary points. The correlation function is symbolized by $R_w(\bar{r}_1,\bar{r}_2)$.

It may be that the correlation function of $w(\bar{r})$ depends only upon the difference of the arguments, i.e., $R_w(\bar{r}_1,\bar{r}_2) = R_w(\bar{r}_1-\bar{r}_2)$, and that the mean of $w(\bar{r})$, defined as $\langle w(\bar{r})\rangle$, is independent of $\bar{r}$. In such a case the random field is said to be homogeneous. Furthermore, the correlation of a homogeneous field w might only depend on the magnitude of the difference of the arguments, i.e., $R_w(\bar{r}_1,\bar{r}_2) = R_w(|\bar{r}_1-\bar{r}_2|)$. In this case the field is isotropic as well as homogeneous.

In the problems dealt with in this work, homogeneous, isotropic random fields will be encountered, and the correlation function will prove to be very useful. The Fourier transform of the correlation function, known as the spectral density, will also prove to be of considerable value in this case. Letting S represent the spectral density, and letting R be the correlation function of the homogeneous isotropic random field, S is given by

$$S(\bar{\kappa}) = \frac{1}{(2\pi)^3} \int R(r)\, e^{-i\bar{\kappa}\cdot\bar{r}}\, d\bar{r} \tag{3.1}$$

where the integration is performed over all space. Since R depends only on the magnitude of $\bar{r}$, the angular integrations in the preceding expression can be performed at once and it can be seen that S will depend only on the magnitude of $\bar{\kappa}$. The complete transformation is thus specified by

$$R(r) = \int S(\kappa)\, e^{i\bar{\kappa}\cdot\bar{r}}\, d\bar{\kappa} \tag{3.2}$$

where the integration is performed over all $\bar{\kappa}$ space, and

$$S(\kappa) = \frac{1}{(2\pi)^3} \int R(r)\, e^{-i\bar{\kappa}\cdot\bar{r}}\, d\bar{r} \, . \tag{3.3}$$

If $\bar{r} = \bar{r}_1 - \bar{r}_2$, then equation (3.2) yields

$$R(|\overline{r}_1 - \overline{r}_2|) = \int S(\kappa)\, e^{i\overline{\kappa}\cdot(\overline{r}_1-\overline{r}_2)}\, d\overline{\kappa}\ . \tag{3.4}$$

As a final remark on random functions, it can be seen that any homogeneous, isotropic random field can be expressed as the sum of a constant plus another homogeneous, isotropic field whose mean is zero. In particular, if $w(\overline{r})$ is a homogeneous, isotropic random field, one observes that

$$w(\overline{r}) = \langle w(\overline{r})\rangle + [w(\overline{r}) - \langle w(\overline{r})\rangle]\ .$$

Letting $w_1 = \langle w(\overline{r})\rangle$, which is a constant because $w(\overline{r})$ is homogeneous, and letting $\tilde{w}(\overline{r}) = w(\overline{r}) - \langle w(\overline{r})\rangle$, one obtains

$$w(\overline{r}) = w_1 + \tilde{w}(\overline{r}) \tag{3.5}$$

where

$$\langle \tilde{w}(\overline{r})\rangle = \langle w(\overline{r}) - \langle w(\overline{r})\rangle\rangle = \langle w(\overline{r})\rangle - \langle w(\overline{r})\rangle = 0$$

and where

$$R_{\tilde{w}}(\overline{r}_1,\overline{r}_2) = \langle \tilde{w}(\overline{r}_1)\, \tilde{w}(\overline{r}_2)\rangle = \langle w(\overline{r}_1)\, w(\overline{r}_2) - w(\overline{r}_1)\, w_1$$

$$- w(\overline{r}_2)\, w_1 + w_1^2\rangle = R_w(|\overline{r}_1 - \overline{r}_2|) - w_1^2 = R_{\tilde{w}}(|\overline{r}_1 - \overline{r}_2|)\ .$$

IV. SCATTERED FIELD OF A DIPOLE LOCATED AT THE CENTER OF A RANDOMLY-INHOMOGENEOUS DIELECTRIC SPHERE

The problem of electromagnetic wave scattering by a sphere which has a dipole located at its center will now be considered. The sphere is composed of an inhomogeneous medium having random variations in its permittivity. These variations are assumed to arise from turbulent motion of the medium; however, the physical factors responsible for the turbulence and for the shape of the volume are beyond the scope of this work.

At any instant of time, the variations in the permittivity are assumed to constitute a homogeneous, isotropic random field. Furthermore, for the purposes of this work, the velocities characterizing the turbulence are assumed to be sufficiently low so that temporal variations in the permittivity may be neglected.

Inside the sphere, the permittivity $\varepsilon(\overline{r})$ is given by $\varepsilon_1[1 + \tilde{\varepsilon}(\overline{r})]$ where $\varepsilon_1 = \langle\varepsilon(\overline{r})\rangle$ and where $\tilde{\varepsilon}(\overline{r})$ is a homogeneous, isotropic random field with $\langle\tilde{\varepsilon}(\overline{r})\rangle = 0$, $|\tilde{\varepsilon}(\overline{r})|_{max} \lll 1$. Furthermore, $\tilde{\varepsilon}(\overline{r})$ is assumed to have a Gaussian correlation function of the form $\nu^2 \exp[-r^2/\ell^2]$ where $\nu^2 = \langle\tilde{\varepsilon}^2\rangle = \langle\varepsilon^2\rangle - \varepsilon_1^2$ and where ℓ, the correlation distance, is much less than the radius R of the sphere, i.e., $\ell/R \ll 1$. Although it is more common now to characterize a turbulent medium through the use of the structure function derived in Kolmogorov's theory of turbulence [11], the Gaussian correlation function is still frequently employed [12,13,14] and it will be assumed here that its use is justified.

Outside the sphere the permittivity is equal to the constant ε_1, the mean value of the permittivity inside the sphere. The permeability μ is assumed to be a constant everywhere.

The coordinate system employed in the problem has its origin at the center of the sphere and has its z axis aligned with the dipole. By virtue of the symmetry of the problem, the observation point, i.e., the point at which the fields are examined, may be chosen to lie in the x-z plane without loss of generality. The situation is illustrated in Fig. 4.1. In this figure, the vectors $\overline{e}_{r_o}$, $\overline{e}_{\theta_o}$, $\overline{e}_{\phi_o}$ at the observation point are unit vectors corresponding to a spherical coordinate system. Spherical coordinates are illustrated in Fig. 4.2.

4.1 Preliminary Development

Some preliminary expressions to be used in dealing with the problem will now be developed.

Let a current source $\overline{J}$ be situated inside the sphere. The quantities to be sought are the components of the electric field in the far zone (the precise meaning far zone for this problem will be considered shortly). Now it was stated earlier that the permittivity of the medium of the sphere was given by $\varepsilon_1[1 + \tilde{\varepsilon}(\overline{r})]$. Provided that $|\tilde{\varepsilon}(\overline{r})|_{max}$ is sufficiently small, the Born approximation developed in Section 2.3 can be used to obtain a solution. In particular, using equations (2.25), (2.26), (2.27), the total electric field will be assumed to be given by

$$\overline{E} = \overline{E}_o + \overline{E}_1 \tag{4.1}$$

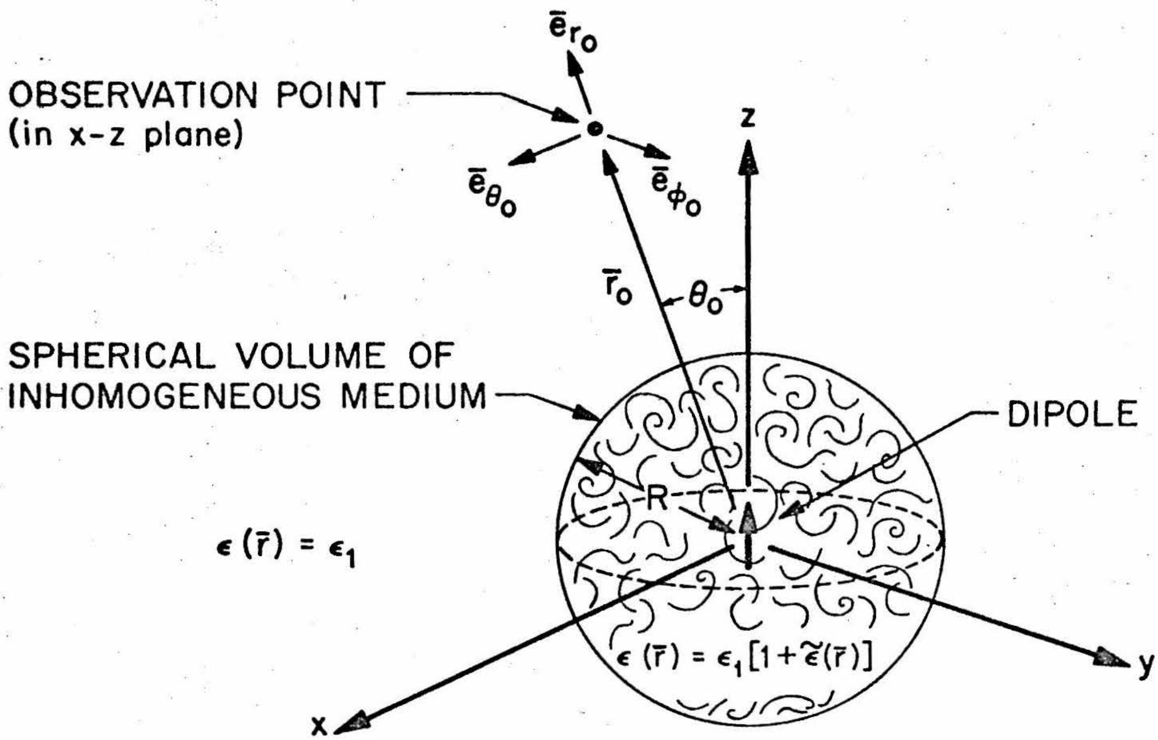

Figure 4.1. Randomly-Inhomogeneous Sphere with a Dipole at the Center

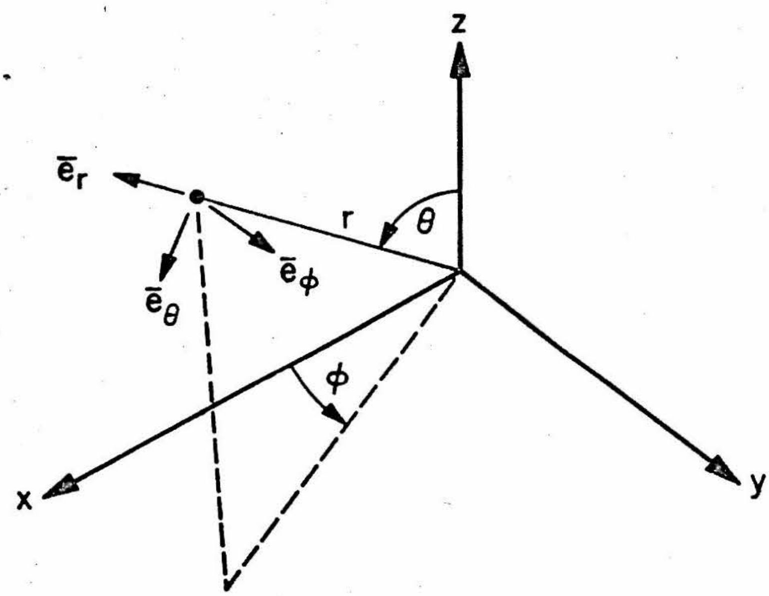

Figure 4.2. Spherical Coordinate System

where $\overline{E}_o, \overline{E}_1$ satisfy

$$\nabla \times \nabla \times \overline{E}_o - k^2 \overline{E}_o = i\omega\mu \overline{J} \tag{4.2}$$

$$\nabla \times \nabla \times \overline{E}_1 - k^2 \overline{E}_1 = k^2 \tilde{\varepsilon} \overline{E}_o \quad (k^2 = \omega^2 \mu \varepsilon_1) . \tag{4.3}$$

In obtaining equations (4.1), (4.2), the parameter ν that appears in Section 2.3 is assumed to be incorporated into both $\overline{E}_1$, the scattered field, and $\tilde{\varepsilon}$. Reiterating part of the discussion of that section, $\overline{E}_o$ is the electric field generated by the source $\overline{J}$ when all space is homogeneous with permittivity ε_1, and $\overline{E}_1$ is the field generated when the source is given by $k^2 \tilde{\varepsilon} \overline{E}_o / i\omega\mu$. Thus, the inhomogeneities of the sphere, when acted upon by the zeroth order field, become the new sources. Physically, $\overline{E}_o$ may be thought of as generating electric dipoles in the medium of the sphere which in turn produce $\overline{E}_1$.

In Section 2.2 the far zone was defined as the region for which $r >> r'$, $kr'^2/r << 1$ at every point $\overline{r}'$ for which the source $\overline{J}(\overline{r}') \neq 0$. In this case, for the $\overline{E}_1$ field, the entire sphere acts as the source. Noting that the radius of the sphere is R and letting $\overline{r}_o$ be the location of the observation point, the far zone is thus defined by $r_o >> R$, $kR^2/r_o << 1$ or $r_o >> kR^2$.

Under the assumption that the observation point lies in the far zone, equations (2.29) and (2.30) may be applied. Thus at the observation point $\overline{r}_o$ one obtains the following expressions for the scattered field:

$$\overline{E}_1(\overline{r}_o) = \frac{-k^2 e^{ikr_o}}{4\pi r_o} \int_V \overline{e}_{r_o} \times [\overline{e}_{r_o} \times \overline{E}_o(\overline{r})] \, \tilde{\varepsilon}(r) \, e^{-ik\overline{e}_{r_o} \cdot \overline{r}} \, d\overline{r} \quad (4.4)$$

$$\overline{E}_1(\overline{r}_o) = \frac{-k^2 e^{ikr_o}}{4\pi r_o} \int_{V'} \overline{e}_{r_o} \times [\overline{e}_{r_o} \times \overline{E}_o(\overline{r})] \, \tilde{\varepsilon}(\overline{r}) \, e^{-ik\overline{e}_{r_o} \cdot \overline{r}} \, d\overline{r}$$

$$- \frac{e^{ikr_o}}{4\pi r_o} \oint_{S''} \{ i\omega\mu \, \overline{n} \times \overline{H}_1(\overline{r}) - ik[\overline{n} \times \overline{E}_1(\overline{r})] \times \overline{e}_{r_o}$$

$$-ik[\overline{n} \cdot (\overline{E}_1(\overline{r}) + \overline{E}_o(\overline{r})] \, \overline{e}_{r_o} \, \tilde{\varepsilon}(\overline{r}) \} \, e^{-ik\overline{e}_{r_o} \cdot \overline{r}} \, dS \, . \quad (4.5)$$

The V in equation (4.4) is the entire volume of the sphere, while the V' in equation (4.5) is the volume of the sphere that remains when an arbitrary volume V'' with surface S'' is deleted from it.

Let $\overline{E}_1$ be given by equation (4.4). It will be found that even in those instances where calculation for $\overline{E}_1$ begins with equation (4.5), most of the following development will still be applicable.

Let $\overline{e}_\alpha$ represent either one of the unit vectors $\overline{e}_{\theta_o}$, $\overline{e}_{\phi_o}$ and let $E_{1\alpha}$ be the component of E_1 in the direction of $\overline{e}_\alpha$. Then from equation (4.4) one obtains

$$\overline{E}_{1\alpha}(\overline{r}_o) = \frac{-k^2 e^{ikr_o}}{4\pi r_o} \int_V \overline{e}_\alpha \cdot \{\overline{e}_{r_o} \times [\overline{e}_{r_o} \times \overline{E}_o(\overline{r})]\} \, \tilde{\varepsilon}(\overline{r}) \, e^{-ik\overline{e}_{r_o} \cdot \overline{r}} \, d\overline{r} \, .$$

One notes that $E_{1r_o}(\overline{r}_o) = 0$ since $\overline{e}_{r_o} \cdot \{\overline{e}_{r_o} \times [\overline{e}_{r_o} \times \overline{E}_o(\overline{r})]\} \equiv 0$ for any $\overline{E}_o$. Furthermore, if $\overline{e}_\alpha$ equals $\overline{e}_{\theta_o}$ or $\overline{e}_{\phi_o}$, then it is easy to show that

$$\overline{e}_\alpha \cdot \{\overline{e}_{r_o} \times [\overline{e}_{r_o} \times \overline{E}_o(\overline{r})]\} = -\overline{e}_\alpha \cdot E_o(\overline{r})$$

for any $\overline{E}_o(\overline{r})$. Hence,

$$E_{1\alpha}(\bar{r}_o) = \frac{k^2 e^{ikr_o}}{4\pi r_o} \int_V \bar{e}_\alpha \cdot \bar{E}_o(\bar{r})\, \tilde{\varepsilon}(\bar{r})\, e^{-ik\bar{e}_{r_o} \cdot \bar{r}}\, d\bar{r} \,. \tag{4.6}$$

Now $\tilde{\varepsilon}(\bar{r})$ is a random field whose spatial dependence cannot be explicitly given. Hence, equation (4.6) cannot be evaluated as it stands. However, the mean and correlation functions of $\tilde{\varepsilon}$ are known and because of this it will now be shown that it is possible to calculate certain averages of the field components.

Taking the ensemble average of equation (4.6) yields

$$\langle E_{1\alpha} \rangle = \left\langle \frac{k^2 e^{ikr_o}}{4\pi r_o} \int_V \bar{e}_\alpha \cdot \bar{E}_o(\bar{r})\, \tilde{\varepsilon}(\bar{r})\, e^{-ik\bar{e}_{r_o} \cdot \bar{r}}\, d\bar{r} \right\rangle$$

or

$$\langle E_1 \rangle = \frac{k^2 e^{ikr_o}}{4\pi r_o} \int_V \bar{e}_\alpha \cdot \bar{E}_o(\bar{r})\, \langle \tilde{\varepsilon}(\bar{r}) \rangle\, e^{-ik\bar{e}_{r_o} \cdot \bar{r}}\, d\bar{r} \,.$$

But $\langle \varepsilon(\bar{r}) \rangle = 0$. Hence, one finds that

$$\langle \bar{E}_{1\alpha}(\bar{r}_o) \rangle = 0 \,. \tag{4.7}$$

Since equation (4.7) holds for all the components of $\bar{E}_1$, it is apparent that

$$\langle \bar{E}_1(\bar{r}_o) \rangle = 0 \,. \tag{4.8}$$

This will be true for all the problems considered in this work.

Now let $\bar{e}_\beta$ also represent either one of the unit vectors $\bar{e}_{\theta_o}$, $\bar{e}_{\phi_o}$, and let $E_{1\beta}$ be the corresponding component of $\bar{E}_1$. Then just as in equation (4.6) one may write

$$E_{1\beta}(\bar{r}_o) = \frac{k^2 e^{ikr_o}}{4\pi r_o} \int_V \bar{e}_\beta \cdot \bar{E}_o(\bar{r}') \, \tilde{\varepsilon}(\bar{r}') \, e^{-ik\bar{e}_{r_o} \cdot \bar{r}'} \, d\bar{r}'. \qquad (4.9)$$

The expression for $\langle E_{1\alpha}(\bar{r}_o) \, E^*_{1\beta}(\bar{r}_o) \rangle$ can now be formed:

$$\langle E_{1\alpha}(\bar{r}_o) \, E^*_{1\beta}(\bar{r}_o) \rangle = \Big\langle \frac{k^4}{16\pi^2 r_o^2} \int_V \bar{e}_\alpha \cdot \bar{E}_o(\bar{r}) \, \tilde{\varepsilon}(\bar{r}) \, e^{-ik\bar{e}_{r_o} \cdot \bar{r}} \, d\bar{r}$$

$$\times \int_V \bar{e}_\beta \cdot \bar{E}^*_o(\bar{r}') \, \tilde{\varepsilon}(\bar{r}') \, e^{ik\bar{e}_{r_o} \cdot \bar{r}'} \, d\bar{r}' \Big\rangle$$

or

$$\langle E_{1\alpha}(\bar{r}_o) E^*_{1\beta}(\bar{r}_o) \rangle = \frac{k^4}{16\pi^2 r_o^2} \int_V \int_V \bar{e}_\alpha \cdot \bar{E}_o(\bar{r}) \, \bar{e}_\beta \cdot \bar{E}^*_o(\bar{r}') \, e^{-ik\bar{e}_{r_o} \cdot (\bar{r}-\bar{r}')}$$

$$\times \, \langle \varepsilon(\bar{r}) \, \varepsilon(\bar{r}') \rangle \, d\bar{r} \, d\bar{r}' .$$

But $\langle \tilde{\varepsilon}(\bar{r}) \, \tilde{\varepsilon}(r') \rangle = R_{\tilde{\varepsilon}}(|\bar{r}-\bar{r}'|)$ where $R_{\tilde{\varepsilon}}$ is the correlation function of $\tilde{\varepsilon}$. Thus,

$$\langle E_{1\alpha} E^*_{1\beta} \rangle = \frac{k^4}{16\pi^2 r_o^2} \int_V \int_V \bar{e}_\alpha \cdot \bar{E}_o(\bar{r}) \, \bar{e}_\beta \cdot \bar{E}^*_o(\bar{r}') \, e^{-ik\bar{e}_{r_o} \cdot (\bar{r}-\bar{r}')}$$

$$\times \, R_{\tilde{\varepsilon}}(|\bar{r}-\bar{r}'|) \, d\bar{r} \, d\bar{r}' . \qquad (4.10)$$

Since the correlation function $R_{\tilde{\varepsilon}}$ is assumed known, the integrals in equation (4.10) may be evaluated at least in principle.

When $\alpha = \beta$, equation (4.10) gives the mean square value of the α component of the scattered field. When $\alpha \neq \beta$, equation (4.10) yields an average of the product of the two transverse components of the field. These are the specific quantities which will be sought in the analysis of the field.

In certain cases it will prove useful to express the correlation function used in equation (4.10) as a Fourier integral of the spectral density S_ε . From equation (3.3) one has

$$R_{\tilde{\varepsilon}}(|\bar{r}-\bar{r}'|) = \int S_{\tilde{\varepsilon}}(\kappa)\, e^{i\bar{\kappa}\cdot(\bar{r}-\bar{r}')}\, d\bar{\kappa}$$

where the region of integration is all of $\bar{\kappa}$ space. Substitution of this into equation (4.10) yields

$$\langle E_{1\alpha}E^*_{1\beta}\rangle = \frac{k^4}{16\pi^2 r_o^2}\int_V\int_V\int \bar{e}_\alpha\cdot\bar{E}_o(\bar{r})\, \bar{e}_\beta\cdot\bar{E}^*_o(\bar{r}')\, e^{-ik\bar{e}_{r_o}\cdot(\bar{r}-\bar{r}')}$$

$$S_{\tilde{\varepsilon}}(\kappa)\, e^{i\bar{\kappa}\cdot(\bar{r}-\bar{r}')}\, d\bar{\kappa}\, d\bar{r}\, d\bar{r}' . \qquad (4.11)$$

By changing the order of integration and rearranging terms, one obtains

$$\langle E_{1\alpha}E^*_{1\beta}\rangle = \frac{k^2}{16\pi^2 r_o^2}\int S_{\tilde{\varepsilon}}(\kappa)\int_V\int_V [\bar{e}_\alpha\cdot\bar{E}_o(\bar{r})e^{-i\bar{r}\cdot(k\bar{e}_{r_o}-\bar{\kappa})}]$$

$$\times\ [\bar{e}_\beta\cdot\bar{E}^*_o(\bar{r}')\, e^{i\bar{r}'\cdot(k\bar{e}_{r_o}-\bar{\kappa})}]\, d\bar{r}\, d\bar{r}'\, d\bar{\kappa} \qquad (4.12)$$

or

$$\langle E_{1\alpha}E^*_{1\beta}\rangle = \frac{k^4}{16\pi^2 r_o^2}\int S_{\tilde{\varepsilon}}(\kappa)\left\{\int_V \bar{e}_\alpha\cdot\bar{E}_o(\bar{r})\, e^{-i\bar{r}\cdot(k\bar{e}_{r_o}-\bar{\kappa})}\, d\bar{r}\right.$$

$$\left.\times\int_V \bar{e}_\beta\cdot\bar{E}^*_o(\bar{r}')\, e^{i\bar{r}'\cdot(k\bar{e}_{r_o}-\bar{\kappa})}\, d\bar{r}'\right\} d\bar{\kappa} . \qquad (4.13)$$

If $\alpha = \beta$, then equation (4.13) becomes

$$<|E_{1\alpha}|^2> = \frac{k^4}{16\pi^2 r_o^2} \int_\varepsilon S_\nu(\kappa) \left| \int_V \bar{e}_\alpha \cdot \bar{E}_o(\bar{r})\, e^{-i\bar{r}\cdot(k\bar{e}_{r_o} - \bar{\kappa})} d\bar{r} \right|^2 d\bar{\kappa}\, . \tag{4.14}$$

In performing some of the operations indicated in equations (4.10) and (4.14) it will be helpful if $\bar{E}_o$ is expressed in terms of the unit vectors at the observation point $\bar{e}_{r_o}$, $\bar{e}_{\theta_o}$, $\bar{e}_{\phi_o}$. Now E_o will be given initially in terms of the unit vectors $\bar{e}_r$, $\bar{e}_\theta$, $\bar{e}_\phi$. Thus, one must express these latter vectors in terms of the unit vectors at the observation point. This is done in the following manner. Referring to Fig. 4.2, it can be easily shown that

$$\bar{e}_r = \sin\theta \cos\phi\, \bar{e}_x + \sin\theta \sin\phi\, \bar{e}_y + \cos\theta\, \bar{e}_z \tag{4.15}$$

$$\bar{e}_\theta = \cos\theta \cos\phi\, \bar{e}_x + \cos\theta \sin\phi\, \bar{e}_y - \sin\theta\, \bar{e}_z \tag{4.16}$$

$$\bar{e}_\phi = -\sin\phi\, \bar{e}_x + \cos\phi\, \bar{e}_y\, . \tag{4.17}$$

In particular, at $\bar{r} = \bar{r}_o$, one has $\theta = \theta_o$, $\phi = \phi_o = 0$, and hence

$$\bar{e}_{r_o} = \sin\theta_o\, \bar{e}_x + \cos\theta_o\, \bar{e}_z \tag{4.18}$$

$$\bar{e}_{\theta_o} = \cos\theta_o\, \bar{e}_x - \sin\theta_o\, \bar{e}_z \tag{4.19}$$

$$\bar{e}_{\phi_o} = \bar{e}_y\, . \tag{4.20}$$

Now $\bar{e}_r$, $\bar{e}_\theta$, $\bar{e}_\phi$ can be expressed as

$$\bar{e}_r = (\bar{e}_r \cdot \bar{e}_{r_o})\bar{e}_{r_o} + (\bar{e}_r \cdot \bar{e}_{\theta_o})\bar{e}_{\theta_o} + (\bar{e}_r \cdot \bar{e}_{\phi_o})\bar{e}_{\phi_o} \tag{4.21}$$

$$\bar{e}_\theta = (\bar{e}_\theta \cdot \bar{e}_{r_o})\bar{e}_{r_o} + (\bar{e}_\theta \cdot \bar{e}_{\theta_o})\bar{e}_{\theta_o} + (\bar{e}_\theta \cdot \bar{e}_{\phi_o})\bar{e}_{\phi_o} \tag{4.22}$$

$$\bar{e}_\phi = (\bar{e}_\phi \cdot \bar{e}_{r_o})\bar{e}_{r_o} + (\bar{e}_\phi \cdot \bar{e}_{\theta_o})\bar{e}_{\theta_o} + (\bar{e}_\phi \cdot \bar{e}_{\phi_o})\bar{e}_{\phi_o} . \tag{4.23}$$

From equations (4.15) and (4.18) one finds that $\bar{e}_r \cdot \bar{e}_{r_o} =$ $(\sin\theta\cos\phi\,\bar{e}_x + \sin\theta\sin\phi\,\bar{e}_y + \cos\theta\,\bar{e}_z)\cdot(\sin\theta_o\,\bar{e}_x + \cos\theta_o\,\bar{e}_y) =$ $\sin\theta_o\sin\theta\cos\phi + \cos\theta_o\cos\theta$. Substituting this and similar expressions for $\bar{e}_r \cdot \bar{e}_{\theta_o}$, $\bar{e}_\theta \cdot \bar{e}_{r_o}$, etc. into equations (4.21) - (4.23), one obtains

$$\begin{aligned}\bar{e}_r &= (\sin\theta_o\sin\theta\cos\phi + \cos\theta_o\cos\theta)\bar{e}_{r_o} \\ &+ (\cos\theta_o\sin\theta\cos\phi - \sin\theta_o\cos\theta)\bar{e}_{\theta_o} + \sin\theta\sin\phi\,\bar{e}_{\phi_o}\end{aligned} \tag{4.24}$$

$$\begin{aligned}\bar{e}_\theta &= (\sin\theta_o\cos\theta\cos\phi - \cos\theta_o\sin\theta)\bar{e}_{r_o} \\ &+ (\cos\theta_o\cos\theta\cos\phi + \sin\theta_o\sin\theta)\bar{e}_{\theta_o} + \cos\theta\sin\phi\,\bar{e}_{\phi_o}\end{aligned} \tag{4.25}$$

$$\bar{e}_\phi = -\sin\theta_o\sin\phi\,\bar{e}_{r_o} - \cos\theta_o\sin\phi\,\bar{e}_{\theta_o} + \cos\phi\,\bar{e}_{\phi_o} . \tag{4.26}$$

The material of this section can now be applied to calculating the mean-square components and the mean of the product of the transverse components of the problems at hand.

4.2 Scattering for the Case of $k\ell >> 1$

It will now be assumed that $k\ell >> 1$, which, because $k = \omega\sqrt{\mu\varepsilon_1} = 2\pi/\lambda$, is equivalent to $\lambda << \ell$. Since it was given that $R/\ell >> 1$, it follows that $kR >>> 1$.

The case of the magnetic dipole will be considered first. Then the case of the electric dipole will be treated.

A. Scattering with the Magnetic Dipole

(i) Derivation

Let a z-oriented magnetic dipole be situated at the center of the randomly-inhomogeneous sphere. From Papas [15], the electric field of such a dipole in a homogeneous medium with permittivity ε_1 and permeability μ is given by

$$\overline{E}_o(\overline{r}) = \frac{i\omega\mu m}{4\pi}\left(\frac{ik}{r} + \frac{1}{r^2}\right) e^{ikr} \sin\theta\, \overline{e}_\phi$$

where m is the strength of the dipole, and where $k = \omega\sqrt{\mu\varepsilon_1}$. Letting $\mathcal{E} = \omega\mu m/4\pi$, this equation becomes

$$\overline{E}_o(\overline{r}) = \mathcal{E}\left(\frac{1}{r} + \frac{i}{kr^2}\right) e^{ikr} \sin\theta\, \overline{e}_\phi \, . \tag{4.27}$$

It will be assumed that $\mathcal{E}$ is real. Next, one finds from equation (4.26) that

$$\overline{e}_\phi = -\sin\theta_o \sin\phi\, \overline{e}_{r_o} - \cos\theta_o \sin\phi\, \overline{e}_{\theta_o} + \cos\phi\, \overline{e}_{\phi_o} \, .$$

Substituting this into equation (4.27), one obtains

$$\begin{aligned}\overline{E}_o(\overline{r}) = &-\mathcal{E}\sin\theta_o\left(\frac{1}{r} + \frac{i}{kr^2}\right) e^{ikr} \sin\theta \sin\phi\, \overline{e}_{r_o} \\ &-\mathcal{E}\cos\theta_o\left(\frac{1}{r} + \frac{i}{kr^2}\right)\sin\theta \sin\phi\, \overline{e}_{\theta_o} + \mathcal{E}\left(\frac{1}{r}+\frac{i}{kr^2}\right)\sin\theta\cos\phi\, \overline{e}_{\phi_o} \, .\end{aligned} \tag{4.28}$$

Now $\overline{E}_o(\overline{r})$ has a $1/r^2$ singularity at the origin which, however, is integrable. Therefore, the scattered field $\overline{E}_1(\overline{r})$ may be given in the far-zone by equation (4.4), i.e.,

$$\overline{E}_1(\overline{r}_o) = \frac{-k^2 e^{ikr_o}}{4\pi r_o} \int_V \overline{e}_{r_o} \times [\overline{e}_{r_o} \times \overline{E}_o(\overline{r})] \, \tilde{\varepsilon}(\overline{r}) \, e^{-ik\overline{e}_{r_o} \cdot \overline{r}} \, d\overline{r}$$

where V is the volume of the sphere. Hence, the expression for the mean square of the components or the mean of the product of the transverse components may be obtained immediately from equation (4.13):

$$<E_{1\alpha}E^*_{1\beta}> = \frac{k^4}{16\pi^2 r_o^2} \int S_{\tilde{\varepsilon}}(\kappa) \left\{ \int_V \overline{e}_\alpha \cdot \overline{E}_o(\overline{r}) \, e^{-i\overline{r}\cdot(\overline{k}-\overline{\kappa})} \, d\overline{r} \right.$$

$$\left. \times \int_V \overline{e}_\beta \cdot \overline{E}^*_o(\overline{r}') \, e^{i\overline{r}'\cdot(\overline{k}-\overline{\kappa})} \, d\overline{r}' \right\} d\overline{\kappa}$$

where $\overline{k} = k\,\overline{e}_{r_o}$ and where the $\overline{\kappa}$ integration is performed over all $\overline{\kappa}$ space. Let

$$L_\alpha(\overline{\kappa}) \equiv \int_V \overline{e}_\alpha \cdot \overline{E}_o(\overline{r}) \, e^{i\overline{\kappa}'\cdot\overline{r}} \, d\overline{r} \tag{4.29}$$

where $\overline{\kappa}' = \overline{\kappa} - \overline{k}$. Then

$$< E_{1\alpha}(\overline{r}_o) \, E^*_{1\beta}(\overline{r}_o) > = \frac{k^4}{16\pi^2 r_o^2} \int S_{\tilde{\varepsilon}}(\kappa) \, L_\alpha(\overline{\kappa}) L^*_\beta(\overline{\kappa}) \, d\overline{\kappa} \, . \tag{4.30}$$

Equation (4.29) will now be considered. Substituting equation (4.28) into equation (4.29) and noting that $\overline{e}_\alpha \cdot \overline{e}_{r_o} = 0$ since $\overline{e}_\alpha = \overline{e}_{\theta_o}$ or $\overline{e}_{\phi_o}$, one finds that

$$L_\alpha(\overline{\kappa}) = \int_V \overline{e}_\alpha \cdot \left\{ - \mathcal{E} \cos\theta_o \left(\frac{1}{r} + \frac{i}{kr^2}\right) e^{ikr} \sin\theta \sin\phi \, \overline{e}_{\theta_o} \right.$$

$$\left. + \mathcal{E}\left(\frac{1}{r} + \frac{i}{kr^2}\right) e^{ikr} \sin\theta \cos\phi \, \overline{e}_{\phi_o} \right\} e^{i\overline{\kappa}'\cdot\overline{r}} \, d\overline{r} \, . \tag{4.31}$$

In particular,

$$L_{\theta_o}(\overline{\kappa}) = -E \cos\theta_o \int_V \left(\frac{1}{r}+\frac{i}{kr^2}\right) e^{ikr}\, e^{i\overline{\kappa}\cdot\overline{r}} \sin\theta \sin\phi \, d\overline{r} \qquad (4.32)$$

$$L_{\phi_o}(\overline{\kappa}) = E \int_V \left(\frac{1}{r}+\frac{i}{kr^2}\right) e^{ikr}\, e^{i\overline{\kappa}'\cdot\overline{r}} \sin\theta \cos\phi \, d\overline{r} \,. \qquad (4.33)$$

The integrations in equations (4.32), (4.33) will be facilitated by expressing the integrand in terms of associated Legendre polynomials and spherical Hankel and Bessel functions. From Appendix A,

$$h_1(kr) = \left[\frac{-1}{kr} - \frac{i}{(kr)^2}\right] e^{ikr} \;, \quad P_1^1(\cos\theta) = \sin\theta \,.$$

Expressing equations (4.32), (4.33) in terms of these yields

$$L_{\theta_o}(\overline{\kappa}) = Ek \cos\theta_o \int_V h_1(kr)\, P_1^1(\cos\theta) \sin\phi \, e^{i\overline{\kappa}'\cdot\overline{r}} \, d\overline{r} \qquad (4.34)$$

$$L_{\phi_o}(\overline{\kappa}) = -Ek \int_V h_1(kr)\, P_1^1(\cos\theta)\, e^{i\overline{\kappa}'\cdot\overline{r}} \, d\overline{r} \,. \qquad (4.35)$$

Let $\overline{\kappa}'$ be given by

$$\overline{\kappa}' = \kappa_x' \overline{e}_x + \kappa_y' \overline{e}_y + \kappa_z' \overline{e}_z = \kappa' \sin\psi \cos\gamma\, \overline{e}_x + \kappa' \sin\psi \sin\gamma\, \overline{e}_y + \kappa' \cos\psi\, \overline{e}_z$$

where ψ, γ correspond to the angles θ, ϕ, respectively, in spherical coordinates. Then $e^{i\overline{\kappa}'\cdot\overline{r}}$ may be expanded as the sum of spherical waves. By using formula (A.18), one obtains

$$e^{i\overline{\kappa}\cdot\overline{r}} = \sum_{n=0}^{\infty} i^n (2n+1) j_n(\kappa' r) \left[\sum_{m=0}^{n} \alpha_m \frac{(n-m)!}{(n+m)!} P_n^m(\cos\psi) P_n^m(\cos\theta) \cos m(\gamma-\phi)\right] \qquad (4.36)$$

where

$$\alpha_m = \begin{cases} 1 & m = 0 \\ 2 & m \neq 0 \end{cases} .$$

Substitution of equation (4.36) into equations (4.34) and (4.35) yields

$$L_{\theta_o} = Ek \cos \theta_o \int_V h_1(kr)\, P_1^1(\cos\theta) \sin \phi \sum_{n=0}^{\infty} \{ i^n (2n+1) j_n(\kappa' r)$$

$$\times \left[\sum_{m=0}^{n} \alpha_m \frac{(n-m)!}{(n+m)!} P_n^m(\cos\psi) P_n^m(\cos\theta) \cos m(\gamma-\phi) \right] \} \, d\bar{r} \qquad (4.37)$$

$$L_{\phi_o} = -Ek \int_V h_1(kr)\, P_1^1(\cos\theta) \cos \phi \sum^{\infty} \{ i^n (2n+1) j_n(\kappa' r)$$

$$\times \left[\sum_{m=0}^{n} \alpha_m \frac{(n-m)!}{(n+m)!} P_n^m(\cos\psi) P_n^m(\cos\theta) \cos m(\gamma-\phi) \right] \} \, d\bar{r} \, . \qquad (4.38)$$

The evaluation of equation (4.37) will now be considered. Changing the order of summation and integration yields

$$L_{\theta_o} = Ek \cos \theta_o \sum_{n=0}^{\infty} \{ i^n (2n+1) \left[\sum_{m=0}^{n} \alpha_m \frac{(n-m)!}{(n+m)!} P_1^m(\cos\psi) \int_V h_1(kr) \right.$$

$$\times \left. P_1^1(\cos\theta) \sin\phi \, j_n(\kappa' r) P_n^m(\cos\theta) \cos m(\phi-\gamma) \, d\bar{r} \right] \}$$

or

$$L_{\theta_o} = Ek \cos \theta_o \sum_{n=0}^{\infty} \{ i^n (2n+1) \left[\sum_{m=0}^{n} \alpha_m \frac{(n-m)!}{(n+m)!} P_n^m(\cos\psi) \int_0^R \int_0^{\pi} \int_0^{2\pi} h_1(kr) \right.$$

$$\times \left. P_1^1(\cos\theta) \sin\phi \, j_n(\kappa' r) P_n^m(\cos\theta) \cos m(\gamma-\phi) \cdot r^2 \sin\theta \, d\phi \, d\theta \, dr \right] \} \, .$$

$$(4.39)$$

Now it is easily verified that

$$\int_0^{2\pi} \sin \phi \cos m(\gamma-\phi)\, d\phi = \begin{cases} \pi \sin \gamma & m = 1 \\ 0 & m \neq 1 \end{cases} .$$

Thus, the ϕ integration in equation (4.39) yields

$$L_{\theta_o} = -E k \cos \theta_o \sum_{n=0}^{\infty} \left\{ i^n (2n+1) \cdot 2\pi \frac{(n-1)!}{(n+1)!} P_n^1(\cos \psi) \sin\gamma \int_0^R \int_0^\pi h_1(kr) \right.$$

$$\times \quad j_n(\kappa' r)\, P_1^1(\cos \theta)\, P_n^1(\cos \theta) r^2 \sin\theta\, d\theta\, dr \,. \qquad (4.40)$$

But from equation (A.17) one has the following orthogonality relationship for the associated Legendre polynomials:

$$\int_0^\pi P_1^1(\cos \theta)\, P_n^1(\cos \theta) \sin \theta\, d\theta = \begin{cases} 4/3 & n = 1 \\ 0 & n \neq 1 \end{cases} .$$

When this is utilized in equation (4.40), the result is

$$L_{\theta_o} = E k \cos \theta_o \cdot 4\pi i\, P_1^1(\cos \psi) \sin\gamma \int_0^R h_1(kr) j_1(\kappa' r) r^2\, dr$$

or

$$L_{\theta_o} = 4\pi i\, E\, k \cos \theta_o \sin\psi \sin\gamma \int_0^R h_1(kr)\; j(\kappa' r) r^2 dr \qquad (4.41)$$

since $P_1^1(\cos \psi) = \sin \psi$. In an identical fashion one can also show that

$$L_{\phi_o} = -4\pi i\, E k \sin\psi \cos\gamma \int_0^R h_1(kr)\; j_1(\kappa' r) r^2\, dr \,. \qquad (4.42)$$

The integrals in equations (4.41) and (4.42) can be evaluated by using equation (A.9) which states

$$\int^{r} t^2 a_n(kt)\, b_n(k_1 t)dt = \frac{r^2}{k^2 - k_1^2}\,[k a_{n+1}(kr)\, b_n(k_1 r) - k_1 a_n(kr) b_{n+1}(k_1 r)]$$

where a_n, b_n are any linear combinations of n^{th} order spherical Bessel or Hankel functions.

Letting $a_n = h_1$, $b_n = j_1$, $k_1 = \kappa'$, one obtains

$$\int_0^R h_1(kr)\, j_2(\kappa' r) r^2 dr = \frac{R^2}{k^2 - \kappa'^2}\,[k h_2(kR) j_1(\kappa' R) - \kappa' h_1(kR) j_2(\kappa' R)]$$

$$- \left\{ \frac{r^2}{k^2 - \kappa'^2}\,[k h_2(kr) j_2(\kappa' r) - \kappa' h_1(kr) j_2(\kappa' r)] \right\}_{r=0} .$$

Evaluation of the term in braces in the preceding expression can be easily done by using the asymptotic approximations for $h_1(kr)$, $h_2(kr)$, $j_1(\kappa' r)$, $j_2(\kappa' r)$ which are valid for $r \to 0$. From Section A.1 in the appendices, one finds that

$$j_1(\kappa' r) \sim \frac{\kappa' r}{3}\ , \quad j_2(\kappa' r) \sim \frac{\kappa'^2 r^2}{15}\ , \quad h_1(kr) \sim \frac{-i}{k^2 r^2}\ , \quad h_2(kr) \sim \frac{-3i}{k^3 r^3}$$

as $r \to 0$. Then

$$\left\{ \frac{r^2}{k^2 - \kappa'^2}\,[k h_2(kr) j_1(\kappa' r) - \kappa' h_1(kr) j_2(\kappa' r)] \right\}_{r=0}$$

$$= \left\{ \frac{r^2}{k^2 - \kappa'^2}\,[k \cdot \frac{-3i}{k^3 r^3} \cdot \frac{\kappa' r}{3} - \kappa' \cdot \frac{-i}{k^2 r^2} \cdot \frac{\kappa'^2 r^2}{15}] \right\}_{r=0} = \frac{-i\kappa'}{(k^2 - \kappa'^2) k^2} .$$

Thus,

$$\int_0^R h_1(kr)j_1(\kappa'r)r^2dr = \frac{1}{k^2-\kappa'^2}\left\{R^2[kh_2(kR)j_1(\kappa'R)-\kappa'h_1(kR)j_2(\kappa'R)] + \frac{i\kappa'}{k^2}\right\} . \tag{4.43}$$

Equation (4.43) is now substituted into equations (4.41) and (4.42). The results are

$$L_{\theta_o}(\bar{\kappa}) = 4\pi i\, \mathcal{E}k \cos\theta_o \frac{\sin\psi \sin\gamma}{k^2-\kappa'^2}\left\{R^2[kh_2(kR)j_1(\kappa'R)-\kappa'h_1(kR)j_2(\kappa'R)] + \frac{i\kappa'}{k^2}\right\} \tag{4.44}$$

$$L_{\phi_o}(\bar{\kappa}) = -4\pi i\, \mathcal{E}k \frac{\sin\psi \cos\gamma}{k^2-\kappa'^2}\left\{R^2[kh_2(kR)j_1(\kappa'R)-\kappa'h_1(kR)j_2(\kappa'R)+\frac{i\kappa'}{k^2}\right\} . \tag{4.45}$$

Now L_{θ_o}, L_{ϕ_o} were found as a preliminary step in evaluating integrals of the form

$$\int S_{\tilde{\varepsilon}}(\kappa)\, L_\alpha(\bar{\kappa})\, L_\beta^*(\bar{\kappa})\, d\bar{\kappa} \tag{4.46}$$

which appears in equation (4.30). The region of integration here is all of $\bar{\kappa}$ space. It will now be shown that if a spherically-shaped volume centered at $\bar{\kappa} = \bar{k}$ and having a radius of $\frac{1}{2}k$ is excluded from the region of integration, then the value of the integral will not be significantly affected. That is, the error incurred by restricting the region of integration of all $\bar{\kappa}$ space to the region where $|\bar{\kappa} - \bar{k}| \doteq \frac{1}{2}k$ will be negligible. Restricting the region of integration in this manner will allow simplification of the integrand which in turn will expedite evaluation of the integral.

Recall that the correlation function of $\tilde{\varepsilon}$ is given as being Gaussian, i.e.,

$$R_{\tilde{\varepsilon}}(r) = \nu^2 e^{-r^2/\ell^2} \quad .$$

From Tatarski [16], the spectral density of this correlation function is

$$S_{\tilde{\varepsilon}}(\kappa) = \frac{\nu^2 \ell^3}{8\pi^{3/2}} e^{-\kappa^2\ell^2/4} \quad . \tag{4.47}$$

Let u denote the region where $|\bar{\kappa} - \bar{k}| < \frac{1}{2}k$, or, since $\bar{\kappa}' = \bar{\kappa} - \bar{k}$, the region where $\kappa' < \frac{1}{2}k$. Then the interest here will center on estimating

$$I_u = \int_u S_{\tilde{\varepsilon}}(\kappa)\ L_\alpha(\bar{\kappa})\ L_\beta^*(\kappa)\ d\bar{\kappa} \ ,$$

the value of the integral in the region which is to be excluded. Now

$$|I_u| = \left| \int_u S_{\tilde{\varepsilon}}(\kappa)\ L_\alpha(\bar{\kappa}) L_\beta^*(\bar{\kappa}) d\bar{\kappa} \right| \leq \int_u |S_{\tilde{\varepsilon}}(\kappa)| \cdot |L_\alpha(\bar{\kappa})| \cdot |L_\beta(\bar{\kappa})|\ d\bar{\kappa} \ .$$

Using equations (4.44), (4.45), (4.46), one can show that

$$|I_u| \leq 2\sqrt{\pi}\ E^2 k^2 \nu^2 \ell^3 \int_u e^{-\kappa^2\ell^2/4} \times \left| \frac{Re^{ikR}[ij_1(\kappa' R) + \frac{\kappa'}{k} j_2(\kappa' R)] + \frac{i\kappa'}{k^2}}{k^2 - \kappa'^2} \right|^2 d\bar{\kappa} \ .$$

Next, since $\kappa' < \frac{1}{2}k$ in u , and since $|j_1(\kappa' R)|$, $|j_2(\kappa' R)| < .5$ for all κ' , it is easily seen that

$$\left| \frac{Re^{ikR}[ij_1(\kappa' R) + \frac{\kappa'}{k} j_2(\kappa' R)] + \frac{i\kappa'}{k^2}}{k^2 - \kappa'^2} \right| < \frac{4R}{3k^2} .$$

Hence,

$$|I_u| < \frac{32\sqrt{\pi}\, E^2 \nu^2 \ell^3 R^2}{9k^2} \int_u e^{-\kappa^2 \ell^2/4}\, d\bar{\kappa} < \frac{10E^2 \nu^2 \ell^3 R^2}{k^2} \int_u e^{-\kappa^2 \ell^2/4}\, d\bar{\kappa} .$$

Now u is a spherical volume in $\bar{\kappa}$ space centered at the point $\bar{\kappa} = \bar{k}$ and having a diameter of k . Because of the spherical symmetry of the integrand, $\bar{k}$ may be taken to lie along the κ_x axis. Let u' be the region contained in a cube whose center is at $\bar{\kappa} = k\bar{e}_x$ and whose sides of length k are parallel to the respective coordinate axes. Then, since u is contained in u', one has

$$|I_u| < \frac{10E^2 \nu^2 \ell^3 R^2}{k^2} \int_{u'} e^{-\kappa^2 \ell^2/4}\, d\bar{\kappa} = \frac{10E^2 \nu^2 \ell^3 R^2}{k^2}$$

$$\times \int_{k/2}^{3k/2} \int_{-k/2}^{k/2} \int_{-k/2}^{k/2} e^{-(\kappa_x^2 + \kappa_y^2 + \kappa_z^2)\ell^2/4}\, d\kappa_y\, d\kappa_z\, d\kappa_x .$$

Then

$$|I_u| < \frac{10E^2 \nu^2 \ell^3 R^2}{k^2} \int_{k/2}^{\infty} e^{-\kappa_x^2 \ell^2/4}\, d\kappa_x \int_{-\infty}^{\infty} e^{-\kappa_y^2 \ell^2/4}\, d\kappa_y \int_{-\infty}^{\infty} e^{-\kappa_z^2 \ell^2/4}\, d\kappa_z .$$

Using the fact that

$$\int_{-\infty}^{\infty} e^{-t^2\ell^2/4}\, dt = \frac{2\sqrt{\pi}}{\ell} \quad ,$$

one has

$$|I_u| < \frac{40\pi E^2 \nu^2 \ell\, R^2}{k^2} \int_{k/2}^{\infty} e^{-\kappa_x^2 \ell^2/4}\, d\kappa_x = \frac{40\pi E^2 \nu^2\, \ell\, R^2}{k^2} \int_0^{\infty} e^{-(\tau^2 + k\tau + \frac{k^2}{4})\ell^2/4}\, d\tau$$

where the variable of integration has been changed from κ_x to $\tau = \kappa_x - \frac{1}{2}k$. Next,

$$|I_u| < \frac{40\pi E^2 \nu^2 \ell\, R^2}{k^2}\, e^{-k^2\ell^2/16} \int_0^{\infty} e^{-\tau^2\ell^2/4}\, e^{-\tau k \ell^2/4}\, d\tau$$

$$\leq \frac{40\pi E^2 \nu^2 \ell\, R^2}{k^2}\, e^{-k^2\ell^2/16} \int_0^{\infty} e^{-\tau^2\ell^2/4}\, d\tau \ .$$

Thus,

$$|I_u| < \frac{40\pi E^2 \nu^2 \ell\, R^2}{k^2}\, e^{-k^2\ell^2/16} \cdot \frac{\sqrt{\pi}}{\ell} < \frac{250 E^2 \nu^2\, R^2}{k^2}\, e^{-k^2\ell^2/16} \quad .$$

But if $k\ell$ is sufficiently large, and it is given that $k\ell \gg 1$, then it is safe to assume that the exponential will reduce $|I_u|$ to a negligible value. Letting U be the region where $|\bar{\kappa} - \bar{k}| \geq \frac{1}{2}k$, i.e., the region where $\kappa' \geq \frac{1}{2}k$, equation (4.30) can thus be replaced by

$$< E_{1\alpha} E_{1\beta}^{*} > = \frac{k^4}{16\pi^2 r_o^2} \int_U S_{\nu\atop\varepsilon}(\kappa)\, L_\alpha(\bar{\kappa})\, L_\beta(\bar{\kappa})\, d\bar{\kappa} \ . \tag{4.48}$$

Now since $\kappa' \geq \frac{1}{2}k$ in U, and since $kR \gg 1$, then $\kappa' R \gg 1$ in U . Therefore, referring to equations (4.44), (4.45), the Bessel and Hankel functions can be replaced by their asymptotic approximations valid for large arguments. From Section A.1 one finds

$$h_1(kr) \sim \frac{-1}{kR} e^{ikR},\ h_2(kR) \sim \frac{i}{kR} e^{ikR},\quad j_1(\kappa'R) \sim \frac{-1}{\kappa'R} \cos \kappa'R ,$$

$$j_2(\kappa'R) \sim \frac{-1}{\kappa'R} \sin \kappa'R$$

as $kR \to \infty$. Substitution of these into equations (4.44) and (4.45) yields

$$L_{\theta_o}(\bar{\kappa}) = 4\pi i \mathcal{E} k \cos \theta_o \frac{\sin\psi \sin\gamma}{k^2 - \kappa'^2} \{e^{ikR}[\frac{-i}{\kappa'} \cos \kappa'R - \frac{1}{k} \sin \kappa'R] + \frac{i\kappa'}{k^2}\} \tag{4.49}$$

$$L_{\phi_o}(\bar{\kappa}) = -4\pi i \mathcal{E} k \frac{\sin\psi \cos\gamma}{k^2 - \kappa'^2} \{e^{ikR}[\frac{-i}{\kappa'} \cos \kappa'R - \frac{1}{k} \sin \kappa'R] + \frac{i\kappa'}{k^2}\}. \tag{4.50}$$

It is now possible to obtain more explicit expressions for $<|E_{1\theta_o}|^2>$, $<|E_{1\phi_o}|^2>$, and $<|E_{1\theta_o} E^*_{1\phi_o}|>$. From equation (4.48) it can be seen that

$$<|E_{1\theta_o}|^2> = \frac{k^4}{16\pi^2 r_o^2} \int_U S_{\underset{\sim}{\varepsilon}}(\kappa)\ |L_{\theta_o}(\bar{\kappa})|^2\ d\bar{\kappa} \tag{4.51}$$

$$<|E_{1\phi_o}|^2> = \frac{k^4}{16\pi^2 r_o^2} \int_U S_{\underset{\sim}{\varepsilon}}(\kappa)\ |L_{\phi_o}(\bar{\kappa})|^2\ d\bar{\kappa} \tag{4.52}$$

$$< E_{1\theta_o} E^*_{1\phi_o} > = \frac{k^4}{16\pi^2 r_o^2} \int_U S_{\underset{\sim}{\varepsilon}}(\kappa)\ L_{\theta_o}(\bar{\kappa})\ L^*_{\phi_o}(\bar{\kappa})\ d\kappa . \tag{4.53}$$

By substituting equations (4.49) and (4.50) into equations (4.51)-(4.53) and noting that $S_{\underset{\sim}{\varepsilon}}(\kappa)$ is given by (4.47), one obtains

$$<|E_{1\theta_o}|^2> = \frac{E^2 k^6 \ell^3 \nu^2 \cos^2\theta_o}{8\pi^{3/2} r_o^2} \int_U e^{-\kappa^2\ell^2/4} \sin^2\psi \sin^2\gamma \, |F(\kappa')|^2 \, d\bar{\kappa} \quad (4.54)$$

$$<|E_{1\phi_o}|^2> = \frac{E^2 k^6 \ell^3 \nu^2}{8\pi^{3/2} r_o^2} \int_U e^{-\kappa^2\ell^2/4} \sin^2\psi \cos^2\gamma \, |F(\kappa')|^2 \, d\bar{\kappa} \quad (4.55)$$

$$< E_{1\theta_o} E^*_{1\phi_o} > = \frac{E^2 k^6 \ell^3 \nu^2 \cos\theta_o}{8\pi^{3/2} r_o^2} \int_U e^{-\kappa^2\ell^2/4} \sin^2\psi \cos\gamma \sin\gamma \, |F(\kappa')|^2 d\bar{\kappa} \quad (4.56)$$

where

$$F(\kappa') = \frac{1}{k^2 - \kappa'^2} \{e^{ikR}[\frac{-i}{\kappa'} \cos \kappa'R - \frac{1}{k} \sin \kappa'R] + \frac{i\kappa'}{k^2} \} . \quad (4.57)$$

Two transformations of coordinates will simplify the computation of the integrals in equations (4.54) - (4.56). Recall that $\bar{\kappa}' = \bar{\kappa} - \bar{k}$ and that $\bar{\kappa}' = \kappa' \sin\psi \cos\gamma \, \bar{e}_x + \kappa' \sin\psi \sin\gamma \, \bar{e}_y + \kappa' \cos\psi \, \bar{e}_z$. Then

$$\sin\psi \cos\gamma = \frac{\kappa'_x}{\kappa'} , \quad \sin\psi \sin\gamma = \frac{\kappa'_y}{\kappa'} .$$

Substitution of these expressions into equations (4.54) - (4.56) and changing the variables of integration from $\bar{\kappa}$ to $\bar{\kappa}'$ yields

$$<|E_{1\theta_o}|^2> = \frac{E^2 k^6 \nu^2 \ell^3 \cos^2\theta_o}{8\pi^{3/2} r_o^2} \int_U e^{-|\bar{\kappa}'+\bar{k}|^2\ell^2/4} \frac{\kappa'^2_y}{\kappa'^2} |F(\kappa')|^2 \, d\bar{\kappa}' \quad (4.58)$$

$$<|E_{1\phi_o}|^2> = \frac{E^2 k^6 \nu^2 \ell^3}{8\pi^{3/2} r_o^2} \int_U e^{-|\bar{\kappa}'+\bar{k}|^2\ell^2/4} \frac{\kappa'^2_x}{\kappa'^2} |F(\kappa')|^2 \, d\bar{\kappa}' \quad (4.59)$$

$$<E_{1\theta_o} E^*_{1\phi_o}> = \frac{E^2 k^6 \nu^2 \ell^3 \cos\theta_o}{8\pi^{3/2} r_o^2} \int_U e^{-|\bar{\kappa}'+\bar{k}|^2 \ell^2/4} \frac{\kappa'_x \kappa'_y}{\kappa'^2} |F(\kappa')|^2 d\bar{\kappa}'. \quad (4.60)$$

The variables of integration will now be transformed from $\bar{\kappa}'$ coordinates to $\bar{\zeta}$ coordinates which are defined by the following axis rotation:

$$\left.\begin{aligned} \zeta_1 &= \kappa'_x \cos\theta_o - \kappa'_z \sin\theta_o \\ \zeta_2 &= \kappa'_y \\ \zeta_3 &= \kappa'_x \sin\theta_o + \kappa'_z \cos\theta_o \,. \end{aligned}\right\} \quad (4.61)$$

The inverse of the transformation is given by

$$\left.\begin{aligned} \kappa'_x &= \zeta_1 \cos\theta_o + \zeta_3 \sin\theta_o \\ \kappa'_y &= \zeta_2 \\ \kappa'_z &= -\zeta_1 \sin\theta_o + \zeta_3 \cos\theta_o \,. \end{aligned}\right\} \quad (4.62)$$

It should be noted that $|\bar{\zeta}| = |\bar{\kappa}'|$. Now let $\bar{\zeta}$ be given by

$$\bar{\zeta} = \zeta \sin\eta \cos\xi \, \bar{e}_1 + \zeta \sin\eta \sin\xi \, \bar{e}_2 + \zeta \cos\eta \, \bar{e}_3$$

where ζ, η, and ξ are spherical coordinates. Thus,

$$\begin{aligned} \zeta_1 &= \zeta \sin\eta \cos\xi \\ \zeta_2 &= \zeta \sin\eta \sin\xi \\ \zeta_3 &= \zeta \cos\eta \,. \end{aligned}$$

Then equations (4.62) become

$$\left.\begin{aligned} \kappa'_x &= \zeta \sin\eta \cos\xi \cos\theta_o + \zeta \cos\eta \sin\theta_o \\ \kappa'_y &= \zeta \sin\eta \sin\xi \\ \kappa'_z &= -\zeta \sin\eta \cos\xi \sin\theta_o + \zeta \cos\eta \cos\theta_o . \end{aligned}\right\} \qquad (4.63)$$

Now

$$|\bar{\kappa}' + \bar{k}|^2 = \kappa'^2 + k^2 + 2\bar{\kappa}' \cdot \bar{k} = \zeta^2 + k^2 + 2\bar{\kappa}' \cdot \bar{k} . \qquad (4.64)$$

Recall that $\bar{k} = k\bar{e}_{r_o} = k \sin\theta_o \bar{e}_x + k \cos\theta_o \bar{e}_z$. Substituting this and equations (4.63) into equation (4.64) yields

$$|\bar{\kappa}' + \bar{k}|^2 = \zeta^2 + k^2 + 2(k\zeta \sin\eta \cos\xi \cos\theta_o \sin\theta_o + k\zeta \cos\eta \sin^2\theta_o$$
$$- k\zeta \sin\eta \cos\xi \cos\theta_o \sin\theta_o + k\xi \cos\eta \cos^2\theta_o)$$

or

$$|\bar{\kappa}' + \bar{k}|^2 = \zeta^2 + k^2 + 2\zeta k \cos\eta . \qquad (4.65)$$

Writing equations (4.58) - (4.60) in terms of the new variable of integration $\bar{\zeta}$ and utilizing equations (4.63) and (4.65) yields

$$<|E_{1\theta_o}|^2> = \frac{E^2 k^6 \ell^3 \nu^2 \cos^2\theta_o}{8\pi^{3/2} r_o^2} \int_U e^{-[\zeta^2+k^2+2\zeta k \cos\eta]\ell^2/4} \times (\sin\eta \sin\xi)^2 |F(\zeta)|^2 \, d\bar{\zeta} \qquad (4.66)$$

$$<|E_{1\phi_o}|^2> = \frac{E^2 k^6 \ell^3 \nu^2}{8\pi^{3/2} r_o^2} \int_U e^{-[\zeta^2+k^2+2\zeta k \cos\eta]\ell^2/4} (\sin\eta \cos\xi \cos\theta_o + \cos\eta \sin\theta_o)^2 |F(\zeta)|^2 \, d\bar{\zeta} \qquad (4.67)$$

$$< E_{1\theta_o} E^*_{1\phi_o} > = \frac{\mathcal{E}^2 k^6 \ell^3 \nu^2 \cos\,\theta_o}{8\pi^{3/2}\, r_o^2} \int_U e^{-[\zeta^2+k^2+2\,k\,\cos\,\eta]\ell^2/4}$$

$$\times\,(\sin\eta\cos\xi\cos\,\theta_o + \cos\eta\sin\,\theta_o)(\sin\eta\,\sin\,\xi)|F(\zeta)|^2\,d\overline{\zeta}\;. \tag{4.68}$$

More explicitly,

$$<|E_{1\theta_o}|^2> = \frac{\mathcal{E}^2 k^6 \nu^2 \ell^3 \cos^2\theta_o}{8\pi^{3/2}\, r_o^2} \int_{k/2}^{\infty}\int_0^{\pi}\int_0^{2\pi} e^{-[\zeta^2+k^2+2\zeta k\,\cos\,\eta]\ell^2/4}$$

$$\sin^2\eta\,\sin^2\xi|F(\zeta)|^2\,\zeta^2\sin\eta\,d\xi d\eta d\zeta \tag{4.69}$$

$$<|E_{1\phi_o}|^2> = \frac{\mathcal{E}^2 k^6 \ell^3 \nu^2}{8\pi^{3/2}\, r_o^2} \int_{k/2}^{\infty}\int_0^{\pi}\int_0^{2\pi} e^{-[\zeta^2+k^2+2\zeta k\,\cos\,\eta]\ell^2/4}$$

$$\times\,[\sin^2\eta\,\cos^2\xi\,\cos^2\theta_o + 2\,\sin\,\eta\cos\,\xi\,\cos\eta\cos\,\theta_o\,\sin\,\theta_o$$

$$+\,\cos^2\eta\,\sin^2\theta_o]|F(\zeta)|^2\zeta^2\sin\eta\,d\xi d\eta d\zeta \tag{4.70}$$

$$< E_{1\theta_o} E^*_{1\phi_o} > = \frac{\mathcal{E}^2 k^6 \nu^2 \ell^3 \cos\,\theta_o}{8\pi^{3/2}\, r_o^2} \int_{k/2}^{\infty}\int_0^{\pi}\int_0^{2\pi} e^{-[\zeta^2+k^2+2\zeta k\,\cos\,\eta]\ell^2/4}$$

$$\times\,[\sin^2\eta\,\cos\xi\sin\xi\cos\,\theta_o + \cos\eta\sin\xi\,\sin\,\theta_o]|F(\zeta)|^2\zeta^2\sin\eta\,d\xi d\eta d\zeta\,. \tag{4.71}$$

The ξ integrations in the preceding equations can be performed at once. Noting that

$$\int_0^{2\pi} \cos^2\xi \, d\xi = \int_0^{2\pi} \sin^2\xi \, d\xi = \pi, \quad \int_0^{2\pi} \sin\xi \, d\xi = \int_0^{2\pi} \sin\xi \cos\xi \, d\xi = 0 ,$$

one finds

$$<|E_{1\theta_o}|^2> = \frac{E^2 k^6 \ell^3 \nu^2 \cos^2\theta_o}{8\sqrt{\pi}\, r_o^2} \int_{k/2}^{\infty} \int_0^{\pi} e^{-[\zeta^2+k^2+2\zeta k \cos\eta]\ell^2/4}$$

$$\times \sin^2\eta \, |F(\zeta)|^2 \zeta^2 \sin\eta \, d\eta d\zeta \qquad (4.72)$$

$$<|E_{1\phi_o}|^2> = \frac{E^2 k^6 \ell^3 \nu^2}{8\sqrt{\pi}\, r_o^2} \int_{k/2}^{\infty} \int_0^{\pi} e^{-[\zeta^2+k^2+2\zeta k \cos\eta]\ell^2/4} [\sin^2\eta \cos^2\theta_o$$

$$+ 2 \cos^2\eta \sin^2\theta_o] \, |F(\zeta)|^2 \zeta^2 \sin\eta \, d\eta d\zeta \qquad (4.73)$$

$$<E_{1\theta_o} E^*_{1\phi_o}> = 0 . \qquad (4.74)$$

It is apparent from equation (4.74) that the transverse components of the scattered field are uncorrelated.

A change in variables will now be made. Letting $t = \cos\eta$ and $w = \zeta\ell$, equations (4.72) and (4.73) become

$$<|E_{1\theta_o}|^2> = \frac{E^2 k^6 \nu^2 \cos^2\theta_o}{8\sqrt{\pi}\, r_o^2} \int_{k\ell/2}^{\infty} \int_{-1}^{1} e^{-[w^2+k^2\ell^2+2wk\ell t]/4}$$

$$\times (1-t^2) \, |F(\tfrac{w}{\ell})|^2 w^2 \, dt \, dw \qquad (4.75)$$

$$<|E_{1\phi_o}|^2> = \frac{E^2 k^6 \nu^2}{8\sqrt{\pi}\, r_o^2} \int_{k\ell/2}^{\infty} \int_{-1}^{1} e^{-[w^2+k^2\ell^2+2wk\ell t]/4} [\cos^2\theta_o + t^2(3\sin^2\theta_o - 1)]\,|F(\tfrac{w}{\ell})|^2\, w^2 dt\, dw \, . \quad (4.76)$$

The t integrations in equations (4.75) and (4.76) will be done first. Performing these integrations is equivalent to finding the two integrals

$$\int_{-1}^{1} e^{-wk\ell t/2}\, dt \quad , \quad \int_{-1}^{1} t^2\, e^{-wk\ell t/2}\, dt \; .$$

The first one can be done immediately, yielding

$$\int_{-1}^{1} e^{-wk\ell t/2} = \frac{2[e^{wk\ell/2} - e^{-wk\ell/2}]}{wk\ell} \quad .$$

In the second integral, two integrations by parts yields

$$\int_{-1}^{1} t^2\, e^{-wk\ell t/2}\, dt = \frac{2[e^{wk\ell/2} - e^{-wk\ell/2}]}{wk\ell} - \frac{8[e^{wk\ell/2} - e^{-wk\ell/2}]}{(wk\ell)^2} + \frac{16[e^{wk\ell/2} - e^{-wk\ell/2}]}{(wk\ell)^3} \quad .$$

Now in equations (4.75) and (4.76) it can be seen that $w \geq \frac{1}{2} k\ell$. Hence, $wk\ell \geq \frac{1}{2}(k\ell)^2$. But $\frac{1}{2}(k\ell)^2 >> 1$. Thus $e^{wk\ell/2} >>> e^{-wk\ell/2}$, and therefore the $e^{-wk\ell/2}$ terms in the preceding equations can be neglected. Hence these equations become

$$\int_{-1}^{1} e^{-wk\ell t/2}\, dt \sim \frac{2e^{wk\ell/2}}{wk\ell} \qquad (4.77)$$

$$\int_{-1}^{1} t^2 e^{-wk\ell t/2}\, dt \sim \left[\frac{2}{wk\ell} - \frac{8}{(wk\ell)^2} + \frac{16}{(wk\ell)^3}\right] e^{wk\ell/2} . \tag{4.78}$$

Equations (4.77) and (4.78) are then substituted into equations (4.75) and (4.76). Now since $w \geq \frac{1}{2} k\ell >> 1$, one has $1/w^2$, $1/w^3 <<< 1$. Therefore, as an approximation, only the highest order term in w will be retained in equation (4.75) and only the two highest order terms in w will be retained in equation (4.76). The results are

$$<|E_{1\theta_o}|^2> \sim \frac{E^2 k^4 \nu^2 \cos^2\theta_o}{\sqrt{\pi}\; r_o^2\, \ell^2} \int_{k\ell/2}^{\infty} e^{-(w-k\ell)^2/4}\, |F(\tfrac{w}{\ell})|^2\, dw \tag{4.79}$$

$$<|E_{1\phi_o}|^2> \sim \frac{E^2 k^4 \nu^2}{4\sqrt{\pi}\; r_o^2 \ell^2} \int_{k\ell/2}^{\infty} e^{-(w-k\ell)^2/4} [(2wk\ell - 12)\sin^2\theta_o + 4] \times |F(\tfrac{w}{\ell})|^2\, dw . \tag{4.80}$$

But $2wk\ell \geq (k\ell)^2$, and assuming $(k\ell)^2 >> 12$, equation (4.80) becomes

$$<|E_{1\phi_o}|^2> \sim \frac{E^2 k^4 \nu^2}{\sqrt{\pi}\; r_o^2 \ell^2} \int_{k\ell/2}^{\infty} e^{-(w-k\ell)^2/4} [\tfrac{1}{2} wk\ell \sin^2\theta_o + 1] |F(\tfrac{w}{\ell})|^2\, dw . \tag{4.81}$$

From equation (4.57) one has

$$F(\kappa') = \frac{1}{k^2 - \kappa'^2} \left\{Re^{ikR} \left[\frac{-i}{\kappa'} \cos \kappa' R - \frac{1}{k} \sin \kappa' R\right] + \frac{i\kappa'}{k}\right\} .$$

In a straightforward manner one can then show that

$$|F(\tfrac{w}{\ell})|^2 = \frac{\ell^6}{(w^2-k^2\ell^2)^2}\left\{\frac{\sin^2\frac{wR}{\ell}}{(k\ell)^2}+\frac{\cos^2\frac{wR}{\ell}}{w^2}-\frac{2}{(k\ell)^2}\left[\cos kR\cos\frac{wR}{\ell}+\frac{w}{k\ell}\sin kR\sin\frac{wR}{\ell}\right]+\frac{w^2}{(k\ell)^4}\right\}. \quad (4.82)$$

Referring to (4.79) and (4.81), it can be seen that because the exponential in the integrand decreases very rapidly for $|w-k\ell| > 1$, most of the contribution to the integral comes from a relatively small neighborhood of $w = k\ell$. Therefore it is reasonable to see if $|F(\frac{w}{\ell})|^2$ can be replaced by some approximation in this neighborhood which would be less complex than the expression in equation (4.82) and hopefully easier to integrate. A simple way to find this approximation is to let $w \to k\ell$ in equation (4.82). One finds that

$$|F(\tfrac{w}{\ell})|^2 \sim \frac{\ell^6}{(2k\ell)^2\,(w-k\ell)^2}\left\{\frac{\sin^2\frac{wR}{\ell}}{(k\ell)^2}+\frac{\cos^2\frac{wR}{\ell}}{(k\ell)^2}-\frac{2}{(k\ell)^2}\left[\cos kR\cos\frac{wR}{\ell}+\frac{k\ell}{k\ell}\sin kR\sin\frac{wR}{\ell}\right]+\frac{(k\ell)^2}{(k\ell)^4}\right\} \quad \text{as } |w-k\ell| \to 0 ,$$

or, after combining terms,

$$|F(\tfrac{w}{\ell})|^2 \sim \frac{\ell^2}{2k^4}\cdot\frac{1-\cos[(w-k\ell)\frac{R}{\ell}]}{(w-k\ell)^2} . \quad (4.83)$$

Equation (4.83) gives the approximation which will be used in evaluating the integrals in equations (4.79) and (4.81). However, before this can be done, one ought to have some estimate of the error incurred by using the approximation. To do this, the approximation will be obtained in a different manner which will yield terms with which the error can be calculated. First it can be seen that

$$\frac{1}{w^2} = \frac{1}{(k\ell)^2} + \frac{1}{w^2} - \frac{1}{(k\ell)^2} = \frac{1}{(k\ell)^2} - \frac{w^2 - (k\ell)^2}{w^2 (k\ell)^2} ,$$

$$\frac{w^2}{(k\ell)^2} = \frac{w^2 - (k\ell)^2 + (k\ell)^2}{(k\ell)^4} = \frac{1}{(k\ell)^2} + \frac{w^2 - (k\ell)^2}{(k\ell)^2} ,$$

$$\frac{w}{k\ell} = \frac{w - k\ell + k\ell}{k\ell} = 1 + \frac{w - k\ell}{k\ell} .$$

When these expressions are substituted into equation (4.82), the result is

$$|F(\tfrac{w}{\ell})|^2 = \frac{\ell^6}{(w^2 - k^2\ell^2)^2} \left\{ \frac{\sin^2 \frac{wR}{\ell}}{(k\ell)^2} + \frac{\cos^2 \frac{wR}{\ell}}{(k\ell)^2} - \frac{(w^2 - k^2\ell^2)\cos^2 \frac{wR}{\ell}}{w^2 (k\ell)^2} \right.$$

$$- \frac{2}{(k\ell)^2} [\cos kR \cos \frac{wR}{\ell} + \sin kR \sin \frac{wR}{\ell} + \frac{(w - k\ell)}{(k\ell)^3} \sin kR \sin \frac{wR}{\ell}]$$

$$\left. + \frac{1}{(k\ell)^2} + \frac{w^2 - (k\ell)^2}{(k\ell)^4} \right\}$$

or

$$|F(\tfrac{w}{\ell})|^2 = \frac{2\ell^6 \{1 - \cos[(w - k\ell)\frac{R}{\ell}]\}}{(k\ell)^2 (w + k\ell)^2 (w - k\ell)^2} + \frac{\ell^6}{(k\ell)^2 (w + k\ell)^2 (w - k\ell)}$$

$$\times \left\{ \frac{-(w + k\ell) \cos^2 \frac{wR}{\ell}}{w^2} - \frac{2 \sin kR \sin \frac{wR}{\ell}}{k\ell} + \frac{w + k\ell}{(k\ell)^2} \right\} .$$

Next, since

$$\frac{1}{(w + k\ell)^2} = \frac{1}{(2k\ell)^2} + \frac{1}{(w + k\ell)^2} - \frac{1}{(2k\ell)^2} = \frac{1}{4(k\ell)^2} - \frac{(w + 3k\ell)(w - k\ell)}{4(k\ell)^2 (w + k\ell)^2} ,$$

the preceding equation for $|F(\frac{w}{\ell})|^2$ can be put into the form

$$|F(\tfrac{w}{\ell})|^2 = \frac{\ell^2}{2k^4}\cdot\frac{1-\cos[(w-k\ell)\frac{R}{\ell}]}{(w-k\ell)^2} - \frac{\ell^2}{2k^4}\cdot\frac{(w+k\ell)\{1-\cos[(w-k\ell)\frac{R}{\ell}]\}}{(w+k\ell)^2\,(w-k\ell)}$$

$$+\frac{\ell^6}{(k\ell)^2\,(w+k\ell)^2\,(w-k\ell)}\left\{\frac{w+k\ell}{(k\ell)^2} - \frac{2\sin kR\sin\frac{wR}{\ell}}{k\ell} - \frac{(w+k\ell)\cos^2\frac{wR}{\ell}}{w^2}\right\}.$$

Let

$$f(w) = \frac{\ell^2}{2k^4}\cdot\frac{1-\cos[(w-k\ell)\frac{R}{\ell}]}{(w-k\ell)^2}$$

$$g_1(w) = \frac{\ell^2}{2k^4}\cdot\frac{(w+3k\ell)\,\{1-\cos[(w-k\ell)\frac{R}{\ell}]\}}{(w+k\ell)^2\,(w-k\ell)}$$

$$g_2(w) = \frac{\ell^4}{k^2(w+k\ell)^2(w-k\ell)}\left\{\frac{w+k\ell}{(k\ell)^2} - \frac{2\sin kR\sin\frac{wR}{\ell}}{k\ell} - \frac{(w+k\ell)\cos^2\frac{wR}{\ell}}{w^2}\right\}$$

so that

$$|F(\tfrac{w}{\ell})|^2 = f(w) - g_1(w) + g_2(w)\ .$$

It can be seen that $f(w)$ is identical to the approximation for $|F(\frac{w}{\ell})|^2$ given by equation (4.83). Hence, the error involved in using the approximation results from dropping the terms $g_1(w)$ and $g_2(w)$. In particular, for equation (4.79), the error is given by the integrals

$$\int_{k\ell/2}^{\infty} e^{-(w-k\ell)^2/4}\, g_1(w)\,dw \quad , \quad \int_{k\ell/2}^{\infty} e^{-(w-k\ell)^2/4}\, g_2(w)\,dw \quad .$$

The magnitude of these integrals will now be determined. Let the range of integration $[\frac{1}{2}\,k\ell,\infty]$ be divided into the three intervals $[\frac{1}{2}\,k\ell,$ $k\ell - \pi/30]$, $[k\ell - \pi/30,\ k\ell + \pi/30]$, $[k\ell + \pi/30,\infty]$. Thus

$$\left|\int_{k\ell/2}^{\infty}\right| = \left|\int_{k\ell/2}^{k\ell-\pi/30} + \int_{k\ell-\pi/30}^{k\ell+\pi/30} + \int_{k\ell+\pi/30}^{\infty}\right|$$

$$\leq \left|\int_{k\ell/2}^{k\ell-\pi/30}\right| + \left|\int_{k\ell-\pi/30}^{k\ell+\pi/30}\right| + \left|\int_{k\ell+\pi/30}^{\infty}\right| .$$

Now using the expressions for $g_1(w)$ and $g_2(w)$ it can be seen without too much difficulty that if $|w - k\ell| \geq \frac{\pi}{30}$, then

$$|g_1(w)|,\ |g_2(w)| < \frac{50\ell}{k^5}$$

Hence, if $g(w)$ represents either $g_1(w)$ or $g_2(w)$, one finds that

$$\left|\int_{k\ell/2}^{k\ell-\pi/30} e^{-(w-k\ell)^2/4}\, g(w)\,dw\right| \leq \int_{k\ell}^{k\ell-\pi/30} e^{-(w-k\ell)^2/4}\, |g(w)|\,dw$$

$$< \frac{50\ell}{k^5}\int_{k\ell/2}^{k\ell-\pi/30} e^{-(w-k\ell)^2/4}\,dw < \frac{50\ell}{k^5}\int_{-\infty}^{k\ell} e^{-(w-k\ell)^2/4}\,dw < \frac{100\ell}{k^5} .$$

In an identical fashion one can also show that

$$\left|\int_{k\ell+\pi/30}^{\infty} e^{-(w-k\ell)^2/4}\, g(w)\,dw\right| < \frac{50\ell}{k^5}\int_{k\ell+\pi/30}^{\infty} e^{-(w-k\ell)^2/4}\,dw < \frac{100\ell}{k^5} .$$

Then

$$\left|\int_{k\ell/2}^{k\ell-\pi/30} e^{-(w-k\ell)^2/4}\, g(w)\,dw\right| + \left|\int_{k\ell+\pi/30}^{\infty} e^{-(w-k\ell)^2/4}\, g(w)\,dw\right|$$

$$< \frac{100\ell}{k^5} + \frac{100\ell}{k^5} = \frac{200\ell}{k^5} \,.$$

Next, for $|w-k\ell| \leq \frac{\pi}{30}$ and $k\ell >> 1$, one can show that

$$g_1(w) \sim \frac{\ell}{2k^5} \frac{1 - \cos[(w-k\ell)\frac{R}{\ell}]}{(w-k\ell)} - \frac{3}{8k^6}\{1 - \cos[(w-k\ell)\frac{R}{\ell}]\}$$

$$g_2(w) \sim \frac{\ell}{k^5} \frac{\sin w\frac{R}{\ell} \sin[(w-k\ell)\frac{R}{2\ell}]}{(w-k\ell)} \,.$$

Then

$$\left|\int_{k\ell-\pi/30}^{k\ell+\pi/30} e^{-(w-k\ell)^2/4}\, g_1(w)\,dw\right| \sim \left|\frac{\ell}{2k^5}\int_{k\ell-\pi/30}^{k\ell+\pi/30} e^{-(w-k\ell)^2/4}\right.$$

$$\frac{\{1 - \cos[(w-k\ell)\frac{R}{\ell}\}}{(w-k\ell)}\,dw - \frac{3}{8k^6}\int_{k\ell-\pi/30}^{k\ell+\pi/30} e^{-(w-k\ell)^2/4}$$

$$\left.\times\ \{1 - \cos[(w-k\ell)\frac{R}{\ell}]\}\ dw\right|$$

$$= \left|\frac{3}{8k^6}\int_{k\ell-\pi/30}^{k\ell+\pi/30} e^{-(w-k\ell)^2/4}\{1 - \cos[(w-k\ell)]\}\ dw\right|$$

$$\leq \frac{3}{4k^6}\int_{k\ell-\pi/30}^{k\ell+\pi/30} dw < \frac{1}{k^6} \,,$$

and

$$\left|\int_{k\ell-\pi/30}^{k\ell+\pi/30} e^{-[w-k\ell)^2/4} g_2(w)dw\right| \sim \frac{\ell}{k^5}\left|\int_{k\ell-\pi/30}^{k\ell+\pi/30} e^{-(w-k\ell)^2/4}\right.$$

$$\left.\times \quad \frac{\sin\frac{wR}{\ell}\,\sin[(w-k\ell)\frac{R}{\ell}]}{(w-k\ell)}\,dw\right|$$

$$\leq \frac{\ell}{k^5}\left[\int_{k\ell-\pi/30}^{k\ell+\pi/30} e^{-(w-k\ell)^2/2}\sin^2\frac{wR}{\ell}\,dw \cdot \int_{k\ell-\pi/30}^{k\ell+\pi/30} \frac{\sin^2[(w-k\ell)\frac{R}{\ell}]}{(w-k\ell)^2}\,dw\right]^{1/2}$$

$$\leq \frac{\ell}{k^5}\left[\frac{\pi}{15}\cdot\frac{\pi R}{\ell}\right]^{1/2} < \frac{\ell}{k^5}\sqrt{\frac{R}{\ell}} \quad .$$

In deriving the estimate for the integral with $g_2(w)$, use was made of the Schwartz inequality

$$\left|\int_a^b h_1(x)\, h_2(x)\, dx\right| \leq \left|\int_a^b h_1^2(x)\, dx \cdot \int_a^b h_2^2(x)\, dx\right|^{1/2}$$

and the fact that

$$\int_{-\infty}^{\infty} \frac{\sin^2 ax}{x^2}\, dx = \pi a \; .$$

Combining results, the error involved in using $f(w)$ as the approximation for $|F(\frac{w}{\ell})|^2$ in the integral

$$\int_{k\ell/2}^{\infty} e^{-(w-k\ell)^2/4} \, |F(\tfrac{w}{\ell})|^2 \, dw$$

is less than

$$\frac{200\ell}{k^5} + \frac{200\ell}{k^5} + \frac{1}{k^6} + \frac{\ell}{k^5}\sqrt{\frac{R}{\ell}} \cong \frac{\ell}{k^5}\left(400 + \sqrt{\frac{R}{\ell}}\right) . \tag{4.84}$$

An identical estimate can be obtained for the integral in equation (4.81) in which the additional approximation is made of replacing the $\frac{1}{2}\, wk\ell \sin^2\theta_o$ term with $\frac{1}{2}(k\ell)^2 \sin^2\theta_o$.

The expressions for the scattered field are now

$$<|E_{1\theta_o}|^2> \sim \frac{E^2 k^4 \nu^2 \cos^2\theta_o}{\sqrt{\pi}\; r_o^2\, \ell^2} \int_{-k\ell/2}^{\infty} e^{-(w-k\ell)^2/4} f(w)dw \tag{4.85}$$

$$<|E_{1\phi_o}|^2> \sim \frac{E^2 k^4 \nu^2}{\sqrt{\pi}\; r_o^2 \ell^2} \left[1+\frac{1}{2}(k\ell)^2 \sin^2\theta_o\right] \int_{-k\ell/2}^{\infty} e^{-(w-k\ell)^2/4} f(w)\, dw. \tag{4.86}$$

Let

$$I_f = \int_{-k\ell/2}^{\infty} e^{-(w-k\ell)^2/4} f(w)\, dw . \tag{4.87}$$

Then once I_f is computed, the final expressions for $<|E_{1\theta_o}|^2>$, $<|E_{1\phi_o}|^2>$ can be obtained. Of course, the magnitude of I_f must be compared with the value in expression (4.84) to determine the relative error incurred by using the approximation for $|F(\frac{w}{\ell})|^2$. Substitution of the explicit expression for $f(w)$ into equation (4.87) yields

$$I_f = \frac{\ell^2}{2k^4} \int_{k\ell/2}^{\infty} e^{-(w-k\ell)^2/4} \frac{\{1 - \cos[(w-k\ell)\frac{R}{\ell}]\}}{(w-k\ell)^2} dw . \tag{4.87'}$$

One can extend the lower limit on the integral in equation (4.87') to $-\infty$ with negligible effect on the value of I_f . Then making the change of variables $\tau = (w-k\ell)\frac{R}{\ell}$, one finds that

$$I_f = \frac{\ell R}{2k^4} \int_{-\infty}^{\infty} e^{-\tau^2 \ell^2/4R^2} \frac{\{1 - \cos\tau\}}{\tau^2} \, d\tau \, . \tag{4.88}$$

One integration by parts yields

$$I_f = \frac{\ell R}{2k^4} \left\{ \int_{-\infty}^{\infty} \frac{1}{\tau} \left\{ \sin\tau - \frac{\ell^2}{4R^2} \tau [1 - \cos\tau] \right\} e^{-\tau^2\ell^2/4R^2} \, d\tau \right\}$$

$$= \frac{\ell R}{2k^4} \left\{ \int_{-\infty}^{\infty} \frac{\sin\tau}{\tau} e^{-\tau^2\ell^2/4R^2} \, d\tau - \frac{\ell^2}{4R^2} \int_{-\infty}^{\infty} e^{-\tau^2\ell^2/4R^2} \, d\tau \right.$$

$$\left. + \frac{\ell^2}{4R^2} \int_{-\infty}^{\infty} \cos\tau \; e^{-\tau^2\ell^2/4R^2} \, d\tau \right\} .$$

But

$$\int_{-\infty}^{\infty} e^{-\tau^2\ell^2/4R^2} \, d\tau = \sqrt{\pi} \, \frac{2R}{\ell} \, ,$$

and from the N.B.S. "Handbook" [17],

$$\frac{1}{2} \int_{-\infty}^{\infty} \cos 2xt \; e^{-at^2} \, dt = \int_{0}^{\infty} \cos 2xt \; e^{-at^2} dt = \frac{1}{2}\sqrt{\frac{\pi}{a}} \, e^{-x^2/a} \tag{4.89}$$

so that one finds

$$I_f = \frac{\ell^5 R}{2(k\ell)^4} \left\{ \int_{-\infty}^{\infty} \frac{\sin\tau}{\tau} e^{-\tau^2\ell^2/4R^2} \, d\tau - \frac{\ell}{2R}\sqrt{\pi} + \frac{\ell}{2R}\sqrt{\pi} \; e^{-R^2/\ell^2} \right\} .$$

But

$$\frac{\sin\tau}{\tau} = \int_{0}^{1} \cos x\tau \, dx \, .$$

Substitution of this into the integral in the expression for I_f and reversal of the order of integration yields

$$I_f = \frac{\ell R}{2k^4}\left\{\int_0^1 \int_{-\infty}^{\infty} \cos x\tau \; e^{-\tau^2\ell^2/4R^2} d\tau - \frac{\ell}{2R}\sqrt{\pi} + \frac{\ell}{2R}\sqrt{\pi}\; e^{-R^2/\ell^2}\right\}$$

Applying equation (4.89) again, one finds

$$I_f = \frac{\ell R}{2k^4}\left\{\frac{2R}{\ell}\sqrt{\pi}\int_0^1 e^{-x^2R^2/\ell^2}\, dx - \frac{\ell}{2R}\sqrt{\pi} + \frac{\ell}{2R}\sqrt{\pi}\; e^{-R^2/\ell^2}\right\}.$$

Changing of the variable of integration in the preceding expression by letting $y = \frac{R}{\ell} x$ yields

$$I_f = \frac{\ell R}{2k^4}\left\{2\sqrt{\pi}\int_0^{R/\ell} e^{-y^2}\, dy - \frac{\ell}{2R}\sqrt{\pi} + \frac{\ell}{2R}\sqrt{\pi}\; e^{-R^2/\ell^2}\right\}.$$

But since it is given that $R/\ell >> 1$, it can be seen that

$$\int_0^{R/\ell} e^{-y^2}\, dy \cong \int_0^{\infty} e^{-y^2}\, dy = \frac{\sqrt{\pi}}{2}.$$

Thus,

$$I_f = \frac{\pi \ell R}{2k^4}\;\{1 - \frac{\ell}{2R\sqrt{\pi}} + \frac{\ell}{2R\sqrt{\pi}}\; e^{-R^2/\ell^2}\}.$$

But again, since $R/\ell >> 1$, then $\frac{\ell}{2R\sqrt{\pi}} << 1$ and $\frac{\ell}{2R\sqrt{\pi}}\, e^{-R^2/\ell^2} <<< 1$.

Hence,

$$I_f \sim \frac{\pi \ell R}{2k^4}. \qquad (4.90)$$

Comparison of the value of I_f with the estimate for the error in expression (4.84) reveals that the error does indeed become negligible as $k\ell\sqrt{R/\ell}$ becomes increasingly large.

Using equation (4.90), the final expression for $<|E_{1\theta_o}|^2>$ and $<|E_{1\theta_o}|^2>$ can be obtained from equations (4.85) and (4.86). The results are

$$<|E_{1\theta_o}|^2> \sim \frac{E^2\nu^2 R\sqrt{\pi}}{2r_o^2\,\ell}\cos^2\theta_o \tag{4.91}$$

$$<|E_{1\phi_o}|^2> \sim \frac{E^2\nu^2 R\sqrt{\pi}}{2r_o^2\,\ell}\left[1+\frac{1}{2}(k\ell)^2\sin^2\theta_o\right]. \tag{4.92}$$

(ii) Discussion of results

The results of the examination of scattering and propagation with the magnetic dipole are represented by equations (4.74), (4.91), and (4.92). They are repeated here for reference:

$$<|E_{1\theta_o}|^2> \sim \frac{E^2\nu^2 R\sqrt{\pi}}{2r_o^2\,\ell}\cos^2\theta_o$$

$$<|E_{1\phi_o}|^2> \sim \frac{E^2\nu^2 R\sqrt{\pi}}{2r_o^2\,\ell}\left[1+\frac{1}{2}(k\ell)^2\sin^2\theta_o\right]$$

$$<E_{1\theta_o}E^*_{1\phi}> \;=\; 0\,.$$

A plot of the root-mean-square values of $E_{1\theta_o}$, $E_{1\phi_o}$ for $k\ell = 10$ is given in Figure 4.3.

In interpreting the equations for the mean square field components, it should be noted from equation (4.27) that the far-zone magnetic dipole field for a homogeneous medium is

$$\overline{E}_o(\overline{r}) \;=\; E\,\frac{e^{ikr}}{r}\sin\theta\,\overline{e}_\phi\,. \tag{4.93}$$

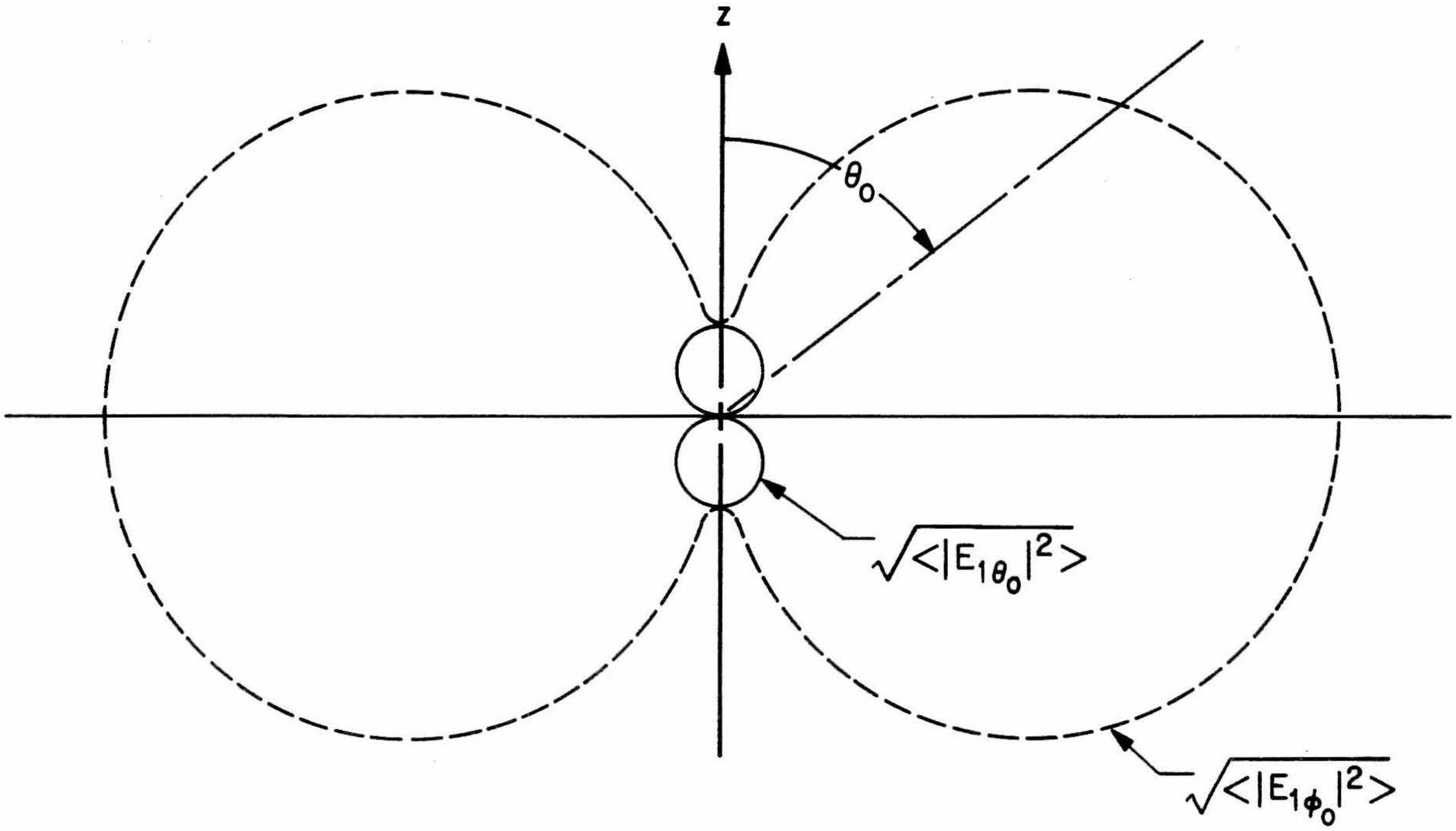

Figure 4.3. Root-Mean-Square Scattered Field Components for Magnetic Dipole with $k\ell = 10$

Using the Poynting vector method, one can easily show that the power radiated by this dipole is $\frac{3\pi}{4}\sqrt{\varepsilon_1/\mu}\,\mathcal{E}^2$. The radiated power is thus proportional to $\mathcal{E}^2$ and independent of the frequency. Hence, the expressions for $<|E_{1\theta_o}|^2>$, $<|E_{1\phi_o}|^2>$ give the behavior of these quantities for constant power in the zero'th order field of the dipole.

Now the method used in treating this scattering and propagation problem was the Born approximation, and it should be recalled that in order for this method to give reliable results, the first order field must be much smaller than the original field. Owing to the random nature of the physical situation of the problem, the first order fields themselves cannot be calculated and so any criterion for determining the validity of the Born method must rely on the mean square components of the field. A condition that seems plausible in insuring the applicability of the Born approximation is to require that $<|\overline{E}_1|^2> << |\overline{E}_o|^2$. Since $(k\ell)^2 >>> 1$, one has

$$<|\overline{E}_1|^2> \sim <|E_{1\phi_o}|^2> \sim \frac{\mathcal{E}^2\nu^2 R\sqrt{\pi}\,k\ell}{4r_o^2}\sin^2\theta_o$$

Using equation (4.92), the condition is thus

$$\frac{\sqrt{\pi}\mathcal{E}^2\nu^2 k^2 R}{4r_o^2}\sin^2\theta_o << \frac{\mathcal{E}^2}{r_o^2}\sin^2\theta_o$$

or simply

$$\nu^2 << \frac{1}{k^2 R\ell} \tag{4.94}$$

where it is recalled that $\nu^2 = <\varepsilon^2> - \varepsilon_1^2$.

If this condition is satisfied, it is reasonably safe to assume that the Born method will yield reliable results.

Some aspects of the results will now be examined. From the expressions for $<|E_{1\theta_o}|^2>$ and $<|E_{1\phi_o}|^2>$ it can be seen that the magnitude of the latter is for the most part $(k\ell)^2$ times the magnitude of the former. However, the magnitudes of the two are equal at $\theta_o = 0$ which is a phenomenon required by the ϕ-symmetry and isotropy of the physical situation.

The vanishing of $<|E_{1\theta_o}|^2>$ at $\theta_o = \pi/2$ has a simple physical explanation. Recall that the $\overline{E}_1$ field can be considered to be generated by electric dipoles which are induced in the inhomogeneities of the medium of the sphere by the $\overline{E}_o$ field. The dipole moment induced in an infinitesimal volume element dv located at the point $\overline{r}$ will be $\varepsilon_1\tilde{\varepsilon}(\overline{r})\overline{E}_o(\overline{r})dv$. This induced dipole is parallel (or anti-parallel) to $\overline{E}_o$. Furthermore, since $\overline{E}_o = E_{o\phi}\,\overline{e}_\phi$ for the original magnetic dipole field, all of the induced dipoles will be directed in the $\overline{e}_\phi$ direction, and hence the vectors representing these dipoles will all lie in planes parallel to the x-y plane (see Figure 4.4). As shown in Figure 4.5, the far-zone field of an electric dipole is in the ξ direction, where ξ is the angle between the axis of the dipole and the ray extending from the dipole to the observation point; the important thing to note is that the vectors representing the dipole and its far-zone field both lie in the same plane. Therefore, if the observation point lies in the far-zone and in the x-y plane, the field at that point due to any one of the induced dipoles in the sphere will be

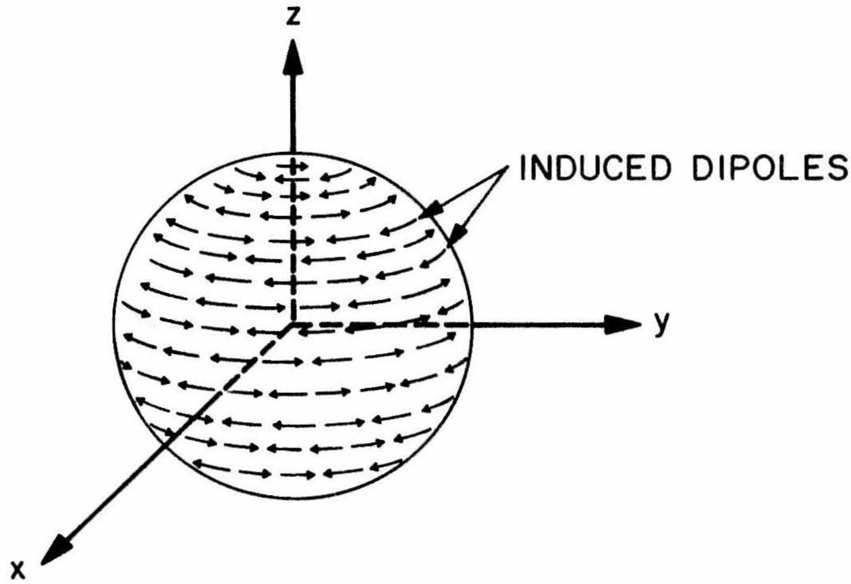

Figure 4.4. Electric Dipoles Induced in the Sphere by the Magnetic Dipole Field

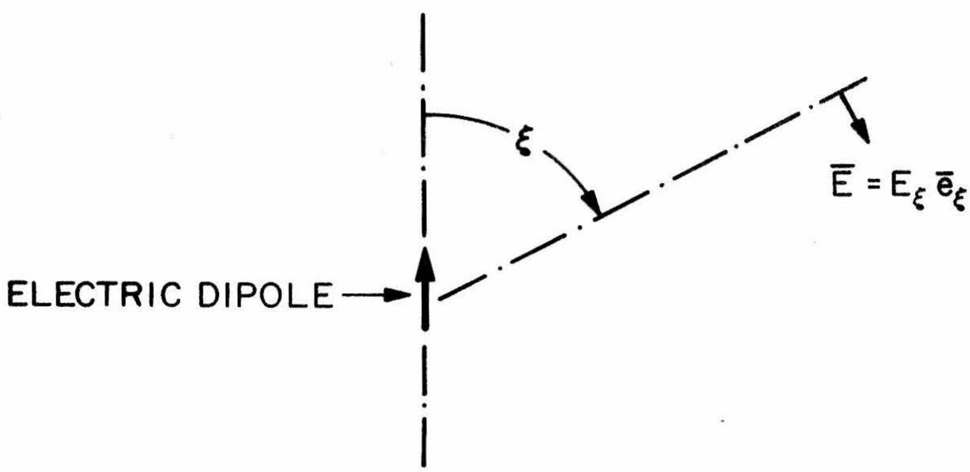

Figure 4.5. Far-Zone Field of an Electric Dipole

directed in some x-y direction and will have no z component. (Actually there would be a slight z-component due to the fact that the observation point is not in the same $z =$ constant plane that the dipole lies in, but rather it lies in a parallel plane. However, this z-component decreases as $1/r_o^2$ and for an observation point in the far-zone the component is negligibly small). Since the total electric field at the observation point is the sum of the fields generated by all the induced dipoles, the total electric field will have no z-component. Noting that the $\overline{e}_\theta$ and $-\overline{e}_z$ directions coincide for a point in the x-y plane, the cause of the vanishing of $<|E_{1\theta_o}|^2>$ for $\theta_o = \pi/2$ is established.

It is of interest to examine the polarization of the $\overline{E}_1$ field. Using the Stokes parameters, this will be done in the same manner that is employed to analyze the polarization of a polychromatic wave [18], except that time averages will be replaced by ensemble averages.

A plane wave may be separated into two parts, an unpolarized part and a polarized part. At any given point in space, the tip of the $\overline{E}$ vector of the unpolarized part of the wave moves in a completely random fashion. On the other hand, the tip of the $\overline{E}$ vector of the polarized part periodically sweeps out an ellipse, circles or straight lines being special cases of the ellipse. Referring to Figure 4.6, two characteristics of this ellipse are the orientation angle ψ $(0 \leq \psi < \pi)$, and the eccentricity angle χ $(|\chi| \leq \pi/4)$ which is defined by $\tan^{-1}(\pm a/A)$ where A is the major axis of the ellipse and a is the minor axis. For a circularly polarized wave, $a = A$ and $\chi = \pm \pi/4$, and for a linearly polarized wave, $a = 0$ and $\chi = 0$.

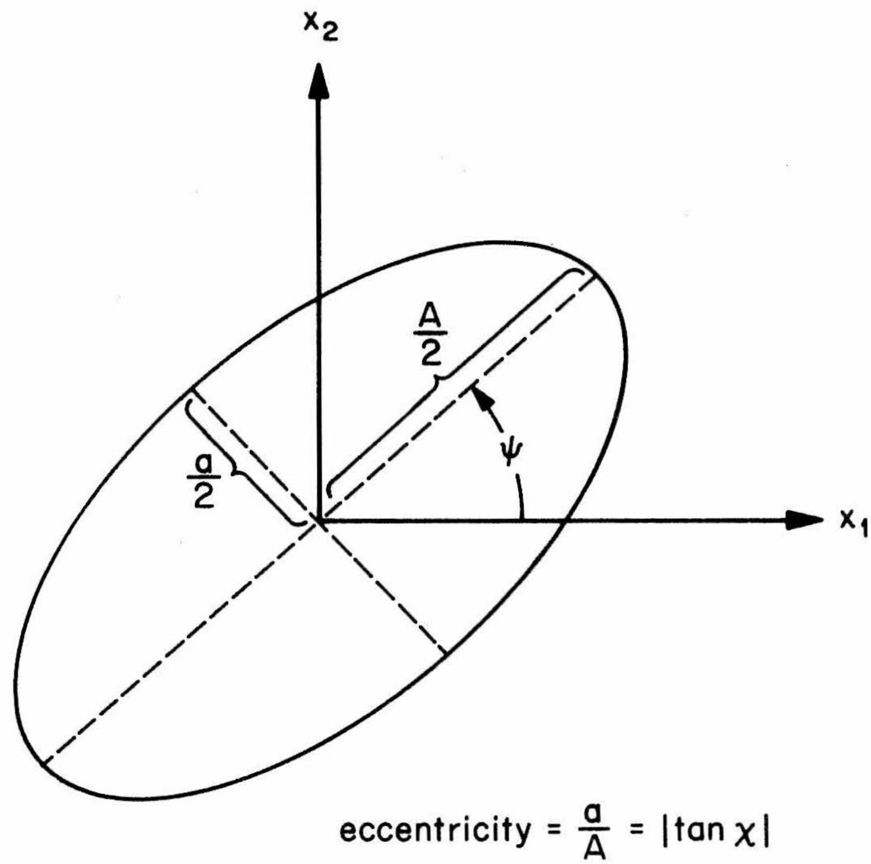

Figure 4.6. Polarization Ellipse

The sign of χ depends upon the direction of rotation of the $\overline{E}$ vector; for a wave receding from the observer, the sign is negative if the rotation is clockwise, positive if counterclockwise.

Referring back to the coordinates of Figure 4.6, the Stokes parameters are defined by

$$s_0 = <|E_{x_1}|^2> + <|E_{x_2}|^2> \tag{4.95}$$

$$s_1 = <|E_{x_1}|^2> - <|E_{x_2}|^2> \tag{4.96}$$

$$s_2 = <E_{x_1} E^*_{x_2}> - <E^*_{x_1} E_{x_2}> \tag{4.97}$$

$$s_3 = i<E^*_{x_1} E_{x_2}> - i<E_{x_1} E^*_{x_2}> \ . \tag{4.98}$$

The relations between the Stokes parameters and the angles ψ and χ are given by

$$\frac{s_1}{\sqrt{s_1^2 + s_2^2 + s_3^2}} = \cos 2\chi \cos 2\psi \tag{4.99}$$

$$\frac{s_2}{\sqrt{s_1^2 + s_2^2 + s_3^2}} = \cos 2\chi \sin 2\psi \tag{4.100}$$

$$\frac{s_3}{\sqrt{s_1^2 + s_2^2 + s_3^2}} = \sin 2\chi \ . \tag{4.101}$$

The degree of polarization m is given by

$$m = \frac{\sqrt{s_1^2 + s_2^2 + s_3^2}}{s_0} \ . \tag{4.102}$$

For a completely polarized wave, $m = 1$, and for an unpolarized wave, $m = 0$.

Now let $x_1 = \theta_o$, $x_2 = \phi_o$. Then from equations (4.95) - (4.99) the Stokes parameters for the $\overline{E}_1$ field are

$$s_0 = <|E_{1\theta_o}|^2> + <|E_{1\phi_o}|^2> \sim \frac{E^2\nu^2R\sqrt{\pi}}{2r_o^2\,\ell}\left[\cos^2\theta_o + 1 + \tfrac{1}{2}(k\ell)^2\sin^2\theta_o\right]$$

$$= \frac{E^2\nu^2R\sqrt{\pi}}{2r_o^2\,\ell}\left[2 + (\tfrac{1}{2}(k\ell)^2 - 1)\sin^2\theta_o\right] \sim \frac{E^2\nu^2R\sqrt{\pi}}{2r_o^2\,\ell}\left[2 + \tfrac{1}{2}(k\ell)^2\sin^2\theta_o\right]$$

$$s_1 = <|E_{1\theta_o}|^2> - <|E_{1\phi_o}|^2> \sim \frac{E^2\nu^2R\sqrt{\pi}}{2r_o^2\,\ell}\left[\cos^2\theta_o - 1 - \tfrac{1}{2}(k\ell)^2\sin^2\theta_o\right]$$

$$= -\frac{E^2\nu^2R\sqrt{\pi}}{2r_o^2\,\ell}\left[1 + \tfrac{1}{2}(k\ell)^2\right]\sin^2\theta_o \sim \frac{-E^2\nu^2R\sqrt{\pi}(k\ell)^2}{4r_o^2\,\ell}\sin^2\theta_o$$

$$s_2 = <E_{1\theta_o}E^*_{1\phi_o}> - <E^*_{1\theta_o}E_{1\phi_o}> = <E_{1\theta_o}E^*_{1\phi_o}> - <E_{1\theta_o}E^*_{1\phi_o}>^* = 0$$

$$s_3 = i<E^*_{1\theta_o}E_{1\phi_o}> - i<E_{1\theta_o}E^*_{1\phi_o}> = i<E_{1\theta_o}E^*_{1\phi_o}>^* - i<E_{1\theta_o}E^*_{1\phi_o}> = 0\,.$$

From equation (4.102) the degree of polarization is

$$m = \frac{[s_1^2 + s_2^2 + s_3^2]^{1/2}}{s_0} \sim \frac{\dfrac{E^2\nu^2R\sqrt{\pi}(k\ell)^2}{4r_o^2\,\ell}\sin^2\theta_o}{\dfrac{E^2\nu^2R\sqrt{\pi}}{2r_o^2\,\ell}\left[2 + \tfrac{1}{2}(k\ell)^2\sin^2\theta_o\right]} = \frac{(k\ell)^2\sin^2\theta_o}{4 + (k\ell)^2\sin^2\theta_o}$$

or

$$m \sim \frac{1}{1 + \frac{4}{(k\ell)^2} \csc^2\theta_o} .$$

Thus $0 \leq m < 1$, and hence the wave is partially polarized. At $\theta_o = 0$ the wave is unpolarized.

The polarized part of the wave will now be examined. Since $s_3 = 0$, it can be seen from equation (4.101) that $\chi = 0$. Hence the polarization is linear. Next, considering equation (4.100), it can be seen that $\sin 2\psi = 0$ since $s_2 = 0$ and $\chi = 0$. Therefore, either $\psi = 0$ or $\psi = \pi/2$. From equation (4.99) one has

$$\frac{s_1}{|s_1|} = \cos 2\psi$$

since $s_2 = s_3 = \chi = 0$. But $s_1 < 0$ for $\theta_o > 0$. Hence $\psi = \pi/2$. Thus, for the polarized part of the wave, the polarization is linear and in the $\emptyset$ direction.

B. Scattering with the Electric Dipole

(i) Derivation

Let a z-oriented electric dipole be situated at the center of the randomly-inhomogeneous sphere. The electric field of such a dipole in a homogeneous medium with permittivity ε_1 and permeability μ is given by [19]

$$\overline{E}_o(\overline{r}) = \frac{m}{4\pi\varepsilon_1}\left(-\frac{2ik}{r^2} + \frac{2}{r^3}\right) e^{ikr} \cos\theta\, \overline{e}_r - \frac{m}{4\pi\varepsilon_1}\left(\frac{k^2}{r} + \frac{ik}{r^2} - \frac{1}{r^3}\right) \times e^{ikr} \sin\theta\, \overline{e}_\theta \qquad (4.103)$$

where m is the dipole strength and where $k = \omega\sqrt{\mu\varepsilon_1}$.

Now $\overline{E}_o(\overline{r})$ has a $1/r^3$ singularity at the origin which is non-integrable. Therefore equation (4.4) cannot be used to express the scattered electric field. Instead, the electric field will be given by equation (4.5):

$$\overline{E}_1(\overline{r}_o) = \frac{-k^2 e^{ikr_o}}{4\pi r_o} \int_{V'} \overline{e}_{r_o} \times [\overline{e}_{r_o} \times \overline{E}_o(\overline{r})] \, \tilde{\varepsilon}(\overline{r}) \, e^{i\overline{k}\cdot\overline{r}} \, d\overline{r}$$

$$- \frac{e^{ikr_o}}{4\pi r_o} \oint_{S''} \{ i\omega\mu \overline{n} \times H_1(\overline{r}) - ik[\overline{n} \times E_1(\overline{r})] \times \overline{e}_{r_o}$$

$$- ik[\overline{n} \cdot (\overline{E}_1(\overline{r}) + (\overline{E}_1(\overline{r}) \, \tilde{\varepsilon}(\overline{r})] \, \overline{e}_{r_o} \} \, e^{-i\overline{k}\cdot\overline{r}} \, dS$$

where S" is the surface of a small spherical volume V" of radius r_1 centered at the origin, V' is the volume of the homogeneous sphere external to S", and where $\overline{k} = k\overline{e}_{r_o}$.

In the far zone $\overline{E}_1(\overline{r}_o)$ is transverse to $\overline{e}_{r_o}$. Therefore it is easy to show that $\overline{e}_{r_o} \times [\overline{e}_{r_o} \times \overline{E}_1(\overline{r}_o)] = -\overline{E}_1(\overline{r}_o)$. Noting that $\overline{e}_{r_o} \times \overline{e}_{r_o} = 0$, one cross multiplies the preceding expression twice by $\overline{e}_{r_o}$ to find

$$\overline{E}_1(\overline{r}_o) = \frac{-k^2 e^{ikr_o}}{4\pi r_o} \int_{V'} \overline{e}_{r_o} \times [\overline{e}_{r_o} \times \overline{E}_o(\overline{r})] \, \tilde{\varepsilon}(\overline{r}) \, e^{-i\overline{k}\cdot\overline{r}} \, d\overline{r}$$

$$- \frac{e^{ikr_o}}{4\pi r_o} \oint_{S''} \{ - i\omega\mu \, \overline{e}_{r_o} \times [\overline{e}_{r_o} \times \overline{H}_1(\overline{r})]$$

$$- ik[\overline{n} \times \overline{E}_1(\overline{r})] \times \overline{e}_{r_o} \} \, e^{-i\overline{k}\cdot\overline{r}} \, dS$$

or

$$\overline{E}_1(\overline{r}_o) = \frac{-k^2 e^{ikr_o}}{4\pi r_o} \int_{V'} \overline{e}_{r_o} \times [\overline{e}_{r_o} \times \overline{E}_o(\overline{r})] \, \tilde{\varepsilon}(\overline{r}) \, e^{-i\overline{k}\cdot\overline{r}} \, d\overline{r}$$

$$+ \frac{i\omega\mu e^{ikr_o}}{4\pi r_o} \overline{e}_{r_o} \times [\overline{e}_{r_o} \times \oint_{S''} \overline{n} \times \overline{H}_1(\overline{r}) \, e^{-i\overline{k}\cdot\overline{r}} \, dS]$$

$$- \frac{ike^{ikr_o}}{4\pi r_o} \overline{e}_{r_o} \times \oint_{S''} \overline{n} \times \overline{E}_1(\overline{r}) \, e^{-i\overline{k}\cdot\overline{r}} \, dS . \qquad (4.104)$$

The behavior of the integrals in equation (4.104) will now be examined as r_1 tends to zero. It is noted immediately that if $r_1 \to 0$, then $V' \to V$, where V is the entire volume of the inhomogeneous sphere; hence, it is understood that one is to find the principal value of the volume integral (which was encountered earlier in connection with equation (2.16)) when the region of integration becomes the entire volume of the inhomogeneous sphere.

Approximate values can be obtained for the surface integrals as $r_1 \to 0$. First, the expression for the magnetic field of the electric dipole in a homogeneous medium will be needed. In a medium with constitutive parameters ε_1 and μ , the magnetic field is [20]

$$\overline{H}_o(\overline{r}) = \frac{i\omega m}{4\pi} \left(\frac{ik}{r} - \frac{1}{r^2}\right) \sin\theta \, e^{ikr} \, \overline{e}_\phi . \qquad (4.105)$$

Let $r_1 << \frac{1}{k}$. Then since $k\ell >> 1$, where ℓ is the correlation distance of the random field $\tilde{\varepsilon}(\overline{r})$, one has $r_1 <<< \ell$. But if r_1 is much less than the correlation distance of $\tilde{\varepsilon}(\overline{r})$, then $\tilde{\varepsilon}(\overline{r})$ can be considered constant on and inside of the surface S'' and may be replaced by $\tilde{\varepsilon}(0)$. Let $\tilde{\varepsilon}(0)$ be denoted by $\tilde{\varepsilon}_o$ where it should be noted that $\tilde{\varepsilon}_o$ is a random variable. Close to the electric dipole, i.e., for $r \leq r_1$, the total electric and magnetic fields are very nearly those which would be generated by the dipole if the medium were homogeneous with permittivity $\varepsilon_1 (1 + \tilde{\varepsilon}_o)$ and permeability μ . Let $k'^2 = \omega^2 \mu \varepsilon_1 (1 + \tilde{\varepsilon}_o)$ so that $k' = \omega \sqrt{\mu \varepsilon_1 (1 + \tilde{\varepsilon}_o)} = k \sqrt{1 + \tilde{\varepsilon}_o}$. Let $\overline{E}(\overline{r})$ and $\overline{H}(\overline{r})$ be the total electric and magnetic fields. Then to obtain the total fields for $r \leq r_1$, one substitutes k' , the expression for ε , and the approximation $e^{ik'r} \sim 1 + ik'r - \frac{1}{2} k'^2 r^2$ into equations (4.103) and (4.105) to find

$$\overline{E}(\overline{r}) \sim \frac{m}{4\pi\varepsilon_1 (1 + \tilde{\varepsilon}_o)} \left(\frac{k'^2}{r^2} + \frac{2}{r^3}\right) \cos\theta \, \overline{e}_r - \frac{m}{4\pi\varepsilon_1 (1 + \tilde{\varepsilon}_o)} \times \left(\frac{1}{2} \frac{k'^2}{r} - \frac{1}{r^3}\right) \sin\theta \, \overline{e}_\theta$$

$$\overline{H}(\overline{r}) \sim \frac{i\omega m}{4\pi} \left(- \frac{1}{2} k'^2 - \frac{1}{r^2}\right) \sin\theta \, \overline{e}_\phi$$

where only the two terms of highest order of magnitude in r are retained for each component. Using the same approximation for the e^{ikr} factor, the following asymptotic approximations can be obtained for the original fields when $r \leq r_1$:

$$\overline{E}_o(\overline{r}) \sim \frac{m}{4\pi\varepsilon_1} \left(\frac{k^2}{r} + \frac{2}{r^3}\right) \cos\theta \, \overline{e}_r - \frac{m}{4\pi\varepsilon_1} \left(\frac{1}{2} \frac{k^2}{r} - \frac{1}{r^3}\right) \sin\theta \, \overline{e}_\theta$$

$$\overline{H}_o(\overline{r}) \sim \frac{i\omega m}{4\pi} \left(- \frac{1}{2} k^2 - \frac{1}{r^2}\right) \sin\theta \, \overline{e}_\phi \, .$$

Now the total field is equal to the original field plus the scattered field, i.e.,

$$\overline{E}(\overline{r}) = \overline{E}_o(\overline{r}) + \overline{E}_1(\overline{r})$$

$$\overline{H}(\overline{r}) = \overline{H}_o(\overline{r}) + \overline{H}_1(\overline{r}) .$$

Hence,

$$\overline{E}_1(\overline{r}) = \overline{E}(\overline{r}) - \overline{E}_o(\overline{r})$$

$$\overline{H}_1(\overline{r}) = \overline{H}(\overline{r}) - \overline{H}_o(\overline{r}) .$$

Using the approximate expressions for $\overline{E}$, $\overline{H}$, $\overline{E}_o$ and $\overline{H}_o$, one finds that

$$\overline{E}_1(\overline{r}) \sim \frac{-m\tilde{\varepsilon}_o}{2\pi\varepsilon_1(1+\tilde{\varepsilon}_o)r^3}\cos\theta\,\overline{e}_r - \frac{m\tilde{\varepsilon}_o}{4\pi\varepsilon_1(1+\tilde{\varepsilon}_o)r^3}\sin\theta\,\overline{e}_\theta$$

$$\overline{H}_1(\overline{r}) \sim \frac{-i\omega m k^2\tilde{\varepsilon}_o}{8\pi}\sin\theta\,\overline{e}_\phi \tag{4.106}$$

for $r \leq r_1$. Using the first order approximation $\dfrac{\tilde{\varepsilon}_o}{1+\tilde{\varepsilon}_o} \sim \tilde{\varepsilon}_o$, the expression for $\overline{E}_1(\overline{r})$ becomes

$$\overline{E}_1(\overline{r}) \sim \frac{-m\tilde{\varepsilon}_o}{2\pi\varepsilon_1 r^3}\cos\theta\,\overline{e}_r - \frac{m\tilde{\varepsilon}_o}{4\pi\varepsilon_1 r^3}\sin\theta\,\overline{e}_\theta . \tag{4.106'}$$

Utilizing these approximate expressions for $\overline{E}_1$ and $\overline{H}_1$, the surface integrals in equation (4.104) become

$$\oint_{S''} \overline{n}\times\overline{H}_1(\overline{r})\, e^{-i\overline{k}\cdot\overline{r}}\, dS \sim \frac{-i\omega\mu m k^2\tilde{\varepsilon}_o}{8\pi}\oint_{S''} \overline{n}\times\overline{e}_\phi \sin\theta\, e^{-i\overline{k}\cdot\overline{r}}\, dS$$

$$\oint_{S''} n\times\overline{E}_1(\overline{r})\, e^{-i\overline{k}\cdot\overline{r}}\, dS \sim \frac{-m\tilde{\varepsilon}_o}{4\pi\varepsilon_1 r_1^3}\oint_{S''} \overline{n}\times[2\cos\theta\,\overline{e}_r + \sin\theta\,\overline{e}_\theta] \times e^{-i\overline{k}\cdot\overline{r}}\, dS .$$

These integrals will be evaluated by using spherical coordinates. Thus $\bar{n} = \bar{e}_r$, $dS = r_1^2 \sin\theta \, d\phi \, d\theta$. Next, making the approximation $e^{-i\bar{k}\cdot\bar{r}} \sim 1 - i\bar{k}\cdot\bar{r} = 1 - ikr_1\bar{e}_{r_o}\cdot\bar{e}_r = 1 - ikr_1[\sin\theta_o \sin\theta\cos\phi + \cos\theta_o \cos\theta]$ on the surface S'', the surface integrals become

$$\oint_{S''} \bar{n}\times\bar{H}_1(r)\, e^{-i\bar{k}\cdot\bar{r}}\, dS \sim \frac{i\omega\mu m k^2\tilde{\varepsilon}_o}{8\pi} r_1^2 \int_0^{\pi}\int_0^{2\pi} \bar{e}_\theta [\sin^2\theta$$

$$- ikr_1(\sin\theta_o\sin^3\theta\cos\phi + \cos\theta_o\cos\theta\sin^3\theta)]\, d\phi\, d\theta$$

$$\oint_{S''} \bar{n}\times\bar{E}_1(\bar{r})\, e^{-i\bar{k}\cdot\bar{r}}\, dS \sim \frac{-m\tilde{\varepsilon}_o}{4\pi\varepsilon_1}\int_0^{\pi}\int_0^{2\pi} \bar{e}_\phi [\frac{1}{r_1}\sin^2\theta$$

$$- ik(\sin\theta_o\sin^3\theta\cos\phi + \cos\theta_o\cos\theta\sin^3\theta)]\, d\phi\, d\theta .$$

As $r_1 \to 0$, the integral with $\bar{H}_1$ obviously tends to zero. Noting that $\bar{e}_\phi = -\sin\phi\,\bar{e}_x + \cos\phi\,\bar{e}_y$ from equation (4.17), the surface integral with $\bar{E}_1$ becomes

$$\oint_{S''} \bar{n}\times\bar{E}_1(\bar{r})\, e^{-i\bar{k}\cdot\bar{r}}\, dS \sim \frac{-m\tilde{\varepsilon}_o}{4\pi\varepsilon_1}\int_0^{\pi}\int_0^{2\pi} [-\sin\phi\,\bar{e}_x + \cos\phi\,\bar{e}_y]$$

$$\times [\frac{1}{r_1}\sin^2\theta - ik(\sin\theta_o\sin^3\theta\cos\phi - \cos\theta_o\cos\theta\sin^3\theta)]\, d\phi\, d\theta .$$

Performing the integrations, one finds that

$$\oint_{S''} \bar{n}\times\bar{E}_1(\bar{r})\, e^{-i\bar{k}\cdot\bar{r}}\, dS \sim \frac{ikm\tilde{\varepsilon}_o\sin\theta_o}{3\varepsilon_1}\bar{e}_y$$

as $r_1 \to 0$.

Using the values obtained for the surface integrals, equation (4.104) now becomes

$$\overline{E}_1(\overline{r}_o) = \frac{-k^2 e^{ikr_o}}{4\pi r_o} \int_V \overline{e}_{r_o} \times [\overline{e}_{r_o} \times \overline{E}_1(\overline{r})] \, \tilde{\varepsilon}(\overline{r}) \, e^{-i\overline{k}\cdot\overline{r}} \, d\overline{r} + \frac{k^2 m \tilde{\varepsilon}_o \sin\theta_o}{12\pi\varepsilon_1 r_o} \overline{e}_{r_o} \times \overline{e}_y$$

or, since $\overline{e}_y = \overline{e}_{\phi_o}$ from equation (4.20) and $\overline{e}_{r_o} \times \overline{e}_{\phi_o} = -\overline{e}_{\theta_o}$,

$$\overline{E}_1(\overline{r}_o) = \frac{-k^2 e^{ikr_o}}{4\pi r_o} \int_V \overline{e}_{r_o} \times [\overline{e}_{r_o} \times \overline{E}_1(\overline{r})] \, \tilde{\varepsilon}(\overline{r}) \, e^{-i\overline{k}\cdot\overline{r}} \, d\overline{r} - \frac{k^2 m \tilde{\varepsilon}_o \sin\theta_o}{12\pi\varepsilon_1 r_o} \overline{e}_{\theta_o} \qquad (4.107)$$

where it is understood that one is to find the principal value of the integral (to find the principal value one merely performs the integration over the angular variables of spherical coordinates first). Let $\overline{e}_\alpha$ represent either one of the unit vectors $\overline{e}_{\theta_o}$, $\overline{e}_{\phi_o}$. Then taking the dot product of $\overline{e}_\alpha$ and $\overline{E}_1(\overline{r}_o)$ yields

$$E_{1\alpha}(\overline{r}_o) = \frac{k^2 e^{ikr_o}}{4\pi r_o} \int_V \overline{e}_\alpha \cdot \overline{E}_o(\overline{r}) \, \tilde{\varepsilon}(\overline{r}) \, e^{-i\overline{k}\cdot\overline{r}} \, d\overline{r} + \frac{E\tilde{\varepsilon}_o \sin\theta_o}{3r_o} \overline{e}_\alpha \cdot \overline{e}_{\theta_o}$$

where $E = -\dfrac{k^2 m}{4\pi\varepsilon_1}$. Similarly, if $\overline{e}_\beta$ also represents either $\overline{e}_{\theta_o}$ or $\overline{e}_{\phi_o}$, then one has

$$E^*_{1\beta}(\overline{r}_o) = \frac{k^2 e^{-ikr_o}}{4\pi r_o} \int_V \overline{e}_\alpha \cdot \overline{E}^*_o(\overline{r}') \, \tilde{\varepsilon}(\overline{r}') \, e^{i\overline{k}\cdot\overline{r}'} \, d\overline{r}' + \frac{E\tilde{\varepsilon}_o \sin\theta_o}{3r_o} \overline{e}_\alpha \cdot \overline{e}_{\theta_o}.$$

Multiplying these expressions together and taking the ensemble average, one finds that

$$< E_{1\alpha}(\bar{r}_o) E^*_{1\beta}(\bar{r}_o) > = \frac{k^4}{16\pi^2 r_o^2} \int_V \int_V \bar{e}_\alpha \cdot \bar{E}_o(\bar{r})\ \bar{e}_\beta \cdot \bar{E}^*_o(\bar{r}')$$

$$\times \ <\tilde{\varepsilon}(\bar{r})\tilde{\varepsilon}(\bar{r}')> e^{-i\bar{k}\cdot(\bar{r}-\bar{r}')}\ d\bar{r}\ d\bar{r}'$$

$$+ \frac{Ek^2 e^{ikr_o} \sin\theta_o\ \bar{e}_\beta \cdot \bar{e}_{\theta_o}}{12\pi\ r_o^2} \int_V \bar{e}_\alpha \cdot \bar{E}_o(\bar{r}) < \tilde{\varepsilon}(\bar{r})\tilde{\varepsilon}_o > \ e^{-\bar{k}\cdot\bar{r}}\ d\bar{r}$$

$$+ \frac{Ek^2 e^{-ikr_o} \sin\theta_o\ \bar{e}_\alpha \cdot \bar{e}_{\theta_o}}{12\pi\ r_o^2} \int_V \bar{e}_\beta \cdot \bar{E}^*_o(\bar{r}') < \tilde{\varepsilon}(\bar{r}')\tilde{\varepsilon}_o > e^{i\bar{k}\cdot\bar{r}'} d\bar{r}'$$

$$+ \frac{E^2 \sin^2\theta_o < \tilde{\varepsilon}_o^2 >}{9 r_o^2} \bar{e}_\alpha \cdot \bar{e}_{\theta_o}\ \bar{e}_\beta \cdot \bar{e}_{\theta_o} \ .$$

But $<\tilde{\varepsilon}(\bar{r})\ \tilde{\varepsilon}(\bar{r}')> = R_{\tilde{\varepsilon}}(|\bar{r}-\bar{r}'|)$. Furthermore, since $\tilde{\varepsilon}_o = \tilde{\varepsilon}(0)$, one has $<\tilde{\varepsilon}(\bar{r})\tilde{\varepsilon}_o> = R_{\tilde{\varepsilon}}(r)$, $<\tilde{\varepsilon}(\bar{r}')\tilde{\varepsilon}_o> = R_{\tilde{\varepsilon}}(r')$, and $<\tilde{\varepsilon}_o^2> = R_{\tilde{\varepsilon}}(0) = \nu^2$. Hence,

$$< E_{1\alpha} E^*_{1\beta} > = \frac{k^4}{16\pi^2 r_o^2} \int_V \int_V \bar{e}_\alpha \cdot \bar{E}_o(\bar{r})\ \bar{e}_\beta \cdot \bar{E}^*_o(\bar{r}') R_{\tilde{\varepsilon}}(|\bar{r}-\bar{r}'|) e^{-i\bar{k}\cdot(\bar{r}-\bar{r}')} d\bar{r}\ d\bar{r}'$$

$$+ \frac{Ek^2 e^{ikr_o} \sin\theta_o\ \bar{e}_\beta \cdot \bar{e}_{\theta_o}}{12\pi r_o^2} \int_V \bar{e}_\alpha \cdot \bar{E}_o(\bar{r})\ R_{\tilde{\varepsilon}}(r)\ e^{-i\bar{k}\cdot\bar{r}}\ d\bar{r}$$

$$+ \frac{Ek^2 e^{-ikr_o} \sin\theta_o\ \bar{e}_\alpha \cdot \bar{e}_{\theta_o}}{12\pi\ r_o^2} \int_V \bar{e}_\beta \cdot \bar{E}^*_o(\bar{r}')\ R_{\tilde{\varepsilon}}(r')\ e^{i\bar{k}\cdot\bar{r}'}\ d\bar{r}'$$

$$+ \frac{E^2 \nu^2 \sin^2\theta_o}{9 r_o^2} \bar{e}_\alpha \cdot \bar{e}_{\theta_o}\ \bar{e}_\beta \cdot \bar{e}_{\theta_o} \ . \qquad (4.108)$$

The correlation functions in equation (4.108) will now be expressed as Fourier integrals of the spectral density. The results are

$$< E_{1\alpha} E^*_{1\beta} > = \frac{k^4}{16\pi^2 r_o^2} \int_V \int_V \int \bar{e}_\alpha \cdot \bar{E}_o(\bar{r}) \; \bar{e}_\beta \cdot \bar{E}^*_o(\bar{r}') \; S_{\tilde{\varepsilon}}(\kappa) \; e^{i\bar{k}\cdot(\bar{r}-\bar{r}')} \times d\bar{\kappa}\, d\bar{r}\, d\bar{r}'$$

$$+ \frac{\varepsilon k^2 e^{ikr_o} \sin\theta_o \bar{e}_\beta \cdot \bar{e}_{\theta_o}}{12\pi r_o^2} \int_V \int \bar{e}_\alpha \cdot \bar{E}_o(\bar{r}) S_{\tilde{\varepsilon}}(\kappa)\, e^{-i\bar{k}\cdot\bar{r}} \times e^{i\bar{\kappa}\cdot\bar{r}}\, d\bar{\kappa}\, d\bar{r}$$

$$+ \frac{\varepsilon k^2 e^{-ikr_o} \sin\theta_o \bar{e}_\alpha \cdot \bar{e}_{\theta_o}}{12\pi r_o^2} \int_V \int \bar{e}_\beta \cdot \bar{E}^*_o(\bar{r}') S^*_{\tilde{\varepsilon}}(\kappa) e^{i\bar{k}\cdot\bar{r}'}\, e^{-i\bar{\kappa}\cdot\bar{r}'}\, d\bar{\kappa}\, d\bar{r}'$$

$$+ \frac{\varepsilon k^2\, {}^2 \sin^2\theta_o}{9 r_o^2}\, \bar{e}_\alpha \cdot \bar{e}_{\theta_o}\, \bar{e}_\beta \cdot \bar{e}_{\theta_o}$$

or

$$< E_{1\alpha} E^*_{1\beta} > = \frac{k^4}{16\pi^2 r_o^2} \int S_{\tilde{\varepsilon}}(\kappa) \left\{ \int_V \bar{e}_\alpha \cdot \bar{E}_o(\bar{r}) e^{-i\bar{r}\cdot(\bar{k}-\bar{\kappa})}\, d\bar{r} \times \int_V \bar{e}_\beta \cdot \bar{E}^*_o(\bar{r}') e^{i\bar{r}'\cdot(\bar{k}-\bar{\kappa})}\, d\bar{r}' \right\} d\bar{\kappa}$$

$$+ \frac{\varepsilon k^2 e^{ikr_o} \sin\theta_o \bar{e}_\beta \cdot \bar{e}_{\theta_o}}{12\pi r_o^2} \int S_{\tilde{\varepsilon}}(\kappa) \left\{ \int_V \bar{e}_\alpha \cdot \bar{E}_o(\bar{r}) e^{-i\bar{r}\cdot(\bar{k}-\bar{\kappa})}\, d\bar{r} \right\} d\bar{\kappa}$$

$$+ \frac{\varepsilon k^2 e^{-ikr_o} \sin\theta_o \bar{e}_\alpha \cdot \bar{e}_{\theta_o}}{12\pi r_o^2} \int S_{\tilde{\varepsilon}}(\kappa) \left\{ \int_V \bar{e}_\beta \cdot \bar{E}^*_o(\bar{r}') e^{i\bar{r}'(\bar{k}-\bar{\kappa})}\, d\bar{r}' \right\} d\bar{\kappa}$$

$$+ \frac{\varepsilon^2 \nu^2 \sin^2\theta_o}{9 r_o^2} \bar{e}_\alpha \cdot \bar{e}_{\theta_o} \bar{e}_\beta \cdot \bar{e}_{\theta_o} \qquad (4.109)$$

where the $\overline{\kappa}$ integration is performed over all $\overline{\kappa}$ space.

Let

$$L_\alpha(\overline{\kappa}) = \int_V \overline{e}_\alpha \cdot \overline{E}_o(\overline{r})\, e^{-i\overline{r}\cdot\overline{\kappa}'}\, d\overline{r} \tag{4.110}$$

where $\overline{\kappa}' = \overline{\kappa} - \overline{k}$. Then equation (4.109) becomes

$$\begin{aligned}< E_{1\alpha}E^*_{1\beta} > = \frac{k^4}{16\pi^2 r_o^2} \int S_{\tilde{\varepsilon}}(\kappa)\, L_\alpha(\overline{\kappa})\, L^*_\beta(\overline{\kappa})\, d\overline{\kappa} \\ + \frac{\mathcal{E} k^2 e^{ikr_o} \sin\theta_o\, \overline{e}_\beta \cdot \overline{e}_{\theta_o}}{12\pi\, r_o^2} \int S_{\tilde{\varepsilon}}(\kappa)\, L_\alpha(\overline{\kappa})\, d\overline{\kappa}\end{aligned}$$

$$+ \frac{\mathcal{E} k^2 e^{-ikr_o} \sin\theta_o\, \overline{e}_\alpha \cdot \overline{e}_{\theta_o}}{12\pi\, r_o^2} \int S_{\tilde{\varepsilon}}(\kappa)\, L^*_\beta(\overline{\kappa}) d\overline{\kappa} + \frac{\mathcal{E}^2 \nu^2 \sin^2\theta_o}{9 r_o^2}\, \overline{e}_\alpha \cdot \overline{e}_{\theta_o}\, \overline{e}_\beta \cdot \overline{e}_{\theta_o} . \tag{4.111}$$

In particular,

$$\begin{aligned}<|E_{1\theta_o}|^2> = \frac{k^4}{16\pi\, r_o^2} \int S_{\tilde{\varepsilon}}(\kappa) |L_{\theta_o}(\overline{\kappa})|^2\, d\overline{\kappa} \\ + 2\,\mathrm{Re}\left\{ \frac{\mathcal{E} k^2 e^{ikr_o} \sin\theta_o}{12\pi\, r_o^2} \int S_{\tilde{\varepsilon}}(\kappa)\, L_{\theta_o}(\kappa) d\overline{\kappa} \right\} + \frac{\mathcal{E}^2 \nu^2 \sin^2\theta_o}{9 r_o^2}\end{aligned} \tag{4.112}$$

$$<|E_{1\phi_o}|^2> = \frac{k^4}{16\pi^2\, r_o^2} \int S_{\tilde{\varepsilon}}(\kappa) |L_{\phi_o}(\overline{\kappa})|^2\, d\overline{\kappa} \tag{4.113}$$

$$\begin{aligned}< E_{1\theta_o} E^*_{1\phi_o} > = \frac{k^4}{16\pi^2\, r_o^2} \int S_{\tilde{\varepsilon}}(\kappa) L_{\theta_o}(\overline{\kappa}) L^*_{\phi_o}(\overline{\kappa}) d\overline{\kappa} + \frac{\mathcal{E} k^2 \sin\theta_o e^{-ikr_o}}{12\pi\, r_o^2} \\ \times \int S_{\tilde{\varepsilon}}(\kappa)\, L^*_{\phi_o}(\kappa)\, d\overline{\kappa} .\end{aligned} \tag{4.114}$$

Now $\overline{E}_o(\overline{r})$ is given by equation (4.103). Recalling that $\mathcal{E} = -k^2m/4\pi\varepsilon_1$, one finds that this equation becomes

$$\overline{E}_o(\overline{r}) = -\mathcal{E}\left(\frac{-2i}{kr^2} + \frac{2}{k^2r^3}\right)e^{ikr}\cos\theta\,\overline{e}_r + \mathcal{E}\left(\frac{1}{r} + \frac{i}{kr^2} - \frac{1}{k^2r^3}\right)e^{ikr}\sin\theta\,\overline{e}_\theta . \tag{4.115}$$

But from equations (4.24) and (4.25) one has

$$\overline{e}_r = (\sin\theta_o\sin\theta\cos\phi + \cos\theta_o\cos\theta)\overline{e}_{r_o} + (\cos\theta_o\sin\theta\cos\phi - \sin\theta_o\cos\theta)\overline{e}_{\theta_o} + \sin\theta\sin\phi\,\overline{e}_{\phi_o}$$

$$\overline{e}_\theta = (\sin\theta_o\cos\theta\cos\phi - \cos\theta_o\sin\theta)\overline{e}_{r_o} + (\cos\theta_o\cos\theta\cos\phi + \sin\theta_o\sin\theta)\overline{e}_{\theta_o} + \cos\theta\sin\phi\,\overline{e}_{\phi_o} .$$

Substituting these expressions into equation (4.115) and using the fact that

$$h_o(kr) = \frac{-i}{kr}e^{ikr}, \quad h_2(kr) = \left[\frac{-i}{kr} - \frac{3}{(kr)^2} - \frac{3i}{(kr)^3}\right]e^{ikr}$$

$$P_2(\cos\theta) = \frac{3}{2}\left[\cos^2\theta - \frac{1}{3}\right], \quad P_2^1(\cos\theta) = 3\sin\theta\cos\theta,$$

one finds that the expression for $\overline{E}_o$ becomes

$$\overline{E}_o(\overline{r}) = -\frac{i\mathcal{E}k}{3}\{2h_o(kr)\cos\theta_o + h_2(kr)[\sin\theta_o P_2^1(\cos\theta)\cos\phi + 2P_2(\cos\theta)\cos\theta_o]\}\,\overline{e}_{r_o} +$$

$$+ \frac{i\mathcal{E}k}{3}\{2h_o(kr)\sin\theta_o - h_2(kr)[\cos\theta_o P_2^1(\cos\theta)\cos\phi - 2P_2(\cos\theta)\sin\theta_o]\}\,\overline{e}_{\theta_o}$$

$$- \frac{i\mathcal{E}k}{3} h_2(kr)\, P_2^1(\cos\theta)\sin\phi\,\overline{e}_{\phi_o} . \tag{4.116}$$

Substitution of equation (4.116) into equation (4.110) yields

$$L_{\theta_o}(\overline{\kappa}) = \frac{i\mathcal{E}k}{3}\int_V \{2h_o(kr)\sin\theta_o - h_2(kr)[\cos\theta_o\, P_2^1(\cos\theta)\cos\phi - 2P_2(\cos\theta)\sin\theta_o]\}\, e^{i\overline{\kappa}'\cdot\overline{r}}\, d\overline{r} \tag{4.117}$$

$$L_{\phi_o}(\overline{\kappa}) = \frac{-i\mathcal{E}k}{3}\int_V h_2(kr)\, P_2^1(\cos\theta)\sin\phi\, e^{i\overline{\kappa}'\cdot\overline{r}}\, d\overline{r} . \tag{4.118}$$

Computation of L_{θ_o} and L_{ϕ_o} proceeds in the same manner as in the case of the magnetic dipole. First, if $\overline{\kappa}' = \kappa'\sin\psi\cos\gamma\,\overline{e}_x + \kappa'\sin\psi\sin\gamma\,\overline{e}_y + \kappa'\cos\psi\,\overline{e}_z$, then from equation (4.36) one has

$$e^{i\overline{\kappa}'\cdot\overline{r}} = \sum_{n=0}^{\infty} i^n(2n+1)j_n(\kappa' r)\left[\sum_{m=0}^{n}\alpha_m\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta) \times P_n^m(\cos\psi)\cos m(\gamma-\phi)\right].$$

Substitution of this into equations (4.117) and (4.118), which are expressed in terms of spherical coordinates, and reversal of the order of summation and integration yields

$$L_{\theta_o}(\overline{\kappa}) = \frac{i\mathcal{E}k}{3}\sum_{n=0}^{\infty}\Bigg\{ i^n(2n+1)\left[\sum_{m=0}^{n}\alpha_m\frac{(n-m)!}{(n+m)!}P_n^m(\cos\psi)\right.$$

$$\times \int_0^R\int_0^\pi\int_0^{2\pi}\{2h_o(kr)\sin\theta_o - h_2(kr)[\cos\theta_o P_2^1(\cos\theta)\cos\phi -$$

$$- 2P_2(\cos\theta)\sin\theta_o]\Big\} j_n(\kappa' r)P_n^m(\cos\theta)\cos m(\gamma-\phi)r^2\sin\theta\, d\phi\, d\theta\, dr\Bigg]\Bigg\}$$

$$L_{\phi_o}(\bar{\kappa}) = \frac{-i\mathcal{E}k}{3}\sum_{n=0}^{\infty}\Bigg\{ i^n(2n+1)\Bigg[\sum_{m=0}^{n}\alpha_m\frac{(n-m)!}{(n+m)!}P_n^m(\cos\psi)$$

$$\times\int_0^R\int_0^\pi\int_0^{2\pi}[h_2(kr)P_2^1(\cos\theta)\sin\phi]\, j_n(\kappa' r)P_n^m(\cos\theta)$$

$$\cos m(\gamma-\phi)r^2\sin\theta\, d\phi\, d\theta\, dr\Bigg]\Bigg\}.$$

Performing the angular integrations with the help of the orthogonality relation (A.19), one finds that

$$L_{\theta_o}(\bar{\kappa}) = \frac{-4\pi i\mathcal{E}k}{3}\int_0^R \{h_2(kr)j_2(\kappa' r)[P_2^1(\cos\psi)\cos\theta_o\cos\gamma$$

$$- 2P_2(\cos\psi)\sin\theta_o] + 2h_o(kr)j_o(\kappa' r)\sin\theta_o\}\, r^2 dr \qquad (4.119)$$

$$L_{\phi_o}(\bar{\kappa}) = \frac{4\pi i\mathcal{E}k}{3}P_2^1(\cos\psi)\sin\gamma\int_0^R h_2(kr)\, j_2(\kappa' r)r^2 dr\,. \qquad (4.120)$$

Using equation (A.9) to evaluate the integrals in equations (4.119) and (4.120), one obtains

$$\int_0^R h_o(kr)j_o(\kappa' r)r^2 dr = \frac{1}{k^2-\kappa'^2}\{R^2[kh_1(kR)j_o(\kappa' R)$$

$$- \kappa' h_o(kR)j_o(\kappa' R)] + \frac{i}{k}\} \qquad (4.121)$$

$$\int_0^R h_2(kr)j_2(\kappa' r)r^2 dr = \frac{1}{k^2-\kappa'^2}\{R^2[kh_3(kR)j_2(\kappa'R) - \kappa' h_2(kR)j_3(\kappa'R)] + \frac{i\kappa'^2}{k^3}\} . \quad (4.122)$$

Substitution of these expressions into equations (4.119) and (4.120) yields

$$L_{\theta_o}(\overline{\kappa}) = \frac{-4\pi i E k}{k^2-\kappa'^2}\left\{\left[R^2[kh_3(kR)j_2(\kappa'R) - \kappa' h_2(kR)j_3(\kappa'R)] + \frac{i\kappa'^2}{k^3}\right]\right.$$

$$\times\left[\cos\theta_o \cos\psi\sin\psi\cos\gamma - (\cos^2\psi - \frac{1}{3})\sin\theta_o\right]$$

$$\left. + \frac{2}{3}\left[R^2[kh_1(kR)j_o(\kappa'R) - \kappa' h_o(kR)j_1(\kappa'R)] + \frac{i}{k}\right]\sin\theta_o\right\} \quad (4.123)$$

$$L_{\phi_o}(\overline{\kappa}) = \frac{4\pi i E k}{k^2-\kappa'^2}\cos\psi\sin\psi\sin\gamma\{R^2[kh_3(kR)j_2(\kappa'R) - \kappa' h_2(kR)j_3(\kappa'R)] + \frac{i\kappa'^2}{k^3}\} \quad (4.124)$$

where the expressions for the Legendre polynomials have been written out explicitly.

Now just as in the case of the magnetic dipole, the region of integration in $\overline{\kappa}$ space for the integrals in equations (4.112) - (4.114) can be restricted from all of $\overline{\kappa}$ space to the region where $|\overline{\kappa} - \overline{k}| = \kappa' \triangleq \frac{1}{2}k$ with negligible error, provided $k\ell >> 1$. Letting U represent the region where $|\overline{\kappa} - \overline{k}| \triangleq \frac{1}{2}k$, equations (4.112) - (4.114) become

$$<|E_{1\theta_o}|^2> = \frac{k^4}{16\pi^2 r_o^2}\int_U S_{\underset{\sim}{\varepsilon}}(\kappa)|L_{\theta_o}(\overline{\kappa})|^2 d\overline{\kappa} + 2\mathrm{Re}\left\{\frac{\mathcal{E}k^2 e^{ikr_o}\sin\theta_o}{12\pi r_o^2}\right.$$

$$\left.\times \int_U S_{\underset{\sim}{\varepsilon}}(\kappa)\, L_{\theta_o}(\overline{\kappa})d\overline{\kappa}\right\} + \frac{\mathcal{E}^2\nu^2\sin^2\theta_o}{9r_o^2} \tag{4.125}$$

$$<|E_{1\phi_o}|^2> = \frac{k^4}{16\pi^2 r_o^2}\int_U S_{\underset{\sim}{\varepsilon}}(\kappa)|L_{\phi_o}(\overline{\kappa})|^2\, d\overline{\kappa} \tag{4.126}$$

$$<E_{1\theta_o}E^*_{1\phi_o}> = \frac{k^4}{16\pi^2 r_o^2}\int_U S_{\underset{\sim}{\varepsilon}}(\kappa)\, L_{\theta_o}(\overline{\kappa})L^*_{\phi_o}(\overline{\kappa})d\overline{\kappa} + \frac{\mathcal{E}k^2\sin\theta_o e^{-ikr_o}}{12\pi r_o^2}$$

$$\times \int_U S^*(\kappa)\, L^*_{\phi_o}(\overline{\kappa})\, d\overline{\kappa}\,. \tag{4.127}$$

Since $\kappa' = |\overline{\kappa} - \overline{k}| \geq \frac{1}{2}k$ in U, and since $kR >>> 1$, one may use the asymptotic expansions which are valid for large arguments for the Hankel and Bessel functions found in equations (4.123) and (4.124). From Section A.1, one finds

$$h_1(kR) \sim \frac{-1}{kR}e^{ikR}\ , \quad h_2(kR) \sim \frac{i}{kR}e^{ikR}\ , \quad h_3(kR) \sim \frac{1}{kR}e^{ikR}$$

$$j_1(\kappa'R) \sim \frac{-1}{\kappa'R}\cos\kappa'R, \quad j_2(\kappa'R) \sim \frac{-1}{\kappa'R}\sin\kappa'R\ , \quad j_3(\kappa'R) \sim \frac{1}{\kappa'R}\cos\kappa'R$$

as $kR \to \infty$. Substitution of these into equations (4.123), (4.124) and rearranging terms yields

$$L_{\theta_o}(\bar{\kappa}) = \frac{4\pi i E k}{k^2-\kappa'^2}(\cos\theta_o \cos\psi \sin\psi \cos\gamma + \sin^2\psi \sin\theta_o)$$

$$\times \left[e^{ikR}\left(\frac{\sin\kappa' R}{\kappa'} + i\,\frac{\cos\kappa' R}{k}\right) - i\,\frac{\kappa'^2}{k^3}\right] + \frac{8\pi E \sin\theta_o}{3k^2} \qquad (4.128)$$

$$L_{\phi_o}(\bar{\kappa}) = \frac{-4\pi i E k}{k^2-\kappa'^2}\cos\psi \sin\psi \sin\gamma \left[e^{ikR}\left(\frac{\sin\kappa' R}{\kappa'} + i\,\frac{\cos\kappa' R}{k}\right) - i\,\frac{\kappa'^2}{k^3}\right]. \qquad (4.129)$$

In the neighborhood of $\kappa' = k$, the last term on the right hand side of equation (4.128) will be quite small in comparison to the rest of the expression. Since it is known from the derivation with the magnetic dipole that most of the contribution to the integrals in equations (4.125) and (4.126) comes from this neighborhood, the last term in eq. (4.128) may be neglected. Then equations (4.128) and (4.129) become

$$L_{\theta_o}(\bar{\kappa}) = 4\pi i E k(\cos\psi \sin\psi \cos\gamma \cos\theta_o + \sin^2\psi \sin\theta_o)\, F(\kappa') \qquad (4.130)$$

$$L_{\phi_o}(\bar{\kappa}) = -4\pi i E k \cos\psi \sin\psi \sin\gamma\, F(\kappa') \qquad (4.131)$$

where

$$F(\kappa') = \frac{1}{k^2-\kappa'^2}\left| e^{ikR}\left(\frac{\sin\kappa' R}{\kappa'} + i\,\frac{\cos\kappa' R}{k}\right) - \frac{i\kappa'^2}{k^3}\right|. \qquad (4.132)$$

Substituting equations (4.130) and (4.131) into equations (4.125) - (4.127), one finds that

$$<|E_{1\theta_o}|^2> = \frac{\mathcal{E}^2 k^6 \ell^3 \nu^2}{8\pi^{3/2} r_o^2} \int_U e^{-\kappa^2\ell^2/4} (\cos\psi \sin\psi \cos\gamma \cos\theta_o + \sin^2\psi \sin\theta_o)^2 |F(\kappa')|^2 d\bar{\kappa}$$

$$+ \frac{\mathcal{E}^2 k^3 \ell^3 \nu^3 \sin\theta_o}{12\pi^{3/2} r_o^2} \mathrm{Re}\left\{ ie^{ikr_o} \int_U e^{-\kappa^2\ell^2/4} (\cos\psi \sin\psi \cos\gamma \cos\theta_o + \sin^2\psi \sin\theta) F(\kappa') d\bar{\kappa} \right\} + \frac{\mathcal{E}^2\nu^2 \sin^2\theta_o}{9r_o^2}$$

(4.133)

$$<|E_{1\phi_o}|^2> = \frac{\mathcal{E}^2 k^6 \ell^3 \nu^2}{8\pi^{3/2} r_o^2} \int_U e^{-\kappa^2\ell^2/4} (\cos\psi \sin\psi \sin\gamma)^2 |F(\kappa')|^2 d\bar{\kappa} \tag{4.134}$$

$$< E_{1\theta_o} E^*_{1\phi_o} > = -\frac{\mathcal{E}^2 k^6 \ell^3 \nu^2}{8\pi^{3/2} r_o^2} \int_U e^{-\kappa^2\ell^2/4} (\cos\psi \sin\psi \cos\gamma \cos\theta_o + \sin^2\psi \sin\theta_o)(\cos\psi \sin\psi \sin\gamma) |F(\kappa')|^2 d\bar{\kappa}$$

$$+ \frac{i\mathcal{E}^2 k^3 \ell^3 \nu^2 \sin\theta \, e^{-ikr_o}}{24\pi^{3/2} r_o^2} \int_U e^{-\kappa^2\ell^2/4} \cos\psi \sin\psi \sin\gamma \, F^*(\kappa') \, d\bar{\kappa} \tag{4.135}$$

where the explicit equation for $S_{\tilde{\varepsilon}}(\kappa)$, given by equation (4.47), has been substituted into the equations.

Changing the integration variables from $\bar{\kappa}$ to $\bar{\kappa}'$, recalling that $\kappa' = \bar{\kappa} - \bar{k}$, and noting that $\cos\psi = \kappa'_z/\kappa'$, $\sin\psi\cos\gamma = \kappa'_x/\kappa'$, $\sin\psi\sin\gamma = \kappa'_y/\kappa'$, one finds that equations (4.133) - (4.135) become

$$<|E_{1\theta_o}|^2> = \frac{\mathcal{E}^2 k^6 \ell^3 \nu^2}{8\pi^{3/2}\, r_o^2} \int_U e^{-|\bar{\kappa}'+\bar{\kappa}|^2 \ell^2/4} \Big[\frac{\kappa'_z}{\kappa'^2}(\kappa'_x \cos\theta_o - \kappa'_z \sin\theta_o)$$

$$+ \sin\theta_o\Big]^2 \, |F(\kappa')|^2 \, d\bar{\kappa}' - \frac{\mathcal{E}^2 k^3 \ell^3 \nu^2 \sin\theta_o}{12\pi^{3/2}\, r_o^2}$$

$$\times \operatorname{Im}\left\{ e^{ikr_o} \int_U e^{-|\bar{\kappa}'+\bar{k}|^2 \ell^2/4} \Big[\frac{\kappa'_z}{\kappa'^2}(\kappa'_x \cos\theta_o - \kappa'_z \sin\theta_o)\right.$$

$$\left. + \sin\theta_o\Big] F(\kappa')\, d\bar{\kappa}' \right\} + \frac{\mathcal{E}^2 \nu^2 \sin^2\theta_o}{9 r_o^2} \tag{4.136}$$

$$<|E_{1\phi_o}|^2> = \frac{\mathcal{E}^2 k^6 \ell^3 \nu^2}{8\pi^{3/2}\, r_o^2} \int_U e^{-|\bar{\kappa}'+\bar{k}|^2 \ell^2/4} \frac{\kappa'^2_z \kappa'^2_y}{\kappa'^4} |F(\kappa')|^2 \, d\bar{\kappa}' \tag{4.137}$$

$$< E_{1\theta_o} E^*_{1\phi_o} > = \frac{-\mathcal{E}^2 k^6 \ell^3 \nu^2}{8\pi^{3/2}\, r_o^2} \int_U e^{-|\bar{\kappa}'+\bar{k}|^2 \ell^2/4} \Big[\frac{\kappa'_z}{\kappa'^2}(\kappa'_x \cos\theta_o - \kappa'_z \sin\theta_o)$$

$$+ \sin\theta_o\Big] \frac{\kappa'_z \kappa'_x}{\kappa'^2} |F(\kappa')|^2 \, d\bar{\kappa}'$$

$$+ \frac{i\mathcal{E}^2 k^3 \ell^3 \nu^2 \sin\theta_o e^{-ikr_o}}{24\pi^{3/2}\, r_o^2} \int_U e^{-|\bar{\kappa}'+\bar{k}|^2} \frac{\kappa'_z \kappa'_x}{\kappa'^2} F^*(\kappa')\, d\bar{\kappa}' . \tag{4.138}$$

Now a second change of variables will be made from the $\bar{\kappa}'$ variables to the $\bar{\zeta}$ variables, where the $\bar{\zeta}$ variables are defined by equations (4.61) - (4.63). Noting that $\zeta = \kappa'$, $\kappa'_x \cos\theta_o - \kappa'_z \sin\theta_o = \zeta_1 = \zeta \sin\eta \cos\xi$, etc., equations (4.136) - (4.138) become

$$<|E_{1\theta_o}|^2> = \frac{E^2k^6\ell^3\nu^2}{8\pi^{3/2}\, r_o^2} \int_{k/2}^{\infty}\int_0^{\pi}\int_0^{2\pi} e^{-[\zeta^2+k^2+2\zeta k \cos\eta]\ell^2/4}$$

$$\times\ [\sin^4\eta\ \cos^4\xi\ \sin^2\theta_o + \cos^2\eta\ \sin^2\eta\ \cos^2\xi\ \cos^2\theta_o\ + \sin^2\theta_o$$

$$-\ 2\cos\eta\,\sin^3\eta\ \cos^3\xi\ \cos\theta_o\sin\theta_o - 2\sin^2\eta\ \cos^2\xi\ \sin^2\theta_o$$

$$+\ 2\cos\eta\,\sin\eta\,\cos\xi\,\cos\theta_o\sin\theta_o]\ |F(\zeta)|^2\ \zeta^2\sin\eta\, d\xi\ d\eta\ d\zeta$$

$$-\ \frac{E^2k^3\ell^3\nu^2\sin\theta_o}{12\pi^{3/2}\, r_o^2}\ \mathrm{Im}\left\{e^{ikr_o}\int_{k/2}^{\infty}\int_0^{\pi}\int_0^{2\pi} e^{-[\zeta^2+k^2+2\zeta k\cos\eta]\ell^2/4}\right.$$

$$\times\ [-\sin^2\eta\ \cos^2\xi\ \sin\theta_o + \cos\eta\,\sin\eta\,\cos\xi\,\cos\theta_o$$

$$\left. +\ \sin\theta_o]\ F(\zeta)\ \zeta^2\sin\eta\, d\xi\ d\eta\ d\zeta\right\} + \frac{E^2\nu^2\sin^2\theta_o}{9r_o^2} \tag{4.139}$$

$$<|E_{1\phi_o}|^2> = \frac{E^2k^6\ell^3\nu^2}{8\pi^{3/2}\, r_o^2}\int_{k/2}^{\infty}\int_0^{\pi}\int_0^{2\pi} e^{-[\zeta^2+k^2+2\zeta k\cos\eta]\ell^2/4}$$

$$\times\ [\sin^4\eta\ \cos^2\xi\ \sin^2\xi\ \sin^2\theta_o\ - 2\cos\eta\,\sin^3\eta\ \cos\xi\,\sin\xi\,\cos\theta_o\sin\theta_o$$

$$+\ \cos^2\eta\ \sin^2\eta\ \sin^2\xi\ \cos^2\theta_o]\,|F(\zeta)|^2\ \zeta^2\sin\eta\, d\xi\ d\eta\ d\zeta \tag{4.140}$$

$$< E_{1\theta_o} E^*_{1\phi_o} > = \frac{-E^2 k^6 \ell^3 \nu^2}{8\pi^{3/2}\, r_o^2} \int_{k/2}^{\infty} \int_0^{\pi} \int_0^{2\pi} e^{-[\zeta^2 + 2\zeta k \cos\eta]\ell^2/4}$$

$$\times \; [\sin^4\eta \cos^3\xi \sin\xi \sin^2\theta_o - \cos\eta \sin^3\eta \cos^2\xi \cos\theta_o \sin\theta_o$$

$$- \; \sin^2\eta \cos\xi \sin\xi \sin^2\theta_o - \cos\eta \sin^3\eta \cos^2\xi \sin\xi \cos\theta_o \sin\theta_o$$

$$+ \cos^2\eta \sin^2\eta \cos\xi \sin\xi \cos^2\theta_o + \cos\eta \sin\eta \sin\xi \cos\theta_o \sin\theta_o]$$

$$\times \; |F(\zeta)|^2 \, \zeta^2 \sin\eta \, d\xi \, d\eta \, d\zeta + \frac{iE^2 k^3 \ell^3 \nu^2 \sin\theta_o e^{-ikr_o}}{24\pi^{3/2}\, r_o^2}$$

$$\times \int_{k/2}^{\infty} \int_0^{\pi} \int_0^{2\pi} e^{-[\zeta^2 + k^2 + 2\zeta k \cos\eta]\ell^2/4} \; [-\sin^2\eta \cos\xi \sin\xi +$$

$$+ \cos\eta \sin\eta \sin\xi \cos\theta_o] \; F^*(\zeta)\zeta^2 \sin\eta \, d\eta \, d\xi \, d\zeta \, . \qquad (4.141)$$

Performing the ξ integrations in eqs. (4.139) - (4.141), one finds that

$$<|E_{1\theta_o}|^2> = \frac{E^2 k^6 \ell^3 \nu^2}{32\sqrt{\pi}\, r_o^2} \int_{k/2}^{\infty} \int_0^{\pi} e^{-[\zeta^2 + k^2 + 2\zeta k \cos\eta]\ell^2/4} \; [3 \sin^4\eta \sin^2\theta_o$$

$$+ \; 4 \cos^2\eta \sin^2\eta \cos^2\theta_o + 8 \cos^2\eta \sin^2\theta_o] \; |F(\zeta)|^2 \, \zeta^2 \sin\eta \, d\eta \, d\zeta$$

$$- \frac{E^2 k^3 \ell^3 \nu^2 \sin^2\theta_o}{12\pi\, r_o^2} \, \mathrm{Im} \left\{ e^{ikr_o} \int_{k/2}^{\infty} \int_0^{\pi} e^{-[\zeta^2 + k^2 + 2\zeta k \cos\eta]\ell^2/4} \right.$$

$$\left. \times \; [1 + \cos^2\eta] \; F(\zeta)\zeta^2 \sin\eta \, d\eta \, d\zeta \right\} + \frac{E^2 \nu^2 \sin^2\theta_o}{9 r_o^2} \qquad (4.142)$$

$$<|E_{1\phi_o}|^2> = \frac{E^2 k^6 \ell^3 \nu^2}{32\sqrt{\pi}\, r_o^2} \int_{k/2}^{\infty} \int_0^{\pi} e^{-[\zeta^2+k^2+2\zeta k \cos\eta]\ell^2/4} [\sin^4\eta \sin^2\theta_o$$

$$+ 4 \cos^2\eta \sin^2\eta \cos^2\theta_o] \, |F(\zeta)|^2 \zeta^2 \sin\eta \, d\eta \, d\zeta \qquad (4.143)$$

$$< E_{1\theta_o} E^*_{1\phi_o} > = 0 \, . \qquad (4.144)$$

It can be seen from equation (4.144) that the transverse components of the field are uncorrelated, just as in the case of the magnetic dipole.

Making the change of variables $w = \zeta\ell$, $t = \cos\eta$, and rearranging terms, one finds that equations (4.142) and (4.143) become

$$<|E_{1\theta_o}|^2> = \frac{E^2 k^6 \nu^2}{32\sqrt{\pi}\, r_o^2} \int_{k\ell/2}^{\infty} \int_{-1}^{1} e^{-[w^2+k^2\ell^2+2wk\ell t]/4} [3 \sin^2\theta_o +$$

$$+ (4 - 2 \sin^2\theta_o) t^2 + (7 \sin^2\theta_o - 4) t^4] \, |F(\tfrac{w}{\ell})|^2 \, w^2 dt \, dw$$

$$- \frac{E^2 k^3 \nu^2 \sin^2\theta_o}{12\sqrt{\pi}\, r_o^2} \, \mathrm{Im} \left\{ e^{ikr_o} \int_{k\ell/2}^{\infty} \int_{-1}^{1} e^{-[w^2+k^2\ell^2+2wk\ell t]/4} \right.$$

$$\left. \times \; [1 + t^2] \, F(\tfrac{w}{\ell}) \, w^2 dt \, dw \right\}$$

$$+ \frac{E^2 \nu^2 \sin^2\theta_o}{9 r_o^2} \qquad (4.145)$$

$$<|E_{1\phi_o}|^2> = \frac{\mathcal{E}^2 k^6 \nu^2}{32\sqrt{\pi}\, r_o^2} \int_{k\ell/2}^{\infty} \int_{-1}^{1} e^{-[w^2+k^2\ell^2+2wk\ell t]/4} \, [\sin^2\theta_o +$$

$$+ 6\cos^2\theta_o - 2)t^2 + (1 - 5\cos^2\theta_o)t^4] \, |F(\tfrac{w}{\ell})|^2 \, w^2 dt\, dw. \quad (4.146)$$

Now through repeated integration by parts, one can show that

$$\int_{-1}^{1} t^4 e^{-wk\ell t/2} dt \sim [\frac{2}{wk\ell} - \frac{16}{(wk\ell)^2} + \frac{96}{(wk\ell)^3}] \, e^{wk\ell/2} \quad (4.147)$$

for $wk\ell >> 1$. Using this equation and equations (4.77) and (4.78), one can perform the t integrations in equations (4.145) and (4.146). Keeping only the higher order terms in $wk\ell$, one obtains

$$<|E_{1\theta_o}|^2> \sim \frac{\mathcal{E}^2 k^4 \nu^2}{\sqrt{\pi}\, \ell^2 r_o^2} \int_{k\ell/2} e^{-[w-k\ell]^2/4} \, (1 + \tfrac{1}{2} wk\ell \, \sin^2\theta_o) |F(\tfrac{w}{\ell})|^2 dw$$

$$- \frac{\mathcal{E}^2 k^2 \nu^2 \sin^2\theta_o}{3\sqrt{\pi}\, \ell r_o^2} \, \mathrm{Im} \left\{ e^{-ikr_o} \int_{k\ell/2}^{\infty} e^{-[w-k\ell]^2/4} \, wF(\tfrac{w}{\ell}) dw \right\}$$

$$+ \frac{\mathcal{E}^2 \nu^2 \sin^2\theta_o}{9 r_o^2} \quad (4.148)$$

$$<|E_{1\phi_o}|^2> \sim \frac{2\mathcal{E}^2 k^3 \nu^2}{\sqrt{\pi}\ell^3 r_o^2} \int_{k\ell/2}^{\infty} e^{-[w-k\ell]^2/4} \left[\frac{1 + \tfrac{1}{2} wk\ell \, \cos^2\theta_o}{w}\right] |F(\tfrac{w}{\ell})|^2 \, dw . \quad (4.149)$$

Owing to the fact that most of the contribution to the integrals in equations (4.148) and (4.149) comes from a relatively small neighborhood about the point $w = k\ell$, the same type of approximations can be made here that were made in the case of the magnetic dipole. These approximations are

$$1 + \frac{1}{2}\, wk\ell\ \sin^2\theta_o \sim 1 + \frac{1}{2}(k\ell)^2 \sin^2\theta_o$$

$$\frac{1 + \frac{1}{2}\, wk\ell\ \sin^2\theta_o}{w} \sim \frac{1 + \frac{1}{2}(k\ell)^2 \sin^2\theta_o}{k\ell}$$

$$F(\tfrac{w}{\ell}) \sim -\frac{\ell}{2k^2}\left\{\frac{\sin[(w-k\ell)\frac{R}{\ell}]}{(w-k\ell)} + i\,\frac{\cos[(w-k\ell)\frac{R}{\ell}] - 1}{(w-k\ell)}\right\} \tag{4.150}$$

$$|F(\tfrac{w}{\ell})|^2 \sim \frac{\ell^2}{2k^4}\left\{\frac{1 - \cos[(w-k\ell)\frac{R}{\ell}]}{(w-k\ell)^2}\right\} . \tag{4.151}$$

When these approximations are made in equations (4.148) and (4.149) and if at the same time the change of variables $\tau = (w-k\ell)\frac{R}{\ell}$ is made, the results are

$$\begin{aligned} <|E_{1\theta_o}|^2> \sim\ & \frac{E^2\nu^2 R}{2\sqrt{\pi}\,\ell r_o^2}\,[1 + \tfrac{1}{2}(k\ell)^2\sin^2\theta_o] \int_{-\infty}^{\infty} e^{-\tau^2\ell^2/4R^2}\,[\frac{1 - \cos\tau}{\tau^2}]\, d\tau \\ & + \frac{E^2 k\ell\nu^2\sin^2\theta_o}{6\sqrt{\pi}\ r_o^2}\ \mathrm{Im}\left\{e^{ikr_o}\int_{-\infty}^{\infty} e^{-\tau^2\ell^2/4R^2}\,[\frac{\sin\tau}{\tau} + i\,\frac{\cos\tau - 1}{\tau}]\,d\tau\right\} \\ & + \frac{E^2\nu^2\sin^2\theta_o}{9r_o^2} \end{aligned} \tag{4.152}$$

$$<|E_{1\phi_o}|^2> \sim \frac{E^2\nu^2 R}{2r_o^2\ell}\,[\cos^2\theta_o + \frac{2}{(k\ell)^2}]\int_{-\infty}^{\infty} e^{-\tau^2\ell^2/4R^2}\,[\frac{1-\cos\tau}{\tau^2}]\,d\tau \tag{4.153}$$

where the lower limit on the integrals has been extended from $\frac{1}{2}\, k\ell$ to $-\infty$.

It has been shown in evaluating equation (4.88) that

$$\int_{-\infty}^{\infty} e^{-\tau^2 \ell^2/4R^2} \left[\frac{1-\cos\tau}{\tau^2}\right] d\tau \sim \pi . \tag{4.154}$$

Furthermore, it can be seen that

$$\int_{-\infty}^{\infty} e^{-\tau^2 \ell^2/4R^2} \left[\frac{\cos\tau - 1}{\tau}\right] d\tau = 0 \tag{4.155}$$

since the integrand is an odd function. Hence, equations (4.152) and (4.153) become

$$\begin{aligned} <|E_{1\theta_o}|^2> \sim \; & \frac{E^2\nu^2 R\sqrt{\pi}}{2r_o^2\,\ell}\left[1 + \frac{1}{2}(k\ell)^2\sin^2\theta_o\right] \\ & + \frac{E^2 k\ell\nu^2\sin^2\theta_o \sin kr_o}{6\sqrt{\pi}\, r_o^2}\int_{-\infty}^{\infty} e^{-\tau^2\ell^2/4R^2}\,\frac{\sin\tau}{\tau}\,d\tau \\ & + \frac{E^2\nu^2\sin^2\theta_o}{9r_o^2} \end{aligned} \tag{4.156}$$

$$<|E_{1\phi_o}|^2> \sim \frac{E^2\nu^2 R\sqrt{\pi}}{2r_o^2\,\ell}\left[\cos^2\theta_o + \frac{2}{(k\ell)^2}\right] . \tag{4.157}$$

Comparing the magnitudes of the first and last terms on the right-hand side of equation (4.156), it can be seen that the former is on the order of $\frac{R}{\ell}(k\ell)^2$ times the latter. Since $k\ell \gg 1$, $R/\ell \gg 1$, the last term should be neglected. Furthermore, the integral in the second term on the right-hand side of equation (4.156) was calculated earlier in evaluating the integral in equation (4.88). The result was

$$\int_{\infty}^{\infty} e^{-\tau^2 \ell^2 / 4R^2} \frac{\sin \tau}{\tau} \, d\tau \sim \pi \, . \tag{4.158}$$

Thus it can be seen that the first term of equation (4.156) is on the order of kR times the second term and hence the second term should also be neglected. Therefore, one has

$$<|E_{1\theta_o}|^2> \sim \frac{\mathcal{E}^2 \nu^2 R\sqrt{\pi}}{2r_o^2 \ \ell} [1 + \tfrac{1}{2}(k\ell)^2 \sin^2\theta_o] \, . \tag{4.159}$$

(ii) <u>Discussion of results</u>

Summarizing the results of the derivation for the electric dipole situated at the center of the randomly-inhomogeneous sphere, one has

$$<|E_{1\theta_o}|^2> \sim \frac{\mathcal{E}^2 \nu^2 R\sqrt{\pi}}{2r_o^2 \ \ell} [1 + \tfrac{1}{2}(k\ell)^2 \sin^2\theta_o]$$

$$<|E_{1\phi_o}|^2> \sim \frac{\mathcal{E}^2 \nu^2 R\sqrt{\pi}}{2r_o^2 \ \ell} [\cos^2\theta_o + \frac{2}{(k\ell)^2}]$$

$$<E_{1\theta_o} E^*_{1\phi_o}> \ = \ 0 \, .$$

A plot of the root-mean-square field components for $k\ell = 10$ is given in Figure 4.7.

It can be seen that the results for the electric dipole are nearly the same as those for the magnetic dipole, except that $E_{1\theta_o}$ and $E_{1\phi_o}$ are interchanged. However, it should be noted that

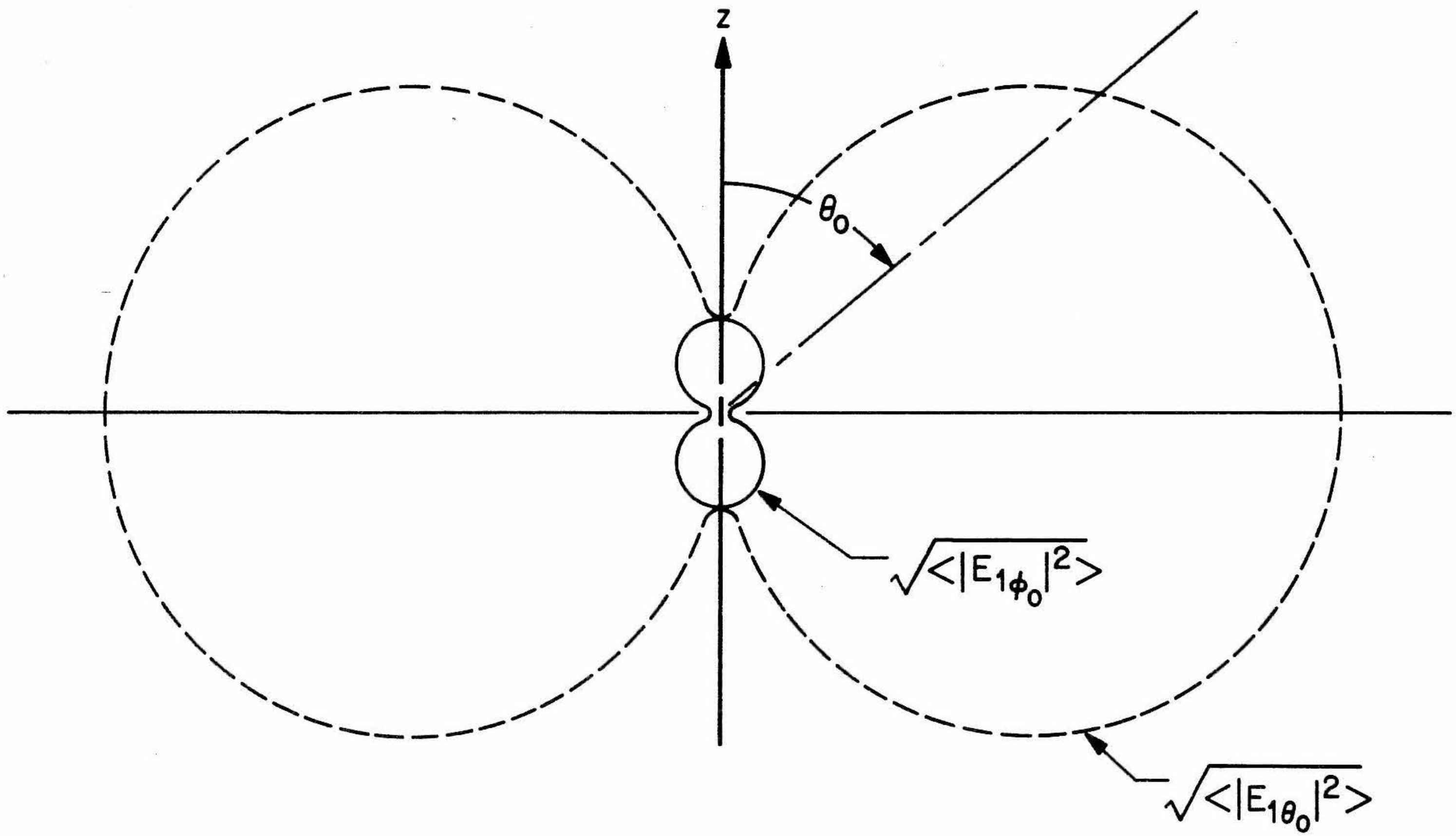

Figure 4.7. Root-Mean-Square Scattered Field Components for Electric Dipole with $k\ell = 10$

$<|E_{1\phi_o}|^2>$ does not vanish at $\theta_o = \pi/2$, unlike the case of $<|E_{1\theta_o}|^2>$ for the magnetic field. It is also seen that $<|E_{1\phi_o}|^2>$ and $<|E_{1\theta_o}|^2>$ are not precisely equal at $\theta_o = 0$ but rather they differ by a term of the order $1/(k\ell)^2$. The reason for this is that in making the approximations in the calculation for $<|E_{1\theta_o}|^2>$, terms of this order were neglected. More exact calculations would show that $<|E_{1\theta_o}|^2>$ and $<|E_{1\phi_o}|^2>$ are in fact equal at $\theta_o = 0$.

One can show that the condition required on the parameters in order that the Born approximation yield reliable results is the same as for the magnetic dipole, namely

$$\nu^2 << \frac{1}{k^2 R\ell} \quad .$$

The polarization of the scattered wave can be analyzed with the help of the Stokes parameters just as in the previous case. The degree of polarization m is given by

$$m \sim \frac{1}{1 + \frac{4}{(k\ell)^2} \csc^2\theta_o}$$

and the polarization of the polarized part of the wave is linear and in the θ direction.

4.3 Scattering for the Case of $k\ell << 1$

The far-zone scattered field will now be determined under the assumption that the wavelength is much greater than the correlation distance of the randomly-inhomogeneous medium, i.e., it is assumed that

$k\ell << 1$. The case of the magnetic dipole will be considered first, to be followed by the case of the electric dipole.

A. Scattering with the Magnetic Dipole

(i) Derivation

The zeroth order electric field of the magnetic dipole in terms of the unit vectors $\bar{e}_{r_o}$, $\bar{e}_{\theta_o}$, and $\bar{e}_{\phi_o}$ was given by equation (4.28) which is repeated here:

$$\bar{E}_o(\bar{r}) = -\mathcal{E} \sin \theta_o (\frac{1}{r} + \frac{i}{kr^2}) e^{ikr} P_1^1(\cos \theta) \sin \phi \, \bar{e}_{r_o}$$

$$-\mathcal{E} \cos \theta_o (\frac{1}{r} + \frac{i}{kr^2}) e^{ikr} P_1^1(\cos \theta) \sin \phi \, \bar{e}_{\theta_o}$$

$$+ \mathcal{E} (\frac{1}{r} + \frac{i}{kr^2}) e^{ikr} P_1^1(\cos \theta) \cos \phi \, \bar{e}_{\phi_o}$$

where the fact that $P_1^1(\cos \theta) = \sin \theta$ has been employed.

Since the singularities of $\bar{E}_o$ at the origin are integrable, the expression for $\bar{E}_1$ or either of its components is given by equation (4.4) or (4.6). Then utilizing the development in Section 4.1, the expression for the mean square components and mean of the product of the components is obtained from equation (4.10):

$$\langle E_{1\alpha} E_{1\beta}^* \rangle = \frac{k^4}{16\pi^2 r_o^2} \int_V \int_V \bar{e}_\alpha \cdot \bar{E}_o(\bar{r}) \, \bar{e}_\beta \cdot \bar{E}_o^*(\bar{r}') e^{-i\bar{k} \cdot (\bar{r}-\bar{r}')}$$

$$\times \; R_\varepsilon(|\bar{r}-\bar{r}'|) d\bar{r} \, d\bar{r}'$$

where V is the entire volume of the randomly-inhomogeneous sphere and where $\bar{k} = k\bar{e}_{r_o}$. In particular,

$$<|E_{1\theta_o}|^2> = \frac{k^4}{16\pi^2 r_o^2} \int_V \int_V \bar{e}_{\theta_o} \cdot \bar{E}_o(\bar{r})\, \bar{e}_{\theta_o} \cdot \bar{E}^*(\bar{r}') e^{-i\bar{k}\cdot(\bar{r}-\bar{r}')}$$

$$\times\ R_{\tilde{\varepsilon}}(|\bar{r}-\bar{r}'|)\, d\bar{r}\, d\bar{r}' \qquad (4.160)$$

$$<|E_{1\emptyset_o}|^2> = \frac{k^4}{16\pi^2 r_o^2} \int_V \int_V \bar{e}_{\emptyset_o} \cdot \bar{E}_o(\bar{r})\, \bar{e}_{\emptyset_o} \cdot \bar{E}^*(\bar{r}') e^{-i\bar{k}\cdot(\bar{r}-\bar{r}')}$$

$$\times\ R_{\tilde{\varepsilon}}(|\bar{r}-\bar{r}|)\, d\bar{r}\, d\bar{r}' \qquad (4.161)$$

$$< E_{1\theta_o} E_{1\emptyset_o} > = \frac{k^4}{16\pi^2 r_o^2} \int_V \int_V \bar{e}_{\theta_o} \cdot \bar{E}_o(\bar{r})\, \bar{e}_{\emptyset_o} \cdot \bar{E}_o^*(\bar{r}') e^{-i\bar{k}\cdot(\bar{r}-\bar{r}')}$$

$$\times\ R_{\tilde{\varepsilon}}(|\bar{r}-\bar{r}'|)\, d\bar{r}\, d\bar{r}' . \qquad (4.162)$$

Substituting the explicit expression for $\bar{E}_o$ into equations (4.160) - (4.162) and recalling that $R_{\tilde{\varepsilon}}(|\bar{r}-\bar{r}'|) = \nu^2 e^{-|\bar{r}-\bar{r}'|^2/\ell^2}$, one finds that

$$<|E_{1\theta_o}|^2> = \frac{E^2 k^4 \nu^2 \cos^2\theta_o}{16\pi^2\, r_o^2} \int_V \int_V e^{-|\bar{r}-\bar{r}'|^2/\ell^2} \left[\frac{1}{rr'} + \frac{i}{k}\left(\frac{1}{r^2 r'}\right.\right.$$

$$\left.\left. - \frac{1}{rr'^2}\right) + \frac{1}{k^2 r^2 r'^2}\right] e^{ik(r-r')}\, P_1^1(\cos\theta)$$

$$\times\ P_1^1(\cos\theta')\sin\emptyset \sin\emptyset' e^{-i\bar{k}\cdot(\bar{r}-\bar{r}')} d\bar{r}\, d\bar{r}'$$

$$(4.163)$$

$$< |E_{1\phi_o}|^2 > = \frac{E^2 k^4 \nu^2}{16\pi^2 r_o^2} \int_V \int_V e^{-|\bar{r}-\bar{r}'|^2/\ell^2} \left[\frac{1}{rr'} + \frac{i}{k}\left(\frac{1}{r^2 r'} - \frac{1}{rr'^2}\right)\right.$$

$$\left. + \frac{1}{k^2 r^2 r'^2}\right] e^{ik(r-r')} P_1^1(\cos\theta)\, P_1^1(\cos\theta')\cos\phi\,\cos\phi'$$

$$\times\; e^{i\bar{k}\cdot(\bar{r}-\bar{r}')}\, d\bar{r}\, d\bar{r}' \quad (4.164)$$

$$< E_{1\theta_o} E^*_{1\phi_o} > = \frac{-E^2 k^4 \nu^2 \cos\theta_o}{16\pi^2 r_o^2} \int_V \int_V e^{-|\bar{r}-\bar{r}'|^2/\ell^2} \left[\frac{1}{rr'} + \frac{i}{k}\left(\frac{1}{r^2 r'} - \frac{1}{rr'^2}\right)\right.$$

$$\left. + \frac{1}{k^2 r^2 r'^2}\right] e^{ik(r-r')} P_1^1(\cos\theta)\, P_1^1(\cos\theta')\cos\phi'\,\sin\phi$$

$$\times\; e^{-i\bar{k}\cdot(\bar{r}-\bar{r}')}\, d\bar{r}\, d\bar{r}'. \quad (4.165)$$

By expressing the correlation function in equation (4.165) as the Fourier transform of the spectral density, one can proceed to show that $< E_{1\theta_o} E^*_{1\phi_o} > = 0$ by using virtually the same method that was employed in Section 4.2. Referring to equation (4.53), the only difference is that one would not reduce the region of integration over all $\bar{\kappa}$ space to the region U where $|\bar{\kappa} - \bar{k}| = \frac{1}{2} k$. It can be seen by following the development in Section 4.2 that this would not affect the demonstration of the fact that $< E_{1\theta_o} E^*_{1\phi_o} > = 0$.

Attention can now be focused on equations (4.163) and (4.164). Asymptotic expressions will be obtained for $< |E_{1\theta_o}|^2 >$ and $< |E_{1\phi_o}|^2 >$ for two cases.

Making the change of variables $\bar{\rho} = \frac{1}{\ell}\,\bar{r}$ and $\bar{\rho}' = \frac{1}{\ell}\,\bar{r}'$ so that $d\bar{r} = \ell^3 d\bar{\rho}$ and $d\bar{r}' = \ell^3 d\bar{\rho}'$, one finds that equations (4.163) and (4.164) become

$$<|E_{1\theta_o}|^2> = \frac{\mathcal{E}^2k^4\ell^4\nu^2\cos^2\theta_o}{16\pi^2\ r_o^2} \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} \cdot \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} e^{-|\bar{\rho}-\bar{\rho}'|^2} \Big[\frac{1}{\rho\rho'}$$

$$+ \frac{i}{k\ell}\Big(\frac{1}{\rho^2\rho'} - \frac{1}{\rho\rho'^2}\Big) + \frac{1}{k^2\ell^2\rho^2\rho'^2}\Big] e^{ik\ell(\rho-\rho')} P_1^1(\cos\theta)$$

$$\times \quad P_1^1(\cos\theta')\sin\emptyset\ \sin\emptyset'\ e^{i\ell\bar{k}\cdot(\bar{\rho}-\bar{\rho}')}\ \rho^2\rho'^2\sin\theta\ \sin\theta'$$

$$\times \quad d\emptyset\ d\theta\ d\rho\ d\emptyset'\ d\theta'\ d\rho' \qquad (4.166)$$

$$<|E_{1\emptyset_o}|^2> = \frac{\mathcal{E}^2k^4\ell^4\nu^2}{16\pi^2\ r_o^2} \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} e^{-|\bar{\rho}-\bar{\rho}'|^2} \Big[\frac{1}{\rho\rho'} + \frac{i}{k\ell}\Big(\frac{1}{\rho^2\rho'}$$

$$- \frac{1}{\rho\rho'^2}\Big) + \frac{1}{k^2\ell^2\rho^2\rho'^2}\Big] e^{ik\ell(\rho-\rho')} P_1^1(\cos\theta\ P_1^1(\cos\theta')\cos\emptyset\ \cos\emptyset'$$

$$\times \quad e^{-i\ell\bar{k}\cdot(\bar{\rho}-\bar{\rho}')}\ \rho^2\rho'^2\sin\theta\ \sin\theta'\ d\emptyset\ d\theta\ d\rho\ d\emptyset'\ d\theta'\ d\rho'. \quad (4.167)$$

Now it can be seen from these equations that as $k \to 0$ while R and ℓ remain constant, then

$$<|E_{1\theta_o}|^2> \sim \frac{\mathcal{E}^2k^4\ell^4\nu^2\cos^2\theta_o}{16\pi^2\ r_o^2} \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} \cdot \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} e^{-|\bar{\rho}-\bar{\rho}'|^2} \frac{1}{k^2\ell^2\rho^2\rho'^2}$$

$$\times \quad e^{ik\ell(\rho-\rho')}\ e^{-i\ell\bar{k}\cdot(\bar{\rho}-\bar{\rho}')}\ P_1^1(\cos\theta)\ P_1^1(\cos\theta')$$

$$\times \quad \sin\emptyset\ \sin\emptyset'\rho^2\rho'^2\sin\theta\ \sin\theta'\ d\emptyset\ d\theta\ d\rho\ d\emptyset'\ d\theta'\ d\rho' \qquad (4.168)$$

$$<|E_{1\phi_o}|^2> \sim \frac{E^2k^4\ell^4\nu^2}{16\pi^2\, r_o^2} \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} \cdot \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} e^{-|\bar{\rho}-\bar{\rho}'|^2} \frac{1}{k^2\ell^2\rho^2\rho'^2}$$

$$\times \quad e^{ik\ell(\rho-\rho')}\, e^{-i\ell\bar{k}\cdot(\bar{\rho}-\bar{\rho}')}\, P_1^1(\cos\theta)\, P_1^1(\cos\theta')\cos\phi\cos\phi'$$

$$\times \quad \rho^2\rho'^2\sin\theta\sin\theta'\, d\phi\, d\theta\, d\rho\, d\phi'\, d\theta'\, d\rho' . \qquad (4.169)$$

Now $e^{-|\bar{\rho}-\bar{\rho}'|^2}$ decreases very rapidly for $|\bar{\rho}-\bar{\rho}'| > 1$. Therefore most of the contribution to the integrals in equations (4.168) and (4.169) will be obtained when $|\bar{\rho}-\bar{\rho}'|$ is small. But $|\rho-\rho'| \leq |\bar{\rho}-\bar{\rho}'|$ and $|\bar{e}_r \cdot(\bar{\rho}-\bar{\rho}')| \leq |\bar{\rho}-\bar{\rho}'|$. Then since it is given that $k\ell << 1$, it will be true that $|k\ell(\rho-\rho')| << 1$, $|\ell\bar{k}\cdot(\bar{\rho}-\bar{\rho}')| << 1$ in the region where most of the contribution to the integrals in (4.168) and (4.169) is derived. Hence one can make the approximations $e^{ik\ell(\rho-\rho')} \sim 1$, $e^{i\ell\bar{k}\cdot(\bar{\rho}-\bar{\rho}')} \sim 1$. Thus, equations (4.168) and (4.169) become

$$<|E_{1\theta_o}|^2> \sim \frac{E^2k^2\ell^2\nu^2\cos^2\theta_o}{16\pi^2\, r_o^2} \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} \cdot \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} e^{-|\bar{\rho}-\bar{\rho}'|^2}$$

$$\times \quad P_1^1(\cos\theta)P_1^1(\cos\theta')\sin\phi\sin\phi'\sin\theta\sin\theta'$$

$$\times \quad d\phi\, d\theta\, d\rho\, d\phi'\, d\theta'\, d\rho' \quad (4.170)$$

$$<|E_{1\phi_o}|^2> \sim \frac{E^2k^2\ell^2\nu^2}{16\pi^2 r_o^2} \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} \cdot \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} e^{-|\bar{\rho}-\bar{\rho}'|^2}$$

$$\times \quad P_1^1(\cos\theta)P_1^1(\cos\theta')\cos\phi\cos\phi'\sin\theta\sin\theta'$$

$$\times \quad d\phi\, d\theta\, d\rho\, d\phi'\, d\theta'\, d\rho' \quad (4.171)$$

as $k \to 0$ while R and ℓ remain fixed.

Returning to equations (4.163) and (4.164), let the change of variables $\bar{\eta} = \frac{1}{R}\bar{r}$, $\bar{\eta}' = \frac{1}{R}\bar{r}'$ now be made. Then the expressions for $<|E_{1\theta_o}|^2>$ and $<|E_{1\phi_o}|^2>$ become

$$<|E_{1\theta_o}|^2> = \frac{E^2k^4R^4\nu^2\cos^2\theta_o}{16\pi^2 r_o^2} \int_0^1 \int_0^\pi \int_0^{2\pi} \cdot \int_0^1 \int_0^\pi \int_0^{2\pi} e^{-|\bar{\eta}-\bar{\eta}'|^2R^2/\ell^2}$$

$$\times \left[\frac{1}{\eta\eta'} + \frac{i}{kR}\left(\frac{1}{\eta^2\eta'} - \frac{1}{\eta\eta'^2}\right) + \frac{1}{k^2R^2\eta^2\eta'^2}\right] e^{kR(\eta-\eta')} e^{iR\bar{k}\cdot(\bar{\eta}-\bar{\eta}')}$$

$$\times P_1^1(\cos\theta)P_1^1(\cos\theta')\sin\phi \sin\phi'\eta^2\eta'^2\sin\theta \sin\theta' d\phi\, d\theta\, d\eta\, d\phi'\, d\theta'\, d\eta' \tag{4.172}$$

$$<|E_{1\phi_o}|^2> = \frac{E^2k^4R^4\nu^2}{16\pi^2 r_o^2} \int_0^1 \int_0^\pi \int_0^{2\pi} \cdot \int_0^1 \int_0^\pi \int_0^{2\pi} e^{-|\bar{\eta}-\bar{\eta}'|^2R^2/\ell^2}$$

$$\times \left[\frac{1}{\eta\eta'} + \frac{i}{kR}\left(\frac{1}{\eta^2\eta'} - \frac{1}{\eta\eta'^2}\right) + \frac{1}{k^2R^2\eta^2\eta'^2}\right] e^{ikR(\eta-\eta')} e^{iR\bar{k}\cdot(\bar{\eta}-\bar{\eta}')}$$

$$\times P_1^1(\cos\theta)P_1^1(\cos\theta')\cos\phi \cos\phi'\eta^2\eta'^2\sin\theta \sin\theta' d\phi\, d\theta\, d\eta\, d\phi'\, d\theta'\, d\eta'. \tag{4.173}$$

These equations suggest that as $R \to \infty$ while k and ℓ remain constant, then one need only consider the $1/\eta\eta'$ term in the brackets. This is because the other terms have coefficients of $1/kR$ or $1/k^2R^2$ which tend to zero as R tends to infinity. The fact that R also appears in the exponential complicates matters, but a careful analysis shows that the major contribution to the integral does come from the

$1/\eta\eta'$ term as R becomes increasingly large. Thus, as $R \to \infty$, one has

$$<|E_{1\theta_o}|^2> \sim \frac{\mathcal{E}^2 k^4 R^4 \nu^2 \cos^2\theta_o}{16\pi^2 r_o^2} \int_0^1 \int_0^\pi \int_0^{2\pi} \cdot \int_0^1 \int_0^\pi \int_0^{2\pi} e^{-|\overline{\eta}-\overline{\eta}'|^2 R^2/\ell^2}$$

$$\times \quad e^{ikR(\eta-\eta')} e^{-iR\overline{k}\cdot(\overline{\eta}-\overline{\eta}')} P_1^1(\cos\theta) P_1^1(\cos\theta') \sin\phi \sin\phi'$$

$$\times \quad \eta\eta' \sin\theta \sin\theta' \, d\phi \, d\theta \, d\eta \, d\phi' \, d\theta' \, d\eta'$$

$$<|E_{1\phi_o}|^2> \sim \frac{\mathcal{E}^2 k^4 R^4 \nu^2}{16\pi^2 r_o^2} \int_0^1 \int_0^\pi \int_0^{2\pi} \cdot \int_0^1 \int_0^\pi \int_0^{2\pi} e^{-|\overline{\eta}-\overline{\eta}'|^2 R^2/\ell^2} e^{ikR(\eta-\eta')}$$

$$\times \; e^{-iR\overline{k}\cdot(\overline{\eta}-\overline{\eta}')} P_1^1(\cos\theta) P_1^1(\cos\theta') \cos\phi \cos\phi'$$

$$\times \quad \eta\eta' \sin\theta \sin\theta' \, d\phi \, d\theta \, d\eta \, d\phi' \, d\theta' \, d\eta'$$

or, in terms of the $\overline{\rho}$ and $\overline{\rho}'$ variables,

$$<|E_{1\theta_o}|^2> \sim \frac{\mathcal{E}^2 k^4 \ell^4 \nu^2 \cos^2\theta_o}{16\pi^2 r_o^2} \int_0^{R/\ell} \int_0^\pi \int_0^{2\pi} \cdot \int_0^{R/\ell} \int_0^\pi \int_0^{2\pi} e^{-|\overline{\rho}-\overline{\rho}'|^2} e^{ik\ell(\rho-\rho')}$$

$$\times \; e^{-i\ell\overline{k}\cdot(\overline{\rho}-\overline{\rho}')} P_1^1(\cos\theta) P_1^1(\cos\theta') \sin\phi \sin\phi'$$

$$\times \quad \rho\rho' \sin\theta \sin\theta' \, d\phi \, d\theta \, d\rho \, d\phi' \, d\theta' \, d\rho' \qquad (4.174)$$

$$<|E_{1\phi_o}|^2> \sim \frac{E^2 k^4 \ell^4 \nu^2}{16\pi^2 r_o^2} \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} \cdot \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} e^{-|\bar{\rho}-\bar{\rho}'|^2} e^{ik\ell(\rho-\rho')}$$

$$\times \quad e^{-i\ell\bar{k}\cdot(\bar{\rho}-\bar{\rho}')} P_1^1(\cos\theta) P_1^1(\cos\theta')$$

$$\times \cos\phi \cos\phi' \sin\theta \sin\theta' d\phi\, d\theta\, d\rho\, d\phi'\, d\theta'\, d\rho' .$$

(4.175)

It should be noted that the observation point must still be in the far-zone, i.e., r_o must satisfy $r_o >> kR^2$.

As before, since $e^{-|\bar{\rho}-\bar{\rho}'|^2}$ decreases very rapidly for $|\bar{\rho}-\bar{\rho}'| > 1$, and since it is still assumed that $k\ell << 1$, the approximations $e^{ik\ell(\rho-\rho')} \sim 1$, $e^{-i\ell\bar{k}\cdot(\bar{\rho}-\bar{\rho}')} \sim 1$ will be employed. Hence equations (4.174) and (4.175) become

$$<|E_{1\theta_o}|^2> \sim \frac{E^2 k^4 \ell^4 \nu^2 \cos^2\theta_o}{16\pi^2\ r_o^2} \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} \cdot \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} e^{-|\bar{\rho}-\bar{\rho}'|^2}$$

$$\times \quad P_1^1(\cos\theta) P_1^1(\cos\theta') \sin\phi \sin\phi' \rho\rho' \sin\theta \sin\theta' d\phi\, d\theta\, d\rho\, d\phi' d\theta' d\rho'$$

(4.176)

$$<|E_{1\phi_o}|^2> \sim \frac{E^2 k^4 \ell^4 \nu^2}{16\pi^2 r_o^2} \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} \cdot \int_0^{\ell}\int_0^{\pi}\int_0^{2\pi} e^{-|\bar{\rho}-\bar{\rho}'|^2}$$

$$\times \quad P_1^1(\cos\theta) P_1^1(\cos\theta') \cos\phi \cos\phi' \rho\rho' \sin\theta \sin\theta' d\phi\, d\theta\, d\rho\, d\phi'\, d\theta'\, d\rho'$$

(4.177)

as $R \to \infty$ while k and ℓ remain fixed.

Thus the problem of finding the far-zone scattered fields when $k\ell << 1$ has been split into two cases, the first being when $k \to 0$ with R and ℓ remaining constant, the second being when $R \to \infty$ with k and ℓ remaining constant.

Now the integrals in equations (4.170), (4.171), (4.176) and (4.177) will be evaluated. It is noted that

$$|\overline{\rho}-\overline{\rho}'|^2 = \rho^2 + \rho'^2 + 2\overline{\rho} \cdot \overline{\rho}' .$$

Hence

$$e^{-|\overline{\rho}-\overline{\rho}'|^2} = e^{-(\rho^2+ \rho'^2)} e^{2\overline{\rho} \cdot \overline{\rho}'} = e^{-(\rho^2+\rho'^2)} e^{i(-i2\overline{\rho}\cdot\overline{\rho}')} .$$

But equation (A.20) gives the expansion of a plane wave of the form $e^{i\overline{k} \cdot \overline{r}}$. Equating $-i2\overline{\rho}\cdot\overline{\rho}'$ with $\overline{k} \cdot \overline{r}$ and using the aforementioned equation, one has

$$e^{i(-i2\overline{\rho}\cdot\overline{\rho}')} = \sum_{n=0}^{\infty} i^n(2n+1)\, j_n(-i2\rho\rho')\left[\sum_{m=0}^{n} \alpha_m \frac{(n-m)!}{(n+m)!}\right.$$

$$\left. \times\; P_n^m(\cos\,\theta)P_n^m(\cos\,\theta')\cos\, m(\phi-\phi')\right] \qquad (4.178)$$

where $\alpha_m = \begin{cases} 1 & m = 0 \\ 2 & m \neq 0 \end{cases}$.

When this is substituted into equations (4.170), (4.171), (4.176) and (4.177), and when the order of summation and integration are changed, these equations become

$$<|E_{1\theta_o}|^2> \sim \frac{E^2k^2\ell^2\nu^2\cos^2\theta_o}{16\pi^2\ r_o^2} \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{n} i^n(2n+1)\ \alpha_m \frac{(n-m)!}{(n+m)!} \right.$$

$$\times \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} \cdot \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} e^{-(\rho^2+\rho'^2)}\ P_1^1(\cos\theta)P_1^1(\cos\theta')\sin\phi\ \sin\phi'$$

$$\times j_n(-i2\rho\rho')P_n^m(\cos\theta')P_n^m(\cos\theta')\cos m(\phi-\phi')\sin\theta\ \sin\theta'$$

$$\left. \times \quad d\phi\ d\theta\ d\rho\ d\phi' d\theta' d\rho' \right\} \quad (4.179)$$

$$<|E_{1\phi_o}|^2> \sim \frac{E^2k^2\ell^2\nu^2}{16\pi^2\ r_o^2} \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{n} i^n(2n+1)\ \alpha_m \frac{(n-m)!}{(n+m)!} \right.$$

$$\times \int_0^{R/\ell}\int_0^{\pi}\int_0^{\pi} \cdot \int_0^{R/\ell}\int_0^{\pi}\int_0^{\pi} e^{-(\rho^2+\rho'^2)}\ P_1^1(\cos\theta)P_1^1(\cos\theta')\cos\phi\ \cos\phi'$$

$$\times j_n(-i2\rho\rho')P_n^m(\cos\theta)P_n^m(\cos\theta')\cos m(\phi-\phi')\sin\theta\ \sin\theta'$$

$$\left. \times \quad d\phi\ d\theta\ d\rho\ d\phi'\ d\theta'\ d\rho \right\} \quad (4.180)$$

as $k \to 0$ with R and ℓ fixed, and

$$<|E_{1\theta_o}|^2> \sim \frac{E^2k^4\ell^4\nu^2\cos^2\theta_o}{16\pi^2\ r_o^2} \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{n} i^n(2n+1)\ \alpha_m \frac{(n-m)!}{(n+m)!} \right.$$

$$\times \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} \cdot \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} e^{-(\rho^2+\rho'^2)}\ P_1^1(\cos\theta)P_1^1(\cos\theta')\sin\phi\ \sin\phi'$$

$$\times j_n(-i2\rho\rho')P_n^m(\cos\theta)P_n^m(\cos\theta')\cos m(\phi-\phi')\rho\rho'\sin\theta\ \sin\theta'$$

$$\left. \times \quad d\phi\ d\theta\ d\rho\ d\phi' d\theta' d\rho' \right\} \quad (4.181)$$

$$<|E_{1\emptyset_o}|^2> \sim \frac{E^2 k^4 \ell^4 \nu^2}{16\pi^2\ r_o^2} \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{n} i^n (2n+1)\ \alpha_m \frac{(n-m)!}{(n+m)!} \right.$$

$$\times \int_0^{R/\ell} \int_0^{\pi} \int_0^{2\pi} \cdot \int_0^{R/\ell} \int_0^{\pi} \int_0^{2\pi} e^{-(\rho^2+\rho'^2)}\ P_1^1(\cos\ \theta) P_1^1(\cos\ \theta') \cos\ \emptyset\ \cos\ \emptyset'$$

$$\times\ j_n(-i2\rho\rho') P_n^m(\cos\ \theta) P_n^m(\cos\ \theta') \cos\ m(\emptyset-\emptyset')\rho\rho' \sin\ \theta\ \sin\ \theta'$$

$$\left. \times\ d\emptyset\ d\theta\ d\rho\ d\emptyset'\ d\theta'\ d\rho' \right\} \qquad (4.182)$$

as $R \to \infty$ while k and ℓ remain constant.

Performing the angular integrations in equations (4.179)- (4.182), one finds that

$$<|E_{1\theta_o}|^2> \sim \frac{iE^2 k^2 \ell^2 \nu^2 \cos^2\theta_o}{3r_o^2} \int_0^{R/\ell} \int_0^{R/\ell} e^{-(\rho^2+\rho'^2)}\ j_1(-i2\rho\rho') d\rho d\rho' \quad (4.183)$$

$$<|E_{1\emptyset_o}|^2> \sim \frac{iE^2 k^2 \ell^2 \nu^2}{3r_o^2} \int_0^{R/\ell} \int_0^{R/\ell} e^{-(\rho^2+\rho'^2)}\ j_1(-i2\rho\rho') d\rho\ d\rho' \qquad (4.184)$$

as $k \to 0$ while k and ℓ remain constant, and

$$<|E_{1\theta_o}|^2> \sim \frac{iE^2 k^4 \ell^4 \nu^2 \cos^2\theta_o}{3r_o^2} \int_0^{R/\ell} \int_0^{R/\ell} e^{-(\rho^2+\rho'^2)} \rho\rho' J_1(-i2\rho\rho') d\rho\ d\rho' \qquad (4.185)$$

$$<|E_{1\emptyset_o}|^2> \sim \frac{iE^2 k^4 \ell^4 \nu^2}{3r_o^2} \int_0^{R/\ell} \int_0^{R/\ell} e^{-(\rho^2+\rho'^2)} \rho\rho' j_1(-i2\rho\rho') d\rho\ d\rho' \qquad (4.186)$$

as $R \to \infty$ while k and ℓ remain fixed.

The integrals in equations (4.183) - (4.186) will now be evaluated. Let

$$I_1 = \int_0^{R/\ell} \int_0^{R/\ell} e^{-(\rho^2+\rho'^2)} j_1(-i2\rho\rho')d\rho \, d\rho' \tag{4.187}$$

$$I_2 = \int_0^{R/\ell} \int_0^{R/\ell} e^{-(\rho^2+\rho'^2)} j_1(-i2\rho\rho') \rho\rho' \, d\rho \, d\rho' \tag{4.188}$$

so that

$$<|E_{1\theta_o}|^2> \sim \frac{iE^2k^2\ell^2\nu^2\cos^2\theta_o}{3r_o^2} I_1 \tag{4.189}$$

$$<|E_{1\phi_o}|^2> \sim \frac{iE^2k^2\ell^2\nu^2}{3r_o^2} I_1 \tag{4.190}$$

as $R \to 0$ while R and ℓ remain constant, and

$$<|E_{1\theta_o}|^2> \sim \frac{iE^2k^4\ell^4\nu^2\cos^2\theta_o}{3r_o^2} I_2 \tag{4.191}$$

$$<|E_{1\phi_o}|^2> \sim \frac{iE^2k^4\ell^4\nu^2}{3r_o^2} I_2 \tag{4.192}$$

as $R \to \infty$ while k and ℓ remain fixed. Integral I_1 will be considered first. To begin with, it is shown in Appendix B that the limits on the integral in equation (4.187) may be extended to infinity with negligible error provided that $R/\ell >>> 1$. Thus,

$$I_1 \sim \int_0^\infty \int_0^\infty e^{-(\rho^2+\rho'^2)} j_1(-i2\rho\rho')\, d\rho\, d\rho' \tag{4.193}$$

Next, $j_1(-i2\rho\rho')$ will be expressed as a Taylor series. Using a formula from the N.B.S. "Handbook" [21], one can show that

$$j_n(z) = \frac{\sqrt{\pi}\, z^n}{2^{n+1}} \sum_{m=0}^{\infty} \frac{(-1)^m\, (z/2)^{2m}}{\Gamma(m+n+\frac{3}{2})m!} \quad . \tag{4.194}$$

Hence,

$$j_1(-i2\rho\rho') = \frac{-i\sqrt{\pi}}{2} \sum_{m=0}^{\infty} \frac{\rho^{2m+1}\, \rho'^{2m+1}}{(m+\frac{5}{2})\; m!} \quad . \tag{4.194'}$$

Substituting equation (4.194') into equation (4.193) and changing the order of summation and integration, one has

$$I_1 \sim \frac{-i\sqrt{\pi}}{2} \sum_{m=0}^{\infty} \left\{ \frac{1}{\Gamma(m+\frac{5}{2})m!} \int_0^\infty e^{-\rho^2} \rho^{2m+1}\, d\rho \int_0^\infty e^{-\rho'^2} \rho'^{2m+1}\, d\rho' \right\}$$

or

$$I_1 \sim \frac{-i\sqrt{\pi}}{2} \sum_{m=0}^{\infty} \left\{ \frac{1}{\Gamma(m+\frac{5}{2})m!} \left[\int_0^\infty e^{-\rho^2} \rho^{2m+1}\, d\rho \right]^2 \right\} \quad .$$

But again from the N.B.S. "Handbook" [22] , one finds that

$$\int_0^\infty e^{-\rho^2} \rho^{2m+1}\, d\rho = \frac{m!}{2} \quad . \tag{4.195}$$

Hence,

$$I_1 \sim \frac{-i\sqrt{\pi}}{8} \sum_{m=0}^{\infty} \frac{m!}{\Gamma(m+\frac{5}{2})} \quad . \tag{4.196}$$

The infinite sum in equation (4.196) may be evaluated with the aid of some properties of the hypergeometric function. From the N.B.S. "Handbook" [23], one finds that

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\ \Gamma(b)} \sum_{m=0}^{\infty} \frac{\Gamma(m+a)\ \Gamma(m+b)}{\Gamma(m+c)\ m!} z^m , \tag{4.197}$$

and

$$F(a,b;c;1) = \frac{\Gamma(c)\ \Gamma(c-b-a)}{\Gamma(c-a)\ \Gamma(c-b)} \tag{4.198}$$

provided $c \neq 0,-1,-2,\cdots$, $\mathrm{Re}(c-a-b) > 0$. When $z = 1$, equation (4.197) becomes

$$F(a,b;c;1) = \frac{\Gamma(c)}{\Gamma(a)\ \Gamma(b)} \sum_{m=0}^{\infty} \frac{\Gamma(m+a)\ \Gamma(m+b)}{\Gamma(m+c)\ m!} .$$

Utilizing equation (4.198), one finds that

$$\frac{\Gamma(c)\ \Gamma(c-b-a)}{\Gamma(c-a)\ \Gamma(c-b)} = \frac{\Gamma(c)}{\Gamma(a)\ \Gamma(b)} \sum_{m=0}^{\infty} \frac{\Gamma(m+a)\ \Gamma(m+b)}{\Gamma(m+c)\ m!} .$$

Hence,

$$\sum_{m=0}^{\infty} \frac{\Gamma(m+a)\ \Gamma(m+b)}{\Gamma(m+c)\ m!} = \frac{\Gamma(a)\ \Gamma(b)\ \Gamma(c-a-b)}{\Gamma(c-a)\ \Gamma(c-b)} \tag{4.199}$$

provided that $c \neq 0,-1,-2,\cdots$, $\mathrm{Re}(c-a-b) > 0$.

Noting that $\Gamma(m+1) = m!$, equation (4.199) can be applied to equation (4.196) to show that

$$I_1 \sim \frac{-i\sqrt{\pi}}{8} \cdot \frac{\Gamma(1)\ \Gamma(1)\ \Gamma(\frac{5}{2}-1-1)}{\Gamma(\frac{5}{2}-1)\ \Gamma(\frac{5}{2}-1)} = \frac{-i\sqrt{\pi}}{8} \frac{\Gamma(\frac{1}{2})}{\Gamma^2(\frac{3}{2})} .$$

But

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi} \quad , \quad \Gamma(\tfrac{3}{2}) = \tfrac{1}{2}\,\Gamma(\tfrac{1}{2}) = \tfrac{1}{2}\sqrt{\pi} \quad .$$

Hence,

$$I_1 \sim -\frac{i}{2} \; . \tag{4.200}$$

Now integral I_2 will be evaluated. Using equation (A.6), the explicit formula for j_1 , and noting that

$$\left.\begin{aligned} \sin(-iz) &= -i \sinh z \\ \cos(-iz) &= \cosh z \;, \end{aligned}\right\} \tag{4.201}$$

one finds that

$$I_2 = \int_0^{R/\ell}\int_0^{R/\ell} e^{-(\rho^2+\rho'^2)} \left[\frac{i \sinh 2\rho\rho'}{4\rho^2\ \rho'^2} - \frac{i \cosh 2\rho\rho'}{2\rho\rho'} \right] \rho\rho'\ d\rho\ d\rho'$$

or

$$I_2 = \frac{-i}{2} \int_0^{R/\ell}\int_0^{R/\ell} e^{-(\rho^2+\rho'^2)} [\cosh 2\rho\rho' - \frac{\sinh 2\rho\rho'}{2\rho\rho'}]\ d\rho\ d\rho' \; .$$

With the aid of equations (A.5) and equations (4.201) it can be seen that

$$\frac{\sinh 2\rho\rho'}{2\rho\rho'} = j_o(-i2\rho\rho') \; .$$

Hence,

$$I_2 = -\frac{i}{2}\int_0^{R/\ell}\int_0^{R/\ell} e^{-(\rho^2+\rho'^2)} \cosh 2\rho\rho' \, d\rho \, d\rho'$$

$$+ \frac{i}{2}\int_0^{R/\ell}\int_0^{R/\ell} e^{-(\rho^2+\rho'^2)} j_o(-i2\rho\rho')d\rho \, d\rho' . \tag{4.202}$$

The second integral on the right hand side of equation (4.202) can be evaluated by the same method that was employed in evaluating integral I_1 . First, the limits are extended to infinity. Then $j_o(-i2\rho\rho')$ is expressed as a Taylor series by means of equation (4.194). Thus,

$$\int_0^{R/\ell}\int_0^{R/\ell} e^{-(\rho^2+\rho'^2)} j_o(-i2\rho\rho')d\rho \, d\rho' \sim \int_0^{\infty}\int_0^{\infty} e^{-(\rho^2+\rho'^2)} \frac{\sqrt{\pi}}{2}$$

$$\times \sum_{m=0}^{\infty} \frac{\rho^{2m}\rho'^{2m}}{\Gamma(m+\frac{3}{2})m!} d\rho \, d\rho' = \frac{\sqrt{\pi}}{2}\sum_{m=0}^{\infty}\left\{\frac{1}{\Gamma(m+\frac{3}{2})m!}\left[\int_0^{\infty} e^{-\rho^2}\rho^{2m}d\rho\right]^2\right\}.$$

Now from the N.B.S. "Handbook" [24], one finds that

$$\int_0^{\infty} e^{-\rho^2} \rho^{2m} \, d\rho = \frac{1}{2}\Gamma(m+\frac{1}{2}) . \tag{4.203}$$

Hence,

$$\int_0^{R/\ell}\int_0^{R/\ell} e^{-(\rho^2+\rho'^2)} j_o(-i2\rho\rho')d\rho \, d\rho' \sim \frac{\sqrt{\pi}}{8}\sum_{m=0}^{\infty}\frac{\Gamma^2(m+\frac{1}{2})}{\Gamma(m+\frac{3}{2})m!} .$$

Next, utilizing equation (4.199), one finds that

$$\int_0^{R/\ell}\int_0^{R/\ell} e^{-(\rho^2+\rho'^2)} j_o(-i2\rho\rho')d\rho \, d\rho' \sim \frac{\sqrt{\pi}}{8} \cdot \frac{\Gamma^3(\frac{1}{2})}{\Gamma^2(1)} = \frac{\pi^2}{8} . \tag{4.204}$$

Thus, equation (4.202) becomes

$$I_2 \sim -\frac{i}{2}\int_0^{R/\ell}\int_0^{R/\ell} e^{-(\rho^2+\rho'^2)}\cosh 2\rho\rho'\,d\rho\,d\rho' + \frac{i\pi^2}{16}\ . \qquad (4.205)$$

But

$$\int_0^{R/\ell}\int_0^{R/\ell} e^{-(\rho^2+\rho'^2)}\cosh 2\rho\rho' d\rho\,d\rho' = \frac{1}{2}\int_0^{R/\ell}\int_0^{R/\ell} e^{-(\rho-\rho')^2}\,d\rho\,d\rho'$$

$$+\frac{1}{2}\int_0^{R/\ell}\int_0^{R/\ell} e^{-(\rho+\rho')^2}\,d\rho\,d\rho'$$

where the fact that $\cosh 2\rho\rho' = \frac{1}{2}[e^{2\rho\rho'} + e^{-2\rho\rho'}]$ has been employed.

Now

$$\int_0^{R/\ell}\int_0^{R/\ell} e^{-(\rho+\rho')^2}\,d\rho\,d\rho' \leq \int_0^{R/\ell} e^{-\rho^2}d\rho\int_0^{R/\ell} e^{-\rho'^2}d\rho'$$

$$\leq \int_0^{\infty} e^{-\rho^2}d\rho\int_0^{\infty} e^{-\rho'^2}d\rho = \frac{\pi}{4}\ .$$

Next, assuming $R/\ell >>> 1$, one has

$$\int_0^{R/\ell} e^{-(\rho-\rho')^2}\,d\rho \sim \int_{-\infty}^{\infty} e^{-\rho^2}\,d\rho = \sqrt{\pi}$$

except when ρ' is very close to 0 or R/ℓ (e.g., $\rho' < 5$ or $\rho' > \frac{R}{\ell} - 5$) which is a negligible part of the interval $[0,R/\ell]$.

Thus

$$\int_0^{R/\ell}\int_0^{R/\ell} e^{-(\rho-\rho')^2}\, d\rho\, d\rho' \sim \sqrt{\pi}\int_0^{R/\ell} d\rho' = \frac{\sqrt{\pi}\, R}{\ell} .$$

But if $R/\ell >>> 1$, then $R\sqrt{\pi}/\ell >>> \pi/4$, and hence

$$\int_0^{R/\ell}\int_0^{R/\ell} e^{-(\rho^2+\rho'^2)} \cosh 2\rho\rho'\, d\rho\, d\rho' \sim \frac{R\sqrt{\pi}}{2\ell} .$$

Thus, equation (4.205) becomes

$$I_2 \sim \frac{-R\sqrt{\pi}}{4\ell} + \frac{i\pi^2}{16} .$$

But since $R\sqrt{\pi}/4\ell >>> \pi^2/16$, one has

$$I_2 \sim \frac{-iR\sqrt{\pi}}{4\ell} . \qquad (4.206)$$

Substituting equations (4.200) and (4.206) into equations (4.189) - (4.192), one finds that

$$<|E_{1\theta_o}|^2> \sim \frac{\mathcal{E}^2 k^2 \ell^2 \nu^2 \cos^2\theta_o}{6r_o^2} \qquad (4.207)$$

$$<|E_{1\phi_o}|^2> \sim \frac{\mathcal{E}^2 k^2 \ell^2 \nu^2}{6r_o^2} \qquad (4.208)$$

as $k \to 0$ while R and ℓ remain constant, and

$$<|E_{1\theta_o}|^2> \sim \frac{\mathcal{E}^2 k^4 \ell^3 R \nu^2 \sqrt{\pi}\, \cos^2\theta_o}{12r_o^2} \qquad (4.209)$$

$$\langle |E_{1\phi_o}|^2 \rangle \sim \frac{E^2 k^4 \ell^3 R \nu^2 \sqrt{\pi}}{12 r_o^2} \tag{4.210}$$

as $R \to \infty$ while k and ℓ remain fixed.

(ii) Discussion of results

Equations (4.207), (4.208), and (4.209), (4.210), together with the fact that $\langle E_{1\theta_o} E^*_{1\phi_o} \rangle = 0$, represent the results of the examination of the far zone scattered field with the magnetic dipole for $k\ell << 1$. Comparing equations (4.207) and (4.208) with equations (4.209) and (4.210), one finds that the ratio of the magnitudes of the former to the latter is of the order $1:(k\ell)^2 \frac{R}{\ell}$. Thus, if $\frac{R}{\ell}(k\ell)^2 << 1$, equations (4.207) and (4.208) must be used to characterize the field, while if $\frac{R}{\ell}(k\ell)^2 >> 1$, then equations (4.209) and (4.210) are employed. If $\frac{R}{\ell}(k\ell)^2 \sim 1$, then neither set of equations would be reliable. This is due to the fact that they are asymptotic approximations and in their derivation terms were neglected whose magnitudes would be significant if $\frac{R}{\ell}(k\ell)^2 \sim 1$.

Further examination of equations (4.207) and (4.208) or (4.209) and (4.210) reveals that the mean square values of the two components of the scattered field are of the same magnitude, unlike the case of $k\ell >> 1$ where the magnitudes differed by a factor of $(k\ell)^2$. It is also significant to note that the mean square value of $E_{1\phi_o}$ is independent of θ_o.

The condition required to insure the validity of the Born approximation will be different now than for the case where $k\ell >> 1$.

Recall that it is assumed that in order for the Born method to give reliable results, the inequality $|\overline{E}_o|^2 >> <|\overline{E}_1|^2>$ should hold. Assuming that $\frac{R}{\ell}(k\ell)^2 >> 1$, and noting that $|\overline{E}_o|^2 \sim \mathcal{E}^2/r_o^2$, the condition becomes

$$\frac{\mathcal{E}^2}{r_o^2} >> \frac{\mathcal{E}^2 \nu^2 k^4 \ell^3 R}{r_o^2}$$

or

$$\nu^2 << \frac{1}{k^4 \ell^3 R} \qquad (4.211)$$

Since $k\ell << 1$, this is a much weaker condition than in the previous case where $k\ell >> 1$. Hence, for a given value of ν^2 , the Born method will be valid for much greater values of R than previously. Now it may happen that $1/k^4\ell^3 R > 1$, but nevertheless the derivation of the Born approximation in Section 2.3 still requires that $\nu^2 << 1$.

The polarization of the scattered field may be analyzed by means of the Stokes parameters just as in the case of $k\ell >> 1$. It is not difficult to show that the degree of polarization m is given by

$$m \sim \frac{1}{2 \csc^2 \theta_o - 1}$$

and that the polarization (of the polarized part of the wave) is linear and in the ϕ direction.

B. Scattering with the Electric Dipole

(i) Derivation

The electric field of the electric dipole in a homogeneous medium was given by equation (4.16):

$$\overline{E}_o(\overline{r}) = -\frac{1}{3}\, i\mathcal{E}k\, \{2h_o(kr)\cos\theta_o + h_2(kr)[\sin\theta_o P_2^1(\cos\theta)\cos\phi + 2P_2(\cos\theta)\cos\theta_o]\}\, \overline{e}_{r_o}$$
$$+ \frac{1}{3}\, i\mathcal{E}k\, \{2h_o(kr)\sin\theta_o - h_2(kr)[\cos\theta_o P_2^1(\cos\theta)\cos\phi - 2P_2(\cos\theta)\sin\theta_o]\}\, \overline{e}_{\theta_o}$$
$$- \frac{1}{3}\, i\mathcal{E}k\, h_2(kr) P_2^1(\cos\theta)\sin\phi\, \overline{e}_{\phi_o}$$

Utilizing the derivation in Section 4.2B, the expression for the ensemble average of two field components is given by equation (4.108):

$$< E_{1\alpha}E_{1\beta}^* > = \frac{k^4}{16\pi^2 r_o^2}\int_V\int_V \overline{e}_\alpha \cdot \overline{E}_o(\overline{r})\overline{e}_\beta \cdot \overline{E}_o^*(\overline{r}')R_{\tilde{\varepsilon}}(|\overline{r}-\overline{r}'|)e^{-i\overline{k}\,(\overline{r}-\overline{r}')}d\overline{r}\,d\overline{r}'$$
$$+ \frac{\mathcal{E}k^2 e^{ikr_o}\sin\theta_o\,\overline{e}_\beta\cdot\overline{e}_{\theta_o}}{12\pi\, r_o^2}\int_V \overline{e}_\alpha\cdot\overline{E}_o(\overline{r})R_{\tilde{\varepsilon}}(r)e^{-i\overline{k}\cdot\overline{r}}\,d\overline{r}$$
$$+ \frac{\mathcal{E}k^2 e^{-ikr_o}\sin\theta_o\,\overline{e}_\alpha\cdot\overline{e}_{\theta_o}}{12\pi\, r_o^2}\int_V \overline{e}_\beta\cdot\overline{E}_o^*(\overline{r}')R_{\tilde{\varepsilon}}(r')e^{i\overline{k}\cdot\overline{r}'}\,d\overline{r}'$$
$$+ \frac{\mathcal{E}^2\nu^2\sin^2\theta_o}{9r_o^2}\,\overline{e}_\alpha\cdot\overline{e}_{\theta_o}\;\overline{e}_\beta\cdot\overline{e}_{\theta_o}$$

where V is the volume of the randomly-inhomogeneous sphere. It should be recalled that it is the principal value of the volume integrals that is to be calculated (one performs the angular integrations first). By substituting the explicit expressions for $\overline{E}_o$ and $R_{\tilde{\varepsilon}}$ into this equation, it can be seen that

$$
\begin{aligned}
<|E_{1\theta_o}|^2> = \frac{E^2 k^6 \ell^6 v^2}{144\pi^2 r_o^2} \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} \cdot \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} e^{-|\overline{\rho}-\overline{\rho}|^2} \Bigg\{ 4\left[\frac{e^{ik\ell(\rho-\rho')}}{(k\ell)^2\rho\rho'}\right]\sin^2\theta_o \\
-2\left[\frac{-1}{(k\ell)^2\rho\rho'} + \frac{3i}{(k\ell)^3\rho\rho'^2} + \frac{3}{(k\ell)^4\rho\rho'^3}\right] e^{ik\ell(\rho-\rho')}[\cos\theta_o \sin\theta_o \\
\times P_2^1(\cos\theta')\cos\phi' - 2P_2(\cos\theta')\sin^2\theta_o] - 2\left[\frac{-1}{(k\ell)^2\rho\rho'} - \frac{3i}{(k\ell)^3\rho^2\rho'}\right. \\
\left. + \frac{3}{(k\ell)^4\rho^3\rho'}\right] e^{ik\ell(\rho-\rho')}[\cos\theta_o \sin\theta_o P_2^1(\cos\theta)\cos\phi \\
- 2P_2(\cos\theta)\sin^2\theta_o] + \left[\frac{1}{(k\ell)^2\rho\rho'} + \frac{9}{(k\ell)^4\rho^2\rho'^2} + \frac{9}{(k\ell)^6\rho^3\rho'^3}\right. \\
- \frac{3}{(k\ell)^4\rho\rho'^3} - \frac{3}{(k\ell)^4\rho^3\rho'} + \frac{i}{(k\ell)^3}\left(\frac{3}{\rho^2\rho'} - \frac{3}{\rho\rho'^2} + \frac{9}{(k\ell)^2\rho^3\rho'^2}\right. \\
\left.\left. - \frac{9}{(k\ell)^2\rho^2\rho'^3}\right)\right] e^{ik\ell(\rho-\rho')}\ [\cos^2\theta_o P_2^1(\cos\theta)P_2^1(\cos\theta')\cos\phi\cos\phi' \\
- 4\cos\theta_o\sin\theta_o P_2^1(\cos\theta)\ P_2(\cos\theta')\cos\phi - 4\cos\theta_o\sin\theta_o P_2^1(\cos\theta') \\
\times P_2(\cos\theta)\cos\phi' + 4P_2(\cos\theta)P_2(\cos\theta')\sin^2\theta_o] \Bigg\}\ e^{-i\ell\overline{k}\cdot(\overline{\rho}-\overline{\rho}')}
\end{aligned}
$$

$$\times\ \rho^2\rho'^2 \sin\theta \sin\theta'\, d\phi\, d\theta\, d\rho\, d\phi' d\theta' d\rho' - \frac{E^2k^3\ell^3\nu^2}{18\pi r_o^2}$$

$$\times\ \mathrm{Im}\left\{ e^{ikr_o} \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} e^{-\rho^2} \left\{ 2\left[\frac{-ie^{ik\ell\rho}}{k\ell\rho}\right] \sin\theta_o - \left[\frac{i}{k\ell\rho} - \frac{3}{(k\ell)^2\rho^2}\right.\right.\right.$$

$$\left. - \frac{3i}{(k\ell)^3\rho^3}\right] e^{ik\ell\rho} \left[\cos\theta_o P_2^1(\cos\theta)\cos\phi - 2P_2(\cos\theta)\sin\theta_o\right]\Big\}$$

$$\times\ e^{-i\ell\bar{k}\cdot\bar{\rho}}\, \rho^2 \sin\theta\, d\phi\, d\theta\, d\rho \Big\} + \frac{E^2\nu^2\sin^2\theta_o}{9r_o^2} \qquad (4.212)$$

$$\langle|E_{1\phi_o}|^2\rangle = \frac{E^2k^6\ell^6\nu^2}{144\pi^2 r_o^2} \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} \cdot \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} e^{-|\bar{\rho}-\bar{\rho}'|^2} \left[\frac{1}{(k\ell)^2\rho\rho'} - \frac{3}{(k\ell)^4\rho\rho'^3}\right.$$

$$+ \frac{9}{(k\ell)^3\rho^2\rho'^2} + \frac{9}{(k\ell)^6\rho^3\rho'^3} - \frac{3}{(k\ell)^4\rho^3\rho'} + \frac{i}{(k\ell)^3}\left(\frac{3}{\rho^2\rho'} - \frac{3}{\rho\rho'^2}\right.$$

$$\left.\left. + \frac{9}{(k\ell)^2\rho^3\rho'^2} - \frac{9}{(k\ell)^2\rho^2\rho'^3}\right)\right] e^{ik\ell(\rho-\rho')}\ P_2^1(\cos\theta)P_2^1(\cos\theta')$$

$$\times\ \sin\phi\, \sin\phi' e^{-i\ell\bar{k}\cdot(\bar{\rho}-\bar{\rho}')}\rho^2\rho'^2 \sin\theta \sin\theta' d\phi\, d\theta\, d\rho\, d\phi' d\theta' d\rho' \qquad (4.213)$$

$$\langle E_{1\theta_o} E^*_{1\phi_o}\rangle = -\frac{E^2k^6\ell^6\nu^2}{144\pi^2 r_o^2} \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} \cdot \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} e^{-|\bar{\rho}-\bar{\rho}'|^2}$$

$$\times\ \{2\sin\theta_o h_o(k\ell\rho)h_2^*(k\ell\rho')P_2^1(\cos\theta')\sin\phi' - h_2(k\ell\rho)h_2^*(k\ell\rho')$$

$$\times\ [\cos\theta_o P_2^1(\cos\theta)P_2^1(\cos\theta')\cos\phi\,\sin\phi' - 2\sin\theta_o P_2(\cos\theta)$$

$$\times \; P_2^1(\cos\theta')\sin\phi']\} \; e^{-i\ell\bar{k}\cdot(\bar{\rho}-\bar{\rho}')}\rho^2\rho'^2\sin\theta\sin\theta' d\phi\, d\theta\, d\rho\, d\phi' d\theta' d\rho'$$

$$+ \frac{iE^2k^3\ell^3\nu^2 e^{-ikr_o}\sin\theta_o}{36\pi\, r_o^2} \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} e^{-\rho'^2} h_2^*(k\ell\rho')P_2^1(\cos\theta')$$

$$\times \; \sin\phi' \; e^{i\ell\bar{k}\cdot\bar{\rho}'} \; \rho'^2\sin\theta'\, d\phi'\, d\theta'\, d\rho' \qquad (4.214)$$

where the change of variables $\bar{\rho} = \frac{1}{\ell}\bar{r}$, $\bar{\rho}' = \frac{1}{\ell}\bar{r}'$ has been made and where explicit expressions for the Hankel functions have been used in equations (4.212) and (4.213).

Using the same method that was employed in Section 4.2 (replacing the correlation function by the Fourier transform of the spectral density, etc.), one can demonstrate that $<E_{1\theta_o} E_{1\phi_o}> = 0$.

Asymptotic approximations for $<|E_{1\theta_o}|^2>$ and $<|E_{1\phi_o}|^2>$ will now be found for two special cases, just as was done with the magnetic dipole. As $k \to 0$ while R and ℓ remain fixed, the only terms in the expressions for the Hankel functions or their products which contribute significantly to the integrals are those with the $1/\rho^3$ or $1/\rho^3\rho'^3$ factors. Thus, making the additional approximations $e^{ik\ell(\rho-\rho')} \sim 1$, $e^{i\ell\bar{k}\cdot(\bar{\rho}-\bar{\rho}')} \sim 1$ in the first integrals on the right in equations (4.212) and (4.213) and making the approximation e $e^{ik\ell\rho} \sim 1$, $e^{i\ell\bar{k}\cdot\bar{\rho}} \sim 1$ in the second integral in equation (4.212), one finds that

$$<|E_{1\theta_o}|^2> \sim \frac{E^2k^6\ell^6\nu^2}{144\pi^2\ r_o^2} \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} \cdot \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} e^{-|\bar{\rho}-\bar{\rho}'|^2} [\frac{9}{(k\ell)^6\rho^3\rho'^3}]$$

$$\times\ [\cos^2\theta_o P_2^1(\cos\theta)P_2^1(\cos\theta')\cos\phi\ \cos\phi' - 4\cos\theta_o\sin\theta_o P_2^1(\cos\theta)$$

$$\times\ P_2(\cos\theta')\cos\phi - 4\cos\theta_o\sin\theta_o P_2^1(\cos\theta')P_2(\cos\theta)\cos\phi'$$

$$+\ 4P_2(\cos\theta)P_2(\cos\theta')\sin^2\theta_o]\rho^2\rho'^2\sin\theta\ \sin\theta' d\phi\ d\theta\ d\rho\ d\phi' d\theta' d\rho'$$

$$-\ \frac{E^2k^3\ell^3\nu^2}{18\pi\ r_o^2}\ \mathrm{Im}\left\{e^{ikr_o}\int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} e^{-\rho^2}[\frac{3i}{(k\ell)^3\rho^3}][\cos\theta_o P_2^1(\cos\theta)\cos\phi\right.$$

$$\left.-\ 2P_2(\cos\theta)\sin\theta_o]\rho^2\sin\theta\ d\phi\ d\theta\ d\rho\right\} + \frac{E^2\nu^2\sin^2\theta_o}{9r_o^2} \tag{4.215}$$

$$<|E_{1\phi_o}|^2> \sim \frac{^2k^6\ell^6\nu^2}{144\pi^2\ r_o^2} \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} \cdot \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} e^{-|\bar{\rho}-\bar{\rho}'|^2} [\frac{9}{(k\ell)^6\rho^3\rho'^3}]$$

$$\times\ [P_2^1(\cos\theta)P_2^1(\cos\theta')\sin\phi\ \sin\phi']\rho^2\rho'^2\sin\theta\ \sin\theta' d\phi\ d\theta\ d\rho\ d\phi' d\theta' d\rho' \tag{4.216}$$

as $k \to 0$ while R and ℓ remain constant. Next, as $R \to \infty$ while k and ℓ remain fixed, the significant terms in the Hankel functions or their products are those with the $1/\rho$ or $1/\rho\rho'$ factors. Again using the approximations that $e^{ik\ell(\rho-\rho')} \sim 1$, etc., since it is still assumed that $k\ell << 1$, one finds that

$$<|E_{1\theta_o}|^2> \sim \frac{\mathcal{E}^2k^6\ell^6\nu^2}{144\pi^2 r_o^2}\int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi}\cdot\int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} e^{-|\bar{\rho}\,\bar{\rho}'|^2}\Bigg(\frac{4\sin^2\theta_o}{(k\ell)^2\rho\rho'}+\frac{2}{(k\ell)^2\rho\rho'}$$

$$\times [\cos\theta_o\sin\theta_o P_2^1(\cos\theta)\cos\phi' - 2P_2(\cos\theta')\sin^2\theta_o + \cos\theta_o\sin\theta_o$$

$$\times P_2^1(\cos\theta)\cos\phi - 2P_2(\cos\theta)\sin^2\theta_o] + \frac{1}{(k\ell)^2\rho\rho'}[\cos^2\theta_o P_2^1(\cos\theta)$$

$$\times P_2^1(\cos\theta')\cos\phi\cos\phi' - 4\cos\theta_o\sin\theta_o P_2(\cos\theta')P_2^1(\cos\theta)\cos\phi$$

$$- 4\cos\theta_o\sin\theta_o P_2^1(\cos\theta')P_2(\cos\theta)\cos\phi' + 4P_2(\cos\theta)P_2(\cos\theta')$$

$$\times \sin^2\theta_o]\Bigg)\rho^2\rho'^2\sin\theta\sin\theta' d\phi\, d\theta\, d\rho\, d\phi' d\theta' d\rho' - \frac{\mathcal{E}^2k^3\ell^3\nu^2}{18\pi r_o^2}$$

$$\times \mathrm{Im}\Bigg\{e^{ikr_o}\int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} e^{-\rho^2}\Bigg(\frac{-2i}{k\ell\rho}\sin\theta_o - \frac{i}{k\ell\rho}[\cos\theta_o P_2^1(\cos\theta)\cos\phi$$

$$-2P_2(\cos\theta)\sin\theta_o]\Bigg)\rho^2\sin\theta\, d\phi\, d\theta\, d\rho\Bigg\} + \frac{\mathcal{E}^2\nu^2\sin^2\theta_o}{9r_o^2} \qquad (4.217)$$

$$<|E_{1\phi_o}|^2> \sim \frac{\mathcal{E}^2k^6\ell^6\nu^2}{144\pi^2 r_o^2}\int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} e^{-|\bar{\rho}-\bar{\rho}'|^2}\left[\frac{1}{(k\ell)^2\rho\rho'}\right]P_2^1(\cos\theta)P_2^1(\cos\theta')$$

$$\times \quad \sin\phi\sin\phi'\,\rho^2\rho'^2\sin\theta\sin\theta'\, d\phi\, d\theta\, d\rho\, d\phi'\, d\theta'\, d\rho' \qquad (4.218)$$

as $R \to \infty$ while k and ℓ remain constant.

Now from equation (4.178) one obtains

$$e^{-|\bar{\rho}-\bar{\rho}'|^2} = e^{-(\rho^2+\rho'^2)} e^{i(-2i\bar{\rho}\cdot\bar{\rho}')} = e^{(\rho^2+\rho'^2)} \sum_{n=0}^{\infty} i^n(2n+1)j_n(-i2\rho\rho')$$

$$\times \sum_{m=0}^{n} \alpha_m \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta)\, P_n^m(\cos\theta') \cos m(\emptyset-\emptyset')]$$

Substitution of this expression into equations (4.215) - (4.218) makes it possible for one to perform the angular integrations. The results are

$$<|E_{1\theta_o}|^2> \sim -\frac{E^2\nu^2}{5r_o^2}[3+\sin^2\theta_o]\int_0^{R/\ell}\int_0^{R/\ell} e^{-(\rho^2+\rho'^2)}\, j_2(-i2\rho\rho')\frac{1}{\rho\rho'}\, d\rho\, d\rho' + \frac{E^2\nu^2\sin^2\theta_o}{9r_o^2} \tag{4.219}$$

$$<|E_{1\emptyset_o}|^2> \sim -\frac{3E^2\nu^2}{5r_o^2}\int_0^{R/\ell}\int_0^{R/\ell} e^{-(\rho^2+\rho'^2)}\, j_2(-i2\rho\rho')\,\frac{1}{\rho\rho'}\, d\rho\, d\rho' \tag{4.220}$$

as $k \to 0$ while R and ℓ remain constant, and

$$<|E_{1\theta_o}|^2> \sim \frac{E^2k^4\ell^4\nu^2}{9r_o^2}[4\sin^2\theta_o\int_0^{R/\ell}\int_0^{R/\ell} e^{-(\rho^2+\rho'^2)} j_o(-i2\rho\rho')\rho\rho' d\rho\, d\rho'$$

$$-\frac{1}{5}(3+\sin^2\theta_o)\int_0^{R/\ell}\int_0^{R/\ell} e^{-(\rho^2+\rho'^2)}\, j_2(-i2\rho\rho')\rho\rho' d\rho\, d\rho']$$

$$+\frac{2E^2k^2\ell^2\nu^2\cos kr_o\sin\theta_o}{9r_o^2} + \frac{E^2\nu^2\sin^2\theta_o}{9r_o^2} \tag{4.221}$$

$$<|E_{1\emptyset_o}|^2> \sim -\frac{E^2k^4\ell^4\nu^2}{15r_o^2}\int_0^{R/\ell}\int_0^{R/\ell} e^{-(\rho^2+\rho'^2)}\, j_2(-i2\rho\rho')\rho\rho' d\rho\, d\rho' \tag{4.222}$$

as $R \to \infty$ while k and ℓ remain constant.

Employing the method that was used to evaluate equation (4.187) one can show that

$$\int_0^{R/\ell}\int_0^{R/\ell} e^{-(\rho^2+\rho'^2)}\, j_2(-i2\rho\rho')\,\frac{1}{\rho\rho'}\,d\rho\,d\rho' \sim -\frac{1}{9}\,. \qquad (4.223)$$

Furthermore, employing the method that was used to evaluate equation (4.188), one finds that

$$\int_0^{R/\ell}\int_0^{R/\ell} e^{-(\rho^2+\rho'^2)}\, j_o(-i2\rho\rho')\rho\rho'\,d\rho\,d\rho' \sim \frac{R\sqrt{\pi}}{4\ell} \qquad (4.224)$$

$$\int_0^{R/\ell}\int_0^{R/\ell} e^{-(\rho^2+\rho'^2)}\, j_2(-i2\rho\rho')\rho\rho'\,d\rho\,d\rho' \sim -\frac{R\sqrt{\pi}}{4\ell}\,. \qquad (4.225)$$

When these values are substituted into equations (4.219) - (4.222), the results are

$$<|E_{1\theta_o}|^2> \sim \frac{E^2\nu^2}{15r_o^2}\,[1+2\sin^2\theta_o] \qquad (4.226)$$

$$<|E_{1\phi_o}|^2> \sim \frac{E^2\nu^2}{15r_o^2} \qquad (4.227)$$

as $k \to 0$ while R and ℓ remain constant, and

$$<|E_{1\theta_o}|^2> \sim \frac{E^2k^4\ell^3R\nu^2\sqrt{\pi}}{60r_o^2}[1+7\sin^2\theta_o] + \frac{2E^2\nu^2k^2\ell^2\cos kr_o\sin\theta_o}{9r_o^2}$$

$$+\frac{E^2\nu^2\sin^2\theta_o}{9r_o^2} \qquad (4.228)$$

$$<|E_{1\phi_o}|^2> \sim \frac{\mathcal{E}^2 k^4 \ell^3 R \nu^2 \sqrt{\pi}}{60 r_o^2} \qquad (4.229)$$

as $R \to \infty$ while k and ℓ remain constant. Since the last two terms on the right hand side of equation (4.228) become negligible in comparison to the first term as $R \to \infty$, this equation becomes

$$<|E_{1\theta_o}|^2> \sim \frac{\mathcal{E}^2 k^4 \ell^3 R \nu^2 \sqrt{\pi}}{60 r_o^2} [1 + 7 \sin^2\theta_o] \quad . \qquad (4.230)$$

(ii) Discussion of results

The results of the examination of the far-zone scattered field with the electric dipole for $k\ell << 1$ are given by equations (4.226), (4.227), and (4.229), (4.230), plus the fact that $<E_{1\theta_o} E^*_{1\phi_o}> = 0$. Comparing equations (4.226), (4.227) with equations (4.229), (4.230), it can be seen that the magnitude of the latter is on the order of $k^4\ell^3R$ times the magnitude of the former. Thus, if $k^4\ell^3R << 1$, the mean square field components are given by equations (4.226) and (4.227), while for $k^4\ell^3R >> 1$ the mean square components are given by equations (4.229) and (4.230). If $k^4\ell^3R \sim 1$, then neither set of equations gives accurate values for the mean square field components.

The condition required in order for there to be confidence in the validity of Born method is obtained in the same way as for the case of the magnetic dipole. Assuming that $k^4\ell^3R >> 1$, one finds that the requirement $|\overline{E}_o|^2 >> <|\overline{E}_1|^2>$ is equivalent to

$$\nu^2 \ll \frac{1}{k^4 \ell^3 R} \tag{4.231}$$

which is identical to the condition found for the magnetic dipole. There is, however, an additional requirement here. Determination of the mean square field components began with equation (4.108). Reexamining the derivation of this equation in Section 4.2, it can be seen that it was obtained in part by the use of approximate values for the scattered fields "close to" the dipole. In Section 4.2, "close to" meant distances much less than the wavelength which was much less than the correlation distance, but in this section "close to" will be assumed to mean distances much less than the correlation distance. The approximate values for the fields, given by equations (4.106) and (4.106'), were derived by assuming that $\tilde{\varepsilon}(\overline{r})$ could be replaced by the spatial constant $\tilde{\varepsilon}(0)$ in order to determine the fields near the dipole. This procedure is presumably valid, provided that the change in $\tilde{\varepsilon}$ per wavelength is small, i.e., provided $\lambda|\nabla\tilde{\varepsilon}| \ll 1$. Noting that $k = 2\pi/\lambda$, this is equivalent to $|\nabla\tilde{\varepsilon}|^2 \lll k^2$. By taking the ensemble average one then obtains the requirement

$$<|\nabla\tilde{\varepsilon}|^2> \lll k^2 .$$

For the purposes of estimation, one can assume that $|\nabla\tilde{\varepsilon}|^2 \sim \tilde{\varepsilon}^2/\ell^2$ where ℓ is the correlation distance. Thus it is required that $<\tilde{\varepsilon}^2>/\ell^2 \lll k^2$. But since $<\tilde{\varepsilon}^2> = \nu^2$, the requirement becomes

$$\nu^2 \lll k^2\ell^2 . \tag{4.232}$$

Equations (4.231) and (4.232) must both be satisfied before the Born solution can be considered reliable.

Through the use of the Stokes parameters, the degree of polarization m of the scattered field is found to be

$$m \sim \begin{cases} \dfrac{1}{1+\csc^2\theta_o} & \text{as } k \to 0 \text{ while } R \text{ and } \ell \text{ remain const.} \\[2ex] \dfrac{1}{1+\frac{2}{7}\csc^2\theta_o} & \text{as } R \to \infty \text{ while } k \text{ and } \ell \text{ remain constant.} \end{cases}$$

The polarization of the polarized part of the wave is found to be linear and in the θ direction.

V. SCATTERED FIELD AT THE CENTER OF A RANDOMLY-INHOMOGENEOUS DIELECTRIC SPHERE UPON WHICH A PLANE WAVE IS INCIDENT

This part is concerned with examining the scattered electric field at the center of a randomly-inhomogeneous sphere when a plane wave is incident upon it. The sphere is assumed to have the same characteristics which were delineated in the beginning of Part IV. The plane wave is taken to be linearly polarized in the x direction and propagates in the negative z direction. The situation is illustrated in Figure 5.1.

The problem of finding the field will be separated into two cases, the first being when the correlation distance is much greater than the wavelength and the second being when the correlation distance is much less than the wavelength. As in the previous part, it is the mean-square values of the components and the mean of the products of any two transverse components which will be sought. Now, however, there will be three components to deal with, not just two as in the last part. It will be found that the methods needed to handle the problem are virtually the same as those used in the previous part.

5.1 Preliminary Development

Let the incident plane wave be given by

$$\overline{E}_o(\overline{r}) = E_o e^{-ik\overline{e}_z \cdot \overline{r}} \, \overline{e}_x \qquad (5.1)$$

where E_o is taken to be real.

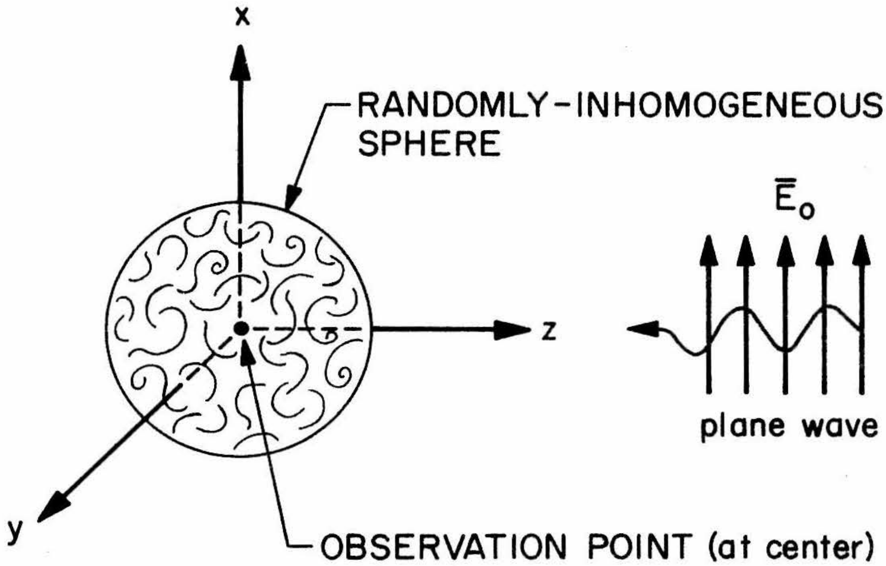

Figure 5.1. Randomly-Inhomogeneous Sphere with Incident Plane Wave

Assuming the validity of the Born approximation, the scattered field at an interior point of the inhomogeneous sphere is given by equation (2.25):

$$\overline{E}_1(\overline{r}_o) = k^2 \int_V \overline{\overline{\Gamma}}(\overline{r}_o,\overline{r}) \cdot \overline{E}_o(\overline{r})\, \tilde{\varepsilon}(\overline{r})d\overline{r} - \frac{1}{3}\tilde{\varepsilon}(\overline{r}_o)\overline{E}_o(\overline{r}_o)$$

where $\overline{r}_o$ is the observation point and where V is the entire volume of the inhomogeneous sphere. It should be recalled that it is understood that one is to find the principal value of the volume integral.

Since the observation point is taken to lie at the center of the sphere and since the origin of the coordinate system is also taken to be at the center of the sphere, then $\overline{r}_o = 0$. Using the preceding equation and noting that $\overline{E}_o(\overline{r})$ is given by equation (5.1), one finds that

$$\overline{E}_1(0) = E_o k^2 \int_V \overline{\overline{\Gamma}}(0,\overline{r}) \cdot \overline{e}_x\, e^{-i\overline{k}\cdot\overline{r}}\, \tilde{\varepsilon}(\overline{r})d\overline{r} - \frac{1}{3}\tilde{\varepsilon}(0)\overline{E}_o(0)\overline{e}_x \quad . \tag{5.2}$$

Now from equation (2.14) one has

$$\Gamma(\overline{r},\overline{r}') = (\overline{\overline{I}} + \frac{1}{k^2}\nabla\nabla)\,\frac{e^{ik|\overline{r}-\overline{r}'|}}{4\pi|\overline{r}-\overline{r}'|}$$

or, alternately,

$$\Gamma(\overline{r},\overline{r}') = (\overline{\overline{I}} + \frac{1}{k^2}\nabla'\nabla')\,\frac{e^{ik|\overline{r}-\overline{r}'|}}{4\pi|\overline{r}-\overline{r}'|} \quad .$$

Hence

$$\overline{\overline{\Gamma}}(0,\overline{r}) = (\overline{\overline{I}} + \frac{1}{k^2}\nabla\nabla)\ \frac{e^{ikr}}{4\pi r} \quad . \tag{5.3}$$

Straightforward computation then reveals that

$$\overline{\overline{\Gamma}}(0,\overline{r}) = \frac{1}{4\pi}[-\ \frac{1}{r} - \frac{3i}{kr^2} + \frac{3}{k^2r^3}]e^{ikr}\ \overline{e}_r\overline{e}_r + \frac{1}{4\pi}[\frac{1}{r} + \frac{i}{kr^2} - \frac{1}{k^2r^3}]e^{ikr}\ \overline{\overline{I}} \ . \tag{5.4}$$

But

$$h_o(kr) = \frac{-ie^{ikr}}{kr}\ , \qquad h_2(kr) = [\frac{i}{kr} - \frac{3}{k^2r^2} - \frac{3i}{k^3r^3}]\ e^{ikr} \quad .$$

Thus, after a bit of manipulation, equation (5.4) can be put into the form

$$\overline{\overline{\Gamma}}(0,\overline{r}) = \frac{ik}{4\pi}\ h_2(kr)\overline{e}_r\overline{e}_r + \frac{ik}{12\pi}[2h_o(kr) - h_2(kr)]\ \overline{\overline{I}} \quad . \tag{5.5}$$

Substitution of equation (5.5) into equation (5.2) yields

$$\overline{E}_1(0) = \frac{E_o ik^3}{4\pi} \int_V \{h_2(kr)\ \overline{e}_r\overline{e}_r \cdot \overline{e}_x + \frac{1}{3}[2h_o(kr) - h_2(kr)]\ e_x\}$$

$$\times \quad \tilde{\varepsilon}(\overline{r})e^{-i\overline{k}\cdot\overline{r}}\ d\overline{r} - \frac{1}{3}\ \tilde{\varepsilon}(0)\ E_o\ \overline{e}_x \quad . \tag{5.6}$$

Now from equation (4.15) one has

$$\overline{e}_r = \sin\theta\cos\phi\ \overline{e}_x + \sin\theta\sin\phi\ \overline{e}_y + \cos\theta\ \overline{e}_z \quad .$$

Then

$$\overline{e}_r\overline{e}_r \cdot \overline{e}_x = \sin^2\theta\cos^2\phi\ \overline{e}_x + \sin^2\theta\cos\phi\sin\phi\ \overline{e}_y$$

$$+ \cos\theta\sin\theta\cos\phi\ \overline{e}_z \quad .$$

Hence equation (5.6) becomes

$$\overline{E}_1(0) = \frac{iE_o k^3}{4\pi} \int_V \{[\tfrac{2}{3} h_o(kr) - h_2(kr)(\tfrac{1}{3} - \sin^2\theta \cos^2\phi)] \overline{e}_x$$

$$+ h_2(kr) \sin^2\theta \cos\phi \sin\phi \, \overline{e}_y + h_2(kr) \cos\theta \sin\theta \cos\phi \, \overline{e}_z\}$$

$$\times \, \tilde{\varepsilon}(\overline{r}) \, e^{-i\overline{k} \cdot \overline{r}} \, d\overline{r} - \frac{1}{3} \tilde{\varepsilon}(0) \, \overline{E}_o \, \overline{e}_x \; . \qquad (5.7)$$

Next, from equations (A.17), (A.18), and (A.19) one finds that

$$P_o(\cos\theta) = 1 \; , \quad P_2(\cos\theta) = \tfrac{1}{2}(3 \cos^2\theta - 1) \; ,$$

$$P_2^1(\cos\theta) = 3 \cos\theta \sin\theta, \quad P_1^2(\cos\theta) = 3 \sin^2\theta \; .$$

In addition, it is known that

$$\cos\phi \sin\phi = \frac{1}{2} \sin 2\phi \; , \qquad \cos^2\phi = \tfrac{1}{2}(1 + \cos 2\phi) \; .$$

Using these expressions, it is not difficult to show that equation (5.7) can be put into the form

$$\overline{E}_1(0) = \frac{iE_o k^3}{4\pi} \int_V \{[\tfrac{2}{3} h_o(kr) \, P_o(\cos\theta) - h_2(kr)(\tfrac{1}{3} P_2(\cos\theta)$$

$$- \frac{1}{6} P_2^2(\cos\theta)\cos 2\phi)]\overline{e}_x + \frac{1}{6} h_2(kr) \, P_2^2(\cos\theta)\sin 2\phi$$

$$+ \frac{1}{3} h_2(kr) \, P_2^1(\cos\theta)\cos\phi \, \overline{e}_z\} \, \tilde{\varepsilon}(\overline{r}) e^{-i\overline{k} \cdot \overline{r}} \, d\overline{r} - \tilde{\varepsilon}(0) \, E_o \overline{e}_x \; . \qquad (5.8)$$

For simplicity, let

$$\overline{C}(\overline{r}) = [\frac{2}{3} h_o(kr)\ P_o(\cos\theta) - h_2(kr)(\frac{1}{3} P_2(\cos\theta)$$

$$- \frac{1}{6} P_2^2(\cos\theta)\cos 2\phi)]\overline{e}_x + \frac{1}{6} h_2(kr)\ P_2^2(\cos\theta)\sin 2\phi\ \overline{e}_y$$

$$+ \frac{1}{3} h_2(kr)\ P_2^1(\cos\theta)\cos\phi\ \overline{e}_z \quad . \qquad (5.9)$$

Then

$$\overline{E}_1(0) = \frac{iE_o k^3}{4\pi} \int_V \overline{C}(\overline{r})\ \tilde{\varepsilon}(\overline{r})\ e^{-i\overline{k}\cdot\overline{r}}\ d\overline{r} - \frac{1}{3}\tilde{\varepsilon}(0)\ E_o\overline{e}_x \quad . \qquad (5.10)$$

Let $\overline{e}_\alpha$ be any of the unit vectors $\overline{e}_x$, $\overline{e}_y$, and $\overline{e}_z$, and let $\overline{E}_1(0)\cdot\overline{e}_\alpha$ be denoted by $E_{1\alpha_o}$. Then from equation (5.10) one has

$$E_{1\alpha_o} = \frac{iE_o k^3}{4\pi} \int_V \overline{e}_\alpha \cdot \overline{C}(\overline{r})\ \tilde{\varepsilon}(\overline{r})\ e^{-i\overline{k}\cdot\overline{r}}\ d\overline{r} - \frac{1}{3}\tilde{\varepsilon}(0)E_o\overline{e}_x\cdot\overline{e}_\alpha \ . \qquad (5.11)$$

Similarly, if $\overline{e}_\beta$ is also one of the unit vectors $\overline{e}_x$, $\overline{e}_y$, and $\overline{e}_z$, then

$$E^*_{1\beta_o} = \frac{-iE_o k^3}{4\pi} \int_V \overline{e}_\beta \cdot \overline{C}^*(\overline{r}')\ \tilde{\varepsilon}(\overline{r}')e^{i\overline{k}\cdot\overline{r}'}\ d\overline{r}'$$

$$- \frac{1}{3}\tilde{\varepsilon}(0)E_o\overline{e}_x\cdot\overline{e}_\beta \quad . \qquad (5.12)$$

Multiplying equation (5.11) by equation (5.12) and taking the ensemble average, one finds that

$$< E_{1\alpha_o} E^*_{1\beta_o} > = \frac{E_o^2 k^6}{16\pi^2} \int_V \int_V \overline{e}_\alpha \cdot \overline{C}(\overline{r})\overline{e}_\beta \cdot \overline{C}^*(\overline{r}')\ <\tilde{\varepsilon}(\overline{r})\ \tilde{\varepsilon}(\overline{r}')>$$

$$\times\ e^{-i\overline{k}\cdot(\overline{r}-\overline{r}')}\ d\overline{r}\ d\overline{r}' - \frac{iE_o^2 k^3}{12\pi}\ \overline{e}_x\cdot\overline{e}_\beta \int_V \overline{e}_\alpha\cdot\overline{C}(\overline{r})<\tilde{\varepsilon}(\overline{r})\tilde{\varepsilon}(0)>\ e^{-i\overline{k}\cdot\overline{r}}\ d\overline{r}$$

$$+ \frac{iE_o^2 k^3}{12\pi} \bar{e}_x \cdot \bar{e}_\alpha \int_V \bar{e}_\beta \cdot \bar{C}^*(\bar{r}')\langle\tilde{\varepsilon}(\bar{r}')\ \tilde{\varepsilon}(0)\rangle\ e^{i\bar{k}\cdot\bar{r}'}\ d\bar{r}'$$

$$+ \frac{1}{9} \langle\tilde{\varepsilon}^2(0)\rangle\ E_o^2\ \bar{e}_x \cdot \bar{e}_\alpha \bar{e}_x \cdot \bar{e}_\beta \quad . \tag{5.13}$$

But

$$\langle\tilde{\varepsilon}(\bar{r})\ \tilde{\varepsilon}(\bar{r}')\rangle = R_{\tilde{\varepsilon}}(|\bar{r}-\bar{r}'|) \ , \qquad \langle\tilde{\varepsilon}(\bar{r})\ \tilde{\varepsilon}(0)\rangle = R_{\tilde{\varepsilon}}(r) \ ,$$

$$\langle\tilde{\varepsilon}(\bar{r}')\ \tilde{\varepsilon}(0)\rangle = R_{\tilde{\varepsilon}}(r') \qquad , \qquad \langle\tilde{\varepsilon}^2(0)\rangle = \nu^2 \quad .$$

Hence, equation (5.13) becomes

$$\langle E_{1\alpha_o} E^*_{1\beta_o}\rangle = \frac{E_o^2 k^6}{16\pi^2} \int_V \int_V \bar{e}_\alpha \cdot \bar{C}(\bar{r})\bar{e}_\beta \cdot \bar{C}^*(\bar{r}')\ R_{\tilde{\varepsilon}}(|\bar{r}-\bar{r}'|) e^{-i\bar{k}\cdot(\bar{r}-\bar{r}')}\ d\bar{r}\ d\bar{r}'$$

$$- \frac{iE_o^2 k^3}{12\pi} \bar{e}_x \cdot \bar{e}_\beta \int_V \bar{e}_\alpha \cdot \bar{C}(\bar{r}) R_{\tilde{\varepsilon}}(r)\ e^{-i\bar{k}\cdot\bar{r}}\ d\bar{r}$$

$$+ \frac{iE_o^2 k^3}{12\pi} \bar{e}_x \cdot \bar{e}_\alpha \int_V \bar{e}_\beta \cdot \bar{C}^*(\bar{r}')\ R_{\tilde{\varepsilon}}(r') e^{i\bar{k}\cdot\bar{r}'}\ d\bar{r}'$$

$$+ \frac{1}{9}\ \nu^2 E_o^2\ \bar{e}_x \cdot \bar{e}_\alpha\ \bar{e}_x \cdot \bar{e}_\beta \quad . \tag{5.14}$$

Equation (5.14) will serve as the starting point for the work in the next two sections.

5.2 Scattered Field for the Case of $k\ell \gg 1$

Expressing the correlation function as the Fourier transform of the spectral density, one finds that equation (5.14) becomes

$$< E_{1\alpha_o} E^*_{1\beta_o} > = \frac{E_o^2 k^6}{16\pi^2} \int_V \int_V \int S_{\tilde{\varepsilon}}(\kappa) \bar{e}_\alpha \cdot \bar{C}(\bar{r}) \bar{e}_\beta \cdot \bar{C}^*(\bar{r}') e^{i\bar{\kappa}\cdot(\bar{r}-\bar{r}')}$$

$$\times e^{-i\bar{k}\cdot(\bar{r}-\bar{r}')} d\bar{\kappa}\, d\bar{r}\, d\bar{r}' - \frac{iE_o^2 k^3}{12\pi} \bar{e}_x \cdot \bar{e}_\beta \int_V \int S_{\tilde{\varepsilon}}(\kappa) \bar{e}_\alpha \cdot \bar{C}(\bar{r}) e^{i\bar{\kappa}\cdot\bar{r}}$$

$$\times e^{-i\bar{k}\cdot\bar{r}}\, d\bar{\kappa}\, d\bar{r} + \frac{iE_o^2 k^3}{12\pi} \bar{e}_x \cdot \bar{e}_\alpha \int_V \int S_{\tilde{\varepsilon}}(\kappa) \bar{e}_\beta \cdot \bar{C}(\bar{r}') e^{-i\bar{\kappa}\cdot\bar{r}'}$$

$$\times e^{i\bar{k}\cdot\bar{r}'}\, d\bar{\kappa}\, d\bar{r}' + \frac{1}{9} \nu^2 E_o^2\, \bar{e}_x \cdot \bar{e}_\alpha\, \bar{e}_x \cdot \bar{e}_\beta \tag{5.15}$$

where the $\bar{\kappa}$ integration is performed over all $\bar{\kappa}$ space. By changing the order of integration in equation (5.15) and rearranging terms, one can show that

$$< E_{1\alpha_o} E^*_{1\beta_o} > = \frac{E_o^2 k^6}{16\pi^2} \int S_{\tilde{\varepsilon}}(\kappa) \left\{ \int_V \bar{e}_\alpha \cdot \bar{C}(\bar{r})\; e^{i(\bar{\kappa}-\bar{k})\cdot\bar{r}}\, d\bar{r} \right.$$

$$\left. \times \int_V \bar{e}_\beta \cdot \bar{C}^*(r') e^{-i(\bar{\kappa}-\bar{k})\cdot\bar{r}'} d\bar{r}' \right\} d\bar{\kappa} - \frac{iE_o^2 k^3}{12\pi} \bar{e}_x \cdot \bar{e}_\beta \int S_{\tilde{\varepsilon}}(\kappa)$$

$$\times \left\{ \int_V \bar{e}_\alpha \cdot \bar{C}(\bar{r}) e^{i(\bar{\kappa}-\bar{k})\cdot\bar{r}}\, d\bar{r} \right\} d\bar{\kappa} + \frac{iE_o^2 k^3}{12\pi} \bar{e}_x \cdot \bar{e}_\alpha \int S_{\tilde{\varepsilon}}(\kappa)$$

$$\times \left\{ \int_V \bar{e}_\beta \cdot \bar{C}(\bar{r}) e^{i(\bar{\kappa}-\bar{k})\cdot\bar{r}}\, d\bar{r} \right\} d\bar{\kappa} + \frac{1}{9} \nu^2 E_o^2\, \bar{e}_x \cdot \bar{e}\;\; \bar{e}_x \cdot \bar{e}_\beta \quad . \tag{5.16}$$

Letting

$$L_\alpha(\bar{\kappa}) = \int_V \bar{e}_\alpha \cdot \bar{C}(\bar{r})\; e^{i(\bar{\kappa}-\bar{k})\cdot\bar{r}}\, d\bar{r} \tag{5.17}$$

with an analogous definition for $L_\beta(\overline{\kappa})$, equation (5.16) becomes

$$
\begin{aligned}
< E_{1\alpha_o} E^*_{1\beta_o} > = \frac{E_o^2 k^6}{16\pi^2} \int S_{\underset{\sim}{\varepsilon}}(\kappa)\, L_\alpha(\overline{\kappa})\, L^*_\beta(\overline{\kappa})\, d\overline{\kappa} \\
- \frac{iE_o^2 k^3}{12\pi}\, \overline{e}_x \cdot \overline{e}_\beta \int S_{\underset{\sim}{\varepsilon}}(\kappa)\, L_\alpha(\overline{\kappa})\, d\overline{\kappa} \\
+ \frac{iE_o^2 k^3}{12\pi}\, \overline{e}_x \cdot \overline{e}_\alpha \int S_{\underset{\sim}{\varepsilon}}(\kappa)\, L^*_\beta(\overline{\kappa})\, d\overline{\kappa} + \frac{1}{9} \nu^2 E_o^2\, \overline{e}_x \cdot \overline{e}_\alpha \overline{e}_x \cdot \overline{e}_\beta .
\end{aligned}
\tag{5.18}
$$

Using equation (5.17) and the explicit expression for $\overline{C}(\overline{r})$ given in equation (5.9), one finds that

$$
\begin{aligned}
L_x(\overline{\kappa}) = \int_0^R \int_0^\pi \int_0^{2\pi} [\tfrac{2}{3}\, h_o(kr) P_o(\cos\theta) - h_2(kr)(\tfrac{1}{3}\, P_2(\cos\theta) \\
- \tfrac{1}{6}\, P_2^2(\cos\theta)\cos 2\phi)] e^{i(\overline{\kappa}-\overline{k})\cdot\overline{r}}\, r^2 \sin\theta\, d\phi\, d\theta\, dr
\end{aligned}
\tag{5.19}
$$

$$
L_y(\overline{\kappa}) = \int_0^R \int_0^\pi \int_0^{2\pi} \frac{1}{6}\, h_2(kr) P_2^2(\cos\theta) \sin 2\phi\; e^{i(\overline{\kappa}-\overline{k})\cdot\overline{r}}\, r^2 \sin\theta\, d\phi\, d\theta\, dr
\tag{5.20}
$$

$$
L_z(\overline{\kappa}) = \int_0^R \int_0^\pi \int_0^{2\pi} \frac{1}{3}\, h_2(kr) P_2^1(\cos\theta) \cos\phi\; e^{i(\overline{\kappa}-\overline{k})\cdot\overline{r}}\, r^2 \sin\theta\, d\phi\, d\theta\, dr .
\tag{5.21}
$$

Equations (5.19) - (5.21) can be evaluated by the same techniques used in Section 4.2. Let $\overline{\kappa}' = \overline{\kappa} - \overline{k}$, and let

$$
\overline{\kappa}' = \kappa' \sin\psi \cos\gamma\, \overline{e}_x + \kappa' \sin\psi \sin\gamma\, \overline{e}_y + \kappa' \cos\psi\, \overline{e}_z
$$

where ψ and γ correspond to the angles θ and ϕ , respectively, in spherical coordinates. Then

$$e^{i(\overline{\kappa}-\overline{k})\cdot\overline{r}} = e^{i\overline{\kappa}'\cdot\overline{r}} ,$$

and using equation (A.20), $e^{i\overline{\kappa}'\cdot\overline{r}}$ can be expressed as a sum of spherical waves:

$$e^{i\overline{\kappa}'\cdot\overline{r}} = \sum_{n=0}^{\infty} i^n(2n+1)j_n(\kappa' r)\Big[\sum_{m=0}^{n} \alpha_m \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta)P_n^m(\cos\psi) \times \cos m(\phi-\gamma)\Big] .$$

Substitution of this expression into equations (5.19) - (5.21) and reversal of the order of summation and integration yields

$$L_x(\overline{\kappa}) = \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\left[i^n(2n+1)\alpha_m \frac{(n-m)!}{(n+m)!} P_n^m(\cos\psi)\int_0^R\int_0^\pi\int_0^{2\pi}\Big[\frac{2}{3}h_o(kr)P_o(\cos\theta) - h_2(kr)\Big(\frac{1}{3}P_2(\cos\theta) - \frac{1}{6}P_2^2(\cos\theta)\cos 2\phi\Big)\Big]j_n(\kappa' r) \times P_n^m(\cos\theta)\cos m(\phi-\gamma)r^2\sin\theta\, d\phi\, d\theta\, dr\right]\right\} \quad (5.22)$$

$$L_y(\overline{\kappa}) = \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\left[i^n(2n+1)\alpha_m \frac{(n-m)!}{(n+m)!} P_n^m(\cos\psi)\int_0^R\int_0^\pi\int_0^{2\pi}\frac{1}{6}h_2(kr) \times P_2^2(\cos\theta)\sin 2\phi\, j_n(\kappa' r)P_n^m(\cos\theta)\cos m(\phi-\gamma)r^2\sin\theta\, d\phi\, d\theta\, dr\right]\right\} \quad (5.23)$$

$$L_z(\overline{\kappa}) = \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\left[i^n(2n+1)\alpha_m \frac{(n-m)!}{(n+m)!} P_n^m(\cos\psi)\int_0^R\int_0^\pi\int_0^{2\pi}\frac{1}{3}h_2(kr) \times P_2^1(\cos\theta)\cos\phi\, j_n(\kappa' r)P_n^m(\cos\theta)\cos m(\phi-\gamma)r^2\sin\theta\, d\phi\, d\theta\, dr\right]\right\} . \quad (5.24)$$

Performing the angular integrations by using the orthogonality relations for the Legendre polynomials and the trigonometric functions, one finds that

$$L_x(\bar{\kappa}) = \frac{8\pi}{3}\int_0^R h_o(kr)\, j_o(\kappa' r) r^2 dr + \frac{4\pi}{3}[P_2(\cos\psi) - \frac{1}{2}P_2^2(\cos\psi)\cos 2\gamma]$$

$$\times \int_0^R h_2(kr)\, j_2(\kappa' r) r^2\, dr \tag{5.25}$$

$$L_y(\bar{\kappa}) = \frac{-2\pi}{3} P_2^2(\cos\psi)\sin 2\gamma \int_0^R h_2(kr)\, j_2(\kappa' r)\, r^2\, dr \tag{5.26}$$

$$L_z(\bar{\kappa}) = \frac{-4\pi}{3} P_2^1(\cos\psi)\cos\gamma \int_0^R h_2(kr)\, j_2(\kappa' r)\, r^2\, dr \quad . \tag{5.27}$$

The integrals in equations (5.25) - (5.27) are evaluated in equations (4.121) and (4.122). Let

$$F_o(\kappa') = \int_0^R h_o(kr)\, j_o(\kappa' r)\, r^2\, dr \tag{5.28}$$

$$F_2(\kappa') = \int_0^R h_2(kr)\, j_2(\kappa' r)\, r^2\, dr \; . \tag{5.29}$$

When this notation is employed in equations (5.25), (5.26), and (5.27), and when the explicit expressions for the Legendre polynomials in terms of sin and cos functions are used, the results are

$$L_x(\bar{\kappa}) = \frac{8\pi}{3} F_o(\kappa') + 2\pi[\cos^2\psi - \frac{1}{3} - \sin^2\psi\, \cos 2\gamma]F_2(\kappa') \tag{5.30}$$

$$L_y(\bar{\kappa}) = -2\pi \sin^2\psi \sin 2\gamma \, F_2(\kappa') \tag{5.31}$$

$$L_z(\bar{\kappa}) = -4\pi \cos\psi \sin\psi \cos\gamma \, F_2(\kappa') \, . \tag{5.32}$$

Using equation (5.18) and the explicit expressions for the spectral density (equation (4.47)), the following equations for the mean-square components and the mean of the products of the transverse components can now be obtained:

$$\begin{aligned}<|E_{1x_o}|^2> = \frac{E_o^2 k^6 \ell^3 \nu^2}{32\pi^{3/2}} \int e^{-\kappa^2\ell^2/4} \{\tfrac{16}{9}|F_o(\kappa')|^2 + [\cos^4\psi + \tfrac{1}{9} + \sin^4 \\
\times \cos^2 2\gamma - \tfrac{2}{3}\cos^2\psi - 2\cos^2\psi \sin^2\psi \cos 2\gamma + \tfrac{2}{3}\sin^2\psi \cos 2\gamma] \\
\times |F_2(\kappa')|^2 + \tfrac{4}{3}[\cos^2\psi - \tfrac{1}{3} - \sin^2\psi \cos 2\gamma][F_o(\kappa')F_2^*(\kappa') \\
+ F_o^*(\kappa')F_2(\kappa')]\} \, d\bar{\kappa} + \frac{E_o^2 k^3 \ell^3 \nu^2}{24\pi^{3/2}} \, \mathrm{Im} \left\{ \int e^{-\kappa^2\ell^2/4} [\tfrac{4}{3} F_o(\kappa') + (\cos^2\psi \right. \\
\left. - \tfrac{1}{3} - \sin^2\psi \cos 2\gamma)] F_2(\kappa') d\bar{\kappa} \right\} + \tfrac{1}{9} E_o^2 \nu^2\end{aligned} \tag{5.33}$$

$$<|E_{1y_o}|^2> = \frac{E_o^2 k^6 \ell^3 \nu^2}{32\pi^{3/2}} \int e^{-\kappa^2\ell^2/4} \sin^4\psi \sin^2 2\gamma \, |F_2(\kappa')|^2 \, d\bar{\kappa} \tag{5.34}$$

$$<|E_{1z_o}|^2> = \frac{E_o^2 k^6 \ell^3 \nu^2}{8\pi^{3/2}} \int e^{-\kappa^2\ell^2/4} \cos^2\psi \sin^2\psi \cos^2\gamma \, |F_2(\kappa')|^2 \, d\bar{\kappa} \tag{5.35}$$

$$<E_{1x_o} E^*_{1y_o}> = - \frac{E_o^2 k^6 \ell^3 \nu^2}{32\pi^{3/2}} \int e^{-\kappa^2\ell^2/4} \left[\frac{4}{3} \sin^2\psi \sin 2\gamma \; F_o(\kappa') F_2^*(\kappa')\right.$$

$$+ (\cos^2\psi \sin^2\psi \sin 2\gamma - \frac{1}{3} \sin^2\psi \sin 2\gamma - \sin^4 \cos 2\gamma \sin 2\gamma)$$

$$\times |F_2(\kappa')|^2 \, d\overline{\kappa} - \frac{iE_o^2 k^3 \ell^3 \nu^2}{48\pi^{3/2}} \int e^{-\kappa^2\ell^2/4} \sin^2\psi \sin 2\gamma \; F_2^*(\kappa') d\overline{\kappa} \qquad (5.36)$$

$$< E_{1x_o} E^*_{1z_o} > = - \frac{E_o^2 k^6 \ell^3 \nu^2}{16\pi^{3/2}} \int e^{-\kappa^2\ell^2/4} \left[\frac{4}{3} \cos\psi \sin\psi \cos\gamma \; F_o(\kappa') F_2^*(\kappa')\right.$$

$$+ (\cos^3\psi \sin\psi \cos\gamma - \frac{1}{3} \cos\psi \sin\psi \cos\gamma - \cos\psi \sin^3\psi \cos\gamma \cos 2\gamma)$$

$$\times |F_2(\kappa')|^2 \, d\overline{\kappa} + \frac{iE_o^2 k^3 \ell^3 \nu^2}{24\pi^{3/2}} \int e^{-\ell^2\kappa^2/4} \cos\psi \sin\psi \cos\gamma \; F_2^*(\kappa') d\overline{\kappa} \qquad (5.37)$$

$$< E_{1y_o} E^*_{1z_o} > = \frac{E_o^2 k^6 \ell^3 \nu^2}{16\pi^{3/2}} \int e^{-\kappa^2\ell^2/4} \cos\psi \sin^3\psi \cos\gamma \sin 2\gamma \; |F_2(\kappa')|^2 \, d\overline{\kappa} . \qquad (5.38)$$

It will now be demonstrated that the mean of the products of the transverse components vanish. Changing the integration variables in equations (5.36) - (5.38) from $\overline{\kappa}$ to $\overline{\kappa}'$, where it is recalled that $\overline{\kappa}' = \overline{\kappa} - \overline{k}$, and noting that $\kappa^2 = |\overline{\kappa}' - \overline{k}|^2 = k^2 + 2\overline{\kappa}' \cdot \overline{k} + \kappa'^2 = k^2 + \kappa'^2 + 2\kappa' k \cos\psi$, yields

$$< E_{1x_o} E^*_{1y_o} > = - \frac{E_o^2 k^6 \ell^3 \nu^2}{32\pi^{3/2}} \int_0^\infty \int_0^\pi \int_0^{2\pi} e^{-[\kappa'^2 + k^2 + 2\kappa' k \cos\psi]\ell^2/4}$$

$$\times \left[\frac{4}{3} \sin^2\psi \sin 2\gamma \; F_o(\kappa') F_2^*(\kappa') + (\cos^2\psi \sin^2\psi \sin 2\gamma - \right.$$

$$- \frac{1}{3}\sin^2\psi \sin 2\gamma - \sin^4\psi \cos 2\gamma \sin 2\gamma) \; |F_2(\kappa')|^2 \kappa'^2 \sin\psi \, d\gamma \, d\psi \, d\kappa'$$

$$- \frac{iE_o^2 k^3 \ell^3 \nu^2}{48\pi^{3/2}} \int_0^\infty \int_0^\pi \int_0^{2\pi} e^{-[\kappa'^2 + k^2 + 2\kappa' k \cos\psi]\ell^2/4} \sin^2\psi \sin 2\gamma$$

$$\times \; F_2^*(\kappa')\kappa'^2 \sin\psi \, d\gamma \, d\psi \, d\kappa' \qquad (5.39)$$

$$\langle E_{1x_o} E_{1z_o}^* \rangle = - \frac{E_o^2 k^6 \ell^3 \nu^2}{16\pi^{3/2}} \int_0^\infty \int_0^\pi \int_0^{2\pi} e^{-[\kappa'^2 + k^2 + 2\kappa' k \cos\psi]\ell^2/4}$$

$$\times \; [\frac{4}{3} \cos\psi \sin\psi \cos\gamma \, F_o(\kappa') F_2^*(\kappa') + (\cos^3\psi \sin\psi \cos\gamma$$

$$- \frac{1}{3} \cos\psi \sin\psi \cos\gamma - \cos\psi \sin^3\psi \cos\gamma \cos 2\gamma) \; |F_2(\kappa')|^2 \kappa'^2 \sin\psi$$

$$\times \; d\gamma \, d\psi \, d\kappa'$$

$$+ \frac{iE_o^2 k^3 \ell^3 \nu^2}{24\pi^{3/2}} \int_0^\infty \int_0^\pi \int_0^{2\pi} e^{-[\kappa'^2 + k^2 + 2k\kappa' \cos\psi]\ell^2/4}$$

$$\times \; \cos\psi \sin\psi \cos\gamma \, F_2^*(\kappa')\kappa'^2 \sin\psi \, d\gamma \, d\psi \, d\kappa' \qquad (5.40)$$

$$\langle E_{1y_o} E_{1z_o}^* \rangle = \frac{E_o^2 k^6 \ell^3 \nu^2}{16\pi^{3/2}} \int_0^\infty \int_0^\pi \int_0^{2\pi} e^{-[\kappa'^2 + k^2 + 2\kappa' k \cos\psi]\ell^2/4}$$

$$\times \; \cos\psi \sin^3\psi \cos\gamma \sin 2\gamma \, |F_2(\kappa')|^2 \kappa'^2 \sin\psi \, d\gamma \, d\psi \, d\kappa' . \qquad (5.41)$$

Integration of equations (5.39) - (5.41) with respect to γ yields

$$\left.\begin{aligned} &< E_{1x_o} E^*_{1y_o} > = 0 \\ &< E_{1x_o} E^*_{1z_o} > = 0 \\ &< E_{1y_o} E^*_{1z_o} > = 0 \quad . \end{aligned}\right\} \qquad (5.42)$$

It should be observed that no use was made of the fact that $k\ell >> 1$ in deriving equations (5.42). Hence they are valid for both $k\ell >> 1$ and $k\ell << 1$.

The expressions for the mean-square components of the scattered field will now be obtained. Drawing on the development of Section 4.2, it is known that the $\overline{\kappa}$ integration in equations (5.33) - (5.35) can be restricted to the region where $|\overline{\kappa} - \overline{k}| = \kappa' \geq \frac{1}{2} k$ with negligible error, provided $k\ell >> 1$. By restricting the $\overline{\kappa}$ integration in this manner and then changing the variables of integration from $\overline{\kappa}$ to $\overline{w}$, where $\overline{w} = \ell(\overline{\kappa} - \overline{k})$, one finds that equations (5.33) - (5.35) assume the following form:

$$<|E_{1x_o}|^2> = \frac{E_o^2 k^6 \nu^2}{32\pi^{3/2}} \int_{k\ell/2}^{\infty} \int_0^{\pi} \int_0^{2\pi} e^{-[w^2 + k^2\ell^2 + 2wk\ell \cos\psi]/4}$$

$$\times \; \{ \tfrac{16}{9} |F_o(\tfrac{w}{\ell})|^2 + [\cos^4\psi + \tfrac{1}{9} + \sin^4\psi \cos^2 2\gamma - \tfrac{2}{3} \cos^2\psi$$

$$- 2 \cos^2\psi \sin^2\psi \cos 2\gamma + \tfrac{2}{3} \sin^2\psi \cos 2\gamma] |F_2(\tfrac{w}{\ell})|^2 + \tfrac{4}{3}[\cos^2\psi - \tfrac{1}{3}$$

$$- \sin^2\psi \cos 2\gamma][F_o(\tfrac{w}{\ell}) F_2^*(\tfrac{w}{\ell}) + F_o^*(\tfrac{w}{\ell}) F_2(\tfrac{w}{\ell})]\} w^2 \sin\psi \, d\gamma \, d\psi \, dw$$

$$+ \frac{E_o^2 k^3 \nu^2}{24\pi^{3/2}} \, \mathrm{Im} \left\{ \int_{k\ell/2}^{\infty} \int_0^{\pi} \int_0^{2\pi} e^{-[w^2 + k^2\ell^2 + 2wk\ell \cos\psi]/4} \, [\tfrac{4}{3} F_o(\kappa') + \right.$$

$$+ \left(\cos^2\psi - \frac{1}{3} - \sin^2\psi \cos 2\gamma\right) F_2\left(\frac{w}{\ell}\right) w^2 \sin\psi \, d\psi \, d\gamma \, d\kappa' \Bigg\}$$

$$+ \frac{1}{9} E_o^2 \nu^2 \tag{5.43}$$

$$<|E_{1y_o}|^2> = \frac{E_o^2 k^6 \nu^2}{32\pi^{3/2}} \int_{k\ell/2}^{\infty} \int_0^{\pi} \int_0^{2\pi} e^{-[w^2 + k^2\ell^2 + 2wk\ell \cos\psi]/4}$$

$$\times \quad \sin^4\psi \sin^2 2\gamma \left|F_2\left(\frac{w}{\ell}\right)\right|^2 w^2 \sin\psi \, d\gamma \, d\psi \, dw \tag{5.44}$$

$$<|E_{1z_o}|^2> = \frac{E_o^2 k^6 \nu^2}{8\pi^{3/2}} \int_{k\ell/2}^{\infty} \int_0^{\pi} \int_0^{\pi} e^{-[w^2 + k^2\ell^2 + 2wk\ell \cos\psi]/4}$$

$$\times \cos^2\psi \sin^2\psi \cos^2\gamma \left|F_2\left(\frac{w}{\ell}\right)\right|^2 w^2 \sin\psi \, d\gamma \, d\psi \, dw \quad . \tag{5.45}$$

Performing the γ integrations in equations (5.43) - (5.45) and making the change of variables $t = \cos\psi$, one finds that

$$<|E_{1x_o}|^2> = \frac{E_o^2 k^6 \nu^2}{32\sqrt{\pi}} \int_{k\ell/2}^{\infty} \int_{-1}^{1} e^{-[w^2 + k^2\ell^2 + 2wk\ell t]/4} \left\{\frac{32}{9} \left|F_o\left(\frac{w}{\ell}\right)\right|^2\right.$$

$$\left. + \left[3t^4 - \frac{10}{3} t^2 + \frac{11}{9}\right] \left|F_2\left(\frac{w}{\ell}\right)\right|^2 + \left[\frac{8}{3} t^2 - \frac{8}{9}\right] \left[F_o^*\left(\frac{w}{\ell}\right) F_2\left(\frac{w}{\ell}\right) + F_o\left(\frac{w}{\ell}\right) F_2^*\left(\frac{w}{\ell}\right)\right]\right\} w^2 \, dt \, dw$$

$$+ \frac{E_o^2 k^3 \nu^2}{12\sqrt{\pi}} \, \mathrm{Im}\left\{ \int_{k\ell/2}^{\infty} \int_{-1}^{1} e^{-[w^2 + k^2\ell^2 + 2wk\ell t]/4} \left[\frac{4}{3} F_o\left(\frac{w}{\ell}\right)\right.\right.$$

$$\left.\left. + \left(t^2 - \frac{1}{3}\right) F_2\left(\frac{w}{\ell}\right)\right] w^2 \, dt \, dw \right\} + \frac{1}{9} E_o^2 \nu^2 \tag{5.46}$$

$$<|E_{1y_o}|^2> = \frac{E_o^2 k^6 \nu^2}{32\sqrt{\pi}} \int_{k\ell/2}^{\infty} \int_{-1}^{1} e^{-[w^2+k^2\ell^2+ 2wk\ell t]/4} [1 - 2t^2+ t^4]$$

$$\times \; |F_2(\tfrac{w}{\ell})|^2 \, w^2 \, dt \, dw \tag{5.47}$$

$$<|E_{1z_o}|^2> = \frac{E_o^2 k^6 \nu^2}{8\sqrt{\pi}} \int_{k\ell/2}^{\infty} \int_{-1}^{1} e^{-[w^2+ k^2\ell^2+ 2wk\ell t]/4} [t^2-t^4] |F_2(\tfrac{w}{\ell})|^2 w^2 dt \, dw \,. \tag{5.48}$$

By employing equations (4.77), (4.78), and (4.147), one can perform the t integrations in equations (5.46) - (5.48). Keeping only the highest order terms in w, the following asymptotic expressions are obtained:

$$<|E_{1x_o}|^2> \sim \frac{E_o^2 k^5 \nu^2}{18\sqrt{\pi}\,\ell} \int_{k\ell/2}^{\infty} e^{-[w-k\ell]^2/4} \{4|F_o(\tfrac{w}{\ell})|^2 + |F_2(\tfrac{w}{\ell})|^2$$

$$+ 2[F_o^*(\tfrac{w}{\ell})F_2(\tfrac{w}{\ell})+ F_o(\tfrac{w}{\ell})F_2^*(\tfrac{w}{\ell})]\}w \, dw + \frac{E_o^2 k^2 \nu^2}{9\sqrt{\pi}\;\ell} \, \mathrm{Im}\left\{ \int_{k\ell}^{\infty} e^{-[w-k\ell]^2/4} \right.$$

$$\left. \times \; [2F_o(\tfrac{w}{\ell}) + F_2(\tfrac{w}{\ell})]w \, dw \right\} + \frac{1}{9} E_o^2 \, \nu^2 \tag{5.49}$$

$$<|E_{1y_o}|^2> \sim \frac{2E_o^2 k^3 \nu^2}{\sqrt{\pi}\;\ell^3} \int_{k\ell}^{\infty} e^{-[w-k\ell]^2/4} |F_2(\tfrac{w}{\ell})|^2 \frac{1}{w} \, dw \tag{5.50}$$

$$<|E_{1z_o}|^2> \sim \frac{E_o^2 k^4 \nu^2}{\sqrt{\pi}\;\ell^2} \int_{k\ell/2}^{\infty} e^{-[w-k\ell]^2/4} |F_2(\tfrac{w}{\ell})|^2 \, dw \tag{5.51}$$

Again drawing on the development in Section 4.2, it is known that the major contribution to the integrals in equations (5.49) - (5.51) is derived from a relatively small neighborhood of $w = k\ell$, and one can use the same type of approximation for $F_o(\frac{w}{\ell})$ and $F_2(\frac{w}{\ell})$ that was used for $F(\frac{w}{\ell})$ in that section. In fact, the approximations for $F_o(\frac{w}{\ell})$ and $F_2(\frac{w}{\ell})$ are both given by equation (4.151):

$$F_o(\tfrac{w}{\ell}) \sim F_2(\tfrac{w}{\ell}) \sim -\frac{\ell}{2k^2}\left\{\frac{\sin[(w-k\ell)\frac{R}{\ell}]}{w-k\ell} + i\,\frac{\cos[w-k\ell)\frac{R}{\ell}]-1}{w-k\ell}\right\} .$$

Using the additional approximations

$$w|F_o(\tfrac{w}{\ell})|^2 \sim k\ell|F_o(\tfrac{w}{\ell})|^2 \ , \ \frac{1}{w}|F_2(\tfrac{w}{\ell})|^2 \sim \frac{1}{k\ell}|F_2(\tfrac{w}{\ell})|^2, \text{ etc.},$$

one finds that equations (5.49) - (5.51) become

$$\begin{aligned}<|E_{1x_o}|^2> \sim \frac{E^2k^2\ell^2\nu^2}{4\sqrt{\pi}} \int_{k\ell/2}^{\infty} e^{-[w-k\ell]^2/4}\left\{\frac{1-\cos[(w-k\ell)\frac{R}{\ell}]}{(w-k\ell)^2}\right\} dw \\ -\frac{E_o^2k\ell\nu^2}{6\sqrt{\pi}}\,\mathrm{Im}\left\{\int_{k\ell/2}^{\infty} e^{-[w-k\ell]^2/4}\left\{\frac{\sin[(w-k\ell)\frac{R}{\ell}]}{(w-k\ell)} + i\,\frac{\cos[(w-k\ell)\frac{R}{\ell}]-1}{(w-k\ell)}\right\} dw\right\} \\ +\frac{1}{9}E_o^2\,\nu^2 \qquad (5.52)\end{aligned}$$

$$<|E_{1y_o}|^2> \sim \frac{E_o^2\nu^2}{\sqrt{\pi}k^2\ell^2}\int_{k\ell/2}^{\infty} e^{-[w-k\ell]^2/4}\left\{\frac{1-\cos[(w-k\ell)\frac{R}{\ell}]}{(w-k\ell)^2}\right\} dw \qquad (5.53)$$

$$<|E_{1z_o}|^2> \sim \frac{E_o^2\nu^2}{2\sqrt{\pi}}\int_{k\ell/2}^{\infty} e^{-[w-k\ell]^2/4}\left\{\frac{1-\cos[(w-k\ell)\frac{R}{\ell}]}{(w-k\ell)^2}\right\} dw \qquad (5.54)$$

The lower limits on the integrals in equations (5.52) - (5.53) may be extended to $-\infty$ without causing any appreciable error. Then making the change of variables $\tau = (w-k\ell)\frac{R}{\ell}$, one obtains

$$<|E_{1x_o}|^2> \sim \frac{E_o^2 k^2 \ell R \nu^2}{4\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\tau^2\ell^2/4R^2} \{\frac{1-\cos\tau}{\tau^2}\} d\tau$$

$$- \frac{E_o^2 k \ell \nu^2}{6\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\tau^2\ell^2/4R^2} \{\frac{\cos\tau - 1}{\tau}\} d\tau + \frac{1}{9} E_o^2 \nu^2 \tag{5.55}$$

$$<|E_{1y_o}|^2> \sim \frac{E_o^2 R \nu^2}{k_o^2 \ell^3 \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\tau^2\ell^2/4R^2} \{\frac{1-\cos\tau}{\tau^2}\} d\tau \tag{5.56}$$

$$<|E_{1z_o}|^2> \sim \frac{E_o^2 R \nu^2}{2\ell\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\tau^2\ell^2/4R^2} \{ \frac{1-\cos\tau}{\tau^2} \} d\tau \ . \tag{5.57}$$

Now from equations (4.154) and (4.155) one has

$$\int_{-\infty}^{\infty} e^{-\tau^2\ell^2/4R^2} \{\frac{1-\cos\tau}{\tau^2}\} d\tau \sim \pi \qquad \text{for} \quad R/\ell >> 1 \ ,$$

$$\int_{-\infty}^{\infty} e^{-\tau^2\ell^2/4R^2} \{\frac{\cos\tau - 1}{\tau}\} d\tau = 0 \ .$$

Thus, equations (5.55) - (5.57) become

$$<|E_{1x_o}|^2> \sim \frac{\sqrt{\pi} E_o^2 k^2 \ell R \nu^2}{4} + \frac{1}{9} E_o^2 \nu^2 \tag{5.58}$$

$$<|E_{1y_o}|^2> \sim \frac{\sqrt{\pi} E_o^2 R \nu^2}{k_o^2 \ell^3} \tag{5.59}$$

$$<|E_{1z_o}|^2> \sim \frac{\sqrt{\pi} E_o^2 R \nu^2}{2\ell} \ . \tag{5.60}$$

Since $k^2 \ell R = (k\ell)^2 \frac{R}{\ell} >>> 1$, the second term on the right in equation (5.58) should be dropped so that the equation becomes

$$<|E_{1x_o}|^2> \sim \frac{\sqrt{\pi}\, E_o^2 k^2 \ell R \nu^2}{4} \quad . \tag{5.61}$$

Equations (5.59), (5.60), and (5.61) give the final expressions for the mean-square field components. These equations, together with equations (5.42), represent the results of the examination of the scattered field at the center of the randomly-inhomogeneous sphere.

Comparing the magnitudes of the field components, it can be seen that the magnitude of $<|E_{1x_o}|^2>$ is greater than that of $<|E_{1y_o}|^2>$ by a factor of $(k\ell)^4$, and furthermore that the magnitude of $<|E_{1x_o}|^2>$ is greater than that of $<|E_{1z_o}|^2>$ by a factor of $(k\ell)^2$.

The Born approximation is assumed to yield reliable results provided $|\overline{E}_o|^2 >> <|\overline{E}_1(0)|^2>$. Since $|\overline{E}_o|^2 = E_o^2$, and since $<|\overline{E}_1(0)|^2> \sim <|E_{1x_o}|^2>$, this implies that

$$\nu^2 << \frac{1}{k^2 \ell R} \quad .$$

This inequality is identical to the one which was derived in Section 4.2A.

5.3 Scattered Field for the Case of $k\ell << 1$

The mean-square components of the scattered field will now be determined for the case where the correlation distance of the inhomogeneous dielectric is much less than the wavelength (it was shown in the previous section that the ensemble averages of all the products of pairs of transverse components vanish). Asymptotic expressions for the mean-square components will be obtained for two cases, one being for $k \to 0$ as R and ℓ remain fixed, and the other being for $R \to \infty$ as k and ℓ remain fixed.

Using equations (5.14), (5.9), and the explicit expression for the correlation function, the following expressions can be obtained for the mean-square field components:

$$
\begin{aligned}
<|E_{1x_o}|^2> = {} & \frac{E_o^2 k^6 \ell^6 \nu^2}{144\pi^2} \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} \cdot \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} e^{-|\overline{\rho}-\overline{\rho}'|^2} e^{-i\ell\overline{k}\cdot(\overline{\rho}-\overline{\rho}')} \\
& \times \Big\{4h_o(k\ell\rho)h_o^*(k\ell\rho')P_o(\cos\theta)P_o^*(\cos\theta') + \frac{1}{4} h_2(k\ell\rho)h_2^*(k\ell\rho')[4P_2(\cos\theta) \\
& \times P_2(\cos\theta') + P_2^2(\cos\theta)P_2^2(\cos\theta')\cos 2\phi \cos 2\phi' - 2P_2(\cos\theta)P_2^2(\cos\theta') \\
& \times \cos 2\phi' - 2P_2(\cos\theta')P_2^2(\cos\theta)\cos 2\phi] - h_o(k\ell\rho)h_2^*(k\ell\rho')[2P_o(\cos\theta) \\
& \times P_2(\cos\theta') - P_o(\cos\theta)P_2^2(\cos\theta')\cos 2\phi'] - h_o^*(k\ell\rho')h_2(k\ell\rho)[2P_o(\cos\theta') \\
& \times P_2(\cos\theta) - P_o(\cos\theta')P_2^2(\cos\theta)\cos 2\phi]\Big\}\rho^2\rho'^2 \sin\theta \sin\theta' d\phi d\theta d\, d\phi' d\theta' d\rho' \\
& + \frac{E_o^2 k^3 \ell^3 \nu^2}{18\pi} \, \mathrm{Im}\Big\{\int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} e^{-\rho^2} e^{i\ell\overline{k}\cdot\overline{\rho}} [2h_o(k\ell\rho)P_o(\cos\theta) - h_2(k\ell\rho) \\
& \times (P_2(\cos\theta) - \frac{1}{2} P_2^2(\cos\theta)\cos 2\phi)]\rho^2 \sin\theta \, d\phi \, d\theta \, d\rho\Big\} + \frac{1}{9} E_o^2 \nu^2
\end{aligned}
\tag{5.62}
$$

$$<|E_{1y_o}|^2> = \frac{E_o^2 k^6 \ell^6 \nu^2}{576\pi^2} \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} \cdot \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} e^{-|\bar{\rho}-\bar{\rho}'|^2} e^{-i\ell\bar{k}\cdot(\bar{\rho}-\bar{\rho}')}$$

$$\times \quad h_2(k\ell\rho)h_2^*(k\ell\rho')P_2^2(\cos\theta)P_2^2(\cos\theta')\sin 2\phi \sin 2\phi' \rho^2\rho'^2$$

$$\times \quad \sin\theta \sin\theta' \, d\phi \, d\theta \, d\rho \, d\phi' \, d\theta' \, d\rho' \quad (5.63)$$

$$<|E_{1z_o}|^2> = \frac{E_o^2 k^6 \ell^6 \nu^2}{144\pi^2} \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} \cdot \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} e^{-|\bar{\rho}-\bar{\rho}'|^2} e^{-i\ell\bar{k}\cdot(\bar{\rho}-\bar{\rho}')}$$

$$\times \quad h_2(k\ell\rho)h_2^*(k\ell\rho')P_2^1(\cos\theta)P_2^1(\cos\theta')\cos\phi \cos\phi' \rho^2\rho'^2$$

$$\times \quad \sin\theta \sin\theta' \, d\phi \, d\theta \, d\rho \, d\phi' \, d\theta' \, d\rho' . \quad (5.64)$$

It should be noted that in using equation (5.14) the change in variables $\bar{\rho} = \frac{1}{\ell}\bar{r}$, $\bar{\rho}' = \frac{1}{\ell}\bar{r}'$ was made.

Now

$$h_o(k\ell\rho) = \frac{-i}{k\ell\rho} e^{ik\ell\rho} , \quad h_2(k\ell\rho) = [\frac{i}{k\ell\rho} - \frac{3}{(k\ell\rho)^2} - \frac{3i}{(k\ell\rho)^3}]e^{k\ell\rho} \quad .$$

Then in a manner similar to that used in Section 4.3, it can be shown that as $k \to 0$, while R and ℓ remain constant, the major contribution to the integrals in equations (5.62) - (5.64) is obtained from the $\frac{-3i}{(k\ell\rho)^3} e^{ik\ell\rho}$, $\frac{-3i}{(k\ell\rho')^3} e^{ik\ell\rho'}$ terms of the second order Hankel function. On the other hand, as $R \to \infty$ while k and ℓ remain constant, the major contribution to the integrals comes from the $\frac{1}{k\ell\rho} e^{ik\ell\rho}$, $\frac{1}{k\ell\rho'} e^{ik\ell\rho'}$ terms in the first and second order Hankel functions. Thus,

$$<|E_{1x_o}|^2> \sim \frac{E_o^2\nu^2}{64\pi^2} \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} \cdot \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} e^{-|\overline{\rho}-\overline{\rho}'|^2}\, e^{i\ell\overline{k}\cdot(\overline{\rho}-\overline{\rho}')}\, e^{ik\ell(\rho-\rho')}$$

$$\times\ [4P_2(\cos\theta)P_2(\cos\theta') + P_2^2(\cos\theta)P_2^2(\cos\theta')\cos 2\phi\cos 2\phi'$$

$$-\ 2P_2(\cos\theta')P_2^2(\cos\theta')\cos 2\phi' - 2P_2(\cos\theta')P_2^2(\cos\theta)\cos 2\phi]$$

$$\times\ \frac{1}{\rho\rho'}\sin\theta\sin\theta' d\phi\, d\theta\, d\rho\, d\phi'\, d\theta'\, d\rho'$$

$$+\ \frac{E_o^2\nu^2}{6\pi}\,\mathrm{Re}\left\{\int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} e^{-\rho^2} e^{i\ell\overline{k}\cdot\overline{\rho}}\, e^{ik\ell\rho}\,[P_2(\cos\theta) - \frac{1}{2}P_2^2(\cos\theta)\cos 2\phi]\right.$$

$$\left.\times\ \frac{1}{\rho}\sin\theta\, d\phi\, d\theta\, d\rho\right\} + \frac{1}{9}E_o^2\nu^2 \qquad (5.65)$$

$$<|E_{1y_o}|^2> \sim \frac{E_o^2\nu^2}{64\pi^2} \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} \cdot \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} e^{-|\overline{\rho}-\overline{\rho}'|^2}\, e^{-i\ell\overline{k}\cdot(\overline{\rho}-\overline{\rho}')}\, e^{ik\ell(\rho-\rho')}$$

$$\times\ P_2^2(\cos\theta)P_2^2(\cos\theta')\sin 2\phi\sin 2\phi'\,\frac{1}{\rho\rho'}\sin\theta\sin\theta'$$

$$\times\ d\phi\, d\theta\, d\rho\, d\phi'\, d\theta'\, d\rho' \qquad (5.66)$$

$$<|E_{1z_o}|^2> \sim \frac{E_o^2\nu^2}{16\pi^2} \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} \cdot \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} e^{-|\overline{\rho}-\overline{\rho}'|^2}\, e^{-i\ell\overline{k}\cdot(\overline{\rho}-\overline{\rho}')}\, e^{ik\ell(\rho-\rho')}$$

$$\times\ P_2^1(\cos\theta)P_2^1(\cos\theta')\cos\phi\cos\phi'\,\frac{1}{\rho\rho'}\sin\theta\sin\theta' d\phi\, d\theta\, d\rho\, d\phi' d\theta' d\rho' \qquad (5.67)$$

as $k \to 0$ while R and ℓ remain constant, and

$$
\begin{aligned}
<|E_{1x_o}|^2> \sim \; & \frac{E_o^2 k^4 \ell^4 \nu^2}{144\pi^2} \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} \cdot \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} e^{-|\overline{\rho}-\overline{\rho}'|^2}\, e^{-i\ell\overline{k}\cdot(\overline{\rho}-\overline{\rho}')} \\
& \times \; e^{ik\ell(\rho-\rho')} \Big\{4P_o(\cos\theta)P_o(\cos\theta') + \tfrac{1}{4}[4P_2(\cos\theta)P_2(\cos\theta') \\
& + P_2^2(\cos\theta)P_2^2(\cos\theta')\cos 2\phi\cos 2\phi' - 2P_2(\cos\theta)P_2^2(\cos\theta')\cos 2\phi' \\
& - 2P_2(\cos\theta')P_2^2(\cos\theta)\cos 2\phi] + [2P_o(\cos\theta)P_2(\cos\theta') \\
& + 2P_o(\cos\theta')P_2(\cos\theta) - P_o(\cos\theta)P_2^2(\cos\theta')\cos 2\phi' \\
& - P_o(\cos\theta)P_2^2(\cos\theta')\cos 2\phi'\Big\}\rho\rho'\sin\theta\sin\theta'\, d\phi\, d\theta\, d\rho\, d\phi' d\theta' d\rho' \\
& - \frac{E_o^2 k^2 \ell^2 \nu^2}{18\pi}\,\mathrm{Re}\left\{\int_0^{R/\ell} e^{-\rho^2}\, e^{-i\ell\overline{k}\cdot\overline{\rho}}\, e^{ik\ell\rho}[2 + P_2(\cos\theta) \right. \\
& \left. - \tfrac{1}{2}P_2^2(\cos\theta)\cos 2\phi]\rho\sin\theta\, d\phi\, d\theta\, d\rho\right\} + \tfrac{1}{9}E_o^2\,\nu^2
\end{aligned}
\qquad (5.68)
$$

$$
\begin{aligned}
<|E_{1y_o}|^2> \sim \; & \frac{E_o^2 k^4 \ell^4 \nu^2}{576\ ^2} \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} \cdot \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} e^{-|\overline{\rho}-\overline{\rho}'|^2}\, e^{-i\ell\overline{k}\cdot(\overline{\rho}-\overline{\rho}')} \\
& \times \; e^{ik\ell(\rho-\rho')}\, P_2^2(\cos\theta)P_2^2(\cos\theta')\sin 2\phi\sin 2\phi'\rho\rho' \\
& \times \;\sin\theta\sin\theta' d\phi\, d\theta\, d\rho\, d\phi' d\theta'\, d\rho'
\end{aligned}
\qquad (5.69)
$$

$$<|E_{1z_o}|^2> \sim \frac{E_o^2 k^4 \ell^4 \nu^2}{144\pi^2} \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} \cdot \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} e^{-|\bar{\rho}-\bar{\rho}'|^2} e^{-i\ell\bar{k}\cdot(\bar{\rho}-\bar{\rho}')}$$

$$\times \; e^{ik\ell(\rho-\rho')} P_2^1(\cos\theta)P_2^1(\cos\theta')\cos\phi\cos\phi'\rho\rho'\sin\theta\sin\theta'$$

$$\times \; d\phi \, d\theta \, d\rho \, d\phi' \, d\theta' \, d\rho' \qquad (5.70)$$

as $R \to \infty$ while k and ℓ remain constant.

Now because $e^{-|\bar{\rho}-\bar{\rho}'|^2}$ decreases very rapidly for $|\bar{\rho}-\bar{\rho}'| > 1$, most of the contribution to integrals in equations (5.66), (5.67), (5.69), (5.70) and the first integrals on the right-hand sides of equations (5.65) and (5.68) will occur in the region where $|\bar{\rho} - \bar{\rho}'|$ is relatively small. Then since $k\ell << 1$, one can make the approximations $e^{i\ell\bar{k}\cdot(\bar{\rho}-\bar{\rho}')} \sim 1$, $e^{ik\ell(\rho-\rho')} \sim 1$ in those equations. For similar reasons, one can make the approximations $e^{i\ell\bar{k}\cdot\bar{\rho}} \sim 1$, $e^{ik\ell\rho} \sim 1$ in the last integrals on the right in equations (5.65) and (5.68). Then using the expansion (see equation (4.178))

$$e^{-|\bar{\rho}-\bar{\rho}'|^2} = e^{-(\rho^2+\rho'^2)} \sum_{n=0}^{\infty} \Bigg\{ i^n(2n+1)j_n(-i2\rho\rho')$$

$$\times \left[\sum_{m=0}^{n} \alpha_m \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta)P_n^m(\cos\theta')\cos m(\phi-\phi') \right] \Bigg\} ,$$

one finds that equations (5.65) - (5.70) become

$$<|E_{1x_o}|^2> \sim \frac{E_o^2\nu^2}{64\pi^2} \sum_{n=0}^{\infty} \Bigg\{ \sum_{m=0}^{n} i^n(2n+1)\alpha_m \frac{(n-m)!}{(n+m)!} \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} \cdot \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} e^{-(\rho^2+\rho'^2)}$$

$$\times \; j_n(-i2\rho\rho')P_n^m(\cos\theta)P_n^m(\cos\theta')\cos m(\phi-\phi')[4P_2(\cos\theta)P_2(\cos\theta')$$

$$+ P_2^2(\cos\theta)P_2^2(\cos\theta')\cos 2\phi \cos 2\phi' - 2P_2(\cos\theta)P_2^2(\cos\theta')\cos 2\phi'$$

$$\left. - 2P_2(\cos\theta')P_2^2(\cos\theta)\cos 2\phi\right]\frac{1}{\rho\rho'} \sin\theta \sin\theta'\, d\phi\, d\theta\, d\rho\, d\phi' d\theta' d\rho' \Bigg\}$$

$$+ \frac{E_o^2\nu^2}{6\pi}\int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} e^{-\rho^2}[P_2(\cos\theta) - \frac{1}{2}P_2^2(\cos\theta)\cos 2\phi]$$

$$\times \quad \frac{1}{\rho}\sin\theta\, d\phi\, d\theta\, d\rho + \frac{1}{9}E_o^2\nu^2 \tag{5.71}$$

$$<|E_{1y_o}|^2> \sim \frac{E_o^2\nu^2}{64\pi^2}\sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n} i^n(2n+1)\alpha_m \frac{(n-m)!}{(n+m)!}\int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi}\cdot\int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi}\right.$$

$$\times e^{-(\rho^2+\rho'^2)}\; j_n(-i2\rho\rho')P_n^m(\cos\theta)P_n^m(\cos\theta')\cos m(\phi-\phi')P_2^2(\cos\theta)$$

$$\left.\times \quad P_2^2(\cos\theta')\sin 2\phi \sin 2\phi' \frac{1}{\rho\rho'}\sin\theta \sin\theta' d\phi\, d\theta\, d\rho\, d\phi' d\theta' d\rho'\right\} \tag{5.72}$$

$$<|E_{1z_o}|^2> \sim \frac{E_o^2\nu^2}{16\pi^2}\sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n} i^n(2n+1)\alpha_m \frac{(n-m)!}{(n+m)!}\int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi}\cdot\int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi}\right.$$

$$\times e^{-(\rho^2+\rho'^2)}\; j_n(-i2\rho\rho')P_n^m(\cos\theta)P_n^m(\cos\theta')\cos m(\phi-\phi')P_2^1(\cos\theta)$$

$$\left.\times P_2^1(\cos\theta')\cos\phi \cos\phi' \frac{1}{\rho\rho'}\sin\theta \sin\theta' d\phi\, d\theta\, d\rho\, d\phi'\, d\theta'\, d\rho'\right\} \tag{5.73}$$

as $k \to 0$ while R and ℓ remain constant, and

$$<|E_{1x_o}|^2> \sim \frac{E_o^2k^4\ell^4\nu^2}{144\pi^2} \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{n} i^n(2n+1)\alpha_m \frac{(n-m)!}{(n+m)!} \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} \cdot \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} \right.$$

$$\times \; e^{-(\rho^2+\rho'^2)} j_n(-i2\rho\rho')P_n^m(\cos\theta)P_n^m(\cos\theta')\cos m(\phi-\phi') \{4P_o(\cos\theta)$$

$$\times \; P_o(\cos\theta') + \tfrac{1}{4}[4P_2(\cos\theta)P_2(\cos\theta') + P_2^2(\cos\theta)P_2^2(\cos\theta')\cos 2\phi \cos 2\phi'$$

$$- 2P_2(\cos\theta)P_2^2(\cos\theta')\cos 2\phi' - 2P_2(\cos\theta')P_2^2(\cos\theta)\cos 2\phi]$$

$$+ 2P_o(\cos\theta)P_2(\cos\theta') + 2P_o(\cos\theta')P_2(\cos\theta) - P_o(\cos\theta)P_2^2(\cos\theta')$$

$$\left. \times \; \cos 2\phi' - P_o(\cos\theta')P_2^2(\cos\theta)\cos 2\phi\} \rho\rho'\sin\theta \sin\theta' d\phi\, d\theta\, d\rho\, d\phi' d\theta' d\rho' \right\}$$

$$- \frac{E_o^2k^2\ell^2\nu^2}{18\pi} \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} e^{-\rho^2}[2 + P_2(\cos\theta) - \tfrac{1}{2} P_2^2(\cos\theta)\cos 2\phi]$$

$$\times \; \rho \sin\theta\, d\phi\, d\theta\, d\rho + \tfrac{1}{9} E_o^2 \nu^2 \qquad (5.74)$$

$$<|E_{1y_o}|^2> \sim \frac{E_o^2k^4\ell^4\nu^2}{576\pi^2} \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{n} i^n(2n+1)\alpha_m \frac{(n-m)!}{(n+m)!} \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} \cdot \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} \right.$$

$$\times \; e^{-(\rho^2+\rho'^2)} j_n(-i2\rho\rho')P_n^m(\cos\theta)P_n^m(\cos\theta')\cos m(\phi-\phi')P_2^2(\cos\theta)P_2^2(\cos\theta')$$

$$\left. \times \; \sin 2\phi \sin 2\phi'\rho\rho' \sin\theta \sin\theta'\, d\phi\, d\theta\, d\rho\, d\phi' d\theta' d\rho' \right\} \qquad (5.75)$$

$$<|E_{1z_o}|^2> \sim \frac{E_o^2 k^4 \ell^4 \nu^2}{144\pi^2} \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{n} i^n (2n+1)\alpha_m \frac{(n-m)!}{(n+m)!} \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} \cdot \int_0^{R/\ell}\int_0^{\pi}\int_0^{2\pi} \right.$$

$$\times \; e^{-(\rho^2+\rho'^2)} \; j_n(-i2\rho\rho')P_n^m(\cos\theta)P_n^m(\cos\theta')\cos m(\phi-\phi')P_2^1(\cos\theta)$$

$$\left. \times \; P_2^1(\cos\theta')\cos\phi\,\cos\phi'\;\rho\rho'\sin\theta\,\sin\theta'\;d\phi\,d\theta\,d\rho\,d\phi' d\theta' d\rho' \right\} \quad (5.76)$$

as $R \to \infty$ while k and ℓ remain constant. When the angular integrations are performed, equations (5.71) - (5.76) become

$$<|E_{1x_o}|^2> \sim \frac{-4E_o^2\nu^2}{5} \int_0^{R/\ell}\int_0^{R/\ell} e^{-(\rho^2+\rho'^2)} \; j_2(-i2\rho\rho')\frac{1}{\rho\rho'} d\rho\, d\rho' + \frac{1}{9} E_o^2\nu^2 \quad (5.77)$$

$$<|E_{1y_o}|^2> \sim -\frac{3E_o^2\nu^2}{5} \int_0^{R/\ell}\int_0^{R/\ell} e^{-(\rho^2+\rho'^2)} \; j_2(-i2\rho\rho')\frac{1}{\rho\rho'} d\rho\, d\rho' \quad (5.78)$$

$$<|E_{1z_o}|^2> \sim \frac{-3E_o^2\nu^2}{5} \int_0^{R/\ell}\int_0^{R/\ell} e^{-(\rho^2+\rho'^2)} \; j_2(-i2\rho\rho')\frac{1}{\rho\rho'} d\rho\, d\rho' \quad (5.79)$$

as $k \to 0$ while R and ℓ remain constant, and

$$<|E_{1x_o}|^2> \sim \frac{4E_o^2 k^4 \ell^4 \nu^2}{45} \int_0^{R/\ell}\int_0^{R/\ell} e^{-(\rho^2+\rho'^2)} [5j_o(-i2\rho\rho') - j_2(-i2\rho\rho')]\rho\rho' d\rho\, d\rho' + \frac{E_o^2\nu^2(1-k^2\ell^2)}{9} \quad (5.80)$$

$$<|E_{1y_o}|^2> \sim \frac{-E_o^2k^4\ell^4\nu^2}{15} \int_0^{R/\ell}\int_0^{R/\ell} e^{-(\rho^2+\rho'^2)}\, j_2(-i2\rho\rho')\rho\rho' d\rho\, d\rho' \quad (5.81)$$

$$<|E_{1z_o}|^2> \sim \frac{-E_o^2k^4\ell^4\nu^2}{15} \int_0^{R/\ell}\int_0^{R/\ell} e^{-(\rho^2+\rho'^2)}\, j_2(-i2\rho\rho')\rho\rho' d\rho\, d\rho' \quad (5.82)$$

as $R \to \infty$ while k and ℓ remain constant.

The integrals that appear in equations (5.77) - 5.82) were encountered earlier and their values are given by equations (4.223), (4.224), and (4.225). Substitution of these values into equations (5.77) - (5.82) yields

$$<|E_{1x_o}|^2> \sim \frac{E_o^2\nu^2}{5} \quad (5.83)$$

$$<|E_{1y_o}|^2> \sim \frac{E_o^2\nu^2}{15} \quad (5.84)$$

$$<|E_{1z_o}|^2> \sim \frac{E_o^2\nu^2}{15} \quad (5.85)$$

as $k \to 0$ while R and ℓ remain constant, and

$$<|E_{1x_o}|^2> \sim \frac{2\sqrt{\pi}\ E_o^2k^4R\nu^2\ell^3}{15} + \frac{E_o^2\nu^2(1-k^2\ell^2)}{9} \quad (5.86)$$

$$<|E_{1y_o}|^2> \sim \frac{\sqrt{\pi}\ E_o^2k^4\ell^3R\nu^2}{60} \quad (5.87)$$

$$<|E_{1z_o}|^2> \sim \frac{\sqrt{\pi}E_o^2 k^4 \ell^3 R \nu^2}{60} \tag{5.88}$$

as $R \to \infty$ while k and ℓ remain constant. Now the second term on the right-hand side of equation (5.86) becomes negligible in comparison to the first as $R \to \infty$ and should be dropped. Thus,

$$<|E_{1x_o}|^2> \sim \frac{2\sqrt{\pi}\ E_o^2 k^4 \ell^3 R \nu^2}{15} \tag{5.89}$$

as $R \to \infty$ while k and ℓ remain constant.

Equations (5.83) - (5.85) and (5.87) - (5.89) are the final expressions for the mean-square components at the center of the randomly-inhomogeneous sphere for the case where $k\ell << 1$. To complete the results it should also be recalled that the ensemble average of any pair of transverse components vanishes, which means that the transverse components are uncorrelated.

Comparing equations (5.83) - (5.85) with (5.87) - (5.89), it can be seen that the ratio of their magnitudes is on the order of $1:k^4\ell^3R$. Thus, if $1 >> k^4\ell^3R$, one should use equations (5.83) - (5.85) to obtain the mean square field components, while if $k^4\ell^3R >> 1$ then equations (5.87) - (5.89) should be employed. Neither set of equations will be accurate for the case where $k^4\ell^3R \sim 1$.

Finally, in order to have confidence in the reliability of the Born approximation, one should require that $|\overline{E}_o|^2 >> |\overline{E}_1(0)|^2$. Assuming $k^4\ell^3R >> 1$, this requirement is equivalent to

$$\nu^2 << \frac{1}{k^4\ell^3R} \ .$$

VI. CONCLUSIONS

Using the Born approximation in solving the vector wave equation, this work studies the far-zone scattered electric field when electric and magnetic dipoles are surrounded by spherical volumes of randomly-inhomogeneous dielectric media, and it also examines the electric field at the center of a randomly-inhomogeneous sphere when a plane wave is incident upon it.

It is found that the inhomogeneities give rise to depolarized components in the electric field (components transverse to the original field vector). It is also found that the transverse components of the scattered field are uncorrelated.

For the problem in which a dipole is situated at the center of the inhomogeneous sphere, it is shown that the magnitude of the mean-square depolarized component (which is transverse to the direction of propagation) is less than that of the polarized component by a factor of $(k\ell)^2$ when $k\ell \gg 1$. When $k\ell \ll 1$, the mean-square magnitudes of the components are found to be nearly the same and the mean-square ϕ component is found to be independent of the polar angle θ_o. In both cases, the presence of inhomogeneities results in the elimination of nulls in the radiation pattern.

For the problem in which the scattered electric field is being examined at the center of the sphere upon which a plane wave is incident, two depolarized components are produced, one parallel to the direction of propagation of the plane wave and one transverse to it. When $k\ell \gg 1$, the magnitude of the mean-square depolarized component parallel to the direction of propagation is a factor of $(k\ell)^2$ less than that of the polarized component, and the magnitude of the other mean-square depolarized component is less than that of the polarized component by a factor of $(k\ell)^4$. When $k\ell \ll 1$, all three components have nearly the same mean-square magnitudes.

In both problems, it is found that the magnitudes of the mean-square polarized components are greater when $k\ell \gg 1$ than when $k\ell \ll 1$.

APPENDIX A

USEFUL EXPRESSIONS AND EXPANSIONS

A.1 Spherical Hankel and Bessel Functions

In the text of this work, use is made of spherical Hankel and Bessel functions. Listed below are explicit expressions and asymptotic approximations for some of these functions.

(i) Spherical Hankel Functions of the First Kind

$$h_o(kr) = \frac{-i}{kr} e^{ikr} \tag{A.1}$$

$$h_1(kr) = [-\frac{1}{kr} - \frac{i}{(kr)^2}] e^{ikr} \tag{A.2}$$

$$h_2(kr) = [\frac{i}{kr} - \frac{3}{(kr)^2} - \frac{3i}{(kr)^3}] e^{ikr} \tag{A.3}$$

$$h_3(kr) = [\frac{1}{kr} + \frac{6i}{(kr)^2} - \frac{15}{(kr)^3} - \frac{15i}{(kr)^4}] e^{ikr} \tag{A.4}$$

(ii) Spherical Bessel Functions

$$j_o(kr) = \frac{1}{kr} \sin kr \tag{A.5}$$

$$j_1(kr) = \frac{1}{(kr)^2} \sin kr - \frac{1}{kr} \cos kr \tag{A.6}$$

$$j_2(kr) = [-\frac{1}{kr} + \frac{3}{(kr)^3}] \sin kr - \frac{3}{(kr)^2} \cos kr \tag{A.7}$$

$$j_3(kr) = [-\frac{6}{(kr)^2} + \frac{15}{(kr)^4}] \sin kr + [\frac{1}{kr} - \frac{15}{(kr)^3}] \cos kr \tag{A.8}$$

(iii) Asymptotic Expressions for Spherical Hankel and Bessel Functions

1. $kr \to \infty$

$$h_o(kr) = -\frac{i}{kr} e^{ikr} \qquad j_o(kr) = \frac{1}{kr} \sin kr$$

$$h_1(kr) \sim -\frac{1}{kr} e^{ikr} \qquad j_1(kr) \sim \frac{-1}{kr} \cos kr$$

$$h_2(kr) \sim \frac{i}{kr} e^{ikr} \qquad j_2(kr) \sim \frac{-1}{kr} \sin kr$$

$$h_3(kr) \sim \frac{1}{kr} e^{ikr} \qquad j_3(kr) \sim \frac{1}{kr} \cos kr$$

2. $kr \to 0$

$$h_o(kr) \sim \frac{-i}{kr} \qquad j_o(kr) \sim 1$$

$$h_1(kr) \sim \frac{-i}{(kr)^2} \qquad j_1(kr) \sim \frac{kr}{3}$$

$$h_2(kr) \sim \frac{-3i}{(kr)^3} \qquad j_2(kr) \sim \frac{(kr)^2}{15}$$

$$h_3(kr) \sim \frac{-15i}{(kr)^4} \qquad j_3(kr) \sim \frac{(kr)^3}{105}$$

An integral involving spherical Hankel and Bessel functions which will prove to be quite useful is

$$\int^{r} t^2 a_n(kt) b_n(k_1 t) dt = \frac{r^2}{k^2 - k_1^2} [k a_{n+1}(kt) b_n(k_1 t) - k_1 a_n(kt) b_{n+1}(k_1 t)] \tag{A.9}$$

where a_n and b_n denote any two linear combinations of spherical Bessel or Hankel functions of order n, and where $k^2 \neq k_1^2$.

Expression (A.9) can be verified by using the following equation from the N.B.S. Handbook [25]:

$$\int^{r} (k^2 - k_1^2) t \, A_\nu(kt) \, B_\nu(k_1 t) dt = r[k \, A_{\nu+1}(kr) \, B_\nu(k_1 r)$$

$$- k_1 \, A_\nu(kr) \, B_{\nu+1}(k_1 r)]$$

where A_ν, B_ν are any linear combinations of cylindrical Bessel or Hankel functions of order ν.

Dividing this by $k^2 - k_1^2$, assuming $k^2 \neq k_1^2$, one obtains

$$\int^{r} t \, A_\nu(kt) \, B_\nu(k_1 t) dt = \frac{r}{k^2 - k_1^2} [k \, A_{\nu+1}(kr) \, B_\nu(kr) - k_1 \, A_\nu(kr) \, B_{\nu+1}(k_1 r)]. \tag{A.10}$$

Now spherical Bessel and Hankel functions of order n are related to cylindrical Bessel and Hankel functions by

$$d_n(kr) = \sqrt{\frac{\pi}{2kr}} \, D_{n+1/2}(kr) \tag{A.11}$$

where d_n is a spherical Bessel or Hankel function and $D_{n+1/2}$ is a cylindrical one. Then by letting $\nu = n + \frac{1}{2}$ in equation (A.10) and using equation (A.11), one obtains

$$\int^{r} t[\sqrt{\frac{2kt}{\pi}} \, a_n(kt)][\sqrt{\frac{2k_1 t}{\pi}} \, b_n(k_1 t)] dt = \frac{r}{k^2 - k_1^2} \Big\{ k[\sqrt{\frac{2kr}{\pi}} \, a_{n+1}(kr)]$$

$$\times [\sqrt{\frac{2k_1 r}{\pi}} \, b_n(k_1 r)] - k_1[\sqrt{\frac{2kr}{\pi}} \, a_n(kr)][\sqrt{\frac{2k_1 r}{\pi}} \, b_{n+1}(k_1 r)] \Big\} \tag{A.12}$$

or, after simplification,

$$\int^{r} t^2 a_n(kt) b_n(k_1 t) dt = \frac{r^2}{k^2 - k_1^2} [k\, a_{n+1}(kr)\ b_n(k_1 r) - k_1 a_n(kr) b_{n+1}(k_1 r)]$$

which is equation (A.9).

A.2 Legendre Polynomials and Associated Legendre Polynomials

A number of Legendre polynomials and associated Legendre polynomials are used in the text. Explicit expressions for these polynomials are listed here.

$$P_o(\cos\theta) = 1 \tag{A.13}$$

$$P_1(\cos\theta) = \cos\theta \tag{A.14}$$

$$P_1^1(\cos\theta) = \sin\theta \tag{A.15}$$

$$P_2(\cos\theta) = \frac{1}{2}[3\cos^2\theta - 1] \tag{A.16}$$

$$P_2^1(\cos\theta) = 3\sin\theta\cos\theta \tag{A.17}$$

$$P_2^2(\cos\theta) = 3\sin^2\theta \tag{A.18}$$

The following orthogonality relationship for Legendre polynomials is frequently employed:

$$\int_0^{\pi} P_n^m(\cos\theta)\ P_{n_1}^m(\cos\theta)\sin\theta\, d\theta = \begin{cases} \dfrac{2}{2n+1}\dfrac{(n+m)!}{(n-m)!} & \text{if}\quad n = n_1 \\ 0 & \text{if}\quad n \neq n_1\,. \end{cases} \tag{A.19}$$

A.3 Expansion of a Plane Wave in Spherical Waves

Consider a plane wave given by $e^{i\bar{k}\cdot\bar{r}}$ where $\bar{r} = r \sin\theta \cos\phi\, \bar{e}_x + r \sin\theta \sin\phi\, \bar{e}_y + r \cos\theta\, \bar{e}_z$, $\bar{k} = k \sin\theta' \times \cos\phi'\, \bar{e}_x + k \sin\theta' \sin\phi'\, \bar{e}_y + k \cos\theta'\, \bar{e}_z$. Then from Stratton [26] one obtains the following expansion:

$$e^{i\bar{k}\cdot\bar{r}} = \sum_{n=0}^{\infty} i^n (2n+1)\, j_n(kr) \left[\sum_{m=0}^{n} \alpha_m \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta) P_n^m(\cos\theta') \times \cos m(\phi - \phi') \right] \tag{A.20}$$

where

$$\alpha_m = \begin{cases} 1 & m = 0 \\ 2 & m \neq 0 \end{cases} .$$

APPENDIX B

JUSTIFICATION FOR THE EXTENSION OF THE LIMITS OF A CERTAIN INTEGRAL

In evaluating the integral

$$I = \int_0^{R/\ell} \int_0^{R/\ell} e^{-(x^2 + y^2)} \; j_1(-i2xy) \; dx \; dy \qquad \text{(B.1)}$$

in Section 4.3A, it was asserted that the upper limits of the integral could be extended from R/ℓ to ∞ without producing a significant error, provided that $R/\ell >>> 1$. This will now be demonstrated.

Let

$$I' = \int_0^{\infty} \int_0^{\infty} e^{-(x^2 + y^2)} \; j_1(-i2xy) dx \; dy \qquad \text{(B.2)}$$

and let

$$I'' \equiv I - I' \qquad \text{(B.3)}$$

so that $I = I' - I''$. If it can be shown that the magnitude of I'' is negligible in comparison to that of I', then the assertion will be proved. To begin with, one has

$$I'' = \iint_S e^{-(x^2 + y^2)} \; j_1(-i2xy) \; dx \; dy \qquad \text{(B.4)}$$

where S is the region of the first quadrant of the x-y plane exterior to the square whose vertices are the points $(0,0)$, $(0,R/\ell)$, $(R/\ell,R/\ell)$, $(R/\ell,0)$. Now

$$j_1(-i2xy) = i[\frac{\sinh 2xy}{4x^2 y^2} - \frac{\cosh 2xy}{2xy}] = \frac{-i}{2} e^{2xy} [\frac{1+e^{-4xy}}{2xy} - \frac{1-e^{-4xy}}{4x^2 y^2}] .$$

Let

$$g(t) = \frac{1+e^{-2t}}{t} - \frac{1-e^{-2t}}{t^2} \tag{B.5}$$

so that

$$j_1(-i2xy) = -\frac{i}{2} e^{2xy} g(2xy) .$$

Then

$$I'' = -\frac{i}{2} \iint_S e^{-(x-y)^2} g(2xy)\,dx\,dy . \tag{B.6}$$

Let region S be divided into three subregions as shown in Figure B.1 . Then

$$I'' = \frac{-i}{2} \iint_{S_1} e^{-(x-y)^2} g(2xy)\,dx\,dy - \frac{i}{2} \iint_{S_2} e^{-(x-y)^2} g(2xy)\,dx\,dy$$

$$- \frac{i}{2} \iint_{S_3} e^{-(x-y)^2} g(2xy)\,dx\,dy$$

and hence

$$|I''| \leq \frac{1}{2} \iint_{S_1} e^{-(x-y)^2} |g(2xy)|\,dx\,dy + \frac{1}{2} \iint_{S_2} e^{-(x-y)^2} |g(2xy)|\,dx\,dy$$

$$+ \frac{1}{2} \iint_{S_3} e^{-(x-y)^2} |g(2xy)|\,dx\,dy .$$

Consider

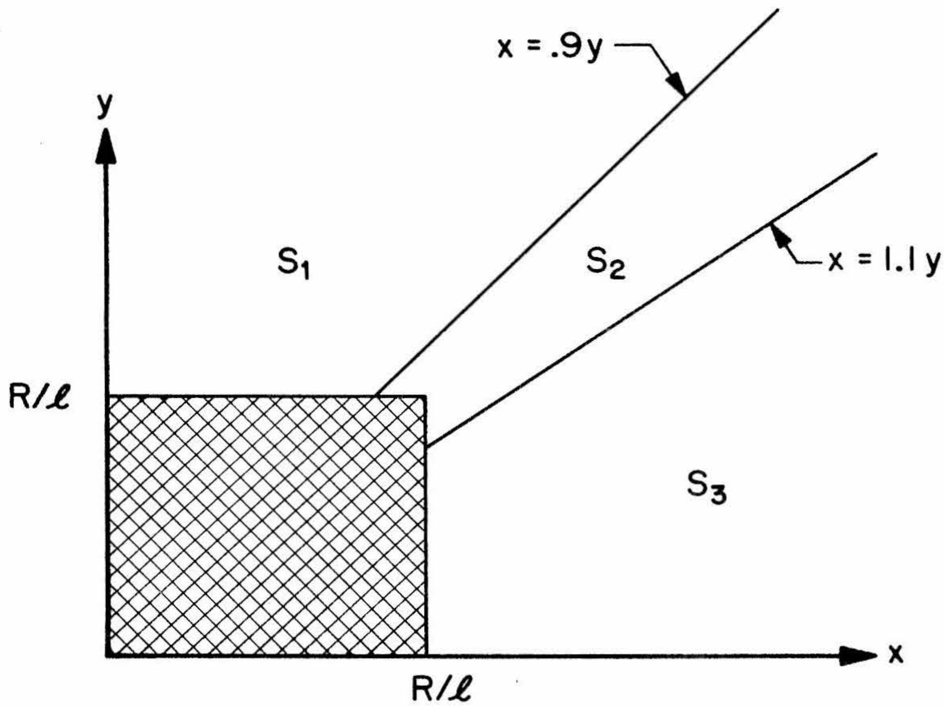

Figure B.1. Subregions of Integration for Integral I''

$$\iint_{S_1} e^{-(x-y)^2}|g(2xy)|\,dx\,dy = \int_{R/\ell}^{\infty}\int_0^{.9y} e^{-(x-y)^2}|g(2xy)|\,dx\,dy .$$

A few computations reveal that $0 \leq g(t) < 2$ for $t \geq 0$. Hence,

$$\iint_{S_1} e^{-(x-y)^2}|g(2xy)|\,dx\,dy < 2\int_{R/\ell}^{\infty}\int_0^{.9y} e^{-(x-y)^2}\,dx\,dy .$$

Making the change of variables $t = .9y - x$, $y = y$, one obtains

$$\iint_{S_1} e^{-(x-y)^2}|g(2xy)|\,dx\,dy < 2\int_{R/\ell}^{\infty}\int_0^{.9y} e^{-(t+.1y)^2}\,dt\,dy$$

$$\leq 2\int_{R/\ell}^{\infty} e^{-.01y^2}\,dy \int_0^{.9y} e^{-t^2}\,dt \leq 2\int_{R/\ell}^{\infty} e^{-.01y^2}\,dy \int_0^{\infty} e^{-t^2}\,dt$$

$$= \sqrt{\pi}\int_{R/\ell}^{\infty} e^{-.01y^2}\,dy .$$

Making the change of variable $\tau = y - R/\ell$, one has

$$\iint_{S_1} e^{-(x-y)^2}|g(2xy)|\,dx\,dy < \sqrt{\pi}\int_0^{\infty} e^{-.01(\tau - R/\ell)^2}\,d\tau$$

$$\leq e^{-.01R^2/\ell^2}\int_0^{\infty} e^{-.01\tau^2}\,d\tau = 5\pi\, e^{-.01R^2/\ell^2} < 20e^{-.01R^2/\ell^2} . \quad (B.8)$$

In exactly the same manner, one can show that

$$\iint_{S_3} e^{-(x-y)^2} |g(2xy)| dx\, dy \;<\; 20e^{-.01R^2/\ell^2} \quad . \tag{B.9}$$

Now consider

$$\iint_{S_2} e^{-(x-y)^2} |g(2xy)| \;\leq\; \int_{.91\, R/\ell}^{\infty} \int_{.9y}^{1.1y} e^{-(x-y)^2} |g(2xy)| dx\, dy \quad .$$

It is apparent from examining the expression for $g(t)$ given in equation (B.5) that $0 < g(t) < 2/t$ for $t > 5$. Then since it is assumed that $R/\ell >> 5$, one has

$$\iint_{S_2} e^{-(x-y)^2} |g(2xy)| dx\, dy < \int_{.91\, R/\ell}^{\infty} \int_{.9y}^{1.1y} e^{-(x-y)^2} \frac{1}{xy}\, dx\, dy \;.$$

But since $x \geq .9y$ in S_2, then $1/xy \leq 1.1/y^2$ in S_2, and hence

$$\iint_{S_2} e^{-(x-y)^2} |g(2xy)| dx\, dy \;<\; \int_{.91\, R/\ell}^{\infty} \int_{.9y}^{1.1y} e^{-(x-y)^2} \frac{1.1}{y^2}\, dx\, dy$$

$$< \; 1.1 \int_{.91R/\ell}^{\infty} \frac{dy}{y^2} \cdot \int_{-\infty}^{\infty} e^{-x^2} dx = 1.21\, \sqrt{\pi}\; \ell/R < 4\ell/R \quad . \tag{B.10}$$

Substituting the values of equations (B.8), (B.9), and (B.10) into the inequality (B.7), one finds that

$$|I''| \;<\; 20e^{-.01R^2/\ell^2} + 2\ell/R \;. \tag{B.11}$$

Now if $R/\ell > 30$ (and it will be assumed that this is the case), then

$20e^{-.01R^2/\ell^2} < \ell/R$. Hence, from equation (B.11), one has

$$|I''| < \frac{3\ell}{R} \quad . \tag{B.12}$$

From equation (4.200) one finds that $I = -i/2$. Thus if $R/\ell >>> 1$, the magnitude of I'' is negligible when compared to that of I , and therefore the assertion on extending the limits of the integral is proved.

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26. Stratton, loc. cit., p. 409.