THE CONJUGATE REPRESENTATION OF SIGNAL ENVELOPE

Thesis by

Theodore H. Davey

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ABSTRACT

The problem of the representation of signal envelope is treated, motivated by the classical Hilbert representation in which the envelope is represented in terms of the received signal and its Hilbert transform. It is shown that the Hilbert representation is the proper one if the received signal is strictly bandlimited but that some other filter is more appropriate in the bandunlimited case. A specific alternative filter, the conjugate filter, is proposed and the overall envelope estimation error is evaluated to show that for a specific received signal power spectral density the proposed filter yields a lower envelope error than the Hilbert filter.

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Chapter 1

Introduction

A quantity of great importance in many communications systems is the signal envelope. For example, in an amplitude modulation radio transmission the amplitude of the carrier signal is varied according to some other signal which contains the information to be transmitted. In order to retrieve the information in a usable form at the receiver we must employ some form of envelope detection. Or in an automatic gain control system, where we desire a constant power level output as the input level varies, the forward gain of the system is an inverse function of the input signal envelope.

In any information transfer there are three basic entities of concern; the sender, the transmission path or medium, and the receiver. We will be interested only in the sender and receiver and will assume perfect transmission. That is, whatever signal $x(t)$ is received is exactly the signal whose reception was intended by the sender. We further assume that in his modulation process the sender has endowed the signal envelope with all the intended information and that, whatever the situation regarding the phase of $x(t)$, the information is to be retrieved by detection of the signal envelope. Specifically the sender has generated and transmitted the signal, which we shall assume is a stationary random process,

$$x(t) = A_o(t) \cos [\omega_o t - \phi_o(t)] \qquad (1.01)$$

where the various quantities in (1.01) have the following interpretations. The carrier frequency $\omega_o = 2\pi f_o$ is assumed absolutely fixed in time. The signal envelope is $A_o(t)$ where the subscript o indicates that it is the intended envelope, the envelope which we are intended to retrieve. The phase $\phi_o(t)$, also an intended quantity, may or may not vary with time and may or may not contain a portion of the information. The various cases regarding the nature of $\phi_o(t)$, namely pure amplitude modulation, AM with phase drift, and simultaneous amplitude-phase modulation, will be discussed separately in Chapters 4 and 5, 6, and 7.

The crux of the problem is that, having received $x(t)$ perfectly, we are still unable to reconstruct the intended envelope, for in (1.01) $A_o(t)$ is not uniquely specified. That is, an infinite number of pairs $\{A,\phi\}$ satisfy (1.01) and without further information regarding the phase, or the relationship between the intended envelope and phase, any one of these pairs may qualify as the intended quantities.

The ultimate goal is the representation of $A_o(t)$ in terms of $x(t)$ where, in view of the preceding discussion, such an envelope representation must certainly depend upon the phase-envelope relationship supplied by the sender. It is interesting that several envelope representations have been advanced, but without mention of the necessity for the phase-envelope relationship.

By far the predominant approach to the representation of the envelope has been through the "analytic signal" representation. In his classic treatise [1], Rice considers signals describable as the sum of a number of constant amplitude fixed phase sinusoids and defines a unique envelope function. In a later paper [2] Dugundji introduces the concept of a pre-envelope m(t) of the signal x(t) defined in terms of the Hilbert transform.

$$m(t) \equiv x(t) + j\,\hat{x}(t) \qquad (1.02)$$

He succeeds in showing that

$$|m(t)| = +\left[x^2(t) + \hat{x}^2(t)\right]^{\frac{1}{2}} \qquad (1.03)$$

is identical with the Rice envelope where the latter is applicable, namely in the case of strictly bandlimited signals. However he goes on to assume that $|m(t)|$ is also <u>the</u> signal envelope even when the Rice definition is not applicable. In a number of other treatments [3-9] the analytic signal is used with varying degrees of emphasis on the necessity of strict bandlimitedness for exact representation. In [10] the envelope of noise is defined as $|m(t)|$ with no assumption of bandlimitedness. In [11-12] it is assumed that the correct representation is $|m(t)|$ and the effort is to find conditions on the signal x(t) for this to be true, leading to minimum phase signals. What began with Rice to be the well defined concept of envelope for a restricted class of signals has become a loosely defined term applied to all varieties of modulated signals and even to random noise [10] in which the envelope is automatically defined in terms of the Hilbert transform with little justification. The reasons given for its use are the independence of the Hilbert filter on the input statistics, that it yields the correct $A_o(t)$ in the strictly narrowband case, and that it is at least a unique representation.

In this thesis we instead seek an envelope representation in terms of $x(t)$ in which the emphasis is placed upon the intent of the sender. The representation to be used is similar to (1.03) but with $\hat{x}(t)$ replaced by the signal $y(t)$ which is obtained from $x(t)$ by linear filtering, so that

$$A_y(t) = [x^2 + y^2]^{\frac{1}{2}} \tag{1.04}$$

Ultimately we desire to find $y(t)$ such that the error, appropriately defined, between the output envelope $A_y(t)$ and the intended envelope $A_o(t)$ is a minimum. This optimization with respect to $A_o(t)$ has not been solved however and we are only able to propose an alternative scheme to the Hilbert representation based upon optimization of $y(t)$. It will be shown that this alternative representation results in lower envelope error than the classical Hilbert representation in many practical cases.

The basic purpose of the thesis is to answer the following three questions for several different types of signal modulation.

1. What information (phase-envelope relationship) is sufficient to remove the ambiguity concerning $A_o(t)$ and $\phi_o(t)$ in (1.01)?
2. Given the information in 1. for a specific input case, is the Hilbert representation the best representation of the envelope and if not, can we find another representation which results in lower envelope estimation error?

3. What are the estimation errors of the Hilbert representation, of the alternative representation, and how do the errors compare as functions of the input signal characteristics?

The thesis is arranged so that the detailed calculations, integral evaluations and algebraic manipulations are isolated from the main body in appendices which are indexed according to the chapters to which they pertain. A knowledge of the elementary theory of stochastic processes is assumed.

Chapter 2

Signal Characterization

2.1 Statistical Properties of the Received Signal

The concept of signal envelope is intuitively linked with the notion of a narrowband waveform. One feels that the envelope can be identified, say on an oscilloscope trace, when through squinted eyes one can ignore the instantaneous variation of the signal and observe that the signal peaks follow a smooth curve, which curve we label the modulation or envelope of the signal. The difficulty in representing the envelope analytically in spite of our obvious intuitive grasp arises out of the condition of squintedness which obscures almost all of the signal behavior, and from our ignorance of just which of the many possible smooth curves passing through the signal peaks should actually represent the envelope.

The problem to which we address ourselves is this: a wide-sense stationary signal $x(t)$ has been received, is exactly that which the sender intended to send (noise-free transmission), and was generated by him through an amplitude (and possibly phase) modulation process such that $x(t)$ has the following mathematical representation

$$x(t) = A_o(t) \cos [\omega_o t - \phi_o(t)] \tag{2.01}$$

The function $A_o(t)$ is the intended envelope and contains all the information intended by the sender. The envelope is a strictly positive signal, except in the trivial case when $x(t)$ is identically zero, and hence the zero crossings of $x(t)$ are those of the cosine factor. The function $\phi_o(t)$ is the <u>intended phase</u> and may contain all, some, or none of the information. The phase may for example be a random variable in which case ϕ_o contains no information and $x(t)$ is ideally amplitude-modulated. Or $\phi_o(t)$ may be a random process independent of $A_o(t)$ (for example due to phase drift in the modulation oscillator) in which case $\phi_o(t)$ interferes with the detection of $A_o(t)$ but contains no intended information. Finally the sender may have generated the phase modulation through some functional dependence on $A_o(t)$ [11-13] in which case $\phi_o(t)$ contains some or all of the intended information and we are faced with the situation of simultaneous amplitude and phase modulation. Whatever the situation regarding the phase, the problem is to operate on $x(t)$ in such a way that we obtain as much of the intended information contained in $A_o(t)$ as possible.

We observe that by itself knowledge of the signal $x(t)$, no matter how exact, is not sufficient to determine $A_o(t)$, for in (2.01) $A_o(t)$ and $\phi_o(t)$ are not unique, given only $x(t)$. That is $x(t)$ may be modeled in the form (2.01) by an infinite number of pairs $A(t)$, $\phi(t)$. This is the manifestation of our ignorance regarding the correct smooth curve joining the peaks of $x(t)$. We must therefore by supplied by the sender with further information regarding the relationship between $A_o(t)$ and $\phi_o(t)$ to resolve the ambiguity.

The representation (2.01) is inconvenient for mathematical operations since $x(t)$ appears as a nonlinear combination of the functions of interest. A more convenient representation of $x(t)$ is

$$x(t) = x_c(t) \cos \omega_o t + x_s(t) \sin \omega_o t \qquad (2.02)$$

in which we identify the relationships

$$x_c \equiv A_o \cos \phi_o \quad , \qquad x_s \equiv A_o \sin \phi_o \qquad (2.03)$$

and conversely

$$A_o \equiv + [x_c^2 + x_s^2]^{\frac{1}{2}} \qquad (2.04)$$

where the + sign emphasizes the strictly positive nature of $A_o(t)$. In the representation (2.02) $x(t)$ is a linear combination of the functions $x_c(t)$ and $x_s(t)$, which functions will be referred to as the in-phase and quadrature components of $x(t)$ respectively. Most of the mathematical operations will be performed in terms of x_c and x_s and the results translated into corresponding results for A_o and ϕ_o. Note that in (2.02) there is present the same nonuniqueness concerning the components of $x(t)$ as exists concerning the envelope and phase in (2.01). Specification of the relationship between A_o and ϕ_o amounts to specification of x_c and x_s. The representations (2.01) and (2.02) have been used by Rice [1], and later by Middleton [5] and many others [8-9],[14]. The statement is usually made in connection with these representations that x_c, x_s, A_o, and ϕ_o are all assumed to be "slowly varying" with respect to the carrier frequency f_o. What precisely is meant, or may be meant, by slowly varying will be discussed later on.

If (2.01) and (2.02) are to represent a random process which is stationary the sender is not at perfect freedom to specify $\{A_o, \phi_o\}$ (or $\{x_c, x_s\}$). The condition of stationarity imposes certain restrictions on the signals. The expected value of $x(t)$, denoted by $\langle x(t)\rangle$, is

$$\langle x(t)\rangle = \langle x_c(t)\rangle \cos \omega_o t + \langle x_s(t)\rangle \sin \omega_o t$$

For $\langle x(t)\rangle$ to be independent of t we must have the expected values of $x_c(t)$ and $x_s(t)$ equal to zero since cos and sin are orthogonal functions. This also implies that $x(t)$ is zero mean.

$$\langle x_c(t)\rangle = \langle x_s(t)\rangle = 0 \Rightarrow \langle x(t)\rangle = 0 \qquad (2.05)$$

Computation of the autocorrelation function $R_x(\tau) = \langle x_1 x_2\rangle$, where $x_1 = x(t)$ and $x_2 = x(t + \tau)$, leads to

$$R_x(\tau) = \frac{1}{2}\left[\langle x_{c1}x_{c2}\rangle + \langle x_{s1}x_{s2}\rangle\right] \cos \omega_o\tau + \frac{1}{2}\left[\langle x_{c1}x_{s2}\rangle - \langle x_{s1}x_{c2}\rangle\right] \sin \omega_o\tau$$

$$+ \frac{1}{2}\left[\langle x_{c1}x_{c2}\rangle - \langle x_{s1}x_{s2}\rangle\right] \cos \omega_o(2t + \tau) + \frac{1}{2}\left[\langle x_{c1}x_{s2}\rangle + \langle x_{s1}x_{c2}\rangle\right] \sin \omega_o(2t+\tau)$$

For $R_x(\tau)$ to be independent of t the coefficients of the sinusoidal functions of t must vanish. Defining two new correlation functions, namely the auto- and crosscorrelation functions of the components of $x(t)$, this condition is

$$\langle x_{c1}x_{c2}\rangle = \langle x_{s1}x_{s2}\rangle \equiv R_c(\tau) \tag{2.06a}$$

$$\langle x_{c1}x_{s2}\rangle = -\langle x_{s1}x_{c2}\rangle \equiv R_s(\tau) \tag{2.06b}$$

From these relations we see that the autocorrelation of the components is even while the crosscorrelation between them is odd.

$$R_c(\tau) = R_c(-\tau) \quad ; \quad R_s(\tau) = -R_s(-\tau) \tag{2.07}$$

Using (2.06) the autocorrelation function of the entire process $x(t)$ is

$$R_x(\tau) = R_c(\tau) \cos \omega_o\tau + R_s(\tau) \sin \omega_o\tau \tag{2.08}$$

so that $R_c(\tau)$ and $R_s(\tau)$ are the in-phase and quadrature components of $R_x(\tau)$. We note in (2.08) that $R_x(\tau)$ is an even function as is required of the autocorrelation of a real signal.

The power spectral density (PSD) of the components will be denoted by $S_c(f)$ and is related to $R_c(\tau)$ by the Fourier Transform (FT).

$$S_c(f) \equiv \int_{-\infty}^{\infty} R_c(\tau)e^{-j2\pi f\tau}d\tau = 2\int_{0}^{\infty} R_c(\tau) \cos 2\pi f\tau \; d\tau \tag{2.09a}$$

We may also define the cross spectral density between the components as $S_s(f)$, the FT of $R_s(\tau)$. Since $R_s(\tau)$ is odd its standard FT would be imaginary. In order to deal with real functions as spectral densities we insert a factor $j = \sqrt{-1}$ in our definition of $S_s(f)$.

$$S_s(f) \equiv j \int_{-\infty}^{\infty} R_s(\tau)\, e^{-j2\pi f\tau} d\tau = 2\int_{0}^{\infty} R_s(\tau) \sin 2\pi f\tau \, d\tau \qquad (2.09b)$$

The expressions of the correlation functions in terms of the spectral densities follow from (2.09) and the properties of the FT.

$$R_c(\tau) = \int_{-\infty}^{\infty} S_c(f) e^{j2\pi f\tau} df \qquad (2.10a)$$

$$R_s(\tau) = -j\int_{-\infty}^{\infty} S_s(f) e^{j2\pi f\tau} df \qquad (2.10b)$$

From (2.09) it is clear that $S_c(f)$ is an even real function while $S_s(f)$ is an odd real function. Finally the PSD of the entire process $x(t)$ is obtained from the combination of (2.08) and (2.09).

$$S_x(f) = \frac{1}{2}\left[S_c(f-f_o) - S_s(f-f_o)\right] + \frac{1}{2}\left[S_c(f+f_o) + S_s(f+f_o)\right] \quad (2.11)$$

2.2 Separation of Signal Components

From (2.04) we see that the envelope would be specified if the components could be calculated from knowledge of $x(t)$. But we have also remarked that the components are not uniquely specified by (2.02), although they are completely determined by the sender, and that further criteria must be supplied along with $x(t)$ to separate them correctly. As a crude example of such a criterion consider that $x_c(t)$ and $x_s(t)$ are each "slowly varying" [20]. More specifically, multiply (2.02) first by $\cos \omega_o t$ and then by $\sin \omega_o t$ and integrate over one period of the carrier, i.e. for $1/f_o$ seconds. If by slowly varying we mean that $x_c(t)$ and $x_s(t)$ may be taken as constant in the interval $1/f_o$ with negligible (in some sense) error then the results of our integrations are

$$x_c(t) \approx 2f_o \int_{t-1/f_o}^{t} x(u) \cos \omega_o u \, du \tag{2.12a}$$

$$x_s(t) \approx 2f_o \int_{t-1/f_o}^{t} x(u) \sin \omega_o u \, du \tag{2.12b}$$

Of course error has crept in here as we have only performed an approximate integration. Furthermore, and more fundamentally, we are not even certain that these components are the ones intended by the sender since we do not know that he used this "slowly varying" condition in generating them.

A different approach develops if we consider the value of $s(t)$ in (2.02) at two different times t and $t-\varepsilon$. If by "slowly varying" we now mean that $x_c(t-\varepsilon) \approx x_c(t)$ and $x_s(t-\varepsilon) \approx x_s(t)$ with negligible (again) error the resulting equations may be solved approximately for the components

$$x_c(t) \approx x(t) \cos \omega_o t - \left[\frac{x(t) \cos \omega_o \varepsilon - x(t-\varepsilon)}{\sin \omega_o \varepsilon}\right] \sin \omega_o t \quad (2.13a)$$

$$x_s(t) \approx \left[\frac{x(t) \cos \omega_o \varepsilon - x(t-\varepsilon)}{\sin \omega_o \varepsilon}\right] \cos \omega_o t + x(t) \sin \omega_o t \quad (2.13b)$$

If ε is small then the components are required to be approximately constant over a shorter interval than in the preceding approach involving integration over $1/f_o$. Indeed if ε is allowed to approach zero (2.13) becomes

$$x_c(t) \approx x(t) \cos \omega_o t - \frac{\dot{x}(t)}{\omega_o} \sin \omega_o t \quad (2.14a)$$

$$x_s(t) \approx \frac{\dot{x}(t)}{\omega_o} \cos \omega_o t + x(t) \sin \omega_o t \quad (2.14b)$$

It must be emphasized again that (2.14) may or may not represent the intent of the sender, depending upon his rule for relating A_o and ϕ_o (or x_c and x_s). Once the rule is known the errors in this envelope detector scheme can be evaluated and it can be compared to other schemes. If the components in (2.14) are combined as indicated in (2.04) and the result denoted by $A_d(t)$ (for differentiator envelope since the result is in general not equal to $A_o(t)$) we have

$$A_d(t) = \left\{x^2(t) + \left[\frac{\dot{x}(t)}{\omega_o}\right]^2\right\}^{\frac{1}{2}} \quad (2.15)$$

Of course we are assuming in this treatment that the derivative of $x(t)$ exists (which it does not, for example, in the case of single-RC noise).

2.3 The Conjugate Signal

While the detector indicated by (2.15) is one which could conceivably be constructed, the preceding development was not so much an attempt to derive a correct representation of the envelope as a motivation for the present section in which we introduce the concept of the conjugate signal [9] (not to be confused with the complex conjugate). In (2.14) for example, the components x_c and x_s are expressed in terms of the received signal x and another signal $-\dot{x}/\omega_o$. Solving (2.14) for $-\dot{x}/\omega_o$ we have

$$-\frac{\dot{x}(t)}{\omega_o} = x_c(t)\sin\omega_o t - x_s(t)\cos\omega_o t \tag{2.16}$$

This may be written in terms of the envelope and phase, $A_d(t)$ and $\phi_d(t)$, using (2.03), (2.15), and (2.16) as

$$-\frac{\dot{x}(t)}{\omega_o} = A_d(t)\sin\left[\omega_o t - \phi_d(t)\right] \tag{2.17}$$

We are led to generalize this result and to identify what we shall call the <u>conjugate signal</u> to $x(t)$, namely

$$x_o(t) \equiv A_o(t)\sin\left[\omega_o t - \phi_o(t)\right] \tag{2.18}$$

On expanding the sinusoid this may also be written

$$x_o(t) \equiv x_c(t) \sin \omega_o t - x_s(t) \cos \omega_o t \qquad (2.19)$$

The conjugate signal, like the received signal, is represented in terms of the intended quantities $\{A_o, \phi_o\}$ or $\{x_c, x_s\}$. The sender, having specified $A_o(t)$ and $\phi_o(t)$, has also specified $x_o(t)$. But we at the receiver have only $x(t)$ and not $x_o(t)$. However we shall see later that if we are given some apriori information about $x_o(t)$ and its relationship to $x(t)$ we are in a good position to represent the envelope $A_o(t)$ correctly. The relationships inverse to (2.02) and (2.19) are

$$x_c(t) = x(t) \cos \omega_o t + x_o(t) \sin \omega_o t \qquad (2.20a)$$

$$x_s(t) = x(t) \sin \omega_o t - x_o(t) \cos \omega_o t \qquad (2.20b)$$

Substitution of (2.20) into (2.04) for the intended envelope yields

$$A_o(t) = \left[x_c^{\,2}(t) + x_s^{\,2}(t)\right]^{\frac{1}{2}} = \left[x^2(t) + x_o^{\,2}(t)\right]^{\frac{1}{2}} \qquad (2.21)$$

This is a very interesting and important result and is the basis of what we shall call the radius detector approach to envelope representation (or demodulation). Originally in (2.04) the envelope was represented in terms of the "slowly varying" quantities $x_c(t)$, $x_s(t)$ and it was natural that the envelope be a slowly varying quantity. However (2.21) represents the slowly varying envelope in terms of rapidly varying quantities, the narrowband input $x(t)$ and its conjugate $x_o(t)$. If it is possible to physically operate on $x(t)$ to derive $x_o(t)$ then the envelope $A_o(t)$ may be detected exactly from $x(t)$. However if it is not possible to derive $x_o(t)$ exactly, but only an estimate, say $s(t)$, of $x_o(t)$, then of course we are able to reconstruct only an approximation, say $A_s(t)$, to $A_o(t)$. Our quest in that case would be for the approximation to $x_o(t)$ which results in the best approximation $A_s(t)$ to the intended envelope $A_o(t)$. Notice that if (2.21) were satisfied the envelope would be obtained without any post-detection low pass filtering, a consequence of the conjugacy property. Also, inasmuch as the sum of two squares is nonnegative, the square root function introduces no ambiguity.

Both the received signal $x(t)$ and its conjugate $x_o(t)$ are known to the sender since he generates both $A_o(t)$ and $\phi_o(t)$. Only the received signal is available to the receiver but we suspect that some information about $x_o(t)$ may be derived from $x(t)$ due to their mutual dependence upon $A_o(t)$ and $\phi_o(t)$. That is, a transformation of $x(t)$ exists which provides at least an approximation of $x_o(t)$. From the form of (2.20) we are motivated to restrict the transformation to be a linear one and shall model the transformation in the form of a linear filter with transfer function $G(f)$. We proceed now to the study of the properties of $x_o(t)$ and then to the appropriate forms for the linear filter in Chapter 3.

2.4 Statistical Properties of the Conjugate Signal

Development of the properties of the conjugate signal proceeds most readily in terms of the in-phase and quadrature components of $x(t)$. The autocorrelation function of $x_o(t)$ is obtained from (2.06) and (2.19).

$$R_{x_o}(\tau) \equiv \langle x_{o1} x_{o2} \rangle = R_c(\tau) \cos \omega_o \tau + R_s(\tau) \sin \omega_o \tau = R_x(\tau) \tag{2.22}$$

The conjugate signal has an autocorrelation function, and hence a PSD, identical with those of $x(t)$. The crosscorrelation function between $x(t)$ and $x_o(t)$ is also easily obtained.

$$R_{xx_o}(\tau) \equiv \langle x_1 x_{o2} \rangle = R_c(\tau) \sin \omega_o \tau - R_s(\tau) \cos \omega_o \tau \tag{2.23}$$

As was mentioned following (2.08) $R_c(\tau)$ and $R_s(\tau)$ represent the in-phase and quadrature components of $R_x(\tau)$ [and hence of $R_{x_o}(\tau)$]. Equation (2.23) shows that the crosscorrelation function $R_{xx_o}(\tau)$ is the conjugate function to $R_x(\tau)$. We also note that $R_{xx_o}(\tau)$ is an odd function which implies

$$R_{xx_o}(-\tau) = - R_{xx_o}(\tau) \Longrightarrow R_{xx_o}(0) = 0 \tag{2.24}$$

and therefore that $x(t)$ and $x_o(t)$ are uncorrelated random variables. It should be noted that they are not however samples of uncorrelated random processes. The spectral densities corresponding to (2.22) and (2.23) are

$$S_{x_o}(f) = S_x(f) = \frac{1}{2}\left[S_c(f-f_o)-S_s(f-f_o)\right] + \frac{1}{2}\left[S_c(f+f_o) + S_s(f+f_o)\right] \quad (2.25a)$$

$$S_{xx_o}(f) = \frac{1}{2j}\left[S_c(f-f_o)-S_s(f-f_o)\right] - \frac{1}{2j}\left[S_c(f+f_o) + S_s(f+f_o)\right] \quad (2.25b)$$

Finally the crosscorrelation between $x_o(t)$ and $x(t)$ is given by

$$R_{x_ox}(\tau) \equiv \langle x_{o1}x_2\rangle = R_{xx_o}(-\tau) = - R_{xx_o}(\tau) \quad (2.26)$$

The properties of $x(t)$ and $x_o(t)$ are now well characterized and we are ready to proceed with the derivation of the appropriate linear filters for the estimation of $x_o(t)$ from $x(t)$. Before doing so however it is appropriate in this chapter on signal characterization to state a few of the results of generalized harmonic analysis which will be of use later on.

2.5 Generalized Signal Characterization

In common amplitude modulation (AM) transmission systems a carrier component is transmitted along with the information-bearing envelope-modulated signal. This amounts to a nonzero mean value for the envelope in (2.01). In these AM systems the phase term is constant with time and we write the signal representation (2.01) to emphasize the presence of both carrier and envelope modulated terms,

$$x(t) = [\sigma_c + a(t)] \cos [\omega_o t - \phi_o] = x_1(t) + x_2(t) \quad (2.27)$$

where we have taken

$$A_o(t) = \sigma_c + a(t) \tag{2.28a}$$

$$x_1(t) = \sigma_c \cos [\omega_o t - \phi_o] \tag{2.28b}$$

$$x_2(t) = a(t) \cos [\omega_o t - \phi_o] \tag{2.28c}$$

The mean value of $A_o(t)$ is σ_c and the mean value of $a(t)$ is zero. Although the phase ϕ_o in (2.27) is a random variable we may still view $x(t)$ as a mixed process. That is $x(t)$ is the sum of a deterministic process $x_1(t)$ and a random process $x_2(t)$. The entire future of the process $x_1(t)$ is determined by a single observation of the peak value σ_c and a zero crossing $\omega_o t - \phi_o = (2n+1)\pi$. The effect of this deterministic component in the PSD of $x(t)$ is the introduction of delta functions at $\pm f_o$ corresponding to infinite power density at the carrier frequency. Of course in a strictly realistic sense the carrier contribution to the PSD is spread over a finite though narrow band of frequencies since the frequency sources used are not perfectly stable for infinite time. However the representation by a delta function in the spectrum is a valuable idealization for engineering use.

In later chapters we shall have occasion to consider the ratio of two different expressions containing such delta functions. To avoid conceptual difficulties associated with such quantitites and to prepare the way for the solution of the optimum filter problem we digress briefly here to introduce some of the concepts of generalized harmonic analysis. In doing so we are interested only in the point of view and not in rigorous mathematical proofs. Such rigor is available in the standard references [14-16].

2.5.1 Generalized Fourier Transforms

Signals which go on forever in time but have finite power possess in general no Fourier transform. Such signals $x(t)$ are characterized by the relation

$$0 < \lim_{T\to\infty} \int_{-T}^{T} |x(u)|^2 du < \infty$$

It is possible to define for such signals a generalized Fourier transform (GFT) which is given in terms of a Stieltjes integral.

$$x(t) = \int_{-\infty}^{\infty} e^{j2\pi ft} d\Lambda(f) \qquad (2.29)$$

The function $\Lambda(f)$ is the GFT of $x(t)$. The inversion relation is [14]

$$\Lambda(f_1) - \Lambda(f_2) = \int_{-\infty}^{\infty} \frac{e^{-j2\pi f_1 u} - e^{-j2\pi f_2 u}}{-j2\pi u} x(u) du \qquad (2.30)$$

or more strictly [15]

$$\frac{\Lambda(f_1+) + \Lambda(f_1-)}{2} - \frac{\Lambda(f_2+) + \Lambda(f_2-)}{2} = \lim_{T\to\infty} \int_{-T}^{T} \frac{e^{-j2\pi f_1 u} - e^{-j2\pi f_2 u}}{-j2\pi u} x(u)\,du \tag{2.31}$$

Equation (2.30), or (2.31), determines $\Lambda(f)$ to within a constant. $\Lambda(f)$ is usually normalized so that $\Lambda(-\infty) = 0$. The autocorrelation of $x(t)$ also possesses a GFT, $\Sigma_x(f)$, which is known as the spectral distribution function (SDF) of $x(t)$.

$$R_x(\tau) = \int_{-\infty}^{\infty} e^{j2\pi f\tau} d\Sigma_x(f) \tag{2.32}$$

$\Sigma_x(f)$ is given in terms of $R_x(\tau)$ by an inversion relation similar to (2.31). The GFT and SDF are related by the surprisingly simple formula

$$E\left\{|d\Lambda(f)|^2\right\} = d\Sigma_x(f) \tag{2.33}$$

from which it is clear that $\Sigma_x(f)$ is a nondecreasing function.

If the derivative of $\Sigma_x(f)$ exists we may write (2.32) as an ordinary Riemann integral

$$R_x(\tau) = \int_{-\infty}^{\infty} e^{j2\pi f\tau} \Sigma_x'(f)\, df$$

and we see that $\Sigma_x'(f)$ is the PSD of $x(t)$, namely $S_x(f)$. In this case we may write (2.33) in the alternative form

$$E\left\{\left|\Lambda(f_1) - \Lambda(f_2)\right|^2\right\} = \int_{f_1}^{f_2} S_x(f)\, df$$

Furthermore, if and only if $x(t)$ is a stationary process then $\Lambda(f)$ is a process with uncorrelated increments. That is, if $f_1 > f_2 \geq f_3 > f_4$ we have

$$E\left\{\left[\Lambda(f_1) - \Lambda(f_2)\right]\left[\Lambda^*(f_3) - \Lambda^*(f_4)\right]\right\} = 0$$

or

$$E\left\{d\Lambda(f_1)\, d\Lambda^*(f_2)\right\} = \begin{cases} 0 \quad , & f_1 \neq f_2 \\ d\Sigma(f_2), & f_1 = f_2 \end{cases} \tag{2.34}$$

If the random process $x(t)$ is the input to a linear filter with transfer function $G(f)$ in the frequency domain and if the output is $y(t)$ with GFT $\Omega(f)$, then

$$y(t) = \int_{-\infty}^{\infty} e^{j2\pi ft}\, d\Omega(f) \tag{2.35}$$

$$= \int_{-\infty}^{\infty} e^{j2\pi ft} G(f)\; d\Lambda(f) \tag{2.36}$$

from which we deduce

$$d\Omega(f) = G(f)d\Lambda(f) \tag{2.37}$$

If the SDF of $y(t)$ is denoted by $\Sigma_y(f)$ then from (2.33) and (2.37)

$$d\Sigma_y(f) = E\left\{|d\Omega(f)|^2\right\} = |G(f)|^2 E\left\{|d\Lambda(f)|^2\right\} = |G(f)|^2 d\Sigma_x(f) \tag{2.38}$$

The autocorrelation function of $y(t)$ is given by

$$R_y(\tau) \equiv \int_{-\infty}^{\infty} e^{j2\pi f\tau} d\Sigma_y(f) = \int_{-\infty}^{\infty} e^{j2\pi f\tau} |G(f)|^2 d\Sigma_x(f) \tag{2.39}$$

If $\Lambda_o(f)$ is defined as the GFT of the conjugate process $x_o(t)$ we may identify the cross SDF between $x(t)$ and $x_o(t)$ as $\Sigma_{xx_o}(f)$ where

$$R_{xx_o}(\tau) = \int_{-\infty}^{\infty} e^{j2\pi f\tau} d\Sigma_{xx_o}(f) \tag{2.40a}$$

and

$$d\Sigma_{xx_o}(f) = E\left\{d\Lambda^*(f)d\Lambda_o(f)\right\} \tag{2.40b}$$

If $\Sigma_{xx_o}(f)$ possesses a derivative we may identify $\Sigma'_{xx_o}(f) = S_{xx_o}(f)$, the cross PSD between $x(t)$ and $x_o(t)$. Equation (2.22) implies that

$$d\Sigma_{x_o}(f) = d\Sigma_x(f)$$

If we define the SDF's $\Sigma_c(f)$ and $\Sigma_s(f)$ for the in-phase and quadrature components in the obvious way to correspond to $R_c(\tau)$ and $R_s(\tau)$, then (2.25) may be written in terms of the SDF's

$$\Sigma_{x_o}(f) = \Sigma_x(f) = \frac{1}{2}\left[\Sigma_c(f-f_o)-\Sigma_s(f-f_o)\right] + \frac{1}{2}\left[\Sigma_c(f+f_o) + \Sigma_s(f+f_o)\right] \quad (2.41a)$$

$$\Sigma_{xx_o}(f) = \frac{1}{2j}\left[\Sigma_c(f-f_o)-\Sigma_s(f-f_o)\right] - \frac{1}{2j}\left[\Sigma_c(f+f_o) + \Sigma_s(f+f_o)\right] \quad (2.41b)$$

2.5.2 Spectral Decomposition

We have seen in the case where $\Sigma(f)$ is everywhere absolutely continous with a finite derivative that $\Sigma'(f)$ is identical with $S(f)$ and all the analysis can be carried through without recourse to generalized harmonic analysis or the theory of Stieltjes integrals. However many important practical cases occur in which these conditions on $\Sigma(f)$ are not satisfied, at least on f-sets of measure zero, and it is helpful to decompose the f-axis into sets which are determined by the continuity properties of Σ. We shall outline very briefly the idea and a few results of the spectral decomposition theory to be used in later chapters. Again no great rigor is intended, the subject being well documented [15].

Let the f-axis be decomposed into n disjoint f-sets $D_1, D_2, \ldots, D_n$ in the interval $(-\infty, +\infty)$ whose union is the whole f-axis. If $x(t)$ is a stationary random process with GFT $\Lambda(f)$ and SDF $\Sigma_x(f)$ then it is possible to exhibit $x(t)$ as a sum of mutually orthogonal stationary processes $x_1(t), x_2(t),\ldots, x_n(t)$ whose individual spectral distributions are confined to the sets $D_1, D_2,\ldots, D_n$ respectively.

The decomposition of particular interest here is that into two disjoint sets on each of which $\Sigma_x(f)$ exhibits behavior peculiar to that set. If Σ_1, Σ_2 are the distribution functions on D_1, D_2 then we may write

$$\Sigma_x(f) = \Sigma_1(f) + \Sigma_2(f) \tag{2.42}$$

where Σ_1 is the discontinuous part or "jump function" of Σ_x and Σ_2 is the absolutely continous part of Σ_x. Usually D_1 is a set of measure zero. We may write for the random process $x(t)$ as in (2.27)

$$x(t) = x_1(t) + x_2(t) \tag{2.43}$$

where $x_1(t)$ and $x_2(t)$ are mutually orthogonal. That is,

$$E\left\{x_1(t)x_2^*(t+\tau)\right\} \doteq 0 \text{ , all } \tau \tag{2.44}$$

The constituent process $x_1(t)$ is easily written as a simple sum from (2.29). If $f_1, f_2,\ldots, f_n$ are the points of discontinuity of $\Sigma_x(f)$, supposed continuous from the right at these points, then

$$x_1(t) = \int_{-\infty}^{\infty} e^{j2\pi ft} d\Lambda_1(f) = \sum_{j=1}^{n} e^{j2\pi ft}\left[\Lambda(f_j) - \Lambda(f_j-)\right] \tag{2.45}$$

where $[\Lambda(f_j) - \Lambda(f_j-)]$ is the amount of the jump in $\Lambda_1(f)$ at f_j and the random variables $[\Lambda(f_j) - \Lambda(f_j-)]$ form an orthogonal set. For example if $x_1(t) = \sigma_c \cos(\omega_o t - \phi_o)$ then

$$\Lambda(f_o) - \Lambda(f_o-) = \frac{\sigma_c}{2} e^{j\phi_o}, \quad \Lambda(-f_o) - \Lambda(-f_o-) = \frac{\sigma_c}{2} e^{-j\phi_o}$$

and

$$d\Sigma_1(f) = \frac{\sigma_c^2}{4}$$

on the f-set $\{-f_o, +f_o\}$.

The constituent $x_2(t)$ may be written

$$x_2(t) = \int_{-\infty}^{\infty} e^{j2\pi ft} d\Lambda_2(f) \tag{2.46}$$

where

$$\Lambda(f) = \Lambda_1(f) + \Lambda_2(f) \tag{2.47}$$

with

$$E\left\{|d\Lambda_{1,2}(f)|^2\right\} = d\Sigma_{1,2}(f) \tag{2.48}$$

This completes the characterization of the signal processes and the extension to signals with infinite power densities on sets of measure zero.

2.6 Singular and Regular Processes

A random process $s(t)$ with PSD $S(f)$ is classified as either singular or regular according as the HT of $\ln S(f)$ diverges or converges. That is, defining the quantity J,

$$J = \mathcal{H}\{\ln S(f)\} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\ln S(u)}{f - u}\, du \tag{2.49}$$

we say that $s(t)$ is regular if and only if $|J| < \infty$. Realizing that $S(f)$ is an even function (2.49) may be written

$$J = 2f\int_{0}^{\infty} \frac{\ln S(u)}{f^2 - u^2}\, du \tag{2.50}$$

To emphasize that the convergence of J is dependent upon the behavior of $\ln S(f)$ as $f \to \infty$ and not the pole at $|u| = |f|$ in (2.50) we may define a new integral

$$J' = \int_{0}^{\infty} \frac{\ln S(u)}{1 + u^2}\, du \tag{2.51}$$

and remark that regularity of $s(t)$ depends upon convergence of J'. This is the regularity criterion usually given [5], [17].

Generally $S(f)$ must decay more slowly than $\exp(-\alpha|f|)$ as $|f| \to \infty$ in order for $s(t)$ to be regular. For example

$S(f) = \exp(-\alpha|f|)$ and $\exp(-\alpha f^2)$ both correspond to singular processes. Of interest later on will be processes whose PSD is zero over a set of nonzero measure (for example strictly bandlimited processes). From (2.51) it is clear that such processes are singular. In contrast all processes with PSD's which are rational functions of frequency are regular processes.

It may be shown that regular processes correspond to truly random processes, and singular processes to quasi-random or deterministic processes. It is a fundamental result that a singular process whose past is known can be exactly predicted arbitrarily far into the future [5], [15]. Such prediction is not possible for regular processes, those which are actually met in practice.

Chapter 3

Envelope Models

3.1 Radius Detector Representation

We saw in Chapter 2 that the introduction of the conjugate signal $x_o(t)$ allows the intended envelope $A_o(t)$ to be represented directly in terms of $x_o(t)$ and the narrowband input $x(t)$ [see (2.21)]. It was remarked that if the conjugate signal could be constructed from knowledge of $x(t)$ and some information about the intended phase-envelope relationship, then the intended envelope $A_o(t)$ could be computed directly from the input. This approach to the representation (or demodulation) of the envelope will be called the radius detector. The name is motivated by the form of (2.21) in which A_o is the length of the vector $x_c + jx_s$ and also of the vector $x + jx_o$.

In later chapters we shall find that, even with perfect knowledge of the intended phase-envelope relationship, it will be possible to construct $x_o(t)$ exactly for only a limited class of input processes, a singular class which in fact will not be encountered in any practical situation. That is, in any practical situation the radius detector will not produce $A_o(t)$ but some other output, say $A_s(t)$, where the envelope representation we shall pursue has the form

$$A_s(t) = + \left[x^2(t) + s^2(t)\right]^{\frac{1}{2}} \tag{3.01}$$

and $s(t)$ is the output of the linear filter $G(f)$ with $x(t)$ as input. The radius detector (3.01) has the schematic form shown in

Figure 1.

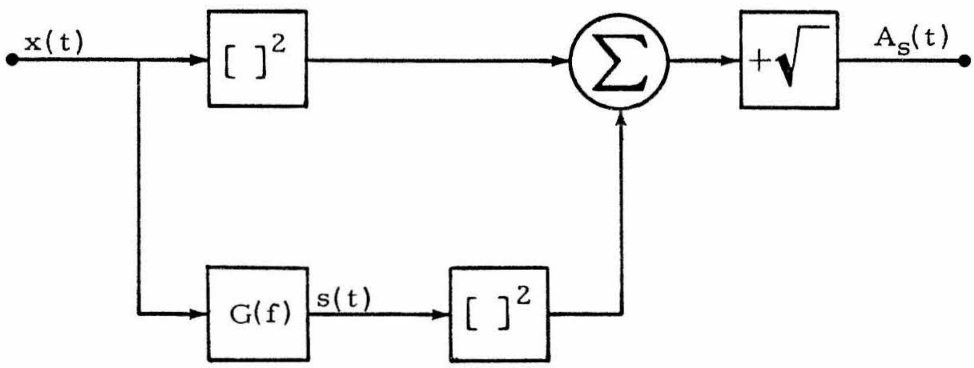


Figure 1

Radius Detector Envelope Representation

Obviously the ultimate design goal is to construct G(f) so that the error, defined in some meaningful way, between the represented, or detected, envelope A_s and that intended, A_o, is a minimum. Unfortunately the nonlinear dependence of such an error measure upon the filter transfer function renders the solution of this problem beyond the resources of the author. The best solution, in this overall envelope error sense, is beyond our reach but a good solution is possible based on the following qualitative reasoning. It is desired to minimize the "distance" between A_o and A_s. From (2.21) and (3.01) for $A_o(t)$ and $A_s(t)$ it seems probable that this "distance" is related in some monotonic way to a "distance" between $x_o(t)$ and s(t). We shall attempt a solution based on an optimization of G(f) to minimize the "distance" between $x_o(t)$ and $s(t)$ and then apply an overall envelope error criterion to evaluate this solution in comparison with competing representations. Specifically we will obtain $G(f)$ which minimizes the mean square (MS) error between x_o and s. This specific filter solution will be called the conjugate filter, denoted by $C(f)$, and its output, the MS estimate of $x_o(t)$, will be denoted by $y(t)$.

The radius detector representation (3.01) is really a generalization of the Hilbert representation used by many authors [2-13] for the envelope. In that representation $s(t)$ is taken to be $\hat{x}(t)$, the Hilbert transform process, where

$$\hat{x}(t) \equiv \mathcal{H}\{x(t)\} \equiv \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(u)}{t-u} \, du \tag{3.02}$$

and the resultant envelope representation is

$$A_{\hat{x}}(t) = + \left[x^2(t) + \hat{x}^2(t) \right]^{\frac{1}{2}} \tag{3.03}$$

The filter transfer function which yields $\hat{x}(t)$ is

$$H(f) = -j \operatorname{sgn}(f) \tag{3.04}$$

independent of the properties of the received signal. We shall find that this filter **yields** $x_o(t)$ exactly, but only for the previously cited singular case, and that in a practical situation the conjugate representation

$$A_y(t) = +\left[x^2(t) + y^2(t)\right]^{\frac{1}{2}} \tag{3.05}$$

will actually yield the lower overall envelope error.

Both realizable and unrealizable filters will be considered for a number of input phase-envelope relationships. The results of the investigation for unrealizable filters will provide a direct comparison between the conjugate representation and the classical Hilbert representation for a specific received signal case. The results for realizable filters will indicate what may be expected in an actual radius detector implementation.

3.2 Filter Optimization for the Conjugate Output

Let the input to the linear filter $G(f)$ be $x(t)$ and the output $s(t)$. The filter output is, using the notation of Section 2.5

$$s(t) = \int_{-\infty}^{\infty} e^{j2\pi\lambda t} G(\lambda)\, d\Lambda(\lambda) \tag{3.06}$$

where $\Lambda(f)$ is the GFT of $x(t)$. If the filter is such that we may identify an impulse response $g(t)$ such that

$$G(f) = \int_{-\infty}^{\infty} g(u) e^{-j2\pi f u} du \tag{3.07}$$

then (3.06) may be written

$$s(t) = \int_{-\infty}^{\infty} g(u)x(t-u)\, du \tag{3.08}$$

We identify as the mean square error between $x_o(t)$ and $s(t)$

$$\mathcal{E}_{x_o s} = E\left\{\left[s(t) - x_o(t)\right]^2\right\} \tag{3.09}$$

where the subscript $x_o s$ is used to identify the argument functions of the error. Recognizing that we are dealing with real signals and using (3.06)

$$\mathcal{E}_{x_o s} = E\left\{\iint_{-\infty}^{\infty} e^{j2\pi\lambda_1 t} e^{-j2\pi\lambda_2 t} G(\lambda_1)G^*(\lambda_2)d\Lambda(\lambda_1)d\Lambda^*(\lambda_2) + x_o^2(t)\right.$$

$$\left. -x_o^*(t)\int_{-\infty}^{\infty} e^{j2\pi\lambda t}G(\lambda)d\Lambda(\lambda) - x_o(t)\int_{-\infty}^{\infty} e^{-j2\pi\lambda t}G^*(\lambda)d\Lambda^*(\lambda)\right\}$$

If $\Lambda_o(f)$ is defined as the GFT of $x_o(t)$ this becomes

$$\mathcal{E}_{x_o s} = E\left\{\iint_{-\infty}^{\infty}\left[e^{j2\pi(\lambda_1-\lambda_2)t} G(\lambda_1)G^*(\lambda_2)d\Lambda(\lambda_1)d\Lambda^*(\lambda_2) + d\Lambda_o(\lambda_1)d\Lambda_o^*(\lambda_2)\right.\right.$$

$$\left.\left. - G(\lambda_1)d\Lambda(\lambda_1)d\Lambda_o^*(\lambda_2) - G^*(\lambda_1)d\Lambda^*(\lambda_1)d\Lambda_o(\lambda_2)\right]\right\}$$

Using (2.34) and (2.40) and interchanging the integration and expectation,

$$\mathcal{E}_{x_o s} = \int_{-\infty}^{\infty}\left[|G(\lambda)|^2 d\Sigma_x(\lambda) + d\Sigma_x(\lambda) - G(\lambda)d\Sigma_{xx_o}^*(\lambda) - G^*(\lambda)d\Sigma_{xx_o}(\lambda)\right]$$

Since $R_{xx_o}(\tau)$ is odd we deduce that $d\Sigma_{xx_o}^*(f) = -d\Sigma_{xx_o}(f)$ and the error becomes

$$\mathcal{E}_{x_o s} = \int_{-\infty}^{\infty}\left[1 + |G(\lambda)|^2\right]d\Sigma_x(\lambda) + \int_{-\infty}^{\infty}\left[G(\lambda) - G^*(\lambda)\right] d\Sigma_{xx_o}(\lambda)$$

Denoting the real and imaginary parts of $G(f)$ by $G_r(f)$ and $G_i(f)$ we may write $G(f) = G_r(f) + jG_i(f)$ and the error is

$$\mathcal{E}_{x_o s} = \int_{-\infty}^{\infty}\left[1 + G_r^2(\lambda) + G_i^2(\lambda)\right]d\Sigma_x(\lambda) + 2j\int_{-\infty}^{\infty} G_i(\lambda)\, d\Sigma_{xx_o}(\lambda) \qquad (3.10)$$

3.3 Unrealizable Conjugate Filter

If we do not impose the restriction of realizability upon $G(f)$ we are at liberty to independently specify $G_r(f)$ and $G_i(f)$. From (3.10) $G_r(f)$ makes a strictly nonnegative contribution to the error since $\Sigma_x(f)$ is a nondecreasing function. The minimization of $\mathcal{E}_{x_o s}$ therefore specifies $G_r(f) = 0$ for all f and $\mathcal{E}_{x_o s}$ becomes

$$\mathcal{E}_{x_o s} = \int_{-\infty}^{\infty} \left[1 + G_i^2(\lambda)\right] d\Sigma_x(\lambda) + 2j \int_{-\infty}^{\infty} G_i(\lambda)\, d\Sigma_{xx_o}(\lambda) \qquad (3.11)$$

Note that if $G(f)$ were required to be realizable we could no longer specify $G_r(f)$ and $G_i(f)$ independently. In fact the realizability of $G(f)$ implies a particular functional relationship between $G_r(f)$ and $G_i(f)$, a well known result of classical circuit theory. It is clear that the realizability restriction will result in a larger minimum error. The subject of realizable filters will be pursued in a later section.

The minimization of $\mathcal{E}_{x_o s}$ for unrealizable $G(f)$ may be effected by a simple argument based on the calculus of variations. We observe first that if $x(t)$ is a real signal then $\Lambda(f)$ must be real-even, imaginary-odd. From (3.06), for $s(t)$ to be real, $G(f)$ must also be real-even, imaginary-odd. That is, for any filter $G(f)$, $G_r(f)$ must be even and $G_i(f)$ odd. If $G_{io}(f)$ denotes the particular filter which minimizes $\mathcal{E}_{x_o s}$ and $\Delta(f)$ is an odd but otherwise arbitrary function, then from (3.06) we have

$$\mathcal{E}_{x_os} = \mathcal{E}_{x_os,min} + 2\varepsilon\left[\int_{-\infty}^{\infty}\Delta(\lambda)G_{io}(\lambda)d\Sigma_x(\lambda) + j\int_{-\infty}^{\infty}\Delta(\lambda)d\Sigma_{xx_o}(\lambda)\right] + \varepsilon^2\int_{-\infty}^{\infty}\Delta^2(\lambda)d\Sigma_x(\lambda) \tag{3.12}$$

where $\mathcal{E}_{x_os,min}$ is the minimum value of $\mathcal{E}_{x_os}$ and we have set

$$G_i(f) = G_{io}(f) + \varepsilon\Delta(f)$$

where ε is a completely arbitrary real constant. Since $\mathcal{E}_{x_os} \geqslant \mathcal{E}_{x_os,min}$ and since the last term of (3.12) is nonnegative, $G_{io}(f)$ must be chosen so that the term in brackets vanishes. If it failed to vanish ε could be chosen so that $\mathcal{E}_{x_os} < \mathcal{E}_{x_os,min}$ in violation of the assumption that $G_{io}(f)$ is the optimum filter. Therefore we must have

$$\int_{0}^{\infty}\Delta(\lambda)\left[G_{io}(\lambda)\ d\Sigma_x(\lambda) + j\ d\Sigma_{xx_o}(\lambda)\right] = 0 \tag{3.13}$$

where the lower integration limit can be set to zero due to the evenness of the integrand. Since the function $\Delta(f)$, except for being odd, is arbitrary $G_{io}(f)$ must satisfy at each point f the equation

$$G_{io}(f)d\Sigma_x(f) = -j\ d\Sigma_{xx_o}(f) \tag{3.14}$$

which is the equation describing the unrealizable conjugate filter.

$$C(f) = 0 + j\, G_{io}(f) = \frac{d\Sigma_{xx_o}(f)}{d\Sigma_x(f)} \tag{3.15}$$

Substituting $C(f)$ into (3.11) the error becomes, since $s(t) = y(t)$,

$$\mathcal{E}_{x_o y} = \int_{-\infty}^{\infty} \left[1 - |C(\lambda)|^2\right] d\Sigma_x(\lambda) \tag{3.16}$$

in terms of the input SDF.

It may be shown (see Appendix 3.3) that (3.15) implies $|C(f)| \leq 1$. If equality holds, $|C(f)|^2 = 1$, then $\mathcal{E}_{x_o y} = 0$ and the filter produces $x_o(t)$ with zero MS error. If $|C(f)|^2$ is not identically unity, the more general case, then $\mathcal{E}_{x_o y} > 0$ and we are unable to produce $x_o(t)$ from a linear operation on $x(t)$. These two cases will be discussed in later sections.

Of course in the case where $\Sigma_x(f)$, $\Sigma_{xx_o}(f)$ are absolutely continuous (3.15) and (3.16) may be written

$$C(f) = \frac{\Sigma'_{xx_o}(f)}{\Sigma'_x(f)} = \frac{S_{xx_o}(f)}{S_x(f)} \tag{3.17a}$$

$$\mathcal{E}_{x_o y} = \int_{-\infty}^{\infty} \left[1 - |C(f)|^2\right] S_x(f)\, df \tag{3.17b}$$

where $S_{xx_o}(f)$, $S_x(f)$ are the cross and self PSD's of $x(t)$, $x_o(t)$ as defined in (2.25).

3.4 Realizable Conjugate Filter

Having carried through the solution for the unrealizable filter in terms of the generalized signal characterization of Section 2.5, and in the interests of notational convenience and familiarity, we shall assume in this section that $\Sigma_{xx_o}(f)$ and $\Sigma_x(f)$ are absolutely continuous functions and that we may therefore carry through this analysis in terms of PSD's without recourse to the generalized treatment. It will be seen later that this assumption incurs no loss of generality, for if $\Sigma_x(f)$ has a discontinuous part, the spectral decomposition theory outlined in Section 2.5.2 allows the minimization to be performed separately for the continuous and discontinuous parts of $\Sigma_x(f)$. Addition of a "jump function" to $\Sigma_x(f)$ will require modification of the filter only at the discrete points of increase, or jumps. The resulting solution will provide minimum error for the singular and regular components separately as well as for the mixed process. This will be an important consideration in Chapters 4 and 5 where the filters will be calculated and evaluated for input processes whose spectral distributions contain a discontinuous part.

Here we seek the filter $G(f)$ which minimizes (3.10) subject to the condition of realizability. This condition can be specified in a number of ways. It is a condition of causality and requires that the inverse FT of $G(f)$ be zero for $t < 0$. Equivalently the requirement is that $G(z)$, as a function of the complex variable z, be analytic in the lower half of the z-plane. Or the requirement may be stated by specifying that $G_r(f)$ be the Hilbert transform (HT) of $G_i(f)$ so that $G(f)$ may be written

$$G = G_r + j\, G_i = \hat{G}_i + j\, G_i = G_r - j\, \hat{G}_r \quad ; \; G_r = \hat{G}_i \tag{3.18}$$

where as before $\hat{}$ denotes the Hilbert transform. These conditions are all equivalent and any one of them is sufficient to derive the remainder. As remarked earlier this additional restriction on $G(f)$ results in a greater error $\mathcal{E}_{x_o s}$ for the realizable filter than for the previously derived unrealizable filter. This is due to the fact that more data are available in the unrealizable case, which makes use of $x(t)$ infinitely far into both the past and future, than in the realizable case, in which only the past behavior, $-\infty < u \leqslant t$, of $x(u)$ is available for the estimation of $x_o(t)$.

The form of the realizable filter for the estimation of one process by another is a well known result of the Wiener filter theory [5], [15], [17-18]. A rederivation of the filter will not be presented here. In shorthand notation the optimum realizable filter for the estimation of $x_o(t)$ from $x(t)$ may be written [18]

$$C(f) = \frac{1}{S_x^-(f)} \left[\frac{S_{xx_o}(f)}{S_x^+(f)} \right]_- \tag{3.19}$$

where the various quantities are to be interpreted as follows. The symbol $[\;]_-$ may be interpreted as the realizable part of the function appearing within the brackets. For example if within the brackets appears a function Q(f) defined on the f-axis then

$$[Q(f)]_- = \int_0^\infty e^{-j2\pi fu}\left[\int_{-\infty}^\infty e^{j2\pi\lambda u}Q(\lambda)d\lambda\right]du \tag{3.20}$$

The $[\ ]_-$ operator computes the time response (inverse FT) corresponding to Q(f), truncates it to include only positive time and transforms back to the frequency domain. The result is a function $[Q(f)]_-$ whose FT will be nonzero only for $t \geqslant 0$ and which will be analytic in the lower half plane when considered as a function of the complex variable. An alternative interpretation of this operator develops if we interchange the order of integration in (3.20) so that

$$[Q(f)]_- = \int_{-\infty}^\infty Q(\lambda)\left[\int_0^\infty e^{-j2\pi(f-\lambda)u}du\right]d\lambda = \int_{-\infty}^\infty Q(\lambda)D(\lambda-f)\ d\lambda \tag{3.21}$$

where

$$D(\lambda-f) \equiv \int_0^\infty e^{j2\pi(\lambda-f)u}du \tag{3.22}$$

is a distribution, or generalized function. It is shown in Appendix 3.4 that D(f) is

$$D(f) = \frac{1}{2}\left[\delta(f) - \frac{j}{\pi f}\right] \tag{3.23}$$

where $\delta(f)$ is the Dirac delta function and $1/\pi f$ is the Hilbert kernel in the f-domain. The result of (3.21) is

$$[Q(f)]_{-} = \frac{1}{2}[Q(f) - j\,\hat{Q}(f)] \tag{3.24}$$

where, as before, $\hat{Q}(f)$ is the HT of $Q(f)$.

Returning to (3.19), $S_x^-(f)$ and $S_x^+(f)$ are the strictly positive factors of $S_x(f)$

$$S_x(f) = S_x^-(f)\, S_x^+(f) \tag{3.25}$$

such that S_x^-, as a function of the complex variable, is analytic in the lower half plane and S_x^+ is analytic in the upper half plane. We may pursue this further by defining

$$L(f) = \ln S_x(f) = \ln S_x^-(f) + \ln S_x^+(f) = L^-(f) + L^+(f)$$

$$L^-(f) = \ln S_x^-(f) \quad , \quad L^+(f) = \ln S_x^+(f)$$

where $L^-(f)$ is the realizable part of $L(f)$, $L^-(f) = [L(f)]_-$, and from (3.24)

$$L^-(f) = \frac{1}{2}[L(f) - j\,\hat{L}(f)]$$

We may thus write for $S_x^-(f)$

$$S_x^-(f) = \exp\{L^-(f)\} = [S_x(f)]^{\frac{1}{2}}\, e^{-\frac{j}{2}\mathcal{H}\left\{\ln S_x(f)\right\}} \tag{3.26a}$$

and similarly

$$S_x^+(f) = [S_x(f)]^{\frac{1}{2}} e^{\frac{j}{2} \mathcal{H}\left\{\ln S_x(f)\right\}} \tag{3.26b}$$

Since $S_x(f)$ is everywhere nonnegative (except possibly on a set of measure zero) the square root introduces no difficulty. These relations are quite general, in fact more general than necessary in the case of spectral densities which are rational functions of frequency. In this case $S_x(f)$ is easily decomposed by assigning the poles and zeros of S_x in the LHP to S_x^- and those in the UHP to S_x^+ as will be seen in Chapter 5.

3.5 Singular Processes - Hilbert Representation

Since $S_x(f) \geq 0$ and $|C(f)|^2 \leq 1$, (3.17b) indicates that $\mathcal{E}_{x_o y} \neq 0$, and the estimate $y(t)$ is not equal in mean square to the process $x_o(t)$, unless $|C(f)|^2 = 1$ or $S_x(f) = 0$ for each f. Consider $x(t)$ to be strictly narrowband with bandwidth $2f_b$ where $f_b < f_o$.

$$S_x(f) = 0 \quad ; \quad f_o - f_b \geq |f| \;,\quad f_o + f_b \leq |f| \tag{3.27}$$

Then both $S_c(f)$ and $S_s(f)$ [see (2.11)] are strictly bandlimited

$$S_c(f) = S_s(f) = 0 \quad , \quad |f| \geq f_b \tag{3.28}$$

In this case $S_x(f)$ and $S_{xx_o}(f)$ [see (2.25b)] reduce to

$$S_x(f) = \begin{cases} \frac{1}{2}\left[S_c(f-f_o) - S_s(f-f_o)\right] & , f > 0 \\ \frac{1}{2}\left[S_c(f+f_o) + S_s(f+f_o)\right] & , f < 0 \end{cases} \tag{3.29a}$$

$$S_{xx_o}(f) = \begin{cases} \frac{1}{2j}\left[S_c(f-f_o) - S_s(f-f_o)\right] & , f > 0 \\ \frac{-1}{2j}\left[S_c(f+f_o) + S_s(f+f_o)\right] & , f < 0 \end{cases} \tag{3.29b}$$

and from (3.17a) the unrealizable filter solution is

$$C(f) = \begin{cases} -j\ \mathrm{sgn}(f) & , S_x(f) \neq 0 \\ \text{arbitrary} & , \text{otherwise} \end{cases} \tag{3.30}$$

where

$$\mathrm{sgn}(f) \equiv \begin{cases} +1 & , f > 0 \\ 0 & , f = 0 \\ -1 & , f < 0 \end{cases} \tag{3.31}$$

Thus either $|C(f)| = 1$ or $S_x(f) = 0$ for all f and $\mathcal{E}_{x_o y} = 0$. The filter defined by (3.30) is just the Hilbert filter, for

$$d\Omega(f) = -j\ \mathrm{sgn}(f)\ d\Lambda(f) \tag{3.32a}$$

implies that

$$y(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(u)}{t-u}\,du \equiv \hat{x}(t) \tag{3.32b}$$

We have discovered therefore that for a strictly narrowband process the optimum estimator of $x_o(t)$ is the Hilbert filter (such that $\hat{x}(t) = x_o(t)$ in mean square) and the radius detector yields the classical Hilbert representation. As we have mentioned however, this singular input case is not to be met in practice, for a strictly narrowband signal may be predicted arbitrarily far into the future from knowledge of its past.

In Chapters 4 and 5 we shall investigate the filter solution and corresponding radius detector representations for a class of amplitude-modulated regular processes with a rational spectral density. For these processes the expressions for $S_x(f)$, $S_{xx_o}(f)$ in terms of $S_c(f)$, $S_s(f)$ do not reduce and the conjugate filter for $x_o(t)$ is no longer the Hilbert filter. It will also develop that the overall envelope error is lower for the conjugate representation than for the Hilbert representation.

Chapter 4

Amplitude Modulation -- Unrealizable Filters

4.1 Phase-envelope Relationship

The foregoing development is applicable to the general modulated process. We turn now to several special cases for which the optimum filters and corresponding detection errors will be computed. In all cases where the filter is optimum for the estimation of the conjugate signal in mean square we shall refer to it as the conjugate filter $C(f)$. This filter will be specified by (3.15) for the unrealizable case discussed in this chapter and by (3.19) for the realizable case to be treated in Chapter 5.

An input signal case of great practical importance is that of pure amplitude modulation. In this case the phase-envelope relationship supplied by the sender specifies that the phase and envelope are statistically independent and further, that the phase is a constant for each sample function of the random process $x(t)$. That is, the phase is a random variable denoted by ϕ_o with a suitably defined probability density function. This information, in conjunction with knowledge of $x(t)$, is sufficient to correctly identify the in-phase and quadrature components, at least mathematically, and hence to correctly represent the intended envelope $A_o(t)$ if the phase ϕ_o is described by an appropriate probability density function.

The received signal given by (2.01) is written in this case

$$x(t) = A_o(t) \cos [\omega_o t - \phi_o] \qquad (4.01)$$

where from (2.03) the in-phase and quadrature components are proportional to $A_o(t)$.

$$x_c(t) = A_o(t) \cos \phi_o \tag{4.02a}$$

$$x_s(t) = A_o(t) \sin \phi_o \tag{4.02b}$$

The stationarity condition (2.06a) and the condition of independence between ϕ_o and A_o specify that

$$E \left\{\cos^2 \phi_o\right\} = E \left\{\sin^2 \phi_o\right\} = \frac{1}{2}$$

since $\sin^2 + \cos^2 = 1$. Application of a trigonmetric identity yields

$$E \left\{\cos 2\phi_o\right\} = 0 \tag{4.03a}$$

Equation (2.06b) specifies that

$$E \left\{\cos \phi_o \sin \phi_o\right\} = - E \left\{\sin \phi_o \cos \phi_o\right\}$$

or, upon application of a similar identity

$$E \left\{\sin 2\phi_o\right\} = 0 \tag{4.03b}$$

Using (4.02) and (4.03), (2.06) leads to

$$R_c(\tau) = \frac{1}{2} R_{A_o}(\tau) \tag{4.04a}$$

$$R_s(\tau) = 0 \tag{4.04b}$$

where

$$R_{A_o}(\tau) \equiv E\left\{A_o(t)\ A_o(t+\tau)\right\}$$

If ϕ_o is distributed such that (4.03) is satisfied (for example, if ϕ_o is uniformly distributed in an interval of length 2π) then $x(t)$ as represented by (4.01) is a stationary amplitude-modulated random process. The correlation functions defined in (2.08) and (2.23) are

$$R_x(\tau) = \frac{1}{2} R_{A_o}(\tau) \cos \omega_o \tau \tag{4.05a}$$

$$R_{xx_o}(\tau) = \frac{1}{2} R_{A_o}(\tau) \sin \omega_o \tau \tag{4.05b}$$

Inasmuch as the unrealizable filter was presented in Chapter 3 in terms of the generalized signal characterization and in order to demonstrate the usefulness of that approach we shall continue in that vein in this chapter. The spectral distribution functions are, from (2.41) and (4.04),

$$\Sigma_x(f) = \frac{1}{4}\left[\Sigma_{A_o}(f-f_o) + \Sigma_{A_o}(f+f_o)\right] \tag{4.06a}$$

$$\Sigma_{xx_o}(f) = \frac{1}{4j}\left[\Sigma_{A_o}(f-f_o) - \Sigma_{A_o}(f+f_o)\right] \tag{4.06b}$$

or, if Σ'_{A_o} exists,

$$S_x(f) = \frac{1}{4}\left[S_{A_o}(f-f_o) + S_{A_o}(f+f_o)\right] \tag{4.07a}$$

$$S_{xx_o}(f) = \frac{1}{4j}\left[S_{A_o}(f-f_o) - S_{A_o}(f+f_o)\right] \tag{4.07b}$$

4.2 Hilbert Filter -- Singular Input Processes

From (4.07) and the discussion in Section 3.5 we observe that with pure amplitude modulation the optimum filter is the Hilbert filter if $A_o(t)$ is strictly bandlimited with bandwidth $2f_b$, $f_b < f_o$. We have seen that the MS error in estimating $x_o(t)$ is zero for this case. But we have also noted that such a modulation process will not be encountered in practice as its future is completely determined by its past and no new information can ever be introduced into the signal.

It is interesting to note that there are singular cases for which the optimum filter is not Hilbert, in fact any singular but bandunlimited process results in a non-Hilbert filter. For example if we suppose the PSD of the received signal to have a Gaussian shape

$$S_{A_o}(f) \propto e^{-[f/f_1]^2}$$

then the optimum filter for the estimation of $x_o(t)$ is, from (4.07),

$$G(f) = -j \tanh \frac{2f_o f}{f_1^2}$$

This filter transfer function approaches $-j \operatorname{sgn}(f)$ as f_o increases without bound, that is as $x(t)$ approaches strict narrowbandedness. It is a general property of the singular case that the corresponding optimum filter transfer functions tend, in absolute value, to unity as the frequency increases without bound. In general the filter transfer functions for the regular case decay to zero as $|f|$ becomes large.

4.3 Conjugate Filter -- Regular Input Processes

The more interesting situation in practice is that in which the regularity condition is satisfied. The spectral densities of regular processes in general decay more slowly than those of nonregular processes; so slowly that $\ln S(f)$ is dominated by f^2 so that the integral criterion J' in (2.51) converges. This greater high frequency content results in an uncertainty of the future given even perfect knowledge of past behavior.

In many cases the spectral densities are, or can be closely approximated by, rational functions of frequency. All such processes are strictly regular. We shall employ as our spectral density model a rational function but with a modification. For demodulation in AM transmission systems either a carrier component must be transmitted with the modulated signal or the carrier must be restored at the receiver before detection if it has been suppressed at the transmitter. In either case the signal to be presented to the detector contains the carrier component which may be considered as due to a nonzero mean value of the envelope. Specifically in (4.01), if we define σ_c as the mean value of $A_o(t)$ and $a(t)$ as the zero mean difference between $A_o(t)$ and σ_c, we may write the intended envelope as

$$A_o(t) = \sigma_c + a(t) \tag{4.08}$$

where $E\{A_o(t)\} = \sigma_c$, $E\{a(t)\} = 0$ and the spectral density of $a(t)$ will be taken as a rational function of frequency. This is the situation mentioned in Section 2.5 in discussing some of the concepts of generalized harmonic analysis. The process $x(t)$ is thus a mixed process containing a deterministic component $x_1(t)$ and a random component $x_2(t)$ where

$$x(t) = x_1(t) + x_2(t) \tag{4.09a}$$

$$x_1(t) = \sigma_c \cos [\omega_o t - \phi_o] \tag{4.09b}$$

$$x_2(t) = a(t) \cos [\omega_o t - \phi_o] \tag{4.09c}$$

This mixture corresponds to the decomposition of the f-axis into two disjoint sets as discussed in Section 2.5. A set of measure zero, namely $\{-f_o, +f_o\}$, is associated with the deterministic component and the spectral distribution function of this component is a step or "jump" function having as its points of increase the point set $\{-f_o, +f_o\}$. The remainder of the f-axis is associated with the random component, the spectral distribution being absolutely continuous on this set.

Inasmuch as the spectral distribution function of $x(t)$ is defined separately on disjoint sets, the transfer function $C(f)$ which satisfies (3.15) is also defined on disjoint sets. The spectral distribution $\Sigma_{A_o}(f)$ of $A_o(t)$ consists of a jump σ_c^2 at $f = 0$ and a continuous function $\Sigma_a(f)$.

$$\Sigma_{A_o}(f) = \sigma_c^2 U(f) + \Sigma_a(f) \tag{4.10}$$

where $U(f)$ is the unit step function, defined here to be continuous on the right.

$$U(f) = \begin{cases} 0 \quad , \quad f < 0 \\ 1 \quad , \quad f \geq 0 \end{cases} \tag{4.11}$$

Substituting for $\Sigma_{A_o}(f)$ into (4.06)

$$\Sigma_x(f) = \frac{1}{4}\left\{\sigma_c^2[U(f-f_o) + U(f+f_o)] + \Sigma_a(f-f_o) + \Sigma_a(f+f_o)\right\} \tag{4.12a}$$

$$\Sigma_{xx_o}(f) = \frac{1}{4j}\left\{\sigma_c^2[U(f-f_o) - U(f+f_o)] + \Sigma_a(f-f_o) - \Sigma_a(f+f_o)\right\} \tag{4.12b}$$

At the point $+f_o$ the jumps in Σ_x, Σ_{xx_o} are

$$d\Sigma_x(f_o) = \frac{1}{4}\,\sigma_c^2 \;, \quad d\Sigma_{xx_o}(f_o) = \frac{1}{4j}\,\sigma_c^2$$

At the point $-f_o$,

$$d\Sigma_x(-f_o) = \frac{1}{4}\,\sigma_c^2 \;, \quad d\Sigma_{xx_o}(-f_o) = -\frac{1}{4j}\,\sigma_c^2$$

Therefore at these points we have for $C(f)$

$$C(+f_o) = -j \quad , \quad C(-f_o) = +j$$

In the set which is the remainder of the f-axis, $|f| \neq f_o$, where Σ_a and Σ_{A_o} are absolutely continuous we have

$$d\Sigma_x(f) = \frac{1}{4}\left\{d\Sigma_a(f-f_o) + d\Sigma_a(f+f_o)\right\}$$

$$, \quad |f| \neq f_o$$

$$d\Sigma_{xx_o}(f) = \frac{1}{4j}\left\{d\Sigma_a(f-f_o) - d\Sigma_a(f+f_o)\right\}$$

Since Σ_a is continuous we may write this in the more familiar PSD notation

$$d\Sigma_x(f) = \frac{1}{4}\left\{S_a(f-f_o) + S_a(f+f_o)\right\}\, df$$

$$, \quad |f| \neq f_o$$

$$d\Sigma_{xx_o}(f) = \frac{1}{4j}\left\{S_a(f-f_o) - S_a(f+f_o)\right\}\, df$$

so that in this set we have for $C(f)$

$$C(f) = -j\,\frac{S_a(f-f_o) - S_a(f+f_o)}{S_a(f-f_o) + S_a(f+f_o)} \qquad , \quad |f| \neq f_o$$

Thus the conjugate filter for regular mixed input processes is discontinuous and given by

$$C(f) = \begin{cases} -j\,\mathrm{sgn}(f) & , \quad |f| = f_o \\ -j\,\dfrac{S_a(f-f_o) - S_a(f+f_o)}{S_a(f-f_o) + S_a(f+f_o)} & , \quad |f| \neq f_o \end{cases} \qquad (4.13)$$

The phenomenon of discontinuity in the filter transfer function is similar to that found in the theory of stochastic prediction in the presence of a deterministic component [15], [19]. Since all the energy in the carrier component is concentrated in the point set $\{-f_o, +f_o\}$ and the total energy of the random component in the point set is zero, by allowing C(f) to have discontinuities we isolate the carrier without disturbing the information content of the random component. To indicate the performance of C(f) with respect to the carrier component alone assume that $S_a(f) = 0$ and the received signal is $x_1(t)$. The calculation of the filter output takes the following form:

$$x_1(t) = \int_{-\infty}^{\infty} e^{-j2\pi ft} d\Lambda_1(f) \Rightarrow \Lambda_1(f) = \frac{\sigma_c}{2}\left[e^{j\phi_o} U(f+f_o) + e^{-j\phi_o} U(f-f_o)\right]$$

so that

$$d\Lambda_1(-f_o) = \frac{\sigma_c}{2} e^{j\phi_o} \quad , \quad d\Lambda_1(+f_o) = \frac{\sigma_c}{2} e^{-j\phi_o}$$

The output of C(f) with input $x_1(t)$ is therefore

$$\int_{-\infty}^{\infty} e^{j2\pi ft} C(f)\, d\Lambda_1(f) = \frac{\sigma_c}{2}\left[C(-f_o) e^{-j(\omega_o t - \phi_o)} + C(f_o) e^{j(\omega_o t - \phi_o)}\right] = \sigma_c \sin[\omega_o t - \phi_o]$$

The filter is therefore effective in exactly producing the conjugate carrier. This is not surprising since the filter acts separately on the carrier and random components (spectral decomposition) and the carrier component is a strictly bandlimited process which we have seen admits of exact conjugate filtering. Thus the error in estimating $x_o(t)$ by the output of $C(f)$ is the same as the error in estimating $x_{2o}(t) = a(t) \sin[\omega_o t - \phi_o]$ by the output of $C(f)$ with only $x_2(t) = a(t) \cos[\omega_o t - \phi_o]$ as the input since $x_1(t)$ and $x_2(t)$ are orthogonal processes and the estimation error for $x_{1o}(t)$ alone is zero. The carrier does not contribute to the error in estimation of $x_o(t)$, but we shall see that it does contribute to the error in estimation of the intended envelope $A_o(t)$ due to the nonlinear character of the radius detector.

We may interpret the filter $C(f)$ as the parallel connection of two filters separately defined on disjoint f-sets with mutually orthogonal inputs, and hence outputs. One filter, $C_1(f)$, defined on the point set $\{-f_o, +f_o\}$, is responsible for filtering the carrier, and is able to do so exactly. The other filter $C_2(f)$, defined everywhere else on the f-axis, is responsible for the random component, and is capable of approximate estimation only. $C(f)$ may then be represented

$$C(f) = C_1(f) + C_2(f) \tag{4.14a}$$

$$C_1(f) = \begin{cases} -j\,\mathrm{sgn}(f) & , \quad |f| = f_o \\ 0 & , \quad |f| \neq f_o \end{cases} \tag{4.14b}$$

$$C(f) = \begin{cases} -j\,\dfrac{S_a(f-f_o) - S_a(f+f_o)}{S_a(f-f_o) + S_a(f+f_o)} & , \quad |f| \neq f_o \\ 0 & , \quad |f| = f_o \end{cases} \tag{4.14c}$$

An apparent contradiction is indicated if we attempt to interpret this result in the time domain. If the filter transfer function $C(f)$ is represented by its inverse FT, the impulse response $c(t)$, then $C_1(f)$ and $C_2(f)$ may be represented by the time functions $c_1(t)$ and $c_2(t)$. But $C_1(f)$ is nonzero at only a finite number of points and hence $c_1(t)$ must be identically zero. Formally we write for the output of the filter,

$$y(t) = \int_{-\infty}^{\infty} e^{j2\pi ft} C(f)\, d\Lambda(f) = \int_{-\infty}^{\infty} c(u)x(t-u)\, du$$

and since $c_1(t) = 0$,

$$y(t) = \int_{-\infty}^{\infty} c_2(u)x(t-u)\, du$$

Apparently when the filter is viewed in the time domain it is completely incapable of filtering the carrier component since $C_1(f)$ has disappeared in the inversion process. But in the f-domain formulation of (4.14) the filter is completely capable of filtering the carrier component. The resolution of this paradox is based on a suggestion by Dr. H. C. Martel and provides an interesting insight. The basic point is that since we are including in our consideration frequency intervals of zero length for $C(f)$, we must include in the specification of the time domain impulse response signals (specifically sinusoids) of zero amplitude. To see the way in which this statement is to be interpreted consider approaching $C(f)$ in (4.14) as a limit. Specifically, consider

$C_1(f)$ to be $-j\ \mathrm{sgn}(f)$ in a band of frequency $[f_1, f_2]$ where $0 < f_1 < f_o < f_2$ (there is no need to consider $C_2(f)$ in resolving the paradox). Define the function

$$C_1(f_1,f_2;f) = \begin{cases} -j\ \mathrm{sgn}(f) & , \quad f_1 \le |f| \le f_2 \\ 0 & , \quad \text{otherwise} \end{cases}$$

so that, in the limit, $C_1(f)$ is given by

$$C_1(f) = \lim_{\substack{f_2 \to f_o + \\ f_1 \to f_o -}} C_1(f_1,f_2;f)$$

The impulse response, say $\overline{c}_1(t)$, of $C_1(f_1,f_2;f)$ is easily found to be

$$\overline{c}_1(t) = \frac{1}{\pi t}\left[\cos 2\pi f_1 t - \cos 2\pi f_2 t\right]$$

The time domain filter output in response to $x_1(t)$ is the convolution of $\overline{c}_1(t)$ with $x_1(t) = \sigma_c \cos[\omega_o t - \phi_o]$ and can be shown to be

$$\overline{c}_1(t) * x_1(t) = \frac{\sigma_c}{2} \sin[\omega_o t - \phi_o]\left[\mathrm{sgn}(\omega_o - \omega_1) - \mathrm{sgn}(\omega_2 - \omega_o)\right]$$

$$= \sigma_c \sin[\omega_o t - \phi_o] = x_{1o}(t)$$

since $f_1 < f_o < f_2$. As $f_1 \to f_o-$ and $f_2 \to f_o+$, the output remains $x_{1o}(t)$, the conjugate of the carrier component. The impulse response $\overline{c}_1(t)$ before taking the limit may be written

$$\bar{c}_1(t) = \frac{2}{\pi t} \sin \pi(f_1 + f_2)t \sin \pi(f_2 - f_1)t$$

where, in the limit

$$c_1(t) = \lim_{\substack{f_2 \to f_o + \\ f_1 \to f_o -}} \bar{c}_1(t) = \lim_{\substack{f_2 \to f_o + \\ f_1 \to f_o -}} 2(f_2 - f_1) \sin \pi f_o t = 0$$

This is the sinusoid of zero amplitude referred to previously. When the convolution is performed before the limit is taken, the conjugate carrier is correctly produced.

4.4 PSD of the Received Signal

To evaluate the effectiveness of the conjugate filter and compare it with competing representations we must choose a specific rational PSD for $a(t)$. The simplest such function, and the one which we shall use in our investigation is that of "single RC" noise for which the PSD is given by

$$S_a(f) = \frac{\sigma_a^2/\pi\beta}{1 + \left(\frac{f}{\beta}\right)^2} \implies \Sigma_a(f) = \frac{\sigma_a^2}{\pi}\left[\frac{\pi}{2} + \tan^{-1}\frac{f}{\beta}\right] \tag{4.15}$$

where $\sigma_a^2 = E\left\{a^2(t)\right\} = R_a(0)$. The SDF of the entire envelope process $A_o(t)$ is, from (4.10),

$$\Sigma_{A_o}(f) = \sigma_c^2 U(f) + \frac{\sigma_a^2}{\pi}\left[\frac{\pi}{2} + \tan^{-1}\frac{f}{\beta}\right] \tag{4.16}$$

and is sketched roughly in Figure 2.

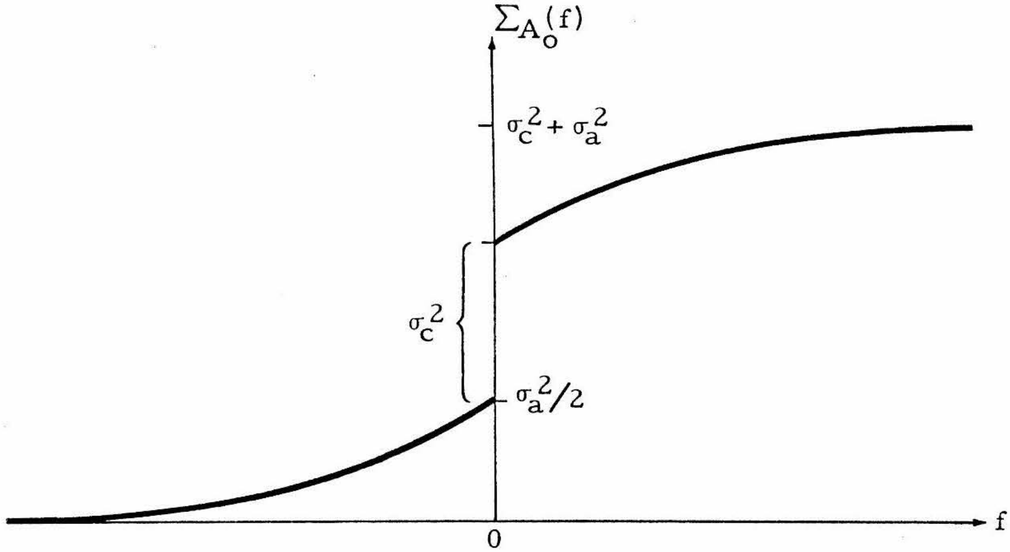


Figure 2

Spectral Distribution Function of $A_o(t)$

The mean square value of $A_o(t)$ is

$$E\left\{A_o^2(t)\right\} = \sigma_c^2 + E\left\{a^2(t)\right\} = \sigma_c^2 + \sigma_a^2 \qquad (4.17)$$

4.5 Unrealizable Conjugate Filter

4.5.1 Filter Transfer Function

The filter component $C_2(f)$ given by (4.14c) with $S_a(f)$ defined in (4.15) takes the form

$$C_2(f) = -j\,\frac{2f_o f}{f^2 + f_o^2 + \beta^2} = -j\,\frac{2f_o f}{f^2 + \gamma^2} \qquad (4.18)$$

where we have defined the parameter $\gamma \equiv +\sqrt{f_o^2 + \beta^2}$. It will also be convenient at a later stage to have the parameter $\eta \equiv \beta/\gamma$. The carrier frequency f_o may also be written in terms of γ and η.

$$\gamma \equiv \sqrt{f_o^2 + \beta^2} \quad , \quad \eta \equiv \beta/\gamma \quad , \quad f_o = \gamma\sqrt{1 - \eta^2} \quad , \quad 0 \leqslant \eta \leqslant 1 \tag{4.19}$$

The parameter η is a measure of the narrowness of the spectrum. As $\eta \to 0$, the spectrum approaches the spectrum of a cosine wave; that is, the process approaches a single frequency, strictly narrowband process.

4.5.2 Computation of Second-Order Statistics

In the calculation of the estimation error for the various cases to be considered it will be convenient to work in terms of the second-order statistical parameters of the signals x, x_o, y, and $\hat{x}$ where $y(t)$ is the output of the conjugate filter and $\hat{x}(t)$ is the output of the Hilbert filter. We shall need the Hilbert signal parameters since we will compare the performance of the conjugate representation with that of the classical Hilbert representation. The RMS values of x, x_o, and $\hat{x}$ are obtained from (2.22), (3.04), (4.05) and (4.17)

$$\sigma_{\hat{x}}^2 = \sigma_{x_o}^2 = \sigma_x^2 = R_x(0) = \frac{1}{2} R_{A_o}(0) = \frac{1}{2}\left[\sigma_c^2 + \sigma_a^2\right] \tag{4.20}$$

The RMS value of $y(t)$ and the cross moment $E\{x_o(t)y(t)\}$ are given by

$$\sigma_y^2 = \langle x_o y \rangle = \int_{-\infty}^{\infty} |C(f)|^2 d\Sigma_x(f) = \frac{1}{2}\left[\sigma_c^2 + \sigma_a^2(1-\eta)\right] \qquad (4.21)$$

where as before the symbol $\langle\ \rangle$ stands for expectation. The cross moment $E\{x_o(t)\hat{x}(t)\}$ is given by

$$\langle x_o \hat{x} \rangle = \frac{1}{2}\left[\sigma_c^2 + \sigma_a^2\left(\frac{2}{\pi}\cos^{-1}\eta\right)\right] = \frac{1}{2}\left[\sigma_c^2 + \sigma_a^2\left(1 - \frac{2}{\pi}\sin^{-1}\eta\right)\right] \qquad (4.22)$$

The details of these calculations are somewhat lengthy and appear in Appendix 4.5.2.

4.5.3 Error in Estimation of $x_o(t)$

The MS error $\mathcal{E}_{x_o s}$ between $x_o(t)$ and the signal $s(t)$ is given by

$$\mathcal{E}_{x_o s} = E\left\{(x_o - s)^2\right\} = \sigma_{x_o}^2 + \sigma_s^2 - 2\langle x_o s \rangle \qquad (4.23)$$

Using (4.23) we may compute the error in estimation for the conjugate and Hilbert filters by letting $s(t)$ be equal to $y(t)$ and $\hat{x}(t)$ respectively. Of course the conjugate filter will result in the lower error because it was specifically designed to estimate $x_o(t)$. The error of the conjugate filter is obtained from(4.20), (4.21) and (4.23)

$$\mathcal{E}_{x_o y} = \sigma_x^2 - \sigma_y^2 = \frac{\sigma_a^2}{2}\eta \qquad (4.24)$$

The error due to Hilbert filtering is, using (4.22) in (4.23),

$$\mathcal{E}_{x_o\hat{x}} = 2\left(\sigma_x^2 - \langle x_o\hat{x}\rangle\right) = \sigma_a^2 \frac{2}{\pi} \sin^{-1}\eta \qquad (4.25)$$

4.5.4 Discussion and Comparison of Conjugate Signal Estimators

As mentioned before the error in estimating $x_o(t)$ is independent of the carrier term and this is verified for the conjugate and Hilbert estimators by (4.24) and (4.25). The ratio of the error for Hilbert estimation to that for conjugate estimation is given by

$$\frac{\mathcal{E}_{x_o\hat{x}}}{\mathcal{E}_{x_o y}} = \frac{4}{\pi} \frac{\sin^{-1}\eta}{\eta} \qquad (4.26)$$

This function increases monotonically in value from $4/\pi$ when $\eta = 0$ to 2 as $\eta \to 1$. The Hilbert filter gives its best performance, namely zero error, as $\eta \to 0$, which is consistent with our earlier results since the process becomes narrowband in this limit.

The calculation for conjugate filtering fortifies the claim that a nonzero error is inherent in the linear estimation of $x_o(t)$ for a regular process $x(t)$. Of course the error measure in which we are ultimately interested is the overall error in estimating the intended envelope $A_o(t)$ by the output $A_s(t)$ of the radius detector system depicted in Figure 1.

4.6 Overall Detector Error

4.6.1 Error Criteria

The comparison of competing representations with respect to envelope error is considerably more complicated than the simple comparison of filters in estimating $x_o(t)$. One difficulty is the choice of an appropriate envelope error criterion. The most natural choice, the mean square error between the output envelope $A_s(t)$ and the intended envelope $A_o(t)$, is not a simple expression. To appreciate this we compute the MS error,

$$\mathcal{E}_{A_oA_s} = E\left\{[A_o - A_s]^2\right\} = \langle A_o^2\rangle + \langle A_s^2\rangle - 2\langle A_oA_s\rangle$$
$$= 3\sigma_x^2 + \sigma_s^2 - 2E\left\{\sqrt{x^2 + x_o^2}\ \sqrt{x^2 + s^2}\right\} \tag{4.27}$$

The last term in (4.27) is not directly expressible in terms of the second-order statistics of x, s and x_o due to the square root within the expectation operator. To proceed further with the calculation of $\mathcal{E}_{A_oA_s}$ it would be necessary to derive the joint probability density function of x, s and x_o and actually perform the expected value operation indicated. This procedure is an interesting exercise and will be discussed in more detail at a later point for the case of normal processes. It is possible however to define an envelope error criterion which is directly expressible in terms of the second-order statistics if the input process is assumed to be normally distributed. Consider the mean square error $\mathcal{E}_{2S}$ between the squared envelope functions. This is referred to for convenience as the MSS envelope error and is specified by

$$\mathcal{E}_{2S} = E\left\{\left[A_o^2(t) - A_s^2(t)\right]^2\right\} \tag{4.28}$$

where it is to be understood that the comparison of $A_s(t)$ is always to be made with the intended envelope $A_o(t)$. From (2.21) and (3.01) this may be written

$$\mathcal{E}_{2S} = E\left\{\left[x_o^2(t) - s^2(t)\right]^2\right\}$$

$$= \langle x_o^4 \rangle + \langle s^4 \rangle - 2\langle x_o^2 s^2 \rangle \tag{4.29}$$

These quantities may be expressed in terms of the second-order statistics if the assumption is made that the random process $x(t)$ is normally distributed. If this is so, both $x_o(t)$ and $s(t)$ are normally distributed and $\mathcal{E}_{2S}$ may be written

$$\mathcal{E}_{2S} = 3\sigma_x^4 + 3\sigma_s^4 - 2\sigma_x^2\sigma_s^2 - 4\langle x_o s \rangle^2 \tag{4.30}$$

in terms of the quantities already calculated in Section 4.5.2.

4.6.2 Computation of $\mathcal{E}_{2S}$ Errors -- Normal Processes

We may now calculate in detail the errors $\mathcal{E}_{2S}$ for the cases $s = \hat{x}$ and $s = y$ using (4.30) and the second-order statistics. The error $\mathcal{E}_{2H}$ in the Hilbert representation is

$$\mathcal{E}_{2H} = 4\left[\sigma_x^4 - \langle x_o \hat{x} \rangle^2\right]$$

since $\sigma_{\hat{x}}^2 = \sigma_x^2$. From (4.22) we may write $\langle x_o \hat{x} \rangle$ as

$$\langle x_o \hat{x} \rangle = \frac{1}{2}\left[\sigma_c^2 + \sigma_a^2(1-q)\right] \tag{4.31}$$

where it is convenient notationally to define the function

$$q = q(\eta) \equiv \frac{2}{\pi} \sin^{-1}\eta \quad ; \quad 1 - q = \frac{2}{\pi} \cos^{-1}\eta \tag{4.32}$$

and where q ranges monotonically from 0 to 1 as η ranges from 0 to 1. Then $\mathcal{E}_{2H}$ becomes

$$\mathcal{E}_{2H} = \sigma_a^4 \, q\left[2 - q + 2(\sigma_c/\sigma_a)^2\right] \tag{4.33}$$

It is also useful to define the effective modulation index m

$$m \equiv \sigma_a^2/\sigma_c^2 \quad , \quad 0 \leqslant m < \infty \tag{4.34}$$

where the range of m is from zero modulation $(\sigma_a^2 = 0, \; m = 0)$ to complete carrier suppression $(\sigma_c^2 = 0, \; m = \infty)$. In terms of q and m (4.33) may be written

$$\mathcal{E}_{2H} = \sigma_c^4 \, m \, q\left[2(1 + m) - m \, q\right] \tag{4.35}$$

To evaluate the MSS error $\mathcal{E}_{2C}$ for the conjugate representation we use $\langle x_o y \rangle$ and σ_y^2 from (4.21) in (4.30) so that the error is

$$\mathcal{E}_{2C} = \sigma_c^4 \, m \, \eta\left[1 + m - \frac{m}{4}\, \eta\right] \tag{4.36}$$

These two error coefficients are plotted in Figure 3 as functions of η for various values of the modulation index m. We see that the conjugate representation yields the lower MSS error in all cases.

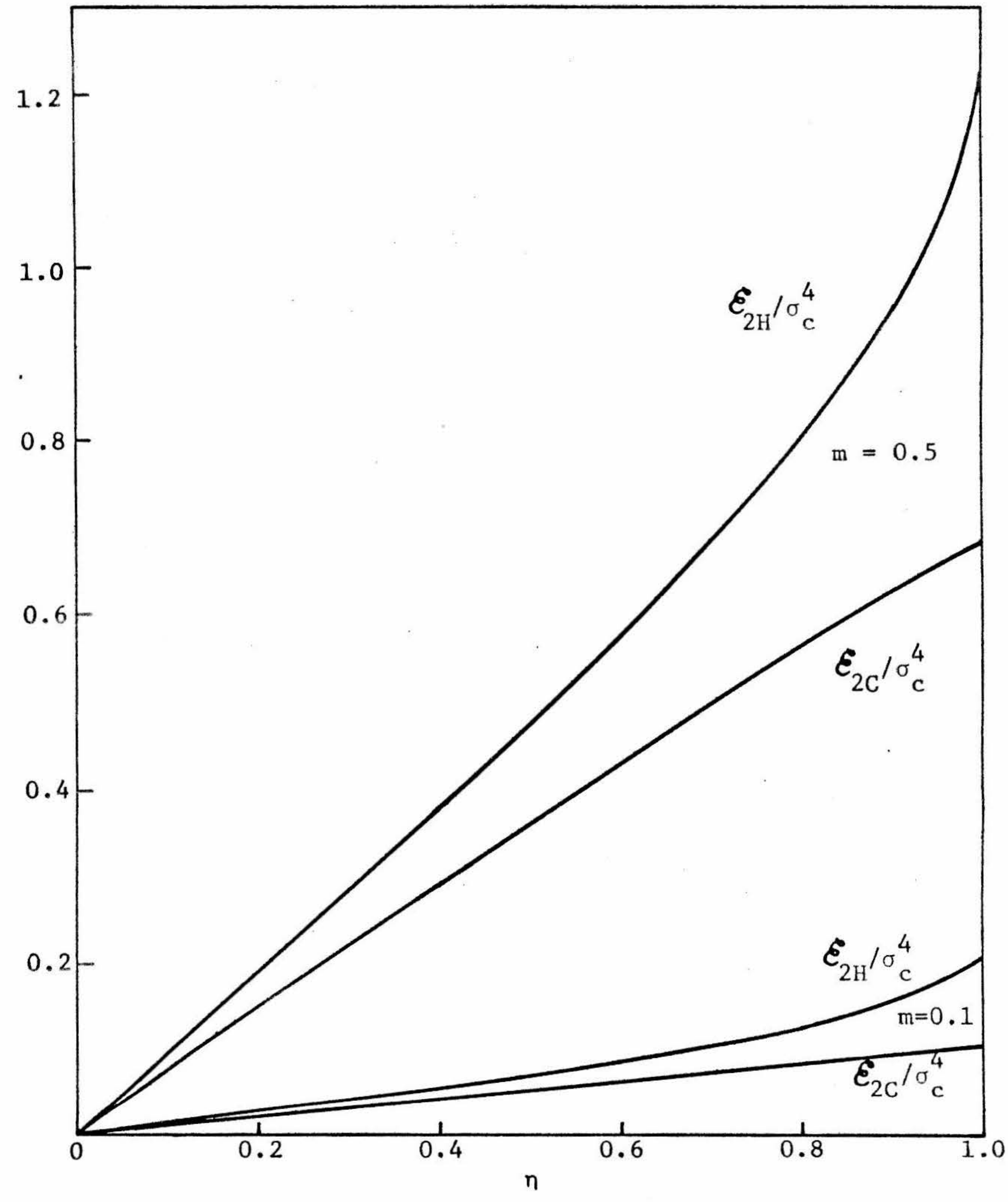


Figure 3

$\mathcal{E}_{2C}/\sigma_c^4$, $\mathcal{E}_{2H}/\sigma_c^4$

for m = 0.1, 0.5

4.6.3 Detector Modification

It has been stated that we are not necessarily dealing with filters optimum (in any sense) for the estimation of $A_o(t)$. Whatever adjustments can be made in the filter to yield lower envelope error are justified. A simple modification of the filter which can be optimized with respect to the estimation of $A_o(t)$ is the insertion of an ideal amplifier with gain $\sqrt{k}$ in the filter path. This is equivalent to a change in the gain or level of the filter. The envelope $A_s(t)$ is now given by

$$A_s(t) = +\left[x^2(t) + ks^2(t)\right]^{\frac{1}{2}} \tag{4.37}$$

and the error, denoted by $\bar{\mathcal{E}}_{2S}$ for the modified representation, is

$$\bar{\mathcal{E}}_{2S} \equiv E\left\{[x_o^2 - ks^2]^2\right\} = k^2 \langle s^4 \rangle - 2k \langle x_o^2 s^2 \rangle + \langle x_o^4 \rangle$$

a quadratic in k, so that $\bar{\mathcal{E}}_{2S}$ may be minimized by the proper choice of k. The value of k for which this expression is a minimum is

$$k = \frac{\langle x_o^2 s^2 \rangle}{\langle s^4 \rangle}$$

and the corresponding error is

$$\bar{\mathcal{E}}_{2S} = \langle x_o^4 \rangle - \frac{\langle x_o^2 s^2 \rangle^2}{\langle s^4 \rangle}$$

With the assumption of normal processes these become, in terms of the second-order statistics,

$$k = \frac{\sigma_x^2\sigma_s^2 + 2\langle x_o s\rangle^2}{3\sigma_s^4} \tag{4.38}$$

and

$$\bar{\mathcal{E}}_{2S} = \frac{4}{3}\left[\sigma_x^2 - \frac{\langle x_o s\rangle^2}{\sigma_s^2}\right]\left[2\sigma_x^2 + \frac{\langle x_o s\rangle^2}{\sigma_s^2}\right] \tag{4.39}$$

4.6.4 Computation of $\bar{\mathcal{E}}_{2S}$ Errors -- Modified Filter

Using (4.39) we may compute the errors $\bar{\mathcal{E}}_{2S}$ for the modified representation (4.37). Strictly speaking the classical Hilbert representation does not allow for such a modification since the modification depends on some knowledge of the input spectrum, and one appeal of the Hilbert representation is its independence of the input signal. We shall calculate $\bar{\mathcal{E}}_{2H}$ anyway out of curiosity and for comparison purposes. For the Hilbert case,

$$\bar{\mathcal{E}}_{2H} = \frac{\sigma_c^4\, q\, m^2}{3\left(1 + \frac{1}{m}\right)^2}\left[2 + \frac{2}{m} - q\right]\left[3 - 2q + q^2 + \frac{2}{m}(3-q) + \frac{3}{m^2}\right] \tag{4.40}$$

It is interesting to note that $\bar{\mathcal{E}}_{2H}$ can be written in terms of the quantities $1 + m$ and mq,

$$\bar{\mathcal{E}}_{2H} = \frac{\sigma_c^4}{3}\, mq\left[2(1 + m) - mq\right]\left[2 + \left(1 - \frac{mq}{1+m}\right)^2\right] \tag{4.41}$$

The MSS error for the modified conjugate representation becomes, since $\sigma_y^2 = \langle x_o y \rangle$,

$$\bar{\mathcal{E}}_{2C} = \sigma_c^4\, m\eta\left[1 + m - \frac{m}{3}\,\eta\right] \tag{4.42}$$

The MSS error coefficients $\bar{\mathcal{E}}_{2H}$ and $\bar{\mathcal{E}}_{2S}$ for the modified representations are displayed in Figure 4.

4.6.5 Limiting Values of MSS Error

Several particular limiting cases are of interest and can be derived rather easily from (4.35), (4.36), (4.41) and (4.42). Here we shall indicate only the various results, saving their comparison and further discussion for Section 4.6.6.

In the extreme narrowband limit, as $\eta \to 0$,

$$\lim_{\eta\to 0} \mathcal{E}_{2H} = \lim_{\eta\to 0} \bar{\mathcal{E}}_{2H} = \frac{4m(1+m)\sigma_c^4}{\pi}\,\eta \tag{4.43a}$$

$$\lim_{\eta\to 0} \mathcal{E}_{2C} = \lim_{\eta\to 0} \bar{\mathcal{E}}_{2C} = m(1+m)\sigma_c^4\eta \tag{4.43b}$$

In the limit as the spectrum of $x(t)$ becomes low pass; i.e., as the carrier frequency f_o becomes small with respect to the spectral corner frequency β, we have $\eta \to 1$ and

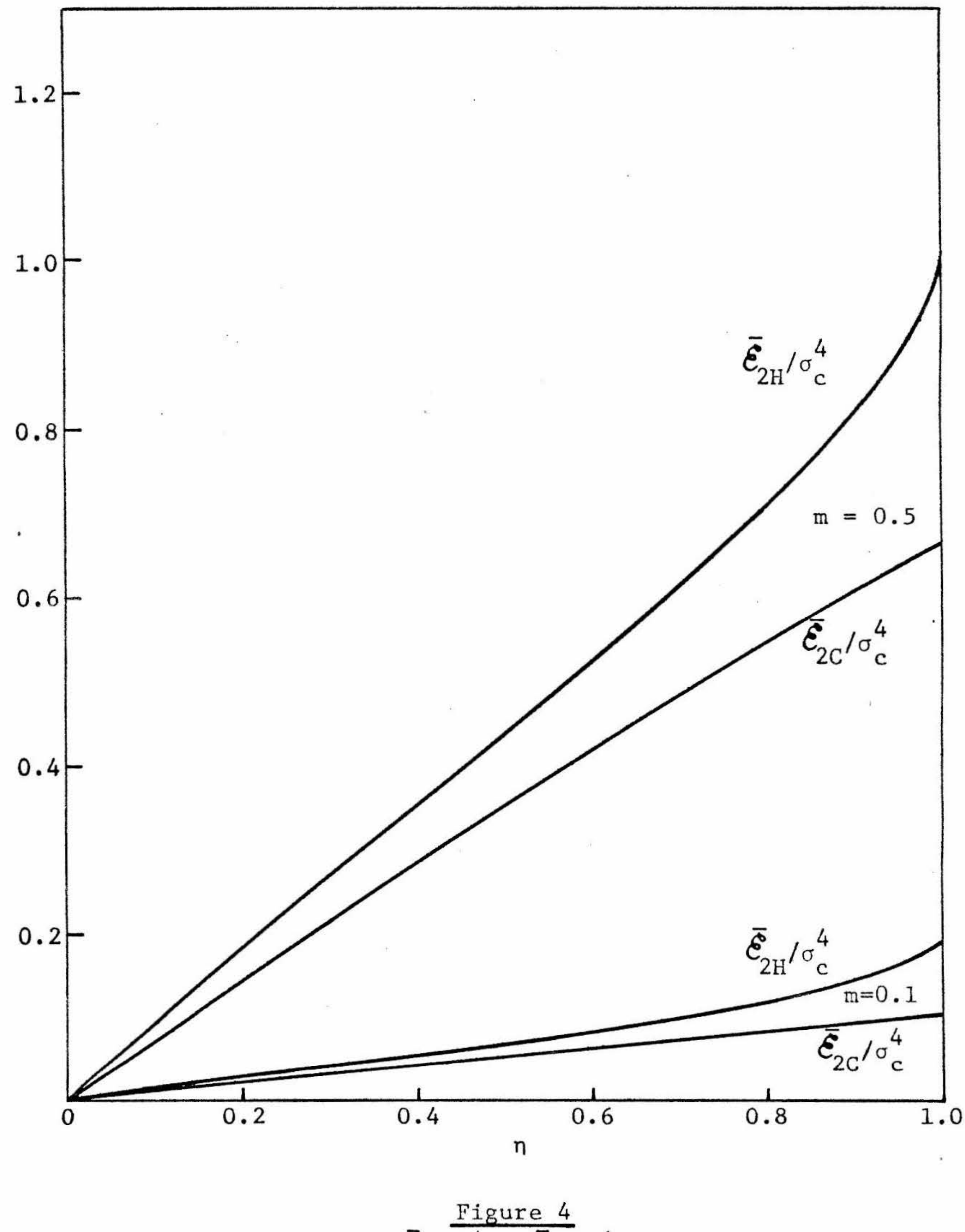


Figure 4

$\bar{\mathcal{E}}_{2C}/\sigma_c^4$, $\bar{\mathcal{E}}_{2H}/\sigma_c^4$

for $m = 0.1, 0.5$

$$\lim_{\eta\to 1} \mathcal{E}_{2H} = \sigma_c^4 m(m+2)$$

$$\lim_{\eta\to 1} \mathcal{E}_{2C} = \sigma_c^4 m(1+3m/4)$$

$$\lim_{\eta\to 1} \bar{\mathcal{E}}_{2H} = \frac{\sigma_c^4}{3} m[m + 2]\left[2 + \left(\frac{1}{1+m}\right)^2\right] \tag{4.44}$$

$$\lim_{\eta\to 1} \bar{\mathcal{E}}_{2C} = \sigma_c^4 m(1+2m/3)$$

Finally, in the limit as the degree of modulation becomes very small, $m \to 0$,

$$\lim_{m\to 0} \mathcal{E}_{2H} = \lim_{m\to 0} \bar{\mathcal{E}}_{2H} = \frac{4}{\pi}\sigma_c^4 m \sin^{-1}\eta \tag{4.45a}$$

$$\lim_{m\to 0} \mathcal{E}_{2C} = \lim_{m\to 0} \bar{\mathcal{E}}_{2C} = \sigma_c^4 m\eta \tag{4.45b}$$

4.6.6 Discussion and Comparison of Results

The MSS error coefficients, normalized by σ_c^4, are plotted in Figures 3 and 4 for $m = 0.1$ and 0.5. In Figure 3 are plotted the unmodified MSS errors and in Figure 4 are plotted the MSS errors for the gain-modified representation. In all cases the Hilbert representation results in greater error than does the conjugate representation.

A better quantity for the comparison of the two envelope representations is the MSS error ratio. Three such ratios are plotted in Figures 5, 6 and 7. In Figure 5 is plotted the ratio $\mathcal{E}_{2H}/\mathcal{E}_{2C}$ of the unmodified errors for values of the modulation index $m = 0, 0.1$ and 0.5. The curves all meet at $4/\pi = 1.273$ when $\eta = 0$ and increase as η increases. Interestingly the Hilbert representation competes more successfully as the modulation index is increased. In Figure 6 is plotted the ratio $\bar{\mathcal{E}}_{2H}/\bar{\mathcal{E}}_{2C}$ in which both representations are gain-modified. This plot has a generally different character than Figure 5. For m greater than about 0.1 the curves begin at $\eta = 0$ with negative slopes so that the error ratio actually decreases at first. That is, there is a range of η in which the Hilbert representation competes more successfully than it does in the narrowband limit. That this is due to the advantage of gain-modification in the Hilbert representation can be seen in Figure 7.

In Figure 7 is plotted the ratio $\mathcal{E}_{2H}/\bar{\mathcal{E}}_{2C}$ of unmodified Hilbert error to modified conjugate error. Here we are considering the true Hilbert representation in which no information regarding the input signal is used and $k = 1$ for all η. In this case the error ratio is monotonic as η increases and there is much less m-dependent variation between the curves than in Figure 6.

Once again, in all cases the Hilbert representation exhibits the lower MSS error and, with this error criterion, proves to be the superior envelope representation.

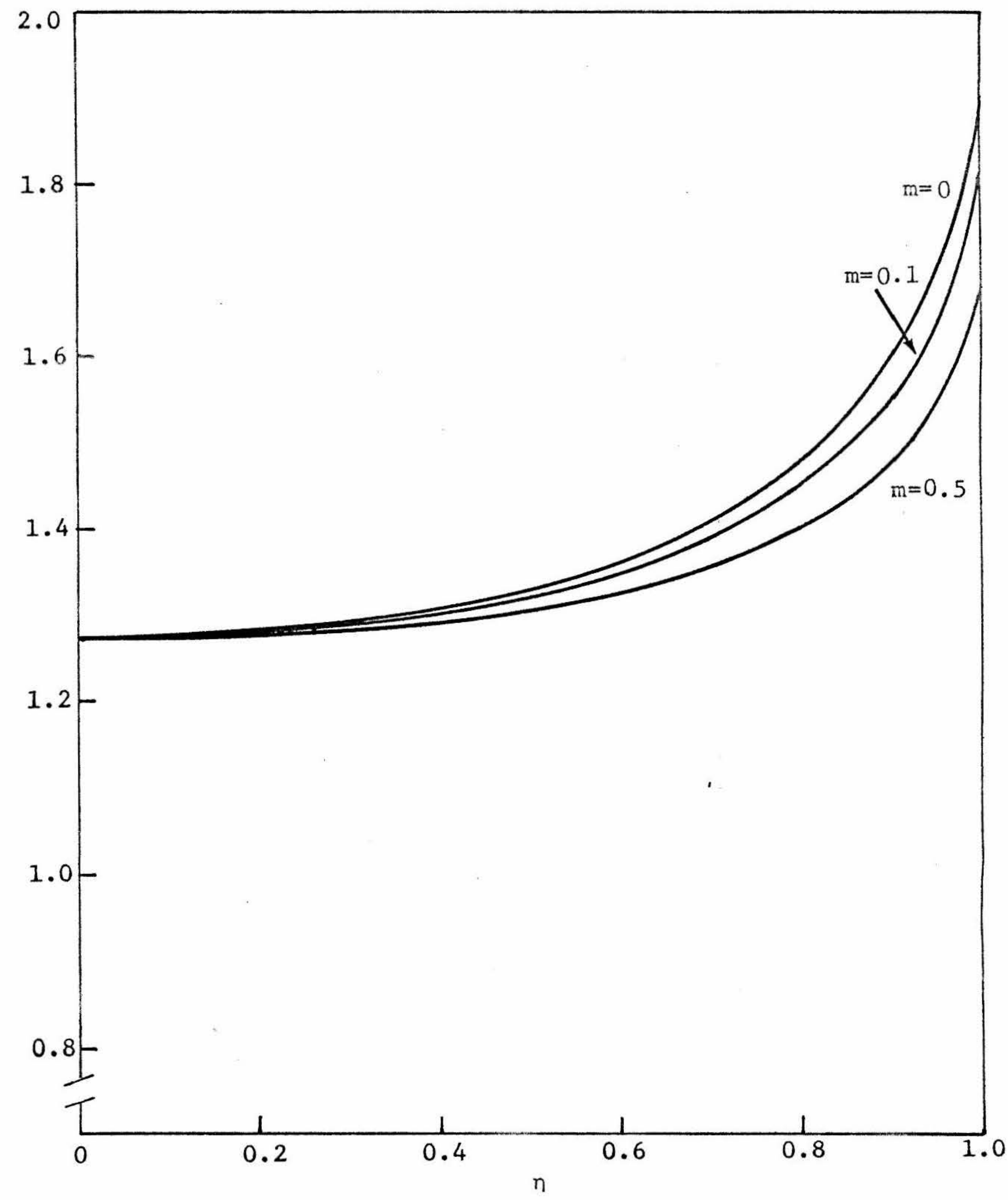


Figure 5

$\mathcal{E}_{2H}/\mathcal{E}_{2C}$ for m = 0, 0.1, 0.5

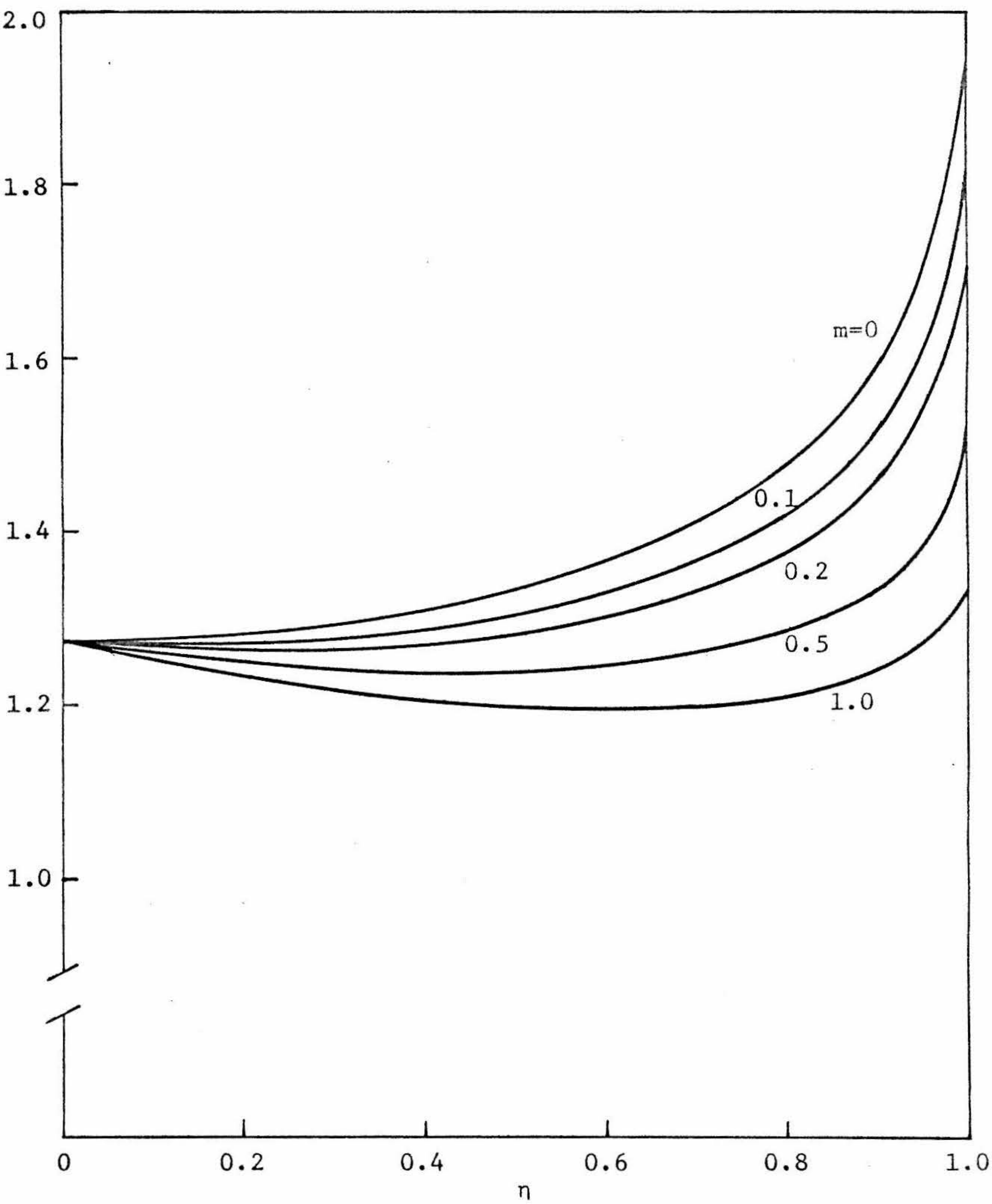


Figure 6

$\bar{\mathcal{E}}_{2H}/\bar{\mathcal{E}}_{2C}$ for m = 0, 0.1, 0.2, 0.5, 1.0

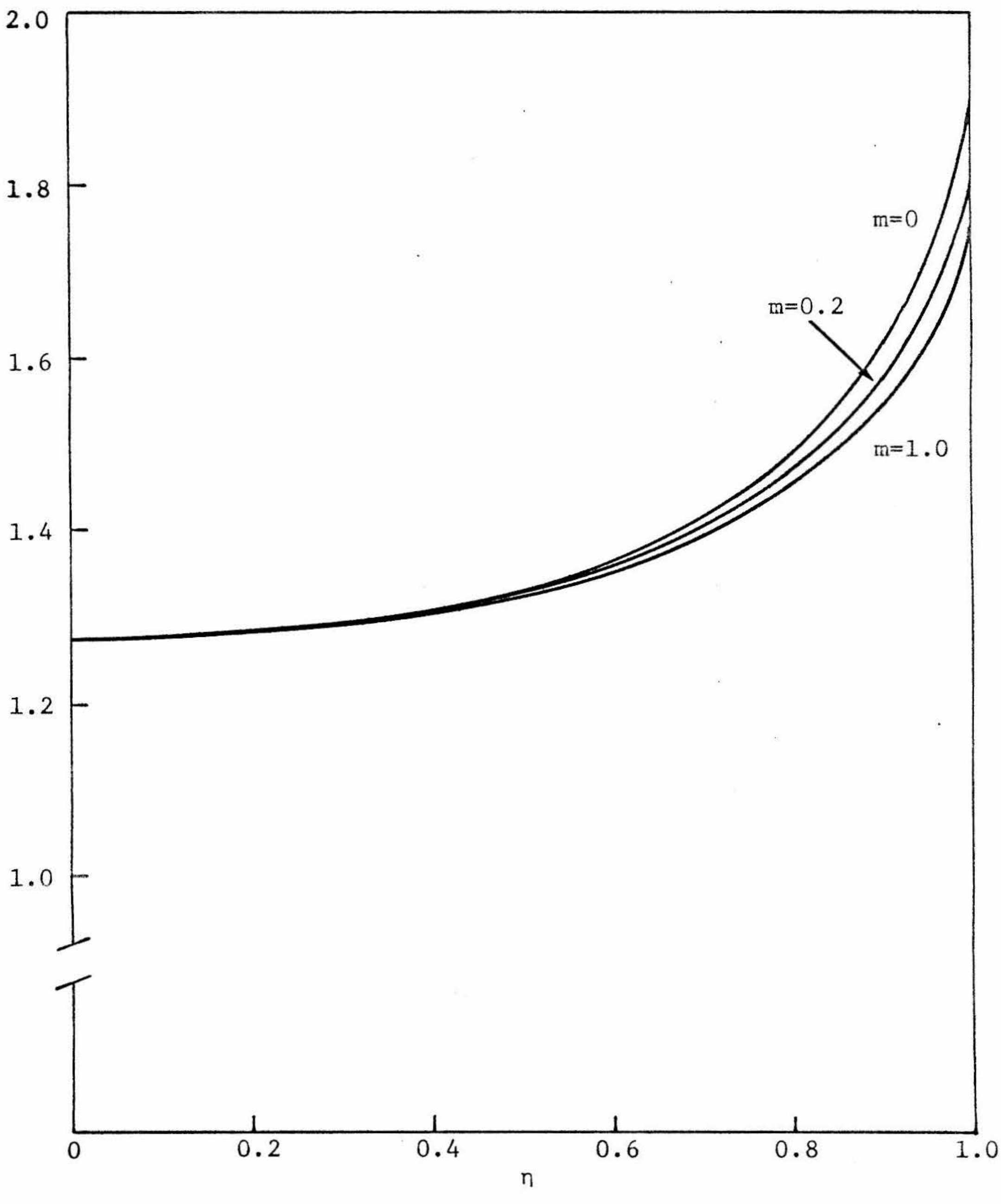


Figure 7

$\mathcal{E}_{2H}/\bar{\mathcal{E}}_{2C}$ for m = 0, 0.2, 1.0

Chapter 5

Amplitude Modulation -- Realizable Filters

The previous chapter has provided a direct comparison, in terms of the overall detector MSS envelope error, between the classical widely used Hilbert envelope representation and the proposed conjugate representation for a specific input spectral density. The basic conclusion drawn there is that while the Hilbert filter is mathematically simpler (in the unmodified case the filter transfer function does not even depend upon the input signal spectral characteristics), the MSS error between the detector output and the intended envelope is lower for the conjugate representation. The analysis performed in Chapter 4 treated unrealizable filters so that a direct comparison could be made. It is of interest to consider the situation for realizable, or causal filters, those whose outputs depend only upon the past behavior of the input signal. The results of this investigation will give a feeling for what may be expected in a practical detector.

The underlying thesis of the conjugate representation has been that a filter which is good for estimating the conjugate signal $x_o(t)$ will result in a radius detector which is good for estimating the intended envelope $A_o(t)$. The results of Chapter 4 confirm that supposition, at least in comparison with the widely proposed Hilbert representation, for a specific input process in the unrealizable case. In the estimation of $x_o(t)$ the deterministic carrier component, defined spectrally on an f-set of measure zero, and the random information-bearing component are treated separately by the filter. The condition of realizability will leave unaffected the treatment of the deterministic component and will restrict the processing of the random component to past behavior only. In this way the filter will have available less information concerning the signal to be estimated and will therefore result in a larger error in estimating $x_o(t)$. Further, we shall find a larger MSS envelope error as well.

The input signal conditions to be considered in the present chapter are identical with those assumed in Chapter 4. That is, we are considering a pure amplitude-modulated process with a single-RC spectral density for the random component. Furthermore the error criteria, both for x_o estimation and overall envelope estimation are the same as in Chapter 4. The difference is in the condition of mathematical realizability, or causality, which the filters must satisfy.

5.1 Realizable Conjugate Filter

5.1.1 Spectral Decomposition

As in Chapter 4 we consider the conjugate filter $C(f)$ to comprise two parallel filters $C_1(f)$ and $C_2(f)$ defined on disjoint f-sets. C_1 is defined on an f-set of measure zero $\{-f_o, +f_o\}$ as determined by the carrier component $x_1(t)$. C_2 is defined on the remainder of the f-axis and is determined by the random component $x_2(t)$. The sum $C(f)$ of these must be realizable. Stated alternatively $c(t)$, the inverse FT of $C(f)$, must be nonzero only for $t \geqslant 0$. But we have seen in Chapter 4 that the inverse FT of $C_1(f)$ is, in the limit, a sinusoid with zero amplitude. Hence $C(f)$ is realizable if and only if $C_2(f)$ is realizable. This is fortunate from a calculational point of view since C_2, depending only upon the random component, is easily derived from the continuous parts of the spectral densities $S_x(f)$ and $S_{xx_o}(f)$ using (3.19). $C_1(f)$ remains the same as in the unrealizable case, namely $-j\ \mathrm{sgn}(f)$ on the set $\{-f_o, +f_o\}$.

5.1.2 Computation of the Realizable Conjugate Filter

The filter $C_2(f)$ is determined by the continuous parts $S_{xc}(f)$ and $S_{xx_oc}(f)$ of the spectral densities, where the subscript c indicates consideration of the continuous part only. From (4.12) these are

$$S_{xc}(f) = \frac{1}{4}\left[S_a(f-f_o) + S_a(f+f_o)\right] \qquad (5.01a)$$

$$S_{xx_oc}(f) = \frac{1}{4j}\left[S_a(f-f_o) - S_a(f+f_o)\right] \qquad (5.01b)$$

We shall give the expressions for the various stages in the filter computation to illustrate the procedure used for calculating the various terms in (3.19) when $S_a(f)$ is a rational function. From (4.15) for the form of $S_a(f)$ these become

$$S_{xc}(f) = \frac{\sigma_a^2}{4\pi\beta}\left[\frac{1}{1+\left(\frac{f-f_o}{\beta}\right)^2} + \frac{1}{1+\left(\frac{f+f_o}{\beta}\right)^2}\right]$$

$$= \frac{\beta\sigma_a^2}{2\pi}\ \frac{(f - j\gamma)(f + j\gamma)}{(f-f_o-j\beta)(f+f_o-j\beta)(f-f_o+j\beta)(f+f_o+j\beta)} \qquad (5.02a)$$

$$S_{xx_oc}(f) = \frac{\sigma_a^2}{4\pi\beta j}\left[\frac{1}{1+\left(\frac{f-f_o}{\beta}\right)^2} - \frac{1}{1+\left(\frac{f+f_o}{\beta}\right)^2}\right]$$

$$= \frac{\beta\sigma_a^2}{\pi j}\ \frac{f_of}{(f-f_o-j\beta)(f+f_o-j\beta)(f-f_o+j\beta)(f+f_o+j\beta)} \tag{5.02b}$$

Based on the discussion of Section 3.4 the factors of $S_{xc}(f)$ are

$$S_{xc}^-(f) = \sqrt{\frac{\beta}{2\pi}}\,\sigma_a\ \frac{f - j\gamma}{(f-f_o-j\beta)(f+f_o-j\beta)} \tag{5.03a}$$

$$S_{xc}^+(f) = \sqrt{\frac{\beta}{2\pi}}\,\sigma_a\ \frac{f + j\gamma}{(f-f_o+j\beta)(f+f_o+j\beta)} \tag{5.03b}$$

The realizable filter $C_2(f)$ is given from (3.19) by

$$C_2(f) = \frac{1}{S_{xc}^-(f)}\left[\frac{S_{xx_oc}(f)}{S_{xc}^+(f)}\right]_- \tag{5.04}$$

where the quantity in brackets is given from (5.02b) and (5.03b) as

$$\frac{S_{xx_oc}}{S_{xc}^+} = -j\sqrt{\frac{2\beta}{\pi}}\,\sigma_a\ \frac{f_of}{(f+j\gamma)(f-f_o-j\beta)(f+f_o-j\beta)}$$

This quantity can be expanded by the method of partial fractions

$$\frac{S_{xx_oc}}{S^+_{xc}} = -j\sqrt{\frac{\beta}{2\pi}}\,\sigma_a\left\{\frac{\frac{-j\gamma f_o}{2\beta(\gamma+\beta)}}{f+j\gamma} + \frac{\frac{f_o+j\beta}{2[f_o+j(\gamma+\beta)]}}{f-f_o-j\beta} - \frac{\frac{f_o-j\beta}{2[f_o-j(\gamma+\beta)]}}{f+f_o-j\beta}\right\}$$

The realizable part of this expression, indicated by $[\;]_-$ in (5.04), excludes the first term, the **pole** at $-j\gamma$, which is of course not analytic in the LHP.

$$\left[\frac{S_{xx_oc}}{S^+_{xc}}\right]_- = -j\sqrt{\frac{\beta}{2\pi}}\,\sigma_a\left\{\frac{\frac{f_o+j\beta}{f_o+j(\gamma+\beta)}}{f-f_o-j\beta} - \frac{\frac{f_o-j\beta}{f_o-j(\gamma+\beta)}}{f+f_o-j\beta}\right\}$$

or

$$\left[\frac{S_{xx_oc}}{S^+_{xc}}\right]_- = -\sqrt{\frac{\beta}{2\pi}}\,\sigma_a\,\frac{f_o(f+j\gamma)}{(\beta+\gamma)(f-f_o-j\beta)(f+f_o-j\beta)}$$

Using (5.03a) for $S^-_{xc}(f)$ the filter $C_2(f)$ is obtained from (5.04),

$$C_2(f) = -\frac{f_o}{\beta+\gamma}\,\frac{f+j\gamma}{f-j\gamma} = -\frac{f_o}{\beta+\gamma}\left[1+\frac{2j\gamma}{f-j\gamma}\right] \qquad (5.05)$$

In terms of the spectral parameter $\eta \equiv \beta/\gamma$ previously defined this is

$$C_2(f) = -\sqrt{\frac{1-\eta}{1+\eta}}\,\frac{f+j\gamma}{f-j\gamma} \qquad (5.06)$$

which is clearly realizable since the only pole is in the UHP. It is interesting to note that $C_2(f)$ is an "all-pass" network. That is, the gain $|C_2(f)|$ of the filter is independent of frequency. The total filter is given by

$$C(f) = C_1(f) + C_2(f) \tag{5.07a}$$

$$C_1(f) = \begin{cases} -j\,\mathrm{sgn}(f) & , \quad |f| = f_o \\ 0 & , \quad |f| \neq f_o \end{cases} \tag{5.07b}$$

$$C_2(f) = \begin{cases} 0 & , \quad |f| = f_o \\ -\sqrt{\frac{1-\eta}{1+\eta}}\,\frac{f+j\gamma}{f-j\gamma} & , \quad |f| \neq f_o \end{cases} \tag{5.07c}$$

5.1.3 Second-Order Statistics for Conjugate Filtering

Using the notation developed for the disjoint filters and the continuous parts of the spectral densities, we may write the second-order statistics as

$$\sigma_x^2 = \frac{1}{2}\left[\sigma_c^2 + \sigma_a^2\right] = \frac{\sigma_c^2}{2}\,(1+m) \tag{5.08a}$$

$$\sigma_y^2 = \langle x_o y \rangle = \frac{1}{2}\left[\sigma_c^2 + \frac{1-\eta}{1+\eta}\;\sigma_a^2\right] = \frac{\sigma_c^2}{2}\left[1 + m\,\frac{1-\eta}{1+\eta}\right] \tag{5.08b}$$

The details of the corresponding calculations appear in Appendix 5.1.3.

5.2 "Realizable" Hilbert Filter

5.2.1 Filter Computation

As a comparison with the realizable conjugate representation we may inquire into the possibility of a realizable Hilbert filter. We may certainly apply the machinery of Section 3.4 to the minimization of the error between the filter output and $\hat{x}(t)$. This should give the realizable form of the Hilbert filter. The calculation is carried out in Appendix 5.2.1 and the result is

$$H_{real.}(f) = H_1(f) + H_2(f) \tag{5.09a}$$

$$H_1(f) = \begin{cases} -j\,\mathrm{sgn}(f) , & |f| = f_o \\ 0 , & |f| \neq f_o \end{cases} \tag{5.09b}$$

$$H_2(f) = \begin{cases} -\frac{j}{2}\,\mathrm{sgn}(f) - \frac{1}{\pi}\ln\frac{|f|}{\gamma} - \sqrt{\frac{1-\eta}{1+\eta}}\,\frac{\cos^{-1}\eta}{\pi}\,\frac{f+j\gamma}{f-j\gamma} , & |f| \neq f_o \\ 0 , & |f| = f_o \end{cases} \tag{5.09c}$$

This filter expression has several interesting features. The Hilbert representation apparently loses its simplicity in the realizable case. The filter (5.09c) depends just as strongly upon the input signal parameters as does the conjugate filter (5.07c). In fact the last term of (5.09c) is proportional to the realizable conjugate filter (5.07c). Furthermore $H_2(f)$ is not well behaved in the sense that it has no impulse response. That is, the inverse FT, which should be the filter impulse response, does not exist. Also the realizability condition requires that the real and imaginary parts of $H_2(f)$ be mutual Hilbert transforms. But $\mathrm{sgn}(f)$ and $\ln[|f|/\gamma]$ are functions which have no Hilbert transforms (however, it is interesting that the derivatives of these functions are mutual Hilbert transforms). Physically it is an altogether unsatisfactory filter, but mathematically it may be written down and its second-order statistics may be calculated.

5.2.2 Second-Order Statistics -- "Realizable" Hilbert Filter

The output of the filter $H_{real.}$ will be denoted by $z(t)$. The detailed calculations of σ_z^2 and $\langle x_o z \rangle$ appear in Appendix 5.2.2. The results are

$$\sigma_z^2 = \frac{\sigma_c^2}{2} + \frac{\sigma_a^2}{4}\left[1 + \frac{1}{1+\eta}\left(\frac{2\cos^{-1}\eta}{\pi}\right)^2\right] \tag{5.10a}$$

$$\langle x_o z \rangle = \frac{\sigma_c^2}{2} + \frac{\sigma_a^2}{2(1+\eta)}\left(\frac{2\cos^{-1}\eta}{\pi}\right) \tag{5.10b}$$

or, in terms of m and q,

$$\sigma_z^2 = \frac{\sigma_c^2}{2}\left\{1 + \frac{m}{2}\left[1 + \frac{(1-q)^2}{1+\eta}\right]\right\} \tag{5.11a}$$

$$\langle x_o z \rangle = \frac{\sigma_c^2}{2}\left[1 + \frac{m(1-q)}{1+\eta}\right] \tag{5.11b}$$

5.3 Overall Realizable MSS Envelope Error

Considering that both the realizable Hilbert and realizable conjugate filters employ information about the input signal we shall compute and compare only the most optimistic error coefficient for each representation, the gain-modified error given by (4.39). The realizable errors are computed by substituting first (5.08) and then (5.11) into (4.39). The results are

$$\bar{\mathcal{E}}_{2C} = \sigma_c^4 \frac{2m\eta}{1+\eta}\left[1 + m\,\frac{1+\frac{1}{3}\eta}{1+\eta}\right] \tag{5.12}$$

for the conjugate representation and

$$\bar{\mathcal{E}}_{2H} = \frac{\sigma_c^4}{3}\left[a - \frac{b^2}{c}\right]\left[2a + \frac{b^2}{c}\right]$$

where

$$a = 1 + m$$

$$b = 1 + \frac{m(1-q)}{1+\eta} \tag{5.13}$$

$$c = 1 + \frac{m}{2}\left[1 + \frac{(1-q)^2}{1+\eta}\right]$$

for the Hilbert representation. The error coefficients, normalized by σ_c^2, are plotted in Figure 8 for values of the modulation index $m = 0.1$, 0.5, and 1.0. As in the unrealizable case the conjugate representation results in the lower MSS error, although the margin is not so great. To compare the representations the error ratio $\bar{\mathcal{E}}_{2H}/\bar{\mathcal{E}}_{2C}$ is plotted in Figure 9 for $m = 0.1$, 0.2, 0.5, and 1.0. As $\eta \to 0$ the ratio approaches $3/4 + 1/\pi = 1.068$ for all values of m, a narrowband limit considerably smaller than that found in the unrealizable case. The narrowband error limits are

$$\lim \bar{\mathcal{E}}_{2C} = 2m(1+m)\sigma_c^2\eta \tag{5.14a}$$

$$\lim \bar{\mathcal{E}}_{2H} = (3/2 + 2/\pi)m(1+m)\sigma_c^2\eta \tag{5.14b}$$

$$\lim (\bar{\mathcal{E}}_{2H}/\bar{\mathcal{E}}_{2C}) = 3/4 + 1/\pi = 1.068 \tag{5.14c}$$

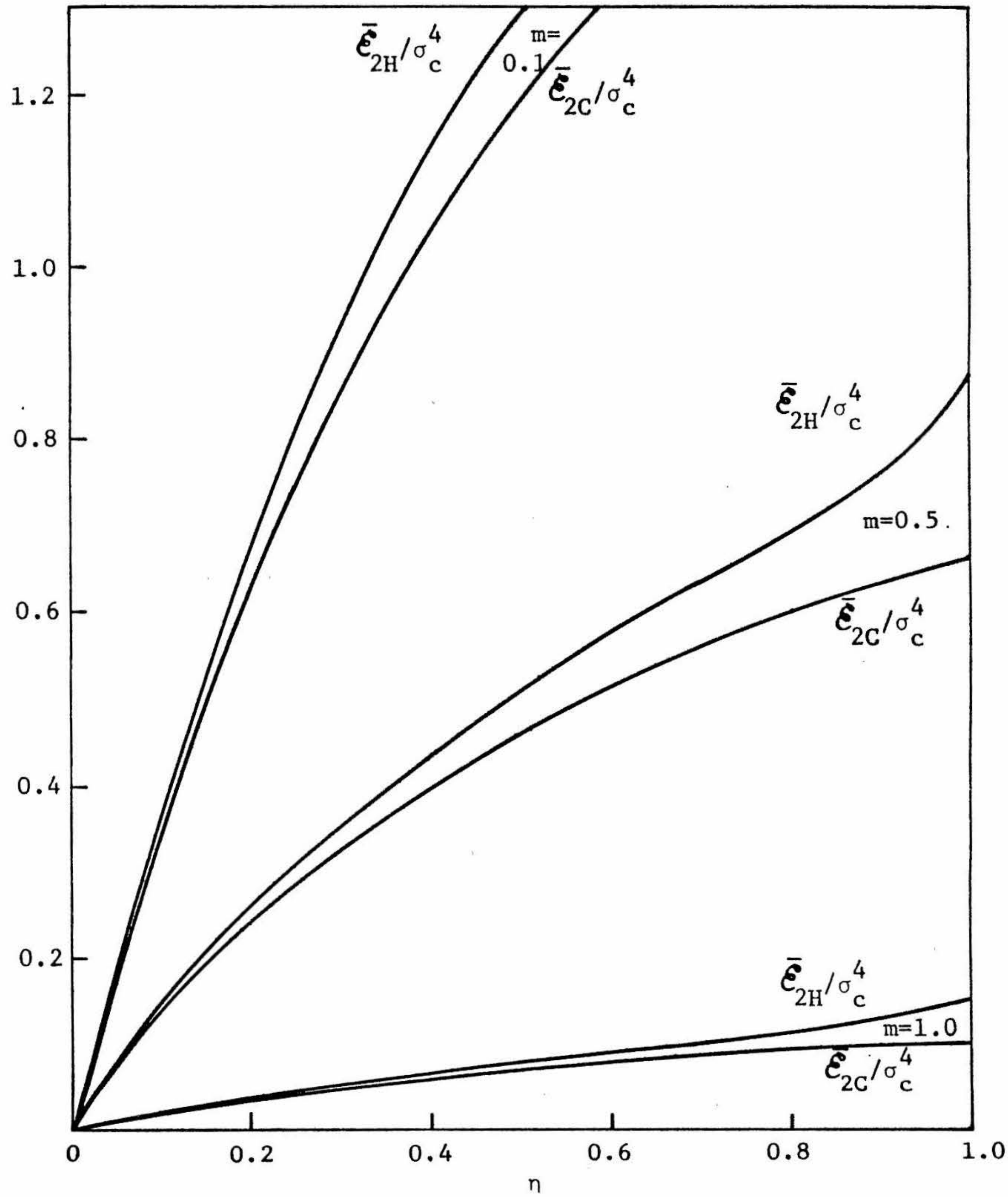


Figure 8

$\bar{\mathcal{E}}_{2C}/\sigma_c^4$, $\bar{\mathcal{E}}_{2H}/\sigma_c^4$ for m = 0.1, 0.5, 1.0

Realizable Filters

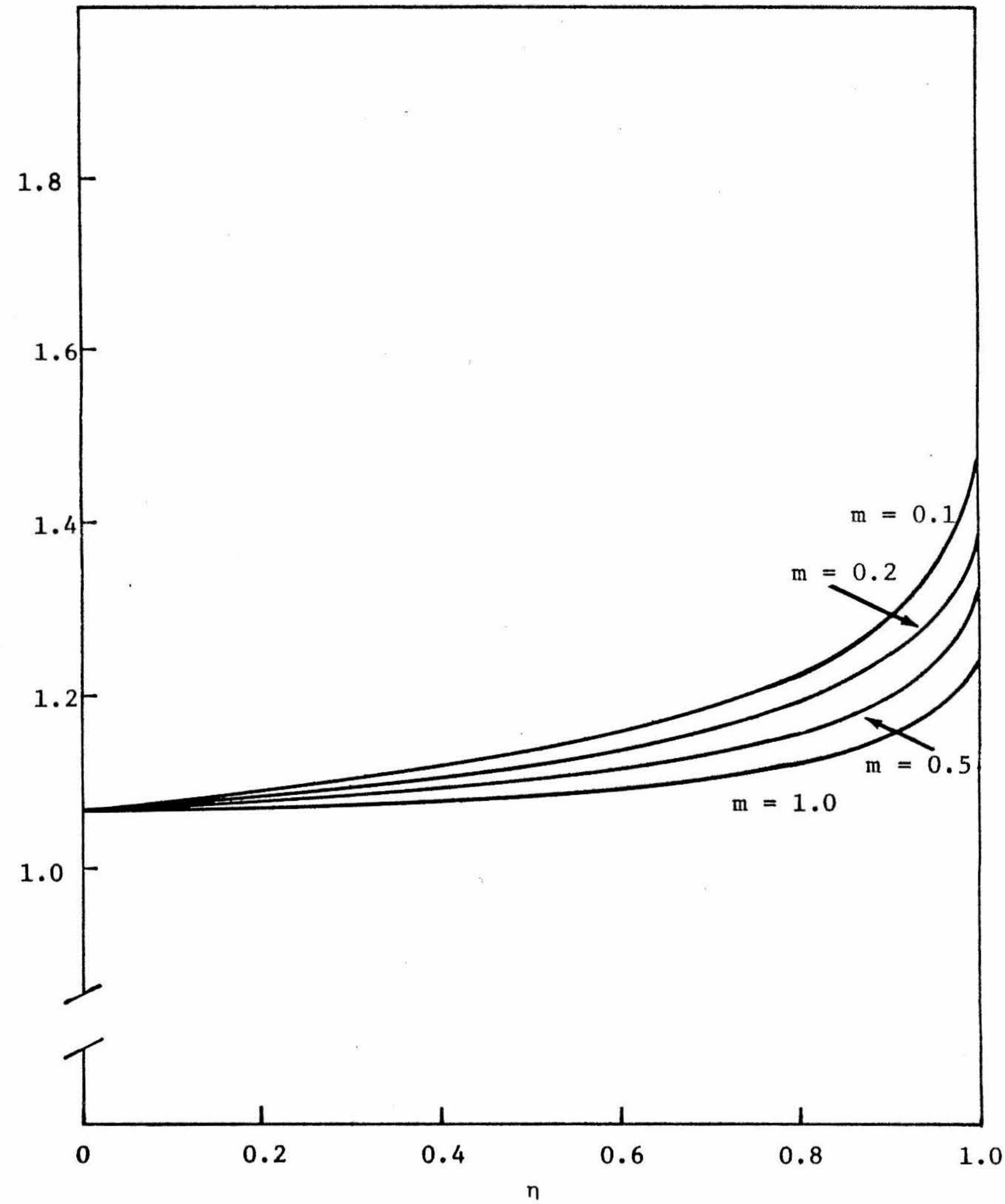


Figure 9

$\bar{\mathcal{E}}_{2H}/\bar{\mathcal{E}}_{2C}$ for m = 0.1, 0.2, 0.5, 1.0

Realizable Filters

Chapter 6

Phase Drift

In chapters 4 and 5 we considered amplitude modulation by an ideal modulator. That is, we assumed that the phase was fixed, a random variable for each sample process of the ensemble. In this chapter we introduce modulator nonideality in the form of phase drift.

6.1 Phase-Envelope Relationship

We shall take the phase $\phi_o(t)$ in (2.01) to be the sum of a random variable ϕ_o and a time varying random process $\theta(t)$. The received waveform $x(t)$ may then be written

$$x(t) = A_o(t) \cos [\omega_o t - \phi_o - \theta(t)] \tag{6.01}$$

The quantities $A_o(t)$, ϕ_o and $\theta(t)$ are mutually statistically independent, each arising from a separate and distinct physical cause. Specifically, the sender has not intended that the phase vary with time and we are not to interpret the phase variation as information. In this sense it is noise, and we desire the detector to provide the best measure of $A_o(t)$ in spite of $\theta(t)$. This is the phase-envelope relationship for independent phase drift, knowledge of which makes possible the separation of the in-phase and quadrature components and allows the correct mathematical representation of the intended envelope.

The in-phase and quadrature components are, from (6.01),

$$x_c(t) = A_o(t) \cos [\phi_o + \theta(t)] \tag{6.02a}$$

$$x_s(t) = A_o(t) \sin [\phi_o + \theta(t)] \tag{6.02b}$$

Under the same stationarity conditions on ϕ_o as were found necessary in Section 4.1 we may write the in-phase and quadrature components of $R_x(\tau)$ defined in (2.06) with the help of (6.02)

$$R_c(\tau) = E\{x_{c1}x_{c2}\} = E\{x_{s1}x_{s2}\} = \frac{1}{2} R_{A_o}(\tau)\ E\left\{\cos[\theta(t+\tau)-\theta(t)]\right\} \tag{6.03a}$$

$$R_s(\tau) = E\{x_{c1}x_{s2}\} = - E\{x_{s1}x_{c2}\} = \frac{1}{2} R_{A_o}(\tau)\ E\left\{\sin[\theta(t+\tau)-\theta(t)]\right\} \tag{6.03b}$$

If we define for notational convenience the phase-dependent quantities $b_c(t_1,t_2)$ and $b_s(t_1,t_2)$,

$$b_c(t_1,t_2) \equiv E\left\{\cos[\theta(t_2) - \theta(t_1)]\right\} \tag{6.04a}$$

$$b_s(t_1,t_2) \equiv E\left\{\sin[\theta(t_2) - \theta(t_1)]\right\} \tag{6.04b}$$

we may write (6.03) as

$$R_c(\tau) = \frac{1}{2} b_c(t,t+\tau)\ R_{A_o}(\tau) \tag{6.05a}$$

$$R_s(\tau) = \frac{1}{2} b_s(t,t+\tau)\ R_{A_o}(\tau) \tag{6.05b}$$

To this point there is no guarantee that $\theta(t)$ is distributed such that $x_c(t)$ and $x_s(t)$ are stationary in (6.03). This will require $b_c(t_1,t_2)$ and $b_s(t_1,t_2)$ to be functions of $t_2 - t_1$, or τ, only, and will in general depend upon the statistical characteristics of $\theta(t)$. We now proceed to the discussion of a specific random phase process which does result in the stationarity of $x_c(t)$ and $x_s(t)$.

6.2 Statistics of the Independent Phase Process

Let the phase process $\theta(t)$ be stationary, zero-mean and normally distributed with autocorrelation function

$$R_\theta(\tau) = E\left\{\theta(t)\ \theta(t+\tau)\right\} \tag{6.06}$$

From (6.05) we see that in order to determine the optimum conjugate filter we must compute $b_c(t_1,t_2)$ and $b_s(t_1,t_2)$. Both functions may be conveniently discussed through the definition of the complex quantity $b(t_1,t_2)$ defined as

$$b(t_1,t_2) = E\left\{e^{j[\theta(t_2)-\theta(t_1)]}\right\} = b_c(t_1,t_2) + jb_s(t_1,t_2) \tag{6.07a}$$

so that

$$b_c(t_1,t_2) = \mathrm{Re}\left\{b(t_1,t_2)\right\}\ ;\ b_s(t_1,t_2) = \mathrm{Im}\left\{b(t_1,t_2)\right\} \tag{6.07b}$$

since $b_c(t_1,t_2)$ and $b_s(t_1,t_2)$ are real functions. The calculation of $b(t_1,t_2)$ is most easily carried out with the aid of the characteristic function $F_\theta(\xi_1,\xi_2)$ of $\theta(t)$. If $p(\theta_1,\theta_2)$ is the second order probability density function of $\theta(t)$ then

$$F_\theta(\xi_1,\xi_2) = \iint_{-\infty}^{\infty} p(\theta_1,\theta_2) e^{j[\xi_1\theta_1 + \xi_2\theta_2]} d\theta_1 d\theta_2 \qquad (6.08)$$

From (6.07a) and (6.08) we see that

$$b(t_1,t_2) = F_\theta(-1,+1) \qquad (6.09)$$

The second order characteristic function of a zero-mean normal process can be derived from (6.08) and is a well known result [5], [14], [18].

$$F_\theta(\xi_1,\xi_2) = \exp\left[-\frac{1}{2}\bar{\xi}\, K_\theta\ \xi\right] \qquad (6.10)$$

where ξ is the column vector with elements ξ_1 and ξ_2, $\bar{\xi}$ is the transpose of ξ, and K_θ is the covariance matrix of the process $\theta(t)$ which, in view of the zero-mean and stationarity conditions on $\theta(t)$, is given by

$$K_\theta \equiv \begin{bmatrix} E\{\theta_1\theta_1\} & E\{\theta_1\theta_2\} \\ E\{\theta_2\theta_1\} & E\{\theta_2\theta_2\} \end{bmatrix} = \begin{bmatrix} R_\theta(0) & R_\theta(\tau) \\ R_\theta(\tau) & R_\theta(0) \end{bmatrix} \qquad (6.11)$$

Thus

$$\bar{\xi}\, K_\theta\ \xi = R_\theta(0)\left\{\xi_1^2 + \xi_2^2\right\} + 2R_\theta(\tau)\ \xi_1\xi_2$$

and

$$F_\theta(\xi_1,\xi_2) = \exp\left[-\frac{1}{2} R_\theta(0)\left(\xi_1^2 + \xi_2^2\right) - R_\theta(\tau)\xi_1\xi_2\right] .$$

Therefore (6.09) yields for $b(t_1,t_2)$, setting $\xi_1 = -1$, $\xi_2 = +1$,

$$b(t_1,t_2) = e^{-\left[R_\theta(0) - R_\theta(\tau)\right]} = b(\tau) \tag{6.12}$$

The function $b(t_1,t_2) = b(t_2-t_1) = b(\tau)$ is independent of the time origin, and is real. From (6.07b) we may write

$$b_c(t_1,t_2) = b_c(\tau) = e^{-\left[R_\theta(0) - R_\theta(\tau)\right]} \tag{6.13a}$$

$$b_s(t_1,t_2) = 0 \tag{6.13b}$$

and from (6.05)

$$R_c(\tau) = \frac{1}{2} b(\tau) R_{A_o}(\tau) = \frac{1}{2} R_{A_o}(\tau) e^{-\left[R_\theta(0) - R_\theta(\tau)\right]} \tag{6.14a}$$

$$R_s(\tau) = 0 \tag{6.14b}$$

so that $x_c(t)$ and $x_s(t)$ are stationary processes and $\theta(t)$ is therefore appropriately distributed. Just as in the case of pure amplitude modulation the in-phase and quadrature components, $x_c(t)$ and $x_s(t)$, are uncorrelated random processes; that is, $R_s(\tau)$ is identically zero. From (2.08) and (2.23) we have

$$R_x(\tau) = R_c(\tau)\cos\omega_o\tau = \frac{1}{2} e^{-[R_\theta(0) - R_\theta(\tau)]} R_{A_o}(\tau)\cos\omega_o\tau \quad (6.15a)$$

$$R_{xx_o}(\tau) = R_c(\tau)\sin\omega_o\tau = \frac{1}{2} e^{-[R_\theta(0) - R_\theta(\tau)]} R_{A_o}(\tau)\sin\omega_o\tau \quad (6.15b)$$

and, finally, from (2.41)

$$\Sigma_x(f) = \frac{1}{2}\left\{\Sigma_c(f-f_o) + \Sigma_c(f+f_o)\right\} \quad (6.16a)$$

$$\Sigma_{xx_o}(f) = \frac{1}{2j}\left\{\Sigma_c(f-f_o) - \Sigma_c(f+f_o)\right\} \quad (6.16b)$$

for the spectral distribution functions where $R_c(\tau)$ and $\Sigma_c(f)$ are a generalized FT pair as discussed in Section 2.5.

Comparing (6.14) with (4.04) for the case of pure amplitude modulation we observe that phase drift causes $R_c(\tau)$ to be multiplied by $b(\tau)$ and hence $d\Sigma_c(f)$ to be convolved with $B(f)$, the FT of $b(\tau)$. The power in $x(t)$, described by (6.1), is obtained from (6.15a)

$$\sigma_x^2 = R_x(0) = R_c(0) = \frac{1}{2} R_{A_o}(0) \quad (6.17)$$

which is the same as for pure amplitude modulation, so that the presence of independent phase drift does not alter the power in the signal. Inasmuch as $d\Sigma_{A_o}(f)$ is convolved with $B(f)$ we expect the continuous part of the spectrum to be spread out to higher frequencies and of lower power at low frequencies.

Chapter 7

Simultaneous Envelope and Phase Modulation

This short chapter is intended to discuss very briefly the situation wherein both the envelope and phase are intentionally modulated by the sender. Generally the phase $\phi_o(t)$ and envelope $A_o(t)$ are related in such a way that the overall signal $x(t)$ is more efficient or economical in some sense than is either the amplitude-modulated or phase-modulated wave alone. We shall be interested in the case in which the sender provides $A_o(t)$ with all the information and intends that we receive the information by envelope detection of $A_o(t)$. The sender also modulates $\phi_o(t)$ to realize some other advantage for the transmitted signal. This area of inquiry seems almost completely untapped, apparently the only comprehensive attempts to treat simultaneous modulation being those of Voelcker [11-12] and Bedrosian [13]. Unfortunately, these treatments are marred by the restrictive assumptions of Fourier transformability and strict narrowbandedness of the signals, assumptions which as we have seen exclude information-bearing random processes. Here is given only a brief introduction to the problem motivated by the work in previous chapters of this thesis.

Briefly stated the result of our investigation so far is that for pure AM the Hilbert representation is appropriate only if the envelope is strictly bandlimited. If the signal is not strictly bandlimited, which is the practical case, then some other representation is optimum. A particular alternative representation has been proposed which while perhaps not optimum yields a lower envelope error. We now ask if the

Hilbert representation can be the appropriate one for a signal whose envelope is band-unlimited but whose phase may vary according to some functional of the envelope. Specifically we ask: for what signal $x(t)$ of the form (2.01) is the optimum estimator of $x_o(t)$ also the Hilbert transform $\hat{x}(t)$?

In view of (2.21) and the definition of the Hilbert representation (3.03) we are seeking a phase-envelope relationship in which $x_o(t)$ is identical with $\hat{x}(t)$ within a sign. That is, given $x(t)$, we desire the functional $f[\ ]$ such that if $\phi_o(t) = f[A_o(t)]$ then

$$\hat{x}(t) = \pm\, x_o(t) = \pm\, A_o(t) \left[\sin \omega_o t - \phi_o(t)\right] \tag{7.01}$$

To assist us in deriving $f[\ ]$, define the complex-valued signal $p(t)$

$$p(t) \equiv A_o(t)\, e^{j\left[\omega_o t - \phi_o(t)\right]} \tag{7.02}$$

We then have

$$x(t) = \mathrm{Re}\left\{p(t)\right\} \tag{7.03a}$$

$$x_o(t) = \mathrm{Im}\left\{p(t)\right\} \tag{7.03b}$$

$$p(t) = x(t) + jx_o(t) \tag{7.03c}$$

Without any loss of generality in choosing the minus sign in (7.01) we require

$$\hat{x}(t) = -\, x_o(t)$$

or, from (7.03)

$$\mathrm{Im}\{p(t)\} = -\mathcal{H}\{\mathrm{Re}[p(t)]\} \tag{7.04}$$

This is just the relationship between the real and imaginary parts of a complex signal which qualifies it to be an analytic signal in the sense of Voelcker. That is, taken as a function of the complex variable z, $p(z)$ is an analytic function everywhere in the complex z-plane. To see the consequences of p-analyticity for $A_o(t)$ and $\phi_o(t)$ consider that the logarithm of a complex signal is analytic in the same region as is the function with the exception of the locations of the zeros of the function. Ignoring the zeros of $p(z)$ for the moment we have from (7.02) that, since $e^{-j\omega_o t}$ is bounded,

$$\ln\left[p(t)e^{-j\omega_o t}\right] = \ln\left[A_o(t)e^{-j\phi_o(t)}\right] = \ln A_o(t) - j\phi_o(t) \tag{7.05}$$

is an analytic function. But this requires that the real and imaginary parts of (7.05) satisfy a relation similar to (7.04).

$$\phi_o(t) = \mathcal{H}\{\ln A_o(t)\} \tag{7.06}$$

This is the phase-envelope relationship for a "minimum-phase" signal derived by Voelcker [11] (see also [4], [12], [13]). In his paper [12] he proceeds to include the effects of the zeros of $p(z)$. We shall not pursue the topic further and refer the reader to the papers cited.

Our result is the phase-envelope relationship for which the Hilbert detector is appropriate. The transmitted signal $x(t)$ takes the form

$$x(t) = A_o(t) \cos \left[\omega_o t - \mathcal{H}\left\{\ln A_o(t)\right\} \right] \tag{7.07}$$

It should be noted however that any attempt to mechanize (7.07) exactly is doomed to failure for, as we have previously mentioned, the Hilbert transform operator is mathematically unrealizable, requiring knowledge of its argument infinitely far into the future as well as past. There is no realizable form of the Hilbert transform except in a formal mathematical sense (see Section 5.2).

Chapter 8

Envelope Statistics -- Unrealizable AM Representations

As was mentioned in Section 4.6.1, while the MS envelope error is a natural quantity of interest, due to the nonlinear nature of the radius detector representation that particular error criterion is not simply expressible in terms of the second-order statistics as is the MSS envelope error. Instead we must actually compute $E\left\{[A_s(t) - A_o(t)]^2\right\}$ (see (4.27) and the associated discussion) from the statistical characteristics of the signals $x(t)$, $x_o(t)$ and $s(t)$. If the received signal $x(t)$ is normally distributed and amplitude-modulated then the threefold joint probability density function of x, x_o and s is completely specified by the second-order statistics previously discussed. The details of deriving joint density functions of normally distributed processes are widely available [5], [14], [18] and only the highlights with needed results are presented here.

8.1 Joint Normal Probability Density of x, x_o, s

Given that x, x_o and s are jointly normal random variables [samples from the jointly normal random processes $x(t)$, $x_o(t)$ and $s(t)$] we define the random column vector

$$V_s = \begin{bmatrix} x \\ x_o \\ s \end{bmatrix} \tag{8.01}$$

where the subscript s indicates that only the variable s will change from case to case (for example, $s = \hat{x}$ or $s = y$). The covariance matrix K_s associated with V_s is given in general by

$$K_s = \begin{bmatrix} \sigma_x^2 & \langle xx_o \rangle & \langle xs \rangle \\ \langle xx_o \rangle & \sigma_{xo}^2 & \langle x_o s \rangle \\ \langle xs \rangle & \langle x_o s \rangle & \sigma_s^2 \end{bmatrix} \qquad (8.02)$$

The matrix may be considerably simplified for our cases of interest. From (2.22) we have $\sigma_{x_o}^2 = \sigma_x^2$. From (2.24) we have

$$\langle xx_o \rangle = R_{xx_o}(0) = 0$$

Since $s(t)$ is the output of a filter $G(f)$ with $x(t)$ as input

$$\langle xs \rangle = \int_{-\infty}^{\infty} G(f) S_x(f)\, df$$

If $s(t) = \hat{x}(t)$, the unrealizable HT of $x(t)$, then $G(t) = H(f) = -j\,\text{sgn}(f)$ so that $G(f)$ is odd and $\langle x\hat{x} \rangle = 0$. If $s(t) = y(t)$, the output of the unrealizable conjugate filter, then $G(f) = C(f)_{\text{unreal.}}$ which is odd [see (4.18)] and therefore $\langle xy \rangle = 0$. Under these conditions the matrix becomes

$$K_s = \begin{bmatrix} \sigma_x^2 & 0 & 0 \\ 0 & \sigma_x^2 & \langle x_o s\rangle \\ 0 & \langle x_o s\rangle & \sigma_s^2 \end{bmatrix} \quad , \quad s = \hat{x} \text{ or } y \tag{8.03}$$

where σ_x^2, σ_s^2 and $\langle x_o s\rangle$ are the second-order statistics for pure AM computed in Section 4.5.2. The determinant of K_s is $|K_s|$

$$|K_s| = \sigma_x^2\left[\sigma_x^2\sigma_s^2 - \langle x_o s\rangle^2\right] \tag{8.04}$$

The matrix inverse to K_s is K_s^{-1} where

$$K_s^{-1} = \frac{1}{|K_s|}\begin{bmatrix} \sigma_x^2\sigma_s^2 - \langle x_o s\rangle^2 & 0 & 0 \\ 0 & \sigma_x^2\sigma_s^2 & -\sigma_x^2\langle x_o s\rangle \\ 0 & -\sigma_x^2\langle x_o s\rangle & \sigma_x^4 \end{bmatrix} \tag{8.05}$$

If $\overline{V}_s$ is the row vector transpose of V_s the quantity $\overline{V}_s K_s^{-1} V_s$ is

$$\overline{V}_s K_s^{-1} V_s = \frac{x^2}{\sigma_x^2} + \frac{1}{\sigma_x^2\sigma_s^2 - \langle x_o s\rangle^2}\left[\sigma_s^2 x_o^2 + \sigma_x^2 s^2 - 2\langle x_o s\rangle x_o s\right]$$

The threefold probability density $p(V_s) = p(x, x_o, s)$ is therefore given by

$$p(V_s) = p(x,x_o,s) = \frac{1}{\left[(2\pi)^3 |K_s|\right]^{\frac{1}{2}}} e^{-\frac{1}{2}\bar{V}_s K_s^{-1} V_s}$$

$$= \frac{1}{\left\{(2\pi)^3 \sigma_x^2 (\sigma_x^2 \sigma_s^2 - \langle x_o s\rangle^2)\right\}^{\frac{1}{2}}} \exp\left[-\frac{x^2}{2\sigma_x^2} - \frac{\sigma_s^2 x_o^2 + \sigma_x^2 s^2 - 2\langle x_o s\rangle x_o s}{2(\sigma_x^2\sigma_s^2 - \langle x_o s\rangle^2)}\right] \quad (8.06)$$

We have arrived at the desired threefold probability density from which can, in principle at least, be computed $E\left\{(A_s - A_o)^2\right\}$. We defer further computation of the MS error to a later section and take up now the question of the first order probability density function assoicated with the envelope $A_s(t)$ where $s = x_o$, $\hat{x}$ or y denoting the intended envelope, the Hilbert representation and the conjugate representation respectively.

8.2 First Order Density of $A_s(t)$

Since

$$A_s(t) = \left[x^2(t) + k\, s^2(t)\right]^{\frac{1}{2}}$$

in the modified representation, the first order density of $A_s(t)$ may be obtained for a particular k from the joint density of $x(t)$ and $s(t)$. This may be derived from (8.06) by integration with respect of x_o, or it may simply be written directly, keeping in mind that $x(t)$ and $s(t)$ are independent (uncorrelated and normally distributed) for the signals $s(t)$ of interest.

$$p(x,s) = p(x)p(s) = \frac{1}{2\pi\sigma_x\sigma_s} e^{-\frac{1}{2}\left[\frac{x^2}{\sigma_x{}^2} + \frac{s^2}{\sigma_s{}^2}\right]} \tag{8.07}$$

Details of the following calculations are given in Appendix 8.2. The probability distribution function of $A_s(t)$ is

$$\text{Prob}\left\{A_s \leqslant R\right\} = \frac{2}{\pi\sqrt{k}\sigma_x\sigma_s}\int_o^R du \int_o^{\sqrt{R^2-u^2}} e^{-\frac{1}{2}\left[\frac{x^2}{\sigma_x{}^2} + \frac{u^2}{k\sigma_s{}^2}\right]} dx \tag{8.08}$$

The corresponding probability density function is

$$p(A_s) = \left.\frac{\partial Pr\left\{A_s \leqslant R\right\}}{\partial R}\right|_{R=A_s\geqslant 0} = \frac{A_s U(A_s)}{\sqrt{k}\sigma_x\sigma_s} e^{-\frac{A_s^2}{4}\left[\frac{1}{k\sigma_s^2} + \frac{1}{\sigma_x^2}\right]} I_o\left[\frac{A_s^2}{4}\left(\frac{1}{k\sigma_s^2} - \frac{1}{\sigma_x^2}\right)\right] \tag{8.09}$$

where $I_o(\ )$ is the modified zero order Bessel function of the first kind [21].

8.2.1 Intended Envelope Probability Density

As a special case, if $s(t) = x_o(t)$ then $\sigma_s^2 = \sigma_x^2$ and, if $k = 1$, (8.09) reduces to

$$p(A_o) = \frac{A_o U(A_o)}{\sigma_x^2} e^{-\frac{A_o^2}{2\sigma_x^2}} \tag{8.10}$$

since $I_o(0) = 1$. This is the commonly quoted Rayleigh first order density of the envelope [5], [14], [18]. In our present perspective we see that it is the first order density of the intended envelope for an AM normally distributed process.

8.2.2 Hilbert Representation

If the filter in the radius detector is such that $s(t) = \hat{x}(t)$, then $\sigma_s^2 = \sigma_{\hat{x}}^2 = \sigma_x^2$. If no gain adjustment is made, $k = 1$ and $p(A_{\hat{x}}) = p(A_o)$ as given by (8.10). If however the gain adjustment is made according to Section 4.6.3, and if we use the second-order statistics calculated in Section 4.5.2, we have

$$k = \frac{1}{3}\left[1 + 2\left(1 - \frac{mq}{1+m}\right)^2\right] \tag{8.11a}$$

$$p(A_{\hat{x}}) = \frac{A_{\hat{x}} U(A_{\hat{x}})}{\sqrt{k}\,\sigma_x^2}\, e^{-\frac{A_{\hat{x}}^2}{4\sigma_x^2}\left(\frac{1+k}{k}\right)} I_o\left[\frac{A_{\hat{x}}^2}{4\sigma_x^2}\left(\frac{1-k}{k}\right)\right] \tag{8.11b}$$

where we recall from (4.34) that $m \equiv \sigma_a^2/\sigma_c^2$ and from (4.32) that $q \equiv [2/\pi]\sin^{-1}\eta$.

8.2.3 Conjugate Representation

If $s(t) = y(t)$ and the second-order statistics of Section 4.5.2 are used, the gain adjustment k is

$$k = \frac{1 + m - \frac{m}{3}\eta}{1 + m - m\eta} \tag{8.12}$$

and the probability density of the conjugate detector output is

$$p(A_y) = \frac{(1+m)A_y U(A_y)}{\sigma_c^2 (1+m-\frac{m}{3}\eta)^2} e^{-\frac{A_y^2}{(1+m)\sigma_c^2}\left[\frac{1+m-\frac{m}{6}\eta}{1+m-\frac{m}{3}\eta}\right]} I_o\left[\frac{A_y^2}{(1+m)\sigma_c^2} \frac{\frac{m}{6}\eta}{1+m-\frac{m}{3}\eta}\right] \tag{8.13}$$

As the process becomes very narrowband, $\eta \to 0$ and $p(A_c) \to p(A_o)$.

8.3 MS Envelope Error -- Amplitude Modulation

According to (4.27) the evaluation of the MS envelope error $\mathcal{E}_{A_oA_s}$ requires the computation of $E\left\{(A_o - A_s)^2\right\}$ which, with the insertion of the gain factor k introduced in Section 4.6.3, amounts to the computation of

$$\mathcal{E}_S \equiv E\left\{(A_o - A_s)^2\right\} = E\left\{\left[\sqrt{x^2 + x_o^2} - \sqrt{x^2 + ks^2}\right]^2\right\} \tag{8.14}$$

where $s = \hat{x}$ for the Hilbert representation, $s = y$ for the conjugate representation, and we denote the MS error as $\mathcal{E}_S$ to distinguish from the MSS error $\mathcal{E}_{2S}$. Since we are evaluating a different envelope error criterion the value of k which minimizes $\mathcal{E}_S$ may be given by some relation other than (4.38). The error $\mathcal{E}_S$ may be written with the aid of (8.06)

$$E\left\{(A_o-A_s)^2\right\} = \frac{1}{\left[(2\pi)^3\sigma_x^2(\sigma_x^2\sigma_s^2 - \langle x_o s\rangle^2)\right]^{\frac{1}{2}}} \tag{8.15}$$

$$\times \iiint_{-\infty}^{\infty} \left[\sqrt{x^2+x_o^2} - \sqrt{x^2+ks^2}\right]^2 e^{-\frac{x^2}{2\sigma_x^2} - \frac{\sigma_s^2 x_o^2 + \sigma_x^2 s^2 - 2\langle x_o s\rangle x_o s}{2\left[\sigma_x^2\sigma_s^2 - \langle x_o s\rangle^2\right]}} \, dx\,dx_o\,ds$$

which may be written

$$E\left\{(A_o-A_s)^2\right\} \tag{8.16}$$

$$= \frac{2\sigma_x^2}{\pi^{3/2}\sqrt{1-\rho_s^2}} \iiint_{-\infty}^{\infty} \left[\sqrt{u^2+v^2} - \sqrt{u^2+\alpha_s w^2}\right]^2 e^{-u^2} e^{-\frac{1}{1-\rho_s^2}(v^2+w^2-2\rho_s vw)} \, du\,dv\,dw$$

upon normalization and definition of the parameters

$$\rho_s \equiv \frac{E\{x_o s\}}{\sigma_x\sigma_s} \qquad \alpha_s = k\frac{\sigma_s^2}{\sigma_x^2} \tag{8.17}$$

Solution of (8.16) would yield the MS error $\mathcal{E}_S$ in terms of σ_x^2, ρ_s and α_s. However the author has not been able to obtain the closed form solution of the integral. We shall have to settle for the evaluation of a limiting case, namely the narrowband limit as $\eta \to 0$, and base on this limited information our initial comparison between the Hilbert and conjugate representations. In the Hilbert case we shall require k to be identically unity; that is, we shall consider only the classical input-independent Hilbert representation.

8.3.1 Narrowband Limit -- Unrealizable Filters

The received signal spectrum becomes narrowband as $\eta \to 0$. In (8.16) the parameters ρ_s and α_s tend to limits as $\eta \to 0$ such that the integral is sufficiently simplified to admit an exact solution. In addition to the value of $\mathcal{E}_S$ for $\eta = 0$ we are interested in the first order term as $\eta \to 0$. It will turn out that this term is proportional to η. To differentiate between the conjugate and Hilbert representations we shall refer to the quantities ρ_c, α_c and ρ_h, α_h respectively.

In the conjugate representation the parameters in (8.17) tend, in the limit as $\eta \to 0$, to

$$\left.\begin{aligned} \rho_c &\equiv \frac{\langle x_o y\rangle}{\sigma_x \sigma_y} = \frac{\sigma_y}{\sigma_x} = \sqrt{1 - \frac{m}{1+m}\eta} \to 1 - \frac{m}{2(1+m)}\eta \\ 1-\rho_c^2 &= \frac{m}{1+m}\eta \\ \alpha_c &\equiv k\frac{\sigma_y^2}{\sigma_x^2} = k\left(1 - \frac{m}{1+m}\eta\right) \end{aligned}\right\} \qquad (8.18)$$

In the Hilbert representation the parameters tend to

$$\left.\begin{aligned}
\rho_h &\equiv \frac{\langle x_o \hat{x} \rangle}{\sigma_x \sigma_{\hat{x}}} = 1 - \frac{2m}{\pi(1+m)} \sin^{-1}\eta \to 1 - \frac{2m}{\pi(1+m)}\eta \\
1-\rho_h^2 &\to \frac{4m}{\pi(1+m)}\eta \\
\alpha_h &\equiv k \frac{\sigma_{\hat{x}}^2}{\sigma_x^2} = k \equiv 1
\end{aligned}\right\} \quad (8.19)$$

We see that in the limit $\eta = 0$, $\rho_c = \rho_h = 1$, $\alpha_c = \alpha_h = k$ so that $E\{A_o A_{\hat{x}}\} = E\{A_o A_y\}$ and $\mathcal{E}_H = \mathcal{E}_C$. It will turn out, as it did for the MSS envelope error that $\lim \mathcal{E}_S = 0$. We are especially anxious therefore to compute the first order η-dependence to provide a comparison. Since the corresponding parameters in (8.18) and (8.19) differ only in the coefficient of η we shall carry through the calculations in terms of

$$\rho_s = 1 - \tfrac{1}{2}\mu_s\eta \ ; \quad 1-\rho_s^2 \to \mu_s\eta \ ; \quad \mu_s = \begin{cases} \mu_c = \dfrac{m}{1+m} \\ \mu_h = \dfrac{4}{\pi}\dfrac{m}{1+m} \end{cases} \quad (8.20)$$

and α_s where $\mu_s = \mu_c$ or μ_h. The details of the calculation appear in Appendix 8.3.1. The solution of (8.16) to first order in η is

$$\mathcal{E}_S = \sigma_x^2 \left\{ 3 + \alpha_s - \frac{2}{\pi}\left[4E(1-\alpha_s) + \mu_s \eta \frac{\alpha_s}{1-\alpha_s}\left\{ E(1-\alpha_s) - K(1-\alpha_s) \right\} \right] \right\} \quad (8.21)$$

the next term being proportional to η^2 where $K(\ )$ and $E(\ )$ are the complete elliptic integrals of the first and second kinds respectively, and where $\mathcal{E}_S$ is the conjugate or the Hilbert MS error depending upon the choice of μ_s in (8.20). The limit as $\eta \to 0$ is

$$\lim_{\eta \to 0} \mathcal{E}_S = \sigma_x^2 \left[3 + k - \frac{8}{\pi} E(1 - k) \right] \quad (8.22)$$

Since $\mathcal{E}_S$ is a MS quantity we must have $\mathcal{E}_S \geq 0$. Indicating that k might depend upon η (as in the MSS case but with perhaps a different functional dependence) by writing $k(\eta)$ we seek $k(0)$ such that $\lim_{\eta \to 0} \mathcal{E}_S$ is minimum. If $k(0) = 1$ then

$$\lim_{\eta \to 0} \mathcal{E}_S = \sigma_x^2 \left[4 - \frac{8}{\pi} E(0) \right] = 0$$

and since this is the minimum value possible for $\mathcal{E}_S$, $k(0) = 1$ is the proper choice. Thus the two envelope representations are indistinguishable at $\eta = 0$, both having zero MS envelope error, consistent with the results of Chapter 4 for MSS error. As η increases (8.21) indicates that the corresponding increase in $\mathcal{E}_S$ depends upon μ_s and hence upon the particular representation. In Appendix 8.3.1 it is shown that the behavior of $\mathcal{E}_S$ to first order as $\eta \to 0$ is

$$\mathcal{E}_S \longrightarrow \frac{k}{2} \sigma_s^2 \mu_s \eta \quad (8.23)$$

Substituting for μ_s from (8.20) and for σ_y^2 from (4.21) we have for the conjugate representation

$$\mathcal{E}_C \rightarrow \frac{k}{2} \sigma_x^2 \frac{m}{1+m} \eta \tag{8.24}$$

and for the Hilbert representation, since $\sigma_{\hat{x}}^2 = \sigma_x^2$ from (4.20),

$$\mathcal{E}_H \rightarrow \frac{2}{\pi} \sigma_x^2 \frac{m}{1+m} \eta \tag{8.25}$$

from which it is clear that the conjugate representation gives the lower MS error as $\eta \rightarrow 0$. The error ratio (Hilbert to conjugate) is, in the limit, since $k(0) = 1$,

$$\lim_{\eta \rightarrow 0} \frac{\mathcal{E}_H}{\mathcal{E}_C} = \sqrt{\frac{4}{\pi}} = 1.13 \tag{8.26}$$

Based on the results of the MSS error investigation in Chapter 4 and on our intuitive feeling for the way in which the Hilbert filter operates we suspect that this is the best performance of the Hilbert relative to the conjugate detector and that the error ratio $\mathcal{E}_H/\mathcal{E}_C$ increases as η increases. This is of course only a guess and we have definite information only for η very small. It is interesting to recall from Section 4.6.5 the limiting values of the MSS error in the same limit as $\eta \rightarrow 0$. From (4.43) they are

$$\mathcal{E}_{2C} \longrightarrow m(1+m)\ \sigma_c^4 \eta \tag{8.27}$$

$$\mathcal{E}_{2H} \longrightarrow \frac{4m(1+m)}{\pi}\ \sigma_c^4 \eta \tag{8.28}$$

$$\lim_{\eta \to 0} \frac{\mathcal{E}_{2H}}{\mathcal{E}_{2C}} = \frac{4}{\pi} = 1.27 \tag{8.29}$$

Thus, in the limit as $\eta \to 0$, $\mathcal{E}_{2H}/\mathcal{E}_{2C} = [\mathcal{E}_H/\mathcal{E}_C]^2$.

8.3.2 Low Modulation Limit

If we study the error in the limit as $m \to 0$ in which the ratio of modulation power σ_a^2 to carrier power σ_c^2 becomes very small, we find that the parameters given in (8.20) have exactly the same form as in the preceding section. The errors and error ratio are thus the same but are to be interpreted as limits as $m \to 0$. Actually no new information is obtained by this consideration since we may view the vanishing of modulation as an equivalent narrowband limit.

8.3.3 Conclusions

While the general solution to the evaluation of the MS error (8.16) has not been obtained, we have seen that in specific limits the conjugate representation exhibits lower MS envelope error than does the Hilbert representation. This result is in general agreement with the results of Chapter 4 where the evaluation of the MSS envelope error for unrealizable representations was performed.

Chapter 9

Summary and Conclusion

We have approached the question of envelope representation emphasizing the intent of the sender and allowing the possibility that the signal phase as well as amplitude may exhibit modulation. The envelope of the signal $x(t)$ is not a unique property of $x(t)$ alone. Some independent information concerning the behavior of the phase, or the relationship between envelope and phase, must be supplied in order to uniquely represent the intended envelope in terms of the received signal $x(t)$. Since the envelope is not uniquely determined by $x(t)$ alone, at least in some instances it must be a different function than the one given by the classical Hilbert representation. We have seen, in fact, that the Hilbert representation provides an exact estimate of the envelope if and only the received signal is strictly bandlimited.

The envelope representation investigation has been limited to the radius detector approach to provide a direct comparison with the Hilbert representation. A fundamental signal related to $x(t)$, the conjugate signal $x_o(t)$, was identified and the intended envelope was shown to be representable in terms of $x(t)$ and $x_o(t)$. However the conjugate signal can be derived exactly from $x(t)$ only when $x(t)$ is strictly bandlimited. In the bandunlimited case, which is the practically interesting case, the conjugate signal and hence the intended envelope can be only inexactly estimated. The radius detector filter would ideally be designed to minimize the error in estimating the envelope

$A_o(t)$. Due to the nonlinear nature of the detector this problem has not been solved. The approach taken was the optimization of the filter with respect to the conjugate signal, the argument being that a filter which produces a good estimate of $x_o(t)$ will result in a radius detector which produces a good estimate of $A_o(t)$. Specifically we have sought the filter which minimizes the MS error between its output $s(t)$ and the conjugate signal $x_o(t)$. This is the conjugate filter. The supposition that a detector employing such a filter produces a good estimate of $A_o(t)$ was verified through the calculation of the MSS estimation error, or MS error between $A_o^2(t)$ and the square of the detector output in Chapters 4 and 5. This error criterion was chosen for its relative tractability. The calculation of the MSS error was based on the assumptions of amplitude modulation, both unrealizable and realizable conjugate filters, and a particular bandunlimited received signal PSD, namely the single-RC spectrum. Also a filter modification was introduced wherein the gain of the filter path was altered by a constant factor to produce a lower overall envelope error. In all cases the conjugate representation exhibited lower overall envelope error than the corresponding Hilbert filter. The Hilbert filter competed most successfully in the narrowband limit where, in the unrealizable case, the ratio of Hilbert to conjugate error was $4/\pi = 1.27$. In the realizable case this ratio limit was $3/4 + 1/\pi = 1.07$. It should be noted that, even in the realizable case, the conjugate filter is not directly synthesizable, due to the discontinuous nature of its transfer function. Of course the Hilbert filter is also not synthesizable. No attempt to compare synthesizable versions of these two filters was made in the thesis.

It was shown in Chapter 6 that even when the phase is time varying the envelope may be appropriately represented in the instance that the phase is a process independent of the intended envelope and appropriately distributed. This is the case, for example, when the transmitting oscillator is subject to phase drift. Given such a phase-envelope relationship it is possible to derive the spectral parameters $S_c(f)$ and $S_s(f)$ with which to compute the corresponding conjugate filter.

A brief discussion was given of the simultaneous modulation of amplitude and phase in Chapter 7. The modulation process was derived for which the conjugate filter is identical with the Hilbert filter.

In Chapter 8 the MS envelope error was computed in the narrowband limit. Again the conjugate filter resulted in the lower envelope error.

The conjugate filtering approach to the representation of signal envelope has merit in that it yields a lower overall envelope estimation error than the classical Hilbert representation in the band-limited case. At least we have shown this to be true for the single-RC spectrum. The actual error calculations for higher order rational spectra would follow the patterns set in Chapters 4 and 5 but with a corresponding increase in computational complexity. Inasmuch as the higher order spectra approach the narrowband case we would expect the advantage of conjugate over Hilbert filtering to decrease as the order of the PSD increases.

Appendix 3.3

Proof that $|C(f)| \leqslant 1$

$C(f)$ is given in (3.15) as

$$C(f) = \frac{d\Sigma_{xx_o}(f)}{d\Sigma_x(f)}$$

Defining the quantities

$$J_1(f) \equiv \frac{1}{2}\left[d\Sigma_c(f) - d\Sigma_s(f)\right]$$

$$J_2(f) \equiv \frac{1}{2}\left[d\Sigma_c(f) + d\Sigma_s(f)\right]$$

we may write from (2.41)

$$d\Sigma_x(f) = J_1(f - f_o) + J_2(f + f_o)$$

$$d\Sigma_{xx_o}(f) = -j\left[J_1(f - f_o) - J_2(f + f_o)\right]$$

Since $\Sigma_x(f)$ is the SDF of a real process, it is strictly nondecreasing and hence $d\Sigma_x(f)$ is strictly nonnegative. This implies that $J_1(f)$ and $J_2(f)$ are each nonnegative. To see this assume $J_1(f)$ is actually negative for some f_1. Then $J_2(2f_o + f_1)$ must be enough positive that the sum $d\Sigma_x(f_o + f_1) = J_1(f_1) + J_2(2f_o + f_1)$ is nonnegative. However, $J_1(f)$ and $J_2(f)$ both tend to zero as $|f| \to \infty$ since $\Sigma_x(f)$ represents a finite power process. Also, $J_1(f)$ and $J_2(f)$ are independent of f_o. Therefore it is possible to choose f_o

large enough such that $J_2(f_1 + 2f_o)$ fails to cancel the negative contribution of $J_1(f_1)$, leading to a negative value for $d\Sigma_x(f_o + f_1)$, a contradiction. Therefore $J_1(f) \geqslant 0$, $J_2(f) \geqslant 0$. The filter $C(f)$ may be written

$$C(f) = -j \frac{J_1(f - f_o) - J_2(f + f_o)}{J_1(f - f_o) + J_2(f + f_o)}$$

$$= -j \frac{\dfrac{J_1(f - f_o)}{J_2(f + f_o)} - 1}{\dfrac{J_1(f - f_o)}{J_2(f + f_o)} + 1}$$

The range of $J_1(f - f_o)/J_2(f + f_o)$ is 0 to ∞ so that $j\,C(f)$ ranges from -1 through 0 to +1 and hence $|C(f)| \leqslant 1$ as was to be proved.

Appendix 3.4

Evaluation of D(f)

The generalized function D(f) was defined in (3.22) as

$$D(f) = \int_0^{\infty} e^{j2\pi ft} \, dt$$

The evaluation of D(f) is carried out by integrating the product of D(f − u) and a test function χ(f)

$$I(f) = \int_{-\infty}^{\infty} D(u-f)\, \chi(u)\, du$$

$$= \int_{-\infty}^{\infty} \chi(u) \int_0^{\infty} e^{j2\pi(u-f)t} \, dt \; du$$

$$\lim_{T \to \infty} \int_{-\infty}^{\infty} \chi(u) \int_0^{T} e^{j2\pi(u-f)t} \, dt \; du$$

$$= \lim_{T \to \infty} \int_{-\infty}^{\infty} \chi(u) \, \frac{e^{j2\pi(u-f)T} - 1}{j2\pi(u-f)} \, du$$

$$= \frac{-1}{j2\pi} \int_{-\infty}^{\infty} \frac{\chi(u)}{u-f} \, du + \lim_{T \to \infty} \int_{-\infty}^{\infty} \frac{\chi(u)\, e^{j2\pi(u-f)T}}{j2\pi(u-f)} \, du$$

$$= -j \, \frac{\hat{\chi}(f)}{2} + \lim_{T \to \infty} \int_{-\infty}^{\infty} \frac{\chi(\frac{\omega}{T} + f)\, e^{j2\pi\,\omega}}{j2\pi\,\omega} \, d\omega$$

$$= -\frac{j}{2}\hat{\chi}(f) + \chi(f)\int_{-\infty}^{\infty}\frac{e^{j2\pi\omega}}{j2\pi\omega}\,d\omega = -\frac{j}{2}\hat{\chi}(f) + \frac{1}{2}\chi(f)$$

$$= \frac{1}{2}\,[\chi(f) - j\hat{\chi}(f)]$$

which is the result shown in (3.24). This implies (3.23)

$$D(f) = \frac{1}{2}\left\{\delta(f) - \frac{j}{\pi f}\right\}.$$

Appendix 4.5.2

Second-Order Statistics -- Unrealizable Filters

A. The cross moment $\langle x_o y\rangle$ may be written

$$\langle x_o y\rangle = \int_{-\infty}^{\infty} C^*(f)\,d\Sigma_{xx_o}(f) = \int_{-\infty}^{\infty} C^*(f)\frac{d\Sigma_{xx_o}(f)}{d\Sigma_x(f)}\,d\Sigma_x(f) = \int_{-\infty}^{\infty} |C(f)|^2 d\Sigma_x(f)$$

$$= \sigma_y^2 = \frac{\sigma_c^2}{4}\left\{|C(f_o)|^2 + |C(-f_o)|^2\right\} + \int_{|f|\neq f_o}^{\infty} |C_2(f)|^2 d\Sigma_x(f)$$

$$= \frac{\sigma_c^2}{2} + \frac{1}{4}\int_{-\infty}^{\infty} \frac{\left\{S_a(f-f_o) - S_a(f+f_o)\right\}^2}{S_a(f-f_o) + S_a(f+f_o)}\,df$$

$$= \frac{\sigma_c^2}{2} + \frac{2\beta\sigma_a^2 f_o^2}{\pi}\int_{-\infty}^{\infty} \frac{f^2}{\left(f^2+\gamma^2\right)\left(f^2-2f_of+\gamma^2\right)\left(f^2+2f_of+\gamma^2\right)}\,df$$

The integral is of the form

$$-\int_{-\infty}^{\infty} \frac{g_3(f)}{h_3(f)h_3(-f)}\,df$$

where

$$h_3(f) \equiv a_o f^3 + a_1 f^2 + a_2 f + a_3$$

$$g_3(f) \equiv b_o f^4 + b_1 f^2 + b_2$$

so that $a_o = 1$, $a_1 = -j(\gamma + 2\beta)$, $a_2 = -\gamma(\gamma + 2\beta)$

$a_3 = j\gamma^3$, $b_o = b_2 = 0$, $b_1 = 1$

The solution to the integral is [17]

$$-\pi j \frac{-a_2 a_3 b_o + a_o a_3 b_1 - a_o a_1 b_2}{a_o a_3(a_o a_3 - a_1 a_2)} = \frac{\pi}{4\beta\gamma(\gamma+\beta)}$$

so that

$$\langle x_o y \rangle = \sigma_y^2 = \frac{1}{2}\left[\sigma_c^2 + \frac{f_o^2}{\gamma(\gamma+\beta)}\sigma_a^2\right] = \frac{1}{2}\left[\sigma_c^2 + (1-\eta)\sigma_a^2\right]$$

as in (4.21).

B. The cross moment $\langle x_o \hat{x} \rangle$ may be written

$$\langle x_o \hat{x} \rangle = \int_{-\infty}^{\infty} H^*(f)\, d\Sigma_{xx_o}(f) = \frac{\sigma_c^2}{4j}\left[H^*(f_o) - H^*(-f_o)\right] + \int_{|f|\neq f_o}^{\infty} H^*(f)\, d\Sigma_{xx_o}(f)$$

$$= \frac{\sigma_c^2}{2} + \frac{\beta\sigma_a^2}{4\pi}\int_{-\infty}^{\infty} \operatorname{sgn}(f)\left[\frac{1}{(f-f_o)^2+\beta^2} - \frac{1}{(f+f_o)^2+\beta^2}\right] df$$

$$= \frac{\sigma_c^2}{2} + \frac{\beta\sigma_a^2}{2\pi}\int_{o}^{\infty} \left[\frac{1}{(f-f_o)^2+\beta^2} - \frac{1}{(f+f_o)^2+\beta^2}\right] df$$

$$= \frac{\sigma_c^2}{2} + \frac{\beta\sigma_a^2}{2\pi}\left\{\int_{-f_o}^{\infty} \frac{du}{u^2+\beta^2} - \int_{f_o}^{\infty} \frac{du}{u^2+\beta^2}\right\}$$

$$= \frac{\sigma_c^2}{2} + \frac{\sigma_a^2}{\pi}\int_{o}^{f_o/\beta} \frac{du}{1+u^2} = \frac{\sigma_c^2}{2} + \frac{\sigma_a^2}{\pi} \tan^{-1}\frac{f_o}{\beta}$$

$$= \frac{1}{2}\left[\sigma_c^2 + \sigma_a^2\left(\frac{2}{\pi}\cos^{-1}\eta\right)\right]$$

as in (4.22).

Appendix 5.1.3

Second-Order Statistics--Realizable Conjugate Filter

From

$$C_2(f) = -\sqrt{\frac{1-\eta}{1+\eta}}\;\frac{f+j\gamma}{f-j\gamma}$$

we have

$$\sigma_y^2 = \frac{\sigma_c^2}{2} + \int_{\substack{-\infty \\ |f|\neq f_o}}^{\infty} |C_2(f)|^2 \, d\Sigma_x(f)$$

$$= \frac{\sigma_c^2}{2} + \frac{1-\eta}{1+\eta} \int_{\substack{-\infty \\ |f|\neq f_o}}^{\infty} d\Sigma_x(f)$$

$$= \frac{1}{2}\left[\sigma_c^2 + \frac{1-\eta}{1+\eta}\,\sigma_a^2\right]$$

as in (5.08b). Also

$$\langle x_o y\rangle = \int_{-\infty}^{\infty} C^*(f)\, d\Sigma_{xx_o}(f) = \frac{\sigma_c^2}{4j}\,[C^*(f_o) - C^*(-f_o)] + \int_{\substack{-\infty \\ |f|\neq f_o}}^{\infty} C_2^*(f)\, d\Sigma_{xx_o}(f)$$

$C_2^*(f)$ is given by

$$C_f^*(f) = -\sqrt{\frac{1-\eta}{1+\eta}}\;\frac{f-j\gamma}{f+j\gamma} = -\sqrt{\frac{1-\eta}{1+\eta}}\;\frac{f^2-\gamma^2-2j\gamma f}{f^2+\gamma^2}$$

The quantity $d\Sigma_{xx_o}(f)$ is odd so that the real part of $C_2^*(f)$ drops out

$$\langle x_o y\rangle = \frac{\sigma_c^2}{2} + 2\sqrt{\frac{1-\eta}{1+\eta}}\,\frac{\gamma f_o \beta \sigma_a^2}{\pi} \int_{-\infty}^{\infty} \frac{f^2\, df}{[f^2+\gamma^2][f^2-2f_o f+\gamma^2][f^2+2f_o f+\gamma^2]}\, df$$

This integral was evaluated in Appendix 4.5.2.

$$\langle x_o y\rangle = \frac{\sigma_c^2}{2} + \frac{\sigma_a^2}{2}\sqrt{\frac{1-\eta}{1+\eta}}\,\frac{f_o}{\gamma+\beta} = \frac{\sigma_c^2}{2} + \frac{\sigma_a^2}{2}\,\frac{1-\eta}{1+\eta}$$

$$= \frac{1}{2}\left[\sigma_c^2 + \frac{1-\eta}{1+\eta}\,\sigma_a^2\right] = \sigma_y^2$$

as in (5.08b).

Appendix 5.2.1

Computation of the "Realizable" Hilbert Filter Transfer Function

The transfer function to be computed is given by

$$H_r(f) = \frac{1}{S_{xc}^{-}(f)}\left[\frac{S_{x\hat{x}c}(f)}{S_{xc}^{+}(f)}\right]_{-} \quad , \; |f| \neq f_o$$

where

$$S_{x\hat{x}}(f) = H(f)\; S_x(f) = -j\;\mathrm{sgn}(f)\; S_x^{+}(f) S_x^{-}(f)$$

so that

$$H_r(f) = \frac{-j}{S_{xc}^{-}(f)}\left[\mathrm{sgn}(f)\; S_{xc}^{-}(f)\right]_{-} , \quad |f| \neq f_o$$

From (3.24) the term $[\;]_{-}$ is given by

$$\left[\mathrm{sgn}(f) S_{xc}^{-}(f)\right]_{-} = \frac{1}{2}\,\mathrm{sgn}(f) S_{xc}^{-}(f) - \frac{j}{2}\mathcal{H}\left\{\mathrm{sgn}(f) S_{xc}^{-}(f)\right\}$$

so that

$$H_r(f) = -\frac{j}{2}\,\mathrm{sgn}(f) - \frac{1}{2S_{xc}^{-}(f)}\;\mathcal{H}\left\{\mathrm{sgn}(f) S_{xc}^{-}(f)\right\} \quad , \quad |f| \neq f_o$$

and $S_{xc}^{-}(f)$ is given from (5.03a) by

$$S_{xc}^{-}(f) = \sqrt{\frac{\beta}{2\pi}}\,\sigma_a \frac{f-j\gamma}{[f-f_o-j\beta][f+f_o-j\beta]} = \sqrt{\frac{\beta}{8\pi}}\,\sigma_a\left[\frac{1+j\frac{\beta-\gamma}{f_o}}{f-f_o-j\beta} + \frac{1-j\frac{\beta-\gamma}{f_o}}{f+f_o-j\beta}\right]$$

We therefore need to compute terms of the form

$$\mathcal{H}\left\{\frac{\mathrm{sgn}(f)}{f-z}\right\} = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{sgn}(u)}{(u-z)(f-u)}\,du = \frac{1}{\pi}\left[\int_{0}^{\infty}\frac{du}{(u+z)(u+f)} - \int_{0}^{\infty}\frac{du}{(u-z)(u-f)}\right]$$

where z is complex, and in our application has positive imaginary part ($\beta \geqslant 0$). The first integral may be written

$$\frac{1}{\pi}\int_{0}^{\infty}\frac{du}{(u+z)(u+f)} = \frac{1}{\pi(f-z)}\lim_{L\to\infty}\int_{0}^{L}\left[\frac{1}{u+z} - \frac{1}{u+f}\right]du$$

where we have

$$\lim_{L\to\infty}\int_{0}^{L}\frac{du}{u+f} = \lim_{L\to\infty}\ln\left|\frac{L+f}{f}\right| \quad \text{since } f \text{ is real,}$$

and, letting $z = f_o + j\beta$,

$$\lim_{L\to\infty}\int_{0}^{L}\frac{du}{u+z} = \lim_{L\to\infty}\left[\int_{0}^{L}\frac{(u+f_o)}{(u+f_o)^2+\beta^2}\,du - j\beta\int_{0}^{L}\frac{du}{(u+f_o)^2+\beta^2}\right]$$

$$= \lim_{L\to\infty}\int_{f_o/\beta}^{(L+f_o)/\beta}\frac{y\,dy}{y^2+1} - j\lim_{L\to\infty}\int_{f_o/\beta}^{(L+f_o)/\beta}\frac{dy}{y^2+1}$$

$$= \ln\sqrt{\frac{\beta^2+(L+f_o)^2}{\beta^2+f_o^2}} + j\left[\tan^{-1}\frac{f_o}{\beta} - \tan^{-1}\frac{L+f_o}{\beta}\right]$$

so that we may write

$$(f-z)\int_0^\infty \frac{du}{(u+z)(u+f)} = \lim_{L\to\infty}\left\{\ln\sqrt{\frac{\beta^2+(L+f_o)^2}{\beta^2+f_o^2}}\left|\frac{f}{L+f}\right| + j\left[\tan^{-1}\frac{f_o}{\beta} - \tan^{-1}\frac{L+f_o}{\beta}\right]\right\}$$

$$= \ln\frac{|f|}{\gamma} + j\tan^{-1}\frac{f_o}{\beta} - j\frac{\pi}{2}$$

since $\beta \geqslant 0$ and where $\gamma \equiv \sqrt{\beta^2 + f_o^2} = |z|$. The complex number z may be written

$$z \equiv f_o + j\beta = \gamma\, e^{j\tan^{-1}\frac{\beta}{f_o}}$$

and since $\pi/2 - \tan^{-1} f_o/\beta = \cot^{-1} f_o/\beta = \tan^{-1} \beta/f_o$ we may write

$$z = \gamma\, e^{j\left(\frac{\pi}{2} - \tan^{-1}\frac{f_o}{\beta}\right)} \Rightarrow \ln z = \ln\gamma + j\left(\frac{\pi}{2} - \tan^{-1}\frac{f_o}{\beta}\right)$$

so that

$$\frac{1}{\pi}\int_0^\infty \frac{du}{(u+z)(u+f)} = \frac{1}{\pi(f-z)}\ln\frac{|f|}{z}$$

Similarly $$\frac{1}{\pi}\int_0^\infty \frac{du}{(u-z)(u-f)} = \frac{-1}{\pi(f-z)}\ln\frac{|f|}{-z} = \frac{-1}{\pi(f-z)}\left[\ln\frac{|f|}{z} + j\pi\right]$$

so that $$\mathcal{H}\left\{\frac{\operatorname{sgn}(f)}{f-z}\right\} = \frac{2}{\pi(f-z)}\left[\ln\frac{|f|}{z} + j\frac{\pi}{2}\right]$$

or
$$\mathcal{H}\left\{\frac{\text{sgn}(f)}{f-f_o-j\beta}\right\} = \frac{2}{\pi(f-f_o-j\beta)}\left[\ln\frac{|f|}{\gamma} + j\tan^{-1}\frac{f_o}{\beta}\right]$$

$$\mathcal{H}\left\{\frac{\text{sgn}(f)}{f+f_o-j\beta}\right\} = \frac{2}{\pi(f+f_o-j\beta)}\left[\ln\frac{|f|}{\gamma} - j\tan^{-1}\frac{f_o}{\beta}\right]$$

We may therefore write

$$\mathcal{H}\left\{\text{sgn}(f)S_{xc}^{-}(f)\right\} = \sqrt{\frac{\beta}{2\pi^3}}\,\sigma_a\left\{\frac{1+j\frac{\beta-\gamma}{f_o}}{f-f_o-j\beta}\left[\ln\frac{|f|}{\gamma} + j\tan^{-1}\frac{f_o}{\beta}\right] + \right.$$

$$\left.\frac{1-j\frac{\beta-\gamma}{f_o}}{f+f_o-j\beta}\left[\ln\frac{|f|}{\gamma} - j\tan^{-1}\frac{f_o}{\beta}\right]\right\}$$

$$= \sqrt{\frac{\beta}{2\pi^3}}\,\sigma_a\,\frac{1}{(f-f_o-j\beta)(f+f_o-j\beta)}\left\{\left(1 + j\frac{\beta-\gamma}{f_o}\right)[f+f_o-j\beta]\left[\ln\frac{|f|}{\gamma} + j\tan^{-1}\frac{f_o}{\beta}\right] + \right.$$

$$\left.\left(1-j\frac{\beta-\gamma}{f_o}\right)[f-f_o-j\beta]\left[\ln\frac{|f|}{\gamma} - j\tan^{-1}\frac{f_o}{\beta}\right]\right\}$$

$$= \sqrt{\frac{\beta}{2\pi^3}}\,\sigma_a\,\frac{1}{(f-f_o-j\beta)(f+f_o-j\beta)}\left\{2(f-j\gamma)\ln\frac{|f|}{\gamma} - \frac{2(\beta-\gamma)}{f_o}(f+j\gamma)\tan^{-1}\frac{f_o}{\beta}\right\}$$

so that

$$\frac{1}{2S_{xc}^{-}(f)}\,\mathcal{H}\left\{\text{sgn}(f)S_{xc}^{-}(f)\right\} = \frac{1}{\pi}\ln\frac{|f|}{\gamma} + \frac{\gamma-\beta}{\pi f_o}\tan^{-1}\frac{f_o}{\beta}\,\frac{f+j\gamma}{f-j\gamma}$$

or in terms of η this is

$$\frac{1}{\pi} \ln \frac{|f|}{\gamma} + \frac{1}{\pi} \sqrt{\frac{1-\eta}{1+\eta}} \cos^{-1}\eta \quad \frac{f+j\gamma}{f-j\gamma}$$

The entire transfer function is

$$H_r(f) = -\frac{j}{2} \operatorname{sgn}(f) - \frac{1}{\pi} \ln \frac{|f|}{\gamma} - \frac{1}{\pi} \sqrt{\frac{1-\eta}{1+\eta}} \cos^{-1}\eta \left(\frac{f+j\gamma}{f-j\gamma}\right); |f| \neq f_o$$

as in (5.09c).

Appendix 5.2.2

Second-Order Statistics -- "Realizable" Hilbert Filter

A. From (5.09c) we have

$$H_2(f) = -\frac{j}{2}\,\mathrm{sgn}(f) - \frac{1}{\pi}\ln\frac{|f|}{\gamma} - K\,\frac{f+j\gamma}{f-j\gamma}$$

where $K \equiv \frac{1}{2}\sqrt{\frac{1-\eta}{1+\eta}}\left(\frac{2}{\pi}\cos^{-1}\eta\right)$. The mean square output is

$$\sigma_z^2 = \int_{-\infty}^{\infty} d\Sigma_z(f) = \frac{\sigma_c^2}{2} + \int_{\substack{-\infty \\ |f|\neq f_o}}^{\infty} |H_2(f)|^2 d\Sigma_x(f)$$

in which

$$|H_2(f)|^2 = \left(\frac{1}{4} + K^2\right) + \frac{1}{\pi^2}\ln^2\frac{|f|}{\gamma} + 2K\gamma\,\frac{f\,\mathrm{sgn}(f)}{f^2+\gamma^2}$$

$$+ \frac{2K}{\pi}\,\frac{f^2-\gamma^2}{f^2+\gamma^2}\ln\frac{|f|}{\gamma}$$

Inserting this function into the integral and calling the four resulting terms a, b, c and d respectively we may calculate $\sigma_z^2 = a + b + c + d$.

a. $$\int_{\substack{-\infty \\ |f|\neq f_o}}^{\infty}\left(\frac{1}{4} + K^2\right)d\Sigma_x(f) = \left(\frac{1}{4} + K^2\right)\frac{\sigma_a^2}{2} = \frac{\sigma_a^2}{8}\left[1 + \frac{1-\eta}{1+\eta}\left(\frac{2}{\pi}\cos^{-1}\eta\right)^2\right]$$

b. $$\frac{1}{\pi^2} \int\limits_{\substack{-\infty \\ |f|\neq f_o}}^{\infty} \ln^2 \frac{|f|}{\gamma}\, d\Sigma_x(f) = \frac{\beta\sigma_a^2}{2\pi^3}\int_o^\infty \ln^2 \frac{f}{\gamma}\left[\frac{1}{f^2-2f_of+\gamma^2} + \frac{1}{f^2+2f_of+\gamma^2}\right] df$$

Consider only the second term for a moment

$$\int_o^\infty \frac{\ln^2(f/\gamma)}{f^2+2f_of+\gamma^2}\, df = \frac{1}{\gamma}\int_o^\infty \frac{\ln^2 u}{1+2f_ou/\gamma+u^2}\, du$$

which is equal [17] to $\frac{1}{\gamma}\frac{\theta_1(\pi^2-\theta_1^2)}{3\sin\theta_1}$ where $\cos\theta_1 = f_o/\gamma$

The other term in b is then $\frac{1}{\gamma}\frac{\theta_2(\pi^2-\theta_2^2)}{3\sin\theta_2}$ where $\cos\theta_2 = -f_o/\gamma$

so that $\theta_1 + \theta_2 = \pi$, $\sin\theta_1 = \sin\theta_2 = \beta/\gamma$. The sum of the two terms is $\frac{\pi\theta_1}{\beta}(\pi-\theta_1) = \frac{\pi}{\beta}\sin^{-1}\eta(\pi-\sin^{-1}\eta) = \frac{\pi^3}{4\beta}\left[1-(\frac{2}{\pi}\cos^{-1}\eta)^2\right]$ and the b contribution to σ_z^2 is

$$\frac{\sigma_a^2}{8}\left[1-\left(\frac{2}{\pi}\cos^{-1}\eta\right)^2\right]$$

c. $$2K\gamma \int_{\substack{-\infty \\ |f|\neq f_o}}^{\infty} \frac{f\ \mathrm{sgn}(f)}{f^2+\gamma^2}\, d\Sigma_x(f) = \frac{K\gamma\beta\sigma_a^2}{\pi}\int_o^{\infty} \frac{f}{f^2+\gamma^2}\left[\frac{1}{f^2-2f_of+\gamma^2}+\frac{1}{f^2+2f_of+\gamma^2}\right] df$$

$$= \frac{2K\gamma\beta\sigma_a^2}{\pi}\int_o^{\infty} \frac{f}{\left[f^2-2f_of+\gamma^2\right]\left[f^2+2f_of+\gamma^2\right]}\, df$$

$$= \frac{K\gamma\beta\sigma_a^2}{2\pi f_o}\int_o^{\infty} \left[\frac{1}{f^2-2f_of+\gamma^2}-\frac{1}{f^2+2f_of+\gamma^2}\right] df$$

The integral was evaluated in Appendix 4.5.2 and is equal to $\frac{2}{\beta}\cos^{-1}\eta$ so that the c term is $\frac{K\gamma\sigma_a^2}{\pi f_o}\cos^{-1}\eta$. Substituting the value of K this becomes

$$\frac{\sigma_a^2}{4(1+\eta)}\left(\frac{2}{\pi}\cos^{-1}\eta\right)^2$$

d. $$\frac{2K}{\pi}\int_{\substack{-\infty \\ |f|\neq f_o}}^{\infty} \frac{f^2-\gamma^2}{f^2+\gamma^2}\ln\frac{|f|}{\gamma}\, d\Sigma_x(f) = \frac{2\beta K\sigma_a^2}{\pi^2}\int_o^{\infty} \frac{(f^2-\gamma^2)\ln(f/\gamma)}{\left[f^2-2f_of+\gamma^2\right]\left[f^2+2f_of+\gamma^2\right]}\, df$$

$$= \frac{2\beta K\sigma_a^2}{\pi^2\gamma}\int_o^{\infty} \frac{(u^2-1)\ln u}{\left[u^2-2f_ou/\gamma+1\right]\left[u^2+2f_ou/\gamma+1\right]}\, du \equiv \frac{2\beta K\sigma_a^2}{\pi^2\gamma} M$$

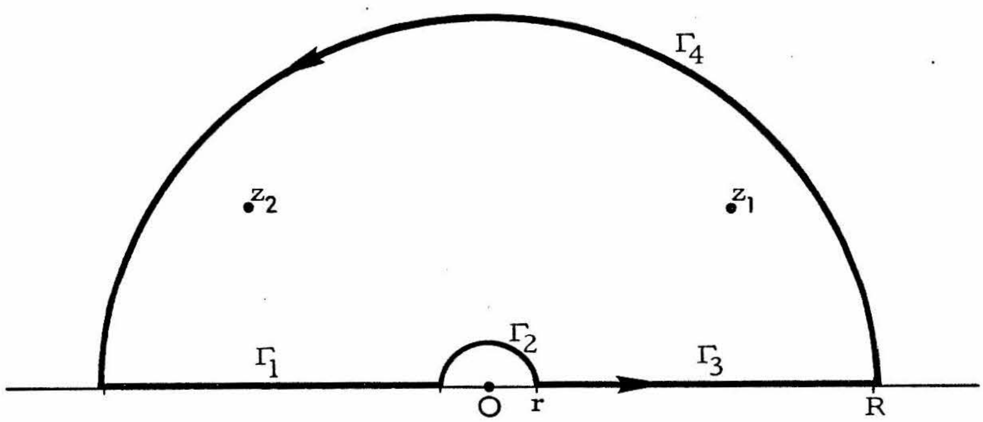
The integral M may be evaluated by the method of residues. Define the contour integral I in the complex z-plane

$$I \equiv \oint_{\Gamma} \frac{(z^2-1)\ln z}{\left[z^2-2f_o z/\gamma+1\right]\left[z^2+2f_o z/\gamma+1\right]}\, dz$$

where the contour is the UHP semicircle indented upward at the origin to avoid the branch point of ln z. The various segments $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ correspond to the integrals I_1, I_2, I_3, I_4 where

$$I = I_1 + I_2 + I_3 + I_4$$

The UHP poles are simple and located at $z_{1,2} = \pm\, f_o/\gamma + j\eta$



I_4: As $R \to \infty$, $I_4 \to 0$

I_2: On Γ_2, $z = re^{j\theta}$, $\pi > \theta > 0$, and as $r \to 0$

$$I_2 \longrightarrow jr \int_o^{\pi} \left[\ln r + j\theta\right] e^{i\theta} d\theta \to 0$$

I_3: On Γ_3, $z = u$ so that $I_3 = M$

I_1: On Γ_1, $z = ue^{j\pi}$ so that

$$I_1 = M + j\pi \int_o^{\infty} \frac{u^2 - 1}{\left[u^2+2f_o u/\gamma+1\right]\left[u^2-2f_o u/\gamma+1\right]} du$$

The integral may be written $\int_o^1 + \int_1^{\infty}$ and setting $v = 1/u$ in the $\int_1^{\infty}$ term we see that the integral vanishes and $I_1 = M$ so that

$$M = \frac{1}{2}\left(I_1 + I_3\right) = \frac{1}{2} I = \pi j \sum \text{Res (UHP)}$$

The UHP poles may be written $z_{1,2} = \pm f_o/\gamma + j\eta = \pm e^{\pm j \sin^{-1}\eta}$ and the integrand of I may be written

$$\frac{(z^2-1)\ln z}{(z-f_o/\gamma+j\eta)(z-f_o/\gamma-j\eta)(z+f_o/\gamma+j\eta)(z+f_o/\gamma-j\eta)}$$

so that summing the two residues and algebraic simplification yield

$$M = -\frac{\pi\beta}{4\eta f_o}\left[2 \sin^{-1}\eta - \pi\right]$$

so that the d term of σ_z^2 becomes

$$\frac{\eta}{4(1+\eta)} \sigma_a^2 \left(\frac{2}{\pi} \cos^{-1}\eta\right)^2$$

The sum of the a, b, c, d terms is

$$\frac{\sigma_a^2}{8}\left[1 + \frac{1-\eta}{1+\eta}\left(\frac{2}{\pi} \cos^{-1}\eta\right)^2 + 1 - \left(\frac{2}{\pi} \cos^{-1}\eta\right)^2 + \frac{2}{1+\eta}\left(\frac{2}{\pi} \cos^{-1}\eta\right)^2 + \frac{2\eta}{1+\eta}\left(\frac{2}{\pi} \cos^{-1}\eta\right)^2\right]$$

so that

$$\sigma_z^2 = \frac{\sigma_c^2}{2} + \frac{\sigma_a^2}{4}\left[1 + \frac{1}{1+\eta}\left(\frac{2}{\pi} \cos^{-1}\eta\right)^2\right]$$

as in (5.10a), or

$$\sigma_z^2 = \frac{\sigma_c^2}{2}\left\{1 + \frac{m}{2}\left[1 + \frac{(1-q)^2}{1+\eta}\right]\right\}$$

as in (5.11a).

B. The cross moment $\langle x_o z\rangle$ is given by

$$\langle x_o z\rangle = \int_{-\infty}^{\infty} H^*_{real}(f)\, d\Sigma_{xx_o}(f) = \frac{\sigma_c^2}{2} + \int_{\substack{-\infty \\ |f|\neq f_o}}^{\infty} H_2^*(f) S_{xx_o}(f)\, df$$

where

$$H_2^*(f) = \frac{j}{2}\,\mathrm{sgn}(f) - \frac{1}{\pi} \ln \frac{|f|}{\pi} - K\, \frac{f^2-\gamma^2-2j\gamma f}{f^2+\gamma^2}$$

$$S_{xx_o}(f)\Big|_{|f|\neq f_o} = \frac{\beta f_o \sigma_a^2}{j\pi} \frac{f}{\left[f^2-2f_o f+\gamma^2\right]\left[f^2+2f_o f+\gamma^2\right]}$$

The even parts of $H_2^*(f)$ drop out of the integral and we have

$$\langle x_o z\rangle = \frac{\sigma_c^2}{2} + \frac{\beta f_o \sigma_a^2}{2\pi} \int_{-\infty}^{\infty} \frac{f\ \mathrm{sgn}(f) + 4\gamma K f^2/(f^2+\gamma^2)}{\left[f^2-2f_o f+\gamma^2\right]\left[f^2+2f_o f+\gamma^2\right]}\, df$$

The first integral term was evaluated in Appendix 4.5.2(B) and found to equal $\frac{\pi}{\beta}\left(\frac{2}{\pi}\cos^{-1}\eta\right)$. The second integral term was evaluated in Appendix 4.5.2(A) and found to equal $\pi(1-\eta)/4\beta f_o^2$. Combination of these terms yields

$$\langle x_o z\rangle = \frac{\sigma_c^2}{2} + \frac{\sigma_a^2}{2(1+\eta)}\left(\frac{2}{\pi}\cos^{-1}\eta\right)$$

as in (5.10b), or

$$\langle x_o z\rangle = \frac{\sigma_c^2}{2}\left[1 + \frac{m(1-q)}{1+\eta}\right]$$

as in (5.11b).

Appendix 8.2

Calculation of the First Order Probability Density of $A_s(t)$

We have the relations

$$A_s(t) = +\left[x^2(t) + ks^2(t)\right]^{\frac{1}{2}}$$

$$p(x,s) = \frac{1}{2\pi\sigma_x\sigma_s} e^{-\frac{1}{2}\left[\frac{x^2}{\sigma_x^2} + \frac{s^2}{\sigma_s^2}\right]}$$

from which can be calculated the first order density $p(A_s)$ of $A_s(t)$. The probability $\Pr\{A_s \leqslant R\}$, $R \geqslant 0$, is the integral of $p(x,s)$ inside the ellipse generated by $x = \pm\sqrt{R^2 - ks^2}$ or four times the integral in the first quadrant

$$\Pr\{A_s^2 \leq R\} = \frac{2}{\pi\sigma_x\sigma_s} \int_0^{R/\sqrt{k}} ds \int_0^{\sqrt{R^2-ks^2}} e^{-\frac{1}{2}\left[\frac{x^2}{\sigma_x^2} + \frac{s^2}{\sigma_s^2}\right]} dx$$

$$= \frac{2}{\pi\sigma_x\sigma_s\sqrt{k}} \int_0^{R} du \int_0^{\sqrt{R^2-u^2}} e^{-\frac{1}{2}\left[\frac{x^2}{\sigma_x^2} + \frac{u^2}{k\sigma_s^2}\right]} dx$$

as in (8.08). The probability density function is obtained by differentiating $\Pr\{A_s \leqslant R\}$ with respect to R and setting $R = A_s$ in the resulting expression

$$p(A_s) = \frac{2}{\pi\sigma_x\sigma_s\sqrt{k}} \frac{\partial P_r\{A_s \le R\}}{\partial R} = \int_o^R e^{-\frac{1}{2}\left[\frac{R^2-u^2}{\sigma_x^2} + \frac{u^2}{k\sigma_s^2}\right]} \frac{R}{\sqrt{R^2-u^2}}\, dR$$

$$= \frac{2\,R}{\pi\sigma_x\sigma_s\sqrt{k}}\, e^{-\frac{R^2}{2\sigma_x^2}} \int_o^R \frac{1}{\sqrt{R^2-u^2}}\, e^{-\frac{1}{2}\left[\frac{1}{k\sigma_s^2} - \frac{1}{\sigma_x^2}\right]u^2} du$$

Substituting $u = R \sin\theta$ $\left(u^2 = R^2 \sin^2\theta = \frac{R^2}{2}[1-\cos 2\theta],\right.$ $\sqrt{R^2-u^2} = R\cos\theta$, $du = R\cos\theta)$ this becomes,

$$\frac{2R}{\pi\sigma_x\sigma_s\sqrt{k}}\, e^{-\frac{R^2}{2\sigma_x^2}} \int_o^{\pi/2} e^{-\frac{R^2}{4}\left[\frac{1}{k\sigma_s^2} - \frac{1}{\sigma_x^2}\right][1 - \cos 2\theta]} d\theta$$

$$= \frac{R}{\pi\sigma_x\sigma_s\sqrt{k}}\, e^{-\frac{R^2}{4}\left[\frac{1}{k\sigma_s^2} + \frac{1}{\sigma_x^2}\right]} \int_o^{\pi} e^{\frac{R^2}{4}\left[\frac{1}{k\sigma_s^2} - \frac{1}{\sigma_x^2}\right]\cos\beta} d\beta$$

and from [17] this is

$$\frac{R}{\sigma_x\sigma_s\sqrt{k}}\, e^{-\frac{R^2}{4}\left[\frac{1}{k\sigma_s^2} + \frac{1}{\sigma_x^2}\right]} I_o\left[\frac{R^2}{4}\left(\frac{1}{k\sigma_s^2} - \frac{1}{\sigma_x^2}\right)\right]$$

Recognizing that $p(A_s) = 0$, $A_s < 0$, we may write

$$p(A_s) = \frac{A_s U(A_s)}{\sqrt{k}\sigma_x\sigma_s} e^{-\frac{A_s^2}{4}\left[\frac{1}{k\sigma_s^2} + \frac{1}{\sigma_x^2}\right]} I_o\left[\frac{A_s^2}{4}\left(\frac{1}{k\sigma_s^2} - \frac{1}{\sigma_x^2}\right)\right]$$

as in (8.09).

Appendix 8.3.1

Calculation of $\lim_{\eta \to 0} \mathcal{E}_s$

The probability density function of x, x_o, s is

$$p(x,x_o,s) = \frac{1}{\left[(2\pi)^3 \sigma_x^2 (\sigma_x^2 \sigma_s^2 - \langle x_o s\rangle^2)\right]^{1/2}} e^{-\frac{x^2}{2\sigma_x^2}}$$

$$\times e^{-\frac{\sigma_s^2 x_o^2 + \sigma_x^2 s^2 - 2\langle x_o s\rangle x_o s}{2(\sigma_x^2 \sigma_s^2 - \langle x_o s\rangle^2)}}$$

The MS error $\mathcal{E}_s$ is

$$\mathcal{E}_s \equiv E\left\{(A_o - A_s)^2\right\} = E\left\{\left[\sqrt{x^2 + x_o^2} - \sqrt{x^2 + ks^2}\right]^2\right\}$$

$$= \frac{1}{\left[(2\pi)^3 \sigma_x^2 (\sigma_x^2 \sigma_s^2 - \langle x_o s\rangle^2\right]^{1/2}} \iiint_{-\infty}^{\infty} \left[\sqrt{x^2 + x_o^2} - \sqrt{x^2 + ks^2}\right]^2$$

$$\times e^{-\frac{x^2}{2\sigma_x^2}} e^{-\frac{\sigma_s^2 x_o^2 + \sigma_x^2 s^2 - 2\langle x_o s\rangle x_o s}{2(\sigma_x^2 \sigma_s^2 - \langle x_o s\rangle^2)}} dx\, dx_o\, ds$$

Substituting $\sqrt{2}\,\sigma_x u = x, \sqrt{2}\,\sigma_x v = x_o, \sqrt{2}\,\sigma_s w = s$, $\rho_s = \frac{\langle x_o s\rangle}{\sigma_x \sigma_s}$, $\alpha_s = k\frac{\sigma_s^2}{\sigma_x^2}$ and writing ρ, α for ρ_s, α_s, this becomes

$$\frac{2\sigma_x^2}{\pi^{3/2}\sqrt{1-\rho^2}} \iiint_{-\infty}^{\infty} \left[\sqrt{u^2+v^2} - \sqrt{u^2+\alpha w^2}\right]^2 e^{-u^2} \times e^{-\frac{1}{1-\rho^2}[v^2 - 2\rho vw + w^2]} \, du\, dv\, dw$$

Completing the square in the exponent,

$$\frac{2\sigma_x^2}{\pi^{3/2}\sqrt{1-\rho^2}} \iiint_{-\infty}^{\infty} \left[\sqrt{u^2+v^2} - \sqrt{u^2+\alpha w^2}\right]^2 e^{-u^2} \, e^{-\frac{(v-\rho w)^2}{1-\rho^2}} \, e^{-w^2} \, du\, dv\, dw$$

Change the variable v to $z = \frac{v-\rho w}{\sqrt{1-\rho^2}} \Longrightarrow v = \sqrt{1-\rho^2}\, z + \rho w$

$$\mathcal{E}_S = \frac{2\sigma_x^2}{\pi^{3/2}} \iiint_{-\infty}^{\infty} \left[\sqrt{u^2 + \left[\sqrt{1-\rho^2}\, z + \rho w\right]^2} - \sqrt{u^2+\alpha w^2}\right]^2 \times e^{-[u^2+z^2+w^2]} \, du\, dz\, dw$$

Substituting $1-\rho_s^2 = \mu_s \eta$ and dropping the subscript again,

$$\mathcal{E}_S = \frac{2\sigma_x^2}{\pi^{3/2}} \iiint_{-\infty}^{\infty} \left[\sqrt{u^2 + \left(\sqrt{\mu\eta}\, z + \sqrt{1-\mu\eta}\, w\right)^2} - \sqrt{u^2+\alpha w^2}\right]^2 \times e^{-[u^2+z^2+w^2]} \, du\, dz\, dw$$

and we seek the behavior of $\mathcal{E}_S$ as $\eta \to 0$.

Squaring the bracket and grouping the terms

$$[\ ]^2 = 2u^2 + \mu\eta z^2 + (1+\alpha-\mu\eta)w^2 + 2\sqrt{\mu\eta}\sqrt{1-\mu\eta}\ zw$$

$$- 2\sqrt{u^2+\alpha w^2}\sqrt{u^2+\mu\eta z^2+2\sqrt{\mu\eta}\sqrt{1-\mu\eta}\ zw+(1-\mu\eta)w^2}$$

The first three terms of $[\ ]^2$ are easily integrated, yielding

$$\frac{2\sigma_x^2}{\pi^{3/2}}\frac{\pi^{3/2}}{2}[2+\mu\eta+1+\alpha-\mu\eta] = \sigma_x^2[3+\alpha]$$

The fourth term of $[\ ]^2$ drops out in the integration due to its oddness. The fifth term may be written

$$-2\sqrt{u^2+\alpha w^2}\sqrt{u^2+w^2+\mu\eta(z^2-w^2)+2\sqrt{\mu\eta}\sqrt{1-\mu\eta}\ zw}$$

$$= -2\sqrt{u^2+\alpha w^2}\sqrt{u^2+w^2}\sqrt{1+\frac{\mu\eta(z^2-w^2)+2\sqrt{\mu\eta}\sqrt{1-\mu\eta}\ zw}{u^2+w^2}}$$

which may be expanded about $\eta = 0$.

$$-2\sqrt{u^2+\alpha w^2}\sqrt{u^2+w^2}\left[1+\frac{\mu\eta(z^2-w^2)+2\sqrt{\mu\eta}\sqrt{1-\mu\eta}\ zw}{2(u^2+w^2)}\right.$$
$$\left.-\frac{[\mu\eta(z^2-w^2)+2\sqrt{\mu\eta}\sqrt{1-\mu\eta}\ zw]^2}{8(u^2+w^2)^2}+\cdots\right]$$

The term in zw drops out due to its oddness. Keeping only terms to first order in η

$$-2\sqrt{u^2+\alpha w^2}\sqrt{u^2+w^2}\left[1+\frac{\mu\eta(z^2-w^2)}{2(u^2+w^2)}-\frac{\mu\eta\ z^2w^2}{2(u^2+w^2)^2}\right]$$

Inserting this in the integral,

$$\frac{-4\sigma_x^2}{\pi^{3/2}} \iiint_{-\infty}^{\infty} \sqrt{u^2+w^2}\sqrt{u^2+\alpha w^2}\left\{1+\mu\eta\left[\frac{z^2-w^2}{2(u^2+w^2)}-\frac{z^2 w^2}{2(u^2+w^2)^2}\right]\right\} \times e^{-[u^2+z^2+w^2]}\, du\, dz\, dw$$

Performing the z-integration,

$$\frac{-4\sigma_x^2}{\pi} \iint_{-\infty}^{\infty} \sqrt{u^2+w^2}\sqrt{u^2+\alpha w^2}\left\{1+\mu\eta\left[\frac{\frac{1}{2}-w^2}{2(u^2+w^2)}-\frac{\frac{1}{2}w^2}{2(u^2+w^2)^2}\right]\right\} \times e^{-(u^2+w^2)}\, du\, dw$$

$$=\frac{-4\sigma_x^2}{\pi} \iint_{0}^{\infty} \sqrt{u^2+w^2}\sqrt{u^2+\alpha w^2}\left\{4+\mu\eta\left[\frac{1-2w^2}{u^2+w^2}-\frac{w^2}{(u^2+w^2)^2}\right]\right\} \times e^{-(u^2+w^2)}\, du\, dw$$

Make the change of variables $u = r\cos\theta$, $w = r\sin\theta$, $du\,dw = r\,dr\,d\theta$

$$-\frac{4\sigma_x^2}{\pi}\int_0^{\infty}\int_0^{\pi/2} r^3 e^{-r^2}\sqrt{\cos^2\theta+\alpha\sin^2\theta}\left\{4+\mu\eta\left[\frac{1-2r^2\sin^2\theta}{r^2}-\frac{\sin^2\theta}{r^2}\right]\right\} dr\,d\theta$$

Performing the r-integration

$$- \frac{2\sigma_x^2}{\pi} \int_0^{\pi/2} \sqrt{1-(1-\alpha)\sin^2\theta} \left\{4 + \mu\eta\left[1 - 2\sin^2\theta - \sin^2\theta\right]\right\} \; d\theta$$

$$= - \frac{2\sigma_x^2}{\pi} \int_0^{\pi/2} \sqrt{1-(1-\alpha)\sin^2\theta} \; \left[4 + \mu\eta - 3\mu\eta \sin^2\theta\right] \; d\theta$$

$$= - \frac{2\sigma_x^2}{\pi} \left\{ (4+\mu\eta) \; E(1-\alpha) - \frac{\mu\eta}{(1-\alpha)} \left[\alpha K(1-\alpha) + (1-2\alpha) \; E(1-\alpha) \right] \right\}, \qquad \alpha \leq 1 \qquad [22]$$

$$= - \frac{2\sigma_x^2}{\pi} \left\{ 4E(1-\alpha) + \mu\eta \frac{\alpha}{1-\alpha} \left[E(1-\alpha) - K(1-\alpha) \right] \right\}$$

Combining all terms we have

$$\mathcal{E}_S = \sigma_x^2 (3+\alpha) - \frac{2}{\pi} \sigma_x^2 \left\{ 4E(1-\alpha) + \mu\eta \frac{\alpha}{1-\alpha} \left[E(1-\alpha) - K(1-\alpha) \right] \right\}$$

or

$$\mathcal{E}_S = \sigma_x^2 \left[3 + \alpha - \frac{2}{\pi} \left\{ 4E(1-\alpha) + \mu\eta \frac{\alpha}{1-\alpha} \left[E(1-\alpha) - K(1-\alpha) \right] \right\} \right]$$

as in (8.21).

As $\eta \to 0$ we have $\alpha \to 1$ and the elliptic integrals tend to

$$\lim_{\alpha \to 1} E(1-\alpha) \longrightarrow \frac{\pi}{2}\left(1 - \frac{1-\alpha}{4}\right)$$

$$\lim_{\alpha \to 1} K(1-\alpha) \longrightarrow \frac{\pi}{2}\left(1 + \frac{1-\alpha}{4}\right)$$

so that $\mathcal{E}_S$, in the limit as $\eta \to 0$, approaches

$$\mathcal{E}_S \longrightarrow \sigma_x^2 \left\{3 + \alpha - \left[4 - (1-\alpha) + \mu\eta \frac{\alpha}{1-\alpha}\left(1 - \frac{1-\alpha}{4} - 1 - \frac{1-\alpha}{4}\right)\right]\right\}$$

$$= \frac{\alpha}{2}\,\sigma_x^2\,\mu\eta = \frac{\alpha_s}{2}\,\sigma_x^2\,\mu_s\eta = \frac{k}{2}\,\sigma_s^2\,\mu_s\eta$$

as in (8.23).

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Glossary of Principal Symbols and Abbreviations

$A(t)$	General envelope
$A_o(t)$	Intended envelope
$A_s(t)$	Output of radius detector in general
$A_{\hat{x}}(t)$	Output of Hilbert detector
$A_y(t)$	Output of conjugate detector
α_s	$k\ \sigma_s^2/\sigma_x^2$
β	Corner frequency of single-RC spectrum
$C(f)$	Conjugate filter transfer function
γ	$\sqrt{f_o^2 + \beta^2}$
D_n	Decomposition set on f-axis
$D(f)$	Distribution function
$\delta(\ )$	Dirac delta function
$E(\ )$	Complete elliptic integral of the second kind
$E\{\ \}$	Expected value of $\{\ \}$ (also $\langle\ \rangle$)
$\bar{\mathcal{E}}$	Envelope error for gain-modified representation
$\mathcal{E}_S$	Mean square error between $A_o(t)$ and $A_s(t)$
$\mathcal{E}_{2S}$	MSS error, or MS error between $A_o^2(t)$ and $A_s^2(t)$
$\mathcal{E}_{x_o s}$	Mean square error between x_o and s
η	β/γ
f	Frequency in general (Hertz)
f_o	Carrier frequency (Hertz)
$F_\theta(\xi_1,\xi_2)$	Second order characteristic function of $\theta(t)$
FT	Fourier transform
$\mathfrak{F}\{\ \}$	Fourier transform operator
$G(f)$	General filter transfer function

$G_i(f)$	Imaginary part of $G(f)$
$G_r(f)$	Real part of $G(f)$
HT	Hilbert transform
$H(f)$	Hilbert filter transfer function
$\mathcal{H}\{\ \}$	Hilbert transform operator
$I_o(\ )$	Modified zero-order Bessel function of the first kind
k	Square of gain modification factor
$\mathbf{K}$	Covariance matrix
$K(\ )$	Complete elliptic integral of the first kind
ln	Natural logarithm
$\Lambda(f)$	Generalized Fourier transform of $x(t)$
$\Lambda_o(f)$	Generalized Fourier transform of $x_o(t)$
m	Modulation index $\equiv \sigma_a^2/\sigma_c^2$
MS	Mean square
ω	Frequency in general (radians/sec.)
ω_o	Carrier frequency (radians/sec.)
$\Omega(f)$	Generalized Fourier transform of $y(t)$
$p(\ )$	First order probability density function
$p(\ ,\ )$	Second order probability density function
$Pr\{\ \}$	Probability distribution function
PSD	Power spectral density
$\phi(t)$	Phase, in general
$\phi_o(t)$	Intended phase
$q(\eta)$	$[2/\pi]\ \sin^{-1}\eta$
$R_c(\tau)$	Autocorrelation function of $x_o(t)$ and $x_s(t)$
$R_s(\tau)$	Crosscorrelation function between $x_c(t)$ *and* $x_s(t)$

$R_x(\tau)$	Autocorrelation function of $x(t)$
$R_{xx_o}(\tau)$	Crosscorrelation function between $x(t)$ and $x_o(t)$
ρ_s	Normalized crosscorrelation between $x_o(t)$ and $s(t)$
$s(t)$	General filter output
$\mathrm{sgn}(\ )$	Signum function
$\sin^{-1}(\ )$	Arcsine function
$S_c(f)$	PSD of $x_c(t)$ and $x_s(t)$
$S_s(f)$	Cross spectral density between $x_c(t)$ and $x_s(t)$
$S_x(f)$	PSD of $x(t)$
σ_a	RMS value of random component of $x(t)$
σ_c	RMS value of carrier component of $x(t)$
σ_x	RMS value of $x(t)$
$\Sigma_x(f)$	Spectral distribution function of $x(t)$
$\Sigma_{xx_o}(f)$	Cross spectral distribution function between $x(t)$ and $x_o(t)$
t	Time in general
$\tanh(\ )$	Hyperbolic tangent function
τ	Time, argument of correlation functions
$\theta(t)$	Random phase drift process
$U(\ )$	Unit step function
V_s	Random column vector
$x_c(t)$	In-phase component of $x(t)$
$x_o(t)$	Conjugate signal to $x(t)$
$x_s(t)$	Quadrature component of $x(t)$
$\hat{x}(t)$	Hilbert transform of $x(t)$
$y(t)$	Output of conjugate filter
$z(t)$	Output of "realizable" Hilbert filter