A Direct Approach to Robustness Optimization

Thesis by

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To My Family

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Abstract

This dissertation reformulates and streamlines the core tools of robustness analysis for linear time invariant systems using now-standard methods in convex optimization. In particular, robust performance analysis can be formulated as a primal convex optimization in the form of a semidefinite program using a semidefinite representation of a set of Gramians. The same approach with semidefinite programming duality is applied to develop a linear matrix inequality test for well-connectedness analysis, and many existing results such as the Kalman-Yakubovich–Popov lemma and various scaled small gain tests are derived in an elegant fashion. More importantly, unlike the classical approach, a decision variable in this novel optimization framework contains all inner products of signals in a system, and an algorithm for constructing an input and state pair of a system corresponding to the optimal solution of robustness optimization is presented based on this information. This insight may open up new research directions, and as one such example, this dissertation proposes a semidefinite programming relaxation of a cardinality constrained variant of the $\mathcal{H}_\infty$ norm, which we term sparse $\mathcal{H}_\infty$ analysis, where an adversarial disturbance can use only a limited number of channels. Finally, sparse $\mathcal{H}_\infty$ analysis is applied to the linearized swing dynamics in order to detect potential vulnerable spots in power networks.

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$\mathbb{C}^n$	A set of n-dimensional complex vectors.
$\mathbb{C}^{n\times m}$	A set of n-by-m complex matrices.
$\mathbb{H}^n$	A set of n-by-n Hermitian matrices.
$[x]_i$	The ith entry of a vector $x \in \mathbb{C}^n$.
$[\mathbf{x}]_i$	The ith entry of a signal $\mathbf{x}$.
$[A]_{ij}$	The (i,j) th entry of a matrix $A \in \mathbb{C}^{n\times m}$.
$\|x\|_2$	The two norm of a vector $x \in \mathbb{C}^n$.
$\|A\|_{2,2}$	The maximum singular value of a matrix $A \in \mathbb{C}^{n\times m}$.
$\rho(A)$	The spectral radius of a matrix $A \in \mathbb{C}^{n\times m}$.
$\mathbf{diag}\,(x)$	n-by-n matrix of the form: $\begin{bmatrix} [x]_1 & & \\ & \ddots & \\ & & [x]_n \end{bmatrix}$.
$\mathbf{diag}\,(A)$	An n dimensional vector of the form $\begin{bmatrix} [A]_{11} \\ \vdots \\ [A]_{nn} \end{bmatrix}$.
$\mathbf{diag}\,(A_1, \cdots, A_m)$	A block diagonal matrix of the form $\begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_n \end{bmatrix}$.
$\mathbf{1}_n$	n dimensional vector whose entries are all ones.
$0_{m\times n}$	m-by-n zero matrix.
I_n	n-by-n identity matrix.

Chapter 1

Introduction

Convex optimization, especially Semidefinite Programming (SDP), is an essential tool in robust control theory, from system analysis with internal and external uncertainties to robust feedback controller synthesis. Many existing results can be stated as a feasibility problem with Linear Matrix Inequality (LMI) constraints, which can be solved via SDP. For example, the celebrated Kalman–Yakubovich–Popov (KYP) lemma concerns the following frequency domain condition, which plays a crucial role in robust control theory:

$$\sigma_{\max}(C(j\omega I - A)^{-1}B + D) < 1, \quad \text{for all } \omega \in \mathbb{R}, \tag{1.1}$$

where $\sigma_{\max}$ is the maximum singular value of a matrix, and (A, B, C, D) are matrices that represent a linear time invariant system. Since the condition (1.1) consists of an infinite number of inequalities parametrized by ω, its exact verification seems formidable. Remarkably, the aforementioned infinite number of inequalities can be easily checked by the following finite dimensional single LMI:

$$\text{there exists } P \succeq 0 \text{ such that } \begin{bmatrix} A^*P + PA + C^*C & PB + C^*D \\ B^*P + D^*C & D^*D - I \end{bmatrix} \prec 0. \tag{1.2}$$

In other words, if one can find a positive semidefinite matrix P that satisfies (1.2), then (1.1) is also true.

Recent advances in numerical methods for solving SDPs, especially the development of interior point methods in the late 1980s and early 90s, render such a feasibility test (1.2) relatively simple. Despite the infinite dimensionality of the condition (1.1), we can easily check it via the condition (1.2) in a tractable manner using an existing SDP solver. This tractability is arguably the most important reason why SDPs are so popular in robust control theory.

In order to obtain an LMI feasibility test for robustness analysis, the popular $\mathcal{S}$-procedure is often used. However, the $\mathcal{S}$-procedure only provides an implication from an LMI feasibility test to robustness of a system. For example, in the KYP lemma, the $\mathcal{S}$-procedure can only be used to prove the implication (1.2) $\Rightarrow$ (1.1), although the converse direction, (1.1) $\Rightarrow$ (1.2), is also true. When an LMI feasibility test is also a necessary condition for robustness of a system, the corresponding $\mathcal{S}$-procedure is said to be *lossless*. Not surprisingly, checking whether the $\mathcal{S}$-procedure is lossless or not is not a trivial task.

For Linear Time Invariant (LTI) systems, a shift-invariant quadratic form is used to establish losslessness results. Consider the following shift-invariant quadratic form $\phi : \mathcal{L}_2(-\infty,\infty) \to \mathbf{R}$ on signals,

$$\phi(\mathbf{u}) = \int_{-\infty}^{\infty} u(t)^* H u(t),$$

where H is a given Hermitian matrix. It turns out that the closure of the following set

$$\mathcal{F} := \{\phi(\mathbf{u}) : \mathbf{u} \in \mathcal{L}_2(-\infty,\infty)\}$$

is *convex*. Let $\mathbf{u}, \mathbf{v} \in \mathcal{L}_2(-\infty,\infty)$ and $\phi_1 := \phi(\mathbf{u})$, $\phi_2 := \phi(\mathbf{v})$. Using the time delayed version of $\mathbf{v}$, $v_\tau(t) := v(t-\tau)$, we can show that

$$\lim_{\tau\to\infty} (\phi(\mathbf{u}) + \phi(\mathbf{v}_\tau)) = \phi_1 + \phi_2,$$

and the above observation is an essential idea to show the convexity of $\mathbf{cl}\,(\mathcal{F})$.

In many cases, the convexity of $\mathbf{cl}\,(\mathcal{F})$ together with the separating hyperplane theorem establishes a sufficient and necessary LMI condition for robustness analysis. This gives rise to a very important question: why SDPs? There are other types of convex programming, such as linear programing and second order cone programming, which are more efficient in practice and can often handle more decision variables than SDPs.

This dissertation shows that the closure of a set of Gramains, which contains all the quadratic information of an LTI system, is semidefinite representable and has an explicit algebraic characterization. Since any shift-invariant quadratic form ϕ can be represented by a Gramian, this semidefinite representation result subsumes the convexity of $\mathbf{cl}(\mathcal{F})$ and this fact may explain why SDPs naturally arise in many robustness analysis results. In addition, based on this observation, this dissertation proposes a novel mathematical foundation of robust control theory with SDPs as a modeling tool, extends many existing results, and unifies technical proofs in an elegant manner. More importantly, since the decision variable in this formulation, a Gramian, contains inner products between input and internal states of a system, an input-state signal pair that solves the corresponding robustness optimization can be constructed.

The key contributions of this dissertation are

- Establishing a semidefinite representation of the closure of a set of Gramians generated by a stable linear time invariant system.
- Presenting an algorithm that constructs an optimal input-state signal pair for robustness optimization.
- Extending $\mathcal{H}_\infty$ analysis to capture various types of disturbance models and cost functions.
- Extending well-connectedness analysis to various types of operators.
- Presenting an SDP based method for a cardinality constrained variant of $\mathcal{H}_\infty$ analysis, "sparse" $\mathcal{H}_\infty$ analysis, and synthesis.

As an example, this dissertation illustrates how to use sparse $\mathcal{H}_\infty$ analysis to check the stability of power networks.

1.1 Some historical remarks

This section briefly introduces relevant literature in robust control theory. Textbooks such as [25, 95, 26] introduce basic concepts and detailed treatments of the central subjects. Practical applications to many engineering disciplines from aerospace engineering [43] to power systems [62] are also available.

Among many exciting results, arguably the most important ones are in robustness analysis and robust controller synthesis. Robustness analysis quantifies the effect of internal uncertainties in a system model (*e.g.*, errors in parameters, unmodeled dynamics, etc.) on the stability of a system and the effect of external disturbances on the performance of a system. Robust controller synthesis designs a feedback controller that compensates for undesirable effects from uncertainties.

There exist many robustness measures that quantify the effect of external disturbances on the output. Among them, the $\mathcal{H}_\infty$ norm stands out for many reasons [25]. To compute the $\mathcal{H}_\infty$ norm of a system, we assume a unit energy disturbance and measure the output energy. However, depending on disturbance models and cost criterions, other norms can also be considered for analysis and synthesis. For example, the $\mathcal{H}_2$ norm [24] assumes a unit peak disturbance with an output energy cost, and the $\mathcal{L}_1$ norm [18, 17] assumes a unit peak disturbance with an output peak cost. More sophisticated robustness measures, such as the entropy of a system, [33, 51], also exist and have a connection to risk-sensitive control [32]. Combinations of some of those measures are also available, such as the mixed $\mathcal{H}_2$-$\mathcal{H}_\infty$ norm [41], and multiple performance criterions [67, 71]. A sinusoidal disturbance that only contains low frequency components is considered by Iwasaki *et al.* [38], and an extension of this result with more sophisticated frequency domain descriptions appeared in [37].

The most closely related robustness measure to the one in this dissertation is from D'Andrea [20]. D'Andrea introduces squared $\mathcal{H}_\infty$ analysis in [19], where each component of disturbance has unit energy, and this result is extended to capture more general disturbance models, as well as cost criterions [20]. This dissertation extends these results even further to any disturbance models that can be represented by a set of Gramians and any arbitrary cost function that is convex in Gramians.

For analysis of systems with internal uncertainties, Zames showed that the $\mathcal{H}_\infty$ norm in conjunction with the small gain theorem can be used to verify stability of a system with a norm bounded intrinsic uncertainty [93], and proposed a general framework for system analysis with internal uncertainties. Safonov generalized this concept in [70, 69]. Doyle introduced μ-analysis [56], a novel computational framework for computing the stability margin of a system that contains internal uncertainties in [23], and provided a scaled small gain test, which is a computationally tractable upper bound of μ. Although computing μ is NP-complete in general [12], Shamma proved that for a linear time varying block diagonal structure, the scaled small gain test provides a tight upper bound [74]. Subsequently, Paganini showed that a scalar linear time varying operator also has this tightness property [57], and Meinsma *et al.* considered a scalar self-adjoint time varying operator [48].

In the robust control literature, the $\mathcal{S}$-procedure [84, 64] is often used to derive a sufficient condition for stability of a system with internal uncertainties. For the necessity of such conditions, losslessness of the $\mathcal{S}$-procedure in shift invariant quadratic forms over l_2 (or $\mathcal{L}_2$) space in [47] plays a crucial role. Although stated in a different way, the aforementioned papers [57, 20, 48] and system analysis using integral quadratic constraints [46] rely on this losslessness result.

In the paper [65], Rantzer proved the KYP lemma using elementary linear algebra and basic convex analysis. In addition, excellent research monograph [3] from Balakrishnan and Vandenberghe attempted to simplify the essential mathematical tools for systems theory. Further, Scherer pointed out that the $\mathcal{S}$-procedure in robust control theory is nothing but a Lagrangian relaxation [72, 73], and Ebihara presented the dual of those programs in [28]. One of the difference of this dissertation is our reinterpretation of those problems as primal problems, since those optimizations can be directly derived from the definition of robustness analysis. The underlying philosophy of all these works is to use optimization as a main mathematical tool, and this dissertation takes the same approach.

A covariance formulation of optimal control theory is extensively covered in [75] and later extended in [76]. Recently, Gattami and Bamieh in [30] leveraged this covariance formulation to derive an SDP for computing the $\mathcal{H}_\infty$ norm, and the paper [89] considered finite frequency disturbance models using the same approach.

This dissertation combines all these ideas to present a novel framework of robust control theory for linear time invariant systems. In particular, Rantzer's idea on decomposing matrices in [65] and Shamma's idea on padding signals in [74] are used to construct a semidefinite representation of the closure of a set of Gramians, a deterministic counterpart of covariance matrices. Based on this Gramian representation, this dissertation shows how existing results can be elegantly stated using SDPs.

1.2 Organization of this dissertation

This dissertation is organized as follows. Chapter 2 gives the minimal background on signals, systems, and SDPs. Chapter 3 presents key lemmas involving the semidefinite representation of the closure of a set of Gramians. Chapter 4 extends $\mathcal{H}_\infty$ analysis using the key results from Chapter 3. Chapter 5 uses SDP duality theory to provide alternative, unified arguments for well-connectedness analysis. Chapter 6 proposes sparse $\mathcal{H}_\infty$ analysis, and provides an SDP relaxation of analysis and synthesis. Chapter 7 illustrates a potential application of sparse $\mathcal{H}_\infty$ analysis to power networks. Finally, Chapter 8 concludes the dissertation and proposes future research directions.

The preliminary results of Chapters 3, 4, 5 can be found in [90], and the conference version of Chapter 6 can be found in [91].

1.3 Miscellaneous published works

- *A Fast Linear Consensus Protocol on an Asymmetric Directed Graph [85]*

  In this paper, a linear consensus protocol design with a strongly connected directed communication topology is considered. A non-convex optimization problem that minimizes the second largest eigenvalue of a system matrix is formulated, and the cone complementary linearization method is applied to design a procedure to find a sub-optimal solution of the corresponding non-convex optimization. The effectiveness of the procedure is verified through numerical simulations.

- *Convex-concave Procedure for Weighted Sum-rate Maximization in a MIMO Interference Network [87]*

  This paper concerns maximizing the weighted sum of communication rates in a multi-input multi-output Gaussian communication channel subject to various constraints, such as total power, per-link power, maximum beam power, and condition number. Interference between channels introduces non-convexity in the objective, and the convex-concave procedure is applied to handle this non-convexity. The convergence of the procedure is shown by adding a damping term, and numerical simulations show its effectiveness. In some cases, it is shown that the damping term helps to find a solution with a better objective value.

- *Reverse and Forward Engineering of Frequency Control in Power Networks [86]*

  In this paper, we analyze the stability of frequency dynamics in a power network. In particular, the existing frequency feedback control scheme is shown to be a distributed algorithm that solves a well-defined global optimization problem, and a novel framework is developed to analyze the effect of a deadband in the control scheme. In addition, a small modification to the existing control architecture is proposed in order to unify the secondary frequency control scheme, which brings the frequency deviation back to the nominal operating frequency range.

- *A Non-convex Alternating Direction Method of Multipliers Heuristic for Optimal Power Flow [92]*

In this paper, an optimal power flow optimization problem that schedules power supplies is considered. The optimal power flow problem is a non-convex quadratic program, which is transformed to a semidefinite program with a rank constraint. To handle the rank constraint, this paper proposes a non-convex alternating direction method. Specifically, an iterative method, which requires an optimal solution of a convex optimization and its projection onto a low rank space, is proposed, and the local optimality of a stationary point is analyzed. In addition, numerical simulations verify effectiveness of the proposed procedure.

Chapter 2

Preliminaries on signals, systems and semidefinite programming

This chapter introduces notations used throughout this dissertation and provides a brief summary of relevant materials. More in-depth treatments can be found in [15, 55, 52, 26] for signals and systems and [4, 11] for semidefinite programs

2.1 Signals and systems

A signal $\mathbf{x}$ is a function that maps a non-negative integer, or a non-negative real number to a complex vector in $\mathbb{C}^n$. For a discrete time signal, we use the bracket notation, $x[k]$, to refer the value at time k, and for a continuous time signal, we use the parenthesis notation, $x(t)$, to refer the value at time t.

The Hilbert space of square summable sequences is denoted as

$$l_2^n := \left\{ \mathbf{x} : \sum_{k=0}^{\infty} x[k]^* x[k] < \infty, x[k] \in \mathbb{C}^n \right\},$$

with the inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{k=0}^{\infty} x[k]^* y[k],$$

where $x^* = (\bar{x})^\top$ is a conjugate transpose of x. Similarly, the Hilbert space of square integrable functions is denoted as

$$\mathcal{L}_2^n := \left\{ \mathbf{x} : \int_0^\infty x(t)^* x(t) < \infty, x(t) \in \mathbb{C}^n \right\},$$

with the inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_0^\infty x(t)^* y(t) dt.$$

We often drop the superscript n if the dimension can be easily inferred in the context.

The 2 norm of a signal is given by

$$\|\mathbf{x}\|_2 := \begin{cases} \sqrt{\sum_{k=0}^\infty x[k]^* x[k]} & \text{if } \mathbf{x} \in l_2 \\ \sqrt{\int_0^\infty x(t)^* x(t) dt} & \text{if } \mathbf{x} \in \mathcal{L}_2. \end{cases}$$

In this dissertation, we consider a causal Linear Time Invariant (LTI) system with a state space realization (A, B, C, D).

A discrete time LTI system has the following description:

$$\begin{aligned} x[k+1] &= Ax[k] + Bw[k] \\ z[k] &= Cx[k] + Dw[k], \end{aligned}$$

where $w[k]$ is the input (or the disturbance), $x[k]$ is the state, and $z[k]$ is the output of the system.

Similarly, a continuous time LTI system has the following description:

$$\begin{aligned} \frac{dx(t)}{dt} &= Ax(t) + Bw(t) \\ z(t) &= Cx(t) + Dw(t). \end{aligned}$$

Since this dissertation assumes that the initial condition of the state is zero, the following relationship holds between the state and the input [15]:

$$x[k] = \sum_{n=0}^{k-1} A^{k-n-1} B w[n], \quad k \geq 1 \tag{2.1}$$

$$x(t) = \int_0^t e^{A(t-\tau)} B w(\tau) d\tau, \quad t \geq 0, \tag{2.2}$$

where e^A is the matrix exponential of A. The above relationship reveals the linearity of the LTI system. Therefore, it is natural to consider a linear operator which maps the input signal $\mathbf{w} \in l_2$ (or $\mathcal{L}_2$) to the state $\mathbf{x}$. However, in order to determine the co-domain of this linear operator, the notion of stability is required. For example, the state $x[k]$ may approach to ∞ as $k \to \infty$, if A contains the eigenvalue of which magnitude is larger than 1. To avoid this situation, we introduce the Schur stability and Hurwitz stability of a matrix A:

$$\begin{aligned} A \text{ is Schur stable} &\Leftrightarrow \rho(A) < 1 \\ A \text{ is Hurwitz stable} &\Leftrightarrow \text{All eigenvalues of } A \text{ have the negative real part.} \end{aligned}$$

A discrete time LTI system is **stable** if A is Schur stable. In addition, for a Schur stable matrix A, it can be shown that $\| \sum_{n=0}^{k-1} A^{k-n-1} B w[n] \|_2 < \infty$ for all $\mathbf{w} \in l_2$ [40, 26]. Similarly, a continuous time LTI system is **stable** if A is Hurwitz stable, and $\mathbf{x}$ in (2.2) is in $\mathcal{L}_2$ for all $\mathbf{w} \in \mathcal{L}_2$.

Therefore, for a stable LTI system, *i.e.*, when A is stable, the following linear operator is well-defined. For a discrete time signal, let us define a linear operator $\mathbf{M}_{AB} : l_2 \to l_2$ such that $\mathbf{x} = \mathbf{M}_{AB}(\mathbf{w})$ if

$$\begin{aligned} x[k+1] &= Ax[k] + Bw[k], \\ x[0] &= 0, \end{aligned}$$

and for a continuous time signal, $\mathbf{x} = \mathbf{M}_{AB}(\mathbf{w})$, if

$$\begin{aligned} \frac{dx(t)}{dt} &= Ax(t) + Bw(t), \\ x(0) &= 0. \end{aligned}$$

Before concluding this section, we introduce an important concept in control theory: the controllability of the system. The pair (A, B) is controllable if we can reach an arbitrary final state $x^f \in \mathbb{C}^n$ from any initial state $x^i \in \mathbb{C}^n$ using the input signal $\mathbf{w}$ in a finite time, where n is the dimension of the state vector. In order to check the controllability, the controllability matrix $\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$ is often used. That is, (A, B) is controllable if and only if the controllability matrix has full row rank. The following theorem provides equivalent conditions of the controllability.

Theorem 2.1: *The following statements are equivalent.*

1. (A, B) *is conrollable.*
2. $\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$ *has full row rank.*
3. $(TAT^{-1}, T^{-1}B)$ *is controllable for some invertible matrix* T*.*
4. $(A + BK, B)$ *is controllable for some matrix* K*.*
5. (A, BT) *is controllable for some invertible matrix* T*.*

PROOF: The proof can be found in the standard textbook, *e.g.*, [26, 15] except the last condition. The controllability matrix for the pair (A, BT) is given by

$$\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} T.$$

Since T is invertible, the controllability of (A, BT) is equivalent to the second condition. ∎

2.2 Semidefinite program

Let $\mathbb{H}^n$ be the Hilbert space of n-by-n Hermitian matrices with the trace inner product

$$\langle X, Y\rangle = \mathbf{Tr}(X^*Y) = \mathbf{Tr}(XY).$$

The Frobenius norm is induced by this inner product

$$\|X\|_F = \sqrt{\mathbf{Tr}(X^*X)} = \sqrt{\mathbf{Tr}(X^2)}.$$

A matrix $X \in \mathbb{H}$ is positive semidefinite (PSD) if all of its eigenvalues are non-negative, and we use a generalized inequality $X \succeq 0$ as a shorthand notation for X being PSD. If all of eigenvalues of X are positive, then X is positive definite and $X \succ 0$.

A standard semidefinite program (SDP) is a convex optimization which has the following form [4]:

$$(P) \qquad \begin{aligned} \underset{X}{\text{minimize}} \quad & \langle C, X\rangle \\ \text{subject to} \quad & \langle A_i, X\rangle = b_i, \qquad i = 1, \cdots n_c \\ & X \succeq 0, \end{aligned} \tag{2.3}$$

where X, C, A_i are real symmetric matrices, and b_i is a real number. The feasible set of this optimization is a set of X such that $X \succeq 0$, and $\langle A_i, X\rangle = b_i$. The above program (P) is strictly feasible if there exists $X \succ 0$ in the feasible set.

A dual of SDP is given by

$$(D) \qquad \begin{aligned} \underset{y_1, \cdots, y_{n_c}}{\text{maximize}} \quad & -\sum_{i=1}^{n_c} b_i y_i \\ \text{subject to} \quad & C - \sum_{i=1}^{n_c} y_i A_i \succeq 0, \end{aligned} \tag{2.4}$$

where $b = \begin{bmatrix} b_1, \cdots, b_{n_c} \end{bmatrix}^T$. The above program (D) is strictly feasible if there exists

$(y_1, \cdots, y_{n_c})$ such that $C - \sum_{i=1}^{n_c} y_i A_i \succ 0$.

The optimization (P) is called the primal program, and (D) is called the dual program. An SDP conic duality theorem relates this primal-dual pair.

Theorem 2.2: *Let $p^\star$ be the optimal value of the optimization (P), and $d^\star$ be the optimal value of (D). Then we have,*

- *Weak duality: $p^\star \geq d^\star$.*
- *Strong duality from (P): If $p^\star$ is bounded by below, and (P) is strictly feasible, then $p^\star = d^\star$. In addition, there exists $(y_1^\star, \cdots, y_{n_c}^\star)$ in the feasible set of (D) such that $-\sum_{i=1}^{n_c} b_i y_i^\star = d^\star$.*
- *Strong duality from (D): If $d^\star$ is bounded by above, and (D) is strictly feasible, then $p^\star = d^\star$. In addition, there exists $X^\star$ in the feasible set of (P) such that $\langle C, X^\star \rangle = p^\star$.*

The proof can be found in [4]. The strictly feasibility condition is often called *Slater's constraint qualification* for strong duality.

Although we use the term *minimize* and *maximize* in the optimization, finding an infimum/supremum is more precise since a feasible point that achieves such an extreme value may not exist. In addition, there are primal-dual SDPs where the dual program is solvable (there exists a dual optimal solution), but the primal program is not solvable even though strong duality holds. Strum's thesis [77] contains many interesting examples on this subject and extensively studies the SDP duality.

SDP becomes an essential tool for many engineering disciplines. See [11] and references there in for the recent progress and modern engineering applications.

Part of the reasons, if not complete, are from both practical and theoretical sides.

1. There exists a polynomial time solver for SDPs, such as the interior point method [53], that is reliable.
2. Modeling capability of SDP is powerful enough to capture real world engineering problems.

3. SDP duality theorem, Theorem 2.2, often provides new theoretical insights.

In this dissertation, we use the primal-dual picture of SDP to provide interesting theoretical insights on robust control theory.

An important remark in here is that standard SDP duality theory concerns the real numbers, whereas our main result will rely on the complex numbers. Extending real SDP results to the complex setting can be done by decomposing an Hermitian matrix $X = X_R + \mathbf{i}X_I$ where X_R is real symmetric and X_I is a real skew-symmetric. Since $X \succeq 0$ is equivalent to $\begin{bmatrix} X_R & X_I \\ -X_I^\top & X_R \end{bmatrix} \succeq 0$, and we can convert any complex SDP to the standard real SDP (2.3). This approach can also be found in [34], and therefore we can apply the existing software package for solving real SDPs, such as [80], to solve complex SDPs. In addition, SDP duality theory can be extended to the complex setting by considering the set of PSD complex matrices as the underlying convex cone. Since all the results from [4] can be applied to a convex cone, this extension is trivial, so this dissertation uses the complex SDPs as a primary modeling tool.

Finally, we present the theorem of alternatives for SDPs, which is a special form of a conic duality theorem. The proofs and related concepts can be found in the excellent monograph from Balakrishnan and Vandenberghe [3].

Let $\mathbb{V}$ be a finite dimensional Hilbert space with an inner product $\langle \cdot, \cdot \rangle_{\mathbb{V}}$ and $\mathbb{W}$ be a finite dimensional Hilbert space with an inner product $\langle \cdot, \cdot \rangle_{\mathbb{W}}$. In addition, let $\mathcal{A} : \mathbb{V} \to \mathbb{H}$ be a linear operator that maps an element in $\mathbb{V}$ to a set of Hermitian matrices, and $\mathcal{B} : \mathbb{V} \to \mathbb{W}$ be a linear operator that maps an element in $\mathbb{V}$ to $\mathbb{W}$. Then we have the following results:

Theorem 2.3 (ALT1): *Exactly one of the following is true.*

(i) There exists an $x \in \mathbb{V}$ *with* $\mathcal{A}(x) + A_0 \succ 0$.

(ii) There exists a $Z \gneq 0$ *such that* $\mathcal{A}^*(Z) = 0,\ \boldsymbol{Tr}(A_0 Z) \leq 0$.

Here A_0 is a given matrix, and $Z \gneq 0$ means $Z \succeq 0$ but $Z \neq 0$, and $\mathcal{A}^*$ is the adjoint of $\mathcal{A}$. The following results is without the matrix A_0.

Theorem 2.4 (ALT2a): *Exactly one of the following is true.*

(i) There exists an $x \in \mathbb{V}$ *with* $\mathcal{A}(x) \gneqq 0$.

(ii) There exists a $Z \succ 0$ *such that* $\mathcal{A}^*(Z) = 0$.

With an additional affine constraint from $\mathcal{B}$, we have the following results.

Theorem 2.5 (ALT4): *Exactly one of the following is true.*

(i) There exists an $x \in \mathbb{V}$ *with* $\mathcal{A}(x) + A_0 \succ 0$, *and* $\mathcal{B}(x) = 0$.

(ii) There exists a $Z \gneqq 0$, $w \in \mathbb{W}$, $\mathcal{A}^*(Z) + \mathcal{B}^*(w) = 0$, *and* $\boldsymbol{Tr}(A_0 Z) \leq 0$.

Theorem 2.6 (ALT5a): *Exactly one of the following is true.*

(i) There exists an $x \in \mathbb{V}$ *with* $\mathcal{A}(x) \gneqq 0$, *and* $\mathcal{B}(x) = 0$.

(ii) There exist $Z \succ 0$, $w \in \mathbb{W}$ *such that* $\mathcal{A}^*(Z) + \mathcal{B}^*(w) = 0$.

We often call the above pair as a strong alternative to each other because exactly one of each pair is true.

The above theorems may seem restrictive because only one $\mathcal{A}$ and $\mathcal{B}$ are being used. Suppose we have multiple constraints,

$$\mathcal{A}_i(x) + A_{i,0} \succ 0 \qquad \text{for } i = 1, \cdots, n_a \tag{2.5}$$

In this case we can form a block diagonal matrix to use Theorem 2.3.

$$\begin{aligned} \mathcal{A}(x) &= \textbf{diag}\left(\mathcal{A}_1(x), \cdots, \mathcal{A}_{n_a}(x)\right) = \begin{bmatrix} \mathcal{A}_1(x) & & \\ & \ddots & \\ & & \mathcal{A}_{n_a}(x) \end{bmatrix} \\ A_0 &= \textbf{diag}\left(A_{1,0}, \cdots, A_{n_a,0}\right) = \begin{bmatrix} A_{1,0} & & \\ & \ddots & \\ & & A_{n_a,0} \end{bmatrix} \end{aligned}$$

Then $\mathcal{A}(x) + A_0 \succ 0$ holds if and only if (2.5) holds. Before applying Theorem 2.3, let us consider the adjoint operator of this block diagonal form of $\mathcal{A}(x), A_0$. Notice that

$$\begin{aligned}\mathbf{Tr}(Z\mathcal{A}(x)) &= \mathbf{Tr}\left(\begin{bmatrix} Z_{11} & \cdots & Z_{1n_a} \\ \vdots & \ddots & \vdots \\ Z_{n_a1} & \cdots & Z_{n_an_a}\end{bmatrix}\begin{bmatrix}\mathcal{A}_1(x) & & \\ & \ddots & \\ & & \mathcal{A}_{n_a}(x)\end{bmatrix}\right) \\ &= \sum_{i=1}^{n_a}\mathbf{Tr}(Z_{ii}\mathcal{A}_i(x)) = \sum_{i=1}^{n_a}\langle\mathcal{A}_i^*(Z_{ii}), x\rangle = \langle\sum_{i=1}^{n_a}\mathcal{A}_i^*(Z_{ii}), x\rangle,\end{aligned}$$

where Z_{ij} are properly partitioned according to the dimensions of $\mathcal{A}_i(x)$ and $\mathcal{A}_j(x)$. Therefore we can conclude that

$$\mathcal{A}^*(Z) = \sum_{i=1}^{n_a}\mathcal{A}_i^*(Z_{ii}),$$

and a strong alternative to (2.5) is given by

$$\begin{aligned} Z &\succeq\neq 0 \\ \sum_{i=1}^{n_a}\mathcal{A}_i^*(Z_{ii}) &= 0 \\ \mathbf{Tr}(A_0Z) = \sum_{i=1}^{n_a}\mathbf{Tr}(A_{i,0}Z_{ii}) &\leq 0.\end{aligned}$$

Notice that in the second and third inequality the non-diagonal part of Z is completely irrelevant. Therefore the above condition is equivalent to the existence of $Z_1, \cdots, Z_{n_a}$ such that

$$\begin{aligned} Z_i &\succeq 0 \\ \sum_{i=1}^{n_a}\mathcal{A}_i^*(Z_i) &= 0 \\ \sum_{i=1}^{n_a}\mathbf{Tr}(A_{i,0}Z_i) &\leq 0,\end{aligned}$$

where at least one Z_i is not zero. This is because if $(Z_1, \cdots, Z_{n_a})$ exist then $Z = \mathbf{diag}\,(Z_1, \cdots, Z_{n_a})$ satisfies the first condition, and for the converse, we can set $Z_i = Z_{ii}$. This leads us to the following corollary of Theorem 2.3.

Corollary 2.1: *Exactly one of the following is true.*

(i) There exists an $x \in \mathbb{V}$ *with* $\mathcal{A}_i(x) + A_{i,0} \succ 0$ *for* $i = 1, \cdots, n_a$.

(ii) There exist $Z_i \succeq 0$ *such that* $\sum_{i=1}^{n_a} \mathcal{A}_i^*(Z_i) = 0$, $\sum_{i=1}^{n_a} \boldsymbol{Tr}(A_{i,0}Z_i) \leq 0$ *where at least one* Z_i *is not zero.*

Similar arguments can be used to show the following corollaries of Theorem 2.5, Thoerem 2.4, and Theorem 2.6.

Corollary 2.2: *Exactly one of the following is true.*

(i) There exists an $x \in \mathbb{V}$ *with* $\mathcal{A}_i(x) + A_{i,0} \succ 0$ *for* $i = 1, \cdots, n_a$, $\mathcal{B}_i(x) = 0$, *for* $i = 1, \cdots, n_b$.

(ii) There exist $Z_i \succeq 0$, $w \in \mathbb{W}_i$ *such that* $\sum_{i=1}^{n_a} \boldsymbol{Tr}(A_{i,0}Z_i) \leq 0$, $\sum_{i=1}^{n_a} \mathcal{A}_i^*(Z_i) + \sum_{i=1}^{n_b} \mathcal{B}_i^*(w_i) = 0$ *where at least one* Z_i *is not zero.*

Corollary 2.3: *Exactly one of the following is true.*

(i) There exists an $x \in \mathbb{V}$ *with* $\mathcal{A}_i(x) \succeq 0$ *for* $i = 1, \cdots, n_a$, *where at least one* $\mathcal{A}_i(x)$ *is not zero.*

(ii) There exist $Z_i \succ 0$, $w \in \mathbb{W}_i$ *such that* $\sum_{i=1}^{n_a} \mathcal{A}_i^*(Z_i) + \sum_{i=1}^{n_b} \mathcal{B}_i^*(w_i) = 0$.

Corollary 2.4: *Exactly one of the following is true.*

(i) There exists an $x \in \mathbb{V}$ *with* $\mathcal{A}_i(x) \succeq 0$ *for* $i = 1, \cdots, n_a$, $\mathcal{B}_i(x) = 0$, *for* $i = 1, \cdots, n_b$, *where at least one* $\mathcal{A}_i(x)$ *is not zero.*

(ii) There exist $Z_i \succ 0$, $w \in \mathbb{W}_i$ *such that* $\sum_{i=1}^{n_a} \mathcal{A}_i^*(Z_i) + \sum_{i=1}^{n_b} \mathcal{B}_i^*(w_i) = 0$.

The same trick will be used when we apply the theorem of alternatives to the block diagonal form of operators in the later chapter.

2.3 Linear matrix inequalities in linear system theory

A Linear Matrix Inequality (LMI) has the following form

$$\sum_{i=1}^{n} F_i x_i + F_0 \succeq 0, \tag{2.6}$$

for Hermitian matrices $F_0, \cdots, F_n$ and $x_1, \cdots, x_n \in \mathbb{R}$. Here x_i's are variables, and the question is to find x_i which satisfies the above generalized inequality. Notice that this is nothing but a feasibility problem, and can be formulated as an SDP with the objective function being 0. This means that a set of x that satisfies (2.6) is a convex set, and an element in the set can be obtained by solving an SDP. Many results in system and control theory can be written in the form of (2.6). The excellent book from Boyd *et. al.* [10] contains both the historical perspective and the role of LMIs in system and control theory.

The following results use the LMI, often called Lyapunov LMI because the results are originated from Lyapunov stability theorem, to verify the stability of an LTI system,

Theorem 2.7: *A matrix A is Hurwitz stable if and only if there exists $P \succ 0$ such that $AP + PA^* \prec 0$.*

Theorem 2.8: *A matrix A is Schur stable if and only if there exists $P \succ 0$ such that $APA^* - P \prec 0$.*

These are standard results and the proofs can be found in [3].

In Theorem 2.7 and 2.8, the matrix P itself is a variable. Although this may not seem to be equivalent to (2.6) but we can convert it into the standard form (2.6) by using a standard basis for Hermitian matrices. To this end, for $P \in \mathbb{C}^{n\times n}$, let $M_{ij} = \frac{1}{2}(e_i e_j^* + e_j e_i^*)$ and $N_{ij} = \frac{\mathbf{i}}{2}(e_i e_j^* - e_j e_i^*)$, where $\{e_1, \cdots, e_n\}$ is the standard basis for $\mathbb{C}^n$. Then, we can represent any Hermitian matrix by $P = \sum_{i,j} M_{ij}x_{ij} + N_{ij}y_{ij}$, where $x_{ij}, y_{ij} \in \mathbb{R}$. In addition, $A^*P + PA = \sum_{i,j}(AM_{ij} + M_{ij}A^*)x_{ij} + (AN_{ij} + N_{ij}A^*)y_{ij}$.

Finally, by embedding P and $AP + PA^*$ to a large matrix $\begin{bmatrix} P & 0 \\ 0 & A^*P + PA \end{bmatrix}$, we can clearly see that the condition in Theorem 2.7 is equivalent to $\begin{bmatrix} P & 0 \\ 0 & AP + PA^* \end{bmatrix} \succ 0$. This shows that we can treat the matrix as a variable and we can always convert the corresponding constraint to the standard LMI by using the standard basis. Therefore we call any expression of the form $f(X) \succeq 0$, where X is a matrix variable and $f(X)$ is an affine operator that maps X to an Hermitian matrix, to be an LMI.

Moreover, we often encounter a non-linear version of (2.6). For example, in the feedback controller synthesis problem, we require $A^*P + PA \prec 0$, where A and P are both variables. We call this type of constraint a Matrix Inequality (MI) since the dependency is not linear. Unlike an LMI constraint, an optimization with matrix inequality constraints is non-convex in general unless the corresponding matrix inequality has a special structure.

The following Lyapunov equation is closely related to Lyapunov LMI in Theorem 2.7 and 2.8:

$$(CLE) \qquad AP + PA^* + Q = 0 \tag{2.7}$$

$$(DLE) \qquad APA^* - P + Q = 0. \tag{2.8}$$

Here A, Q are given matrices, and we would like to find an Hermitian P satisfies (CLE) or (DLE). Notice that Q has to be Hermitian, since $AP + PA^*$ and $APA^* - P$ are Hermitian.

Using the vectorization operator $\mathbf{vec}\,(P) = \begin{bmatrix} [P]_{11} & [P]_{12} & \cdots & [P]_{nn} \end{bmatrix}^\top$ for $P \in \mathbb{C}^{n\times n}$, and the Kronecker product

$$A \otimes B = \begin{bmatrix} [A]_{11}B & \cdots & [A]_{1n}B \\ \vdots & \ddots & \vdots \\ [A]_{n1}B & \cdots & [A]_{nn}B \end{bmatrix},$$

where $A \in \mathbb{C}^{n\times n}$, we can convert (CLE) and (DLE) to the systems of linear equation.

$$(CLE) \qquad (I_n \otimes A + \bar{A} \otimes I_n)\mathbf{vec}\,(P) = -\mathbf{vec}\,(Q)$$
$$(DLE) \qquad (\bar{A} \otimes A - I_n \otimes I_n)\mathbf{vec}\,(P) = -\mathbf{vec}\,(Q)\,.$$

Therefore the existence and uniqueness of P is completely determined by the invertibility of $I_n \otimes A + \bar{A} \otimes I_n$ and $\bar{A} \otimes A - I_n \otimes I_n$. By identifying $\otimes$ with a tensor product, we can see that the eigenvalues of $I_n \otimes A + \bar{A} \otimes I_n$ are given by $\lambda_i(A) + \bar{\lambda}_j(A)$, where $\lambda_i(A)$ is the ith eigenvalue of A. Therefore if A is Hurwitz stable, *i.e.*, $\mathbf{Re}(\lambda_i(A)) < 0$ for all i, then $\mathbf{Re}\big(\lambda_i(A) + \bar{\lambda}_j(A)\big) < 0$. This shows that all the eigenvalues of $I_n \otimes A + \bar{A} \otimes I_n$ is not zero, and therefore $I_n \otimes A + \bar{A} \otimes I_n$ is invertible and P uniquely exists. In fact, $P = \int_0^\infty e^{At}Qe^{A^*t}dt$ solves (CLE), and we can easily see that if $Q \succeq 0$ then $P \succeq 0$.

Similarly, for (DLE), since the eigenvalues of $\bar{A} \otimes A - I_n \otimes I_n$ are given by $\bar{\lambda}_i(A)\lambda_j(A) - 1$, if A is Schur stable, then there is no zero eigenvalue. Therefore $\bar{A} \otimes A - I_n \otimes I_n$ becomes invertible, and P uniquely exists. The analytical expression of P is given by $P = \sum_{k=0}^\infty A^kQ(A^*)^k$.

We summarize all these results in the following and the formal proof can be found in [26].

Theorem 2.9: *If A is Hurwitz stable, then there exists the unique PSD matrix $P = \int_0^\infty e^{At}Qe^{A^*t}dt$, which satisfies $AP + PA^* + Q = 0$.*

Theorem 2.10: *If A is Schur stable, then there exists the unique PSD matrix $P = \sum_{k=0}^\infty A^kQ(A^*)^k$, which satisfies $APA^* - P + Q = 0$.*

Before concluding this chapter, we introduce a controllability Gramian. For a stable A, we define a controllability Gramian which gives an alternative characterization of controllability.

$$W_c := \begin{cases} \sum_{k=0}^\infty A^kBB^*(A^*)^k & \text{if } A \text{ is Schur stable} \\ \int_0^\infty e^{At}BB^*e^{A^*t}dt & \text{if } A \text{ is Hurwitz stable.} \end{cases}$$

From Theorem 2.9 and 2.10 we can easily see that the controllability Gramian satisfies the following Lyapunov equation:

$$\begin{aligned} \text{Discrete:} \qquad & AW_cA^* - W_c + BB^* = 0 \\ \text{Continuous:} \qquad & AW_c + W_cA^* + BB^* = 0 \end{aligned}$$

Notice that since $BB^* \succeq 0$, $W_c \succeq 0$.

Finally, we have the following result, and the proof can be found in [26].

Theorem 2.11: *The pair* (A, B) *is controllable if and only if the controllability Gramian* $W_c \succ 0$*.*

So far we present the minimal background on the concept used in this dissertation. Complete and detail treatment of the subject can be found in the references given in the beginning of this chapter.

Chapter 3

A semidefinite representation of Gramians

This chapter introduces Gramians generated by a stable LTI system which contains complete quadratic information of the system. In turns out that these Gramians admit a semidefinite representation. This chapter presents this result, and provides an algorithm which constructs an input-state pair generating a Gramian. This algorithm can be used to find the optimal solution of robustness optimization.

3.1 A semidefinite representation of Gramians: Discrete time case

3.1.1 Set of Gramians

For a signal $\mathbf{u} \in l_2^n$, we define the Gramian $\Lambda : l_2 \to \mathbb{H}^n$ such that

$$\Lambda(\mathbf{u}) = \sum_{k=0}^{\infty} u[k]u[k]^*.$$

The Gramian is well-defined because each entry is finite. In addition, it should be clear that $\Lambda(\mathbf{u}) \succeq 0$, for all $\mathbf{u} \in l_2^n$. For a notational convienience, we often use $\Lambda(\mathbf{u}_1, \mathbf{u}_2)$ for $\Lambda\left(\begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix}\right)$.

Consider the following stable discrete time LTI system, $\mathbf{P}$, which maps an input

$\mathbf{w}$ to an output $\mathbf{z}$,

$$x[k+1] = Ax[k] + Bw[k] \tag{3.1}$$

$$x[0] = 0 \tag{3.2}$$

$$z[k] = Cx[k] + Dw[k], \tag{3.3}$$

where $x \in \mathbb{C}^{n_x}, w \in \mathbb{C}^{n_w}, z \in \mathbb{C}^{n_z}$ and A is Schur stable. Since A is Schur stable, we can consider the linear operator $\mathbf{M}_{AB} : l_2^{n_w} \to l_2^{n_x}$ such that $\mathbf{x} = \mathbf{M}_{AB}(\mathbf{w})$, where $\mathbf{x}$ is the state of $\mathbf{P}$, *i.e.*,

$$x[k] = \sum_{i=0}^{k} A^{k-i} Bw[i] \qquad \text{for } k \geq 0. \tag{3.4}$$

We introduce the following set of discrete-time Gramians which is a primary object in this dissertation.

$$\mathcal{D}_{\text{Gram}} := \left\{ V \in \mathbb{H}^{n_x+n_w} \ : V = \Lambda\left(\mathbf{M}_{AB}(\mathbf{w}), \mathbf{w}\right) \text{ for some } \mathbf{w} \in l_2 \right\}$$

In other words, $V \in \mathcal{D}_{\text{Gram}}$ if and only if there exists an input $\mathbf{w} \in l_2$ such that

$$V = \sum_{k=0}^{\infty} \begin{bmatrix} x[k] \\ w[k] \end{bmatrix} \begin{bmatrix} x[k] \\ w[k] \end{bmatrix}^*,$$

where the state $\mathbf{x} = \mathbf{M}_{AB}(\mathbf{w})$ is from (3.4). Since $\mathbf{x} \in l_2$ for all $\mathbf{w} \in l_2$, a Gramian $\Lambda(\mathbf{x}, \mathbf{w})$ is well-defined for all $\mathbf{w} \in l_2$.

Notice that V contains the complete second order information of the LTI system.

For example, since $\begin{bmatrix} 0_{n_x \times n_x} & I_{n_w} \end{bmatrix} \begin{bmatrix} x[k] \\ w[k] \end{bmatrix} = w[k]$,

$$\begin{aligned}
\|\mathbf{w}\|_2^2 &= \sum_{k=0}^{\infty} w[k]^* w[k] = \sum_{k=0}^{\infty} \mathbf{Tr}(w[k]w[k]^*) \\
&= \sum_{k=0}^{\infty} \mathbf{Tr}\left(\left(\begin{bmatrix} 0_{n_w \times n_x} & I_{n_w} \end{bmatrix} \begin{bmatrix} x[k] \\ w[k] \end{bmatrix}\right)\left(\begin{bmatrix} 0_{n_w \times n_x} & I_{n_w} \end{bmatrix} \begin{bmatrix} x[k] \\ w[k] \end{bmatrix}\right)^*\right) \\
&= \mathbf{Tr}\left(\begin{bmatrix} 0_{n_w \times n_x} & I_{n_w} \end{bmatrix} V \begin{bmatrix} 0_{n_w \times n_x} & I_{n_w} \end{bmatrix}^*\right) = \mathbf{Tr}\left(\begin{bmatrix} 0_{n_x \times n_x} & 0_{n_x \times n_w} \\ 0_{n_w \times n_x} & I_{n_w} \end{bmatrix} V\right),
\end{aligned}$$

where the last equality is from the cyclic property of the trace operator. The examples of the second order information include

- The norm of the input: $\|\mathbf{w}\|_2^2 = \mathbf{Tr}\left(\begin{bmatrix} 0_{n_x \times n_x} & 0_{n_x \times n_w} \\ 0_{n_w \times n_x} & I_{n_w} \end{bmatrix} V\right)$.
- The norm of the state: $\|\mathbf{x}\|_2^2 = \mathbf{Tr}\left(\begin{bmatrix} I_{n_x} & 0_{n_x \times n_w} \\ 0_{n_w \times n_x} & 0_{n_w \times n_w} \end{bmatrix} V\right)$.
- The norm of the output: $\|\mathbf{z}\|_2^2 = \mathbf{Tr}\left(\begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D \end{bmatrix} V\right)$.
- Passivity of the system: $\langle \mathbf{z}, \mathbf{w} \rangle = \frac{1}{2}\mathbf{Tr}\left(\begin{bmatrix} 0_{n_x \times n_w} & C^* \\ C & D + D^* \end{bmatrix} V\right)$.

Since the Gramians contain the quadratic information, we can use the Gramians for robustness analysis. For example, the $\mathcal{H}_\infty$ norm of $\mathbf{P}$ is given by $\|\mathbf{P}\|_\infty := \sup_{\|\mathbf{w}\|_2=1} \|\mathbf{z}\|_2$, *i.e.*, an l_2-l_2 induced gain. From the definition, we can easily see that

$$\|\mathbf{P}\|_\infty^2 := \sup_{V \in \mathcal{D}_{\text{Gram}}} \left\{ \mathbf{Tr}\left(\begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D \end{bmatrix} V\right) : \mathbf{Tr}\left(\begin{bmatrix} 0_{n_x \times n_x} & 0_{n_x \times n_w} \\ 0_{n_w \times n_x} & I_{n_w} \end{bmatrix} V\right) = 1 \right\}.$$

In order to compute $\|\mathbf{P}\|_\infty^2$ from this definition, we need to search over the infinite dimensional space l_2 to check $V \in \mathcal{D}_{\text{Gram}}$, which is not a trivial task. In addition, many system analyses rely on the quadratic information of the system having the same issue. This being said, it is desirable to find a convenient way to characterize the set $\mathcal{D}_{\text{Gram}}$.

From the equation (3.1),

$$x[k+1]x[k+1]^* = (Ax[k]+Bw[k])(Ax[k]+Bw[k])^*.$$

By taking the infinite sum, we have

$$\sum_{k=0}^{\infty} x[k+1]x[k+1]^* = \sum_{k=0}^{\infty}(Ax[k]+Bw[k])(Ax[k]+Bw[k])^*, \tag{3.5}$$

and since $x[0]=0$, $\sum_{k=0}^{\infty} x[k+1]x[k+1]^* = \sum_{k=0}^{\infty} x[k]x[k]^*$.

Moreover,

$$\begin{aligned}\sum_{k=0}^{\infty} x[k]x[k]^* &= \sum_{k=0}^{\infty}\left(\begin{bmatrix} I_{n_x} & 0_{n_x\times n_w}\end{bmatrix}\begin{bmatrix} x[k] \\ w[k]\end{bmatrix}\right)\left(\begin{bmatrix} I_{n_x} & 0_{n_x\times n_w}\end{bmatrix}\begin{bmatrix} x[k] \\ w[k]\end{bmatrix}\right)^* \\ &= \begin{bmatrix} I_{n_x} & 0_{n_x\times n_w}\end{bmatrix}\sum_{k=0}^{\infty}\begin{bmatrix} x[k] \\ w[k]\end{bmatrix}\begin{bmatrix} x[k] \\ w[k]\end{bmatrix}^*\begin{bmatrix} I_{n_x} & 0_{n_x\times n_w}\end{bmatrix}^*\end{aligned}$$

$$\begin{aligned}\sum_{k=0}^{\infty}(Ax[k]+Bw[k])(Ax[k]+Bw[k])^* &= \sum_{k=0}^{\infty}\left(\begin{bmatrix} A & B\end{bmatrix}\begin{bmatrix} x[k] \\ w[k]\end{bmatrix}\right)\left(\begin{bmatrix} A & B\end{bmatrix}\begin{bmatrix} x[k] \\ w[k]\end{bmatrix}\right)^* \\ &= \begin{bmatrix} A & B\end{bmatrix}\sum_{k=0}^{\infty}\begin{bmatrix} x[k] \\ w[k]\end{bmatrix}\begin{bmatrix} x[k] \\ w[k]\end{bmatrix}^*\begin{bmatrix} A & B\end{bmatrix}^*.\end{aligned}$$

This shows that any $V = \sum_{k=0}^{\infty}\begin{bmatrix} x[k] \\ w[k]\end{bmatrix}\begin{bmatrix} x[k] \\ w[k]\end{bmatrix}^* \in \mathcal{D}_{\text{Gram}}$ satisfies the following constraints:

$$V \succeq 0 \tag{3.6}$$

$$\begin{bmatrix} I_{n_x} & 0_{n_x \times n_w} \end{bmatrix} V \begin{bmatrix} I_{n_x} & 0_{n_x \times n_w} \end{bmatrix}^* = \begin{bmatrix} A & B \end{bmatrix} V \begin{bmatrix} A & B \end{bmatrix}^*. \tag{3.7}$$

By defining the set

$$\mathcal{D}_{\mathrm{SDP}} := \left\{ V \in \mathbb{H} : V \succeq 0, \begin{bmatrix} I_{n_x} & 0_{n_x \times n_w} \end{bmatrix} V \begin{bmatrix} I_{n_x} & 0_{n_x \times n_w} \end{bmatrix}^* = \begin{bmatrix} A & B \end{bmatrix} V \begin{bmatrix} A & B \end{bmatrix}^* \right\},$$

we can easily see that $\mathcal{D}_{\mathrm{Gram}} \subset \mathcal{D}_{\mathrm{SDP}}$. Therefore we obtain the following upper bound of $\|\mathbf{P}\|_\infty^2$ by replacing $\mathcal{D}_{\mathrm{Gram}}$ with $\mathcal{D}_{\mathrm{SDP}}$:

$$\begin{aligned} \|\mathbf{P}\|_\infty^2 &= \sup_{V \in \mathcal{D}_{\mathrm{Gram}}} \left\{ \mathbf{Tr}\left(\begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D \end{bmatrix} V \right) : \mathbf{Tr}\left(\begin{bmatrix} 0_{n_x \times n_x} & 0_{n_x \times n_w} \\ 0_{n_w \times n_x} & I_{n_w} \end{bmatrix} V \right) = 1 \right\} \\ &\leq \sup_{V \in \mathcal{D}_{\mathrm{SDP}}} \left\{ \mathbf{Tr}\left(\begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D \end{bmatrix} V \right) : \mathbf{Tr}\left(\begin{bmatrix} 0_{n_x \times n_x} & 0_{n_x \times n_w} \\ 0_{n_w \times n_x} & I_{n_w} \end{bmatrix} V \right) = 1 \right\}. \end{aligned}$$

More importantly, since $\mathcal{D}_{\mathrm{SDP}}$ is an intersection of a subspace in $\mathbb{H}$ and a semidefinite cone, $\mathcal{D}_{\mathrm{SDP}}$ is a finite dimensional closed, convex cone that is SDP representable. Therefore, computing the upper bound can be solved via finite dimensional SDP.

Based on this observation, the immediate, important question arises: Is $\mathcal{D}_{\mathrm{SDP}}$ equal to $\mathcal{D}_{\mathrm{Gram}}$? If so, the above upper bound becomes tight, and $\mathcal{H}_\infty$ norm can be exactly computed using SDP. However, the answer to this question is negative. Let us see the following example.

Example 3.1: Let $A = \frac{1}{2}$, $B = 1$, and $V = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$. Then $V \in \mathcal{D}_{\mathrm{SDP}}$, and $\mathbf{rank}\,(V) = 1$.

Let $\mathbf{w} \in l_2$ with $w[0] = 1$. Then $x[0] = 0$, and $x[1] = 1$.

$$\Lambda(\mathbf{x}, \mathbf{w}) \succeq \begin{bmatrix} 0 \\ w[0] \end{bmatrix} \begin{bmatrix} 0 \\ w[0] \end{bmatrix}^* + \begin{bmatrix} x[1] \\ w[1] \end{bmatrix} \begin{bmatrix} x[1] \\ w[1] \end{bmatrix}^* = \begin{bmatrix} 1 & w[1]^* \\ w[1] & w[1]^* w[1] + 1 \end{bmatrix} \succ 0.$$

This shows for any $\mathbf{w} \neq 0$, the corresponding Gramian is positive definite. Since any non-trivial matrix in $\mathcal{D}_{\text{Gram}}$ is full rank, $V \notin \mathcal{D}_{\text{Gram}}$.

Although $\mathcal{D}_{\text{SDP}} \neq \mathcal{D}_{\text{Gram}}$, it turns out that we can go beyond the relationship $\mathcal{D}_{\text{Gram}} \subset \mathcal{D}_{\text{SDP}}$: the closure of $\mathcal{D}_{\text{Gram}}$, $\mathbf{cl}\,(\mathcal{D}_{\text{Gram}})$, is $\mathcal{D}_{\text{SDP}}$, which is one of the main contributions of this dissertation.

Lemma 3.1: *For all $V \in \mathcal{D}_{SDP}$, $\varepsilon > 0$, there exists a Gramian $\tilde{V} \in \mathcal{D}_{Gram}$ such that*

$$\|V - \tilde{V}\|_F < \varepsilon \tag{3.8}$$

$$\begin{bmatrix} 0_{n_x} & I_{n_w} \end{bmatrix} (V - \tilde{V}) \begin{bmatrix} 0_{n_x} & I_{n_w} \end{bmatrix}^* = 0. \tag{3.9}$$

The proof can be found in the Appendix A. The equation (3.9) implies that the right bottom block of V, which corresponds to the Gram matrix of $\mathbf{w}$, $\Lambda(\mathbf{w})$, can be matched exactly. However, as we have seen in Example 3.1, it may not be possible to match other blocks. Nonetheless, our proof is constructive, which means that we can always find an input $\mathbf{w} \in l_2$ whose Gramian $\Lambda(\mathbf{x}, \mathbf{w})$ is arbitrary close to a given matrix $V \in \mathcal{D}_{\text{SDP}}$. Therefore the following consequence is immediate.

Lemma 3.2: $\mathcal{D}_{SDP} = \mathbf{cl}\,(\mathcal{D}_{Gram})$.

Proof: Since $\mathcal{D}_{\text{SDP}}$ is closed and $\mathcal{D}_{\text{Gram}} \subset \mathcal{D}_{\text{SDP}}$, $\mathbf{cl}\,(\mathcal{D}_{\text{Gram}}) \subset \mathcal{D}_{\text{SDP}}$. In addition, from Lemma 3.1, $\mathcal{D}_{\text{SDP}} \subset \mathbf{cl}\,(\mathcal{D}_{\text{Gram}})$. ∎

The above two lemmas are the key results of this chapter.

3.1.2 Controllability Gramian and relative interior

In this section, we investigate the relative interior of $\mathcal{D}_{\text{SDP}}$, $\{V : V \in \mathcal{D}_{\text{SDP}}, V \succ 0\}$. Recall that the controllability Gramian $W_c \succ 0$ if and only if (A, B) is controllable. Since $V = \begin{bmatrix} W_c & 0_{n_x \times n_w} \\ 0_{n_w \times n_x} & I_{n_w} \end{bmatrix} \in \mathcal{D}_{\text{SDP}}$, $W_c \succ 0$ implies $V \succ 0$. This means that $\mathcal{D}_{\text{SDP}}$ contains a positive definite matrix, which guarantees strict feasibility of optimization in the later chapter. Remarkably, the converse is also true.

Proposition 3.1: *There exists $V \in \mathcal{D}_{SDP}$ such that $V \succ 0$ if and only if (A, B) is controllable.*

PROOF: Suppose (A, B) is controllable, then the controllability Gramian $W_c \succ 0$. Recall that $AW_cA^* - W_c + BB^* = 0$. Therefore $V = \begin{bmatrix} W_c & 0_{n_x \times n_w} \\ 0_{n_w \times n_x} & I_{n_w} \end{bmatrix} \succ 0$ is in $\mathcal{D}_{\text{SDP}}$, because

$$\begin{bmatrix} A & B \end{bmatrix} V \begin{bmatrix} A & B \end{bmatrix}^* - \begin{bmatrix} I_{n_x} & 0_{n_x \times n_w} \end{bmatrix} V \begin{bmatrix} I_{n_x} & 0_{n_x \times n_w} \end{bmatrix}^* = AW_cA^* - W_c + BB^* = 0.$$

Now suppose that there exists a positive definite $V \in \mathcal{D}_{\text{SDP}}$. Let us partition $V = \begin{bmatrix} X & R \\ R^* & W \end{bmatrix}$, where $X \in \mathbb{C}^{n_x \times n_x}, R \in \mathbb{C}^{n_x \times n_w}, W \in \mathbb{C}^{n_w \times n_w}$. Then, since $V \in \mathcal{D}_{\text{SDP}}$,

$$AXA^* - X + BR^*A^* + ARB^* + BWB^* = 0.$$

Let $W = TT^*$, and $\tilde{B} = BT$, $K = T^{-1}R^*X^{-1}$. Then,

$$(A + \tilde{B}K)X(A + \tilde{B}K)^* - X + \tilde{B}\tilde{B}^* = 0.$$

Since $X \succ 0$, $(A + \tilde{B}K, \tilde{B})$ is controllable. This is equivalent to $(A, \tilde{B}) = (A, BT)$ is controllable, and therefore (A, B) is controllable. ∎

3.2 A semidefinite representation of Gramians: Continuous time case

This section presents the continuous time result on Gramians.

3.2.1 Set of Gramians

For a signal $\mathbf{u} \in \mathcal{L}_2$, we define the Gramian $\Lambda : \mathcal{L}_2 \to \mathbb{H}$ such that

$$\Lambda(\mathbf{u}) := \int_0^\infty u(t)u(t)^* dt.$$

The Gramian is well-defined because each entry is finite.

Consider the following stable LTI system, $\mathbf{P}$, which maps an input $\mathbf{w}$ to an output $\mathbf{z}$,

$$\frac{dx(t)}{dt} = Ax(t) + Bw(t) \tag{3.10}$$

$$x(0) = 0 \tag{3.11}$$

$$z(t) = Cx(t) + Dw(t), \tag{3.12}$$

where A is Hurwtiz stable, $x(t) \in \mathbb{C}^{n_x}$, $w(t) \in \mathbb{C}^{n_w}$, $z(t) \in \mathbb{C}^{n_z}$. Since A is Hurwitz stable, we can consider the linear operator $\mathbf{M}_{AB} : \mathcal{L}_2^{n_w} \to \mathcal{L}_2^{n_x}$ such that $\mathbf{x} = \mathbf{M}_{AB}(\mathbf{w})$, where $\mathbf{x}$ is the state of $\mathbf{P}$, *i.e.*,

$$x(t) = \int_0^t e^{A(t-\tau)} Bw(\tau)d\tau. \tag{3.13}$$

Consider the following set of continuous-time Gramians:

$$\mathcal{C}_{\text{Gram}} := \{V \in \mathbb{H} \ : V = \Lambda\left(\mathbf{M}_{AB}(\mathbf{w}), \mathbf{w}\right) \text{ for some } \mathbf{w} \in \mathcal{L}_2^{n_w}\}$$

For the semidefinite representation of continuous-time Gramians, consider the following set:

$$\mathcal{C}_{\text{SDP}} := \{V \in \mathbb{H}^{n_x+n_w} : V \succeq 0, \begin{bmatrix} A & B \end{bmatrix} V \begin{bmatrix} I_{n_x} \\ 0_{n_w \times n_x} \end{bmatrix} + \begin{bmatrix} I_{n_x} & 0_{n_x \times n_w} \end{bmatrix} V \begin{bmatrix} A^* \\ B^* \end{bmatrix} = 0\}$$

Notice that $\mathcal{C}_{\text{SDP}}$ is a closed convex cone that is semidefinite representable.

Now we have the following proposition.

Proposition 3.2: *The set $\mathcal{C}_{Gram} \subset \mathcal{C}_{SDP}$.*

PROOF: For all $V \in \mathcal{C}_{\text{Gram}}$, there exists $\mathbf{w} \in \mathcal{L}_2$ such that $V = \Lambda(\mathbf{x}, \mathbf{w})$ where $\mathbf{x} = \mathbf{M}_{AB}(\mathbf{w})$. Since $\dot{x} = Ax + Bw$, we have

$$\frac{d}{dt}(x(t)x(t)^*) = (Ax + Bw)x^* + x(AX + Bw)^*.$$

By taking an integration from 0 to ∞, we have

$$\int_0^\infty \frac{d}{dt}(x(t)x(t)^*)dt = \int_0^\infty (Ax + Bw)x^* + x(Ax + Bw)^* dt$$
$$= \begin{bmatrix} A & B \end{bmatrix} V \begin{bmatrix} I_{n_x} \\ 0_{n_w \times n_x} \end{bmatrix} + \begin{bmatrix} I_{n_x} & 0_{n_x \times n_w} \end{bmatrix}^* V \begin{bmatrix} A^* \\ B^* \end{bmatrix}.$$

Since $\mathbf{w} \in \mathcal{L}_2$, $\mathbf{x} \in \mathcal{L}_2$. Therefore $x(\infty) = 0$, and

$$\int_0^\infty \frac{d}{dt}(x(t)x(t)^*)dt = x(\infty)x(\infty)^* - x(0)x(0)^* = 0.$$

This shows $\begin{bmatrix} A & B \end{bmatrix} V \begin{bmatrix} I_{n_x} \\ 0_{n_w \times n_x} \end{bmatrix} + \begin{bmatrix} I_{n_x} & 0_{n_x \times n_w} \end{bmatrix}^* V \begin{bmatrix} A^* \\ B^* \end{bmatrix} = 0$, and $V \in \mathcal{C}_{\text{SDP}}$. ∎

Likewise in the discrete time case, we have the converse result.

Lemma 3.3: *For all $V \in \mathcal{C}_{SDP}$, $\varepsilon > 0$, there exists a Gramian $\tilde{V} \in \mathcal{C}_{Gram}$ such that*

$$\|V - \tilde{V}\|_F < \varepsilon \tag{3.14}$$

$$\begin{bmatrix} 0_{n_w \times n_x} & I_{n_w} \end{bmatrix} (V - \tilde{V}) \begin{bmatrix} 0_{n_w \times n_x} & I_{n_w} \end{bmatrix}^* = 0. \tag{3.15}$$

The proof can be found in the Appendix A.

In addition, the following consequence is immediate.

Lemma 3.4: $\mathcal{C}_{SDP} = \mathbf{cl}\left(\mathcal{C}_{Gram}\right)$.

PROOF: Since $\mathcal{C}_{\text{SDP}}$ is closed and $\mathcal{C}_{\text{Gram}} \subset \mathcal{C}_{\text{SDP}}$, $\mathbf{cl}\left(\mathcal{C}_{\text{Gram}}\right) \subset \mathcal{C}_{\text{SDP}}$. In addition, from Lemma 3.3, $\mathcal{C}_{\text{SDP}} \subset \mathbf{cl}\left(\mathcal{C}_{\text{Gram}}\right)$. ∎

3.2.2 Controllability Gramian and relative interior

As in the discrete time case, the relative interior of $\mathcal{C}_{\text{SDP}}$ is not empty if and only if when the pair (A, B) is controllable.

Proposition 3.3: *There exists a positive definite $V \in \mathcal{C}_{SDP}$ if and only if (A, B) is controllable.*

PROOF: Suppose (A, B) is controllable. Since A is Hurwitz stable, the controllability gramian W_c

$$AW_c + W_cA^* + BB^* = 0$$

is positive definite. Let

$$V = \begin{bmatrix} W_c & \frac{1}{2}B \\ \frac{1}{2}B^* & tI \end{bmatrix}.$$

Then $\begin{bmatrix} A & B \end{bmatrix} V \begin{bmatrix} I & 0 \end{bmatrix}^* + \begin{bmatrix} I & 0 \end{bmatrix} V \begin{bmatrix} A & B \end{bmatrix}^* = 0$. Since $W_c \succ 0$, $V \succ 0$ if and only if $tI - \frac{1}{4}B^*W_c^{-1}B \succ 0$. Therefore, by taking sufficiently large t, we can make $V \succ 0$.

Now suppose there exists $V \in \mathcal{C}_{\text{SDP}}$ such that $V \succ 0$. Let us partition $V = \begin{bmatrix} X & R \\ R^* & W \end{bmatrix}$, where $X \in \mathbb{C}^{n_x \times n_x}$, $R \in \mathbb{C}^{n_x \times n_w}$, $W \in \mathbb{C}^{n_w \times n_w}$. Since $V \in \mathcal{C}_{\text{SDP}}$,

$$AX + XA^* + BR^* + RB^* = 0,$$

and $X \succ 0$. Let $K = \left(R - \frac{1}{2}B\right)^* X^{-1}$, then $(A+BK)X + X(A+BK)^* + BB^* = 0$. This shows that the pair $(A+BK, B)$ is controllable, which implies that (A, B) is controllable. ∎

3.3 Input construction algorithm: Discrete time case

This section provides an algorithm for constructing an l_2 signal $\mathbf{w} \in l_2^{n_w}$ that approximates $V \in \mathcal{D}_{\text{SDP}}$. Detailed analysis can be found in the proof of Lemma 3.1 in the appendix. The first step is to decompose V in to rank one matrices $V_i \in \mathcal{D}_{\text{SDP}}$. This in turn relies on the construction of an unitary matrix U which satisfies $\begin{bmatrix} A & B \end{bmatrix} V^{1/2} = \begin{bmatrix} I_{n_x} & 0_{n_x \times n_w} \end{bmatrix} V^{1/2} U$. To this end, we set $F = \begin{bmatrix} A & B \end{bmatrix} V^{1/2}$, and $G = \begin{bmatrix} I_{n_x} & 0_{n_x \times n_w} \end{bmatrix} V^{1/2}$, then find U such that $F = GU$.

Suppose we have two matrices $F, G \in \mathbb{C}^{n \times (n+m)}$ such that $FF^* = GG^*$. Now the goal is to find U such that $F = GU$. From the proof of Lemma A.1, we can conclude that the left singular vectors F, G are the same, and so are singular values. Let r be the rank of G. Since the number of non-zero singular values of F are same as G, the rank of F is also r. Then, from singular value decomposition,

$$\begin{aligned} G &= \sum_{i=1}^{r} \sigma_i u_i v_i^* \\ F &= \sum_{i=1}^{r} \sigma_i u_i t_i^*, \end{aligned}$$

where $\sigma_i > 0$, and $\{u_1, \cdots, u_r\}, \{v_1, \cdots, v_r\}$, and $\{t_1, \cdots, t_r\}$ are orthonormal vectors. Since the psuedo-inverse of G, $G^\dagger = \sum_{i=1}^{r} \frac{1}{\sigma_i} v_i u_i^*$, by multiplying it to F, we can obtain

$$G^\dagger F = \sum_{i=1}^{r} v_i t_i^*.$$

Let $\{v_{r+1}, \cdots, v_{n+m}\}, \{t_{r+1}, \cdots, t_{n+m}\}$ be orthonormal bases for the null space of G and F, respectively. Since their ranks are the same, so are the dimensions of their null spaces.

Then a matrix $U = \sum_{i=1}^{n+m} v_i t_i^*$ is unitary, since $UU^* = \sum_{i,j} v_i t_i^* t_j v_j^* = \sum_i v_i v_i^* = I_{n+m}$. In addition, $GU = (\sum_{i=1}^{r} \sigma_i u_i v_i^*)(\sum_{j=1}^{n+m} v_j t_j^*) = \sum_{i=1}^{r} \sigma_i u_i t_i^* = F$. Therefore, once we prepare $G^\dagger F$ and orthonormal bases for null spaces, we can sum them up to obtain a unitary U such that $F = GU$. We summarize this observation in the following algorithm.

Algorithm 1: Unitary matrix construction

Input: Complex matrices, F, G such that $FF^* = GG^*$

Output: A unitary matrix U such that $F = GU$

---

1. Obtain an orthonormal basis, $\{v_i\}$, for the null space of G
2. Obtain an orthonormal basis, $\{t_i\}$, for the null space of F
3. $U = G^\dagger F + \sum_i v_i t_i^*$

Notice that $GU = G(G^\dagger F + \sum_i v_i t_i^*) = GG^\dagger F = F$, because the left singular vectors of F and G are the same.

Now the next step is to find a rank one decomposition of $V \in \mathcal{D}_{\text{SDP}}$. Since $\begin{bmatrix} A & B \end{bmatrix} V \begin{bmatrix} A & B \end{bmatrix}^* = \begin{bmatrix} I_{n_x} & 0_{n_x \times n_w} \end{bmatrix} V \begin{bmatrix} I_{n_x} & 0_{n_x \times n_w} \end{bmatrix}^*$, we set $F = \begin{bmatrix} A & B \end{bmatrix} V^{1/2}$, $G = \begin{bmatrix} I_{n_x} & 0_{n_x \times n_w} \end{bmatrix} V^{1/2}$, then find an unitary matrix U using the Algorithm 1. Using the eigen-decomposition of $U = \sum_i e^{\mathbf{i}\theta_i} u_i u_i^*$, we can obtain $V_i = V^{1/2} u_i u_i^* V^{1/2}$. Then we use eigen-decomposition of $V_i = \begin{bmatrix} x_i \\ w_i \end{bmatrix} \begin{bmatrix} x_i \\ w_i \end{bmatrix}^*$ and use w_i to construct a sinusoid. These are the constructions in the proof of Proposition A.2. In fact, we can merge some of these steps to obtain a simpler algorithm.

Recall that the ultimate goal of the above procedure is to find a pair $(x, w) \in \mathbb{C}^{n_x} \times \mathbb{C}^{n_w}$ and θ such that $e^{\mathbf{i}\theta} x = Ax + Bw$. It turns out this can be done by partitioning $V^{1/2} u_i = \begin{bmatrix} x_i \\ w_i \end{bmatrix}$ and setting $\theta = \theta_i$. Since $V_k = V^{1/2} u_i u_i^* V^{1/2}$, the above (x_i, w_i) results in $V_k = \begin{bmatrix} x_i \\ w_i \end{bmatrix} \begin{bmatrix} x_i \\ w_i \end{bmatrix}^*$. In addition, from $\begin{bmatrix} A & B \end{bmatrix} V^{1/2} = \begin{bmatrix} I_{n_x} & 0_{n_x \times n_w} \end{bmatrix} V^{1/2} U$, we have $\begin{bmatrix} A & B \end{bmatrix} V^{1/2} u_i = \begin{bmatrix} I_{n_x} & 0_{n_x \times n_w} \end{bmatrix} V^{1/2} U u_i = e^{\mathbf{i}\theta_i} \begin{bmatrix} I_{n_x} & 0_{n_x \times n_w} \end{bmatrix} V^{1/2} u_i$, which shows

$$Ax_i + Bw_i = e^{\mathbf{i}\theta_i} x_i.$$

Therefore the following algorithm constructs the frequency of the sinusoid, θ, and the corresponding steady state vector x, and the input vector w.

Algorithm 2: Sinusoid construction

Input: $A \in \mathbb{C}^{n_x \times n_x}, B \in \mathbb{C}^{n_x \times n_w}$, and $V \in \mathcal{D}_{\text{SDP}}$

Output: $n_x + n_w$ number of triplets (θ_i, x_i, w_i) such that $e^{\mathbf{i}\theta_i} x_i = Ax_i + Bw_i$

---

1. Computing $V^{1/2}$ such that $V = V^{1/2} V^{1/2}$.
2. Using Algorithm 1, obtaining U with $F = \begin{bmatrix} A & B \end{bmatrix} V^{1/2}, G = \begin{bmatrix} I_{n_x} & 0_{n_x \times n_w} \end{bmatrix} V^{1/2}$.
3. Performing eigenvalue decomposition $U = \sum_{i=1}^{n_x + n_w} e^{\mathbf{i}\theta_i} u_i u_i^*$.
4. Partitioning $V^{1/2} u_i = \begin{bmatrix} x_i \\ w_i \end{bmatrix}$, where $x_i \in \mathbb{C}^{n_x}$ and $w_i \in \mathbb{C}^{n_w}$.

Now the final algorithm constructs an input $\mathbf{w}$ whose Gramian $\Lambda(\mathbf{M}_{AB}(\mathbf{w}), \mathbf{w})$ is close enough to V with the desired accuracy ε.

Algorithm 3: Input construction

Input: $A \in \mathbb{C}^{n_x \times n_x}, B \in \mathbb{C}^{n_x \times n_w}$, $V \in \mathcal{D}_{\text{SDP}}$, and $\varepsilon > 0$

Output: A signal $\mathbf{w}$ such that $\|\Lambda(\mathbf{M}_{AB}(\mathbf{w}), \mathbf{w}) - V\|_F < \varepsilon$

---

1. Obtaining (θ_i, x_i, w_i) using the Algorithm 2.
2. Set $N = 10, T = 10$.

3. Let $\hat{w}_i[k] = \begin{cases} \frac{1}{\sqrt{N}} e^{\mathbf{i}\theta_i k} w_i & \text{if } 0 \le k < N \\ 0 & \text{if } N \le k \end{cases}.$

4. Let $w[k] = \begin{cases} \hat{w}_1[k] & \text{if } 0 \le k < N+T \\ \hat{w}_2[k-N-T] & \text{if } N+T \le k < 2(N+T) \\ \vdots & \\ \hat{w}_{n+m}[k-(n+m-1)(N+T)] & \text{if } (n+m-1)(N+T) \\ & \quad \le k < (n+m)(N+T) \\ 0 & \text{if } (n+m)(N+T) \le k \end{cases}.$

5. If $\|\Lambda(\mathbf{M}_{AB}\mathbf{w}, \mathbf{w}) - V\|_F < \varepsilon$ then stop. Otherwise $N \leftarrow 10N$, $T \leftarrow 10T$, and go to the step 3.

3.3.1 Numerical example

To verify the effectiveness of our algorithm, we present a numerical example here. Consider

$$\begin{aligned} A &= \begin{bmatrix} 0.8 & 0.1 & 0 \\ 0.1 & 0.8 & 0.1 \\ 0 & 0.1 & 0.8 \end{bmatrix} \\ B &= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\top \\ V &= \begin{bmatrix} 7.1020 & 9.0062 & 7.1020 & 0 \\ 9.0062 & 11.5724 & 9.0062 & 0 \\ 7.1020 & 9.0062 & 7.1020 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Since $\rho(A) = 0.9414 < 1$, A is Schur stable. In addition, we can check that $\begin{bmatrix} A & B \end{bmatrix} V \begin{bmatrix} A & B \end{bmatrix}^* = \begin{bmatrix} I_{n_x} & 0_{n_x \times n_w} \end{bmatrix} V \begin{bmatrix} I_{n_x} & 0_{n_x \times n_w} \end{bmatrix}^*$, and $\mathbf{rank}\,(V) = 3$.

By using algorithm 2, we obtain

$$\theta_1 = 0, \qquad x_1 = \begin{bmatrix} 2.4019 \\ 3.2026 \\ 2.4019 \end{bmatrix}, \qquad w_1 = 0.1601$$
$$\theta_2 = 0.9147, \qquad x_2 = \begin{bmatrix} -0.2755 - \mathbf{i}0.7684 \\ -0.3674 - \mathbf{i}0.7232 \\ -0.2755 - \mathbf{i}0.7684 \end{bmatrix}, \quad w_2 = 0.6980$$
$$\theta_3 = -0.9147, \quad x_3 = \begin{bmatrix} -0.2755 + \mathbf{i}0.7684 \\ -0.3674 + \mathbf{i}0.7232 \\ -0.2755 + \mathbf{i}0.7684 \end{bmatrix}, \quad w_3 = 0.6980$$
$$\theta_4 = 0.8, \qquad x_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad w_4 = 0.$$

Notice that there are only 3 non-trivial (x, w) because the rank of V is 3. Since $w_4 = 0$, w_r is not needed to approximate V. So from now on, we only consider w_1, w_2, and w_3.

Let $V_i = \begin{bmatrix} x_i \\ w_i \end{bmatrix} \begin{bmatrix} x_i \\ w_i \end{bmatrix}^*$. Then $V_i \in \mathcal{D}_{\mathrm{SDP}}$, and $\mathbf{rank}\,(V_i) = 1$. Therefore by using the algorithm 3, we should be able to approximate each V_i arbitrarily close. In the proof of Proposition A.9 and in the algorithm 3, we use the construction $\hat{w}[k] = \begin{cases} \frac{1}{\sqrt{N}} e^{\mathbf{i}\theta_i k} w_i & \text{if } 0 \le k < N \\ 0 & \text{if } N \le k \end{cases}$ when $\mathbf{rank}\,(V) = 1$.

Then we claim that $\lim_{N\to\infty} \|\Lambda(\mathbf{M}_{AB}(\hat{\mathbf{w}}), \hat{\mathbf{w}}) - V\|_F = 0$. Therefore we should be able to verify this fact in the current example. Since $\Lambda(\mathbf{M}_{AB}(\hat{\mathbf{w}}), \hat{\mathbf{w}})$ requires the infinite summation, which is hard to obtain numerically, we use sufficiently large K to approximate $\Lambda(\mathbf{M}_{AB}(\hat{\mathbf{w}}), \hat{\mathbf{w}})$:

$$\Lambda(\mathbf{M}_{AB}(\hat{\mathbf{w}}), \hat{\mathbf{w}}) \approx \sum_{k=0}^{K} \begin{bmatrix} \hat{x}[k] \\ \hat{w}[k] \end{bmatrix} \begin{bmatrix} \hat{x}[k] \\ \hat{w}[k] \end{bmatrix}^*,$$

where $\hat{\mathbf{x}} = \mathbf{M}_{AB}(\hat{\mathbf{w}})$. Since A^k converges to zero exponentially fast, this numerical approximation is fine.

Figure 3.1 - 3.3 shows the log-log plots of the approximation error between V_i and $\Lambda(\mathbf{M}_{AB}(\hat{\mathbf{w}}_{\mathbf{i}}), \hat{\mathbf{w}}_{\mathbf{i}})$. As we can see, the error decreases as N increases. In fact, the figures suggest that $\|\Lambda(\mathbf{M}_{AB}(\hat{\mathbf{w}}_{\mathbf{i}}), \hat{\mathbf{w}}_{\mathbf{i}}) - V_i\|_F \approx \mathcal{O}(\frac{1}{N})$, which is consistent with our analysis in the proof of Proposition A.9.

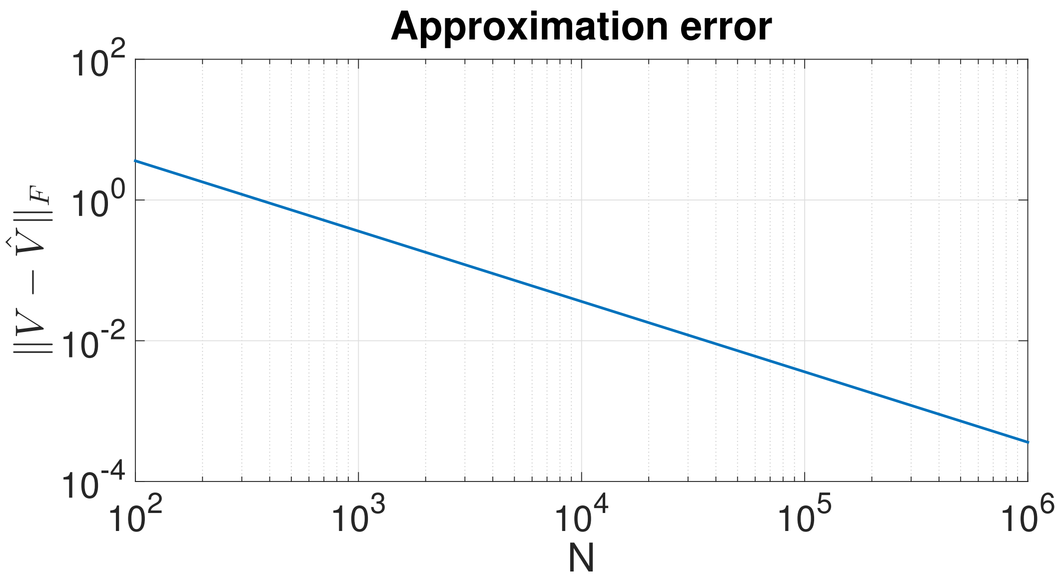

Figure 3.1: Approximation of V_1 using $\hat{w}_1$.

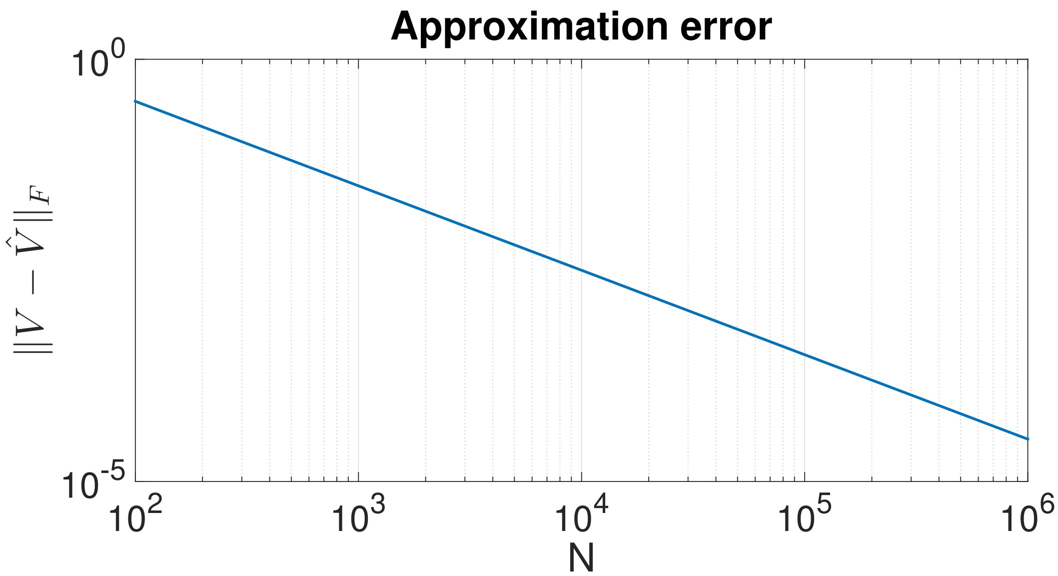

Figure 3.2: Approximation of V_2 using $\hat{w}_2$.

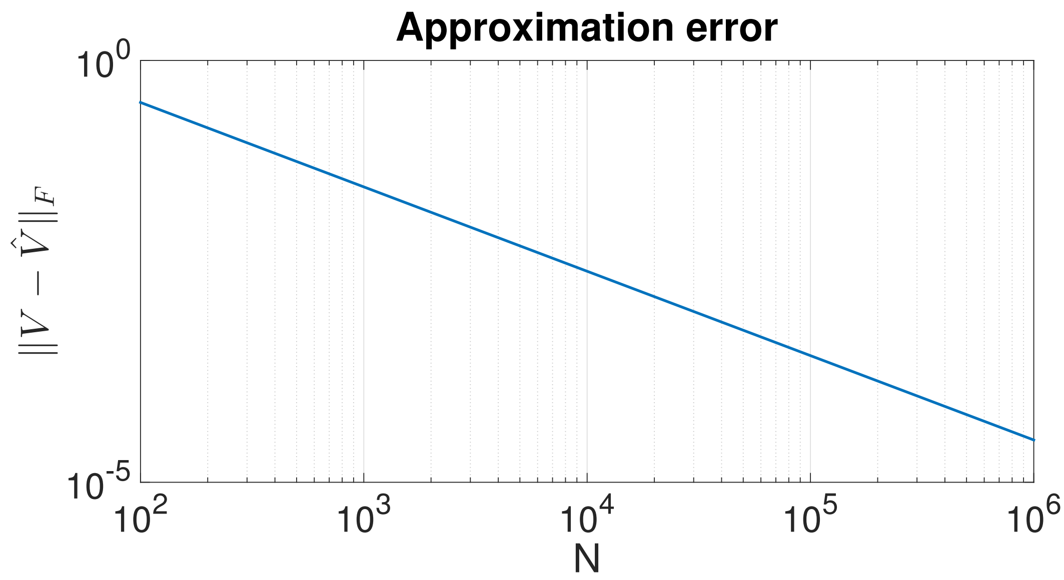

Figure 3.3: Approximation of V_3 using $\hat{w}_3$.

The above construction only concerns rank one part of V. The signal for $V = V_1+V_2+V_3$ is obtained by padding $\hat{w}_1, \hat{w}_2, \hat{w}_3$. In the algorithm 3, the spacing between each signal is set to be T, and in the simulation we choose $T = 1000$. Figure 3.4 shows the error $\|\Lambda(\mathbf{M}_{AB}(\mathbf{w}), \mathbf{w}) - V_i\|_F \approx \mathcal{O}(\frac{1}{N})$ which is consistent with our analysis in the proof of Lemma 3.1.

3.4 Input construction algorithm: Continuous time case

In the discrete time case, the goal is to find $(x, w) \in \mathbb{C}^{n_x} \times \mathbb{C}^{n_w}$ and θ such that $e^{\mathbf{i}\theta}x = Ax + Bw$. On the other hand, in the continuous time case, the goal is to find (x, w) and ω such that $\mathbf{i}\omega x = Ax + Bw$. To this end, we modify the above algorithms to accommodate the continuous time setting.

Recall that $V \in \mathcal{C}_{\text{SDP}}$ satisfies $\begin{bmatrix} A & B \end{bmatrix} V \begin{bmatrix} I_{n_x} & 0_{n_x \times n_w} \end{bmatrix}^* + \begin{bmatrix} I_{n_x} & 0_{n_x \times n_w} \end{bmatrix} V \begin{bmatrix} A & B \end{bmatrix}^* = 0$, which is equivalent to

$$\begin{bmatrix} A + I_{n_x} & B \end{bmatrix} V \begin{bmatrix} A + I_{n_x} & B \end{bmatrix}^* = \begin{bmatrix} A - I_{n_x} & B \end{bmatrix} V \begin{bmatrix} A - I_{n_x} & B \end{bmatrix}^*.$$

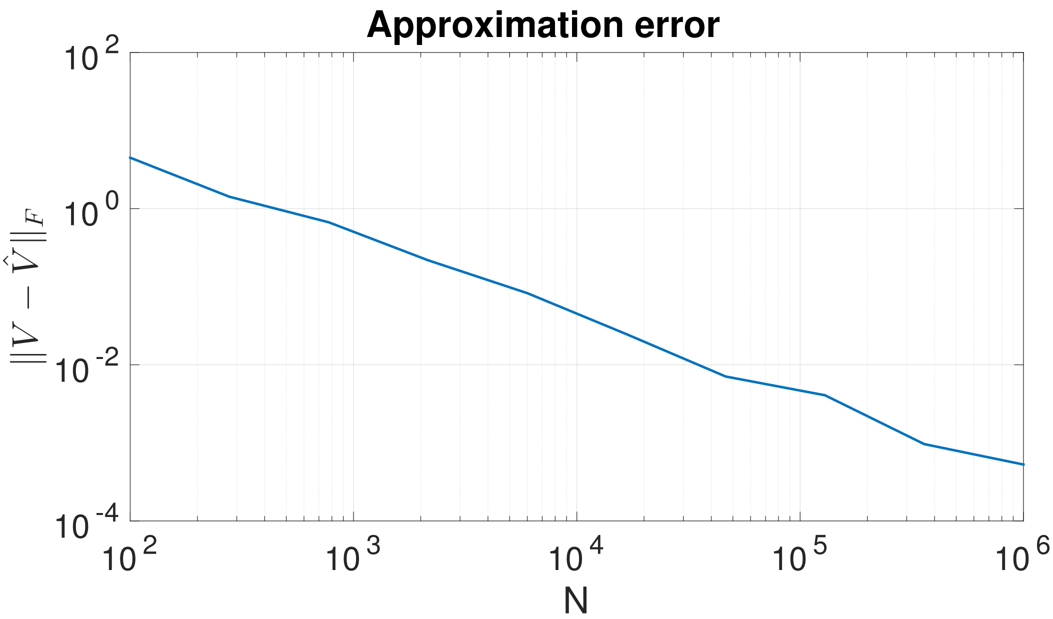

Figure 3.4: Approximation of V using w

Therefore, by using the algorithm 1, we can find an unitary matrix U such that

$$\begin{bmatrix} A+I_{n_x} & B \end{bmatrix} V^{1/2} = \begin{bmatrix} A-I_{n_x} & B \end{bmatrix} V^{1/2}U.$$

Let $e^{\mathbf{i}\theta_i}$ be the ith eigenvalue of U and u_i be the corresponding eigenvector. Then by partitioning $V^{1/2}u_i = \begin{bmatrix} x_i \\ w_i \end{bmatrix}$, we have

$$\begin{aligned} \begin{bmatrix} A+I_{n_x} & B \end{bmatrix} V^{1/2}u_i &= \begin{bmatrix} A-I_{n_x} & B \end{bmatrix} V^{1/2}Uu_i \\ \begin{bmatrix} A+I_{n_x} & B \end{bmatrix} \begin{bmatrix} x_i \\ w_i \end{bmatrix} &= e^{\mathbf{i}\theta_i} \begin{bmatrix} A-I_{n_x} & B \end{bmatrix} \begin{bmatrix} x_i \\ w_i \end{bmatrix}. \end{aligned}$$

This shows that

$$(e^{\mathbf{i}\theta_i}+1)x_i = (e^{\mathbf{i}\theta_i}-1)(Ax_i + Bw_i).$$

Therefore, if $\theta_i \neq 0$, then we can set $\omega_i = \frac{1}{\mathbf{i}}\frac{e^{\mathbf{i}\theta_i}+1}{e^{\mathbf{i}\theta_i}-1}$. Since $\frac{e^{\mathbf{i}\theta_i}+1}{e^{\mathbf{i}\theta_i}-1}$ is purely imaginary, $\omega_i \in \mathbb{R}$, and we have $\mathbf{i}\omega_i x_i = Ax_i + Bw_i$ which is desired.

If $\theta_i = 0$, then this implies $x_i = 0$. In terms of $\mathbf{i}\omega x = Ax + Bw$, this corresponds to $\omega = +\infty$. Therefore we will use a large enough ω to generate this type of w, but with some abuse of notation, let us use $+\infty$ if $\theta_i = 0$.

Algorithm 4: Sinusoid construction (continuous time case)

Input: $A \in \mathbb{C}^{n_x \times n_x}, B \in \mathbb{C}^{n_x \times n_w}$, and $V \in \mathcal{D}_{\text{SDP}}$

Output: $n_x + n_w$ number of triplets (ω_i, x_i, w_i) such that $\mathbf{i}\omega_i x_i = Ax_i + Bw_i$

---

1. Computing $V^{1/2}$ such that $V = V^{1/2}V^{1/2}$.
2. Using Algorithm 1, obtaining U with $F = \begin{bmatrix} A + I_{n_x} & B \end{bmatrix} V^{1/2}$, $G = \begin{bmatrix} A - I_{n_x} & B \end{bmatrix} V^{1/2}$.
3. Performing eigenvalue decomposition $U = \sum_{i=1}^{n_x+n_w} e^{\mathbf{i}\theta_i} u_i u_i^*$.
4. Let $\mathbf{i}\omega_i = \frac{e^{\mathbf{i}\theta_i}+1}{e^{\mathbf{i}\theta_i}-1}$.
5. Partitioning $V^{1/2}u_i = \begin{bmatrix} x_i \\ w_i \end{bmatrix}$, where $x_i \in \mathbb{C}^{n_x}$ and $w_i \in \mathbb{C}^{n_w}$.

Then we use these (ω_i, x_i, w_i) to construct the signal $\mathbf{w}$ such that $\Lambda(\mathbf{M}_{AB}(\mathbf{w}), \mathbf{w})$ is close enough to V.

Algorithm 5: Input construction (continuous time case)

Input: $A \in \mathbb{C}^{n_x \times n_x}, B \in \mathbb{C}^{n_x \times n_w}$, $V \in \mathcal{C}_{\text{SDP}}$, and $\varepsilon > 0$

Output: A signal $\mathbf{w}$ such that $\|\Lambda(\mathbf{M}_{AB}(\mathbf{w}), \mathbf{w}) - V\|_F < \varepsilon$

---

1. Obtaining (ω_i, x_i, w_i) using the Algorithm 4.
2. Set $N = 10, T = 10$, and $\omega_c = +10^6$.
3. If $\omega_i \neq \infty$, then $\hat{w}_i(t) = \begin{cases} \frac{1}{\sqrt{N}} e^{\mathbf{i}\omega_i t} w_i & \text{if } 0 \leq t < N \\ 0 & \text{if } N \leq t \end{cases}$.

If $\omega_i = \infty$, then $\hat{w}_i(t) = \begin{cases} \frac{1}{\sqrt{N}} e^{\mathbf{i}\omega_c t} w_i & \text{if } 0 \leq t < N \\ 0 & \text{if } N \leq t \end{cases}$.

4. Let $w(t) = \begin{cases} \hat{w}_1(t) & \text{if } 0 \leq t < N + T \\ \hat{w}_2(t - N - T) & \text{if } N + T \leq t < 2(N + T) \\ \vdots & \\ \hat{w}_{n+m}(t - (n+m-1)(N+T)) & \text{if } (n+m-1)(N+T) \\ & \quad \leq t < (n+m)(N+T) \\ 0 & \text{if } (n+m)(N+T) \leq t \end{cases}$.

5. If $\|\Lambda(\mathbf{M}_{AB}\mathbf{w}, \mathbf{w}) - V\|_F < \varepsilon$ then stop. Otherwise $N \leftarrow 10N$, $T \leftarrow 10T$, $\omega_c \leftarrow 10\omega_c$, and go to the step 3.

Chapter 4

Extended $\mathcal{H}_\infty$ analysis

4.1 $\mathcal{H}_\infty$ analysis

4.1.1 $\mathcal{H}_\infty$ analysis

In $\mathcal{H}_\infty$ analysis, we would like to find the worst-case disturbance that maximizes the output norm. Specifically consider the stable LTI system $\mathbf{M}$:

$$\begin{aligned} x^+ &= Ax + Bw \\ z &= Cx + Dw, \end{aligned}$$

where x^+ is $x[k+1]$ for discrete time, $\dot{x}$ for the continuous time with initial condition being zero, $x \in \mathbb{C}^{n_x}$, $w \in \mathbb{C}^{n_w}$, $z \in \mathbb{C}^{n_z}$. Then the $\mathcal{H}_\infty$ norm of $\mathbf{M}$ is given by

$$\|\mathbf{M}\|_\infty := \sup_{\mathbf{w}} \{\|\mathbf{M}\mathbf{w}\|_2 : \|\mathbf{w}\|_2 = 1\},$$

which is an $l_2 - l_2$ ($\mathcal{L}_2 - \mathcal{L}_2$) induced norm. Using Gramian $V = \Lambda(\mathbf{x}, \mathbf{w})$, we have

$$\begin{aligned} f_0(V) &:= \mathbf{Tr}\left(\begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D \end{bmatrix} V \right) = \|\mathbf{M}\mathbf{w}\|_2^2 \\ g_1(V) &:= \mathbf{Tr}\left(\begin{bmatrix} 0_{n_x \times n_x} & 0_{n_x \times n_w} \\ 0_{n_w \times n_x} & I_{n_w} \end{bmatrix} V \right) = \|\mathbf{w}\|_2^2. \end{aligned}$$

Therefore, the square of the $\mathcal{H}_\infty$ norm can be obtained:

$$\begin{aligned} \text{Discrete time:} \quad & \|\mathbf{M}\|_\infty^2 &= \sup\{f_0(V) : V \in \mathcal{D}_{\text{Gram}},\ g_1(V) = 1\} \\ \text{Continuous time:} \quad & \|\mathbf{M}\|_\infty^2 &= \sup\{f_0(V) : V \in \mathcal{C}_{\text{Gram}},\ g_1(V) = 1\}. \end{aligned}$$

As we pointed out, obtaining the optimal solution of the above optimization is not trivial because $\mathcal{D}_{\text{Gram}}$ and $\mathcal{C}_{\text{Gram}}$ are involved with the infinite dimensional space. The celebrated Kalman–Yakubovich–Popov (KYP) lemma shows that computing the $\mathcal{H}_\infty$ norm can be done via SDP. In this section, we leverage the results from Chatper 3, Lemma 3.1 and 3.3, to show how to use SDP to compute the $\mathcal{H}_\infty$ norm without relying on the KYP lemma.

Indeed, from Lemma 3.1 and 3.3, we have the following result.

Proposition 4.1: *Let* $\mathcal{F} = \{V \in \mathbb{H} : g_1(V) = 1\}$. *Then* $\mathbf{cl}\,(\mathcal{D}_{Gram} \cap \mathcal{F}) = \mathcal{D}_{SDP} \cap \mathcal{F}$, *and* $\mathbf{cl}\,(\mathcal{C}_{Gram} \cap \mathcal{F}) = \mathcal{C}_{SDP} \cap \mathcal{F}$.

PROOF: Since $\mathcal{F}$ is closed, and $\mathbf{cl}\,(\mathcal{D}_{\text{Gram}}) = \mathcal{D}_{\text{SDP}}$, $\mathbf{cl}\,(\mathcal{D}_{\text{Gram}} \cap \mathcal{F}) \subset \mathcal{D}_{\text{SDP}} \cap \mathcal{F}$. Consider $V \in \mathcal{D}_{\text{SDP}} \cap \mathcal{F}$. From Lemma 3.1, for any $\varepsilon > 0$, there exists $\tilde{V} \in \mathcal{D}_{\text{Gram}}$ such that $\|V - \tilde{V}\|_F < \varepsilon$, and $\begin{bmatrix} 0_{n_x \times n_x} & 0_{n_x \times n_w} \\ 0_{n_w \times n_x} & I_{n_w} \end{bmatrix} (V - \tilde{V}) = 0$, which shows $g_1(V) = g_1(\tilde{V}) = 1$. This shows $\tilde{V} \in \mathcal{D}_{\text{Gram}} \cap \mathcal{F}$. For $\mathcal{C}_{\text{Gram}}$, we can apply similar arguments, and this concludes the proof. ∎

The above proposition has a great implication on $\mathcal{H}_\infty$ analysis. Since the objective $f_0(V)$ is a continuous function, we can replace the feasible set by its closure, which leads us to following optimizations.

$$\textbf{(Discrete-time)} \quad \begin{aligned} \underset{V \succeq 0}{\text{maximize}} \quad & \mathbf{Tr}\left(\begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D \end{bmatrix} V \right) \\ \text{subject to} \quad & \begin{bmatrix} A & B \end{bmatrix} V \begin{bmatrix} A & B \end{bmatrix}^* = \begin{bmatrix} I_{n_x} & 0_{n_x \times n_w} \end{bmatrix} V \begin{bmatrix} I_{n_x} & 0_{n_x \times n_w} \end{bmatrix}^* \\ & \mathbf{Tr}\left(\begin{bmatrix} 0_{n_x \times n_x} & 0_{n_x \times n_w} \\ 0_{n_w \times n_x} & I_{n_w} \end{bmatrix} V \right) = 1. \end{aligned} \tag{4.1}$$

$$\begin{array}{lll} & \underset{V\succeq 0}{\text{maximize}} & \mathbf{Tr}\left(\begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D \end{bmatrix} V\right) \\ \textbf{(Continuous-time)} & \text{subject to} & \begin{bmatrix} A & B \end{bmatrix} V \begin{bmatrix} I_{n_x} & 0_{n_x\times n_w} \end{bmatrix}^* + \begin{bmatrix} I_{n_x} & 0_{n_x\times n_w} \end{bmatrix} V \begin{bmatrix} A & B \end{bmatrix}^* = 0 \\ & & \mathbf{Tr}\left(\begin{bmatrix} 0_{n_x\times n_x} & 0_{n_x\times n_w} \\ 0_{n_w\times n_x} & I_{n_w} \end{bmatrix} V\right) = 1 \end{array} \tag{4.2}$$

Clearly, the above optimizations are SDPs, and therefore we can solve them very efficiently. In addition, once we obtain the optimal solution, Algorithm 3 and 5 in Chapter 3 can be used to construct an optimal input $\mathbf{w}$, which achieves the optimal value asymptotically. This is very different with the KYP lemma based approach, where an optimal input is hard to find.

Since we have an SDP, it is natural to derive the SDP dual of the problem. Before deriving the dual, let us firstly unify the discrete-time $\mathcal{H}_\infty$ analysis (4.1), and the continuous-time $\mathcal{H}_\infty$ analysis (4.2) using the following observation:

$$\begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x\times n_w} \end{bmatrix} V \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x\times n_w} \end{bmatrix}^* = \begin{bmatrix} \begin{bmatrix} A & B \end{bmatrix} V \begin{bmatrix} A & B \end{bmatrix}^* & \begin{bmatrix} A & B \end{bmatrix} V \begin{bmatrix} I_{n_x} & 0_{n_x\times n_w} \end{bmatrix}^* \\ \begin{bmatrix} I_{n_x} & 0_{n_x\times n_w} \end{bmatrix} V \begin{bmatrix} A & B \end{bmatrix}^* & \begin{bmatrix} I_{n_x} & 0_{n_x\times n_w} \end{bmatrix} V \begin{bmatrix} I_{n_x} & 0_{n_x\times n_w} \end{bmatrix}^* \end{bmatrix}.$$

Therefore, the linear equality constraint in (4.1) can be written as

$$\Pi_d\left(\begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x\times n_w} \end{bmatrix} V \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x\times n_w} \end{bmatrix}^*\right) = 0,$$

where $\Pi_d\left(\begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix}\right) = X - Z,\ X \in \mathbb{C}^{n_x\times n_x}, Y \in \mathbb{C}^{n_x\times n_x}, Z \in \mathbb{C}^{n_x\times n_x}$. In the continuous time case,

$$\Pi_c\left(\begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x\times n_w} \end{bmatrix} V \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x\times n_w} \end{bmatrix}^*\right) = 0,$$

where $\Pi_c\left(\begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix}\right) = Y+Y^*$. This shows that by defining appropriate linear map $\Pi : \mathbb{H}^{2n_x} \to \mathbb{H}^{n_x}$ of $\begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_w} \end{bmatrix} V \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_w} \end{bmatrix}^*$ we can unify the optimizations (4.1) and (4.2) in the following form.

$$\begin{aligned} \underset{V \succeq 0}{\text{maximize}} \quad & \mathbf{Tr}\left(\begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D \end{bmatrix} V\right) \\ \text{subject to} \quad & \Pi\left(\begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_w} \end{bmatrix} V \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_w} \end{bmatrix}^*\right) = 0 \\ & \mathbf{Tr}\left(\begin{bmatrix} 0_{n_x \times n_x} & 0_{n_x \times n_w} \\ 0_{n_w \times n_x} & I_{n_w} \end{bmatrix} V\right) = 1 \end{aligned} \tag{4.3}$$

Here we choose $\Pi = \Pi_d$ for the discrete time case, and $\Pi = \Pi_c$ for the continuous time case. Since Π is a linear operator, the optimization (4.3) is still an SDP.

To derive the SDP dual of (4.3), we need the adjoint of Π, Π^*. For the discrete-time case,

$$\mathbf{Tr}\left(P\Pi_d\left(\begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix}\right)\right) = \mathbf{Tr}(P(X-Z)) = \mathbf{Tr}\left(\begin{bmatrix} P & 0_{n_x \times n_x} \\ 0_{n_x \times n_x} & -P \end{bmatrix} \begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix}\right),$$

which shows the adjoint $\Pi_d^*(P) = \begin{bmatrix} P & 0_{n_x \times n_x} \\ 0_{n_x \times n_x} & -P \end{bmatrix}$. We can further simplify the notation using the Kronecker Product. Notice that $\begin{bmatrix} P & 0_{n_x \times n_x} \\ 0_{n_x \times n_x} & -P \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes P$, and therefore $\Pi_d^*(P) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes P$.

For the continuous-time case,

$$\mathbf{Tr}\left(P\Pi_c\left(\begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix}\right)\right) = \mathbf{Tr}(P(Y+Y^*)) = \mathbf{Tr}\left(\begin{bmatrix} 0_{n_x \times n_x} & P \\ P & 0_{n_x \times n_x} \end{bmatrix} \begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix}\right),$$

	$\Pi\left(\begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix}\right)$	Ψ
Discrete-time	$X - Z$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Continuous-time	$Y + Y^*$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Table 4.1: The linear operator Π and its adjoint $\Pi^*(P) = \Psi \otimes P$ for $\mathcal{H}_\infty$ analysis.

which shows $\Pi_c^*(P) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes P$.

Therefore the adjoint operator is given by $\Pi^*(P) = \Psi \otimes P$ with the appropriate choice of Ψ. We summarize this result in Table 4.1.

Now from the following Lagrangian,

$$\begin{aligned} \mathcal{L}(V, P, t) &= \mathbf{Tr}\left(\begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D \end{bmatrix} V\right) + \mathbf{Tr}\left(P\Pi\left(\begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_w} \end{bmatrix} V \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_w} \end{bmatrix}^*\right)\right) \\ &\quad + t\left(1 - \mathbf{Tr}\left(\begin{bmatrix} 0_{n_x \times n_x} & 0_{n_x \times n_w} \\ 0_{n_w \times n_x} & I_{n_w} \end{bmatrix} V\right)\right) \\ &= t + \mathbf{Tr}\left(\left(\begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_w} \end{bmatrix}^* (\Psi \otimes P) \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_w} \end{bmatrix} + \begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D - tI_{n_w} \end{bmatrix}\right) V\right), \end{aligned}$$

we can conclude that the SDP dual of (4.3) is given by

$$\begin{array}{ll} \underset{P=P^*, t}{\text{minimize}} & t \\ \text{subject to} & \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_w} \end{bmatrix}^* (\Psi \otimes P) \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_w} \end{bmatrix} + \begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D - tI_{n_w} \end{bmatrix} \preceq 0. \end{array} \tag{4.4}$$

For the discrete time case, the above optimization becomes

$$\begin{array}{ll} \underset{P=P^*, t}{\text{minimize}} & t \\ \text{subject to} & \begin{bmatrix} A^*PA - P & A^*PB \\ B^*PA & -tI_{n_w} \end{bmatrix} + \begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D \end{bmatrix} \preceq 0, \end{array} \tag{4.5}$$

which is the same optimization derived from the KYP lemma. This shows that the KYP lemma is not the only way to obtain the SDP representation of $\mathcal{H}_\infty$ analysis, and it is also a dual of our well-defined primal optimization (4.1).

Moreover, we can show that the dual program, (4.4), is strictly feasible, and strong duality holds.

Proposition 4.2: (4.4) *is strictly feasible.*

PROOF: The upper left block of LMI constraint in (4.4) is given by $A^*PA - P + C^*C \preceq 0$ for the discrete time case, $A^*P + PA + C^*C \preceq 0$ for the continuous time case. For the discrete time case, since A is Schur stable, $P = \tau \sum_{k=0}^{\infty} (A^*)^k A^k \succeq 0$ satisfies $A^*PA - P = -\tau I_{n_x}$. For the continuous time case, $P = \tau \int_0^\infty e^{A^*t} e^{At} dt \succeq 0$ satisfies $A^*P + PA = -\tau I_{n_x}$. Therefore, by taking sufficiently large τ and t, we can make the LMI strictly feasible. ∎

As a result, we have the following corollary from the Conic duality theorem [4].

Corollary 4.1: *The duality gap between* (4.3) *and* (4.4) *is zero, and the primal problem* (4.3) *is solvable.*

However, this does not guarantee the solvability of the dual program (4.4). Consider the following example.

Example 4.1: Let $A = \frac{1}{2}, B = 0, C = 1, D = 1$. Then the optimal solution of (4.1) is given by $V^\star = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, and the corresponding optimal value is $+1$. The optimal value of the dual (4.5) is also $+1$, by taking $\lambda^\star = 1$, and $P^\star \to \infty$. Clearly, the dual optimum is not attained.

The pair (A, B) in the above example is not controllable, and this phenomena is closely related to the controllability assumption in the KYP lemma. In order to ensure the existence of a multiplier P (a dual optimal solution), we need the controllability assumption.

Proposition 4.3: *The primal program* (4.3) *is strictly feasible if and only if* (A, B) *is controllable.*

PROOF: From Proposition 3.1, there exists $V \in \mathcal{D}_{\text{SDP}}$ such that $V \succ 0$ if and only if (A, B) is controllable. Also, from Proposition 3.3, there exists $V \in \mathcal{C}_{\text{SDP}}$ such that $V \succ 0$ if and only if (A, B) is controllable. ■

As a corollary, we have the following result on strong duality.

Corollary 4.2: *Suppose* (A, B) *is controllable. Then both the primal* (4.3) *and dual program* (4.4) *are solvable, and strong duality holds.*

4.1.2 A proof of bounded real lemma

Bounded real lemma is the special form of the KYP lemma which states the $\mathcal{H}_\infty$ norm of the system is less than 1. Recall that the $\mathcal{H}_\infty$ norm is from the primal optimization (4.3), whereas the KYP lemma based optimization is the dual of (4.3). Therefore it is easy to prove the bounded real lemma using SDP duality.

Theorem 4.1 (Bounded real lemma, strict inequality): *For the stable LTI system* $\mathbf{M}$ *whose state space form is* (A, B, C, D), $\|\mathbf{M}\|_\infty < 1$ *if and only if there exists* $P \in \mathbb{H}$ *such that*

$$\begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_w} \end{bmatrix}^* (\Psi \otimes P) \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_w} \end{bmatrix} + \begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D - I_{n_w} \end{bmatrix} \prec 0. \tag{4.6}$$

PROOF: In order to use the theorem of alternatives, let us consider

$$\begin{aligned} \mathcal{A}(P) &= -\begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_w} \end{bmatrix}^* (\Psi \otimes P) \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_w} \end{bmatrix}, \\ A_0 &= -\begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D - I_{n_w} \end{bmatrix}. \end{aligned}$$

Then $\mathbf{Tr}(V\mathcal{A}(P)) = \mathbf{Tr}\left(\Pi\left(\begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x\times n_w}\end{bmatrix} V \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x\times n_w}\end{bmatrix}^*\right) P\right)$, which shows $\mathcal{A}^*(V) = \Pi\left(\begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x\times n_w}\end{bmatrix} V \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x\times n_w}\end{bmatrix}^*\right)$.

From Theorem 2.3, there exists P with (4.6) if and only if there is no $V \gneq 0$ such that $\Pi\left(\begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x\times n_w}\end{bmatrix} V \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x\times n_w}\end{bmatrix}^*\right) = 0, \mathbf{Tr}\left(\begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D - I_{n_w}\end{bmatrix} V\right) \geq 0.$
Since any feasible point in (4.3) satisfies $\mathbf{Tr}\left(\begin{bmatrix} 0_{n_x\times n_x} & 0_{n_x\times n_w} \\ 0_{n_w\times n_x} & I_{n_w}\end{bmatrix} V\right) = 1$, we have

$$\mathbf{Tr}\left(\begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D\end{bmatrix} V\right) \geq \mathbf{Tr}\left(\begin{bmatrix} 0_{n_x\times n_x} & 0_{n_x\times n_w} \\ 0_{n_w\times n_x} & I_{n_w}\end{bmatrix} V\right) = 1.$$

Therefore, there is no V in the feasible set of (4.3) such that the objective value $f_0(V) \geq 1$. Since the primal program (4.3) is solvable, this shows that the optimal value of (4.3) is less than 1 so as $\|\mathbf{M}\|_\infty^2$. By reversing the argument, we can conclude that proof. ■

The corresponding non-strict inequality version is easier to prove thanks to the strong duality.

Theorem 4.2 (Bounded real lemma, non-strict inequality): *For the stable LTI system* $\mathbf{M}$ *whose state space form is* (A, B, C, D) *with controllable pair* (A, B), $\|\mathbf{M}\|_\infty \leq 1$ *if and only if there exists* $P \in \mathbb{H}$ *such that*

$$\begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x\times n_w}\end{bmatrix}^* (\Psi \otimes P) \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x\times n_w}\end{bmatrix} + \begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D - I_{n_w}\end{bmatrix} \preceq 0. \tag{4.7}$$

Proof: If $\|\mathbf{M}\|_\infty < 1$, them from Theorem 4.1 the result is obvious. Suppose $\|\mathbf{M}\|_\infty = 1$. Then the optimal value of (4.3) is 1. From strong duality, Corollary 4.2, the optimal value of (4.3) is 1 if and only if there exists a dual optimal solution $(t^\star, P^\star)$ with $t^\star = 1$. ■

4.2 Extended $\mathcal{H}_\infty$ analysis

In $\mathcal{H}_\infty$ analysis, a disturbance $\mathbf{w}$ is assumed to have an unit energy, $\|\mathbf{w}\|_2 = 1$. Suppose more information about a disturbance is known beforehand. Then $\mathcal{H}_\infty$ norm becomes conservative since the analysis does not exploit this additional information. Therefore, it is natural to ask the question of whether we can capture more general disturbance sets beyond $\|\mathbf{w}\|_2 = 1$, and formulate appropriate $\mathcal{H}_\infty$ optimization.

The same argument may apply to the robustness measure. In $\mathcal{H}_\infty$ analysis, we seek the maximum output norm $\|\mathbf{z}\|_2^2$ in order to measure the impact of the worst-case disturbance. However, suppose we would like to measure the minimum level of each output channel $\|\mathbf{z}_i\|_2^2$ instead of the lumped sum $\|\mathbf{z}\|_2^2 = \sum_i \|\mathbf{z}_i\|_2^2$, which allows us to measure the worst-case simultaneous excitation of the output channel.

Finally, we may also want to include the correlation-like information between the state $\mathbf{x}$ and the input $\mathbf{w}$. For example, the passivity $\langle \mathbf{x}, \mathbf{w} \rangle \leq 0$ implies that the disturbance is not supplying the energy to the system, which allows us to model the passive type disturbance.

In this chapter, we make an extension of $\mathcal{H}_\infty$ analysis to capture various disturbance models as well as cost criterions when such a model and criterion can be expressed in terms of the Gramian $V = \Lambda(\mathbf{x}, \mathbf{w})$. Some of these results are known from [19] in the form of the scaled small gain test. On the contrary, we explicitly propose a well-defined optimization problem that can handle various disturbance models/ In addition, the optimal input can be constructed using algorithms in Chapter 3.

Specifically, let us consider the following optimization with a continuous convex function $f_0(V)$, and affine functions $f_i : \mathbb{H} \rightarrow \mathbb{H}$, for $i = 1, \cdots, n_1$.

$$\begin{aligned} \underset{V}{\text{minimize}} \quad & f_0(V) \\ \text{subject to} \quad & f_i(V) \preceq 0, \qquad i = 1, \cdots n_1 \\ & V \in \mathcal{D}_{\text{Gram}} \text{ (or } \mathcal{C}_{\text{Gram}}). \end{aligned} \tag{4.8}$$

As in the $\mathcal{H}_\infty$ analysis case, it may be natural to replace $\mathcal{D}_{\text{Gram}}$ with $\mathcal{D}_{\text{SDP}}$ to obtain a convex program. However, the following example shows that this procedure can be problematic.

Example 4.2: Consider the LTI system with $A = \frac{1}{2}, B = 1$. Let $V_0 = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$, and consider the following optimization:

$$\begin{aligned} \underset{V}{\text{minimize}} \quad & 0 \\ \text{subject to} \quad & V = V_0 \\ & V \in \mathcal{D}_{\text{Gram}}. \end{aligned} \tag{4.9}$$

Clearly, the constraint $V = V_0$ can be represented by two affine function; $f_1(V) = V - V_0$, and $f_2(V) = -V + V_0$, and the objective is trivially continuous and convex. Therefore it is in the form of (4.8). In Example 3.1, we show that $V_0 \notin \mathcal{D}_{\text{Gram}}$, which shows the optimal value is $+\infty$ because the feasible set is empty. However, since $V_0 \in \mathcal{D}_{\text{SDP}}$, if we replace $\mathcal{D}_{\text{Gram}}$ by $\mathcal{D}_{\text{SDP}}$ then the optimal value becomes 0. In fact, if we perturb the optimization (4.9)

$$\begin{aligned} \underset{V}{\text{minimize}} \quad & 0 \\ \text{subject to} \quad & V_0 - \epsilon I \preceq V \preceq V_0 + \epsilon I \\ & V \in \mathcal{D}_{\text{Gram}}, \end{aligned} \tag{4.10}$$

then the optimal value becomes 0, because we can arbitrarily approximate $V_0 \in \mathcal{D}_{\text{SDP}}$ by $V \in \mathcal{D}_{\text{Gram}}$.

The difference between the above example and $\mathcal{H}_\infty$ analysis, where replacing $\mathcal{D}_{\text{Gram}}$ with $\mathcal{D}_{\text{SDP}}$ is not an issue, is that the optimization (4.9) specifies entire components of V. According to Lemma 3.1, only the right bottom block of V, which corresponds to $\Lambda(\mathbf{w})$, can be exactly matched, and not the entire component. In order to avoid this situation, we need *constraint qualification* for the feasible set.

To this end, let us modify the extended $\mathcal{H}_\infty$ analysis as the following optimization:

$$\begin{array}{lll} \underset{V}{\text{minimize}} & f_0(V) & \\ \text{subject to} & f_i(V) \preceq 0, \qquad i = 1, \cdots n_1 & \\ & g_i\left(\begin{bmatrix}0_{n_w\times n_x} & I_{n_w}\end{bmatrix} V \begin{bmatrix}0_{n_w\times n_x} & I_{n_w}\end{bmatrix}^*\right) \preceq 0, \qquad i = 1, \cdots, n_2 & \\ & V \in \mathcal{D}_{\text{Gram}} \text{ (or } \mathcal{C}_{\text{Gram}}), & \end{array} \tag{4.11}$$

where $f_0(V)$ is a continuous, convex function and $f_i : \mathbb{H}^{n_x+n_w} \to \mathbb{H}$ and $g_i : \mathbb{H}^{n_w} \to \mathbb{H}$ are affine functions. Here g_i is used to model the disturbance information which is not related to the state or the output. For example, in $\mathcal{H}_\infty$ analysis, only $g_1(W) = \mathbf{Tr}(W) - 1$ and $g_2(W) = 1 - \mathbf{Tr}(W)$ are used, and there are no f_i.

The following examples illustrate the problems that can be handled using the above optimization (4.11).

Example 4.3 ($\mathcal{H}_\infty$ analysis): In the $\mathcal{H}_\infty$ analysis, the disturbance model is given by $\|\mathbf{w}\|_2 = 1$. Therefore, by setting

$$\begin{aligned} f_0(V) &= -\mathbf{Tr}\left(\begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D \end{bmatrix} V\right) \\ g_1(W) &= \mathbf{Tr}(W) - 1 \\ g_2(W) &= 1 - \mathbf{Tr}(W), \end{aligned}$$

$\mathcal{H}_\infty$ analysis can be represented as (4.11).

Example 4.4 (Square $\mathcal{H}_\infty$ analysis): In square $\mathcal{H}_\infty$ analysis [19], the disturbance model is given by $\|[\mathbf{w}]_i\|_2 \leq 1$, for $i = 1, \cdots, n_w$, where $[\mathbf{w}]_i$ is the ith component of $\mathbf{w}$. Therefore,

$$\begin{aligned} f_0(V) &= -\mathbf{Tr}\left(\begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D \end{bmatrix} V\right) \\ g_i(W) &= [W]_{ii} - 1, \text{for } i = 1, \cdots, n_w. \end{aligned}$$

Example 4.5 (Grouped square $\mathcal{H}_\infty$ analysis): Suppose $\mathbf{w} \in l_2^4$, and the disturbance model is given by $\|[\mathbf{w}]_1\|_2^2 + \|[\mathbf{w}]_2\|_2^2 \le 1$, $\|[\mathbf{w}]_3\|_2^2 + \|[\mathbf{w}]_4\|_2^2 \le 1$. In this case, we have

$$\begin{aligned} f_0(V) &= -\mathbf{Tr}\left(\begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D \end{bmatrix} V\right) \\ g_1(W) &= [W]_{11} + [W]_{22} - 1 \\ g_2(W) &= [W]_{33} + [W]_{44} - 1. \end{aligned}$$

Example 4.6 (Principal component bound): Suppose we know that the maximum eigenvalue of the Gramian $\Lambda(\mathbf{W})$ is bounded by one. Then, we have

$$\begin{aligned} f_0(V) &= -\mathbf{Tr}\left(\begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D \end{bmatrix} V\right) \\ g_1(W) &= W - I_{n_w}. \end{aligned}$$

See [58] and references therein for the application of this disturbance model.

Example 4.7 (Minimum gain): In $\mathcal{H}_\infty$ analysis, one would like to *maximize* the output norm. Instead, if we want to find a *minimal* gain of the system, then we have,

$$\begin{aligned} f_0(V) &= \mathbf{Tr}\left(\begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D \end{bmatrix} V\right) \\ g_1(W) &= \mathbf{Tr}(W) - 1 \\ g_2(W) &= 1 - \mathbf{Tr}(W), \end{aligned}$$

Example 4.8 (Simultaneous excitation): Suppose we have multiple output channel, $\mathbf{z}_i = C_i\mathbf{x} + D_i\mathbf{w}$, for $i = 1, \cdots, n_o$, and would like to measure the impact of disturbance on the *minimum* level of output. In this case, we can use

$$
\begin{aligned}
f_0(V) &= -\min_{i=1,\cdots,n_o} \mathbf{Tr}\left(\begin{bmatrix} C_i^*C_i & C_i^*D_i \\ D_i^*C_i & D_i^*D_i \end{bmatrix} V\right) \\
g_1(W) &= \mathbf{Tr}(W) - 1 \\
g_2(W) &= 1 - \mathbf{Tr}(W).
\end{aligned}
$$

By replacing $\mathcal{D}_{\text{Gram}}$ and $\mathcal{C}_{\text{Gram}}$ with $\mathcal{D}_{\text{SDP}}$ and $\mathcal{C}_{\text{SDP}}$, respectively, we obtain the following convex program, which provides the lower bound of (4.11), since the feasible set of the following program is larger than (4.11).

$$
\begin{aligned}
\underset{V\succeq 0}{\text{minimize}} \quad & f_0(V) \\
\text{subject to} \quad & f_i(V) \preceq 0, \qquad i = 1,\cdots n_1 \\
& g_i\left(\begin{bmatrix} 0_{n_w\times n_x} & I_{n_w} \end{bmatrix} V \begin{bmatrix} 0_{n_w\times n_x} & I_{n_w} \end{bmatrix}^*\right) \preceq 0, \qquad i = 1,\cdots,n_2 \\
& \Pi\left(\begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x\times n_w} \end{bmatrix} V \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x\times n_w} \end{bmatrix}^*\right) = 0,
\end{aligned}
\tag{4.12}
$$

where Π is from Table 4.1,

$$
\Pi\left(\begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix}\right) = \begin{cases} X - Z & \text{Discrete time case} \\ Y + Y^* & \text{Continuous time case.} \end{cases}
$$

Let

$$
\begin{aligned}
\mathcal{F} := \Bigg\{ V : \quad & f_i(V) \preceq 0, \text{ for } i = 1,\cdots n_1, \\
& g_i\left(\begin{bmatrix} 0_{n_w\times n_x} & I_{n_w} \end{bmatrix} V \begin{bmatrix} 0_{n_w\times n_x} & I_{n_w} \end{bmatrix}^*\right) \preceq 0, \text{ for } i = 1,\cdots,n_2 \Bigg\}
\end{aligned}
$$

$$\mathcal{F}^\circ := \left\{ V : \quad f_i(V) \prec 0, \text{ for } i = 1, \cdots n_1, \right.$$
$$\left. g_i\left(\begin{bmatrix} 0_{n_w \times n_x} & I_{n_w} \end{bmatrix} V \begin{bmatrix} 0_{n_w \times n_x} & I_{n_w} \end{bmatrix}^*\right) \preceq 0, \text{ for } i = 1, \cdots, n_2 \right\}.$$

Notice that $\mathcal{F}^\circ$ requires $f_i(V)$ to be strictly feasible. Then the following result provides a sufficient condition for the exactness of relaxation (4.12).

Proposition 4.4: *Suppose $\mathcal{F}^\circ \cap \mathcal{D}_{SDP}$ is not empty. Then* $\mathbf{cl}\,(\mathcal{D}_{Gram} \cap \mathcal{F}) = \mathcal{D}_{SDP} \cap \mathcal{F}$*. Similarly, if $\mathcal{F}^\circ \cap \mathcal{C}_{SDP}$ is not empty, then* $\mathbf{cl}\,(\mathcal{C}_{Gram} \cap \mathcal{F}) = \mathcal{C}_{SDP} \cap \mathcal{F}$*.*

PROOF: It is clear that $\mathbf{cl}\,(\mathcal{D}_{\text{Gram}} \cap \mathcal{F}) \subset \mathcal{D}_{\text{SDP}} \cap \mathcal{F}$, since $\mathcal{F}$ is closed. Let $V_0 \in \mathcal{D}_{\text{SDP}} \cap \mathcal{F}^\circ$. Since $\mathcal{D}_{\text{SDP}} \cap \mathcal{F}$ is convex, for any $V \in \mathcal{D}_{\text{SDP}} \cap \mathcal{F}$, $0 < \theta \leq 1$, $\hat{V} = (1-\theta)V + \theta V_0 \in \mathcal{D}_{\text{SDP}} \cap \mathcal{F}^\circ$. Let us pick θ such that $\|V - \hat{V}\|_F < \frac{\varepsilon}{2}$. In addition, from Lemma 3.1 and continuity of f_i, we can find $\tilde{V} \in \mathcal{D}_{\text{Gram}}$ such that $\|\hat{V} - \tilde{V}\|_F < \frac{\varepsilon}{2}$, and $\tilde{V} \in \mathcal{F}^\circ$. Therefore, $\|V - \tilde{V}\|_F \leq \|V - \hat{V}\|_F + \|\hat{V} - \tilde{V}\|_F < \varepsilon$, and $\tilde{V} \in \mathcal{D}_{\text{Gram}} \cap \mathcal{F}$ because $\mathcal{F}^\circ \subset \mathcal{F}$, which shows $\mathcal{D}_{\text{SDP}} \cap \mathcal{F} \subset \mathbf{cl}\,(\mathcal{D}_{\text{Gram}} \cap \mathcal{F})$. For $\mathcal{C}_{\text{SDP}}$, we can apply the same argument and this concludes the proof. ∎

Since $f_0(V)$ is continuous, we obtain the following corollary.

Corollary 4.3: *If $\mathcal{F}^\circ \cap \mathcal{D}_{SDP}$ (or $\mathcal{C}_{SDP}$) is not empty, then the optimal value of* (4.12) *is equal to* (4.11)*.*

Recall that our proof of Lemma 3.1 and 3.3 explicitly constructs l_2 ($\mathcal{L}_2$) signal approximates the optimal solution $V^\star$ of (4.12). Therefore we can easily construct the optimal disturbance and check whether the resulting disturbance is reasonable or not.

4.2.1 Dual problem

In this section, we derive the Conic dual program of (4.12). Let us consider the Lagrangian

$$\begin{aligned}\mathcal{L}(V,P,S_i,Y_i) \;:=\; & f_0(V)+\sum_{i=1}^{n_1}\mathbf{Tr}(S_i f_i(V))+\sum_{i=1}^{n_2}\mathbf{Tr}\Big(Y_i g_i\left(\begin{bmatrix}0_{n_w\times n_x} & I_{n_w}\end{bmatrix} V \begin{bmatrix}0_{n_w\times n_x} & I_{n_w}\end{bmatrix}^*\right)\Big)\\ &+\mathbf{Tr}\left(P\Pi\left(\begin{bmatrix}A & B\\ I_{n_x} & 0_{n_x\times n_w}\end{bmatrix} V \begin{bmatrix}A & B\\ I_{n_x} & 0_{n_x\times n_w}\end{bmatrix}^*\right)\right),\end{aligned}$$

where $P=P^*, S_i\succeq 0, Y_i\succeq 0$. Since f_i is affine, let $\tilde{f}_i(V)=f_i(V)-f_i(0)$, so that $\tilde{f}_i$ becomes linear. Like-wise, for g_i, let $\tilde{g}_i(W)=g_i(W)-g_i(0)$, so that $\tilde{g}_i$ is linear. Then,

$$\begin{aligned}\mathcal{L}(V,P,S_i,Y_i) &= f_0(V)-\sum_{i=1}^{n_1}\mathbf{Tr}(S_i f_i(0))-\sum_{i=1}^{n_2}\mathbf{Tr}(Y_i g_i(0))\\ &+\mathbf{Tr}\left(\left(\sum_{i=1}^{n_1}\tilde{f}_i^*(S_i)+\sum_{i=1}^{n_2}\begin{bmatrix}0_{n_x\times n_x} & 0_{n_x\times n_w}\\ 0_{n_w\times n_x} & \tilde{g}_i^*(Y_i)\end{bmatrix}+\begin{bmatrix}A & B\\ I_{n_x} & 0_{n_x\times n_w}\end{bmatrix}^*(\Psi\otimes P)\begin{bmatrix}A & B\\ I_{n_x} & 0_{n_x\times n_w}\end{bmatrix}\right)V\right).\end{aligned}$$

Therefore, the dual problem of (4.12) is given by

$$\begin{aligned}\underset{P=P^*,S_i\succeq 0,Y_i\succeq 0}{\text{maximize}} \; & -\sum_{i=1}^{n_1}\mathbf{Tr}(S_i f_i(0))-\sum_{i=1}^{n_2}\mathbf{Tr}(Y_i g_i(0))\\ &-f_0^\star\left(\sum_{i=1}^{n_1}\tilde{f}_i^*(S_i)+\sum_{i=1}^{n_2}\begin{bmatrix}0_{n_x\times n_x} & 0_{n_x\times n_w}\\ 0_{n_w\times n_x} & \tilde{g}_i^*(Y_i)\end{bmatrix}+\begin{bmatrix}A & B\\ I_{n_x} & 0_{n_x\times n_w}\end{bmatrix}^*(\Psi\otimes P)\begin{bmatrix}A & B\\ I_{n_x} & 0_{n_x\times n_w}\end{bmatrix}\right),\end{aligned} \tag{4.13}$$

where

$$f_0^\star(U)=\sup_{V\succeq 0}\{\mathbf{Tr}(UV)-f_0(V)\},$$

the convex conjugate of f_0 over the domain $V\succeq 0$.

In addition, using this dual program, it is possible to derive an LMI test for bounding the optimal value of (4.12). That is, if there exists a dual feasible point

(P, S_i, Y_i) such that the objective function in the dual program (4.13) is greater than α, then the optimal value of (4.12) is also greater than α from weak duality. Furthermore, if strong duality holds between the dual program (4.13) and the primal program (4.12), then this LMI test provides a tight lower bound.

4.2.2 Examples

In this section, we provide some specific examples of the extended $\mathcal{H}_\infty$ analysis, both the primal (4.12) and the dual (4.13).

4.2.2.1 Squared $\mathcal{H}_\infty$ analysis

In the square $\mathcal{H}_\infty$ analysis, we assume the two norm of the each disturbance channel is independently bounded by a certain amount $\sqrt{c_i}$. In addition, the cost criterion is the output norm. Let n_x be the dimension of the state $\mathbf{x}$, and n_w be the dimension of the disturbance $\mathbf{w}$. Then, SDP version of square $\mathcal{H}_\infty$ analysis can be formulated as

$$\begin{aligned}
\underset{V}{\text{maximize}} \quad & \mathbf{Tr}\left(\begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D \end{bmatrix} V\right) \\
\text{subject to} \quad & V_{i+n_x,i+n_x} \le c_i, \qquad i = 1, \cdots, n_w, \\
& \Pi\left(\begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_w} \end{bmatrix} V \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_w} \end{bmatrix}^*\right) = 0, \\
& V \succeq 0.
\end{aligned} \tag{4.14}$$

Here all the constraints can be represented by $g_i(W)$, and therefore this SDP relaxation is exact.

The SDP dual of (4.14) is as follows:

$$\begin{aligned}
\underset{P=P^*,y}{\text{minimize}} \quad & \sum_{i=1}^{n_w} c_i y_i \\
\text{subject to} \quad & \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_w} \end{bmatrix}^* (\Psi \otimes P) \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_w} \end{bmatrix} \\
& + \begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D - \mathbf{diag}\,(y_1, \cdots, y_{n_w}) \end{bmatrix} \preceq 0, \\
& y_i \geq 0, \qquad i = 1, \cdots, n_w,
\end{aligned} \tag{4.15}$$

where Ψ is from Table 4.1,

$$\Psi = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & \text{Discrete time case} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \text{Continuous time case.} \end{cases}$$

Notice that the LMI constraint is very similar to the bounded real lemma case (4.4), except the identity block is replaced by the diagonal matrix. In addition, strong duality holds because the dual program (4.15) is strictly feasible. This can be easily checked by choosing P so that the upper left block of the LMI is strictly feasible and choosing $y_i = \tau$, where τ is sufficiently large.

From strong duality, the following LMI test from the dual program (4.15) can be used to bound the optimal value of (4.14).

There exists $(y_1, \cdots, y_{n_w}, P)$ such that

$$\sum_{i=1}^{n_w} c_i y_i < 1 \tag{4.16}$$

$$\begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_w} \end{bmatrix}^* (\Psi \otimes P) \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_w} \end{bmatrix} + \begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D - \mathbf{diag}\,(y_1, \cdots, y_{n_w}) \end{bmatrix} \prec 0.$$

Proposition 4.5: *The optimal value of the primal program* (4.14) *is strictly less than* 1 *if and only if* (4.16) *holds.*

Proof: Suppose (4.16) holds. Then the optimal value of the dual program (4.15) is strictly less than 1, and from weak duality, the optimal value of the primal program is also strictly less than 1. Now suppose the optimal value of the primal program is strictly less than 1. From strong duality, the optimal value of the dual program is also strictly less than 1.

Therefore, there exists $(y_1, \cdots, y_{n_w}, P)$ such that $\sum_{i=1}^{n_w} c_i y_i < 1$, and

$$\begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_w} \end{bmatrix}^* (\Psi \otimes P) \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_w} \end{bmatrix} + \begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D - \mathbf{diag}\,(y_1, \cdots, y_{n_w}) \end{bmatrix} \prec 0.$$

In addition, since the dual program has a strictly feasible point, by taking a convex combination of $(y_1, \cdots, y_{n_w}, P)$ and the strictly feasible point of the dual program, we can conclude the proof. ■

The above proposition in the discrete time case firstly appeared in [19], but the proof is significantly simplified in here, and the above proposition contains the continuous time result.

4.2.2.2 Principal component bound

Suppose we have the bound on the Gramian, $\Lambda(\mathbf{w}) \preceq I_{n_w}$. Then we have,

$$\begin{array}{ll} \underset{V}{\text{maximize}} & \mathbf{Tr}\left(\begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D \end{bmatrix} V\right) \\ \text{subject to} & \begin{bmatrix} 0_{n_w \times n_x} & I_{n_w} \end{bmatrix} V \begin{bmatrix} 0_{n_w \times n_x} & I_{n_w} \end{bmatrix}^* \preceq I \\ & \Pi\left(\begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_w} \end{bmatrix} V \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_w} \end{bmatrix}^*\right) = 0 \\ & V \succeq 0. \end{array} \tag{4.17}$$

Here all the constraints can be represented by $g_i(W)$, and therefore this SDP relaxation is exact.

The SDP dual of (4.17) is as follows:

$$\begin{array}{ll} \underset{P=P^*,Y}{\text{minimize}} & \mathbf{Tr}(Y) \\ \text{subject to} & \begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D-Y \end{bmatrix} + \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x\times n_w} \end{bmatrix}^* (\Psi\otimes P) \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x\times n_w} \end{bmatrix} \preceq 0 \\ & Y \succeq 0. \end{array} \tag{4.18}$$

4.2.2.3 Simultaneous excitation

Suppose we have multiple output variables, $\mathbf{z}_i = C_i\mathbf{x} + D_i\mathbf{w}$, for $i = 1\cdots, n_o$. In this case, the extended $\mathcal{H}_\infty$ analysis is given by,

$$\begin{array}{ll} \underset{V}{\text{maximize}} & \underset{i=1,\cdots,n_o}{\min} \mathbf{Tr}\left(\begin{bmatrix} C_i^*C_i & C_i^*D_i \\ D_i^*C_i & D_i^*D_i \end{bmatrix} V \right) \\ \text{subject to} & \mathbf{Tr}\left(\begin{bmatrix} 0_{n_x\times n_x} & 0_{n_x\times n_w} \\ 0_{n_w\times n_x} & I_{n_w} \end{bmatrix} V \right) = 1 \\ & \Pi\left(\begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x\times n_w} \end{bmatrix} V \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x\times n_w} \end{bmatrix}^* \right) = 0 \\ & V \succeq 0. \end{array} \tag{4.19}$$

Here the objective function is continuous and all the constraints can be represented by $g_i(W)$, and therefore this SDP relaxation is exact. By introducing a slack variable

t, we have

$$
\begin{aligned}
\underset{V,t}{\text{maximize}} \quad & t \\
\text{subject to} \quad & \mathbf{Tr}\left(\begin{bmatrix} 0_{n_x\times n_x} & 0_{n_x\times n_w} \\ 0_{n_w\times n_x} & I_{n_w} \end{bmatrix} V\right) = 1 \\
& t \le \mathbf{Tr}\left(\begin{bmatrix} C_i^*C_i & C_i^*D_i \\ D_i^*C_i & D_i^*D_i \end{bmatrix} V\right), \qquad \text{for } i = 1, \cdots, n_o \\
& \Pi\left(\begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x\times n_w} \end{bmatrix} V \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x\times n_w} \end{bmatrix}^*\right) = 0 \\
& V \succeq 0.
\end{aligned}
\tag{4.20}
$$

The SDP dual of (4.20) is as follows:

$$
\begin{aligned}
\underset{P=P^*,s_i,y}{\text{minimize}} \quad & y \\
\text{subject to} \quad & \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x\times n_w} \end{bmatrix}^* (\Psi \otimes P) \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x\times n_w} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -yI \end{bmatrix} \\
& + \sum_{i=1}^{n_o} s_i \begin{bmatrix} C_i^*C_i & C_i^*D_i \\ D_i^*C_i & D_i^*D_i \end{bmatrix} \preceq 0 \\
& s_i \ge 0, \sum_{i=1}^{n_o} s_i = 1.
\end{aligned}
\tag{4.21}
$$

4.2.2.4 Minimal gain

The $\mathcal{H}_\infty$ norm can be seen as the maximum gain of the LTI system. Instead, suppose we would like to compute

$$\min_{\|\mathbf{w}\|_2=1} \|\mathbf{z}\|_2.$$

In this case, we have

$$\begin{array}{ll} \underset{V}{\text{minimize}} & \mathbf{Tr}\left(\begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D \end{bmatrix} V\right) \\ \text{subject to} & \mathbf{Tr}\left(\begin{bmatrix} 0_{n_x \times n_x} & 0_{n_x \times n_w} \\ 0_{n_w \times n_x} & I_{n_w} \end{bmatrix} V\right) = 1 \\ & \Pi\left(\begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_w} \end{bmatrix} V \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_w} \end{bmatrix}^*\right) = 0 \\ & V \succeq 0. \end{array} \tag{4.22}$$

The dual problem is given by

$$\begin{array}{ll} \underset{P=P^*,t}{\text{maximize}} & t \\ \text{subject to} & \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_w} \end{bmatrix}^* (\Psi \otimes P) \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_w} \end{bmatrix} + \begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D - tI_{n_w} \end{bmatrix} \succeq 0. \end{array} \tag{4.23}$$

Chapter 5

Robust stability analysis

5.1 Well-connectedness analysis with uncertain systems

5.1.1 Well-connectedness

In this chapter, we would like to investigate when

$$\mathbf{I} - \mathbf{M}\Delta$$

is invertible for all $\Delta \in \mathbf{\Delta}$. Here $\mathbf{M}$ is a stable LTI system that has a state space realization (A, B, C, D), Δ is a bounded linear operator such that $\Delta : l_2 \to l_2$ (or $\mathcal{L}_2 \to \mathcal{L}_2$), and $\mathbf{I}$ is the identity operator. Consider the following quantity:

$$\eta_{\mathbf{\Delta}}(\mathbf{M}) := \inf_{\Delta \in \mathbf{\Delta}, \|\mathbf{p}\|_2 = 1} \|\mathbf{p} - \mathbf{M}(\Delta(\mathbf{p}))\|_2.$$

Since $\mathbf{M}, \Delta$ are linear operators, $\mathbf{I} - \mathbf{M}\Delta$ is invertible if $\eta_{\mathbf{\Delta}}(\mathbf{M}) > 0$. Therefore, we say $\mathbf{M}$ is *robustly well-connected* with $\mathbf{\Delta}$ if $\eta_{\mathbf{\Delta}}(\mathbf{M}) > 0$. [1]

In fact, robust stability with an intrinsic uncertainty can be checked using robust well-connectedness. For example, consider the following feedback configuration in

[1] If M, Δ are matrices, then well-connectedness is equivalent to the invertibility of $I - M\Delta$. However since M, Δ are linear operators in an infinite dimensional space, the well-connectedness is stronger than the invertibility of $I - M\Delta$. See [60], and Chapter 8 in [26]. Our definition is in fact *uniform* robust well-connectedness in [74].

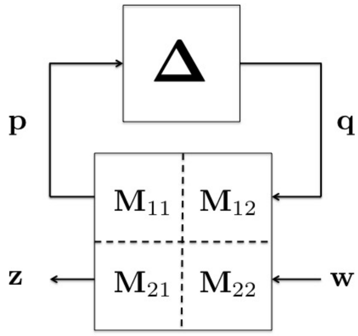

Figure 5.1: Feedback configuration with an uncertain block Δ.

Fig. 5.1.

From the configuration we have

$$\begin{aligned} \mathbf{q} &= \Delta \mathbf{p} \\ \begin{bmatrix} \mathbf{p} \\ \mathbf{z} \end{bmatrix} &= \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \mathbf{w} \end{bmatrix}. \end{aligned}$$

We use n_q, n_p, n_z, n_w for the dimension of signal $\mathbf{q}, \mathbf{p}, \mathbf{z}, \mathbf{w}$, respectively.

Observe that if $\mathbf{I} - \mathbf{M}_{11}\Delta$ is non-singular, then the map from $\mathbf{w}$ to $\mathbf{z}$ is given by

$$\mathbf{M}_{22} + \mathbf{M}_{21}\Delta(\mathbf{I} - \mathbf{M}_{11}\Delta)^{-1}\mathbf{M}_{12},$$

which shows the map $\mathbf{w}$ to $\mathbf{z}$ is well defined. In addition, since all operators are bounded, we can conclude that the system remains stable for Δ. This shows that if we can prove $\mathbf{I} - \mathbf{M}_{11}\Delta$ is non-singular for all $\Delta \in \boldsymbol{\Delta}$, then we can ensure that the system $\mathbf{M}$ remains stable for all $\Delta \in \boldsymbol{\Delta}$. Chapter 8 and 9 of the standard textbook [26] covers this topic and presents a systematic way to convert questions in robust control theory to the above well-connectedness question.

5.1.2 An SDP for well-connectedness analysis

In order to check well-connectedness of a system, we need a method for checking $\eta_{\boldsymbol{\Delta}}(\mathbf{M}) > 0$. One direct way is to sample many $\Delta \in \boldsymbol{\Delta}$ and compute $\eta_{\boldsymbol{\Delta}}(\mathbf{M})$. This approach provides an upper bound of $\eta_{\boldsymbol{\Delta}}(\mathbf{M})$, but this straightforward approach has two challenges:

1. It may not be clear how to choose Δ among $\boldsymbol{\Delta}$, especially, when there are an infinite number of Δ in $\boldsymbol{\Delta}$.

2. Computing $\eta_{\boldsymbol{\Delta}}(\mathbf{M})$ is not trivial because the underlying space is infinite dimensional.

Instead, we use our key lemmas (SDP representation of a set of Gramians) to show that the optimization $\eta_{\boldsymbol{\Delta}}(\mathbf{M})$ becomes an SDP if $\boldsymbol{\Delta}$ can be specified by a Gramian, $\Lambda(\mathbf{p}, \mathbf{q})$, where $\mathbf{p}$ is the input of Δ and $\mathbf{q}$ is the output of Δ. The underlying assumptions in this section are

1. A system $\mathbf{M}$ has a state-space representation (A, B, C, D).

2. A system $\mathbf{M}$ is stable.

3. There exists a constant $\alpha_{\boldsymbol{\Delta}}$ such that $\|\Delta\|_{2,2} \leq \alpha_{\boldsymbol{\Delta}}$ for all $\Delta \in \boldsymbol{\Delta}$.

Since many engineering system models have a state-space representation, the first assumption is not restrictive. The second assumption states that the nominal system without the uncertainty is required to be stable. Otherwise, since the nominal system is already unstable, it may not make sense to discuss the robustness of the system. The third assumption restricts the size of the perturbation from the uncertainty $\boldsymbol{\Delta}$. Recall that we would like to ensure $\mathbf{I} - \mathbf{M}\Delta$ to be invertible for all $\Delta \in \boldsymbol{\Delta}$. If there is no bound on Δ, it is always possible to find Δ, which makes $\mathbf{I} - \mathbf{M}\Delta$ singular, unless the range space of $\boldsymbol{\Delta}$ is a subset of the kernel of $\mathbf{M}$, a very exceptional case.

The first step is to eliminate dependency on Δ by providing an equivalent characterization of $\mathbf{q} = \Delta\mathbf{p}$. To this end, let

$$\mathcal{R}_{\boldsymbol{\Delta}} := \{(\mathbf{p}, \mathbf{q}) : \mathbf{q} = \Delta\mathbf{p}, \text{for some } \Delta \in \boldsymbol{\Delta}\},$$

which is a subset of $l_2^{n_p} \times l_2^{n_q}$ (or, $\mathcal{L}_2^{n_p} \times \mathcal{L}_2^{n_q}$). From linearity of Δ, it should be obvious that $(0, \mathbf{q}) \in \mathcal{R}_{\boldsymbol{\Delta}}$ implies $\mathbf{q} = 0$.

Using this set notation, the infimum can be restated as

$$\eta_{\mathbf{\Delta}}(\mathbf{M}) = \inf_{(\mathbf{p},\mathbf{q})\in\mathcal{R}_{\mathbf{\Delta}}, \|\mathbf{p}\|_2=1} \|\mathbf{p} - \mathbf{M}\mathbf{q}\|_2.$$

Further, suppose that the relationship $(\mathbf{p}, \mathbf{q}) \in \mathcal{R}_{\mathbf{\Delta}}$ can be completely characterized by a Gramian $\Lambda(\mathbf{p}, \mathbf{q})$ in the following form:

$$\begin{aligned} \mathcal{R}_{\mathbf{\Delta}} := \{(\mathbf{p}, \mathbf{q}) : \mathcal{A}_{i\mathbf{\Delta}}(\Lambda(\mathbf{p}, \mathbf{q})) \succeq 0,\ \text{for } i = 1, \cdots n_a, \\ \mathcal{B}_{i\mathbf{\Delta}}(\Lambda(\mathbf{p}, \mathbf{q})) = 0,\ \text{for } i = 1, \cdots, n_b\}, \end{aligned}$$

where $\mathcal{A}_i$, $\mathcal{B}_i$ are linear operators such that $\mathcal{A}_{i\mathbf{\Delta}} : \mathbb{H}^{n_p+n_q} \to \mathbb{H}$ and $\mathcal{B}_{i\mathbf{\Delta}} : \mathbb{H}^{n_p+n_q} \to \mathbb{V}$, for some Hilbert space $\mathbb{V}$. In addition, since $(\Lambda(\mathbf{p}) = 0) \Rightarrow (\mathbf{p} = 0) \Rightarrow (\mathbf{q} = 0)$ from $\mathbf{q} = \Delta\mathbf{p}$, if the (1,1) block of $\Lambda(\mathbf{p}, \mathbf{q})$, $\Lambda(\mathbf{p})$, is zero then it should imply that the (2,2) block of $\Lambda(\mathbf{p}, \mathbf{q})$, $\Lambda(\mathbf{q})$, is zero.

Since we often rely on the partition of $\Lambda(\mathbf{p}, \mathbf{q})$ to describe $\mathcal{A}_{i\mathbf{\Delta}}$ and $\mathcal{B}_{i\mathbf{\Delta}}$, let us denote a partition of $\Lambda(\mathbf{p}, \mathbf{q})$ by

$$\Lambda(\mathbf{p}, \mathbf{q}) = Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^* & Z_{22} \end{bmatrix},$$

where $Z_{11} \in \mathbb{C}^{n_p \times n_p}$, $Z_{12} \in \mathbb{C}^{n_p \times n_q}$, $Z_{22} \in \mathbb{C}^{n_q \times n_q}$ so that $Z_{11} = \Lambda(\mathbf{p})$, $Z_{22} = \Lambda(\mathbf{q})$ and $[Z_{12}]_{ij} = \langle [\mathbf{p}]_i, [\mathbf{q}]_j \rangle$. Using this notation, it should be obvious that if Z has the form $Z = \begin{bmatrix} 0_{n_p\times n_p} & 0_{n_p\times n_q} \\ 0_{n_q\times n_p} & Z_{22} \end{bmatrix}$ then

$$\mathcal{A}_{i\mathbf{\Delta}}\left(\begin{bmatrix} 0_{n_p\times n_p} & 0_{n_p\times n_q} \\ 0_{n_q\times n_p} & Z_{22} \end{bmatrix}\right) \succeq 0, \mathcal{B}_{i\mathbf{\Delta}}\left(\begin{bmatrix} 0_{n_p\times n_p} & 0_{n_p\times n_q} \\ 0_{n_q\times n_p} & Z_{22} \end{bmatrix}\right) = 0 \quad \Rightarrow \quad Z_{22} = 0, \tag{5.1}$$

from the fact that $\mathbf{q} = \Delta\mathbf{p}$.

The following example shows how to use $\mathcal{A}_{i\mathbf{\Delta}}$ to describe $\mathbf{\Delta}$.

Example 5.1: Let $\mathbf{\Delta} := \{\Delta : \|\Delta\|_{2,2} \leq 1\}$. Later, we will show that $\mathcal{R}_{\mathbf{\Delta}} = \{(\mathbf{p}, \mathbf{q}) :$

$\|\mathbf{p}\|_2 \geq \|\mathbf{q}\|_2\}$. In this case, using the Gramian $\Lambda(\mathbf{p},\mathbf{q})$, we can easily see that $\mathcal{A}_{\mathbf{\Delta}}\left(\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^* & Z_{22}\end{bmatrix}\right) = \mathbf{Tr}(Z_{11} - Z_{22})$.

Finally, let us find an expression for $\mathbf{Mq}-\mathbf{p}$. Recall that $\mathbf{M}$ is a stable LTI system with the state space realization (A, B, C, D). Consider the following artificial stable LTI system:

(Discrete time)
$$\begin{aligned} x[k+1] &= Ax[k] + \begin{bmatrix} B & 0_{n_x \times n_p}\end{bmatrix}\begin{bmatrix} q[k] \\ p[k]\end{bmatrix} \\ x[0] &= 0, \end{aligned}$$

(Continuous time)
$$\begin{aligned} \dot{x} &= Ax(t) + \begin{bmatrix} B & 0_{n_x \times n_p}\end{bmatrix}\begin{bmatrix} q(t) \\ p(t)\end{bmatrix} \\ x(0) &= 0, \end{aligned}$$

where n_x is the dimension of $\mathbf{x}$. Here we augment the signal $\mathbf{p}$ as an input that has no effect on the state $\mathbf{x}$.

From the following Gramian

$$V := \Lambda(\mathbf{x},\mathbf{q},\mathbf{p}),$$

we have

$$\begin{aligned} \|\mathbf{Mq}-\mathbf{p}\|_2^2 &= \mathbf{Tr}\left(\begin{bmatrix} C & D & -I_{n_p}\end{bmatrix} V \begin{bmatrix} C & D & -I_{n_p}\end{bmatrix}^*\right) \\ \Lambda(\mathbf{p},\mathbf{q}) &= \begin{bmatrix} 0_{n_q\times n_x} & 0_{n_q\times n_p} & I_{n_q} \\ 0_{n_p\times n_x} & I_{n_p} & 0_{n_p\times n_q}\end{bmatrix} V \begin{bmatrix} 0_{n_q\times n_x} & 0_{n_q\times n_p} & I_{n_q} \\ 0_{n_p\times n_x} & I_{n_p} & 0_{n_p\times n_q}\end{bmatrix}^*. \end{aligned}$$

Therefore, using the above Gramian, $\eta_{\mathbf{\Delta}}(\mathbf{M})$ can be computed by

(Discrete time)
$$\eta_{\mathbf{\Delta}}(\mathbf{M})^2 = \inf_V \left\{ \mathbf{Tr}\left(\begin{bmatrix} C & D & -I_{n_p}\end{bmatrix} V \begin{bmatrix} C & D & -I_{n_p}\end{bmatrix}^*\right) : V \in \mathcal{D}_{\text{Gram}}, \begin{bmatrix} 0_{n_q\times n_x} & 0_{n_q\times n_p} & I_{n_q} \\ 0_{n_p\times n_x} & I_{n_p} & 0_{n_p\times n_q}\end{bmatrix} V \begin{bmatrix} 0_{n_q\times n_x} & 0_{n_q\times n_p} & I_{n_q} \\ 0_{n_p\times n_x} & I_{n_p} & 0_{n_p\times n_q}\end{bmatrix}^* \in \mathcal{F}_{\mathbf{\Delta}} \right\}$$

$$\text{(Continuous time)} \qquad \eta_{\mathbf{\Delta}}(\mathbf{M})^2 = \inf_V \Big\{ \mathbf{Tr}\Big(\begin{bmatrix} C & D & -I_{n_p} \end{bmatrix} V \begin{bmatrix} C & D & -I_{n_p} \end{bmatrix}^* \Big) : V \in \mathcal{C}_{\text{Gram}}, \begin{bmatrix} 0_{n_q\times n_x} & 0_{n_q\times n_p} & I_{n_q} \\ 0_{n_p\times n_x} & I_{n_p} & 0_{n_p\times n_q} \end{bmatrix} V \begin{bmatrix} 0_{n_q\times n_x} & 0_{n_q\times n_p} & I_{n_q} \\ 0_{n_p\times n_x} & I_{n_p} & 0_{n_p\times n_q} \end{bmatrix}^* \in \mathcal{F}_{\mathbf{\Delta}} \Big\}$$

where

$$\begin{aligned} \mathcal{F}_{\mathbf{\Delta}} = \{ Z \in \mathbb{H} : \mathcal{A}_{i\mathbf{\Delta}}(Z) \succeq 0, \quad & i = 1, \cdots, n_a, \\ \mathcal{B}_{i\mathbf{\Delta}}(Z) = 0, \quad & i = 1, \cdots, n_b, \\ & \mathbf{Tr}(Z_{11}) = 1 \}. \end{aligned}$$

Here, $\mathcal{F}_{\mathbf{\Delta}}$ is being used for the constraint $(\mathbf{p}, \mathbf{q}) \in \mathcal{R}_{\mathbf{\Delta}}$ and $\|\mathbf{p}\|_2^2 = \mathbf{Tr}(Z_{11}) = 1$.

Notice that we only have the constraint on

$$\begin{bmatrix} 0_{n_q\times n_x} & 0_{n_q\times n_p} & I_{n_q} \\ 0_{n_p\times n_x} & I_{n_p} & 0_{n_p\times n_q} \end{bmatrix} V \begin{bmatrix} 0_{n_q\times n_x} & 0_{n_q\times n_p} & I_{n_q} \\ 0_{n_p\times n_x} & I_{n_p} & 0_{n_p\times n_q} \end{bmatrix}^*,$$

and this constraint does not contain $\mathbf{x}$. Therefore, from Proposition 4.4, we can conclude that the closure of the above set is given by replacing $\mathcal{D}_{\text{Gram}}$ by $\mathcal{D}_{\text{SDP}}$, and $\mathcal{C}_{\text{Gram}}$ by $\mathcal{C}_{\text{SDP}}$, respectively.

Proposition 5.1: *The closure of the set*

$$\{V : V \in \mathcal{D}_{Gram}, \begin{bmatrix} 0_{n_q\times n_x} & 0_{n_q\times n_p} & I_{n_q} \\ 0_{n_p\times n_x} & I_{n_p} & 0_{n_p\times n_q} \end{bmatrix} V \begin{bmatrix} 0_{n_q\times n_x} & 0_{n_q\times n_p} & I_{n_q} \\ 0_{n_p\times n_x} & I_{n_p} & 0_{n_p\times n_q} \end{bmatrix}^* \in \mathcal{F}_{\mathbf{\Delta}}\}$$

is given by

$$\{V : V \in \mathcal{D}_{SDP}, \begin{bmatrix} 0_{n_q\times n_x} & 0_{n_q\times n_p} & I_{n_q} \\ 0_{n_p\times n_x} & I_{n_p} & 0_{n_p\times n_q} \end{bmatrix} V \begin{bmatrix} 0_{n_q\times n_x} & 0_{n_q\times n_p} & I_{n_q} \\ 0_{n_p\times n_x} & I_{n_p} & 0_{n_p\times n_q} \end{bmatrix}^* \in \mathcal{F}_{\mathbf{\Delta}}\}$$

Similarly, the closure of the set

$$\{V : V \in \mathcal{C}_{Gram}, \begin{bmatrix} 0_{n_q\times n_x} & 0_{n_q\times n_p} & I_{n_q} \\ 0_{n_p\times n_x} & I_{n_p} & 0_{n_p\times n_q} \end{bmatrix} V \begin{bmatrix} 0_{n_q\times n_x} & 0_{n_q\times n_p} & I_{n_q} \\ 0_{n_p\times n_x} & I_{n_p} & 0_{n_p\times n_q} \end{bmatrix}^* \in \mathcal{F}_{\boldsymbol{\Delta}}\}$$

is given by

$$\{V : V \in \mathcal{C}_{Gram}, \begin{bmatrix} 0_{n_q\times n_x} & 0_{n_q\times n_p} & I_{n_q} \\ 0_{n_p\times n_x} & I_{n_p} & 0_{n_p\times n_q} \end{bmatrix} V \begin{bmatrix} 0_{n_q\times n_x} & 0_{n_q\times n_p} & I_{n_q} \\ 0_{n_p\times n_x} & I_{n_p} & 0_{n_p\times n_q} \end{bmatrix}^* \in \mathcal{F}_{\boldsymbol{\Delta}}\}$$

PROOF: Let us partition $V = \begin{bmatrix} X & R \\ R^* & W \end{bmatrix}$ where $X \in \mathbb{C}^{n_x\times n_x}$, $R \in \mathbb{C}^{n_x\times(n_q+n_p)}$, $W \in \mathbb{C}^{(n_q+n_p)\times(n_q+n_p)}$. Then the constraint

$$\begin{bmatrix} 0_{n_q\times n_x} & 0_{n_q\times n_p} & I_{n_q} \\ 0_{n_p\times n_x} & I_{n_p} & 0_{n_p\times n_q} \end{bmatrix} V \begin{bmatrix} 0_{n_q\times n_x} & 0_{n_q\times n_p} & I_{n_q} \\ 0_{n_p\times n_x} & I_{n_p} & 0_{n_p\times n_q} \end{bmatrix}^* \in \mathcal{F}_{\boldsymbol{\Delta}}$$

can be modeled using $g_i(W) = \mathcal{A}_{i\boldsymbol{\Delta}}\left(\begin{bmatrix} 0_{n_p\times n_q} & I_{n_p} \\ I_{n_q} & 0_{n_q\times n_p} \end{bmatrix} W \begin{bmatrix} 0_{n_p\times n_q} & I_{n_p} \\ I_{n_q} & 0_{n_q\times n_p} \end{bmatrix}\right)$ for $i = 1, \cdots, n_a$, and $g_{j+n_a}(W) = \mathcal{B}_{i\boldsymbol{\Delta}}\left(\begin{bmatrix} 0_{n_p\times n_q} & I_{n_p} \\ I_{n_q} & 0_{n_q\times n_p} \end{bmatrix} W \begin{bmatrix} 0_{n_p\times n_q} & I_{n_p} \\ I_{n_q} & 0_{n_q\times n_p} \end{bmatrix}\right)$, $g_{j+n_a+n_b} = -\mathcal{B}_{i\boldsymbol{\Delta}}\left(\begin{bmatrix} 0_{n_p\times n_q} & I_{n_p} \\ I_{n_q} & 0_{n_q\times n_p} \end{bmatrix} W \begin{bmatrix} 0_{n_p\times n_q} & I_{n_p} \\ I_{n_q} & 0_{n_q\times n_p} \end{bmatrix}\right)$ for $j = 1, \cdots, n_b$. Since there is no f_i in this modeling, from Proposition 4.4, we can conclude the proof. ∎

This remarkable feature allows us to compute $\eta_{\boldsymbol{\Delta}}(\mathbf{M})^2$ using the following SDP.

$$\begin{aligned}
\underset{V\succeq 0}{\text{minimize}} \quad & \mathbf{Tr}\Big(\begin{bmatrix} C & D & -I_{n_p}\end{bmatrix} V \begin{bmatrix} C & D & -I_{n_p}\end{bmatrix}^*\Big) \\
\text{subject to} \quad & \Pi\left(\begin{bmatrix} A & B & 0_{n_x\times n_p} \\ I_{n_x} & 0_{n_x\times n_q} & 0_{n_x\times n_p}\end{bmatrix} V \begin{bmatrix} A & B & 0_{n_x\times n_p} \\ I_{n_x} & 0_{n_x\times n_q} & 0_{n_x\times n_p}\end{bmatrix}^*\right) = 0, \\
& \mathbf{Tr}\left(\begin{bmatrix} 0_{(n_x+n_q)\times(n_x+n_q)} & 0_{(n_x+n_q)\times n_p} \\ 0_{n_p\times(n_x+n_q)} & I_{n_p}\end{bmatrix} V\right) = 1, \\
\mathcal{A}_{i\boldsymbol{\Delta}} & \left(\begin{bmatrix} 0_{n_q\times n_x} & 0_{n_q\times n_p} & I_{n_q} \\ 0_{n_p\times n_x} & I_{n_p} & 0_{n_p\times n_q}\end{bmatrix} V \begin{bmatrix} 0_{n_q\times n_x} & 0_{n_q\times n_p} & I_{n_q} \\ 0_{n_p\times n_x} & I_{n_p} & 0_{n_p\times n_q}\end{bmatrix}^*\right) \succeq 0, \text{ for } i = 1, \cdots n_a \\
\mathcal{B}_{i\boldsymbol{\Delta}} & \left(\begin{bmatrix} 0_{n_q\times n_x} & 0_{n_q\times n_p} & I_{n_q} \\ 0_{n_p\times n_x} & I_{n_p} & 0_{n_p\times n_q}\end{bmatrix} V \begin{bmatrix} 0_{n_q\times n_x} & 0_{n_q\times n_p} & I_{n_q} \\ 0_{n_p\times n_x} & I_{n_p} & 0_{n_p\times n_q}\end{bmatrix}^*\right) = 0, \text{ for } i = 1, \cdots n_b,
\end{aligned} \tag{5.2}$$

where Π is from Table 4.1 in Chapter 4,

$$\Pi\left(\begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22}\end{bmatrix}\right) = \begin{cases} X_{11} - X_{22} & \text{Discrete time case} \\ X_{12} + X_{12}^* & \text{Continuous time case}\end{cases}$$

with $X_{11} \in \mathbb{C}^{n_x\times n_x}$, $X_{12} \in \mathbb{C}^{n_x\times(n_q+n_p)}$, $X_{22} \in \mathbb{C}^{(n_q+n_p)\times(n_q+n_p)}$.

Therefore, if the optimal value of (5.2) is greater than 0, we can conclude that $\mathbf{M}$ is robustly well-connected to $\boldsymbol{\Delta}$. In addition, $\eta_{\boldsymbol{\Delta}}(\mathbf{M})$ can be obtained using SDP, and $\eta_{\boldsymbol{\Delta}}(\mathbf{M})$ quantifies the distance from not being robustly well-connected. In other words, if $\eta_{\boldsymbol{\Delta}}(\mathbf{M})$ is not zero, but very small, then we may conclude that the feedback configuration is practically fragile, because there exists $\Delta \in \boldsymbol{\Delta}$ that makes $\mathbf{I} - \mathbf{M}\Delta$ almost singular.

In summary, our argument shows that if the relationship $\mathbf{q} = \boldsymbol{\Delta}(\mathbf{p})$ can be completely characterized by the Gramian $\Lambda(\mathbf{p}, \mathbf{q})$, then the corresponding well-connectedness problem becomes an finite dimensional SDP (5.2), although the original problem, $\eta_{\boldsymbol{\Delta}}(\mathbf{M})$, is seemingly impossible to solve.

5.2 LMI test for well-connectedness

Recall that $\eta_{\mathbf{\Delta}}(\mathbf{M})$ can be *exactly* computed using the optimization (5.2). In this section, we derive an LMI test for checking $\eta_{\mathbf{\Delta}}(\mathbf{M}) > 0$ using SDP duality.

5.2.1 SDP dual problem

Let us derive the SDP dual of (5.2). Firstly, from the Lagrangian

$$
\begin{aligned}
\mathcal{L}(P,t,Y_i,S_i) &= \mathbf{Tr}\left(\begin{bmatrix} C & D & -I_{n_p}\end{bmatrix} V \begin{bmatrix} C & D & -I_{n_p}\end{bmatrix}^*\right) \\
&\quad +t+\mathbf{Tr}\left(\begin{bmatrix} 0_{(n_x+n_q)\times(n_x+n_q)} & 0_{(n_x+n_q)\times n_p} \\ 0_{n_p\times(n_x+n_q)} & -tI_{n_p}\end{bmatrix} V\right) \\
&\quad +\mathbf{Tr}\left(P\Pi\left(\begin{bmatrix} A & B & 0_{n_x\times n_p} \\ I_{n_x} & 0_{n_x\times n_q} & 0_{n_x\times n_p}\end{bmatrix} V \begin{bmatrix} A & B & 0_{n_x\times n_p} \\ I_{n_x} & 0_{n_x\times n_q} & 0_{n_x\times n_p}\end{bmatrix}^*\right)\right) \\
&\quad -\sum_{i=1}^{n_a}\mathbf{Tr}\left(Y_i\mathcal{A}_{i\mathbf{\Delta}}\left(\begin{bmatrix} 0_{n_q\times n_x} & 0_{n_q\times n_p} & I_{n_q} \\ 0_{n_p\times n_x} & I_{n_p} & 0_{n_p\times n_q}\end{bmatrix} V \begin{bmatrix} 0_{n_q\times n_x} & 0_{n_q\times n_p} & I_{n_q} \\ 0_{n_p\times n_x} & I_{n_p} & 0_{n_p\times n_q}\end{bmatrix}^*\right)\right) \\
&\quad -\sum_{i=1}^{n_b}\mathbf{Tr}\left(Q_i\mathcal{B}_{i\mathbf{\Delta}}\left(\begin{bmatrix} 0_{n_q\times n_x} & 0_{n_q\times n_p} & I_{n_q} \\ 0_{n_p\times n_x} & I_{n_p} & 0_{n_p\times n_q}\end{bmatrix} V \begin{bmatrix} 0_{n_q\times n_x} & 0_{n_q\times n_p} & I_{n_q} \\ 0_{n_p\times n_x} & I_{n_p} & 0_{n_p\times n_q}\end{bmatrix}^*\right)\right) \\
&= t+\mathbf{Tr}\left(\begin{bmatrix} C & D & -I_{n_p}\end{bmatrix}^* \begin{bmatrix} C & D & -I_{n_p}\end{bmatrix} V\right) \\
&\quad +\mathbf{Tr}\left(\begin{bmatrix} 0_{(n_x+n_q)\times(n_x+n_q)} & 0_{(n_x+n_q)\times n_p} \\ 0_{n_p\times(n_x+n_q)} & -tI_{n_p}\end{bmatrix} V\right) \\
&\quad +\mathbf{Tr}\left(\begin{bmatrix} A & B & 0_{n_x\times n_p} \\ I_{n_x} & 0_{n_x\times n_q} & 0_{n_x\times n_p}\end{bmatrix}^* (\Psi\otimes P) \begin{bmatrix} A & B & 0_{n_x\times n_p} \\ I_{n_x} & 0_{n_x\times n_q} & 0_{n_x\times n_p}\end{bmatrix} V\right) \\
&\quad -\sum_{i=1}^{n_a}\mathbf{Tr}\left(\begin{bmatrix} 0_{n_q\times n_x} & 0_{n_q\times n_p} & I_{n_q} \\ 0_{n_p\times n_x} & I_{n_p} & 0_{n_p\times n_q}\end{bmatrix}^* \mathcal{A}_{i\mathbf{\Delta}}^*(Y_i) \begin{bmatrix} 0_{n_q\times n_x} & 0_{n_q\times n_p} & I_{n_q} \\ 0_{n_p\times n_x} & I_{n_p} & 0_{n_p\times n_q}\end{bmatrix} V\right) \\
&\quad -\sum_{i=1}^{n_b}\mathbf{Tr}\left(\begin{bmatrix} 0_{n_q\times n_x} & 0_{n_q\times n_p} & I_{n_q} \\ 0_{n_p\times n_x} & I_{n_p} & 0_{n_p\times n_q}\end{bmatrix}^* \mathcal{B}_{i\mathbf{\Delta}}^*(Q_i) \begin{bmatrix} 0_{n_q\times n_x} & 0_{n_q\times n_p} & I_{n_q} \\ 0_{n_p\times n_x} & I_{n_p} & 0_{n_p\times n_q}\end{bmatrix} V\right),
\end{aligned}
$$

where Ψ is from Table 4.1 in Chapter 4,

$$\Psi = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & \text{(Discrete time case)} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \text{(Continuous time case)}, \end{cases}$$

and $Y_i \succeq 0$. This shows that the dual problem of (5.2) is given by,

$$\begin{aligned}
&\underset{t,P=P^*,Y_i\succeq 0,Q_i=Q_i^*}{\text{maximize}} \quad t \\
&\text{subject to} \quad \begin{bmatrix} C & D & -I_{n_p} \end{bmatrix}^* \begin{bmatrix} C & D & -I_{n_p} \end{bmatrix} + \begin{bmatrix} 0_{(n_x+n_q)\times(n_x+n_q)} & 0_{(n_x+n_q)\times n_p} \\ 0_{n_p\times(n_x+n_q)} & -tI_{n_p} \end{bmatrix} \\
&+ \begin{bmatrix} A & B & 0_{n_x\times n_p} \\ I_{n_x} & 0_{n_x\times n_q} & 0_{n_x\times n_p} \end{bmatrix}^* (\Psi \otimes P) \begin{bmatrix} A & B & 0_{n_x\times n_p} \\ I_{n_x} & 0_{n_x\times n_q} & 0_{n_x\times n_p} \end{bmatrix} \\
&- \sum_{i=1}^{n_a} \begin{bmatrix} 0_{n_q\times n_x} & 0_{n_q\times n_p} & I_{n_q} \\ 0_{n_p\times n_x} & I_{n_p} & 0_{n_p\times n_q} \end{bmatrix}^* \mathcal{A}_{i\boldsymbol{\Delta}}^*(Y_i) \begin{bmatrix} 0_{n_q\times n_x} & 0_{n_q\times n_p} & I_{n_q} \\ 0_{n_p\times n_x} & I_{n_p} & 0_{n_p\times n_q} \end{bmatrix} \\
&- \sum_{i=1}^{n_b} \begin{bmatrix} 0_{n_q\times n_x} & 0_{n_q\times n_p} & I_{n_q} \\ 0_{n_p\times n_x} & I_{n_p} & 0_{n_p\times n_q} \end{bmatrix}^* \mathcal{B}_{i\boldsymbol{\Delta}}^*(Q_i) \begin{bmatrix} 0_{n_q\times n_x} & 0_{n_q\times n_p} & I_{n_q} \\ 0_{n_p\times n_x} & I_{n_p} & 0_{n_p\times n_q} \end{bmatrix} \succeq 0.
\end{aligned} \tag{5.3}$$

Suppose there exists a dual feasible point $(t, P, Y_1, \cdots, Y_{n_a}, Q_1, \cdots, Q_{n_b})$ of (5.3) such that $t > 0$, then the optimal value of (5.3) is greater than 0. From weak duality, we can immediately conclude that $\eta_{\boldsymbol{\Delta}}(\mathbf{M}) > 0$ and a system $\mathbf{M}$ is robustly well-connected to the uncertainty $\boldsymbol{\Delta}$. Since finding a dual feasible point is an LMI problem, we have a *sufficient* LMI test for the well-connectedness analysis. However, this is not in the usual form, *e.g.*, a scaled small gain test such as [74, 19, 57, 48], and the dimension of this LMI test is larger than the usual scaled small gain test. Moreover, this LMI test is only sufficient, not necessary. In the following section, we derive a sufficient and necessary LMI test for $\eta_{\boldsymbol{\Delta}}(\mathbf{M}) > 0$ using the theorem of alternatives, and this recovers the scaled small gain test in [74, 19, 57, 48].

5.2.2 An LMI test for well-connectedness

Although (5.3) provides an LMI test for $\eta_{\mathbf{\Delta}}(\mathbf{M}) > 0$, using the theorem of alternatives, we can simplify the LMI test even further. Firstly, let us consider the following result.

Proposition 5.2: $\eta_{\mathbf{\Delta}}(\mathbf{M}) = 0$ *if and only if there exists* V *in the feasible set of* (5.2) *such that*

$$\boldsymbol{Tr}\left(\begin{bmatrix} C & D & -I_{n_p} \end{bmatrix} V \begin{bmatrix} C & D & -I_{n_p} \end{bmatrix}^*\right) = 0. \tag{5.4}$$

The proof can be found in the appendix, and the underlying idea is to show the compactness of the feasible set. The above proposition states that the optimization (5.2) is solvable. Based on this result, we can find a low dimensional representation of V which achieves $\eta_{\mathbf{\Delta}}(\mathbf{M}) = 0$, since (5.4) introduces the redundancy in V.

Proposition 5.3: *For all* V *in the feasible set of* (5.2) *such that* (5.4) *holds, there exists* $W \succeq 0$ *in* $\mathbb{H}^{n_x+n_q}$ *such that*

$$V = \begin{bmatrix} I_{n_x} & 0_{n_x \times n_q} \\ 0_{n_q \times n_x} & I_{n_q} \\ C & D \end{bmatrix} W \begin{bmatrix} I_{n_x} & 0_{n_x \times n_q} \\ 0_{n_q \times n_x} & I_{n_q} \\ C & D \end{bmatrix}^*.$$

The proof can be found in the appendix. Using this representation of V, we have

$$\begin{aligned}
&\begin{bmatrix} A & B & 0_{n_x \times n_p} \\ I_{n_x} & 0_{n_x \times n_q} & 0_{n_x \times n_p} \end{bmatrix} V \begin{bmatrix} A & B & 0_{n_x \times n_p} \\ I_{n_x} & 0_{n_x \times n_q} & 0_{n_x \times n_p} \end{bmatrix}^* \\
&= \begin{bmatrix} A & B & 0_{n_x \times n_p} \\ I_{n_x} & 0_{n_x \times n_q} & 0_{n_x \times n_p} \end{bmatrix} \begin{bmatrix} I_{n_x} & 0_{n_x \times n_q} \\ 0_{n_q \times n_x} & I_{n_q} \\ C & D \end{bmatrix} W \begin{bmatrix} I_{n_x} & 0_{n_x \times n_q} \\ 0_{n_q \times n_x} & I_{n_q} \\ C & D \end{bmatrix}^* \begin{bmatrix} A & B & 0_{n_x \times n_p} \\ I_{n_x} & 0_{n_x \times n_q} & 0_{n_x \times n_p} \end{bmatrix}^* \\
&= \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_q} \end{bmatrix} W \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_q} \end{bmatrix}^*.
\end{aligned}$$

$$\begin{bmatrix} 0_{n_q\times n_x} & 0_{n_q\times n_p} & I_{n_q} \\ 0_{n_p\times n_x} & I_{n_p} & 0_{n_p\times n_q} \end{bmatrix} V \begin{bmatrix} 0_{n_q\times n_x} & 0_{n_q\times n_p} & I_{n_q} \\ 0_{n_p\times n_x} & I_{n_p} & 0_{n_p\times n_q} \end{bmatrix}^*$$

$$= \begin{bmatrix} 0_{n_q\times n_x} & 0_{n_q\times n_p} & I_{n_q} \\ 0_{n_p\times n_x} & I_{n_p} & 0_{n_p\times n_q} \end{bmatrix} \begin{bmatrix} I_{n_x} & 0_{n_x\times n_q} \\ 0_{n_q\times n_x} & I_{n_q} \\ C & D \end{bmatrix} W \begin{bmatrix} I_{n_x} & 0_{n_x\times n_q} \\ 0_{n_q\times n_x} & I_{n_q} \\ C & D \end{bmatrix}^* \begin{bmatrix} 0_{n_q\times n_x} & 0_{n_q\times n_p} & I_{n_q} \\ 0_{n_p\times n_x} & I_{n_p} & 0_{n_p\times n_q} \end{bmatrix}^*$$

$$= \begin{bmatrix} C & D \\ 0_{n_q\times n_x} & I_{n_q} \end{bmatrix} W \begin{bmatrix} C & D \\ 0_{n_q\times n_x} & I_{n_q} \end{bmatrix}^* .$$

$$\begin{aligned} &\begin{bmatrix} 0_{n_p\times(n_x+n_q)} & I_{n_p} \end{bmatrix} V \begin{bmatrix} 0_{n_p\times(n_x+n_q)} & I_{n_p} \end{bmatrix}^* \\ &= \begin{bmatrix} 0_{n_p\times(n_x+n_q)} & I_{n_p} \end{bmatrix} \begin{bmatrix} I_{n_x} & 0_{n_x\times n_q} \\ 0_{n_q\times n_x} & I_{n_q} \\ C & D \end{bmatrix} W \begin{bmatrix} I_{n_x} & 0_{n_x\times n_q} \\ 0_{n_q\times n_x} & I_{n_q} \\ C & D \end{bmatrix}^* \begin{bmatrix} 0_{n_p\times(n_x+n_q)} & I_{n_p} \end{bmatrix}^* \\ &= \begin{bmatrix} C & D \end{bmatrix} W \begin{bmatrix} C & D \end{bmatrix}^* . \end{aligned}$$

Therefore, $\eta_{\mathbf{\Delta}}(\mathbf{M}) = 0$ if and only there exists $W \succeq 0$ such that

$$\Pi\left(\begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x\times n_q} \end{bmatrix} W \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x\times n_q} \end{bmatrix}^*\right) = 0 \tag{5.5}$$

$$\mathcal{A}_{i\mathbf{\Delta}}\left(\begin{bmatrix} C & D \\ 0_{n_q\times n_x} & I_{n_q} \end{bmatrix} W \begin{bmatrix} C & D \\ 0_{n_q\times n_x} & I_{n_q} \end{bmatrix}^*\right) \succeq 0, \text{ for } i = 1, \cdots, n_a \tag{5.6}$$

$$\mathcal{B}_{i\mathbf{\Delta}}\left(\begin{bmatrix} C & D \\ 0_{n_q\times n_x} & I_{n_q} \end{bmatrix} W \begin{bmatrix} C & D \\ 0_{n_q\times n_x} & I_{n_q} \end{bmatrix}^*\right) = 0, \text{ for } i = 1, \cdots, n_b \tag{5.7}$$

$$\mathbf{Tr}\left(\begin{bmatrix} C & D \end{bmatrix} W \begin{bmatrix} C & D \end{bmatrix}^*\right) = 1. \tag{5.8}$$

Recall that our goal is to find a condition for $\eta_{\mathbf{\Delta}}(\mathbf{M}) > 0$. Since existence of W satisfying (5.5) - (5.8) guarantees $\eta_{\mathbf{\Delta}}(\mathbf{M}) = 0$, if one can prove that there exists no such W, then $\eta_{\mathbf{\Delta}}(\mathbf{M}) > 0$, *i.e.*, the system $\mathbf{M}$ is robustly well-connected to the uncertain set $\mathbf{\Delta}$. Therefore it is natural to apply the theorem of alternative to find a condition for non-existence of W. Before applying the theorem of alternative, let us firstly replace the condition (5.8) with more appropriate form. Notice that all conditions (5.5) - (5.7) are homogeneous in W. Therefore if there exists $W \succeq 0$ such that $\mathbf{Tr}\left(\begin{bmatrix} C & D \end{bmatrix} W \begin{bmatrix} C & D \end{bmatrix}^*\right) > 0$ then we can rescale W so that W satisfies (5.5) - (5.8). On the contrary, if $\mathbf{Tr}\left(\begin{bmatrix} C & D \end{bmatrix} W \begin{bmatrix} C & D \end{bmatrix}^*\right) = 0$, then W has to be zero according to the following proposition.

Proposition 5.4: *If* $W \succeq 0$ *satisfies* (5.5) - (5.7) *and* $\mathbf{Tr}\left(\begin{bmatrix} C & D \end{bmatrix} W \begin{bmatrix} C & D \end{bmatrix}^*\right) = 0$, *then* $W = 0$.

PROOF: Since $W \succeq 0$ and $\mathbf{Tr}\left(\begin{bmatrix} C & D \end{bmatrix} W \begin{bmatrix} C & D \end{bmatrix}^*\right) = 0$, we can easily see that

$$\begin{bmatrix} C & D \end{bmatrix} W = 0,$$

which implies

$$\begin{bmatrix} C & D \\ 0_{n_q \times n_x} & I_{n_q} \end{bmatrix} W \begin{bmatrix} C & D \\ 0_{n_q \times n_x} & I_{n_q} \end{bmatrix}^* = \begin{bmatrix} 0_{n_x \times n_x} & 0_{n_x \times n_q} \\ 0_{n_q \times n_x} & W_{22} \end{bmatrix},$$

where W_{22} is the (2,2) block of W. Since

$$\mathcal{A}_{i\mathbf{\Delta}}\left(\begin{bmatrix} 0_{n_x \times n_x} & 0_{n_x \times n_q} \\ 0_{n_q \times n_x} & W_{22} \end{bmatrix}\right) \succeq 0, \mathcal{B}_{i\mathbf{\Delta}}\left(\begin{bmatrix} 0_{n_x \times n_x} & 0_{n_x \times n_q} \\ 0_{n_q \times n_x} & W_{22} \end{bmatrix}\right) = 0$$

and from the requirement on the operators $(\mathcal{A}_{i\mathbf{\Delta}}, \mathcal{B}_{i\mathbf{\Delta}})$, (5.1), we have $W_{22} = 0$. Moreover since $W \succeq 0$, from Schur complement, we can easily see that W should have the form $W = \begin{bmatrix} W_{11} & 0_{n_x \times n_q} \\ 0_{n_q \times n_x} & 0_{n_q \times n_q} \end{bmatrix}$. For the discrete time case, since W satisfies (5.5), $AW_{11}A^* - W_{11} = 0$. Notice that $W_{11} = 0$ is the unique solution for this Lyapunov equation, since A is Schur stable. Similarly, for the continuous time case, we have $AW_{11} + W_{11}A^* = 0$, and $W_{11} = 0$ is the unique solution for this Lyapunov equation since A is Hurwitz stable. ∎

The above proposition states that $\mathbf{Tr}\left(\begin{bmatrix} C & D \end{bmatrix} W \begin{bmatrix} C & D \end{bmatrix}^*\right) = 1$ is essentially equivalent to the condition $W \neq 0$, which is the following result.

Proposition 5.5: *There exists* $W \succeq 0$ *such that* (5.5) - (5.8) *if and only if there exists* $W \gneq 0$ *such that* (5.5) - (5.7).

Proof: Suppose there exists $W \succeq 0$ such that (5.5) - (5.8). From Proposition 5.4, we can conclude that $W \neq 0$. For the converse direction, suppose there exists $W \gneq 0$ such that (5.5) - (5.7). Since $W \neq 0$, from Proposition 5.4, $\mathbf{Tr}\left(\begin{bmatrix} C & D \end{bmatrix} W \begin{bmatrix} C & D \end{bmatrix}^*\right) = \alpha > 0$. Then $\frac{1}{\alpha} W \succeq 0$ satisfies (5.5) - (5.8). ■

Now we are ready to apply the theorem of alternative to derive an LMI test for $\eta_{\mathbf{\Delta}}(\mathbf{M}) > 0$.

Proposition 5.6: *Exactly one of the following is true.*

(i) $\eta_{\mathbf{\Delta}}(\mathbf{M}) = 0$

(ii) There exist $Y_1, \cdots, Y_{n_a} \succ 0$, $Q_1, \cdots, Q_{n_b}$, *and* P *such that*

$$\begin{bmatrix} C & D \\ 0_{n_q \times n_x} & I_{n_q} \end{bmatrix}^* \left(\sum_{i=1}^{n_a} \mathcal{A}_{i\mathbf{\Delta}}^*(Y_i) + \sum_{i=1}^{n_b} \mathcal{B}_{i\mathbf{\Delta}}^*(Q_i)\right) \begin{bmatrix} C & D \\ 0_{n_q \times n_x} & I_{n_q} \end{bmatrix} + \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_q} \end{bmatrix}^* (\Psi \otimes P) \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_q} \end{bmatrix} \prec 0. \tag{5.9}$$

Proof: From Proposition 5.5, $\eta_{\mathbf{\Delta}}(\mathbf{M}) = 0$ if and only if there exists $W \gneq 0$ such that (5.5) - (5.7). This is equivalent to $\mathcal{A}(W) \gneq 0$, and $\mathcal{B}(W) = 0$ where

$$\mathcal{A}(W) = \begin{bmatrix} W & 0 \\ 0 & \mathcal{A}_{\mathbf{\Delta}}\left(\begin{bmatrix} C & D \\ 0_{n_q \times n_x} & I_{n_q} \end{bmatrix} W \begin{bmatrix} C & D \\ 0_{n_q \times n_x} & I_{n_q} \end{bmatrix}^*\right) \end{bmatrix}$$

$$\mathcal{B}(W) = \begin{bmatrix} \Pi\left(\begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_q} \end{bmatrix} W \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_q} \end{bmatrix}^*\right) & 0 \\ 0 & \mathcal{B}_{\mathbf{\Delta}}\left(\begin{bmatrix} C & D \\ 0_{n_q \times n_x} & I_{n_q} \end{bmatrix} W \begin{bmatrix} C & D \\ 0_{n_q \times n_x} & I_{n_q} \end{bmatrix}^*\right) \end{bmatrix},$$

and

$$\begin{aligned}
\mathcal{A}_{\mathbf{\Delta}}(X) &= \mathbf{diag}\left(\mathcal{A}_{1\mathbf{\Delta}}(X), \cdots, \mathcal{A}_{n_a\mathbf{\Delta}}(X)\right) \\
\mathcal{B}_{\mathbf{\Delta}}(X) &= \mathbf{diag}\left(\mathcal{B}_{1\mathbf{\Delta}}(X), \cdots, \mathcal{B}_{n_b\mathbf{\Delta}}(X)\right).
\end{aligned}$$

Since the adjoints of $\mathcal{A}^*, \mathcal{B}^*$ are given by

$$\begin{aligned}
\mathcal{A}^*(S, Y_1, \cdots, Y_{n_a}) &= S + \begin{bmatrix} C & D \\ 0_{n_q\times n_x} & I_{n_q} \end{bmatrix}^* \left(\sum_{i=1}^{n_a} \mathcal{A}_{i\mathbf{\Delta}}^*(Y_i)\right) \begin{bmatrix} C & D \\ 0_{n_q\times n_x} & I_{n_q} \end{bmatrix} \\
\mathcal{B}^*(P, Q_1, \cdots, Q_{n_b}) &= \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x\times n_q} \end{bmatrix}^* (\Psi \otimes P) \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x\times n_q} \end{bmatrix} \\
&\quad + \begin{bmatrix} C & D \\ 0_{n_q\times n_x} & I_{n_q} \end{bmatrix}^* \left(\sum_{i=1}^{n_b} \mathcal{B}_{i\mathbf{\Delta}}^*(Q_i)\right) \begin{bmatrix} C & D \\ 0_{n_q\times n_x} & I_{n_q} \end{bmatrix},
\end{aligned}$$

from the theorem of alternative, Corollary 2.4, the strong alternative of (i) is that there exist $S \succ 0$, $Y_i \succ 0$, and P, Q_i such that

$$\begin{aligned}
& S + \begin{bmatrix} C & D \\ 0_{n_q\times n_x} & I_{n_q} \end{bmatrix}^* \left(\sum_{i=1}^{n_a} \mathcal{A}_{i\mathbf{\Delta}}^*(Y_i) + \sum_{i=1}^{n_b} \mathcal{B}_{i\mathbf{\Delta}}^*(Q_i)\right) \begin{bmatrix} C & D \\ 0_{n_q\times n_x} & I_{n_q} \end{bmatrix} \\
& + \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x\times n_q} \end{bmatrix}^* (\Psi \otimes P) \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x\times n_q} \end{bmatrix} = 0.
\end{aligned}$$

By eliminating S, we obtain (ii). ■

Since $\mathbf{M}$ is *not* robustly well-connected with $\mathbf{\Delta}$ if (i) holds and vice versa, we have the following consequence on robust well-connectedness.

Corollary 5.1: *The system* $\mathbf{M}$ *is robustly stable with respect to* $\mathbf{\Delta}$ *if and only if there exist* $Y_1, \cdots, Y_{n_a}$, $Q_1, \cdots, Q_{n_b}$, *and* P *satisfying* (5.9).

Unlike the SDP dual approach in the previous section, the above LMI test (5.9) is sufficient and necessary. In addition, the total dimension of LMI is smaller than the test from the dual program (5.3). More importantly, the LMI test (5.9) includes the existing results as a special case, as we will see in the following examples.

5.3 Examples of $\mathbf{\Delta}$

In this section, we provide examples of $\mathbf{\Delta}$ and how $\mathcal{A}_{\mathbf{\Delta}}$, $\mathcal{B}_{\mathbf{\Delta}}$ is being used to describe this set. All the proofs can be found in the appendix. In addition, we also derive the LMI test (5.9) to illustrate the procedure. Some of these results turn out to be the scaled small gain test [74, 19, 57, 48], a well-known LMI test for robust well-connectedness analysis. In this point of view, Proposition 5.6 provides the proof of well-known results while making a greater extension to the general $\mathbf{\Delta}$ that can be expressed using Gramians.

5.3.1 Full block Linear Time Varying Uncertainty

Consider the case when $\mathbf{\Delta} = \{\Delta : \|\Delta\|_{2,2} \leq 1\}$, *i.e.*, arbitrary uncertain linear operator with the norm condition. Here $\|\Delta\|_{2,2} = \sup_{\|\mathbf{x}\|_2=1} \|\Delta(\mathbf{x})\|_2$.

The following proposition provides the set description of $\mathbf{\Delta}$.

Proposition 5.7: *[26] There exists* $\|\Delta\|_{2,2} \leq 1$ *such that* $\mathbf{q} = \Delta\mathbf{p}$ *if and only if* $\|\mathbf{q}\|_2 \leq \|\mathbf{p}\|_2$.

In fact, the convolution operator can be used to describe Δ. For simplicity, consider the discrete time case with $\|\mathbf{p}\|_2 = 1$. Let $\mathbf{s} = \Delta(\mathbf{r}) = \langle \mathbf{p}, \mathbf{r} \rangle \mathbf{q}$. Then

$$s[k] = \sum_{i=0}^{\infty} p[i]^* r[i] q[k] = \sum_{i=0}^{\infty} q[k] p[i]^* r[i] = \sum_{i=0}^{\infty} H[k,i] r[i],$$

where $H[k,i] = q[k]p[i]^*$. Unlike the time invariant convolution kernel which only depends on the difference, $k - i$, the kernel $H[k,i]$ depends on k and i. This is the reason why such Δ block is called the Linear Time Varying (LTV) uncertainty block, because its convolution kernel is time-varying.

Now, $\mathcal{A}_{\mathbf{\Delta}}(Z) = \mathbf{Tr}(Z_{11}) - \mathbf{Tr}(Z_{22})$, and the adjoint is given by $\mathcal{A}_{\mathbf{\Delta}}^*(y) = \begin{bmatrix} yI_{n_p} & 0_{n_p \times n_q} \\ 0_{n_q \times n_p} & -yI_{n_q} \end{bmatrix}$. The LMI test (5.9) becomes

$$\begin{bmatrix} C & D \\ 0_{n_q \times n_x} & I_{n_q} \end{bmatrix}^* \begin{bmatrix} yI_{n_p} & 0_{n_p \times n_q} \\ 0_{n_q \times n_p} & -yI_{n_q} \end{bmatrix} \begin{bmatrix} C & D \\ 0_{n_q \times n_x} & I_{n_q} \end{bmatrix} + \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_q} \end{bmatrix}^* (\Psi \otimes P) \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_q} \end{bmatrix} \prec 0,$$

for some $y > 0$, and $P = P^*$.

By dividing y both sides and expanding the first term, we obtain

$$\begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D - I_{n_q} \end{bmatrix} + \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_q} \end{bmatrix}^* (\Psi \otimes P) \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_q} \end{bmatrix} \prec 0,$$

which is precisely the $\mathcal{H}_\infty$ norm condition, $\|\mathbf{M}\|_\infty < 1$. This connection is pointed out in the seminal paper from Zames [93].

5.3.2 Self-adjoint LTV

Consider $\mathbf{\Delta} = \{\Delta : \Delta = \Delta^*, \|\Delta\|_{2,2} \leq 1\}$. Here Δ is a self-adjoint operator, and the following proposition provides a set description of $\mathbf{\Delta}$.

Proposition 5.8: *There exists* $\|\Delta\|_{2,2} \leq 1, \Delta = \Delta^*$ *such that* $\mathbf{q} = \Delta \mathbf{p}$ *if and only if* $\|\mathbf{q}\|_2 \leq \|\mathbf{p}\|_2$ *and* $\langle \mathbf{p}, \mathbf{q} \rangle = \langle \mathbf{q}, \mathbf{p} \rangle$.

From the above proposition, $\mathcal{A}_{\mathbf{\Delta}}(Z) = \mathbf{Tr}(Z_{11} - Z_{22})$, and $\mathcal{B}_{\mathbf{\Delta}}(Z) = \mathbf{Tr}(Z_{12} - Z_{12}^*)$, and $\mathcal{A}_{\mathbf{\Delta}}^*(y) = \begin{bmatrix} yI_{n_p} & 0_{n_p \times n_q} \\ 0_{n_q \times n_p} & -yI_{n_p} \end{bmatrix}$ and $\mathcal{B}_{\mathbf{\Delta}}^*(q) = \begin{bmatrix} 0_{n_p \times n_q} & \mathbf{i}qI_{n_p} \\ -\mathbf{i}qI_{n_q} & 0_{n_q \times n_p} \end{bmatrix}$, where $y, q \in \mathbb{R}$ and $y > 0$. Therefore, the LMI test (5.9) becomes

$$\begin{bmatrix} C & D \\ 0_{n_q \times n_x} & I_{n_q} \end{bmatrix}^* \begin{bmatrix} yI_{n_q} & \mathbf{i}qI_{n_q} \\ -\mathbf{i}qI_{n_q} & -yI_{n_q} \end{bmatrix} \begin{bmatrix} C & D \\ 0_{n_q \times n_x} & I_{n_q} \end{bmatrix} + \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_q} \end{bmatrix}^* (\Psi \otimes P) \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_q} \end{bmatrix} \prec 0.$$

By expanding the first term and dividing y, we have

$$\begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D - I_{n_q} \end{bmatrix} + \mathbf{i}q \begin{bmatrix} 0_{n_q \times n_q} & C \\ -C^* & D^* - D \end{bmatrix} + \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_q} \end{bmatrix}^* (\Psi \otimes P) \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_q} \end{bmatrix} \prec 0.$$

5.3.3 Skew-Hermitian LTV

Consider $\mathbf{\Delta} = \{\Delta : \Delta + \Delta^* = 0, \|\Delta\|_{2,2} \leq 1\}$. Here Δ is a skew-Hermitian operator, and the following proposition provides a set description of $\mathbf{\Delta}$.

Proposition 5.9: *There exists* $\|\Delta\|_{2,2} \leq 1, \Delta + \Delta^* = 0$ *such that* $\mathbf{q} = \Delta\mathbf{p}$ *if and only if* $\|\mathbf{q}\|_2 \leq \|\mathbf{p}\|_2$ *and* $\langle \mathbf{p}, \mathbf{q} \rangle + \langle \mathbf{q}, \mathbf{p} \rangle = 0$.

Now, $\mathcal{A}_{\mathbf{\Delta}}(Z) = \mathbf{Tr}(Z_{11} - Z_{22})$, and $\mathcal{B}_{\mathbf{\Delta}}(Z) = \mathbf{Tr}(Z_{12} + Z_{12}^*)$, and $\mathcal{A}_{\mathbf{\Delta}}^*(y) = \begin{bmatrix} yI_{n_p} & 0_{n_p \times n_q} \\ 0_{n_q \times n_p} & -yI_{n_p} \end{bmatrix}$ and $\mathcal{B}_{\mathbf{\Delta}}^*(q) = \begin{bmatrix} 0_{n_p \times n_p} & qI_{n_p} \\ qI_{n_q} & 0_{n_q \times n_q} \end{bmatrix}$, where $y, q \in \mathbb{R}$ and $y > 0$. Therefore, (5.9) becomes

$$\begin{bmatrix} C & D \\ 0_{n_q \times n_x} & I_{n_q} \end{bmatrix}^* \begin{bmatrix} yI_{n_q} & qI_{n_q} \\ qI_{n_q} & -yI_{n_q} \end{bmatrix} \begin{bmatrix} C & D \\ 0_{n_q \times n_x} & I_{n_q} \end{bmatrix} + \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_q} \end{bmatrix}^* (\Psi \otimes P) \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_q} \end{bmatrix} \prec 0.$$

By expanding the first term and dividing y, we have

$$\begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D - I_{n_q} \end{bmatrix} + q \begin{bmatrix} 0_{n_q \times n_q} & C \\ C^* & D^* + D \end{bmatrix} + \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_q} \end{bmatrix}^* (\Psi \otimes P) \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_q} \end{bmatrix} \prec 0.$$

5.3.4 Scalar LTV

Consider the scalar uncertainty structure $\mathbf{\Delta} = \{\Delta : \Delta = \delta I, \|\Delta\|_{2,2} \leq 1\}$. Here δI is a shorthand notation for $[\mathbf{q}]_i = \delta([\mathbf{p}]_i)$ for all i, where δ is a linear operator which maps $l_2^1 \to l_2^1$ or $\mathcal{L}_2^1 \to \mathcal{L}_2^1$.

Proposition 5.10: *There exists* $\|\delta\|_{2,2} \leq 1$ *such that* $\mathbf{q} = \delta I(\mathbf{p})$ *if and only if* $\Lambda(\mathbf{q}) \preceq \Lambda(\mathbf{p})$.

Now since $\Lambda(\mathbf{p}) \succeq \Lambda(\mathbf{q})$, $\mathcal{A}_{\mathbf{\Delta}}(Z) = Z_{11} - Z_{22}$. Since the adjoint is given by $\mathcal{A}^*_{\mathbf{\Delta}}(Y) = \begin{bmatrix} Y & 0_{n_p \times n_q} \\ 0_{n_q \times n_p} & -Y \end{bmatrix}$, the test (5.9) becomes

$$\begin{bmatrix} C & D \\ 0_{n_q \times n_x} & I_{n_q} \end{bmatrix}^* \begin{bmatrix} Y & 0_{n_p \times n_q} \\ 0_{n_q \times n_p} & -Y \end{bmatrix} \begin{bmatrix} C & D \\ 0_{n_q \times n_x} & I_{n_q} \end{bmatrix} + \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_q} \end{bmatrix}^* (\Psi \otimes P) \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_q} \end{bmatrix} \prec 0,$$

where $Y \succ 0$. By left and right multiplying $\begin{bmatrix} I_{n_x} & 0_{n_x \times n_q} \\ 0_{n_q \times n_x} & Y^{-1/2} \end{bmatrix}$, we can conclude that

$$\|Y^{1/2} \mathbf{M} Y^{-1/2}\|_\infty < 1,$$

which is the scaled small gain test with a full complex matrix Y. This result can be found in [59].

5.3.5 Scalar self-adjoint LTV

Consider $\mathbf{\Delta} = \{\Delta : \Delta = \delta I, \delta = \delta^*, \|\Delta\|_{2,2} \leq 1\}$. Here δ is a self-adjoint operator.

Proposition 5.11: *The following conditions are equivalent:*

(i) $\Lambda(\mathbf{q}) \preceq \Lambda(\mathbf{p})$, $\langle [\mathbf{q}]_i, [\mathbf{p}]_j \rangle = \langle [\mathbf{p}]_i, [\mathbf{q}]_j \rangle$ *for all* (i, j).

(ii) There exists $\|\delta\|_{2,2} \leq 1$, $\delta = \delta^*$ *such that* $\mathbf{q} = \delta I(\mathbf{p})$

Now, $\mathcal{A}_{\mathbf{\Delta}}(Z) = Z_{11} - Z_{22}$, and $\mathcal{B}_{\mathbf{\Delta}}(Z) = Z_{12} - Z^*_{12}$, and $\mathcal{A}^*_{\mathbf{\Delta}}(y) = \begin{bmatrix} Y & 0_{n_p \times n_q} \\ 0_{n_q \times n_p} & -Y \end{bmatrix}$ and $\mathcal{B}^*_{\mathbf{\Delta}}(Q) = \begin{bmatrix} 0_{n_p \times n_p} & Q \\ -Q & 0_{n_q \times n_q} \end{bmatrix}$, where $Y \succ 0$, and Q is a skew-Hermitian matrix.

Therefore, the LMI test (5.9) becomes

$$\begin{bmatrix} C & D \\ 0_{n_q \times n_x} & I_{n_q} \end{bmatrix}^* \begin{bmatrix} Y & Q \\ -Q & -Y \end{bmatrix} \begin{bmatrix} C & D \\ 0_{n_q \times n_x} & I_{n_q} \end{bmatrix} + \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_q} \end{bmatrix}^* (\Psi \otimes P) \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_q} \end{bmatrix} \prec 0.$$

This result in the infinite dimensional form is appeared in [48]. In contrast, the above LMI contains the finite dimensional matrices.

5.3.6 Scalar pure imaginary LTV

Consider $\mathbf{\Delta} = \{\Delta : \Delta = \delta I, \delta + \delta^* = 0, \|\Delta\|_{2,2} \leq 1\}$. Here δ is a skew-Hermitian operator. In this case, $(\mathbf{p}, \mathbf{q}) \in \mathcal{R}_{\mathbf{\Delta}}$ if and only if $\Lambda(q) \preceq \Lambda(p)$, $\langle [\mathbf{q}]_i, [\mathbf{p}]_j \rangle + \langle [\mathbf{p}]_i, [\mathbf{q}]_j \rangle = 0$ for all (i, j). This can be shown by Proposition 5.10 with the pair $(\mathbf{p}, \mathbf{iq})$. Now, $\mathcal{A}_{\mathbf{\Delta}}(Z) = Z_{11} - Z_{22}$, and $\mathcal{B}_{\mathbf{\Delta}}(Z) = Z_{12} + Z_{12}^*$, and $\mathcal{A}_{\mathbf{\Delta}}^*(y) = \begin{bmatrix} Y & 0_{n_p \times n_q} \\ 0_{n_q \times n_p} & -Y \end{bmatrix}$ and $\mathcal{B}_{\mathbf{\Delta}}^*(Q) = \begin{bmatrix} 0_{n_p \times n_p} & Q \\ Q & 0_{n_q \times n_q} \end{bmatrix}$, where $Y \succ 0$, and Q is Hermitian.

Therefore, (5.9) becomes

$$\begin{bmatrix} C & D \\ 0_{n_q \times n_x} & I_{n_q} \end{bmatrix}^* \begin{bmatrix} Y & Q \\ Q & -Y \end{bmatrix} \begin{bmatrix} C & D \\ 0_{n_q \times n_x} & I_{n_q} \end{bmatrix} + \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_q} \end{bmatrix}^* (\Psi \otimes P) \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_q} \end{bmatrix} \prec 0.$$

5.3.7 Integral Quadratic Constraints

Consider the case where $(\mathbf{p}, \mathbf{q}) \in \mathcal{R}_{\mathbf{\Delta}}$ can be represented by the integral quadratic constraints

$$\text{(Discrete-time):} \qquad \sum_{k=0}^{\infty} \begin{bmatrix} p[k] \\ q[k] \end{bmatrix}^* H \begin{bmatrix} p[k] \\ q[k] \end{bmatrix} \geq 0$$

$$\text{(Continuous-time):} \qquad \int_{t=0}^{\infty} \begin{bmatrix} p(t) \\ q(t) \end{bmatrix}^* H \begin{bmatrix} p(t) \\ q(t) \end{bmatrix} dt \geq 0$$

for some constant matrix H and the norm bound condition, $\|\mathbf{q}\|_2 \leq \|\mathbf{p}\|_2$. The latter requirement is not severely restrictive, since one can always rescale p and H appropriately to make $\|\Delta\|_{2,2} \leq 1$. In this case, $\mathcal{A}_{\mathbf{\Delta}}(Z) = \mathbf{diag}\left(\mathbf{Tr}(HZ), \mathbf{Tr}(Z_{11} - Z_{22})\right)$, and $\mathcal{A}^*_{\mathbf{\Delta}}(y_1, y_2) = y_1 H + \mathbf{diag}\,(y_2 I, -y_2 I)$. Therefore, (5.9) becomes

$$\begin{bmatrix} C & D \\ 0_{n_q \times n_x} & I_{n_q} \end{bmatrix}^* \left(\begin{bmatrix} y_2 I_{n_p} & 0 \\ 0 & -y_2 I_{n_q} \end{bmatrix} + y_1 H \right) \begin{bmatrix} C & D \\ 0_{n_q \times n_x} & I_{n_q} \end{bmatrix}$$
$$+ \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_q} \end{bmatrix}^* (\Psi \otimes P) \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_q} \end{bmatrix} \prec 0.$$

By dividing y_2, and re-label y_1 by $y > 0$ we have

$$\begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D \end{bmatrix} + yH + \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_q} \end{bmatrix}^* (\Psi \otimes P) \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_q} \end{bmatrix} \prec 0.$$

5.3.8 Block-diagonal structure

Let us consider the block diagonal structure, that is $\mathbf{\Delta} = \{\Delta : \mathbf{diag}\,(\Delta_1, \cdots, \Delta_n)\,, i = 1, \cdots, n\}$. Here all of the previous modeling on Δ can be used, for example $\Delta_i = \Delta_i^*$, or $\Delta_i = \delta_i I$ to name a few. Let E_i be the projection matrix such that $\Delta_i = E_i \Delta E_i^*$. Then $(\mathbf{p}, \mathbf{q}) \in \mathcal{R}_{\mathbf{\Delta}}$ if and only if $(E_i \mathbf{p}, E_i \mathbf{q}) \in \mathcal{R}_{\mathbf{\Delta}_i}$.

For example, if

$$\Delta = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix},$$

where $\delta_2 = \delta_2^*$, then $(\mathbf{p}, \mathbf{q}) \in \mathcal{R}_{\mathbf{\Delta}}$ if and only if $\|[\mathbf{q}]_1\|_2 \le \|[\mathbf{p}]_1\|_2$, $\|[\mathbf{q}]_2\|_2 \le \|[\mathbf{p}]_2\|_2$, and $\langle [\mathbf{q}]_2, [\mathbf{p}]_2 \rangle = \langle [\mathbf{p}]_2, [\mathbf{q}]_2 \rangle$. In this case,

$$\mathcal{A}_{\mathbf{\Delta}}(Z) = \begin{bmatrix} [Z]_{11} - [Z]_{33} & 0 \\ 0 & [Z]_{22} - [Z]_{44} \end{bmatrix}, \mathcal{B}_{\mathbf{\Delta}}(Z) = [Z]_{24} - \overline{[Z]_{24}},$$

and the test (5.9) becomes

$$\begin{bmatrix} C & D \\ 0_{n_q \times n_x} & I_{n_q} \end{bmatrix}^* \begin{bmatrix} Y & Q \\ -Q & -Y \end{bmatrix} \begin{bmatrix} C & D \\ 0_{n_q \times n_x} & I_{n_q} \end{bmatrix} + \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_q} \end{bmatrix}^* (\Psi \otimes P) \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_q} \end{bmatrix} \prec 0,$$

with $Y = \begin{bmatrix} y_1 & 0 \\ 0 & y_2 \end{bmatrix}$ where $y_1 > 0, y_2 > 0$, and $Q = \begin{bmatrix} 0 & 0 \\ 0 & q \end{bmatrix}$ for some imaginary q.

In the general case, $\mathcal{A}_{\mathbf{\Delta}}(Z) = \mathbf{diag}\left(\mathcal{A}_{\Delta_i}\left(\begin{bmatrix} E_i & 0 \\ 0 & E_i \end{bmatrix} Z \begin{bmatrix} E_i^* & 0 \\ 0 & E_i^* \end{bmatrix}\right)\right)$, and $\mathcal{B}_{\mathbf{\Delta}}(Z) = \mathbf{diag}\left(\mathcal{B}_{\Delta_i}\left(\begin{bmatrix} E_i & 0 \\ 0 & E_i \end{bmatrix} Z \begin{bmatrix} E_i^* & 0 \\ 0 & E_i^* \end{bmatrix}\right)\right)$.

Therefore,

$$\mathcal{A}_{\mathbf{\Delta}}^*(y_1, \cdots, y_n) = \sum_{i=1}^{n} \begin{bmatrix} E_i & 0 \\ 0 & E_i \end{bmatrix}^* \mathcal{A}_{i\mathbf{\Delta}}^*(y_i) \begin{bmatrix} E_i & 0 \\ 0 & E_i \end{bmatrix}$$

$$\mathcal{B}_{\mathbf{\Delta}}^*(q_1, \cdots, q_n) = \sum_{i=1}^{n} \begin{bmatrix} E_i & 0 \\ 0 & E_i \end{bmatrix}^* \mathcal{B}_{i\mathbf{\Delta}}^*(q_i) \begin{bmatrix} E_i & 0 \\ 0 & E_i \end{bmatrix}$$

and the test (5.9) becomes

$$\begin{bmatrix} C & D \\ 0_{n_q \times n_x} & I_{n_q} \end{bmatrix}^* \left(\mathcal{A}_{\mathbf{\Delta}}^*(y_1, \cdots, y_n) + \mathcal{B}_{\mathbf{\Delta}}^*(q_1, \cdots, q_n)\right) \begin{bmatrix} C & D \\ 0_{n_q \times n_x} & I_{n_q} \end{bmatrix} + \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_q} \end{bmatrix}^* (\Psi \otimes P) \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_q} \end{bmatrix} \prec 0.$$

For example, assume $\mathbf{\Delta} = \{\Delta : \mathbf{diag}\left(\Delta_1, \cdots, \Delta_n\right), \|\Delta_i\| \le 1, i = 1, \cdots, n\}$, *i.e.*, the repeated full blcok LTV structure in [74]. Then

$$\mathcal{A}_{\mathbf{\Delta}}^*(y_1, \cdots, y_n) = \sum_{i=1}^{n} \begin{bmatrix} y_i E_i^* E_i & 0 \\ 0 & -y_i E_i^* E_i \end{bmatrix},$$

since $\mathcal{A}_{\mathbf{\Delta}_i}^*(y_i) = \begin{bmatrix} y_i I & 0 \\ 0 & -y_i I \end{bmatrix}$. Therefore the test (5.9) is given by

$$\begin{bmatrix} C & D \\ 0_{n_q \times n_x} & I_{n_q} \end{bmatrix}^* \begin{bmatrix} \Theta & 0_{n_p \times n_q} \\ 0_{n_q \times n_p} & -\Theta \end{bmatrix} \begin{bmatrix} C & D \\ 0_{n_q \times n_x} & I_{n_q} \end{bmatrix} + \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_q} \end{bmatrix}^* (\Psi \otimes P) \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_q} \end{bmatrix} \prec 0,$$

where $\Theta = \sum_i y_i E_i^* E_i$. Since $y_i > 0$, clearly $\Theta \succ 0$.

By left and right multiplying $\begin{bmatrix} I_{n_x} & 0_{n_x \times n_q} \\ 0_{n_q \times n_x} & \Theta^{-1/2} \end{bmatrix}$ to the above LMI and expanding the first term, we can obtain

$$\begin{bmatrix} C^*\Theta C & C^*\Theta D \Theta^{-1/2} \\ \Theta^{-1/2} D^* \Theta C & \Theta^{-1/2} D^* \Theta D \Theta^{-1/2} - I_{n_q} \end{bmatrix} + \begin{bmatrix} A & B\Theta^{-1/2} \\ I_{n_x} & 0_{n_x \times n_q} \end{bmatrix}^* (\Psi \otimes P) \begin{bmatrix} A & B\Theta^{-1/2} \\ I_{n_x} & 0_{n_x \times n_q} \end{bmatrix} \prec 0.$$

From the bounded real lemma, this is equivalent to

$$\|\Theta^{1/2} \mathbf{M} \Theta^{-1/2}\|_\infty < 1,$$

the famous scaled small gain test with the form $\Theta = \sum_i y_i E_i E_i^* = \mathbf{diag}\,(y_1 I, \cdots, y_n I)$, where $y_i > 0$.

Similarly, for the repeated scalar block LTV structure in [57],

$$\boldsymbol{\Delta} = \{\Delta : \mathbf{diag}\,(\delta_1 I, \cdots, \delta_n I)\,, \|\Delta_i\| \leq 1, i = 1, \cdots, n\}$$

we have

$$\mathcal{A}^*_{\boldsymbol{\Delta}}(Y_1, \cdots, Y_n) = \sum_{i=1}^{n} \begin{bmatrix} E_i Y_i E_i^* & 0 \\ 0 & -E_i Y_i E_i^* \end{bmatrix},$$

since $\mathcal{A}^*_{\boldsymbol{\Delta}_i}(Y_i) = \begin{bmatrix} Y_i & 0 \\ 0 & -Y_i \end{bmatrix}$.

Using the similar argument, we can obtain

$$\begin{bmatrix} C^*\Theta C & C^*\Theta D \Theta^{-1/2} \\ \Theta^{-1/2} D^* \Theta C & \Theta^{-1/2} D^* \Theta D \Theta^{-1/2} - I_{n_q} \end{bmatrix} + \begin{bmatrix} A & B\Theta^{-1/2} \\ I_{n_x} & 0_{n_x \times n_q} \end{bmatrix}^* (\Psi \otimes P) \begin{bmatrix} A & B\Theta^{-1/2} \\ I_{n_x} & 0_{n_x \times n_q} \end{bmatrix} \prec 0.$$

From the bounded real lemma, this is equivalent to

$$\|\Theta^{1/2} \mathbf{M} \Theta^{-1/2}\|_\infty < 1,$$

the famous scaled small gain test with the form $\Theta = \sum_i E_i Y_i E_i^* = \mathbf{diag}\,(Y_1, \cdots, Y_n)$ where $Y_i \succ 0$.

5.4 Numerical example

We consider the following dynamics from Example 5.4 in [43]

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -1.3046(1 \pm \delta_1) & 1 & -0.2142(1 \pm \delta_2) & 0 \\ 0 & 47.7110(1 \pm \delta_3) & 0 & -104.83(1 \pm \delta_4) & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 275100 & 49059 & 3753 & -16861 & -163 \end{bmatrix},$$

where each δ_i is a fixed but unknown real parameter. In [43], the author tries to verify stability of an autonomous dynamics $\frac{d}{dt}x = A_0 x$, but here we modify this example to illustrate how to use well-connectedness analysis.

Consider the following autonomous system

$$\dot{x} = A_0 x.$$

By introducing additional variable p, q, we have

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -1.3046 & 1 & -0.2142 & 0 \\ 0 & 47.7110 & 0 & -104.83 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 275100 & 49059 & 3753 & -16861 & -163 \end{bmatrix}}_{A} x$$

$$+ \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ -1.3046 & -0.2142 & 0 & 0 \\ 0 & 0 & 47.7110 & -104.83 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{B} q$$

$$p = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}}_{C} x + \underbrace{0}_{D} q$$

$$q = \underbrace{\begin{bmatrix} \delta_1 & 0 & 0 & 0 \\ 0 & \delta_2 & 0 & 0 \\ 0 & 0 & \delta_3 & 0 \\ 0 & 0 & 0 & \delta_4 \end{bmatrix}}_{\Delta} .p$$

Although in [43], δ_i is assumed to be a fixed but unknown real number, but we assume that each δ_i is an operator which maps $\mathcal{L}_2^1 \to \mathcal{L}_2^1$ to illustrate how to use our analysis tool. It should be clear that the above representation of autonomous dynamics is

Structure	$\mu_{\mathbf{\Delta}}(\mathbf{M})$
Full block LTV	0.3427
Full block self-adjoint LTV	0.3427
Scalar block diagonal LTV	0.4644
Scalar self-adjoint block diagonal LTV	0.4644

Table 5.1: $\mu_{\mathbf{\Delta}}(\mathbf{M})$ for various structures. If we assume δ_i are real numbers, $\mu_{\mathbf{\Delta}}(\mathbf{M}) = 0.5381$ is given by the exhaustive search [43].

a feedback connection between p and q. As we discussed in the beginning of this chapter, stability of this feedback connection is equivalent to well-connectedness of $\mathbf{M}$, which has a state representation (A, B, C, D) with respect to the uncertain operator δ. Further, we would like to obtain the allowable size of uncertainty

$$\mu_{\mathbf{\Delta}}(\mathbf{M}) = \inf\{\|\Delta\|_{2,2} : \mathbf{I} - \mathbf{M}\Delta, \Delta \in \mathbf{\Delta}\}.$$

Recall that in all previous examples, we assume $\|\Delta\|_{2,2} \leq 1$, but the above quantity requires to handle different norm bound $\|\Delta\|_{2,2}$. This can be easily done by rescaling C, D matrices.

Suppose $\|\Delta\|_{2,2} \leq \mu$. Since well-connectedness only concerns the product (or cascading) $M\Delta$, (M, Δ) is robustly well-connected if and only if $(\mu M, \frac{1}{\mu}\Delta)$ is robustly well-connected. Therefore, in order to check that $\mathbf{M}$ is robustly well-connected to Δ with $\|\Delta\|_{2,2} \leq \mu$, it suffices to check μM is robustly well connected to $\left\|\tilde{\Delta}\right\|_{2,2} = \left\|\frac{1}{\mu}\Delta\right\|_{2,2} \leq 1$,

Since $\mu\mathbf{M}$ can be easily obtained by scaling (C, D) by $(\mu C, \mu D)$, in order to find $\mu_{\mathbf{\Delta}}(\mathbf{M}) = \inf\{\|\Delta\|_{2,2} : \mathbf{I} - \mathbf{M}\Delta, \Delta \in \mathbf{\Delta}\}$, it suffices to find the minimum μ such that $\mu\mathbf{M}$ is not robustly well-connected to $\mathbf{\Delta}$ assuming the norm bound $\|\mathbf{\Delta}\|_{2,2} \leq 1$. Since we can easily check that whether $\mu\mathbf{M}$ is robustly well-connected to $\mathbf{\Delta}$ or not using the test (5.9) for the class of structures, we combine the bisection search on μ to provide $\mu_{\mathbf{\Delta}}(\mathbf{M})$.

Here $\mathbf{\Delta}$ has a block diagonal structure, but we apply the LMI test from the full block LTV, full block self-adjoint LTV, scalar block diagonal LTV, and scalar self-adjoint block diagonal LTV structures for an illustrative purpose.

The table 5.1 shows $\mu_{\mathbf{\Delta}}(\mathbf{M})$ assuming various structures on $\mathbf{\Delta}$. Since full block LTV does not exploit the block diagonal nature of $\mathbf{\Delta}$, with relatively small norm bound 0.3427, $\mathbf{M}$ lost robust well-connectedness. Here since all problem parameters are real numbers, exploiting the self-adjoint nature of $\mathbf{\Delta}$ does not change $\mu_{\mathbf{\Delta}}(\mathbf{M})$.

Chapter 6

Sparse $\mathcal{H}_\infty$ analysis and synthesis

This chapter proposes a new robust analysis tool motivated by large-scale systems by leveraging a novel, primal formulation of $\mathcal{H}_\infty$ analysis in Chapter 4. The $\mathcal{H}_\infty$ norm of a system measures its robustness by quantifying the worst-case behavior of a system perturbed by a unit-energy disturbance. However, the disturbance that induces such worst-case behavior requires perfect coordination among all disturbance channels. Given that many systems of interest, such as the power grid, the internet, and automated vehicle platoons, are large-scale and spatially distributed, such coordination may not be possible, and hence the $\mathcal{H}_\infty$ norm, used as a measure of robustness, may be too conservative. We therefore propose a cardinality constrained variant of the $\mathcal{H}_\infty$ norm in which an adversarial disturbance can use only a limited number of channels. As this problem is inherently combinatorial, we present a semidefinite programming (SDP) relaxation based on the ℓ_1 norm that yields an upper bound on the cardinality constrained robustness problem. We further propose a simple rounding heuristic based on the optimal solution of SDP relaxation, which provides a lower bound. Motivated by privacy in large-scale systems, we also extend these relaxations to computing the minimum gain of a system subject to a limited number of inputs. Finally, we also present an SDP based optimal controller synthesis method for minimizing the SDP relaxation of our novel robustness measure. The effectiveness of our semidefinite relaxation is demonstrated through numerical examples.

6.1 Sparse $\mathcal{H}_\infty$ norm

Consider the stable LTI system $\mathbf{M}$

$$\begin{aligned} x^+ &= Ax + Bw \\ z &= Cx + Dw, \end{aligned}$$

where x^+ is $x[k+1]$ for discrete time, $\dot{x}$ for the continuous time, with initial condition being zero. In addition, $x \in \mathbb{C}^{n_x}, w \in \mathbb{C}^{n_w}, z \in \mathbb{C}^{n_z}$.

Recall that the $\mathcal{H}_\infty$ norm of $\mathbf{M}$ is given by,

$$\|\mathbf{M}\|_\infty := \sup_{\mathbf{w}} \{\|\mathbf{M}\mathbf{w}\|_2 : \|\mathbf{w}\|_2 = 1\}.$$

The $\mathcal{H}_\infty$ norm has well known implications on the robust stability of the system with uncertain blocks [26], as well as many practical interpretations [25].

One such interpretation is that an attacker seeks to maximize their disruption of the system using the disturbance $\mathbf{w}$ – in this case, the optimal disturbance $\mathbf{w}^\star$ is precisely a disturbance that maximizes the attacker's impact on the system. Taking an opposite perspective, from the viewpoint of a system designer, the maximizing disturbance denotes a weak point of the system that may need to be addressed.

A seemingly innocuous assumption in the above analysis is that the attacker can simultaneously coordinate all of the disturbance channels: although reasonable in a centralized setting, this assumption may prove to be quite conservative if $\mathbf{M}$ is a distributed system. In particular, if there are many possible disturbances (B has many columns), and these disturbances enter through channels that are physically separated, it may be overly conservative to consider the response of the system to a centralized attack. In order to alleviate this conservativeness, we propose a *cardinality constrained* variation of $\mathcal{H}_\infty$ analysis, (4.3), in which we assume that at most k disturbance channels can have non-zero energy. Here we define the cardinality of the input signal, $\mathbf{Card}(\mathbf{w})$, as the number of components that has non-zero energy, *i.e.*, the number of indices i such that $\|[\mathbf{w}]_i\|_2 > 0$. This means the adversary can choose up to k number of input channels to disturb the system, which is equivalent to choose up to k number of columns of B to form the low dimensional input in l_2^k $(\mathcal{L}_2^k)$.

$$\bar{\mu}_k(\mathbf{M}) := \sup_{\mathbf{w}} \{\|\mathbf{M}\mathbf{w}\|_2 : \|\mathbf{w}\| \leq 1, \mathbf{Card}(\mathbf{w}) \leq k\}.$$

We refer to $\bar{\mu}_k(\mathbf{M})$ as the k-sparse $\mathcal{H}_\infty$ norm of system $\mathbf{M}$.

It should be clear that $\bar{\mu}_k(\mathbf{M}) \leq \|\mathbf{M}\|_\infty$ for all k, but what is not clear is whether the gap between the two quantities can be made large. Indeed, if the gap is in general small, then the additional effort needed to accommodate the cardinality constraint on the disturbance may not be justified. Before elaborating on other interpretations of the k-sparse $\mathcal{H}_\infty$ norm of a system, we show that the gap between $\bar{\mu}_k(\mathbf{M})$ and $\|\mathbf{M}\|_\infty$ can be made arbitrarily large for a fixed k by increasing the state dimension of the underlying system.

Example 6.1: Consider the stable discrete time system $\mathbf{M}$ with the state space realization $(A, I_n, I_n, 0_{n,n})$ where $A = 0.99(I_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^\top) + 0.1\frac{1}{n}\mathbf{1}\mathbf{1}^\top$. Thanks to the permutation invariance of the system, $\|\mathbf{M}\|_\infty$ and $\bar{\mu}_k(\mathbf{M})$ can be easily obtained, and it can be shown that the ratio $\frac{\bar{\mu}_k(\mathbf{M})}{\|\mathbf{M}\|_\infty} = O\left(\sqrt{\frac{k}{n}}\right)$. Thus for a fixed k, the gap between $\bar{\mu}_k(\mathbf{M})$ and $\|\mathbf{M}\|_\infty$ ca ben made arbitrarily large by letting $n \to \infty$. Fig. 6.1 shows $\frac{\bar{\mu}_k(\mathbf{M})}{\|\mathbf{M}\|_\infty}$ for $k = 5$, and $n = 5, \ldots, 30$.

Example 6.1 shows the existence of systems for which standard $\mathcal{H}_\infty$ analysis can be overly conservative if indeed only at most k disturbances can be expected to coordinate their efforts to disrupt the system. We now outline two concrete engineering applications in which such an analysis may arise.

Robustness analysis for distributed system: Quantifying the robustness of a distributed system, such as the power grid, allows the system designer to plan for and mitigates the worst case effects of un-modeled dynamics and disturbances. The need for robustness is increasingly important in the context of the power grid as it becomes more reliant on intermittent distributed energy resources, such as renewables. However, as mentioned, $\mathcal{H}_\infty$ analysis assumes that all such distributed energy

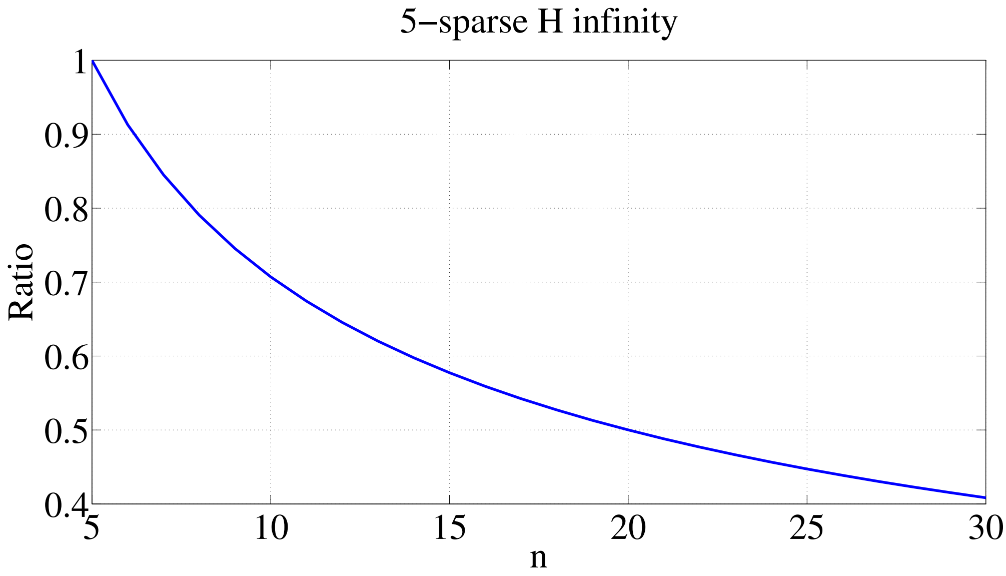

Figure 6.1: The ratio, $\bar{\mu}_5(\mathbf{M})/\|\mathbf{M}\|_\infty$ for $n = 5, \cdots, 30$.

resources coordinate with each other to destabilize the power network, which may be overly conservative and lead to loss of efficiency. Rather, we propose using the k-sparse $\mathcal{H}_\infty$ norm of the system to identify and quantify vulnerabilities of the system to potentially more realistic disturbances.

Robustness analysis for consensus network: The well-studied problem of consensus (or synchronization) [54, 9, 39] is one in which a set of agents seek to converge to a common value using simple local averaging rules. When these local rules are linear and time invariant, the consensus protocol can be modeled as an LTI system. In this case, a system dynamics A satisfies the following properties [83]: $A\mathbf{1} = \mathbf{1}$, $A^\top\mathbf{1} = \mathbf{1}$, and $\rho(A - \frac{1}{n}\mathbf{1}\mathbf{1}^\top) < 1$, where n is the number of nodes in the network.

Although typically considered in a disturbance free setting, it is also natural to ask how much local disturbances applied to individual agents can affect the system's ability to reach consensus. Concretely, assume that each agent can be corrupted by a separate disturbance, i.e., $B = I_n$, and we measure the effect of the disturbances on the deviation of each state x_k^j from the consensus value, as encoded by $z_k^j = x_k^j - \frac{1}{n}\sum_i x_k^i$, such that $C = I_n - \frac{1}{n}\mathbf{1}\mathbf{1}^\top$, and $D = 0$. Note that the marginally stable mode of A is unobservable with respect to the measured output defined by C, and the system has a finite $\mathcal{H}_\infty$ norm and k-sparse $\mathcal{H}_\infty$ norm.

Whereas the $\mathcal{H}_\infty$ norm of the resulting system measures the effects of a worst-case attack on all agents, the k-sparse $\mathcal{H}_\infty$ norm measures the effects of worst-case attack on only k agents. From an attacker's perspective, this may result in a more realistically implementable strategy, and from a system designer's perspective, this provides valuable information as to which agents should be most closely monitored and protected from attack.

6.1.1 The k-sparse minimal gain of a system

We can also define the minimal k-sparse gain of system $\mathbf{M}$, which we denote by $\underline{\mu}_k(\mathbf{M})$ as

$$\begin{aligned} \{\underline{\mu}_k(\mathbf{M})\}^2 := \underset{\mathbf{w},\mathbf{x}}{\text{minimize}} \quad & \|C\mathbf{x} + D\mathbf{w}\|_2^2 \\ \text{subject to} \quad & x^+ = Ax + Bw \\ & x[0] = 0 \\ & \|\mathbf{w}\|_2^2 \geq 1 \\ & \textbf{Card}(\mathbf{w}) \leq k. \end{aligned} \tag{6.1}$$

Privacy: An immediate interpretation of this optimization problem is in terms of *privacy*. Suppose that a publicly available variable is defined by $\mathbf{z} = C\mathbf{x}$, and that a user wishes to transfer at least γ units of power to $\mathbf{y} = G\mathbf{x} + H\mathbf{w}$ while minimizing their effect on the public variable. The optimal action for the user to take can be determined by solving optimization problem (6.1) with the added constraint

$$\|G\mathbf{x} + H\mathbf{w}\|_2^2 \geq \gamma^2. \tag{6.2}$$

System security: One can also view the user in the above scenario as an attacker, and the publicly available variable as a system monitor: in this case, the optimal input $\mathbf{w}_\star$ corresponds to the least detectable input that still disrupts the $\mathbf{y}$ by γ units of power. Allowing for sparse optimal inputs $\mathbf{w}_\star$ makes for more realistically implementable actions by either a user or an attacker.

6.1.2 Connections to the Restricted Isometry Property and Regularization for Design

Our problem formulation seeks the minimal and maximal gains of a linear operator restricted to k-sparse subspaces. When the linear operator is a static matrix D, instead of a dynamical system ($A = B = C = 0$), then the cardinality constrained optimization problems (6.3) and (6.1) compute precisely the maximal and minimal *restricted eigenvalues* [6] of the matrix $D^\top D$, that is the maximal and minimal gains of D restricted to sparse subspaces. They are also closely linked to the Restricted Isometry Property (RIP) constant of the matrix, which can be used to state conditions for the recovery of sparse vectors [13] via convex optimization, and can be used to perform sparse principal component analysis (sPCA) [21]. We can therefore view optimization problem (6.3) as a tool for bounding the restricted eigenvalues of an infinite dimensional LTI operator acting on signals in ℓ_2 and $\mathcal{L}_2$.

Moreover, the k-sparse $\mathcal{H}_\infty$ norm and the k-sparse minimal gain of a system also have natural connections to the Regularization for Design (RFD) framework developed in [45]. In the RFD framework, atomic norms [14] are added as convex penalties to traditional model matching problems in order to design architecturally simple controllers. Further, control theoretic analogs to the recovery conditions found in the structured inference literature are stated in terms of restricted gains that are closely related to the k-sparse $\mathcal{H}_\infty$ norm and k-sparse minimal gain of a system.

6.2 SDP relaxation of k-sparse $\mathcal{H}_\infty$ analysis

Although we propose k-sparse $\mathcal{H}_\infty$ analysis $\bar{\mu}_k(\mathbf{M})$ in a concrete manner, the problem formulation has two intrinsic challenges. First of all, the variables are in an infinite dimensional space, and therefore the optimization is an infinite dimensional optimization. Secondly, the combinatorial nature of the cardinality constraint on $\mathbf{w}$ may require the exhaustive search. In order to overcome these challenges, we propose an SDP-based convex relaxation of the k-sparse $\mathcal{H}_\infty$ analysis.

Since $[\Lambda(\mathbf{w})]_{ii} = \|\mathbf{w}_i\|_2^2$, $\mathbf{Card}\,(\mathbf{w}) \leq k$ if and only if $\Lambda(\mathbf{w})$ has up to k number of non-zero diagonal entries. This shows that we can formulate k-sparse $\mathcal{H}_\infty$ analysis using Gramian

$$\begin{aligned}
\underset{V}{\text{maximize}} \quad & \mathbf{Tr}\left(\begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D \end{bmatrix} V\right) \\
\text{subject to} \quad & V \in \mathcal{D}_{\text{Gram}} \text{ (or } \mathcal{C}_{\text{Gram}}) \\
& \mathbf{Tr}\left(\begin{bmatrix} 0_{n_x \times n_x} & 0_{n_x \times n_w} \\ 0_{n_w \times n_x} & I_{n_w} \end{bmatrix} V\right) \leq 1 \\
& \mathbf{Card}\,([V]_{n_x+1,n_x+1}, \cdots, [V]_{n_x+n_w,n_x+n_w}) \leq k,
\end{aligned} \tag{6.3}$$

where n_x is the state dimension, and n_w is the input dimension. Since the cardinality constraints only affects the right bottom block of V, $\begin{bmatrix} 0_{n_w \times n_x} & I_{n_w} \end{bmatrix} V \begin{bmatrix} 0_{m,n} & I_m \end{bmatrix}^*$ we can easily see that Lemma 3.1 and 3.3 justify the replacement of $\mathcal{D}_{\text{Gram}}$ and $\mathcal{C}_{\text{Gram}}$ by $\mathcal{D}_{\text{SDP}}$ and $\mathcal{C}_{\text{SDP}}$.

$$\begin{aligned}
\underset{V}{\text{maximize}} \quad & \mathbf{Tr}\left(\begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D \end{bmatrix} V\right) \\
\text{subject to} \quad & \Pi\left(\begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_w} \end{bmatrix} V \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_w} \end{bmatrix}^*\right) = 0, V \succeq 0 \\
& \mathbf{Tr}\left(\begin{bmatrix} 0_{n_x \times n_x} & 0_{n_x \times n_w} \\ 0_{n_w \times n_x} & I_{n_w} \end{bmatrix} V\right) \leq 1 \\
& \mathbf{Card}\,([V]_{n_x+1,n_x+1}, [V]_{n_x+2,n_x+2}, \cdots, [V]_{n_x+n_w,n_x+n_w}) \leq k,
\end{aligned} \tag{6.4}$$

where $\Pi\left(\begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix}\right) = X_{11} - X_{22}$ for $\mathcal{D}_{\text{SDP}}$, and $\Pi\left(\begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix}\right) = X_{12} + X_{12}^*$ for $\mathcal{C}_{\text{SDP}}$. The optimal value of (6.4) is equal to $\bar{\mu}_k\,(\mathbf{M})^2$.

In order to circumvent the intractability of the cardinality constraint, we use an l_1 norm relaxation [79]. This approach is inspired by [21], in which the authors consider the l_1 relaxation of an analogous cardinality constraint to obtain an SDP relaxation of the sparse PCA problem, in which one seeks the leading sparse singular vector of a matrix (as mentioned previously, this is closely related to the RIP constant of a matrix and to analogous quantities in RFD). In order to adapt this idea to our problem formulation, we need the following observation.

Proposition 6.1: *Consider $W \in \mathbb{C}^{n\times n}$ such that $W \succeq 0$, $\textbf{Tr}(W) \leq 1$. Then, $\mathbf{1}^T|W|\mathbf{1} \leq n$.*

Proof: Consider a Hermitian matrix H where

$$H_{ij} \;=\; \begin{cases} 1 & \text{if } i = j \\ e^{\mathbf{i}\theta_{ij}} & \text{if } i \neq j, \end{cases}$$

for some θ_{ij}. If we construct H such that $H_{ij} = e^{\mathbf{i}\angle W_{ij}}$, then $\mathbf{1}^T|W|\mathbf{1} = \mathbf{Tr}(H^*W)$. This shows that $\mathbf{1}^T|W|\mathbf{1} \leq \sup_H \mathbf{Tr}(H^*W)$, and from the Von Neumann's trace inequality [49], we have

$$\mathbf{Tr}(H^*W) \leq \sum_i \sigma_i(W)\sigma_i(H),$$

where σ_i is the ith singular value of the matrix. Furthermore, by definition of H we have $\sigma_1(H) \leq \sum_i \sigma_i(H) = \mathbf{Tr}(H) = n$. Therefore,

$$\begin{aligned} \mathbf{Tr}(H^*W) &\leq \sum_i \sigma_i(W)\sigma_i(H) \leq \sigma_1(H)\sum_i \sigma_i(W) \\ &\leq \; n\mathbf{Tr}(W) \leq n, \end{aligned}$$

and $\mathbf{1}^T|W|\mathbf{1} \leq \sup_H \mathbf{Tr}(H^*W) \leq n$. Notice that this upper bound is achieved by $W = \frac{1}{n}\mathbf{1}_{n_w}\mathbf{1}_{n_w}^\top$, which shows the inequality is tight. ∎

Now we are ready to connect the l_1 norm bound with the cardinality constraint.

Proposition 6.2: *Consider a positive semidefinite matrix W with $\textbf{Tr}(W) \leq 1$ and $\mathbf{Card}\,(\mathbf{diag}\,(W)) \leq k$. Then, $\mathbf{1}^T|W|\mathbf{1} \leq k$.*

PROOF: Without loss of generality, we can assume that $W_{11}, \cdots, W_{ii}$ are not zero, where $i \le k$. Then from the Schur complement, we can easily check that W should have the form

$$W = \left[\begin{array}{c|c} \tilde{W} & 0 \\ \hline 0 & 0 \end{array}\right],$$

where $\tilde{W}$ is a $i \times i$ Hermitian matrix. Therefore, from the Proposition 6.1, $\mathbf{1}^T|W|\mathbf{1} = \mathbf{1}^T|\tilde{W}|\mathbf{1} \le i \le k$, which concludes the proof. ■

In the cardinality constrained problem (6.4), the right bottom part of V, $W = \begin{bmatrix} 0_{n_w \times n_x} & I_{n_w} \end{bmatrix} V \begin{bmatrix} 0_{m,n} & I_m \end{bmatrix}^*$ matrix satisfies the requirement in Proposition 6.2. This shows that if we replace the cardinality constraint by a l_1 norm bound, then we have a larger feasible set. Although this procedure provides an upper bound of (6.4), the resulting optimization becomes a semidefinite program, so we can solve it very efficiently [11]. Therefore, we propose the following l_1 relaxation of (6.4), $\bar{\mu}_k^{sdp}(\mathbf{M})$, which is the main optimization problem in this section.

$$\begin{array}{ll} \underset{V,W}{\text{maximize}} & \mathbf{Tr}\left(\begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D \end{bmatrix} V\right) \\ \text{subject to} & \Pi\left(\begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_w} \end{bmatrix} V \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_w} \end{bmatrix}^*\right) = 0 \\ & V \succeq 0 \\ & W = \begin{bmatrix} 0_{n_w \times n_x} & I_{n_w} \end{bmatrix} V \begin{bmatrix} 0_{n_w \times n_x} & I_{n_w} \end{bmatrix}^* \\ & \mathbf{Tr}(W) \le 1 \\ & \mathbf{1}^T|W|\mathbf{1} \le k, \end{array} \tag{6.5}$$

and we denote the square root of the optimal value as $\bar{\mu}_k^{sdp}(\mathbf{M})$. It should be obvious that $\bar{\mu}_k(\mathbf{M}) \le \bar{\mu}_k^{sdp}(\mathbf{M})$.

A careful remark is that for a complex matrix, $|W|$ should be treated as a second order cone constraint, not a linear programming constraint [42].

6.2.1 Extension to k-sparse minimal gain

In the previous section, we introduced a k-sparse minimal gain. A similar approach can be used to obtain the following SDP relaxation of $\underline{\mu}_k(\mathbf{M})$.

$$\begin{aligned} \underset{V,W}{\text{minimize}} \quad & \mathbf{Tr}\left(\begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D \end{bmatrix} V\right) \\ \text{subject to} \quad & \Pi\left(\begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x\times n_w} \end{bmatrix} V \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x\times n_w} \end{bmatrix}^*\right) = 0 \\ & V \succeq 0 \\ & W = \begin{bmatrix} 0_{n_w\times n_x} & I_{n_w} \end{bmatrix} V \begin{bmatrix} 0_{n_w\times n_x} & I_{n_w} \end{bmatrix}^* \\ & \mathbf{Tr}(W) \leq 1 \\ & \mathbf{1}^T|W|\mathbf{1} \leq k, \end{aligned} \tag{6.6}$$

and we denote the square root of the optimal value as $\{\underline{\mu}_k^{sdp}(\mathbf{M})\}$.

6.2.2 Rounding heuristic for solution refinement

Let $W^\star$ be the optimal solution of (6.5). Since this matrix contains an information on the worst-case disturbance, we can extract potential candidate for the worst case disturbance channels that solves (6.3). The basic idea is to obtain the top k entries in $\mathbf{diag}\,(W)$, say $\{W_{i_1i_1}, W_{i_2i_2}, \cdots, W_{i_ki_k}\}$, and then adjust B and D such that only these disturbance channels can be used. Then we solve $\mathcal{H}_\infty$ analysis with the new B and D. Since we choose a specific disturbance channel, we obtain a lower bound of (6.3). We summarize the procedure as follows.

Rounding heuristic:

---

1. Solve (6.5) to obtain $W^\star$.

2. Find the indices $\{i_1, \cdots, i_k\}$ such that $W^\star_{i_1 i_1} \geq \cdots \geq W^\star_{i_k i_k} \geq \cdots \geq W^\star_{i_n i_n}$.

3. Construct $E := \begin{bmatrix} e_{i_1} & \cdots & e_{i_k} \end{bmatrix} \in \mathbb{C}^{n_w \times k}$ using a standard basis $\{e_i\} \in \mathbb{R}^{n_w}$.

4. Let $\tilde{B} := BE$, $\tilde{D} = DE$, and the stable LTI system $\tilde{\mathbf{M}}$ with the state space realization $(A, \tilde{B}, C, \tilde{D})$. Then define $\bar{\mu}_k^{round}(\mathbf{M}) := \|\tilde{\mathbf{M}}\|_\infty$.

Notice that the third step chooses $i_1, \cdots i_k$ to be the active disturbance channels. From this rounding procedure we obtain the inequality

$$\bar{\mu}_k^{round}(\mathbf{M}) \leq \bar{\mu}_k(\mathbf{M}) \leq \bar{\mu}_k^{sdp}(\mathbf{M})$$

Therefore, if the gap between $\bar{\mu}_k^{round}(\mathbf{M})$ and $\bar{\mu}_k^{sdp}(\mathbf{M})$ is not large, then $\bar{\mu}_k^{round}(\mathbf{M})$ effectively solves the k-sparse $\mathcal{H}_\infty$ problem and returns the candidate the disturbance channels.

6.2.3 SDP Dual problem

As optimization problem (6.5), it is natural to consider its SDP dual problem. To do this, let us begin with the following observation.

Proposition 6.3: *For $w \geq 0$, $\lambda \in \mathbb{C}$,*

$$\sup_{x \in \mathbb{C}} \{-w|x| + \mathbf{Re}(\lambda x)\} = \begin{cases} 0 & \text{if } |\lambda| \leq w \\ +\infty & \text{otherwise} \end{cases}.$$

Proof: Suppose $|\lambda| > w$. Let $x = \alpha \lambda^*$. Then

$$-w|x| + \mathbf{Re}(\lambda x) = \alpha|\lambda|(|\lambda| - w).$$

By taking $\alpha \to \infty$, we obtain the result.

Suppose $|\lambda| \leq w$. From Cauchy-Schwartz inequality,

$$-w|x| + \mathbf{Re}(\lambda x) \leq -w|x| + |\lambda||x| \leq (|\lambda| - w)|x| \leq 0,$$

for all $x \in \mathbb{C}$. Since the upper bound is achieved by $x = 0$, we can conclude the proof. ■

With this technical tool in hand, we may proceed to derive the dual to optimization problem (6.5). First, we form the Lagrangian function in terms of V,

$$\begin{aligned}
& L(V, P, Q, \lambda, t) \\
:= \; & \mathbf{Tr}(QV) + \mathbf{Tr}\left(\begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D \end{bmatrix} V\right) \\
& + \mathbf{Tr}\left(P\Pi\left(\begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_w} \end{bmatrix} V \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_w} \end{bmatrix}^*\right)\right) \\
& + \lambda\left(1 - \mathbf{Tr}\left(\begin{bmatrix} 0_{n_x \times n_x} & 0_{n_x \times n_w} \\ 0_{n_w \times n_x} & I_{n_w} \end{bmatrix} V\right)\right) + t\left(k - \mathbf{Tr}\left(\begin{bmatrix} 0_{n_x \times n_x} & 0_{n_x \times n_w} \\ 0_{n_w \times n_x} & \mathbf{1}_{n_w}\mathbf{1}_{n_w}^\top \end{bmatrix} |V|\right)\right),
\end{aligned}$$

where $P = P^*$, $Q \succeq 0$, $\lambda \geq 0$, $t \geq 0$.

Using cyclic property of the trace operator and from Proposition 6.3, we can obtain the dual function $d(Q, P, \lambda, t) := \sup_{V=V^*} L(V, P, \lambda, t)$, which becomes $\lambda + k \cdot t$ when,

$$\left| Q + \begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D - \lambda I_{n_w} \end{bmatrix} + \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_w} \end{bmatrix}^* (\Psi \otimes P) \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_w} \end{bmatrix} \right| \leq \begin{bmatrix} 0_{n_x \times n_x} & 0_{n_x \times n_w} \\ 0_{n_w \times n_x} & t\mathbf{1}_{n_w}\mathbf{1}_{n_w}^\top \end{bmatrix}, \tag{6.7}$$

where the inequality $\leq$ is a component-wise inequality, and $\Psi = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ for the discrete-time case, and $\Psi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ for the continuous-time case. In addition,

$d(P, \lambda, t) = +\infty$ if (Q, P, λ, t) does not satisfy (6.7). By defining $Y = Y^*$ to be a right bottom block of (6.7), we obtain the following dual program of (6.5).

$$\begin{aligned}
\underset{P,Y,\lambda,t}{\text{minimize}} \quad & \lambda + k \cdot t \\
\text{subject to} \quad & \begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D - \lambda I_m - Y \end{bmatrix} + \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_w} \end{bmatrix}^* (\Psi \otimes P) \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_w} \end{bmatrix} \preceq 0 \\
& |Y| \leq t \mathbf{1}_{n_w} \mathbf{1}_{n_w}^\top \\
& P = P^*, Y = Y^*, t \geq 0, \lambda \geq 0.
\end{aligned} \tag{6.8}$$

Notice that if we set $t = 0$, then we recover the SDP derived from the bounded real lemma which computes $\mathcal{H}_\infty$ norm of the system. It is clear that $t = 0$ is a suboptimal solution of (6.8), and therefore we can easily see that the $\mathcal{H}_\infty$ norm is an upper bound of (6.8), and this is consistent with the definition of k-sparse $\mathcal{H}_\infty$ norm.

Another observation is that if we assume (A, B, C, D) are real matrices, then a similar argument as in [88] shows that all matrices in (6.8) can be taken as real matrices. In this case, the absolute value constraint becomes $-t\mathbf{1}_{n_w}\mathbf{1}_{n_w}^\top \leq Y \leq t\mathbf{1}_{n_w}\mathbf{1}_{n_w}^\top$, a familiar LP constraint for l_1 optimization.

6.2.4 Strong duality and Lyapunov stability

In this section, we investigate strong duality between the primal (6.5) and the dual (6.8), and modify the dual program in order to ensure the stability of A. This modification plays an important role in the following synthesis method, in which the optimal controller that minimizes (6.5) is to be found.

We can show that when A is stable, the dual program (6.8) is *strictly* feasible, and therefore the strong duality holds between (6.5) and (6.8).

Proposition 6.4: *Suppose A is stable. Then the dual program* (6.8) *is strictly feasible.*

Proof: In the discrete-time case, the LMI constraint in the dual problem (6.8) becomes

$$\begin{bmatrix} A^*PA - P + C^*C & C^*D + A^*PB \\ D^*C + B^*PA & B^*PB + D^*D - \lambda I_m - Y \end{bmatrix} \preceq 0.$$

If A is Schur stable, $P = \tau \sum_{k=0}^{\infty} (A^*)^k A^k$ is well-defined and satisfies the Lyapunov equation $A^*PA - P = -\tau I$. Therefore, by taking sufficiently large τ, we can make $A^*PA - P + C^*C$ strictly negative definite, and with sufficiently large λ, the above LMI constraint can be made strictly feasible.

For the continuous-time case, we have

$$\begin{bmatrix} A^*P + PA + C^*C & C^*D + PB \\ D^*C + B^*P & D^*D - \lambda I_m - Y \end{bmatrix} \preceq 0.$$

If A is Hurwitz stable, then by letting $P = \tau \int_0^{\infty} e^{A^*t} e^{At} dt$, together with sufficiently large τ, we obtain the same conclusion. ∎

This shows that the dual program (6.8) is strictly feasible, and from the Conic duality theorem we have the following strong duality result.

Corollary 6.1: *The duality gap between the primal* (6.5) *and its dual* (6.8) *is zero, and the primal* (6.5) *is solvable.*

If A is Schur stable, any Hermitian P such that $A^*PA - P \preceq 0$ is necessarily positive semidefinite. Similarly, if A is Hurwitz stable, any P such that $A^*P + PA \preceq 0$ is necessarily positive semidefinite. This shows that in the dual program (6.8), we can add the constraint $P \succeq 0$ without changing the problem. In fact, we can obtain the stability of A by means of $P \succ 0$.

Proposition 6.5: *The following are equivalent.*

(i) A is stable, and $\bar{\mu}_k^{sdp}(\mathbf{M}) < \alpha$.

(ii) There exist $P \succ 0$, λ, t, Y such that

$$\begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_w} \end{bmatrix}^* (\Psi \otimes P) \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_w} \end{bmatrix} + \begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D - \lambda I_m - Y \end{bmatrix} \prec 0 \tag{6.9}$$

$$|Y|_{ij} \leq t, \qquad \textit{for all } i, j \tag{6.10}$$

$$\lambda + kt < \alpha \tag{6.11}$$

PROOF: Suppose (i) holds. From Strong duality, there exists $P = P^*, \lambda, t, Y$ which satisfies (6.10)-(6.11) and

$$\begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_w} \end{bmatrix}^* (\Psi \otimes P) \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_w} \end{bmatrix} + \begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D - \lambda I_m - Y \end{bmatrix} \preceq 0.$$

Since the dual (6.8) is strictly feasible, there exists a strictly feasible point $(P_0, \lambda_0, t_0, Y_0)$. Then we can find $\epsilon > 0$ such that $(\tilde{P}, \tilde{\lambda}, \tilde{t}, \tilde{Y}) = (1-\epsilon)(P, \lambda, t, Y) + \epsilon(P_0, \lambda_0, t_0, Y_0)$ satisfies (6.9) - (6.11). Furthermore, for the discrete time case, the upper left block becomes $A^*\tilde{P}A - \tilde{P} \prec 0$, and since A is Schur stable, $\tilde{P} \succ 0$. Similarly, for the continuous time case, we have $A^*\tilde{P} + \tilde{P}A \prec 0$, and since A is Hurwitz stable, $\tilde{P} \succ 0$.

Now suppose (ii) is true. For the discrete-time case, since $A^*PA - P \prec 0$ for some $P \succ 0$, A is Schur stable. For the continuous-time case, since $A^*P + PA \prec 0$ for some $P \succ 0$, A is Hurwitz stable. Therefore, from strong duality, $\bar{\mu}_k^{sdp}(\mathbf{M}) < \alpha$.■

From the above proposition, we have the following dual problem *without* the stability assumption on A.

$$
\begin{aligned}
\underset{P,Y,\lambda,t}{\text{minimize}} \quad & \lambda + k \cdot t \\
\text{subject to} \quad & \begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D - \lambda I_m - Y \end{bmatrix} + \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_w} \end{bmatrix}^* (\Psi \otimes P) \begin{bmatrix} A & B \\ I_{n_x} & 0_{n_x \times n_w} \end{bmatrix} \prec 0 \\
& |Y| \leq t \mathbf{1}_{n_w} \mathbf{1}_{n_w}^\top \\
& P \succ 0, Y = Y^*, t \geq 0, \lambda \geq 0.
\end{aligned}
\tag{6.12}
$$

Here if the optimal value of (6.12) is finite, then A is stable, and $\bar{\mu}_k^{sdp}(\mathbf{M})$ is equal to the optimal value of (6.12). This formulation is very important for the synthesis method where the stability of A is unknown and required to be stable.

6.3 k-sparse $\mathcal{H}_\infty$ synthesis

In this section, we consider the following system:

$$
\begin{aligned}
x_p^+ &= Ax_p + B_1 w + B_2 u \\
z &= C_1 x + D_{11} w + D_{12} u \\
y &= C_2 x + D_{21} w
\end{aligned}
$$

with the following linear feedback controller

$$
\begin{aligned}
x_c^+ &= A_K x_c + B_K y \\
u &= C_K x_c + D_K y,
\end{aligned}
$$

where $x_p \in \mathbb{C}^{n_p}$ is the state of the plant, $w \in \mathbb{C}^{n_w}$ is the disturbance, $u \in \mathbb{C}^{n_u}$ is the control input, $z \in \mathbb{C}^{n_z}$ is the performance output, $y \in \mathbb{C}^{n_y}$ is the measured output, and $x_c \in \mathbb{C}^{n_c}$ is the state of the controller. Here x^+ is $x[k+1]$ in the discrete time case, and $\frac{dx}{dt}$ in the continuous time case.

After the closing the loop, *i.e.*, substituting $\mathbf{u}$ back to $\mathbf{x}$, we have the following closed loop system, $\mathbf{M}_{cl}$, with the state space realization:

$$\begin{aligned}\begin{bmatrix} x_p \\ x_c \end{bmatrix}^+ &= \underbrace{\begin{bmatrix} A + B_2 D_K C_2 & B_2 C_K \\ B_K C_2 & A_K \end{bmatrix}}_{A_{cl}} \begin{bmatrix} x_p \\ x_c \end{bmatrix} + \underbrace{\begin{bmatrix} B_1 + B_2 D_K D_{21} \\ B_K D_{21} \end{bmatrix}}_{B_{cl}} w \\ z &= \underbrace{\begin{bmatrix} C_1 + D_{12} D_K C_2 & D_{12} C_K \end{bmatrix}}_{C_{cl}} \begin{bmatrix} x_p \\ x_c \end{bmatrix} + \underbrace{\begin{bmatrix} D_{11} + D_{12} D_K D_{21} \end{bmatrix}}_{D_{cl}} w.\end{aligned}$$

The goal in this section is to design the controller A_K, B_K, C_K, D_K such that

- Internally stabilizes $\mathbf{M}_{cl}$, *i.e.*, A_{cl} is stable.
- Minimizes $\bar{\mu}_k^{sdp}(\mathbf{M}_{cl})$.

The actual performance metric we are interested in is $\bar{\mu}_k(\mathbf{M}_{cl})$, but due to combinatorial nature of this performance metric, we change our attention to its upper bound, $\bar{\mu}_k^{sdp}(\mathbf{M}_{cl})$, which can be obtained via SDP. Using the dual formulation of computing $\bar{\mu}_k^{sdp}(\mathbf{M}_{cl})$, we can state this optimal control problem as follows:

$$\begin{aligned} &\underset{A_K, B_K, C_K, D_K, P_{cl}, Y, \lambda, t}{\text{minimize}} \quad \lambda + k \cdot t \\ &\text{subject to} \quad |Y| \leq t \mathbf{1}_{n_w} \mathbf{1}_{n_w}^\top \\ &\begin{bmatrix} A_{cl} & B_{cl} \\ I_n & 0_{n,m} \end{bmatrix}^* (\Psi \otimes P_{cl}) \begin{bmatrix} A_{cl} & B_{cl} \\ I_n & 0_{n,m} \end{bmatrix} + \begin{bmatrix} C_{cl}^* C_{cl} & C_{cl}^* D_{cl} \\ D_{cl}^* C_{cl} & D_{cl}^* D_{cl} - \lambda I - Y \end{bmatrix} \prec 0, \\ &\quad P_{cl} \succ 0, Y = Y^*, t \geq 0, \lambda \geq 0 \end{aligned} \tag{6.13}$$

where $n = n_p + n_c$, *i.e.*, the total sum of number of states in the plant and the controller, $m = n_w$, and

$$\Psi = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & \text{Discrete time case} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \text{Continuous time case.} \end{cases}$$

Notice that we use strictly inequality and $P_{cl} \succ 0$ in (6.13) to ensure the stability of A_{cl}. As long as the optimal value of (6.13) is finite, the corresponding (A_K, B_K, C_K, D_K) successfully stabilizes the plant. In other words, if there is no feasible point of (6.13), then there exists no controller stabilizes the system, so the synthesis problem makes no sense. Therefore we assume that there exists at least one feasible point of (6.13), *i.e.*, (A, B_2, C_2) is stabilizable and detectable.

The above synthesis problem (6.13) is not convex, because the matrix inequality constraint contains many products between (A_{cl}, B_{cl}, P_{cl}), which is not an LMI in terms of decision variables $(A_K, B_K, C_K, D_K, P_{cl})$. In order to circumvent this difficulty, we follow the LMI based approach for $\mathcal{H}_\infty$ synthesis [29], but we use the refined results from [26].

The first step is to eliminate A_K, B_K, C_K, D_K in the matrix inequality constraint in (6.13).

$$\begin{bmatrix} A_{cl} & B_{cl} \\ I_n & 0_{n,m} \end{bmatrix}^* (\Psi \otimes P_{cl}) \begin{bmatrix} A_{cl} & B_{cl} \\ I_n & 0_{n,m} \end{bmatrix} + \begin{bmatrix} C_{cl}^* C_{cl} & C_{cl}^* D_{cl} \\ D_{cl}^* C_{cl} & D_{cl}^* D_{cl} - T \end{bmatrix} \prec 0, \tag{6.14}$$

where $P_{cl} \succ 0$, and $T = \lambda I_m + Y$. Since $D_{cl}^* D_{cl} - T \prec 0$, $T \succ 0$, so we can decompose $T = T^{1/2} T^{1/2}$. By left and right multiplying $\begin{bmatrix} I & 0 \\ 0 & T^{-1/2} \end{bmatrix}$ to (6.14), we have

$$\begin{bmatrix} A_{cl} & B_{cl} T^{-1/2} \\ I_n & 0_{n,m} \end{bmatrix}^* (\Psi \otimes P_{cl}) \begin{bmatrix} A_{cl} & B_{cl} T^{-1/2} \\ I_n & 0_{n,m} \end{bmatrix} + \begin{bmatrix} C_{cl}^* C_{cl} & C_{cl}^* D_{cl} T^{-1/2} \\ T^{-1/2} D_{cl}^* C_{cl} & T^{-1/2} D_{cl}^* D_{cl} T^{-1/2} - I \end{bmatrix} \prec 0. \tag{6.15}$$

We can easily see that the above condition is in the bounded real lemma form, and the condition (6.15) is equivalent to the system $(A_{cl}, B_{cl}T^{-1/2}, C_{cl}, D_{cl}T^{-1/2})$ has $\mathcal{H}_\infty$ norm less than 1. Therefore, for a given $T = \lambda I + Y$, there exists $(A_K, B_K, C_K, D_K, P_{cl})$ that satisfies (6.14) if and only if there exists an $\mathcal{H}_\infty$ suboptimal controller of which $\mathcal{H}_\infty$ norm is less than 1, and we can use the standard suboptimal $\mathcal{H}_\infty$ controller synthesis results.

Let $\tilde{B}_1 = B_1 T^{-1/2}$, $\tilde{D}_{11} = D_{11}T^{-1/2}$, and $\tilde{D}_{21} = D_{21}T^{-1/2}$. Then, since

$$\begin{aligned} B_{cl}T^{-1/2} &= \begin{bmatrix} B_1T^{-1/2} + B_2D_KD_{21}T^{-1/2} \\ B_KD_{21}T^{-1/2} \end{bmatrix} \\ D_{cl}T^{-1/2} &= \begin{bmatrix} D_{11}T^{-1/2} + D_{12}D_KD_{21}T^{-1/2} \end{bmatrix}, \end{aligned}$$

from Lemma 7.8 and Lemma 11.3 in [26] there exists $(A_K, B_K, C_K, D_K, P_{cl})$ that satisfies (6.14) if and only if there exists (P_{cl}, P, Q) such that

$$\left[\begin{array}{c|c} N_c & 0 \\ \hline 0 & I_{n_z} \end{array}\right]^* \left[\begin{array}{c|c} \begin{bmatrix} A & I_{n_p} \\ C_1 & 0 \end{bmatrix} (\Psi \otimes Q) \begin{bmatrix} A & I_{n_p} \\ C_1 & 0 \end{bmatrix}^* - \begin{bmatrix} 0 & 0 \\ 0 & I_{n_w} \end{bmatrix} & \begin{bmatrix} \tilde{B}_1 \\ \tilde{D}_{11} \end{bmatrix} \\ \hline \begin{bmatrix} \tilde{B}_1^* & \tilde{D}_{11}^* \end{bmatrix} & -I_{n_z} \end{array}\right] \left[\begin{array}{c|c} N_c & 0 \\ \hline 0 & I_{n_z} \end{array}\right] \prec 0 \tag{6.16}$$

$$\left[\begin{array}{c|c} \tilde{N}_o & 0 \\ \hline 0 & I_{n_w} \end{array}\right]^* \left[\begin{array}{c|c} \begin{bmatrix} A & \tilde{B}_1 \\ I_{n_p} & 0 \end{bmatrix}^* (\Psi \otimes P) \begin{bmatrix} A & \tilde{B}_1 \\ I_{n_p} & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & I_{n_z} \end{bmatrix} & \begin{bmatrix} C_1^* \\ \tilde{D}_{11}^* \end{bmatrix} \\ \hline \begin{bmatrix} C_1 & \tilde{D}_{11} \end{bmatrix} & -I_{n_w} \end{array}\right] \left[\begin{array}{c|c} \tilde{N}_o & 0 \\ \hline 0 & I_{n_w} \end{array}\right] \prec 0, \tag{6.17}$$

where

$$\begin{aligned} \mathbf{Range}\,(N_c) &= \mathbf{Ker}\left(\begin{bmatrix} B_2^* & D_{12}^* \end{bmatrix}\right) && N_c \text{ has full column rank} \\ \mathbf{Range}\left(\tilde{N}_o\right) &= \mathbf{Ker}\left(\begin{bmatrix} C_2 & \tilde{D}_{21} \end{bmatrix}\right) && \tilde{N}_o \text{ has full column rank} \end{aligned}$$

$$\begin{aligned} P &= \begin{bmatrix} I_{n_p} & 0_{n_c} \end{bmatrix} P_{cl} \begin{bmatrix} I_{n_p} & 0_{n_c} \end{bmatrix}^* \\ Q &= \begin{bmatrix} I_{n_p} & 0_{n_c} \end{bmatrix} P_{cl}^{-1} \begin{bmatrix} I_{n_p} & 0_{n_c} \end{bmatrix}^* \\ P_{cl} &\succ 0, \end{aligned}$$

n_p is the dimension of the state of the plant, x_p, and n_c is the dimension of the state of the controller, x_c, *i.e.*,

$$\begin{aligned} P_{cl} &= \begin{bmatrix} P & * \\ * & * \end{bmatrix} \\ P_{cl}^{-1} &= \begin{bmatrix} Q & * \\ * & * \end{bmatrix}. \end{aligned}$$

Notice that we completely eliminate A_K, B_K, C_K, D_K, and the condition (6.17) and (6.16) are LMIs in P and Q, respectively. However, we still have following two challenges.

1. $\tilde{B}_1, \tilde{D}_{21}$ and $\tilde{N}_o$ depend on $T = \lambda I + Y$ which is a decision variable.
2. P, Q are from P_{cl}, P_{cl}^{-1} which are also decision variables. Therefore an arbitrary pair (P, Q) causes the problem because there may not exist P_{cl} results in P, Q.

To resolve the first part, let us find the equivalent conditions of (6.16) and (6.17). For (6.16), since $\tilde{B}_1 = B_1 T^{-1/2}$, $\tilde{D}_{11} = D_{11} T^{-1/2}$, we have

$$\left[\begin{array}{c|c} \begin{bmatrix} A & I_{n_p} \\ C_1 & 0 \end{bmatrix} (\Psi \otimes Q) \begin{bmatrix} A & I_{n_p} \\ C_1 & 0 \end{bmatrix}^* - \begin{bmatrix} 0 & 0 \\ 0 & I_{n_w} \end{bmatrix} & \begin{bmatrix} \tilde{B}_1 \\ \tilde{D}_{11} \end{bmatrix} \\ \hline \begin{bmatrix} \tilde{B}_1^* & \tilde{D}_{11}^* \end{bmatrix} & -I_{n_z} \end{array}\right]$$

$$= \left[\begin{array}{c|c} I & 0 \\ \hline 0 & T^{-1/2} \end{array}\right]^* \left[\begin{array}{c|c} \begin{bmatrix} A & I_{n_p} \\ C_1 & 0 \end{bmatrix} (\Psi \otimes Q) \begin{bmatrix} A & I_{n_p} \\ C_1 & 0 \end{bmatrix}^* - \begin{bmatrix} 0 & 0 \\ 0 & I_{n_w} \end{bmatrix} & \begin{bmatrix} B_1 \\ D_{11} \end{bmatrix} \\ \hline \begin{bmatrix} B_1^* & D_{11}^* \end{bmatrix} & -T \end{array}\right] \left[\begin{array}{c|c} I & 0 \\ \hline 0 & T^{-1/2} \end{array}\right].$$

Furthermore, using the following identity,

$$\left[\begin{array}{c|c} I & 0 \\ \hline 0 & T^{-1/2} \end{array}\right] \left[\begin{array}{c|c} N_c & 0 \\ \hline 0 & I_{n_z} \end{array}\right] = \left[\begin{array}{c|c} N_c & 0 \\ \hline 0 & I_{n_z} \end{array}\right] \left[\begin{array}{c|c} I & 0 \\ \hline 0 & T^{-1/2} \end{array}\right]$$

together with the invertibility of T, we can conclude that the matrix inequality (6.16) is equivalent to

$$\left[\begin{array}{c|c} N_c & 0 \\ \hline 0 & I_{n_z} \end{array}\right]^* \left[\begin{array}{c|c} \begin{bmatrix} A & I_{n_p} \\ C_1 & 0 \end{bmatrix} (\Psi \otimes Q) \begin{bmatrix} A & I_{n_p} \\ C_1 & 0 \end{bmatrix}^* - \begin{bmatrix} 0 & 0 \\ 0 & I_{n_w} \end{bmatrix} & \begin{bmatrix} B_1 \\ D_{11} \end{bmatrix} \\ \hline \begin{bmatrix} B_1^* & D_{11}^* \end{bmatrix} & -T \end{array}\right] \left[\begin{array}{c|c} N_c & 0 \\ \hline 0 & I_{n_z} \end{array}\right] \prec 0.$$

For (6.17), let us investigate the null space of $\begin{bmatrix} C_2 & \tilde{D}_{21} \end{bmatrix}$ which spans the range space of $\tilde{N}_o$. Since $D_{21} = \tilde{D}_{21} T^{1/2}$, we have $\begin{bmatrix} C_2 & D_{21} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = C_2 v_1 + \tilde{D}_{21} T^{1/2} v_2$. This shows that $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbf{Ker}\left(\begin{bmatrix} C_2 & D_{21} \end{bmatrix}\right)$ if and only if $\begin{bmatrix} v_1 \\ T^{1/2} v_2 \end{bmatrix} \in \mathbf{Ker}\left(\begin{bmatrix} C_2 & \tilde{D}_{21} \end{bmatrix}\right)$. Therefore, for any $\tilde{N}_o$, there exists N_o such that

$$\tilde{N}_o = \begin{bmatrix} I_{n_x} & 0 \\ 0 & T^{1/2} \end{bmatrix} N_o,$$

where $\mathbf{Range}\,(N_o) = \mathbf{Ker}\left(\begin{bmatrix} C_2 & D_{21} \end{bmatrix}\right)$, and $N_o^* N_o = I$.

Thus, (6.17) is equivalent to

$$\left[\begin{array}{c|c} \begin{bmatrix} I_{n_x} & 0 \\ 0 & T^{1/2} \end{bmatrix} N_o & 0 \\ \hline 0 & I_{n_w} \end{array}\right]^* \left[\begin{array}{c|c} \begin{bmatrix} A & \tilde{B}_1 \\ I_{n_p} & 0 \end{bmatrix}^* (\Psi \otimes P) \begin{bmatrix} A & \tilde{B}_1 \\ I_{n_p} & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & I_{n_z} \end{bmatrix} & \begin{bmatrix} C_1^* \\ \tilde{D}_{11}^* \end{bmatrix} \\ \hline \begin{bmatrix} C_1 & \tilde{D}_{11} \end{bmatrix} & -I_{n_w} \end{array}\right] \left[\begin{array}{c|c} \begin{bmatrix} I_{n_x} & 0 \\ 0 & T^{1/2} \end{bmatrix} N_o & 0 \\ \hline 0 & I_{n_w} \end{array}\right]$$

$$= \left[\begin{array}{c|c} N_o & 0 \\ \hline 0 & I \end{array}\right]^* \left[\begin{array}{c|c} \begin{bmatrix} A & B_1 \\ I_{n_p} & 0 \end{bmatrix}^* (\Psi \otimes P) \begin{bmatrix} A & B_1 \\ I_{n_p} & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & T \end{bmatrix} & \begin{bmatrix} C_1^* \\ D_{11}^* \end{bmatrix} \\ \hline \begin{bmatrix} C_1 & D_{11} \end{bmatrix} & -I_{n_w} \end{array}\right] \left[\begin{array}{c|c} N_o & 0 \\ \hline 0 & I \end{array}\right].$$

In the above two expressions, T shows up in the LMI, and all other matrices are completely determined by the problem parameters.

The second challenge is to find a condition on (P, Q) such that there exists $P_{cl} \succ 0$ such that $P = \begin{bmatrix} I_{n_p} & 0_{n_c,n_c} \end{bmatrix} P_{cl} \begin{bmatrix} I_{n_p} & 0_{n_c,n_c} \end{bmatrix}^*$ and $Q = \begin{bmatrix} I_{n_p} & 0_{n_c,n_c} \end{bmatrix} P_{cl}^{-1} \begin{bmatrix} I_{n_p} & 0_{n_c,n_c} \end{bmatrix}^*$. From Lemma 7.9 in [26], a necessary and sufficient condition for this is

$$\begin{aligned} \begin{bmatrix} P & I_{n_p} \\ I_{n_p} & Q \end{bmatrix} &\succeq 0 \\ \mathbf{rank}\left(\begin{bmatrix} P & I_{n_p} \\ I_{n_p} & Q \end{bmatrix} \right) &\leq n_c + n_p \\ P \succ 0, Q \succ 0 \end{aligned}$$

Furthermore, if (P, Q) satisfies the above conditions, we can construct P_{cl} in a following manner. Since $P - Q^{-1} \succeq 0$ from Schur complement, there exists R such that $RR^* = P - Q^{-1}$. Then $P_{cl} = \begin{bmatrix} P & R \\ R^* & I_{n_c} \end{bmatrix}$, and $P_{cl}^{-1} = \begin{bmatrix} Q & -QR \\ -R^*Q & I_{n_c} + R^*QR \end{bmatrix}$.

However, the rank constraint $\mathbf{rank}\left(\begin{bmatrix} P & I_{n_p} \\ I_{n_p} & Q \end{bmatrix} \right) \leq n_c + n_p$ is not a convex constraint, and hard to handle in general.

One way to fix this is to require $n_p \geq n_c$, which makes the rank constraint trivially hold. In other words, when the controller order, n_p, is greater than equal to the plant order n_c, then entire synthesis problem becomes convex. Therefore we assume that $n_p \geq n_c$ so that the resulting synthesis problem is convex.

With all these tools in hand, we obtain the following SDP, which computes the achievable limit of $\bar{\mu}_k^{sdp}(\mathbf{M}_{cl})$ and constructs the corresponding P_{cl}:

$$\begin{aligned}
&\underset{P,Q,Y,\lambda,t}{\text{minimize}} \quad \lambda + k\cdot t\\
&\text{subject to} \quad \begin{bmatrix} P & I_{n_p} \\ I_{n_p} & Q \end{bmatrix} \succeq 0\\
&\left[\begin{array}{c|c} N_c & 0 \\ \hline 0 & I_{n_z}\end{array}\right]^* \left[\begin{array}{c|c} \begin{bmatrix} A & I_{n_p} \\ C_1 & 0 \end{bmatrix} (\Psi \otimes Q) \begin{bmatrix} A & I_{n_p} \\ C_1 & 0 \end{bmatrix}^* - \begin{bmatrix} 0 & 0 \\ 0 & I_{n_w} \end{bmatrix} & \begin{bmatrix} B_1 \\ D_{11} \end{bmatrix} \\ \hline \begin{bmatrix} B_1^* & D_{11}^* \end{bmatrix} & -\lambda I_{n_z} + Y \end{array}\right] \left[\begin{array}{c|c} N_c & 0 \\ \hline 0 & I_{n_z}\end{array}\right] \prec 0\\
&\left[\begin{array}{c|c} N_o & 0 \\ \hline 0 & I\end{array}\right]^* \left[\begin{array}{c|c} \begin{bmatrix} A & B_1 \\ I_{n_p} & 0 \end{bmatrix}^* (\Psi \otimes P) \begin{bmatrix} A & B_1 \\ I_{n_p} & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & \lambda I_{n_z} + Y \end{bmatrix} & \begin{bmatrix} C_1^* \\ D_{11}^* \end{bmatrix} \\ \hline \begin{bmatrix} C_1 & D_{11} \end{bmatrix} & -I_{n_w} \end{array}\right] \left[\begin{array}{c|c} N_o & 0 \\ \hline 0 & I\end{array}\right] \prec 0\\
&\qquad |Y| \leq t \mathbf{1}_{n_w} \mathbf{1}_{n_w}^\top\\
&\qquad P \succ 0, Q \succ 0, Y = Y^*, t \geq 0, \lambda \geq 0.
\end{aligned} \tag{6.18}$$

Notice that the above optimization (6.18) is clearly an SDP. As we mentioned, the optimal value of (6.18) is the achievable lower bound of (6.13). More importantly, we can recover $P_{cl} = \begin{bmatrix} P & R \\ R^* & I_{n_c} \end{bmatrix}$, where $RR^* = P - Q^{-1}$. and this choice of P_{cl} guarantees the existence of controller (A_K, B_K, C_K, D_K) that achieves the optimal value of (6.18).

In order to construct the controller (A_K, B_K, C_K, D_K) that achieves this limit, let us revisit the problem (6.13). Recall that the matrix inequality constraint in (6.13) is given by

$$\textbf{Discrete time:} \quad \begin{bmatrix} A_{cl}^* P_{cl} A_{cl} - P_{cl} + C_{cl}^* C_{cl} & A_{cl}^* P_{cl} B_{cl} + C_{cl}^* D_{cl} \\ B_{cl}^* P_{cl} A_{cl} + D_{cl}^* C_{cl} & D_{cl}^* D_{cl} - \lambda I - Y \end{bmatrix} \prec 0$$

$$\textbf{Continuous time:} \quad \begin{bmatrix} A_{cl}^* P_{cl} + P_{cl} A_{cl} + C_{cl}^* C_{cl} & P_{cl} B_{cl} + C_{cl}^* D_{cl} \\ B_{cl}^* P_{cl} + D_{cl}^* C_{cl} & D_{cl}^* D_{cl} - \lambda I - Y \end{bmatrix} \prec 0.$$

For a given $P_{cl} \succ 0$, by taking Schur complement, we have the following equivalent

LMIs:

$$\textbf{Discrete time:} \quad \begin{bmatrix} -P_{cl}^{-1} & A_{cl} & B_{cl} & 0 \\ A_{cl}^* & -P_{cl} & 0 & C_{cl}^* \\ B_{cl}^* & 0 & -I_{n_w} & D_{cl}^* \\ 0 & C_{cl} & D_{cl} & -\lambda I_{n_z} - Y \end{bmatrix} \prec 0 \tag{6.19}$$

$$\textbf{Continuous time:} \quad \begin{bmatrix} A_{cl}^* P_{cl} + P_{cl} A_{cl} & P_{cl} B_{cl} & C_{cl}^* \\ B_{cl}^* P_{cl} & -I_{n_w} & D_{cl}^* \\ C_{cl} & D_{cl} & -\lambda I_{n_z} - Y \end{bmatrix} \prec 0. \tag{6.20}$$

Therefore, the following SDP constructs (A_K, B_K, C_K, D_K), which achieves the optimal value of (6.18).

$$\textbf{Discrete time:} \quad \begin{aligned} \underset{A_K, B_K, C_K, D_K, Y, \lambda, t}{\text{minimize}} \quad & \lambda + k \cdot t \\ \text{subject to} \quad & (6.19) \\ & A_{cl} = \begin{bmatrix} A + B_2 D_K C_2 & B_2 C_K \\ B_K C_2 & A_K \end{bmatrix} \\ & B_{cl} = \begin{bmatrix} B_1 + B_2 D_K D_{21} \\ B_K D_{21} \end{bmatrix} \\ & C_{cl} = \begin{bmatrix} C_1 + D_{12} D_K C_2 & D_{12} C_K \end{bmatrix} \\ & D_{cl} = \begin{bmatrix} D_{11} + D_{12} D_K D_{21} \end{bmatrix} \\ & |Y| \le t \mathbf{1}_{n_w} \mathbf{1}_{n_w}^\top \\ & Y = Y^*, t \ge 0, \lambda \ge 0. \end{aligned} \tag{6.21}$$

$$\begin{aligned}
&\underset{A_K,B_K,C_K,D_K,Y,\lambda,t}{\text{minimize}} && \lambda + k\cdot t \\
&\text{subject to} && (6.20) \\
& && A_{cl} = \begin{bmatrix} A + B_2D_KC_2 & B_2C_K \\ B_KC_2 & A_K \end{bmatrix} \\
\textbf{Continuous time:}\quad & && B_{cl} = \begin{bmatrix} B_1 + B_2D_KD_{21} \\ B_KD_{21} \end{bmatrix} \\
& && C_{cl} = \begin{bmatrix} C_1 + D_{12}D_KC_2 & D_{12}C_K \end{bmatrix} \\
& && D_{cl} = \begin{bmatrix} D_{11} + D_{12}D_KD_{21} \end{bmatrix} \\
& && |Y| \le t\mathbf{1}_{n_w}\mathbf{1}_{n_w}^\top \\
& && Y = Y^*, t \ge 0, \lambda \ge 0.
\end{aligned} \tag{6.22}$$

We summarize the overall synthesis procedure in which two SDPs is required to be solved as follows:

Optimal sparse $\mathcal{H}_\infty$ norm controller synthesis

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1. Solve (6.18) to obtain P, Q.
2. Construct $P_{cl}^\star = \begin{bmatrix} P & R \\ R^* & I_{n_c} \end{bmatrix}$, where $RR^* = P - Q^{-1}$.
3. Fix $P_{cl}^\star$, and solve (6.21) or (6.22) to construct (A_K, B_K, C_K, D_K).

6.4 Numerical examples

In this section, we demonstrate the usefulness of our approach on various types of systems. As will be seen, in many cases, the gap between our upper and lower bounds

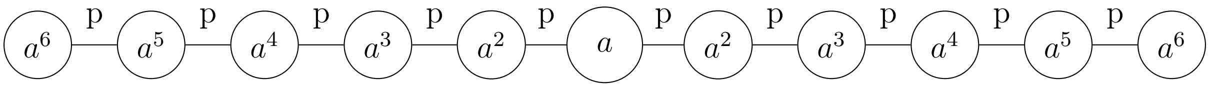


Figure 6.2: Homogeneous chain with $n = 5$.

is very small, indicating that our relaxations are fairly tight. For the optimization, we use CVX [35] with SDPT3 [80].

6.4.1 A linear chain

Here we consider a linear chain with $2n + 1$ nodes. Each subsystem has one internal state, and these states interact with adjacent states. We assume that a disturbance can hit each state with unity gain, and the performance is the total sum of energy in each state. This results in $B = I$, $C = I$, and $D = 0$, and $A \in \mathbb{R}^{2n+1 \times 2n+1}$ has the following form:

$$A_{ij} := \begin{cases} a^{|n+1-i|} & \text{if } i = j \\ p & \text{if } i = j + 1 \text{ or } i = j - 1 \\ 0 & \text{otherwise} \end{cases},$$

where we pick $a = 0.8$, and $p = 0.1$. See the Fig. 6.2.

Thanks to the system's symmetry, we can easily obtain the k-sparse $\mathcal{H}_\infty$ maximizing disturbance analytically. The solution is to select the center node disturbance channel when $k = 1$, and as k increases, including the right (or left) closest node from the center. See Fig. 6.3 for the result. Here the semi-definite relaxation provides an upper bound and our rounding scheme provides a lower bound. Due to its symmetry, the SDP relaxation has a hard time to find the actual solution, but interestingly enough, our rounding scheme returns the true optimal value. We also compute the $\mathcal{H}_\infty$ norm of the system, and the ratio between 1-sparse norm and $\mathcal{H}_\infty$ norm is around 0.85.

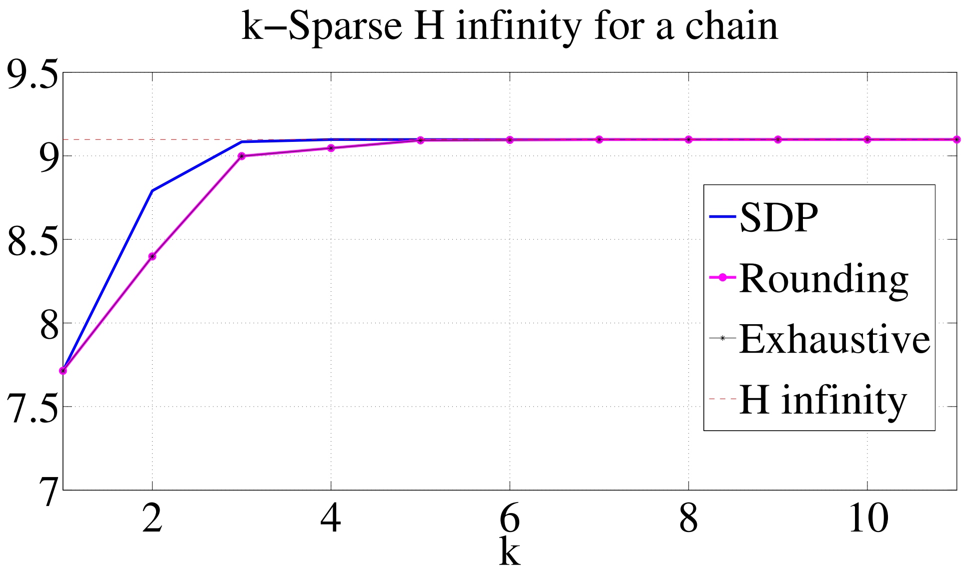


Figure 6.3: Sparse $\mathcal{H}_\infty$ norm of a linear chain computed by SDP, rounding heuristic and exhaustive search.

6.4.2 Random dynamical system

For this example, we construct an Erdős-Renyi graph with (n, p). The weight of each edge is drawn from the standard normal distribution to construct A. Here, $B = 0.1I_n$, $C = I_n$, and $D = 0$.

After obtaining values of 20 samples, we plot the mean of the upper bound and the lower bound (see Fig. 6.4). We also perform exhaustive searches up to $k = 5$ to find the true optimal value. In this case, we can see that the exact solution has matched with our rounding scheme.

6.4.3 Synchronization network

To construct the example of a synchronization network, we choose the Petersen graph for the graph topology.

Based on this topology, we generate two synchronization networks. The first one is based on the maximum degree rule, and the second one is based on the fastest protocol synthesis method via semidefinite programming [83]. See Fig. 6.6 and 6.7 for the result.

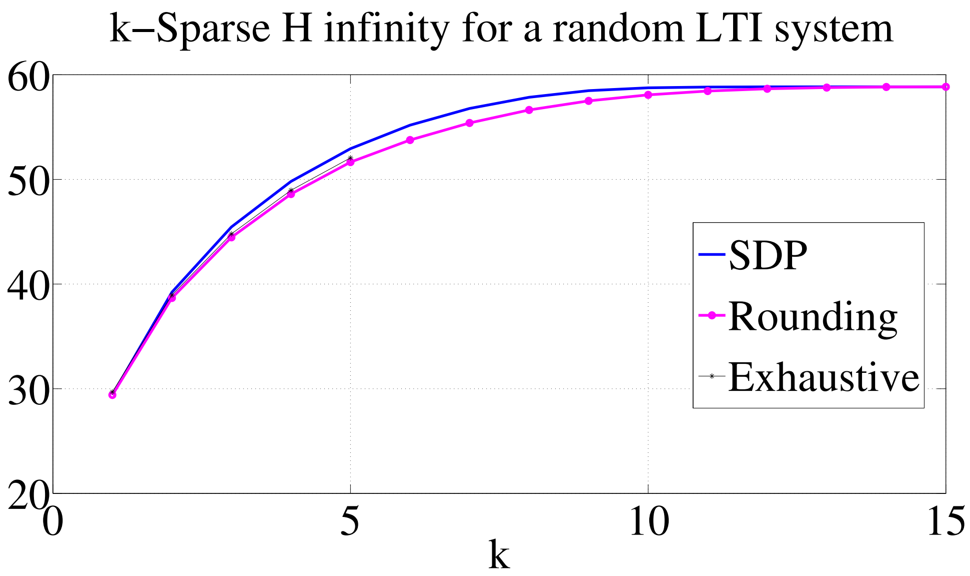


Figure 6.4: Average of sparse $\mathcal{H}_\infty$ norm of random LTI systems computed by SDP, rounding heuristic and exhaustive search.

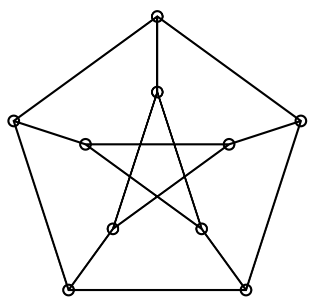


Figure 6.5: The Petersen graph.

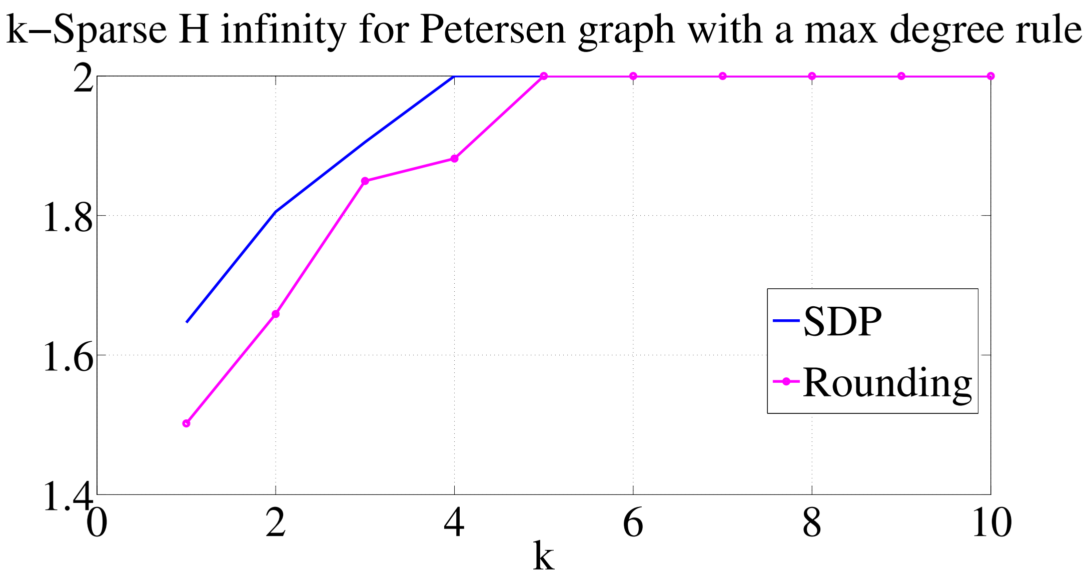


Figure 6.6: Sparse $\mathcal{H}_\infty$ norm of the Petersen synchronization network with the maximum degree rule computed by SDP, rounding heuristic and exhaustive search.

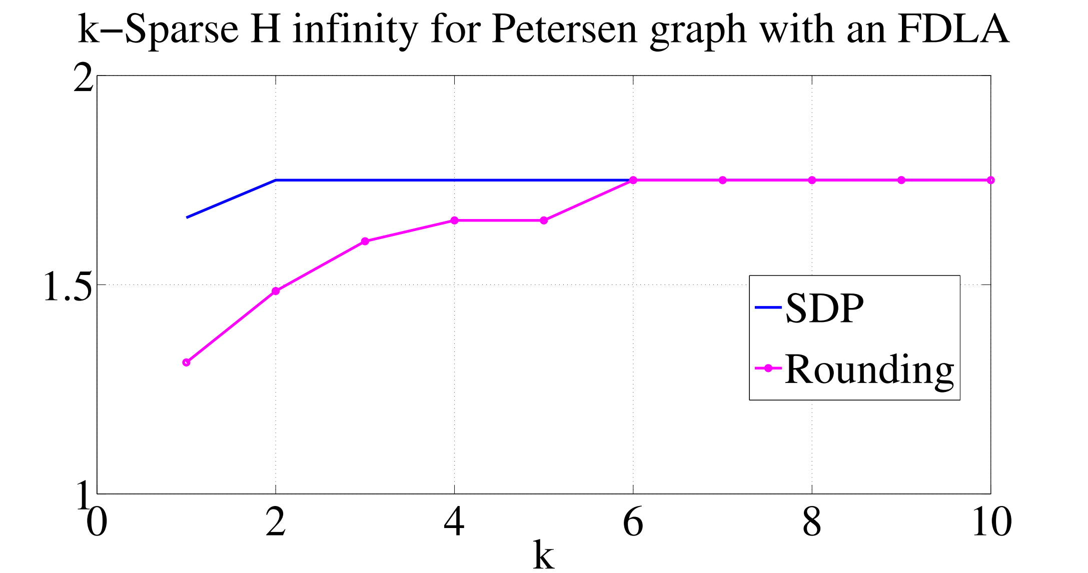


Figure 6.7: Sparse $\mathcal{H}_\infty$ norm of the Petersen synchronization network with the maximum spectral gap computed by SDP, rounding heuristic and exhaustive search.

k-sparse controller	k-sparse $\mathcal{H}_\infty$ norm 1	2	3	$\mathcal{H}_\infty$
1	**1.1826**	1.3939	1.5078	1.7152
2	1.2289	**1.3340**	1.4116	1.5258
3	1.2509	1.3539	**1.4053**	1.5159
$\mathcal{H}_\infty$	1.3832	1.4172	1.4389	**1.5050**

Table 6.1: The k-sparse $\mathcal{H}_\infty$ norms of the closed loop dynamics. The controller is constructed using proposed synthesis method. The minimum value in each column is highlighted.

6.4.4 k-sparse $\mathcal{H}_\infty$ synthesis

To illustrate the effectiveness of our synthesis approach, we apply our method to the following system:

$$A = \begin{bmatrix} 0.5 & 0.2 & 0 \\ 0.2 & 0.5 & 0.2 \\ 0 & 0.2 & 0.5 \end{bmatrix}, \quad B_1 = \begin{bmatrix} I_3 & 0_{3\times3} \end{bmatrix}, \quad B_2 = I_3$$
$$C_1 = \begin{bmatrix} I_3 \\ 0_{3\times3} \end{bmatrix}, \quad D_{11} = 0_{6\times6}, \quad D_{12} = \begin{bmatrix} 0_{3\times3} \\ I_3 \end{bmatrix}$$
$$C_2 = I_3, \quad D_{21} = \begin{bmatrix} 0_{3\times3} & I_3 \end{bmatrix}, \quad D_{22} = 0_{3\times3}.$$

Based on our approach, we obtain the controller that minimizes the SDP relaxation of the k-sparse $\mathcal{H}_\infty$ norm using convex optimization. Then we compute the actual k-sparse $\mathcal{H}_\infty$ norm via exhaustive search to compare the results; see the Table 6.1 for the result.

Since our synthesis method is based on the SDP relaxation of the k-sparse $\mathcal{H}_\infty$ norm, the resulting controller may not be the true optimal controller. However, as we can see, the controllers computed with respect to relaxations of the k-sparse $\mathcal{H}_\infty$ norm exhibit better performance with respect to k disturbances than the general $\mathcal{H}_\infty$ optimal controller. In particular, if only k disturbances are allowed to coordinate their attack, then we see that if a controller is designed to mitigate the worst case effect of a larger number of disturbances, this can in fact lead to a degradation in the closed loop k-spare $\mathcal{H}_\infty$ norm of the system.

Chapter 7

Power network analysis

Power network consists of set of electrical power sources, loads and the transmission lines connecting them. Upon successful steady-state operation, the power network supplies the demand power from the load. The dynamics of power network has a different notion of stability depending on the time-scale. For example, in the sub-second scale, the voltage stability is a primary concern, whereas in the hour level scale, the power flow balance is more important to guarantee the stability of power network.

In this chapter, we focus on the stability of the swing dynamics which captures the transient response of the power network in the second scale. The swing dynamics concerns the power balance in the power network and failure of such assessment may result in the significant impact on the power network: cascading failure, a major source of massive black out.

The stability of the swing dynamics often relies on the large inertia in the synchronous generator and local power system stabilizer, which provides a strong damping in the power network. However, due to recent integration of intermittent, low inertia power sources such as solar panel and wind farm, the stability may be compromised. These sources are essentially the disturbances to the power network.

Using sparse $\mathcal{H}_\infty$ analysis tool, we propose a method to locate the vulnerable nodes in the linearized swing dynamics. Then we verify this result with the more detailed nonlinear swing dynamics model.

7.1 System model

Consider a power network modeled by a connected graph $(\mathcal{N}, \mathcal{E})$, with a set $\mathcal{N}$ of buses and a set $\mathcal{E}$ of power lines connecting the buses. We assume that the power network is initially at the steady state (or equilibrium). The nominal frequency is ω^0, the voltage magnitude at the ith bus is v_i, and the nominal phase angle at the ith bus is θ_i^0. All the variables introduced in the following are deviations from this nominal steady state.

We consider the *linearized* swing dynamics in the following form [5, 94]. At the ith bus,

$$h_i \dot{\omega}_i = -d_i \omega_i + w_i + \sum_{j \in \mathcal{E}} P_{ji}, \qquad \text{for the generator bus} \tag{7.1}$$

$$0 = -d_i \omega_i + w_i + \sum_{j \in \mathcal{E}} P_{ji}, \qquad \text{for the load bus} \tag{7.2}$$

where ω_i is the frequency deviation from the nominal frequency ω^0, h_i is the inertia of the synchronous generator, d_i is a damping term from the frequency sensitive load and power system stabilizer, and w_i is the sum of active power injection at the ith bus. If $w_i > 0$, then ith bus has the additional power injection from renewable sources or unplugged loads, and $w_i < 0$, then ith bus has the excessive power consumption due to the shortage of the renewable sources or plugged loads.

In addition, P_{ij} is the branch power flow from the ith bus to the jth bus, which is given by the following dynamics:

$$\dot{P}_{ij} = y_{ij}(\omega_i - \omega_j), \tag{7.3}$$

where $y_{ij} = \frac{v_i v_j}{x_{ij}} \cos(\theta_i^0 - \theta_j^0)$, v_i is the nominal bus voltage at ith bus, and x_{ij} the reactance of power line (i, j).

In order to apply our analysis method, we firstly need to eliminate the algebraic equation (7.2). This procedure is typically done by the Kron reduction method (see *e.g.*, [22]) which generates a reduced, equivalent circuit model of power dynamics, but this reduced model does not admit the disturbance from the load side. Since we assume that the load can also be subject to the disturbance, let us develop another equivalent model.

Let $M \in \mathbb{R}^{|\mathcal{N}|\times|\mathcal{E}|}$ be the signed incidence matrix of the graph $(\mathcal{N},\mathcal{E})$, P be the branch power flow vector, and ω_G, ω_L be the frequency at the generator bus and the load bus, respectively. Let n_G be the number of generator buses, and n_L be the number of load buses. Then, by partitioning M by $M_1 = \begin{bmatrix} I_{n_G} & 0_{n_L\times n_L} \end{bmatrix} M$, $M_2 = \begin{bmatrix} 0_{n_G\times n_G} & I_{n_L} \end{bmatrix} M$, we have

$$\begin{aligned} H\dot{\omega_G} &= -D_G\omega_G + w_1 + M_1 P \\ 0 &= -D_L\omega_L + w_2 + M_2 P \\ \dot{P} &= -(YM_1^\top \omega_G + YM_2^\top \omega_L), \end{aligned}$$

where H, D_G, D_L, Y are diagonal matrices whose entries are given by h_i, d_i, y_{ij} according to the indices of ω_G, ω_L and P.

From the second equation, we have

$$\omega_L = D_L^{-1}(w_2 + M_2 P),$$

and by substituting this back

$$\begin{aligned} \dot{P} &= YM_1^\top \omega_G + YM_2^\top D_L^{-1}(w_2 + M_2 P) \\ &= YM_1^\top \omega_G + YM_2^\top D_L^{-1} M_2 P + YM_2^\top D_L^{-1} w_2. \end{aligned}$$

Therefore we obtain the following state space model of the linearized swing dynamics.

$$\frac{d}{dt}\begin{bmatrix} \omega_G \\ P \end{bmatrix} = \underbrace{\begin{bmatrix} -H^{-1}D_G & H^{-1}M_1 \\ -YM_1^\top & -YM_2^\top D_L^{-1} M_2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \omega_G \\ P \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} H^{-1} & 0 \\ 0 & -YM_2^\top D_L^{-1} \end{bmatrix}}_{B_w} \underbrace{\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}}_{w}$$

Before presenting the output variable we use, let us investigate the stability of autonomous dynamics $\dot{x} = Ax$. Consider the following Lyapunov function:

$$V(\omega_G, P) := \frac{1}{2}\begin{bmatrix}\omega_G \\ P\end{bmatrix}^\top \begin{bmatrix}H & 0 \\ 0 & Y^{-1}\end{bmatrix}\begin{bmatrix}\omega_G \\ P\end{bmatrix} = \frac{1}{2}\omega_G^\top H\omega_G + \frac{1}{2}P^\top Y^{-1}P.$$

Here the first term is the kinetic energy stored in the synchronous generators, and the second term can be seen as the potential energy in the power network. In fact, this Lyapunov function is closely related to the energy function used in [5] to analyze post-fault transient stability of power network. Notice that since H and Y are positive diagonal matrices, the above Lyapunov function is positive definite.

By taking the time derivative of V, we have

$$\begin{aligned}\frac{dV}{dt} = \frac{1}{2}\begin{bmatrix}\omega_G \\ P\end{bmatrix}^\top &\left(\begin{bmatrix}H & 0 \\ 0 & Y^{-1}\end{bmatrix}\begin{bmatrix}-H^{-1}D_G & H^{-1}M_1 \\ -YM_1^\top & -YM_2^\top D_L^{-1}M_2\end{bmatrix}\right. \\ &\left.+\begin{bmatrix}-D_GH^{-1} & -M_1Y \\ M_1^\top H^{-1} & -M_2^\top D_L^{-1}M_2Y\end{bmatrix}\begin{bmatrix}H & 0 \\ 0 & Y^{-1}\end{bmatrix}\right)\begin{bmatrix}\omega_G \\ P\end{bmatrix} \\ &= \begin{bmatrix}\omega_G \\ P\end{bmatrix}^\top \begin{bmatrix}-D_G & 0 \\ 0 & -M_2^\top D_L^{-1}M_2\end{bmatrix}\begin{bmatrix}\omega_G \\ P\end{bmatrix} \leq 0\end{aligned}$$

for all (ω_G, P), since D_G, D_L are positive diagonal matrices. Therefore, from LaSalle's invariance principle (see *e.g.*, [40]), the autonomous system $\frac{d}{dt}\begin{bmatrix}\omega_G \\ P\end{bmatrix} = A\begin{bmatrix}\omega_G \\ P\end{bmatrix}$ converges to the invariance set

$$\mathcal{I} := \{(\omega_G, P) : \dot{V}(\omega_G, P) = 0\}.$$

Notice that $\dot{V}(\omega_G, P) = 0$ when $\omega_G = 0$ and $M_2P = 0$. In addition, from the dynamics, $\dot{\omega_G} = -D_G\omega_G + M_1P = 0$, we can conclude that $M_1P = 0$ for all elements in $\mathcal{I}$. Therefore,

$$\mathcal{I} = \{(\omega_G, P) : \omega_G = 0, MP = 0\}.$$

Since MP is the sum of the branch power flow at each node bus, in the invariant set, the branch power flow is totally balanced at each bus.

In fact, the above invariant set is the null space of A, $\mathbf{Ker}\,(A)$, and the above Lyapunov function proves that except this 0 eigenvalue, all other eigenvalues of A have negative real part, *i.e.*, A is marginally stable. Since our proposed analysis tool assumes that stable A, it may seem prohibitive to apply our tool to linearized swing dynamics. However it can be shown that this marginally stable mode is uncontrollable. Since we assume that the initial condition is zero, a minimal representation of linearized swing dynamics, which is stable, can be used. See the appendix of this chapter for detailed arguments.

For the output variable, since $V(\omega_G, P)$ can be seen as the energy stored in the power network, we use the following choice of the output:

$$z = \underbrace{\begin{bmatrix} H^{1/2} & 0 \\ 0 & Y^{-1/2} \end{bmatrix}}_{C} \begin{bmatrix} \omega_G \\ P \end{bmatrix},$$

so that

$$\int_0^t z(\tau)^* z(\tau) d\tau = 2 \int_0^t V(\tau) d\tau.$$

7.2 Case study: New England benchmark power network

In this section, we apply k-sparse $\mathcal{H}_\infty$ analysis to linearized swing dynamics, then identify the potential weak spots of the New England 39 bus system [2], a widely used benchmark power network. In addition, we use Power System Toolbox [16], a more realistic power system dynamics simulator which captures nonlinear phenomena, to simulate the New England 39 bus system with disturbances returned by our analysis tool.

7.2.1 Power network data

The New England 39 bus system originally used in [2] and all the parameters can be found online [81] or in the book [61].

Fig. 7.1, 7.2 show the configuration of the New England 39 bus system and active power flow in MW. As the name suggests, it consists of 39 buses, and 9 of them contain the synchronous generator, and the 39th bus is connected to the rest of the US power grid. The totaled sum of supplied power is 61.9293 MW, and consumed power is 61.5050 MW. Detail operating conditions, as well as system parameters, can be found in [2], and we attach those parameters to the appendix of this chapter.

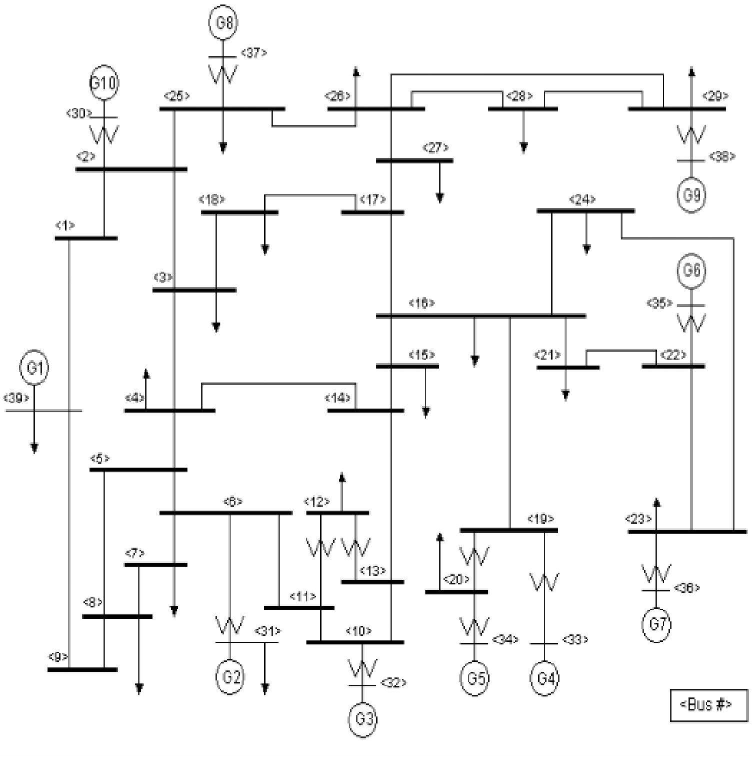


Figure 7.1: New England 39 power network configuration. Figure from [81].

7.2.2 Linearized swing dynamics

Although most of the necessary parameters for (A, B_w, C) can be found in the appendix, the damping terms in the load buses are not provided. In practice, those

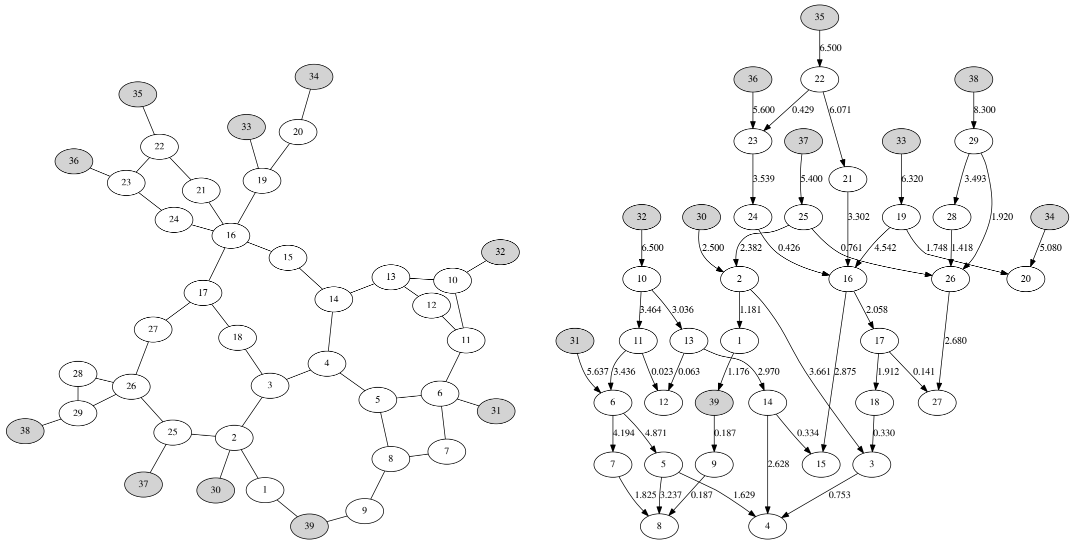
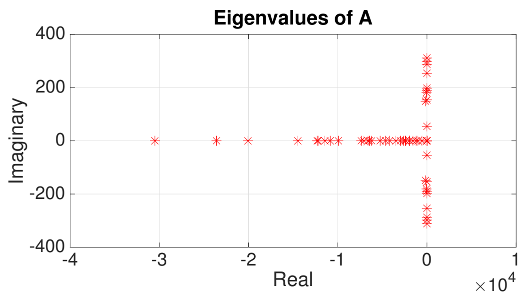


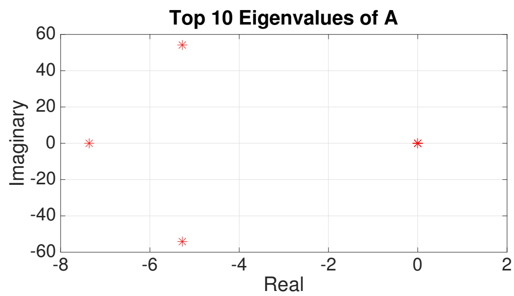
Figure 7.2: Topology of New England 39 power network and active power flow in MW. Grey nodes are synchronous generator buses.

terms are hard to be found exactly, so we assume a small damping term at each load bus, $d_i = 0.1$ in order to construct (A, B_w, C). All those matrices can be found in the appendix of this chapter.

For the eigenvalues of A, see Fig. 7.3. There are 7 zero eigenvalues, and real parts of all other eigenvalues are negative.



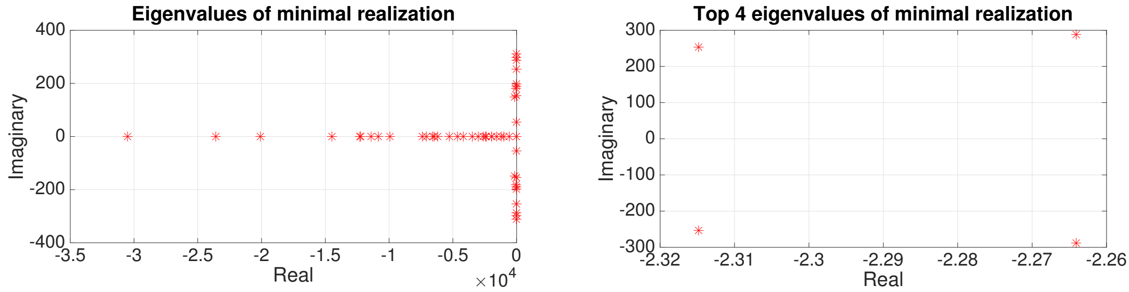
(a) All eigenvalues of A.



(b) Top 10 eigenvalues of A.

Figure 7.3: Eigenvalues of A in the linearized swing dynamics of the New England power network. There are 7 of zero eigenvalues coming from $\mathbf{Ker}\,(M)$, and the system is marginally stable.

We then find the minimal realization of (A, B_w, C). The invariant subspace $\mathbf{Ker}\,(A)$ is eliminated because it is uncontrollable, and this minimal representation is stable. See Fig. 7.4 for the eigenvalues of this minimal representation.



(a) All eigenvalues of the system matrix in a minimal realization.

(b) Top 4 eigenvalues of the system matrix in a minimal realization.

Figure 7.4: Eigenvalues of the system matrix in a minimal realization (A, B_w, C).

However, when we use a semidefinite program to compute $\mathcal{H}_\infty$ norm, CVX [35] together with SDPT3 [80] and MOSEK [1], the solvers fail to find a solution although the system has the $\mathcal{H}_\infty$ norm as 9.8383, which can be obtained other methods, *e.g.*, [7]. We conjecture that this is related to the volume of the relative interior, which is an empty set when (A, B_w) is not controllable. In order to increase numerical stability of the interior point method, we apply the balanced model reduction [50] to the minimal realization.

Fig. 7.5 shows the distribution of Hankel singular values of minimal realization of (A, B_w, C). We truncate the system after the 21th Hankel singular value, and the resulting $\mathcal{H}_\infty$ norm is given by 9.8390. Here the relative error is given by 1%, and we proceed with our analysis with this truncated system. After this treatment, SDPT3 is able to solve all semidefinite programs for $\mathcal{H}_\infty$ analysis as well as k-sparse $\mathcal{H}_\infty$ analysis successfully.

Fig. 7.6 shows the k-sparse $\mathcal{H}_\infty$ norm obtained from rounding heuristic and $\mathcal{H}_\infty$ norm of the system. The gap between k-sparse $\mathcal{H}_\infty$ norm and $\mathcal{H}_\infty$ norm is almost negligible even with a relatively small number of disturbance channel.

We also extract the weak spots in the network based on the solution of each k-sparse $\mathcal{H}_\infty$ analysis and corresponding disturbance using our algorithm in Chapter 3. Fig. 7.7 shows the time simulation with the disturbance from each analysis. Since the extracted worst-case disturbance is always sinusoid, after a transient response, the frequency deviation settles down to the sinusoidal steady state response.

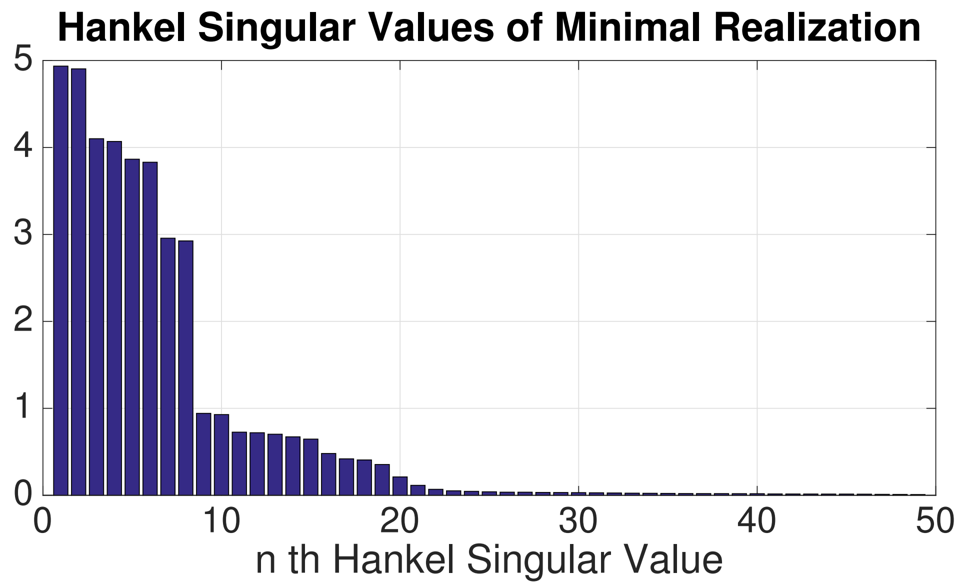


Figure 7.5: Hankel singular values of minimal realizations of (A, B_w, C).

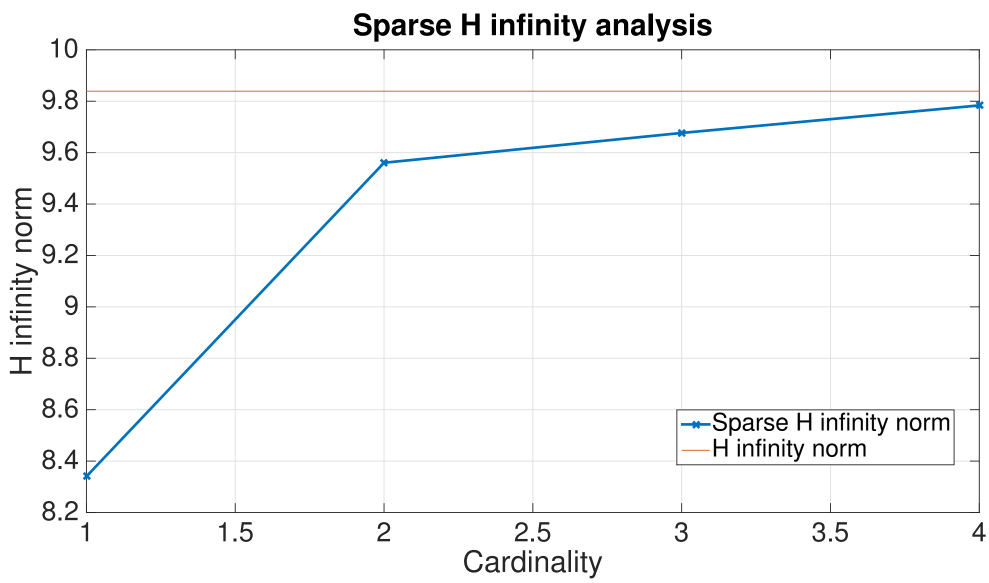
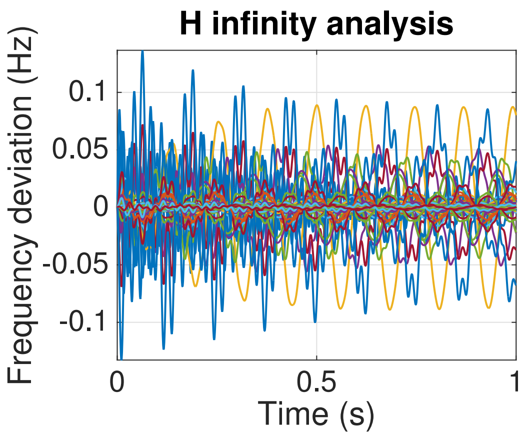


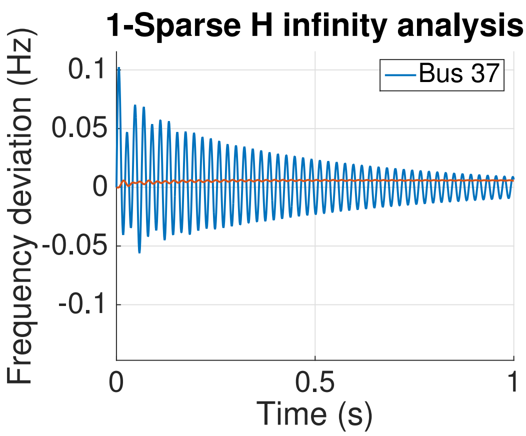
Figure 7.6: $\mathcal{H}_\infty$ and k-sparse $\mathcal{H}_\infty$ norm of the system.

7.2.3 Nonlinear system analysis

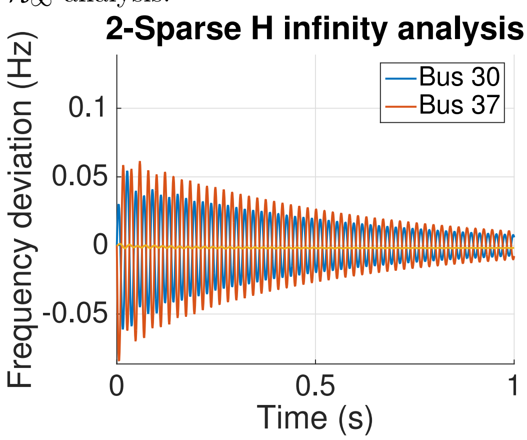
In this section, we simulate nonlinear swing dynamics, a more realistic dynamics model for power grid, of the New England 39 bus system using the Power System Toolbox [16]. In addition, we apply the worst-case sinusoidal disturbance from 1-sparse and 3-sparse $\mathcal{H}_\infty$ analysis of linearized swing dynamics to check the effectiveness of proposed approach, locating the weak spots of the system. For comparison,



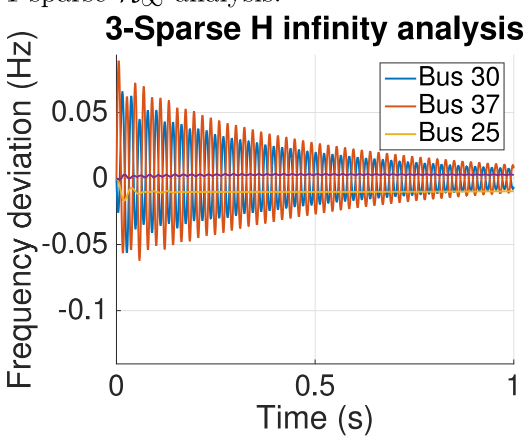
(a) Response with the disturbance from $\mathcal{H}_\infty$ analysis.



(b) Response with the disturbance from 1-sparse $\mathcal{H}_\infty$ analysis.



(c) Response with the disturbance from 2-sparse $\mathcal{H}_\infty$ analysis.



(d) Response with the disturbance from 3-sparse $\mathcal{H}_\infty$ analysis.

Figure 7.7: Frequency deviation at each bus in the linearized swing dynamics with the worst-case disturbance. In the k-sparse $\mathcal{H}_\infty$ analysis case, the buses subject to the disturbance are in the legend in each plot.

we also design two additional sinusoidal disturbances. All these disturbances have the form

$$w(t) = \beta \cos(\theta t) w_0,$$

where β controls the size of the disturbance, $w_0 \in \mathbb{R}^{39}$ with $\|w_0\|_2 = 1$, and $\theta \in \mathbb{R}$. We summarize the disturbances in the Table 7.1. Here the value in Bus i is the value of $[w_0]_i$, otherwise $[w_0]_i = 0$.

We then apply these disturbances to the system using the Power System Toolbox

	1-Sparse	3-Sparse
θ	-0.245	0.3013
Bus 37	1	0.8799
Bus 30	-	-0.4593
Bus 25	-	-0.1222

	1-Arbitrary	3-Arbitrary
θ	0.1	0.1
Bus 1	1	0.8243
Bus 10	-	-0.2085
Bus 30	-	0.5264

Table 7.1: Disturbance configurations. The bus value in the table is the value in the vector w_0, otherwise 0.

up to $T = 30s$. Fig. 7.8 shows the size of the output measured by $\sqrt{\int_{t=0}^{T} z(t)^* z(t) dt}$. The impact of the disturbances from 1-sparse $\mathcal{H}_\infty$ analysis and the arbitrarily designed disturbance hitting Bus 1 is negligible compared to the disturbance from 3-sparse $\mathcal{H}_\infty$ analysis.

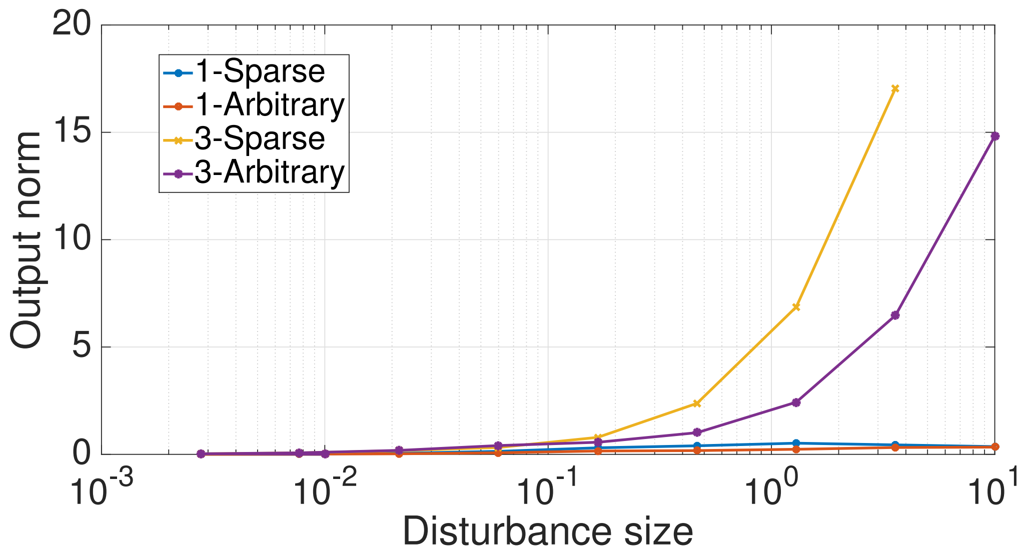
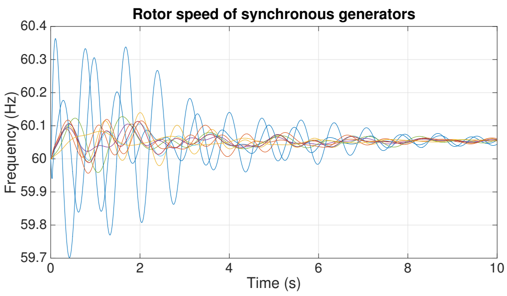


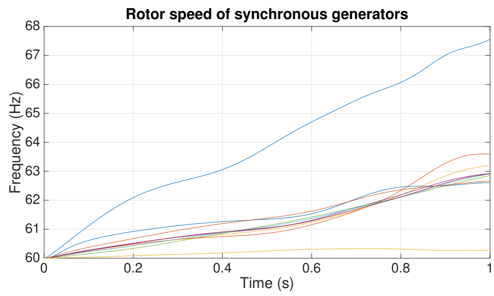
Figure 7.8: Output norm comparison with various disturbance size, β.

As we can see, the disturbance from 3-sparse $\mathcal{H}_\infty$ analysis has the most significant impact on the output norm which shows the great potential of proposed approach. In fact, in the 3-sparse $\mathcal{H}_\infty$ analysis case, for the large size of disturbance $\beta = 10$, the Power System Toolbox stops the simulation because there is no solution of the power flow equation. In other words, the system becomes unstable when we apply the large size of the disturbance from 3-sparse $\mathcal{H}_\infty$ analysis.

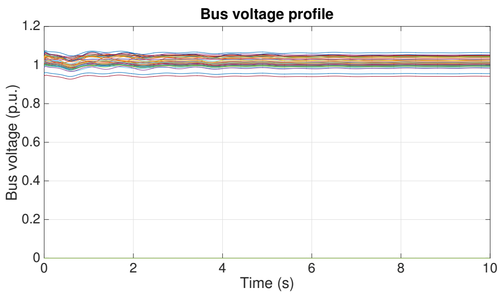
Fig. 7.9 shows the angular velocity of synchronous generators, ω_G, and the voltage profile at each bus when we apply the disturbance from 3-sparse $\mathcal{H}_\infty$ analysis with size $\beta = 1, 10$. When the disturbance size is small ($\beta = 1$), the system remains stable, and converges to another equilibrium point, whereas when the size is large ($\beta = 10$), the simulation breaks down after $t = 1.0s$. As we can see, when $\beta = 10$, the bust voltage starts to collapse, which causes the global instability of the power network. This phenomena is observed that when we apply the disturbance from 3-sparse $\mathcal{H}_\infty$ analysis but not when we apply the arbitrarily designed disturbances. This suggests that the identified buses from 3-sparse $\mathcal{H}_\infty$ analysis are potential weak spots in the power network, and if this type of disturbance is expected to happen, the power system operator should design the additional protection mechanism in order to stop the voltage collapsing, a potential source of massive black out.



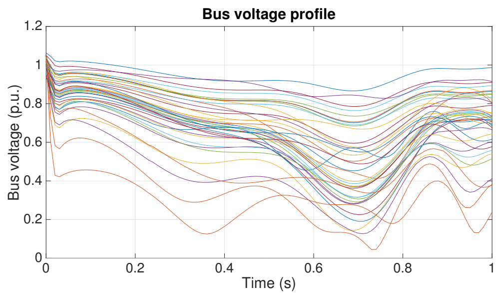
(a) ω_G with $\beta = 1$.



(b) ω_G with $\beta = 10$.



(c) $|V|$ with $\beta = 1$.



(d) $|V|$ with $\beta = 10$

Figure 7.9: Angular velocity of synchronous generators, ω_G, and the magnitude of bus voltages, $|V|$, at each bus when applying the disturbance from 3-sparse $\mathcal{H}_\infty$ analysis. With $\beta = 8$, the system becomes unstable.

In summary, we identify the $25, 30, 37$ buses as the vulnerable spots of New England 39 power network as well as the worst-case disturbance using our proposed sparse $\mathcal{H}_\infty$ analysis tool, and show that with large size of disturbance, we can collapse the voltage which may cause the global instability of the corresponding power network.

This case study suggests that our proposed analysis tool may serve as a tool for addressing robustness of the power network by identifying vulnerable buses, although more in-depth study with various real power network operation data is needed to confirm the effectiveness of our approach.

Chapter 8

Conclusion and Future work

So far we have developed a novel mathematical foundation for robust control theory. Based on our new observation on a set of Gramians, extended $\mathcal{H}_\infty$ analysis is proposed to handle various disturbance models that can be described by the quadratic information of a system. Moreover, this Gramian based approach is also proven to be effective for well-connectedness analysis. An SDP duality theory together with our key lemmas on Gramians gives us a fruitful, rich extension of existing results and unifies the proofs in an elegant fashion. Therefore, it may be possible to open up new research directions by leveraging this novel formulation. As one such example, sparse $\mathcal{H}_\infty$ analysis is proposed, and its SDP relaxation as well as an optimal controller synthesis method is developed. Finally, sparse $\mathcal{H}_\infty$ analysis is used to identify the potential weak spots in the power networks, and the case study shows the potential of this approach.

Here are possible future research directions.

- **New robustness measure**

  In $\mathcal{H}_\infty$ analysis, the objective function is an output norm, which is a linear function of Gramians and this objective function results in an SDP. Since the feasible set is convex, any convex function of Gramains results in a convex program to solve. For example, the following objective

$$\log\det\left(\sum_{k=0}^{\infty} x[k]x[k]^*\right)$$

can be represented by a Gramian $V = \sum_{k=0}^{\infty} \begin{bmatrix} x[k] \\ w[k] \end{bmatrix} \begin{bmatrix} x[k] \\ w[k] \end{bmatrix}^*$

$$\log\det\left(\begin{bmatrix} I & 0 \end{bmatrix} V \begin{bmatrix} I & 0 \end{bmatrix}^*\right).$$

Compared to the norm type objective such as $\sum_{k=0}^{\infty} x[k]^* x[k] = \mathbf{Tr}\left(\begin{bmatrix} I & 0 \end{bmatrix} V \begin{bmatrix} I & 0 \end{bmatrix}^*\right)$, the above log-det objective quantifies the "area" of ellipsoid spanned by the state $\mathbf{x}$, instead of focusing on the total size of each directions. With the unit norm disturbance, $\|\mathbf{w}\|_2 = 1$, the corresponding robustness analysis becomes

$$\begin{aligned} \underset{V}{\text{maximize}} \quad & \log\det\left(\begin{bmatrix} I & 0 \end{bmatrix} V \begin{bmatrix} I & 0 \end{bmatrix}^*\right) \\ \text{subject to} \quad & \begin{bmatrix} A & B \end{bmatrix} V \begin{bmatrix} A & B \end{bmatrix}^* = \begin{bmatrix} I & 0 \end{bmatrix} V \begin{bmatrix} I & 0 \end{bmatrix}^* \\ & \mathbf{Tr}\left(\begin{bmatrix} 0 & I \end{bmatrix} V \begin{bmatrix} 0 & I \end{bmatrix}^*\right) = 1 \\ & V \succeq 0, \end{aligned} \tag{8.1}$$

which is a max-det optimization [82], which is a convex program that can be solved using a polynomial time algorithm. In addition, the dual program of (8.1) is given by

$$\begin{aligned} \underset{S,P,\lambda}{\text{minimize}} \quad & \lambda - \log\det(S) - n \\ \text{subject to} \quad & \begin{bmatrix} A^*PA - P + S & A^*PB \\ B^*PA & B^*PB - \lambda I \end{bmatrix} \preceq 0 \\ & S \succeq 0, \end{aligned} \tag{8.2}$$

where n is the dimension of the state $\mathbf{x}$. For comparison, consider the dual program of $\mathcal{H}_\infty$ analysis with $C = I_n$, and $D = 0$:

$$\begin{aligned} \underset{P,\lambda}{\text{minimize}} \quad & \lambda \\ \text{subject to} \quad & \begin{bmatrix} A^*PA - P + C^*C & A^*PB \\ B^*PA & B^*PB - \lambda I \end{bmatrix} \preceq 0. \end{aligned} \tag{8.3}$$

Since $C^*C \succeq 0$, we can easily see that if we choose $S = C^*C$ in (8.2), then the (8.2) and (8.3) are equivalent. In this point of view, the adversary not only tries to design the disturbance to maximize the overall gain of the system, but also design the output measurement matrix C. In terms of the primal optimization (8.1), since the goal is to maximize the area of the ellipsoid $\sum_{k=0}^{\infty} x[k]x[k]^*$, the adversary does not know which output is going to be chosen, and so maximizes the entire area, hoping that any measurements are equally bad. Depending on the application, this new type of robustness analysis may be useful. Other objective functions can be used, and as long as the objective is convex in terms of a Gramian, the corresponding optimization remains as a convex program.

- **Efficient algorithm for robustness analysis**

  Although all the optimization and LMI test in this dissertation in the form of convex optimization, it heavily relies on the semidefinite programming. A generic semidefinite programming solver can handle only up to hundreds of variables. This being said, for a large scale system, the existing off-the-shelf algorithm may not be an appropriate tool for robustness analysis. Fortunately, in the KYP lemma case, exploiting special structures in the LMI results in an efficient algorithm [8, 63]. Therefore, it may be possible to exploit the structure in the LMI from extended $\mathcal{H}_\infty$ analysis, sparse $\mathcal{H}_\infty$ analysis, and well-connectedness analysis, to design an efficient algorithm. More importantly, if the system has special properties, such as positivity, it has been shown that the variables in the LMI can be reduced significantly [78, 66]. Therefore exploiting structure may allow us more scalable, and efficient algorithm for robustness analysis.

- **Extension to nonlinear systems**

  For the polynomial system, for example, $x[k+1] = a_1x[k] + a_2x^2[k] + w[k]$, we can also define the Gram matrix $V = \begin{bmatrix} \langle x^2, x^2\rangle & \langle x^2, x\rangle & \langle x^2, w\rangle \\ \langle x, x^2\rangle & \langle x, x\rangle & \langle x, w\rangle \\ \langle w, x^2\rangle & \langle w, x\rangle & \langle w, w\rangle \end{bmatrix} \succeq 0$, and find a necessary condition for V. In the linear system case, this also provides a sufficient condition, but in the nonlinear system case, we may have to go up higher dimension including x^3, x^4, and so on. This may give us the natural hierarchical approach to a nonlinear input-output analysis.

Appendix A

Proof of Lemma 3.1 and 3.3

A.1 Useful facts from linear algebra

The first formula we need is Gelfand's formula [31], which relates a matrix norm $\|A\|$ and the spectral radius $\rho(A)$. The proof can also be found in [44].

Theorem A.1 (Gelfand, 1941): *For all $A \in \mathbb{C}^{n\times n}$, $\lim_{k\to\infty} \|A^k\|^{1/k} = \rho(A)$.*

An example of a matrix norm includes the operator norm of A,

$$\|A\|_{2,2} = \sup_{\|x\|_2=1} \|Ax\|_2 = \sigma_{\max}(A),$$

where $\|x\|_2 = \sqrt{x^*x}$, and $\sigma_{\max}(A)$ is the maximum singular value of A. Throughout in this appendix, we shall use this norm for a matrix.

The following observation is also useful.

Proposition A.1: *For $x, y \in \mathbb{C}^n$, $\|xy^*\|_F = \|x\|_2\|y\|_2$.*

Proof: $\|xy^*\|_F^2 = \mathbf{Tr}((xy^*)^*(yx^*)) = \|x\|_2^2\|y\|_2^2$. ■

A.2 Extreme points of $\mathcal{D}_{\text{SDP}}$

The following lemma is crucial to analyze the extreme points of $\mathcal{D}_{\text{SDP}}$.

Lemma A.1 (Rantzer, 1996): *Let F, G complex matrices with same dimensions. Then $FF^* = GG^*$ if and only if there exists a unitary matrix U such that $F = GU$.*

PROOF: The original proof can be found in [65], but we present alternative proof based on singular value decomposition. Since $FF^* = GG^*$, the left singular vectors of F and G are the same and so are singular values. Therefore $F = U_L \Sigma U_F^*$, $G = U_L \Sigma U_G^*$, where U_L, U_F, U_G are unitary matrices. By setting $U = U_G U_F^*$, we can conclude that $F = GU$. For the converse direction, $FF^* = (GU)(GU)^* = GUU^*G^* = GG^*$. ■

From the above lemma, we can conclude that any non-zero element in $\mathcal{D}_{\mathrm{SDP}}$ can be represented as a sum of rank one matrices in $\mathcal{D}_{\mathrm{SDP}}$.

Proposition A.2 (Rank one decomposition): *For all $V \in \mathcal{D}_{SDP} \subset \mathbb{H}^{n_x+n_w}$, there exists $V_1, \cdots, V_{n_x+n_w} \in \mathcal{D}_{SDP}$ such that $V = \sum_{k=1}^{n_x+n_w} V_k$ and $\mathbf{rank}\,(V_k) \leq 1$ for $k = 1, \cdots, n_x + n_w$.*

PROOF: Suppose $V \in \mathcal{D}_{\mathrm{SDP}}$. Since $V \succeq 0$, we can decompose $V = V^{1/2}V^{1/2}$. Then, from

$$\underbrace{\begin{bmatrix} I_{n_x} & 0_{n_x \times n_w} \end{bmatrix} V^{1/2}}_{F} \underbrace{V^{1/2} \begin{bmatrix} I_{n_x} & 0_{n_x \times n_w} \end{bmatrix}^*}_{F^*} = \underbrace{\begin{bmatrix} A & B \end{bmatrix} V^{1/2}}_{G} \underbrace{V^{1/2} \begin{bmatrix} A & B \end{bmatrix}^*}_{G^*},$$

there must exist a unitary matrix U such that

$$\begin{bmatrix} I_{n_x} & 0_{n_x \times n_w} \end{bmatrix} V^{1/2} = \begin{bmatrix} A & B \end{bmatrix} V^{1/2} U.$$

Being unitary, $U = \sum_{k=1}^{r} e^{\mathbf{i}\theta_k} u_k u_k^*$. Let $V_k = V^{1/2} u_k u_k^* V^{1/2} \succeq 0$. Clearly, $\mathbf{rank}\,(V_k) \leq 1$. Moreover, $\sum_k V_k = V^{1/2} \sum_k u_k u_k^* V^{1/2} = V^{1/2}V^{1/2} = V$.

Finally, since $\begin{bmatrix} I_{n_x} & 0_{n_x \times n_w} \end{bmatrix} V^{1/2} u_k = \begin{bmatrix} A & B \end{bmatrix} V^{1/2} U u_k = e^{\mathbf{i}\theta_k} \begin{bmatrix} A & B \end{bmatrix} V^{1/2} u_k$, we have $\begin{bmatrix} I_{n_x} & 0_{n_x \times n_w} \end{bmatrix} V_k \begin{bmatrix} I_{n_x} & 0_{n_x \times n_w} \end{bmatrix}^* = \begin{bmatrix} A & B \end{bmatrix} V_k \begin{bmatrix} A & B \end{bmatrix}^*$ which shows $V_k \in \mathcal{D}_{\mathrm{SDP}}$. ■

Note that the eigenvalue decomposition of V may not work because there is no guarantee that each eigenvector of V is in $\mathcal{D}_{\mathrm{SDP}}$.

A.3 Extreme points of $\mathcal{C}_{\text{SDP}}$

The following lemma is crucial to analyze the extreme points of $\mathcal{C}_{\text{SDP}}$.

Lemma A.2 (Rantzer, 1996): *Let F, G complex matrices with same dimension. Then $FG^* + GF^* = 0$ if and only if there exists a unitary matrix U such that $F + G = (F - G)U$.*

Proof: Notice that $FG^* + GF^* = 0$ is equivalent to $(F+G)(F+G)^* = (F-G)(F-G)^*$. Invoking Lemma A.1, we can conclude the proof. ∎

As in the discrete time case, any element in $\mathcal{C}_{\text{SDP}}$ can be expressed by a sum of rank one matrices.

Proposition A.3 (Rank one decomposition): *For all $V \in \mathcal{C}_{SDP} \subset \mathbb{H}^{n_x+n_w}$, there exists $V_1, \cdots, V_{n_x+n_w} \in \mathcal{C}_{SDP}$ such that $V = \sum_{k=1}^{n_x+n_w} V_k$ and* **rank** $(V_k) \leq 1$ *for $k = 1, \cdots, n_x + n_w$.*

Proof: Suppose $V \in \mathcal{C}_{\text{SDP}}$. Since $V \succeq 0$, we can decompose $V = V^{1/2}V^{1/2}$. Then, from

$$\underbrace{\begin{bmatrix} I_{n_x} & 0_{n_x \times n_w} \end{bmatrix} V^{1/2}}_{F} \underbrace{V^{1/2} \begin{bmatrix} A & B \end{bmatrix}^*}_{G^*} + \underbrace{\begin{bmatrix} A & B \end{bmatrix} V^{1/2}}_{G} \underbrace{V^{1/2} \begin{bmatrix} I_{n_x} & 0_{n_x \times n_w} \end{bmatrix}^*}_{F^*} = 0,$$

there must exist a unitary matrix U such that

$$\begin{bmatrix} A + I_{n_x} & B \end{bmatrix} V^{1/2} = \begin{bmatrix} A - I_{n_x} & B \end{bmatrix} V^{1/2} U.$$

Since U is unitary, $U = \sum_{k=1}^{r} e^{\mathbf{i}\theta_k} u_k u_k^*$.

Let $V_k = V^{1/2} u_k u_k^* V^{1/2} \succeq 0$. Clearly, **rank** $(V_k) \leq 1$. Moreover, $\sum_k V_k = V^{1/2} \sum_k u_k u_k^* V^{1/2} = V^{1/2}V^{1/2} = V$. Finally, from

$$\begin{bmatrix} A + I_{n_x} & B \end{bmatrix} V^{1/2} u_k = \begin{bmatrix} A - I_{n_x} & B \end{bmatrix} V^{1/2} U u_k = e^{\mathbf{i}\theta_k} \begin{bmatrix} A - I_{n_x} & B \end{bmatrix} V^{1/2} u_k,$$

we have

$$\begin{aligned}&\left(\begin{bmatrix}A + I_{n_x} & B\end{bmatrix} V^{1/2} u_k\right)\left(\begin{bmatrix}A + I_{n_x} & B\end{bmatrix} V^{1/2} u_k\right)^* \\ = \;&\left(\begin{bmatrix}A - I_{n_x} & B\end{bmatrix} V^{1/2} u_k\right)\left(\begin{bmatrix}A - I_{n_x} & B\end{bmatrix} V^{1/2} u_k\right)^*.\end{aligned}$$

By rearranging terms, we can conclude that

$$\begin{bmatrix}A & B\end{bmatrix} V_k \begin{bmatrix}I_{n_x} & 0_{n_x \times n_w}\end{bmatrix}^* + \begin{bmatrix}I_{n_x} & 0_{n_x \times n_w}\end{bmatrix} V_k \begin{bmatrix}A & B\end{bmatrix}^* = 0,$$

and this shows $V_k \in \mathcal{C}_{\text{SDP}}$. ∎

A.4 Some useful bounds on the size of A^k and e^{At}

In this section, we will show that for all positive integers p, $\sum_{k=0}^{\infty} \left\|A^k\right\|_{2,2}^p < \infty$ when A is Schur stable, and $\int_0^{\infty} \left\|e^{At}\right\|_{2,2}^p dt < \infty$ when A is Hurwitz stable. These facts are being used to control the transient error terms in the main proof.

From Gelfand's formula, Theorem A.1, the following consequence is immediate.

Proposition A.4: *Suppose $\rho(A) < 1$. For all integers $p \geq 1$, $\sum_{k=0}^{\infty} \left\|A^k\right\|_{2,2}^p < \infty$.*

PROOF: Let $\epsilon = (1 - \rho(A))/2 > 0$. Then from Theorem A.1, there exists $N \in \mathbb{N}$ such that

$$\left\|A^k\right\|_{2,2} < (\rho(A) + \epsilon)^k,$$

for all $k \geq N$. Then,

$$\sum_{k=0}^{\infty} \left\|A^k\right\|_{2,2}^p = \sum_{k=0}^{N} \left\|A^k\right\|_{2,2}^p + \sum_{k=N+1}^{\infty} \left\|A^k\right\|_{2,2}^p < \sum_{k=0}^{N} \left\|A^k\right\|_{2,2}^p + \sum_{k=N+1}^{\infty} (\rho(A) + \epsilon)^{kp}.$$

Since $\rho(A) + \epsilon < 1$, the second term in the last inequality is finite. ∎

For the continuous time case, we need the following bound on e^{At}. The summary of this type of result can also be found in [36].

Proposition A.5: *For a square matrix A, let $\mu = \max_i\{\mathbf{Re}(\lambda_i(A))\}$, where $\mathbf{Re}(\lambda_i(A))$ are the real part of ith eigenvalue of A. Then for any $\alpha > \mu$, there exists a constant β such that*

$$\left\|e^{At}\right\|_{2,2} \leq \beta e^{\alpha t}.$$

PROOF: Let the Jordan canonical form of $A = TJT^{-1}$. Then $e^{At} = Te^{Jt}T^{-1}$. Recall that the matrix exponential of the ith Jordan block with the multiplicity k is given by

$$e^{J_i t} = e^{\lambda_i t}\begin{bmatrix} 1 & \cdots & \frac{1}{(k-1)!}t^{k-1} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}.$$

Since $\mathbf{Re}(\lambda_i) - \alpha < 0$, we can conclude that $\lim_{t\to\infty} e^{-\alpha t}e^{J_i t} = 0$, and therefore $\lim_{t\to\infty} e^{-\alpha t}e^{Jt} = 0$. Notice that

$$\left\|e^{-\alpha t}e^{At}\right\|_{2,2} = \left\|e^{-\alpha t}Te^{Jt}T^{-1}\right\|_{2,2} \leq \|T\|_{2,2}\left\|T^{-1}\right\|_{2,2}\left\|e^{-\alpha t}e^{Jt}\right\|_{2,2}.$$

Therefore $\lim_{t\to\infty}\left\|e^{-\alpha t}e^{At}\right\|_{2,2} = 0$, and from continuity, there exists β such that

$$\left\|e^{-\alpha t}e^{At}\right\|_{2,2} \leq \beta.$$

■

The above proposition shows the exponential convergence nature of $\left\|e^{At}\right\|_{2,2}$ when A is Hurwitz stable. By leveraging this fact, we can show that $\left\|e^{At}\right\|_{2,2}$ is in $\mathcal{L}_p$, a set of pth power integrable functions.

Proposition A.6: *Suppose A is Hurwitz stable. For all integers $p \geq 1$, $\int_0^\infty \left\|e^{At}\right\|_{2,2}^p dt < \infty$.*

PROOF: Let $\mu = \max_i\{\mathbf{Re}(\lambda_i(A))\}$, where $\mathbf{Re}(\lambda_i(A))$ are the real part of ith eigenvalue of A. Since A is Hurwitz, $\mu < 0$, we can therefore find α such that $\mu < \alpha < 0$.

From Proposition A.5, there exists β such that

$$\left\|e^{At}\right\|_{2,2} \leq \beta e^{\alpha t}.$$

Then,

$$\int_0^\infty \left\|e^{At}\right\|_{2,2}^p dt \leq \int_0^\infty \beta^p e^{\alpha p t} dt = -\frac{\beta^p}{\alpha p},$$

since $\alpha < 0$. ∎

A.5 Proof of Lemma 3.1

A.5.1 Technical lemmas

In this section, we provide technical results to find a bound on the terms appears in the proof of our key lemma, Lemma 3.1. Since we only consider a stable LTI system, the system dynamics A is assumed to be Schur stable. Recall that $\mathbf{x} = \mathbf{M}_{AB}(\mathbf{w})$, if

$$x[k+1] = Ax[k] + Bw[k],$$

where $x[0] = 0$, $x[k] \in \mathbb{C}^{n_x}$, $w[k] \in \mathbb{C}^{n_w}$.

The first result concerns a finite truncation of $\Lambda(\mathbf{u})$:

$$\Lambda_N(\mathbf{u}) = \sum_{k=0}^{N-1} u[k]u[k]^*.$$

Proposition A.7: *Suppose* $\mathbf{w}$ *has finite number of non-zero entries. For any* $\varepsilon > 0$*, there exists* $N \in \mathbb{N}$*, such that for all* $n \geq N$*,*

$$\left\|\Lambda(\mathbf{M}_{AB}(\mathbf{w}), \mathbf{w}) - \Lambda_n(\mathbf{M}_{AB}(\mathbf{w}), \mathbf{w})\right\|_F < \varepsilon.$$

PROOF: Let $\mathbf{x} = \mathbf{M}_{AB}\mathbf{w}$, and $T = \text{argmin}\{T : w[k] = 0, \text{for all } k \geq T\}$. Then $x[k] = A^{k-T}x[T]$ for all $k \geq T$.

For $N \geq T$, we have

$$\|\Lambda(\mathbf{M}_{AB}(\mathbf{w}), \mathbf{w}) - \Lambda_N(\mathbf{M}_{AB}(\mathbf{w}), \mathbf{w})\|_F = \left\| \sum_{k=N}^{\infty} A^{k-T}x[T](A^{k-T}x[T])^* \right\|_F$$

$$\leq \sum_{k=N}^{\infty} \|A^{k-T}x[T]\|_2^2 \leq \|A^{N-T}x[T]\|_2^2 \sum_{k=0}^{\infty} \left\|A^k\right\|_{2,2}^2.$$

From Proposition A.4, $\sum_{k=0}^{\infty} \left\|A^k\right\|_{2,2}^2 < \infty$. Therefore

$$\lim_{N\to\infty} \|\Lambda(\mathbf{M}_{AB}(\mathbf{w}), \mathbf{w}) - \Lambda_N(\mathbf{M}_{AB}(\mathbf{w}), \mathbf{w})\|_F = 0,$$

since $\lim_{N\to\infty} \|A^{N-T}x[T]\|_2^2 = 0$. ∎

Recall that any signal in $\mathbf{x} \in l_2$, $\lim_{k\to\infty} \|x[k]\|_2 = 0$. Therefore, the infinity norm $\|\mathbf{x}\|_\infty = \max_k \|x[k]\|_2 < \infty$.

Proposition A.8: *Let* $\mathbf{x}, \mathbf{w} \in l_2$*, and* $\mathbf{y} \in l_2$ *such that* $y[k] = A^k y_0$ *for all* $k \geq 0$*. Then there exists a constant* C *such that*

$$\|\Lambda(\mathbf{x} + \mathbf{y}, \mathbf{w}) - \Lambda(\mathbf{x}, \mathbf{w})\| \leq C \max\{(\|\mathbf{x}\|_\infty + \|\mathbf{w}\|_\infty)\|y_0\|_2, \|y_0\|_2^2\}.$$

Proof: Notice that,

$$\begin{aligned}
&\|\Lambda(\mathbf{x}+\mathbf{y},\mathbf{w})-\Lambda(\mathbf{x},\mathbf{w})\|_F \\
=\;& \left\|\sum_{k=0}^{\infty}\begin{bmatrix}x[k]+y[k]\\ w[k]\end{bmatrix}\begin{bmatrix}x[k]+y[k]\\ w[k]\end{bmatrix}^* - \begin{bmatrix}x[k]\\ w[k]\end{bmatrix}\begin{bmatrix}x[k]\\ w[k]\end{bmatrix}^*\right\|_F \\
=\;& \left\|\sum_{k=0}^{\infty}\begin{bmatrix}y[k]\\ 0\end{bmatrix}\begin{bmatrix}x[k]\\ w[k]\end{bmatrix}^* + \begin{bmatrix}x[k]\\ w[k]\end{bmatrix}\begin{bmatrix}y[k]\\ 0\end{bmatrix}^* + \begin{bmatrix}y[k]\\ 0\end{bmatrix}\begin{bmatrix}y[k]\\ 0\end{bmatrix}^*\right\|_F \\
\leq\;& \sum_{k=0}^{\infty} 2\left\|\begin{bmatrix}y[k]\\ 0\end{bmatrix}\begin{bmatrix}x[k]\\ w[k]\end{bmatrix}^*\right\|_F + \left\|\begin{bmatrix}y[k]\\ 0\end{bmatrix}\begin{bmatrix}y[k]\\ 0\end{bmatrix}^*\right\|_F \\
=\;& \sum_{k=0}^{\infty} 2\|y[k]\|_2\sqrt{\|x[k]\|_2^2+\|w[k]\|_2^2}+\|y[k]\|_2^2.
\end{aligned}$$

Since $\mathbf{x},\mathbf{w}\in l_2$, we have

$$\sqrt{\|x[k]\|_2^2+\|w[k]\|_2^2}\leq\|\mathbf{x}\|_\infty+\|\mathbf{w}\|_\infty,$$

for all k. Moreover, since $y[k]=A^k y_0$, we have

$$\begin{aligned}
&\|\Lambda(\mathbf{x}+\mathbf{y},\mathbf{w})-\Lambda(\mathbf{x},\mathbf{w})\|_F \\
\leq\;& \sum_{k=0}^{\infty} 2(|\mathbf{x}\|_\infty+\|\mathbf{w}\|_\infty)\|A^k y_0\|_2+\|A^k y_0\|_2^2 \\
\leq\;& \sum_{k=0}^{\infty} 2(|\mathbf{x}\|_\infty+\|\mathbf{w}\|_\infty)\left\|A^k\right\|_{2,2}\|y_0\|_2+\left\|A^k\right\|_{2,2}^2\|y_0\|_2^2 \\
\leq\;& C\max\{(\|\mathbf{x}\|_\infty+\|\mathbf{w}\|_\infty)\|y_0\|_2,\|y_0\|_2^2\},
\end{aligned}$$

where $C=\sum_{k=0}^{\infty}2\|A^k\|_2+\left\|A^k\right\|_{2,2}^2$. From Proposition A.4, $C<\infty$, and this concludes the proof. ■

A.5.2 Proof of Lemma 3.1

Now we are ready to prove Lemma 3.1. The main idea of the proof is as follows.

- Firstly, we will show that any rank one matrix in $\mathcal{D}_{\mathrm{SDP}}$ can be generated by a

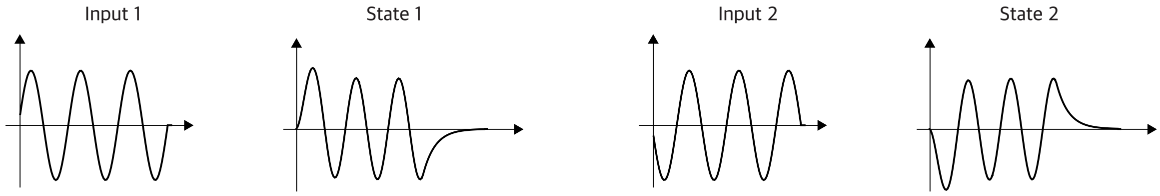


Figure A.1: Examples of inputs and states for V in Proposition A.9. By controlling the size of the input, the contribution from the transient response can be made small.

sinusoidal input $\mathbf{w}$. However since sinusoids are not in l_2, we construct an l_2 signal which approximates the sinusoidal input using finite number of non-zero entries. In $\mathcal{H}_\infty$ analysis [25], the worst case signal is sinusoid which is not in l_2. So one has to approximate this sinusoid using an l_2 signal and the supremum is not achieved. Therefore, this procedure should not be surprising.

- For a matrix in $\mathcal{D}_{\mathrm{SDP}}$ with arbitrary rank, we use Proposition A.2 to decompose it to rank one matrices in $\mathcal{D}_{\mathrm{SDP}}$. Since rank one matrices in $\mathcal{D}_{\mathrm{SDP}}$ can be approximated by an input with finite non-zero entries, we pad them together to approximate the target matrix in $\mathcal{D}_{\mathrm{SDP}}$. This padding idea also has been used in robustness analysis with linear time-varying uncertainty [74].

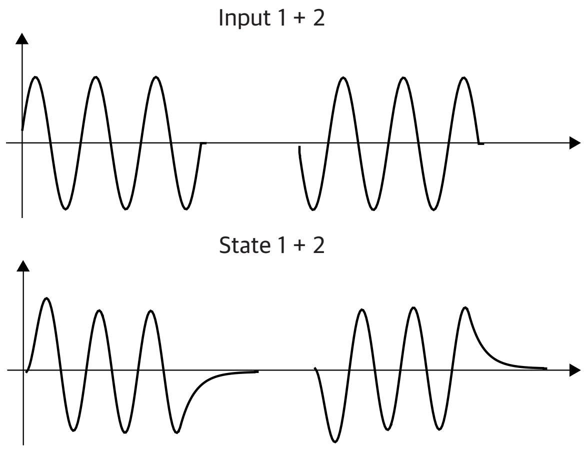


Figure A.2: Example of inputs and states for V in Lemma 3.1. By padding inputs, we can approximate a matrix in $\mathcal{D}_{\mathrm{SDP}}$ arbitrarily close.

Proposition A.9: *Suppose $V \in \mathcal{D}_{SDP}$, and* $\mathbf{rank}\,(V) \leq 1$. *Then for all $\varepsilon > 0$, there exists* $\mathbf{w}$ *with a finite number of non-zero entries such that*

$$\|\Lambda(\mathbf{Mw}, \mathbf{w}) - V\|_F < \varepsilon \tag{A.1}$$

$$\Lambda(\mathbf{w}) = \begin{bmatrix} 0_{n_w \times n_x} & I_{n_w} \end{bmatrix} V \begin{bmatrix} 0_{n_w \times n_x} & I_{n_w} \end{bmatrix}^*. \tag{A.2}$$

PROOF: Suppose $V = 0$. Then $\mathbf{w} = 0$ satisfies (A.1) - (A.2).

Now, suppose $V \in \mathcal{D}_{\mathrm{SDP}}$ and $\mathbf{rank}\,(V) = 1$. Then there exists x_s, w_s such that $V = \begin{bmatrix} x_s \\ w_s \end{bmatrix} \begin{bmatrix} x_s \\ w_s \end{bmatrix}^*$. Since $V \in \mathcal{D}_{\mathrm{SDP}}$, we have

$$\underbrace{\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} x_s \\ w_s \end{bmatrix}}_{f} \underbrace{\begin{bmatrix} x_s \\ w_s \end{bmatrix}^* \begin{bmatrix} A & B \end{bmatrix}^*}_{f^*} = \underbrace{\begin{bmatrix} I_{n_x} & 0_{n_x \times n_w} \end{bmatrix} \begin{bmatrix} x_s \\ w_s \end{bmatrix}}_{g} \underbrace{\begin{bmatrix} x_s \\ w_s \end{bmatrix}^* \begin{bmatrix} I_{n_x} & 0_{n_x \times n_w} \end{bmatrix}^*}_{g^*},$$

and from Lemma A.1, there exists $\theta \in \mathbb{R}$ such that $e^{\mathbf{i}\theta} x_s = Ax_s + Bw_s$.

Let $\mathbf{w}_N$ be

$$w_N[k] \;=\; \begin{cases} \frac{1}{\sqrt{N}} e^{\mathbf{i}\theta k} w_s & \text{if } 0 \leq k < N \\ 0 & \text{if } N \leq k. \end{cases}$$

Then w_N has N number of non-zero entries. In addition, $\Lambda(\mathbf{w}_N) = \sum_{k=0}^{\infty} w_N[k] w_N[k]^* = \sum_{k=0}^{N-1} w_N[k] w_N[k]^* = w_s w_s^* = \begin{bmatrix} 0_{n_w \times n_x} & I_{n_w} \end{bmatrix} V \begin{bmatrix} 0_{n_w \times n_x} & I_{n_w} \end{bmatrix}^*$, and therefore $\mathbf{w}_N$ satisfies (A.2).

For $\mathbf{x} = \mathbf{M}_{AB}(\mathbf{w}_N)$, let us define the following signals, $\mathbf{x}_p$ and $\mathbf{x}_h$:

$$x_p[k] \;=\; \begin{cases} \frac{1}{\sqrt{N}} e^{\mathbf{i}\theta k} x_s & \text{if } 0 \leq k < N \\ \frac{1}{\sqrt{N}} A^{k-N} e^{\mathbf{i}\theta N} x_s & \text{if } N \leq k, \end{cases}$$

$$x_h[k] \;=\; -\frac{1}{\sqrt{N}} A^k x_s,$$

where $\mathbf{x}_p$ is the particular solution due to the input $\mathbf{w}_N$, and $\mathbf{x}_h$ is the homogeneous solution due to the initial condition $x[0] = 0$. We can easily check $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$ by substituting it to $x[k+1] = Ax[k] + Bw[k]$.

Notice that

$$\Lambda_N(\mathbf{x}_p, \mathbf{w}_N) = \sum_{k=0}^{N-1} \begin{bmatrix} x_p[k] \\ w_N[k] \end{bmatrix} \begin{bmatrix} x_p[k] \\ w_N[k] \end{bmatrix}^* = \frac{1}{N} \sum_{k=0}^{N-1} \begin{bmatrix} x_s \\ w_s \end{bmatrix} \begin{bmatrix} x_s \\ w_s \end{bmatrix}^* = V.$$

Therefore,

$$\begin{aligned} \|\Lambda(\mathbf{x}_p, \mathbf{w}_N) - V\|_F &= \left\| \sum_{k=N}^{\infty} \begin{bmatrix} x_p[k] \\ w_N[k] \end{bmatrix} \begin{bmatrix} x_p[k] \\ w_N[k] \end{bmatrix}^* \right\|_F \\ \leq \frac{1}{N} \sum_{k=0}^{\infty} \left\| \begin{bmatrix} A^k x_s \\ 0 \end{bmatrix} \begin{bmatrix} A^k x_s \\ 0 \end{bmatrix}^* \right\|_F &\leq \frac{1}{N} \sum_{k=0}^{\infty} \|A^k x_s\|_2^2 \\ \leq \underbrace{\sum_{k=0}^{\infty} \|A^k\|_{2,2}^2}_{C_1} \frac{\|x_s\|_2^2}{N}, \end{aligned}$$

where $C_1 < \infty$ from Proposition A.4. Finally,

$$\begin{aligned} & \|\Lambda(\mathbf{x}, \mathbf{w}_N) - V\|_F = \|\Lambda(\mathbf{x}_p + \mathbf{x}_h, \mathbf{w}_N) - V\|_F \\ \overset{(a)}{\leq} \; & \|\Lambda(\mathbf{x}_p, \mathbf{w}_N) - V\|_F + \|\Lambda(\mathbf{x}_p + \mathbf{x}_h, \mathbf{w}_N) - \Lambda(\mathbf{x}_p, \mathbf{w}_N)\|_F \\ \overset{(b)}{\leq} \; & C_1 \frac{\|x_s\|_2^2}{N} + C_2 \max\{(\|\mathbf{x}_p\|_\infty + \|\mathbf{w}_N\|_\infty)\|x_h[0]\|_2, \|x_h[0]\|_2^2\}, \end{aligned}$$

for some positive constant $C_2 < \infty$. Here (a) is from the triangle inequality and (b) is from Proposition A.8.

Since the initial condition of the homogenous solution $x_h[0] = -\frac{1}{\sqrt{N}} x_s$, we have $\|x_h[0]\|_2 = \frac{1}{\sqrt{N}} \|x_s\|_2$. In addition, from the definition of $\mathbf{x}_p$, we can easily see that $\|\mathbf{x}_p\|_\infty \propto \frac{1}{\sqrt{N}}$. By combining all these bounds, we can conclude that there exists a positive constant C which only depends on A, x_s, w_s such that

$$\|\Lambda(\mathbf{M}_{AB}(\mathbf{w}_N), \mathbf{w}_N) - V\|_F \leq \frac{C}{N}.$$

Therefore, $\lim_{N\to\infty} \|\Lambda(\mathbf{M}_{AB}(\mathbf{w})_N, \mathbf{w}_N) - V\|_F = 0$. ■

Now we are ready to present the proof of our key lemma, Lemma 3.1, which considers V with an arbitrary rank.

Proof (Proof of Lemma 3.1): From Proposition A.2, $V = \sum_{i=1}^{n+m} V_i$ where $V_i \in \mathcal{D}_{\text{SDP}}$, and $\mathbf{rank}\,(V_i) \leq 1$. Let us rearrange these terms, so that $V = \sum_{i=1}^{r} V_i$ where $\mathbf{rank}\,(V_i) = 1$. We now use an induction argument on r, that is there exists $\mathbf{w}$ with finite non-zero entries such that

$$\|\Lambda(\mathbf{M}_{AB}(\mathbf{w}), \mathbf{w}) - V\|_F < \varepsilon$$
$$\begin{bmatrix} 0_{n_w \times n_x} & I_{n_w} \end{bmatrix} V \begin{bmatrix} 0_{n_w \times n_x} & I_{n_w} \end{bmatrix}^* = \Lambda(\mathbf{w}),$$

for all $\varepsilon > 0$.

Suppose $r \leq 1$, then from Proposition A.9, the proof is done.

Assume the induction hypothesis holds. Then for $\sum_{i=1}^{r-1} V_i \in \mathcal{D}_{\text{SDP}}$, there exists $\tilde{\mathbf{w}}$ with finite non-zero entries such that

$$\left\|\Lambda(\mathbf{M}_{AB}(\tilde{\mathbf{w}}), \tilde{\mathbf{w}}) - \sum_{i=1}^{r-1} V_i\right\|_F < \frac{1}{4}\varepsilon,$$
$$\Lambda(\tilde{\mathbf{w}}) = \sum_{i=1}^{r-1} \begin{bmatrix} 0_{n_w \times n_x} & I_{n_w} \end{bmatrix} V_i \begin{bmatrix} 0_{n_w \times n_x} & I_{n_w} \end{bmatrix}^*.$$

Similarly, for $V_r \in \mathcal{D}_{\text{SDP}}$, there exists $\hat{\mathbf{w}}$ with finite non-zero entries such that

$$\|\Lambda(\mathbf{M}\hat{\mathbf{w}}, \hat{\mathbf{w}}) - V_r\|_F < \frac{1}{4}\varepsilon,$$
$$\Lambda(\hat{\mathbf{w}}) = \begin{bmatrix} 0_{n_w \times n_x} & I_{n_w} \end{bmatrix} V_r \begin{bmatrix} 0_{n_w \times n_x} & I_{n_w} \end{bmatrix}^*.$$

Let $T = \text{argmin}\{T : \tilde{w}[k] = 0, \text{ for all } k \geq T\}$. From Proposition A.7, we can find N_1 such that

$$\|\Lambda(\mathbf{M}_{AB}(\tilde{\mathbf{w}}), \tilde{\mathbf{w}}) - \Lambda_{n+T}(\mathbf{M}_{AB}(\tilde{\mathbf{w}}), \tilde{\mathbf{w}})\|_F < \frac{1}{4}\epsilon,$$

for all $n \geq N_1$.

Consider the following signal $\mathbf{w}$

$$w[k] = \begin{cases} \tilde{w}[k] & \text{if } 0 \leq k < N+T \\ \hat{w}[k-N-T] & \text{if } N+T \leq k, \end{cases}$$

where $N \geq N_1$. Clearly, $\mathbf{w}$ has a finite number of non-zero entries, and $\Lambda(\mathbf{w}) = \Lambda(\tilde{\mathbf{w}}) + \Lambda(\hat{\mathbf{w}})$, which shows

$$\Lambda(\mathbf{w}) = \begin{bmatrix} 0_{n_w \times n_x} & I_{n_w} \end{bmatrix} V \begin{bmatrix} 0_{n_w \times n_x} & I_{n_w} \end{bmatrix}^*.$$

Let $\tilde{\mathbf{x}} = \mathbf{M}_{AB}(\tilde{\mathbf{w}})$, $\hat{\mathbf{x}} = \mathbf{M}_{AB}(\hat{\mathbf{w}})$, and $\mathbf{x} = \mathbf{M}_{AB}(\mathbf{w})$. Then,

$$x[k] = \begin{cases} \tilde{x}[k] & \text{if } 0 \leq k < T \\ A^{k-T}\tilde{x}[T] & \text{if } T \leq k < N+T \\ \hat{x}[k-N-T] + A^{k-T}\tilde{x}[T] & \text{if } N+T \leq k. \end{cases}$$

Notice that

$$\begin{aligned} \Lambda(\mathbf{x}, \mathbf{w}) &= \Lambda_{N+T}(\tilde{\mathbf{x}}, \tilde{\mathbf{w}}) \\ &\quad + \sum_{k=N+T}^{\infty} \begin{bmatrix} \hat{x}[k-N-T] + A^{k-T}\tilde{x}[T] \\ \hat{w}[k-N-T] \end{bmatrix} \begin{bmatrix} \hat{x}[k-N-T] + A^{k-T}\tilde{x}[T] \\ \hat{w}[k-N-T] \end{bmatrix}^* \\ &= \Lambda_{N+T}(\tilde{\mathbf{x}}, \tilde{\mathbf{w}}) + \Lambda(\hat{\mathbf{x}} + \mathbf{y}, \hat{\mathbf{w}}), \end{aligned}$$

where $y[k] = A^{k+N}\tilde{x}[T]$.

Therefore,

$$\begin{aligned}
& \|\Lambda(\mathbf{x},\mathbf{w}) - V\|_F \\
\le\ & \left\|\Lambda_{N+T}(\tilde{\mathbf{x}},\tilde{\mathbf{w}}) - \sum_{i=1}^{r-1} V_i\right\|_F + \|\Lambda(\hat{\mathbf{x}}+\mathbf{y},\hat{\mathbf{w}}) - V_r\|_F \\
\le\ & \|\Lambda_{N+T}(\tilde{\mathbf{x}},\tilde{\mathbf{w}}) - \Lambda(\tilde{\mathbf{x}},\tilde{\mathbf{w}})\|_F + \left\|\Lambda(\tilde{\mathbf{x}},\tilde{\mathbf{w}}) - \sum_{i=1}^{r-1} V_i\right\|_F \\
& + \|\Lambda(\hat{\mathbf{x}}+\mathbf{y},\hat{\mathbf{w}}) - \Lambda(\hat{\mathbf{x}},\hat{\mathbf{w}})\|_F + \|\Lambda(\hat{\mathbf{x}},\hat{\mathbf{w}}) - V_r\|_F \\
\le\ & \frac{3}{4}\epsilon + C\max\{(\|\hat{\mathbf{x}}\|_\infty + \|\hat{\mathbf{w}}\|_\infty)\|y[0]\|_2, \|y[0]\|_2^2\},
\end{aligned}$$

where the first and second inequalities are from the triangle inequality, and the last inequality is from Proposition A.8.

Notice that $\hat{\mathbf{x}}, \hat{\mathbf{w}}$ do not depend on N, the spacing between $\tilde{\mathbf{w}}$ and $\hat{\mathbf{w}}$ in $\mathbf{w}$, whereas $\|y[0]\|_2 = \|A^N\tilde{x}[T]\|$. Therefore, the second term in the last inequality goes to zero as $N \to \infty$, and by taking sufficiently large N, we can bound the last term by $\frac{1}{4}\epsilon$. Now it is easy to see that $\tilde{V} = \Lambda(\mathbf{x},\mathbf{w})$ gives the result. ■

Remark A.1: *The above proof constructs an input* $\mathbf{w}$ *which approximates* $V \in \mathcal{D}_{SDP}$*. In terms of optimization, once we obtain the optimal solution* $V^\star \in \mathcal{D}_{SDP}$*, we can always find an input* $\mathbf{w} \in l_2$ *that achieves the same optimal value in the limit.*

A.6 Proof of Lemma 3.3

We basically need to modify the previous result to the continuous-time setting, but the proof strategy is identical.

A.6.1 Technical lemmas

For a Hurwtiz stable A, recall that $\mathbf{x} = \mathbf{M}_{AB}(\mathbf{w})$ if

$$\frac{dx(t)}{dt} = Ax(t) + Bw(t),$$

where $x(0) = 0$.

Likewise, the first result considers the finite truncation of $\Lambda(\mathbf{u})$:

$$\Lambda_T(\mathbf{u}) = \int_0^T u(t)u(t)^* dt.$$

Proposition A.10: *Suppose $\mathbf{w} \in \mathcal{L}_2$ has finite support. Then, for any $\varepsilon > 0$, there exists $T \geq 0$, such that for all $t \geq T$,*

$$\|\Lambda(\mathbf{M}_{AB}(\mathbf{w}), \mathbf{w}) - \Lambda_t(\mathbf{M}_{AB}(\mathbf{w}), \mathbf{w})\|_F < \varepsilon.$$

PROOF: Let $\mathbf{x} = \mathbf{M}_{AB}(\mathbf{w})$, and $\tau = \operatorname{argmin}\{\tau : w(t) = 0, \text{for all } t \geq \tau\}$. Then $x(t) = e^{A(t-\tau)}x(\tau)$, for all $t \geq \tau$.

For $T \geq \tau$, we have

$$\begin{aligned} & \|\Lambda(\mathbf{M}_{AB}(\mathbf{w}), \mathbf{w}) - \Lambda_T(\mathbf{M}_{AB}(\mathbf{w}), \mathbf{w})\|_F \\ = & \left\| \int_T^\infty e^{A(t-\tau)} x(\tau) x(\tau)^* e^{A^*(t-\tau)} dt \right\|_F \leq \int_T^\infty \left\| e^{A(t-\tau)} x(\tau) \right\|_2^2 dt \\ \leq & \left\| e^{A(T-\tau)} x(\tau) \right\|_2^2 \int_0^\infty \left\| e^{At} \right\|_{2,2}^2 dt. \end{aligned}$$

From Proposition A.6, $\int_0^\infty \left\| e^{At} \right\|_{2,2}^2 dt < \infty$. Therefore

$$\lim_{T\to\infty} \|\Lambda(\mathbf{M}_{AB}(\mathbf{w}), \mathbf{w}) - \Lambda_T(\mathbf{M}_{AB}(\mathbf{w}), \mathbf{w})\|_F = 0,$$

and together with continuity, this concludes the proof. ∎

Recall that any signal in $\mathbf{x} \in \mathcal{L}_2$, the infinity norm $\|\mathbf{x}\|_\infty = \text{ess. sup}_t \|x(t)\|_2$ is finite.

Proposition A.11: *Let $\mathbf{x}, \mathbf{w} \in \mathcal{L}_2$, and $\mathbf{y} \in \mathcal{L}_2$ such that $y(t) = e^{At}y_0$ for all $t \geq 0$.*

Then there exists a constant C such that

$$\begin{aligned} & \|\Lambda(\mathbf{x}+\mathbf{y},\mathbf{w})-\Lambda(\mathbf{x},\mathbf{w})\| \\ \leq\ & C\max\{(|\mathbf{x}\|_\infty+\|\mathbf{w}\|_\infty)\|y_0\|_2, \|y_0\|_2^2\}. \end{aligned}$$

PROOF: Notice that,

$$\begin{aligned} & \|\Lambda(\mathbf{x}+\mathbf{y},\mathbf{w})-\Lambda(\mathbf{x},\mathbf{w})\|_F \\ = & \left\| \int_0^\infty \begin{bmatrix} x(t)+y(t) \\ w(t) \end{bmatrix} \begin{bmatrix} x(t)+y(t) \\ w(t) \end{bmatrix}^* - \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}^* dt \right\|_F \\ = & \left\| \int_{t=0}^\infty \begin{bmatrix} y(t) \\ 0 \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}^* + \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \begin{bmatrix} y(t) \\ 0 \end{bmatrix}^* + \begin{bmatrix} y(t) \\ 0 \end{bmatrix} \begin{bmatrix} y(t) \\ 0 \end{bmatrix}^* dt \right\|_F \\ \leq & \int_0^\infty 2 \left\| \begin{bmatrix} y(t) \\ 0 \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}^* \right\|_F + \left\| \begin{bmatrix} y(t) \\ 0 \end{bmatrix} \begin{bmatrix} y(t) \\ 0 \end{bmatrix}^* \right\|_F dt \\ = & \int_0^\infty 2\|y(t)\|_2 \sqrt{\|x(t)\|_2^2+\|w(t)\|_2^2} + \|y(t)\|_2^2 dt. \end{aligned}$$

Since $\mathbf{x}, \mathbf{w} \in \mathcal{L}_2$, we have

$$\text{ess. sup}_t \sqrt{\|x(t)\|_2^2+\|w(t)\|_2^2} \leq \|\mathbf{x}\|_\infty + \|\mathbf{w}\|_\infty.$$

Moreover, since $y(t) = e^{At}y_0$, we have

$$\begin{aligned} & \|\Lambda(\mathbf{x}+\mathbf{y},\mathbf{w})-\Lambda(\mathbf{x},\mathbf{w})\|_F \\ \leq & \int_0^\infty 2(|\mathbf{x}\|_\infty+\|\mathbf{w}\|_\infty)\|e^{At}y_0\|_2 + \|e^{At}y_0\|_2^2 dt \\ \leq & \int_0^\infty 2(|\mathbf{x}\|_\infty+\|\mathbf{w}\|_\infty) \left\|e^{At}\right\|_{2,2} \|y_0\|_2 + \left\|e^{At}\right\|_{2,2}^2 \|y_0\|_2^2 dt \\ \leq & \ C\max\{(|\mathbf{x}\|_\infty+\|\mathbf{w}\|_\infty)\|y_0\|_2, \|y_0\|_2^2\}, \end{aligned}$$

where $C = \int_0^\infty 2\ \left\|e^{At}\right\|_{2,2} + \left\|e^{At}\right\|_{2,2}^2 dt$. From Proposition A.6, it is clear that $C < \infty$.∎

A.6.2 Proof of Lemma 3.3

Now we are ready to prove Lemma 3.3.

Proposition A.12: *Suppose $V \in \mathcal{C}_{SDP}$, and* $\textbf{rank}\,(V) \le 1$. *Then for all $\varepsilon > 0$, there exists* $\mathbf{w}$ *with finite support such that*

$$\|\Lambda(\mathbf{M}_{AB}(\mathbf{w}), \mathbf{w}) - V\|_F < \varepsilon \tag{A.3}$$

$$\Lambda(\mathbf{w}) = \begin{bmatrix} 0_{n_w \times n_x} & I_{n_w} \end{bmatrix} V \begin{bmatrix} 0_{n_w \times n_x} & I_{n_w} \end{bmatrix}^*. \tag{A.4}$$

PROOF: Suppose $V = 0$. Then $\mathbf{w} = 0$ satisfies (A.3) - (A.4).

Now, consider $V \in \mathcal{C}_{\text{SDP}}$ such that $\textbf{rank}\,(V) = 1$. Then there exists x_s, w_s such that $V = \begin{bmatrix} x_s \\ w_s \end{bmatrix} \begin{bmatrix} x_s \\ w_s \end{bmatrix}^*$. Since $V \in \mathcal{C}_{\text{SDP}}$, we have

$$\underbrace{\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} x_s \\ w_s \end{bmatrix}}_{f} \underbrace{\begin{bmatrix} x_s \\ w_s \end{bmatrix}^* \begin{bmatrix} I_{n_x} & 0_{n_x \times n_w} \end{bmatrix}^*}_{g^*} + \underbrace{\begin{bmatrix} I_{n_x} & 0_{n_x \times n_w} \end{bmatrix} \begin{bmatrix} x_s \\ w_s \end{bmatrix}}_{g} \underbrace{\begin{bmatrix} x_s \\ w_s \end{bmatrix}^* \begin{bmatrix} A & B \end{bmatrix}^*}_{f^*} = 0,$$

and from Lemma A.2, there exists $\theta \in \mathbb{R}$ such that $Ax_s + Bw_s + x_s = e^{\mathbf{i}\theta}(Ax_s + Bw_s - x_s)$. In other words, $(e^{\mathbf{i}\theta} + 1)x_s = (e^{\mathbf{i}\theta} - 1)(Ax_s + Bw_s)$.

Firstly, let us consider the case where $\theta \neq 0$. In this case, $\frac{e^{\mathbf{i}\theta}+1}{e^{\mathbf{i}\theta}-1}x_s = Ax_s + Bw_s$. In addition, since $\frac{e^{\mathbf{i}\theta}+1}{e^{\mathbf{i}\theta}-1}$ is purely imaginary, let $\omega = \mathbf{Im}\left(\frac{e^{\mathbf{i}\theta}+1}{e^{\mathbf{i}\theta}-1}\right)$, then

$$\mathbf{i}\omega x_s = Ax_s + Bw_s.$$

Let us define $\mathbf{w}_N$

$$w_T(t) = \begin{cases} \frac{1}{\sqrt{T}} e^{j\omega t} w_s & \text{if } 0 \le t < T \\ 0 & \text{if } T \le t. \end{cases}$$

It is easy to see that $\Lambda(\mathbf{w}) = \int_0^\infty w(t)w(t)^* dt = w_s w_s^*$, and therefore $\mathbf{w}$ satisfies (3.15). For $\mathbf{x} = \mathbf{M}_{AB}(\mathbf{w})$, let us define $\mathbf{x}_p$ and $\mathbf{x}_h$ such that

$$\begin{aligned} x_p(t) &= \begin{cases} \frac{1}{\sqrt{T}} e^{j\omega t} x_s & \text{if } 0 \le t < T \\ \frac{1}{\sqrt{T}} e^{A(t-T)} e^{j\omega T} x_s & \text{if } T \le t, \end{cases} \\ x_h(t) &= -\frac{1}{\sqrt{T}} e^{At} x_s, \end{aligned}$$

then $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$. Notice that

$$\begin{aligned} \Lambda_T(\mathbf{x}_p, \mathbf{w}_T) &= \int_0^T \begin{bmatrix} x_p(t) \\ w_T(t) \end{bmatrix} \begin{bmatrix} x_p(t) \\ w_T(t) \end{bmatrix}^* dt \\ &= \frac{1}{T} \int_0^T \begin{bmatrix} x_s \\ w_s \end{bmatrix} \begin{bmatrix} x_s \\ w_s \end{bmatrix}^* dt = V \end{aligned}$$

and this shows

$$\begin{aligned} \|\Lambda(\mathbf{x}_p, \mathbf{w}_T) - V\|_F &= \left\| \int_T^\infty \begin{bmatrix} x_p(t) \\ w_T(t) \end{bmatrix} \begin{bmatrix} x_p(t) \\ w_T(t) \end{bmatrix}^* dt \right\|_F \\ &\le \frac{1}{T} \int_T^\infty \left\| \begin{bmatrix} e^{At} x_s \\ 0 \end{bmatrix} \begin{bmatrix} e^{At} x_s \\ 0 \end{bmatrix}^* \right\|_F dt \\ &\le \frac{1}{T} \int_T^\infty \|e^{A(t-T)} x_s\|_2^2 dt \\ &\le \frac{1}{T} \|x_s\|_2^2 \underbrace{\int_0^\infty \|e^{At}\|_{2,2}^2 \, dt}_{C_1} . \end{aligned}$$

Here $C_1 < \infty$ from Proposition A.6. Now we have

$$\begin{aligned} & \|\Lambda(\mathbf{x}, \mathbf{w}_T) - V\|_F = \|\Lambda(\mathbf{x}_p + \mathbf{x}_h, \mathbf{w}_T) - V\|_F \\ \overset{(a)}{\le} \; & \|\Lambda(\mathbf{x}_p, \mathbf{w}_T) - V\|_F + \|\Lambda(\mathbf{x}_p + \mathbf{x}_h, \mathbf{w}_T) - \Lambda(\mathbf{x}_p, \mathbf{w}_T)\|_F \\ \overset{(b)}{\le} \; & C_1 \frac{\|x_s\|_2^2}{T} + C_2 \max\{(\|\mathbf{x}_p\|_\infty + \|\mathbf{w}_T\|_\infty)\|x_h(0)\|_2, \|x_h(0)\|_2^2\}, \end{aligned}$$

for some positive constant C_2. Here, (a) is from the triangle inequality and (b) is from Proposition A.11. Recall that $\|\mathbf{x}_p\|_\infty \propto \frac{1}{\sqrt{T}}$, $\|\mathbf{w}_T\|_\infty = \frac{1}{\sqrt{T}}\|w_s\|$, $\|x_h(0)\|_2 = \frac{1}{\sqrt{T}}\|x_s\|$. Therefore,

$$\max\{(|\mathbf{x}_p\|_\infty + \|\mathbf{w}_T\|_\infty)\|h_0\|_2, \|h_0\|_2^2\} \propto \frac{1}{T}.$$

By combining all these bounds, we can conclude that there exists a positive constant C which only depends on A, x_s, w_s such that

$$\|\Lambda(\mathbf{x}, \mathbf{w}_T) - V\|_F \leq \frac{C}{T}.$$

Therefore

$$\lim_{T\to\infty} \|\Lambda(\mathbf{x}, \mathbf{w}_T) - V\|_F = 0,$$

and this concludes the proof for $\theta \neq 0$.

Now consider the case $\theta = 0$. Then, $Ax_s + Bw_s + x_s = e^{\mathbf{i}\theta}(Ax_s + Bw_s - x_s)$ implies $x_s = 0$. This shows the rank one matrix $V = \begin{bmatrix} 0 \\ w_s \end{bmatrix}\begin{bmatrix} 0 \\ w_s \end{bmatrix}^*$ for some w_s. In fact, $\theta = 0$ corresponds to $\omega = +\infty$. So the main idea is to choose large enough ω to approximate $+\infty$.

Since A is Hurwitz stable, $\mathbf{i}\omega I - A$ is invertible for all $\omega \in \mathbb{R}$. Let $\hat{x}_s = (\mathbf{i}\omega I - A)^{-1}Bw_s$, and $\hat{V} = \begin{bmatrix} \hat{x}_s \\ w_s \end{bmatrix}\begin{bmatrix} \hat{x}_s \\ w_s \end{bmatrix}^* \succeq 0$. Then $\hat{V} \in \mathcal{C}_{\text{SDP}}$, and $\textbf{rank}\,(V) \leq 1$. Furthermore, there exists $\hat{\theta} \neq 0$ such that $\omega = \frac{\sin\hat{\theta}}{\cos\hat{\theta}-1}$. Therefore, we can apply the construction in the case $\theta \neq 0$ ($\omega \neq +\infty$) to find $\mathbf{w}$ with finite support such that $\|\Lambda(\mathbf{M}_{AB}(\mathbf{w}), \mathbf{w}) - \hat{V}\|_F < \frac{\varepsilon}{2}$ and satisfies (A.4).

Finally, notice that

$$\|\hat{V} - V\|_F = \left\| \begin{bmatrix} \hat{x}_s\hat{x}_s^* & \hat{x}_s w_s^* \\ w_s\hat{x}_s^* & 0 \end{bmatrix} \right\|_F \leq \|\hat{x}_s\|_2^2 + 2\|\hat{x}_s\|_2\|w_s\|_2.$$

Since $\lim\limits_{\omega\to\infty} \|(\mathbf{i}\omega I - A)^{-1}\|_{2,2} = 0$, and $\|\hat{x}_s\|_2 \leq \|\mathbf{i}\omega I - A)^{-1}\|_{2,2} \|Bw_s\|_2$, there exists ω such that $\|\hat{V} - V\|_F < \frac{1}{2}\varepsilon$.

Therefore, we can conclude that

$$\|\Lambda(\mathbf{M}_{AB}(\mathbf{w}), \mathbf{w}) - V\|_F \leq \|\Lambda(\mathbf{M}_{AB}(\mathbf{w}), \mathbf{w}) - \hat{V}\|_F + \|\hat{V} - V\|_F < \varepsilon. \qquad \blacksquare$$

Now we are ready to prove Lemma 3.3.

Proof (Proof of Lemma 3.3): From Proposition A.3, we can decompose $V = \sum_{i=1} V_i$ where $V_i \in \mathcal{C}_{\text{SDP}}$, and $\textbf{rank}\,(V_i) \leq 1$. Let us rearrange these terms, so that $V = \sum_{i=1}^{r} V_i$, where $\textbf{rank}\,(V_i) = 1$. We now use an induction on r, that is, for any r and $\varepsilon > 0$, there exists $\mathbf{w}$ with finite support such that

$$\|\Lambda(\mathbf{M}_{AB}(\mathbf{w}), \mathbf{w}) - V\|_F < \varepsilon \tag{A.5}$$

$$\Lambda(\mathbf{w}) = \begin{bmatrix} 0_{n_w \times n_x} & I_{n_w} \end{bmatrix} V \begin{bmatrix} 0_{n_w \times n_x} & I_{n_w} \end{bmatrix}^*. \tag{A.6}$$

Suppose $r \leq 1$, then from Proposition A.12, the proof is done.

Now assume the induction hypothesis holds. Then for $\sum_{i=1}^{r-1} V_i \in \mathcal{C}_{\text{SDP}}$, there exists $\tilde{\mathbf{w}}$ with a finite support such that (A.5) with $\frac{\varepsilon}{4}$ and (A.6) hold. Similarly, for V_r let $\hat{\mathbf{w}}$ satisfies (A.5)-(A.6). Let $\tau = \text{argmax}\{\tau : \tilde{w}(t) = 0, \text{for all } t \geq \tau\}$. Then from Proposition A.10, there exists T such that for all $t \geq T$,

$$\|\Lambda(\mathbf{M}\tilde{\mathbf{w}}, \tilde{\mathbf{w}}) - \Lambda_t(\mathbf{M}\tilde{\mathbf{w}}, \tilde{\mathbf{w}})\| < \frac{1}{4}\epsilon.$$

Consider the following signal $\mathbf{w}_N$:

$$w_N(t) = \begin{cases} \tilde{w}(t) & \text{if } 0 \leq t < N + T \\ \hat{w}(t - N - T) & \text{if } N + T \leq t. \end{cases}$$

Clearly, $\mathbf{w}_N$ has finite support, and $\Lambda(\mathbf{w}_N) = \Lambda(\tilde{\mathbf{w}}) + \Lambda(\hat{\mathbf{w}})$, which shows

$$\Lambda(\mathbf{w}_N) = \begin{bmatrix} 0_{n_w \times n_x} & I_{n_w} \end{bmatrix} V \begin{bmatrix} 0_{n_w \times n_x} & I_{n_w} \end{bmatrix}^*.$$

Now, let $\tilde{\mathbf{x}} = \mathbf{M}\tilde{\mathbf{w}}$, $\hat{\mathbf{x}} = \mathbf{M}\hat{\mathbf{w}}$, and $\mathbf{x}_N = \mathbf{M}_{AB}(\mathbf{w})$. Then,

$$x_N(t) = \begin{cases} \tilde{x}(t) & \text{if } 0 \leq t < T \\ e^{A(t-T)}\tilde{x}(T) & \text{if } T \leq t < N+T \\ \hat{x}(t-N-T) + e^{A(t-T)}\tilde{x}(T) & \text{if } N+T \leq t. \end{cases}$$

Notice that

$$\begin{aligned} & \Lambda(\mathbf{x}, \mathbf{w}) \\ = \; & \Lambda_{N+T}(\tilde{\mathbf{x}}, \tilde{\mathbf{w}}) \\ & + \int_{N+T}^{\infty} \begin{bmatrix} \hat{x}(t-N-T) + e^{A(t-T)}\tilde{x}(T) \\ \hat{w}(t) \end{bmatrix} \begin{bmatrix} \hat{x}(t-N-T) + e^{A(t-T)}\tilde{x}(T) \\ \hat{w}(t) \end{bmatrix}^* \\ = \; & \Lambda_{N+T}(\tilde{\mathbf{x}}, \tilde{\mathbf{w}}) + \Lambda(\hat{\mathbf{x}} + \mathbf{y}, \hat{\mathbf{w}}), \end{aligned}$$

where $y(t) = e^{A(t+N)}\tilde{x}(T)$.

Finally,

$$\begin{aligned} & \|\Lambda(\mathbf{x}_N, \mathbf{w}_N) - V\|_F \\ \overset{(a)}{\leq} \; & \left\| \Lambda_{N+T}(\tilde{\mathbf{x}}, \tilde{\mathbf{w}}) - \sum_{i=1}^{r-1} V_i \right\|_F + \|\Lambda(\hat{\mathbf{x}} + \mathbf{y}, \hat{\mathbf{w}}) - V_r\|_F \\ \overset{(b)}{\leq} \; & \|\Lambda_{N+T}(\tilde{\mathbf{x}}, \tilde{\mathbf{w}}) - \Lambda(\tilde{\mathbf{x}}, \tilde{\mathbf{w}})\|_F + \left\| \Lambda(\tilde{\mathbf{x}}, \tilde{\mathbf{w}}) - \sum_{i=1}^{r-1} V_i \right\|_F \\ & + \|\Lambda(\hat{\mathbf{x}} + \mathbf{y}, \hat{\mathbf{w}}) - \Lambda(\hat{\mathbf{x}}, \hat{\mathbf{w}})\|_F + \|\Lambda(\hat{\mathbf{x}}, \hat{\mathbf{w}}) - V_r\|_F \\ \overset{(c)}{\leq} \; & \frac{3}{4}\epsilon + C \max\{(\|\hat{\mathbf{x}}\|_\infty + \|\hat{\mathbf{w}}\|_\infty)\|y(0)\|_2, \|y(0)\|_2^2\}, \end{aligned}$$

where (a) and (b) are from the triangle inequality, and (c) is from Proposition A.11.

Therefore,

$$\lim_{N \to} \|\Lambda(\mathbf{M}_{AB}(\mathbf{w}_N), \mathbf{w}_N) - V\|_F = 0,$$

and this concludes the proof. ∎

Appendix B

Proof of results in Chapter 5

B.1 Proof of Proposition 5.2

The proof of Proposition 5.2 relies on the compactness of the feasible set of the optimization (5.2). Recall that the feasible set of (5.2) is the closure of the following set

$$\mathcal{F}_{\text{gram}} = \begin{cases} \{V : V \in \mathcal{D}_{\text{Gram}}, \begin{bmatrix} 0_{n_p \times n_x} & 0_{n_p \times n_q} & I_{n_p} \\ 0_{n_q \times n_x} & I_{n_q} & 0_{n_q \times n_p} \end{bmatrix} V \begin{bmatrix} 0_{n_p \times n_x} & 0_{n_p \times n_q} & I_{n_p} \\ 0_{n_q \times n_x} & I_{n_q} & 0_{n_q \times n_p} \end{bmatrix}^* \in \mathcal{F}_{\boldsymbol{\Delta}}\} & \text{Discrete time} \\ \{V : V \in \mathcal{C}_{\text{Gram}}, \begin{bmatrix} 0_{n_p \times n_x} & 0_{n_p \times n_q} & I_{n_p} \\ 0_{n_q \times n_x} & I_{n_q} & 0_{n_q \times n_p} \end{bmatrix} V \begin{bmatrix} 0_{n_p \times n_x} & 0_{n_p \times n_q} & I_{n_p} \\ 0_{n_q \times n_x} & I_{n_q} & 0_{n_q \times n_p} \end{bmatrix}^* \in \mathcal{F}_{\boldsymbol{\Delta}}\} & \text{Continuous time,} \end{cases}$$

where

$$\begin{aligned} \mathcal{F}_{\boldsymbol{\Delta}} = \{Z \in \mathbb{H} : \mathcal{A}_{i\boldsymbol{\Delta}}(Z) \succeq 0, \quad & i = 1, \cdots, n_a, \\ \mathcal{B}_{i\boldsymbol{\Delta}}(Z) = 0, \quad & i = 1, \cdots, n_b, \\ & \mathbf{Tr}(Z_{11}) = 1\}. \end{aligned}$$

Here, $\mathcal{F}_{\boldsymbol{\Delta}}$ is being used for the constraint $(\mathbf{p}, \mathbf{q}) \in \mathcal{R}_{\boldsymbol{\Delta}}$ and $\|\mathbf{p}\|_2^2 = \mathbf{Tr}(Z_{11}) = 1$.

The following proposition states that $\mathcal{F}_{\text{gram}}$ is bounded.

Proposition B.1: *$\mathcal{F}_{gram}$ is bounded.*

PROOF: Let $V \in \mathcal{F}_{\text{gram}}$. Then there exists a signal triplet $(\mathbf{x}, \mathbf{p}, \mathbf{q})$ and $\Delta \in \mathbf{\Delta}$ such that

$$\begin{aligned} V &= \Lambda(\mathbf{x}, \mathbf{q}, \mathbf{p}) \\ \mathbf{x} &= \mathbf{M}\mathbf{q} \\ \mathbf{q} &= \Delta\mathbf{p} \\ \|\mathbf{p}\|_2 &= 1. \end{aligned}$$

Since $\|\Delta\|_{2,2} \leq \alpha_\Delta$, we have

$$\|\mathbf{q}\|_2 \leq \alpha_\Delta.$$

In addition, since $\mathbf{M}$ is stable, $\|\mathbf{M}\|_{2,2} = \beta < \infty$ (see *e.g.*, [26]), and

$$\|\mathbf{x}\|_2 \leq \beta\|\mathbf{q}\| \leq \alpha_\Delta\beta.$$

Therefore,

$$\mathbf{Tr}(V) = \mathbf{Tr}(\Lambda(\mathbf{x}, \mathbf{q}, \mathbf{p})) = \|\mathbf{x}\|_2^2 + \|\mathbf{q}\|_2^2 + \|\mathbf{p}\|_2^2 \leq \alpha_\Delta\beta + \alpha_\Delta + 1 < \infty,$$

which concludes the proof. ■

Now we are ready to present the proof.

Proposition B.2: $\eta_{\mathbf{\Delta}}(\mathbf{M}) = 0$ *if and only if there exists* V *in the feasible set of* (5.2) *such that*

$$\boldsymbol{Tr}\Big(\begin{bmatrix} C & D & -I_{n_p} \end{bmatrix} V \begin{bmatrix} C & D & -I_{n_p} \end{bmatrix}^*\Big) = 0.$$

PROOF: Since the feasible set of (5.2) is the closure of $\mathcal{F}_{\text{gram}}$, the feasible set of (5.2) is compact. Since the optimal value of (5.2) is bounded below by 0, from the extreme value theorem (see *e.g.*, [68]), we can conclude that (5.2) has the solution. ■

B.2 Proof of Proposition 5.3

This section contains the proof of Proposition 5.3.

Proposition B.3: *For all V in the feasible set of* (5.2) *such that* (5.4) *holds, there exists W such that*

$$V = \begin{bmatrix} I_{n_x} & 0_{n_x \times n_q} \\ 0_{n_q \times n_x} & I_{n_q} \\ C & D \end{bmatrix} W \begin{bmatrix} I_{n_x} & 0_{n_x \times n_q} \\ 0_{n_q \times n_x} & I_{n_q} \\ C & D \end{bmatrix}^* .$$

PROOF: Recall that any V in the feasible set of (5.2) is positive semidefinite. Let us decompose $V = \sum_k \begin{bmatrix} x_k \\ q_k \\ p_k \end{bmatrix} \begin{bmatrix} x_k \\ q_k \\ p_k \end{bmatrix}^*$. Here x_k, q_k, p_k are vectors, not signals. Then from (5.4),

$$\mathbf{Tr}\left(\begin{bmatrix} C & D & -I_{n_p} \end{bmatrix} V \begin{bmatrix} C & D & -I_{n_p} \end{bmatrix}^*\right) = 0,$$

we have

$$\sum_k \|Cx_k + Dq_k - p_k\|_2^2 = 0,$$

which implies $Cx_k + Dq_k = p_k$. Therefore,

$$\begin{aligned} V &= \sum_k \begin{bmatrix} x_k \\ q_k \\ Cx_k + q_k \end{bmatrix} \begin{bmatrix} x_k \\ q_k \\ Cx_k + Dq_k \end{bmatrix}^* \\ &= \begin{bmatrix} I_{n_x} & 0_{n_x \times n_q} \\ 0_{n_q \times n_x} & I_{n_q} \\ C & D \end{bmatrix} \underbrace{\left(\sum_k \begin{bmatrix} x_k \\ q_k \end{bmatrix} \begin{bmatrix} x_k \\ q_k \end{bmatrix}^*\right)}_{W} \begin{bmatrix} I_{n_x} & 0_{n_x \times n_q} \\ 0_{n_q \times n_x} & I_{n_q} \\ C & D \end{bmatrix}^* . \end{aligned}$$

■

B.3 Proof of Proposition 5.7

Proposition B.4: *[26] There exists* $\|\Delta\|_{2,2} \leq 1$ *such that* $\mathbf{q} = \Delta\mathbf{p}$ *if and only if* $\|\mathbf{q}\|_2 \leq \|\mathbf{p}\|_2$.

PROOF: Suppose there exists $\|\Delta\|_{2,2} \leq 1$. Then,

$$\|\mathbf{q}\|_2 = \|\Delta\mathbf{p}\|_2 \leq \|\Delta\|_{2,2} \|\mathbf{p}\|_2 \leq \|\mathbf{p}\|_2.$$

Conversely, suppose $\|\mathbf{q}\|_2 \leq \|\mathbf{p}\|_2$. For $\mathbf{p} \neq 0$, let $\Delta(\mathbf{r}) = \frac{\langle \mathbf{p}, \mathbf{r} \rangle}{\|\mathbf{p}\|_2^2}\mathbf{q}$. Then $\Delta(\mathbf{p}) = \mathbf{q}$, and

$$\begin{aligned} \|\Delta\|_{2,2} = \sup_{\|\mathbf{r}\|_2=1} \|\Delta(\mathbf{r})\|_2 = \sup_{\|\mathbf{r}\|_2=1} \left\| \frac{\langle \mathbf{p}, \mathbf{r} \rangle}{\|\mathbf{p}\|_2^2}\mathbf{q} \right\|_2 &= \sup_{\|\mathbf{r}\|_2=1} \left| \frac{\langle \mathbf{p}, \mathbf{r} \rangle}{\|\mathbf{p}\|_2^2} \right| \|\mathbf{q}\|_2 \\ &\leq \sup_{\|\mathbf{r}\|_2=1} \frac{\|\mathbf{r}\|_2 \|\mathbf{p}\|_2}{\|\mathbf{p}\|_2^2} \|\mathbf{q}\|_2 \leq \frac{\|\mathbf{q}\|_2}{\|\mathbf{p}\|_2} \leq 1. \end{aligned}$$

If $\mathbf{p} = 0$, then $\mathbf{q} = 0$ and $\Delta = 0$. ∎

B.4 Proof of Proposition 5.8

Firstly, let us consider the following result.

Proposition B.5: *Let* $u, v \in \mathbb{C}^n$. *Then the following statements are equivalent.*

(i) $u^*u \geq v^*v$ *and* $u^*v = v^*u$.

(ii) There exists $H \in \mathbb{C}^{n\times n}$ *such that* $H = H^*$, $\|H\|_{2,2} \leq 1$, *and* $v = Hu$.

PROOF: (ii) ⇒ (i) is easy to check. So let us consider (i) ⇒ (ii). If $u^*u = 0$, then $H = 0$ satisfies (ii).

Now suppose $u^*u > 0$. Without loss of generality, we can assume $u^*u = 1$. Otherwise we can scale $\tilde{u} = \frac{u}{\|u\|_2}$, $\tilde{v} = \frac{v}{\|u\|_2}$, and $\tilde{H} = \frac{H}{\|u\|_2}$.

Let $\alpha = u^*v$. Then $\bar{\alpha} = \overline{u^*v} = v^*u = u^*v = \alpha$, which shows α is a real scalar. In addition, $|\alpha| = |u^*v| \leq \|u\|_2\|v\|_2 \leq 1$, since $\|v\|_2 \leq \|u\|_2 = 1$.

Consider $r = v - \alpha u$. Clearly, $r^*u = 0$, *i.e.*, r and u are orthogonal to each other. If $r = 0$, then $v = \alpha u$, and $H := \alpha uu^*$ satisfies (ii). Suppose $r \neq 0$, then the following choice of H,

$$H := \alpha \left(uu^* - \frac{rr^*}{r^*r} \right) + ur^* + ru^*,$$

satisfies (ii). The reason is as follows. Firstly, since α is real, $H^* = H$. In addition, since $r^*u = 0$, $Hu = \alpha u + r = v$. Lastly,

$$\begin{aligned} H^2 &= (\alpha uu^* - \frac{\alpha}{r^*r} rr^* + ur^* + ru^*)(\alpha uu^* - \frac{\alpha}{r^*r} rr^* + ur^* + ru^*) \\ &= \alpha^2 uu^* + \alpha ur^* + \frac{\alpha^2}{r^*r} rr^* - \alpha ru^* - \alpha ur^* + (r^*r)uu^* + \alpha ru^* + rr^* \\ &= (r^*r + \alpha^2)uu^* + \frac{r^*r + \alpha^2}{r^*r} rr^*. \end{aligned}$$

Since $r^*r = (v - \alpha u)^*(v - \alpha u) = v^*v - \alpha^2$, we have

$$H^2 = (v^*v)uu^* + (v^*v)\frac{r}{\|r\|_2}\frac{r^*}{\|r\|_2},$$

which shows that the maximum singular values of H is $\|v\|_2^2$. Since $\|v\|_2 \leq \|u\|_2 = 1$, we can conclude that $\|H\|_{2,2} \leq 1$. ■

Now we are ready to present the proof.

Proposition B.6: *There exists* $\|\Delta\|_{2,2} \leq 1, \Delta = \Delta^*$ *such that* $\mathbf{q} = \Delta\mathbf{p}$ *if and only if* $\|\mathbf{q}\|_2 \leq \|\mathbf{p}\|_2$ *and* $\langle \mathbf{p}, \mathbf{q} \rangle = \langle \mathbf{q}, \mathbf{p} \rangle$.

Proof: Suppose $\mathbf{q} = \Delta\mathbf{p}$ where $\Delta \in \mathbf{\Delta}$. Then from Proposition 5.7, $\|\mathbf{q}\|_2 \leq \|\mathbf{p}\|_2$. Moreover, $\langle \mathbf{q}, \mathbf{p} \rangle = \langle \Delta\mathbf{p}, \mathbf{p} \rangle = \langle \mathbf{p}, \Delta^*\mathbf{p} \rangle = \langle \mathbf{p}, \Delta\mathbf{p} \rangle = \langle \mathbf{p}, \mathbf{q} \rangle$.

Conversely, consider an orthonormal basis $\mathcal{V}$ for $\mathcal{T} := \text{span}\{\mathbf{p}, \mathbf{q}\}$, subspace of l_2 or $\mathcal{L}_2$. Let $d = \dim \mathcal{T} \leq 2$, and $\mathbf{L}_\mathcal{V} : \mathcal{T} \to \mathbb{C}^d$ be a coordinate representation of a vector in $\mathcal{T}$ with respect to the basis $\mathcal{V}$. For $u = \mathbf{L}_\mathcal{V}(\mathbf{p}), v = \mathbf{L}_\mathcal{V}(\mathbf{q})$, since $\mathcal{V}$ is an orthonormal basis, we have $u^*u \geq v^*v$ and $u^*v = v^*u$. Therefore from Proposition B.5, there exists H such that $\|H\|_{2,2} \leq 1$, $H = H^*$, and $v = Hu$. Then for $\Delta_\mathcal{T} := \mathbf{L}_\mathcal{V}^{-1} H \mathbf{L}_\mathcal{V}$, we have

$\|\Delta_{\mathcal{T}}\| \leq 1, \Delta_{\mathcal{T}} = \Delta_{\mathcal{T}}^*$ and $\Delta_{\mathcal{T}}(\mathbf{p}) = \mathbf{q}$. We can extend the linear operator $\Delta_{\mathcal{T}}$ to the entire space using the projection operator $\Pi_{\mathcal{T}} : l_2 \to \mathcal{T}$. Let

$$\Delta(\mathbf{r}) = \Delta_{\mathcal{T}}(\Pi_{\mathcal{T}}(\mathbf{r})).$$

It is routine to check Δ satisfies (ii). ■

B.5 Proof of Proposition 5.9

Proposition B.7: *There exists* $\|\Delta\|_{2,2} \leq 1, \Delta + \Delta^* = 0$ *such that* $\mathbf{q} = \Delta\mathbf{p}$ *if and only if* $\|\mathbf{q}\|_2 \leq \|\mathbf{p}\|_2$ *and* $\langle \mathbf{p}, \mathbf{q}\rangle + \langle \mathbf{q}, \mathbf{p}\rangle = 0$.

PROOF: Suppose $\mathbf{q} = \Delta\mathbf{p}$. Then $\|\mathbf{q}\|_2 = \|\Delta\mathbf{p}\|_2 \leq \|\Delta\|_{2,2}\|\mathbf{p}\|_2 \leq \|\mathbf{p}\|_2$. Morover, $\langle \mathbf{p}, \mathbf{q}\rangle = \langle \mathbf{p}, \Delta\mathbf{p}\rangle = \langle \Delta^*\mathbf{p}, \mathbf{p}\rangle = -\langle \Delta\mathbf{p}, \mathbf{p}\rangle = -\langle \mathbf{q}, \mathbf{p}\rangle$, which shows $\langle \mathbf{p}, \mathbf{q}\rangle + \langle \mathbf{q}, \mathbf{p}\rangle = 0$.

For the converse direction, suppose $\|\mathbf{q}\|_2 \leq \|\mathbf{p}\|_2$ and $\langle \mathbf{p}, \mathbf{q}\rangle + \langle \mathbf{q}, \mathbf{p}\rangle = 0$. Consider $\tilde{\mathbf{q}} = \mathbf{i}\mathbf{q}$. Then $\|\tilde{\mathbf{q}}\|_2 \leq \|\mathbf{p}\|_2$, and $\langle \mathbf{p}, \tilde{\mathbf{q}}\rangle = \langle \tilde{\mathbf{q}}, \mathbf{p}\rangle$. From Proposition 5.8, there exists $\tilde{\Delta}$ such that $\|\tilde{\Delta}\| \leq 1$, $\tilde{\mathbf{q}} = \tilde{\Delta}\mathbf{p}$, and $\tilde{\Delta}^* = \tilde{\Delta}$. By setting $\Delta = -\mathbf{i}\tilde{\Delta}$, we can conclude the proof. ■

B.6 Proof of Proposition 5.10

Like in Self-adjoint LTV case, the following auxiliary result turns out to be useful.

Proposition B.8: *Let* $F, G \in \mathbb{C}^{n\times m}$. *Then the following statements are equivalent.*

(i) $F^*F \succeq G^*G$.

(ii) There exists $H \in \mathbb{C}^{n\times n}$ *such that* $\|H\|_{2,2} \leq 1$, *and* $G = HF$.

PROOF: Following proof is from [65].

For (ii) $\Rightarrow$ (i), notice that

$$\|Gx\|_2 = \|HFx\|_2 \leq \|Fx\|_2,$$

for all x, which shows $x^*G^*Gx \leq x^*F^*Fx$.

For (i) $\Rightarrow$ (ii), since $F^*F \succeq G^*G$, there exists a matrix X such that $G^*G = F^*F + X^*X$. This shows

$$\begin{bmatrix} F \\ X \end{bmatrix}^* \begin{bmatrix} F \\ X \end{bmatrix} = \begin{bmatrix} G \\ 0 \end{bmatrix}^* \begin{bmatrix} G \\ 0 \end{bmatrix},$$

and from Lemma A.1, there exists an unitary matrix $U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$ such that

$$\begin{bmatrix} F & X \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} G \\ 0 \end{bmatrix}.$$

Therefore $F = U_{11}G$, since U is unitary, $\|U_{11}\|_{2,2} \leq 1$. ■

Proposition B.9: *There exists* $\|\delta\|_{2,2} \leq 1$ *such that* $\mathbf{q} = \delta I(\mathbf{p})$ *if and only if* $\Lambda(\mathbf{q}) \preceq \Lambda(\mathbf{p})$.

PROOF: Firstly, suppose $\mathbf{q} = \delta I(\mathbf{p})$. Then, for any $u \in \mathbb{C}^n$,

$$\|u^*\mathbf{q}\|_2 = \|u^*\delta I(\mathbf{p})\|_2 = \|\delta I(u^*\mathbf{p})\|_2 \leq \|\Delta\|_{2,2} \|u^*\mathbf{p}\|_2 \leq \|u^*\mathbf{p}\|_2.,$$

which shows $u^*(\Lambda(\mathbf{p}) - \Lambda(\mathbf{q}))u \geq 0$. Therefore, $\Lambda(\mathbf{q}) \preceq \Lambda(\mathbf{p})$.

Conversly, let us consider an orthonormal basis $\mathcal{V}$ for

$$\mathcal{T} := \operatorname{span}\{[\mathbf{p}]_1, \cdots, [\mathbf{p}]_n, [\mathbf{q}]_1, \cdots, [\mathbf{q}]_n\},$$

a subspace of l_2^1 or $\mathcal{L}_2^1$. Let $d = \dim \mathcal{T} \leq 2n$, and $\mathbf{L}_{\mathcal{V}} : \mathcal{T} \to \mathbb{C}^d$ be a coordinate representation of a vector in $\mathcal{T}$ with respect to the basis $\mathcal{V}$. Define $u_i := \mathbf{L}_{\mathcal{V}}([\mathbf{p}]_i) \in \mathbb{C}^d$

and $v = \mathbf{L}_{\mathcal{V}}([\mathbf{q}]_i) \in \mathbb{C}^d$. Since $\mathcal{V}$ is an orthonormal basis, we have

$$\begin{aligned} \langle [\mathbf{p}]_i, [\mathbf{p}]_j \rangle &= u_i^* u_j \\ \langle [\mathbf{q}]_i, [\mathbf{q}]_j \rangle &= v_i^* v_j \\ \langle [\mathbf{p}]_i, [\mathbf{q}]_j \rangle &= u_i^* v_j. \end{aligned}$$

Let us define following matrices

$$\begin{aligned} U &= \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} \\ V &= \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}. \end{aligned}$$

Then since

$$\begin{aligned} e_i^*(\Lambda(\mathbf{p}))e_j &= \sum_{k=0}^{\infty} e_i^* p[k] p[k]^* e_j = \sum_{k=0}^{\infty} \overline{[p[k]]_j} [p[k]]_i \\ &= \langle [\mathbf{p}]_j, [\mathbf{p}]_i \rangle = u_j^* u_i = (Ue_j)^*(Ue_i) = e_j^* U^* U e_i, \end{aligned}$$

we have $\Lambda(\mathbf{p})^\top = \overline{\Lambda(\mathbf{p})} = U^*U$. Similarly, we have $\Lambda(\mathbf{q})^\top = \overline{\Lambda(\mathbf{q})} = V^*V$. Since $U^*U \succeq V^*V$, there exists H such that $\|H\|_{2,2} \leq 1$, and $V = HU$.

Since $v_i = Ve_i = HUe_i = Hu_i$, we can easily see that $\delta_{\mathcal{T}} := \mathbf{L}_{\mathcal{V}}^{-1} H \mathbf{L}_{\mathcal{V}}$ satisfies $\|\delta_{\mathcal{T}}\| \leq 1, \delta_{\mathcal{T}}^* = \delta_{\mathcal{T}}$ and $\delta_{\mathcal{T}}([\mathbf{p}]_i) = [\mathbf{q}]_i$. We can extend the linear operator $\delta_{\mathcal{T}}$ to the entire space using the projection operator $\Pi_{\mathcal{T}}$. Let

$$\delta(\mathbf{r}) := \delta_{\mathcal{T}}(\Pi_{\mathcal{T}}(\mathbf{r})),$$

and it is routine to check if $\|\delta\|_{2,2} \leq 1$. ■

An alternative proof can be found in [59].

B.7 Proof of Proposition 5.11

We firstly need a matrix extension of Proposition B.5.

Proposition B.10: *Let $F, G \in \mathbb{C}^{n\times m}$. Then the following statements are equivalent.*

(i) $F^*F \succeq G^*G$ *and* $F^*G = G^*F$

(ii) There exists $H \in \mathbb{C}^{m\times m}$ *such that* $H = H^*$, $\|H\|_{2,2} \leq 1$, *and* $G = HF$.

PROOF: The proof can be found in [38, 27]. ■

Now we are ready to present the proof.

Proposition B.11: *The following conditions are equivalent:*

(i) $\Lambda(\mathbf{q}) \preceq \Lambda(\mathbf{p})$, $\sum_{k=0}^{\infty} q_k p_k^* = \sum_{k=0}^{\infty} p_k q_k^*$.

(ii) There exists $\|\delta\|_{2,2} \leq 1$, $\delta = \delta^*$ *such that* $\mathbf{q} = \delta I(\mathbf{p})$

PROOF: (ii) $\Rightarrow$ (i) is trivial. For (i) $\Rightarrow$ (ii), consider an orthonormal basis $\mathcal{V}$ for $\mathcal{T} := \text{span}\{[\mathbf{p}]_1, \cdots, [\mathbf{p}]_n, [\mathbf{q}]_1, \cdots, [\mathbf{q}]_n\}$, a subspace of l_2^1. Let $d = \dim \mathcal{T} \leq 2n$, and $\mathbf{L}_{\mathcal{V}} : \mathcal{T} \to \mathbb{C}^d$ be a coordinate representation of a vector in $\mathcal{T}$ with respect to the basis $\mathcal{V}$. Define $u_i := \mathbf{L}_{\mathcal{V}}([\mathbf{p}]_i) \in \mathbb{C}^d$ and $v = \mathbf{L}_{\mathcal{V}}([\mathbf{q}]_i) \in \mathbb{C}^d$. Since $\mathcal{V}$ is an orthonormal basis, we have

$$\begin{aligned}
\langle [\mathbf{p}]_i, [\mathbf{p}]_j \rangle &= u_i^* u_j \\
\langle [\mathbf{q}]_i, [\mathbf{q}]_j \rangle &= v_i^* v_j \\
\langle [\mathbf{p}]_i, [\mathbf{q}]_j \rangle &= u_i^* v_j.
\end{aligned}$$

Let us define following matrices:

$$\begin{aligned}
U &= \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} \\
V &= \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}.
\end{aligned}$$

Then since

$$\begin{aligned}
e_i^*(\Lambda(\mathbf{p}))e_j &= \sum_{k=0}^{\infty} e_i^* p[k] p[k]^* e_j = \sum_{k=0}^{\infty} \overline{[p[k]]_j} [p[k]]_i \\
&= \langle [\mathbf{p}]_j, [\mathbf{p}]_i \rangle = u_j^* u_i = (Ue_j)^*(Ue_i) = e_j^* U^* U e_i,
\end{aligned}$$

we have $\Lambda(\mathbf{p})^\top = \overline{\Lambda(\mathbf{p})} = U^*U$. Similarly, we have $\Lambda(\mathbf{q})^\top = \overline{\Lambda(\mathbf{q})} = V^*V$, and since $\Lambda(\mathbf{q}) \preceq \Lambda(\mathbf{p})$, $V^*V \preceq U^*U$. In addition, from$\sum_{k=0}^{\infty} q_k p_k^* = \sum_{k=0}^{\infty} p_k q_k^*$, we have

$$\begin{aligned} e_i^* \sum_{k=0}^{\infty} q[k]p[k]^* e_j &= e_i^* \sum_{k=0}^{\infty} p[k]q[k]^* e_j \\ \langle [\mathbf{p}]_j, [\mathbf{q}]_i \rangle &= \langle [\mathbf{q}]_j, [\mathbf{p}]_i \rangle \\ (Ue_j)^*(Ve_i) &= (Ve_j)^*(Ue_i) \\ e_j^* U^* V e_i &= e_j^* V^* U e_i \end{aligned}$$

which shows $U^*V = V^*U$. Therefore, from Proposition B.10, there exists H such that $\|H\|_{2,2} \leq 1$, $H = H^*$ and $V = HU$. Since $v_i = Ve_i = HUe_i = Hu_i$, we can easily see that $\delta_{\mathcal{T}} := \mathbf{L}_{\mathcal{V}}^{-1} H \mathbf{L}_{\mathcal{V}}$ satisfies $\|\delta_{\mathcal{T}}\| \leq 1, \delta_{\mathcal{T}}^* = \delta_{\mathcal{T}}$ and $\delta_{\mathcal{T}}([\mathbf{p}]_i) = [\mathbf{q}]_i$. We can extend the linear operator $\delta_{\mathcal{T}}$ to the entire space using the projection operator $\Pi_{\mathcal{T}}$,

$$\delta(\mathbf{r}) = \delta_{\mathcal{T}}(\Pi_{\mathcal{T}}(\mathbf{r})),$$

and it is routine to check if δ satisfies (ii).

For the continuous time case, we replace l_2 by $\mathcal{L}_2$ and the infinite sum by the integral to reach the same conclusion. ∎

Appendix C

Appendix to chapter 7

C.1 Minimal representation of linearized swing dynamics

This section derives the null space of A, and $A^\top$, and show that (A, B_w) is uncontrollable.

Proposition C.1: *The null space of A is given by*

$$\mathbf{Ker}\,(A) := \left\{ \begin{bmatrix} \omega_G \\ P \end{bmatrix} : \omega_G = 0, MP = 0 \right\}.$$

Proof: If $\begin{bmatrix} \omega_G \\ P \end{bmatrix} \in \mathbf{Ker}\,(A)$, then

$$-D_G \omega_G + M_1 P = 0 \tag{C.1}$$

$$M_1^\top \omega_G + M_2^\top D_L^{-1} M_2 P = 0. \tag{C.2}$$

From (C.1), we have $\omega_G = D_G^{-1} M_1 P$. By substituting this to (C.2), we have

$$M^\top \begin{bmatrix} D_G^{-1} & 0 \\ 0 & D_L^{-1} \end{bmatrix} MP = 0, \tag{C.3}$$

where we use the identity

$$M_1 = \begin{bmatrix} I & 0 \end{bmatrix} M$$
$$M_2 = \begin{bmatrix} 0 & I \end{bmatrix} M.$$

Recall that the null space of $M^\top$ is spanned by $\mathbf{1}$, the all one vector, because M is the incidence matrix. This implies that

$$\begin{bmatrix} D_G^{-1} & 0 \\ 0 & D_L^{-1} \end{bmatrix} MP = \alpha \mathbf{1},$$

for some α, and

$$MP = \alpha \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{\mathcal{N}} \end{bmatrix},$$

where d_i is the damping term at each bus. In order to obtain α, let us multiply $\mathbf{1}^\top$ to to the both sides. Then,

$$\begin{aligned} \mathbf{1}^\top MP &= \alpha \mathbf{1}^\top \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{\mathcal{N}} \end{bmatrix} \\ 0 &= \alpha \sum_{i=1}^{\mathcal{N}} d_i, \end{aligned}$$

and since $\sum_{i=1}^{\mathcal{N}} d_i > 0$, $\alpha = 0$. Therefore $MP = 0$, and $\omega_G = 0$ if $(\omega_G, P) \in \mathbf{Ker}\,(A)$.$\blacksquare$

Since all trajectories from the autonomous dynamics $\dot{x} = Ax$ converge to $\mathbf{Ker}\,(A)$, we can conclude that the real parts of all other eigenvalues are negative. In addition, we can prove that the eigenspace of A corresponding to 0 eigenvalue is uncontrollable under B_w. To begin with, let us present the null space of $\mathbf{Ker}\left(A^\top\right)$.

Proposition C.2:

$$\mathbf{Ker}\left(A^\top\right) := \left\{ \begin{bmatrix} 0 \\ u \end{bmatrix} : MYu = 0 \right\}.$$

PROOF: Let $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ be a vector in $\mathbf{Ker}\left(A^\top\right)$. Then,

$$A^\top \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -D_G H^{-1} & -M_1 Y \\ M_1^\top H^{-1} & -M_2^\top D_L^{-1} M_2 Y \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -D_G H^{-1} u_1 - M_1 Y u_2 \\ M_1^\top H^{-1} u_1 - M_2^\top D_L^{-1} M_2 Y u_2 \end{bmatrix} = 0.$$

Thus $u_1 = -H D_G^{-1} M_1 Y u_2$ and

$$(M_1^\top D_G^{-1} M_1 Y + M_2^\top D_L^{-1} M_2 Y) u_2 = M^\top \begin{bmatrix} D_G & 0 \\ 0 & D_L \end{bmatrix}^{-1} MY u_2 = 0.$$

Since $\mathbf{Ker}\left(M^\top\right)$ is spanned by $\mathbf{1}$, there exists α such that

$$\begin{bmatrix} D_G & 0 \\ 0 & D_L \end{bmatrix}^{-1} MY u_2 = \alpha \mathbf{1}.$$

Therefore

$$MY u_2 = \alpha \begin{bmatrix} d_1 \\ \vdots \\ d_{\mathcal{N}} \end{bmatrix}.$$

Since $\mathbf{1}^\top M = 0$, we have

$$0 = \alpha \sum_{i=1}^{\mathcal{N}} d_i,$$

which implies $\alpha = 0$ since $\sum_{i=1}^{\mathcal{N}} d_i > 0$. Now we can conclude that $MYu_2 = 0$ and $u_1 = 0$. ■

Now we have the following proposition.

Proposition C.3: *Consider* $\dot{x} = Ax + B_w w$. *Then* $\mathbf{Ker}\,(A)$ *is an uncontrollable invariant subspace of* A.

PROOF: Using Jordan decomposition of $A^\top$, we can find a matrix J and an invertible matrix T such that

$$A^\top = T \begin{bmatrix} 0_{k\times k} & 0 \\ 0 & J \end{bmatrix} T^{-1},$$

where k is the dimension of $\mathbf{Ker}\left(A^\top\right)$ and the first k columns of T form a basis for $\mathbf{Ker}\left(A^\top\right)$. In other words, if $\{u_1, \cdots, u_k\}$ form a basis for $\mathbf{Ker}\,(MY)$, then

$$T \begin{bmatrix} I_k \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 \\ u_1 & \cdots & u_k \end{bmatrix}.$$

Therefore,

$$\begin{bmatrix} I_k & 0 \end{bmatrix} T^\top B_w = (T \begin{bmatrix} I_k \\ 0 \end{bmatrix})^\top B_w = \begin{bmatrix} 0 & u_1^\top \\ \vdots & \vdots \\ 0 & u_k^\top \end{bmatrix} \begin{bmatrix} H^{-1} & 0 \\ 0 & -YM_2^\top D_L^{-1} \end{bmatrix} = 0$$

since $M_2 Y u_i = 0$ for all $i = 1, \cdots k$. Therefore, using the following coordinate transformation,

$$\begin{aligned} T^\top A (T^\top)^{-1} &= \begin{bmatrix} 0 & 0 \\ 0 & J \end{bmatrix} \\ T^\top B_w &= \begin{bmatrix} 0 \\ \tilde{B}_w \end{bmatrix}, \end{aligned}$$

where $\tilde{B}_w = \begin{bmatrix} 0 & I \end{bmatrix} T^\top B_w$, it is clear that $\mathbf{Ker}\,(A)$ is uncontrollable. ■

C.2 New England 39 power network data

Gen No.	H	R_a	x'_d	x'_q	x_d	x_q	T'_{do}	T'_{qo}	x_l
1	500.0	0	0.006	0.008	0.02	0.019	7.0	0.7	0.003
2	30.3	0	0.0697	0.170	0.295	0.282	6.56	1.5	0.035
3	35.8	0	0.0531	0.0876	0.2495	0.237	5.7	1.5	0.0304
4	28.6	0	0.0436	0.166	0.262	0.258	5.69	1.5	0.0295
5	26.0	0	0.132	0.166	0.67	0.62	5.4	0.44	0.054
6	34.8	0	0.05	0.0814	0.254	0.241	7.3	0.4	0.0224
7	26.4	0	0.049	0.186	0.295	0.292	5.66	1.5	0.0322
8	24.3	0	0.057	0.0911	0.290	0.280	6.7	0.41	0.028
9	34.5	0	0.057	0.0587	0.2106	0.205	4.79	1.96	0.0298
10	42.0	0	0.031	0.008	0.1	0.069	10.2	0.0	0.0125

Table C.1: Parameters of synchronous generators in the New England 39 power network. 60Hz, 100 MVA base. Table is from [81]

Line Data					Transformer Tap	
From Bus	To Bus	R	X	B	Magnitude	Angle
1	2	0.0035	0.0411	0.6987	0.000	0.00
1	39	0.0010	0.0250	0.7500	0.000	0.00
2	3	0.0013	0.0151	0.2572	0.000	0.00
2	25	0.0070	0.0086	0.1460	0.000	0.00
3	4	0.0013	0.0213	0.2214	0.000	0.00
3	18	0.0011	0.0133	0.2138	0.000	0.00
4	5	0.0008	0.0128	0.1342	0.000	0.00
4	14	0.0008	0.0129	0.1382	0.000	0.00
5	6	0.0002	0.0026	0.0434	0.000	0.00
5	8	0.0008	0.0112	0.1476	0.000	0.00
6	7	0.0006	0.0092	0.1130	0.000	0.00
6	11	0.0007	0.0082	0.1389	0.000	0.00
7	8	0.0004	0.0046	0.0780	0.000	0.00
8	9	0.0023	0.0363	0.3804	0.000	0.00
9	39	0.0010	0.0250	1.2000	0.000	0.00
10	11	0.0004	0.0043	0.0729	0.000	0.00
10	13	0.0004	0.0043	0.0729	0.000	0.00
13	14	0.0009	0.0101	0.1723	0.000	0.00
14	15	0.0018	0.0217	0.3660	0.000	0.00
15	16	0.0009	0.0094	0.1710	0.000	0.00
16	17	0.0007	0.0089	0.1342	0.000	0.00
16	19	0.0016	0.0195	0.3040	0.000	0.00
16	21	0.0008	0.0135	0.2548	0.000	0.00
16	24	0.0003	0.0059	0.0680	0.000	0.00
17	18	0.0007	0.0082	0.1319	0.000	0.00
17	27	0.0013	0.0173	0.3216	0.000	0.00
21	22	0.0008	0.0140	0.2565	0.000	0.00
22	23	0.0006	0.0096	0.1846	0.000	0.00
23	24	0.0022	0.0350	0.3610	0.000	0.00
25	26	0.0032	0.0323	0.5130	0.000	0.00
26	27	0.0014	0.0147	0.2396	0.000	0.00
26	28	0.0043	0.0474	0.7802	0.000	0.00
26	29	0.0057	0.0625	1.0290	0.000	0.00
28	29	0.0014	0.0151	0.2490	0.000	0.00
12	11	0.0016	0.0435	0.0000	1.006	0.00
12	13	0.0016	0.0435	0.0000	1.006	0.00
6	31	0.0000	0.0250	0.0000	1.070	0.00
10	32	0.0000	0.0200	0.0000	1.070	0.00
19	33	0.0007	0.0142	0.0000	1.070	0.00
20	34	0.0009	0.0180	0.0000	1.009	0.00
22	35	0.0000	0.0143	0.0000	1.025	0.00
23	36	0.0005	0.0272	0.0000	1.000	0.00
25	37	0.0006	0.0232	0.0000	1.025	0.00
2	30	0.0000	0.0181	0.0000	1.025	0.00
29	38	0.0008	0.0156	0.0000	1.025	0.00
19	20	0.0007	0.0138	0.0000	1.060	0.00

Table C.2: Parameters of transmission lines in the New England 39 power network. 60Hz, 100 MVA base. Table is from [81]

Bus	Type	Voltage [p.u.]	Load		Generator		
			MW	Mvar	MW	MVar	Gen No.
1	PQ	-	0.0	0.0	0.0	0.0	
2	PQ	-	0.0	0.0	0.0	0.0	
3	PQ	-	322.0	2.4	0.0	0.0	
4	PQ	-	500.0	184.0	0.0	0.0	
5	PQ	-	0.0	0.0	0.0	0.0	
6	PQ	-	0.0	0.0	0.0	0.0	
7	PQ	-	233.8	84.0	0.0	0.0	
8	PQ	-	522.0	176.0	0.0	0.0	
9	PQ	-	0.0	0.0	0.0	0.0	
10	PQ	-	0.0	0.0	0.0	0.0	
11	PQ	-	0.0	0.0	0.0	0.0	
12	PQ	-	7.5	88.0	0.0	0.0	
13	PQ	-	0.0	0.0	0.0	0.0	
14	PQ	-	0.0	0.0	0.0	0.0	
15	PQ	-	320.0	153.0	0.0	0.0	
16	PQ	-	329.0	32.3	0.0	0.0	
17	PQ	-	0.0	0.0	0.0	0.0	
18	PQ	-	158.0	30.0	0.0	0.0	
19	PQ	-	0.0	0.0	0.0	0.0	
20	PQ	-	628.0	103.0	0.0	0.0	
21	PQ	-	274.0	115.0	0.0	0.0	
22	PQ	-	0.0	0.0	0.0	0.0	
23	PQ	-	247.5	84.6	0.0	0.0	
24	PQ	-	308.6	-92.0	0.0	0.0	
25	PQ	-	224.0	47.2	0.0	0.0	
26	PQ	-	139.0	17.0	0.0	0.0	
27	PQ	-	281.0	75.5	0.0	0.0	
28	PQ	-	206.0	27.6	0.0	0.0	
29	PQ	-	283.5	26.9	0.0	0.0	
30	PV	1.0475	0.0	0.0	250.0	-	Gen10
31	PV	0.9820	9.2	4.6	-	-	Gen2
32	PV	0.9831	0.0	0.0	650.0	-	Gen3
33	PV	0.9972	0.0	0.0	632.0	-	Gen4
34	PV	1.0123	0.0	0.0	508.0	-	Gen5
35	PV	1.0493	0.0	0.0	650.0	-	Gen6
36	PV	1.0635	0.0	0.0	560.0	-	Gen7
37	PV	1.0278	0.0	0.0	540.0	-	Gen8
38	PV	1.0265	0.0	0.0	830.0	-	Gen9
39	PV	1.0300	1104.0	250.0	1000.0	-	Gen1

Table C.3: Power and voltage set points in the New England 39 power network. 60Hz, 100 MVA base. Table is from [81]

C.3 Linearized swing dynamics of the New England 39 power network

$$A = \left[\begin{array}{cccccccccccccccccccccccccccccccccccccccccccccccccccccccc}
0&0&0&0&0&0&0&0&0&0 & -448.799&0&0&0&448.799&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&-622.0976 \\
0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&526.5239&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 659.0754&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&724.9829&0&0&0&0&0&0&0 & 0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&541.6539&0&0&0&0 & 0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&713.9983&0&0 & 0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&775.7019 & 0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&546.3639&0 \\
0&0&0&0&0&0&0&0&0&0 & 0&37.6991&0&0&0&0&0&0&0&0 & 0&0&0&0&0&37.6991&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0 \\
79.7131&0&0&0&0&0&0&0&0&0 & -1594.261&-797.1305&797.1305&797.1305&797.1305&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&-129.246 & -1292.4598&-1292.4598&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 & 2119.352&0&-4238.704&-2119.352&-2119.352&2119.352&2119.352&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 & 3841.0726&0&-3841.0726&-7682.1453&-3841.0726&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&3841.0726&3841.0726 & 0&0&0&0&0&0 \\
-181.2749&0&0&0&0&0&0&0&0&0 & 1812.7489&0&-1812.7489&-1812.7489&-1812.7489&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 & 0&0&1437.7234&0&0&-2875.4469&-1437.7234&1437.7234&1437.7234&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 & 0&0&2344.475&0&0&-2344.475&-4688.95&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&-2344.475&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&2327.1232&0&-4654.2464&-2327.1232&2327.1232 & -2327.1232&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&2326.2757&0&-2326.2757&-4652.5514&0 & 0&0&0&0&0&0&0&0&0&0 & 0&-2326.2757&2326.2757&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&2542.4827&0&-5084.9653 & 2542.4827&0&0&-2542.4827&2542.4827&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&-11438.8801&0&11438.8801 & -22877.7602&-11438.8801&-11438.8801&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&11438.8801 \\
0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & -3059.317&-6118.634&-3059.317&3059.317&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&3059.317 \\
0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & -3661.0195&-3661.0195&-7322.0391&0&0&0&-3661.0195&0&0&-3661.0195 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&3661.0195 \\
0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&-5865.2996 & 0&5865.2996&0&-11730.5993&5865.2996&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&798.4812 & 0&0&0&798.4812&-1596.9623&798.4812&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&-125.0095 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&1250.0949&-1250.0949&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&-7068.0138&0&0&0&-14136.0277&-7068.0138&-7068.0138&-7068.0138 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&-7081.522&-14163.044&-7081.522&0 & -7081.522&7081.522&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0 \\
0&0&-147.358&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&-1473.5801&-1473.5801&-1473.5801&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&-686.3085&0&0&0&-686.3085&0&0&-1372.6171 & -686.3085&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&-687.6033&0&-687.6033 & -1375.2066&687.6033&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&-2992.2304&0 & 0&0&0&0&0&0&0&2992.2304&0&0 & 2992.2304&-5984.4608&2992.2304&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&1378.7572&0 & 0&0&0&0&0&0&0&0&0&0 & 0&1378.7572&-2757.5145&1378.7572&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&3199.5126&-6399.0252&3199.5126&3199.5126&3199.5126&3199.5126&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0 \\
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0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&1608.1087&-1608.1087&-3216.2173&-1608.1087&-1608.1087&0&0 & 1608.1087&1608.1087&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&2269.8513&-2269.8513&-2269.8513&-4539.7025&-2269.8513&0&0 & 0&0&0&2269.8513&0&0&0&0&0&0 & 0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&5201.3106&-5201.3106&-5201.3106&-5201.3106&-10402.6212&0&0 & 0&0&0&0&0&0&-5201.3106&0&0&0 & 0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&-3779.8153&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&3779.8153&0&0&0&-7559.6306&-3779.8153 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&1807.2375&0&0&0&-1807.2375&-3614.475 & 0&0&0&0&0&0&0&0&0&0 & -1807.2375&0&0&0&0&0 \\
0&0&0&-218.383&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&2183.8297&0&0&0&0 & -2183.8297&-2183.8297&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&2230.5051&0&0&0&0 & -2230.5051&-4461.0101&2230.5051&0&0&0&0&0&0&0 & 0&0&0&0&0&0 \\
0&0&0&0&-165.6698&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&1656.6978&-1656.6978&0&0&0&0&0&0&0 & 0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&2255.7175&0&0&0 & 0&0&0&-4511.4351&2255.7175&2255.7175&0&0&0&0 & 0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&3366.9881&-6733.9762&-3366.9881&3366.9881&3366.9881&0&0 & 0&0&0&0&0&0 \\
0&0&0&0&0&-228.2112&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&2282.1116&-2282.1116&-2282.1116&0&0&0&0 & 0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&-894.8154&0&0 & 0&0&0&0&894.8154&0&-1789.6309&-894.8154&0&0 & 0&0&0&0&0&0 \\
0&0&0&0&0&0&-120.2861&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&1202.8607&0&-1202.8607&-1202.8607&0&0 & 0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 & 0&0&0&1022.7294&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&-2045.4589&-1022.7294 & 1022.7294&1022.7294&1022.7294&0&0&0 \\
0&0&0&0&0&0&0&-139.0896&0&0 & 0&0&0&1390.896&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&-1390.896&-1390.896 & 0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&-2186.2114 & 0&0&0&0&0&0&0&0&2186.2114&0 & -4372.4228&-2186.2114&-2186.2114&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&692.3953&0 & -692.3953&-1384.7906&-692.3953&692.3953&0&0 \\
0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&523.7893&0 & -523.7893&-523.7893&-1047.5785&-523.7893&523.7893&0 \\
0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&2186.476&-2186.476&-4372.9521&2186.476&0 \\
0&0&0&0&0&0&0&0&-205.7999&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&2057.9991&2057.9991&-2057.9991&0 \\
0&123.5121&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 1235.1214&1235.1214&1235.1214&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&-1235.1214
\end{array}\right]$$

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B_s = \left[\begin{array}{ccccccccccccccccccccccccccccccccccccccc}
448.799 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 622.0976 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 526.5239 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 659.0754 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 724.9829 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 541.6539 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 713.9983 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 775.7019 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 546.3639 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 37.6991 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -797.1305 & 797.1305 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1292.4598 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2119.352 & 2119.352 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3841.0726 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3841.0726 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1812.7489 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1437.7234 & 1437.7234 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2344.475 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2344.475 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2327.1232 & 2327.1232 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2326.2757 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2326.2757 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2542.4827 & 0 & 0 & 2542.4827 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 11438.8801 & -11438.8801 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3059.317 & 3059.317 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3661.0195 & 0 & 0 & 0 & 0 & 3661.0195 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5865.2996 & 5865.2996 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -798.4812 & 798.4812 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1250.0949 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -7068.0138 & 7068.0138 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -7081.522 & 0 & 0 & 7081.522 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1473.5801 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 686.3085 & -686.3085 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -687.6033 & 687.6033 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2992.2304 & 2992.2304 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1378.7572 & 1378.7572 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3199.5126 & 3199.5126 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3450.1364 & 3450.1364 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1608.1087 & 0 & 0 & 1608.1087 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2269.8513 & 0 & 0 & 0 & 0 & 2269.8513 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5201.3106 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5201.3106 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3779.8153 & 3779.8153 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1807.2375 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1807.2375 & 0 & 0 \\
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