PARANORMAL OPERATORS ON A HILBERT SPACE

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ABSTRACT

In this thesis an extensive study is made of the set $\mathcal{P}$ of all paranormal operators in $B(\mathcal{H})$, the set of all bounded endomorphisms on the complex Hilbert space $\mathcal{H}$. $T \in B(\mathcal{H})$ is paranormal if for each z contained in the resolvent set of T, $d(z,\sigma(T))\|(T-zI)^{-1}\| = 1$ where $d(z,\sigma(T))$ is the distance from z to $\sigma(T)$, the spectrum of T. $\mathcal{P}$ contains the set $\mathcal{N}$ of normal operators and $\mathcal{P}$ contains the set of hyponormal operators. However, $\mathcal{P}$ is contained in $\mathcal{L}$, the set of all $T \in B(\mathcal{H})$ such that the convex hull of the spectrum of T is equal to the closure of the numerical range of T. Thus, $\mathcal{N} \subseteq \mathcal{P} \subseteq \mathcal{L}$.

If the uniform operator (norm) topology is placed on $B(\mathcal{H})$, then the relative topological properties of $\mathcal{N}$, $\mathcal{P}$, and $\mathcal{L}$ can be discussed. In Section IV, it is shown that: 1) $\mathcal{N}$, $\mathcal{P}$, and $\mathcal{L}$ are arc-wise connected and closed, 2) $\mathcal{N}$, $\mathcal{P}$, and $\mathcal{L}$ are nowhere dense subsets of $B(\mathcal{H})$ when $\dim \mathcal{H} \geq 2$, 3) $\mathcal{N} = \mathcal{P}$ when $\dim \mathcal{H} < \infty$, 4) $\mathcal{N}$ is a nowhere dense subset of $\mathcal{P}$ when $\dim \mathcal{H} = \infty$, 5) $\mathcal{P}$ is not a nowhere dense subset of $\mathcal{L}$ when $\dim \mathcal{H} < \infty$, and 6) it is not known if $\mathcal{P}$ is a nowhere dense subset of $\mathcal{L}$ when $\dim \mathcal{H} = \infty$.

The spectral properties of paranormal operators are of current interest in the literature. Putnam [22, 23] has shown that certain points on the boundary of the spectrum of a paranormal operator are either normal eigenvalues or normal approximate eigenvalues. Stampfli [26] has shown that a hyponormal operator with countable spectrum is normal. However, in Theorem 3.3, it is shown that a paranormal operator T with countable spectrum can be written as the direct sum, $N \oplus A$,

of a normal operator N with $\sigma(N) = \sigma(T)$ and of an operator A with $\sigma(A)$ a subset of the derived set of $\sigma(T)$. It is then shown that A need not be normal. If we restrict the countable spectrum of $T \in \mathcal{P}$ to lie on a C^2-smooth rectifiable Jordan curve G_o, then T must be normal [see Theorem 3.5 and its Corollary]. If T is a scalar paranormal operator with countable spectrum, then in order to conclude that T is normal the condition $\sigma(T) \subseteq G_o$ can be relaxed [see Theorem 3.6]. In Theorem 3.7 it is then shown that the above result is not true when T is not assumed to be scalar. It was then conjectured that if $T \in \mathcal{P}$ with $\sigma(T) \subseteq G_o$, then T is normal. The proof of Theorem 3.5 relies heavily on the assumption that T has countable spectrum and cannot be generalized. However, the Corollary to Theorem 3.9 states that if $T \in \mathcal{P}$ with $\sigma(T) \subseteq G_o$, then T has a non-trivial lattice of invariant subspaces. After the completion of most of the work on this thesis, Stampfli [30, 31] published a proof that a paranormal operator T with $\sigma(T) \subseteq G_o$ is normal. His proof uses some rather deep results concerning numerical ranges whereas the proof of Theorem 3.5 uses relatively elementary methods.

INTRODUCTION

In this thesis an extensive study is made of the topological and spectral properties of a subset $\mathcal{P}$ of $B(\mathcal{H})$, the set of bounded endomorphisms on the Hilbert space $\mathcal{H}$. An element T of $\mathcal{P}$, called a paranormal operator, is defined by the relationship $d(z,\sigma(T))\|(T-zI)^{-1}\| = 1$ for all $z \in \rho(T)$, where $\rho(T)$ is the resolvent set of T, and $d(z,\sigma(T))$ is the distance from z to $\sigma(T)$, the spectrum of T. Operators with the above growth condition on $\|(T-zI)^{-1}\|$ arise naturally in spectral operator theory [8].

$\mathcal{P}$ is a very large set in that it contains all normal operators, $\mathcal{N}$, and all hyponormal operators, yet $\mathcal{P}$ is small enough to be contained in the set $\mathcal{L}$ of all T in $B(\mathcal{H})$ such that the convex hull of the spectrum of T is equal to the closure of the numerical range of T. Thus $\mathcal{N} \subseteq \mathcal{P} \subseteq \mathcal{L}$. If the uniform operator (norm) topology is placed on $B(\mathcal{H})$, then the relative topological properties of $\mathcal{N}$, $\mathcal{P}$, and $\mathcal{L}$ can be discussed. As an illustration, it is shown [see Section IV] that $\mathcal{N}$, $\mathcal{P}$, and $\mathcal{L}$ are arc-wise connected, closed, nowhere dense subsets of $B(\mathcal{H})$ when $\dim \mathcal{H} \geqslant 2$, $\mathcal{N} = \mathcal{P}$ when $\dim \mathcal{H} < \infty$, and $\mathcal{N}$ is a nowhere dense subset of $\mathcal{P}$ when $\dim \mathcal{H} = \infty$.

The spectral properties of paranormal and hyponormal operators have received considerable attention in the current literature [22, 23, 30, 31]. Stampfli [25, 26, 27] has shown that a hyponormal operator whose spectrum is a sufficiently "thin" subset of the complex plane, must indeed be normal. In particular, a hyponormal operator with countable spectrum is normal. On the other hand, a paranormal operator

T with countable spectrum can be decomposed as the direct sum, $N \oplus A$, of a normal operator N with $\sigma(N) = \sigma(T)$, and of an operator A with $\sigma(A)$ a subset of the derived set of $\sigma(T)$ [Theorem 3.3]. However, A need not be normal. Stampfli [26, 27] has also proved that a hyponormal operator whose spectrum is a subset of a C^2-smooth rectifiable Jordan curve G_o, is in fact normal. It was then conjectured that this result would hold for T paranormal. This conjecture with the added assumption that T has a countable spectrum is proved here [see Theorem 3.4 and its Corollary], but the conjecture without the countability assumption on the spectrum remained unproved. However, using the Dunford spectral operator theory, it was shown that a paranormal operator whose spectrum is not countable and is a subset of G_o, must indeed have a nontrivial lattice of invariant subspaces [see Theorem 3.9 and its Corollary].

After having completed most of the work for this thesis, Stampfli [30] announced a solution to a problem that generalizes a result stated above. Namely, he proved that a paranormal operator whose spectrum is a subset of G_o, is normal. In his proof [31] several rather deep results about numerical ranges are crucial, whereas the result proved in this thesis [Theorem 3.4] uses relatively elementary methods which depend heavily on the assumption that the spectrum is countable. It would be interesting to see a relatively simple proof of Stampfli's result.

Orland [19] has characterized operators T for which the convex hull of $\sigma(T)$, co $\sigma(T)$, equals the closure of the numerical range of T by the condition $d(z, \text{co}\,\sigma(T))\|(T-zI)^{-1}\| \leq 1$ for all $z \notin \text{co}\,\sigma(T)$, where $d(z, \text{co}\,\sigma(T))$ is the distance from z to $\text{co}\,\sigma(T)$. Thus, an extremal condition on $\|(T-zI)^{-1}\|$ has been characterized by an extremal geometric condition, namely $\text{co}\,\sigma(T)$ is equal to the closure of the numerical range on T. In general [7, p. 556] $d(z,\sigma(T))\|(T-zI)^{-1}\| \geq 1$ for each $z \in \rho(T)$, so that the condition, $d(z,\sigma(T))\|(T-zI)^{-1}\| = 1$, for paranormality is an extremal condition on $\|(T-zI)^{-1}\|$. Therefore, it seems reasonable to expect that the set of paranormal operators can be characterized by some extremal property; however, this remains an open question.

Section I: Definitions and Basic Hilbert Space Properties

Let $\mathcal{H}$ be a complex Hilbert space with the inner product of two elements $x, y \in \mathcal{H}$ denoted by (x,y). An operator on $\mathcal{H}$ is a continuous linear transformation mapping $\mathcal{H}$ into $\mathcal{H}$. Denote the set of all operators on $\mathcal{H}$ by $B(\mathcal{H})$. $B(\mathcal{H})$ will always be given the uniform operator topology. $I \in B(\mathcal{H})$ will always denote the identity operator. For $T \in B(\mathcal{H})$, let $\|T\| = \sup\{\|Tx\|: x \in \mathcal{H}, \|x\| = 1\}$ denote the norm of T and T* denote the adjoint of T. $T \in B(\mathcal{H})$ is invertible if and only if there exists $S \in B(\mathcal{H})$ such that $ST = TS = I$. In this event S is unique and is denoted by S^{-1}. $\rho(T) = \{z \in \mathbb{C} : T-zI \text{ is invertible}\}$ where $\mathbb{C}$ denotes the complex field. $\rho(T)$ is called the resolvent set of T and $\sigma(T) = \mathbb{C} - \rho(T)$ is called the spectrum of T. $\sigma_p(T)$ is the point spectrum of T and $\sigma_\pi(T)$ is the approximate point spectrum of T, [see 11]. The boundry of a set S is denoted by ∂S. It is an important fact that $\partial\sigma(T) \subseteq \sigma_\pi(T)$, [11, problem 63]. $z \in \sigma_\pi(T)$ is a normal approximate eigenvalue for T when: 1) $\|x_n\| = 1$ and $\|(T-zI)x_n\| \to 0$ as $n \to \infty$ imply $\|(T^*-\bar{z}I)x_n\| \to 0$ as $n \to \infty$, and 2) $\|y_n\| = 1$ and $\|(T^*-\bar{z}I)y_n\| \to 0$ as $n \to \infty$ imply $\|(T-zI)y_n\| \to 0$ as $n \to \infty$. $z \in \sigma_p(T)$ is a normal eigenvalue for T when $\{x \in \mathcal{H} : Tx = zx\} = \{x \in \mathcal{H} : T^*x = \bar{z}x\}$. It is well known that the eigenspaces of distinct normal eigenvalues of an operator T are orthogonal [14, p. 233]. If z is a normal eigenvalue for T, then it is not necessarily true that z is a normal approximate eigenvalue for T.

An operator T is quasi-nilpotent if $\sigma(T) = \{0\}$. T is a projection if $T^2 = T$. T is an orthogonal projection if $T^2 = T$ and

$T^* = T$. It is well known that T is an orthogonal projection if and only if $T^2 = T$ and $\|T\| \leq 1$. T is a positive operator, denoted by $T \geq 0$, if $(Tx,x) \geq 0$ for all $x \in \mathcal{H}$. Let $R_{sp}(T) = \sup\{|z|: z \in \sigma(T)\}$ denote the spectral radius of an operator T. By [7, lemma 4, p. 567],

$$R_{sp}(T) = \lim_{n\to\infty} \|T^n\|^{1/n} \qquad \text{and} \qquad R_{sp}(T) \leq \|T\|.$$

For $z \in \rho(T)$, $R(T,z) = (T-zI)^{-1}$ is called the resolvent of T at z.

An operator T is normal if $T^*T - TT^* = 0$. It is well known that if T is normal then $\|T\| = R_{sp}(T)$, [11, p. 115]. T is hyponormal if $T^*T - TT^* \geq 0$ or if $TT^* - T^*T \leq 0$, and T is paranormal if $\|R(T,z)\| = 1/d(z,\sigma(T))$ for all $z \in \rho(T)$ where $d(z,\sigma(T)) = \inf\{|z-w|: w \in \sigma(T)\}$. Let $\mathcal{P}$ denote the set of all paranormal operators on $\mathcal{H}$.

For T an operator on $\mathcal{H}$ the numerical range of T, $W(T)$, is equal to $\{(Tx,x): x \in \mathcal{H}, \|x\| = 1\}$. $W(T)$ is a convex, bounded subset of the complex plane and $\sigma(T) \subseteq \overline{W(T)}$, [see 6]. In [6] Donoghue proves the following theorem.

<u>Proposition 1.1</u> If T is an operator on a two dimensional Hilbert space $\mathcal{H}$, then $W(T)$ is the convex hull of an ellipse whose foci are the eigenvalues of T. If the eigenvalues are distinct, then the eccentricity of the ellipse is $\sin t$ where $0 \leq t \leq \pi/2$ such that $\cos t = |(x,y)|$ where x and y are unit vectors in $\mathcal{H}$ each generating a

distinct eigenspace. If the eigenvalues are the same, say z, then $W(T)$ is the closed circular disc about z of diameter $\|T-zI\|$.

A. Wintner [32] has shown that $\|R(T,z)\| \leq 1/d(z,W(T))$ for all $z \notin \overline{W(T)}$. It is well known [7, corollary 3, p. 566] that $\|R(T,z)\| \geq 1/d(z,\sigma(T))$ for all $z \in \rho(T)$. If S is a set of complex numbers, then let co S denote the convex hull of S. Let $\mathcal{L} = \{T \in B(\mathcal{H}): \text{co}\,\sigma(T) = \overline{W(T)}\}$. G. Orland [19] has proved the following characterization of $\mathcal{L}$.

<u>Proposition 1.2</u> $T \in \mathcal{L}$ if and only if $\|R(T,z)\| \leq 1/d(z,\text{co}\,\sigma(T))$ for all $z \notin \text{co}\,\sigma(T)$.

If $T \in B(\mathcal{H})$, then T is the direct sum of $A \in B(M)$ and $B \in B(M^{\perp})$, written $T = A \oplus B$, if M is a closed subspace of $\mathcal{H}$ that reduces T, $M \neq (0)$ and $M \neq \mathcal{H}$, T restricted to M equals A, and T restricted to $M^{\perp}$ equals B. The following theorem will be used several times [see 11].

<u>Proposition 1.3</u> If the operator T is the direct sum of the operators A and B, then

1. $\|T\| = \text{Max}\{\|A\|, \|B\|\}$
2. $\sigma(T) = \sigma(A) \cup \sigma(B)$
3. $W(T) = \text{co}(W(A) \cup W(B))$.

If T is an operator on $\mathcal{H}$ and $u(z)$ is a rational function of z with no poles lying in the spectrum of T, then we can represent $u(T)$ by

$$u(T) = -\frac{1}{2\pi i}\int_S u(z)R(T,z)\,dz$$

where S is the union of a finite number of closed rectifiable curves that form the boundry of an open bounded set D of complex numbers, where $\sigma(T) \subseteq D$ and $\overline{D}$ contains no singularities of $u(z)$ [see 24].

The mapping E from the Borel sets B in the complex plane $\mathbb{C}$ into the set of projection operators on $\mathcal{H}$ is a resolution of the identity for $T \in B(\mathcal{H})$ [see 8, p. 219] if and only if

1. $E(\mathbb{C}) = I$ and $E(S_1 \cap S_2) = E(S_1)E(S_2)$ for each $S_1, S_2 \in B$,
2. there exists $M > 0$ such that for all $S \in B$

$$\| E(S) \| \leq M,$$

3. if $\{S_n\}$ is a sequence of disjoint Borel sets, then for each $x \in \mathcal{H}$

$$E(\bigcup_n S_n)x = \sum_n E(S_n)x, \quad \text{and}$$

4. for each $S \in B$,

$$TE(S) = E(S)T \quad \text{and} \quad \sigma(T|_{E(S)}) \subseteq \overline{S}.$$

An operator T is a spectral operator if T has a resolution of the identity. If $T \in B(\mathcal{H})$ is a spectral operator with resolution of the identity E, then we can form the following operator on $\mathcal{H}$ [8, p. 226]

$$\hat{T} = \int_{\sigma(T)} z\,dE_z.$$

A spectral operator T is scalar if and only if $T = \hat{T}$. If T is a scalar operator and if f is a rational function with no poles in $\sigma(T)$, then

$$f(T) = \int_{\sigma(T)} f(z)\, dE_z.$$

If T is a scalar operator, then for each fixed $x,y \in \mathcal{H}$ there exists a unique complex Borel measure m such that

$$(Tx,y) = \int_{\sigma(T)} z\, dm(z)$$

and m is given by $m(S) = (E(S)x,y)$ for each Borel set S. We will need the following theorem [8, p. 230].

<u>Proposition 1.4</u> T is a scalar spectral operator if and only if there exists a normal operator N and a self-adjoint operator Q such that $T = QNQ^{-1}$.

We shall always assume that the arc G_o is C^2-smoothly imbedded in a one parameter family of closed rectifiable Jordan curves G_t, $-1 \leq t \leq +1$.

Let T be an operator on $\mathcal{H}$ and let x be a fixed element of $\mathcal{H}$. Then $R(T,z)x$ is a vector valued analytic function of z on $\rho(T)$ and may possible be extended analytically to a larger open set. If the spectrum of T is a nowhere dense set of complex numbers, then $R(T,z)x$ must have unique analytic extensions since then the resolvent set is dense so that any two analytic, or even continuous, extensions must agree on their common domain of definition. Therefore, if the spectrum of T is a subset of G_0, then $R(T,z)x$ has unique analytic extensions. By taking the union of all open sets each of which is the domain of an analytic extension of $R(T,z)x$, we obtain a maximal open set on which $R(T,z)x$ can be analytically extended. Let $\rho(T,x)$ be this open set and let $\sigma(T,x)$ be the complement of $\rho(T,x)$ in the complex plane [8].

An operator T on $\mathcal{H}$ is said to satisfy Dunford's Boundedness Condition (B) [8, p. 226], if there exists a constant K depending only on T such that if $x,y \in \mathcal{H}$ and $\sigma(T,x) \cap \sigma(T,y) = \emptyset$, then $\|x\| \leq K\|x+y\|$.

Section II: Basic Properties of Paranormal Operators

An operator T on the Hilbert space $\mathcal{H}$ is paranormal if for each $z \in \rho(T)$, $\|R(T,z)\| = 1/d(z,\sigma(T))$. Let $\mathcal{P}$ denote the class of all paranormal operators on $\mathcal{H}$. Every normal operator is clearly hyponormal [see Section I for definition]. Stampfli [26] has shown that every hyponormal operator is paranormal:

Proposition 2.1 If the operator T is hyponormal, then T is paranormal.

Proof. The proof follows from these facts about hyponormal operators [26]: (1) If a and b are complex numbers and T is hyponormal, then $aT + bI$ is hyponormal. (2) If T is hyponormal, then $\|T\| = R_{sp}(T)$. (3) If T is hyponormal and invertible, then T^{-1} is hyponormal.

Let $z \in \rho(T)$ then

$$\|R(T,z)\| = R_{sp}((T - zI)^{-1})$$

$$= \frac{1}{d(0,\sigma(T - zI))}$$

$$= \frac{1}{d(z,\sigma(T))} .$$

Therefore T is paranormal.

Theorem 2.1 If T is a paranormal operator and if a and b are complex numbers, then $aT + bI$ and T^* are paranormal.

Proof. If $a = 0$, the proof is trivial. Suppose $a \neq 0$ and let $z \in \rho(aT + bI)$. Then

$$\|((aT + bI) - zI)^{-1}\| = \frac{1}{|a|}\|(T - \frac{z - b}{a}I)^{-1}\|$$

$$= \frac{1}{|a| d((z - b)/a, \sigma(T))}$$

$$= \frac{1}{d(z, \sigma(aT + bI))} .$$

Therefore $aT + bI$ is paranormal.

If $\bar{z} \in \rho(T^*)$, then $z \in \rho(T)$ and $\|(T^* - \bar{z}I)^{-1}\| = \|(T - zI)^{-1}\| = 1/d(z, \sigma(T)) = 1/d(\bar{z}, \sigma(T^*))$. Therefore T^* is paranormal.

The fact that $\text{co}\ \sigma(T) = \overline{W(T)}$ when T is paranormal follows immediately from Proposition 1.2. An elementary proof, independent of Proposition 1.2 is given below.

Theorem 2.2 If T is a paranormal operator, then $\text{co}\ \sigma(T) = \overline{W(T)}$.

Proof. For any operator $\text{co}\ \sigma(T) \subseteq \overline{W(T)}$. Suppose the theorem were false, then by a translation and rotation, if necessary, we may assume [see Theorem 2.1] $\sup \text{Re co}\ \sigma(T) \leq 0$ and $\sup \text{Re}\ W(T) > 0$. Let $z > 0$ and x be a unit vector in $\mathcal{H}$. Then since $d(z, \sigma(T))\|R(T,z)\| = 1$, we have

$$z^2 \leq d(z,\sigma(T))^2 \leq \|(T - zI)x\|^2$$

$$= \|Tx\|^2 + z^2 - 2z\mathrm{Re}(Tx,x).$$

Therefore for all $z > 0$, $z\mathrm{Re}(Tx,x) \leq 0$ so that $\mathrm{Re}(Tx,x) \leq 0$. Hence, $\sup \mathrm{Re}\, W(T) \leq 0$. Contradiction.

The following two theorems will be useful in constructing examples.

Theorem 2.3 If A is any operator on $\mathcal{H}$, then there exists a Hilbert space $\mathcal{K}$ and a normal operator N on $\mathcal{K}$ with $\sigma(N) = \overline{W(A)}$ such that $T = A \oplus N \in B(\mathcal{H} \oplus \mathcal{K})$ is paranormal.

Proof. Since $\overline{W(A)}$ is a compact set of complex numbers, there exists a Hilbert space $\mathcal{K}$ and a normal operator N on $\mathcal{K}$ such that $\sigma(N) = \overline{W(A)}$ [7, p. 581]. Let $T = A \oplus N$. Then by Proposition 1.3 $\sigma(T) = \sigma(N)$. If $z \in \rho(T)$, then

$$\|R(A,z)\| \leq \frac{1}{d(z,W(A))} = \frac{1}{d(z,\sigma(T))},$$

and

$$\|R(N,z)\| = \frac{1}{d(z,\sigma(N))} = \frac{1}{d(z,\sigma(T))}.$$

Therefore

$$\|R(T,z)\| = \mathrm{Max}\{\|R(A,z)\|, \|R(N,z)\|\} = \frac{1}{d(z,\sigma(T))},$$

so that T is paranormal.

Later on we will need the fact that the normal operator N in Theorem 2.3 can be chosen so that $\sigma(N) - \sigma(A)$ is a countable set.

Theorem 2.4 If A is any operator, then there exists a normal operator N such that

1. $A \oplus N$ is paranormal,
2. $\sigma(N) \supseteq \sigma(A)$, and
3. $\sigma(N) - \sigma(A)$ is a countable set whose points of accumulation are contained in $\sigma(A)$.

Proof. Assume $\|A\| = 1$. For $n = 1,2,3,\ldots,$ let

$$S_n = \{z: \ 2/(n+1) \leq d(z,\sigma(A)) \leq 2/n\},$$

$$M_n = \mathrm{Max}\{4, \sup\{\|R(A,z)\|: \ z \in S_n\}\},$$

and let $B(z,r)$ be the open disc of radius r about z. Since S_n is compact and

$$S_n \subseteq \bigcup_{z \in S_n} B(z, 1/M_n),$$

there exists $z_{n_i} \in S_n$, $1 \leq i \leq m_n$ such that

$$S_n \subseteq \bigcup_{i=1}^{m_n} B(z_{n_i}, 1/M_n) .$$

Let N be a normal operator such that [see 10, p. 581]

$$\sigma(N) = \sigma(A) \cup \{z_{n_i}: 1 \leq i \leq m_n, n = 1,2,3,\ldots\}.$$

Let $T = A \oplus N$. If $z \in \rho(A)$ and if $|z| \leq 2$, then there exists n and i such that $z \in S_n \cap B(z_{n_i}, 1/M_n)$. Hence $|z - z_{n_i}| \leq 1/M_n$. Then

$$d(z,\sigma(T)) \, \|R(A,z)\| \leq |z - z_{n_i}| \, 1/M_n \leq 1.$$

If $|z| > 2$, then

$$d(z,\sigma(T)) = d(z,\sigma(N)) \leq d(z,W(A)).$$

Therefore

$$d(z,\sigma(T)) \, \|R(A,z)\| \leq d(z,W(A)) \, \|R(A,z)\| \leq 1.$$

Since N is normal, $\|R(N,z)\| = 1/d(z,\sigma(N)) = 1/d(z,\sigma(T))$, for all $z \in \rho(T)$. Let $z \in \rho(T)$, then

$$\|R(T,z)\| = \text{Max}\{\|R(A,z)\|, \|R(N,z)\|\} = \frac{1}{d(z,\sigma(T))} .$$

Therefore T is paranormal.

As we shall see, the class of all hyponormal operators on $\mathcal{H}$ is distinct, in general, from the class $\mathcal{P}$ of paranormal operators. We know [25] that if T is hyponormal, then $\|T\| = R_{sp}(T)$ and if T is also invertible, then T^{-1} is hyponormal. These properties do not generalize to paranormal operators.

Theorem 2.5 There exists an invertible paranormal operator T such that

1. T is not hyponormal
2. T^2 is not paranormal
3. $\|T\| > R_{sp}(T)$, and
4. T^{-1} is not paranormal.

Proof. Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Let N be a normal operator such that $\sigma(N) = \overline{W(A)}$, and let $T = A \oplus N$. Then by Theorem 2.3, T is paranormal. We know from [25] that a hyponormal operator is hyponormal on invariant subspaces. Therefore, since A is not hyponormal, T is not hyponormal. By Proposition 1.1, W(A) is the closed disc of radius ½ about $z = 1$, and $W(A^2)$ is the closed disc of radius 1 about $z = 1$. Therefore

$$0 \in \overline{W(A^2)} \subseteq \overline{W(T^2)} .$$

But

$$0 \notin co(\sigma(T)^2) = co\sigma(T^2).$$

Therefore, $co\sigma(T^2) \neq \overline{W(T^2)}$ and so by Theorem 2.2 T^2 is not paranormal. Let $x = \begin{pmatrix} \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} \end{pmatrix}$, then $\|x\| = 1$ and $\|Ax\| = \frac{1}{2}\sqrt{10}$. Then

$$\|T\| \geqslant \|Ax\| = \tfrac{1}{2}\sqrt{10} > 3/2 = R_{sp}(T).$$

Therefore $\|T\| > R_{sp}(T)$. If T^{-1} were paranormal, then

$$\|T\| = \|R(T^{-1},0)\| = \frac{1}{d(0,\sigma(T^{-1}))} = R_{sp}(T).$$

Contradiction. Hence T^{-1} is not paranormal.

Section III: Spectral Properties of Paranormal Operators

In addition to discussing the relatively elementary spectral properties of paranormal operators, the goal of this section is to find out what can be said about a paranormal operator whose spectrum is a "thin" subset of the complex plane. In particular, if T has countable spectrum, then T can be decomposed as the direct sum, $N \oplus A$, of a normal operator N with $\sigma(N) = \sigma(T)$, and of an operator A with $\sigma(A)$ a subset of the derived set of $\sigma(T)$. In general the operator $T = N \oplus A$ need not be normal, however if $\sigma(T)$ lies on a C^2-smooth rectifiable Jordan curve G_o, then T is indeed normal. If T is a scalar paranormal operator with countable spectrum, then in order to show that T is normal, the condition that $\sigma(T) \subseteq G_o$ can be weakened to the following condition: For each $z \in \sigma(T)$ there exists $w \in \rho(T)$ such that $|w-z| = d(w,\sigma(T))$. An example is then given to show that a (non-scalar) paranormal operator T with countable spectrum satisfying the above property, need not be normal.

If the assumption that the paranormal operator T has countable spectrum is dropped, then T has a non-trivial lattice of closed invariant subspaces when $\sigma(T) \subseteq G_o$. I was unable to prove the more general statement that a paranormal operator T with $\sigma(T) \subseteq G_o$ is normal. Just recently Stampfli [31] published a proof of this statement (see the introduction for a more detailed discussion of this).

In general the points on the boundary of the spectrum of an operator T are not normal eigenvalues nor normal approximate eigenvalues (although $\partial\sigma(T) \subseteq \sigma_{\pi}(T)$, [11]). However, certain points on $\partial\sigma(T)$ when T is paranormal must be either normal eigenvalues or normal approximate eigenvalues. The theorem is due to Putnam [22, 23], but the proof given here utilizes a trick of von Neumann and is shorter.

Theorem 3.1 If T is an operator with z an eigenvalue or approximate eigenvalue of T and if there exists $w \in \rho(T)$ such that $|w-z| \|R(T,w)\| = 1$, then z is a normal eigenvalue or normal approximate eigenvalue, respectively.

Proof. Suppose z is an approximate eigenvalue of T. Let w be as stated in the theorem. Then there exists a sequence $\{x_n\}$ of unit vectors in $\mathcal{H}$ such that $\|(T-zI)x_n\| \to 0$. Let $S = (z-w)R(T,w)$. Then $I-S = R(T,w)(T-zI)$ so that $\|(I-S)x_n\| \to 0$ and $\|Sx_n\| \to 1$. Moreover, $\|S\| = |z-w| \|R(T,w)\| = 1$. Now

$$\|(I-S^*)x_n\|^2 - \|(I-S)x_n\|^2 = \|S^*x_n\|^2 - \|Sx_n\|^2 \leq 1 - \|Sx_n\|^2,$$

so that $\|(I-S^*)x_n\| \to 0$. Then since $I-S = R(T,w)(T-zI) = (T-zI)R(T,w)$,

$$\begin{aligned}\|(T^*-\bar{z}I)x_n\| &= \|(T^*-\bar{w}I)(I-S^*)x_n\| \\ &\leq \|T^*-\bar{w}I\| \, \|(I-S^*)x_n\| \to 0.\end{aligned}$$

Therefore $\|(T^*-\bar{z}I)x_n\| \to 0$. If $\{y_n\}$ is a sequence of unit vectors in $\mathcal{H}$ such that $\|(T^*-\bar{z}I)y_n\| \to 0$, then by a similar argument $\|(T-zI)y_n\| \to 0$. Therefore z is a normal approximate eigenvalue. The proof for z an eigenvalue of T is similar.

Corollary If T is a paranormal operator with z an eigenvalue or approximate eigenvalue of T and if there exists $w \in \rho(T)$ such that $|w-z| = d(z,\sigma(T))$, then z is a normal eigenvalue or normal approximate eigenvalue, respectively.

Proof. Let z and w be as stated. Then since T is paranormal

$$|w-z| \, \|R(T,w)\| = \frac{|w-z|}{d(w,\sigma(T))} = 1.$$

The corollary now follows from Theorem 3.1.

If T is an operator, then in general isolated points of the spectrum of T are not eigenvalues [see 11, problem 147]. If T is paranormal, then the following is true.

Theorem 3.2 Isolated points of the spectrum of a paranormal operator are normal eigenvalues.

Proof. Let T be paranormal and let z be an isolated point of $\sigma(T)$. Let C be a circle about z of small enough radius so that if $w \in C$, then $|w-z| = d(w,\sigma(T))$. Since $\|(w-z)R(T,w)\| = 1$ as $w \to z$, $(w-z)R(T,w)$ is an analytic function of w at $w = z$. Therefore $(w-z)R(T,w)$ is analytic on an open disc containing C. Let

$$P = -\frac{1}{2\pi i}\int_C R(T,w)\, dw.$$

Then

$$TP - zP = -\frac{1}{2\pi i}\int_C (w-z)R(T,w)\,dw = 0,$$

so that $TP = zP$. Since $P \neq 0$ [see 24, p. 421], $z \in \sigma_p(T)$. By the corollary to Theorem 3.1, z is a normal eigenvalue.

Corollary If T is a paranormal operator on a finite dimensional Hilbert space, then T is normal.

Proof. Since the dimension of the Hilbert space is finite, $\sigma(T)$ is a finite set. By Theorem 3.2, each element of $\sigma(T)$ is a normal eigenvalue. Recall that the eigenspaces of distinct normal eigenvalues are orthogonal [14, p. 233]. If $\sigma(T) = \{z_1, z_2, \ldots, z_n\}$ and if I_i is the identity operator on the eigenspace of z_i, $i = 1,2,\ldots,n$, then T can be written as

$$T = z_1 I_1 \oplus z_2 I_2 \oplus \cdots \oplus z_n I_n.$$

Therefore T is normal.

Paranormal Operators with $\sigma(T)$ Countable

Only paranormal operators with countable spectra will be considered in this part. It will be shown that a paranormal operator with a countable spectrum is not necessarily a normal operator. However, in general we can make the following statement:

Theorem 3.3 If T is a paranormal operator on $\mathcal{H}$ with countable spectrum, then either T is normal or $T = A \oplus N$ where N is a normal operator with $\sigma(N) = \sigma(T)$ and $\sigma(A)$ is a subset of the derived set of $\sigma(T)$.

Proof. If z is an isolated point of $\sigma(T)$, then by Theorem 3.2 z is a normal eigenvalue of T; let $E(z)$ be the eigenspace of z. Let $\sigma_o(T)$ denote the set of all isolated points of $\sigma(T)$. Let

$$M = \text{closed span} \bigcup_{z \in \sigma_o(T)} E(z).$$

By the definition of a normal eigenvalue [see Section I] $E(z)$ reduces T and T is normal on $E(z)$, for each $z \in \sigma_o(T)$. Consequently, M reduces T and T is normal on M. Since $\sigma(T)$ must have at least one isolated point, $M \neq (0)$. If $M = \mathcal{H}$, then T is normal.

Suppose $M \neq \mathcal{H}$. Then write $\mathcal{H} = K \oplus M$ and $T = A \oplus N$ where A is T restricted to K and N is T restricted to M. Clearly $\sigma(N) = \sigma(T)$ and N is normal. Suppose to the contrary that $\sigma(A)$ is not a subset of the set of all accumulation points of $\sigma(T)$, then there exists $w \in \sigma(A)$ such that w is an isolated point of $\sigma(T)$. Then w is an isolated point of $\sigma(A)$, so there exists a circle C about w such that if $z \in C$, then $|z-w| = d(z,\sigma(T)) = d(z,\sigma(A))$. Then for $z \in C$

$$\|R(A,z)\| \leq \text{Max}\{\|R(A,z)\|, \|R(N,z)\|\} = \|R(T,z)\|$$

$$= \frac{1}{d(z,\sigma(T))} = \frac{1}{d(z,\sigma(A))}.$$

Therefore we may use the method of proof of Theorem 3.2 to conclude that $w \in \sigma_p(A)$. But $\sigma_p(A) \subseteq \sigma_p(T)$, so that w is a normal eigenvalue of T. Contradiction.

With Theorem 3.3 we can easily classify all compact paranormal operators.

Corollary If T is a compact paranormal operator, then either T is normal or $T = A \oplus N$ where N is compact and normal and A is compact and quasi-nilpotent.

Proof. The spectrum of a compact operator is countable with zero the only possible point of accumulation.

To show that there exists a non-normal paranormal operator with countable spectrum, simply let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and choose a normal operator N with $\sigma(N)$ countable as in Theorem 2.4 so that $A \oplus N$ is paranormal. Then $A \oplus N$ has countable spectrum and is not normal since A is not normal.

However, if the countable set $\sigma(T)$ for $T \in \mathcal{P}$ is further restricted by assuming that $\sigma(T)$ lies on the C^2-smooth rectifiable Jordan curve G_o, then T must indeed be normal. First, a more general theorem will be proved.

Theorem 3.4 If A is an operator on $\mathcal{H}$ with $\sigma(A) \subseteq G_o$ and with w an isolated point of $\sigma(A)$ and if there exists an open set U containing G_o such that for $z \in U-G_o$, $d(z,G_o)\|R(A,z)\| \leq 1$, then w is a normal eigenvalue of A.

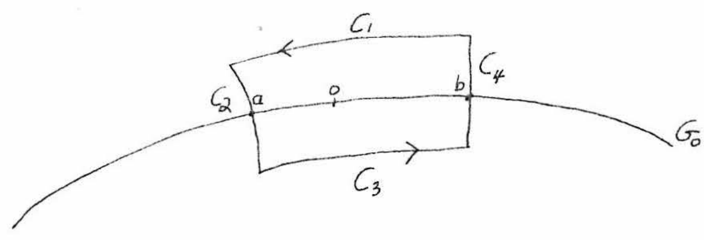
Proof. Without loss of generality assume that w = 0. Since zero is an isolated point of $\sigma(A)$, $R(A,z)$ is analytic in a deleted open neighborhood D of zero with $D \subseteq U$. Let C be a circle about zero contained in D. Then for each $z \in D$

$$R(A,z) = \sum_{n=-\infty}^{\infty} A_n z^n$$

where

$$A_n = -\frac{1}{2\pi i}\int_C z^{-n-1} R(A,z)\, dz.$$

Recall that G_o is C^2-smoothly embedded in a one parameter family of closed rectifiable Jordan curves G_t, $-1 \leq t \leq 1$.



For $a,b \in G_o$ define $C_{a,b} = C_1 + C_2 + C_3 + C_4$ where the C_i's are as shown above and such that

1. for $z \in C_2$, $|z-a| = d(z,G_o)$,

2. for $z \in C_4$, $|z-b| = d(z,G_o)$, and

3. $C_1 \subseteq G_t$ and $C_3 \subseteq G_{-t}$ for some $0 < t < 1$.

We shall always assume that $C_{a,b} \subseteq D$ and that zero is contained in the interior of the region determined by $C_{a,b}$. Now define

$$I(a,b) = -\frac{1}{2\pi i}\int_{C_{a,b}} (z-a)(z-b)R(A,z)\,dz$$

$$= -\frac{1}{2\pi i}\int_{C} (z-a)(z-b)R(A,z)\,dz.$$

Then $I(a,b)$ is independent of t and we have

$$I(a,b) - A_{-3} = \frac{1}{2\pi i}\int_C (-z^2 + (a+b)z - ab + z^2)R(A,z)\,dz$$

$$= (a+b)\frac{1}{2\pi i}\int_C zR(A,z)\,dz - (ab)\frac{1}{2\pi i}\int_C R(A,z)\,dz.$$

Therefore

$$\lim_{\substack{a,b \to 0 \\ a,b \in G_o}} I(a,b) = A_{-3}.$$

Therefore, to show $A_{-3} = 0$, it suffices to show

$$\lim_{\substack{a,b \to 0 \\ a,b \in G_o}} I(a,b) = 0.$$

Let $\epsilon > 0$, and fix $0 < t < 1$ so that the arcs C_2 and C_4 have lengths less than ϵ for every $a,b \in G_o$. Let

$$p(\epsilon) = \sup\{\|R(A,z)\| : \; z \in G_t \cup G_{-t}\} < \infty .$$

Then there exists $q(\epsilon) > 0$ such that whenever $|a|, |b| < q(\epsilon)$, we have

1. the length of C_1 and C_3 is less than $1/p(\epsilon)$, and
2. if $z \in C_{a,b}$, then $|z| < 2\epsilon$.

If $z \in C_2$, then $\|R(A,z)\| \leq 1/d(z,G_o) = 1/|z-a|$. Similarly, if $z \in C_4$ then $\|R(A,z)\| \leq 1/|z-b|$. Making the obvious estimates we obtain

$$\| I(a,b)\| \quad (5/\pi)(2\epsilon)^2,$$

whenever $|a|, |b| < q(\epsilon)$. Therefore $I(a,b) \to 0$ as $a,b \to 0$; $a,b \in G_o$. Hence $A_{-3} = 0$. Therefore $A_{-n} = 0$ for $n = 3,4,5,\ldots$ [7, Theorem 18, p. 573]. Thus $R(A,z)$ has a pole at zero and hence $0 \in \sigma_p(A)$ [see 7, p. 573].

To see that $0 \in \sigma_p(A)$ is a normal eigenvalue, we choose $u \in D$ such that $d(u,G_o) = |u| = d(u,\sigma(A))$. Then since $\|R(A,u)\| \leq 1/d(u,G_o) = 1/|u|$, and since $\|R(A,u)\| \geq 1/d(u,\sigma(A)) = 1/|u|$, we obtain $|u| \|R(A,u)\| = 1$. Thus by Theorem 3.1, zero is a normal eigenvalue of A.

<u>Theorem 3.5</u> If T is an operator with countable spectrum and with $\sigma(T) \subseteq G_o$, and if there exists an open set U containing G_o such that for $z \in U - G_o$, $d(z,G_o)\|R(T,z)\| \leq 1$, then T is a normal operator.

Proof. Let M be the largest closed subspace of $\mathcal{H}$ such that M reduces T and T is normal on M. By Theorem 3.4 each isolated point z of

$\sigma(T)$ is a normal eigenvalue. If $E(z)$ is the eigenspace of z, then $E(z) \subseteq M$. Hence $M \neq (0)$.

If $M = \mathcal{H}$ then we are done, so assume $M \neq \mathcal{H}$. Write $\mathcal{H} = M^{\perp} \oplus M$ and $T = A \oplus N$, where A is T restricted to $M^{\perp}$ and N is T restricted to M. Since $\sigma(A) \subseteq \sigma(T)$ [see Proposition 1.3], we know $\sigma(A)$ is countable and hence has an isolated point z. By Theorem 3.5, z is a normal eigenvalue of A and consequently the eigenspace, $E(z)$, of z reduces A. Thus $E(z)$ reduces T and T is normal on $E(z)$. By the maximality of M, $E(z) \subseteq M$. Contradiction, since $E(z) \subseteq M^{\perp}$.

<u>Corollary</u> If T is a paranormal operator with countable spectrum and if $\sigma(T) \subseteq G_o$, then T is normal.

Proof. For $z \in \rho(T)$, $\|R(T,z)\| = 1/d(z,\sigma(T)) \leq 1/d(z,G_o)$. The corollary now follows from Theorem 3.5.

In the Corollary to Theorem 3.5 it was shown that a paranormal operator T with countable spectrum and with $\sigma(T) \subseteq G_o$, must be normal. If it is assumed that T is a scalar paranormal operator with $\sigma(T)$ countable, then to show that T is normal, the condition $\sigma(T) \subseteq G_o$ can be weakened.

<u>Theorem 3.6</u> If T is a scalar paranormal operator with countable spectrum and if for each $u \in \sigma(T)$ there exists $w \in \rho(T)$ such that $|w-u| = d(w,\sigma(T))$, then T is normal.

Proof. Let $u \in \sigma(T)$, then there exists a sequence $\{u_n\} \subseteq \rho(T)$ such that $u_n \to u$ and $|u_n - u| = d(u_n, \sigma(T))$. Since T is scalar

$$T = \int_{\sigma(T)} z \, dE_z .$$

Therefore

$$(u_n - u) R(T, u_n) = \int_{\sigma(T)} \frac{u-u_n}{z-u_n} \, dE_z .$$

Let $x, y \in \mathcal{H}$ be fixed and define m to be the complex Borel measure $m(S) = (E(S)x, y)$ for each Borel set S in $\sigma(T)$. For each $z \in \sigma(T)$ let

$$f_n(z) = \frac{u-u_n}{z-u_n} \quad \text{and} \quad f(z) = \begin{cases} 1 & \text{if } z = u \\ 0 & \text{if } z \neq u \end{cases} .$$

Then $|f_n(z)| \leq 1$ and $f_n(z) \to f(z)$. Therefore we may apply the Lebesgue dominated convergence theorem:

$$\begin{aligned} |m(\{u\})| &= \left| \int_{\sigma(T)} f(z) \, dm(z) \right| \\ &= \lim_{n \to \infty} \left| \int_{\sigma(T)} f_n(z) \, dm(z) \right| \\ &= \lim_{n \to \infty} |((u-u_n) R(T, u_n) x, y)| \\ &\leq |u - u_n| \, \|R(T, u_n)\| \, \|x\| \, \|y\| = \|x\| \, \|y\| . \end{aligned}$$

Since $m(\{u\}) = (E(\{u\})x, y)$, we have that

$$|(E(\{u\})x, y)| \leq \|x\| \, \|y\| .$$

Letting $y = E(\{u\})x$, we obtain $\|E(\{u\})x\| \leq \|x\|$, and hence $\|E(\{u\})\| \leq 1$. Therefore $E(\{u\})$ is an orthogonal projection for each $u \in \sigma(T)$ [see Section I].

Let $S \subseteq \sigma(T)$ be a Borel set, then S is a countable set so write $S = \{z_1, z_2, z_3, \ldots\}$. Then for each $x, y \in \mathcal{H}$, we have

$$(E(S)x,y) = \sum_{n=1}^{\infty} (E(\{z_n\})x,y) = \sum_{n=1}^{\infty} (x,E(\{z_n\})y)$$

$$= \operatorname{conj} \sum_{n=1}^{\infty} (E(\{z_n\})y,x) = \overline{(E(S)y,x)} = (x,E(S)y).$$

Therefore $E(S) = E(S)^*$ and hence $E(S)$ is an orthogonal projection. Consequently, T is a scalar operator with a resolution of the identity of orthogonal projections; and hence T is normal.

To show that the condition $\sigma(T) \subseteq G_o$ is stronger than the condition stated in Theorem 3.6, let C be the following countable compact set of complex numbers:

$$C = \{0\} \cup \{1/n + i \frac{\sin n}{n} : n = 1,2,3,\ldots\}.$$

Then C does not lie on a C^2-smooth closed arc, but C does satisfy the condition in Theorem 3.6, i.e. for each $z \in C$ there exists $w \notin C$ such that $|w-z| = d(w,C)$. To see this, observe that this condition is equivalent to the following: For each $z \in C$ there exists an open disc D not intersecting C such that $z \in \partial D$.

We will now see that the following condition on $\sigma(T)$ cannot be omitted from Theorem 3.6: For each $z \in \sigma(T)$ there exists $w \in \rho(T)$ such that $|w-z| = d(w,\sigma(T))$. Stampfli [25] has shown that a scalar hyponormal operator whose spectrum has zero area in the complex plane, must be normal. Stampfli's result does not generalize to paranormal operators.

Theorem 3.7 There exists a non-normal, scalar paranormal operator T such that $\sigma(T)$ is countable. Moreover, $\sigma(T)$ has exactly two points of accumulation.

Proof. Let N_o be the normal operator $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, let Q be the self-adjoint operator $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, and let A be the non-normal operator $QN_oQ^{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. By Theorem 2.4 there exists a normal operator N such that:
1) $\sigma(N)$ is countable with zero and one the only points of accumulation, and 2) $T = A \oplus N$ is paranormal. Since A is not normal, T is not normal. Let $B = Q \oplus I$. Then $B = B^*$ and

$$B(N_o \oplus N)B^{-1} = QN_oQ^{-1} \oplus N = A \oplus N = T.$$

Therefore, by Proposition 1.4, T is scalar.

In light of Theorem 3.6, it seems reasonable to conjecture the following theorem: If T is a paranormal operator with countable spectrum such that for each $z \in \sigma(T)$ there exists $w \in \rho(T)$ such that $|w-z| = d(w,\sigma(T))$, then T is normal. This conjecture is false.

Stampfli [25] has shown that a hyponormal operator with countable spectrum is normal, however this result does not generalize to the paranormal case.

Theorem 3.8 There exists a paranormal operator T such that

1. $\sigma(T)$ is countable with zero the only point of accumulation,
2. if $z \in \sigma(T)$, then $|z-2| \leq 2$, and
3. T is not normal.

Proof. Let D_n be the closed disc of radius n about n, for $n = 1,2$. Let V be the Volterra integration operator, i.e. for $f \in L^2(0,1)$,

$$(Vf)(t) = \int_0^t f(x)\,dx.$$

Let $B = (I + V)^{-1}$, and let $A = I - B$. By [11, problem 150], $\sigma(B) = \{1\}$ and $\|B\| = 1$. Hence $\sigma(A) = \{0\}$ and $W(B)$ is contained in the closed disc of radius $\|B\| = 1$. Therefore $W(A) \subseteq D_1$. For $n = 1,2,\ldots$, let

1. $F_n = \{z \in D_2 : 4/(n+1) \leq |z| \leq 4/n\}$.
2. $M_n = \sup\{\|R(A,z)\| : z \in F_n\}$,
3. $d_n = \inf\{d(z,W(A)) : z \in (\partial D_2) \cap F_n\} > 0$,
4. $P_n = \text{Max}\{M_n, 1/d_n\}$, and
5. $B(z,r)$ be the open disc of radius r about z.

Then

$$F_n \subseteq \bigcup_{z \in F_n} B(z, 1/P_n).$$

Since F_n is compact, there exists $z_{n_i} \in F_n$, $1 \leq i \leq m_n$, such that

$$F_n \subseteq \bigcup_{i=1}^{m_n} B(z_{n_i}, 1/P_n).$$

Let N be a normal operator with $\sigma(N) = \{0\} \cup \{z_{n_i} : 1 \leq i \leq m_n,$ $n = 1,2,3,\ldots\}$, then $\sigma(N)$ is a countable set with zero the only point of accumulation. Let $T = A \oplus N$, then $\sigma(T) = \sigma(N)$. We now verify that T is paranormal.

If $z \in D_2$, $z \neq 0$, then there exists n and i such that $z \in F_n \cap B(z_{n_i}, 1/P_n)$. Then

$$\begin{aligned} d(z,\sigma(N))\|R(A,z)\| &\leq |z-z_{n_i}| \; \|R(A,z)\| \\ &\leq (1/P_n)\|R(A,z)\| \\ &\leq (1/M_n)\|R(A,z)\| \leq 1. \end{aligned}$$

If z is real and negative, then

$$d(z,\sigma(N))\|R(A,z)\| \leq |z| \frac{1}{d(z,W(A))} = 1.$$

Suppose $z \notin D_2$ and that z is not real and negative. Let x be the point of intersection of ∂D_2 with the shortest line segment connecting z and $\overline{W(A)}$. Observe that $x \neq 0$. Then $d(z,W(A)) = |z-x| + d(x,W(A))$. There exists n and i such that $x \in F_n \cap B(z_{n_i}, 1/P_n)$. Then $|x-z_{n_i}| \leq 1/P_n$, and so

$$|z-z_{n_i}| \leq |z-x| + 1/P_n \leq |z-x| + d_n$$

$$\leq |z-x| + d(x,W(A)) = d(z,W(A)).$$

Therefore,

$$d(z,\sigma(N))\|R(A,z)\| \leq |z-z_{n_i}| \, \|R(A,z)\|$$

$$\leq d(z,W(A))\frac{1}{d(z,W(A))} = 1.$$

Therefore, for each complex number $z \neq 0$,

$$d(z,\sigma(N))\|R(A,z)\| \leq 1.$$

Since N is normal, for each $z \in \rho(T) = \rho(N)$, $\|R(N,z)\| = 1/d(z,\sigma(N)) = 1/d(z,\sigma(T))$. Hence, for $z \in \rho(T)$

$$\|R(T,z)\| = \text{Max}\{\|R(A,z)\|, \|R(N,z)\|\} = \frac{1}{d(z,\sigma(T))}.$$

Therefore T is paranormal.

For the rest of this section, the assumption that T has countable spectrum will be dropped. If T is an operator on $\mathcal{H}$, then T is said to have a lattice of closed subspaces on the closed subsets of $\sigma(T)$ if for each closed $S \subseteq \sigma(T)$ there exists a closed subspace, M(S), of $\mathcal{H}$ such that: 1) $M(\emptyset) = 0$, 2) $M(\sigma(T)) = \mathcal{H}$, and 3) if S_1 and S_2 are closed subsets of $\sigma(T)$, then $M(S_1 \cap S_2) = M(S_1) \cap M(S_2)$. The lattice is non-trivial if there exists a closed set $S \subseteq \sigma(T)$ such that $M(S) \neq 0$ and $M(S) \neq \mathcal{H}$.

An operator T is said to satisfy Dunford's Boundedness condition (B) [8, p. 226], if there exists a constant k depending only on T such that if $x, y \in \mathcal{H}$ and $\sigma(T,x) \cap \sigma(T,y) = \emptyset$ [see Section I], then $\|x\| \leq k\|x+y\|$.

<u>Theorem 3.9</u> If $T \in B(\mathcal{H})$ with $\sigma(T) \subseteq G_o$ and if there exists an open set U containing G_o such that for each $z \in U - G_o$, $d(z,G_o)\|R(T,z)\| \leq 1$, then either there exists a complex number b such that $T = bI$, or T has a non-trivial lattice of invariant subspaces on the closed subsets of $\sigma(T)$.

If T is a paranormal operator with $\sigma(T) \subseteq G_o$, then for each $z \notin G_o$, $d(z,G_o)\|R(T,z)\| \leq d(z,\sigma(T))\|R(T,z)\| = 1$. Thus, as an immediate consequence of Theorem 3.9, we have the following:

<u>Corollary</u> If T is a paranormal operator with $\sigma(T) \subseteq G_o$, then either there exists a complex number b such that $T = bI$, or T has a non-trivial lattice of invariant subspaces on the closed subsets of $\sigma(T)$.

The following lemma comes from [7, Lemma 2, p. 240; and Theorem 18, p. 264].

<u>Lemma 1</u> (a) $\sigma(T,x) = \emptyset$ if and only if $x = 0$.

(b) If $T \in B(\mathcal{H})$ with $\sigma(T) \subseteq G_o$ and if there exists an open set U containing G_o such that for each $z \in U - G_o$,

$d(z,G_0)\|R(T,z)\| \leq 1$, then T is scalar if and only if T satisfies condition (B).

Lemma 2 If T is an operator on $\mathcal{H}$ with $\sigma(T) \subseteq G_0$, then

$$\sigma(T) = \text{closure} \bigcup_{x \in \mathcal{H}} \sigma(T,x).$$

Proof. Clearly $\sigma(T,x) \subseteq \sigma(T)$ for each $x \in \mathcal{H}$. Suppose the lemma to be false, then there exists $a \in \sigma(T)$ and $\epsilon > 0$ such that $N(a,\epsilon) \cap \rho(T,x) = \emptyset$ for all $x \in \mathcal{H}$, where $N(a,\epsilon) = \{z: |z-a| < \epsilon\}$. Let $f_x(z)$ be the maximal analytic extension of $R(T,z)x$ for each $x \in \mathcal{H}$. Then f_x is analytic in $N(a,\epsilon) \cup \rho(T)$. For $z \in \rho(T) \cup N(a,\epsilon)$, define $B_z: \mathcal{H} \longrightarrow \mathcal{H}$ as $B_z x = f_x(z)$. Fix $x,y \in \mathcal{H}$, then for $z \in \rho(T)$

$$B_z(x + y) = f_{x+y}(z) = R(T,z)(x + y)$$

$$= R(T,z)x + R(T,z)y = B_z x + B_z y.$$

Since B is a continuous function of z and since $\sigma(T) \subseteq G_0$, we have that $B_z(x + y) = B_z x + B_z y$ for all $z \in \rho(T) \cup N(z,\epsilon)$. Similarly one shows B_z is homogeneous and $B_z(T-zI) = (T-zI)B_z = I$ for all $z \in \rho(T) \cup N(a,\epsilon)$. Therefore, $B_a(T-aI) = (T-aI)B_a = I$, so that $T-aI$ is one-to-one and onto. Thus, by [11, problem 41], $T-aI$ is invertible and $a \in \rho(T)$. Contradiction.

We now state the following lemma from [7, Lemma 4, p. 254].

Lemma 3 If $T \in B(\mathcal{H})$ with $\sigma(T) \subseteq G_o$ and if there exists an open set U containing G_o so that for each $z \in U - G_o$, $d(z, G_o) \| R(T,z) \| \leq 1$, and if S is a closed subset of $\sigma(T)$, then $M(S) = \{x \in \mathcal{H} : \sigma(T,x) \subseteq S\}$ is a closed subspace invariant under T.

Proof of Theorem 3.9. If $\sigma(T) = \{b\}$ for some complex number b, then by Theorem 3.4 $T = bI$. For the rest of the proof assume that $\sigma(T)$ has at least two points.

Let M(S) be defined as in Lemma 3. By Lemma 1, $\sigma(T,x) = 0$ if and only if $x = 0$, so that $M(\emptyset) = 0$. Since $\sigma(T,x) \subseteq \sigma(T)$ for each $x \in \mathcal{H}$, $M(\sigma(T)) = \mathcal{H}$. It is obvious that $M(S_1 \cap S_2) = M(S_1) \cap M(S_2)$ whenever S_1 and S_2 are closed subsets of $\sigma(T)$. Therefore, these sets form a lattice of closed, invariant subspaces on the closed subsets of $\sigma(T)$. If this lattice is non-trivial, then the proof is complete.

Suppose that this lattice is trivial, i.e. $M(S) = 0$ or $M(S) = \mathcal{H}$ for every closed subset S of $\sigma(T)$. Then, since $x \in M(\sigma(T,x))$, $M(\sigma(T,x)) = \mathcal{H}$ for every nonzero $x \in \mathcal{H}$. Consequently, by Lemma 2, $\sigma(T,x) = \sigma(T)$ for all nonzero $x \in \mathcal{H}$. Hence whenever $\sigma(T,x) \cap \sigma(T,y) = \emptyset$, either $x = 0$ or $y = 0$. Thus T satisfies condition (B) trivially so that, by Lemma 1, T is scalar. Therefore, since $\sigma(T)$ has at least two elements, T has a non-trivial lattice (formed from the resolution of the identity of T [see Section I]) of closed, invariant subspaces on the closed subsets of $\sigma(T)$.

Section IV: Topological Properties of Paranormal Operators

Let $\mathcal{P}$ denote the set of all paranormal operators on Hilbert space $\mathcal{H}$. Let

$$\mathcal{L} = \{T \in B(\mathcal{H}): \ \mathrm{co}\,\sigma(T) = \overline{W(T)}\},$$

and let $\mathcal{N}$ be the set of all normal operators on $\mathcal{H}$. It will always be assumed that $B(\mathcal{H})$ has the uniform operator (norm) topology. In this section, the topological properties of $\mathcal{P}$ relative to $\mathcal{N}$, $\mathcal{L}$, and $B(\mathcal{H})$ will be discussed.

It will be shown that $\mathcal{N}$, $\mathcal{P}$, and $\mathcal{L}$ are closed, arc-wise connected subsets of $B(\mathcal{H})$. In Section II, it was shown that $\mathcal{N} \subseteq \mathcal{P} \subseteq \mathcal{L}$. When the dimension of $\mathcal{H}$ is finite, then by the corollary to Theorem 3.2, $\mathcal{N} = \mathcal{P}$. When $\dim \mathcal{H} = \infty$, then it will be shown that $\mathcal{N}$ is a nowhere dense subset of $\mathcal{P}$. When $\dim \mathcal{H} \leq 4$, then $\mathcal{P} = \mathcal{L}$ and when $5 \leq \dim \mathcal{H} < \infty$, then $\mathcal{P}$ has a nonempty interior in $\mathcal{L}$. When $\dim \mathcal{H} = \infty$, then $\mathcal{P} \neq \mathcal{L}$ and it is not known if $\mathcal{P}$ is a nowhere dense subset of $\mathcal{L}$. Finally, it will be shown that $\mathcal{L}$ is a nowhere dense subset of $B(\mathcal{H})$, when $\dim \mathcal{H} \geq 2$.

The following notation will be used in this section: If S is a compact set in the complex plane $\mathbb{C}$ and if $\epsilon > 0$, then let

$$S + (\epsilon) = \{z \in \mathbb{C} : \ d(z,S) < \epsilon\}.$$

If S and S_n, $n = 1,2,3,\ldots$ are compact sets in $\mathbb{C}$, then the sequence $\{S_n\}$ approaches S, written $S_n \to S$, if for every $\epsilon > 0$ there exists a positive integer N such that for $n \geq N$, $S_n \subseteq S + (\epsilon)$ and $S \subseteq S_n + (\epsilon)$.

In general $\sigma(T)$ is not a continuous function of T in $B(\mathcal{H})$ [see 11, problem 85], but $\sigma(T)$ is continuous if we restrict T to $\mathcal{P}$.

<u>Theorem 4.1</u> If $\{T_n\}$ is a sequence of paranormal operators approaching the operator T in norm, then

$$\sigma(T_n) \to \sigma(T) \qquad \text{as} \qquad n \to \infty .$$

To prove this theorem we need the following lemma from [11, problem 86].

<u>Lemma</u> If $T \in B(\mathcal{H})$ and $\epsilon > 0$, then there exists $\delta > 0$ such that if $S \in B(\mathcal{H})$ and $\|T-S\| < \delta$, then $\sigma(S) \subseteq \sigma(T) + (\epsilon)$.

Proof of Theorem 4.1 We know by the lemma that for each $\epsilon > 0$ there exists a positive integer N such that for $n \geq N$, $\sigma(T_n) \subseteq \sigma(T) + (\epsilon)$. Therefore, to show $\sigma(T_n) \to \sigma(T)$, it suffices to show that for each $\epsilon > 0$ there exists a positive integer N such that $\sigma(T) \subseteq \sigma(T_n) + (\epsilon)$ for all $n \geq N$. If this does not hold, then without loss of generality we may assume that there exists $\epsilon > 0$ and a sequence $\{z_n\} \subseteq \sigma(T)$ such that $d(z_n, \sigma(T_n)) \geq \epsilon$ for all n. Since $\sigma(T)$ is compact, we may assume $z_n \to z \in \sigma(T)$. If $|z_n - z| < \epsilon/2$, then

$$d(z,\sigma(T_n)) \geq d(z_n,\sigma(T_n)) - |z - z_n|$$

$$\geq \epsilon - \epsilon/2$$

$$= \epsilon/2.$$

Hence

$$\| R(T_n,z) \| = \frac{1}{d(z,\sigma(T_n))} \leq 2/\epsilon.$$

Now choose m so that $\|(T_m - T)R(T_m,z)\| < 1$, then $I - (T_m - T)R(T_m,z)$ is invertible [11, problem 173]. Let

$$A = R(T_m,z)(I - (T_m - T)R(T_m,z))^{-1}.$$

Then $A(T - zI) = (T - zI)A = I$ so that $z \in \rho(T)$. Contradiction.

Theorem 4.2 $\mathcal{P}$ is an arc-wise connected, closed subset of $B(\mathcal{H})$.

Proof. Since $T \in \mathcal{P}$ implies $aT \in \mathcal{P}$ for every complex number a, we see that the ray in $B(\mathcal{H})$ through T is contained in $\mathcal{P}$. Therefore $\mathcal{P}$ is arc-wise connected.

Suppose $T_n \to T$, $\{T_n\}$ a sequence of operators in $\mathcal{P}$, and $T \in B(\mathcal{H})$. Let $z \in \rho(T)$. By the lemma to Theorem 4.1,

$$\lim_{n\to\infty} \sup \frac{1}{d(z,\sigma(T_n))} \leq \frac{1}{d(z,\sigma(T))} .$$

Therefore, since $\|R(T_n,z)\| = 1/d(z,\sigma(T_n))$ whenever $z \in \rho(T_n)$, there exists a positive integer N such that the sequence $\{\|R(T_n,z)\| : n \geq N\}$ is bounded. Then, since $R(T,z) - R(T_n,z) = R(T,z)(T-T_n)R(T_n,z)$, $\|R(T_n,z)\| \to \|R(T,z)\|$ as $n \to \infty$. Consequently,

$$\|R(T,z)\| = \lim_{n\to\infty} \|R(T_n,z)\|$$

$$= \lim_{n\to\infty} \frac{1}{d(z,\sigma(T_n))}$$

$$\leq \frac{1}{d(z,\sigma(T))} .$$

Since in general $\|R(T,z)\| \geq 1/d(z,\sigma(T))$, T is paranormal.

<u>Theorem 4.3</u> $\mathcal{L}$ is an arc-wise connected, closed subset of $B(\mathcal{H})$.

Proof. Since $T \in \mathcal{L}$ implies that $aT \in \mathcal{L}$ for every complex number a, $\mathcal{L}$ is arc-wise connected.

Let $T_n \to T$, $\{T_n\} \subseteq \mathcal{L}$ and $T \in B(\mathcal{H})$. Since $|(T_n x,x)-(Tx,x)| \leq \|T_n - T\|$ for $\|x\| = 1$, $W(T_n) \subseteq W(T)+(2\|T-T_n\|)$ and $W(T) \subseteq W(T_n)+(2\|T-T_n\|)$. Consequently, $\overline{W(T_n)} \to \overline{W(T)}$. Let $\epsilon > 0$, then by the lemma to Theorem 4.1 there exists a positive integer N such that $\sigma(T_n) \subseteq \sigma(T) + (\epsilon)$ for all $n \geq N$. Therefore, for $n \geq N$, $\text{co}\ \sigma(T_n) \subseteq \text{co}\ \sigma(T) + (\epsilon)$ and hence

$$\overline{W(T)} = \lim_{n\to\infty} \overline{W(T_n)}$$

$$= \lim_{n\to\infty} \text{co}\ \sigma(T_n)$$

$$\subseteq \text{co}\ \sigma(T) + (\epsilon).$$

Since $\epsilon > 0$ is arbitrary, $\overline{W(T)} \subseteq \text{co}\ \sigma(T)$. Since in general $\text{co}\ \sigma(T) \subseteq \overline{W(T)}$, $T \in \mathcal{L}$.

Let $\mathcal{N}$ be the set of all normal operators on $\mathcal{H}$. Since $\|T_n - T\| \to 0$ implies $\|T_n^* - T^*\| \to 0$, $\mathcal{N}$ is closed in the uniform operator topology on $B(\mathcal{H})$. Since $T \in \mathcal{N}$ implies $aT \in \mathcal{N}$ for any complex a, $\mathcal{N}$ is arc-wise connected.

We know that

$$\mathcal{N} \subseteq \mathcal{P} \subseteq \mathcal{L} \subseteq B(\mathcal{H}).$$

Much more can be said about how the above four sets are related.

It has already been shown [see the corollary to Theorem 3.2] that $\mathcal{N} = \mathcal{P}$ when $\dim \mathcal{H} < \infty$. When $\dim \mathcal{H} = \infty$, then $\mathcal{N}$ is a very "thin" subset of $\mathcal{P}$. The following theorem makes this more precise.

Theorem 4.4 $\mathcal{N}$ is a nowhere dense subset of $\mathcal{P}$ when $\dim \mathcal{H} = \infty$.

Proof. Since $\mathcal{N}$ is closed, to show that $\mathcal{N}$ is a nowhere dense subset of $\mathcal{P}$, it suffices to show that $\mathcal{N}$ has empty interior in $\mathcal{P}$. Let $T \in \mathcal{N}$ and let $\epsilon > 0$.

First suppose that T has an eigenvalue of infinite multiplicity. We may assume that the eigenvalue is zero. Let M be the eigenspace of zero. Then $\dim M = \infty$, M reduces T, and we can write $T = B \oplus Z$ where Z is the zero operator on M. Let

$$S = \begin{pmatrix} 0 & \epsilon \\ 0 & 0 \end{pmatrix} \oplus N$$

be a non-normal paranormal operator [see Theorem 2.3] on M with N a normal operator such that

$$\sigma(N) = \text{closure } W\begin{pmatrix} 0 & \epsilon \\ 0 & 0 \end{pmatrix}.$$

Then $B \oplus S$ is a non-normal paranormal operator such that

$$\|T - B \oplus S\| = \|B \oplus Z - B \oplus S\| = \|S\| = \epsilon.$$

The last equality holds since

$$\|N\| = R_{sp}(N) = \epsilon/2 \qquad \text{and} \qquad \left\|\begin{pmatrix} 0 & \epsilon \\ 0 & 0 \end{pmatrix}\right\| = \epsilon .$$

Therefore, since $\epsilon > 0$ is arbitrary, T is not contained in the interior of $\mathcal{N}$ in $\mathcal{P}$.

If $\sigma(T)$ is finite and $T \in \mathcal{N}$, then $\sigma(T) = \sigma_p(T)$ and T has an eigenvalue of infinite multiplicity. We therefore assume that $\sigma(T)$ is infinite and that zero is an accumulation point of $\sigma(T)$. Let D be the open disc about zero of radius $\epsilon/2$. Let E be the resolution of the identity for T [see Section I] so that

$$T = \int_{\sigma(T)} z dE_z .$$

Let $M = E(\overline{D})$, $P = \sigma(T) - D$, and let

$$A = \int_P z dE_z .$$

Then M reduces T, $\dim M = \infty$, and A is a normal operator. Let Z be the zero operator on M. Then $A \oplus Z$ is a normal operator with zero an eigenvalue of infinite multiplicity, and

$$\| T - A \oplus Z \| = \left\| \int_{\overline{D}} z dE_z \right\| \leq \epsilon/2 .$$

By the first part of this proof, there exists a non-normal paranormal operator S such that $\| A \oplus Z - S \| < \epsilon/2$. Then

$$\| T - S \| \leq \| T - A \oplus Z \| + \| A \oplus Z - S \| < \epsilon .$$

Therefore, since $\epsilon > 0$ is arbitrary, T is not contained in the interior of $\mathcal{N}$ in $\mathcal{P}$. Hence the interior of $\mathcal{N}$ in $\mathcal{P}$ is empty.

Next, the relationship between $\mathcal{P}$ and $\mathcal{L}$ will be discussed. To do this define C_2 to be the set of all operators $T \in \mathcal{L}$ with $W(T)$ a closed line segment or a point. For $k = 3,4,5,\ldots,$ let C_k be the set of all operators $T \in \mathcal{L}$ such that $W(T)$ is the convex hull of a polygon with k sides. If $T \in C_k$, $k = 2,3,\ldots,$ then each vertex of $W(T)$ must be in the spectrum of T. S. Hildebrandt [15] has shown that if $z \in \sigma_p(T) \cap \partial W(T)$ for $T \in \mathcal{L}$, then z is a normal eigenvalue of T. Thus for $T \in C_k$ the verticies of $W(T)$ are normal eigenvalues of T, when $\dim \mathcal{H} < \infty$. Hence, all the operators in $C_n \cup C_{n-1}$ are normal operators when $\dim \mathcal{H} = n < \infty$.

S. Hildebrandt [15] has shown that $\mathcal{N} = \mathcal{P} = \mathcal{L}$ when $\dim \mathcal{H} \leq 4$, and that $\mathcal{N} \neq \mathcal{L}$ for $5 \leq \dim \mathcal{H} < \infty$. The following theorem says much more about how $\mathcal{P}$ and $\mathcal{L}$ are related when $5 \leq \dim \mathcal{H} < \infty$. Recall that $\mathcal{P} = \mathcal{N}$ for $\dim \mathcal{H} < \infty$.

<u>Theorem 4.5</u> If $5 \leq \dim \mathcal{H} = n < \infty$, then the interior of $\mathcal{P}$ in $\mathcal{L}$ equals $C_n \cup C_{n-1}$.

Proof. Suppose $T \in C_n \cup C_{n-1}$. Since $C_n \cup C_{n-1} \subseteq \mathcal{N}$, T is normal. There exists $\epsilon > 0$ such that whenever $S \in \mathcal{L}$, $\| T - S \| < \epsilon$, then $S \in C_n \cup C_{n-1}$. To show this, suppose the statement were false. Then there would exist $\{S_n\} \subseteq \mathcal{L}$ such that $\| T - S_n \| \to 0$ and

$$S_n \in \bigcup_{i=2}^{n-2} C_i.$$

Then, since $W(S_n) \to W(T)$,

$$T \in \bigcup_{i=2}^{n-2} C_i.$$

Contradiction. Hence, T is an interior point of $\mathcal{P}$ in $\mathcal{L}$.

Let T be contained in the interior of $\mathcal{P} = \mathcal{N}$ in $\mathcal{L}$. Suppose $T \notin C_n \cup C_{n-1}$. Let $\epsilon > 0$. Since $\operatorname{co} \sigma(T) = W(T)$, $T \in C_k$, for some $k \leq n - 2$. Since $\dim \mathcal{H} \geq 5$ and since $T \in C_k$, there exists a normal operator N such that

1. $\|T - N\| < \epsilon/2$
2. $W(N)$ is a polygon with at least three sides, and
3. N has at least two eigenvalues z, w contained in the interior of $W(N)$.

Write $N = A \oplus B$ where B can be written as $B = \begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix}$.
Let $a > 0$ and let

$$C = \begin{pmatrix} z & a \\ 0 & w \end{pmatrix}.$$

then

$$\|B - C\| = \left\|\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\right\| = a .$$

Choose $a > 0$ small enough so that $W(C) \subseteq W(N)$ and so that $a < \epsilon/2$. Then since $W(A) = W(N)$ and $\sigma(A \oplus C) = \sigma(N)$, $\operatorname{co} \sigma(A \oplus C) = W(A \oplus C)$. Hence $A \oplus C \in \mathcal{L}$. Since $A \oplus C$ is not normal and since

$$\|T - A \oplus C\| \leq \|T - N\| + \|N - A \oplus C\| \leq \epsilon/2 + \epsilon/2 = \epsilon,$$

T is not an interior point of $\mathcal{P}$ in $\mathcal{L}$. Contradiction. Hence $T \in C_n \cup C_{n-1}$.

It is an open question as to what the interior of $\mathcal{P}$ in $\mathcal{L}$ is when $\dim \mathcal{H} = \infty$. However, it can be shown that $\mathcal{P} \neq \mathcal{L}$ when $\dim \mathcal{H} = \infty$.

Theorem 4.6 $\mathcal{P} \neq \mathcal{L}$ when $\dim \mathcal{H} = \infty$.

Proof. Write $\mathcal{H} = M \oplus M^{\perp}$ where $\dim M = 5$. Let

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

where a,b,c are three complex numbers that form a triangle with W(A) contained in the interior of the triangle. Consider $A \oplus N$ as an operator on M and observe that $\operatorname{co} \sigma(A \oplus N) = W(N) = W(A \oplus N)$. Since $A \oplus N$ is not normal and since $\dim M < \infty$, $A \oplus N$ is not paranormal. Hence there exists $z \in \rho(A \oplus N)$ such that

$$\|R(A \oplus N, z)\| > \frac{1}{d(z, \sigma(A \oplus N))}.$$

Let I be the identity operator on $M^{\perp}$ and let $T = A \oplus N \oplus aI$.

Then, $\sigma(T) = \sigma(A \oplus N)$ and $W(T) = W(A \oplus N)$. Therefore $T \in \mathcal{L}$.
Since $d(z,\sigma(T)) \leq |z-a|$,

$$\|R(T,z)\| = \text{Max}\left\{\|R(A \oplus N,z)\|, \frac{1}{|z-a|}\right\}$$

$$= \|R(A \oplus N,z)\|$$

$$> \frac{1}{d(z,\sigma(T))} .$$

Therefore T is not paranormal.

The main result of this section is to show that if $\dim \mathcal{H} \geq 2$, then $\mathcal{L}$ is a nowhere dense subset of $B(\mathcal{H})$ in the uniform operator (norm) topology. Once this is shown, it follows immediately that $\mathcal{N}$ and $\mathcal{P}$ are nowhere dense subsets of $B(\mathcal{H})$.

Theorem 4.7 $\mathcal{L}$ is a nowhere dense subset of $B(\mathcal{H})$ when $\dim \mathcal{H} \geq 2$.

To prove this theorem we need the following two technical lemmas.

Lemma 1 If z_1 and z_2 are distinct, normal approximate eigenvalues of $T \in B(\mathcal{H})$, then there exists sequences $\{x_n\}$ and $\{y_n\}$ of unit vectors in $\mathcal{H}$ such that

1. $(x_n, y_n) = 0$ for all n,
2. $\|(T-z_1I)x_n\| \to 0$ as $n \to \infty$, and
3. $\|(T-z_2I)y_n\| \to 0$ as $n \to \infty$.

Lemma 2 If $T \in \mathcal{L}$ such that there exists distinct $a, b \in \partial W(T) \cap \sigma_p(T)$, then T is not contained in the interior of $\mathcal{L}$.

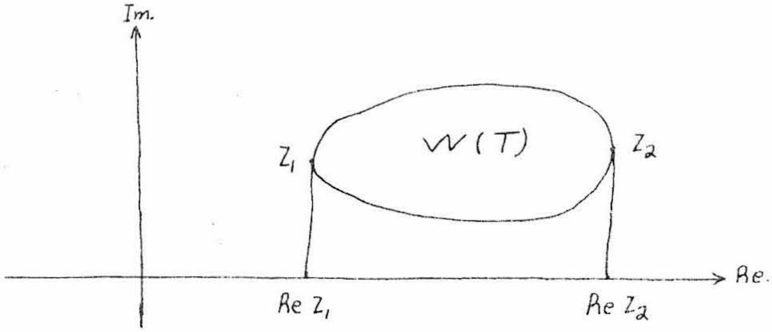
Proof of Theorem 4.7 Since $\mathcal{L}$ is closed [Theorem 4.3], to show that $\mathcal{L}$ is nowhere dense it suffices to show that $\mathcal{L}$ has empty interior.

We first remark that if T is in the interior of $\mathcal{L}$, then $\sigma(T)$ must contain at least two points. Suppose $T \in \mathcal{L}$ and $\sigma(T) = \{a\}$. Then $((T-aI)x,x) = 0$ for all $x \in \mathcal{H}$ so that $T = aI$. Since $\dim \mathcal{H} \geq 2$, write $\mathcal{H} = M \oplus M^{\perp}$ where $\dim M = 2$. Let $b > 0$ and define $A \in B(\mathcal{H})$ as

$$A = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \text{ on } M, \qquad \text{and} \qquad A = 0 \text{ on } M^{\perp}.$$

Then $\sigma(T+A) = \{a\}$ and since $b \neq 0$, $\{a\} \neq \overline{W(T+A)}$. Therefore $T+A \notin \mathcal{L}$. Since $\|A\| = b > 0$ is arbitrary, $T \notin$ interior $\mathcal{L}$.

With the above remark completed, we can now finish the proof of Theorem 4.7. Suppose the theorem were false and there exists $T \in$ interior $\mathcal{L}$. Then there exists $\epsilon > 0$ such that whenever $V \in B(\mathcal{H})$ and $\|T - V\| < \epsilon$, then $V \in \mathcal{L}$. From the above remark $\sigma(T)$ must contain at least two points. There must be at least two extreme points of W(T), since extreme points of W(T) for $T \in \mathcal{L}$ are extreme points of $\sigma(T)$. Hence, after a rotation, if necessary, we may assume there exists $z_1, z_2 \in \sigma_{\pi}(T) \cap \partial W(T)$



such that

1. $\text{Re } z_1 = \inf \text{Re } W(T)$,

2. $\text{Re } z_2 = \sup \text{Re } W(T)$, and

3. $\text{Re } z_1 < \text{Re } z_2$.

Since $z_1, z_2 \in \partial W(T)$, z_1 and z_2 are normal approximate eigenvalues of T [14, Theorem 2, p. 233]. By Lemma 1 there exists unit vectors $x, y \in \mathcal{H}$ such that $(x,y) = 0$, $\|(T-z_1I)x\| < \epsilon/8$, and $\|(T-z_2I)y\| < \epsilon/8$. Let M be the closed subspace spanned by $\{x,y\}$. Define $C \in B(\mathcal{H})$ as

$$Cx = -(\epsilon/4)x,$$

$$Cy = +(\epsilon/4)y, \text{ and}$$

$$Cz = 0 \quad \text{for all} \quad z \in M^{\perp}.$$

Since $\|C\| \leq \epsilon/2$, $T + C \in \mathcal{L}$. Since

$$((T + C)x,x) = (Tx,x) - \epsilon/4 \quad \text{and}$$

$$|(Tx,x) - z_1| \leq \|(T-z_1I)x\| < \epsilon/8,$$

we obtain $\inf \text{Re } W(T + C) < \text{Re } z_1$. Since $T+C \in \mathcal{L}$, there exists $a \in \sigma_{\pi}(T+C) \cap \partial W(T+C)$ such that $\text{Re } a = \inf \text{ Re } W(T+C)$. Since C is a compact operator, Weyl's spectral inclusion Theorem [11, problem 143] yields

$$\sigma(T+C) - \sigma_p(T+C) \subsetneq \sigma(T).$$

Therefore, $a \in \sigma_p(T+C)$, so that

$$a \in \sigma_p(T+C) \cap \partial W(T+C).$$

Similarly one shows there exists

$$b \in \sigma_p(T+C) \cap \partial W(T+C)$$

such that $\text{Re } b = \sup \text{Re } W(T+C) > \text{Re } z_2$, and hence $a \neq b$. By Lemma 2, there exists $S \in B(\mathcal{H})$ such that $\|S\| < \epsilon/2$ and $T+C+S \notin \mathcal{L}$. But $\|T-(T+C+S)\| \leq \|C\| + \|S\| < \epsilon$ so by assumption $T+C+S \in \mathcal{L}$. Contradiction.

Proof of Lemma 1. There exists sequences $\{w_n\}$ and $\{y_n\}$ of unit vectors in $\mathcal{H}$ such that $\|(T-z_1I)w_n\| \to 0$ and $\|(T-z_2I)y_n\| \to 0$ as $n \to \infty$. Then

$$|(z_1-z_2)(w_n,y_n)| = |(z_1w_n,y_n) - (w_n,\bar{z}_2y_n)|$$

$$\leq |((T-z_1I)w_n,y_n)| + |(w_n,(T^*-\bar{z}_2I)y_n)|$$

$$\leq \|(T-z_1I)w_n\| + \|(T^*-\bar{z}_2I)y_n\|.$$

Therefore, $|(z_1-z_2)(w_n,y_n)| \to 0$ as $n \to \infty$. Since $z_1 \neq z_2$, $(w_n,y_n) \to 0$ as $n \to \infty$.

There exists complex numbers a_n and b_n and unit vectors x_n in $\mathcal{H}$ such that $w_n = a_n y_n + b_n x_n$, $|a_n|^2 + |b_n|^2 = 1$, and $(x_n, y_n) = 0$. From the above paragraph we have that $a_n \to 0$, so $|b_n| \to 1$. Therefore, $\|(T-z_1 I)x_n\| \to 0$ as $n \to \infty$.

Proof of Lemma 2. Let $\epsilon > 0$. Since $a, b \in \sigma_p(T) \cap \partial W(T)$, a and b are normal eigenvalues of T [18, Theorem 2, p. 233]. Let $u, v \in \mathcal{H}$ be unit vectors such that

$$Tu = au \qquad \text{and} \qquad Tv = bv$$

Then $(u,v) = 0$ and the closed subspace N spanned by $\{u,v\}$ reduces T. Define $S \in B(\mathcal{H})$ as

$$Su = \epsilon v$$

$$Sv = 0$$

$$Sz = 0 \text{ for all } z \in N^{\perp}.$$

Then we may write $T + S = A \oplus B$ corresponding to $\mathcal{H} = N \oplus N^{\perp}$. Then the matrix representation of A relative to $\{u,v\}$ is

$$A = \begin{pmatrix} a & \epsilon \\ 0 & b \end{pmatrix}.$$

Hence $\sigma(A) = \{a,b\} \subseteq \sigma(T)$. Clearly $\sigma(B) \subseteq \sigma(T)$. Therefore,

$$\text{co } \sigma(T+S) \subseteq \text{co } \sigma(T) = \overline{W(T)}.$$

A is not a normal operator. So by Proposition 1.1, $W(A)$ is the convex hull of a nondegenerate ellipse (i.e., not a straight line) with foci at a and b. Since $W(A) \subseteq \overline{W(T+S)}$, we must have

$$\text{co } \sigma(T+S) \neq \overline{W(T+S)}.$$

Therefore, $T+S \notin \mathcal{L}$. Thus, since $\|S\| = \epsilon > 0$ is arbitrary, $T \notin \text{interior } \mathcal{L}$.

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