THE COMPLEX ANGULAR MOMENTUM THEORY OF THE PRODUCTION OF THREE PARTICLES IN COLLISIONS OF TWO STRONGLY INTERACTING PARTICLES AT HIGH ENERGY

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ABSTRACT

The problem of the continuation to complex values of the angular momentum of the partial wave amplitude is examined for the simplest production process, that of two particles $\rightarrow$ three particles. The presence of so-called "anomalous singularities" complicates the procedure followed relative to that used for quasi two-body scattering amplitudes. The anomalous singularities are shown to lead to exchange-degenerate amplitudes with possible poles in much the same way as "normal" singularities lead to the usual signatured amplitudes. The resulting exchange-degenerate trajectories would also be expected to occur in two-body amplitudes.

The representation of the production amplitude in terms of the singularities of the partial wave amplitude is then developed and applied to the high energy region, with attention being paid to the emergence of "double Regge" terms. Certain new results are obtained for the behavior of the amplitude at zero momentum transfer, and some predictions of polarization and minima in momentum transfer distributions are made. A calculation of the polarization of the ρ^{o} meson in the reaction $\pi^{-}p \rightarrow \pi^{-}\rho^{o}p$ at high energy with small momentum transfer to the proton is compared with data taken at 25 Gev by W. D. Walker and collaborators. The result is favorable, although limited by the statistics of the available data.

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I. INTRODUCTION

The success in recent years in the theory of strong interactions of the idea [1] of Regge pole dominance of high energy elastic and quasi-elastic two-body collisions is well known. The idea lends itself to a generalization [2,3] that leads to the description of high energy collisions with more than two particles or resonances in the final state. The work of Kibble [2] and Ter-Martirosyan [3] on this problem is based on the theory of partial wave amplitudes with complex angular momentum for production processes. These papers have the unsatisfactory feature that the authors assume analyticity properties for the amplitude that are not obvious, and which may be invalid in the unitary S matrix theory. In the present paper we attempt to remove this unsatisfactory feature of the theory.

Chapter II is addressed to the problem of defining complex angular momentum partial waves in the absence of two ad hoc assumptions made in References [2] and [3]. The assumptions are (1) that anomalous (complex) singularities of the amplitude in the plane of a two-particle sub-energy remain bounded as the total energy goes to infinity [4], and (2) that a production amplitude satisfies a double dispersion relation in a sub-energy and a momentum transfer variable. Both assumptions are found to be unnecessary in the derivations (Chapters III, IV) of single and double Regge exchange terms for the high energy behavior of the $2 \rightarrow 3$ amplitude. Attention is also given (Chapter III) to possible fixed singularities of the continued partial wave amplitude.

Assumption (1) is shown to be incorrect for a simple Feynman diagram for the $2 \to 3$ process. It is then shown (Chapter II) that if this result holds in the S matrix theory, there is a unique exchange-degenerate partial wave amplitude, in addition to the usual signatured partial wave amplitudes. The new amplitude permits a Sommerfeld-Watson transformation (Chapter III), and its singularities may be significant in the high energy region of the crossed channel. One would expect these unsignatured singularities to occur also in continued two-body partial wave amplitudes.

Assumption (2) is shown to be false in the equal mass case, if the remaining variables are held fixed at arbitrary values in the physical region. Without this assumption, it is shown possible in Chapter IV to derive double pole terms with the conventional properties. In particular, the formula obtained in Reference [2] for the "propagator" of a Regge particle at high energy in the crossed channel is verified [5]. The results are obtained for external particles with arbitrary spin. The remainder of Chapter IV is devoted to some technical points related to the application of the model, such as factorization and the use of a definite natural parity for Regge poles. The consequences of parity invariance and the contribution of the coupling of a pole to a J plane cut are written out explicitly.

Chapter V deals with applications of the theory to data fitting at special values of the momentum transfers. The recent result of Tan and Wang [6] of the independence of the cross section on the Toller variable ω [7], for a leading double pole term when one of the momentum transfers vanishes is generalized to a single pole term for helicity amplitudes with external particles of arbitrary mass and spin by means of the introduction of Toller's quantum number M [8,9]. It is found that a leading double pole term in the helicity amplitude has only a phase dependence on ω in the asymptotic limit at zero momentum transfer. For double Toller $M = 0$ exchange, it is also found that the particle that couples to the two Regge (Lorentz) poles is longitudinally polarized when one pole couples to two spinless external particles and one momentum transfer vanishes in the limit of infinite energy. This prediction is compared with some data [10] at 25 Bev for the reaction $\pi^- p \to \pi^- \rho^o p$.

Minima in momentum transfer distributions are shown to be expected for the reaction $\pi^- N \to \pi^- \pi^o N$ at zero momentum transfer for either the π^- or the nucleon, and in the reaction $\pi^- p \to \pi^o X n$ when the nucleon momentum transfer t_{pn} passes through a value near $-.5(\mathrm{Gev})^2$, where X is a pseudo-scalar $G = +1$ non-strange meson (η or η').

II. THE ANALYTIC CONTINUATION OF THE PARTIAL WAVE AMPLITUDE FOR THE REACTION IN THE CROSSED CHANNEL

1. Partial Wave Decomposition in the Crossed Channel

We consider the strong interaction process

$$a + b \to 1 + 2 + 3 \tag{1}$$

where the particles have general spins.

From crossing [11,12], the amplitude for (1) is related to the amplitude for the crossed channel reaction:

$$\bar{1} + a \to \bar{b} + 2 + 3 \tag{2}$$

where $\bar{1}$, $\bar{b}$ are the anti-particles of particles "1" and "b", respectively.

From Lorentz invariance, the various helicity amplitudes [13] for (1) are each a function of 5 independent variables, which we take to be the Lorentz invariant set:

$$\begin{aligned} s &= (p_a + p_b)^2 & t_1 &= (p_1 - p_a)^2 \\ s_{23} &= (p_2 + p_3)^2 & t_2 &= (p_3 - p_b)^2 \\ s_{12} &= (p_1 + p_2)^2 \end{aligned} \tag{3}$$

where $p_\sigma = (E_\sigma, q_\sigma)$ $(\sigma = 1,2,a,b,3)$ is the 4-momentum of particle "σ" in some convenient coordinate system for reaction (1). Crossing symmetry relates the helicity amplitudes for (1) and (2) at the same values of the Lorentz invariant variables, with p_1 and p_b regarded as the negatives of the 4-momenta of particles "$\bar{1}$" and "$\bar{b}$" respectively in some fixed coordinate system for reaction (2). p_a, p_2, p_3, are then the 4-momenta of particles "a", "2" and "3", respectively in that coordinate system.

Now we decompose a helicity amplitude for (2) $F_{\lambda_3\lambda_{\bar{b}}\lambda_2;\lambda_{\bar{1}}\lambda_a}(s,s_{12},s_{23},t_1,t_2)$ according to J, the angular momentum of particle pair ($\bar{1}$ + a) in its zero-momentum system (z.m.s.) and J', the angular momentum of particle pair ($\bar{b}$ + 3) in its z.m.s. It is simplest to begin by considering the decomposition of the amplitudes [14] $F_{(\delta_3\delta_{\bar{b}})\lambda_2;\lambda_{\bar{1}}\lambda_a}$ which are amplitudes for the process (2), in which the helicities δ_3, $\delta_{\bar{b}}$ are measured in the z.m.s. of (3 + $\bar{b}$), and the helicities λ_2, $\lambda_{\bar{1}}$, λ_a are measured in the overall z.m.s., i.e., the z.m.s. of ($\bar{1}$ + a). We refer to these amplitudes, first introduced by Wick [15] and Shirokov [16] as pseudo-helicity amplitudes. We use the method of Reference [15] to form the state $|JM,\delta_3\delta_{\bar{b}}\ J'M',\lambda_2\rangle$ consisting of particles (3 + $\bar{b}$), in a state of angular momentum J', and z component of angular momentum M', with the positive z axis taken parallel to $-\vec{q}_2$. J, M are, respectively, the total angular momentum and the z component of the total angular momentum of the system of particles (2 + $\bar{b}$ + 3) in the overall z.m.s., and λ_2 is the helicity of particle "2" in this system.

Inverting Eq. (5) of Reference [15] we obtain the two-particle state (3 + $\bar{b}$) with definite relative momentum and zero total momentum. (We use the normalization of Reference [15].)

$$|p_3'(\theta',\phi'),\delta_3\delta_{\bar{b}}\rangle = \sum_{J'=\nu'}^{\infty}\sum_{M'=-J'}^{J'} |J'M'\delta_3\delta_{\bar{b}}\rangle D^{J'}_{M'\delta}(\phi',\theta',0)\,\eta_{J'}\left(\frac{4\sqrt{t_2}}{p_3'}\right)^{1/2} \qquad (4)$$

where $\eta_{J'} = \left(\frac{2J'+1}{4\pi}\right)^{1/2}$, and $p_3' = \frac{\Delta^{1/2}(t_2, m_b^2, m_3^2)}{2\sqrt{t_2}}$ is the momentum of particle "3" in this coordinate system. (θ', ϕ') are the polar angles of $\vec{p}_3$ in a definite coordinate system that is a z.m.s. of $(3 + \bar{b})$. $\delta = \delta_3 - \delta_{\bar{b}}$; $v = 0$ if $3, \bar{b}$ are both fermions or both bosons, $v = 1/2$ otherwise, and $\Delta^{1/2}(x,y,z) = (x^2 - y^2 - z^2 - 2xy - 2yz - 2zx)^{1/2}$. The continuation of this function is defined in Appendix B.2.

We complete the definition of the coordinate system by choosing $\vec{q}_3$, the 3-momentum of particle "3" in the z.m.s., to lie in the xz plane with positive x component. Then by choosing the axes in the z.m.s. of $(3 + \bar{b})$ to be parallel to this set, we have $\phi' = 0$ in Eq. (4).

The next step is to form the state of $(3 + \bar{b})$ with total momentum $\vec{Q}_{3\bar{b}} = -\vec{q}_2$, where $\vec{q}_2$ is the momentum of particle "2" in the overall z.m.s. By our choice of axes, this state is simply the result of a Lorentz velocity transformation along the positive z axis applied to the state defined in Eq. (4), so that

$$|Q_{3\bar{b}}, \theta', \delta_3 \delta_{\bar{b}}\rangle = \sum_{J', M'} \eta_{J'} Z(Q_{3\bar{b}}) |J'M'\delta_3\delta_{\bar{b}}\rangle\, D^{J'}_{M'\delta}(0,\theta',0) \left(\frac{4\sqrt{t_2}}{p_3'}\right)^{1/2} \tag{5}$$

where $Z(Q_{3\bar{b}})$ is the Lorentz transformation required to take $Q^0_{3\bar{b}} = (\sqrt{t_2}, 0,0,0)$ into the 4-vector $Q_{3\bar{b}} = (\sqrt{t_2 + q_2^2}, 0, 0, |\vec{q}_2|)$.

In Wick's helicity convention for 2 particle states of non-zero total momentum, the state $Z(Q_{3\bar{b}})|J'M'\ \delta_3\delta_{\bar{b}}\rangle$ is denoted $|Q_{3\bar{b}}; J'M'; \delta_3\delta_{\bar{b}}\rangle$. Using this notation, and inverting Eq. (20) of Reference [15], one obtains

$$\left| Q_{3\bar{b}}; J'M'; \delta_3\delta_{\bar{b}} \right\rangle = \sum_{J=v}^{\infty} \sum_{M=-J}^{J} \left(\frac{4\sqrt{t_1}}{q_2}\right)^{1/2} \eta_J \left| P^0 JM; t_2 J'M' \delta_3 \delta_{\bar{b}}, \lambda_2 \right\rangle D^J_{M,M'-\lambda_2}(0,0,0) \tag{6}$$

where the ket on the right side denotes a state of $(2 + 3 + \bar{b})$ with 4-momentum $P^0 = (\sqrt{t_1}, 0)$ and with angular momentum quantum numbers JM, formed from a state of $(3 + \bar{b})$ with angular momentum quantum numbers $J'M'$ in the z.m.s. of these two particles. The argument of $D^J_{M,M'-\lambda_2}$ is (0,0,0) because the momentum $\vec{Q}_{3\bar{b}}$ is the direction of the z axis. ($v = 1/2$ if J = half odd integer, $v = 0$ if J = integer). Combining (5) and (6)

$$\left| Q_{3\bar{b}}; \theta'; \delta_3\delta_{\bar{b}}; q_2\lambda_2 \right\rangle = \sum_{J,J',M'} \eta_J \eta_{J'}\, 4\left(\frac{\sqrt{t_1}t_2}{q_2 p_3'}\right)^{1/2} d^{J'}_{M'\delta}(z') \left| P^0 JM; t_2 J'M' \delta_3\delta_{\bar{b}}, \lambda_2 \right\rangle \tag{7}$$

where $M = M' - \lambda_2$, and $\displaystyle\sum_{J,J',M'} = \sum_{M'=-\infty}^{\infty} \sum_{J=|M'-\lambda_2|}^{\infty} \sum_{J'=\sigma_{max}}^{\infty}$, $\sigma_{\substack{max\\min}} = \substack{max\\min}\{|M'|, |\delta|\}$.

We consider the transition to state (7) of the two particle state $(\bar{1} + a)$ with the same total energy. Let the polar angles of $\vec{q}_a$ in the z.m.s. of $(\bar{1} + a)$ defined above be (θ, ϕ). Then, as in Eq. (4),

$$\left| q_a(\theta,\phi), \lambda_a\lambda_{\bar{1}} \right\rangle = \sum_{J,M} \eta_J \left| a\bar{1}; JM; \lambda_a\lambda_{\bar{1}} \right\rangle D^J_{M,\rho}(\phi,\theta,0) \left(\frac{4\sqrt{t_1}}{q_a}\right)^{1/2} \tag{8}$$

where $\rho = \lambda_a - \lambda_{\bar{1}}$ and $q_a = \Delta^{1/2}(t_1, m_1^2, m_a^2)/2\sqrt{t_1}$. This leads to the following T matrix element for reaction (2):

$$\left\langle Q_{3\bar{b}}, \theta'; \delta_3\delta_{\bar{b}}, q_2\lambda_2 \right| T \left| q_a(\theta,\phi), \lambda_a\lambda_{\bar{1}} \right\rangle$$
$$= \sum_{J,J',M'} \eta_J^2 \eta_{J'}\, 8\left(\frac{t_1\sqrt{t_2}}{q_{\bar{1}} q_2 p_3'}\right)^{1/2} d^{J'}_{M'\delta}(z')\, d^J_{M'-\lambda_2,\rho}(z)\, e^{i(\lambda_2-M')\phi}\, a_{(\delta_3\delta_{\bar{b}})\lambda_2,\lambda_{\bar{1}}\lambda_a}(J,J',M';t_1,t_2) \tag{9}$$

where $a_{(\delta_3\delta_{\bar b})\lambda_2,\lambda_{\bar 1}\lambda_a}(J,J';M';t_1,t_2) = \langle P^0 J M; t_2 J' M'; \delta_3\delta_{\bar b},\lambda_2 | T | P; J M; \lambda_a \lambda_{\bar 1}\rangle$, which is independent of M by the invariance of the T operator under rotations, and $z = \cos\theta$, $z' = \cos\theta'$, and

$$\sum_{J,J',M'} = \sum_{M'=-\infty}^{\infty} \sum_{J=\lambda_{max}}^{\infty} \sum_{J'=\delta_{max}}^{\infty} \quad , \quad \text{where } \lambda_{\substack{max.\\min.}} = \substack{max.\\min.} \{ |M'-\lambda_2| , |\rho| \}$$

Noticing that the dependence of $a_{(\delta_3\delta_{\bar b})\lambda_2,\lambda_{\bar 1}\lambda_a}(J,J';M';t_1,t_2)$ on M' is analogous to its dependence on λ_2, we adopt a new notation (introducing a factor $\sqrt{t_1}/n_{J'}$ for the sake of symmetry):

$b_{M'(\delta_3\delta_{\bar b})\lambda_2,\lambda_{\bar 1}\lambda_a}(J,J',t_1,t_2) = a_{(\delta_3\delta_{\bar b})\lambda_2,\lambda_{\bar 1}\lambda_a}(J,J';M';t_1,t_2)\sqrt{t_1}/n_{J'}$ Then Eq. (9) may be written as:

$$F_{(\delta_3\delta_{\bar b})\lambda_2,\lambda_{\bar 1}\lambda_a}(z,z',\phi,t_1,t_2) = \sum_{J,J',M'} b_{M'(\delta_3\delta_{\bar b})\lambda_2,\lambda_{\bar 1}\lambda_a}(J,J',t_1,t_2)\, d^J_{M'-\lambda_2,\rho}(z)\, d^{J'}_{M',\delta}(z')\, e^{i(\lambda_2-M')\phi}$$
$$\times\, n_J^2 n_{J'}^2\, 8 \left(\frac{\sqrt{t_2}}{q_{\bar 1} q_2 p_3'}\right)^{1/2} \tag{10}$$

We have written the amplitude as a function of the variable set $\{z, z', \phi, t_1, t_2\}$ instead of as a function of the set $\{s, s_{12}, s_{23}, t_1, t_2\}$. The connecting equations between these two sets are given in Appendix B.2. Properties of the $d^J_{\lambda\mu}(z)$ functions are summarized in Appendix A.

To obtain the partial wave decomposition of the helicity amplitude for (2), where all helicities are measured in the z.m.s. of $(3 + \bar b + 2)$, we must rotate the spin directions of particles 3 and $\bar b$ in their respective rest frames relative to the directions of these spins in the state (7). The spin component of particle "3" in state (7) is measured in its rest frame along the direction of $-\vec q\,'_{\bar b}$, where $\vec q\,'_x$ is the 3-momentum of particle "x" $(x = \bar 1, a, \bar b, 2, 3)$ in this coordinate system. The spin component of particle "3" in a helicity state in the z.m.s. of $(3 + \bar{b} + 2)$ is measured along the direction of $-\vec{q}_{\bar{b}} - \vec{q}_2$. To get the second from the first, we rotate the first state through positive angle β_3 about the axis $\hat{n}_3 = \hat{n}_1 \times \hat{n}_2$, where $\hat{n}_1 = -\vec{q}\,'_{\bar{b}} / |\vec{q}\,'_{\bar{b}}|$, $\hat{n}_2 = -\vec{q}_{\bar{b}} + \vec{q}_3 / |\vec{q}_{\bar{b}} + \vec{q}_3|$ and $\cos \beta_3 = \hat{n}_1 \cdot \hat{n}_2$. This rotation, applied to state (7), changes only the spin of particle "3", and affects no other quantities that specify the state. Similarly, to obtain the state in which the helicity of particle "$\bar{b}$" is measured in the z.m.s. of $(3 + \bar{b} + 2)$, one must rotate (7) in the rest frame of particle "$\bar{b}$" through positive angle $\beta_{\bar{b}}$ about the axis $\hat{n}'_3 = \hat{n}'_1 \times \hat{n}'_2$, where $\hat{n}'_1 = -\vec{q}\,''_3 / |\vec{q}\,''_3|$, where $\vec{q}\,''_x$ = 3-momentum of particle "x" in this rest system of particle "b", $\hat{n}'_2 = (\vec{q}\,''_3 + \vec{q}\,''_2) / |\vec{q}\,''_3 + \vec{q}\,''_2|$, and $\cos \beta_{\bar{b}} = \hat{n}'_1 \cdot \hat{n}'_2$. Thus one obtains the helicity state in the z.m.s. of $(3 + \bar{b} + 2)$:

$$|Q_{3\bar{b}}, \theta', \lambda_3 \lambda_{\bar{b}}, \lambda_2\rangle = \sum_{\delta_3=-S_3}^{S_3} \sum_{\delta_{\bar{b}}=-S_b}^{S_b} d^{S_3}_{\delta_3 \lambda_3}(\beta_3)\, d^{S_{\bar{b}}}_{\delta_{\bar{b}} \lambda_{\bar{b}}}(\beta_{\bar{b}})\, |Q_{3\bar{b}}, \theta', \delta_3 \delta_{\bar{b}}, \lambda_2\rangle \tag{11}$$

and, correspondingly, the relation between the helicity amplitudes and the pseudo-helicity amplitudes for (2) is:

$$F_{\lambda_3 \lambda_{\bar{b}} \lambda_2, \lambda_{\bar{1}} \lambda_a} = \sum_{\delta_3, \delta_{\bar{b}}} d^{S_3}_{\delta_3 \lambda_3}(\beta_3)\, d^{S_{\bar{b}}}_{\delta_{\bar{b}} \lambda_{\bar{b}}}(\beta_{\bar{b}})\, F_{(\delta_3 \delta_{\bar{b}})\lambda_2, \lambda_{\bar{1}} \lambda_a} \tag{12}$$

2. Analytic Properties of the Production Amplitude

a. Fourier decomposition in ϕ

The amplitude has been written as a Fourier series in ϕ :

$$F = \sum_{M'=-\infty}^{\infty} F_{M'}(z, z', t_1, t_2)\, e^{i(\lambda_2 - M')\phi} \tag{13a}$$

where

$$F_{M'} = \sum_{J'=\sigma_{max}}^{\infty} \sum_{J=\lambda_{max}}^{\infty} b_{M'}(J,J',t_1,t_2)\, \eta_J^2 \eta_{J'}^2\, 8 \left(\frac{\sqrt{t_2}}{q_T\, q_2\, p_3'}\right)^{1/2} d^{J'}_{M'\delta}(z')\, d^{J}_{M'-\lambda_2,\rho}(z) \tag{13b}$$

Also, from (13a)

$$F_{M'} = \frac{1}{2\pi}\int_0^{2\pi} d\phi\; F(z,z',\phi,t_1,t_2)\, e^{i(M'-\lambda_2)\phi} \tag{13c}$$

Now we assume that the series in (13a) converges when ϕ is physical. According to the result of Appendix E, $0 \leq \phi < 2\pi$ when the other variables are in the physical region for either reaction (1) or reaction (2). We may thus phrase our assumption more precisely as follows: there exists a continuation of $F_{M'}(z,z',t_1,t_2)$, defined in Eq. (13b), to the kinematical region of $\{z,z',t_1,t_2\}$ that is physical for reaction (1), and Eq. (13a) is continuable to this region. Roughly speaking, it is assumed possible to write the amplitude in the physical region for the direct channel as a crossed channel helicity series.

b. Lehmann ellipse in z

From the definition of $d^J_{\lambda\mu}(z)$, Eq. (A.4), one may rewrite Eq. (13b) in the following form:

$$G_{M'} = \frac{F_{M'}(z,z',t_1,t_2)}{\left(\frac{1-z}{2}\right)^{\frac{|M'-\lambda_2-\rho|}{2}} \left(\frac{1+z}{2}\right)^{\frac{|M'-\lambda_2+\rho|}{2}}} \tag{14a}$$

$$= \sum_{J=\lambda_{max}}^{\infty} C_{M'}(J)\, P^{|M'-\lambda_2-\rho|,\,|M'-\lambda_2+\rho|}_{J-\lambda_{max}}(z) \tag{14b}$$

$$C_{M'}(J) = \sum_{J'=\sigma_{max}}^{\infty} b_{M'}(J,J',t_1,t_2)\, \eta_J^2 \eta_{J'}^2\, 8 \left(\frac{\sqrt{t_2}}{p_3' q_T\, q_2}\right)^{1/2} d^{J'}_{M'\delta}(z')\, \frac{N(J,\lambda_{max})}{N(J,\lambda_{min})}\, \mathrm{sign}(M'-\lambda_2,\rho) \tag{15}$$

where $N(J,\lambda) = \{\Gamma(J-\lambda+1)\ \Gamma(J+\lambda+1)\}^{1/2}$

and $\text{sign}\ (\alpha,\beta) = \begin{cases} (-1)^{\alpha-\beta} & \text{if } \alpha \geqq |\beta|, \text{ or } -\beta \geqq |\alpha| \\ +1 & \text{otherwise.} \end{cases}$

Now, as shown in Appendix B.3, a necessary and sufficient condition for a series of Jacobi polynomials $\sum_n a_n P_n^{(\alpha,\beta)}(z)$ to converge inside an ellipse in the z plane with foci at $z = \pm 1$ (i.e., in a domain that encloses the physical region) is that the power series $\sum a_n z^n$ have a radius of convergence $R > 1$. As a consequence of this result, if the series (12) converges in a domain that contains the domain of physical z for reaction (2) (i.e., $-1 \leq z \leq 1$), then it converges uniformly inside an ellipse in the z plane, and is therefore analytic within that ellipse. The equation of the ellipse is $(z = x + iy)$

$$x^2 \left(R - \frac{1}{R}\right)^2 + y^2 \left(R + \frac{1}{R}\right)^2 = \left(R^2 - \frac{1}{R^2}\right)^2 \tag{16}$$

c. Real singularities in z :

The existence of such a z plane ellipse within which the scattering amplitude is analytic has been proved in field theory for two-body spinless scattering by Lehmann [17] and for the $2 \to 3$ amplitude by Ascoli and Minguzzi [18]. Here we will show that the existence of such an ellipse of convergence is consistent with all branch points corresponding to thresholds in two-body channels. That is, we will show that in the S matrix theory, there are no thresholds in two-body invariant mass variables that generate real branch points in z between -1 and 1, provided that the remaining variables (i.e., s_{23}, s, t_1 and t_2), which are held fixed, are taken to satisfy certain inequalities.

To show this, we notice that, as there are 5 particles in the process (2), there are exactly 10 two-body channels, corresponding to 10 invariant mass variables. Four are held fixed: s_{23}, s, t_1 and t_2. The remaining ones are:

$$
\begin{aligned}
s_{13} &= (p_1 + p_3)^2 = (p_3 - p_{\bar{1}})^2 & \qquad t_{2a} &= (p_2 - p_a)^2 \\
s_{12} &= (p_1 + p_2)^2 = (p_2 - p_{\bar{1}})^2 & \qquad t_{1b} &= (p_1 - p_b)^2 \\
t_{3a} &= (p_3 - p_a)^2 & \qquad t_{2b} &= (p_2 - p_b)^2
\end{aligned}
\tag{17}
$$

We replace s_{12} in the variable set (3) by z according to Eqs. (B.3), and (B.4) of Appendix B:

$$
z = \frac{s_{12} - m_1^2 - m_2^2 + 2E_{\bar{1}}E_2}{2\,p_{\bar{1}}\,p_2}
\tag{18}
$$

where

$$
\begin{aligned}
p_2 &= \Delta^{1/2}(t_1, t_2, m_2^2)/2\sqrt{t_1} & \qquad E_2 &= (p_2^2 + m_2^2)^{1/2} = (t_1 + m_2^2 - t_2)/2\sqrt{t_1} \\
p_{\bar{1}} &= \Delta^{1/2}(t_1, m_1^2, m_a^2)/2\sqrt{t_1} & \qquad E_{\bar{1}} &= (p_{\bar{1}}^2 + m_1^2)^{1/2} = (t_1 + m_1^2 - m_a^2)/2\sqrt{t_1}
\end{aligned}
\tag{19}
$$

Two of the variables in the list (17) are not related to z at fixed s_{23}, t_1, t_2, s . These are:

$$
\begin{aligned}
t_{1b} &= s_{23} + m_1^2 + m_a^2 + m_b^2 - s - t_1 \\
t_{2b} &= t_1 - t_2 - s_{23} + m_2^2 + m_b^2 - m_3^2
\end{aligned}
\tag{20}
$$

We now examine the branch points of F in the variables s_{12}, s_{23}, t_{3a}, t_{2a} corresponding to normal thresholds in the corresponding channels. We observe that a branch point in a channel variable occurs at a value > 0 if the threshold is in that channel.

(i) <u>Branch points of F in s_{12} corresponding to a normal threshold in (12) channel:</u>

Since $s_{12}^{th} > 0$, $z > +1$ if (from (18)) $2E_2E_{\bar{1}} - m_1^2 - m_2^2 > 2p_2p_{\bar{1}}$.
We take t_1, t_2 positive, and above threshold for (2). That is,

$$t_1 > \text{Max}\ \{(\sqrt{t_2} + m_2)^2,\ (m_1 + m_a)^2\}$$

$$t_2 > (m_3 + m_b)^2 \tag{21}$$

Then $p_{\bar{1}}$, p_2 are real and positive, and $z > 1$ if

$$(m_1^2 - m_2^2)^2 - 4(m_1^2 + m_2^2)\ E_2E_{\bar{1}} + 4E_2^2\ m_1^2 + 4E_{\bar{1}}^2\ m_2^2 > 0 \tag{22}$$

Inserting the values (19) for $E_{\bar{1}}$, E_2 and simplifying, (22) becomes:

$$t_1[(t_2 - m_a^2)(m_2^2 - m_1^2)] + (m_1^2\ t_2 - m_2^2\ m_a^2)(t_2 + m_1^2 - m_2^2 - m_a^2) > 0 \tag{23}$$

If $m_2 > m_1$, this can be satisfied by choosing

$$t_2 > \text{Max}\ \{\frac{m_2^2\ m_a^2}{m_1^2}\ ,\ m_a^2 + m_2^2 - m_1^2\} \tag{24a}$$

If $m_1 > m_2$, then (23) becomes: $t_1(m_1^2 - m_2^2)(t_2 - m_a^2) < (t_2\ m_1^2 - m_2^2\ m_a^2)$ $\times(t_2 + m_1^2 - m_2^2 - m_a^2)$ which will be satisfied if (for $t_2 > m_a^2$):

$$t_1(m_1^2 - m_2^2) < m_1^2(t_2 + m_1^2 - m_2^2 - m_a^2) \tag{24b}$$

Clearly, if $m_2 > m_1$, the inequality (24a) for t_2 is consistent with the inequalities (21). If $m_1 > m_2$, we must check that (24b) is consistent with (21). Inserting (21) for t_1 in (24b) one obtains:

$$-m_2^2(t_2+ m_2^2) + 2m_2 \sqrt{t_2} < m_1^2(m_1^2 - 2m_2^2 - m_a^2 - 2m_2 \sqrt{t_2})$$

leading to the following inequality for t_2 (if $t_2 > (m_1^2 - m_2^2)^2/m_2^2$) :

$$t_2 > (m_1^2 - m_2^2 + m_1 m_a)^2/ m_2^2 \tag{24c}$$

Thus the (12) channel branch points lead to $z > 1$ if

$$t_2 > \text{Max} \left\{\frac{m_2^2 m_a^2}{m_1^2} , m_a^2 + m_2^2 - m_1^2 , (m_3+ m_b)^2\right\} \quad \text{if} \quad m_2 > m_1$$

or (25a)

$$t_2 > \text{Max} \left\{(m_1^2 - m_2^2 + m_1 m_a)^2/ m_2^2 , (m_3+ m_b)^2, m_a^2\right\}$$

and

$$t_1 < \frac{m_1^2}{(m_1^2 - m_2^2)} (t_2+ m_1^2 - m_2^2 - m_a^2) \quad \text{if} \quad m_1 > m_2 \tag{25b}$$

(ii) <u>Branch points of F corresponding to a normal threshold in the ($\overline{2}$a) channel:</u>

$$\text{Since} \quad s_{12} - m_1^2 - m_2^2 = t_2+ m_a^2 - t_{2a} - t_1 \tag{26}$$

for fixed t_1, t_2 that satisfy the inequalities (25) and (21), it follows from (18) that high lying thresholds in the ($\overline{2}$a) channel leads to branch points in z with large negative values. In order to keep all such branch points outside the interval $-1 \leq z \leq +1$, we impose the condition

$$-t_1 + t_2 + m_a^2 + 2E_2E_{\overline{1}} < -2p_2\, p_{\overline{1}} \tag{27}$$

Then, substituting the values (19) for E_2, $E_{\overline{1}}$, p_2, $p_{\overline{1}}$ leads (see Appendix B.4) to the following inequality:

$$t_1(t_2 - m_1^2)(m_2^2 - m_a^2) + (t_2 + m_a^2 - m_1^2 - m_2^2)(m_a^2 t_2 - m_1^2\, m_2^2) > 0 \tag{28}$$

Incidentally, one may notice the great similarity of (28) and (23); they are the same except for the exchange of m_1 and m_a . Thus it is sufficient that

$$t_2 > \mathrm{Max}\ \left\{(m_3 + m_b)^2,\ m_1^2 + m_2^2 - m_a^2,\ \frac{m_1^2\, m_2^2}{m_a^2}\right\} \qquad \text{if } m_2 \geq m_a \tag{29a}$$

or

$$t_1 < \frac{m_a^2}{m_a^2 - m_2^2}\,(t_2 + m_a^2 - m_1^2 - m_2^2) \tag{29b}$$

$$t_2 > \mathrm{Max}\ \{(m_a^2 - m_2^2 + m_1 m_a)^2/\ m_2^2,\ (m_3 + m_b)^2,\ m_1^2\} \qquad \text{if } m_a > m_2 \tag{29c}$$

Clearly, (25) and (29) are mutually consistent.

(iii) <u>Thresholds in the (13) channel:</u>

Since $s_{12} - m_1^2 - m_2^2 = s - s_{23} - s_{13} + m_3^2$ (30)

Eq. (27) tells us that we can insure that the resulting z plane branch point will occur at $z < -1$ if we impose the condition:

$$s - s_{23} < t_2 + m_a^2 - t_1 - m_3^2 \tag{31}$$

for then $s_{12} - m_1^2 - m_2^2 + 2E_2E_{\overline{1}} - 2p_2\, p_{\overline{1}}$. Since neither s nor s_{23} have been subjected to any previous conditions, we can satisfy (31) by the appropriate choice of s_{23} for arbitrary fixed s .

(iv) Thresholds in the $(\overline{3}a)$ channel:

Since

$$s_{12} - m_1^2 - m_2^2 = s + t_2 + t_{3a} - m_a^2 - m_b^2 - m_1^2 - m_2^2 - m_3^2 \quad (32)$$

then, by choosing

$$s > -t_2 + m_a^2 + m_b^2 + m_1^2 + m_2^2 + m_3^3 \quad (33)$$

one can insure that the resulting z plane branch points are real and occur at $z < -1$. Consistency with (31) is no problem; one must simply have

$$s_{23} < -2t_2 + 2m_3^2 + m_b^2 + m_1^2 + m_2^2 + t_1 \quad (34)$$

Thus we have shown that if Eqs. (31), (33), (25), and (29) are satisfied, then all two-body channel normal threshold branch points of F result in real branch points in z that lie outside the interval $-1 \leq z \leq +1$.

We note that we may not choose s_{23} arbitrarily with both t_1 and t_2 fixed, for then (34) imposes a lower bound. We also note in passing that (25), (29), (31), (33) are satisfied in a kinematical domain that overlaps the physical region for the reaction (2).

d. Complex singularities in z

In production amplitudes, in addition to the real singularities generated by thresholds in two-body channels, there are complex singularities generated by more complicated thresholds. This has been shown explicitly for the $3 \to 3$ amplitude [19]. Complex singularities have also been found in the $2 \to 3$ amplitude in the perturbation theory [20,21].

We now examine some of the properties of these singularities in perturbation theory for the purpose of finding a reasonable assumption for the analytic properties of the $2 \to 3$ amplitude in S matrix theory. Azimov et al [21] show that the two-pole Feynman diagram (Figure 1) has complex singularities in the z plane. The diagram of Figure 1b is shown in Reference [21] to possess the following complex conjugate branch points (Eqs. (14a), (14b) of Reference [21]):

$$z = z_{23}z_{30} \pm i[(1 - z_{23}^2)(z_{30}^2 - 1)]^{1/2}$$

$$z = -z_{12}z_{10} \pm i[(1 - z_{12}^2)(z_{10}^2 - 1)]^{1/2}$$

where z = cosine of the angle between "2" and "5"

z_{23} = cosine of the angle between "2" and "3"

z_{12} = cosine of the angle between "2" and "1"

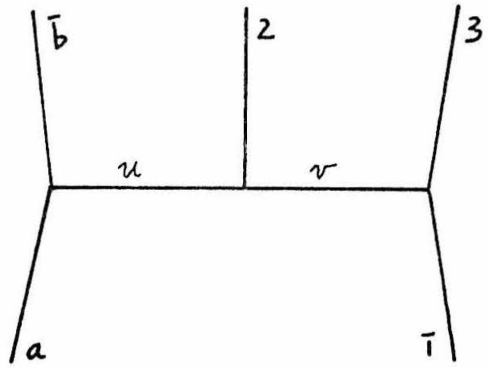
$$z_{10} = \frac{x^2 - m_1^2 - m_4^2 + 2E_1E_4}{2p_1p_4}$$

and

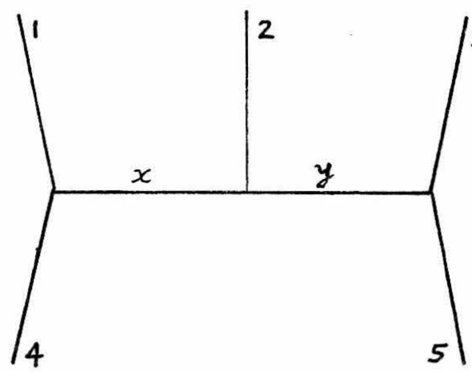
$$z_{30} = \frac{y^2 - m_3^2 - m_5^2 + 2E_3E_5}{2p_3p_5} .$$

where E_k, p_k $(k = 1,2,3,4,5)$ are the energy and momentum respectively of particle "k" in the reaction $4+5 \to 1+2+3$ and x,y are the masses of the exchanged particles in Figure 1b.

We prefer to consider rather the labelling of Figure 1a to which correspond the singularities:



(a)



(b)

Figure 1. A Feynman diagram with a complex branch point in z ; (a) in the notation of this paper, (b) in the notation of Reference [21].

$$z = z_{23}z_{30} \pm i[(1 - z_{23}^2)(z_{30}^2 - 1)]^{1/2} \tag{35a}$$

$$z = -z_{2\bar{b}}\, z_{\bar{b}0} \pm i[(1 - z_{2\bar{b}}^2)(z_{\bar{b}0}^2 - 1)]^{1/2} \tag{35b}$$

where

$$z_{23} = \frac{s_{23} - m_2^2 - m_3^2 - 2E_2E_3}{2p_2\, p_3}$$

$$z_{2\bar{b}} = \frac{t_{2b} - m_2^2 - m_b^2 - 2E_2E_{\bar{b}}}{2p_2\, p_{\bar{b}}}$$

$$z_{\bar{b}o} = \frac{u^2 - m_b^2 - m_a^2 + 2E_{\bar{b}}E_a}{2p_{\bar{b}}\, p_a}$$

$$z_{30} = \frac{v^2 - m_3^2 - m_1^2 + 2E_3E_{\bar{1}}}{2p_3\, p_{\bar{1}}}$$

and $$t_{2b} = t_1 - t_2 - s_{23} + m_2^2 + m_b^2 + m_3^2 \quad .$$

Here E_k, p_k are the energy and momentum respectively of particle "k" ($k = \bar{1}, a, \bar{b}, 2, 3$) in the z.m.s. of the reaction $\bar{1} + a \to \bar{b} + 2 + 3$, and u, v are the masses of the exchanged particles in the Feynman amplitude represented by Figure 1a.

Thus

$$E_2 = \frac{t_1 + m_2^2 - t_2}{2\sqrt{t_1}} \quad , \qquad E_3 = \frac{t_1 + m_3^2 - t_{2b}}{2\sqrt{t_1}}$$

$$E_a = \frac{t_1 + m_a^2 - m_1^2}{2\sqrt{t_1}} \quad , \qquad E_{\bar{b}} = \frac{t_1 + m_b^2 - s_{23}}{2\sqrt{t_1}}$$

If t_1, t_2, s_{23} are held fixed at physical values for the reaction (2), and if the mass u is permitted to become arbitrarily large, then the branch points (35a) tend to infinity in the z plane asymptotic to the rays

$$\theta_1^{\pm} = \pm \tan^{-1} \frac{(1 - z_{23}^2)^{1/2}}{z_{23}} \tag{36a}$$

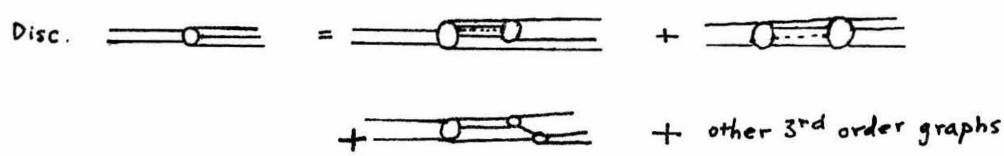
If t_1, t_2, s_{23} are physical for (2), then $\theta_1^{\pm}$ are real. Similarly, if v becomes arbitrarily large, then the branch points (35b) tend to infinity in the z plane with the asymptotes

$$\theta_2^{\pm} = \pm \tan^{-1} \frac{(1 - z_{2\overline{b}}^2)^{1/2}}{z_{2\overline{b}}} \tag{36b}$$

and $\theta_2^{\pm}$ are real in the physical region for (2).

Using the calculated branch points as a guide, one would expect the branch cut structure of the amplitude $G_{M'}(z, s_{23}, t_1, t_2)$ as a function of z to be similar to that displayed in Figure 2, with t_1, t_2, s_{23} fixed in the physical region for $\overline{1} + a \to \overline{b} + 2 + 3$.

It is of interest to investigate the origin of the complex singularities in S matrix theory. Under the assumption of maximal analyticity of the first degree [22], all such singularities are generated by the unitarity equation. Using a simple version of the bubble notation of Olive [23], we write the unitarity equation for the $2 \to 3$ amplitude in the form:



+ 4th and higher order graphs (37)

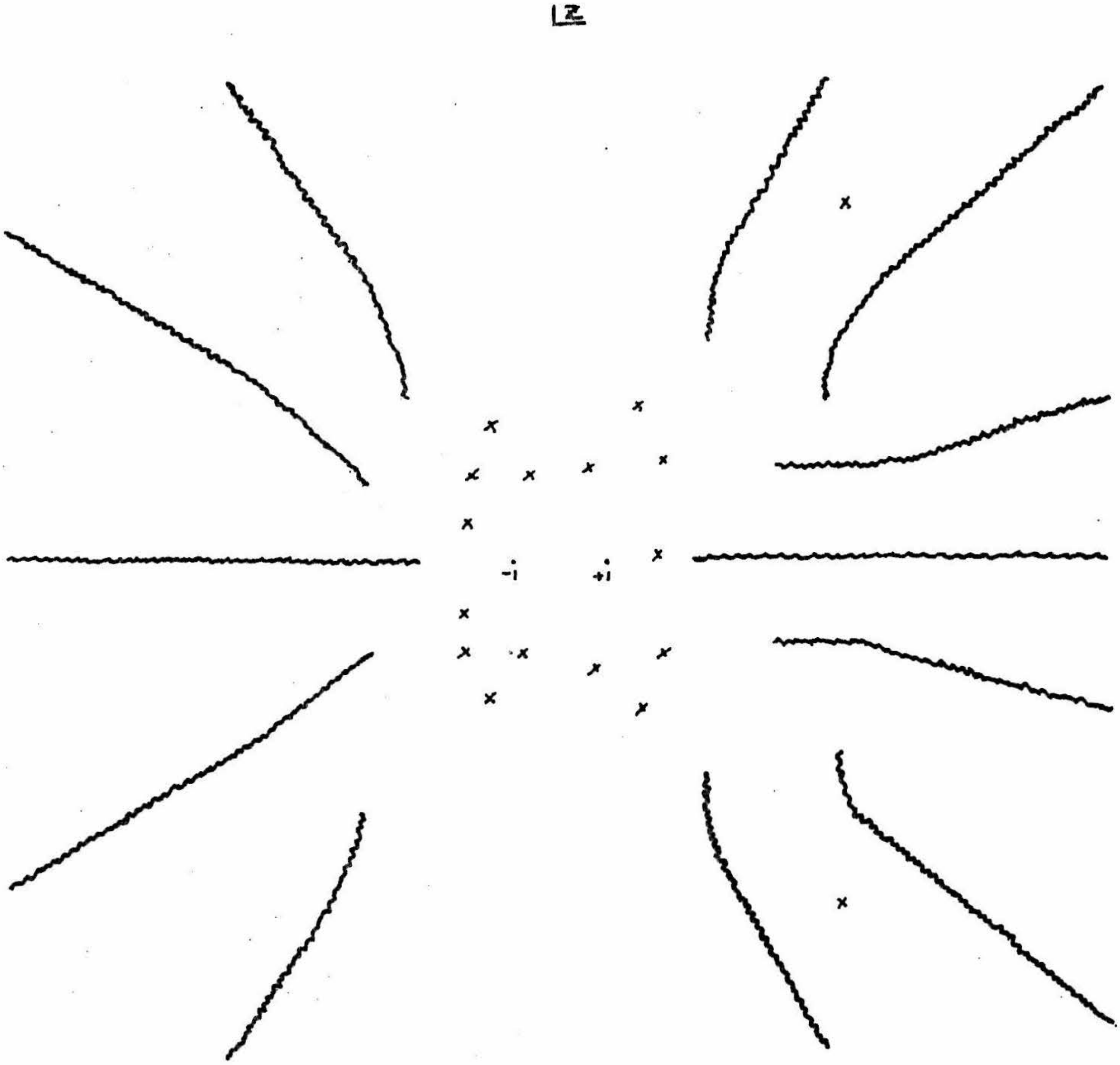


Figure 2. A typical singularity structure for the production amplitude II.(14) in the z plane. The wavy lines represent branch cuts. The x's represent poles.

We define the order of a term in this equation to mean the number of bubbles that appear in the corresponding graph, i.e., the number of times a connected part appears in the term containing the branch cut.

The result of Sec. c is that no second order graphs contain complex singularities, provided inequalities (31), (33), (25), and (29) are satisfied. Thus, the complex singularities arise from third and higher order graphs.

If the analytic structure of $G_{M'}$ is not qualitatively more complicated (e.g., intersecting branch cuts) than that represented in Figure 2, then $G_{M'}$ satisfies a dispersion relation in z. We take this dispersion relation to be without subtractions, to reduce the complexity of the formulas. The problem with a finite number N of subtractions bears the same relation to the problem with no subtractions as in the case of the $2 \to 2$ amplitude. In the case of $2 \to 2$, the continuations with N and 0 subtractions in the dispersion relation, respectively, are the same for $\mathrm{Re}\, J > N$, as shown by Squires [24]. Thus

$$G_{M'}(z, s_{23}, t_1, t_2) = \frac{1}{2\pi i} \int_C ds \, \frac{G_{M'}(s, s_{23}, t_1, t_2)}{s - z} - \sum_{\substack{\text{poles} \\ k}} \frac{g_{M'}(z_k, s_{23}, t_1, t_2)}{z_k - z} \tag{38}$$

where C is the sum of the contours that enclose the branch lines in Figure 2, each contour enclosing its branch line in a clockwise sense, and $g_{M'}$ is the residue of $G_{M'}$ at the pole $z = z_k$.

3. Analytic Continuation of the Partial Wave Amplitude to Complex J

We make use of Eq. (38) to determine the continuation of the partial wave amplitude to complex J , by a method based on that for $2 \to 2$ amplitudes derived by Froissart and Gribov [25]. The partial wave amplitude may be obtained from Eq. (10) by using the orthogonality properties of the $d^J_{\lambda\mu}(z)$ functions for physical J (Appendix A), resulting in

$$b^{(J,J',t_1,t_2)}_{M'(\delta_3\delta_b)\lambda_2;\lambda_T\lambda_a} = (2\pi)^2 \int_{-1}^{1} dz \int_{-1}^{1} dz' \, F^{(z,z',t_1,t_2)}_{M'(\delta_3\delta_b)\lambda_2,\lambda_T\lambda_a} \frac{1}{8}\left(\frac{q_T q_2 p_3'}{\sqrt{t_2}}\right)^{1/2} d^{J'}_{M'\delta}(z') \, d^{J}_{M'-\lambda_2,\rho}(z) \tag{39}$$

Using $G_{M'}(z,s_{23},t_1,t_2)$ as defined in Eq. (14), and the definition of $d^{J}_{M'-\lambda_2,\rho}(z)$ from Appendix A,

$$b_{M'} = \int d\sigma' \, \mathrm{sign}(M'-\lambda_2,\rho) \frac{N(J,\lambda_{max})}{N(J,\lambda_{min})} \frac{1}{2\pi i}$$
$$\times \Bigg\{ \int_C d\zeta \int_{-1}^{1} dz \left(\frac{1+z}{2}\right)^{|M'-\lambda_2+\rho|} \left(\frac{1-z}{2}\right)^{|M'-\lambda_2-\rho|} G_{M'}(\zeta,s_{23}) P^{(|M'-\lambda_2-\rho|,|M'-\lambda_2+\rho|)}_{J-\lambda_{max}}(z) \cdot \frac{1}{\zeta - z}$$
$$+ \frac{2\pi}{i} \sum_{\substack{\text{poles}\\ k}} \int_{-1}^{1} dz \left(\frac{1+z}{2}\right)^{|M'-\lambda_2+\rho|} \left(\frac{1-z}{2}\right)^{|M'-\lambda_2-\rho|} \frac{g_{M'}(z_k,s_{23})}{z_k - z} P^{(|M'-\lambda_2-\rho|,|M'-\lambda_2+\rho|)}_{J-\lambda_{max}}(z_k) \Bigg\} \tag{40}$$

where $\lambda_{\substack{max\\min}} = \substack{max\\min}\{|M'-\lambda_2|,|\rho|\}$, sign (λ,μ), $N(J,\lambda)$ are defined below Eq. (15), and $\int d\sigma' = \int_{-1}^{1} dz' \, (2\pi)^2 \frac{1}{8} \left(\frac{q_T q_2 p_3'}{\sqrt{t_2}}\right)^{1/2} d^{J'}_{M'\delta}(z')$

Then, using the relation (Reference [26], Eq. 10.8(20))

$$2\, Q^{(\alpha,\beta)}_n(z'') \, (z''-1)^{\alpha} (z''+1)^{\beta} = \int_{-1}^{1} dz \, \frac{P^{(\alpha,\beta)}_n(z) \, (1-z)^{\alpha}(1+z)^{\beta}}{z''-z} \tag{41}$$

one obtains

$$b_{M'}(J) = \int d\sigma' \; \mathrm{sign}(M'-\lambda_2,\rho) \frac{N(J,\lambda_{max})}{N(J,\lambda_{min})}$$
$$\times \left\{ \frac{1}{\pi i} \int_C dz \left(\frac{1-z}{2}\right)^{|M'-\lambda_2-\rho|} \left(\frac{1+z}{2}\right)^{|M'-\lambda_2+\rho|} e^{i\pi(M'-\lambda_2-\rho)} G_{M'}(z,s_{23})\, Q_{J-\lambda_{max}}^{(|M'-\lambda_2-\rho|,\,|M'-\lambda_2+\rho|)}(z) \right.$$
$$\left. - 2\sum_k \left(\frac{1-z_k}{2}\right)^{|M'-\lambda_2-\rho|} \left(\frac{1+z_k}{2}\right)^{|M'-\lambda_2+\rho|} e^{i\pi(M'-\lambda_2-\rho)} g_{M'}(z_k,s_{23})\, Q_{J-\lambda_{max}}^{(|M'-\lambda_2-\rho|,\,|M'-\lambda_2+\rho|)}(z_k) \right\} \quad (42)$$

At this point, and for reasons that we discuss below Eq. (51), and again later in connection with the Sommerfeld-Watson transformation, we introduce the unique decomposition:

$$b_{M'}(J,J',t_1,t_2) = B_{M'}(J,J',t_1,t_2) + C_{M'}(J,J',t_1,t_2) \quad (43)$$

where $B_{M'}$ is given by the right side of (42) with C replaced with C_o, the contour that encloses the branch lines on the real axis in the clockwise sense, and the sum over poles replaced by the sum over the poles on the real axis.

Since the continuation is performed in J while J' is physical, it is convenient to introduce the function

$$b_{M'(\delta_3\delta_{\bar{b}})\lambda_2,\lambda_{\bar{1}}\lambda_a}(J,s_{23},t_1,t_2) = \sum_{J'} (2J'+1)\, d^{J'}_{M'\delta}(z') \left(\frac{\sqrt{t_2}}{q_{\bar{1}}q_2 p_3'}\right)^{1/2} b_{M'(\delta_3,\delta_{\bar{b}})\lambda_2,\lambda_{\bar{1}}\lambda_a}(J,J',t_1,t_2) \quad (44)$$

and the corresponding decomposition

$$b_{M'}(J,s_{23}) = B_{M'}(J,s_{23}) + C_{M'}(J,s_{23}) \quad (45)$$

where $B_{M'}(J,s_{23})$ and $C_{M'}(J,s_{23},t_1,t_2)$ are each given by a formula of the same form as (42), with $b_{M'}(J,J')$ replaced by $B_{M'}(J,J')$ and

$C_{M'}(J,J')$, respectively. The relation between z' and s_{23} is given in Appendix B, Eqs. (B.9), (B.10).

The formula that gives the continuation of $B_{M'}(J,s_{23},t_1,t_2)$ to complex J may be written

$$\begin{aligned} B_{M'}(J,s_{23}) = {} & \frac{\pi}{i}\,\mathrm{sign}(M'-\lambda_2,\rho)\frac{N(J,\lambda_{max})}{N(J,\lambda_{min})} \\ & \times\Bigg\{\int_{z_r^0}^{\infty} dz \left(\frac{1+z}{2}\right)^{|M'-\lambda_2+\rho|}\left(\frac{1-z}{2}\right)^{|M'-\lambda_2-\rho|} e^{i\pi(M'-\lambda_2-\rho)}\,\underset{z}{\mathrm{disc.}}\,G_{M'}(z,s_{23})\,Q_{J-\lambda_{max}}^{(|M'-\lambda_2-\rho|,|M'-\lambda_2+\rho|)}(z) \\ & + (-1)^{J+\lambda_{max}}\int_{-z_\ell^0}^{\infty} dz\left(\frac{1+z}{2}\right)^{|M'-\lambda_2-\rho|}\left(\frac{1-z}{2}\right)^{|M'-\lambda_2+\rho|} e^{i\pi(M'-\lambda_2-\rho)}\,\underset{-z}{\mathrm{disc.}}\,G_{M'}(-z,s_{23})\,Q_{J-\lambda_{max}}^{(|M'-\lambda_2+\rho|,|M'-\lambda_2-\rho|)}(z)\Bigg\} \\ & -2\pi^2\,\mathrm{sign}(M'-\lambda_2,\rho)\frac{N(J,\lambda_{max})}{N(J,\lambda_{min})}\Bigg\{\sum_{k_r}\left(\frac{1-z_{k_r}}{2}\right)^{|M'-\lambda_2-\rho|}\left(\frac{1+z_{k_r}}{2}\right)^{|M'-\lambda_2+\rho|}(-1)^{M'-\lambda_2-\rho}\,g_{M'}(z_{k_r},s_{23})\,Q_{J-\lambda_{max}}^{(|M'-\lambda_2-\rho|,|M'-\lambda_2+\rho|)}(z_{k_r}) \\ & + (-1)^{J+\lambda_{max}}\sum_{k_\ell}\left(\frac{1+z_{k_\ell}}{2}\right)^{|M'-\lambda_2+\rho|}\left(\frac{1-z_{k_\ell}}{2}\right)^{|M'-\lambda_2-\rho|}(-1)^{M'-\lambda_2-\rho}\,g_{M'}(z_{k_\ell},s_{23})\,Q_{J-\lambda_{max}}^{(|M'-\lambda_2+\rho|,|M'-\lambda_2-\rho|)}(-z_{k_\ell})\Bigg\} \end{aligned} \tag{46}$$

where $\underset{z}{\mathrm{disc.}}H(z) = H(z+i\varepsilon) - H(z-i\varepsilon)$ for real z.

where z_r^0, z_ℓ^0 are the real branch points $> +1$ and < -1, respectively with the smallest moduli, and z_{k_r}, z_{k_ℓ} are the poles of G on the real axis with $z_{k_r} > 0 > z_{k_\ell}$.

Using the functions $e^J_{\lambda\mu}(z)$ of Gunson and Andrews [27] (some properties of these functions are summarized in Appendix A), Eq. (46) may be written in more compact form as

$$\begin{aligned} B_{M'}(J,s_{23}) = {} & -i\pi\left(\int_{z_r^0}^{\infty} dz\, e^J_{M'-\lambda_2,\rho}(z)\,\underset{z}{\mathrm{disc.}}\,F_{M'}(z,s_{23}) + (-1)^{J-M'+\lambda_2}\int_{-z_\ell^0}^{\infty} dz\, e^J_{M'-\lambda_2,-\rho}(z)\,\underset{-z}{\mathrm{disc.}}\,F_{M'}(-z,s_{23})\right) \\ & -2\pi^2\left(\sum_{k_r} e^J_{M'-\lambda_2,\rho}(z_{k_r})\,f_{M'}(z_{k_r},s_{23}) + (-1)^{J-M'+\lambda_2}\sum_{k_\ell} e^J_{M'-\lambda_2,-\rho}(-z_{k_\ell})\,f_{M'}(z_{k_\ell},s_{23})\right) \end{aligned} \tag{47}$$

where $f_{M'}(z_k,s_{23},t_1,t_2)$ = Residue of $F_{M'}(z,z',t_1,t_2)$ at the pole $z = z_k$.

We've shown in Sec. 2.c that $z_r^o > +1$, $z_\ell^o < -1$ provided that s_{23}, t_2, t_1 and s satisfy the inequalities derived there. In that case, Eq. (47) would provide a unique meremorphic continuation to complex J for the function $B_{M'}(J,s_{23})\, N(J,\lambda_{min})/N(J,\lambda_{max})$ for $\mathrm{Re}\, J > \alpha$ where $\left|\frac{\mathrm{disc.}\, F_{M'}(z,z')}{z^{\alpha+\epsilon}}\right| \to 0$ for arbitrary positive ε as $|z| \to \infty$, except for the factor $(-1)^J$. This is a consequence of Carlson's theorem [28] and the asymptotic behavior of $e^J_{\lambda\mu}(z)$ as $|J| \to \infty$ (see Eqs. (49) and (50) below). This factor of $(-1)^J$ is familiar from the $2 \to 2$ problem, where $(-1)^{J-v}$ ($v=0$ or $1/2$, according as physical J corresponds to J = integer or half odd integer, respectively) is replaced by the "signature" ± 1 and two separate unique continuations, corresponding to signatures + and − , are made into the complex J plane. Clearly, the same prescription works here also, so that the continuations are

$$B^{\pm}_{M'}(J,s_{23}) = -i\pi\left(\int_{z_r^o}^{\infty} dz\, e^J_{M'-\lambda_2,\rho}(z)\, \underset{z}{\mathrm{disc.}}\, F_{M'}(z,z') \pm (-1)^{M'-\lambda_2-v}\int_{-z_\ell^o}^{\infty} dz\, e^J_{M'-\lambda_2,-\rho}(z)\, \underset{-z}{\mathrm{disc.}}\, F_{M'}(-z,z')\right)$$
$$-2\pi^2\left(\sum_{k_r} e^J_{M'-\lambda_2,\rho}(z_{k_r})\, f_{M'}(z_{k_r},s_{23}) \pm (-1)^{M'-\lambda_2-v}\sum_{k_\ell} e^J_{M'-\lambda_2,-\rho}(-z_{k_\ell})\, f_{M'}(z_{k_\ell},s_{23})\right) \tag{48}$$

since, as $J \to \infty$ (Appendix A, and Reference [27])

$$e^J_{\lambda\mu}(z) \simeq \left(\frac{\pi}{2}\right)^{1/2} \frac{e^{i\eta\frac{\pi}{2}(\lambda-\mu)}\left[z-(z^2-1)^{1/2}\right]^{J+1/2}}{\sqrt{J}\;(z^2-1)^{1/4}}, \quad \eta = \mathrm{sign}(\mathrm{Im}\, z) \tag{49}$$

and

$$\left[z-(z^2-1)^{1/2}\right]^J \simeq \exp\left[J\log\left|z-(z^2-1)^{1/2}\right|\right]\exp(i\psi J) \tag{50}$$

where $\psi = \arg(z - (z^2-1)^{1/2})$.

At every point z on the right side of (48) at which a function of the type $e^J_{\lambda\mu}(z)$ is to be evaluated, $\psi = 0$ and $\log|z-(z^2-1)^{1/2}| < 0$

so that $|B^{\pm}_{M'}(J,s_{23})| < K/\sqrt{J}$ as $J \to \infty$, Re $J > \alpha$, for some positive K, and the continuation of $B^{\pm}_{M'}(J)$ to complex J is unique. Of course the integral in (48) does not converge for Re $J < \alpha$ where $\left|\frac{disc.\, F_{M'}(z,z')}{z^{\alpha-\epsilon}}\right| \to \infty$ for arbitrary positive ε as $|z| \to \infty$, so that in this case the continuation formula, Eq. (48), is valid in a right half plane only.

$C_{M'}(J)$ may also be continued uniquely to complex J. From (45), (47), (44) and (42), one obtains

$$C_{M'}(J,s_{23}) = -i\pi \int_{C'} dz\; e^{J}_{M'-\lambda_2,\rho}(z)\, F_{M'}(z,z') - 2\pi^2 \sum_{poles}{}' e^{J}_{M'-\lambda_2,\rho}(z_\kappa)\, f_{M'}(z,s_{23}) \tag{51}$$

where C' encloses in a clockwise sense all the branch lines in Figure 2 except those on the real axis and $\sum'_{\text{poles}}$ is the sum over all the poles indicated in Figure 2 except those on the real axis.

At all points z on the right side of (51) at which the functions $e^{J}_{M'-\lambda_2,\rho}(z)$ are to be evaluated, we have $|\psi| < \pi$ and $|z - (z^2-1)^{1/2}| < 1$, where ψ is defined in Eq. (50) (see Appendix B.3 for some of the properties of $[z - (z^2-1)^{1/2}]$ in the complex z plane), and so from the asymptotic behavior in J of $e^{J}_{\lambda\mu}(z)$, Eq. (49), we have the result

$$\frac{|C_{M'}(J,s_{23})|}{e^{(\pi-\epsilon)|J|}} \to 0 \quad \text{as } |J| \to \infty,\ \text{Re } J > J_0,$$

for some positive ε, and some real J_0.

Thus, from Carlson's theorem [28], $C_{M'}(J,s_{23})N(J,\lambda_{min})/N(J,\lambda_{max})$ is uniquely continued to a meremorphic function of J in a right half plane by Eq. (51). Thus, there is no need to introduce signature to obtain a unique continuation of $C_{M'}(J,s_{23})$ and this amplitude is exchange degenerate [29], since the continuations from even and odd $(J-v)$ coincide. Another way of putting this is to state that the contributions of the continued $C_{M'}(J,s_{23})$ are the same to both signatured continued partial wave amplitudes $b^{\pm}_{M'}(J,s_{23})$ where

$$b^{\pm}_{M'}(J,s_{23}) = B^{\pm}_{M'}(J,s_{23}) + C_{M'}(J,s_{23}) \qquad (52)$$

4. The Continuation to Complex J'

It is not possible to continue Eq. (42) to complex J' by introducing a dispersion relation in z' for $F_{M'}(z,z',t_1,t_2)$ (with the appropriate kinematical factor, viz., $\left(\frac{1+z'}{2}\right)^{\frac{|M'+\delta|}{2}} \left(\frac{1-z'}{2}\right)^{\frac{|M'-\delta|}{2}}$ removed). This is so because threshold branch points, for fixed t_1, t_2 and s can occur in the domain $-1 \leq z' \leq +1$, when z is allowed to vary over the domain of integration in (42). Explicitly, consider the branch points due to thresholds in the (13) channel. Now $s_{23} - m_2^2 - m_3^2 = s - s_{12} - s_{13}^{th} + m_1^2$ where s_{13}^{th} is the value of s_{13} at which the threshold branch point occurs. In Eq. (42), z and therefore s_{12} can assume arbitrarily large positive and negative values, so that for sufficiently large s_{13}^{th} (and branch points s_{13}^{th} occur at arbitrarily large values), $s_{23} - m_2^2 - m_3^2$ can assume arbitrary real values, and corresponding branch points in z' occur everywhere in $-1 \leq z \leq +1$, as z varies over the domain of integration of Eq. (40). This argument is valid when t_1 and t_2 are fixed at physical values for any $2 \to 3$ reaction related to (1) by crossing, at least for the case of all masses being equal. The denominator in (18) is real in this case for all possible values of t_1 and t_2.

On the other hand, we may evidently continue $b_{M'}(J,J')$ in J' when J is held fixed at a physical value in a manner identical with that used to continue $b_{M'}(J,J')$ in J with J' physical. It is most convenient to perform this continuation in J' by exploiting the formal equivalence of J and J' in Eq. (39). In order to make use of a dispersion relation for $G_{M'}(s_{12},z') = F_{M'}(z,z') \Big/ \left(\frac{1+z'}{2}\right)^{\frac{|M'+\delta|}{2}} \left(\frac{1-z'}{2}\right)^{\frac{|M'-\delta|}{2}}$ in z', we must first continue (39) to an appropriate kinematical region in t_1 and t_2.

The appropriate kinematical region is given by inequalities (25), (29), (31), and (33) when the replacements

$$(t_1,t_2,s_{23}) \to (t_2,t_1,s_{12})$$

$$(m_1^2,m_3^2,m_a^2,m_b^2) \to (m_3^2,m_1^2,m_b^2,m_a^2)$$

are made. With this prescription, the inequalities correspond to keeping the branch points in z_1 for the class of singularities considered in Sec. 2.c outside the line $-1 \leq z_1 \leq +1$, where z_1 is the angle between the momentum of particle $\overline{2}$ and particle 3 in the z.m.s. of $3 + \overline{b}$ for the reaction

$$3 + \overline{b} \to \overline{1} + a + \overline{2} \tag{53}$$

From invariance under crossing [30], the helicity amplitude $F^{(z_1,z_2,\phi';t_2,t_1)}_{(\lambda_{\overline{1}}\lambda_a)\lambda_{\overline{2}};\lambda_3\lambda_{\overline{b}}}$ for (53), with $(z_1,z_2,\emptyset')$ defined for (53) analogously to $(z,z',\emptyset)$ for (2), is given by $F^{(z,z',\phi,t_1,t_2)}_{(\lambda_3\lambda_{\overline{b}})\lambda_2;\lambda_{\overline{1}}\lambda_a}$

up to a phase that is a matter of convention, where $\phi' = \phi$, $z_1 = -z'$, $z_2 = -z$, $\lambda_{\bar{2}} = -\lambda_2$. Thus the inequalities leading to the branch points in z_1 with $|z_1| > 1$, then correspond to branch points of the amplitude in z' with $|z'| > 1$.

Then, in this kinematical region in t_1, t_2 the continuation of the partial wave amplitude in J' follows the same lines as the continuation in J. In particular, we make a decomposition similar to (52) for the partial wave amplitude, i.e.,

$$b^{\pm}_{M'}(s_{12}, J', t_1, t_2) = B^{\pm}_{M'}(s_{12}, J', t_1, t_2) + C_{M'}(s_{12}, J', t_1, t_2) \tag{54}$$

where the continuation of $B^{\pm}_{M'}(s_{12}, J')$ is given by:

$$\begin{aligned} B^{\pm}_{M'}(s_{12}, J') = & -i\pi\left(\int_{z_r^{o\prime}}^{\infty} dz'\, e^{J'}_{M'\delta}(z')\, \operatorname*{disc}_{z'} F_{M'}(z,z') \pm (-1)^{M'-\nu'} \int_{-z_\ell^{o\prime}}^{\infty} dz'\, e^{J'}_{M',-\delta}(z')\, \operatorname*{disc}_{-z'} F_{M'}(z,-z')\right) \\ & -2\pi^2\left(\sum_{k_r} e^{J'}_{M'\delta}(z'_{k_r})\, f_{M'}(z, z'_{k_r}) \pm (-1)^{M'-\nu'} \sum_{k_\ell} e^{J'}_{M',-\delta}(-z'_{k_\ell})\, f_{M'}(z, z'_{k_\ell})\right) \end{aligned} \tag{55}$$

with the quantities $[z_r^{o\prime}, z_\ell^{o\prime}, z_{k_r}^{o\prime}, z'_{k_\ell}, f_{M'}(z, z'_k)]$ defined analogously to $[z_r^o, z_\ell^o, z_{k_r}, z_{k_\ell}, f_{M'}(z_k, z')]$ that appear in Eq. (48) for the continuation of $B^{\pm}_{M'}(J, s_{23})$ to complex J. Similarly, $C_{M'}(s_{12}, J')$ is continued to complex J' by a formula that is the analogue of Eq. (51):

$$C_{M'}(s_{12}, J') = -i\pi \int_{C'} dz'\, e^{J'}_{M'\delta}(z')\, F_{M'}(z,z') - 2\pi^2 \sum_{\substack{\text{poles}\\ n}}' e^{J'}_{M'\delta}(z_n)\, f_{M'}(s_{12}, z_n) \tag{56}$$

III. SOMMERFELD-WATSON TRANSFORMATION

1. Transformation of the Sum to an Integral

We follow the usual procedure [24] for transforming the partial wave series into a sum over J plane singularities of the partial wave amplitude, together with background terms. We start with Eq. II.(10), using II.(44) to effect the sum over J :

$$F_{(\delta_3\delta_{\bar{b}})\lambda_2;\lambda_{\bar{T}}\lambda_a}(z,z',\phi,t_1,t_2) = \frac{1}{2\pi^2}\sum_{M'=-\infty}^{\infty}\sum_{J=\lambda_{max}}^{\infty} b_{M'(\delta_3\delta_{\bar{b}})\lambda_2,\lambda_{\bar{T}}\lambda_a}(J,s_{23},t_1,t_2)\,(2J+1)\,e^{i(\lambda_2-M')\phi}\,d^J_{M'-\lambda_2,\rho}(z) \tag{1}$$

$$= \frac{1}{2\pi^2}\sum_{M'}\sum_{J,\tau} b_{M'(\delta_3\delta_{\bar{b}})\lambda_2,\lambda_{\bar{T}}\lambda_a}(J,s_{23},t_1,t_2)\,(2J+1)\,e^{i(\lambda_2-M')\phi}\,d^{\tau}_{M'-\lambda_2,\rho}(J,z) \tag{2}$$

where
$$d^{\tau}_{\lambda\mu}(J,z) = \frac{1}{2}\left[d^J_{\lambda\mu}(z) + \tau(-1)^{\lambda-\nu}d^J_{\lambda,-\mu}(-z)\right] \tag{3}$$

In Eq. (2) τ is summed over the values of the signature introduced in II.(48), and $b^{\tau}_{M'}(J)$ is the signatured partial wave amplitude continued to complex J by means of Eq. (52), (51), and (48) of the preceding chapter.

Eq. (2) can be transformed to an integral in the J plane over a contour C_J that encloses $J \geq \lambda$ max along the real axis. We choose the sense of the contour to be clockwise (Figure 3a) and obtain

$$F = \frac{i}{4\pi^2}\sum_{M'}\int_{C_J} dJ\,\frac{(2J+1)\,b^{\tau}_{M'}(J)\,e^{i(\lambda_2-M')\phi}}{\sin\pi(J-M'+\lambda_2)}\,d^{\tau}_{M'-\lambda_2,-\rho}(J,-z) \tag{4}$$

Using the decomposition II.(52) for $b^{\tau}_{M'}(J)$, F is written as the sum of two contour integrals:

$$F = \frac{i}{4\pi^2} \sum_{M'} e^{i(\lambda_2 - M')\phi} \left[\sum_{\tau} \int_{C_J} dJ \, \frac{(2J+1) B^{\tau}_{M'}(J) \, d^{\tau\,(J,-z)}_{M'-\lambda_2,-\rho}}{\sin\pi(J-M'+\lambda_2)} + \int_{C_J} dJ \, \frac{(2J+1) C_{M'}(J) \, d^{J}_{M'-\lambda_2,-\rho}(-z)}{\sin\pi(J-M'+\lambda_2)} \right] \tag{5a}$$

$$= \quad F_B + F_C \tag{5b}$$

where F_B, F_C are given by the formulas of the form of (5a) with $B^{\tau}_{M'}(J)$ and $C_{M'}(J)$, respectively, appearing in the integrand.

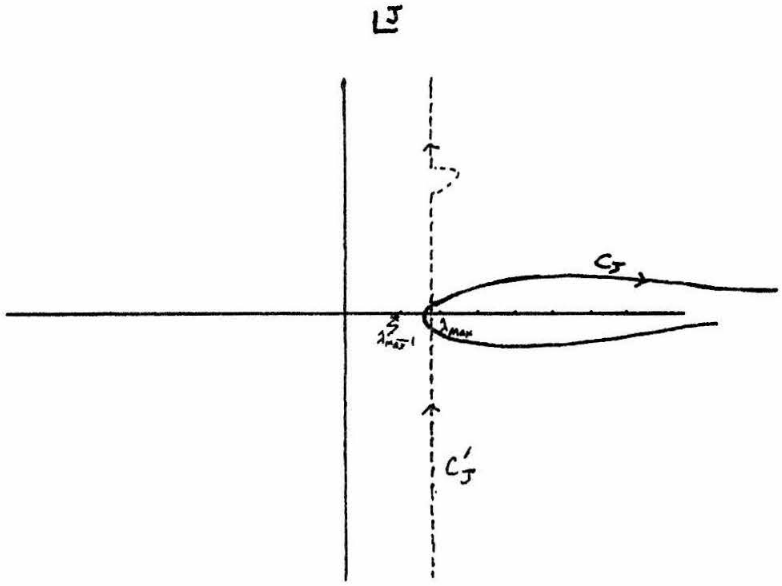
2. Deformation of the Integral: Singularities

Next, we deform the contour C_J into the contour C'_J that is parallel to the Im J axis, crosses the Re J axis at J_o, $\lambda \max > J_o > \lambda \max - 1$, and is deformed to avoid any singularities of the integrand on the line $\mathrm{Re}\, J = J_o$ (broken contour in Figure 3a). We close the contour with a semi-circle in the right half plane, and we show that the contribution of the integral over this semi-circle vanishes in the limit of infinite radius, both for F_B and F_C, if z is taken on the negative real axis, $z < -1$, and within the ellipse II.(16). As $|J| \to \infty$, from Eq. (A.18)

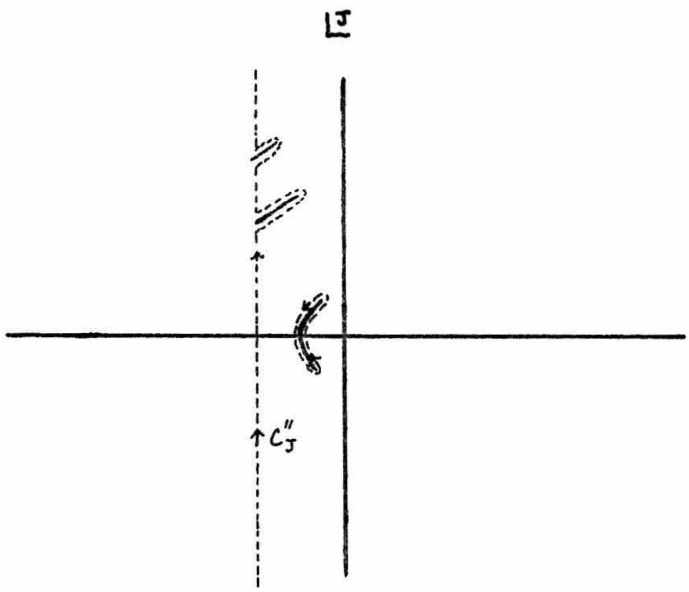
$$\frac{d^{J}_{M'-\lambda_2,-\rho}(-z)}{\sin\pi(J-M'+\lambda_2)} \simeq \frac{C_1}{\sqrt{J}} \, e^{-\pi|\mathrm{Im} J|} e^{|aJ|}$$

where $|C_1|$ is independent of J, and $a = \log|z + (z^2 - 1)^{1/2}|$.

$C_{M'}(J)$ is given by II.(51), and as $|J| \to \infty$, each contribution on the right side of that equation gives a contribution to $C_{M'}(J)$ of the form $C_2 e^{CJ} e^{i\sigma J}$ where C_2 is independent of J and $C = \log|z - (z^2 - 1)^{1/2}|$; $\sigma = \arg[z - (z^2 - 1)^{1/2}]$, where z is in the domain of integration and summation in (51). Since all such points



(a)



(b)

Figure 3. J plane contours

z are outside the ellipse II.(16), $C < -|a|$ and since all points z in Eq. II.(51) have a nonvanishing imaginary part, $|\sigma| < \pi$. Thus,

$$\left| \frac{C_{M'}(J)\, d^{J}_{M'-\lambda_2,-\rho}(-z)}{\sin\pi(J-M'+\lambda_2)} \right| \longrightarrow 0 \qquad \text{exponentially as } \operatorname{Re} J \to +\infty \text{ or as } |\operatorname{Im} J| \to \infty$$

Using the same method, $\sigma = 0$ for all terms on the right side of II.(48) and

$$\left| \frac{B^{\pm}_{M'}(J)\, d^{\pm}_{M'-\lambda_2,-\rho}(J,-z)}{\sin\pi(J-M'+\lambda_2)} \right| \longrightarrow 0 \qquad \text{exponentially as } \operatorname{Re} J \text{ or } |\operatorname{Im} J| \to \infty$$

Thus the contribution of the integral over the contour at infinity vanishes for both terms in Eq. (5a).

In deforming C_J into C_J' in Eq. (5), one must add in terms to the right side that represent the contribution of the poles and branch cuts of the integrand. For simplicity, we will assume that branch lines are absent to the right of $\operatorname{Re} J = J_0$.

Then

$$F_B = \frac{i}{4\pi^2} \sum_{M',\tau} \left(\int_{C_J'} dJ\, \frac{(2J+1)\, B^{\tau}_{M'}(J)\, d^{\tau}_{M'-\lambda_2,-\rho}(-z)}{\sin\pi(J-M'+\lambda_2)} + 2\pi i \sum_{\substack{\text{Poles} \\ \operatorname{Re}\alpha_\ell > J_0}} \frac{d^{\tau}_{M'-\lambda_2,-\rho}(\alpha_\ell,-z)\, \beta^{\tau}_{M'}(t_1,t_2,s_{23})}{\sin\pi(\alpha_\ell - M' + \lambda_2)} \right) e^{i(\lambda_2-M')\phi} \tag{6a}$$

where $\beta^{\tau}_{M'}(t_1,t_2,s_{23})$ is the residue of $B^{\tau}_{M'(\sigma_3\sigma_5)\lambda_2,\lambda_1\lambda_a}(J, s_{23}, t_1, t_2)$ at the pole $J = \alpha_\ell$, and

$$F_C = \frac{i}{4\pi^2} \sum_{M'} e^{i(\lambda_2 - M')\phi} \left(\int_{C_J'} dJ \, \frac{(2J+1)\, C_{M'}(J)\, d^J_{M'-\lambda_2, -\rho}(-z)}{\sin\pi (J - M' + \lambda_2)} + 2\pi i \sum_{\substack{\text{poles} \\ \mathrm{Re}\,\alpha_\ell > J_0}} \frac{d^{\alpha_\ell}_{M'-\lambda_2, -\rho}(-z)\, \gamma_{M'}(t_1, t_2, s_{23})}{\sin\pi (\alpha_\ell - M' + \lambda_2)} \right) \tag{6b}$$

and $\gamma_{M'}$ is the residue of $C_{M'}(J)$ at the pole $J = \alpha_\ell$.

Now we deform the contour C_J' into the contour C_J'' in Eq. (6) where C_J'' is the line $\mathrm{Re}\, J = -1/2$, suitably deformed to avoid any singularities of the integrand on that line, and extended from $-1/2 - i\infty$ to $-1/2 + i\infty$ (Figure 3b). The line $\mathrm{Re}\, J = -1/2$ is chosen by the criterion of the fastest decrease in the limit $z \to \infty$ of the contribution of the integral over C_J'' . In doing this, we must add terms to the right side of Eq. (6) that come from the singularities of the integrand in the region between C_J' and C_J'' .

Before discussing these contributions, we note that it is possible to deform the contour farther to the left in the J plane, and decrease the asymptotic contribution to the amplitude of the background integral by using a device due to Mandelstam [31] and generalized to $2 \to 2$ amplitudes for particles with spin by Drechsler [32]. In doing this, however, one obtains additional contributions that dominate the background integral in the asymptotic region in z . These contributions may be shown to cancel if Mandelstam symmetry is assumed. In our case, Mandelstam symmetry would read

$$b_{M'}(J, s_{23}, t_1, t_2) = (-1)^{M' - \lambda_2 - \rho}\, b_{M'}(-J-1, s_{23}, t_1, t_2) \tag{7}$$

for $(J - M' + \lambda_2)$ = half-odd integer.

The symmetry has been proved in spinless potential scattering with Yukawa potentials by Mandelstam [31] and conditions for it to hold in $2 \to 2$ scattering with general spins are discussed in Drechsler's paper. Corresponding conditions for $2 \to 3$ scattering may not exist, because Drechsler's conditions are based on the N/D equations which, in turn, are based on the existence of only a left hand cut and a right hand cut for the $2 \to 2$ amplitude. In the $2 \to 3$ problem, additional cuts also exist, as discussed earlier. To avoid the complication of seeking an appropriate generalization of the N/D equations to test Mandelstam symmetry, we extend the contour no farther to the left than the line $\mathrm{Re}\, J = -1/2$. Except for this feature, the treatment here for the $2 \to 3$ case parallels that of Drechsler [32] for the $2 \to 2$ case.

In deforming the contour to C_J'', one picks up the following contributions in Eqs. (6):

(i) Moving poles $-1/2 < \mathrm{Re}\, \alpha(t_1) < \lambda\,\mathrm{max}$.

(ii) Poles due to the vanishing of $\sin \pi(J-M'+\lambda_2)$ in the region $-1/2 < J < \lambda$ max.

(iii) Fixed "Kronecker delta" singularities of the partial wave amplitude.

(iv) Branch cuts of the continued partial wave amplitude.

We discuss (i) first. This presents no problem, as the sums $\sum\limits_{\mathrm{Re}\,\alpha_\ell > J_0}$ in (6a) and (6b) are simply rewritten $\sum\limits_{\mathrm{Re}\,\alpha_\ell > -1/2}$. Notice that the trajectory $J = \alpha(t_1)$ is taken to be a function of t_1 only. This is essentially equivalent to assuming that the Regge poles are the same as those that appear in $2 \to 2$ scattering. Another way of stating this is that the position of a pole in the continuation in J, the angular momentum of a pair of particles in their z.m.s., depends only on the invariant mass of the pair, as is the case in $2 \to 2$ scattering.

Next, we consider (iv). For the sake of pragmatic simplicity, we assume that there are finitely many non-intersecting branch cuts C_k. Intersecting branch cuts, as pointed out by Drechsler, can lead to a complicated sheet structure for the J plane, and little is known about the cuts, apart from the position and energy dependence of a certain type of branch point [33].

A branch cut may occur in $B_{M'}(J)$ or in $C_{M'}(J)$. The respective contributions are

$$F_B^{b.c.} = \frac{i}{4\pi^2} \sum_{M'} e^{i(\lambda_2 - M')\phi} \int_{C_k} dJ \ \underset{J}{\mathrm{disc.}} \frac{B^{\tau}_{M'}(J, s_{23}, t_1, t_2) \ d^{\tau}_{M'-\lambda_2, -\rho}(-z)}{\sin\pi(J - M' + \lambda_2)} (2J+1) \tag{8a}$$

$$F_C^{b.c.} = \frac{i}{4\pi^2} \sum_{M'} e^{i(\lambda_2 - M')\phi} \int_{C_k} dJ \ \underset{J}{\mathrm{disc.}} \frac{C_{M'}(J, s_{23}, t_1, t_2) \ d^{J}_{M'-\lambda_2, -\rho}(-z)}{\sin\pi(J - M' + \lambda_2)} (2J+1) \tag{8b}$$

where $\underset{J}{\mathrm{disc}}\{B^{\tau}_{M'}(J), C_{M'}(J)\}$ are the discontinuities in $B^{\tau}_{M'}(J), C_{M'}(J)$ across the branch cuts in the sense implied by Figure 3b.

The remaining contributions are of type (ii) and (iii) and we discuss them together. Equation II.(48) is valid for $\mathrm{Re}\, J > N$ where N may be taken to be the smallest integer $> \alpha$, where $\left|\frac{F_{M'}(z)}{z^{\alpha+\epsilon}}\right| \to 0$ as $|z| \to \infty$ for any positive ε, and we do assume the existence of a power bound for $F_{M'}(z)$, as is generally done for $2 \to 2$ amplitudes [24]. Equation II.(46) says that $B^{\tau}_{M'}(J)$ has a one over square root singularity, that is, a behavior $(J-\Lambda)^{-1/2}$ as $(J-v)$ passes through an integer of the wrong signature with $\lambda_{min} \leq J < \lambda$ max. The representation II.(48) is no longer valid for Re $J < N$, but, as mentioned by Mandelstam and Wang [34], these singularities persist in the $2 \to 2$ amplitude with spin due to their occurrence in the left hand discontinuity function for the partial wave amplitude in the presence of a third double spectral function, as first pointed out by Gribov and Pomeranchuk [35] for the spinless case. These singularities may also persist in the amplitudes $B^{\tau}_{M'}(J)$ in the problem being discussed.

As J passes through such a point, called a "sense-nonsense" point because $J \geq \lambda$ min is "sense" and $J < \lambda$ max is "nonsense" (we abbreviate this by referring to it as an s-n point), the multiplying function $d^{\tau}_{M'-\lambda_2,-\rho}(J,-z)$ vanishes as $(J-\Lambda)^{3/2}$ because the signature factor vanishes as $(J-\Lambda)$ and the $d^{J}_{M'-\lambda_2,-\rho}(-z)$ function vanishes as $\sqrt{J-\Lambda}$ [Reference 26, Table I]. The denominator $\sin \pi(J-M' + \lambda_2)$ also vanishes as $(J-\Lambda)$, so that the integrand is finite at an s-n point of the wrong signature, and there is no contribution from such a point when the contour is deformed from C'_J to C''_J .

As J passes through an s-n point of the right signature, the continued partial wave amplitude $B_{M'}(J)$ vanishes as $\sqrt{J-\Lambda}$, because in this case the continuation reduces to an integral over a $d^{J}_{M'-\lambda_2,\rho}(z)$ function as in II.(40) which vanishes as $\sqrt{J-\Lambda}$. At such a right signature point $d^{\tau}_{M'-\lambda_2,-\rho}(J,-z)$ also vanishes as $\sqrt{J-\Lambda}$, so that the integrand is finite, despite the vanishing of $\sin \pi(J-M'+\lambda_2)$, and no contribution to the right side of Eq. (6a) is obtained from the vanishing of the denominator at s-n values of J , when the contour C'_J is shifted to the left, to the contour C''_J . The same argument applies to the s-n points in the integrand for F_C in Eq. (6b).

When $(J-v)$ passes through an integer of the wrong signature with $J < \lambda$ min, referred to as a "nonsense-nonsense" (abbreviated n-n) point, II.(48) suggests that $B^{\tau}_{M'}(J)$ has a pole, because $e^{J}_{M'\lambda_2,\rho}(z)$ has a pole there. At an n-n point of right signature, II.(48) suggests that $B^{\tau}_{M'}(J)$ is regular, for the same reason that it vanishes at s-n points of the right signature, i.e., the continuation reduces to a formula of the form II.(40). This behavior persists for $2 \to 2$ amplitudes that satisfy the Mandelstam representation as indicated in Reference [32]. At n-n points $d^{\tau(J,-z)}_{M'-\lambda_2,\rho}$ vanishes in the case of wrong signature because of the signature factor, and this cancels the zero in the denominator due to the vanishing of $\sin\pi(J-M'+\lambda_2)$, but this leaves a possible pole in the integrand due to the suggested pole in $B^{\tau}_{M'}(J)$, since $d^{J}_{M'\lambda_2,\rho}(z)$ is regular at n-n points. At right signature points $d^{\tau(J,-z)}_{M'\lambda_2,-\rho}$ does not vanish, and there is a pole in the integrand due to the vanishing of the denominator. There are poles of the integrand, then, at n-n points of both right and wrong signature, and these lead to contributions to F_B, F_C, respectively, of the forms

$$F_B^{n-n} = -\frac{1}{2\pi}\sum_{M'}\sum_{J=v}^{\lambda_{min}-1} e^{i(\lambda_2-M')\phi}(2J+1)\,\mathrm{Res.}\left(\frac{B^{\tau}_{M'}(J)\,d^{\tau(J,-z)}_{M'-\lambda_2,-\rho}}{\sin\pi(J-M'+\lambda_2)}\right) \tag{9a}$$

$$F_C^{n-n} = -\frac{1}{2\pi}\sum_{M'}\sum_{J=v}^{\lambda_{min}-1} e^{i(\lambda_2-M')\phi}\,\frac{(2J+1)}{\pi}\,(-1)^{J-M'+\lambda_2}\,d^{J}_{M'-\lambda_2,-\rho}(-z)\;C_{M'}(J) \tag{9b}$$

For later applications, we note that at n-n values $d^{J}_{\lambda\mu}(z)$ behaves as z^{-J-1} for $|z| \to \infty$, so that these terms are

dominated by the background integral when $|z| \to \infty$.

As a result of the discussion above, the Sommerfeld-Watson transformed series assumes the form

$$
\begin{aligned}
F_B = {} & \frac{i}{4\pi^2} \sum_{M',\tau} e^{i(\lambda_2 - M')\phi} \int_{C_J''} dJ \, \frac{(2J+1)\, B^{\tau}_{M'}(J, s_{23})\, d^{\tau}_{M'-\lambda_2,-\rho}(J,-z)}{\sin\pi(J - M' + \lambda_2)} \\
& - \frac{1}{2\pi} \sum_{M',\tau} \sum_{\substack{\text{poles } \ell \\ \operatorname{Re}\alpha_\ell > -\frac{1}{2}}} \frac{d^{\tau}_{M'-\lambda_2,-\rho}(\alpha_\ell,-z)\, \beta^{\tau}_{M'}(t_1,t_2,s_{23})\, e^{i(\lambda_2 - M')\phi}}{\sin\pi(\alpha_\ell - M' + \lambda_2)} \\
& + \frac{i}{4\pi^2} \sum_{M',\tau} \sum_{\substack{\text{branch} \\ \text{cuts}, k}} e^{i(\lambda_2 - M')\phi} \int_{C_k} dJ \, \frac{(2J+1) \left[\operatorname{disc}_J B^{\tau}_{M'}(J, s_{23})\right] d^{\tau}_{M'-\lambda_2,-\rho}(J,-z)}{\sin\pi(J - M' + \lambda_2)} \\
& - \frac{1}{2\pi} \sum_{M',\tau} \sum_{J=v}^{\lambda_{min}-1} e^{i(\lambda_2 - M')\phi} (2J+1) \operatorname{Res.} \left[\frac{B^{\tau}_{M'}(J)\, d^{\tau}_{M'-\lambda_2,-\rho}(J,-z)}{\sin\pi(J - M' + \lambda_2)} \right]
\end{aligned}
\qquad (10a)
$$

$$
\begin{aligned}
F_C = {} & \frac{i}{4\pi^2} \sum_{M'} e^{i(\lambda_2 - M')\phi} \int_{C_J''} dJ \, \frac{(2J+1)\, C_{M'}(J, s_{23})\, d^{J}_{M'-\lambda_2,-\rho}(-z)}{\sin\pi(J - M' + \lambda_2)} \\
& - \frac{1}{2\pi} \sum_{M'} \sum_{\substack{\text{poles}, m \\ \operatorname{Re}\alpha_m > -\frac{1}{2}}} d^{\alpha_m}_{M'-\lambda_2,-\rho}(-z) \, \frac{\gamma_{M'}(t_1,t_2,s_{23})\, e^{i(\lambda_2 - M')\phi}}{\sin\pi(\alpha_m - M' + \lambda_2)} \\
& + \frac{i}{4\pi^2} \sum_{M'} \sum_{\substack{\text{branch} \\ \text{cuts}, n}} e^{i(\lambda_2 - M')\phi} \int_{C_n} dJ \, \frac{\left[\operatorname{disc}_J C_{M'}(J, s_{23})\right]}{\sin\pi(J - M' + \lambda_2)} \, d^{J}_{M'-\lambda_2,-\rho}(-z) \\
& - \frac{1}{2\pi} \sum_{M'} \sum_{J=v}^{\lambda_{min}-1} e^{i(\lambda_2 - M')\phi} \, \frac{(2J+1)}{\pi} \, C_{M'}(J)\, (-1)^{J-M'+\lambda_2}\, d^{J}_{M'-\lambda_2,-\rho}(-z)
\end{aligned}
\qquad (10b)
$$

Alternatively, we may perform the Sommerfeld-Watson transformation in the partial-wave series in J', by using the equations of II.4 to continue the partial wave amplitude $b_{M'}(s_{12}, J', t_1, t_2)$ to complex J', and perform the procedure corresponding to that which led to Eq. (10a) and (10b). Then we obtain

$$F = \tilde{F}_B + \tilde{F}_C \tag{11}$$

where

$$\begin{aligned}
\tilde{F}_B = {} & \frac{i}{4\pi^2} \sum_{M', \tau'} e^{i(\lambda_2 - M')\phi} \int_{C''_J} dJ' \frac{(2J'+1)\, B^{\tau'}_{M'}(s_{12}, J')\, d^{\tau'}_{M',-\delta}(J', -z')}{\sin\pi(J'-M')} \\
& - \frac{1}{2\pi} \sum_{M', \tau'} \sum_{\substack{\text{poles}, s \\ \mathrm{Re}\,\tilde{\alpha}_s > -\frac{1}{2}}} e^{i(\lambda_2 - M')\phi} \frac{d^{\tau'}_{M',-\delta}(\tilde{\alpha}_s, -z')\, \tilde{\beta}^{\tau'}_{M'}(t_2, t_1, s_{12})}{\sin\pi(\tilde{\alpha}_s - M')} \\
& + \frac{i}{4\pi^2} \sum_{M', \tau'} \sum_{\substack{\text{branch} \\ \text{cuts}, r}} e^{i(\lambda_2 - M')\phi} \int_{C_r} dJ' \frac{(2J'+1)\, [\underset{J'}{\mathrm{disc.}}\, B^{\tau'}_{M'}(s_{12}, J')]\, d^{\tau'}_{M',-\delta}(J', -z')}{\sin\pi(J'-M')} \\
& - \frac{1}{2\pi} \sum_{M', \tau'} \sum_{J' \geq \nu'}^{\sigma_{\min}-1} e^{i(\lambda_2 - M')\phi} (2J'+1)\, \mathrm{Res.} \left[\frac{B^{\tau'}_{M'}(s_{12}, J')\, d^{\tau'}_{M',-\delta}(J', -z')}{\sin\pi(J'-M')} \right]
\end{aligned} \tag{12a}$$

$$\begin{aligned}
\tilde{F}_C = {} & \frac{i}{4\pi^2} \sum_{M'} e^{i(\lambda_2 - M')\phi} \int_{C''_J} dJ' \frac{(2J'+1)\, C_{M'}(s_{12}, J')\, d^{J'}_{M',-\delta}(-z')}{\sin\pi(J'-M')} \\
& - \frac{1}{2\pi} \sum_{M'} \sum_{\substack{\text{poles}, m \\ \mathrm{Re}\,\tilde{\alpha}_m > -\frac{1}{2}}} \frac{d^{\tilde{\alpha}_m}_{M',-\delta}(-z')\, \gamma_{M'}(t_2, t_1, s_{12})\, e^{i(\lambda_2 - M')\phi}}{\sin\pi(\tilde{\alpha}_m - M')} \\
& + \frac{i}{4\pi^2} \sum_{M'} \sum_{\substack{\text{branch} \\ \text{cuts}, n}} e^{i(\lambda_2 - M')\phi} \int_{C_n} dJ' \frac{(2J'+1)\, [\underset{J'}{\mathrm{disc.}}\, C_{M'}(s_{12}, J')]\, d^{J'}_{M',-\delta}(-z')}{\sin\pi(J'-M')} \\
& - \frac{1}{2\pi^2} \sum_{M'} \sum_{J' \geq \nu'}^{\sigma_{\min}-1} e^{i(\lambda_2 - M')\phi} (2J'+1)\, C_{M'}(s_{12}, J')\, (-1)^{J'-M'}\, d^{J'}_{M',-\delta}(-z')
\end{aligned} \tag{12b}$$

IV. APPLICATIONS TO THE HIGH ENERGY REGION

We restrict our discussion of the high energy behavior of the amplitude for the reaction II.(1) to the case in which the two momentum transfers t_1, t_2 (Eq. II.(3)) are held fixed, and $s \to \infty$. Then the pair energy s_{13} necessarily $\to \infty$ (for the proof, and for other kinematical relations in the asymptotic region, see Appendix C). Furthermore, it is not possible for both s_{12} and s_{23} to remain bounded in this limit. Accordingly, we consider separately the cases in which one or both of these sub-energies $\to \infty$ as $s \to \infty$.

1. The Amplitude as One Pair Energy Remains Bounded

a. The amplitude for large s_{12}

The standard method [36] for obtaining the high energy behavior of a $2 \to 2$ amplitude at fixed momentum transfer is based on the assumption that the crossed channel amplitude, after undergoing a Sommerfeld-Watson transformation, may be continued in the kinematical variables to the physical region of the direct channel, and represents the direct channel amplitude through the crossing relations [37]. Likewise, we assume here that the representation III.(10) of the crossed channel amplitudes (i.e., the helicity amplitudes for II.(2)) may be continued to the physical region of the direct channel (i.e., the physical region for II.(1)), and represents the helicity amplitudes of the direct channel scattering by means of the appropriately generalized crossing relations [11,12]. Thus, we use III.(10) to investigate the behavior of the amplitude as $s \to \infty$ for fixed t_1, t_2, s_{23} and $\emptyset$.

Using II.(18), we note that at fixed t_1, t_2 ($t_1 \neq 0$), $|z| \to \infty$ linearly in s_{12}. In fact, in the physical region for II.(1) the denominator of II.(18) has the same sign as t_1 (Appendix B), and $t_1 < 0$ when s_{12} is large (Appendix C), so that $\mathrm{Re}\, z \to -\infty$ as $s_{12} \to \infty$, and sign ($\mathrm{Im}\, z$) = - sign($\mathrm{Im}\, s_{12}$). The asymptotic forms for large $|z|$ of the functions $d^{\alpha}_{M'-\lambda_2,-\rho}(-z)$ and $d^{\tau}_{M'-\lambda_2,-\rho}(\alpha,-z)$ that appear on the right side of II.(10) are found in Appendix A.3 to be

$$d^{\alpha}_{M'-\lambda_2,-\rho}(-z) \simeq e^{i\eta\frac{\pi}{2}(M'-\lambda_2+\rho)} \frac{\Gamma(2\alpha+1)\, e^{-i\eta\pi\alpha}}{N(\alpha,\rho)\, N(\alpha,M'-\lambda_2)} \left(\frac{z}{2}\right)^{\alpha}$$

$$d^{\tau}_{M'-\lambda_2,-\rho}(\alpha,-z) \simeq \frac{e^{i\eta\frac{\pi}{2}(M'-\lambda_2+\rho)}\,\Gamma(2\alpha+1)}{N(\alpha,\rho)\, N(\alpha,M'-\lambda_2)}\, \frac{1}{2}\left(1+\tau e^{i\eta\pi(\alpha-\nu)}\right) e^{-i\eta\pi\alpha}\left(\frac{z}{2}\right)^{\alpha} \tag{1}$$

where we have used $-z = e^{-i\eta\pi\alpha}\, z$, $\eta = \mathrm{sign}\,(\mathrm{Im}\, z)$.

In our case, we are interested in this physical region of II.(1), so that $\mathrm{sign}(\mathrm{Im}\, s_{12}) = +1$ and $\eta = -1$. There are situations, however, in which the more general form (1) would be needed; for example, in the computation of a unitarity integral in which the amplitude on both sides of a cut is involved.

From (1) it is apparent that in the limit of large s_{12}, the behavior of the amplitude is determined by those terms on the right side of III.(10) with the largest values of $\mathrm{Re}\, J$. If that term is a pole, it may occur either in F_B or in F_C, and the leading term in these two cases of pole dominance are the following:

Pole Dominance

(i) Pole in F_B dominates:

$$F_{(\delta_3\delta_{\bar{b}})\lambda_2;\lambda_{\bar{1}}\lambda_a} \simeq -\frac{1}{2\pi}\sum_{M'} \frac{e^{i(\lambda_2-M')\phi}e^{i\eta\frac{\pi}{2}(M'-\lambda_2+\rho)}\Gamma(2\alpha+2)\beta^{(t_1,t_2,s_{23})}_{M'(\delta_3\delta_{\bar{b}})\lambda_2;\lambda_{\bar{1}}\lambda_a}\,\mathcal{S}_\alpha\, e^{-i\eta\pi\nu}}{\sin\pi(\alpha-M'+\lambda_2)\,N(\alpha,M'-\lambda_2)\,N(\alpha,\rho)}\left(\frac{z}{2}\right)^\alpha$$

$$= -\frac{1}{2\pi}\sum_{M'} \frac{e^{i(\lambda_2-M')(\eta\frac{\pi}{2}+\phi)}e^{i\eta\pi\rho/2}\,\Gamma(2\alpha+2)\,\beta^{(t_1,t_2,s_{23})}_{M'(\delta_3\delta_{\bar{b}})\lambda_2;\lambda_{\bar{1}}\lambda_a}}{\sin\pi(\alpha-\nu)\,N(\alpha,M'-\lambda_2)\,N(\alpha,\rho)}\,\mathcal{S}_\alpha\left(\frac{z}{2}\right)^\alpha$$

$$\mathcal{S}_\alpha = \frac{1}{2}\left(\tau + e^{-i\eta\pi(\alpha-\nu)}\right)$$

(2a)

(ii) Pole in F_C dominates:

$$F_{(\delta_3\delta_{\bar{b}})\lambda_2;\lambda_{\bar{1}}\lambda_a} \simeq -\frac{1}{2\pi}\sum_{M'} \frac{e^{i(\lambda_2-M')\phi}\,\Gamma(2\alpha+2)\,\gamma^{(t_1,t_2,s_{23})}_{M'(\delta_3\delta_{\bar{b}})\lambda_2;\lambda_{\bar{1}}\lambda_a}}{\sin\pi(\alpha-\nu)\,N(\alpha,M'-\lambda_2)\,N(\alpha,\rho)}\, e^{-i\eta\pi(\alpha-\nu)}\left(\frac{z}{2}\right)^\alpha \times e^{i\eta\frac{\pi}{2}(\rho+\lambda_2-M')}$$

(2b)

As noted before, F_C is exchange degenerate, and this is evidenced by the absence of the signature factor in the asymptotic behavior of the contribution of a pole in F_C.

The leading term with largest Re J may possibly occur in the integrand of the discontinuity function across a cut in the J plane. This may occur in applications at large momentum transfer, or at small momentum transfer if the Pomeranchukon is a fixed cut. The contribution of such a leading singularity at $J = \alpha_c(t_1)$ is of the form:

Cut Dominance

(i) Cut in F_B dominates:

$$F_{(\delta_3\delta_{\bar{b}})\lambda_2;\lambda_{\bar{1}}\lambda_a} \simeq \frac{i}{4\pi^2}\sum_{M'} \frac{e^{i(\lambda_2-M')\phi}\,\Gamma(2\alpha_c+2)\,e^{\frac{i\eta\pi}{2}(\rho+\lambda_2-M')}\,\underset{J=\alpha_c}{\mathrm{disc.}}\,B_{M'}(J,s_{23},t_1,t_2)}{\sin\pi(\alpha_c-\nu)\,N(\alpha_c,M'-\lambda_2)\,N(\alpha_c,\rho)\,\log|z|}$$

$$+\,O\left(\frac{z^{\alpha_c}}{\log^2|z|}\right) \tag{3a}$$

(ii) Cut in F_C dominates:

$$F_{(\delta_3\delta_{\bar{b}})\lambda_2;\lambda_{\bar{1}}\lambda_a} \simeq \frac{i}{4\pi^2}\sum_{M'} \frac{e^{i(\lambda_2-M')\phi}\,e^{\frac{i\eta\pi}{2}(\rho+\lambda_2-M')}\,e^{-i\eta\pi(\alpha_c-\nu)}\,\underset{J=\alpha_c}{\mathrm{disc.}}\,C^{(J,s_{23},t_1,t_2)}_{M'(\delta_3\delta_{\bar{b}})\lambda_2,\lambda_{\bar{1}}\lambda_a}\,\Gamma(2\alpha_c+2)}{\sin\pi(\alpha_c-\nu)\,N(\alpha_c,\rho)\,N(\alpha_c,M'-\lambda_2)\,\log|z|}\left(\frac{z}{2}\right)^{\alpha_c}$$

$$+\,O\left(\frac{z^{\alpha_c}}{\log^2|z|}\right) \tag{3b}$$

This derivation of the behavior as $s_{12} \to \infty$ breaks down in the case $t_1 = 0$ for then $|z|$ is bounded by unity. This situation is familiar from its occurrence in two-body backward scattering of unequal mass particles, where it was examined by Goldberger and Jones [38], and Freedman and Wang [39]. There it was found that when an analyticity requirement is imposed, either in the form of the Mandelstam representation as in Reference [38], or analyticity in the Khuri representation as in Reference [39], that the behavior $s^{\alpha(u)}$ persists even at $u = 0$. We assume a similar substructure here permits us to use Eqs. (2) and (3) in applications in the limit $t_1 \to 0$.

b. Factorization

The contribution of a Regge pole to a $2 \to 2$ amplitude is generally taken to be factorizable into two parts that depend, respectively, on the variables of the two channels [40]. That is, if the contribution of the Regge pole $J = \alpha(t)$ to the amplitude for $a+\bar{c} \to \bar{b}+d$ with channel energy t is written for large s in the form

$$F_{\lambda_{\bar{b}}\lambda_d;\lambda_a\lambda_{\bar{c}}} \simeq \frac{e^{i\eta\frac{\pi}{2}(\lambda'-\lambda)}\beta^{(t)}_{\lambda_{\bar{b}}\lambda_d;\lambda_a\lambda_{\bar{c}}}\,\Gamma(2\alpha+2)\,\xi_\alpha\,\left(\frac{z}{2}\right)^\alpha}{\sin\pi(\alpha-\nu)\,N(\alpha,\lambda)\,N(\alpha,\lambda')}$$

where

$$\begin{aligned} s &= (p_a - p_{\bar{b}})^2 \\ &= m_a^2+m_b^2-\frac{1}{2t}(t+m_a^2-m_c^2)(t+m_b^2-m_d^2)+\frac{z}{2t}\Delta^{1/2}(t,m_a^2,m_c^2)\Delta^{1/2}(t,m_b^2,m_d^2) \\ \lambda &= \lambda_a-\lambda_{\bar{c}} \\ \lambda' &= \lambda_{\bar{b}}-\lambda_d \\ \eta &= \operatorname{sign}(\operatorname{Im} z) \end{aligned} \tag{4}$$

Then the residue function $\beta_{\lambda_{\bar{b}}\lambda_d;\lambda_a\lambda_{\bar{c}}}$ is taken to factor as

$$\beta^{(t)}_{\lambda_{\bar{b}}\lambda_d;\lambda_a\lambda_c} = \beta^{(t)}_{\lambda_{\bar{b}}\lambda_d}\,\beta^{(t)}_{\lambda_a\lambda_{\bar{c}}} \tag{5}$$

If the helicity residue function $\beta^{(t_1,t_2,s_{23})}_{M'(\delta_3\delta_{\bar{b}})\lambda_2;\lambda_{\bar{1}}\lambda_a}$ of the pole $J = \alpha(t_1)$ in the amplitude for the reaction $\bar{1}+a \to 2+$"object" of helicity M' is taken to factor in the same way as if the "object" were a particle, we can write:

$$\beta^{(t_1,t_2,s_{23})}_{M'(\delta_3\delta_{\bar{b}})\lambda_2;\lambda_{\bar{1}}\lambda_a} = \beta^{(t_1)}_{\lambda_{\bar{1}}\lambda_a}\,\beta^{(t_1,t_2,s_{23})}_{M'(\delta_3\delta_{\bar{b}})\lambda_2} \tag{6}$$

and identify $\beta^{(t_1)}_{\lambda_{\bar{1}}\lambda_a}$ as the same helicity residue function of the two-body channel $(\bar{1}+a)$ at the pole $J = \alpha(t_1)$ as occurs in two-body reactions of the type $\bar{1}+a \to x+y$, where $(x+y)$ represents any two-body channel that has this pole and that couples to $(\bar{1}+a)$.

$\beta^{(t_1,t_2,s_{23})}_{M'(\delta_3\delta_{\bar{b}})\lambda_2}$ may be regarded as defined by (6), and is the residue at the pole associated with the coupling of the three-body channel $\bar{b} + 3 + 2$ with helicity λ_2 and M', respectively, for the particle "2" and the two-particle subsystem $(3 + \bar{b})$ in the z.m.s. of $(\bar{1} + a)$. Furthermore, the notation $\beta_{M'(\delta_3\delta_{\bar{b}})\lambda_2}$ signifies that δ_3, $\delta_{\bar{b}}$ are the helicities of 3 and $\bar{b}$ in the z.m.s. of $(3 + \bar{b})$. Thus, the dependence of the amplitude on the helicities $\lambda_{\bar{1}}$, λ_a factors out of the sum in (2a) or (2b), and we can write it as

$$F_{(\delta_3\delta_{\bar{b}})\lambda_2;\lambda_{\bar{1}}\lambda_a} \simeq -\frac{e^{i\eta\pi\rho/2}\beta^{(t_1)}_{\lambda_{\bar{1}}\lambda_a}}{2\pi N(\alpha,\rho)} \times \sum_{M'} \frac{e^{i(\lambda_2-M')\phi_{\mathfrak{z}_\alpha}} e^{i\eta\frac{\pi}{2}(\lambda_2-M')}\Gamma(2\alpha+2)\,\beta^{(t_1,t_2,s_{23})}_{M'(\delta_3\delta_{\bar{b}})\lambda_2}}{\sin\pi(\alpha-\nu)\,N(\alpha,M'-\lambda_2)}\left(\frac{z}{2}\right)^{\alpha} \tag{7}$$

where we have taken a signatured pole, for definiteness.

c. <u>Poles with definite parity</u>

When $\mathrm{Re}(\alpha(t)-\nu)$ passes through an integer of the correct signature, i.e., $\mathrm{Re}(\alpha(t)-\nu)$ = even or odd integer, corresponding to even or odd signature, a Regge pole in a $2 \to 2$ amplitude gives a contribution that is associated with that of a particle, either a bound state or a resonance [36]. For this to occur, it is necessary that $\mathrm{Re}(\alpha(t)-\nu)$ a "sense" integer, i.e., that $\mathrm{Re}(\alpha(t)-\nu) \geq \mathrm{Max}\{|\lambda|,|\lambda'|\}$ in the two-body reaction described above by Eq. (4). Similarly, particles are associated with right signature integer sense values of $\mathrm{Re}(\alpha-\nu)$ in the $2 \to 3$ case. Bound states and resonances have definite parity, and in the $2 \to 2$ case, the trajectories couple to the following parity combinations [41]:

$$|JM\lambda_1\lambda_2\rangle_\pm = \frac{1}{\sqrt{2}}|JM\lambda_1\lambda_2\rangle \pm \frac{1}{\sqrt{2}}\eta_1\eta_2(-1)^{s_1+s_2-v}|JM-\lambda_1,-\lambda_2\rangle \tag{8}$$

where η_x, s_x refer to the intrinsic parity and angular momentum of particle "x". Ths combination has parity $P = \pm(-1)^{J-v}$, as may easily be seen from Eq. (41) of Reference [13]. The + or - is characteristic of the trajectory, and we refer to $P_\alpha = \pm 1$ as the natural parity, or simply the parity of the trajectory. Exchange nondegenerate trajectories form particles of one parity only (which may be $+P_\alpha$ or $-P_\alpha$, depending on the signature), whereas exchange degenerate trajectories have particles of alternating parity along the trajectory [42].

The pole at $J = \alpha(t_1)$ that gives rise to the asymptotic term (7) in the $2 \to 3$ amplitude occurs in the continued angular momentum of the two-body state $(\bar{1} + a)$, and so the fact that the coupling of a trajectory to two-body states is via one of the combinations (8) means that we must replace $\beta_{\lambda_{\bar{1}}\lambda_a}^{(t_1)}$ in Eq. (7) by $\frac{1}{2}\left[\beta_{\lambda_{\bar{1}}\lambda_a}^{(t_1)} + P_\alpha\eta_{\bar{1}}\eta_a(-1)^{s_{\bar{1}}+s_a-v}\beta_{-\lambda_{\bar{1}},-\lambda_a}^{(t_1)}\right]$. The additional factor of $2^{-1/2}$ comes from the fact that it is the state $\frac{1}{\sqrt{2}}\left[|JM\lambda_1\lambda_2\rangle_+ + |JM\lambda_1\lambda_2\rangle_-\right]$ that occurs in the partial wave decomposition. Thus, with signatured pole dominance, one has the result

$$F_{(\delta_3\delta_{\bar{5}})\lambda_2;\lambda_{\bar{1}}\lambda_a} \simeq -\frac{1}{4\pi}\frac{e^{i\eta\pi\rho/2}}{N(\alpha,\rho)}\left(\beta_{\lambda_{\bar{1}}\lambda_a}^{(t_1)} + P_\alpha\eta_{\bar{1}}\eta_a(-1)^{s_{\bar{1}}+s_a-v}\beta_{-\lambda_{\bar{1}},-\lambda_a}^{(t_1)}\right)\zeta_\alpha$$
$$\times\sum_{M'}\frac{e^{i(\lambda_2-M')\phi}\,\Gamma(2\alpha+2)\,\beta_{M'(\delta_3\delta_{\bar{5}})\lambda_2}^{(t_1,t_2,s_{23})}\,e^{i\eta\frac{\pi}{2}(\lambda_2-M')}}{\sin\pi(\alpha-v)\,N(\alpha,M'-\lambda_2)}\left(\frac{z}{2}\right)^{\alpha(t_1)} \tag{9}$$

d. Consequences of parity invariance

Shirokov [16] has worked out the consequences of parity invariance for processes in terms of the pseudo-helicity amplitudes. From Eq. (12) of Reference [16], the result for the $2 \to 3$ amplitude is [43]

$$F^{(z,z',\phi,t_1,t_2)}_{(\delta_3\delta_{\bar{b}})\lambda_2;\lambda_{\bar{1}}\lambda_a} = \eta_2\eta_3\eta_{\bar{b}}\eta_{\bar{1}}\eta_a (-1)^{s_3+s_{\bar{b}}+s_2+s_{\bar{1}}+s_a} (-1)^{\lambda_a-\lambda_{\bar{1}}+\delta_3-\delta_{\bar{b}}-\lambda_2} \times F^{(z,z',-\phi,t_1,t_2)}_{(-\delta_3,-\delta_{\bar{b}})-\lambda_2;-\lambda_{\bar{1}},-\lambda_a} \tag{10}$$

where s_x = intrinsic spin of particle "x" $= s_{\bar{x}}$.

This relation may be applied to obtain conditions on the residue functions $\beta^{(t_1,t_2,s_{23})}_{M'(\delta_3\delta_{\bar{b}})\lambda_2}$ that appear in Eq. (9). Thus,

$$\beta^{(t_1,t_2,s_{23})}_{M'(\delta_3\delta_{\bar{b}})\lambda_2} = (-1)^{\rho+\delta-\lambda_2}(-1)^{s_3+s_{\bar{b}}+s_2+s_{\bar{1}}+s_a}\eta_2\eta_3\eta_{\bar{b}}\eta_{\bar{1}}\eta_a \; P_\alpha \eta_{\bar{1}}\eta_a \times (-1)^{s_{\bar{1}}+s_a-\nu} e^{-i\eta\pi\rho} e^{-i\eta\pi(\lambda_2-M')} \beta^{(t_1,t_2,s_{23})}_{-M'(-\delta_3,-\delta_{\bar{b}})-\lambda_2} \tag{11}$$

or,
$$\beta^{(t_1,t_2,s_{23})}_{-M'(-\delta_3,-\delta_{\bar{b}})-\lambda_2} = (-1)^{\delta-M'}(-1)^{s_3+s_{\bar{b}}+s_2-\nu}\eta_3\eta_{\bar{b}}\eta_2 P_\alpha \beta^{(t_1,t_2,s_{23})}_{M'(\delta_3,\delta_{\bar{b}})\lambda_2} \tag{12}$$

e. The amplitude for large s_{23}

To investigate the behavior of the continued helicity amplitudes for II.(2) as $s_{23} \to \infty$ at fixed t_1, t_2 $(t_2 \neq 0)$ and fixed s_{12}, one starts with III.(12) and applies essentially the same method to the expression for $|z'| \to \infty$, as in Secs. a. through d. to the expression III.(10) for $|z| \to \infty$. To make the analysis as similar as possible, it is convenient to introduce the second crossed channel:

$$3 + \bar{b} \to \bar{1} + a + \bar{2} \tag{13}$$

and perform the partial wave decomposition corresponding to II.(10)

$$F_{(\delta_{\bar{1}}\delta_a)\lambda_{\bar{2}};\delta_3\delta_{\bar{b}}} = \sum_{\substack{J_1,J_2\\M'}} e^{i(\lambda_{\bar{2}}-M')\phi}\, b^{(J_1,J_2,t_2,t_1)}_{M'(\delta_{\bar{1}}\delta_a)\lambda_{\bar{2}};\delta_3\delta_{\bar{b}}}\; 8\left(\frac{\sqrt{t_1}}{q_3 q_2 q_{\bar{1}}'}\right)^{1/2} d^{J_1}_{M'-\lambda_{\bar{2}},-\delta}(z_1)\, d^{J_2}_{M',-\rho}(z_2) \tag{14}$$

where J_1, J_2, z_1, z_2 are defined with respect to (13) in the same manner as J, J', z, z' are defined with respect to II.(2). That is, $z_1 = \cos\theta_1$, where θ_1 = angle between $p_{\bar{2}}$ and p_3 in the z.m.s. of $3 + \bar{b}$. Then

$$s_{23} = m_2^2 + m_3^2 - \frac{1}{2t_2}(t_2 + m_2^2 - t_1)(t_2 + m_3^2 - m_b^2)$$
$$+ \frac{1}{2t_2}\Delta^{1/2}(t_2, m_3^2, m_b^2)\,\Delta^{1/2}(t_2, t_1, m_2^2)\, z_1 \tag{15}$$

and comparing this with Eq. (B.9), one sees that

$$z_1 = -z' \tag{16}$$

Similarly, $z_2 = \cos\theta_2$, θ_2 = angle between $-\hat{p}_{\bar{2}}$ and $\hat{p}_{\bar{1}}$ in the z.m.s. of $(\bar{1} + a)$ in the reaction (13), and by analogy to Eq. (B.9)

$$s_{12} = m_1^2 + m_2^2 + \frac{1}{2t_1}(t_2 - t_1 - m_2^2)(t_1 + m_1^2 - m_a^2)$$
$$- \frac{1}{2t_1}\Delta^{1/2}(t_1, t_2, m_2^2)\,\Delta^{1/2}(t_1, m_1^2, m_a^2)\, z_2 \tag{17}$$

Thus, from Eq. II.(18)

$$z_2 = -z \tag{18}$$

We note that ϕ in Eq. (14) is the same angle that appears in II.(10), since it is the angle between the planes of the momenta of particle pairs $(\bar{1} + a)$ and $(\bar{b} + 3)$ in the rest system of particle $\bar{2}$, which is the same as the rest system of particle 2 in the reaction II.(2) with the same values of the independent variables.

One then treats Eq. (14) in the same manner as II.(10) which leads to Eq. (19) below, corresponding to Eq. (9). As $s_{23} \to \infty$, for the case of signatured pole dominance (i.e., the pole $J' = \tilde{\alpha}(t_2)$ is the singularity farthest to the right in the J' plane)

$$F_{(\delta_{\bar{1}}\delta_a)\lambda_{\bar{2}};\delta_3\delta_{\bar{b}}} \simeq -\frac{1}{4\pi}\frac{e^{+i\eta'\pi\delta/2}}{N(\tilde{\alpha},\delta)}\left(\tilde{\beta}_{\delta_3\delta_{\bar{b}}}(t_2) + P_{\tilde{\alpha}}\,\tilde{\beta}_{-\delta_3,-\delta_{\bar{b}}}(t_2)\,\eta_3\eta_{\bar{b}}(-1)^{s_3+s_{\bar{b}}-\nu'}\right)\xi_{\tilde{\alpha}}\,\Gamma(2\tilde{\alpha}+2)$$

$$\times \sum_{M'} \frac{e^{i(\lambda_{\bar{2}}-M')(\phi-\eta'\frac{\pi}{2})}\,\tilde{\beta}_{M'(\delta_{\bar{1}},\delta_a)\lambda_{\bar{2}}}(t_2,t_1,s_{12})}{N(\tilde{\alpha},M'-\lambda_2)\,\sin\pi(\tilde{\alpha}-\nu')}\left(\frac{z'}{2}\right)^{\tilde{\alpha}(t_2)}. \qquad \eta' = \mathrm{sign}(\mathrm{Im}\,z') \tag{19}$$

The relation between the amplitudes for II.(2) and (13) assumes a simple form in terms of the pseudo-helicity amplitudes $F_{(\delta_{\bar{1}}\delta_a)\lambda_{\bar{2}};\delta_3\delta_{\bar{b}}}$ and $F_{(\delta_3\delta_{\bar{b}})\lambda_2;\delta_{\bar{1}}\delta_a}$. First, according to the analyticity postulate of Stapp [30], the crossing relation between the amplitudes for reaction (13) and that for reaction

$$2 + 3 + \bar{b} \to \bar{1} + a \tag{20}$$

is

$$F^{c\,(-z',-z,\phi,t_2,t_1)}_{(\delta_{\bar{1}}\delta_a)\lambda_{\bar{2}};\delta_3\delta_{\bar{b}}} = F^{(z,z',\phi,t_1,t_2)}_{(\delta_{\bar{1}}\delta_a;(\delta_3,\delta_{\bar{b}})\lambda_2} \tag{21}$$

where $\lambda_2 = -\lambda_{\bar{2}}$, and $F^c_{(\delta_{\bar{1}}\delta_a)\lambda_{\bar{2}};\delta_3\delta_{\bar{b}}}$ is the continuation of the amplitude for (20) in the 4-momentum of particle "2" from $p_{\bar{2}}$ to $-p_{\bar{2}}$, where $p_{\bar{2}}$ is positive timelike.

Then, the amplitude for (20) is related by time-reversal to the amplitude for II.(2). Time reversal invariance [30] yields

$$\sigma_t\, F^{(z,z',\phi,t_1,t_2)}_{\delta_{\bar{1}}\delta_a;(\delta_3\delta_{\bar{b}})\lambda_2} = F^{(z,z',\phi,t_1,t_2)}_{(\delta_3\delta_{\bar{b}})\lambda_2;\delta_{\bar{1}}\delta_a} \tag{22}$$

where $\sigma_t = \pm 1$ is a phase factor that may be different for non-interfering amplitudes. Thus,

$$F^{(z,z',\phi,t_1,t_2)}_{(\delta_3\delta_{\bar{b}})\lambda_2;\delta_{\bar{1}}\delta_a} = \sigma_t\, F^{c\,(-z',-z,\phi,t_2,t_1)}_{(\delta_{\bar{1}}\delta_a)\lambda_{\bar{2}};\delta_3\delta_{\bar{b}}} \tag{23}$$

σ_t is a matter of convention, and we choose it to be +1 in all cases.

2. The Amplitude for Large Values of Both Pair Energies

When s_{12} and s_{23} are large, and t_1, t_2 are in the physical region for the process II.(1), the assumption we have made about the continuability of the Sommerfeld-Watson transformed amplitude states that III.(10) is a valid representation of the continued amplitude for II.(1). The corresponding assumption about the similarly treated amplitude for the process (13) asserts that the corresponding representation which is distinct is also valid in this kinematical region. Below, we examine the consequences of the simultaneous validity of the two continuations.

(i) Double Pole Dominance

Let us consider first the case in which as $s_{12} \to \infty$ a signatured pole at $J = \alpha(t_1)$ is the term that has the largest value of Re J on the right side of III.(10) and a signatured pole at $J' = \tilde{\alpha}(t_2)$ is the term that has the largest value of Re J' on the right side of Eq. III.(12). Then, from Eqs. (19) and (23), as $s_{23} \to \infty$,

$$F_{(\delta_3 \delta_{\bar{b}})\lambda_2;\, \delta_{\bar{1}}\delta_a} \simeq \left(-\frac{1}{4\pi}\right) \frac{e^{i\eta'\pi\delta/2}}{N(\tilde{\alpha},\delta)} \left(\tilde{\beta}_{\delta_3\delta_{\bar{b}}}(t_2) + P_{\tilde{\alpha}}\eta_3\eta_{\bar{b}}(-1)^{s_3+s_{\bar{b}}-v'}\tilde{\beta}_{-\delta_3,-\delta_{\bar{b}}}(t_2)\right) I_{\tilde{\alpha}}$$
$$\times\, \Gamma(2\tilde{\alpha}+2) \sum_{M'} \frac{e^{i(\lambda_{\bar{2}}-M')(\phi-\eta'\pi/2)}\,\tilde{\beta}_{M'(\delta_{\bar{1}}\delta_a)\lambda_{\bar{2}}}(t_2,t_1,s_{12})}{N(\tilde{\alpha},M'-\lambda_2)\,\sin\pi(\tilde{\alpha}-v')} \left(\frac{z'}{2}\right)^{\tilde{\alpha}(t_2)}$$
$$(\lambda_{\bar{2}} = -\lambda_2) \qquad (24)$$

so that to begin with

$$\frac{N(\tilde{\alpha},\delta)\, F_{(\delta_3,\delta_{\bar{b}})\lambda_2;\,\delta_{\bar{1}}\delta_a}\, e^{-i\eta'\pi\delta/2}}{\left(\frac{z'}{2}\right)^{\tilde{\alpha}(t_2)}\left(\tilde{\beta}_{\delta_3\delta_{\bar{b}}}(t_2) + P_{\tilde{\alpha}}\eta_3\eta_{\bar{b}}(-1)^{s_3+s_{\bar{b}}-v'}\tilde{\beta}_{-\delta_3,-\delta_{\bar{b}}}(t_2)\right)}$$

$$\to \text{ limit that is independent of } s_{23},\ \delta_3,\ \delta_{\bar{b}} \qquad (25)$$

Applying this to the representation III.(10), we see that each of the functions $B^{\tau}_{M'}(J,s_{23},t_1,t_2)$, $\beta_{M'}(t_1,t_2,s_{23})$, $\underset{J}{\mathrm{disc}}.B_{M'}(J,s_{23},t_1,t_2)$, $C_{M'}(J,s_{23},t_1,t_2)$, $\gamma_{M'}(t_1,t_2,s_{23})$, and $\underset{J}{\mathrm{disc}}.C_{M'}(J,s_{23},t_1,t_2)$ have this behavior. In some cases the limit in (25)is zero. We now make the assumption that whenever the internal quantum number selection rules allow the coupling of a particle lying on the trajectory $J' = \tilde{\alpha}(t_2)$ to a particle lying on the trajectory $J = \alpha(t_1)$ together with the particle $\overline{2}$, then the residue $\beta_{M'}(t_1,t_2,s_{23})$ does not vanish in this limit. None of the above functions divided by $s_{23}^{\tilde{\alpha}(t_2)}$ can go to infinity as $s_{23} \to \infty$, for cancellation of this asymptotic behavior is necessary and would take place in two or more terms for a range of values of s_{12}, and so, by analyticity, the sum of these terms would have to vanish identically.

Let us now suppose that the internal quantum numbers permit the coupling of a particle lying on the trajectory $J = \alpha(t_1)$ to the particle "2" together with a particle lying on the trajectory $J' = \tilde{\alpha}(t_2)$, where the two trajectories are the singularities with the largest real parts in the J and J' planes, respectively. Then we have the case of double pole dominance, and in the limit of large s_{12} and large s_{23}, (25) holds, and also

$$\frac{F_{(\delta_2\delta_{\bar{b}})\lambda_2;\delta_T\delta_a} \quad N(\alpha,\rho)}{e^{i\eta\frac{\pi}{2}\rho} \quad s_{12}^{\alpha(t_1)}\left[\beta_{\delta_T\delta_a}(t_1) + P_\alpha\eta_T\eta_a\beta^{(t_1)}_{-\delta_T,-\delta_a}(-1)^{s_T+s_a-\nu}\right]}$$

$$\to \text{ limit that is independent of } s_{12},\ \lambda_T,\ \lambda_a \qquad (26)$$

so that combining (25) and (26) one obtains the following result for double pole dominance:

Double Pole Dominance

$$F_{(\delta_3\delta_{\tilde b})\lambda_2;\delta_T\delta_a} \simeq e^{i\eta\pi\ell/2} e^{i\eta'\pi\delta/2} \xi_\alpha \xi_{\tilde\alpha} \left(\tilde\beta^{(t_2)}_{\delta_3\delta_{\tilde b}} + P_{\tilde\alpha}\eta_3\eta_{\tilde b}(-1)^{S_3+S_{\tilde b}-\nu'}\tilde\beta^{(t_2)}_{-\delta_3,-\delta_{\tilde b}}\right) N(\alpha,\beta) N(\tilde\alpha,\delta)$$
$$\times\left(\beta^{(t_1)}_{\delta_T\delta_a} + P_\alpha\eta_T\eta_a(-1)^{S_T+S_a-\nu}\beta^{(t_1)}_{-\delta_T,-\delta_a}\right)\times\left(\frac{z}{2}\right)^{\alpha(t_1)}\left(\frac{z'}{2}\right)^{\tilde\alpha(t_2)}\Gamma_{\lambda_2}(t_1,t_2,\phi)$$

(27)

where $$\Gamma_{\lambda_2}(t_1,t_2,\phi) = \sum_{M'} e^{i(\lambda_2-M')\phi}\,\Gamma^{2\alpha\tilde\alpha}_{M'\lambda_2}(t_1,t_2)\, e^{i(\eta-\eta')\frac{\pi}{2}(\lambda_2-M')}$$ (28)

with $$\Gamma^{2\alpha\tilde\alpha}_{M'\lambda_2}(t_1,t_2) = \lim_{S_{23}\to\infty} \frac{\beta^{(t_1,t_2,S_{23})}_{M'(\delta_3\delta_{\tilde b})\lambda_2}\left[-4\pi\left(\tilde\beta^{(t_2)}_{\delta_3\delta_{\tilde b}}+P_{\tilde\alpha}\eta_3\eta_{\tilde b}(-1)^{S_3+S_b-\nu'}\tilde\beta^{(t_2)}_{-\delta_3,-\delta_{\tilde b}}\right)\right]^{-1}}{\sin\pi(\alpha-\nu)\, N(\alpha,M'-\lambda_2)\,\left(\frac{z'}{2}\right)^{\tilde\alpha(t_2)}}$$

$$= \lim_{S_{12}\to\infty}\frac{\tilde\beta^{(t_2,t_1,S_{12})}_{(M'-2\lambda_2)(\delta_T,\delta_a)\lambda_2}\left[-4\pi\left(\beta^{(t_1)}_{\delta_T\delta_a}+P_\alpha\eta_a\eta_T(-1)^{S+S-\nu}\beta^{(t_1)}_{-\delta_T,-\delta_a}\right)\right]^{-1}}{\sin\pi(\tilde\alpha-\nu')\, N(\tilde\alpha,M')}$$

(29)

Parity invariance, as expressed in Eq. (10), when applied to Eqs. (27), (28) for the double pole dominance term, leads to the result:

$$F^{(z,z',\phi,t_1,t_2)}_{(\delta_3\delta_{\tilde b})\lambda_2;\delta_T\delta_a} = \eta_2\, e^{-i\eta'\pi\nu'} e^{i\eta\pi\nu}(-1)^{S_2-\lambda_2} F^{(z,z',-\phi,t_1,t_2)}_{(\delta_3\delta_{\tilde b})-\lambda_2;\delta_T\delta_a}\times P_\alpha P_{\tilde\alpha}$$

(30)

In terms of the $\Gamma_{M'\lambda_2}(t_1,t_2)$, one has

$$\Gamma_{-M',-\lambda_2}(t_1,t_2) = P_\alpha P_{\tilde\alpha}\eta_2\, e^{i\eta'\pi\nu'}e^{-i\eta\pi\nu}(-1)^{S_2-\lambda_2}\,\Gamma_{M'\lambda_2}(t_1,t_2)$$ (31)

The formula (27) for the case of double Regge pole dominance of reaction II.(1) in the appropriate kinematical region was first obtained by Kibble [2] in one of the first papers on the extension of the Regge pole model to production amplitudes. The theoretical basis of the double Regge pole term in this work differs from the one in Reference [2] by the inclusion of anomalous thresholds which are known to exist, in general, in the analytic structure of the production amplitude, the knowledge of which forms the basis for the continuation of the partial wave amplitude. Thus, in addition to the usual signatured partial wave amplitude, which may have poles and cuts, we found an unsignatured amplitude, also with possible poles and cuts. In Reference [2] it was further assumed that there exists a simultaneous meremorphic continuation of $b_{\mu'(\delta_3\delta_{\bar{b}})\lambda_2,\lambda_{\bar{1}}\lambda_a}(J,J',t_1,t_2)$ of Eq. II.(10) into the right half J and J' planes. We do not make that assumption here and, in view of the remark at the beginning of Sec. II.[4], such a double continuation seems unlikely to exist. Eq. (27) must be regarded as heuristic, nonetheless, for the helicity sum has not been proved to converge. In Appendix F a proof is presented of the existence of a domain of convergence that is valid for a Regge pole term in the case in which the pole occurs in an unsignatured amplitude.

Eq. (27) represents the leading term for large s_{12}, s_{23} in the case in which the leading J plane singularity, which is taken to be a pole, can couple to the particle 2, together with the leading J' plane singularity, which is also taken to be a pole. Other cases are possible, of course. For example, the leading singularity in either the J or the J' plane may be a cut, rather than a pole. Furthermore, it may occur that no particle on the leading J plane singularity can couple to a particle on the leading J' singularity together with particle "2". These singularities may occur in signatured or unsignatured amplitudes, but the difference between the leading contributions in these cases is trivial; simply the replacement of the signature factor ζ_α by $e^{-i\eta\pi(\alpha-v)}$.

As further illustrations of the possibilities, we compute the leading terms for the following two examples:

1) The leading J plane singularity is a cut, the leading J' plane singularity is a pole and the two can couple to each other via the external particle "2".

2) The leading J and J' plane singularities are both poles, but the quantum numbers do not allow a coupling of particle "2" and the leading J' plane pole to the leading J plane pole. This case is further defined by specifying that the leading singularity in the J' plane that, together with particle "2", can couple to the leading J plane pole is also a pole, and vice-versa.

(ii) Cut-Pole Dominance

From (3), and not assuming factorization for the cut,

$$F_{(\delta_3\delta_{\bar{b}})\lambda_2;\lambda_{\bar{1}}\lambda_a} \simeq \frac{i}{4\pi^2}\sum_{M'}\frac{e^{i(\lambda_2-M')(\eta\frac{\pi}{2}+\phi)}\Gamma(2\alpha_c+2)}{\sin\pi(\alpha_c-\nu)}\underset{J=\alpha_c}{\mathrm{disc.}}\frac{B^{(J,s_{23},t_1,t_2)}_{M'(\delta_3\delta_{\bar{b}})\lambda_2;\lambda_{\bar{1}}\lambda_a}\,\xi_{\alpha_c}\,e^{i\eta\frac{\pi\rho}{2}}}{N(\alpha_c,M'-\lambda_2)\,N(\alpha_c,\rho)\,\log|z|}\left(\frac{z}{2}\right)^{\alpha_c(t_1)}$$

and from (24)

$$F_{(\delta_3\delta_{\bar{b}})\lambda_2;\lambda_{\bar{1}}\lambda_a} \simeq \left(-\frac{1}{4\pi}\right)\frac{e^{i\eta'\pi\delta/2}}{N(\tilde{\alpha},\delta)}\left(\tilde{\beta}_{\delta_3\delta_{\bar{b}}}(t_2)+P_{\tilde{\alpha}}\,\eta_3\eta_{\bar{b}}(-1)^{s_3+s_{\bar{b}}-\nu'}\tilde{\beta}_{-\delta_3,-\delta_{\bar{b}}}(t_2)\right)\xi_{\tilde{\alpha}}\,\Gamma(2\tilde{\alpha}+2)$$
$$\times\sum_{M'}\frac{e^{-i(\lambda_2-M')(\eta'\frac{\pi}{2}-\phi)}\,\tilde{\beta}^{(t_2,t_1,s_{12})}_{(M'-2\lambda_2)(\lambda_{\bar{1}}\lambda_a)-\lambda_2}}{N(\tilde{\alpha},M'-\lambda_2)\,\sin\pi(\tilde{\alpha}-\nu')}\left(\frac{z'}{2}\right)^{\tilde{\alpha}(t_2)}$$

Using the same method as in the double pole case, one obtains

$$F_{(\delta_3\delta_{\bar{b}})\lambda_2;\lambda_{\bar{1}}\lambda_a} \simeq \frac{e^{i\eta'\pi\delta/2}\,\xi_{\tilde{\alpha}}\,\Gamma(2\tilde{\alpha}+2)}{N(\tilde{\alpha},\delta)}\left(\tilde{\beta}_{\delta_3\delta_{\bar{b}}}(t_2)+P_{\tilde{\alpha}}\,\eta_3\eta_{\bar{b}}(-1)^{s_3+s_{\bar{b}}-\nu'}\tilde{\beta}_{-\delta_3,-\delta_{\bar{b}}}(t_2)\right)$$
$$\times\left(\frac{z}{2}\right)^{\alpha_c(t_1)}\left(\frac{z'}{2}\right)^{\alpha(t_2)}\frac{\Gamma^{(t_1,t_2)}_{M'(\lambda_{\bar{1}}\lambda_a)\lambda_2}}{\log|z|}e^{-i\eta'\frac{\pi}{2}(\lambda_2-M')} \tag{34}$$

where

$$\Gamma^{(t_1,t_2)}_{M'(\lambda_{\bar{1}}\lambda_a)\lambda_2} = \lim_{s_{12}\to\infty} \frac{\tilde{\beta}^{(t_2,t_1,s_{12})}_{(M'-2\lambda_2)(\lambda_{\bar{1}},\lambda_a)-\lambda_2} \log|z|}{-4\pi \left(\frac{z}{2}\right)^{\alpha_c} \sin\pi(\tilde{\alpha}-\nu)\, N(\tilde{\alpha}, M'-\lambda_2)} \tag{35}$$

To motivate this calculation we point out that at large momentum transfers, contributions from cuts are expected to be important. As an example, consider the position of the AFS [33,44,45] J plane branch point generated by two Regge trajectories $\alpha_1(t)$, $\alpha_2(t)$. The branch point with largest real part is shown by Rothe [46] to occur at $J = \alpha_1(u) + \alpha_2(\sqrt{t} - \sqrt{u})^2) - 1$ where u is a solution to

$$\alpha_1'(t) - \frac{(\sqrt{t} - \sqrt{u})}{\sqrt{u}}\, \alpha_2'(\sqrt{t} - \sqrt{u})^2) = 0\ .$$

The most common model used presently for Regge poles is that of straight-line trajectories with universal positive slope [47] and there is some theoretical support for this model [48]. In that model the branch point occurs at

$$J = \alpha_c = \alpha_1(t/4) + \alpha_2(t/4) - 1$$
$$= \alpha_1(t) + [\alpha_2(0) - 1 - \tfrac{t}{2}\alpha_1'] > \alpha_1(t)$$

for sufficiently large negative t

Similarly, $\alpha_c > \alpha_2(t)$ for $-t$ sufficiently large.

In the case of two equal trajectories (e.g., ρ-ρ) with

$$\alpha_\rho \simeq \frac{1}{2} + t$$

then,
$$\alpha_c > \alpha_\rho \quad \text{for} \quad t \lesssim -1(\mathrm{Gev})^2$$

If the pole $\alpha_1(t)$ is the Pomeranchon with 0 slope, then the branch point is at $j = \alpha_c = \alpha_2(0)$ which will dominate the pole

$j = \alpha_2(t)$ for $t < 0$, but will in turn be dominated by the Pomeranchon pole.

(iii) In this example, we suppose that the dominant singularity in the J plane is the pole trajectory $J = \alpha_1(t_1)$, and the singularity with the largest real part, apart from this one, is the pole $J = \alpha_2(t_1)$. Likewise in the J' plane, the two singularities with largest real parts are the pole trajectories $J' = \tilde{\alpha}_1(t_2)$, $J' = \tilde{\alpha}_2(t_2)$ with $\mathrm{Re}\,\tilde{\alpha}_1 > \mathrm{Re}\,\tilde{\alpha}_2$. We suppose the quantum numbers to allow the couplings $\alpha_1 \leftrightarrow (2 + \tilde{\alpha}_2)$ and $\alpha_2 \leftrightarrow (2 + \tilde{\alpha}_1)$, but to not allow $\alpha_1 \leftrightarrow (2 + \tilde{\alpha}_1)$.

As an example, one may consider the reaction $\pi N \to \pi' \rho^0 N'$ with the identifications $\{a,b,1,2,3\} \leftrightarrow \{\pi, N, \pi', \rho^0, N'\}$. The leading singularity in the J plane (J = angular momentum of the $\bar{\pi} - \pi$ system) then corresponds to the Pomeron (P) and the next leading trajectory is the ρ among the known trajectories. Also the leading trajectory in the J' plane (J' = angular momentum of $N\bar{N}'$ system) is the Pomeron and the next leading trajectory is the ρ. The quantum numbers do not allow the coupling of the P to (P+ρ) because $I = 0$ for the P and $I = 1$ for the ρ.

In this case for large s_{12},

$$F_{(\delta_3\delta_{\bar{b}})\lambda_2;\delta_{\bar{1}}\delta_a} \simeq \sum_{i=1}^{2} \frac{-e^{i\eta\pi\rho/2}}{2\pi\, N(\alpha_i,\rho)} \left(\beta^{(i)}_{\delta_{\bar{1}}\delta_a}(t_1) + P_{\alpha_i}\beta^{(i)}_{-\delta_{\bar{1}},-\delta_a}(t_1)\right) \mathcal{J}_{\alpha_i}\,\Gamma(2\alpha_i+1)$$
$$* \sum_{M'} \frac{e^{i(\lambda_2-M')(\phi+\eta\pi/2)}\,\beta^{(i)\,(t_1,t_2,s_{23})}_{M'(\delta_3\delta_{\bar{b}})\lambda_2}\left(\frac{z}{2}\right)^{\alpha_i(t_1)}}{\sin\pi(\alpha_i-\nu)\,N(\alpha_i,M'-\lambda_2)} \tag{36}$$

where $\beta^{(i)\,(t_1,t_2\,s_{23})}_{M'(\delta_3\delta_{\bar{b}})\lambda_2}$ refers to the residue at

$$J = \alpha_i(t_1)$$

For large s_{23}

$$F_{(\delta_3\delta_{\bar b})\lambda_2;\delta_T\delta_a} \simeq \sum_{i=1}^{2} \frac{-e^{i\eta'\pi\delta/2}}{2\pi N(\tilde\alpha_i,\delta)} \left(\tilde\beta^{(i)}_{\delta_3,\delta_{\bar b}}(t_2) + P_{\tilde\alpha_i}\eta_3\eta_{\bar b}(-1)^{s_3+s_{\bar b}-v'}\tilde\beta^{(i)}_{-\delta_3,-\delta_{\bar b}}(t_2)\right)\mathcal{J}_{\tilde\alpha_i}$$
$$\times \sum_{M'} \frac{\Gamma(2\tilde\alpha_i+2)\, e^{i(\lambda_2-M')(\phi-\eta'\frac{\pi}{2})}\,\tilde\beta^{(i)}_{(M'-2\lambda_2)(\delta_T\delta_a)-\lambda_2}(t_2,t_1,s_{12})}{\sin\pi(\tilde\alpha_i-v')\, N(\tilde\alpha_i, M'-\lambda_2)} \left(\frac{z'}{2}\right)^{\tilde\alpha_i(t_2)}$$

(37)

Consider now Eq. (36) for large, but finite, s_{12}. Dividing that equation by $(z'/2)^{\tilde\alpha_1(t_2)}$ and taking the limit $s_{23} \to \infty$, one finds

$$F_{(\delta_3\delta_{\bar b})\lambda_2;\delta_T\delta_a} \simeq -\frac{e^{i\eta\pi\rho/2}}{2\pi N(\alpha_2,\rho)} \left(\beta^{(2)}_{\delta_T\delta_a}(t_1) + P_{\alpha_2}\eta_T\eta_a(-1)^{s_T+s_a-v}\beta^{(2)}_{-\delta_T,-\delta_a}(t_1)\right)\mathcal{J}_{\alpha_2}\Gamma(2\alpha_2+2)$$
$$\times \sum_{M'} \frac{e^{i(\lambda_2-M')(\phi+\eta\frac{\pi}{2})}\,\beta^{(2)}_{M'(\delta_3,\delta_{\bar b})\lambda_2}(t_1,t_2,s_{23})}{\sin\pi(\alpha_2-v)\, N(\alpha_2,M'-\lambda_2)\left(\frac{z'}{2}\right)^{\tilde\alpha_1(t_2)}} \left(\frac{z}{2}\right)^{\alpha_2(t_1)}$$

Comparing this with (37), also divided by $(z'/2)^{\tilde\alpha_1(t_2)}$, in the limit $s_{23} \to \infty$, one finds

$$\frac{e^{i\eta\pi\rho/2}}{N(\alpha_2,\rho)} \left(\beta^{(2)}_{\delta_T\delta_a}(t_1) + P_{\alpha_2}\eta_T\eta_a(-1)^{s_T+s_a-v}\beta^{(2)}_{-\delta_T,-\delta_a}(t_2)\right) \frac{\mathcal{J}_{\alpha_2}\Gamma(2\alpha_2+2)\, e^{i\eta\frac{\pi}{2}(\lambda_2-M')}\,\beta^{(2)}_{M'(\delta_3,\delta_{\bar b})\lambda_2}(t_1,t_2,s_{23})}{\sin\pi(\alpha_2-v)\, N(\alpha_2,M'-\lambda_2)\, (z'/2)^{\tilde\alpha_1(t_2)}}$$
$$\simeq \frac{e^{i\eta'\pi\delta/2}}{N(\tilde\alpha,\delta)} \left(\tilde\beta^{(1)}_{\delta_3\delta_{\bar b}}(t_2) + P_{\tilde\alpha_1}\eta_3\eta_{\bar b}(-1)^{s_3+s_{\bar b}-v}\tilde\beta^{(1)}_{-\delta_3,-\delta_{\bar b}}(t_2)\right)\Gamma(2\tilde\alpha_1+2)\, e^{-i\eta'\frac{\pi}{2}(\lambda_2-M')}$$
$$\times \frac{\tilde\beta^{(1)}_{(M'-2\lambda_2)(\delta_T,\delta_a)-\lambda_2}(t_2,t_1,s_{12})}{\sin\pi(\tilde\alpha_1-v')\, N(\tilde\alpha_1,M'-\lambda_2)} \cdot \frac{1}{(z/2)^{\alpha_2(t_1)}}$$

(38)

Then from the reasoning that led to Eq. (27), one obtains that the $i = 2$ term on the right side of Eq. (36) may be written as

$$e^{i\eta\pi\rho/2}\, e^{i\eta'\pi\delta/2}\, \xi_{\alpha_2}\, \xi_{\tilde{\alpha}_1} \left(\tilde{\beta}^{(1)}_{\delta_3 \delta_5}(t_2) + P_{\tilde{\alpha}_1} \eta_3 \eta_5 (-1)^{s_3+s_5-\nu'} \tilde{\beta}^{(1)}_{-\delta_3,-\delta_5}(t_2) \right) e^{i(\eta-\eta')\frac{\pi}{2}(\lambda_2-M')}$$

$$\times \left(\beta^{(2)}_{\delta_7 \delta_a}(t_1) + P_{\alpha_2} \eta_7 \eta_a (-1)^{s_7+s_a-\nu} \beta^{(2)}_{-\delta_7,-\delta_a}(t_1) \right) \times \left(\frac{z}{2}\right)^{\alpha_2(t_1)} \left(\frac{z'}{2}\right)^{\tilde{\alpha}_1(t_2)} \Gamma^{2\alpha_2\tilde{\alpha}_1}_{\lambda_2}(t_1, t_2, \phi) \tag{39}$$

where $\Gamma^{2\alpha_2\tilde{\alpha}_1}_{\lambda_2}$ is given by equations analogous to (28) and (29). Similarly, the $i=1$ term on the right side of (36) may be put in this form, so that when s_{12} and s_{23} are both large, the amplitude is approximately given by the sum of two terms both of the form (39), differing only by the exchange of trajectory labels "1" and "2".

V. SOME EXPERIMENTAL CONSEQUENCES OF DOUBLE POLE DOMINANCE: DEPENDENCE OF THE AMPLITUDE ON THE TOLLER VARIABLE AND POLARIZATION AT ZERO MOMENTUM TRANSFER; PREDICTED MINIMA IN DIFFERENTIAL CROSS SECTIONS

1. Dependence of the Amplitude on a Sub-Energy

The most remarkable feature of the double Regge pole contribution Eq. IV.(27), to the amplitude for the single particle production process is the form of the dependence of the amplitude on the two-particle invariant sub-energies with two momentum transfers held fixed as these sub-energies become large. This dependence has been tested against experiment, using parameters obtained from Regge pole fits to quasi two-body reactions [49,50,51]. The results have been qualitatively successful, but not conclusive. The definitiveness of the comparison is limited most severely by the statistics of the available data.[49]. More events, by a factor of about 5, seem to be necessary to significantly improve the accuracy of the comparison.

The dependence of the double pole amplitude on the two-particle sub-energies is complicated by the fact that there is also a dependence on ϕ , which is shown in Appendix E, to be the same as ω , the Toller variable [7]. From Eq. (C.20) of the Appendix,

$$\cos\phi = \frac{\pm 1}{2|\sqrt{t_1 t_2}|}\left\{\frac{\Delta(t_1,t_2,M_2^2)}{M_2^2+K_2^2} + t_1 + t_2 - M_2^2\right\} \tag{1}$$

where $\underset{\sim}{K}_2$ = the component of the momentum of particle "2" that is perpendicular to the momentum of particle "a", in the z.m.s. of the direct channel process II.(1). And from Eq. (C.18),

$$(m_2^2 + \underset{\sim}{K}_2^2) = \lim_{\substack{s_{12}, s_{23} \to \infty \\ t_1, t_2 \text{ fixed}}} \frac{s_{12}\, s_{23}}{s} \tag{2}$$

From (1) and (2), one observes that at fixed (large) s , the double pole contribution to the amplitude may have a more involved dependence on s_{12}, s_{23} than that given by the form $s_{12}^{\alpha(t_1)}\, s_{23}^{\tilde{\alpha}(t_2)}$. The fits to the data in References [49-51] are carried out, however, with the assumption that there is no dynamical dependence on $\emptyset$. The success of these fits therefore indicates that the dynamical dependence on $\emptyset$ is weak [52]. In the next section it is shown why this may be expected for small t_1 or small t_2 . The condition of the smallness of t_1 or t_2 is fulfilled by most of the events analyzed in References [49-51].

2. Dependence on the Toller Variable at Zero Momentum Transfer

Tan and Wang [6] have shown that in the case of the scattering of equal mass, spinless particles, the leading double pole term for the $2 \to 3$ amplitude is independent of $\emptyset = \omega$ in the limit $t_1 \to 0$ or $t_2 \to 0$ as $s_{12} \to \infty$ and $s_{23} \to \infty$. In that paper, it is also stated that the spin-averaged cross section is independent of ω in this limit with the average being taken over the helicity of particle "2". Below, we prove a generalization of the result of Tan and Wang based on the introduction of Toller's quantum number M [8]. The contribution of a pole term to an arbitrary helicity amplitude, for external particles of any mass, is shown to depend on ω only through a phase factor, in the limit $t_i \to 0$, where the pole occurs in the J plane of the two-particle system with invariant $(\text{mass})^2 = t_i$ $(i = 1,2)$. This result does not depend on the existence of a leading double pole term, but holds for each single Lorentz pole contribution to the amplitude for asymptotic values of the sub-energy s_{12} in the limit $t_1 \to 0$, or s_{23} when $t_2 \to 0$.

The demonstration is based on the result of Appendix D. Let the t_1 channel be

$$t_1 \quad : \quad \bar{1} + a \to \bar{Y} + 2 \tag{3}$$

and let the s and s_{12} channels be:

$$s \quad : \quad a + b \to 1 + 2 + 3 \tag{4a}$$

$$s_{12} \quad : \quad Y + a \to 1 + 2 \tag{4b}$$

where Y is the two-particle system $(b + \bar{3})$ with invariant $(\text{mass})^2 = t_2$. We consider (3) and (4) in the limit s , $s_{12} \to \infty$, t_2 bounded $(t_2 \neq m_2^2)$ and $t_1 \to 0$. As shown in Appendix D, if the limit is taken with the s channel scattering restricted to the forward cone (more precisely, if $z_s = 1 + O(m_1^8/s^4)$ as $s \to \infty$, where z_s = cosine of the angle between $\vec{p}_1$ and $\vec{p}_a$ in the z.m.s. of reaction (4a)), then in this limit $|z_t| \to 1$ where z_t = cosine of the angle between $\vec{p}_{\bar{1}}$ and $\vec{p}_2$ in the z.m.s. of reaction (3).

Now suppose that in this limit the amplitude for (4b) is dominated by a Lorentz pole in the t_1 channel. As shown by Le Bellac [9], if the Lorentz pole has Toller quantum number M , then in the limit $t_1 \to 0$, the s_{12} channel helicity amplitude vanishes [53] unless $|\lambda_1 - \lambda_a| = M$, where λ_1, λ_a are the helicities of "1" and "a" in the s_{12} channel (4b). According to the Trueman-Wick two-body crossing relations [37], the amplitude for the s_{12} channel process with particle helicities $\{\lambda_y, \lambda_a, \lambda_1, \lambda_2\}$ is equal (up to a phase factor of ± 1) to the continued amplitude of the t_1 channel with particle helicities

$\{\lambda'_{\overline{1}},\lambda'_a,\lambda'_{\overline{y}},\lambda'_2\}$ where $\lambda'_{\overline{1}} = \lambda_1$, $\lambda'_a = \lambda_a$, $\lambda'_{\overline{y}} = \lambda_y$, $\lambda'_2 = \lambda_2$, since all the angles in the crossing relation (Eq. (42) of Reference [37]) vanish in this limit. Thus, the residue of a pole term for (3) vanishes unless $\lambda_{\overline{1}} - \lambda_a = \sigma(\lambda_2 - \lambda_{\overline{y}}) = \pm M$, where $\sigma = -1$ if $m_a = m_1$ and $\sigma = \text{sign}\ (m_a^2 - m_1^2)\ (m_2^2 - t_2)$ otherwise.

Any particular t_1 channel helicity amplitude for (3) has $(\lambda_{\overline{1}} - \lambda_a) = +M$ or $-M$, so that $\lambda_{\overline{y}} - \lambda_2 = \pm\sigma M$ in order that the pole may give a nonvanishing contribution to that helicity amplitude. Referring to Eq. IV.(2), and noticing that $\lambda_{\overline{y}} = M'$ in the expression on the right side for the contribution of a pole, one has the result that the helicity series $\sum_{M'}$ collapses to a single term: $M' = \lambda_2 \pm \sigma M$ for a Lorentz pole with Toller quantum number M in the limit $t_1 \to 0$. Thus, the dependence of a pole term in Eq. IV.(2) on ω is simply that of the phase factor $e^{i\sigma M\omega}$, where $\lambda_{\overline{1}} - \lambda_a = \pm M$, and M is the Toller quantum number of the pole.

It is clear that an analogous result holds in the limit $t_2 \to 0$, $s_{23} \to \infty$ for a Lorentz pole in the J plane of particle pair $(3 + \overline{b})$. We may also consider the more general case of a sum of poles in the J plane of, say, the particle pair $(\overline{b} + 3)$. Then, the analysis above allows us to conclude that each Lorentz pole has a simple phase dependence on ω in the limit $t_2 \to 0$ and $s_{23} \to \infty$, and if there are non-dominating poles with different Toller quantum numbers, the non-leading behavior has a more complicated ω-dependence. In the case in which both "3" and "b" have zero spin, $M = 0$ for any Lorentz pole that couples to $(\overline{b} + 3)$ and then it is only the contribution of cuts and background terms in III.(10) that have a more complicated ω-

dependence. The contribution of the sum of pole terms to the amplitude is in that case independent of ω .

3. Polarization of the Particle at the Central Vertex at Zero Momentum Transfer

Consider the case of double pole dominance of the reaction

$$a + b \to 1 + 2 + 3 \tag{5}$$

in the limit $s_{12}, s_{23} \to \infty$, $t_1, t_2 \to 0$. That is, consider the case of reaction (5) in which the leading term has a pole in the J plane of the $(3 + \bar{b})$ system, and also a pole in the J plane of the $(\bar{1} + a)$ system. Then a pseudo-helicity amplitude $F_{(\delta_3 \delta_{\bar{b}})\lambda_2 ; \lambda_{\bar{1}} \lambda_a}$ for the channel reaction

$$\bar{1} + a \to \bar{b} + 2 + 3 \tag{6}$$

will receive a non-vanishing contribution in the above limit from the leading double pole term only if

$$\lambda_{\bar{1}} - \lambda_a = \sigma_1 M_1 \tag{7}$$

and

$$\delta_3 - \delta_{\bar{b}} = \sigma_2 M_2 \tag{8}$$

where M_1, M_2 are the Toller quantum numbers of the two poles, and σ_1, σ_2 may assume the values ± 1. The first condition, Eq. (7), follows from the discussion in Sec. 2, and the second condition, Eq. (8), follows from the corresponding discussion for the reaction

$$\bar{1} + a + 2 \to \bar{b} + 3 \tag{9}$$

and the crossing relation, Eq. IV.(23).

The angles β_3, $\beta_{\bar{b}}$ that appear in Eq. II.(11) which relate the helicity and the pseudo-helicity amplitudes for (6) are computed in Appendix B.2, and in the limit considered here, tend to 0 or π according as $m_b \geqq m_3$, or $m_3 > m_b$.

Thus, the leading double pole term contributes to a helicity amplitude $F_{\lambda_3 \lambda_{\bar{b}} \lambda_2 ; \lambda_{\bar{1}} \lambda_a}$ for (6) if

$$\lambda_3 = \lambda_b + \sigma_2' M_2$$

and (10)

$$\lambda_{\bar{1}} = \lambda_a + \sigma_1 M_1$$

($\sigma_2' = \pm\sigma_2$ with the + sign used, if $m_3 \geqq m_b$)

Since all the momenta are collinear in this limit (Appendix D), angular momentum conservation states that

$$\sigma_3(\lambda_3 - \lambda_{\bar{b}}) + \sigma_3\sigma_4\lambda_2 = \lambda_{\bar{1}} - \lambda_a \tag{11}$$

so that

$$\lambda_2 = \sigma_3\sigma_4(\sigma_1 M_1 - \sigma_2'\sigma_3 M_2) \tag{12}$$

where $\sigma_4 = \mathrm{sign}(m_2^2 - t_2)$, $\sigma_3 = -1$ if $m_1 = m_a$ and $\sigma_3 = \mathrm{sign}(m_a^2 - m_1^2)$ if $m_1 \neq m_a$. Thus, the possible values of $|\lambda_2|$ are $|M_1 \pm M_2|$. In the special case of both Lorentz poles being single Regge poles, i.e., $M_1 = M_2 = 0$, this gives the unique value $\lambda_2 = 0$.

If the particles "3" and "b" are spinless, one may obtain the polarization under a less stringest limit, viz., $s_{12} \to \infty$, $t_1 \to 0$, t_2 bounded. For then (11) reads

$$\sigma_3\sigma_4\lambda_2 = \lambda_{\overline{1}} - \lambda_a \tag{13}$$

Equation (11) may be used since the particle momenta in the t_1 channel (6) still tend to a collinear limit if, as discussed in Appendix D, $z_s = 1 + 0\,(m_1^8/s^4)$ as $s \to \infty$, where z_s is defined below Eq. (4).

Furthermore, from Eq. (7)

$$\lambda_2 = \pm M_1 \tag{14}$$

which gives the polarization. Equation IV.(12) may be then used to obtain the relative phase of the $\lambda_2 = +M_1$ and $\lambda_2 = -M_1$ amplitudes.

<u>Application: Comparison with experiment for the reaction $\pi^- p \to \pi^- \rho^o p$</u>

Consider the reaction

$$\pi_i^- \, p_i \to \pi_f^- \, \rho^o p_f \tag{15}$$

in the kinematical region of large $S_{\rho p_f} = (\rho + p_f)^2$ and vanishingly small $t_p = (p_f - p_i)^2$. (Here we use the particle name to denote the 4-momentum of the particle in a convenient z.m.s. for the reaction.)

The spin component of the ρ-meson in its rest frame along the direction [54] $\hat{n} = (\vec{p}_f^{\,\prime} - \vec{p}_i^{\,\prime})/|\vec{p}_f^{\,\prime} - \vec{p}_i^{\,\prime}|$ taken as the z axis is $\pm M$ where M is the Toller quantum number of the leading trajectory in the angular momentum plane of the $(\overline{p}_i + p_f)$ system, and $\vec{p}_i^{\,\prime}$, $\vec{p}_f^{\,\prime}$ are the momenta of the initial and final proton, respectively, in this reference system. Since the leading trajectory in $p - p$ elastic scattering is the vacuum (Pomeron) trajectory, and this is also the leading trajectory in $\pi^{\pm} - p$ elastic scattering [55], $M = 0$ if the Pomeron is a

Lorentz pole, since a Lorentz pole that occurs in the J plane of a pair of spinless particles necessarily has Toller $M = 0$.

We test $M = 0$ for the Pomeron by examining approximately 2000 events [56] for the following reaction, at the incident π^- laboratory energy of 25 Gev:

$$\pi_i^- \, p \rightarrow \pi_1^- \, \pi^+ \, \pi_2^- \, p_f \tag{16}$$

We scan the events subject to the following criterion to obtain ρ-meson events:

(A) Either $0.49\ (\mathrm{Gev})^2 < S_{\pi_1^-\pi^+} \quad 0.68\ (\mathrm{Gev})^2$

or $0.49\ (\mathrm{Gev})^2 < S_{\pi_2^-\pi^+} < 0.68\ (\mathrm{Gev})^2$

but not both.

Denoting by π_f^- the final state negative pion that does not form a state with π^+ that is in the ρ band, and denoting the other negative pion as π_ρ^- , the following events are excluded:

(B)

$$1.42\ (\mathrm{Gev})^2 < S_{\pi_f^-\pi^+} < 1.72\ (\mathrm{Gev})^2$$

$$2.53\ (\mathrm{Gev})^2 < S_{\pi_f^-\pi^+} < 2.92\ (\mathrm{Gev})^2$$

which eliminates possible f^0 and π_N^0 (1650) events. Furthermore, events which lie in any of the following kinematical regions are also excluded:

(C)

$$1.03\ (\mathrm{Gev})^2 < S_{3\pi} < 1.23\ (\mathrm{Gev})^2$$

$$1.58\ (\mathrm{Gev})^2 < S_{3\pi} < 1.64\ (\mathrm{Gev})^2$$

$$1.69\ (\mathrm{Gev})^2 < S_{3\pi} < 1.77\ (\mathrm{Gev})^2$$

$$2.55\ (\mathrm{Gev})^2 < S_{3\pi} < 2.91\ (\mathrm{Gev})^2$$

and these criteria eliminate possible A1, $A2_H$, $A2_L$ and π_A (1640) events.

To eliminate interference from resonance formation in the πN and $\pi\pi N$ systems, we further exclude events lying in the following regions:

(D)
$$S_{\pi^+\pi^-_\rho p} < 7.0\ (\text{Gev})^2$$
$$S_{\pi^+ p} < 3.0\ (\text{Gev})^2$$
$$S_{\pi^-_\rho p} \text{ or } S_{\pi^-_f p} < 3.0\ (\text{Gev})^2$$

The first condition in (D) is the high energy condition for the (ρ+p) system.

The last condition to be imposed is that of the smallness of the proton momentum transfer, and the boundedness of the pion momentum transfer:

(E)
$$|t_p| < .067\ (\text{Gev})^2$$
$$|t_\pi| = |(\pi^-_f - \pi^-_i)|^2 < 1.5\ (\text{Gev})^2$$

After all these cuts, there are 109 remaining events.

These cuts eliminate interference from the formation of resonances, but do not eliminate interference from non-resonant ($N+3\pi$) background. Thus one would expect the sample of events to arise primarily from the intermediate states ($\pi^-_f + \rho + p$) and ($\pi^-_f + (\pi^-_\rho + \pi^+) + p$) where ($\pi^-_\rho + \pi^+$) has an invariant mass in the ρ region. Now, recent phase shift analyses [57] indicate that π-π scattering in the ρ band is dominated by S and P waves, with negligible contributions from D and higher waves. Using this result, the events consist primarily of intermediate states $(\pi_f^- + \rho^0 + p)$ and $(\pi_f^- + (\pi_\rho^- + \pi^+) + p)$ with $(\pi_\rho^- + \pi^+)$ in an S wave in the energy region of the ρ .

These intermediate states may interfere. Let α, β, γ, δ be the amplitudes respectively for $\pi_\rho^- \pi^+$ in an S wave, in a P wave with $J_z = 0$, in a P wave with $J_z = +1$, and in a P wave with $J_z = -1$. The distribution as a function of $\cos\theta$, where θ = the angle that the momentum of the positive pion makes with the z axis is then given by

$$N(\theta) = N(|\alpha|^2 + |\beta|^2\cos^2\theta + 2\,\mathrm{Re}\,\alpha\,\beta\,\cos\theta + (|\gamma|^2 + |\delta|^2)\sin^2\theta) \qquad (17)$$

where N is a constant of proportionality. There is no interference between the $J_z = \pm 1$ states with either the S wave or the $J_z = 0$ P wave state, because the interference terms sum to zero upon integration over the azimuthal angle.

The 109 events satisfying (A) and (E) and not (B), (C), or (D) are plotted as a histogram in the variable $\cos\theta$ in Fig. 4. The asymmetry between the number of events with positive and negative values of $\cos\theta$ is a clear indication that not all the events involve ρ-meson production.

The theory of this section states that if $M = 0$, then $\gamma = \delta = 0$, so that a two-parameter fit to the shape of the distribution is suggested

$$N_A(\theta) = N'(1 + a^2\cos^2\theta + b\cos\theta) \qquad (18)$$

where N' is a normalization constant chosen so that the area under the curve is equal to the total number of events. A least squares fit

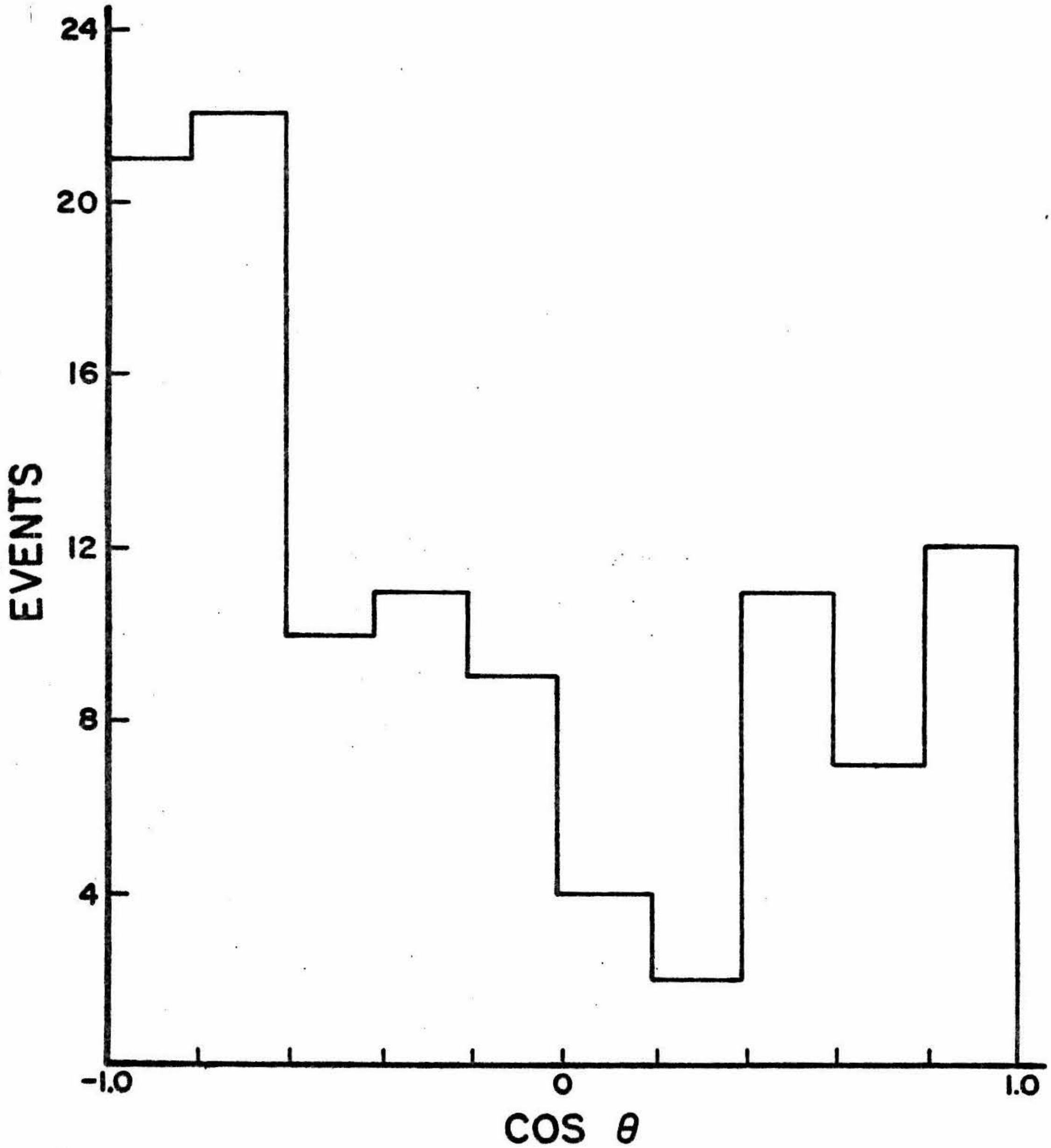


Fig. 4. The distribution of π^+ momenta in the rest frame of the $(\pi^+ + \pi^-)$ system relative to the axis $(\vec{p}_f' - \vec{p}_i') / |\vec{p}_f' - \vec{p}_i'|$ where $\vec{p}_i', \vec{p}_f'$ are the momenta of the initial and final proton in this reference system.

with

$$N' = 4.63$$

$$a = -1.57$$

$$b = 3.51$$

consistent with the condition $|b| \leq 1 + a^2$, necessary for a fit with S and P wave interference only, is drawn as Curve A in Fig. 5. Also shown in Fig. 5 as a broken curve is $N_B(\theta) = 16.3 \sin^2\theta$ which is similarly normalized. If there were $J_z = \pm 1$ P wave states, one would have the resulting curve $N(\theta) = \alpha N_B(\theta) + (1-\alpha) N_A(\theta)$, and it is clear from Fig. 5 that the best fit to the data suggests that α is small, and is consistent with 0 , thus indicating that $M = 0$ for the dominant part of the vacuum singularity.

4. Minima ("Dips") in Momentum Transfer Distributions: Examples

a. Zero Momentum Transfer

Consider the reaction

$$\pi_i N_i \rightarrow \pi_f X N_f \tag{19}$$

where X is a non-strange pseudo-scalar meson (π, η, or η') . Let us introduce the variables t_1, t_2, s_{12}, s_{23}, s by means of the identifications:

$$\begin{aligned} N_i &\leftrightarrow a \\ \pi_i &\leftrightarrow b \\ N_f &\leftrightarrow 1 \\ X &\leftrightarrow 2 \\ \pi_f &\leftrightarrow 3 \end{aligned} \tag{20}$$

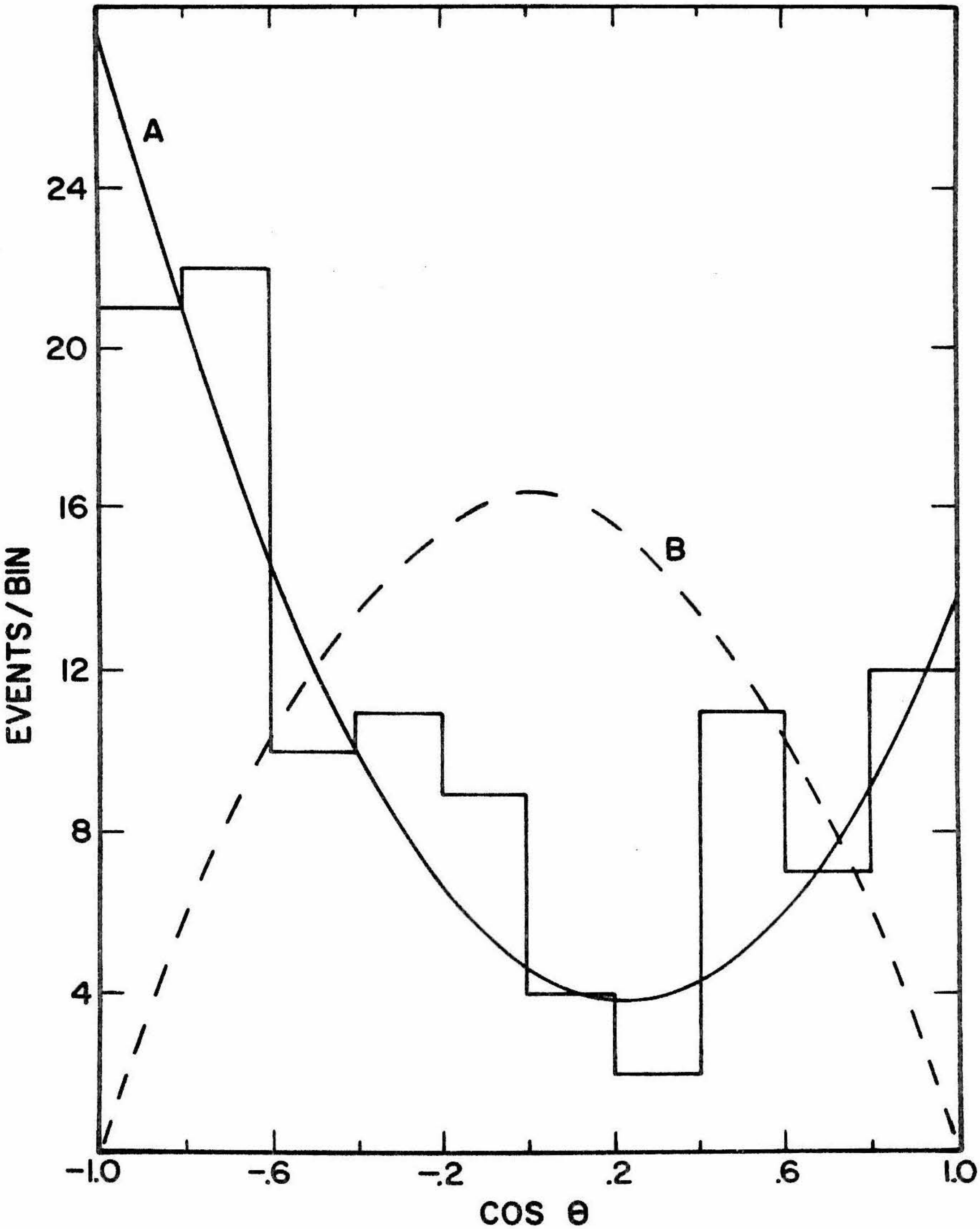


Fig. 5. The data of Fig. 4 with two curves: (A) $\lambda_2 = 0$ + S wave; (B) non-interfering $\lambda_2 = \pm 1$ background.

We will now show that in the limit of large s_{12} and bounded t_2 one would expect a dip in the cross section for (19) as $t_1 \to 0$. Since $\lambda_2 = 0$, it follows from Eq. (13) that $\lambda_{\bar{I}} = \lambda_a$ in this limit for the contribution of a pole in the J plane of the $(\bar{I} + a)$ system to the amplitude for the process (19) to be nonvanishing. Thus only the term $M' = \lambda_2 = 0$ survives in the helicity series, Eq. IV.(7), for the contribution of a pole in the $(\bar{I} + a)$ system, since $M = |\lambda_{\bar{I}} - \lambda_a| = 0$, and from the discussion in Sec. 2, $M' = \lambda_2 \mp M = 0$.

From parity invariance, Eq. IV.(12), one has the result

$$\beta_{0(0,0)0}^{(t_1,t_2,s_{23})} = -P_\alpha \, \beta_{0(0,0)0}^{(t_1,t_2,s_{23})} \tag{21}$$

where P_α is the natural parity of the pole in the J plane of the $(\bar{N}_f + N_i)$ system.

Thus, only odd natural parity trajectories in the $N \bar{N}$ channel can contribute at $t_1 \equiv t_N = 0$. Since the odd natural parity trajectories lie lower in the J plane than the even natural parity ones, one would expect a minimum in the cross-section at $t_N = 0$ in the limit $s_{12} \equiv s_{XN} \to \infty$, since trajectories with even natural parity can contribute away from $t_N = 0$.

b. A Charge Exchange Reaction at a Nonsense, Wrong Signature Point

Consider the reaction (19) where the nucleon undergoes a change of charge. That is, consider

$$\pi_i \, p \to \pi_f \, Xn \tag{22}$$

Now the highest lying known trajectories that can occur in the J plane of the $(\bar{p}\,n)$ system are the ρ and the A2, which pass through the value 0 at approximately the same point $t_o = \alpha_\rho^{-1}(0) \simeq \alpha_{A2}^{-1}(0)$. If X has G parity + 1, the ρ can contribute, but A2 cannot, whereas if X has G parity - 1, the A2 can contribute, but the ρ cannot. Consider the case of positive G parity, i.e., $X = \eta$ or η'.

Then, as t_1 passes through t_o which is a nonsense point of wrong signature for the ρ trajectory, one would expect the ρ pole terms in Eq. III.(10) with $M' \neq \lambda_2$ to give no contribution of the form z^α, in the absence of Mandelstam-Wang fixed singularities [34]. Even in the presence of these singularities, because of the approximate degeneracy of the couplings of the ρ and A2 trajectories to the $\bar{N}-N$ system [58], one would expect, following Finkelstein [58], that the terms in Eq. III.(10) of IV.(7) with $M' \neq \lambda_2$ will give a negligible contribution. Since $\lambda_2 = 0$, $M' = 0$ also, and the only possible non-vanishing coupling of the ρ trajectory occurs through $\beta_{0(0,0)0}^{(t_1,t_2,s_{23})}$ at $t_1 = t_o$. Eq. (21), however, implies that $\beta_{0(0,0)0} = 0$ for the ρ trajectory, since it has even natural parity. Thus, the contribution of the ρ trajectory is expected to vanish at $t_1 = t_o$, resulting in a minimum for the cross section for (22), with $X = \eta$ or η'. There are no data as yet to confirm or contradict this prediction of a minimum in the differential cross-section.

APPENDIX A. Some Properties of the Rotation Functions

The results of this section of this appendix are not new, but are included here for ease of reference. The same comment applies to most of the results in Appendices B.1 and C.

1. The rotation functions of the first kind

$d^J_{\lambda\mu}(\theta)$ are defined in terms of the Jacobi polynomials [59] by means of the following formula [60] ($J - \lambda$, $J - \mu$, and $2J$ are integers), for $\lambda \geqq |\mu|$:

$$d^J_{\lambda\mu}(\cos\theta) = \frac{N(J,\lambda)}{N(J,\mu)} (\cos\tfrac{\theta}{2})^{\lambda+\mu} (\sin\theta/_2)^{\lambda-\mu} P^{(\lambda-\mu,\lambda+\mu)}_{J-\lambda}(\cos\theta) \tag{A.1}$$

where $N(J,\lambda) = \{\Gamma(J - \lambda + 1)\ \Gamma(J + \lambda + 1)\}^{1/2}$ (A.2)

The other cases are given by the symmetry conditions [60]:

$$d^J_{\lambda\mu}(\theta) = (-1)^{\lambda-\mu} d^J_{\mu\lambda}(\theta) = d^J_{-\mu,-\lambda}(\theta) = (-1)^{\lambda-\mu} d^J_{-\lambda,-\mu}(\theta) \tag{A.3}$$

These formulas may be combined to yield

$$d^J_{\lambda\mu}(\theta) = \text{sign}(\lambda,\mu) \frac{N(J,\lambda_{max})}{N(J,\lambda_{min})} (\cos\theta/_2)^{|\lambda+\mu|} (\sin\tfrac{\theta}{2})^{|\lambda-\mu|} P^{(|\lambda-\mu|,|\lambda+\mu|)}_{J-\lambda_{max}}(\cos\theta) \tag{A.4}$$

$$\lambda^{max}_{min} = {}^{max}_{min}\{|\lambda|, |\mu|\}$$

where $\text{sign}(\lambda,\mu) = \begin{cases} +1, & \text{if } \lambda \geq |\mu|, \text{ or } -\mu \geq |\lambda| \\ (-1)^{\lambda-\mu} & \text{otherwise} \end{cases}$

$$= (-1)^{\frac{(\lambda-\mu)-|\lambda-\mu|}{2}}$$

Also, $d^J_{\lambda,-\mu}(-z) = (-1)^{J-\lambda} d^J_{\lambda\mu}(z)$ (A.5)

This leads to the following orthogonality relation for the $d^J_{\lambda\mu}(z)$ when $J-\lambda$, $2J$, $J-\mu$ are all integers:

$$\int_{-1}^{1} dz \; d^J_{\lambda\mu}(z)\, d^{J'}_{\lambda\mu}(z) = \frac{2\,\delta_{JJ'}}{2J+1} \tag{A.6}$$

Equation (A.4) may be continued to complex J and complex $z = \cos\theta$ by using the representation of the Jacobi polynomial in terms of the hypergeometric function [61]

$$P_n^{(\alpha,\beta)}(z) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\,\Gamma(\alpha+1)} \left(\frac{z+1}{2}\right)^n F\left(-n, -n-\beta\,; \alpha+1, \frac{z-1}{z+1}\right) \tag{A.7}$$

Considered as a function of J and z, the $d^J_{\lambda\mu}(z)$ may then be written

$$d^J_{\lambda\mu}(z) = \operatorname{sign}(\lambda,\mu)\, \frac{N(J,\lambda_{max})}{N(J,\lambda_{min})}\, \frac{\Gamma(J-\lambda_{max}+|\lambda-\mu|+1)}{\Gamma(J-\lambda_{max})\,\Gamma(1+|\lambda-\mu|)}$$

$$\times \left(\frac{1+z}{2}\right)^{J-\lambda_{max}+\frac{|\lambda+\mu|}{2}} \left(\frac{1-z}{2}\right)^{\frac{|\lambda-\mu|}{2}} F\left(-J+\lambda_{max}, -J+\lambda_{max}-|\lambda-\mu|, 1+|\lambda-\mu|, \frac{z-1}{z+1}\right) \tag{A.8}$$

Thus, $d^J_{\lambda\mu}(z)$ has a branch cut extending from $-\infty$ to -1, due to that in the hypergeometric function, as well as a possible branch cut from $-\infty$ to -1 due to the factor $\left(\frac{1-z}{2}\right)^{\frac{|\lambda-\mu|}{2}}$ and a branch cut from $+1$ to $+\infty$ due to the factor $\left(\frac{1+z}{2}\right)^{J-\lambda_{max}+\frac{|\lambda+\mu|}{2}}$. As a function of J, $d^J_{\lambda\mu}(z)$ is seen to have square root branch points at "sense-nonsense" values of J. These branch points at $J = J_o$ are of the $\sqrt{J-J_o}$ type.

From (A.8), one can determine the asymptotic behavior of $d^J_{\lambda\mu}(z)$ for large $|z|$. The use of Eq. I.2.8(26) of Reference [26] results in

$$F\left(-J+\lambda_{max}, -J+\lambda_{max}-|\lambda+\mu|, 1+|\lambda-\mu|, \frac{z-1}{z+1}\right)$$

$$\longrightarrow \frac{\Gamma(1+|\lambda-\mu|)\Gamma(2J+1)}{\Gamma(1+J+|\lambda-\mu|-\lambda_{max})\Gamma(J+1+\lambda_{max})} \quad \text{as } |z|\to\infty$$

Defining the principal sheet of $d^J_{\lambda\mu}(z)$ so that $-\pi \leq \arg z \leq \pi$, $1 - z = e^{-i\eta\pi}(z-1)$, where $\eta = \text{sign}\ (\text{Im}\ z)$. Then

$$d^J_{\lambda\mu}(z) \simeq \frac{\text{sign}(\lambda,\mu)\Gamma(2J+1)}{N(J,\lambda_{max})\ N(J,\lambda_{min})} e^{-i\eta\frac{\pi}{2}|\lambda-\mu|} \left(\frac{z}{2}\right)^J$$

$$\text{as } |z|\to\infty$$

Now sign $(\lambda,\mu)\, e^{-i\eta\frac{\pi}{2}|\lambda-\mu|} = e^{-i\eta\frac{\pi}{2}(\lambda-\mu)}$

so that

$$d^J_{\lambda\mu}(z) \simeq \frac{e^{-i\eta\frac{\pi}{2}(\lambda-\mu)}\Gamma(2J+1)}{N(J,\lambda_{max})\ N(J,\lambda_{min})} \left(\frac{z}{2}\right)^J \tag{A.9}$$

a result which agrees with that obtained by Kibble [2] and which disagrees with that obtained by Omnes and Alessandrini [62].

2. The rotation functions of the second kind

These are defined in Eq. (3.1) of Reference [27] by the relation:

$$e^J_{\lambda\mu}(z) = \frac{\pi}{2\sin\pi(J-\lambda)} \left\{e^{-i\eta\pi(J-\lambda)} d^J_{\lambda\mu}(z) - d^J_{\lambda,-\mu}(-z)\right\} \tag{A.10}$$

They are related to Jacobi functions of the second kind by Eq. (A.9) of Reference [32]:

$$e^J_{\lambda\mu}(z) = (-1)^{\lambda-\mu}\text{sign}(\lambda,\mu)\frac{N(J,\lambda_{max})}{N(J,\lambda_{min})}\left(\frac{1+z}{2}\right)^{\frac{|\lambda+\mu|}{2}}\left(\frac{1-z}{2}\right)^{\frac{|\lambda-\mu|}{2}} Q^{|\lambda-\mu|,|\lambda+\mu|}_{J-\lambda_{max}}(z) \tag{A.11}$$

There are also the following useful relations [27] with functions of the first kind:

$$e^{J}_{\lambda\mu}(z) - e^{-J-1}_{-\lambda,-\mu}(z) = \pi \cot \pi(J-\lambda)\, d^{J}_{\lambda\mu}(z) \tag{A.12}$$

and for $-1 < z < +1$

$$e^{J}_{\lambda\mu}(z+i0) - e^{J}_{\lambda\mu}(z-i0) = -i\pi\, d^{J}_{\lambda\mu}(z) \tag{A.13}$$

3. Asymptotic behavior

For large $|\mu|$ [63]

$$|d^{J}_{\lambda\mu}(z)| \simeq \text{const.} \times |\mu|^{\lambda-1/2} \left|\frac{1-z}{1+z}\right|^{|\mu|/2}$$
$$|\arg(1+z)| < \pi-\varepsilon\,, \; |\mathrm{Re}\,\mu| \to \infty \tag{A.14}$$

and

$$|e^{J}_{\lambda\mu}(z)| \simeq \max.\left\{ |\mu|^{\lambda-1/2} \left|\frac{1-z}{1+z}\right|^{\mu/2},\; |\mu|^{-\lambda} \left|\frac{1+z}{1-z}\right|^{\mu/2} \right\} \tag{A.15}$$

For large $|z|$ [27]

$$e^{J}_{\lambda\mu}(z) \simeq \frac{1}{2}\, \frac{N(J,\lambda)\, N(J,\mu)\, e^{i\eta\frac{\pi}{2}(\lambda-\mu)}}{\Gamma(2J+2)} \left(\frac{z}{2}\right)^{-J-1} \tag{A.16}$$

$d^{J}_{\lambda\mu}(z)$ for large (z) is given by Eq. (A.9).

For large $|J|$ [27]

$$e^{J}_{\lambda\mu}(z) \simeq \left(\frac{\pi}{2}\right)^{1/2} \frac{e^{i\eta\frac{\pi}{2}(\lambda-\mu)}\left[z-(z^2-1)^{1/2}\right]^{J+1/2}}{\sqrt{J}\;\;(z^2-1)^{1/4}}$$
$$-\pi+\varepsilon < \arg J < \pi-\varepsilon \tag{A.17}$$

where the last term is cut along the negative z axis only and taken positive for $z > 0$.

$d^J_{\lambda\mu}(z)$ for large $|J|$ is given by (12) and (17) [64]:

$$\frac{d^J_{\lambda\mu}(z)}{\sin\pi(J-\lambda)} \underset{\substack{|J|\to\infty \\ 0<|\arg J|<\pi}}{\sim} \left(\frac{2}{\pi}\right)^{1/2} \frac{1}{(z^2-1)^{1/4}} \exp\left(\frac{-\pi|\mathrm{Im}J| + |(J+\frac{1}{2})\xi|}{\sqrt{J}}\right), \quad \xi = \log\left(z+(z^2-1)^{1/2}\right) \tag{A.18}$$

APPENDIX B. GENERAL KINEMATICS AND DERIVATIONS OMITTED IN THE TEXT

1. Choice of variables for the $2 \to 3$ process

As in the text, we designate the direct channel as

$$a + b \to 1 + 2 + 3 \tag{B.1}$$

and the crossed channel as

$$\bar{1} + a \to \bar{b} + 2 + 3 \tag{B.2}$$

Then the two sets of variables used in the text are

$$\{s, s_{12}, s_{23}, t_1, t_2\} \qquad \text{(A)}$$

defined in Eq. II.(3) and the variables

$$\{z, z', \phi\ ; t_1, t_2\} \qquad \text{(B)}$$

defined in Chapter II in relation to reaction (B.2).

z is the cosine of the angle between $\vec{p}_{\bar{1}}$ and $\vec{p}_2$ in the z.m.s. for (B.2). So that, evaluating $s_{12} = (p_{\bar{1}} - p_2)^2$, one obtains

$$z = \frac{s_{12} - m_1^2 - m_2^2 + 2E_{\bar{1}}E_2}{2p_{\bar{1}}p_2} \tag{B.3}$$

where

$$p_{\bar{1}} = \frac{\Delta^{1/2}(t_1, m_1^2, m_a^2)}{2\sqrt{t_1}} \qquad E_{\bar{1}} = (t_1 + m_1^2 - m_a^2)/2\sqrt{t_1}$$

$$p_2 = \frac{\Delta^{1/2}(t_1, m_2^2, t_2)}{2\sqrt{t_1}} \qquad E_2 = (t_1 + m_2^2 - t_2)/2\sqrt{t_1} \tag{B.4}$$

In order to continue z to the physical region of (B.1) from the physical region of (B.2), we must use a definite continuation of $\Delta^{1/2}(t_1,m_1^2,m_a^2)$, $\Delta^{1/2}(t_1,t_2,m_2)$ and $\sqrt{t_1}$. Now

$$\Delta^{1/2}(x,y,z) = [x^2+y^2+z^2-2xy-2yz-2xz]^{1/2}$$
$$= [(x-(\sqrt{y}+\sqrt{z})^2)(x-(\sqrt{y}-\sqrt{z})^2)]^{1/2} \quad \text{for } y,z > 0 \qquad \text{(B.5)}$$

The form (B.5) indicates that as a function of x for positive y and z, $\Delta^{1/2}(x,y,z)$ has two branch points at $x = (\sqrt{y} \pm \sqrt{z})^2$. We define $\Delta^{1/2}(x,y,z)$ to be positive for $x > (\sqrt{y} + \sqrt{z})$, and draw the branch cuts along the real x axis from $(\sqrt{y} + \sqrt{z})^2$ to $+\infty$ and from $(\sqrt{y} - \sqrt{z})^2$ to $-\infty$. Then for x real and $< (\sqrt{y} - \sqrt{z})^2$, $\Delta^{1/2}(x,y,z)$ is real and negative.

Now we impose the following symmetry properties for the continuation:

$$\Delta^{1/2}(x,y,z) = \Delta^{1/2}(-x,-y,-z) \qquad \text{(B.6)}$$

$$\Delta^{1/2}(x,y,z) = \Delta^{1/2}(Px,Py,Pz) \qquad \text{(B.7)}$$

where (Px,Py,Pz) is any permutation of (x,y,z) when two of the variables are real and positive and the third is complex. Consider the case in which y and z are both negative. Then (B.6) provides the continuation of the function to complex x. In fact, (B.6) and (B.7) give the continuation of the function to arbitrary real values of (x,y,z) and these comprise the cases of interest. To complete the definition of the continuation of the momenta, we define $\sqrt{t_i}$ to be positive for $t_i > 0$, $\text{Im } t_i > 0$ and to possess a branch cut extending from 0 to ∞.

Now we proceed to the definition and continuation of z' and $\cos \phi$. z' is the negative of the cosine of the angle between $\vec{p}_3$ and $\vec{p}_2$ in the z.m.s. of $(3 + \bar{b})$ for the reaction (B.2). This is the same as the cosine of the angle between $-\vec{p}_3$ and $\vec{p}_{\bar{2}}$ in the z.m.s. of $(3 + \bar{b})$ in the crossed reaction:

$$3 + \bar{b} \rightarrow \bar{1} + a + \bar{2} \tag{B.8}$$

for the momentum $\vec{p}_{\bar{2}}$ in the z.m.s. of (B.8) is in the same direction as $\vec{p}_2$ in (B.2) in the z.m.s. of $(3 + \bar{b})$. This may most easily be seen by crossing particle "2" in its rest frame to relate the reactions (B.2) and (B.8). We obtain z' by evaluating $s_{23} = (p_{\bar{2}} - p_3)^2$ for this reaction in its z.m.s. Then,

$$z' = -\frac{s_{23} - m_2^2 - m_3^2 + 2E_{\bar{2}}' E_3'}{2 p_{\bar{2}}' p_3'} \tag{B.9}$$

where

$$p_{\bar{2}}' = \frac{\Delta^{1/2}(t_2, t_1, m_2^2)}{2\sqrt{t_2}} \qquad E_{\bar{2}}' = \frac{t_2 + m_2^2 - t_1}{2\sqrt{t_2}}$$

$$p_3' = \frac{\Delta^{1/2}(t_2, m_3^2, m_b^2)}{2\sqrt{t_2}} \qquad E_3 = \frac{t_2 + m_3^2 - m_b^2}{2\sqrt{t_2}} \tag{B.10}$$

To obtain $\cos \phi$ we note that $(\pi - \theta, \pi + \phi)$ are the polar angles of $\vec{p}_{\bar{1}}$ in the z.m.s. of $(\bar{1} + a)$ for reaction (B.2) with the coordinate system defined in Chapter II of the text (i.e., positive z axis along $-\vec{p}_2$, and $\vec{p}_3$ in the x-z plane with positive x component).

Setting $z_c' = \hat{p}_2 \cdot \hat{p}_{\bar{b}}$ in this coordinate system, one has

$$t_{1b} = (p_{\bar{b}} - p_{\bar{1}})^2 = m_1^2 + m_b^2 - 2E_{\bar{1}} E_{\bar{b}} + 2 p_{\bar{1}} p_{\bar{b}} \left(\sqrt{1-z^2}\sqrt{1-z_c'^2} \cos\phi + z z_c' \right)$$

This equation determines $\cos \phi$ in terms of the variables (A) for

$$t_{1b} = m_1^2 + m_b^2 + m_a^2 - t_1 + s_{23} - s$$

and
$$p_T = \Delta^{1/2}(t_1, m_1^2, m_a^2)/2\sqrt{t_1} \qquad E_T = (t_1 + m_1^2 - m_a^2)/2\sqrt{t_1}$$

$$p_{\bar b} = \Delta^{1/2}(t_1, m_b^2, s_{23})/2\sqrt{t_1} \qquad E_{\bar b} = (t_1 + m_b^2 - s_{23})/2\sqrt{t_1}$$

z is given by (B.3) and (B.4) so that

$$z = \frac{2t_1(s_{12} - m_1^2 - m_2^2) + (t_1 + m_1^2 - m_a^2)(t_1 + m_2^2 - t_2)}{\Delta^{1/2}(t_1, t_2, m_2^2)\ \Delta^{1/2}(t_1, m_1^2, m_a^2)} \tag{B.11}$$

and
$$z_c' = \hat{p}_2 \cdot \hat{p}_{\bar b} = \frac{-t_{2b} + m_2^2 + m_b^2 + 2E_2 E_{\bar b}}{2\, p_2 p_{\bar b}}$$

or
$$z_c' = \frac{2t_1(-t_1 - m_3^2 + t_2 + s_{23}) + (t_1 + m_b^2 - s_{23})(t_1 + m_2^2 - t_2)}{\Delta^{1/2}(t_1, t_2, m_2^2)\ \Delta^{1/2}(t_1, m_b^2, s_{23})} \tag{B.12}$$

and the equation for $\cos\phi$ is

$$m_a^2 - t_1 + s_{23} - s = -\frac{1}{2t_1}(t_1 + m_1^2 - m_a^2)(t_1 + m_b^2 - s_{23}) + \frac{1}{2t_1}\Delta^{1/2}(t_1, m_1^2, m_a^2)\Delta^{1/2}(t_1, m_b^2, s_{23})K \tag{B.13}$$

where $K = \sqrt{1 - z^2}\,\sqrt{1 - z_c'^2}\,\cos\phi + zz_c'$.

B.2 Computation of the Cosines of the Angles that Appear in Eq. II.(11)

β_3 = the angle between the vectors $(\vec{p}_2 + \vec{p}_{\bar b})$ and $\vec{p}_{\bar b}$ in the rest system of particle 3.

Take $\vec{p}_{\bar{b}}$ in the z direction. Then in the z.m.s. of $(3 + \bar{b})$, $\hat{p}_2 \cdot \hat{p}_3 = -z' = -\hat{p}_2 \cdot \hat{p}_{\bar{b}}$ so that the z component of $\vec{p}_2$ in this coordinate system

p_2'(z-component) $= p_2' z'$, where p_2' = momentum of "2" in the z.m.s. of $(3 + \bar{b})$

and $p_{\bar{b}}'$(z-component) $= p_{\bar{b}}' = \dfrac{\Delta^{1/2}(t_2, m_3^2, m_b^2)}{2\sqrt{t_2}}$, denoting particle momenta in this system by primes

The rest frame of particle 3 is obtained from the z.m.s. of $3 + \bar{b}$ by a velocity transformation along the z axis. Then, denoting the momenta in this coordinate system by the superscript "o", one has

$$p_2^o(\text{z-component}) = \gamma(p_2' z' + vE_2') \equiv p_2^o(z) \tag{B.14}$$

where $v = \dfrac{p_3'}{E_3'} = \dfrac{\Delta^{1/2}(t_2, m_3^2, m_b^2)}{t_2 + m_3^2 - m_b^2}$

and $\gamma = (1-v^2)^{-1/2} = \dfrac{t_2 + m_3^2 - m_b^2}{2 m_3 \sqrt{t_2}}$

Taking $\vec{p}_2$ in the x-z plane with positive x-component

$$p_2^o(\text{x-component}) = p_2' \sqrt{1 - z'^2}$$

and

$$\vec{p}_{\bar{b}}^{\,o} = p_b^o \hat{z} = \hat{z}\, \gamma(p_{\bar{b}} + vE_{\bar{b}}) \tag{B.15}$$

Then $\cos\beta_3 = \dfrac{1}{|\vec{p}_2^{\,o} + \vec{p}_b^{\,o}|}(p_b^o + p_2^o(z))$ (B.16)

To compute $p_2^o(z)$ we require p_2', E_2' . p_2' is the same as p_2 in (B.10)

$$p_2' = \frac{\Delta^{1/2}(t_1,t_2,m_2^2)}{2\sqrt{t_2}} \tag{B.17}$$

also
$$E_2' = -E_{\bar{2}} = \frac{t_1-m_2^2-t_2}{2\sqrt{t_2}} \tag{B.18}$$

Thus
$$p_2^o(z) = \frac{t_2+m_3^2-m_b^2}{2m_3\sqrt{t_2}}\left(\frac{z'\Delta^{1/2}(t_1,t_2,m_2^2)}{2\sqrt{t_2}} + \frac{\Delta^{1/2}(t_2,m_3^2,m_b^2)(t_1-m_2^2-t_2)}{(t_2+m_3^2-m_b^2)\ 2\sqrt{t_2}}\right)$$

$$= \frac{t_2+m_3^2-m_b^2}{4m_3\sqrt{t_2}}\left(z'\Delta^{1/2}(t_1,t_2,m_2^2) + \Delta^{1/2}(t_2,m_3^2,m_b^2)\frac{(t_1-m_2^2-t_2)}{(t_2+m_3^2-m_b^2)}\right)$$

and
$$p_{\bar{b}}^o = \frac{t_2+m_3^2-m_b^2}{2m_3\sqrt{t_2}}\left(\frac{\Delta^{1/2}(t_2,m_3^2,m_b^2)}{2\sqrt{t_2}} + \frac{\Delta^{1/2}(t_2,m_3^2,m_b^2)\ (t_2+m_b^2-m_3^2)}{2\sqrt{t_2}\ (t_2+m_3^2-m_b^2)}\right)$$

$$= \frac{\Delta^{1/2}(t_2,m_3^2,m_b^2)}{2m_3} \tag{B.19}$$

Now $E_2^o = \gamma(E_2' + v\, p_2'z')$

$$= \frac{t_2+m_3^2-m_b^2}{4m_3t_2}\left(t_1-m_2^2-t_2 + \frac{\Delta^{1/2}(t_2,m_3^2,m_b^2)\Delta^{1/2}(t_1,t_2,m_2^2)z'}{t_2+m_3^2-m_b^2}\right)$$

so that, using Eqs. (B.9) and (B.10) for z' this last equation becomes

$$E_2^o = \frac{t_2+m_3^2-m_b^2}{4m_3t_2}\left[t_1-m_2^2-t_2 + \frac{2t_2(s_{23}-m_2^2-m_3^2)}{t_2+m_3^2-m_b^2} - (t_1-t_2-m_2^2)\right]$$

$$= (s_{23}-m_2^2-m_3^2)/2m_3$$

and therefore
$$p_2^o = (E_2^{o2}-m_2^2)^{1/2} = \Delta^{1/2}(s_{23},m_2^2,m_3^2)/2m_3$$

thus,

$$\cos\beta_3 = \frac{\dfrac{\Delta^{1/2}(t_2,m_3^2,m_b^2)}{2m_3} + \dfrac{t_2+m_3^2-m_b^2}{4m_3t_2}\left(z'\Delta^{1/2}(t_1,t_2,m_2^2) + \Delta^{1/2}\dfrac{(t_2,m_3^2,m_b^2)(t_1-m_2^2-t_2)}{t_2+m_3^2-m_b^2}\right)}{\left\{\dfrac{\Delta(s_{23},m_2^2,m_3^2)}{4m_3^2} + \dfrac{\Delta(t_2,m_3^2,m_b^2)}{4m_3^2} + 2p_2^0(z)p_{\bar{b}}^0\right\}^{1/2}}$$

$$\frac{\Delta^{1/2}(t_2,m_3^2,m_b^2) + \dfrac{(t_2+m_3^2-m_b^2)(s_{23}-m_2^2-m_3^2)}{\Delta^{1/2}(t_2,m_3^2,m_b^2)} + \dfrac{2(t_2+m_2^2-t_1)m_3^2}{\Delta^{1/2}(t_2,m_3^2,m_b^2)}}{\left\{\Delta(s_{23},m_2^2,m_3^2) + \Delta(t_2,m_3^2,m_b^2) + 2(s_{23}-m_2^2-m_3^2)(t_2+m_3^2-m_b^2) + 4m_3^2(t_2+m_2^2-t_1)\right\}^{1/2}}$$

(B.20)

In the limit as $s_{23} \to \infty$ for fixed $t_1, t_2 \neq 0$,

$$\cos\beta_3 \longrightarrow \frac{t_2+m_3^2-m_b^2}{\Delta^{1/2}(t_2,m_3^2,m_b^2)} \tag{B.21}$$

Similarly, one may compute $\cos\beta_{\bar{b}}$ with the result that $\cos\beta_{\bar{b}}$ is given by an expression of a form similar to (B.20) arising from the replacements $m_b \to m_3$, $m_3 \to m_b$, $s_{23} \to t_{2b}$, $z' \to -z'$. Since for t_1, t_2 fixed, as $s_{23} \to \infty$, $t_{2b} \simeq -s_{23}$, in this limit $\cos\beta_{\bar{b}}$ is given by (B.21) with the exchange of the labels "3" and "b", and an overall minus sign.

B.3 A Lemma Concerning the Lehmann Ellipse

Consider the series of Jacobi polynomials:

$$F(z) = \sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(z) \tag{B.22}$$

and the associated power series

$$G(z) = \sum_{n=0}^{\infty} a_n z^n \tag{B.23}$$

We will show in this appendix that the power series (B.23) converges for $|z| < R, R > 1$ if and only if the series (B.22) of Jacobi polynomials converges within the ellipse

$$|z + 1| + |z - 1| = (R + \frac{1}{R}) \tag{B.24}$$

First, suppose that (B.23) converges for $|z| < R, R > 1$. Then from the ratio test

$$\overline{\lim_{n \to \infty}} \left|\frac{a_{n+1}}{a_n}\right| < \frac{1}{R} \tag{B.25}$$

Then, applying the ratio test to the series of Jacobi polynomials, one observes that the series is convergent if

$$\overline{\lim_{n \to \infty}} \left| \frac{P_{n+1}^{(\alpha,\beta)}(z)}{P_n^{(\alpha,\beta)}(z)} \right| = |z + (z^2-1)^{1/2}| < R \tag{B.26}$$

For $R > 1$, the closed curve

$$|z + (z^2 - 1)^{1/2}| = R \tag{B.27}$$

is the ellipse (B.24), and points that satisfy (B.26) are all interior to the ellipse.

That (B.27) and (B.24) represent the same ellipse may be seen as follows: (B.27) is equivalent to

$$|z - (z^2 - 1)^{1/2}| = \frac{1}{R} \tag{B.28}$$

Adding the two equations, squaring both sides, and simplifying, one obtains

$$|z|^2 + |z^2 - 1| = \frac{1}{2}(R^2 + \frac{1}{R^2}) \tag{B.29}$$

and this is the same equation that one obtains from squaring both sides of (B.24).

Conversely, suppose the series (B.22) converges within the ellipse (B.27), where $R > 1$. Then from the ratio test, it follows that (B.25) is valid, which in turn implies that the power series (B.23) converges for $|z| < R$.

B.4 Derivation of the Inequality Eq. II.(28)

We start from Eq. II.(27)

$$-t_1 + t_2 + m_a^2 + 2E_2 E_{\bar{1}} < -2p_2 p_{\bar{1}}$$

and insert the values II.(19) for E_2, $E_{\bar{1}}$, p_2, $p_{\bar{1}}$ to obtain

$$2t_1(-t_1 + t_2 + m_a^2) + (t_1 + m_2^2 - t_2)(t_1 + m_1^2 - m_a^2) + \Delta^{1/2}(t_1, m_1^2, m_a^2) \times \Delta^{1/2}(t_1, t_2, m_2^2) < 0$$

or

$$\Delta^{1/2}(t_1, m_1^2, m_a^2)\, \Delta^{1/2}(t_1, t_2, m_2^2) < t_1^2 - t_1 t_2 - t_1(m_1^2 + m_a^2 + m_2^2) + (t_2 - m_2^2)(m_1^2 - m_a^2) .$$

Squaring both sides

$$[t_1^2 - 2t_1(m_1^2 + m_a^2) + (m_1^2 - m_a^2)^2][t_1^2 - 2t_1(t_2 + m_2^2) + (t_2 - m_2^2)^2]$$

$$< t_1^4 - 2t_1^3(t_2 + m_1^2 + m_a^2 + m_2^2) + t_1^2(t_2 + m_1^2 + m_a^2 + m_2^2)^2$$

$$+ (t_2 - m_2^2)^2 (m_1^2 - m_a^2)^2 + 2(t_2 - m_2^2)(m_1^2 - m_a^2)[t_1^2 - t_1(t_2 + m_1^2 + m_a^2 + m_2^2)]$$

The terms in t_1^4, t_1^3 cancel, as does the constant term. This leaves

$$t_1^2[(t_2 - m_2^2)^2 + (m_1^2 - m_a^2)^2 + 4(t_2 + m_2^2)(m_1^2 + m_a^2) - 2(t_2 - m_2^2)(m_1^2 - m_a^2)$$

$$-(t_2 + m_1^2 + m_a^2 + m_2^2)^2] < 2t_1[(m_1^2 + m_a^2)(t_2 - m_2^2)^2 + (m_1^2 - m_a^2)^2(t_2 + m_2^2)$$

$$- (t_2 - m_2^2)(m_1^2 - m_a^2)(t_2 + m_1^2 + m_a^2 + m_2^2)]$$

Dividing by $t_1(>0)$ and simplifying,

$$t_1[4t_2(m_a^2 - m_2^2) + 4m_2^2 m_1^2 - 4m_1^2 m_a^2]$$

$$< 2[2m_a^2 t_2^2 + t_2(-2m_2^2(m_1^2 + m_a^2) + 2m_a^4 - 2m_1^2 m_a^2) + 2m_2^4 m_1^2$$

$$+ m_2^2[(m_1^2 - m_a^2)^2 + (m_1^4 - m_a^4)]$$

This simplifies further into

$$t_1[t_2(m_a^2 - m_2^2) + m_2^2 m_1^2 - m_1^2 m_a^2]$$

$$< [m_a^2 t_2^2 + t_2(-m_2^2 m_1^2 - m_2^2 m_a^2 - m_1^2 m_a^2 + m_a^4) + m_1^2 m_2^2(m_2^2 + m_1^2 - m_a^2)]$$

which leads directly to

$$t_1(t_2 - m_1^2)(m_a^2 - m_2^2) < (m_a^2 t_2 - m_1^2 m_2^2)(t_2 + m_a^2 - m_1^2 - m_2^2)$$

which is the desired result.

APPENDIX C. Asymptotic Kinematics with Two Fixed Momentum Transfers

Using the method of Popova and Ter-Martirosyan [3], we decompose the z.m.s. momenta of the 5 particles in the direct channel reaction (B.1) according to their components parallel and perpendicular to the momentum of particle "a". We denote the parallel (longitudinal) components by

$$k_a = k_b = k = \Delta^{1/2}(s, m_a^2, m_b^2)/2\sqrt{s}$$

and

$$k_1,\ k_2,\ k_3\ .$$

We denote the transverse components by $\underset{\sim}{\kappa}_1$, $\underset{\sim}{\kappa}_2$, $\underset{\sim}{\kappa}_3$ where

$$\underset{\sim}{\kappa}_1 + \underset{\sim}{\kappa}_2 + \underset{\sim}{\kappa}_3 = 0 \tag{C.1}$$

We consider the limit $s \to \infty$ for fixed t_1

$$t_1 = (p_1 - p_a)^2 = m_1^2 + m_a^2 - 2E_1E_a + 2k_1k \tag{C.2}$$

where

$$E_1 = (p_1^2 + m_1^2)^{1/2} = \frac{s + m_1^2 - s_{23}}{2\sqrt{s}}$$

$$E_a = (k^2 + m_a^2)^{1/2} = (s + m_a^2 - m_b^2)\ /\ 2\sqrt{s}$$

$$p_1^2 = k_1^2 + \underset{\sim}{\kappa}_1^2 \tag{C.3}$$

Then, in order that t_1 remain bounded as $s \to \infty$, we must have both p_1/k and k/p_1 bounded in this limit. Also, $p_1 - k_1 = O(1/k)$, so that

$$\frac{\underset{\sim}{\kappa}_1^2}{2k_1} = O(1/k) \text{ which implies that } \underset{\sim}{\kappa}_1^2 \text{ is bounded.} \tag{C.4}$$

Thus

$$t_1 \simeq m_1^2 + m_a^2 - 2kk_1 - (m_1^2 + \kappa_1^2)(\frac{k}{k_1}) - m_a^2(\frac{k_1}{k}) + 2k\,k_1 \tag{C.5}$$

and $t_1 \to -\underset{\sim}{\kappa}_1^2$ as $s \to \infty$

also,

$$\frac{k}{k_1}, \frac{k_1}{k} \text{ are bounded as } s \to \infty \tag{C.6}$$

Similarly, if $|t_2|$ is bounded,

$$\frac{k}{k_3} \text{ and } \frac{k_3}{k} \text{ are bounded, and } t_2 \to -\underset{\sim}{\kappa}_2^2 \text{ as } s \to \infty \tag{C.7}$$

In order that t_1, t_2 be bounded, it is necessary that $\vec{p}_3 \cdot \vec{p}_b > 0$ and $\vec{p}_1 \cdot \vec{p}_a > 0$. It follows that, since $\vec{p}_1 + \vec{p}_2 + \vec{p}_3 = \vec{0}$,

$$k_3 = k_1 \pm k_2 \tag{C.8}$$

according to whether $\vec{p}_2 \cdot \vec{p}_a$ is positive or negative. Now,

$$s_{12} = m_1^2 + m_2^2 + 2(m_1^2 + k_1^2 + \underset{\sim}{\kappa}_1^2)^{1/2}(m_2^2 + k_2^2 + \underset{\sim}{\kappa}_2^2)^{1/2} \mp 2k_1k_2 - 2\underset{\sim}{\kappa}_1 \cdot \underset{\sim}{\kappa}_2$$

$$\simeq m_1^2 + m_2^2 + 2k_1k_2(1 \mp 1) + \frac{k_1}{k_2}(m_2^2 + \underset{\sim}{\kappa}_2^2) + \frac{k_2}{k_1}(m_1^2 + \underset{\sim}{\kappa}_1^2) - 2\underset{\sim}{\kappa}_1 \cdot \underset{\sim}{\kappa}_2 \tag{C.9}$$

and

$$s_{23} = m_2^2 + m_3^2 + 2(m_3^2 + k_3^2 + \underset{\sim}{\kappa}_3^2)^{1/2}(m_2^2 + k_2^2 + \underset{\sim}{\kappa}_2^2)^{1/2} \pm 2k_2k_3 - 2\underset{\sim}{\kappa}_2 \cdot \underset{\sim}{\kappa}_3$$

$$\simeq m_2^2 + m_3^2 + 2k_2k_3(1 \pm 1) + \frac{k_3}{k_2}(m_2^2 + \underset{\sim}{\kappa}_2^2) + \frac{k_2}{k_3}(m_3^2 + \underset{\sim}{\kappa}_3^2) - 2\underset{\sim}{\kappa}_2 \cdot \underset{\sim}{\kappa}_3 \tag{C.10}$$

From (C.9) and (C.10),

$$\frac{k_2}{k} \to 0 \quad \text{as } k \to \infty \text{ , if both } s_{12} \text{ and } s_{23} \to \infty \text{ also.} \tag{C.11}$$

Thus, if $s_{12}, s_{23} \to \infty, t_1, t_2$ bounded,

$$s_{12} = m_1^2 + m_2^2 + 2k_1[(m_2^2 + k_2^2 + \underset{\sim}{\kappa}_2^2)^{1/2} \mp k_2] - 2\underset{\sim}{\kappa}_1 \cdot \underset{\sim}{\kappa}_2$$
$$+ \frac{1}{k_1}(m_1^2 + \underset{\sim}{\kappa}_1^2)(m_2^2 + k_2^2 + \underset{\sim}{\kappa}_2^2)^{1/2} + 0(\tfrac{1}{s}) \tag{C.12}$$

$$s_{23} = m_2^2 + m_3^2 + 2k_3[(m_2^2 + k_2^2 + \underset{\sim}{\kappa}_2^2)^{1/2} \pm k_2] - 2\underset{\sim}{\kappa}_3 \cdot \underset{\sim}{\kappa}_1$$
$$+ \frac{1}{k_3}(m_3^2 + \underset{\sim}{\kappa}_3^2)(m_2^2 + k_2^2 + \underset{\sim}{\kappa}_2^2)^{1/2} + 0(\tfrac{1}{s}) \tag{C.13}$$

so that

$$(s_{12} - m_1^2 - m_2^2 + 2\underset{\sim}{\kappa}_1 \cdot \underset{\sim}{\kappa}_2)(s_{23} - m_2^2 - m_3^2 + 2\underset{\sim}{\kappa}_3 \cdot \underset{\sim}{\kappa}_2)$$
$$\simeq 4k_1k_3(m_2^2 + \underset{\sim}{\kappa}_2^2) + (m_1^2 + \underset{\sim}{\kappa}_1^2)[(m_2^2 + k_2^2 + \underset{\sim}{\kappa}_2^2)^{1/2} \pm k_2]$$
$$\times (m_2^2 + k_2^2 + \underset{\sim}{\kappa}_2^2)^{1/2} + (m_3^2 + \underset{\sim}{\kappa}_3^2)[(m_2^2 + k_2^2 + \underset{\sim}{\kappa}_2^2)^{1/2} \mp k_2][m_2^2 + k_2^2 + \underset{\sim}{\kappa}_2^2]^{1/2} \tag{C.14}$$

Now

$$s_{13} \simeq m_1^2 + m_3^2 + 4k_1k_3 + m_1^2 + m_3^2 + \underset{\sim}{\kappa}_1^2 + \underset{\sim}{\kappa}_3^2 - 2\underset{\sim}{\kappa}_1 \cdot \underset{\sim}{\kappa}_3 + 0(\tfrac{1}{s}) \tag{C.15}$$

so that

$$(s_{12} - m_1^2 - m_2^2 + 2\underset{\sim}{\kappa}_1 \cdot \underset{\sim}{\kappa}_2)(s_{23} - m_2^2 - m_3^2 + 2\underset{\sim}{\kappa}_2 \cdot \underset{\sim}{\kappa}_3)$$

$$= (s_{13} - 2m_1^2 - 2m_3^2 - \underset{\sim}{\kappa}_1^2 - \underset{\sim}{\kappa}_3^2 + 2\underset{\sim}{\kappa}_1 \cdot \underset{\sim}{\kappa}_3)(m_2^2 + \underset{\sim}{\kappa}_2^2)$$

$$+ (m_1^2 + \underset{\sim}{\kappa}_1^2)(m_2^2 + k_2^2 + \underset{\sim}{\kappa}_2^2)^{1/2} \, [(m_2^2 + k_2^2 + \underset{\sim}{\kappa}_2^2)^{1/2} \pm k_2]$$

$$+ (m_3^2 + \underset{\sim}{\kappa}_3^2)(m_2^2 + k_2^2 + \underset{\sim}{\kappa}_2^2)^{1/2} \, [(m_2^2 + k_2^2 + \underset{\sim}{\kappa}_2^2)^{1/2} \mp k_2] + O(\frac{k_2}{k}) \quad (C.16)$$

Inserting

$$s_{13} = s - s_{12} - s_{23} + m_1^2 + m_2^2 + m_3^2 \quad (C.17)$$

then dividing by s , and taking the limit, one obtains Ter-Martirosyan's theorem [3] for the case of a three-particle final state:

$$\frac{s_{12} s_{23}}{s} \rightarrow (m_2^2 + \underset{\sim}{\kappa}_2^2) \quad \text{as} \quad s_{12}, s_{23} \rightarrow \infty, \; t_1, t_2 \quad \text{fixed} \quad (C.18)$$

C.2 Evaluation of $\cos \phi$ in the asymptotic region

We use (B.13) in the asymptotic region. Now, from (B.3) and (B.4) one has, for $t_1 \neq 0$,

$$z \simeq \frac{2t_1 \; s_{12}}{\Delta^{1/2}(t_1, m_1^2, m_a^2) \; \Delta^{1/2}(t_1, m_2^2, t_2)} \rightarrow \mp\infty, \text{ for } t_1 \lessgtr 0 \; .$$

and so

$$\sqrt{1 - z^2} \simeq \pm i \left| \sqrt{1 - z^2} \right| \simeq \pm i \, |z| \simeq \mp \frac{i2t_1 \; s_{12}}{\Delta^{1/2}(t_1, m_1^2, m_a^2) \; \Delta^{1/2}(t_1, m_2^2, t_2)}$$

Here we define $\sqrt{1 - z^2}$ with branch cuts extending from ± 1 to $\pm\infty$ respectively, and positive for z real and $-1 < z < 1$.

z_c' , defined in Eq. (B.12), is bounded in the limit $s \to \infty$. Then, dividing both sides of (B.13) by s , and taking the limit as $s \to \infty$, one obtains

$$-1 = \frac{(m_2^2 + \underset{\sim}{\kappa}_2^2)}{\Delta^{1/2}(t_1,m_2^2,t_2)} \quad \{\mp i \cos \phi \sqrt{1 - z_c'^2} + z_c'\} \tag{C.19}$$

Now

$$z_c' \simeq \frac{t_1 + t_2 - m_2^2}{\Delta^{1/2}(t_1,t_2,m_2^2)} > +1 \quad \text{from (B.12)}$$

$$\sqrt{1 - z_c'^2} = \frac{+i \; |\sqrt{4t_1t_2}|}{|\Delta^{1/2}(t_1,t_2,m_2^2)|} = \frac{-2i | \sqrt{t_1t_2}|}{\Delta^{1/2}(t_1,t_2,m_2^2)}$$

So that (C.19) becomes

$$-\frac{\Delta(t_1,t_2,m_2^2)}{m_2^2 + \underset{\sim}{\kappa}_2^2} = \mp 2 \; |\sqrt{t_1t_1}| \cos \phi + t_1 + t_2 - m_2^2$$

or,

$$\cos \phi = \frac{\pm 1}{2|\sqrt{t_1t_2}|} \quad \{\frac{\Delta(t_1,t_2,m_2^2)}{m_2^2 + \;^2_2} + t_1 + t_2 - m_2^2\}$$

$$(|\sqrt{t_1t_2}| = -\sqrt{t_1t_2} \quad \text{when} \quad t_1,t_2 < 0),$$

$$\text{according as} \quad t_1 \lessgtr 0 \; . \tag{C.20}$$

Consider (C.20) in the limit $t_1 \to 0^-$. Now

$$\underset{\sim}{\kappa}_2^2 = (-\underset{\sim}{\kappa}_1 - \underset{\sim}{\kappa}_3)^2 \simeq -t_1 - t_2 + 2\underset{\sim}{\kappa}_1 \cdot \underset{\sim}{\kappa}_3 = -t_1 - t_2 + 2\sqrt{-t_1} \; \sqrt{-t_2} \cos \zeta_{13} \tag{C.21}$$

where ζ_{13} = angle between $\underset{\sim}{\kappa}_1$ and $\underset{\sim}{\kappa}_3$, the transverse parts of the momenta of particles "1" and "3"; so that (C.20) becomes

$$2m_2^2\sqrt{(-t_1)(-t_2)}\cos\phi - 2\sqrt{(-t_1)(-t_2)}\,(t_1+t_2+2\sqrt{(-t_1)(-t_2)}\cos\zeta_{13})\cos\phi$$

$$= \Delta(t_1,t_2,m_2^2) - (t_1+t_2-m_2^2)(t_1+t_2-2\sqrt{(-t_1)(-t_2)}\cos\zeta_{13}+m_2^2) \qquad (C.22)$$

In the limit as $t_1 \to 0^-$, both sides vanish ($\cos\phi$ remains physical). Then, equating coefficients of $\sqrt{(-t_1)}$ leads to

$$2m_2^2\sqrt{(-t_2)}\cos\phi - 2\sqrt{-t_2}\,t_2\cos\phi = 2\sqrt{-t_2}(t_2-m_2^2)\cos\zeta_{13}$$

so that $\qquad \cos\phi = -\cos\zeta_{13}$

i.e., $\qquad \phi = \pi - \zeta_{13} \qquad$ (C.23)

Since in the limit $t_1 \to 0^-$, ζ_{13} may assume arbitrary values between 0 and π, ϕ is similarly unconstrained in this limit.

APPENDIX D. The Asymptotic Kinematics at Zero Momentum Transfer

We show that in the limit $s, s_{12} \to \infty$, $t_1 \to 0$, t_2 fixed, the momenta of all particles in the crossed channel (B.2) are collinear. We show this when the limit is taken with the direct channel scattering angle restricted to a cone (the "forward cone") about the forward direction, the size of the cone depending on s. The angle in question is that between $\vec{p}_1$ and $\vec{p}_a$, the momenta of particles "1" and "a" in the z.m.s. of the direct channel process (B.1).

This is based on a corresponding result for $2 \to 2$ processes, which we describe first.

Consider the process

$$s : \quad a + b \to 1 + 2 \tag{D.1}$$

and the process in the crossed channel

$$t : \quad \bar{1} + a \to \bar{b} + 2 \tag{D.2}$$

Now

$$t = (p_1 - p_a)^2 = m_1^2 + m_a^2 - \frac{1}{2s}(s + m_a^2 - m_b^2)(s + m_1^2 - m_2^2) + \frac{\Delta^{1/2}(s, m_a^2, m_b^2)\, \Delta^{1/2}(s, m_1^2, m_2^2)}{2s} z_s \tag{D.3}$$

where $s = (p_a + p_b)^2$ and $z_s = \hat{p}_1 \cdot \hat{p}_a$ = cosine of the s-channel scattering angle.

Now as $s \to \infty$, $|t| \to \infty$ also, unless $z_s \to +1$.

We are interested in the limit $t \to 0$. So let $|z_s - 1| < |\varepsilon(s)|$ uniformly in t , where $\varepsilon(s) \to 0$ as $s \to \infty$ for arbitrary t .

Expanding (D.3) in powers of $1/s$:

$$\begin{aligned} t \simeq\ & m_1^2 + m_a^2 - \frac{s}{2} - (m_a^2 - m_b^2 + m_1^2 - m_2^2) - (m_a^2 - m_b^2)(m_1^2 - m_2^2) \\ & + \frac{s}{2}\left[1 - \frac{(m_a^2 + m_b^2)}{s} - 2\frac{m_a^2 m_b^2}{s^2} - 2\frac{(m_a^2 + m_b^2)\, m_a^2 m_b^2}{s^3}\right] \\ & \times \left[1 - \frac{(m_1^2 + m_2^2)}{s} - 2\frac{m_1^2 m_2^2}{s^2} - \frac{2 m_1^2 m_2^2 (m_1^2 + m_2^2)}{s^3}\right] z_s \end{aligned} \tag{D.4}$$

Let $\varepsilon(s) = \dfrac{\varepsilon_1}{s} + \dfrac{\varepsilon_2}{s^2} + \dfrac{\varepsilon_3}{s^3} + \ldots$

Then

$$t \simeq -\frac{(m_a^2-m_b^2)(m_1^2-m_2^2)}{2s} + \frac{\epsilon_1}{2} + \frac{\epsilon_2}{2s} - \frac{\epsilon_1(m_a^2+m_b^2+m_1^2+m_2^2)}{s}$$
$$-\frac{\epsilon_2}{2s^2}(m_a^2+m_b^2+m_1^2+m_2^2) + \frac{(m_1^2+m_2^2)(m_a^2+m_b^2)}{2s} - \frac{m_a^2m_b^2}{s} - \frac{m_1^2m_2^2}{s} + \frac{\epsilon_3}{2s^2}$$
$$+\frac{\epsilon_1}{2s^2}\left[(m_a^2+m_b^2)(m_1^2+m_2^2) - 2m_a^2m_b^2 - 2m_1^2m_2^2\right] - \frac{(m_a^2+m_b^2)\,m_a^2m_b^2}{s^2}$$
$$-\frac{m_1^2m_2^2(m_1^2+m_2^2)}{s^2} + \frac{m_a^2m_b^2(m_1^2+m_2^2)}{s^2} + \frac{m_1^2m_2^2(m_a^2+m_b^2)}{s^2}$$

Clearly, in order that $t \to 0$, $\varepsilon_1 = 0$. Then

$$t \simeq \frac{(m_1^2-m_a^2)(m_b^2-m_2^2)+\epsilon_2/2}{s} + \frac{(m_1^2m_2^2-m_a^2m_b^2)(m_a^2+m_b^2-m_1^2-m_2^2)}{s^2}$$
$$+\frac{\epsilon_3}{2s^2} - \frac{\epsilon_2(m_a^2+m_b^2+m_1^2+m_2^2)}{2s^2} \tag{D.5}$$

Now in the case in which $m_1 = m_a$, $m_2 \neq m_b$, take $\varepsilon_2 = \varepsilon_3 = 0$ (ε_n, $n \geq 4$ are still arbitrary), then

$$t\ s^2 \to -m_1^2\ (m_2^2 - m_b^2)^2 \tag{D.6}$$

In the case $m_1 \neq m_a$, $m_2 \neq m_b$, take $\varepsilon_2 = 0$ (then ε_n, for $n \geq 3$ are still arbitrary), and then

$$t\ s \to (m_1^2 - m_a^2)(m_b^2 - m_2^2) \tag{D.7}$$

Now consider the scattering in the t channel. Let $z_t = \hat{p}_{\bar{b}} \cdot \hat{p}_a$ in the z.m.s. of $(\bar{1} + a)$ in reaction (D.2). Then

$$s = (p_{\bar{b}}-p_a)^2 = m_b^2+m_a^2 + \frac{1}{2t}\Delta^{1/2}(t,m_a^2,m_1^2)\Delta^{1/2}(t,m_b^2,m_2^2)\,z_t$$
$$-\frac{1}{2t}(t+m_a^2-m_1^2)(t+m_b^2-m_2^2) \tag{D.8}$$

In the case $m_1 = m_a$, $m_b \neq m_2$

$$s = m_b^2 + m_1^2 + \frac{\sqrt{t - 4m_1^2}}{2\sqrt{t}}\, \Delta^{1/2}(t, m_b^2, m_2^2)\; z_t - \frac{(t + m_b^2 - m_2^2)}{2} \tag{D.9}$$

From (D.6), $s\sqrt{-t} \to m_1 |m_2^2 - m_b^2|$. So that multiplying (D.9) by $\sqrt{-t}$, and taking the limit $s \to \infty$, $t \to 0$ with $z_s = 1 + O(m_1^8/s^4)$, one obtains the result

$$m_1 |m_2^2 - m_b^2| \to - m_1 |m_2^2 - m_b^2| z_t, \quad \text{and so} \quad z_t \to -1 \tag{D.10}$$

For the case of unequal mass $m_1 \neq m_a$, $m_b \neq m_2$, multiplying (D.8) by t and taking the limit $s \to \infty$, $t \to 0$, $z_s = 1 + O(m_1^6/s^3)$, the last condition allowing the use of (D.7), one obtains the result:

$$(m_1^2 - m_a^2)(m_b^2 - m_2^2) \to |m_a^2 - m_1^2|\, |m_b^2 - m_2^2|\, \frac{z_t}{2} - \frac{(m_a^2 - m_1^2)(m_b^2 - m_2^2)}{2} \tag{D.11}$$

so that $1 \mp z_t \to 2$. We note that these results remain valid even if $m_2^2 \to \infty$, provided that $m_2^2/s \to 0$.

Turning now to the five-body case, let the s channel be

$$a + b \to 1 + x \tag{D.12}$$

where x is the particle pair (2 + 3) with invariant $(\text{mass})^2 =$ $s_{23}(s_{23} > m_b^2)$. Let the t channel be

$$\bar{1} + a \to \bar{b} + x = \bar{b} + 2 + 3 \tag{D.13}$$

Let us consider the physical s channel limit $t_1 \to 0$, t_2 bounded, s, $s_{12} \to \infty$. Then $s_{23}/s \to 0$, and we can apply the result just derived, viz., that $\vec{p}_1$, $\vec{p}_a$, $\vec{p}_b$, $\vec{p}_x$ are all parallel or antiparallel in the t channel z.m.s. Also,

$$t_2 = m_b^2 + m_3^2 + \frac{1}{2t_1}(t_1 + m_b^2 - s_{23})(t_1 + m_3^2 - t_{2b})$$

$$- \frac{\Delta^{1/2}(t_1, m_b^2, s_{23})\, \Delta^{1/2}(t_1, m_3^2, t_{2b})\, y}{2t_1} \qquad \text{(D.14)}$$

where $y = \hat{p}_b \cdot \hat{p}_3$ in the z.m.s. of the t channel.

In order that t_2 remain bounded in the limit being considered, $y \to -1$. And

$$s_{23} = m_2^2 + m_3^2 + \frac{1}{2t_1}(t_1 + m_2^2 - t_2)(t_1 + m_3^2 - t_{2b})$$

$$- \frac{1}{2t_1}\Delta^{1/2}(t_1, t_2, m_2^2)\, \Delta^{1/2}(t_1, m_3^2, t_{2b})\, u \qquad \text{(D.15)}$$

where $u = \hat{p}_2 \cdot \hat{p}_3$

and $t_{2b} = -s_{23} + t_1 - t_2 + m_b^2 + m_3^2 + m_2^2$, since $s_{23}/s \to 0$, $u \to \pm 1$, according as $t_2 \lessgtr m_2^2$.

Thus we have proved the following kinematical lemma:

For the reaction $a + b \to 1 + 2 + 3$, in the limit as $s = (p_a + p_b)^2 \to \infty$, $s_{12} = (p_1 + p_2)^2 \to \infty$, $t_1 = (p_1 - p_a)^2 \to 0$, and $z_s = \hat{p}_1 \cdot \hat{p}_a = 1 + O(m_1^8/s^4)$, and $t_2 = (p_3 - p_b)^2$ bounded, all particle momenta in the crossed channel

$$\bar{1} + a \to \bar{b} + 2 + 3$$

tend to a collinear limit.

APPENDIX E. The Equivalence of ϕ and ω

This appendix demonstrates that the angle ϕ appearing in the partial wave decomposition II.(10) is the same as the Toller angle ω introduced by Bali, Chew, and Pignotti [7]. As defined in Chapter II, ϕ is the azimuthal angle of $\vec{p}_{\bar{1}}$ in the z.m.s. of the reaction II.(2) with the positive z axis taken along $-\vec{p}_2$, and $\vec{p}_{\bar{b}}$ taken in the x-z plane, with negative x component. Thus ϕ is the angle between the planes containing $(\vec{p}_{\bar{1}}, \vec{p}_2)$ and $(\vec{p}_{\bar{b}}, \vec{p}_2)$ respectively in the z.m.s. for II.(2).

ϕ is the angle between the plane containing $(\vec{p}_{\bar{1}}, \vec{p}_a, \vec{p}_2)$ and the plane containing $(\vec{p}_2, \vec{p}_3, \vec{p}_{\bar{b}})$ in the z.m.s. of II.(2), and it is invariant under a Lorentz transformation along the direction of $\vec{p}_2$. Thus, ϕ is the angle between the plane containing the momenta of particles "3" and "$\bar{b}$" and the plane containing the momenta of particles "$\bar{1}$" and "a" in the rest frame of particle "2". ϕ is also then the angle between the plane containing the momenta of "1" and "a" and the plane containing the momenta of "3" and "b" in the rest frame of particle "2" in the direct channel scattering II.(1) obtained from the channel II.(2) by the crossing particles $\bar{1}$ and $\bar{b}$. Thus ϕ is the variable ω introduced in Ref. [7].

APPENDIX F. The Convergence of the Helicity Series for the Unsignatured Amplitudes.

In this appendix we show that there exists a domain of convergence in z for the helicity series that represents the contribution of a pole in J to an unsignatured amplitude. The series in question is the second term on the right side of Eq. III.10.b for a particular pole $J = \alpha(t_1)$

$$S = \sum_{M'} \frac{\gamma_{M'}(t_1,t_2,s_{23}) e^{i(\lambda_2 - M')\phi}}{\sin \pi(\alpha - M' + \lambda_2)} d^{\alpha}_{M'-\lambda_2,-\rho}(-z) \tag{F.1}$$

As shown in Appendix E, the angle ϕ is physical in the physical region for the direct channel, so that $|e^{i(\lambda_2 - M')\phi}| = 1$, independent of M'. Now we apply the Cauchy root test [65] to demonstrate the existence of a domain of convergence. Noticing that for $M' \to \infty$

$$|\sin \pi(\alpha - M' + \lambda\nu|^{1/M'} \to 1, \; \alpha - M'+\lambda_2 \neq \text{integer}$$

and from Eq. (A.14)

$$|d^{\alpha}_{M'-\lambda_2,-\rho}(-z)|^{1/M'} \to \left|\frac{1+z}{1-z}\right|^{1/2} \tag{F.2}$$

A bound is then required for $\left|\lambda_{M'}(t_1,t_2,s_{23})\right|^{1/M'}$, where $\gamma_{M'}$ is the residue of $C_{M'}(J,s_{23},t_1,t_2)$, defined in Eq. II.(51), at the pole $J = \alpha(t_1)$. Thus we are led to consider the behavior of $|C_{M'}|^{1/M'}$ as $M' \to \infty$. Let us suppose that the pole occurs in a contour integral around one branch line on the right side of Fig. 2. Then, from II.(51)

$$|C_M|^{1/M} \simeq \left|\int_{z_o}^{\infty} dz \; e^{J}_{M-\lambda_2,\rho}(z) \; F_M(z)\right|^{1/M} \text{ near the pole } J = \alpha(t_1) \tag{F.3}$$

For fixed J, we assume the integral is uniformly convergent with respect to M. That is, a finite point exists on the contour $z = N(J,\varepsilon)$ such that

$$\left| \int_C dz \; e^J_{M-\lambda_2,\rho}(z) \; F_M(z) \right| < \varepsilon \qquad \text{for all } M \tag{F.4}$$

where ε is an arbitrary positive number, and C is the contour from $N(J,\varepsilon)$ to ∞.

To insure the consistency of this assumption with the existence of a pole at $J = \alpha(t_1)$, we must have $N(J) \to \infty$ as $J \to \alpha(t_1)$. Then

$$|C_M| \leq \left| \int_{z_o}^{N} dz \; e^J_{M-\lambda_2,\rho}(z) \; F_M(z) \right| + \varepsilon \quad , \tag{F.5}$$

where ε is arbitrarily small, and independent of M. Then, as $M \to \infty$,

If $\quad \left| \int_{z_o}^{\infty} dz \; e^J_{M-\lambda_2,\rho}(z) \; F_M(z) \right| \to 0 \quad \text{as} \quad M \to \infty$

then

$$|C_M|^{1/M} \to 0 \quad \text{or} \quad 1 \text{ , as } \quad M \to \infty \tag{F.6}$$

If $\quad \left| \int_{z_o} dz \; e^J_{M-\lambda_2,\rho}(z) \; F_M(z) \right|$ is bounded below as $M \to \infty$

then

$$\lim_{M \to \infty} |C_M|^{1/M} \leq \lim_{M \to \infty} \left| \int_{z_o}^{N} dz \; e^J_{M-\lambda_2,\rho}(z) \; F_M(z) \right|^{1/M} \tag{F.7}$$

Now, applying the Schwarz inequality [66] to the right side of (F.7),

$$\lim_{M\to\infty} |C_M|^{1/M} = \lim_{M\to\infty} \left(\int_{z_o}^{N} dz |e^J_{M-\lambda_2,\rho}(z)|^2 \right)^{1/M} \left(\int_{z_o}^{N} dz |F_M(z)|^2 \right)^{1/M} \quad (F.8)$$

Then, using Eq. (A.15), one obtains

$$\lim_{M\to\infty} \left(\int_{z_o}^{N} dz\ |e^J_{M-\lambda_2,\rho}(z)|^2 \right)^{1/M} = \left| \frac{1 + z_1}{1 - z_1} \right| \quad (F.9)$$

where z_1 is some point on the contour from z_o to N .

Since F_M is a Fourier coefficient (see Eq. II.(13.c)), from the Riemann-Lebesgue lemma [67] $F_M(z) \to 0$ as $M \to \infty$ so that

$$\lim_{M\to\infty} \left(\int_{z_o}^{N} dz |F_M(z)|^2 \right)^{1/M} \to 0 \text{ or } 1 \quad (F.10)$$

Thus in the neighborhood of $J = \alpha(t)$, except for the point $J = \alpha(t)$, $\lim\limits_{M\to+\infty} |C_M(J)|^{\frac{1}{M}} \le \left|\frac{1+z_1}{1-z_1}\right| = K$. Since $C_M(J) \simeq \dfrac{\gamma_M}{J - \alpha(t_1)}$ near $J = \alpha(t_1)$, $\lim\limits_{M\to\infty} |\gamma_M|^{1/M} = K$, i.e., $|\gamma_M|^{1/M}$ is bounded as $M \to \infty$.

Applying the Cauchy root test [65] to the series (F.1) results in absolute convergence for

$$K \left| \frac{1 + z}{1 - z} \right|^{1/2} < 1$$

or

$$\left| \frac{1 + z}{1 - z} \right| < 1/K^2$$

If z is real and < -1, this means

$$-z - 1 < \frac{1}{K^2}(1 - z) \quad , \text{ or } \quad 1 < -z < \frac{1 + \frac{1}{K^2}}{1 - \frac{1}{K^2}}$$

is the domain of convergence.

The same analysis may be applied to the series as $M \to -\infty$ to yield another domain of convergence that overlaps this one, and the analysis may also be applied to a pole arising from a contour around a branch line on the left side of Fig. 2.

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The resulting partial wave amplitudes apparently do not permit a

Sommerfeld-Watson transformation with the neglect of the contribution from the semicircle at infinity.

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