EIGENVALUE PROBLEMS ASSOCIATED WITH KORN'S INEQUALITIES IN THE THEORY OF ELASTICITY

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ABSTRACT

Interest in the possible applications of a priori inequalities in linear elasticity theory motivated the present investigation. Korn's inequality under various side conditions is considered, with emphasis on the Korn's constant. In the "second case" of Korn's inequality, a variational approach leads to an eigenvalue problem; it is shown that, for simply-connected two-dimensional regions, the problem of determining the spectrum of this eigenvalue problem is equivalent to finding the values of Poisson's ratio for which the displacement boundary-value problem of linear homogeneous isotropic elastostatics has a non-unique solution.

Previous work on the uniqueness and non-uniqueness issue for the latter problem is examined and the results applied to the spectrum of the Korn eigenvalue problem. In this way, further information on the Korn constant for general regions is obtained.

A generalization of the "main case" of Korn's inequality is introduced and the associated eigenvalue problem is again related to the displacement boundary-value problem of linear elastostatics in two dimensions.

TABLE OF CONTENTS

I. INTRODUCTION

1. Description of Korn Inequalities

We begin this work by describing what is meant by a Korn inequality. Let $\underline{u}(\underline{x})$ be a vector field defined and twice continuously differentiable on the closure R+B of an open, bounded, connected region R with boundary B, in two or three dimensions. We introduce the functionals

$$D(\underline{u}) = \int_R u_{i,j} u_{i,j} dV \ , \tag{1.1}$$

$$S(\underline{u}) = \int_R \frac{1}{4}(u_{i,j}+u_{j,i})(u_{i,j}+u_{j,i}) dV \ , \tag{1.2}$$

where here and throughout this work, Cartesian tensor notation is used, Latin subscripts range from one to n, where n is the number of dimensions, and summation over repeated subscripts is implied. Subscripts preceded by a comma indicate differentiation with respect to the corresponding Cartesian coordinate.

A Korn inequality states the existence of a number $K_1 > 0$, depending only on the shape of the region R, such that

$$D(\underline{u}) \leq K_1 S(\underline{u}) \ , \tag{1.3}$$

for all vector fields $\underline{u}$ satisfying certain side conditions. Since we get equality in (1.3) for any K_1 if $\underline{u}$ is a constant vector, we will agree henceforth to identify vector fields differing only by a constant.

The necessity for imposing some side conditions on $\underline{u}$ in order for (1.3) to hold can be seen from the fact that if $\underline{u}$ is a pure rotation, that is, $u_i = \epsilon_{ijk} x_j \omega_k$, where $\underline{\omega}$ is a constant vector and ϵ_{ijk} is the alternating tensor, then

$$S(\underline{u}) = 0 \ , \qquad D(\underline{u}) > 0 \ .$$

Let

$$R(\underline{u}) = \frac{1}{4} \int_R (u_{i,j} - u_{j,i})(u_{i,j} - u_{j,i}) dV \ . \tag{1.4}$$

Then

$$D(\underline{u}) = S(\underline{u}) + R(\underline{u}) \ , \tag{1.5}$$

and we see that K_1 is not less than one and that the inequality (1.3) is equivalent to either of the following two inequalities:

$$R(\underline{u}) \leq \left(1 - \frac{1}{K_1}\right) D(\underline{u}) \ , \tag{1.6}$$

$$R(\underline{u}) \leq (K_1 - 1) S(\underline{u}) \ . \tag{1.7}$$

K. O. Friedrichs [1] gave a comprehensive formulation and proof of Korn's inequality in the form (1.7), under three different side conditions on $\underline{u}$, namely

First Case: $\underline{u} = 0$ on boundary B of R , (1.8)

Second Case: $\int_R (u_{i,j} - u_{j,i}) dV = 0$, (1.9)

Main Case:

$$\begin{cases} \int_R (u_{i,j} - u_{j,i}) dV = 0 \ , & (1.10) \\ u_{i,jj} + u_{j,ji} = 0 \quad \text{in } R \ . & (1.11) \end{cases}$$

(We note that the conditions (1.8), (1.9) eliminate the possibility of pure rotations.)

Friedrichs in [1] refers to papers by A. Korn [2, 3] in which the inequality (1.7) under conditions (1.8), (1.9) was treated. The class of admissible regions R is specified by Friedrichs [1]: we merely note here that this class includes bounded regions with corners or edges. Friedrichs shows that to prove Korn's inequality in the second case, it is sufficient to prove the inequality in the main case.

We introduce another case of Korn's inequality, which we call the <u>Extended Main Case</u>.

There exists a $K_1(\sigma) > 0$ such that

$$D(\underline{u}) \le K_1(\sigma) S(\underline{u}) \ , \qquad (1.12)$$

for all $\underline{u}$ satisfying

$$\begin{cases} \int_R (u_{i,j} - u_{j,i})\, dV = 0 \ , & (1.13) \\ u_{i,jj} + \dfrac{1}{1-2\sigma} u_{j,ji} = 0 \quad \text{in } R \ , & (1.14) \end{cases}$$

where σ is a real parameter such that $-1 < \sigma < \frac{1}{2}$. When $\sigma = 0$, this reduces to the main case. If the vector field $\underline{u}$ is regarded as an infinitesimal displacement field in an homogeneous, isotropic elastic medium occupying R, then (1.14) are the displacement equations of equilibrium in the case of zero body force, where σ is taken to be Poisson's ratio.

Henceforth, we take R to be an open bounded connected region, admissible in the sense of Friedrichs [1]. As shown by

Friedrichs [1], Korn's inequality in the first case is particularly simple, and we have

$$D(\underline{u}) \leq 2S(\underline{u}) \ , \tag{1.15}$$

for all $\underline{u}$ such that

$$\underline{u} = 0 \quad \text{on} \quad B \ . \tag{1.16}$$

The inequality (1.15) follows immediately from the identity

$$2S(\underline{u}) = D(\underline{u}) + \int_R (\operatorname{div} \underline{u})^2 dV \ , \tag{1.17}$$

which in turn, follows from applying the divergence theorem and boundary condition (1.16) to the identity

$$2S(\underline{u}) = \int_R [u_{i,j}u_{i,j} + (u_i u_{j,i})_{,j} - (u_i u_{j,j})_{,i} + u_{i,i}u_{j,j}] dV \ . \tag{1.18}$$

Note that for vector fields $\underline{u}$ such that div $\underline{u} = 0$ in R and $\underline{u} = 0$ on B, we get equality in (1.15). Thus, in the first case,

$$D(\underline{u}) \leq K_1 S(\underline{u}) \ , \tag{1.19}$$

with $K_1 = 2$ and this constant is the best possible.

If an isotropic homogeneous elastic medium in R is subjected to an infinitesimal displacement $\underline{u}$, the strain energy is proportional to

$$E(\underline{u}) = S(\underline{u}) + \frac{\sigma}{1-2\sigma} \int_R (\operatorname{div} \underline{u})^2 dV \ , \tag{1.20}$$

where σ is Poisson's ratio, which we take to lie in $-1 < \sigma < \frac{1}{2}$. If the inequality

$$D(\underline{u}) \leq K_1 S(\underline{u}) \tag{1.21}$$

holds under some side conditions on $\underline{u}$, then

$$D(\underline{u}) \le K_o E(\underline{u}) , \tag{1.22}$$

where

$$K_o = \begin{cases} K_1 & (0 \le \sigma < \frac{1}{2}) \\ \left(\frac{1-2\sigma}{1+(n-2)\sigma}\right) K_1 & (-1 < \sigma \le 0) , \end{cases}$$

and n is the number of dimensions. For $0 \le \sigma < \frac{1}{2}$, this statement is clearly true, and for $-1 < \sigma < 0$, we use the inequality

$$0 \le \int_R (\text{div } \underline{u})^2 dV \le \int_R n(u_{1,1}^{\ 2} + \ldots + u_{n,n}^{\ 2}) dV \le nS(\underline{u}) . \tag{1.23}$$

We now show how Korn's inequality may be used to obtain lower bounds for the fundamental frequencies of vibration of an elastic solid. The first non-zero frequency ω of the vibration problem for an homogeneous isotropic elastic body, occupying the region R, and fixed on the boundary B, is such that

$$\mu\left[\nabla^2 \underline{u} + \frac{1}{1-2\sigma} \nabla(\nabla \cdot \underline{u})\right] + \rho\omega^2 \underline{u} = 0 \quad \text{in } R , \tag{1.24}$$

$$\underline{u} = 0 \quad \text{on } B , \tag{1.25}$$

where μ is the shear modulus and ρ is the density. The parameter ω may be characterized by the minimum principle:

$$\frac{\rho\omega^2}{2\mu} = \min_{\substack{\underline{v}=0 \\ \text{on } B}} \frac{E(\underline{v})}{\int_R |\underline{v}|^2 dV} , \tag{1.26}$$

where $E(\underline{v})$ is defined by equation (1.20).

Now use inequality (1.22) in the form

$$E(\underline{v}) \geq \frac{D(\underline{v})}{K_o} , \tag{1.27}$$

where

$$K_o = \begin{cases} K_1 & (0 \leq \sigma < \frac{1}{2}) , \\ \left(\frac{1-2\sigma}{1+(n-2)\sigma} \right) K_1 & (-1 < \sigma \leq 0) . \end{cases} \tag{1.28}$$

Since we are dealing with the first case here, we take $K_1 = 2$ in (1.28). The relations (1.26) and (1.27) give

$$\frac{\rho\omega^2}{2\mu} \geq \frac{1}{K_o} \min_{\substack{\underline{v}=0 \\ \text{on } B}} \frac{D(\underline{v})}{\int_R |\underline{v}|^2 dV} . \tag{1.29}$$

Let λ_o be the smallest eigenvalue of the acoustic problem

$$\nabla^2\varphi + \lambda\varphi = 0 \qquad \text{in } R , \tag{1.30}$$

$$\varphi = 0 \qquad \text{on } B . \tag{1.31}$$

The parameter λ_o may be characterized by the minimum principle

$$\lambda_o = \min_{\substack{\varphi=0 \\ \text{on } B}} \frac{\int_R |\nabla\varphi|^2 dV}{\int_R \varphi^2 \, dV} . \tag{1.32}$$

Then relations (1.29) and (1.32) give

$$\frac{\rho\omega^2}{2\mu} \geq \frac{\lambda_o}{K_o} , \tag{1.33}$$

where

$$K_o = \begin{cases} 2 & (0 \leq \sigma < \frac{1}{2}) , \\ \left(\frac{1-2\sigma}{1+(n-2)\sigma} \right) 2 & (-1 < \sigma \leq 0) . \end{cases} \tag{1.34}$$

A lower bound for the first non-zero frequency for the free vibration problem may be obtained in a similar manner, using Korn's inequality in the second case.

In order to illustrate another application of Korn's inequality-one which involves the extended main case - we point out that the question of finding a lower bound for the ratio

$$E(\underline{u}) / \int_R |\underline{u}|^2 dV \qquad (1.35)$$

arises in the problem of estimating the "characteristic decay length" associated with Saint-Venant's principle. In this situation, the vector field $\underline{u}$ is known to satisfy the displacement equations of equilibrium (1.14) as well as the constraints

$$\int_R u_i dV = \int_R (u_{i,j} - u_{j,i}) dV = 0 \ . \qquad (1.36)$$

In the analysis of this issue carried out by R. A. Toupin [4] a lower bound for the ratio (1.35) was taken to be the smallest positive frequency of free elastic vibration of R, thus making no use of the fact that $\underline{u}$ satisfies (1.14).

If, instead, we employ Korn's inequality (1.12) in the extended main case, together with the definition (1.20) of $E(\underline{u})$, we have*

$$E(\underline{u}) \geq S(\underline{u}) \geq \frac{1}{K_1(\sigma)} D(\underline{u}) \ , \qquad (1.37)$$

where use has been made of the side conditions (1.14) and the second

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*In this discussion, we assume $0 \leq \sigma < \frac{1}{2}$ for simplicity.

of (1.36). Since $\underline{u}$ also satisfies the first of (1.36), we have

$$D(\underline{u}) \geq \hat{\lambda} \int_R |\underline{u}|^2 dV \ , \tag{1.38}$$

where $\hat{\lambda}$ is the smallest positive eigenvalue of the problem

$$\nabla^2 \varphi + \lambda\varphi = 0 \qquad \text{in } R \ , \tag{1.39}$$

$$\partial\varphi/\partial n = 0 \qquad \text{on } B \ . \tag{1.40}$$

Combining (1.37), (1.38) we obtain the alternate lower bound

$$\frac{E(\underline{u})}{\int_R |\underline{u}|^2 dV} \geq \frac{\hat{\lambda}}{K_1(\sigma)} \quad , \tag{1.41}$$

for vector fields $\underline{u}$ satisfying (1.14) and (1.36).

It was this possible relevance of Korn's inequality to Saint-Venant's principle which originally motivated this investigation.

In the remainder of Chapter I, we describe some of the previous work on Korn's inequality. In Chapter II, following an approach used by Payne and Weinberger [5], we derive an eigenvalue problem associated with Korn's inequality in the second case and we list some properties of the eigenvalues and eigenfunctions. In Chapter III, we relate this eigenvalue problem (which we call a Korn eigenvalue problem) to the displacement boundary-value problem of linear homogeneous isotropic elastostatics in two dimensions. Chapter IV consists of a summary of results on the displacement boundary-value problem of plane strain. In Chapter V, we apply these results to the Korn eigenvalue problem. Finally, in Chapter VI, we treat the extended main case in a similar fashion.

2. History

Friedrichs, in §7 and §8 of [1] uses the first case of Korn's inequality to prove the existence of a solution of the displacement boundary-value problem of linear, homogeneous, isotropic elastostatics. He also treats the free vibration problem and uses the second case of Korn's inequality to show the existence of a discrete set of frequencies of vibration which tend to infinity. S. G. Mikhlin [6] proves Korn's inequality in the second case: he makes use of the work of D. M. Eidus [7], which is based on Friedrichs' approach. Mikhlin then uses Korn's inequality in the second case to prove the existence of a solution of the traction boundary-value problem of linear anisotropic elastostatics, and he also treats the mixed and mixed-mixed boundary-value problems. Mikhlin's use of Korn's inequality is to prove that the operator of elasticity theory is positive definite for the various boundary conditions considered. G. Fichera [8, 9] has done work of a similar nature.

B. Bernstein and R. A. Toupin [10] in 1960 introduced the terminology 'Korn Constant', which we define as follows: if a Korn inequality of the form (1.21) exists for a given region R, under some side conditions on $\underline{u}$, then the set of real numbers

$$D(\underline{u})/S(\underline{u}) \qquad (2.1)$$

has a least upper bound. We call this least upper bound 'Korn's Constant' for the region R in the first case, second case, main case, or extended main case, and denote these numbers by $K_R^{(1)}$, $K_R^{(2)}$, $K_R^{(M)}$, $K_R^{(EM)}$, respectively. In §1 we showed that $K_R^{(1)} = 2$ for any region R.

Bernstein and Toupin [10] posed the question of obtaining explicit values of the Korn constants $K_R^{(2)}$, $K_R^{(M)}$ for particular regions and considered the problem of finding upper and lower bounds for these constants, paying special attention to the case of a sphere and a circle. Subsequently, Payne and Weinberger [5], using a variational approach which we describe in detail in Chapter II, showed that for a sphere

$$K_R^{(2)} = \frac{56}{13} , \qquad (2.2)$$

and for the circle (two dimensions),

$$K_R^{(2)} = 4 . \qquad (2.3)$$

They also showed that if Korn's inequality in the second case holds for each of the domains $R_1, \ldots, R_n$, then it holds for their union.

C. M. Dafermos [11], using the approach of Payne and Weinberger, considered the case of a two-dimensional circular ring and determined the Korn constant $K_R^{(2)}$ for any fixed $r = R_1/R_2$ $(0 < R_1 < R_2)$. As $r \to 0$, his result checked with that of Payne and Weinberger for a circle. He also gave an upper bound for the Korn constant $K_R^{(2)}$ for the union of two regions in terms of the Korn constants for the separate regions.

The present work is concerned with the question of obtaining information about the Korn constants $K_R^{(2)}$, $K_R^{(M)}$, $K_R^{(EM)}$ for general regions. It will be seen in the sequel that for two-dimensional regions, this question is intimately connected with the issue of uniqueness and non-uniqueness for the displacement boundary-value problem of linear homogeneous isotropic elastostatics.

II. AN EIGENVALUE PROBLEM ASSOCIATED WITH KORN'S INEQUALITY

3. The Eigenvalue Problem in the Second Case

In this section, we consider the second case of Korn's inequality:

$$D(\underline{u}) \leq K_1 S(\underline{u}) \;, \tag{3.1}$$

for all $\underline{u}$ satisfying

$$\int_R (u_{i,j} - u_{j,i}) dV = 0 \;. \tag{3.2}$$

Following Payne and Weinberger [5], we pose the following variational problem. Find

$$\sup \frac{D(\underline{u})}{S(\underline{u})} \;,$$

where the supremum is taken over all vector fields $\underline{u}$ such that $S(\underline{u}) \neq 0$ and (3.2) holds. We will show that the Euler equations and natural boundary conditions associated with this problem are

$$(2-K)u_{j,ii} - K u_{i,ij} = 0 \qquad \text{in } R \;, \tag{3.3}$$

$$[(2-K)u_{j,i} - K u_{i,j}]n_i = 0 \qquad \text{on } B \;, \tag{3.4}$$

where $\underline{n}$ is the unit outward normal on B and K is an eigenvalue parameter.

Following the usual procedure of the calculus of variations, we introduce constant Lagrange multipliers K and μ_{ij} ($\mu_{ij} = 0$ for $i = j$), and consider the functional

$$J[\underline{u}] = D(\underline{u}) - KS(\underline{u}) - \mu_{ij} \int_R (u_{i,j} - u_{j,i}) dV \;.$$

A necessary condition for a stationary value of $J[\underline{u}]$ is that

$$\delta J = \int_R 2u_{j,i}\delta u_{j,i}dV - \frac{K}{2}\int_R (u_{i,j}+u_{j,i})(\delta u_{i,j}+\delta u_{j,i})dV$$

$$- \mu_{ij}\int_R (\delta u_{i,j}-\delta u_{j,i})dV = 0 ,$$

where δ denotes the first variation. Application of the divergence theorem gives

$$\delta J = \int_R \delta u_j\{(K-2)u_{j,ii}+Ku_{i,ij}\}dV+\int_B \delta u_j\{(2-K)u_{j,i}-Ku_{i,j}+\mu_{ij}-\mu_{ji}\}n_i dS = 0 .$$

Standard arguments of the calculus of variations now imply that the extremal field $\underline{u}$ satisfies the Euler equations

$$(2-K)u_{j,ii} - Ku_{i,ij} = 0 \quad \text{in } R , \tag{3.5}$$

and the natural boundary conditions

$$[(2-K)u_{j,i}-Ku_{i,j}+\mu_{ij}-\mu_{ji}]n_i = 0 \quad \text{on } B . \tag{3.6}$$

We show that $\mu_{ij} = \mu_{ji}$ and so (3.6) reduces to (3.4). Multiply (3.5) by $\epsilon_{ijk}x_k$ and integrate over R to get

$$\int_R \epsilon_{ijk}x_k[(2-K)u_{j,mm}-Ku_{m,mj}]dV = 0 .$$

Apply the divergence theorem to get

$$\int_B \epsilon_{ijk}x_k[(2-K)u_{j,m}-Ku_{m,j}]n_m dS-\int_R \epsilon_{ijk}[(2-K)u_{j,k}-Ku_{k,j}]dV = 0 . \tag{3.7}$$

Use the boundary conditions (3.6) in (3.7) to find

$$\int_B \epsilon_{ijk}x_k(\mu_{jm}-\mu_{mj})n_m dS-\int_R \epsilon_{ijk}[(2-K)u_{j,k}-Ku_{k,j}]dV = 0 . \tag{3.8}$$

For each fixed i, the volume integral in (3.8) is zero, as can be seen by expanding the terms and using the constraint (3.2). Thus, equation (3.8) reads

$$\int_B \epsilon_{ijk} x_k (\mu_{jm} - \mu_{mj}) n_m \, dS = 0 \ . \tag{3.9}$$

Now apply the divergence theorem to obtain

$$\int_R \epsilon_{ijk} \delta_{km} (\mu_{jm} - \mu_{mj}) dV = 0 \ , \tag{3.10}$$

where δ_{km} is the Kronecker delta, $\delta_{km} = 1$ if $k = m$, $\delta_{km} = 0$ otherwise. Then (3.10) reads

$$\epsilon_{ijk} (\mu_{jk} - \mu_{kj}) \int_R dV = 0 \ , \tag{3.11}$$

which implies that

$$\mu_{ij} = \mu_{ji} \qquad \text{as required.}$$

We will refer to the problem of finding the values of K for which there exists a non-trivial* vector field $\underline{u}$ satisfying

$$(2-K) u_{j,ii} - K u_{i,ij} = 0 \qquad \text{in } R \ , \tag{3.12}$$

$$[(2-K) u_{j,i} - K u_{i,j}] n_i = 0 \qquad \text{on } B \ , \tag{3.13}$$

as the Korn eigenvalue problem. It was solved explicitly for the case of a sphere and a circle by Payne and Weinberger [5]. For a sphere, they found that the spectrum contains an infinite number of eigenvalues whose supremum is 56/13. For a circle, they found a finite spectrum, consisting of the values 1, 2, 4, each occurring with infinite multiplicity. Moreover, for these cases, Payne and Weinberger proved the

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*A vector field $\underline{u}$ is trivial if it is identically constant.

completeness of the eigenfunctions of the Korn eigenvalue problem with respect to vector fields with finite Dirichlet integrals. It follows from this completeness result that the largest eigenvalue of the Korn problem is in fact the Korn's constant $K_R^{(2)}$ when R is a sphere or a circle. Thus, for a sphere $K_R^{(2)} = 56/13$, while for a circle $K_R^{(2)} = 4$.

We remark here that there is also an eigenvalue problem associated with Korn's inequality in the first case, namely, the differential equations (3.12) together with the boundary condition $\underline{u} = 0$ on B. Since we already know that Korn's constant is equal to two in this case, we will not be concerned with this problem in detail in what follows.

4. Some Properties of the Korn Eigenvalue Problem

We write (3.12), (3.13) in the form

$$(2-K)\nabla^2\underline{u} - K\nabla(\nabla\cdot\underline{u}) = 0 \qquad \text{in } R, \tag{4.1}$$

$$[(2-K)u_{j,i} - Ku_{i,j}]n_i = 0 \qquad \text{on } B. \tag{4.2}$$

We now observe that if we identify K with $1/\nu$ where ν is Poisson's ratio, equations (4.1) are formally equivalent to the displacement equations of equilibrium of linear homogeneous isotropic elasticity. This fact was also noticed by Dafermos [11]. We will return to examine the implications of this equivalence later.

We have the following remarks to make concerning the Korn eigenvalue problem (4.1), (4.2):

Remark 1. K = 1 is an eigenvalue of infinite multiplicity of (4.1), (4.2) for any region.

When K = 1, equations (4.1) and boundary conditions (4.2) become

$$\nabla^2 \underline{u} - \nabla(\nabla \cdot \underline{u}) = 0 \qquad \text{in } R\,, \qquad (4.3)$$

$$\text{curl}\, \underline{u} \times \underline{n} = 0 \qquad \text{on } B\,. \qquad (4.4)$$

Any vector field $\underline{u} = \nabla\psi$, where ψ is an arbitrary smooth function on R+B, satisfies (4.3), (4.4). Note that for this $\underline{u}$, $D(\underline{u}) = S(\underline{u})$.

Remark 2. K = 2 is an eigenvalue of infinite multiplicity of (4.1), (4.2) for any region.

When K = 2, (4.1) and (4.2) read

$$\nabla(\nabla \cdot \underline{u}) = 0 \qquad \text{in } R\,, \qquad (4.5)$$

$$u_{i,j} n_i = 0 \qquad \text{on } B\,. \qquad (4.6)$$

We shall show that any smooth vector field $\underline{u}$ such that

$$\text{div}\, \underline{u} = 0 \qquad \text{in } R+B\,, \qquad (4.7)$$

$$\underline{u} = 0 \qquad \text{on } B\,, \qquad (4.8)$$

satisfies (4.5), (4.6); this will prove the assertion concerning K = 2.

To begin with, we consider the three-dimensional case and we suppose that $\underline{u}$ satisfies (4.7), (4.8). Then equation (4.5) clearly holds, and it remains to show that the boundary condition (4.6) is satisfied. If $\underline{t}^{(1)}$, $\underline{t}^{(2)}$ are two linearly independent unit vectors tangent to B at $\underline{x}$, then from (4.8) we obtain

$$u_{i,j} t_j^{(\alpha)} = 0 \qquad \text{on } B\,. \quad (\alpha = 1,2) \qquad (4.9)$$

Let the matrix U be given by

$$U = (u_{i,j})_{\text{on } B}\,. \qquad (4.10)$$

From the boundary condition (4.9), we obtain

$$U\underline{t}^{(\alpha)} = 0 \ , \tag{4.11}$$

which implies that $\det U = 0$, since $\underline{t}^{(\alpha)}$ is non-trivial. Furthermore, using the fact that $\operatorname{div} \underline{u} = 0$ in R+B, we find that $\operatorname{Tr} U = 0$, where $\operatorname{Tr} U = u_{i,i}$ denotes the trace of U.

In general, the characteristic polynomial associated with a matrix U is given by

$$F(\lambda) \equiv \det(U-\lambda I) = -\lambda^3 + \lambda^2 \operatorname{Tr}U + \tfrac{1}{2}[\operatorname{Tr}(U^2) - (\operatorname{Tr}U)^2]\lambda + \det U \ ,$$

so that in our case

$$F(\lambda) = -\lambda^3 + \tfrac{1}{2}\operatorname{Tr}(U^2)\lambda \ . \tag{4.12}$$

Now $\lambda = 0$ is an eigenvalue of U, and in fact, since $\underline{t}^{(1)}, \underline{t}^{(2)}$ are two linearly independent eigenvectors of U corresponding to the eigenvalue 0, $\lambda = 0$ must be a double eigenvalue.

Thus,

$$F'(0) = \tfrac{1}{2}\operatorname{Tr}(U^2) = 0 \ , \tag{4.13}$$

and so, from (4.12), we obtain

$$F(\lambda) = \det(U-\lambda I) = -\lambda^3 \ . \tag{4.14}$$

Therefore, $\lambda = 0$ is the only eigenvalue of U; it occurs with multiplicity three.

We now let

$$v_j = u_{i,j}n_i \qquad \text{on B} \ . \tag{4.15}$$

Then, from (4.9) we find

$$\underline{v}\cdot\underline{t}^{(\alpha)} = u_{i,j}t_j^{(\alpha)}n_i = 0 \ . \tag{4.16}$$

This implies that there exists a scalar c such that

$$\underline{v} = c\,\underline{n} , \tag{4.17}$$

that is,

$$u_{i,j} n_i = c\, n_j , \tag{4.18}$$

or equivalently,

$$U^T \underline{n} = c\,\underline{n} , \tag{4.19}$$

where U^T denotes the transpose of U. Equation (4.19) implies that c is an eigenvalue of U^T, but since the eigenvalues of U^T are the same as those of U, we have $c = 0$. Thus, from (4.17) and (4.15), we find that

$$u_{i,j} n_i = 0 \qquad \text{on } B , \tag{4.20}$$

and so the boundary condition (4.6) is satisfied. Thus, for the three-dimensional case, we have shown that if the vector field $\underline{u}$ satisfies (4.7), (4.8), then $\underline{u}$ also satisfies (4.5) and (4.6). The same argument can be adapted to the two-dimensional case.

In two dimensions, however, we can prove a stronger result, namely, that <u>all</u> eigenfunctions of (4.5), (4.6) may be characterized in terms of vector fields $\underline{v}$ satisfying

$$\text{div}\,\underline{v} = 0 \qquad \text{in } R+B , \tag{4.21}$$

$$\underline{v} = 0 \qquad \text{on } B . \tag{4.22}$$

Suppose that $\underline{u}$ is a non-trivial vector field satisfying

$$u_{\beta,\beta\alpha} = 0 \qquad \text{in } R+B , \tag{4.23}$$

$$u_{\alpha,\beta} n_\alpha = 0 \qquad \text{on } B \,, \tag{4.24}$$

where the Greek subscripts range over the integers (1, 2). Equation (4.23) implies that

$$u_{\beta,\beta} = \text{constant} = a \qquad \text{in } R+B \,. \tag{4.25}$$

We observe that $\epsilon_{\nu\beta} n_\nu$ is tangent to B, and so we have for any vector field $\underline{u}$ defined on R+B,

$$\frac{du_\rho}{ds} = \epsilon_{\nu\beta} u_{\rho,\beta} n_\nu \qquad \text{on } B \,, \tag{4.26}$$

where $\epsilon_{\nu\beta}$ is the two-dimensional alternator.* Multiplying (4.26) by $\epsilon_{\rho\mu}$ and using the identity

$$\epsilon_{\rho\mu}\epsilon_{\nu\beta} = \delta_{\rho\nu}\delta_{\mu\beta} - \delta_{\rho\beta}\delta_{\mu\nu} \,, \tag{4.27}$$

we obtain

$$\epsilon_{\rho\mu}\frac{du_\rho}{ds} = u_{\nu,\mu} n_\nu - u_{\beta,\beta} n_\mu \tag{4.28}$$

$$= -a n_\mu \qquad \text{on } B \,, \tag{4.29}$$

where we have used (4.24) and (4.25). Multiplying (4.29) by $\epsilon_{\alpha\mu}$ and using the identity

$$\epsilon_{\alpha\mu}\epsilon_{\rho\mu} = \delta_{\alpha\rho} \,, \tag{4.30}$$

we find that

$$\frac{du_\alpha}{ds} = -a\epsilon_{\alpha\mu} n_\mu = a\frac{dx_\alpha}{ds} \qquad \text{on } B \,, \tag{4.31}$$

which implies that

$$u_\alpha = a x_\alpha + d_\alpha \qquad \text{on } B \,, \tag{4.32}$$

where d_α is a constant vector. We now define the vector field $\underline{v}$ by

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* $\epsilon_{11} = \epsilon_{22} = 0$, $\epsilon_{12} = -\epsilon_{21} = 1$.

$$v_\alpha = u_\alpha - ax_\alpha - d_\alpha \qquad \text{in } R+B \ . \tag{4.33}$$

Then, equations (4.25) and boundary conditions (4.32) imply that the vector field $\underline{v}$ satisfies (4.21), (4.22).

Suppose now that $\underline{v}$ is a non-zero vector field satisfying (4.21), (4.22). Define a vector field $\underline{u}$ by

$$\underline{u} = \underline{v} + a\underline{x} + \underline{d} \tag{4.34}$$

where a is an arbitrary constant scalar and $\underline{d}$ is an arbitrary constant vector in R+B. From (4.34), (4.21), and (4.22) it follows that

$$u_{\alpha,\alpha} = a \qquad \text{in } R+B \ , \tag{4.35}$$

$$u_\alpha = ax_\alpha + d_\alpha \qquad \text{on } B \ . \tag{4.36}$$

Equation (4.35) implies that $\underline{u}$ satisfies (4.23), and thus it remains to show that the boundary condition (4.36) implies that $\underline{u}$ satisfies (4.24). The boundary condition (4.36) implies that

$$\frac{du_\rho}{ds} = a\frac{dx_\rho}{ds} = -a\epsilon_{\rho\lambda}n_\lambda \qquad \text{on } B \ . \tag{4.37}$$

Multiplying (4.37) by $\epsilon_{\rho\mu}$ and using the identity (4.30), we obtain

$$\epsilon_{\rho\mu}\frac{du_\rho}{ds} = -a\,n_\mu \qquad \text{on } B \ . \tag{4.38}$$

Using (4.35), we may write (4.38) as

$$\epsilon_{\rho\mu}\frac{du_\rho}{ds} = -u_{\beta,\beta}n_\mu \qquad \text{on } B \ . \tag{4.39}$$

From (4.28), we see that for any vector field $\underline{u}$ defined on R+B,

$$\epsilon_{\rho\mu}\frac{du_\rho}{ds} = u_{\nu,\mu}n_\nu - u_{\beta,\beta}n_\mu \qquad \text{on } B \ . \tag{4.40}$$

Thus, (4.39) and (4.40) imply that

$$u_{\nu,\mu} n_{\nu} = 0 \qquad \text{on } B , \tag{4.41}$$

which is precisely the boundary condition (4.24). This completes the proof of our assertion concerning the eigenvalue K = 2 in the two-dimensional case.

We conclude our discussion of Remark 2 by noting that for the particular vector field $\underline{u}$ defined by (4.7), (4.8), we obtain $D(\underline{u}) = 2S(\underline{u})$, as follows from (1.17).

<u>Remark 3.</u> $K = \infty$ is not an eigenvalue of (4.1), (4.2).

When $K = \infty$, (4.1), (4.2) become*

$$\nabla^2 \underline{u} + \nabla(\nabla \cdot \underline{u}) = 0 \qquad \text{in } R , \tag{4.42}$$

$$[u_{i,j} + u_{j,i}] n_i = 0 \qquad \text{on } B . \tag{4.43}$$

As we remarked at the beginning of this section, equations (4.42) coincide formally with the displacement equations of equilibrium of linear isotropic elasticity when Poisson's ratio is zero. In this case, the stress/displacement relations give

$$\tau_{ij} = \mu(u_{i,j} + u_{j,i}) ,$$

where τ_{ij} is the stress tensor, μ is the shear modulus. Thus, we may interpret the problem (4.42), (4.43) as the traction boundary-

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*The case $K = \infty$ for (4.1), (4.2) is understood to mean the result obtained formally by dividing (4.1), (4.2) respectively by K and letting $K \to \infty$. An analogous interpretation applies to similar situations arising in the sequel.

value problem (with homogeneous boundary conditions) of linear isotropic elasticity when Poisson's ratio is zero. By the classical Kirchhoff uniqueness result, which is valid for Poisson's ratio in the range $(-1, \frac{1}{2})$, we have $\tau_{ij} \equiv 0$ in R, which implies that $\underline{u}$ is at most a rigid body displacement field. The condition

$$\int_R (u_{i,j} - u_{j,i})\, dV = 0$$

eliminates pure rotations, so that $\underline{u}$ is at most a translation and is therefore trivial.

Remark 4. Every solution of (4.1), (4.2) satisfies the condition

$$\int_R (u_{i,j} - u_{j,i})\, dV = 0 \ . \tag{4.44}$$

To prove (4.44), multiply (4.1) by $\epsilon_{ijk} x_k$, integrate over R, and use the divergence theorem to get

$$\int_B \epsilon_{ijk} x_k [(2-K) u_{j,m} - K u_{m,j}] n_m \, dS - \int_R \epsilon_{ijk} [(2-K) u_{j,k} - K u_{k,j}] dV = 0 \ .$$

The first integral is zero by (4.2). Thus,

$$\int_R \epsilon_{ijk} [(2-K) u_{j,k} - K u_{k,j}] dV = 0 \ . \tag{4.45}$$

Expand terms in (4.45) to get (4.44).

Remark 5. If $\underline{u}$, $\underline{v}$ are solutions of (4.1), (4.2) corresponding to distinct values of K, then we have the orthogonality conditions:

$$S(\underline{u}, \underline{v}) \equiv \frac{1}{4} \int_R (u_{i,j} + u_{j,i})(v_{i,j} + v_{j,i}) dV = 0 \ , \tag{4.46}$$

$$D(\underline{u}, \underline{v}) \equiv \int_R u_{i,j} v_{i,j} \, dV = 0 \ , \tag{4.47}$$

$$R(\underline{u}, \underline{v}) \equiv \frac{1}{4} \int_R (u_{i,j} - u_{j,i})(v_{i,j} - v_{j,i}) dV \qquad 0 \ . \tag{4.48}$$

By hypothesis, $\underline{u}$ satisfies

$$(2-K_1)u_{j,ii} - K_1 u_{i,ij} = 0 \qquad \text{in } R \ , \tag{4.49}$$

$$[(2-K_1)u_{j,i} - K_1 u_{i,j}]n_i = 0 \quad \text{on } B \ , \tag{4.50}$$

and $\underline{v}$ satisfies

$$(2-K_2)v_{j,ii} - K_2 v_{i,ij} = 0 \qquad \text{in } R \ , \tag{4.51}$$

$$[(2-K_2)v_{j,i} - K_2 v_{i,j}]n_i = 0 \qquad \text{on } B \ , \tag{4.52}$$

with $K_1 \neq K_2$.

Multiply (4.49) by v_j, (4.51) by u_j, subtract and integrate over R to get

$$\int_R [(2-K_1)v_j u_{j,ii} - (2-K_2)u_j v_{j,ii} - K_1 v_j u_{i,ij} + K_2 u_j v_{i,ij}] dV = 0 \ .$$

Thus, $$\int_R [(2-K_1)(v_j u_{j,i})_{,i} - (2-K_1)v_{j,i}u_{j,i} - (2-K_2)(u_j v_{j,i})_{,i}$$
$$+ (2-K_2)u_{j,i}v_{j,i} - K_1 v_j u_{i,ij} + K_2 u_j v_{i,ij}] dV = 0 \ .$$

Use the divergence theorem and boundary conditions (4.50), (4.52) to get

$$\int_B [K_1 v_j u_{i,j} - K_2 u_j v_{i,j}]n_i dS + \int_R [(K_1 - K_2)u_{j,i}v_{j,i} - K_1 v_j u_{i,ij} + K_2 u_j v_{i,ij}] dV$$
$$= 0 \ .$$

Again use the divergence theorem to obtain

$$\int_R [(K_1 - K_2)u_{j,i}v_{j,i} + K_1 v_{j,i}u_{i,j} - K_2 u_{j,i}v_{i,j}] dV = 0 \ ,$$

that is,

$$\int_R [K_1(u_{i,j}+u_{j,i})v_{j,i}-K_2u_{j,i}(v_{i,j}+v_{j,i})]dV = 0 . \qquad (4.53)$$

Interchange i and j in (4.53) to find

$$\int_R [K_1(u_{i,j}+u_{j,i})v_{i,j}-K_2u_{i,j}(v_{i,j}+v_{j,i})]dV = 0 . \qquad (4.54)$$

Add (4.53) and (4.54) to get

$$(K_1-K_2)\int_R (u_{i,j}+u_{j,i})(v_{i,j}+v_{j,i})dV = 0 ,$$

which implies (4.46) since $K_1 \neq K_2$. Equations (4.47) and (4.48) may be verified in a similar fashion.

Remark 6. The eigenvalues of (4.1), (4.2) are real.

Suppose K, $\overline{K}$ were eigenvalues with corresponding eigenfunctions $\underline{u}$, $\overline{\underline{u}}$, where the bar denotes complex conjugate. Then, if $K \neq \overline{K}$, by (4.47) we have

$$\int_R u_{i,j}\overline{u}_{i,j}\, dV = 0 ,$$

which implies $u_{i,j} \equiv 0$ in R, that is, $\underline{u}$ is a constant vector field. This contradicts the assumption that K is an eigenvalue.

Remark 7. The eigenvalues of (4.1), (4.2) are greater than or equal to one.

Suppose K is an eigenvalue of (4.1), (4.2). Multiplying (4.1) by u_j and integrating over R, we obtain

$$\int_R [(2-K)u_{j,ii}u_j-Ku_{i,ij}u_j]dV = 0 . \qquad (4.55)$$

Now (4.55) may be written as

$$\int_R [(2-K)(u_j u_{j,i})_{,i} - K(u_j u_{i,j})_{,i} - (2-K)u_{j,i}u_{j,i} + Ku_{j,i}u_{i,j}]dV = 0. \quad (4.56)$$

Using the divergence theorem and boundary conditions (4.2), we find from (4.56) that

$$\int_R [(K-2)u_{j,i}u_{j,i} + Ku_{j,i}u_{i,j}]dV = 0 . \quad (4.57)$$

Equation (4.57) may be written

$$-\int_R \left[\frac{(1-K)}{2} (u_{i,j}+u_{j,i})(u_{i,j}+u_{j,i}) + \tfrac{1}{2}(u_{i,j}-u_{j,i})(u_{i,j}-u_{j,i}) \right] dV = 0 ,$$

that is,

$$-2[(1-K)S(\underline{u}) + R(\underline{u})] = 0 , \quad (4.58)$$

where $S(\underline{u}), R(\underline{u})$ are defined by (1.2), (1.4), respectively.

Suppose that $K < 1$. Then, since $S(\underline{u})$ and $R(\underline{u})$ are non-negative, (4.58) implies that

$$S(\underline{u}) = 0 , \quad R(\underline{u}) = 0 . \quad (4.59)$$

But (4.59) implies that $\underline{u}$ is a constant vector, contradicting the assumption that K is an eigenvalue. This verifies the assertion made in Remark 7.

III. TRANSFORMATION OF THE KORN EIGENVALUE PROBLEM IN TWO DIMENSIONS

5. Transformation Theorems

As we remarked at the end of §3, there is an eigenvalue problem associated with Korn's inequality in the first case, namely, that of finding the values of K for which the boundary-value problem

$$(2-K)\nabla^2\underline{u} - K\nabla(\nabla\cdot\underline{u}) = 0 \qquad \text{in } R \,, \tag{5.1}$$

$$\underline{u} = 0 \qquad \text{on } B \,, \tag{5.2}$$

has a non-trivial solution. We recall the observation made at the beginning of §4, that if the vector field $\underline{u}$ is regarded as an infinitesimal displacement field in an homogeneous isotropic elastic medium occupying the region R, then (5.1) may be interpreted as the displacement equations of equilibrium in the case of zero body force, if we identify K with $1/\nu$, where ν is Poisson's ratio. Thus, the eigenvalue problem (5.1), (5.2) may be related directly to the displacement boundary-value problem of linear homogeneous isotropic elastostatics.

In Chapters III and V, we shall be concerned mainly with the Korn eigenvalue problem in the second case, that is, the eigenvalue problem (4.1), (4.2). Unless we mention it explicitly otherwise, we confine our attention to simply-connected two-dimensional regions R, and we use the notation convention that Greek subscripts range over the integers (1, 2). In this section, we shall state and prove transformation theorems which establish a relationship between the Korn eigenvalue problem in the second case and the displacement boundary-value problem of plane isotropic elastostatics.

Theorem 5.1. Suppose $K \neq 1$ and let $\underline{u} \in C^2(R+B)$ be a non-trivial solution of the Korn eigenvalue problem, with eigenvalue K, so that

$$(2-K)u_{\alpha,\beta\beta} - Ku_{\beta,\beta\alpha} = 0 \quad \text{in } R, \tag{5.3}$$

$$[(2-K)u_{\alpha,\beta} - Ku_{\beta,\alpha}]n_\beta = 0 \quad \text{on } B. \tag{5.4}$$

Then there exists a non-trivial vector field $\underline{\xi} \in C^2(R+B)$ such that

(i) $$\xi_{\alpha,\beta} = K(u_{\beta,\alpha} - u_{\alpha,\beta}) + 2(1-K)u_{\beta,\alpha} - 2(1-K)\delta_{\alpha\beta}u_{\gamma,\gamma}, \tag{5.5}$$

(ii) $$\nabla^2\underline{\xi} + \frac{1}{1-2\nu}\nabla(\nabla\cdot\underline{\xi}) = 0 \quad \text{in } R, \tag{5.6}$$

(iii) $$\underline{\xi} = 0 \quad \text{on } B, \tag{5.7}$$

where $\nu = 1 - \frac{1}{K}$.

We also have the following converse of Theorem 5.1.

Theorem 5.2. Suppose $\nu \neq 1$ and let $\underline{\xi} \in C^2(R+B)$ be a non-trivial solution of the problem

$$\nabla^2\underline{\xi} + \frac{1}{1-2\nu}\nabla(\nabla\cdot\underline{\xi}) = 0 \quad \text{in } R, \tag{5.8}$$

$$\underline{\xi} = 0 \quad \text{on } B, \tag{5.9}$$

with eigenvalue ν. Then there exists a non-trivial vector field $\underline{u} \in C^2(R+B)$ such that

$$u_{\alpha,\beta} = -\frac{1}{4\nu}[\xi_{\alpha,\beta} - \xi_{\beta,\alpha} + 2(1-\nu)\xi_{\beta,\alpha} - 2(1-\nu)\delta_{\alpha\beta}\xi_{\gamma,\gamma}], \tag{5.10}$$ *

and $\underline{u}$ satisfies the Korn eigenvalue problem (5.3), (5.4) with $K = \frac{1}{1-\nu}$.

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* It will be seen in Chapter IV that $\nu = 0$ is not an eigenvalue of problem (5.8), (5.9).

The remainder of §5 is taken up with the proof of Theorems 5.1 and 5.2. Some further discussion of these theorems is given in §6.

Proof of Theorem 5.1

We first show that there is a vector field $\underline{\xi}$ satisfying (5.5). Let

$$H_{\alpha\beta} = K(u_{\beta,\alpha}-u_{\alpha,\beta})+2(1-K)u_{\beta,\alpha}-2(1-K)\delta_{\alpha\beta}u_{\gamma,\gamma} \quad . \tag{5.11}$$

Direct calculation shows that

$$\begin{aligned}\epsilon_{\rho\mu}\epsilon_{\beta\lambda}H_{\rho\beta,\lambda} = H_{\beta\beta,\mu}-H_{\lambda\mu,\lambda} &= -2(1-K)u_{\gamma,\gamma\mu}-(2-K)u_{\mu,\lambda\lambda} \\ &+Ku_{\lambda,\mu\lambda}+2(1-K)\delta_{\lambda\mu}u_{\gamma,\gamma\lambda} = -(2-K)u_{\mu,\lambda\lambda}+Ku_{\lambda,\lambda\mu} ,\end{aligned} \tag{5.12}$$

where $\epsilon_{\beta\lambda}$ is the two-dimensional alternator and we have used the identity

$$\epsilon_{\rho\mu}\epsilon_{\beta\lambda} = \delta_{\rho\beta}\delta_{\mu\lambda}-\delta_{\rho\lambda}\delta_{\mu\beta} \quad . \tag{5.13}$$

Since $\underline{u}$ satisfies (5.3), it follows from (5.12) that

$$\epsilon_{\rho\mu}\epsilon_{\beta\lambda}H_{\rho\beta,\lambda} = 0 \quad \text{in } R . \tag{5.14}$$

Multiplying (5.14) by $\epsilon_{\alpha\mu}$ and using (5.13) contracted on μ, λ, we find that

$$\epsilon_{\beta\lambda}H_{\alpha\beta,\lambda} = H_{\alpha 1,2}-H_{\alpha 2,1} = 0 \quad \text{in } R . \tag{5.15}$$

Equation (5.15) and the simple connectivity of R imply that there exists a function $\xi_\alpha \in C^2(R+B)$, unique to within an arbitrary additive constant, such that

$$\xi_{\alpha,\beta} = H_{\alpha\beta} \ . \tag{5.16}$$

In view of the definition (5.11), this is precisely (5.5).

To show that the vector field $\underline{\xi}$ of (5.16) satisfies the differential equation (5.6) with $\nu = 1 - \frac{1}{K}$, we proceed as follows. If $K \neq 2$, (5.16), (5.11) provide

$$\xi_{\alpha,\beta\beta} + \frac{1}{1-2(1-\frac{1}{K})} \xi_{\beta,\beta\alpha} = -Ku_{\alpha,\beta\beta} + \frac{K^2}{2-K} u_{\beta,\beta\alpha} \ ,$$

and hence, by (5.3),

$$\xi_{\alpha,\beta\beta} + \frac{1}{1-2(1-\frac{1}{K})} \xi_{\beta,\beta\alpha} = 0 \quad \text{in } R.$$

Thus, (5.6) holds if $K \neq 2$. When $K = 2$, (5.16), (5.11) yield

$$\xi_{\beta,\beta\alpha} = 2u_{\beta,\beta\alpha} \ ,$$

and hence by (5.3) with $K = 2$,

$$\xi_{\beta,\beta\alpha} = 0 \ .$$

Thus, (5.6) holds* if $K = 2$.

To establish the boundary condition (5.7), we observe that $\epsilon_{\lambda\beta} n_\lambda$ is tangent to B, so that by (5.16)

$$\frac{d\xi_\rho}{ds} = H_{\rho\beta}\epsilon_{\lambda\beta}n_\lambda \quad \text{on } B \ . \tag{5.17}$$

We thus have, with the aid of (5.11), (5.13), and (5.4),

$$\epsilon_{\rho\mu}\frac{d\xi_\rho}{ds} = \epsilon_{\rho\mu}\epsilon_{\lambda\beta}H_{\rho\beta}n_\lambda = [2-K)u_{\mu,\lambda} - Ku_{\lambda,\mu}]n_\lambda = 0 \quad \text{on } B \ . \tag{5.18}$$

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*See footnote, p. 21.

Multiplying (5.18) by $\epsilon_{\alpha\mu}$ and using (5.13) contracted on λ, μ, we obtain

$$\frac{d\xi_\alpha}{ds} = 0 \qquad \text{on } B \ . \tag{5.19}$$

Thus, by choosing the arbitrary additive constant suitably in the construction of $\underline{\xi}$ in (5.16), we may arrange that the boundary condition (5.7) holds.

Finally, we remark that (5.5) may be uniquely inverted to give $u_{\alpha,\beta}$ in terms of $\xi_{\lambda,\mu}$. It follows that $\underline{\xi}$ is constant in R if and only if $\underline{u}$ is constant in R. Thus, $\underline{\xi}$ is non-trivial. This completes the proof of Theorem 5.1.

The proof of Theorem 5.2 may be carried out by using an almost identical argument to that used in proving Theorem 5.1, and consequently will be omitted.

6. Discussion of Theorems 5.1 and 5.2

We recall here the boundary-value problem satisfied by the non-trivial vector field $\underline{\xi}$ appearing in the statement of Theorems 5.1 and 5.2:

$$\nabla^2 \underline{\xi} + \frac{1}{1-2\nu} \nabla(\nabla \cdot \underline{\xi}) = 0 \qquad \text{in } R \ , \tag{6.1}$$

$$\underline{\xi} = 0 \qquad \text{on } B \ . \tag{6.2}$$

If the vector field $\underline{\xi}$ is regarded as an infinitesimal displacement field in an homogeneous, isotropic elastic medium occupying the two-dimensional region R, then (6.1) may be identified with the displacement equations of equilibrium in the case of zero body force, where the parameter ν is taken to be Poisson's ratio. We wish to consider the boundary-value problem (6.1), (6.2) as an eigenvalue problem, with ν the eigenvalue parameter, and we shall be interested in the range of values of ν for which the problem (6.1), (6.2) has a non-trivial solution. Viewed in another manner, this is equivalent to investigating the issue of non-uniqueness for the displacement boundary-value problem of two-dimensional elastostatics. As we shall see in Chapter IV, this question has been considered by various authors.

Theorems 5.1 and 5.2 establish an equivalence between the two-dimensional Korn eigenvalue problem and the displacement boundary-value problem of plane elastostatics in the following sense. Excluding the exceptional values $K = 1$ and $\nu = 1$, the problem of determining the set of eigenvalues K of the boundary-value problem (5.3) and (5.4) for simply-connected plane regions reduces to finding the set of values of ν for which the displacement boundary-value problem of plane elastostatics has a non-unique solution.

We conclude this chapter by giving a brief indication of how one may arrive at the content of Theorems 5.1 and 5.2. As we remarked at the beginning of §4, the differential equations (4.1) of the Korn eigenvalue problem are formally equivalent to the displacement equations of equilibrium of linear isotropic elasticity with Poisson's ratio ν if the parameter K is identified with $1/\nu$. Thus, in two dimensions, solutions of equations (5.3) may be represented in terms of an Airy function φ, which is biharmonic in R, and auxiliary harmonic functions. (See, for example, Muskhelishvili [12].) When substituted into the boundary conditions (5.4), this leads to the observation that the boundary conditions for φ are exactly the boundary conditions for the deflection in the bending problem for an isotropic elastic plate with a free edge, when an appropriate identification is made between the parameter K and the Poisson ratio of the plate. The bending problem for a plate with a free edge may, in turn, be transformed to the homogeneous displacement boundary-value problem of plane elastostatics, with an appropriate identification between the Poisson ratios of the two problems. This was shown by S. G. Lekhnitskii [13]. Lekhnitskii's results are presented in English in the notes of I. S. Sokolnikoff [14]. (See also Green and Zerna [15], page 247.) In this manner, one is led to a transformation from the Korn eigenvalue problem with eigenvalue K, to the homogeneous displacement boundary-value problem of plane elastostatics with $\nu = 1 - \frac{1}{K}$.

It appears that there is no direct analog of Theorems 5.1 and 5.2 in the three-dimensional case.

IV. SOME RESULTS ON THE DISPLACEMENT BOUNDARY-VALUE PROBLEM OF PLANE STRAIN

7. Differential Equation Formulation

We are interested in the range of values of the parameter ν for which the following boundary-value problem has a non-trivial solution. Find $\underline{\xi} \in C^2(R+B)$ such that

$$\nabla^2 \underline{\xi} + \frac{1}{1-2\nu} \nabla(\nabla \cdot \underline{\xi}) = 0 \qquad \text{in } R , \qquad (7.1)$$

$$\underline{\xi} = 0 \qquad \text{on } B , \qquad (7.2)$$

where R is a simply-connected region in two dimensions. Henceforth, the boundary-value problem (7.1), (7.2) will be called the homogeneous displacement boundary-value problem of plane strain. In this chapter, we accumulate some known results from the literature concerning this problem.

Theorem 7.1. $\underline{\xi} \equiv 0$ if $\nu < \frac{1}{2}$ or $\nu > 1$.

Theorem 7.1 is the uniqueness theorem for the displacement boundary-value problem of plane strain for bounded regions. The proof given for three-dimensional bounded regions by M. E. Gurtin and E. Sternberg [16] is also valid in two dimensions.

Theorem 7.2. $\nu = 1$ and $\nu = \frac{1}{2}$ are eigenvalues of infinite multiplicity for (7.1),(7.2) for any region R.*

When $\nu = 1$, (7.1), (7.2) read

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*Cf. Remarks 1 and 2 of §4.

$$\nabla^2 \underline{\xi} - \nabla(\nabla \cdot \underline{\xi}) = 0 \qquad \text{in } R\,, \tag{7.3}$$

$$\underline{\xi} = 0 \qquad \text{on } B\,. \tag{7.4}$$

Any vector $\underline{\xi} = \nabla\varphi$, where φ is an arbitrary smooth function on $R+B$, such that $\underline{\xi} = 0$ on B satisfies (7.3), (7.4).

When $\nu = \frac{1}{2}$, (7.1), (7.2) read

$$\nabla(\nabla \cdot \underline{\xi}) = 0 \qquad \text{in } R\,, \tag{7.5}$$

$$\underline{\xi} = 0 \qquad \text{on } B\,. \tag{7.6}$$

Any smooth vector field $\underline{\xi}$ such that $\nabla \cdot \underline{\xi} = 0$ in R, $\underline{\xi} = 0$ on B satisfies (7.5), (7.6). Theorem 7.2 is also proved by S. G. Mikhlin in [17], for the three-dimensional analog of (7.1), (7.2).

Theorem 7.3. Suppose $\frac{1}{2} < \nu < 1$. Then there exists a region R, boundary B such that a non-trivial vector field $\underline{\xi}$ exists satisfying (7.1), (7.2).

This result was proved by J. L. Ericksen [18] for the n-dimensional case. For our purposes, it is convenient to outline here the argument in two dimensions. It may be verified by direct substitution that

$$\xi_1 = x_1^2 - \frac{2(1-\nu)}{1-2\nu}\, x_2^2 - \epsilon\,, \qquad \xi_2 \equiv 0\,, \tag{7.7}$$

where ϵ is an arbitrary positive constant, satisfies equation (7.1). The curve $\xi_1 = 0$ has the form

$$x_1^2 - \frac{2(1-\nu)}{1-2\nu}\, x_2^2 = \epsilon\,, \tag{7.8}$$

which may be written as

$$\frac{x_1^2}{\epsilon} + \frac{x_2^2}{\epsilon(2\nu-1)/2(1-\nu)} = 1 \quad . \tag{7.9}$$

When $\frac{1}{2} < \nu < 1$, $(2\nu-1)/1-\nu > 0$ and (7.9) describes an ellipse with axes in the ratio $\left(\frac{2\nu-1}{2(1-\nu)}\right)^{\frac{1}{2}}$. Thus, we have shown that Theorem 7.3 holds with the two-dimensional region R consisting of the interior of the ellipse (7.9), where ϵ is an arbitrary positive constant.

Theorem 7.4. If $\nu = 3/4 + \delta$ is an eigenvalue of (7.1), (7.2) for some region R, then $\nu = 3/4 - \delta$ is also an eigenvalue for R, where $0 < \delta < 1/4$.

By hypothesis, $\underline{\xi}$ is non-trivial and satisfies

$$\xi_{\alpha,\beta\beta} + (-\tfrac{1}{2}-2\delta)^{-1}\xi_{\beta,\beta\alpha} = 0 \qquad \text{in R ,} \tag{7.10}$$

$$\xi_\alpha = 0 \qquad \text{on B .} \tag{7.11}$$

The function $\hat{\xi}_\alpha$ defined by $\hat{\xi}_\alpha = \epsilon_{\alpha\beta}\xi_\beta$ is non-trivial and satisfies

$$\hat{\xi}_\alpha = 0 \qquad \text{on B .} \tag{7.12}$$

We now verify that $\hat{\xi}_\alpha$ satisfies

$$\hat{\xi}_{\alpha,\beta\beta} + (-\tfrac{1}{2}+2\delta)^{-1}\hat{\xi}_{\beta,\beta\alpha} = 0 \qquad \text{in R .} \tag{7.13}$$

Using the definition of $\hat{\xi}_\alpha$, we have

$$\hat{\xi}_{\alpha,\beta\beta} + (-\tfrac{1}{2}+2\delta)^{-1}\hat{\xi}_{\beta,\beta\alpha} = \epsilon_{\alpha\gamma}\xi_{\gamma,\beta\beta} + (-\tfrac{1}{2}+2\delta)^{-1}\epsilon_{\beta\gamma}\xi_{\gamma,\beta\alpha} \; . \tag{7.14}$$

Multiplying (7.14) by $\epsilon_{\alpha\lambda}$ and using the identities

$$\epsilon_{\alpha\lambda}\epsilon_{\alpha\gamma} = \delta_{\lambda\gamma} \;, \quad \epsilon_{\alpha\lambda}\epsilon_{\beta\gamma} = \delta_{\alpha\beta}\delta_{\lambda\gamma} - \delta_{\alpha\gamma}\delta_{\lambda\beta} \;, \tag{7.15}$$

we find

$$\varepsilon_{\alpha\lambda}(\hat{\xi}_{\alpha,\beta\beta}+(-\tfrac{1}{2}+2\delta)^{-1}\hat{\xi}_{\beta,\beta\alpha}) = \xi_{\lambda,\beta\beta}+(-\tfrac{1}{2}+2\delta)^{-1}(\xi_{\lambda,\beta\beta}-\xi_{\alpha,\lambda\alpha})$$

$$= (-\tfrac{1}{2}+2\delta)^{-1}[(\tfrac{1}{2}+2\delta)\xi_{\lambda,\beta\beta}-\xi_{\beta,\beta\lambda}] = 0 \ , \tag{7.16}$$

having used equation (7.10). Multiplying (7.16) by $\varepsilon_{\rho\lambda}$ and using the first of (7.15), we find

$$\hat{\xi}_{\rho,\beta\beta}+(-\tfrac{1}{2}+2\delta)^{-1}\hat{\xi}_{\beta,\beta\rho} = 0 \ , \tag{7.17}$$

which is exactly equation (7.13). (We note that a similar proof may be given for the case $\delta = 1/4$, in which case Theorem 7.4 agrees with Theorem 7.2.)

S. G. Mikhlin in [17] and [19] examines the eigenvalue problem (7.1), (7.2) for three-dimensional regions, referring to the papers of E. and F. Cosserat [20, 21, 22] on this subject. By using general methods of functional analysis, Mikhlin proves certain results, of which we note here the following:

(i) Problem (7.1), (7.2) has a countable set of eigenfunctions orthogonal under the scalar products $D(\underline{u}, \underline{v})$ and $\int_R \operatorname{div}\underline{u}\operatorname{div}\underline{v}\,dV$. A completeness result for these eigenfunctions is also established. (See also Chapter V of this work.)

(ii) The eigenvalues are real and may accumulate only at $\nu = 1,\ 3/4,\ 1/2$.

(iii) Eigenvalues not equal to (1, 3/4, 1/2) have finite multiplicity.

For the two-dimensional eigenvalue problem (7. 1), (7. 2) we are considering in the present work, further information on the spectrum may be obtained by using the formulation of (7. 1), (7. 2) in terms of regular functions of a complex variable. This will be presented in the next section.

8. Complex Variable and Integral Equation Formulation

As is well known, the fundamental boundary-value problems of plane elastostatics may be reduced to boundary-value problems in the theory of functions of a complex variable. (See, for example, Muskhelishvili [12].) The problem of finding a vector field $\underline{\xi} \in C^2(R+B)$ satisfying

$$\nabla^2\underline{\xi} + \frac{1}{1-2\nu}\nabla(\nabla\cdot\underline{\xi}) = 0 \qquad \text{in } R, \tag{8.1}$$

$$\underline{\xi} = 0 \qquad \text{on } B, \tag{8.2}$$

is equivalent to a boundary-value problem in complex variable theory, which we state as follows.

Let ν be a real parameter, and define the complex variable z by $z = x_1+ix_2$. Find functions $\varphi(z)$, $\psi(z)$ regular in R, such that the function

$$V(z) = (3-4\nu)\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} \tag{8.3}$$

satisfies

$$V(z) = 0 \qquad \text{on } B\,. \tag{8.4}$$

Assuming the origin is contained in R, we may subject $\varphi(z)$ to the normalization condition $\varphi(0) = 0$. By a non-trivial function $V(z)$ we shall mean a function $V(z)$ not identically zero in R, and by a non-trivial vector field $\underline{\xi}$ we shall mean a vector field $\underline{\xi}$ not identically zero in R. The equivalence between the problem of finding a non-trivial vector field $\underline{\xi}$ satisfying (8.1), (8.2) and the problem of finding a non-trivial function $V(z)$ satisfying (8.3), (8.4), is established by the following theorems.

Theorem 8.1. Let $V(z)$ be non-trivial and satisfy (8.3), (8.4). Let $2(\xi_1+i\xi_2) = V(z)$. Then $\underline{\xi} = (\xi_1, \xi_2)$ is non-trivial and satisfies (8.1), (8.2).

The statements concerning non-triviality and the satisfaction of boundary conditions follow immediately. It may be verified by direct substitution of the real and imaginary parts of $V(z)$ in equation (8.1), and using the Cauchy-Riemann equations for the real and imaginary parts of $\varphi(z)$, $\psi(z)$ respectively, that $\underline{\xi}$ satisfies equation (8.1).

Theorem 8.2. Suppose $\nu \neq \frac{1}{2}$, 1 and let $\underline{\xi}$ be a non-trivial vector field satisfying (8.1), (8.2). Then there exists a non-trivial function $V(z)$ satisfying (8.3), (8.4).

Theorem 8.2 may be proved by observing that equation (8.1) may be written as

$$(1-\nu)^{-1}\xi_{\alpha,\beta\beta}+[(1-2\nu)(1-\nu)]^{-1}\xi_{\beta,\beta\alpha} = 0 \qquad \text{in } R \,. \tag{8.5}$$

Rewriting equation (8.5) in the form

$$2(1-2\nu)^{-1}\xi_{\beta,\beta\alpha}+(1-\nu)^{-1}(\xi_{\alpha,\beta\beta}-\xi_{\beta,\beta\alpha}) = 0 \,, \tag{8.6}$$

and using the identity

$$\epsilon_{\beta\alpha}\epsilon_{\lambda\mu}\xi_{\mu,\lambda\beta} = (\delta_{\beta\lambda}\delta_{\alpha\mu}-\delta_{\beta\mu}\delta_{\alpha\lambda})\xi_{\mu,\lambda\beta} = \xi_{\alpha,\beta\beta}-\xi_{\beta,\alpha\beta} \,,$$

we may write (8.6) as

$$2(1-2\nu)^{-1}\xi_{\beta,\beta\alpha}+(1-\nu)^{-1}\epsilon_{\beta\alpha}\epsilon_{\lambda\mu}\xi_{\mu,\lambda\beta} = 0 \,,$$

that is,

$$P_{,\alpha} + \epsilon_{\beta\alpha}Q_{,\beta} = 0 \qquad \text{in } R \,, \tag{8.7}$$

where

$$P(x_1,x_2) = 2(1-2\nu)^{-1}\xi_{\beta,\beta} \,, \quad Q(x_1,x_2) = (1-\nu)^{-1}\epsilon_{\lambda\mu}\xi_{\mu,\lambda} \,. \tag{8.8}$$

Equations (8.7) are the Cauchy-Riemann equations for P and Q and imply that in a simply-connected region R, there exists a regular function $f(z)$ such that

$$f(z) = P + iQ \qquad \text{in } R \,. \tag{8.9}$$

Let

$$\varphi(z) = \frac{1}{4}\int_0^z f(\rho)d\rho = p + iq \,. \tag{8.10}$$

Then $\varphi(z)$ is regular in R, $\varphi(0) = 0$, p and q are related through the Cauchy-Riemann equations, and

$$\varphi'(z) = p_{,1}+iq_{,1} = q_{,2}-ip_{,2} = \frac{1}{4}(P+iQ) \,. \tag{8.11}$$

We define the functions $U(x_1, x_2)$, $W(x_1, x_2)$ by

$$2U = (3-4\nu)p - \frac{1}{4}(Px_1 + Qx_2) \ , \tag{8.12}$$

$$2W = (3-4\nu)q - \frac{1}{4}(Px_2 - Qx_1) \ . \tag{8.13}$$

By using the Cauchy-Riemann equations for P, Q and for p, q, it may be shown that

$$2(U_{,1} + W_{,2}) = (1-2\nu)P \ , \tag{8.14}$$

$$2(W_{,1} - U_{,2}) = 2(1-\nu)Q \ . \tag{8.15}$$

We define the functions $r(x_1, x_2)$, $s(x_1, x_2)$ by

$$r = -2\xi_1 + 2U \ , \tag{8.16}$$

$$s = 2\xi_2 - 2W \ . \tag{8.17}$$

Using (8.14), (8.15) and (8.8), it may be shown that r and s are related by the Cauchy-Riemann equations

$$r_{,\alpha} + \epsilon_{\beta\alpha} s_{,\beta} = 0 \qquad \text{in } R \ . \tag{8.18}$$

Equations (8.18) imply that in a simply-connected region R, there exists a regular function $\psi(z)$ such that

$$\psi(z) = r + is \qquad \text{in } R \ . \tag{8.19}$$

Equations (8.16), (8.17), (8.12), (8.13) now imply that

$$2\xi_1 = (3-4\nu)p - \frac{1}{4}(Px_1 + Qx_2) - r \qquad \text{in } R \ , \tag{8.20}$$

$$2\xi_2 = (3-4\nu)q - \frac{1}{4}(Px_2 - Qx_1) + s \qquad \text{in } R \ , \tag{8.21}$$

that is,

$$2(\xi_1 + i\xi_2) = (3-4\nu)\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} \qquad \text{in R} . \tag{8.22}$$

Now defining $V(z)$ by

$$V(z) = 2(\xi_1 + i\xi_2) \qquad \text{in R} , \tag{8.23}$$

it follows that $V(z)$ is non-trivial and satisfies (8. 3), (8. 4). This completes the proof of Theorem 8. 2.

Theorem 8. 3. Let $V(z)$ satisfy (8. 3) and (8. 4). If $\nu \leq \frac{1}{2}$ or $\nu \geq 1$, then $V(z) \equiv 0$ in R.

The proof of the uniqueness theorem given by Muskhelishvili on pages 156 - 158 of [12] may be readily adapted to supply a proof of Theorem 8. 3.

We note that the equivalence between the differential equation formulation of the eigenvalue problem and the complex variable formulation does not hold when $\nu = \frac{1}{2}, 1$.

We turn now to the reduction of the boundary-value problem (8. 3), (8. 4) to a Fredholm integral equation of the second kind. We refer the reader to Muskhelishvili [12] for details and we merely record here the final results.

Let the bounded, simply-connected region R have a boundary B with continuous curvature and suppose R is mapped conformally onto the unit disc in the complex ρ-plane by $z = \omega(\rho)$. Denoting the circumference of the unit circle by γ, we have $\omega'(\rho) \neq 0$ inside and on γ. (See §47 of [12].) Assume, without loss of generality, that

$\omega(0) = 0$. Then, the boundary-value problem (8.3), (8.4) can be reduced to the following problem.

Find a regular function $\varphi_o'(\rho)$ in $|\rho| < 1$ such that $\varphi_o'(\rho)$ satisfies

$$\varkappa\, \varphi_o'(\sigma_o) - \frac{1}{2\pi i} \int_\gamma \frac{\partial}{\partial \sigma_o} \left\{ \frac{\omega(\sigma) - \omega(\sigma_o)}{\sigma - \sigma_o} \right\} \frac{\overline{\varphi_o'(\sigma)}}{\overline{\omega'(\sigma)}}\, d\sigma = 0 , \qquad (8.24)$$

where $\varkappa = 3 - 4\nu \neq 0$, and σ, σ_o are arbitrary points on the unit circle γ : $|\sigma| = 1$. The derivation of the integral equation (8.24) may be found in [12]. (See also S. G. Mikhlin [23].) By a non-trivial function $\varphi_o'(\rho)$ we shall mean a function $\varphi_o'(\rho)$ not identically zero.

We state the following theorems establishing the equivalence between the integral equation (8.24) and the boundary-value problem (8.3), (8.4).

Theorem 8.4. Let $\varphi_o'(\rho)$ be non-trivial and satisfy (8.24). Then there exists a non-trivial $V(z)$ satisfying (8.3), (8.4).

Theorem 8.5. Let $V(z)$ be non-trivial and satisfy (8.3), (8.4) with $\nu \neq 3/4$. Then there exists a non-trivial function $\varphi_o'(\rho)$ satisfying (8.24).

Theorems 8.4 and 8.5 may be established by suitably adapting the material of Chapter XIV of [12]. We remark here that the equation (8.24) is not valid when $\nu = 3/4$. The special value $\nu = 3/4$ will be discussed later in this section. The integral equation (8.24) may be written

$$\varphi_o'(\sigma_o) - \frac{\lambda}{2\pi i}\int_\gamma T(\sigma_o,\sigma)\,\overline{\varphi_o'(\sigma)}\,d\sigma = 0 \;, \tag{8.25}$$

where $\lambda = \varkappa^{-1} = (3-4\nu)^{-1}$ and

$$T(\sigma_o,\sigma) = \frac{1}{\overline{\omega'(\sigma)}}\,\frac{\partial}{\partial\sigma_o}\left\{\frac{\omega(\sigma)-\omega(\sigma_o)}{\sigma-\sigma_o}\right\} . \tag{8.26}$$

Separating $T(\sigma_o,\sigma)$ and $\varphi_o'(\sigma)$ into real and imaginary parts and substituting into (8.25), one obtains two real Fredholm equations, which may be reduced to a single Fredholm integral equation of the second kind. We have the following theorems concerning the integral equation (8.25).

<u>Theorem 8.6.</u> The eigenvalues λ of the integral equation (8.25) are real, have finite multiplicity, and may accumulate only at infinity.

The fact that the eigenvalues of (8.25) are real is proved by Mikhlin in §47 of [23], where he uses the work of D. I. Sherman [24]. The remainder of Theorem 8.6 for general Fredholm integral equations is proved in §8 of [23].

In terms of the eigenvalues ν, Theorem 8.6 says that eigenvalues of problem (8.1), (8.2) not equal to 1, 3/4, 1/2 have finite multiplicity, which is the two-dimensional analog of Mikhlin's result given in (iii), §7 here. The assertion concerning the eigenvalues λ possibly accumulating only at infinity says that the eigenvalues of problem (8.1), (8.2) may accumulate only at $\nu = 3/4$. Thus, in two dimensions, one obtains a stronger result than that of Mikhlin for three dimensions, which says that the eigenvalues ν may accumulate only at $\nu = 1$, $3/4$, and $1/2$. (See result (ii) at end of §7 here.)

Theorem 8.7. If the region R can be mapped conformally by a rational function onto the unit disc, then the kernel $T(\sigma_o, \sigma)$ of (8.25) is degenerate and the spectrum is finite.

The proof of this result may be found in §44 and §4 of [23]. For our purposes, it is convenient to outline part of the proof here.

As shown in §44 of [23], if $\omega(\sigma)$ is rational, the kernel $T(\sigma_o, \sigma)$, given by (8.26), may be expressed as

$$T(\sigma_o, \sigma) = \sum_{k=1}^{N} a_k(\sigma_o) b_k(\sigma) , \qquad (8.27)$$

which shows that the kernel is degenerate. Substitute from (8.27) into the integral equation (8.25) to get

$$\varphi'_o(\sigma_o) - \frac{\lambda}{2\pi i} \sum_{k=1}^{N} a_k(\sigma_o) \int_\gamma b_k(\sigma) \overline{\varphi'_o(\sigma)}\, d\sigma = 0 . \qquad (8.28)$$

Let the complex constants c_k be defined by

$$c_k = \int_\gamma b_k(\sigma) \overline{\varphi'_o(\sigma)}\, d\sigma , \qquad (8.29)$$

and write (8.28) as

$$\varphi'_o(\sigma_o) = \frac{\lambda}{2\pi i} \sum_{k=1}^{N} c_k a_k(\sigma_o) . \qquad (8.30)$$

Take the complex conjugate of equation (8.30), multiply both sides of the resulting equation by $b_j(\sigma_o)$, and integrate with respect to σ_o to find

$$\int_\gamma b_j(\sigma_o)\overline{\varphi_o'(\sigma_o)}\, d\sigma = -\frac{\lambda}{2\pi i}\sum_{k=1}^{N}\overline{c_k}\int_\gamma b_j(\sigma_o)\overline{a_k(\sigma_o)}\, d\sigma_o \quad . \tag{8.31}$$

Recall the definition (8.29) of c_j and write (8.31) as

$$c_j = \lambda\sum_{k=1}^{N}\overline{c_k}\,\gamma_{jk} \,, \qquad (j = 1,\ldots, N) \tag{8.32}$$

where the complex constants γ_{jk} are known and are given by

$$\gamma_{jk} = -\frac{1}{2\pi i}\int_\gamma b_j(\sigma_o)\overline{a_k(\sigma_o)}\, d\sigma_o \,. \qquad (j, k = 1,\ldots, N) \tag{8.33}$$

Equation (8.30) gives the solution $\varphi_o'(\sigma_o)$ of (8.25), once the constants c_k are known. The constants c_k are determined from the system of linear equations (8.32), which, when separated into real and imaginary parts, yields a system of 2N equations for the 2N unknowns Re c_k, Im c_k. The necessary condition to be satisfied by the coefficients of this system for a non-trivial solution to exist is that a $2N \times 2N$ determinant of the form $|\delta_{ij} - \lambda a_{ij}|$ should vanish, implying that the spectrum of equation (8.25) is finite.

To conclude this section, we return to the basic eigenvalue problem (8.1), (8.2) and state the following theorem concerning the value $\nu = 3/4$.

Theorem 8.8. The value $\nu = 3/4$ is an eigenvalue of the problem (8.1), (8.2) if and only if the region R can be mapped conformally by a rational function onto the unit disc. If $\nu = 3/4$ is an eigenvalue, it has infinite multiplicity.

The proof of Theorem 8.8 may be found in §9, Chapter IV of A. V. Bitsadze [25], on observing that the system of equations considered by him are identical to the equations resulting from (8.1) here on setting $\nu = 3/4$ and replacing ξ_2 by $-\xi_2$.

V. APPLICATION OF THE RESULTS OF CHAPTER IV TO THE KORN EIGENVALUE PROBLEM

9. The K-spectrum

(i) The circle:

Let the region R be a circle of radius r, with its center at the origin. Then the mapping function $\omega(\rho)$, defined in §8, becomes

$$\omega(\rho) = r\rho \qquad (9.1)$$

and the kernel $T(\sigma_o, \sigma)$ given by equation (8.26) is identically zero. Thus, the integral equation (8.25) has no finite eigenvalues. In terms of the eigenvalues ν, this means that $\nu = 1, 3/4, 1/2$ are the only possible eigenvalues of (8.1), (8.2) for a circle.

Theorem 7.2 says that $\nu = 1$ and $\nu = 1/2$ are eigenvalues of infinite multiplicity for any region, and so it remains to consider the value $\nu = 3/4$. The fact that $\nu = 3/4$ is an eigenvalue of infinite multiplicity for a circle follows from Theorem 8.8 given at the end of the last chapter. It is of interest to give a direct verification here.

When $\nu = 3/4$, V(z) as defined by equation (8.3) is given by

$$V(z) = -z\,\overline{\varphi'(z)} - \overline{\psi(z)} \qquad \text{in } R, \qquad (9.2)$$

and V(z) must satisfy

$$V(z) = 0 \qquad \text{on } B. \qquad (9.3)$$

Taking the complex conjugate of (9.2), we get

$$\overline{V(z)} = -\bar{z}\,\varphi'(z) - \psi(z) \qquad \text{in } R. \qquad (9.4)$$

Let g(z) be an arbitrary regular function in R, and define the regular functions $\varphi'(z)$ and $\psi(z)$ by

$$\varphi'(z) = z\,g(z) , \tag{9.5}$$

$$\psi(z) = -r^2 g(z) . \tag{9.6}$$

Then equation (9.4) gives

$$\overline{V(z)} = (r^2 - z\bar{z})g(z) , \tag{9.7}$$

and we see that

$$\overline{V(z)} = 0 \qquad \text{on } B , \tag{9.8}$$

where $z\bar{z} = r^2$. Thus, the function $V(z)$ given by equation (9.7), where $g(z)$ is an arbitrary regular function in R, has the required form (9.2) and satisfies the boundary condition (9.3). Therefore, $\nu = 3/4$ is an eigenvalue of infinite multiplicity for a circle.

For a circle, we have found that the spectrum of eigenvalues ν consists of the discrete set

$$\nu = 1 , 3/4 , 1/2 , \tag{9.9}$$

each occurring with infinite multiplicity. We now recall the transformation Theorem 5.2, which says that if ν is an eigenvalue for the homogeneous displacement boundary-value problem of plane strain for a region R, then, for $\nu \neq 1$,

$$K = 1/1-\nu \tag{9.10}$$

is an eigenvalue of the Korn eigenvalue problem for R. From (9.10) with $\nu = 3/4,\ 1/2$, we find that $K = 4,\ 2$ are eigenvalues of infinite multiplicity for a circle. Recalling Remark 1 of §4, which says that $K = 1$ is an eigenvalue of infinite multiplicity for any region, we thus obtain the result that the spectrum of eigenvalues K for a circle consists of the set

$$K = 1,\ 2,\ 4\ , \tag{9.11}$$

each occurring with infinite multiplicity. Thus, we recover the result of Payne and Weinberger [5], described at the end of §3 of this work.

(ii) Regions mapped conformally onto the unit disc by a rational mapping.

Let the region R be mapped conformally onto the unit disc by the rational function

$$z = \omega(\rho)\ . \tag{9.12}$$

Theorem 8.7 now says that the kernel of the integral equation (8.25) is degenerate and that the spectrum of eigenvalues λ of (8.25) is finite. The method of proof of Theorem 8.7, as outlined in §8, now affords a means of explicit calculation of the eigenvalues λ. We illustrate the procedure in the following example.

Consider the mapping defined by

$$x_1+ix_2 \equiv z = \omega(\rho) = \rho + m\rho^2\ , \tag{9.13}$$

where the parameter m satisfies $0 \le m < \frac{1}{2}$. When the complex variable $\rho = e^{is}$ describes the unit circle γ, $(0 \le s \le 2\pi)$, (x_1, x_2) describes in the z-plane a curve B, with parametric representation

$$\begin{aligned} x_1 &= \cos s + m \cos 2s\ , \\ x_2 &= \sin s + m \sin 2s\ . \end{aligned} \tag{9.14}$$

This curve is called Pascal's limacon. For $0 \le m < \frac{1}{2}$, $\omega(\rho)$ is regular in $|\rho| < 1$ and $\omega'(\rho) = 1+2m\rho \neq 0$ in $|\rho| \le 1$. Then the relation (9.13) gives the conformal mapping of the region inside Pascal's limacon onto the unit disc. Notice that when $m = 0$, Pascal's limacon reduces to a circle, while when $m = \frac{1}{2}$, the curve becomes a cardioid with a cusp at $\rho = -1$, where $\omega'(\rho) = 0$.

For the mapping (9.13), the kernel $T(\sigma_o, \sigma)$ given by equation (8.26) becomes

$$T(\sigma_o, \sigma) = \frac{1}{1+2m\bar{\sigma}} \frac{\partial}{\partial \sigma_o} \left\{ \frac{\sigma - \sigma_o + m(\sigma^2 - \sigma_o^2)}{\sigma - \sigma_o} \right\} = \frac{m}{1+2m\bar{\sigma}} , \qquad (9.15)$$

which is independent of σ_o. In the notation used in the proof of Theorem 8.7, we have

$$T(\sigma_o, \sigma) = a_1(\sigma_o) b_1(\sigma) , \qquad (9.16)$$

where

$$a_1(\sigma_o) = 1 , \qquad (9.17)$$

$$b_1(\sigma) = \frac{m}{1+2m\bar{\sigma}} . \qquad (9.18)$$

From equation (8.32) we thus get

$$c_1 = \lambda \overline{c_1} \gamma_{11} , \qquad (9.19)$$

where

$$\gamma_{11} = -\frac{1}{2\pi i} \int_\gamma b_1(\sigma_o) \overline{a_1(\sigma_o)} \, d\sigma_o , \qquad (9.20)$$

the latter expression coming from equation (8.33). From equations (9.17), (9.18), and (9.20), using the fact that on the unit circle

$\overline{\sigma_o} = \sigma_o^{-1}$, we obtain

$$\gamma_{11} = -\frac{1}{2\pi i}\int_\gamma \frac{m d\sigma_o}{1+2m\overline{\sigma_o}} = -\frac{1}{2\pi i}\int_\gamma \frac{m\sigma_o d\sigma_o}{\sigma_o+2m} . \qquad (9.21)$$

The integrand in (9.21) has a simple pole inside γ at $\sigma_o = -2m$ and thus we find

$$\gamma_{11} = 2m^2 . \qquad (9.22)$$

Using (9.22), equation (9.19) reads

$$c_1 = \lambda \overline{c_1} 2m^2 , \qquad (9.23)$$

and so, if

$$\lambda = \pm 1/2m^2 , \qquad (9.24)$$

(9.23) admits a non-trivial solution for c_1 , that is, the integral equation (8.25) has a non-trivial solution $\varphi_o'(\rho)$.

We recall from §8 that

$$\lambda = \varkappa^{-1} = (3-4\nu)^{-1} , \qquad (9.25)$$

and so, from (9.24), we find that the values

$$\nu = \frac{3}{4} - \frac{m^2}{2} , \quad \frac{3}{4} + \frac{m^2}{2} \qquad (9.26)$$

are eigenvalues for the basic problem (8.1), (8.2). Since the mapping function $\omega(\rho)$ in (9.13) is rational, Theorem 8.8 implies that $\nu = 3/4$ is an eigenvalue of infinite multiplicity for the basic eigenvalue problem (8.1), (8.2) for Pascal's limacon. Recalling Theorem 7.2, which says that $\nu = 1, \frac{1}{2}$ are eigenvalues of infinite multiplicity for any region, we have the complete spectrum

$$\nu = \frac{1}{2}, \frac{3}{4} - \frac{m^2}{2}, \frac{3}{4}, \frac{3}{4} + \frac{m^2}{2}, 1 \qquad (9.27)$$

for Pascal's limacon, where the values 1/2 , 3/4 , 1 occur with infinite multiplicity and the other eigenvalues have finite multiplicity. (See Theorem 8. 6.)

We now deduce the spectrum for the eigenvalues K of the Korn problem, using relation (9. 10) and the fact that K = 1 is an eigenvalue of infinite multiplicity for any region. In this manner, we obtain the spectrum

$$K = 1 , 2 , \frac{4}{1+2m^2} , 4 , \frac{4}{1-2m^2} , \qquad (9.28)$$

where the values 1, 2, 4 occur with infinite multiplicity, and the other eigenvalues have finite multiplicity. We note from (9. 28) that the largest eigenvalue K for Pascal's limacon is given by

$$K_{max} = \frac{4}{1-2m^2} . \qquad (9.29)$$

We conclude this discussion by stating the analog of Theorem 8. 8 for the Korn eigenvalue problem: the value K = 4 is an eigenvalue of the Korn problem if and only if the region R can be mapped conformally onto the unit disc by a rational function. When K = 4 is an eigenvalue, it has infinite multiplicity.

(iii) Some remarks on the largest eigenvalue K of the Korn problem.

The fact that the eigenvalues ν of the basic problem (8. 1),(8.2) may accumulate only at $\nu = 3/4$, as shown by Theorem 8. 6, means that for a given region R, there exists a largest eigenvalue, say $\nu = \nu_{max}$ in the interval $[\frac{1}{2}, 1)$. Relation (9.10) then implies that for this R there exists a largest eigenvalue K, say $K = K_{max}$, given by

$$K_{max} = \frac{1}{1-\nu_{max}} \quad . \tag{9.30}$$

For a region which can be mapped conformally by a rational function onto the unit disc, from the concluding remark of (ii) here we obtain the lower bound

$$K_{max} \geq 4 \tag{9.31}$$

for all such regions.

The example furnished by Ericksen to prove Theorem 7.3 illustrates the fact that there is no universal upper bound on the largest eigenvalue K_{max} for all plane regions. To see this, we consider a value of ν arbitrarily close to one, say

$$\nu = 1-\delta \ , \tag{9.32}$$

where $\delta > 0$ is arbitrarily small. Then, by Theorem 7.3, we can construct an ellipse R with axes in the ratio

$$\left(\frac{2\nu-1}{2(1-\nu)}\right)^{\frac{1}{2}} = \left(\frac{1}{2\delta} - 1\right)^{\frac{1}{2}} \ , \tag{9.33}$$

such that $\nu = 1-\delta$ is an eigenvalue for R. Relation (9.10) now implies that

$$K = \frac{1}{1-\nu} = \frac{1}{\delta} \tag{9.34}$$

is an eigenvalue for the Korn problem for this ellipse, the ratio of whose axes, as given by (9.33), is arbitrarily large. Thus, for a sufficiently long thin ellipse, the eigenvalue K given by (9.34) is arbitrarily large.

(iv) The relation between Korn's constant and the largest eigenvalue.

We wish to demonstrate here that the largest eigenvalue of the Korn eigenvalue problem

$$(2-K)u_{\alpha,\beta\beta} - Ku_{\beta,\beta\alpha} = 0 \qquad \text{in } R\,, \tag{9.35}$$

$$[(2-K)u_{\alpha,\beta} - Ku_{\beta,\alpha}]n_\beta = 0 \qquad \text{on } B\,, \tag{9.36}$$

is in fact the Korn constant $K_R^{(2)}$, if we assume the completeness of the eigenfunctions of (9.35), (9.36). We recall that Payne and Weinberger [5] have established a completeness result for the special cases of a sphere (three dimensions) and a circle.

From the previous results of this section, we know that the eigenvalues of (9.35), (9.36) may accumulate only at $K = 4$ and that $K = 1, 2$ are eigenvalues of infinite multiplicity for any region. For the purposes of our subsequent discussion, it is convenient to order the eigenvalues in the following way. Let K_n $(n = 1, 2, \ldots)$ denote the sequence of eigenvalues ordered such that

$$K_2 < K_4 < \ldots K_{2n} \ldots < 4 < \ldots K_{2n-1} \ldots < K_3 < K_1\,, \tag{9.37}$$

where $K_2 = 1$, $K_4 = 2$, and K_1 is the largest eigenvalue. We let $K_o = 4$, which may or may not be an eigenvalue, depending on the shape of the region R. (See the comment at the end of §9 (ii).) The eigenfunctions corresponding to the eigenvalues K_k will be denoted by $\underline{u}^{(k)}$ (k = 0, 1, 2, ...) and we note that the special forms of $\underline{u}^{(2)}$, $\underline{u}^{(4)}$ are given in Remarks 1 and 2 of §4.

Let the vector field $\underline{f}$ have square-integrable first derivatives and satisfy the normalization condition

$$\int_R (f_{\alpha,\beta} - f_{\beta,\alpha}) dA = 0 \ . \qquad (9.38)$$

We employ the bilinear forms

$$S(\underline{f}, \underline{g}) = \frac{1}{4} \int_R (f_{\alpha,\beta} + f_{\beta,\alpha})(g_{\alpha,\beta} + g_{\beta,\alpha}) dA \ , \qquad (9.39)$$

$$D(\underline{f}, \underline{g}) = \int_R f_{\alpha,\beta} \, g_{\alpha,\beta} \, dA \ , \qquad (9.40)$$

which appeared in equations (4.46) and (4.47), respectively. We also use the notation

$$S(\underline{f}, \underline{f}) = S(\underline{f}) \quad , \quad D(\underline{f}, \underline{f}) = D(\underline{f}) \ . \qquad (9.41)$$

According to Remark 5 of §4, the eigenfunctions $\underline{u}^{(k)}$ have the orthogonality properties

$$S(\underline{u}^{(i)}, \underline{u}^{(j)}) = D(\underline{u}^{(i)}, \underline{u}^{(j)}) = 0 \ , \qquad i \neq j \ . \qquad (9.42)$$

We define the norm of $\underline{f}$ by

$$\|\underline{f}\| = [S(\underline{f})]^{\frac{1}{2}} \ . \qquad (9.43)$$

The Fourier coefficients of $\underline{f}$ are defined by

$$c_k = S(\underline{f}, \underline{u}^{(k)}) \qquad (k = 0, 1, 2, \ldots) \ , \qquad (9.44)$$

where the eigenfunctions $\underline{u}^{(k)}$ are normalized such that

$$S(\underline{u}^{(k)}, \underline{u}^{(k)}) = S(\underline{u}^{(k)}) = 1 \ . \qquad (9.45)$$

We adopt the convention that $c_o = 0$, if $K_o = 4$ is <u>not</u> an eigenvalue for R. The eigenfunctions $\underline{u}^{(k)}$ are said to be <u>complete</u> if

$$\lim_{n\to\infty} S(\underline{f} - \sum_{k=0}^{n} c_k \underline{u}^{(k)}) = 0 , \qquad (9.46)$$

for all vector fields $\underline{f}$ with square-integrable first derivatives satisfying (9. 38)

Suppose now that the eigenfunctions $\underline{u}^{(k)}$ are complete in the sense of this definition. Equation (9. 46) implies that

$$S(\underline{f}) = \sum_{k=0}^{\infty} c_k^2 , \qquad (9.47)$$

where we have used (9. 39), (9. 42), (9. 44), and (9. 45). Since the vector fields $\underline{f}$ and $\underline{u}^{(k)}$ satisfy the normalization condition (9. 38), we may apply Korn's inequality in the second case to obtain

$$D(\underline{f} - \sum_{k=0}^{n} c_k \underline{u}^{(k)}) \leq K_R^{(2)} S(\underline{f} - \sum_{k=0}^{n} c_k \underline{u}^{(k)}) , \qquad (9.48)$$

where $K_R^{(2)}$ is the Korn constant. The inequality (9. 48), in conjunction with equation (9. 46), implies that

$$\lim_{n\to\infty} D(\underline{f} - \sum_{k=0}^{n} c_k \underline{u}^{(k)}) = 0 . \qquad (9.49)$$

Equation (9. 49) may be written as

$$\lim_{n\to\infty} \left[D(\underline{f}) + \sum_{k=0}^{n} c_k^2 D(\underline{u}^{(k)}) - 2 \sum_{k=0}^{n} c_k D(\underline{f}, \underline{u}^{(k)}) \right] = 0 , \qquad (9.50)$$

where we have used the orthogonality condition (9. 42).

To simplify (9.50), we proceed as follows. By applying the divergence theorem, we obtain, for any eigenfunction $\underline{u}^{(k)}$ with corresponding eigenvalue K_k,

$$(2-K_k)D(\underline{f},\underline{u}^{(k)}) = (2-K_k)\int_B f_\alpha u^{(k)}_{\alpha,\beta} n_\beta ds - (2-K_k)\int_R f_\alpha u^{(k)}_{\alpha,\beta\beta} dA \ .$$

By (9.35), (9.36) this reduces to

$$(2-K_k)D(\underline{f},\underline{u}^{(k)}) = K_k \int_B f_\alpha u^{(k)}_{\beta,\alpha} n_\beta ds - K_k \int_R f_\alpha u^{(k)}_{\beta,\beta\alpha} dA$$

$$= K_k \int_R f_{\alpha,\beta} u^{(k)}_{\beta,\alpha} dA = K_k [2S(\underline{f},\underline{u}^{(k)}) - D(\underline{f},\underline{u}^{(k)})] ,$$

whence

$$D(\underline{f},\underline{u}^{(k)}) = K_k S(\underline{f},\underline{u}^{(k)}) = K_k c_k \ , \qquad (9.51)$$

for any $\underline{f}$ with the assumed properties. It also follows from (9.51) with $\underline{f} = \underline{u}^{(k)}$ that

$$D(\underline{u}^{(k)}) = K_k S(\underline{u}^{(k)}) = K_k \ . \qquad (9.52)$$

Using (9.51) and (9.52) in (9.50), we find that

$$D(\underline{f}) = \sum_{k=0}^{\infty} c_k^2 K_k \ . \qquad (9.53)$$

Comparing (9.47) and (9.53), we conclude that

$$D(\underline{f}) \leq K_1 S(\underline{f}) \ , \qquad (9.54)$$

since K_1 is the largest eigenvalue. (See (9.37).) Thus, if completeness in the form (9.46) is assumed, it follows from (9.54) that the Korn constant $K_R^{(2)}$ coincides with the largest eigenvalue K_1.

It may be noted that Mikhlin in [19] considers the completeness question for the eigenfunctions of the homogeneous displacement boundary-value problem of three-dimensional, linear homogeneous isotropic elastostatics. By adapting his analysis, it seems likely that a completeness result for the Korn eigenvalue problem could be established.

(v) The first case.

The eigenvalue problem associated with Korn's inequality in the first case was described at the beginning of Chapter III. (See equations (5.1), (5.2).) We noted that this problem could be directly identified with the homogeneous displacement boundary-value problem of linear homogeneous isotropic elastostatics (in either two or three dimensions) if the parameter K was taken to be $1/\nu$. Thus, the largest eigenvalue of this problem corresponds to the smallest eigenvalue ν. Since $\nu = \frac{1}{2}$ is the smallest eigenvalue in $[\frac{1}{2}, 1]$ for any region, we find that $K = 2$ is the largest eigenvalue of (5.1), (5.2) for any region. Assuming a completeness result for the eigenfunctions of (5.1), (5.2), we recover the result that the Korn constant for the first case is equal to two for any region (see §1).

10. Upper Bounds on the Eigenvalues K

We recall here the integral equation (8.25), which reads, for $\nu \neq 3/4$,

$$\varphi_o'(\sigma_o) - \frac{\lambda}{2\pi i}\int_\gamma \frac{\partial}{\partial\sigma_o}\left\{\frac{\omega(\sigma)-\omega(\sigma_o)}{\sigma-\sigma_o}\right\}\frac{\overline{\varphi_o'(\sigma)}}{\overline{\omega'(\sigma)}}\, d\sigma = 0\,, \qquad (10.1)$$

where

$$\lambda = (3-4\nu)^{-1} . \qquad (10.2)$$

The uniqueness result of Theorem 8.3 and the equivalence between the complex variable formulation and the integral equation formulation of the basic eigenvalue problem, as established by Theorems 8.4 and 8.5, now imply that the eigenvalues λ of (10.1) must satisfy

$$|\lambda| > 1 . \qquad (10.3)$$

We note that if λ is an eigenvalue of equation (10.1), then $-\lambda$ is also an eigenvalue. This follows from the observation that if $\varphi'_o(\sigma_o)$ is an eigenfunction corresponding to the eigenvalue λ, then $i\varphi'_o(\sigma_o)$ is an eigenfunction corresponding to the eigenvalue $-\lambda$. The corresponding result for the eigenvalues ν of the basic eigenvalue problem is given in Theorem 7.4, which establishes that the eigenvalues ν occur in pairs of the form $3/4 \pm \delta$ $(0 < \delta < 1/4)$.

A lower bound on the eigenvalues λ, of the form

$$|\lambda| \geq \lambda_o > 1 , \qquad (10.4)$$

furnishes an upper bound on the eigenvalues K. This may be seen from the fact that the eigenvalues ν may be written as

$$\nu = \frac{3}{4} \pm \frac{1}{4|\lambda|} . \qquad (10.5)$$

Thus, from (10.4), we obtain

$$\nu = \frac{3}{4} + \frac{1}{4|\lambda|} \leq \frac{3}{4} + \frac{1}{4\lambda_o} , \qquad (10.6)$$

and recalling that the eigenvalues ν and the eigenvalues K are related by

$$K = 1/1-\nu \ , \tag{10.7}$$

we find that

$$K \le \frac{4\lambda_o}{\lambda_o - 1} \ . \tag{10.8}$$

Some methods for obtaining lower bounds on the eigenvalues λ of the integral equation (10. 1) are given in the remainder of this section. For this purpose, it is convenient to rewrite (10. 1) as a system of two real integral equations as follows. We introduce the notation

$$\sigma = e^{is} \quad (0 \le s \le 2\pi) \ , \quad \sigma_o = e^{is_o} \quad (0 \le s_o \le 2\pi) \ ,$$

$$\frac{1}{2\pi} T(\sigma_o, \sigma)\sigma = M(s_o, s) + iN(s_o, s) \ ,$$

$$\varphi_o'(\sigma) = p_o(s) + iq_o(s) \ , \tag{10.9}$$

where $T(\sigma_o, \sigma)$ is defined by equation (8. 26). The integral equation (10. 1) may then be written in the form

$$\underline{u}(s_o) - \lambda \int_0^{2\pi} A(s_o, s)\underline{u}(s)ds = 0 \ , \tag{10.10}$$

where the column vector $\underline{u}(s)$ is given by

$$\underline{u}(s) = \begin{pmatrix} p_o(s) \\ q_o(s) \end{pmatrix} \ , \tag{10.11}$$

and the matrix $A(s_o, s)$ is given by

$$A(s_o, s) \equiv (a_{\alpha\beta}) = \begin{pmatrix} M(s_o, s) & N(s_o, s) \\ N(s_o, s) & -M(s_o, s) \end{pmatrix} . \tag{10.12}$$

Suppose now that λ is an eigenvalue of (10. 10), so that $\underline{u}(s)$ is not identically zero and satisfies (10. 10). Let

$$||\underline{u}|| = (p_o^2+q_o^2)^{\frac{1}{2}} , \tag{10. 13}$$

and

$$||A|| = (\mathrm{Tr}\ A^T A)^{\frac{1}{2}} , \tag{10. 14}$$

where A^T denotes the transpose of the matrix A , that is, $A^T = (a_{\beta\alpha})$ and Tr B denotes the trace of the matrix $B = (b_{\alpha\beta})$, that is, $\mathrm{Tr}\, B = b_{\alpha\alpha}$. From equation (10. 10) we have

$$\|\underline{u}(s_o)\| \le |\lambda| \int_0^{2\pi} \|A(s_o, s)\| \ \|\underline{u}(s)\| ds , \tag{10. 15}$$

and applying the Schwarz inequality we get

$$||\underline{u}(s_o)||^2 \le \lambda^2 \left(\int_0^{2\pi} ||A(s_o, s)||^2 ds \right)\left(\int_0^{2\pi} ||\underline{u}(s)||^2 ds \right) . \tag{10. 16}$$

If we now impose the normalization condition

$$\int_0^{2\pi} ||\underline{u}(s)||^2 ds = 1 , \tag{10. 17}$$

and integrate relation (10. 16) with respect to s_o, we find

$$\lambda^2 \ge \left[\int_0^{2\pi} \int_0^{2\pi} ||A(s_o, s)||^2 ds\, ds_o \right]^{-1} . \tag{10. 18}$$

Using definition (10. 14) and equation (10. 12), we may write (10. 18) in the form

$$\lambda^2 \ge \left[2 \int_0^{2\pi} \int_0^{2\pi} (M^2 + N^2) ds\, ds_o \right]^{-1} . \tag{10. 19}$$

Finally, using (10. 9) to write

$$M^2+N^2 = \frac{1}{4\pi^2} |T(s_o, s)|^2 , \tag{10.20}$$

we express (10. 19) in the form

$$\lambda^2 \geq 2\pi^2 \left[\int_0^{2\pi} \int_0^{2\pi} |T(s_o, s)|^2 ds\, ds_o \right]^{-1} . \tag{10.21}$$

If, instead of (10. 17) we impose the normalization condition

$$\max_{s \in [0,2\pi]} ||\underline{u}(s)|| = 1 , \tag{10.22}$$

then, from (10. 15) and (10.22), we find

$$||\underline{u}(s_o)|| \leq |\lambda| \int_0^{2\pi} ||A(s_o, s)|| \max_s ||u(s)|| ds = |\lambda| \int_0^{2\pi} ||A(s_o, s)|| ds \tag{10.23}$$

and using (10. 22) again, we obtain

$$1 \leq |\lambda| \max_{s_o} \int_0^{2\pi} ||A(s_o, s)|| ds . \tag{10.24}$$

Using (10. 14) and (10. 12), we may write (10.24) as

$$|\lambda| \geq \left[\max_{s_o} \int_0^{2\pi} [2(M^2+N^2)]^{\frac{1}{2}} ds \right]^{-1} , \tag{10.25}$$

or, using (10. 9), as

$$|\lambda| \geq (2)^{\frac{1}{2}} \pi \left[\max_{s_o} \int_0^{2\pi} |T(s_o, s)| ds \right]^{-1} . \tag{10.26}$$

Different lower bounds may be obtained by using iterated kernels, as we now describe. Writing (10. 10) in the form

$$\underline{u}(s_o) = \lambda \int_0^{2\pi} A(s_o, s) \underline{u}(s) ds , \tag{10.27}$$

it follows that

$$\underline{u}(s_o) = \lambda^2 \int_0^{2\pi} \int_0^{2\pi} A(s_o, t)A(t, s)\underline{u}(s)ds\, dt \quad ,$$

that is,

$$\underline{u}(s_o) = \lambda^2 \int_0^{2\pi} A_1(s_o, s)\underline{u}(s)ds \quad , \tag{10.28}$$

where

$$A_1(s_o, s) = \int_0^{2\pi} A(s_o, t)A(t, s)dt \quad . \tag{10.29}$$

Treating equation (10.28) in the same way as we dealt with equation (10.10), we obtain the lower bounds

$$\lambda^4 \geq \left[\int_0^{2\pi} \int_0^{2\pi} ||A_1(s_o, s)||^2 ds\, ds_o \right]^{-1} \quad , \tag{10.30}$$

corresponding to (10.18), and

$$\lambda^2 \geq \left[\max_{s_o} \int_0^{2\pi} ||A_1(s_o, s)|| ds \right]^{-1} \quad , \tag{10.31}$$

which corresponds to inequality (10.24).

It seems difficult to evaluate the relative merits of the lower bounds (10.21), (10.26), (10.30), and (10.31) in general, but some indications may be found from the particular case of Pascal's limacon, treated in §9, for which the exact eigenvalues are known. Thus, from (9.24) we have the eigenvalues

$$\lambda = \pm \frac{1}{2m^2} \quad , \tag{10.32}$$

and the kernel $T(\sigma_o, \sigma)$ is given by (9.15), which reads

$$T(\sigma_o, \sigma) = \frac{m}{1+2m\overline{\sigma}} \quad , \tag{10.33}$$

where $\sigma = e^{is}$ $(0 \le s \le 2\pi)$. Thus, the kernel is independent of $\sigma_o = e^{is_o}$. We now evaluate the integral appearing on the right side of inequality (10.21). Let

$$I = \int_0^{2\pi}\int_0^{2\pi} |T(s_o, s)|^2 ds\, ds_o = 2\pi \int_0^{2\pi} |T(s_o, s)|^2 ds \quad . \tag{10.34}$$

We write I as a contour integral around the unit circle, that is:

$$I = \frac{2\pi}{i} \int_\gamma \frac{T\overline{T}}{\sigma} d\sigma = \frac{m^2 2\pi}{i} \int_\gamma \frac{d\sigma}{(1+2m\sigma)(\sigma+2m)} \quad . \tag{10.35}$$

The integrand has a simple pole inside γ at $\sigma = -2m$, and so, by the Residue Theorem, we obtain

$$I = \frac{4\pi^2 m^2}{1-4m^2} \quad . \tag{10.36}$$

Recalling the definition of I from (10.34), the inequality (10.21) becomes

$$\lambda^2 \ge \frac{1-4m^2}{2m^2} \quad . \tag{10.37}$$

The right hand side of inequality (10.26) contains the integral

$$\int_0^{2\pi} |T(s_o, s)| ds = m \int_0^{2\pi} \frac{ds}{|1+2me^{-is}|} = m \int_0^{2\pi} (1+4m^2+4m\cos s)^{-\frac{1}{2}} ds \, ,$$

which may be written as

$$\frac{4m}{1+2m} K(k) \, , \tag{10.38}$$

where K(k) denotes the complete elliptic integral of the first kind,

$$K(k) = \int_0^{\pi/2} (1-k^2 \sin^2\theta)^{-\frac{1}{2}} d\theta \quad , \tag{10.39}$$

and k is given by

$$k = \frac{2(2m)^{\frac{1}{2}}}{1+2m} \quad . \tag{10.40}$$

Thus, the inequality (10. 26) reads

$$|\lambda| \geq (2)^{\frac{1}{2}}\pi(1+2m)[4m\, K(k)]^{-1} \; . \tag{10.41}$$

For the first iterated kernel $A_1(s_o, s)$ defined by equation (10. 29), it may be shown that

$$||A_1(s_o, s)|| \leq \frac{4m^2 K(k)}{2\pi^2(1+2m)} \quad \frac{1}{|1+2me^{-is}|} \quad , \tag{10.42}$$

for the case of Pascal's limacon, and that the inequality (10. 30) implies that

$$\lambda^2 \geq (1-4m^2)^{\frac{1}{2}}(1+2m)\pi[4m^2 K(k)]^{-1} \quad . \tag{10.43}$$

Using (10. 42), it can be shown that (10. 31) implies

$$|\lambda| \geq (2)^{\frac{1}{2}}\pi(1+2m)[4mK(k)]^{-1} \quad , \tag{10.44}$$

which is the same as (10. 41).

From relation (10. 4), we know that the eigenvalues λ of the integral equation (10. 1) must be such that

$$|\lambda| > 1 \quad . \tag{10.45}$$

For the particular case of Pascal's limacon which we are considering here, the inequality (10. 37) implies (10. 45) only for values of the parameter, m which satisfy

$$m^2 < 1/6 \ , \quad \text{that is} \ , \qquad m < .408 \ . \tag{10.46}$$ *

Inequality (10.43), which uses the first iterated kernel, implies (10.45) for values of m such that

$$m < .42 \ , \tag{10.47}$$

while (10.41) [and (10.44)] imply (10.45) for

$$m < .46 \ . \tag{10.48}$$

The reason why the inequalities (10.37), (10.41), (10.43) fail to predict result (10.45) for all values of m in the interval $0 \le m < \frac{1}{2}$ is that the right hand sides of the above mentioned inequalities tend to zero as m tends to $\frac{1}{2}$.** The underlying reason for this behavior is the presence of the quantity $[\overline{\omega'(\sigma)}]^{-1}$ in the kernel $T(\sigma_o, \sigma)$. (See equation (10.33).) As $m \to \frac{1}{2}$, $\overline{\omega'(\sigma)} \to 0$ when $\sigma = -1$. Attempts to remedy this defect and to obtain useful bounds for general classes of regions, say, for convex regions, have so far proved unsuccessful.

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*We recall from (9.13) that m lies in the interval $0 \le m < \frac{1}{2}$.

**As $m \to \frac{1}{2}$, k in (10.40) tends to one and $K(k) \to \infty$.

VI. THE EXTENDED MAIN CASE OF KORN'S INEQUALITY

11. Derivation of Eigenvalue Problem

In this chapter, we consider the extended main case of Korn's inequality, as formulated in §1:

$$D(\underline{u}) \leq K_1(\sigma)\, S(\underline{u}) \tag{11.1}$$

for all $\underline{u}$ satisfying

$$\int_R (u_{i,j} - u_{j,i})\, dV = 0 \ , \tag{11.2}$$

$$u_{i,jj} + \frac{1}{1-2\sigma}\, u_{j,ji} = 0 \quad \text{in } R \ , \tag{11.3}$$

$$(i, j = 1, 2 \ \text{ or } \ i, j = 1, 2, 3)$$

where σ is a real parameter such that $-1 < \sigma < \frac{1}{2}$. As we remarked in §1, if the vector field $\underline{u}$ is regarded as an infinitesimal displacement field in an homogeneous, isotropic elastic solid occupying R, then (11.3) are the displacement equations of equilibrium in the case of zero body force, where σ is taken to be Poisson's ratio.

For the special value $\sigma = 0$, the extended main case reduces to the main case which was employed by Friedrichs [1]. In the application of Korn's inequality which motivated the present investigation (see Chapter I), the vector field $\underline{u}$ to which an inequality of the form (11.1) is to be applied is known to satisfy the differential equations (11.3). With a view to obtaining the optimum K_1 in (11.1), the side conditions (11.3) are therefore natural ones to consider. Our reason for investigating the extended main case thus differs from

Friedrichs' motivation for introducing the main case in [1], where it was employed solely as a device in the proof of Korn's inequality in the second case.

For the extended main case, we now carry out a program analogous to that developed in Chapters II and III for the second case. To this end, we begin by posing the following variational problem. Find

$$\sup \frac{D(\underline{u})}{S(\underline{u})} \quad , \tag{11.4}$$

where the supremum is taken over all vector fields $\underline{u}$ such that $S(\underline{u}) \neq 0$ and (11.2) and (11.3) hold. Introducing the constant Lagrange multipliers K and μ_{ij} ($\mu_{ij} = 0$ for $i = j$), we consider the functional

$$J[\underline{u}] = D(\underline{u}) - KS(\underline{u}) - \mu_{ij} \int_R (u_{i,j} - u_{j,i}) dV \ . \tag{11.5}$$

A necessary condition for a stationary value of $J[\underline{u}]$ is that

$$\delta J = \int_R 2u_{j,i}\delta u_{j,i} dV - \frac{K}{2} \int_R (u_{i,j}+u_{j,i})(\delta u_{i,j}+\delta u_{j,i}) dV$$

$$- \mu_{ij} \int_R (\delta u_{i,j} - \delta u_{j,i}) dV = 0 \ . \tag{11.6}$$

The first variation $\delta\underline{u}$ must satisfy the constraints (11.3), that is,

$$\delta u_{i,jj} + \frac{1}{1-2\sigma} \delta u_{j,ji} = 0 \quad \text{in } R \ . \tag{11.7}$$

Introducing the Lagrange multipliers $\lambda_i(\underline{x})$ and using equations (11.6) and (11.7), we find that $\delta\underline{u}$ must satisfy

$$\int_R \Big[2u_{j,i}\delta u_{j,i} - \frac{K}{2}(u_{i,j}+u_{j,i})(\delta u_{i,j}+\delta u_{j,i}) - \mu_{ij}(\delta u_{i,j}-\delta u_{j,i})$$

$$+ \lambda_i(\delta u_{i,jj} + \frac{1}{1-2\sigma}\,\delta u_{j,ji})\Big] dV = 0 \;,$$

which we rewrite in the form

$$\int_R \Big[2u_{j,i}\delta u_{j,i} - \frac{K}{2}(u_{i,j}+u_{j,i})(\delta u_{i,j}+\delta u_{j,i}) - \mu_{ij}(\delta u_{i,j}-\delta u_{j,i})$$

$$+ \lambda_i(\delta u_{i,jj} + \frac{2\sigma}{1-2\sigma}\,\delta u_{j,ji} + \delta u_{j,ij})\Big] dV = 0 \;. \qquad (11.8)$$

Applying the divergence theorem to (11.8), we obtain

$$\int_R \Big[\delta u_j\{(K-2)u_{j,ii}+Ku_{i,ij}\} - \delta u_{i,j}\lambda_{i,j} - \frac{2\sigma}{1-2\sigma}\,\delta u_{j,j}\lambda_{i,i} - \lambda_{i,j}\delta u_{j,i}\Big] dV$$

$$+ \int_B \Big[\delta u_j\{(2-K)u_{j,i} - Ku_{i,j} + \mu_{ij} - \mu_{ji}\}n_i + \lambda_i(\delta u_{i,j} + \delta u_{j,i} + \frac{2\sigma}{1-2\sigma}\,\delta u_{k,k}\delta_{ij})n_j\Big] dS$$

$$= 0 \;. \qquad (11.9)$$

Applying the divergence theorem once more, we find from (11.9) that

$$\int_R \Big[\delta u_j\Big\{(K-2)u_{j,ii} + Ku_{i,ij} + \lambda_{j,ii} + \frac{1}{1-2\sigma}\,\lambda_{i,ij}\Big\}\Big] dV$$

$$+ \int_B \Big[\delta u_j\Big\{(2-K)u_{j,i} - Ku_{i,j} + \mu_{ij} - \mu_{ji} - (\lambda_{i,j} + \lambda_{j,i} + \frac{2\sigma}{1-2\sigma}\,\lambda_{k,k}\delta_{ij})\Big\}n_i$$

$$+ \lambda_i(\delta u_{i,j} + \delta u_{j,i} + \frac{2\sigma}{1-2\sigma}\,\delta u_{k,k}\delta_{ij})n_j\Big] dS = 0 \;. \qquad (11.10)$$

We now introduce the "surface traction" $\delta \underline{T}$ associated with $\delta \underline{u}$ as

$$\delta T_i = (\delta u_{i,j} + \delta u_{j,i} + \frac{2\sigma}{1-2\sigma}\,\delta u_{k,k}\delta_{ij})n_j \quad \text{on } B \;. \qquad (11.11)$$

Thus, equation (11.10) may be written

$$\int_R \Big[\delta u_j \{ (K-2)u_{j,ii} + Ku_{i,ij} + \lambda_{j,ii} + \frac{1}{1-2\sigma} \lambda_{i,ij} \} \Big] dV$$

$$+ \int_B \Big[\delta u_j \Big\{ (2-K)u_{j,i} - Ku_{i,j} + \mu_{ij} - \mu_{ji} - (\lambda_{i,j} + \lambda_{j,i} + \frac{2\sigma}{1-2\sigma} \lambda_{k,k} \delta_{ij}) \Big\} n_i$$

$$+ \delta T_i \lambda_i \Big] dS = 0 \ . \qquad (11.12)$$

Since the quantities u_i and T_i may be specified independently on the boundary B, the standard arguments of the calculus of variations may now be applied to (11.12) to show that the extremal field $\underline{u}$ and the Lagrange multiplier $\underline{\lambda}$ must satisfy the Euler equations

$$(2-K)u_{j,ii} - Ku_{i,ij} - \lambda_{j,ii} - \frac{1}{1-2\sigma} \lambda_{i,ij} = 0 \quad \text{in } R \ , \qquad (11.13)$$

and the natural boundary conditions

$$\lambda_i = 0 \qquad \text{on } B \ , \qquad (11.14)$$

and

$$\Big[(2-K)u_{j,i} - Ku_{i,j} + \mu_{ij} - \mu_{ji} - (\lambda_{i,j} + \lambda_{j,i} + \frac{2\sigma}{1-2\sigma} \lambda_{k,k} \delta_{ij}) \Big] n_i = 0 \text{ on B.} \quad (11.15)$$

The extremal field $\underline{u}$ must also satisfy the constraints

$$\int_R (u_{i,j} - u_{j,i}) dV = 0 \ , \qquad (11.16)$$

and

$$u_{i,jj} + \frac{1}{1-2\sigma} u_{j,ji} = 0 \quad \text{in } R \ . \qquad (11.17)$$

In a manner similar to that used in §3, it can be shown that the constant Lagrange multipliers μ_{ij} satisfy

$$\mu_{ij} = \mu_{ji} \ , \qquad (11.18)$$

and thus we arrive at the following Korn eigenvalue problem in the extended main case.

Find a pair of vector fields $\underline{u}$, $\underline{\lambda}$, with $\underline{u}$ non-trivial, such that

$$(2-K)\nabla^2\underline{u}-K\nabla(\nabla\cdot\underline{u})-\nabla^2\underline{\lambda}-\frac{1}{1-2\sigma}\nabla(\nabla\cdot\underline{\lambda}) = 0 \quad \text{in } R\,, \tag{11.19}$$

$$\nabla^2\underline{u}+\frac{1}{1-2\sigma}\nabla(\nabla\cdot\underline{u}) = 0 \quad \text{in } R\,, \tag{11.20}$$

$$\int_R (u_{i,j}-u_{j,i})dV = 0\,, \tag{11.21}$$

$$\underline{\lambda} = 0 \quad \text{on } B\,, \tag{11.22}$$

and

$$\left[(2-K)u_{j,i}-Ku_{i,j}-(\lambda_{i,j}+\lambda_{j,i}+\frac{2\sigma}{1-2\sigma}\delta_{ij}\lambda_{k,k})\right]n_i = 0 \quad \text{on } B\,. \tag{11.23}$$

By a non-trivial vector field $\underline{u}$ we mean $\underline{u}$ non-constant and by a non-trivial vector field $\underline{\lambda}$ we mean $\underline{\lambda}$ not identically equal to zero in R. In the next section, we shall describe some properties of the Korn eigenvalue problem (11.19) - (11.23) for the case of two dimensions.

12. Some Properties of the Eigenvalue Problem in Two Dimensions

Henceforth, we confine our attention to two-dimensional simply-connected regions R and we write equations (11.19) - (11.23) as

$$(2-K)u_{\alpha,\beta\beta}-Ku_{\beta,\beta\alpha}-\lambda_{\alpha,\beta\beta}-\frac{1}{1-2\sigma}\lambda_{\beta,\beta\alpha} = 0 \quad \text{in } R\,, \tag{12.1}$$

$$u_{\alpha,\beta\beta}+\frac{1}{1-2\sigma}u_{\beta,\beta\alpha} = 0 \quad \text{in } R\,, \tag{12.2}$$

$$\int_R (u_{\alpha,\beta}-u_{\beta,\alpha})dA = 0\,, \tag{12.3}$$

$$\lambda_\alpha = 0 \quad \text{on } B\,, \tag{12.4}$$

and

$$\left[(2-K)u_{\alpha,\beta}-Ku_{\beta,\alpha}-(\lambda_{\alpha,\beta}+\lambda_{\beta,\alpha}+\frac{2\sigma}{1-2\sigma}\delta_{\alpha\beta}\lambda_{\gamma,\gamma})\right]n_\beta = 0 \quad \text{on } B\,, \tag{12.5}$$

where the Greek subscripts range over the integers (1, 2). We have the following remarks to make concerning the eigenvalue problem (12.1) - (12.5).

Remark 1. If $K \neq 1$, $1/\sigma$, ∞, then the vector fields $\underline{u}$, $\underline{\lambda}$ satisfying (12.1) - (12.5) are either both trivial or both non-trivial.

Suppose that $\underline{u}$ is trivial. Then from (12.1), (12.4) we find that $\underline{\lambda}$ satisfies

$$\nabla^2 \underline{\lambda} + \frac{1}{1-2\sigma} \nabla(\nabla \cdot \underline{\lambda}) = 0 \qquad \text{in } R, \qquad (12.6)$$

$$\underline{\lambda} = 0 \qquad \text{on } B. \qquad (12.7)$$

Thus, $\underline{\lambda}$ satisfies the homogeneous displacement boundary-value problem of two-dimensional isotropic elastostatics. Since by hypothesis, σ lies in the interval $-1 < \sigma < \frac{1}{2}$, the uniqueness result of Theorem 7.1 implies that $\underline{\lambda}$ is trivial.

Suppose now that the vector field $\underline{\lambda}$ is trivial. Then, from (12.1), (12.2), (12.5) we find that $\underline{u}$ satisfies, for $K \neq \infty$,

$$(2-K)\nabla^2 \underline{u} - K\nabla(\nabla \cdot \underline{u}) = 0 \qquad \text{in } R, \qquad (12.8)$$

$$\nabla^2 \underline{u} + \frac{1}{1-2\sigma} \nabla(\nabla \cdot \underline{u}) = 0 \qquad \text{in } R, \qquad (12.9)$$

and

$$[(2-K)u_{\alpha,\beta} - Ku_{\beta,\alpha}]n_\beta = 0 \qquad \text{on } B. \qquad (12.10)$$

Equations (12.8), (12.9) imply that

$$\nabla^2 \underline{u} = 0 \quad \text{and} \quad \nabla(\nabla \cdot \underline{u}) = 0 \qquad \text{in } R, \qquad (12.11)$$

unless

$$\begin{vmatrix} 2-K & -K \\ 1 & 1/1-2\sigma \end{vmatrix} = \frac{2(1-K\sigma)}{1-2\sigma} = 0 \ . \qquad (12.12)$$

Thus, if $K \neq 1/\sigma$, from (12.11) we obtain

$$(u_{\alpha,\beta} - u_{\beta,\alpha})_{,\beta} = 0 \qquad \text{in } R \ ,$$

which implies that

$$u_{\alpha,\beta} - u_{\beta,\alpha} \equiv \text{constant} \qquad \text{in } R \ .$$

The zero average rotation condition (12.3) now implies that

$$u_{\alpha,\beta} = u_{\beta,\alpha} \qquad \text{in } R \ . \qquad (12.13)$$

Using (12.13), the boundary condition (12.10) may be written as

$$2(1-K)u_{\alpha,\beta}n_\beta = 0 \qquad \text{on } B \ . \qquad (12.14)$$

Thus, if $K \neq 1,\ 1/\sigma$, the vector field $\underline{u}$ must satisfy

$$\nabla^2 \underline{u} = 0 \qquad \text{in } R \qquad (12.15)$$

and

$$u_{\alpha,\beta}n_\beta = \frac{\partial u_\alpha}{\partial n} = 0 \qquad \text{on } B \ . \qquad (12.16)$$

By the uniqueness result for the Neumann problem, we see that $\underline{u}$ must be trivial. This completes the verification of the assertion made in Remark 1.

Remark 2. $K = 1$ is an eigenvalue of infinite multiplicity of (12.1) - (12.5) for any region.

We take $\underline{\lambda}$ to be identically zero and $\underline{u} = (u_1, u_2)$, where $u_1 = \varphi$, $u_2 = -\psi$ and φ, ψ are conjugate harmonic functions. Direct calculation shows that (12.1)-(12.5) are identically satisfied for any choice of φ, ψ.

Remark 3. $K = \infty$ is not an eigenvalue of (12.1)-(12.5).

When $K = \infty$, (12.1)-(12.5) become*

$$u_{\alpha,\beta\beta} + u_{\beta,\beta\alpha} = 0 \qquad \text{in } R, \qquad (12.17)$$

$$u_{\alpha,\beta\beta} + \frac{1}{1-2\sigma} u_{\beta,\beta\alpha} = 0 \qquad \text{in } R, \qquad (12.18)$$

$$\int_R (u_{\alpha,\beta} - u_{\beta,\alpha}) dA = 0, \qquad (12.19)$$

$$\lambda_\alpha = 0 \qquad \text{on } B, \qquad (12.20)$$

and

$$[u_{\alpha,\beta} + u_{\beta,\alpha}] n_\beta = 0 \qquad \text{on } B. \qquad (12.21)$$

If $\sigma = 0$, then (12.17), (12.18) coincide and the proof of Remark 3, §4 may be directly applied in the present case to show that $\underline{u}$ is trivial.

Suppose now that $\sigma \neq 0$. Then (12.17), (12.18) imply that

$$u_{\alpha,\beta\beta} = 0 \quad \text{and} \quad u_{\beta,\beta\alpha} = 0 \qquad \text{in } R. \qquad (12.22)$$

Thus, we obtain

$$(u_{\alpha,\beta} - u_{\beta,\alpha})_{,\beta} = 0 \qquad \text{in } R,$$

which implies that

$$u_{\alpha,\beta} - u_{\beta,\alpha} \equiv \text{constant} \qquad \text{in } R.$$

The zero average rotation condition (12.19) now gives

$$u_{\alpha,\beta} = u_{\beta,\alpha} \qquad \text{in } R. \qquad (12.23)$$

Using (12.23), the boundary condition (12.21) becomes

$$2 u_{\alpha,\beta} n_\beta \equiv 2 \frac{\partial u_\alpha}{\partial n} = 0 \qquad \text{on } B. \qquad (12.24)$$

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*See footnote, page 21.

Equations (12.22) and the boundary conditions (12.24) imply that $\underline{u}$ is trivial.

Remark 4. (i) The eigenvalues K of (12.1)-(12.5) are real and $K \geq 1$.

(ii) Every solution of (12.1), (12.2), (12.4), (12.5) satisfies the normalization condition

$$\int_R (u_{\alpha,\beta} - u_{\beta,\alpha}) dA = 0 .$$

The verification of the assertions made in Remark 4 presents no difficulty and will be omitted.

Remark 5. When the parameter $K = 1/\sigma$, the problem (12.1)-(12.5) reduces to the Korn eigenvalue problem in the second case.

When $K = 1/\sigma$, equations (12.2) and (12.1) imply that the vector field $\underline{\lambda}$ satisfies

$$\lambda_{\alpha,\beta\beta} + \frac{1}{1-2\sigma} \lambda_{\beta,\beta\alpha} = 0 \qquad \text{in } R, \tag{12.25}$$

$$\lambda_\alpha = 0 \qquad \text{on } B. \tag{12.26}$$

Thus, $\underline{\lambda}$ satisfies the homogeneous displacement boundary-value problem of plane elastostatics with $-1 < \sigma < \frac{1}{2}$, and so by Theorem 7.1, $\underline{\lambda}$ is trivial. With $\underline{\lambda}$ trivial and $K = 1/\sigma$, the problem (12.1)-(12.5) is exactly the Korn eigenvalue problem (5.3), (5.4) in the second case.

Using the results obtained in Chapter V on the K-spectrum for the second case, we may assert the following with regard to $K = 1/\sigma$ in the extended main case:

(i) If $-1 < \sigma \leq 0$, $K = 1/\sigma$ is not an eigenvalue of (12.1)-(12.5) for any region.

(ii) If $0 < \sigma < 1/4$ or $1/4 < \sigma < 1/2$, $K = 1/\sigma$ may be an eigenvalue of finite multiplicity.

(iii) If $\sigma = 1/4$, $K = 4$ is an eigenvalue if and only if the region R can be mapped conformally by a rational function onto the unit disc. When $K = 4$ is an eigenvalue, it has infinite multiplicity.

13. Transformation Theorems

In this section we state and prove theorems which establish an equivalence between the Korn eigenvalue problem (12.1)-(12.5) in the extended main case and the displacement boundary-value problem of plane strain in linear isotropic elasticity. We first show that the vector fields $\underline{u}$ and $\underline{\lambda}$ which satisfy (12.1)-(12.5) are expressible in terms of vector fields $\underline{u}^{(1)}$, $\underline{u}^{(2)}$ which satisfy uncoupled homogeneous displacement boundary-value problems with suitable respective Poisson's ratios $\nu^{(1)}$, $\nu^{(2)}$.

Theorem 13.1. Suppose $\underline{u}, \underline{\lambda} \in C^2(R+B)$ and satisfy

$$(2-K)u_{\alpha,\beta\beta} - Ku_{\beta,\beta\alpha} - \lambda_{\alpha,\beta\beta} - \frac{1}{1-2\sigma}\lambda_{\beta,\beta\alpha} = 0 \quad \text{in } R, \qquad (13.1)$$

$$u_{\alpha,\beta\beta} + \frac{1}{1-2\sigma}u_{\beta,\beta\alpha} = 0 \quad \text{in } R, \qquad (13.2)$$

$$\lambda_\alpha = 0 \quad \text{on } B, \qquad (13.3)$$

and

$$[(2-K)u_{\alpha,\beta}-Ku_{\beta,\alpha}-(\lambda_{\alpha,\beta}+\lambda_{\beta,\alpha}+\frac{2\sigma}{1-2\sigma}\delta_{\alpha\beta}\lambda_{\gamma,\gamma})]n_\beta = 0 \quad \text{on } B, \qquad (13.4)$$

where σ is given, $-1 < \sigma < \frac{1}{2}$. Suppose $K \neq 1, 1/\sigma, \infty$ and define the real numbers $A^{(1)}$, $A^{(2)}$ by

$$\left.\begin{aligned} 2(1-2\sigma)(1-K\sigma)A^{(1)} &= -2(\sigma-1)(1-K\sigma)-(1-2\sigma)(K-1) \\ &\quad +[4(\sigma-1)^2(1-K\sigma)^2+(1-2\sigma)^2(K-1)^2]^{\frac{1}{2}}, \\ 2(1-2\sigma)(1-K\sigma)A^{(2)} &= -2(\sigma-1)(1-K\sigma)-(1-2\sigma)(K-1) \\ &\quad -[4(\sigma-1)^2(1-K\sigma)^2+(1-2\sigma)^2(K-1)^2]^{\frac{1}{2}}. \end{aligned}\right\} \qquad (13.5)$$

Then there exist two vector fields $\underline{u}^{(1)}$, $\underline{u}^{(2)}$ ϵ $C^2(R+B)$ such that

(i) $$2A^{(\delta)}u^{(\delta)}_{\alpha,\beta} = K(u_{\beta,\alpha}-u_{\alpha,\beta})+2(1-K)u_{\beta,\alpha}-2(1-K)\delta_{\alpha\beta}u_{\gamma,\gamma}$$
$$-(\lambda_{\beta,\alpha}-\lambda_{\alpha,\beta})-2A^{(\delta)}\lambda_{\alpha,\beta}+\frac{2(1-\sigma)}{1-2\sigma}\delta_{\alpha,\beta}\lambda_{\gamma,\gamma}, \quad (\delta = 1,2)$$

(13.6)

(ii) $$\nabla^2\underline{u}^{(\delta)}+\frac{1}{1-2\nu^{(\delta)}}\nabla(\nabla\cdot\underline{u}^{(\delta)}) = 0 \quad \text{in } R, \qquad (13.7)$$

(iii) $$\underline{u}^{(\delta)} = 0 \quad \text{on } B, \qquad (13.8)$$

where

$$\nu^{(\delta)} = \sigma+\frac{(1-\sigma)}{A^{(\delta)}}. \quad (\delta = 1,2) \qquad (13.9)$$

Proof of Theorem 13.1

We first show that there exist vector fields $\underline{u}^{(1)}, \underline{u}^{(2)}$ satisfying (13.6). For $\delta = 1$ or 2, let

$$H^{(\delta)}_{\alpha\beta} = K(u_{\beta,\alpha}-u_{\alpha,\beta})+2(1-K)u_{\beta,\alpha}-2(1-K)\delta_{\alpha\beta}u_{\gamma,\gamma}$$

$$-(\lambda_{\beta,\alpha}-\lambda_{\alpha,\beta})-2A^{(\delta)}\lambda_{\alpha,\beta}+\frac{2(1-\sigma)}{1-2\sigma}\delta_{\alpha\beta}\lambda_{\gamma,\gamma}\,, \tag{13.10}$$

where $A^{(\delta)}$ is defined by (13.5). Direct calculation shows that for $\delta = 1$ or 2

$$\epsilon_{\rho\mu}\epsilon_{\beta\nu}H^{(\delta)}_{\rho\beta,\nu} = H^{(\delta)}_{\beta\beta,\mu}-H^{(\delta)}_{\nu\mu,\nu} = -2(1-K)u_{\gamma,\gamma\mu}+\left[\frac{4(1-\sigma)}{1-2\sigma}-2A^{(\delta)}\right]\lambda_{\gamma,\gamma\mu}$$

$$-(2-K)u_{\mu,\nu\nu}+Ku_{\nu,\mu\nu}+2(1-K)\delta_{\nu\mu}u_{\gamma,\gamma\nu}-(1-2A^{(\delta)})\lambda_{\nu,\mu\nu}+\lambda_{\mu,\nu\nu}$$

$$-\frac{2(1-\sigma)}{1-2\sigma}\delta_{\nu\mu}\lambda_{\gamma,\gamma\nu} = -(2-K)u_{\mu,\nu\nu}+Ku_{\nu,\nu\mu}+\lambda_{\mu,\nu\nu}+\frac{1}{1-2\sigma}\lambda_{\gamma,\gamma\mu}\,, \tag{13.11}$$

where we have used the identity

$$\epsilon_{\rho\mu}\epsilon_{\beta\nu} = \delta_{\rho\beta}\delta_{\mu\nu}-\delta_{\rho\nu}\delta_{\mu\beta}\ . \tag{13.12}$$

Since $\underline{u}$, $\underline{\lambda}$ satisfy (13.1), it follows from (13.11) that

$$\epsilon_{\rho\mu}\epsilon_{\beta\nu}H^{(\delta)}_{\rho\beta,\nu} = 0 \quad \text{in } R\ . \quad (\delta = 1, 2) \tag{13.13}$$

Multiplying (13.13) by $\epsilon_{\alpha\mu}$ and using (13.12) contracted on μ, ν we find that

$$\epsilon_{\beta\nu}H^{(\delta)}_{\alpha\beta,\nu} = H^{(\delta)}_{\alpha 1,2}-H^{(\delta)}_{\alpha 2,1} = 0 \qquad \text{in } R\ . \tag{13.14}$$

Equation (13.14) and the simple connectivity of R imply that there exist functions $\underline{u}^{(1)}, \underline{u}^{(2)} \in C^2(R+B)$, unique to within an arbitrary additive constant, such that

$$2A^{(\delta)}u^{(\delta)}_{\alpha,\beta} = H^{(\delta)}_{\alpha\beta}\,, \qquad (\delta = 1, 2) \tag{13.15}$$

where $H^{(\delta)}_{\alpha\beta}$ is given by (13.10). Thus, (13.6) holds.

To show that the vector fields $\underline{u}^{(\delta)}$ of (13.15) satisfy the equations (13.7) with $\nu^{(\delta)}$ given by (13.9), we proceed as follows. Equations (13.15), (13.10) provide, for $\delta = 1$ or 2,

$$2A^{(\delta)}\left[u^{(\delta)}_{\alpha,\beta\beta}+A^{(\delta)}\{A^{(\delta)}(1-2\sigma)-2(1-\sigma)\}^{-1}u^{(\delta)}_{\beta,\beta\alpha}\right] = (2-K)u_{\beta,\alpha\beta}-Ku_{\alpha,\beta\beta}$$

$$-2(1-K)u_{\gamma,\gamma\alpha}+(1-2A^{(\delta)})\lambda_{\alpha,\beta\beta}-\lambda_{\beta,\alpha\beta}+\frac{2(1-\sigma)}{1-2\sigma}\lambda_{\gamma,\gamma\alpha}$$

$$+A^{(\delta)}\{A^{(\delta)}(1-2\sigma)-2(1-\sigma)\}^{-1}\left[-2(1-K)u_{\gamma,\gamma\alpha}+\left\{\frac{4(1-\sigma)}{1-2\sigma}-2A^{(\delta)}\right\}\lambda_{\gamma,\gamma\alpha}\right]$$

$$= -Ku_{\alpha,\beta\beta}+\left[K-2(1-K)A^{(\delta)}\{A^{(\delta)}(1-2\sigma)-2(1-\sigma)\}^{-1}\right]u_{\gamma,\gamma\alpha}$$

$$+(1-2A^{(\delta)})(\lambda_{\alpha,\beta\beta}+\frac{1}{1-2\sigma}\lambda_{\gamma,\gamma\alpha}) = [(1-2A^{(\delta)})(2-K)-K]u_{\alpha,\beta\beta}$$

$$+\left[2A^{(\delta)}K-2(1-K)A^{(\delta)}\{A^{(\delta)}(1-2\sigma)-2(1-\sigma)\}^{-1}\right]u_{\gamma,\gamma\alpha}\,, \qquad (13.16)$$

where we have used (13.1). The right hand side of equation (13.16) may be written as

$$[(1-2A^{(\delta)})(2-K)-K][u_{\alpha,\beta\beta}+f(A^{(\delta)},K,\sigma)u_{\beta,\beta\alpha}]\,, \qquad (13.17)$$

where

$$f(A,K,\sigma) =$$

$$2A[KA(1-2\sigma)-2K(1-\sigma)-(1-K)][(1-2A)(2-K)-K]^{-1}[A(1-2\sigma)-2(1-\sigma)]^{-1}\,. \qquad (13.18)$$

Now from the definitions (13.5), it is readily verified that $A = A^{(1)}$ and $A = A^{(2)}$ are the roots of the quadratic equation

$$A^2(1-2\sigma)(1-K\sigma)+A\{2(\sigma-1)(1-K\sigma)+(1-2\sigma)(K-1)\}+(\sigma-1)(K-1) = 0\,. \qquad (13.19)$$

By making use of this observation, it can be shown that for $\delta = 1, 2$, $f(A^{(\delta)}, K, \sigma)$ has the value $(1-2\sigma)^{-1}$, and so, by (13.17), we find that the right hand side of equation (13.16) is equal to

$$[(1-2A^{(\delta)})(2-K)-K][u_{\alpha,\beta\beta} + \frac{1}{1-2\sigma} u_{\beta,\beta\alpha}] \qquad (13.20)$$

$$= 0 , \qquad (13.21)$$

by virtue of equation (13.2). Thus, for $\delta = 1$ or 2 , we have shown that the vector fields $\underline{u}^{(\delta)}$ satisfy equations (13.7), with $\nu^{(\delta)}$ given by (13.9).

To establish the boundary condition (13.8), we note that $\epsilon_{\gamma\beta} n_\gamma$ is tangent to B so that by (13.15) we have for $\delta = 1$ or 2 ,

$$2A^{(\delta)} \frac{du_\rho^{(\delta)}}{ds} = H^{(\delta)}_{\rho\beta} \epsilon_{\gamma\beta} n_\gamma \qquad \text{on B} . \qquad (13.22)$$

We thus obtain, with the aid of (13.10) and (13.12),

$$2A^{(\delta)} \epsilon_{\rho\mu} \frac{du_\rho^{(\delta)}}{ds} = \epsilon_{\rho\mu} \epsilon_{\gamma\beta} H^{(\delta)}_{\rho\beta} n_\gamma = (H^{(\delta)}_{\gamma\mu} - H^{(\delta)}_{\beta\beta} \delta_{\mu\gamma}) n_\gamma$$

$$= [(2-K)u_{\mu,\gamma} - Ku_{\gamma,\mu} + \lambda_{\gamma,\mu} - \lambda_{\mu,\gamma} - \frac{2(1-\sigma)}{1-2\sigma} \lambda_{\rho,\rho} \delta_{\mu\gamma}$$

$$-2A^{(\delta)}(\lambda_{\gamma,\mu} - \lambda_{\rho,\rho} \delta_{\mu\gamma})] n_\gamma \qquad \text{on B} . \qquad (13.23)$$

Since $\epsilon_{\gamma\beta} n_\gamma$ is tangent to B, for any vector field $\underline{\lambda}$ we have

$$\epsilon_{\rho\mu} \frac{d\lambda_\rho}{ds} = \epsilon_{\rho\mu} \epsilon_{\gamma\beta} n_\gamma \lambda_{\rho,\beta} = (\lambda_{\gamma,\mu} - \lambda_{\beta,\beta} \delta_{\mu\gamma}) n_\gamma \qquad \text{on B} , \qquad (13.24)$$

where we have used the identity (13.12). The boundary condition (13.3), in conjunction with (13.24), implies that

$$(\lambda_{\gamma,\mu} - \lambda_{\beta,\beta} \delta_{\mu\gamma}) n_\gamma = 0 \qquad \text{on B} . \qquad (13.25)$$

Thus, we may write (13.23) as

$$2A^{(\delta)}\epsilon_{\rho\mu}\frac{du_{\rho}^{(\delta)}}{ds} = [(2-K)u_{\mu,\gamma}-Ku_{\gamma,\mu}-(\lambda_{\gamma,\mu}+\lambda_{\mu,\gamma}+\frac{2\sigma}{1-2\sigma}\delta_{\mu\gamma}\lambda_{\rho,\rho})]n_{\gamma}$$

$$= 0 \qquad \text{on } B , \qquad (13.26)$$

by virtue of the boundary condition (13.4). Multiplying (13.26) by $\epsilon_{\alpha\mu}$ and using (13.12) contracted on ν,μ we get

$$2A^{(\delta)}\frac{du_{\alpha}^{(\delta)}}{ds} = 0 \qquad \text{on } B . \qquad (13.27)$$

By choosing the arbitrary additive constants suitably in the construction of $\underline{u}^{(\delta)}$ in (13.15), we may arrange that the boundary condition (13.8) holds. This completes the proof of Theorem 13.1.

We also have a converse result to Theorem 13.1 which, when given two vector fields $\underline{u}^{(1)}, \underline{u}^{(2)}$ satisfying displacement boundary-value problems, enables us to construct vector fields $\underline{u}, \underline{\lambda}$ satisfying the Korn eigenvalue problem (13.1)-(13.4). We state this result as follows.

Theorem 13.2. Let $A^{(1)} \neq 1$, $A^{(2)} \neq 1$ be real numbers $(A^{(1)} > A^{(2)})$ such that

(i) $$\frac{(2\sigma-1){A^{(1)}}^2+(3-4\sigma)A^{(1)}+(\sigma-1)}{-\sigma(1-2\sigma){A^{(1)}}^2+(1-2\sigma^2)A^{(1)}+(\sigma-1)} = \frac{(2\sigma-1){A^{(2)}}^2+(3-4\sigma)A^{(2)}+(\sigma-1)}{-\sigma(1-2\sigma){A^{(2)}}^2+(1-2\sigma^2)A^{(2)}+(\sigma-1)} . \qquad (13.28)$$

(ii) For $\delta = 1$ or 2, suppose that $\underline{u}^{(\delta)} \epsilon C^2(R+B)$ and satisfies

$$\nabla^2\underline{u}^{(\delta)} + \frac{1}{1-2\nu^{(\delta)}}\nabla(\nabla\cdot\underline{u}^{(\delta)}) = 0 \quad \text{in } R , \qquad (13.29)$$

$$\underline{u}^{(\delta)} = 0 \qquad \text{on } B , \tag{13.30}$$

where

$$\nu^{(\delta)} = \sigma + \frac{(1-\sigma)}{A^{(\delta)}} . \tag{13.31}$$

Then there exist vector fields $\underline{u}$, $\underline{\lambda} \in C^2(R+B)$ related to $\underline{u}^{(1)}$, $\underline{u}^{(2)}$ by (13.6) and satisfying equations (13.1)-(13.4) of the Korn eigenvalue problem with

$$K = \frac{(2\sigma-1)A^{(\delta)^2}+(3-4\sigma)A^{(\delta)}+(\sigma-1)}{-\sigma(1-2\sigma)A^{(\delta)^2}+(1-2\sigma^2)A^{(\delta)}+(\sigma-1)} . \quad (\delta = 1,2) \tag{13.32}$$

The proof of Theorem 13.2 proceeds along lines similar to that of Theorem 13.1 and will be omitted. We return now to Theorem 13.1 and make the following remarks concerning the range of values of $\nu^{(\delta)}$, (δ = 1 or 2) defined by (13.9). As we shall show presently, depending on the location of K and σ, one of the two parameters $\nu^{(1)}$ or $\nu^{(2)}$ will always lie in the range for which the displacement boundary-value problem has a unique solution. Thus, in Theorem 13.1, either $\underline{u}^{(1)}$ or $\underline{u}^{(2)}$ will be identically zero, so that we arrive at an equivalence between the Korn eigenvalue problem (13.1)-(13.4) and a <u>single</u> displacement boundary-value problem. We state this result as follows.

<u>Theorem 13.3.</u> Let $\underline{u}$, $\underline{\lambda}$ be <u>non-trivial</u> solutions of the Korn eigenvalue problem (13.1)-(13.4) with eigenvalue $K \neq 1/\sigma$, 1. Let the vector fields $\underline{u}^{(1)}$, $\underline{u}^{(2)}$ be as in Theorem 13.1.

(i) If $-1 < \sigma \leq 0$, then $\underline{u}^{(2)} \equiv 0$ and $\underline{u}^{(1)}$ is non-trivial.

(ii) If $0 < \sigma < \frac{1}{2}$ and $1 < K < 1/\sigma$, then $\underline{u}^{(2)} \equiv 0$ and $\underline{u}^{(1)}$ is non-trivial.

(iii) If $0 < \sigma < \frac{1}{2}$ and $K > 1/\sigma$, then $\underline{u}^{(1)} \equiv 0$ and $\underline{u}^{(2)}$ is non-trivial.

Proof of Theorem 13.3

The range of values of $\nu^{(1)}$, $\nu^{(2)}$ defined by (13.9), will depend on the range of values of $A^{(1)}, A^{(2)}$ respectively. We first of all examine the behavior of $A^{(1)}, A^{(2)}$ as functions of K for fixed σ in $-1 < \sigma < \frac{1}{2}$. To this end, we introduce the quantity

$$X = \frac{1}{2}\left(\frac{K-1}{1-K\sigma}\right) . \tag{13.33}$$

Recalling the definitions (13.5) of $A^{(1)}$ and $A^{(2)}$, we have

$$A^{(1)} = \frac{1-\sigma}{1-2\sigma} - X + \left[\left(\frac{1-\sigma}{1-2\sigma}\right)^2 + X^2\right]^{\frac{1}{2}} , \tag{13.34}$$

$$A^{(2)} = \frac{1-\sigma}{1-2\sigma} - X - \left[\left(\frac{1-\sigma}{1-2\sigma}\right)^2 + X^2\right]^{\frac{1}{2}} . \tag{13.35}$$

We set

$$y = \left(\frac{1-2\sigma}{1-\sigma}\right)X , \tag{13.36}$$

and write (13.34), (13.35) respectively as

$$A^{(1)} = \frac{1-\sigma}{1-2\sigma}\left[1-y+(1+y^2)^{\frac{1}{2}}\right] , \tag{13.37}$$

$$A^{(2)} = \frac{1-\sigma}{1-2\sigma}\left[1-y-(1+y^2)^{\frac{1}{2}}\right] . \tag{13.38}$$

The range of K of interest in (13.33) is $K > 1$. From equations (13.33), (13.36), (13.37) and (13.38) it may be verified that $A^{(1)}$ and $A^{(2)}$ satisfy the following inequalities.

(i) If $-1 < \sigma \leq 0$, then

$$A^{(2)} < 0 , \tag{13.39}$$

and

$$1 < A^{(1)} < \frac{2(1-\sigma)}{1-2\sigma} . \tag{13.40}$$

(ii) If $0 < \sigma < \frac{1}{2}$ and $1 < K < 1/\sigma$, then

$$A^{(2)} < 0 , \tag{13.41}$$

and

$$\frac{1-\sigma}{1-2\sigma} < A^{(1)} < \frac{2(1-\sigma)}{1-2\sigma} . \tag{13.42}$$

(iii) If $0 < \sigma < \frac{1}{2}$ and $K > 1/\sigma$, then

$$1 < A^{(2)} < \frac{1-\sigma}{1-2\sigma} , \tag{13.43}$$

and

$$\frac{2(1-\sigma)}{1-2\sigma} < A^{(1)} . \tag{13.44}$$

We recall now the definitions (13.9) of $\nu^{(\delta)}$ ($\delta = 1$ or 2) as

$$\nu^{(\delta)} = \sigma + \frac{(1-\sigma)}{A^{(\delta)}} . \tag{13.45}$$

Using the inequalities (13.39) - (13.44), we deduce the following inequalities for $\nu^{(1)}$ and $\nu^{(2)}$.

(i) If $-1 < \sigma \leq 0$, then

$$\nu^{(2)} < 0 , \tag{13.46}$$

and

$$\tfrac{1}{2} < \nu^{(1)} < 1 \ . \tag{13.47}$$

(ii) If $0 < \sigma < \frac{1}{2}$ and $1 < K < 1/\sigma$, then

$$\nu^{(2)} < \tfrac{1}{2} \ , \tag{13.48}$$

and

$$\tfrac{1}{2} < \nu^{(1)} < 1-\sigma \ . \tag{13.49}$$

(iii) If $0 < \sigma < \frac{1}{2}$ and $K > 1/\sigma$, then

$$1-\sigma < \nu^{(2)} < 1 \ , \tag{13.50}$$

and

$$\nu^{(1)} < \tfrac{1}{2} \ . \tag{13.51}$$

From Theorem 7.1, we have the following conclusion with regard to vector fields $\underline{u}^{(\delta)}$ satisfying the homogeneous displacement boundary-value problem (13.7), (13.8):

$$\text{If } \nu^{(\delta)} < \tfrac{1}{2} \text{ or } \nu^{(\delta)} > 1 \ , \quad \text{then } \underline{u}^{(\delta)} \equiv 0 \ . \tag{13.52}$$

The result (13.52), in conjunction with the inequalities (13.46), (13.48), and (13.51) establishes the statements (i), (ii), and (iii) of Theorem 13.3 concerning the identically zero solutions $\underline{u}^{(\delta)}$. The relations (13.6) of Theorem 13.1 may be uniquely inverted to express $\lambda_{\alpha,\beta}$, $u_{\gamma,\mu}$ in terms of $u^{(\delta)}_{\rho,\lambda}$. Since $\underline{u}$, $\underline{\lambda}$ are non-trivial, we see that both $\underline{u}^{(1)}$ and $\underline{u}^{(2)}$ cannot be trivial. This concludes the proof of Theorem 13.3.

Theorem 13.3, together with Theorem 13.1, enables us to construct a non-trivial solution of the homogeneous displacement boundary-value problem (13.7), (13.8) with a certain eigenvalue ν, when we are given a non-trivial pair $\underline{u}$, $\underline{\lambda}$ satisfying the Korn eigenvalue problem (13.1)-(13.4) with eigenvalue K. A converse to this result may be formulated by using Theorem 13.2. Our interest in this work is mainly concerned with the eigenvalues and we wish to state the following theorem, which may be established by utilizing the preceding results of this section.

Theorem 13.4. Let ν be an eigenvalue of the homogeneous displacement boundary-value problem of plane strain

$$\nabla^2 \underline{w} + \frac{1}{1-2\nu} \nabla(\nabla \cdot \underline{w}) = 0 \qquad \text{in } R, \tag{13.53}$$

$$\underline{w} = 0 \qquad \text{on } B, \tag{13.54}$$

with $\frac{1}{2} \le \nu < 1$. Then

$$K(\nu, \sigma) = \frac{\sigma^2 - 2\sigma\nu - \nu^2 + 3\nu - 1}{(1-\nu)(2\sigma^2 - 2\sigma + \nu)} \tag{13.55}$$

is an eigenvalue of the Korn eigenvalue problem [(13.1)-(13.4)] in the extended main case. Conversely, let K be an eigenvalue of (13.1)-(13.4).

(a) If $-1 < \sigma \le 0$ or if $0 < \sigma < \frac{1}{2}$ and $1 \le K < 1/\sigma$, then

$$\nu = \sigma + \frac{2(1-\sigma)(1-2\sigma)(1-K\sigma)}{-2(\sigma-1)(1-K\sigma) - (1-2\sigma)(K-1) + [4(\sigma-1)^2(1-K\sigma)^2 + (1-2\sigma)^2(K-1)^2]^{\frac{1}{2}}} \tag{13.56}$$

is an eigenvalue of (13.53), (13.54).

(b) If $0 < \sigma < \frac{1}{2}$ and $K > 1/\sigma$, then

$$\nu = \sigma + \frac{2(1-\sigma)(1-2\sigma)(1-K\sigma)}{-2(\sigma-1)(1-K\sigma)-(1-2\sigma)(K-1)-[4(\sigma-1)^2(1-K\sigma)^2+(1-2\sigma)^2(K-1)^2]^{\frac{1}{2}}} \tag{13.57}$$

is an eigenvalue of (13.53), (13.54).

(c) If $0 < \sigma < \frac{1}{2}$ and $K = 1/\sigma$, then

$$\nu = 1 - \sigma \tag{13.58}$$

is an eigenvalue of (13.53), (13.54).

We conclude this section by making some remarks concerning Theorem 13.4. Further discussion of the results obtained here will be given in the next section. When $\nu = \frac{1}{2}$, the equation (13.55) yields $K(\frac{1}{2}, \sigma) = 1$ and the result checks with Remark 2 of §12. Conversely, substitution of $K = 1$ in (13.56) yields $\nu = \frac{1}{2}$, and the result is consistent with the fact that $\nu = \frac{1}{2}$ is an eigenvalue of (13.53), (13.54) for any region. Assertion (c) of Theorem 13.4 states that if $K = 1/\sigma$ is an eigenvalue of (13.1)-(13.4), then $\nu = 1 - \frac{1}{K}$ is an eigenvalue of the displacement boundary-value problem (13.53), (13.54). In view of our Remark 5 of §12 concerning the value $K = 1/\sigma$, this statement is equivalent to Theorem 5.1 concerning the Korn eigenvalue problem in the second case.

14. Discussion of the Results of §13

The results of §13, which establish an equivalence between the Korn eigenvalue problem in the extended main case and the displacement boundary-value problem of plane strain for simply-connected regions, enable us to deduce properties of the K-spectrum from the known results on the ν-spectrum for the displacement boundary-value problem, which were given in Chapter IV. We recall here equation (13.55),

$$K(\nu, \sigma) = \frac{\sigma^2 - 2\sigma\nu - \nu^2 + 3\nu - 1}{(1-\nu)(2\sigma^2 - 2\sigma + \nu)} , \qquad (14.1)$$

which relates the eigenvalues ν to the eigenvalues K. It can be shown that, for fixed σ in $(-1, \frac{1}{2})$, $K(\nu, \sigma)$ is a monotonic increasing function of ν in $[\frac{1}{2}, 1)$. Thus, the largest eigenvalue K of the Korn problem is obtained by using the largest eigenvalue ν in the interval $[\frac{1}{2}, 1)$ of the displacement boundary-value problem in equation (14.1). We consider first the case of a circle and state the following result.

(i) The circle.

From equation (9.9) of Chapter V, we know that for a circle, the ν-spectrum consists of the discrete set $\nu = 1, 3/4, 1/2$, each occurring with infinite multiplicity. Thus, from Theorem 13.4 we deduce that the K-spectrum for a circle in the extended main case consists of the values

$$K(\tfrac{1}{2}, \sigma) = 1 , \quad K(3/4, \sigma) = \frac{16\sigma^2 - 24\sigma + 11}{8\sigma^2 - 8\sigma + 3} , \qquad (14.2)$$

each occurring with infinite multiplicity. The largest eigenvalue is

$$K(3/4, \sigma) = \frac{16\sigma^2 - 24\sigma + 11}{8\sigma^2 - 8\sigma + 3} , \tag{14.3}$$

which lies in the interval

$$51/19 < K(3/4, \sigma) \leq 4 , \tag{14.4}$$

for $-1 < \sigma < \frac{1}{2}$. The maximum value of $K(3/4, \sigma)$ is attained when $\sigma = 1/4$, so that

$$K(3/4, 1/4) = 4 . \tag{14.5}$$

Recalling equation (9.11), we see that the largest eigenvalue for a circle in the extended main case is always less than the corresponding eigenvalue in the second case, unless $\sigma = 1/4$, in which event both eigenvalues are equal. The latter observation is consistent with Remark 5 of §12 concerning the eigenvalue $K = 1/\sigma$.

(ii) Regions mapped conformally onto the unit disc by a rational mapping.

Recalling Theorem 8.8 regarding the eigenvalue $\nu = 3/4$, we deduce the result that $K(3/4, \sigma)$, defined by (14.3), is an eigenvalue for the Korn eigenvalue problem in the extended main case if and only if the region R can be mapped conformally onto the unit disc by a rational function. Moreover, when $K(3/4, \sigma)$ is an eigenvalue, it has infinite multiplicity.

In §9 (ii), we found the spectrum

$$\nu = \frac{1}{2}, \frac{3}{4} - \frac{m^2}{2}, \frac{3}{4}, \frac{3}{4} + \frac{m^2}{2}, 1 \tag{14.6}$$

for Pascal's limacon, where the values 1/2, 3/4, 1 occur with infinite multiplicity and the other eigenvalues have finite multiplicity. The parameter m appears in the mapping function (9.13) and satisfies $0 \le m < \frac{1}{2}$. Using Theorem 13.4, and equation (14.1), we find that the largest eigenvalue for Pascal's lamacon in the extended main case is given by

$$K = \frac{16\sigma^2 - \sigma(24+16m^2)+11+12m^2-4m^4}{(1-2m^2)(8\sigma^2-8\sigma+3+2m^2)}, \tag{14.7}$$

corresponding to the eigenvalue $\nu = \frac{3}{4} + \frac{m^2}{2}$ of (14.6).

(iii) The largest eigenvalue.

As in §9 (iii), we may conclude that for a given region R, there is in fact a largest eigenvalue K in the extended main case, but that there is no upper bound for the eigenvalues K valid for all regions. For regions which can be mapped conformally onto the unit disc by a rational transformation, the largest eigenvalue K in the extended main case is not less than $K(3/4, \sigma)$ as given in (14.3).

REFERENCES

1. Friedrichs, K. O., "On the Boundary-value Problems of the Theory of Elasticity and Korn's Inequality," Annals of Mathematics, 48, 441-471 (1947).

2. Korn, A., "Die Eigenschwingungen eines elastischen Körpers mit ruhender Oberfläche," Akad. der Wissensch., Munich, Math.-phys. Kl., Berichte 36, 351-401 (1906).

3. Korn, A., "Über einige Ungleichungen, welche in der Theorie der elastischen und elektrischen Schwingungen eine Rolle spielen," Bulletin Internationale, Cracovie Akademie Umiejet, Classe des sciences mathématiques et naturelles, 705-724 (1909).

4. Toupin, R. A., "Saint-Venant's Principle," Archive for Rational Mechanics and Analysis, 18, 83-96 (1965).

5. Payne, L. E. and H. F. Weinberger, "On Korn's Inequality," Archive for Rational Mechanics and Analysis, 8, 89-98 (1961).

6. Mikhlin, S. G., The Problem of the Minimum of a Quadratic Functional. San Francisco: Holden-Day, Inc. (1965).

7. Eidus, D. M., "On a Mixed Problem of Elasticity Theory," Dokl. Akad. Nauk SSSR, 76, 181-184 (1951). (Russian)

8. Fichera, G., "Sull'esistenza e sul calcolo delle soluzioni dei problemi al contorno relativi all'equilibrio di un corpo elastico," Annali della Scuola Normale Superiori di Pisa, Ser. III, 4, 35-99 (1950).

9. Fichera, G., Linear Elliptic Differential Systems and Eigenvalue Problems. Berlin: Springer-Verlag (1965).

10. Bernstein, B. and R. A. Toupin, "Korn Inequalities for the Sphere and Circle," Archive for Rational Mechanics and Analysis, 6, 51-64 (1960).

11. Dafermos, C. M., "Some Remarks on Korn's Inequality," Zeitschrift für angewandte Mathematik und Physik, (ZAMP) 19, 913-920 (1968).

12. Muskhelishvili, N. I., Some Basic Problems of the Mathematical Theory of Elasticity. Fourth Edition. Groningen: P. Noordhoff (1963).

13. Lekhnitskii, S. G., "On Some Problems Related to the Theory of Bending of Thin Strips," Prikl. Mat. i Mech., (PMM) 2, 181-210 (1938). (Russian with German summary)

14. Sokolnikoff, I. S., Mathematical Theory of Elasticity. Mimeographed lecture notes, Brown University, pp. 382-386 (1941).

15. Green, A. E. and W. Zerna, Theoretical Elasticity. Oxford: Oxford University Press (1954).

16. Gurtin, M. E. and E. Sternberg, "On the First Boundary-value Problem of Linear Elastostatics," Archive for Rational Mechanics and Analysis, 6, 177-187 (1960).

17. Mikhlin, S. G., "Further Investigation of the Cosserat Functions," Vestnik Leningrad Univ., 22, no. 7, 96-102 (1967). (Russian with English summary)

18. Ericksen, J. L., "On the Dirichlet Problem for Linear Differential Equations," Proc. American Math. Soc., 8, 521-522 (1957).

19. Mikhlin, S. G., "On Cosserat Functions," in Boundary Value Problems and Integral Equations. (Problems in the Mathematical Analysis Series, Vol. 1) Editor: V. I. Smirnov. Izdat. Leningrad. Univ., Leningrad, 1966 (English translation by Consultants Bureau, New York City, 1968 pp. 55-65.)

20. Cosserat, E. and F., "Sur les équations de la théorie de l'élasticité," Comptes Rendus, 126, 1089-1091 (1898).

21. Cosserat, E. and F., "Sur la déformation infiniment petite d'un ellipsoide élastique," Comptes Rendus, 127, 315-318 (1898).

22. Cosserat, E. and F., "Sur la solution des équations de l'élasticité, dans le cas où les valeurs des inconnues à la frontière sont données," Comptes Rendus, 133, 145-147 (1901).

23. Mikhlin, S. G., Integral Equations and Their Applications to Certain Problems in Mechanics, Mathematical Physics and Technology. Second Edition. New York: Pergamon Press (1957).

24. Sherman, D. I., "On the distribution of the eigenvalues of the integral equations of the plane theory of elasticity," Trudy Seism. Inst. Akad. Nauk SSSR, No. 82 (1938). (Works, Institute of Seismology, Ac. Sci. USSR) (Russian).

25. Bitsadze, A. V., Boundary Value Problems for Second Order Elliptic Equations. Amsterdam: North-Holland Publishing Company (1968).