REPRESENTING MEASURES ON THE ROYDEN BOUNDARY FOR SOLUTIONS OF $\Delta u = Pu$ ON A RIEMANNIAN MANIFOLD

Thesis by

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ABSTRACT

Consider the Royden compactification R^* of a Riemannian n-manifold R, $\Gamma = R^* \setminus R$ its Royden boundary, Δ its harmonic boundary and the elliptic differential equation $\Delta u = Pu$, $P \geq 0$ on R. A regular Borel measure m^P can be constructed on Γ with support equal to the closure of $\Delta^P = \{q \in \Delta : q$ has a neighborhood U in R^* with $\int_{U \cap R} P < \infty \}$. Every energy-finite solution u (i.e. $E(u) = D(u) + \int_R u^2 P < \infty$, where $D(u)$ is the Dirichlet integral of u) can be represented by $u(z) = \int_\Gamma u(q) K(z,q) dm^P(q)$ where $K(z,q)$ is a continuous function on $R \times \Gamma$. A $\widetilde{PE}$-function is a nonnegative solution which is the infimum of a downward directed family of energy-finite solutions. A nonzero $\widetilde{PE}$-function is called $\widetilde{PE}$-minimal if it is a constant multiple of every nonzero $\widetilde{PE}$-function dominated by it. THEOREM. There exists a $\widetilde{PE}$-minimal function if and only if there exists a point $q \in \Gamma$ such that $m^P(q) > 0$. THEOREM. For $q \in \Delta^P$, $m^P(q) > 0$ if and only if $m^0(q) > 0$.

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INTRODUCTION

The notion of $\widetilde{HD}$-minimal functions was first introduced by C. Constantinescu and A. Cornea in 1958.

The objective of this dissertation is to give a complete characterization of the more general class of all $\widetilde{PE}$-minimal functions on Riemannian manifolds following the pattern that M. Nakai has established in the case of $\widetilde{HD}$-minimal functions on Riemann surface, and finally to show that $\widetilde{PE}$-minimal functions are closely related to $\widetilde{HD}$-minimal functions.

In finite dimensional real Euclidean space, the boundary of a set has a strong influence over all harmonic functions on the set. A Riemann surface generally can not be embedded in a larger surface, and therefore it has no natural boundary. For this purpose, Royden introduced, and Nakai developed the concept of Royden boundary for Riemann surfaces which is suitable for the study of all harmonic functions with finite Dirichlet integral (called HD-functions). For instance, HD-functions satisfy the maximum principle on the Royden boundary.

Nakai also constructed a bounded positive regular Borel harmonic representing measure on the Royden boundary so that together with a harmonic kernel every HD-function has an integral representation on the boundary and can thus be characterized. Since $\widetilde{HD}$-functions, in particular $\widetilde{HD}$-minimal functions, are infima of downward directed families of nonnegative HD-functions, it is not surprising that they can also be represented, although in a different fashion, as integrals on the Royden boundary. One striking result is that there exists an $\widetilde{HD}$-minimal function if and only if there exists an atom on the Royden boundary with respect to this harmonic representing measure.

Recently, M. Glasner and R. Katz have shown that the concept of Royden boundary can be carried over to Riemannian manifolds, and that the Royden boundary is suitable for the study of not only HD-functions, but PE-functions, i.e. the solutions of the elliptic differential equations $\Delta u = Pu$ with finite energy integral.

In this thesis, we will see that a representing measure together with a kernel for the solutions of $\Delta u = Pu$ can be constructed similarly on the Royden boundary, and most results for HD-, $\widetilde{HD}$- and $\widetilde{HD}$-minimal functions can be generalized to PE-, $\widetilde{PE}$- and $\widetilde{PE}$-minimal functions. The first main result is that there exists a $\widetilde{PE}$-minimal function if and only if there exists an atom on the Royden boundary with respect to the solution representing measure.

What is more significant is that under certain circumstances there exists a $\widetilde{PE}$-minimal function if and only if there exists an $\widetilde{HD}$-minimal function.

In Chapter I, some preliminaries about Riemannian manifolds are given. We will prove that the family of all solutions of $\Delta u = Pu$ on a Riemannian manifold forms a harmonic class in Chapter II. Although the procedure has been referred to, this is the first systematic exposition which brings the local properties of solutions of elliptic differential equations in Euclidean space up to a Riemannian manifold.

In Chapter III, the Royden compactification and the Royden boundary are described. Several maximum and minimum principles are given in Chapter IV. The representing measures on the Royden boundary are constructed in Chapter V. PE- and $\widetilde{PE}$-functions are carefully studied in Chapter VI. Finally, the two main results are given in Chapter VII.

I. PRELIMINARIES

In this chapter the definition of a Riemannian manifold is given. The calculus of exterior differential forms is also sketched. We shall see that the equation $\Delta u = Pu$ on a Riemannian manifold, where P is a C^1 n-form, in terms of local coordinates is a self-adjoint uniformly elliptic second order differential equation. For a C^1-function the Dirichlet integral as well as the energy integral with respect to a n-form P are introduced. Finally, Green's formula and the Dirichlet principle are derived from Stokes' theorem.

Ia. DEFINITIONS. Let R be a connected Hausdorff space. A family $A = \left\{ (U_a, \phi_a) : a \in I \right\}$, where I is some index set, is an atlas for R if

1) for each a, U_a is an open relatively compact subset of R and $\phi_a : U_a \longrightarrow \phi_a(U_a) \subset E^n$ is a homeomorphism, where E^n denotes n-dimensional Euclidean space,

2) $\left\{ U_a : a \in I \right\}$ covers R,

3) if $(U_1, \phi_1), (U_2, \phi_2) \in A$, then $\phi_1 \circ \phi_2^{-1}$ is a C^2-function whenever it is defined.

The space R, with A, is called a C^2 n-manifold. The homeomorphism ϕ associated to each open set U is called a system of local coordinates on U, and the open set U is called a parametric region. U is called a parametric ball if $\phi(U)$ is the open unit ball in E^n. U is also called a parametric cube if $\phi(U)$ is a n-dimensional cube $(a_1, b_1) \times \cdots \times (a_n, b_n)$ in E^n.

A C^2 m-submanifold $(m < n)$ M of R is a connected subset such that for all $x \in M$, there is a pair $(U,\phi) \in A$ with $x \in U$ and $\phi(U \cap M)$ is an open subset of an m-dimensional subspace of E^n.

Ib. DEFINITION. A C^2 n-manifold R is orientable if there is an atlas A on R such that $\det(\phi_1 \circ \phi_2^{-1}) > 0$ for any $\phi_1, \phi_2 \in A$ whenever $\phi_1 \circ \phi_2^{-1}$ is defined.

Ic. THEOREM. Every C^2 n-manifold is arcwise connected.

Id. THEOREM. A C^2 n-manifold is separable if and only if it has countable base of parametric balls.

These theorems are analogues of Theorems 2-17, 2-18 of [18].

Ie. DEFINITION. A Riemannian manifold R is a connected, orientable, separable, noncompact (for our purpose) C^2 n-manifold with a fundamental tensor g_{ij} yielding a positive definite form $g_{ij}r^i r^j$ where $g_{ij} = g_{ji}$ are C^1-Hölder continuous.

From now on, R will always denote a Riemannian manifold.

If. We denote inverse matrix of (g_{ij}) by (g^{ij}) and the determinant of (g_{ij}) by g. If we denote the local coordinates of R by $x^1, \ldots, x^n$, then the arc element ds in each parametric region is given by $ds^2 = g_{ij}dx^i dx^j$.

A (Riemannian) metric can be defined on R in terms of the arc element ds in a natural way.

REMARK. R is a metric space where the metric topology of R is equivalent to the locally Euclidean topology of R.

Indeed, if we take a countable base $B = \{U\}$ of parametric balls for R (see Id). Note that $B' = \{U'\}$, which is the family of parametric balls having same centers and 1/2 radii as those in B, covers R. Then on each $U' \in B'$ in terms of local coordinates there is a constant c such that for every vector $(y^1, \ldots, y^n) \in E^n$ we have

$$(1/c) \sum (y^i)^2 \leq \sum g_{ij} y^i y^j \leq c \sum (y^i)^2$$

because $\Sigma g_{ij} y^i y^j$ is continuous on $U \times S$, where S is the unit ball in E^n.

Ig. Let A^p be the set of all differentiable p-forms $f = \sum_a f_a dx^a$, where $a = (a_1, \ldots, a_p)$, $1 \leq a_i \leq a_{i+1} \leq n$, f_a is a covariant tensor of rank p and $dx^a = dx^{a_1} \wedge \cdots \wedge dx^{a_p}$. An inner product is determined for A^p by the rule

$$dx^a \cdot dx^b = \det(g^{a_i b_j}). \tag{1}$$

In the 1-dimensional space A^n this product gives the volume element w by $w \cdot w = 1$.

The Hodge star operator $* : A^p \longrightarrow A^{n-p}$ is defined as follows. If $f \in A^p$, then $*f \in A^{n-p}$ is determined by its exterior products

$$h \wedge *f = (f\cdot h)w$$

for all $h \in A^p$. This gives an isomorphism between A^p and A^{n-p}.

Ih. The <u>exterior derivative</u> df of a p-form $f = \sum_a f_a dx^a$ is the (p+1)-form

(2) $$df = \sum_a df_a \wedge dx^a$$

where $df = \sum_i (\partial f_a/\partial x^i)dx^i$.

The <u>Laplacian</u> of a function (0-form) u is $\Delta u = *d*du$.

Ii. To determine Δu in local coordinates, we begin by computing the local expression of *f for an arbitrary p-form f. Let

$$f = \sum_a f_a dx^{a_1} \wedge \ldots \wedge dx^{a_p},$$

(3) $$*f = \sum_b (*f)_b dx^{b_1} \wedge \ldots \wedge dx^{b_{n-p}}.$$

For a given $b = (b_1,\ldots,b_{n-p})$, take the complement $b' = (b'_1,\ldots,b'_p)$ of b and left multiply both sides of (3) by $dx^{b'_1} \wedge \ldots \wedge dx^{b'_p}$. On the left we obtain

$$dx^{b'_1} \wedge \ldots \wedge dx^{b'_p} \wedge *f = ((dx^{b'_1} \wedge \ldots \wedge dx^{b'_p})\cdot f)w$$

$$= \sum_a f_a \det(g^{a_i b'_j})w,$$

while on the right we have

$$\mathcal{E}_{b'b}(*f)_b dx^1 \wedge \ldots \wedge dx^n = \mathcal{E}_{b'b}(*f)_b \frac{w}{\sqrt{g}}$$

where $\mathcal{E}_{b'b}$ is the signature of the permutation $b'b$. We conclude that

$$(*f)_b = \mathcal{E}_{b'b}\sqrt{g}\sum_a f_a \det(g^{a_i b'_j}).$$

Since we shall only deal with functions (0-forms) u on R, we proceed as follows.

$$\Delta u = *d*du ,$$

(4) $$du = \sum \frac{\partial u}{\partial x^i} dx^i ,$$

(5) $$*du = \sum_j (-1)^{j-1}\sqrt{g}\sum_i \frac{\partial u}{\partial x^i} g^{ij} dx^1 \wedge \ldots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \ldots \wedge dx^n ,$$

$$d*du = \sum_j \frac{\partial}{\partial x^j}\left(\sqrt{g}\sum_i \frac{\partial u}{\partial x^i} g^{ij}\right) dx^1 \wedge \ldots \wedge dx^n$$

$$= \sum_{i,j} \frac{\partial}{\partial x^j}\left(\sqrt{g}g^{ij}\frac{\partial u}{\partial x^i}\right)\frac{w}{\sqrt{g}} ,$$

$$\Delta u = \frac{1}{\sqrt{g}}\sum_{i,j}\frac{\partial}{\partial x^j}\left(\sqrt{g}g^{ij}\frac{\partial u}{\partial x^i}\right) .$$

In this thesis, we shall study the solutions of the equation $\Delta u = *Pu$, where P is a nonnegative n-form. Locally it is a <u>self-adjoint elliptic second order differential equation</u> as we have just shown.

Furthermore, we shall see that, locally $\Delta u = *Pu$ is a <u>uniformly</u> elliptic differential equation.

Take a countable base $B = \{U\}$ of parametric balls for R (cf. Id). Let $B' = \{U'\}$ be the family of parametric balls which have same centers and 1/2 radii as those in B. Note that B' covers R. In each $U \in B$, by the previous result, the equation $\Delta u = *Pu$ has the form

$$\frac{1}{\sqrt{g}} \sum \frac{\partial}{\partial x^j} \left(\sqrt{g} g^{ij} \frac{\partial u}{\partial x^i} \right) = \frac{1}{\sqrt{g}} pu,$$

where $p dx^1 \wedge \ldots \wedge dx^n$ is a local representation for P. If $U' \in B'$ which has same center as U but 1/2 radius, then there exists a constant $c \geq 1$ such that

$$\sum y_i^2 / c \leq \sum g^{ij} y_i y_j \leq c \sum y_i^2$$

on U' where $(y_1, \ldots, y_n) \in E^n$ is an arbitrary point. This was established in If.

Thus $\Delta u = *Pu$ is a uniformly elliptic second order differential equation. Hence, in particular, the results in [5] and [19] can be applied in parametric balls U .

Ij. <u>DEFINITION</u>. A subset G of R is said to have a <u>piecewise smooth boundary</u> bG if bG consists of a countable number of C^2 (n-1)- and C^2 (n-2)-submanifolds which do not accumulate in R.

<u>DEFINITION</u>. An <u>exhaustion</u> of R is a family $\{R_n : n = 1, 2, \ldots\}$ of relatively compact open sets such that $\bar{R}_n \subset R_{n+1}$ for all n and $\bigcup \{R_n : n = 1, 2, \ldots\} = R$.

<u>THEOREM</u>. There exists an exhaustion $\{R_n\}$ of R such that each R_n is a region with piecewise smooth boundary.

Proof. By Id, we can choose a countable base $\{U_i\}$ of parametric balls for R which covers R. Let $R_1 = U_1$, and define inductively R_n to be a connected open set which is a finite union of U's such that $R_n \supset R_{n-1} \cup U_n$. Clearly $\{R_n\}$ meets the requirement.

Ik. Integration on manifolds can be defined by means of partition of unity. For the details see [17].

Consider any relatively compact subset S of R. For functions $u, v \in C^1(S)$, the Dirichlet integral of u and v over S is defined as

$$D_S(u,v) = \int_S du \wedge {*dv}.$$

In addition, if P is a n-form on R, the energy integral with respect to P of u and v over S is defined as

$$E_S(u,v) = D_S(u,v) + \int_S uvP.$$

If S is any subset of R and $u, v \in C^1(S)$, the Dirichlet and energy integrals of u, v over S are defined as

$$D_S(u,v) = \lim D_{S_n}(u,v)$$

and

$$E_S(u,v) = \lim E_{S_n}(u,v),$$

if these limits exist, where $S_n = S \cap R_n$ and $\{R_n\}$ is an exhaustion of R.

If $S = R$, we simply write

$$D(u,v) \qquad \text{and} \qquad E(u,v)$$

instead of $D_R(u,v)$ and $E_R(u,v)$.

We also denote

$$D_S(u) = D_S(u,u)$$

and

$$E_S(u) = E_S(u,u)$$

and drop the letter S when $S = R$; these are the Dirichlet and energy integrals of u over S.

Since $D_{S_n}(u)$ and $E_{S_n}(u)$ are monotonic, we can conclude by the Schwarz inequality that the definitions of $D_S(u,v)$, $E_S(u,v)$ are independent of the choice of $\{R_n\}$.

Suppose S is contained in a parametric region. We shall calculate the expression of $D_S(u,v)$ in terms of the local coordinates.

By (4) and (5) with v in place of u we see that

$$du \wedge {*dv} = \sum \sqrt{g} g^{ij} \frac{\partial u}{\partial x^i} \frac{\partial v}{\partial x^j} \ dx^1 \wedge \ldots \wedge dx^n.$$

Thus

$$D_S(u,v) = \int_S \sum \sqrt{g} g^{ij} \frac{\partial u}{\partial x^i} \frac{\partial v}{\partial x^j} \ dx^1 \wedge \ldots \wedge dx^n.$$

11. STOKES' THEOREM. Let G be a relatively compact open set in R whose boundary bG is piecewise smooth. Then for all $f \in A^{n-1}$, we have

$$\int_{bG} f = \int_G df$$

COROLLARY (Green's formula). If $u \in C^1(\bar{G})$ and $v \in C^2(\bar{G})$, then

$$D_G(u,v) + \int_G ud{*}dv = \int_{bG} u{*}dv.$$

Proof. In fact, by Stokes' theorem

$$\int_{bG} u{*}dv = \int_G d(u{*}dv)$$

$$= \int_G du \wedge {*}dv + \int_G ud{*}dv$$

$$= D_G(u,v) + \int_G ud{*}dv.$$

Im. DIRICHLET PRINCIPLE. Let G be a relatively compact open set in R whose boundary bG is piecewise smooth. If $u \in C^2(\bar{G})$ and $v \in C^1(\bar{G})$ such that u is equal to v outside G and u satisfies the equation $d{*}du = uP$ in G where P is a C^1 n-form on R, then $E_G(u) \leq E_G(v)$.

Proof. By Green's formula

$$E_G(u - v,u) = D_G(u - v,u) + \int_G (u - v)uP$$

$$= D_G(u - v,u) + \int_G (u - v)d{*}du$$

$$= \int_{bG} (u - v){*}du$$

$$= 0.$$

Hence

$$E_G(u - v, u) = E_G(u) - E_G(v, u).$$

Now

$$0 \leq E_G(u - v)$$

$$= E_G(u) - 2E_G(u, v) + E_G(v)$$

$$= E_G(v) - E_G(u),$$

i.e.

$$E_G(u) \leq E_G(v).$$

II. ABSTRACT HARMONIC CLASSES

Let P be a C^1 n-form on a Riemannian manifold R. By identifying n-forms and 0-forms which correspond under the * isomorphism, we can write the equation $d{*}du = uP$ as

(1) $$\Delta u = Pu.$$

Beginning with a few simple axioms, M. Brelot [1] has developed the abstract theory of harmonic functions (also cf. [9]) which contains the classical results of the theory of certain elliptic differential equations as special cases. In particular, the complete presheaf of solutions of the equation (1) on open subsets of a Riemannian manifold R satisfies the axioms as we shall see in this chapter.

Indeed in this thesis, the concept of abstract harmonic classes will be used as a unifying tool and also as a mean for making the transition from a few well-known local properties of (1) to the global properties that will be needed.

Throughout the exposition we use the following notations:

a) for any function f and G a subset of its domain,

$$f|G$$

is the restriction of f on G,

b) for any functions f and g

$$(f \cup g)(x) = \max(f(x), g(x))$$

$$(f \cap g)(x) = \min(f(x), g(x)),$$

and

c) instead of $\lim_{x\in A, x\to x_o} f(x)$, $\liminf_{x\in A, x\to x_o} f(x)$ and $\limsup_{x\in A, x\to x_o} f(x)$, we simply write

$$\lim_A f(x_o), \quad \liminf_A f(x_o) \quad \text{and} \quad \limsup_A f(x_o)$$

respectively whenever no confusion arises.

IIa. M. Brelot has introduced the following definition of a harmonic class (cf. p. 169, [9]).

Let W be a connected, non-compact, locally compact, Hausdorff space. If G is an open subset of W, then by bG and $\bar{G}$ we mean the boundary and the closure of G respectively.

The functions that we shall consider on W are extended real-valued functions with the usual lattice ordering $\geq$. By an upward directed family of functions we mean a nonempty family such that for any two functions f and g in the family there is a third function h in the family with $h(x) \geq (f \cup g)(x)$ for all $x \in W$. A downward directed family is similarly defined.

DEFINITION. Let $\mathcal{H}$ be a class of real-valued continuous functions with open domains in W such that for each open set $G \subset W$ the family $\mathcal{H}(G)$, consisting of all functions in $\mathcal{H}$ with domains equal to G, is a real vector space. An open set G of W is said to be regular for $\mathcal{H}$ or regular if for every continuous real-valued function f defined on bG there is a unique continuous function $H(f,G)$, or simply $H(f)$, defined on $\bar{G}$ such that $H(f,G)|bG = f$, $H(f,G)|G \in \mathcal{H}$, and $H(f,G) \geq 0$ if $f \geq 0$.

Moreover the class $\mathcal{H}$ is called a <u>harmonic class on</u> W if it satisfies following three axioms.

<u>Axiom</u> I. $\mathcal{H}$ is a complete presheaf, i.e. (1) if $h \in \mathcal{H}$ with domain U, then $h|V \in \mathcal{H}$ for all open subset $V \subset U$, and (2) if $\{h_a : a \in I\} \subset \mathcal{H}$, where h_a has domain U_a and I is some index set, such that $h_a | U_a \cap U_b = h_b | U_a \cap U_b$ for all $a, b \in I$, then there is an $h \in \mathcal{H}$ with domain $\bigcup\{U_a : a \in I\}$ such that $h|U_a = h_a$ for all $a \in I$.

<u>Axiom</u> II. There is a base for the topology of W consisting of regular regions.

<u>Axiom</u> III. If G is a region in W, A a compact subset of G and x a point of A, then there is a constant $M \geq 1$ such that every nonnegative function $h \in \mathcal{H}(G)$ satisfies the inequality $h(z) \leq Mh(x)$ at every point z of A.

IIb. <u>THEOREM</u>. Given Axioms I and II, the following axioms are equivalent to Axiom III.

Axiom III_1. If $\mathcal{F}$ is an upward directed subfamily of $\mathcal{H}(G)$ where G is a region in W, then the upper envelope of $\mathcal{F}$ is either $+\infty$ or a function in $\mathcal{H}(G)$.

Axiom III_2. If $\{h_n\}$ is an increasing sequence in $\mathcal{H}(G)$ where G is a region in W, then either $\lim h_n = +\infty$ or $\lim h_n$ is in $\mathcal{H}(G)$.

Axiom III_3. If G is a region in W, then every nonnegative function in $\mathcal{H}(G)$ is either identically 0 or has no zeros in G. Furthermore, for any point x in G, the set

$$\mathcal{F}_x = \{h \in \mathcal{H}(G) : h \geq 0 \text{ and } h(x) = 1\}$$

is equicontinuous at x.

For the proof, see p. 373 and p. 378 of [2] as well as p. 598 of [10].

IIc. <u>THEOREM</u>. The family $\mathcal{K} = \{h : h$ is a solution of (1) on U and U is an open subset of $R\}$ is a harmonic class.

Proof. Obviously the set $\mathcal{K}(G)$ of all solutions of (1) on an open set G is a real vector space. Axiom I is clearly satisfied. The collection B' of all parametric balls with same centers and $\frac{1}{2}$ radii as those in B, which is a countable base for the topology of R as in Id, is a countable base for the topology of R. To see these parametric balls in B' are regular with respect to $\mathcal{K}$ it is sufficient to see the the corresponding balls in E^n are regular with respect to the corresponding elliptic differential equations. At this point we appeal to the results in [5]. Finally, $\mathcal{K}$ satisfies Axiom III because of the following lemma.

IId. <u>LEMMA</u>. Let G be any region in R and $x, y \in G$. Then there exist constants a and b depending on x and y such that

$$a\,u(y) \leq u(x) \leq b\,u(y)$$

for any nonnegative solution $u \in \mathcal{K}(G)$.

Proof. Indeed, if x and y are in a parametric ball, then the lemma is an easy consequence of the Harnack's theorem of [5]. Otherwise, there exists a finite number of points $z_1, \ldots, z_n \in G$ such that z_i and z_{i+1} are in some parametric ball $U_i \subset G$ where $i = 0,1,2,\ldots,n$ and $z_0 = x$, $z_{n+1} = y$. Hence by previous observation, there are constants $a_1, \dots, a_n$ and $b_1, \dots, b_n$ such that

$$a_{i+1}u(z_{i+1}) \le u(z_i) \le b_{i+1}u(z_{i+1})$$

$i = 0,1,2,\dots,n.$ Consequently

$$a_1 a_2 \dots a_n u(y) \le u(x) \le b_1 b_2 \dots b_n u(y).$$

IIe. Together with Axiom III_3, we draw the following useful consequence from IId.

THEOREM. Let G be any region in R. For any $x,y \in G$, let $k(x,y) = \inf \{ c : (1/c)u(y) \le u(x) \le cu(y)$ for all nonnegative functions $u \in \mathcal{K}(G) \}$, then

i) $k(x,y) \ge 1$,

ii) $u(y)/k(x,y) \le u(x) \le k(x,y)u(y)$ for any nonnegative function $u \in \mathcal{K}(G)$,

iii) $k(x,y) = k(y,x)$ for all $x,y \in G$,

iv) $\lim_{x,y \to z} k(x,y) = 1$ for all $z \in G$,

v) for fixed y, $k(x,y)$ is a continuous function of x.

Proof. i) and ii) are obvious. From ii) we see that

$$u(x)/k(x,y) \le u(y) \le k(x,y)u(x).$$

Thus $k(x,y) \le k(y,x)$. By symmetry we also have $k(y,x) \le k(x,y)$. Hence iii) follows. To prove iv), if $\varepsilon > 0$ is any positive number, let δ be such that $N = \max(1 + \delta, 1/(1 - \delta)) < 1 + \varepsilon$. Recall Axiom III_3 asserts that the family $\mathcal{F}_z = \{ u/u(z) : u \in \mathcal{K}(G),\ u > 0 \}$ is

equicontinuous at z. Thus there is a neighborhood U of z such that for all $x \in U$

$$\left| u(x)/u(z) - 1 \right| < \delta ,$$

or equivalently

$$(1 - \delta)u(z) < u(x) < (1 + \delta)u(z).$$

Hence

$$u(z)/N < u(x) < Nu(z).$$

Consequently,

$$1 \leq k(x,z) \leq N < 1 + \varepsilon ,$$

i.e. $\lim_{x \to z} k(x,z) = 1.$

Now, from the inequalities

$$u(z)/k(x,z) \leq u(x) \leq k(x,z)u(z)$$

$$u(z)/k(y,z) \leq u(y) \leq k(y,z)u(z),$$

we obtain

$$u(y)/(k(x,z)k(y,z)) \leq u(x) \leq k(x,z)k(y,z)u(y).$$

Thus by definition and i)

vi) $$1 \leq k(x,y) \leq k(x,z)k(y,z).$$

Hence $k(x,y)$ tends to 1 as x and y tend to z because $k(x,z)$ and $k(y,z)$ do by the previous argument.

It remains to prove v). Note that vi) is true for all x,y and $z \in G$. Thus

$$1/k(z,x) = k(x,y)/(k(z,x)k(y,x)) \leq k(x,y)/k(z,y) \leq k(x,z)k(y,z)/k(z,y)$$
$$= k(x,z).$$

Hence we have $\lim_{x \to z} k(x,y)/k(z,y) = 1$, or $\lim_{x \to z} k(x,y) = k(z,y)$, i.e. $k(x,y)$ is a continuous function of x.

IIf. From now, a function $u \in \mathcal{K}$ with open domain $U \subset R$ will be called a <u>solution on U</u>.

The particular harmonic class of harmonic functions with open domains in R, i.e. $\{h : h$ satisfies $\Delta h = 0$ on U, U is an open subset of R$\}$ will be denoted by $\mathcal{K}^{\circ}$.

IIg. <u>DEFINITION</u>. A <u>regular exhaustion of R with respect to a harmonic class</u> $\mathcal{H}$ is an exhaustion of R such that each of its members is regular with respect to $\mathcal{H}$.

<u>THEOREM</u>. There is a regular exhaustion on R with respect to both $\mathcal{K}^{\circ}$ and $\mathcal{K}$.

Proof. Consider the exhaustion $\{R_n\}$ constructed in Theorem Ij. Note that for any parametric ball U with local coordinates ϕ, $\phi(bR_n \cap U)$ satisfies the cone condition (see [8], p.329). Hence there exists a barrier function at every point of $\phi(bR_n \cap U)$ on $\phi(R_n \cap U)$. This means that $\phi(bR_n \cap U)$ is regular in E^n for the Laplace equation

$$\partial^2 u/\partial x^{1^2} + \cdot \cdot \cdot \cdot \cdot + \partial^2 u/\partial x^{n^2} = 0.$$

Hence by Theorem 10.2 of [19], $\phi(bR_n \cap U)$ is regular in E^n for the elliptic equation (cf. Ii)

$$\sum \frac{\partial}{\partial x^j} (\sqrt{g} g^{ij} \frac{\partial u}{\partial x^i}) = pu.$$

$bR_n \cap U$ is thus regular with respect to both $\mathcal{K}^{\circ}$ and $\mathcal{K}$ by Theorem 3.3 of [9], and hence bR_n is regular with respect to both $\mathcal{K}^{\circ}$ and $\mathcal{K}$ by Brelot's comparison theorem (cf. Theorem 3.2, [9]).

IIh. Consider again an abstract harmonic class $\mathcal{H}$ on W as in IIa. If G is a regular open set, let C(bG) be the set of all continuous functions on bG. For each $f \in C(bG)$, we consider the function H(f,G) given in the definition of regularity (IIa). For each $x \in G$, it is easy to see that H(f,G)(x), as a function of f, is a positive linear functional. There is a positive regular Borel measure r(x,G) defined on bG such that

$$H(f,G)(x) = \int_{bG} f dr(x,G)$$

for each $f \in C(bG)$. Since $r(x,G)(bG) = H(1,G)(x) < \infty$, this measure, hence also the functional, are bounded.

DEFINITION. r(x,G) is called the harmonic measure with respect to $\mathcal{H}$ for G at x.

IIi. Let f be lower semi-continuous on bG where G is as in IIh.

DEFINITION. f is integrable with respect to $\mathcal{H}$ on bG if it is integrable with respect to r(x,G) for all $x \in G$, i.e. $\int f dr(x,G)$

$= \sup \left\{ \int g dr(x,G) : g \in C(bG),\ g \leq f \right\} < \infty$ for all $x \in G$. In this case, we set

$$H(f,G)(x) = H(f)(x) = \int_{bG} f dr(x,G).$$

IIj. Brelot ([1], p. 65) has proved the following lemma which in fact is an easy consequence of Axiom III_1.

LEMMA. Let f be lower semi-continuous on bG where G is a regular open set. If f is integrable with respect to $\mathcal{H}$ on bG, then $H(f) \in \mathcal{H}(G)$.

IIk. THEOREM. Let G be a regular open subset of W for $\mathcal{H}$, f a lower semi-continuous function on bG and $x \in bG$. If f is bounded from above, then

$$\lim \sup_G H(f)(x) \leq \lim \sup_{bG} f(x).$$

If f is bounded from below, then

$$\lim \inf_{bG} f(x) \leq \lim \inf_G H(f)(x).$$

For the proof, see [9] on p. 173.

IIl. Let W be as in IIa. A family $\overline{\mathcal{H}}$ of extended real-valued lower semi-continuous function with open domains in W is called a superharmonic class with respect to $\mathcal{H}$ if

1) for all $v \in \overline{\mathcal{H}}$ with domain G, $v(x) < \infty$ for some x in each component of G, and

2) for all $x \in G$ such that $v(x) < \infty$ and for any neighborhood U of x, there is a regular region V with $x \in V \subset \overline{V} \subset U$ such that v is integrable on bV and $v(y) \geq H(v,V)(y)$ for all y in V.

A useful observation is

LEMMA. If $u, v \in \overline{\mathcal{H}}$ with same domain, then $u \wedge v$, $u + v$ and au are in $\overline{\mathcal{H}}$ if $a \geq 0$.

Functions in $\overline{\mathcal{H}}$ are called *superharmonic functions with respect to* $\mathcal{H}$.

We say an upper semi-continuous function u belongs to the family $\underline{\mathcal{H}}$, the *subharmonic class with respect to* $\mathcal{H}$, if $-u \in \overline{\mathcal{H}}$, and u is called a *subharmonic function with respect to* $\mathcal{H}$.

If G is an open subset of W, we let $\overline{\mathcal{H}}(G)$ denote the class of functions in $\overline{\mathcal{H}}$ with domain G and $\underline{\mathcal{H}}(G)$ the class of functions in $\underline{\mathcal{H}}$ with domain G.

IIm. THEOREM. $\mathcal{H} = \overline{\mathcal{H}} \cap \underline{\mathcal{H}}$.

Proof. Obviously $\mathcal{H} \subset \overline{\mathcal{H}} \cap \underline{\mathcal{H}}$. Conversely, for any $u \in \overline{\mathcal{H}} \cap \underline{\mathcal{H}}$ with domain G, u being both lower and upper semi-continuous must be continuous on G. Furthermore, for any $x \in G$, there is a regular open neighborhood U of x with $\overline{U} \subset G$. By the super- and sub-harmonicity of u, we have $u|U \geq H(u,U) \geq u|U$, i.e. $u|U = H(u,U)$. By Axiom I, $u \in \mathcal{H}(G)$. Hence $\overline{\mathcal{H}} \cap \underline{\mathcal{H}} \subset \mathcal{H}$.

IIn. THEOREM. Assume $1 \in \overline{\mathcal{H}}(W)$. Let G be a region in W and a, b are constants with $a \leq 0 \leq b$.

1) If $v \in \overline{\mathcal{H}}(G)$ and $v \geq a$, then either $v > a$ or $v \equiv a$.

2) If $u \in \underline{\mathcal{H}}(G)$ and $u \leq b$, then either $u < b$ or $u \equiv b$.

3) A nonconstant function in $\mathcal{H}(G)$ takes neither a nonnegative maximum nor a nonpositive minimum in G.

For the proofs of this and the next theorem, see [9], p. 178.

IIo. <u>THEOREM</u>. Let G be a relatively compact open subset of W and v a nonconstant function in $\overline{\mathcal{H}}(G)$. Let c be a nonpositive number such that $c \leq \liminf_G v(x)$ for each $x \in bG$. If $1 \in \overline{\mathcal{H}}(G)$, then $v > c$ on G.

In view of this, we shall henceforth make the assumption that $1 \in \overline{\mathcal{H}}(W)$.

IIp. Some relations between harmonic functions and solutions on R are given below.

<u>THEOREM</u>. Let $\overline{\mathcal{K}}^{\circ +} = \{ h \in \overline{\mathcal{K}}^{\circ} : h \geq 0 \}$ and $\underline{\mathcal{K}}^{-} = \{ u \in \underline{\mathcal{K}} : u \leq 0 \}$. Then $\overline{\mathcal{K}}^{\circ +} \subset \overline{\mathcal{K}}$ and $\underline{\mathcal{K}}^{-} \subset \underline{\mathcal{K}}^{\circ}$. In particular, $1 \in \overline{\mathcal{K}}(R)$.

Proof. To prove the first assertion it is sufficient to show that $\mathcal{K}^{\circ +} = \{ u \in \mathcal{K}^{\circ} : u \geq 0 \} \subset \overline{\mathcal{K}}$. So let $u \in \mathcal{K}^{\circ +}$ and any z in the domain of u. For any regular neighborhood U of z which is contained in the domain of u, let v be the element of $\mathcal{K}(U)$ such that $v|bU = u|bU$. We want to show that $u(z) \geq v(z)$. If this is false, then let $V = \{ x \in U : u(x) < v(x) \}$. Note that $v \geq 0$ on U by the regularity of U, and $\Delta(v - u) = \Delta v = Pv \geq 0$ in V. Hence $v - u$ is constant on V (see [4], p. 326), which means that V is empty. Thus $u \geq v$ on U, i.e. $u \in \overline{\mathcal{K}}$. The second assertion can be proved similarly.

IIq. From now on, functions in $\overline{\mathcal{K}}^{\circ}$ and $\underline{\mathcal{K}}^{\circ}$ will be called <u>super-</u> and <u>sub-harmonic functions</u> respectively, while functions in $\overline{\mathcal{K}}$ and $\underline{\mathcal{K}}$ will be called <u>super-</u> and <u>sub-solutions</u> respectively.

IIr. Some classical theorems about solutions of $\Delta u = Pu$ are particularly easy when viewed in terms of harmonic classes.

THEOREM. Let $\{u_n\}$ be a sequence in $\mathcal{H}(W)$. If $\{u_n\}$ converges uniformly on each compact subset of W to a function u, then $u \in \mathcal{H}(W)$.

Proof. For all $x \in W$, there is a relatively compact regular open neighborhood U of x. In terms of harmonic measure $r(x,U)$

$$u_n(x) = \int_{bU} u_n dr(x,U).$$

Since u_n converges to u uniformly on compact sets, in particular on bU, we have

$$u(x) = \int_{bU} u dr(x,U).$$

This is true for all $x \in U$, hence u is integrable and $u \in \mathcal{H}(U)$. Hence $u \in \mathcal{H}(W)$ by Axiom I.

IIs. LEMMA. Let U be any relatively compact regular open subset of W. For any $u \in \mathcal{H}(W)$, there exist nonnegative functions $u^+, u^- \in \mathcal{H}(U)$ such that $u = u^+ - u^-$ on U.

Moreover, u^+ and u^- are continuous on $\overline{U}$, and $u^+|bU = u \vee 0|bU$, $u^-|bU = -(u \wedge 0)|bU$.

Proof. Let $u^+ = H(u \vee 0, U)$ and $u^- = H(-(u \wedge 0), U)$.

COROLLARY. $u^+ \leq \sup_U|u|$, and $u^- \leq \sup_U|u|$.

Proof. This is an application of Theorem IIo.

IIt. THEOREM. Let $\{u_n\}$ be a bounded sequence of functions in $\mathfrak{K}(R)$ ($\mathfrak{K}^\circ(R)$, resp.). Then there is a subsequence which converges uniformly on compact subsets to a function $u \in \mathfrak{K}(R)$ ($\mathfrak{K}^\circ(R)$, resp.).

Proof. Consider R as a metric space (cf. If). We are going to prove that $\{u_n\}$ is an equicontinuous sequence of continuous functions on any compact set. Because of the boundedness of $\{u_n\}$, Ascoli's theorem asserts that $\{u_n\}$ is a normal family, i.e. $\{u_n\}$ has a subsequence converging uniformly on compact subsets to a function u. By Theorem IIr, $u \in \mathfrak{K}(R)$.

Indeed, for any point $q \in R$ and any relatively compact regular open neighborhood U about q, we have by Theorem IIe

$$u_n^+(q)/k(p,q) \le u_n^+(p) \le k(p,q)u_n^+(q)$$

for all $p \in U$, where u_n^+ (and u_n^- later on) is defined in the Lemma. Hence

$$(1/k(p,q) - 1)u_n^+(q) \le u_n^+(p) - u_n^+(q) \le (k(p,q) - 1)u_n^+(q).$$

Thus

$$\left|u_n^+(p) - u_n^+(q)\right| \le u_n^+(q) \max(k(p,q) - 1 , 1 - 1/k(p,q))$$

$$\le M \max(k(p,q) - 1 , 1 - 1/k(p,q)),$$

where M is a bound for $\{u_n\}$. Similarly

$$\left|u_n^-(p) - u_n^-(q)\right| \le M \max(k(p,q) - 1 , 1 - 1/k(p,q)).$$

As a consequence,

$$\left| u_n(p) - u_n(q) \right| \leq 2M \max(k(p,q) - 1 , 1 - 1/k(p,q)),$$

where the right hand side tends to 0 independently of n as p tends to q since k(p,q) is a continuous function. Hence $\{u_n\}$ is equicontinuous at q. Since R is a metric space, $\{u_n\}$ is equicontinuous on any compact subset of R.

III. ROYDEN COMPACTIFICATION

In a region of Euclidean space, the boundary of the region plays an essential role in the study of solutions of elliptic differential equations. The Royden compactification R^* of a Riemannian manifold R is used to give the manifold a boundary $\Gamma = R^* \setminus R$, the Royden boundary, with which the class of all solutions with finite energy can be analyzed.

The Royden compactification is constructed by means of Royden algebra which is the collection of all bounded Tonelli functions with finite Dirichlet integrals. In this chapter, we shall study the properties of the Royden algebra and describe the construction of the Royden compactification.

An intrinsic part of the Royden boundary is the harmonic boundary. Its importance will be examined in next chapter.

IIIa. DEFINITION. A real-valued function f on R is a Tonelli function if for each parametric cube U with local coordinates ϕ, where $\phi(U) = (a_1,b_1)\times\cdots\times(a_n,b_n)$, f satisfies the following properties:

1) for each i the function $x_i \longrightarrow f\circ\phi^{-1}(x_1,\ldots,x_n)$ is absolutely continuous for almost all (with respect to the Lebesgue measure) $(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n) \in (a_1,b_1)\times\cdots\times(a_{i-1},b_{i-1})\times(a_{i+1},b_{i+1})\times\cdots\times(a_n,b_n)$.

2) $\partial f/\partial x^i$ are square integrable over any compact subset of $\phi(U)$.

IIIb. Let $M(R)$ be the set of all bounded Tonelli functions on R with finite Dirichlet integrals. $M(R)$ will turn out to be an algebra as we shall see in the next theorem. We call $M(R)$ the <u>Royden algebra</u>.

<u>THEOREM</u>. $M(R)$ is a commutative algebra with identity under the usual algebraic operations.

For any $f \in M(R)$, $1/f \in M(R)$ if and only if $\inf_R |f| > 0$.

Proof. Clearly $M(R)$ is a real vector space. It is also clear that the constant function 1 is also in $M(R)$ serving as the identity. Now we are going to show $M(R)$ is an algebra.

For any $f, g \in M(R)$, fg is a bounded Tonelli function. Let K be a bound for f^2 and g^2. Then on any relatively compact S,

$$
\begin{aligned}
D_S(fg) &= \int_S d(fg) \wedge *d(fg) \\
&= \int_S (gdf + fdg) \wedge (g*df + f*dg) \\
&= \int_S g^2 df \wedge *df + 2fg\, df \wedge *dg + f^2 dg \wedge *dg \\
&\leq K\,(D_S(f) + 2D_S(f,g) + D_S(g)) \\
&\leq K\,(D_S(f) + 2\sqrt{D_S(f)D_S(g)} + D_S(g)) \\
&= K\,(\sqrt{D_S(f)} + \sqrt{D_S(g)})^2 .
\end{aligned}
$$

Thus

$$D(fg) \leq K\,(\sqrt{D(f)} + \sqrt{D(g)})^2 < \infty .$$

i.e. $fg \in M(R)$. Hence $M(R)$ is indeed an algebra.

Finally if $f \in M(R)$ with $\inf_R |f| > 0$. Then $\sup_R |1/f| < L$ for some L. Note on any relatively compact set S

$$D_S(1/f) = \int_S d(1/f) \wedge *d(1/f)$$

$$= \int_S (df \wedge *df)/f^4$$

$$\leq L^4 D_S(f).$$

Thus

$$D(1/f) \leq L^4 D(f) < \infty .$$

i.e. $1/f \in M(R)$.

IIIc. <u>THEOREM</u>. $M(R)$ is a lattice under the usual lattice operations $\cup$ and $\cap$.

Proof. Let $G = \{ x \in R : f(x) > g(x) \}$. Then

$$D(f \cup g) = D_G(f \cup g) + D_{R \setminus G} D(f \cup g)$$

$$= D_G(f) + D_{R \setminus G}(g)$$

$$\leq D(f) + D(g)$$

$$< \infty .$$

Similarly,

$$D(f \cap g) \leq D(f) + D(g) < \infty .$$

IIId. We shall make use of the following notions of convergence in $M(R)$.

Let $\{f_n\}$ be a sequence in $M(R)$. We write

1) $f = C\text{-}\lim f_n$ if $\{f_n\}$ converges to f uniformly on any compact subset of R.

2) $f = B\text{-}\lim f_n$ if $\{f_n\}$ is bounded and $f = C\text{-}\lim f_n$.

3) $f = U\text{-}\lim f_n$ if $\{f_n\}$ converges to f uniformly.

4) $f = D\text{-}\lim f_n$ if $\lim D(f - f_n) = 0$.

5) $f = E\text{-}\lim f_n$ if $\lim E(f - f_n) = 0$.

6) $f = XD\text{-}\lim f_n$ (or $f = XE\text{-}\lim f_n$) if $f = D\text{-}\lim f_n$ (or $f = E\text{-}\lim f_n$) and $f = X\text{-}\lim f_n$, where X can be C, B or U.

IIIe. <u>THEOREM</u>. $M(R)$ is BD-complete, i.e. complete with respect to the BD-convergence.

Cf. [12], Lemma 1.5. This theorem means, in part, if $f_n \in M(R)$, $f = B\text{-}\lim f_n$ and $\lim D(f_n - f_{n+p}) = 0$ for all p, then $f \in M(R)$ and $f = BD\text{-}\lim f_n$.

IIIf. <u>THEOREM</u>. $C^1(R) \cap M(R)$ is dense in $M(R)$ with respect to the UD-convergence.

In fact, this is Theorem 1 of [12]. We are more interested in the following corollaries.

<u>COROLLARY</u> (<u>Green's formula</u>). Let G be a relatively compact

open subset of R with piecewise smooth boundary bG. If $u \in M(R)$ and $v \in C^2(\overline{G})$, then

$$D_G(u,v) + \int_G ud{*}dv = \int_{bG} u{*}dv .$$

COROLLARY (Dirichlet Principle). Let G be a relatively compact open subset of R with piecewise smooth boundary bG. If $u,v \in M(R)$ such that $u = v$ on $R \setminus G$ and u satisfies $\Delta u = Pu$ in $\overline{G}$, then $E_G(u) \leq E_G(v)$.

IIIg. The Royden compactification of R is a compact Hausdorff space R* such that

1) R is open and dense in R*,

2) functions in M(R) can be extended as continuous functions to R*, and

3) M(R) separates points of R*.

The compactification is unique up to homeomorphism fixing R elementwise.

We shall give a description of the construction of the Royden compactification in section IIIi. For the details, see p. 184 of [14] and p. 159 of [13] or cf. any exposition on the Gelfand representation.

IIIh. For all functions $f \in M(R)$, we define a norm

$$\|f\| = \sqrt{D(f)} + \sup_R |f| .$$

Then

THEOREM. M(R) with the norm $\|\cdot\|$ is a Banach algebra.

Cf. IIIb and IIIe.

IIIi. We denote the dual space of $M(R)$ by $M(R)^*$ and consider the weak* topology of $M(R)^*$. Let us consider the set R^* of all multiplicative bounded linear functionals L with $L(1) = 1$. It can be proved that R^* is a subset of the closed unit ball and is closed in the weak* topology of $M(R)^*$. Furthermore R^* is compact and Hausdorff. We define a mapping σ from R to R^* such that for every $p \in R$ we have $\sigma(p)(f) = f(p)$ for all $f \in M(R)$. It is easy to see that σ is one-to-one and continuous. Moreover $\sigma(R)$ is open and dense in R^*. We now identify R with $\sigma(R)$. We can see that the topology of R is the same as the relative topology of R in R^*. Now we are ready to extend every function in $M(R)$ to R^* continuously. For every $f \in M(R)$, we set $\bar{f}(L) = L(f)$ for all $L \in R^*$. In particular if $p \in R$, then $\bar{f}(\sigma(p)) = \sigma(p)(f) = f(p)$. It turns out that the function $\bar{f}$ is indeed a continuous extension of f. That $M(R)$ separates points of R^* is immediate.

R^* constructed above is <u>the Royden compactification of</u> R.

<u>From now on, if</u> G <u>is any subset of</u> R <u>or</u> R^*, $\bar{G}$ <u>always means the closure of</u> G <u>in</u> R^*.

IIIj. <u>THEOREM</u>. $M(R)$ is dense in $C(R^*)$ with respect to the uniform norm.

Indeed, the proof is an application of Stone-Weierstrass theorem.

IIIk. This and the Urysohn lemma lead to the following consequence.

<u>THEOREM</u>. (<u>Urysohn property</u>) Let K_1 and K_2 be any two disjoint compact subsets of R^*. For any two distinct real number r_1, r_2, $r_1 < r_2$, there always exists a function $f \in M(R)$ such that $r_1 \leq f \leq r_2$ and $f|K_i = r_i$, $i = 1,2$.

Proof. By Urysohn lemma, there is a function $g \in C(R^*)$ with $g|K_1 = r_1 - 2$ and $g|K_2 = r_2 + 2$. By IIIj, there exists a function $h \in M(R)$ with $h|K_1 < r_1 - 1$ and $h|K_2 > r_2 + 1$. Then the function $f = (h \cup r_1) \cap r_2 \in M(R)$ meets the requirement.

IIIl. <u>DEFINITION</u>. $\Gamma = R^* \setminus R$ is called <u>the Royden boundary</u> of R.

The following proposition is due to Nakai (Proposition 6, [16]).

<u>PROPOSITION</u>. $\overline{G} \setminus \overline{bG}$ is open in R^* for any open subset G of R.

IIIm. If we let $M_o(R)$ be the set of functions in $M(R)$ with compact support in R, then we have the following theorem.

<u>THEOREM</u>. $\Gamma = \{ q \in R^* : f(q) = 0 \text{ for all } f \in M_o(R) \}$.

Proof. If $g \in M_o(R)$, then $f|\Gamma = f|R^* \setminus R = 0$ because the support of f is in R.

On the other hand, for any $q \in R$, there is a relatively compact open neighborhood U of q in R. Note that $R^* \setminus U$ is compact in R^*. By virtue of Theorem IIIk, there is a function $g \in M(R)$ with $g(q) = 1$ and $g|R^* \setminus U = -1$. Let $f = g \cup 0 \in M(R)$. Note that $\operatorname{supp} f \subset \overline{U} \subset R$ and $f(q) > 0$. Thus $f \in M_0(R)$.

IIIn. Let $M_\Delta(R)$ be the BD-closure of $M_0(R)$ in $M(R)$, then

$$\Delta = \{q \in R^* : f(q) = 0 \text{ for all } f \in M_\Delta(R)\}$$

is called the <u>harmonic boundary</u> of R.

The following is obvious.

<u>LEMMA</u>. Δ is a compact subset of Γ.

IIIo. Now we consider the <u>Royden compactification of subregions</u>.

Given any region G in R, G itself is a Riemannian manifold. Hence we can consider the Royden compactification G^* of G. It is not surprising that there is a canonical relation between the Royden boundaries of G and R.

The next two theorems are Propositions 7 and 8 of [16].

IIIp. <u>THEOREM</u>. There exists a unique continuous mapping j from G^* onto $\overline{G}$ fixing G elementwise, where G^* is the Royden compactification of G and $\overline{G}$ the closure of G in R^*.

IIIq. Clearly $\overline{G} \setminus G = \overline{bG} \cup ((\overline{G} \setminus \overline{bG}) \cap \Gamma)$. If we let

$B(G) = (\overline{G} \setminus \overline{bG}) \cap \Gamma$, then $G^* \setminus G = j^{-1}(\overline{bG}) \cup j^{-1}(B(G))$. Moreover

<u>THEOREM</u>. j is a homeomorphism of $G \cup j^{-1}(B(G))$ onto $G \cup B(G)$.

Later on (cf. VIIe) we shall talk about representing measures on the Royden boundaries, and shall show that a Borel subset E of $B(G)$ has positive measure with respect to R^* if and only if $j^{-1}(E)$ has positive measure with respect to G^*.

IV. HARMONIC BOUNDARY

We now change to the following more traditional notations. For any open subset G of R, we consider the solutions of $\Delta u = Pu$, and let

$$PX(G) = \{ u \in \mathcal{K}(G) : u \text{ satisfies the property } X \text{ on } G \}$$

where X can be a boundedness property like N (nonnegative), B (bounded), D (finite Dirichlet integral), E (finite energy integral) or combinations of these.

Also for harmonic functions, we let

$$HX(G) = \{ u \in \mathcal{K}^{\circ}(G) : u \text{ satisfies the property } X \text{ on } G \}$$

where X can be N, B, D or combinations of these.

We say that u is a PX-function on G if $u \in PX(G)$, or u is an HX-function on G if $u \in HX(G)$.

As we shall see, Δ will play a significant role in determining the energy-finite solutions. Indeed, in this chapter, we shall state that energy-finite solutions, in particular Dirichlet-finite harmonic functions satisfy the maximum principle on Δ. Moreover, an energy-finite Tonelli function f can be uniquely decomposed into a sum as $f = u + g$ where $u \in PE(R)$, $u|\Delta = f|\Delta$, $E(u) \leq E(f)$ and $|u| \leq \max_{\Delta} |f|$.

Glasner and Katz [6] have proved the following result for the harmonic boundary Δ:

$\Delta = \emptyset$ if and only if R is parabolic (i.e. there is no harmonic Green's function on R).

Since we want to study the Riemannian manifolds with nontrivial energy-finite solutions which are always nonparabolic [7], we shall always have nonempty harmonic boundaries.

At the end of this chapter, we shall introduce a subset Δ^P of Δ which will eventually determine all PE-functions.

IVa. LEMMA. Let $\widetilde{M}(R) = \{f : f \text{ is Tonelli and } D(f) < \infty\}$. Then $\widetilde{M}(R)$ is a lattice with respect to the usual lattice operations $\cup$ and $\cap$. Moreover, functions in $\widetilde{M}(R)$ have continuous extended real-valued extension to R^*.

The proof is simple, for if f is a nonnegative function in $\widetilde{M}(R)$, then $h = f/(1 + f)$ is a nonnegative function in $M(R)$ which has a continuous extension $\bar{h}$ to R^* by IIIg. It is easy to see that $\bar{f} = \bar{h}/(1 - \bar{h})$ $(\bar{f}(q) = \infty$ if $\bar{h}(q) = 1)$ is the continuous extension of f on R^*. In general if f is not nonnegative, then f is a difference of two nonnegative ones in $\widetilde{M}(R)$, i.e. $f = (f \cup 0) - ((-f) \cup 0)$.

IVb. Clearly, HD(R) and PD(R) are subsets of $\widetilde{M}(R)$. Furthermore we have the following theorems.

THEOREM (Maximum and minimum principles). Every HD-function on R takes its maximum and minimum on Δ.

Cf. [14], p. 192.

IVc. THEOREM (Maximum principle). 1). Every nonnegative

PD-function on R takes its maximum on Δ.

2). A PD-function on R is nonnegative if it is nonnegative on Δ.

Cf. [7], Theorem 2.

IVd. Let $\widetilde{E}(R) = \{f : f$ is Tonelli on R and $E(f) < \infty\}$. Since $\widetilde{E}(R) \subset \widetilde{M}(R)$, functions in $\widetilde{E}(R)$ have extended real-valued continuous extension to R*.

<u>THEOREM</u> (<u>Royden-Nakai decomposition theorem</u>). Every function $f \in \widetilde{E}(R)$ has the unique decomposition $f = u + g$ with

1) $u \in PE(R)$, $g \in \widetilde{E}(R)$ and $f - u|\Delta = 0$,

2) $|u| \leq \sup_{\Delta}|f|$,

3) if v is a supersolution on R and $v \geq f$, then $v \geq u$,

4) $E(u) \leq E(f)$.

For the proof, see Theorem 3 of [7] and cf. p. 190 of [14].

IVe. <u>DEFINITION</u>. A compact set $K \subset R^*$ is a <u>distinguished compact set</u> if $b(K \cap R)$ is piecewise smooth and $\overline{K \cap R} = K$.

A useful generalization of the previous theorem is

<u>THEOREM</u>. Let K be a distinguished compact set in R*. Every function $f \in \widetilde{E}(R)$ has the unique decomposition $f = u + g$ with

1) $u \in PE(R \setminus K) \cap \widetilde{E}(R)$, $g \in \widetilde{E}(R)$ and $f - u|\Delta \cup K = 0$,

2) $|u| \leq \sup_{(\Delta \setminus K) \cup b(K \cap R)}|f|$,

3) if v is a supersolution on $R \setminus K$ and $v \geq f$ on $R \setminus K$, then $v \geq u$ on $R \setminus K$,

4) $E(u) \leq E(f)$.

Cf. [7], p. 350.

IVf. Let us denote by π_K the correspondence which associates to every $f \in \widetilde{E}(R)$ the unique $u \in PE(R \setminus K)$ in the decomposition theorem IVe, i.e. we have

$$\pi_K f = u.$$

We denote

$$\pi = \pi_\emptyset$$

where $\emptyset$ is the empty set. π_K is called the <u>solution projection</u> on R with respect to K (or simply the solution projection on R if $K = \emptyset$).

When $P \equiv 0$, the corresponding projection is denoted by π_K^o, and is called the <u>harmonic projection</u> on R with respect to K (or simply the harmonic projection on R if $K = \emptyset$ and denoted by π^o).

It is worthwhile to note that projections are linear mappings.

IVg. <u>LEMMA</u>. If F is a closed subset of $\Gamma \setminus \Delta$, then there is a positive superharmonic function v which is in M(R) such that $v = 0$ on Δ and $v = \infty$ on F.

Cf. [3], Hilfsatz 9.1, p. 101.

IVh. <u>MINIMUM PRINCIPLE</u>. Let G be an arbitrary region in R and u a supersolution on G bounded from below. If for some nonpositive number c, $\liminf_G u(q) \geq c$ for all $q \in bG \cup (\Delta \cap \overline{G})$, then $u \geq c$.

Proof. We extend u to u′ as a lower semi-continuous function on $\overline{G}$ by defining

$$u'(p) = \liminf_G u(p)$$

for all $p \in \overline{G} \setminus G$. Note $u'|G = u$. For any $b < c$, let

$$U = \{ p \in \overline{G} \setminus G : u'(p) > b \} .$$

U is open in $\overline{G} \setminus G$ since u′ is lower semi-continuous, and $bG \cup (\Delta \cap \overline{G}) \subset U$. Note $F = (\overline{G} \setminus G) \setminus U$ is a compact subset in R^*, in particular, F is a closed subset of $\Gamma \setminus \Delta$. By the lemma, there is a nonnegative superharmonic function $v \in M(R)$ such that $v|F = \infty$. Since v is also a supersolution by IIp, the function $w = u' + v/n$ is a supersolution on G by Lemma III, where n is a positive integer. Observe that $w > b$ on $\overline{G} \setminus G$. Being a lower semi-continuous function w takes its minimum a on $\overline{G}$. By minimum principle IIn, w can only take its minimum on the boundary $\overline{G} \setminus G$. Hence $w \geq a \geq b$, and indeed $w > b$ on G. Consequently $w \geq c$ on G. As n tends to ∞, we see that $u \geq c$ on G.

IVi. We have seen that the harmonic boundary Δ plays a significant role determining the energy-finite solutions, and hence Dirichlet-finite harmonic functions. However, in [7] it is shown that for energy-finite solutions, Δ is too big. More precisely, an open subset Δ^P of Δ to be given below has a strong influence on solutions.

<u>DEFINITION</u>. The n-form P is said to <u>have finite integral at</u>

$p \in \Delta$ if there is a neighborhood $U \subset R^*$ of p such that $\int_{U \cap R} P < \infty$. The set of all such points $p \in \Delta$ will be denoted by Δ^P.

An immediate observation is

LEMMA. Δ^P is open in Δ.

IVj. LEMMA. If $f \in \tilde{E}(R)$, then $f|\Delta \setminus \Delta^P = 0$.

Proof. If $f(p) > 0$ for some $p \in \Delta \setminus \Delta^P$, then there is a neighborhood U of p such that $f|U > \delta > 0$ for some δ. We have $\infty > E(f) \geq \int_{U \cap R} f^2 P \geq \delta^2 \int_{U \cap R} P$, hence $\int_{U \cap R} P < \infty$, i.e. $p \in \Delta^P$. This is, however, impossible.

IVk. As a corollary of IVc and IVj, we have

THEOREM (Maximum principle). 1). Every nonnegative PE-function on R takes its maximum on Δ^P.

2). A PE-function on R is nonnegative if it is nonnegative on Δ^P.

IVl. It is worthwhile to note that

$\Delta^P = \Delta$ if $\int_R P < \infty$. In particular, $\Delta^P = \Delta$ when $P \equiv 0$.

V. REPRESENTING MEASURES

A representing measure on the Royden boundary for the class of HD-functions was constructed by Nakai [14]. In this chapter we generalize his results to the equation $\Delta u = Pu$. A relation between these two kinds of representing measures is given in Ve.

Va. Let $E(R)$ be the set of all bounded, energy-finite Tonelli functions.

THEOREM. $E(R)$ is a subalgebra as well as a sublattice of $M(R)$. $E(R)$ has identity if and only if $\int_R P < \infty$.

Proof. For any $f,g \in E(R)$, it is obvious that $af \in E(R)$ for any real number a. Also by Schwarz's inequality

$$\begin{aligned} E(f+g) &= E(f) + 2E(f,g) + E(g) \\ &\le E(f) + 2\sqrt{E(f)E(g)} + E(g) \\ &< \infty . \end{aligned}$$

Moreover,

$$\begin{aligned} E(fg) &= D(fg) + \int f^2 g^2 P \\ &\le D(fg) + \sup_R g^2 \int f^2 P \\ &< \infty . \end{aligned}$$

So $E(R)$ is a subalgebra. On the other hand, we let $G = \{x \in R : f(x) > g(x)\}$, then

$$E(f \cup g) = E_G(f \cup g) + E_{R \setminus G}(f \cup g)$$

$$\leq E(f) + E(g)$$

$$< \infty .$$

Similarly $E(f \cap g) < \infty$. Hence $E(R)$ is also a sublattice.

Vb. THEOREM. $E(R)$ is BE-complete.

Proof. Let $\{u_n\}$ be a BE-Cauchy sequence in $E(R)$. Clearly $\{u_n\}$ is also a BD-cauchy sequence in $M(R)$ which is BD-complete. Hence a BD-limit $u \in M(R)$ exists. Consider a measure on R defined by $\lambda(G) = \int_G P$ for any Borel set G and the complete space $L^2(R,\lambda)$ of all square integrable functions with respect to λ on R. Since $\{u_n\}$ is E-Cauchy, it means $\lim E(u_{n+m} - u_n) = \lim(D(u_{n+m} - u_n) + \int(u_{n+m} - u_n)^2 P) = 0$ for all m. In particular, $\lim \int(u_{n+m} - u_n)^2 P = 0$ for all m. Hence $\{u_n\}$ is a Cauchy sequence in $L^2(R,\lambda)$. Since $\lim u_n = u$, we have $\lim \int (u - u_n)^2 P = 0$, i.e. $u = \text{BE-}\lim u_n$. Finally, $E(u) \leq E(u_n) + E(u - u_n) < \infty$ by the triangle inequality. Thus $u \in E(R)$.

Vc. Since $E(R)$ is a subset of $M(R)$ whose elements are continuous on R^*, the restrictions of functions in $E(R)$ on Δ^P are continuous on Δ^P . What we have more is

THEOREM. $E(R) \big| \Delta^P = \left\{ f \big| \Delta^P : f \in E(R) \right\}$ is dense in $C_o(\Delta^P)$ with respect to the uniform norm.

Proof. By Lemma IVj $f = 0$ on $\Delta \setminus \Delta^P$ for all $f \in E(R)$. For any positive ε, the set $K = \{p \in \Delta : f(p) \geq \varepsilon\}$ is a compact subset of Δ and also a compact subset of Δ^P. Obviously $f|\Delta^P \setminus K < \varepsilon$. Hence $E(R)|\Delta^P \subset C_0(\Delta^P)$. We are to show that $E(R)$ separates points of Δ^P. For any two distinct points $q_1, q_2 \in \Delta^P$, there is a neighborhood U in R^* of q_1 which excludes q_2 such that $\int_{U \cap R} P < \infty$. By the Urysohn property (IIIk), there is a function $f \in M(R)$ with $0 \leq f \leq 1$, $f(q_1) = 1$ and $\operatorname{supp} f \subset U$. We have $E(f) = D(f) + \int f^2 P \leq D(f) + \int_{U \cap R} P < \infty$. Hence $f \in E(R)$ and $f(q_1) \neq f(q_2)$. As a by-product, we see that $E(R)$ vanishes identically at no point of Δ^P. The Stone-Weierstrass theorem assures the denseness of $E(R)|\Delta^P$ in $C_0(\Delta^P)$.

Vd. We consider the projection π introduced in IVf. For a fixed $z_0 \in R$, we define a linear functional s on $E(R)|\Delta^P$ by

$$sf = \pi f(z_0)$$

for all f in $E(R)$. By the decomposition theorem IVd and Theorem IVk,

$$|sf| \leq \sup_{\Delta^P} |f|.$$

Hence s is a positive bounded linear functional on $E(R)|\Delta^P$ with norm $\|s\| = \sup\{|sf| : f \in E(R), \sup_{\Delta^P}|f| = 1\} \leq 1$. Since $E(R)|\Delta^P$ is dense in $C_0(\Delta^P)$ which is complete with respect to the sup norm, we can extend s to $C_0(\Delta^P)$ by an obvious limit process. Thus s becomes a positive bounded linear functional on $C_0(\Delta^P)$ with $\|s\| \leq 1$. By the Riesz representation theorem, there is a unique bounded positive regular Borel measure m on Δ^P such that

$$sf = \int_{\Delta^P} f dm.$$

We can regard m as a measure on Γ by defining

$$m(G) = m(G \cap \Delta^P)$$

for all $G \subset \Gamma$.

When $P \equiv 0$, we denote the corresponding measure by m^o.

The measures m and m^o will be called the <u>representing measure with center</u> z_o <u>for solutions and harmonic functions</u> respectively.

Ve. <u>THEOREM</u>. 1) m is a positive bounded regular Borel measure,

2) $v(z_o) \geq \int_\Gamma v dm$ for all supersolutions $v \in \tilde{E}(R)$,

3) $\text{supp}\ m = \Delta^P$ (where support of m is considered relative to the topology of Δ^P and by definition $m(\Gamma \setminus \Delta^P) = 0$),

4) m, satisfying 1), 2) and 3), is unique,

5) $m(\Delta^P) \leq 1$; $m(\Delta^P) = 1$ if and only if $P \equiv 0$.

Cf. [14], Theorem 2.1.

Proof. 1) is true by Riesz representation theorem. For any supersolution v in E(R), $v \geq \pi v$ by IVd. Hence

$$v(z_o) \geq \pi v(z_o) = sv = \int vdm.$$

Thus 2) is proved.

Now suppose $S = \text{supp}\ m$ is a proper subset of Δ^P. Since S is closed in Δ^P, the set $\Delta^P \setminus S$ is open in Δ^P. Thus for an arbitrary $q \in \Delta^P \setminus S$, there is an open neighborhood U in R* of q such that $U \cap \Delta^P$ is disjoint from S and $\int_{U \cap R} P < \infty$. By the Urysohn property, there is a function f in M(R) such that $0 \leq f \leq 1$, $\text{supp}\ f \subset U$ and $f(q) = 1$.

Note that $E(f) = D(f) + \int f^2 P \leq D(f) + \int_{U \cap R} P < \infty$. Thus $f \in E(R)$, so we can apply π to f. Note also that $\pi f > 0$ on R, in particular $sf = \pi f(z_o) > 0$. Hence $\int_{U \cap \Delta^P} dm \geq \int_{\Delta^P} f dm = sf > 0$. But $U \cap \Delta^P \subset \Delta^P \setminus S$. Thus the contradiction establishes 3).

If $\underline{m}$ is a measure with support in Δ^P satisfying 1) and 2), then for all f in E(R),

$$\int_{\Delta^P} f d\underline{m} = \pi f(z_o) = \int_{\Delta^P} f dm$$

by 2) and the decomposition theorem IVd. Since $E(R)|\Delta^P$ is dense in $C_o(\Delta^P)$, such equalities are also true for functions in $C_o(\Delta^P)$. That $\underline{m} = m$ is then a consequence of the uniqueness part of the Riesz representation theorem. Hence 4) is true.

$m(\Delta^P) = \|s\| \leq 1$ by Vd and the Riesz representation theorem. If $P \equiv 0$, then $\Delta^P = \Delta$ and $m = m^o$. We have

$$m(\Delta^P) = m^o(\Delta) = \int_{\Delta} dm^o = \pi^o 1(z_o) = 1,$$

since in this case solutions are harmonic functions. Conversely if $m(\Delta^P) = 1$, then $\|s\| = 1$. For any integer i, however large, there is a nonnegative function f^i in $C_o(\Delta^P)$ such that $\sup f^i = 1$ and $|1 - sf^i| < 1/i$. Since $E(R)|\Delta^P$ is dense in $C_o(\Delta^P)$, to each i there is a sequence $\{f_n^i\} \subset E(R)$ of nonnegative functions such that

$$\lim_{n \to \infty} (\sup_{\Delta^P} |f^i - f_n^i|) = 0,$$

$$\sup_{\Delta^P} f_n^i = 1$$

(replace by $f_n^i/(\sup_{\Delta^P} f_n^i)$ otherwise) and

$$\sup_{\Delta^P}\left|f^i - f_i^i\right| < 1/i.$$

Note that $sf^i = \lim_{n\to\infty} sf_n^i$. $\{\pi f_i^i\}$ is a sequence of bounded solutions.

By IIt, it contains a subsequence, again denoted by $\{\pi f_i^i\}$, converging uniformly on compact subsets to a solution u. Observe that

$$u(z_o) = \lim \pi f_i^i(z_o) = \lim sf_i^i$$

and

$$\begin{aligned}\left|1 - sf_i^i\right| &\le \left|1 - sf^i\right| + \left|sf^i - sf_i^i\right| \\ &< 1/i + \|s\| \sup_{\Delta^P}\left|f^i - f_i^i\right| \\ &< 1/i + 1/i \\ &= 2/i.\end{aligned}$$

Hence $u(z_o) = 1$. Since $\{\pi f_i^i\}$ is a sequence uniformly bounded by 1 (see IVd), u is also bounded by 1. Thus u takes its maximum at $z_o \in R$. Hence $u \equiv 1$ by maximum principle IIn. This is possible only if $P \equiv 0$.

Vf. The representing measure m is obtained with respect to a fixed point $z_o \in R$. Similarly, we can construct a representing measure with respect to any other point $z \in R$ and denote it by m_z (m_z^o if $P \equiv 0$). It is useful to know the following theorem.

<u>THEOREM</u>. If $f \in C_o(\Delta^P)$, then $u(z) = \int_\Gamma f dm_z$ is a solution.

Proof. If f is a PE-function, then $f = \pi f$ and hence $f(z) = \pi f(z) = \int f dm_z$ by Ve,2).

Now if $f \in C_o(\Delta^P)$, then there is a sequence $\{f_n\} \subset B(R)$ such that $f = U\text{-lim } f_n$ on Δ^P. Note that πf_n are PE-functions and

$$\sup_{R^*} |\pi f_m - \pi f_n| \leq \sup_{\Delta^P} |\pi f_m - \pi f_n|$$

$$= \sup_{\Delta^P} |f_m - f_n| \quad ,$$

which tends to 0 as m, n tend to ∞. Thus $\{\pi f_n\}$, being a uniformly convergent sequence of continuous functions on R*, has a limit $\underline{u}$ which is also continuous on R*. By IIr, $\underline{u}$ is a solution and

$$\underline{u}(z) = \lim \pi f_n(z)$$

$$= \lim \int_{\Delta^P} f_n dm_z$$

$$= \int_{\Delta^P} f dm_z$$

$$= u(z) \ ,$$

i.e. u is a solution.

Vg. LEMMA. m_z is absolutely continuous with respect to m.

Proof. For all nonnegative function $f \in C_o(\Delta^P)$, $u(z) = \int f dm_z$ is a nonnegative solution by Vf. By IIe,

$$\int f dm_z \leq k(z, z_o) \int f dm.$$

Since $C_o(\Delta^P)$ is dense in both $L^1(\Delta^P, m_z)$ and $L^1(\Delta^P, m)$, we can see that m_z is absolutely continuous with respect to m.

Vh. LEMMA. For any $z \in R$, there is a Radon-Nikodym derivative $\underline{K}(z,q)$ of m_z with respect to m with the following properties.

1) $\underline{K}(z_o,q) = 1$ for all $q \in \Delta^P$,

2) $\underline{K}(z,q) = 0$ on $R \times (\Gamma \setminus \Delta^P)$,

3) $\underline{K}(z',q)/k(z,z') \leq \underline{K}(z,q) \leq k(z,z')\underline{K}(z',q)$ on $R \times \Delta^P$ except on a set $E(z,z') \subset \Delta^P$ of m-measure zero.

Proof. 1) is satisfied because $m_{z_o} = m$ by definition. 2) is obtained since m is concentrated on Δ^P . It remains to prove 3).

For any nonnegative $f \in C_o(\Delta^P)$ we have by Vf and IIe,

$$(1/k(z,z'))\int f dm_{z'} \leq \int f dm_z \leq k(z,z') \int f dm_{z'}$$

or

$$(1/k(z,z'))\int \underline{K}(z',q)f(q)dm(q)$$

$$\leq \int \underline{K}(z,q)f(q)dm(q)$$

$$\leq k(z,z') \int \underline{K}(z',q)f(q)dm(q).$$

Hence

$$\underline{K}(z',q)/k(z,z') \leq \underline{K}(z,q) \leq k(z,z')\underline{K}(z',q)$$

except on a set $E(z,z') \subset \Delta^P$ of m-measure zero.

COROLLARY. $1/k(z,z_o) \leq \underline{K}(z,q) \leq k(z,z_o)$ on $R \times \Delta^P$ except for q in a set $E(z,z_o)$ of m-measure 0.

Proof. Let $z' = z_o$ in 3) and apply 1).

Vi. THEOREM. There is a real valued function (kernel) $K(z,q)$ defined on $R \times \Gamma$ such that it is continuous on R, and

1) $K(z_o,q) = 1$ on Δ^P,

2) $K(z,q) = 0$ on $R \times (\Gamma \setminus \Delta^P)$,

3) $1/k(z,z_o) \leq K(z,q) \leq k(z,z_o)$ on $R \times \Delta^P$,

4) for any fixed $z \in R$, $K(z,q)$ is a nonnegative Borel function on Γ.

5) for any supersolution $v \in \widetilde{E}(R)$,

$$v(z) \geq \int_\Gamma K(z,q)v(q)dm(q),$$

6) $K(z,q) \in PN(R)$ except for q in a set of m-measure 0.

REMARK. When $P \equiv 0$, the corresponding kernel will be denoted by $K^o(z,q)$ and it was introduced in [14], p. 196.

Proof. Let T be a countable dense subset of R containing z_o. Set $E = \bigcup \{E(z,z') : z,z' \in T\}$ where $E(z,z')$ is as in Vh. Note that $m(E) = 0$. For any $q \in \Delta^P \setminus E$ and $z,z' \in T$, we have

$$\underline{K}(z',q)/k(z,z') \leq \underline{K}(z,q) \leq k(z,z')\underline{K}(z',q),$$

or

$$(1/k(z,z') - 1)\underline{K}(z',q) \leq \underline{K}(z,q) - \underline{K}(z',q) \leq (k(z,z') - 1)\underline{K}(z',q);$$

thus

$$|\underline{K}(z,q) - \underline{K}(z',q)| \leq \underline{K}(z',q) \max(k(z,z') - 1 \ , \ 1 - 1/k(z,z'))$$
$$\leq k(z',z_o) \max(k(z,z') - 1, \ 1 - 1/k(z,z')).$$

We obtain, for $q \in \Delta^P \setminus E$

$$(7) \qquad \lim_{z,z' \in T; z,z' \to z''} |\underline{K}(z,q) - \underline{K}(z',q)| = 0$$

for all $z \in R$ by IIe. Note that (7) is also true for all $q \in \Gamma \setminus \Delta^P$ by Vh, 2). We then define

$$K(z \ ,q) = \begin{cases} \lim_T \underline{K}(z,q) & \text{if } q \in \Gamma \setminus E \\ 1 & \text{if } q \in E. \end{cases}$$

For fixed $q \in \Gamma$, $K(z,q)$ is a continuous function on R. To see this let $q \notin E$ and $z \in R$ be any points. Pick any sequence $\{z_n\}$ converging to z. For each n, let $x_n \in T$ such that the Riemannian distance between x_n and z_n is less than $1/n$, and

$$|\underline{K}(x_n,q) - K(z_n,q)| < 1/n.$$

By the triangle inequality, $\{x_n\}$ is a sequence converging to z. Thus

$$K(z,q) = \lim \underline{K}(x_n,q)$$

by definition. Then

$$|K(z,q) - K(z_n,q)| \leq |K(z,q) - \underline{K}(x_n,q)| + |\underline{K}(x_n,q) - K(z_n,q)|,$$

where both terms on the right tend to 0. We have

$$K(z,q) = \lim K(z_n,q).$$

Thus $K(z,q)$ is continuous on R when $q \notin E$. But obviously, $K(z,q)$ is continuous when $q \in E$.

1) and 2) are satisfied since $K(z,q)$ agrees with $\underline{K}(z,q)$ when $z \in T$ or when $q \notin \Delta^P$.

$K(z,q)$ satisfies 3) because of Corollary Vh and the continuity of $k(z,z_o)$.

4) is true by the construction of $K(z,q)$.

To prove 5), let v be any supersolution in $\tilde{E}(R)$, $z \in R$ and $z_n \in T$ such that $z = \lim z_n$. Note that

$$K(z,q) = \lim \underline{K}(z_n,q)$$

m a. e. and by Ve,2)

$$v(z_n) \geq \int \underline{K}(z_n,q)v(q)dm(q).$$

By the continuity of v and Lebesgue's dominated convergence theorem

$$v(z) \geq \int K(z,q)v(q)dm(q).$$

Thus 5) is proved.

Now it remains to prove 6). For any nonnegative $f \in C_o(\Delta^P)$ $w(z) = \int f(q)dm_z(q)$ is a solution by Vf, and hence is continuous. Thus

$$w(z) = \lim_T w(z)$$

$$= \lim_T \int f(q)\underline{K}(z,q)dm(q) \; .$$

Recall $K(z,q) = \lim_T \underline{K}(z,q)$ and apply the Lebesgue dominated convergence theorem again, we have

$$w(z) = \int f(q)K(z,q)dm(q).$$

Now cover R by a family $\{U_i : i = 1,2,\ldots\}$ of parametric balls. For any $z \in R$, $z \in U_i$ for some i. Recall the harmonic measure $r(z,U_i)$ with respect to the harmonic class $\mathcal{K}$ for U_i at z as defined in IIh, and hence obtain

$$\int_{\Gamma} f(q)K(z,q)dm(q)$$

$$= \int_{bU_i} \int_{\Gamma} f(q)K(x,q)dm(q)dr(z,U_i)(x)$$

$$= \int_{\Gamma} f(q) \int_{bU_i} K(x,q)dr(z,U_i)(x)dm(q)$$

by Fubini's theorem. Thus for any $z \in U_i$

$$K(z,q) = \int_{bU_i} K(x,q)dr(z,U_i)(x)$$

except on a set $A_i(z) \subset \Delta^p$ with $m(A_i(z)) = 0$. Let

$$A_i = \bigcup \{ A_i(z) : z \in T \cap U_i \}$$

and

$$A = E \cup (\bigcup \{ A_i : i = 1,2,\ldots \}).$$

Note $m(A) = 0$. For all $q \in \Gamma \setminus A$ and $z' \in T \cap U_i$,

$$\underline{K}(z',q) = K(z',q)$$

$$= \int_{bU_i} K(x,q)dr(z',U_i)(x),$$

which is the evaluation at z' of the unique solution on U_i with the continuous boundary value $K(x,q)$ (see IIh and IIk). As z' tends to z'' for an arbitrary $z'' \in U_i$, we have $K(z'',q) = \lim_{z' \to z''} K(z',q)$ and

$$K(z'',q) = \int_{bU_i} K(x,q)dr(z'',U_i)(x)$$

for all $q \in \Gamma \setminus A$. Thus $K(z,q) \in PN(U_i)$ if $q \in \Gamma \setminus A$. Since U_i is arbitrary, $K(z,q) \in PN(R)$ m a. e.

VI. THE CHARACTERIZATIONS OF PE- AND $\widetilde{PE}$-FUNCTIONS

A PE-function, as we defined before, is a solution of $\Delta u = Pu$ with finite energy integral. A nonnegative solution is called a $\widetilde{PE}$-function if it can be obtained as the infimum of the family of all PE-functions above it. Both PE- and $\widetilde{PE}$-functions can be represented by the representing measure m and the associated kernel K(z,q) as

$$u(z) = \int_{\Gamma} u(q)K(z,q)dm(q)$$

if $u \in PE(R)$, and

$$u(z) = \int_{\Gamma} (\limsup_R u(q))K(z,q)dm(q)$$

if $u \in \widetilde{PE}(R)$. Furthermore, for each $u \in \widetilde{PE}(R)$, there is an upper semi-continuous f on Γ which is the infimum of the family of all PE-functions above it on Δ such that

$$\limsup_R u(q) = f(q) \qquad m \text{ a. e.}$$

Most of the results mentioned above are generalizations of the work of Nakai [14] for HD- and $\widetilde{HD}$-functions where an $\widetilde{HD}$-function is a nonnegative harmonic function obtained as the infimum of the family of all HD-functions above it.

VIa. For any functions u_1 and u_2 on R we define the following

$$u_1 \wedge u_2 = \text{g.s.m.}(u_1, u_2)$$
$$= \text{the greatest solution minorant of } u_1 \text{ and } u_2,$$

and

$$u_1 \vee u_2 = \text{l.s.m.}(u_1, u_2)$$
$$= \text{the least solution majorant of } u_1 \text{ and } u_2.$$

<u>THEOREM</u>. PE(R) is a vector lattice under $\wedge$, $\vee$. In fact, $v_1 \wedge v_2|\Delta = v_1 \cap v_2|\Delta$ and $v_1 \vee v_2|\Delta = v_1 \cup v_2|\Delta$ for any $v_1, v_2 \in PE(R)$.

Cf. [13] , Theorem 3.11.

Proof. Obviously, PE(R) is a vector space. Note that $PE(R) \subset \widetilde{E}(R)$ which is a lattice under $\cap$ and $\cup$. For any u_1 and u_2 in PE(R), $u_1 \cap u_2$ is a supersolution by Lemma III and is in $\widetilde{E}(R)$ by the previous observation. By the Royden-Nakai decomposition theorem IVd $\pi(u_1 \cap u_2) \in PE(R)$, $\pi(u_1 \cap u_2) \leq u_1 \cap u_2$ and $\pi(u_1 \cap u_2)|\Delta = (u_1 \cap u_2)|\Delta$. If v is any solution which is dominated by both u_1 and u_2, then for all $q \in \Delta$

$$\begin{aligned} &\liminf_R (\pi(u_1 \cap u_2) - v)(q) \\ &= \lim_R \pi(u_1 \cap u_2)(q) - \limsup_R v(q) \\ &= \lim_R (u_1 \cap u_2)(q) - \limsup_R v(q) \\ &= \liminf_R ((u_1 \cap u_2) - v)(q) \\ &\geq 0 \end{aligned}$$

by the continuities of functions in $\widetilde{E}(R)$ on R^*. Theorem IVh implies that

$$\pi(u_1 \cap u_2) \geq v$$

on R. Hence

$$u_1 \wedge u_2 = \pi(u_1 \cap u_2)$$

and is in PE(R). It can be proved similarly that $u_1 \vee u_2 = \pi(u_1 \cup u_2)$ is in PE(R) by observing that $-(u_1 \cup u_2)$ is a supersolution.

COROLLARY. Every PE-function u is a difference of two nonnegative ones. In fact, $u = (u \vee 0) - ((-u) \vee 0)$.

VIb. THEOREM. If $u \in PE(R)$, then

$$u(z) = \int_\Gamma u(q)K(z,q)dm(q).$$

Proof. Note that both u and -u are supersolutions in $\widetilde{E}(R)$, and apply Vi,5).

VIc. THEOREM. If $u \in PE(R)$, then $u|\Gamma \in L^1(\Gamma,m)$.

Proof. By Corollary VIa, it suffices to prove the case where u is nonnegative. Indeed

$$u(z) = \int_\Gamma u(q)K(z,q)dm(q).$$

Thus

$$\int_\Gamma u(q)dm(q) = u(z_o) < \infty \quad ,$$

i.e. $u|\Gamma \in L^1(\Gamma,m)$.

VId. <u>THEOREM</u>. If $f \in L^1(\Gamma, m)$, then

$$u(z) = \int_\Gamma f(q)K(z,q)dm(q)$$

is a solution.

Cf. [14], Theorem 2.2.

Proof. Since $E(R)|\Delta^P$ is dense in $C_o(\Delta^P)$ in sup norm which is in turn dense in $L^1(\Gamma, m)$ in L^1-norm, $E(R)|\Delta^P$ is dense in $L^1(\Gamma, m)$ in L^1-norm. Let us pick a sequence $\{f_n\} \subset E(R)$ such that

$$\lim \int_\Gamma |f_n - f| dm = 0.$$

Let $u_n = \pi f_n$ which is in PBE(R) and $u_n|\Delta^P = f_n|\Delta^P$ by IVd, thus

$$u_n(z) = \int_\Gamma f_n(q)K(z,q)dm(q)$$

by VIc. Now

$$|u(z) - u_n(z)| \leq \int_\Gamma |f(q) - f_n(q)| K(z,q)dm(q)$$

$$\leq k(z,z_o) \int_\Gamma |f(q) - f_n(q)| dm(q)$$

by Vi,3). Hence $\{u_n\}$ tends to u uniformly on each compact subset of R by the continuity of $k(z,z_o)$. Thus u is a solution by IIr.

VIe. By Axiom III_1, the lower envelope of a downward directed family of nonnegative PE-functions is again a nonnegative solution. We denote by $\widetilde{PE}(R)$ the collection of all nonnegative solutions obtained in such fashion. Solutions in $\widetilde{PE}(R)$ are called $\widetilde{PE}$-functions.

$\widetilde{HD}$-functions are defined similarly.

VIf. <u>THEOREM</u>. A nonnegative solution is in the family $\widetilde{PE}(R)$ if and only if it is the C-limit of a decreasing sequence in PE(R).

Proof. The sufficiency is clear. To prove the necessity, let $u \in \widetilde{PE}(R)$ such that

$$u(z) = \inf \{ v(z) : v \in \mathcal{G} \},$$

where $\mathcal{G}$ is some downward directed family in PE(R). For any fixed point $x \in R$, there is a sequence $\{u_n\} \subset \mathcal{G}$ such that $u(x) = \lim u_n(x)$. Let

$$v_1 = u_1$$

and

$$v_n = u_n \wedge v_{n-1} \ .$$

By VIa, $\{v_n\}$ is a decreasing sequence in PE(R). Axiom III_2 insures that $v = \lim v_n$ is a solution and Dini's theorem provides that actually $v = C\text{-}\lim v_n$. Observe that $v - u \geq 0$ and $v(x) = u(x)$, hence $v \equiv u$ by Axiom III. Thus u is the C-limit of a decreasing sequence in PE(R).

VIg. <u>THEOREM</u>. A nonnegative function u is in $\widetilde{PE}(R)$ if and only if $u(z) = \inf \{ v(z) : v \in \mathcal{G}(u) \}$ where $\mathcal{G}(u) = \{v \in PE(R) : v \geq u \}$.

Proof. The sufficiency is obvious because $\mathcal{G}(u)$ is a downward directed family of nonnegative solutions in PE(R) by Theorem VIa.

On the other hand, being in $\widetilde{PE}(R)$, $u = \lim v_n$ where $\{v_n\}$ is a decreasing sequence in PE(R). Clearly $\{v_n\} \subset \mathcal{G}(u)$, hence

$$u(z) = \inf \{ v_n(z) : n = 1,2 ,\ldots \}$$
$$\geq \inf \{ v(z) : v \in \mathcal{G}(u) \} .$$

However, we always have

$$u(z) \leq \inf \{ v(z) : v \in \mathcal{G}(u) \} .$$

Thus

$$u(z) = \inf \{ v(z) : v \in \mathcal{G}(u) \} .$$

REMARK. $\mathcal{G}(u)|\Delta$ is closed under $\cap$.

VIh. For any nonnegative function f on the harmonic boundary Δ, we define

$$\mathcal{F}(f) = \{ v \in PE(R) : v \geq f \text{ on } \Delta \} ,$$

and

$$\mathcal{U}(\Delta) = \{ f : f \text{ is a nonnegative function on } \Delta \text{ and } f(q) = \inf \{ v(q) : v \in \mathcal{F}(f) \} \} .$$

VIi. LEMMA. $\mathcal{F}(f)$ is a downward directed family.

Proof. For any $v_1, v_2 \in \mathcal{F}(f)$, $v_1 \wedge v_2 \leq v_i$, $i = 1,2$. And $(v_1 \wedge v_2)|\Delta = (v_1 \cap v_2)|\Delta \geq f$ by Theorem VIa. Thus $v_1 \wedge v_2 \in \mathcal{F}(f)$.

REMARK. Certainly $u(z) = \inf \{ v(z) : v \in \mathcal{F}(f)\}$ is a $\widetilde{PE}$-function. However, there is no guarantee that $\widetilde{PE}$-functions are continuous on R* or have finite energy. In particular, we can not hope to prove that any function in $\mathcal{U}(\Delta)$ is the restriction of a function on Δ which is the limit of a decreasing sequence of PE-functions.

VIj. LEMMA. The characteristic function of a compact subset of Δ^P is in $\mathcal{U}(\Delta)$.

Proof. Let $K \subset \Delta^P$ be any compact subset and c_K its characteristic function. For any $q_o \in \Delta \setminus K$, we cover K by finite number of open subsets $U_1, \ldots, U_n$ of R* such that $\int_{U_i \cap R} P < \infty$ and $q_o \notin U_i$ for any $i = 1,2,\ldots,n$. The Urysohn peoperty guarantees the existence of a nonnegative function $f \in M(R)$ such that $f \leq 1$, $f|K = 1$ and $\operatorname{supp} f \subset \bigcup\{U_i : i = 1,2,\ldots,n\}$. Observe that $c_K \leq f$ and $E(f) = D(f) + \int f^2 P \leq D(f) + \Sigma \int_{U_i \cap R} P < \infty$. Hence $f \in E(R)$. Thus $u = \pi f \in PE(R)$ and $u|\Delta = f|\Delta \geq c_K$. Now $c_K(q_o) = 0 = f(q_o) = u(q_o)$. Also for any point $q_1 \in K$, $c_K(q_1) = 1 = f(q_1) = u(q_1)$. Since q_o and q_1 are arbitrary and $c_K(q) \leq \inf\{v(q) : v \in \mathcal{F}(c_K)\} \leq u(q)$ for all $q \in \Delta$, we have $c_K(q) = \inf \{ v(q) : v \in \mathcal{F}(c_K) \}$.

VIk. In order to prove the next theorem, we have to prove a lemma for technical purposes.

<u>LEMMA</u>. X is a locally compact Hausdorff space with a bounded positive regular Borel measure k. If $\mathcal{F}$ is a family of nonnegative extended real-valued continuous functions on X which is closed under "$\cap$", i.e. $f_1 \cap f_2 \in \mathcal{F}$ for all $f_1, f_2 \in \mathcal{F}$, then

$$\int_X \inf\{f : f \in \mathcal{F}\}\,dk = \inf\left\{\int_X f\,dk : f \in \mathcal{F}\right\}.$$

Proof. Let $\underline{f}(q) = \inf\{f(q) : f \in \mathcal{F}\}$ for all $q \in X$. First we shall prove that $\underline{f}$ is upper semi-continuous.

For a fixed $q_o \in X$, there is a decreasing sequence $\{f_n\} \subset \mathcal{F}$ such that $\underline{f}(q_o) = \lim f_n(q_o)$. Let $f' = \lim f_n$ which is upper semi-continuous. Clearly $\underline{f} \leq f'$. For any positive ε, there is an open neighborhood U of q_o such that $f'(q) < f'(q_o) + \varepsilon = \underline{f}(q_o) + \varepsilon$ for all $q \in U$. Thus $\underline{f}(q) < \underline{f}(q_o) + \varepsilon$ for all $q \in U$. Hence $\underline{f}$ is upper semi-continuous.

In particular $\underline{f}$ is integrable. Now denote $r = \inf\{\int f dk : f \in \mathcal{F}\}$. There is a decreasing sequence $\{g_n\} \subset \mathcal{F}$ such that $r = \lim \int g_n dk$. If $g = \lim g_n$, then $\int g dk = r$ by monotone convergence theorem. Clearly $\underline{f} \leq g$, hence $\int \underline{f} dk \leq \int g dk = r$. We claim that actually the equality holds.

Suppose on the contrary $\int \underline{f} dk < r$, then there is a positive δ such that the set $A(\delta) = \{q \in X : \underline{f}(q) < g(q) - \delta\}$ has positive measure. By the regularity of the measure k there is a compact subset $A \subset A(\delta)$ with positive measure $k(A) = a$. The Egoroff theorem states that there is a compact set K such that $\{g_n\}$ converges uniformly to g on K and $k(X \setminus K) < a/2$. Hence $k(K \cap A) = k(A) - k(A \setminus K) \geq k(A) - k(X \setminus K) > a - a/2 = a/2$; in particular $K \cap A$ is not empty.

For any $q \in K \cap A$, there is a function $f_q \in \mathcal{F}$ with $f_q(q) < g(q) - \delta/2$. Since g is continuous on $K \cap A$ and f_q is continuous on X, there is an open neighborhood V(q) of q in $K \cap A$ with $\delta/4 \leq g(q') - f_q(q')$ for all $q' \in V(q)$. $K \cap A$ is compact, so there are finite number of such $V(q_1),\ldots,V(q_n)$ which cover $K \cap A$. If we set $h = f_{q_1} \cap \cdots \cap f_{q_n}$, then $h \in \mathcal{F}$ and $g - h > \delta/4$ on $K \cap A$. Note that $g_n \cap h \in \mathcal{F}$ and $g \cap h = \lim g_n \cap h$. However,

$$r = \lim \int (g_n \cap h)dk = \int (g \cap h)dk = \int_{X \setminus (K \cap A)} (g \cap h)dk + \int_{K \cap A} (g \cap h)dk$$

$$\leq \int_{X \setminus (K \cap A)} gdk + \int_{K \cap A} hdk \leq \int_{X \setminus (K \cap A)} gdk + \int_{K \cap A} (g - \delta/4)dk$$

$$= \int_X gdk - (\delta/4) \int_{K \cap A} dk \leq r - (\delta/4)(\delta/2).$$ Hence the contradiction proves the lemma.

VII. <u>THEOREM</u>. $u \in \widetilde{PE}(R)$ if and only if there is a function $f \in \mathcal{U}(\Delta)$ such that $u(z) = \int K(z,q)f(q)dm(q)$. In this case,

$u(z) = \inf \{ v(z) : v \in \mathcal{F}(f) \}$.

Cf. [14] , Lemma 3.2.

Proof. If u is any $\widetilde{PE}$-function, then

$$u(z) = \inf \{ v(z) : v \in \mathcal{G}(u) \}$$

by VIg. Let

$$f(q) = \inf \{ v(q) : v \in \mathcal{G}(u) \}$$

for all $q \in \Delta$. Clearly $\mathcal{G}(u) \subset \mathcal{F}(f)$, hence $f(q) \geq \inf \{ v(q) : v \in \mathcal{F}(f) \}$. But we always have that $f(q) \leq \inf \{ v(q) : v \in \mathcal{F}(f) \}$.

Thus $v = \inf \{ v(q) : v \in \mathfrak{F}(f) \}$ and $f \in \mathcal{U}(\Delta)$. By VIg

$$u(z) = \inf \{ v(z) : v \in \mathcal{G}(u) \}$$

$$= \inf \left\{ \int v(q)K(z,q)dm(q) : v \in \mathcal{G}(u) \right\}.$$

Now observe that, by Theorem VIa, the family of all functions in $\mathcal{G}(u)$ restricted on Δ is closed under "$\cap$". Thus the lemma VIk implies that

$$u(z) = \int \inf \{ v(q) : v \in \mathcal{G}(u) \} K(z,q)dm(q)$$

$$= \int f(q)K(z,q)dm(q).$$

To prove the sufficiency, let f be any function in $\mathcal{U}(\Delta)$,

i.e.

$$f(q) = \inf \{ v(q) : v \in \mathfrak{F}(f) \} .$$

Note that by Theorem VIa the family of all functions in $\mathfrak{F}(f)$ restricted to Δ is closed under "$\cap$". Again by Lemma VIk,

$$u(z) = \int f(q)K(z,q)dm(q)$$

$$= \int \inf \{ v(q) : v \in \mathfrak{F}(f) \} K(z,q)dm(q)$$

$$= \inf \left\{ \int v(q)K(z,q)dm(q) : v \in \mathfrak{F}(f) \right\}$$

$$= \inf \{ v(z) : v \in \mathfrak{F}(f) \}$$

which is in $\widetilde{PE}(R)$ since $\mathfrak{F}(f)$ is a downward directed family of non-negative solutions in $PE(R)$.

VIm. THEOREM. If $u \in \widetilde{PE}(R)$ such that $u(z) = \int f(q)K(z,q)dm(q)$ for some $f \in \mathcal{U}(\Delta)$, then

$\limsup_R u(q) \leq f(q)$ for all $q \in \Delta$, and moreover $\limsup_R u(q) = f(q)$ m a.e. on Δ.

Proof. For any $v \in \mathfrak{F}(f)$, we have that $v \in PE(R)$ and

$$u(z) = \int f(q)K(z,q)dm(q)$$

$$\leq \int v(q)K(z,q)dm(q)$$

$$= v(z).$$

Thus for all $q \in \Delta$,

$$\limsup_R u(q) \leq \limsup_R v(q) = v(q)$$

by the continuity of PE-functions on R^*. Hence

$$\limsup_R u(q) \leq \inf \{ v(q) : v \in \mathfrak{F}(f) \} = f(q)$$

for all $q \in \Delta$.

Observe that $f(q) = 0$ for all $q \in \Delta \setminus \Delta^P$, hence $\limsup_R u(q) = f(q) = 0$ for any $q \in \Delta \setminus \Delta^P$. Now suppose for some positive ε and some compact subset K of Δ^P, $\limsup_R u(q) < f(q) - \varepsilon$ for all $q \in K$. We are going to show that $m(K) = 0$.

Consider the characteristic function c_K of K and

$$w(z) = \varepsilon \int c_K(q)K(z,q)dm(q).$$

Note that $w(z_o) = \varepsilon\, m(K)$. We claim that w is bounded. In fact, we can cover K by open sets $U_1, \ldots, U_n$ in R^* such that $\int_{U_i \cap R} P < \infty$ for every $i = 1,2,\ldots,n$. By the Urysohn property there is a function $g \in M(R)$ such that $0 \leq g \leq \varepsilon$, $g|K = \varepsilon$ and $\operatorname{supp} g \subset \bigcup\{U_i : i = 1,2,\ldots,n\}$. Observe that $E(g) = D(g) + \int g^2 P \leq D(g) + \varepsilon^2 \Sigma \int_{U_i \cap R} P < \infty$. Thus $g \in E(R)$. Hence $\pi g \in PBE(R)$ and $\pi g|\Delta = g|\Delta \geq \varepsilon c_K$. As a consequence $\pi g(z) = \int g(q)K(z,q)dm(q) \geq \varepsilon \int c_K(q)K(z,q)dm(q) = w(z)$. $\pi g \leq \sup_\Delta g \leq \varepsilon$, hence w is bounded by ε.

By Lemma VIj c_K is in $\mathcal{U}(\Delta)$ and therefore w is a $\widetilde{PE}$-function. By the part of the theorem just proved, $\limsup_R w(q) \leq \varepsilon c_K(q)$ for all $q \in \Delta$, in particular

$$\limsup_R w(q) = 0 \qquad \text{for all } q \in \Delta \setminus K$$

and

$$\limsup_R w(q) \leq \varepsilon \qquad \text{for all } q \in K.$$

Thus no matter whether $q \in K$ or $q \in \Delta \setminus K$,

$$\limsup_R (u(q) + w(q)) \leq \limsup_R u(q) + \limsup_R w(q)$$

$$\leq f(q).$$

Consequently for any $v \in \mathcal{F}(f)$

$$\liminf_R (v(q) - u(q) - w(q))$$

$$\geq \liminf_R v(q) - \limsup_R (u(q) + w(q))$$

$$\geq f(q) - f(q)$$

$$= 0.$$

Since $v - u - w$ is bounded from below, we have $v \geq u + w$ by the minimum principle IVh. Thus

$$v(z_o) \geq u(z_o) + w(z_o) = u(z_o) + \varepsilon m(K).$$

Now together with the previous theorem,

$$u(z_o) = \inf \{ v(z_o) : v \in \mathcal{F}(f) \}$$
$$\geq u(z_o) + \varepsilon m(K).$$

Hence $0 \geq m(K)$. Thus $m(K) = 0$.

VII. CHARACTERIZATION OF $\widetilde{PE}$-MINIMAL FUNCTIONS

This chapter consists of two parts. The first is a generalization of Nakai's work [14] on $\widetilde{HD}$-minimal functions to $\widetilde{PE}$-minimal functions. The second is a new result which shows that $\widetilde{HD}$-minimality is closely related to $\widetilde{PE}$-minimality.

VIIa. DEFINITION. A nonzero $\widetilde{PE}$-function u is called a $\widetilde{PE}$-*minimal* function if for any $\widetilde{PE}$-function v such that $u \geq v$ we have $cu = v$ for some constant c.

$\widetilde{HD}$-*minimal* functions are defined similarly.

VIIb. THEOREM. There exists a $\widetilde{PE}$-minimal function u on R if and only if there exists a point $q_o \in \Delta^P$ with positive m-measure. More precisely, if u is $\widetilde{PE}$-minimal, then there is a point $q_o \in \Delta^P$ such that $m(q_o) > 0$ and $u(z) = aK(z,q_o)$ for some positive constant a. Conversely, if $m(q_o) > 0$ for some $q_o \in \Delta^P$, then $K(z,q_o)$ is a $\widetilde{PE}$-minimal function.

Cf. [14], Theorem 3.3.

Proof. Let u be a $\widetilde{PE}$-minimal function and

$$K_i = \left\{ q \in \Delta^P \ : \ \limsup_R u(q) \geq 1/i \ \right\}.$$

Since

$$S = \left\{ q \in \Delta^P \ : \ \limsup_R u(q) > 0 \ \right\}$$
$$= \bigcup \left\{ K_i \ : \ i = 1,2,\ldots \right\}$$

and

$$m(S) > 0,$$

there is some n such that $m(K_n) > 0$. For simplicity, we denote K_n by K. Note that K is compact. By VIj the characteristic function c_K is in $\mathcal{U}(\Delta)$ and by VIl

$$w(z) = (1/n) \int K(z,q)c_K(q)dm(q)$$

is a $\widetilde{PE}$-function. By VIm

$$\lim\sup_R w(q) = (1/n)c_K(q)$$

a.e. on Δ. Now we are going to show that the measure on K is concentrated at an atom in K.

To see this, let A be any compact set in K such that $m(K \setminus A) > 0$. Again we conclude from VIj, VIl and VIm that

$$v(z) = (1/n) \int K(z,q)c_A(q)dm(q)$$

is a $\widetilde{PE}$-function and

$$\lim\sup_R v(q) = (1/n)c_A(q)$$

a.e. on Δ. Clearly $u \geq v$. If $m(A) > 0$, then v is a nontrivial $\widetilde{PE}$-function dominated by u. Thus there is a nonzero positive constant c such that $cu = v$ by the minimality of u. However, note that

$$c/n \leq c \lim\sup_R u(q) = \lim\sup_R v(q) = 0$$

for almost all $q \in K \setminus A$. This is absurd. Hence $m(A) = 0$. If K has no atomic point q_o such that $m(K) = m(q_o)$, then there exist subsets B and C of K such that $K = B \cup C$ with $B \cap C = \emptyset$ and $m(B) > 0$ $m(C) > 0$. By the regularity of m, there are compact subsets L and M of

B and C respectively both with positive measure. However, the foregoing argument implies that both $m(M)$ and $m(L)$ must be zero. This contradiction establishes the existence of q_o such that $m(K) = m(q_o)$.

Since $K_i \subset K_{i+j}$ for all j and $\lim m(K_i) = m(S)$, we have indeed $m(S) = m(q_o)$. By VIl and VIm for some $f \in \mathcal{U}(\Delta)$ with $f(q) = \lim\sup_R u(q)$ a.e. we conclude that

$$u(z) = \int K(z,q)f(q)dm(q)$$

$$= f(q_o)m(q_o)K(z,q_o).$$

To prove the sufficiency, let $q_o \in \Delta^P$ such that $m(q_o) = 1/k > 0$. By VIj and VIl

$$K(z,q_o) = k \int K(z,q)c_{\{q_o\}}(q)dm(q)$$

is a $\widetilde{PE}$-function. We are going to prove that $K(z,q_o)$ is $\widetilde{PE}$-minimal. If v is a $\widetilde{PE}$-function such that

$$v(z) \leq K(z,q_o),$$

then

$$\lim\sup_R v(q) \leq \lim\sup_{z \in R, z \to q} K(z,q_o)$$

$$= kc_{\{q_o\}}(q) \qquad \text{a.e. on } \Delta$$

i.e.

$$\lim\sup v(q) = 0 \qquad \text{a.e. on } \Delta \setminus \{q_o\}.$$

We set $\lim\sup v(q_o) = a$. Then

$$v(z) = \int K(z,q)(\lim\sup_R v(q))dm(q)$$

$$= a \int K(z,q) c_{\{q_o\}}(q) dm(q)$$

$$= (a/k) K(z,q_o).$$

Thus $K(z,q_o)$ is $\widetilde{PE}$-minimal.

VIIc. THEOREM. Let $q_o \in \Delta^P$. Then $m(q_o) > 0$ if and only if $m^o(q_o) > 0$.

Indeed, if $m(q_o) > 0$. Let $\{U_n\}$ be a sequence of open neighborhoods of q_o in R* such that

$$U_n \supset \overline{U}_{n+1} ,$$

$$\lim m(U_n) = m(q_o) ,$$

$$\lim m^o(U_n) = m^o(q_o) ,$$

and

$$\int_{U_n \cap R} P < \infty .$$

By the Urysohn property, there is an $f_n \in E(R)$ such that

$$0 \leq f_n \leq 1,$$

$$f_n | \overline{U}_{n+1} = 1$$

and

$$\operatorname{supp} f_n \subset U_n .$$

Clearly $f_n \in E(R)$ and $f_n \geq f_{n+1}$ for all n. By the choice of $\{U_n\}$, we have $\lim f_n | \Delta = c_{\{q_o\}}$ almost everywhere with respect to both m and m^o. Note that $\pi^o f_n \in HBD(R)$, $\pi f_n \in PBE(R)$ and $\pi^o f_n | \Delta = \pi f_n | \Delta = f_n | \Delta$. By IIp $\pi^o f_n$ is a supersolution, hence so is $\pi^o f_n - \pi f_n$. Observe that for all $q \in \Delta$

$$\liminf_R (\pi^o f_n - \pi f_n)(q) = \lim_R (\pi^o f_n - \pi f_n)(q)$$

$$= f_n(q) - f_n(q)$$

$$= 0.$$

We have $\pi^o f_n \geq \pi f_n$ by the minimum principle IVh.

Note that $\{\pi^o f_n\}$ and $\{\pi f_n\}$ are both decreasing sequences. Since

$$\int K^o(z,q) f_n(q) dm^o(q) = \pi^o f_n(z)$$

$$\geq \pi f_n(z)$$

$$= \int K(z,q) f_n(q) dm(q),$$

as n tends to ∞, we have by the monotone convergence theorem

$$\int K^o(z,q) c_{\{q_o\}}(q) dm^o(q) \geq \int K(z,q) c_{\{q_o\}}(q) dm(q).$$

Setting $z = z_o$ gives

$$m^o(q_o) \geq m(q_o) > 0.$$

Thus the necessity is proved.

To prove the sufficiency, we need the lemmas of VIId - VIIg.

VIId. The following lemma is a modification of a result on Riemann surface due to Nakai ([16], Proposition 9). His result does not generalize completely to Riemannian manifolds. But the following is sufficient for our purposes.

<u>LEMMA</u>. Let $q_o \in \Gamma$ be a point such that $m^o(q_o) > 0$ and U be any neighborhood of q_o in R^*. Then there exists a neighborhood V of q_o in R^* such that $V \subset U$ and $V \cap R$ is a region in R with piecewise smooth boundary.

To prove this lemma, we simply replace the triangulations by parametric balls in Nakai's proof; then everything follows.

VIIe. Consider a region G in R, its closure $\overline{G}$ in R^* and its Royden compactification G^*. We recall (see IIIo - IIIq) that $B(G) = (\overline{G} \setminus G) \cap \Gamma$ and j is the unique continuous mapping from G^* onto $\overline{G}$ fixing G elementwise. We have $G^* \setminus G = j^{-1}(bG) \cup j^{-1}((B(G))$. Moreover, by theorem IIIq, j is a homeomorphism of $G \cup j^{-1}(B(G))$ to $G \cup B(G)$.

Let $z_o \in G$ which is the center of the representing measure m on the Royden boundary Γ. If we denote the Royden boundary of G by Γ_G, harmonic boundary by Δ_G and the representing measure for solutions on Γ_G by m_G, then following is true.

<u>LEMMA</u>. Let G be a region in **R** with piecewise smooth boundary such that $\int_G P < \infty$. Let E be any Borel subset of B(G), then

$m(E) > 0$ if and only if $m_G(j^{-1}(E)) > 0$.

Nakai ([16], Proposition 8) has proved the case where $P = 0$.

Proof. Without loss of generality we may assume that E is compact and $E \subset \Delta^P$. Note that $E \cap bG = \emptyset$ since $E \subset B(G)$.

There exists a sequence $\{V_n\}$ of open sets in R^* such that $E \subset \overline{V}_{n+1} \subset V_n \subset B(G) \cup G$, $\int_{V_n \cap R} P < \infty$ for all n and $m(E) = \lim m(V_n \cap \Gamma)$, $m_G(j^{-1}(E)) = \lim m_G(j^{-1}(V_n \cap \Gamma))$. Take $f_n \in M(R)$ such that $0 \leq f_n \leq 1$, $f_n | \overline{V}_{n+1} = 1$ and $f_n | R^* \setminus V_n = 0$. Thus $\{f_n\}$ is a decreasing sequence and $f_n \in E(R)$ for all n. If we view f_n as functions on G, then $f_n \in E(G)$ because we always have that $E(R) \subset E(G)$. We denote the extension of f_n to G^* by f_n^*, then $f_n^* | j^{-1}(\overline{V}_{n+1}) = 1$, $f_n^* | G^* \setminus j^{-1}(V_n) = 0$ and $0 \leq f_n^* \leq 1$. Now we set

$$u_n(z) = \int_{\Gamma} K(z,q) f_n(q) dm(q)$$

and

$$\underline{v}_n^*(z) = \int_{\Gamma_G} K_G(z,q) f_n^*(q) dm_G(q),$$

where $K_G(z,q)$ is the associated kernel of m_G. Let us consider the exhaustion $\{R_n\}$ of R constructed in Theorem IIg. We set

$$v_{n,m} = \pi_{R^* \setminus (R_m \cap G)} f_n$$

where π is the projection defined in IVf. Then $\{v_{n,m}\}_m$ is a sequence bounded by $\sup_R f_n$ and $E(v_{n,m+1}) \leq E(v_{n,m}) \leq E(f_n)$ by the Dirichlet principle of IIIf, and thus $\{E(v_{n,m})\}_m$ converges. Note that

$$E(v_{n,m+p} - v_{n,m} \cdot v_{n,m+p}) = 0$$

by Green's formula of IIIf. Consequently,

$$E(v_{n,m+p} - v_{n,m}) = E(v_{n,m}) - E(v_{n,m+p}).$$

Note that $v_{n,m}|R\setminus G = 0$. Also, being a bounded sequence of solutions on G, $\{v_{n,m}\}_m$ has a convergent subsequence, again denoted by $\{v_{n,m}\}_m$, which is BE-Cauchy on G. Let

$$v_n = \lim_{m\to\infty} v_{n,m}.$$

Note that $v_n|G \in E(G)$, which is BE-complete, and $v_n|R\setminus G = 0$. We are going to show that v_n is continuous on R and hence $v_n \in E(R)$.

In fact, if x is any point on bG and N is a parametric ball about x, we let w be the strong barrier function for x on $\overline{N\cap G}$ with respect to the differential operator $L = \Delta - P$ (see p. 341, [4]). Note that $L(w) \leq -1$ and $w|\overline{N\cap G}\setminus\{x\}$ is strictly positive while $w(x) = 0$ (see p. 341, [4]). Let c be a positive constant such that

$$c \inf\{w(y) : y \in bN\cap\overline{G}\} \geq \sup_R f_n.$$

Then

$$\begin{aligned} L(cw - v_{n,m}) &= L(cw) - L(v_{n,m}) \\ &\leq -c \\ &\leq 0. \end{aligned}$$

Since $cw - v_{n,m} \geq 0$ on $b(N\cap G)$, we have

$$cw - v_{n,m} \geq 0$$

on $\overline{N \cap G}$ by the maximum principle (p. 326, [4]). Letting $m \to \infty$ gives

$$cw - v_n \geq 0$$

on $\overline{N \cap G}$. Thus

$$0 \leq \underline{\lim}_G\, v_n(x) \leq \underline{\lim}_G\, cw(x) = 0,$$

i.e. v_n is continuous at x. Since x is arbitrary, we see that v_n is continuous at every point of bG and is continuous on R.

Now we let

(1) $$u_{n,m} = \pi_{R^* \setminus R_m} v_n.$$

Similarly, $\{u_{n,m}\}_m$ has a BE-Cauchy subsequence, again denoted by $\{u_{n,m}\}_m$, on R. We set

$$u_n' = \text{BE-}\lim_{m \to \infty} u_{n,m}.$$

Then $u_n' \in PBE(R)$. Note that $u_n'|\Delta = v_n|\Delta = f_n|\Delta$. Thus $u_n' = u_n$ by maximum principle IVc. Note also that $\{u_n\}$ and $\{v_n\}$ are decreasing sequences since $\{f_n\}$ is decreasing on Δ. Clearly v_n is a subsolution on R. Thus $u_{n,m} \geq v_n$ on R_m for all m. Hence $u_n \geq v_n$ on R as well as on R^*. We denote the extension of $v_{n,m}$ to G^* by $v^*_{n,m}$. Since $\{v_{n,m}\}_m$ is a BE-Cauchy sequence on G and $v_{n,m} - f_n = 0$ on $bG \cup B(G)$, we have $v^*_{n,m} - f^*_n = 0$ on Γ_G, $v_n \in PBE(G)$ and $v^*_n|\Delta_G = \underline{v}^*_n|\Delta_G = f^*_n|\Delta_G$, where v^*_n is the extension of v_n to G^*. Therefore we have $v^*_n = \underline{v}^*_n$. Also $v^*_n \geq v^*_{n+1}$ and $u_n \geq \underline{v}^*_n = v^*_n$. Note that

$$m(V_n \cap \Gamma) \geq u_n(z_o) \geq \underline{v}^*_n(z_o) \geq m_G(j^{-1}(V_{n+1} \cap \Gamma_G)).$$

Hence

(2) $$m(E) \geq m_G(j^{-1}(E)).$$

Now suppose $m_G(j^{-1}(E)) = 0$. Note $v_1 - v_n \in PN(G)$ and $v_1 - v_n | R \setminus G = 0$. Thus $v_1 - v_n$ is a subsolution on R. Also note that $u_{1,m} - u_{n,m} \in PN(R_m)$ and $u_{1,m} - u_{n,m} \geq v_1 - v_n$ on $R^* \setminus R_m$. Hence $u_{1,m} - u_{n,m} \geq v_1 - v_n$ on R. In particular, $u_{1,m+1} - u_{n,m+1} \geq v_1 - v_n$ on bR_m implies that

(3) $$u_{1,m+1} - u_{n,m+1} \geq u_{1,m} - u_{n,m}$$

on bR_m. By the minimum principle IIo, (3) is true on R_m. Therefore (3) holds on R as well as on R^*. Consequently,

(4) $$\lim_{m\to\infty} (u_{1,m} - u_{n,m}) = u_1 - u_n$$

$$\geq u_{1,m} - u_{n,m}$$

for all m.

Since $0 = m_G(j^{-1}(E)) = \lim v_n^*(z_o)$, $\{v_n\}$ tends to 0 on G by Axiom III_2. Hence $\lim v_n = 0$ on bR_m. As $u_{n,m} | bR_m = v_n | bR_m$ by (1), we have $\lim_{n\to\infty} u_{n,m} = 0$ on R_m by the maximum principle. (4) implies that

$$\lim_{n\to\infty} (u_1 - u_n) \geq \lim_{n\to\infty} (u_{1,m} - u_{n,m})$$

$$= u_{1,m}$$

on R_m for all m. Since $u_1 = \lim_{m\to\infty} u_{1,m}$, we have

$$\lim_{n\to\infty} (u_1 - u_n) \geq u_1$$

on R. As a result $\lim u_n \leq 0$, and therefore $\lim u_n = 0$. Consequently

$m(E) = \lim u_n(z_o) = 0.$

Together with (2), we see that $m(E) > 0$ if and only if $m_J(j^{-1}(E)) > 0$.

VIIf. The following lemma is due to Nakai [15].

LEMMA. If P is any C^1 n-form on R such that $\int_R P < \infty$, then there exists an isomorphism T of $HB(R)$ onto $PB(R)$ with the following properties:

1) $\{u_n\} \subset HB(R)$ is a decreasing sequence with limit $u \in HB(R)$ if and only if $\{Tu_n\}$ is a decreasing sequence in $PB(R)$ with limit $Tu \in PB(R)$.

2) Let $\{R_i\}$ be a regular exhaustion of R. For any $u \in HB(R)$ let $T_i u$ be a continuous function on R such that $T_i u | R_i \in PB(R_i)$ and $T_i u | R \setminus R_i = u | R \setminus R_i$. Then $Tu = B\text{-}\lim T_i u$.

3) $\sup_R |Tu| = \sup_R |u|$ for all $u \in HB(R)$.

VIIg. COROLLARY. $h \in HBD(R)$ if and only if $Th \in PBD(R)$ for all $h \in HB(R)$. In this case $h | \Delta = Th | \Delta$.

Proof. Note that the sequence $\{T_i h\} \subset PB(R_i)$ converges to Th uniformly on compact subsets.

If $h \in HBD(R)$, then by hypothesis and the Dirichlet principle (IIIf)

$$E(T_{i+j}h) \leq E(T_i h) \leq E(h) < \infty .$$

Moreover Green's formula (IIIf) implies that

$$0 = E_{R_{i+j}}(T_{i+j}u,\ T_{i+j}u - T_i u)$$

$$= E(T_{i+j}u,\ T_{i+j}u - T_i u)$$

$$= E(T_{i+j}u) - E(T_{i+j}u\ ,\ T_i u).$$

Thus

$$0 \leq E(T_{i+j}u - T_i u)$$

$$= E(T_{i+j}u) - 2E(T_{i+j}u\ ,\ T_i u) + E(T_i u)$$

$$= E(T_{i+j}u) - E(T_i u).$$

Hence $\{T_i u\}$ is a BE-Cauchy sequence. Since $E(R)$ is BE-complete, we have $Tu \in PBE(R)$.

Furthermore, it is clear that $T_i u - u \in M_o(R)$ by IIIm. Since $M_\Delta(R)$ is the BD-closure of $M_o(R)$, we have $Tu - u \in M_\Delta(R)$. Hence $(Tu - u)\,|\,\Delta = 0$, i.e. $u\,|\,\Delta = Tu\,|\,\Delta$.

The proof for the sufficiency is exactly the same.

VIIh. Now we are going to complete the proof of Theorem VIIc. Let $q_o \in \Delta^P$ such that $m^o(q_o) > 0$. By VIId, there exists a neighborhood $\underline{G}$ of q_o in R^* $\int_{\underline{G}\cap R} P < \infty$ and $\underline{G} \cap R$ is a region in R with piecewise smooth boundary. Consider the region $G = \underline{G} \cap R$, its closure $\overline{G}$ in R^* and its Royden compactification G^*. We also consider the continuous mapping j from G^* onto $\overline{G}$ given in IIIp which fixes G elementwise. Since j is a homeomorphism from $G \cup j^{-1}(B(G))$ to $G \cup B(G)$ by IIIq, we have $q_1 = j^{-1}(q_o)$ is a point in Γ_G which is the Royden boundary of G. By VIIe, $m^o_G(q_1) > 0$.

By VIIf, HB(G) is isomorphic to PB(G) under the isomorphism T since $\int_G P < \infty$. Note that

$$h(z) = K_G^o(z,q_1) = \int_{\Gamma_G} K_G^o(z,q) c_{\{q_1\}}(q) dm_G^o(q)$$

is a nontrivial function in $\widetilde{HD}(G) \cap HB(G)$ since $h(z_o) = 1$. Let $\{U_n\}$ be a sequence of open sets in G* such that $U_n \supset \overline{U}_{n+1} \ni q_1$,

$U_n \cap \Gamma_G \subset \Delta_G$, $m_G^o(q_1) = \lim m_G^o(U_n \cap \Gamma_G)$ and

$m_G(q_1) = \lim m_G(U_n \cap \Gamma_G)$. For each n, there is an $f_n \in M(G)$ such that $0 \leq f_n \leq 1$, supp $f_n \subset U_n$, $f_n | \overline{U}_{n+1} = 1$ by the Urysohn property of M(G). We see that $\{f_n\}$ is a decreasing sequence converging to $c_{\{q_1\}}$ m_G^o a.e. as well as m_G a.e. on Γ_G. Let

$$u_n = \pi_G^o f_n$$

where π_G^o is the harmonic projection on G (see IVf). Thus $u_n \in HBD(G)$, $u_n | \Delta_G = f_n | \Delta_G$ and hence $\{u_n\}$ is a decreasing sequence by maximum principle IVb. We have

$$\lim u_n(z) = \lim \int_{\Gamma_G} K_G(z,q) u_n(q) dm_G^o(q)$$

$$= \int_{\Gamma_G} K_G(z,q) c_{\{q_1\}}(q) dm_G^o(q)$$

$$= h(z)$$

by the monotone convergence theorem.

Note that $Tu_n \in PBE(G)$ and $Tu_n | \Delta_G = u_n | \Delta_G$ by VIIg. Also

$Th = \lim Tu_n$ by VIIf. Thus

$$\begin{aligned} Th(z) &= \lim Tu_n(z) \\ &= \lim \int_{\Gamma_G} K_G(z,q)u_n(q)dm_G(q) \\ &= \int_{\Gamma_G} K_G(z,q)c_{\{q_1\}}(q)dm_G(q) \\ &= K_G(z,q_1)m_G(q_1) \end{aligned}$$

by the monotone convergence theorem again. Since

$$\sup_G Th = \sup_G h > 0,$$

we have $m_G(q_1) > 0$.

We apply VIIe once more and see that $m(q_o) > 0$ since $j^{-1}(q_o) = q_1$.

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