Chains of Non-regular de Branges Spaces

Thesis by

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To my wife Xintang,

whose love has made this possible.

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Abstract

We consider canonical systems with singular left endpoints, and discuss the concept of a *scalar* spectral measure and the corresponding generalized Fourier transform associated with a canonical system with a singular left endpoint. We use the framework of de Branges' theory of Hilbert spaces of entire functions [dB68] to study the correspondence between chains of non-regular de Branges spaces, canonical systems with singular left endpoints, and spectral measures.

We find sufficient integrability conditions on a Hamiltonian H which ensure the existence of a chain of de Branges functions in the first generalized Pólya class $\mathcal{P}_{\leqslant 1}$ with Hamiltonian H. This result generalizes de Branges' Theorem 41 in [dB68], which showed the sufficiency of stronger integrability conditions on H for the existence of a chain in the Pólya class $\mathcal{P}_0$. We show the conditions that de Branges came up with are also necessary. In the case of Kreĭn's strings, namely when the Hamiltonian is diagonal, we show our proposed conditions are also necessary.

We also investigate the asymptotic conditions on chains of de Branges functions as t approaches its left endpoint. We show there is a one-to-one correspondence between chains of de Branges functions satisfying certain asymptotic conditions and chains in the Pólya class $\mathcal{P}_0$. In the case of Kreĭn's strings, we also establish the one-to-one correspondence between chains satisfying certain asymptotic conditions and chains in the generalized Pólya class $\mathcal{P}_{\leqslant 1}$.

Contents

List of Figures

Chapter 0

Introduction

A *canonical system of differential equations* is a system of the form

$$\Omega \dot{X}(t) = zH(t)X(t), \quad t \in I := (t_-, t_+) \tag{1}$$

where $\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $H(t) \geqslant 0$ is a real symmetric matrix that is locally integrable w.r.t. $t \in I$, and is known as the *Hamiltonian*. By definition, solutions to (1) are absolute continuous functions $X = X(t) : (t_-, t_+) \to \mathbb{C}^2$. We say the Hamiltonian H has a *regular* left endpoint if H is integrable at t_-, namely $\int_{t_-}^{c} H(t)dt$ has finite elements for $c \in I$. Otherwise we say H has a *singular* left endpoint.

The canonical system (1) is of great importance as it has the most complete solution to the inverse spectral problem, namely the one-to-one correspondence between regular spectral measures and Hamiltonian with regular left endpoints (see Theorem A below). This result was obtained by Kreĭn [Kre52]. de Branges [dB59, dB60, dB61a] obtained deep results for Hamiltonian with singular left points. It is well known that Schrödinger equation, Dirac equation, or more generally any second order self-adjoint system of differential equations with real coefficients can be transformed to canonical systems (See Section 1.1).

The main tool de Branges used is the theory of Hilbert spaces of entire functions, which was developed by himself [dB59, dB60, dB61a, dB61b, dB62a, dB68] in the 1960s. de Branges investigated Hilbert spaces $(\mathcal{B}, (\cdot, \cdot))$ which satisfy the following axioms:

(H1) If $F \in \mathcal{B}$ and $F(w) = 0$ for nonreal w, then $\frac{z-\bar{w}}{z-w}F(z) \in \mathcal{B}$ and

$$\left(\frac{z-\bar{w}}{z-w}F(z), \frac{z-\bar{w}}{z-w}G(z) \right) = (F, G), \quad \text{if } F, G \in \mathcal{B}, F(w) = G(w) = 0,$$

(H2) The point evaluation $F \mapsto F(w)$ is a continuous linear functional on $\mathcal{B}$, for all nonreal w, (H3) If $F \in \mathcal{B}$, then $F^\# \in \mathcal{B}$ ($F^\#(z) := \overline{F(\bar{z})}$) and

$$(F^\#, G^\#) = (G, F), \qquad \text{for } F, G \in \mathcal{B}.$$

Such Hilbert spaces of entire functions are called *de Branges spaces (dB-spaces)*.

Denote the set of analytic functions on a region Λ by $\mathcal{A}(\Lambda)$ and Hardy space H^2 on $\mathbb{C}_+$ by $H^2(\mathbb{C}_+)$. de Branges [dB68, Theorem 23] showed any nonzero dB-space can be written as

$$\mathcal{B} = \mathcal{B}(E) := \left\{ F \in \mathcal{A}(\mathbb{C}) : \frac{F}{E}, \frac{F^\#}{E} \in H^2(\mathbb{C}_+) \right\}$$

where $E \in \mathcal{A}(\mathbb{C})$ satisfies

$$|E(z)| > 0, \quad |E(z)| \geqslant |E^\#(z)|, \quad \forall z \in \mathbb{C}_+.$$

Such functions are known as the *Hermite-Biehler functions* or *de Branges functions (dB-functions)*. E is said to be *non-degenerate* if the second inequality is strict.

dB-spaces have a profound inherent chain structure. Let $\mathcal{B}$ be a nonzero dB-space sitting isometrically in $L^2(\mu)$ for some positive measure μ on $\mathbb{R}$, then there exists a *unique chain of dB-spaces* $\{\mathcal{B}_t\}_{t\in(t_-,t_+)}$ (up to re-parametrization of t) s.t.

(i) There exists a unique $b \in (t_-, t_+)$, s.t. $\mathcal{B}_b = \mathcal{B}$,

(ii) $\mathcal{B}_a$ sits almost isometrically in $\mathcal{B}_c$, $\forall t_- < a < c < t_+$,

(iii) $\mathcal{B}_a$ sits almost isometrically in $L^2(\mu)$, $\forall t_- < a < t_+$,

(iv) $\|F\|_{\mathcal{B}_t}$ is a *continuous* non-increasing function of $t \in (a, t_+)$, for $F \in \mathcal{B}_a$ and $a \in I$,

(v) The reproducing kernels $K_{t,z}$ satisfy $\lim_{t\to t_-} K_{t,z}(z) = 0, \quad \forall z \in \mathbb{C}$.

Namely, any nonzero dB-space sitting isometrically in $L^2(\mu)$ can be extended into a chain of dB-spaces. In this case, μ is said to be a *spectral measure* of the chain $\{\mathcal{B}_t\}_{t\in I}$. We point out the conditions (iv) and (v) are assumed to ensure the chain is "saturated," as explained below. For two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, $\mathcal{H}_1 \sqsubseteq \mathcal{H}_2$ means $\mathcal{H}_1$ sits isometrically in $\mathcal{H}_2$. Let $\tilde{\mathcal{B}} \sqsubseteq \mathcal{B}$ be a nonzero dB-space, then $\tilde{\mathcal{B}}$ must be equal to $\mathcal{B}_a$ isometrically for some $a \in (t_-, b]$. Any dB-space $\hat{\mathcal{B}}$ sitting between $\mathcal{B}$ and $L^2(\mu)$, namely $\mathcal{B} \sqsubseteq \hat{\mathcal{B}} \sqsubseteq L^2(\mu)$, must be equal to $\mathcal{B}_c$ isometrically for some $c \in [b, t_+)$. In this sense the chain is "saturated." The formal definition of a chain of dB-spaces and the exact meaning of "almost isometrical" inclusion will be given in Section 1.2.2. The chain structure will be discussed in detail in Section 1.2.4.

The bridge connecting chains of dB-spaces and canonical systems is the corresponding chains of dB-functions. A *real entire function* F is an entire function s.t. $F = F^\#$. A matrix of real entire functions $M(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}$ is said to be a *Nevanlinna matrix* if $\det M(z) = 1$, $\forall z \in \mathbb{C}$, and

$$\frac{M^*(z)\Omega M(z) - \Omega}{z - \bar{z}} \geqslant 0, \quad \forall z \in \mathbb{C}. \tag{2}$$

A Nevanlinna matrix M is said to be *normalized* if $M(0) = I_2$, the 2×2 identity matrix. A family of non-degenerate dB-functions $\{E_t\}_{t\in I}$ is called a *chain of dB-functions (dB-chain)* if there exist normalized non-constant Nevanlinna matrices $(M_{a\to b})_{t_- < a < b < t_+}$ s.t. $M_{a\to b}$ is continuous for $a, b \in I$, and

$$\begin{pmatrix} A_b \\ C_b \end{pmatrix} = M_{a\to b} \begin{pmatrix} A_a \\ C_a \end{pmatrix}, \quad \forall t_- < a < b < t_+, \tag{3}$$

and $\lim_{t\to t_-} K_{t,z}(z) = 0, \forall z \in \mathbb{C}$, where $K_{t,z}$ is the reproducing kernel of $\mathcal{B}(E_t)$ at z. It can be shown for any chain of dB-spaces $\{\mathcal{B}_t\}_{t\in I}$, there exists a (non-unique) chain of dB-functions $\{E_t\}_{t\in I}$ s.t. $\mathcal{B}_t = \mathcal{B}(E_t)$, $\forall t \in I$. Moreover, for a chain of dB-functions, there exists a unique matrix-valued function $H(t) \in \mathbb{R}^{2\times 2}$, s.t. $H(t) \in L^1_{\text{loc}}(I)$, $H(t) \geqslant 0$, and

$$\Omega \begin{pmatrix} A_b \\ C_b \end{pmatrix} - \Omega \begin{pmatrix} A_a \\ C_a \end{pmatrix} = z \int_a^b H(t) \begin{pmatrix} A_t \\ C_t \end{pmatrix} dt. \tag{4}$$

The existence and uniqueness of such an H are shown in Section 1.2.3, and the unique H is called the *Hamiltonian of the chain* $\{E_t\}_{t\in I}$.

de Branges theory plays a pivotal role in the spectral theory of canonical systems for several reasons. Firstly, a chain of dB-spaces has at least one spectral measure (see Section 2.4.2). Secondly, for a chain of dB-functions $\{E_t\}_{t\in I}$ s.t. E_t has no real zeros and $E_t(0) = 1$, $\forall t \in I$ (we will see these assumptions are not restrictive at all), let H be the Hamiltonian of $\{E_t\}_{t\in I}$, then there exists a generalized Fourier transform $\mathcal{W}_{\mathcal{B}}$ which maps $L^2(H; (t_-, c])$ isometrically onto $\mathcal{B}(E_c) = \mathcal{B}_c$ for $c \in (t_-, t_+)$ (with minor technical complications), therefore it maps $L^2(H; (t_-, c])$ into $L^2(\mu)$ isometrically. The significance of these results is that it extends the spectral theory of canonical system substantially beyond the regular case. Traditionally, the (scalar) spectral measure and the generalized Fourier transform are only defined if t_- is a *regular* left endpoint of the Hamiltonian H, while de Branges theory extends the limit to the class of Hamiltonian associated with a chain of dB-functions, whose left endpoint may or may not be regular. In Section 2.3 we will show H is the Hamiltonian of a chain of dB-spaces if and only if the canonical system (1) on (t_-, c) has a discrete spectrum for some $c \in (t_-, t_+)$, or equivalently the corresponding Titchmarsh-Weyl m-function has a meromorphic extension.

We here briefly introduce the results in the regular case. A dB-space $\mathcal{B}$ is said to be regular if

$$\frac{F(z) - F(w)}{z - w} \in \mathcal{B}, \quad \forall F \in \mathcal{B}, \quad \forall w \in \mathbb{C}.$$

One can show that for a chain of dB-spaces $\{\mathcal{B}_t\}_{t\in I}$, if $\mathcal{B}_t$ is regular for *some* $t \in I$, the $\mathcal{B}_t$ is regular for *any* $t \in I$ (see Proposition 1.32). Therefore we call a chain of dB-spaces *regular* if any dB-space in the chain is regular. A positive measure μ on $\mathbb{R}$ is said to be regular if

$$\int_{-\infty}^{+\infty} \frac{d\mu(\lambda)}{1+\lambda^2} < \infty.$$

We have the following one-to-one correspondence between Hamiltonian with regular left endpoints, chains of regular dB-spaces, and regular measures on $\mathbb{R}$. The results stated below are due to de Branges.

Theorem A (Theorem 1.34). *(i) Let $\{\mathcal{B}(E_t)\}_{t\in I}$ be a dB-chain with Hamiltonian H and a spectral measure μ. If $\{\mathcal{B}(E_t)\}$ is regular, then μ is regular, t_- is a regular left endpoint of H, and $\lim_{t\to t_-} E_t(z) \equiv w$ locally uniformly in z for some complex constant w which doesn't depend on z.*

(ii) Let $H = H(t)$, $t \in I$ be a Hamiltonian. If t_- is a regular left endpoint of H, then there exists a regular dB-chain $\{\mathcal{B}(E_t)\}_{t\in I}$ with H as its Hamiltonian. The chain is unique if we specify $E_t(0) = 1$ for $t \in I$.

(iii) Let μ be a regular positive measure on $\mathbb{R}$, then there exists a chain of regular dB-spaces $\{\mathcal{B}_t\}_{t\in I}$ s.t. μ is a spectral measure of $\{\mathcal{B}_t\}$. The chain is unique up to re-parametrization of t.

Traditionally, the case where t_- and t_+ are both singular endpoints of the Hamiltonian H is treated as a full-line problem, where spectral matrices are derived instead of (scalar) spectral measures (cf. [Tit62], [CL55], and [LS75]). In the case that there exists a chain of dB-functions with Hamiltonian H, de Branges' results have the benefit that we can investigate the (scalar) spectral measure, rather than the spectral matrix which is more complicated and contains redundant information (cf. [Lev87, Section 6.1]). However, when t_- is a singular left endpoint of H, the nice one-to-one correspondence in Theorem A doesn't hold anymore, and many questions on the correspondence remain widely open. To name a few, we might have multiple chains of dB-spaces sitting in $L^2(\mu)$ even if μ is regular. For a given Hamiltonian H, there might be multiple chains of dB-spaces with H as its Hamiltonian. Partial results have been obtained and will be discussed in Section 2.4. Nevertheless, the main focus of this dissertation is to construct chains of dB-spaces for a given Hamiltonian with a singular left endpoint.

The existence of a chain of dB-spaces in the regular case (i.e., for Hamiltonian with regular left endpoint) is straightforward. One can consider the matrix solution $M_{t_-\to t}$ to the canonical system (1) with the boundary condition $\lim_{t\to t_-} M_{t_-\to t}(z) = I_2$, the 2×2 identity matrix. The first column of $M_{t_-\to t}$, denoted by $\binom{A_{t_-\to t}}{C_{t_-\to t}}$, generates a chain of dB-functions $E_t(z) := A_{t_-\to t}(z) - iC_{t_-\to t}$ and consequently a chain of dB-spaces $\{\mathcal{B}(E_t)\}_{t\in(t_-,t_+)}$. There have been attempts to construct chains of dB-spaces for Hamiltonian with singular left endpoints, among which de Branges investigated the case that $E_t(z)$ doesn't have a finite limit as $t \to t_-$, while instead $E_t(z)e^{\beta(t)z}$ converges as $t \to t_-$ where $\beta(t)$ is the anti-derivative of $H_{21}(t)$, the lower left element of the Hamiltonian H. Indeed, de Branges [dB61a, Theorem IV] showed the existence and uniqueness of a chain of dB-functions in the *Pólya class* $\mathcal{P}_0$ of entire functions, which is the set of nonzero entire functions that can be approximated locally uniformly by polynomials with no zeros in $\mathbb{C}_+$, with H as its Hamiltonian if the Hamiltonian H satisfies

$$\alpha(t_-) := \lim_{t \to t_-} \alpha(t) = 0, \tag{5}$$

$$\alpha(t) > 0 \text{ for } t \in I, \tag{6}$$

$$\int_{t_-}^{b} \alpha(t) d\gamma(t) < \infty \text{ for some (hence for all) } b \in I, \tag{7}$$

where $h(t) := \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$ is an anti-derivative of H. The precise statements are given in Theorem B(i) below. We make two remarks here: firstly, the Pólya class $\mathcal{P}_0$ of entire functions, which arose when people study the limit functions of polynomials whose zeros lie in $\mathbb{C}_- \cup \mathbb{R}$, is of great importance and interesting for its own sake. The background of the Pólya class is presented by [BJ54] and [Lev64], and we will discuss more of their properties in Section 3.2.1. Secondly, the conditions (5)—(6) are actually necessary conditions on the Hamiltonian H for the existence of a chain of dB-spaces with H as its Hamiltonian, and only (7) is the critical condition that ensures the existence of a chain of dB-functions E_t in the Pólya class $\mathcal{P}_0$.

An alternative proof of Theorem B(i) is obtained in Section 4.4, where the theory of Laguerre classes of entire functions (cf. [dB68, Pages 288-292]) enters the proof. Moreover, we prove the condition (7) is also a necessary condition on the Hamiltonian H for the existence of a chain of dB-functions in the Pólya class $\mathcal{P}_0$ with H as its Hamiltonian. Moreover, we investigate the asymptotic condition E_t satisfies as $t \to t_-$, and show certain asymptotic conditions on E_t are equivalent to the assumption that $E_t \in \mathcal{P}_0$. The precise statements are given in Theorem B(ii) below.

A dB-function E is said to be *strict* if $|E(z)| > |E^{\#}(z)|$, $\forall z \in \mathbb{C}_+$. E is said to be *normalized* if $E(0) = 1$. We then conclude

Theorem B (Theorem 3.15)**.** *(i) Let $H = H(t)$, $t \in I$ be a Hamiltonian. If H satisfies* (5)—(7)*, then there exists a unique dB-chain with H as its Hamiltonian, s.t. E_t is strict, normalized, and* $\lim_{t \to t_-} E_t(z)e^{\beta(t)z} = 1$ *locally uniformly in z. For this unique dB-chain $\{\mathcal{B}(E_t)\}$, we also have $E_t \in \mathcal{P}_0$, $\forall t \in I$.*

(ii) Let $\{\mathcal{B}(E_t)\}_{t \in I}$ be a dB-chain with Hamiltonian $H = H(t)$ s.t. E_t is strict and normalized for some $t \in I$, then:

- *If $E_t \in \mathcal{P}_0$ for some $t \in I$, then $E_t \in \mathcal{P}_0$ for all $t \in I$, $H(t)$ satisfies* (7) *and $E_t(z)e^{\beta(t)z}$ converges to $S(z) := e^{az^2+bz}$ for some $a, b \in \mathbb{R}$ locally uniformly in z, as $t \to t_-$.*
- *If $S(z) := \lim_{t\to t_-} E_t(z)e^{\beta(t)z}$ exists and is real entire, then $H(t)$ satisfies* (7), *and $E_t = S\tilde{E}_t$ where $\tilde{E}_t \in \mathcal{P}_0$, $\forall t \in I$, and $\lim_{t\to t_-} \tilde{E}_t(z)e^{\beta(t)z} = 1$ locally uniformly in z.*

It can be shown that the Pólya class consists of dB-functions of the form $E(z) = e^{-az^2}E_0(z)$ where $a \geqslant 0$ and E_0 is a dB-function of genus at most 1 (cf. [Lev64, Chapter VIII, Theorem 4]). In fact, one can generalize the notion of the Pólya class by considering $E(z) = e^{-az^{2k+2}}E_0(z)$ where $a \geqslant 0$ and E_0 is a dB-function of genus at most $2k+1$. Actually, one of the main results of this dissertation (see Theorem C below) is to generalize Theorem B(i) so that for a Hamiltonian H which satisfies weaker integrability condition than (7), there exists a chain of dB-functions in the first generalized Pólya class $\mathcal{P}_{\leqslant 1}$ with H being its Hamiltonian, where the first generalized Pólya class $\mathcal{P}_{\leqslant 1}$ consists of dB-functions of the form $e^{-az^4}E_0(z)$ where $a \geqslant 0$ and $E_0 \in dB$ has genus at most 3.

Theorem C (Theorem 4.11)**.** *Let $H = H(t)$, $t \in I$ be a Hamiltonian and $h = h(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$ be its anti-derivative. Assume that*

$$\alpha(t_-) := \lim_{t\to t_-} \alpha(t) = 0, \tag{8}$$

$$\alpha(t) > 0 \text{ for } t \in I, \tag{9}$$

$$\int_{t_-}^{b}\int_{t_-}^{t} \alpha(s)^2 d\gamma(s)d\gamma(t) < \infty \text{ for some (hence for all) } b \in I, \tag{10}$$

$$\int_{t_-}^{b}\int_{t_-}^{t} (\beta(t)-\beta(s))^2\, d\alpha(s)d\gamma(t) < \infty \text{ for some (hence for all) } b \in I, \tag{11}$$

then there exists a unique dB-chain $\{\mathcal{B}(E_t)\}_{t\in I}$ with H as its Hamiltonian, s.t. $E_t \in \mathcal{P}_{\leqslant 1}$ is normalized, strict, non-degenerate, and

$$\lim_{t\to t_-} E_t(z)\exp\left(\beta(t)z - \left(\int_t^1 \alpha(s)d\gamma(s)\right)z^2 - 2\left(\int_t^1\int_{t_-}^{s} (\beta(s)-\beta(u))\, d\alpha(u)d\gamma(s)\right)z^3\right) = 1$$

locally uniformly in z.

Again, we point out that the conditions (8)—(9) are just trivial necessary conditions on the Hamiltonian H for it to be associated with a chain of dB-functions, while conditions (10)—(11) are the critical conditions that ensure the existence of a chain of dB-functions E_t in the first generalized Pólya class $\mathcal{P}_{\leqslant 1}$.

Kreĭn's strings are a special type of canonical system where the Hamiltonian H is a diagonal matrix. Due to its comparative simplicity over the general canonical system, symmetry of the associated chain of dB-spaces (cf. [dB62b]), and relevance to other fields including the interpolation problem and diffusion processes (cf. [Man68], [DM08]), Kreĭn's strings have been studied extensively by mathematicians including Gohberg and Kreĭn [GK70], de Branges [dB62b, dB68], Dym [Dym71], Dym and McKean [DM70, DM08], Kats [Kat94], and Kotani [Kot75, Kot07, Kot13]. For Kreĭn's strings, the asymptotic condition in Theorem B(i) becomes

$$\lim_{t\to t_-} E_t(z) = 1$$

locally uniformly in z, which is similar to the regular case.

For Kreĭn's strings, as $\beta(t) \equiv 0$, condition (11) becomes vacuous. We propose another method to prove Theorem C in the case of Kreĭn's strings, where we use the theory of Laguerre classes of entire functions (cf. [dB68, Pages 288-292]). Moreover, for Kreĭn's strings, we also prove the converse of Theorem C, namely for a chain of dB-functions in the first generalized Pólya class $\mathcal{P}_{\leqslant 1}$ that are symmetric about the origin, its associated (diagonal) Hamiltonian satisfies (10). These results are summarized in Theorem D, which is analogous to Theorem B for the Pólya case.

Theorem D (Theorem 4.20)**.** *(i) Let $H = H(t)$, $t \in I$ be a diagonal Hamiltonian. If H satisfies* (8)—(10)*, then there exists a unique dB-chain $\{\mathcal{B}(E_t)\}_{t\in I}$ with H as its Hamiltonian, s.t.*

$$\lim_{t\to t_-} E_t(z)e^{-\left(\int_t^1 \alpha(s)d\gamma(s)\right)z^2} = 1$$

locally uniformly in z. For this unique dB-chain $\{\mathcal{B}(E_t)\}$, we also have $E_t \in \mathcal{P}_{\leqslant 1}$, $\forall t \in I$.

(ii) Let $\{\mathcal{B}(E_t)\}_{t\in I}$ be a dB-chain with diagonal Hamiltonian $H = H(t)$, and $E_t(0) = 1$ for $t \in I$.

- *If $E_t \in \mathcal{P}_{\leqslant 1}$ for some $t \in I$, then $E_t \in \mathcal{P}_{\leqslant 1}$ for all $t \in I$, $H(t)$ satisfies* (10) *and $E_t(z)e^{-\left(\int_t^1 \alpha(s)d\gamma(s)\right)z^2}$ converges to $S(z) := \exp\left(\sum_{n=1}^4 a_n z^n\right)$ for some $a_n \in \mathbb{R}$ locally uniformly in z, as $t \to t_-$.*
- *If $S(z) := \lim_{t\to t_-} E_t(z)e^{-\left(\int_t^1 \alpha(s)d\gamma(s)\right)z^2}$ exists and is real entire, then $H(t)$ satisfies* (10)*, and $E_t = S\tilde{E}_t$ where $\tilde{E}_t \in \mathcal{P}_{\leqslant 1}$, $\forall t \in I$, and $\lim_{t\to t_-} \tilde{E}_t(z)e^{-\left(\int_t^1 \alpha(s)d\gamma(s)\right)z^2} = 1$ locally uniformly in z.*

Last but not the least, we show for a dB-space $\mathcal{B}(E)$, the assumption that $E \in \mathcal{P}_{\leqslant k}$ is closely related to the assumption that the Bezout operator $T_{A,0} \in \mathfrak{S}_{2k+2}$, the $(2k+2)$-th Schatten class, where the Bezout operator $T_{A,0}$ acts on $\mathcal{B}(E)$ and is defined to be

$$T_{A,0}(F)(z) = \frac{A(0)F(z) - A(z)F(0)}{z}, \quad \forall F \in \mathcal{B}(E).$$

Therefore the Pólya class corresponds to the Hilbert-Schmidt class of operators. The exact meaning of the correspondence will be discussed in Section 4.5. For any dB-space $\mathcal{B}(E)$ s.t. E is normalized and strict , the Bezout operator is a compact self-adjoint operator acting on $\mathcal{B}(E)$. Therefore, the possible extensions of Theorem B and Theorem C will lead to a more complete solution to the problem of the existence of chains of dB-functions for a given Hamiltonian. In other words, we should expect to construct chains of dB-functions of generalized Pólya class $\mathcal{P}_{\leqslant k}$ for Hamiltonian satisfying even weaker integrability condition near t_-, and consequently the corresponding Bezout operators $T_{A_t,0}$ belong to the $(2k+2)$-Schatten class.

Even though our main focus is to apply de Branges theory to solve the spectral problems for the canonical system, we should point out that de Branges theory is highly interesting for its own sake, and is relevant to many other fields of mathematics. In particular, de Branges theory seems to be relevant to number theory and the Riemann hypothesis. Actually, let ξ be the Riemann ξ-function, $A_\xi(z) := \xi\left(\frac{1}{2} - iz\right)$, $C_\xi(z) := i\xi'\left(\frac{1}{2} - iz\right)$, and $E_\xi(z) := A_\xi(z) - iC_\xi(z)$, then one can check that the Riemann hypothesis holds if and only if $E_\xi(z)$ is a dB-function (cf. Lagarias [Lag06]). Consequently, this leads to two possible approaches to the Riemann hypothesis. Firstly, one can construct a Hilbert space of entire functions and verify the axioms (H1)—(H3), and then prove $\mathcal{B} = \mathcal{B}(E_\xi)$. Alternatively, one can construct a Hamiltonian, or equivalently a canonical system, and show the chain of dB-spaces are associated with E_ξ. Several interesting examples have emerged from the interaction between de Branges theory and number theory, for instance the Mellin transform and Sonine spaces (cf. [RR69], [dB68], and [Bur02]). A more recent example is the one-dimension Schrödinger equation with the Morse potential, namely

$$-\frac{d}{dt^2} + V_k(t) \text{ on } [t_-, +\infty), \quad V_k(t) = \frac{1}{4}e^{2t} + ke^t$$

with fixed boundary condition at t_-. The eigenfunction corresponding to an eigenvalue λ is the Whittaker function $W_{k,\lambda}$. Lagarias [Lag09] showed $F(z) := W_{k,z-\frac{1}{2}}(t)$ for fixed k and $t > 0$ displays Riemann-ξ behaviors, in the sense that:

- F is a real entire function of order 1 and maximal type, and is real on the critical line $\Re z = \frac{1}{2}$,
- $F(z) = F(1-z)$,
- Number of zeros in $[-T, T] = \frac{2}{\pi} T \log T + \frac{2}{\pi}\left(2\log 2 - 1 - \log_{t_-}\right) T + O(1)$,
- All but finitely many zeros of F are on $\Re z = \frac{1}{2}$. All other zeros are on the real line. All zeros are simple, except possibly at $z = \frac{1}{2}$.

For more discussion on the connection between de Branges theory and number theory we refer the readers to [dB86], [Lag06], and [Suz12]. We also point out that most interesting examples in the application of de Branges theory belong to the non-regular case.

We outline the contents of this dissertation. In Chapter 1, we review the spectral theory of canonical systems with *regular* left endpoints and *singular* right endpoints which was developed by

Titchmarsh and Weyl (see [Wey09, Wey10a, Wey10b] and [Tit62]), and give a brief introduction to de Branges theory of Hilbert spaces of entire functions. We show the de Branges theory is related to canonical systems with *regular* left endpoints, and the one-to-one correspondence between Hamiltonian with *regular* left endpoints, chains of *regular* dB-spaces, and *regular* measures.

In Chapter 2, we discuss the spectral matrix approach for canonical systems with both endpoints being *singular*, then we give a necessary and sufficient condition for the existence of a chain of dB-spaces for a given Hamiltonian, namely Israel Kats' Theorem, which shows the de Branges theory applies (i.e., the scalar spectral measure and the generalized Fourier transform exist) as long as the canonical system has a *compact resolvent* near the left endpoint. We also briefly summarize results from de Branges theory which apply to both the regular case and the non-regular case.

In Chapter 3, we review some known results on the spectral theory of canonical systems. In particular, we discuss Kotani's results on Kreĭn's strings [Kot75, Kot07] and de Branges' results [dB68, Theorem 41] on canonical systems with singular left endpoints under assumptions (5)—(7). In particular, de Branges showed the existence of a chain of dB-spaces in the Pólya class $\mathcal{P}_0$ for a Hamiltonian that satisfies (5)—(7). We prove the converse that the assumptions (5)—(7) are also necessary conditions on H for the existence of the chain of dB-spaces in the Pólya class $\mathcal{P}_0$ with H as its Hamiltonian.

In Chapter 4, we introduce the generalized Pólya classes $\mathcal{P}_{\leqslant k}$, and show how to generalize de Branges' results [dB68, Theorem 41] for canonical systems with *singular* left endpoints from the Pólya class $\mathcal{P}_0$ to the first generalized Pólya class $\mathcal{P}_{\leqslant 1}$. We also show the condition that the chain of dB-spaces are in $\mathcal{P}_{\leqslant k}$ is related to the condition that the Bezout operator $T_{A,0}$ belongs to the $(2k+2)$-th Schatten class $\mathfrak{S}_{2k+2}$.

Chapter 1

Half-line problems and chains of regular de Branges spaces

In this chapter we review the spectral theory of canonical systems with *regular* left endpoints and de Branges theory. In Section 1.1 we explain how to transform any self-adjoint system of differential equations with real coefficients to a canonical system, and briefly review the spectral theory of canonical systems. In Section 1.2 we give an introduction to de Branges theory of Hilbert spaces of entire functions, and define the Fourier transform and spectral measures in the new settings. The one-to-one correspondence between Hamiltonian with regular left endpoints, regular de Branges chains and regular measures are explained in Section 1.3. In Section 1.4 we show that the classical Fourier transform is a special case of the generalized Fourier transform.

1.1 Canonical systems with regular left endpoints

In this section we focus on canonical system, one special kind of self-adjoint system; yet any self-adjoint system with real coefficients can be represented by a canonical system. We will show how to do this momentarily. First let us give the definition of self-adjoint system and canonical system.

Definition 1.1. (i) A *self-adjoint system* of differential equations with real coefficients, or self-adjoint system for short, is given by

$$\Omega \dot{X}(t) = zH(t)X(t) - Q(t)X(t), \quad t \in I, \tag{1.1}$$

where I is an interval on $\mathbb{R}$, $\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $H(t)$, $Q(t) \in L^1_{\text{loc}}(I \to \mathbb{R}^{2\times 2})$ are symmetric and $H(t) \geqslant 0$ (i.e., $H(t)$ is positive-semidefinite).

(ii) A *canonical system* of differential equations, or canonical system for short, is a self-adjoint system where $Q \equiv 0$, i.e.,

$$\Omega \dot{X} = zHX, \quad t \in I, \tag{1.2}$$

and H is called the *Hamiltonian* (of the canonical system).

(iii) A real symmetric 2×2 matrix H is said to be a *Hamiltonian* on the interval I if $H \in L^1_{\text{loc}}(I)$ and $H(t) \geqslant 0$, $\forall t \in I$. A Hamiltonian H is called *normalized* if $\operatorname{tr}(H(t)) \equiv 1$, $\forall t \in I$.

Remark. We always assume there's no interval on which $H \equiv 0$ a.e. for the self-adjoint system (1.1) and canonical system (1.2), otherwise we can just delete the interval and make some trivial changes to the solutions.

We use h to denote some anti-derivative of H through out this dissertation, then the differential equation (1.2) can be re-written as an integral equation

$$\Omega X(b,z) - \Omega X(a,z) = z \int_a^b H(t)X(t,z)dt = z \int_a^b dh(t)X(t,z), \quad \forall t_- < a < b < t_+. \tag{1.3}$$

We will use the two forms of canonical systems interchangeably.

Any self-adjoint system can be transformed into a canonical system. For any self-adjoint system, let $V(t) \in SL(2,\mathbb{R})$ be the solution to the self-adjoint system with $z = 0$, i.e.,

$$\Omega \dot{V} = -QV, \tag{1.4}$$

s.t. $V(t_0) = I_2$ is the identity matrix for some $t_0 \in I$. Let $\tilde{H}(t) := V^*(t)H(t)V(t)$ and $X(t,z)$ be the solution to the canonical system

$$\Omega \dot{X} = z\tilde{H}X,$$

then $Y := VX$ solves the self-adjoint system.

Remark. As $V(t) \in SL(2,\mathbb{R})$, $V(t)^* = V(t)^T$.

Example 1.2. The Sturm-Liouville equation

$$-(p\dot{u})^{\cdot} + qu = zru, \quad t \in I$$

can be re-written in the form of self-adjoint system as

$$\Omega \dot{X} = z \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} X - \begin{pmatrix} q & 0 \\ 0 & -\frac{1}{p} \end{pmatrix} X,$$

where $X = \binom{u}{v}$ and $v = -p\dot{u}$.

From now on, we mainly consider canonical systems instead of general self-adjoint systems, and most of the results on canonical systems can be easily translated to results on self-adjoint systems. WLOG let's consider the case $I = [0, +\infty)$. When we write the interval I as a closed interval or half-closed interval, we assume it's locally integrable at the endpoints that are contained in I.

As H is locally integrable at the left endpoint 0, the solution to the canonical system (1.2) with given boundary values at $t = 0$ exists (using Picard's iterative method, see discussion below Theorem 1.21).

Now we consider the canonical equation with boundary condition at $t = b$:

$$\begin{aligned} &\Omega \dot{X} = zHX, \quad t \in [0, +\infty), \\ &\left\langle X(b,z), \begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix} \right\rangle = 0, \end{aligned} \tag{1.5}$$

where the inner product $\langle \cdot, \cdot \rangle$ is defined to be

$$\left\langle \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle = (u_1, u_2) \cdot \begin{pmatrix} \bar{v}_1 \\ \bar{v}_2 \end{pmatrix} = u_1\bar{v}_1 + u_2\bar{v}_2, \quad \forall u_1, u_2, v_1, v_2 \in \mathbb{C}.$$

Let $Y_1(t,z)$, $Y_2(t,z)$ be the solutions to the canonical system (1.2) with boundary values

$$Y_1(0,z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad Y_2(0,z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

For fixed nonreal z, any solution to the canonical system (1.2) must be a linear combination of Y_1 and Y_2. Suppose

$$\left\langle (Y_1(t,z) - l_b(z)Y_2(t,z)), \begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix} \right\rangle = 0, \quad \beta \in [0, \pi),$$

then

$$l_b(z) = \frac{Y_{11}(b,z)\cot\beta + Y_{12}(b,z)}{Y_{21}(b,z)\cot\beta + Y_{22}(b,z)}.$$

Replace $\cot\beta$ by a complex variable w, and define

$$l_b(z,w) = \frac{Y_{11}(b,z)w + Y_{12}(b,z)}{Y_{21}(b,z)w + Y_{22}(b,z)},$$

then for fixed nonreal z, $l_b(z,w)$ is a meromorphic function of w and maps the real line to a circle in $\mathbb{C}_+ \cup \mathbb{R}$, denoted by $\mathcal{C}_b$, which is known as the *Weyl circle* at $t = b$. $l_b(z, \cdot)$ also maps the closed upper half-plane $\mathbb{C}_+$ to $\mathcal{D}_b \subseteq \mathbb{C}_+ \cup \mathbb{R}$, the closed disk having $\mathcal{C}_b$ as its boundary, which is known as the *Weyl disk* at $t = b$.

Moreover, $\mathcal{D}_c \subseteq \mathcal{D}_b$ if $c > b$, therefore the limit of $\mathcal{D}_b$ as $b \to +\infty$ is either a circle or a point, and we call it the *limit-circle* or *limit-point*, respectively. For nonreal z, let $m(z)$ be the limit point or any point on the limit circle, then $\psi(t,z) := Y_1(t,z) - m(z)Y_2(t,z)$ is a solution to (1.2) for $t \in [0, +\infty)$. Such a solution is called a *Weyl solution*. The function $m(z)$ is analytic on $\mathbb{C}\backslash\mathbb{R}$ and $\overline{m(z)} = m(\bar{z})$

for nonreal z, and is known as the *Titchmarsh-Weyl m-function.* We state the well known results in this section without giving proofs. For proofs and a more detailed introduction to Titchmarsh-Weyl theory we refer the readers to [Tit62], [CL55], and [LS75].

The canonical system (1.2) can be thought as a formal differential operator $H^{-1}\Omega\frac{d}{dt}$ acting on X in the domain of the differential operator. We now define $L^2(H)$ for a Hamiltonian H on I and show it's indeed a Hilbert space. For that purpose we need to classify the points on I into H-ordinary points and H-special points.

Definition 1.3. Let $H(t)$, $t \in I$ be a Hamiltonian. A number $b \in I$ is said to be *H-special* if there are $a, c \in I$ s.t. $b \in (a,c)$ and

$$H(t) = s_t \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}, \quad \forall t \in (a,c),$$

where $s_t \in \mathbb{R}$ and $\theta \in [0,\pi)$ is a constant which does not depend on t. Such an interval (a,c) is said to be *H-indivisible with type θ*. $t \in I$ is called *H-ordinary* if it is not H-special.

Remark. (i) The above assumption on interval (a,c) is actually equivalent to the assumption $\operatorname{rank}(h(c) - h(a)) = 1$ where $h(t)$ is an anti-derivative of $H(t)$. The equivalence can be easily checked using basic linear algebra. It suffices to show that for two 2×2 positive semi-definite matrices A and B, $\det(A+B) = 0$ implies $\det A = \det B = 0$ and $\lambda A + \mu B$ for some $\lambda, \mu \in \mathbb{R}$.

(ii) An H-indivisible interval is called a *jump interval* in some of the literature. The H-ordinary/H-special points were called *regular/singular* by de Branges [dB68, Section 40].

The following definition of $L^2(H;I)$ was given by de Branges [dB68, Section 43]. The more general situation was considered by Kac[1] in [Kac50].

Definition 1.4. $L^2(H;I)$ consists of all pairs $\binom{f_1}{f_2}$ of complex Borel measurable functions of t defined on I, which are constants on each H-indivisible interval, s.t.

$$\left\| \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\|^2_{L^2(H;I)} = \int_I (f_1(t), f_2(t)) H(t) \begin{pmatrix} \overline{f_1(t)} \\ \overline{f_2(t)} \end{pmatrix} dt < \infty$$

with inner product defined to be

$$\left(\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right)_{L^2(H;I)} = \int_I (f_1(t), f_2(t)) H(t) \begin{pmatrix} \overline{g_1(t)} \\ \overline{g_2(t)} \end{pmatrix} dt, \quad \forall \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in L^2(H;I).$$

Remark. (i) We will denote $L^2(H;I)$ by $L^2(H)$ when there's no ambiguity.

[1]Due to inconsistency in translation, Kac and Kats in this dissertation actually refer to the same mathematician Israel Samoilovich Kats.

(ii) It's important to assume f_1 and f_2 are constants on H-indivisible intervals, otherwise the Parseval's identity of the generalized Fourier transform defined in Theorem 1.28 below breaks down.

de Branges [dB68, Theorem 43] showed $L^2(H)$ is a Hilbert space with the inner product defined above. The binary relation (X, Y) where $\Omega\dot{X} = zHY$ becomes a well-defined symmetric operator on subspaces of $L^2(H)$. It's possible to extend it to self-adjoint operators. For discussion along this direction we refer the readers to [Kat83], [Kat84], [Kat02], [HdSW00], and in particular [DM08] for Kreĭn's strings.

Now for the solution $Y_\alpha(t, z)$ (by abuse of notation) to the canonical system (1.2) s.t.

$$Y_\alpha(0, z) = \begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix}, \quad \alpha \in [0, 2\pi), \tag{1.6}$$

there exists a positive measure μ_α on $\mathbb{R}$, s.t.

$$\left\| \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\|_{L^2(H)} = \|F\|_{L^2(\mu)}, \tag{1.7}$$

where F is the limit of

$$F_n(z) := \left(\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \overline{Y_\alpha(\cdot, z)} \right)_{L^2(H;[0,n])} = \int_0^n (f_1(t), f_2(t))H(t)Y_\alpha(t, z)dt$$

in $L^2(\mu_\alpha)$ as $n \to +\infty$.

The measure μ_α is usually referred to as a *spectral measure* of the canonical system (1.2) with boundary value (1.6), and the equality (1.7) is known as the *Parseval's equality*. The transform

$$\mathcal{F} : \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \mapsto F$$

is known as the *generalized Fourier transform*, or the *Titchmarsh-Weyl-Fourier transform*, or the *Fourier transform* for short.

Moreover, the spectral measure μ_α is closely related to the Titchmarsh-Weyl m-function. In particular, if we choose $\alpha = 0$, then $Y(t, z) = Y_2(t, z)$, and we have

$$m(z) = a + bz + \int_{-\infty}^{+\infty} \left(\frac{1}{z - \lambda} - \frac{\lambda}{1 + \lambda^2} \right) d\mu_0(\lambda) \tag{1.8}$$

for some $a \in \mathbb{R}$ and $b > 0$, and

$$\mu_0(b) - \mu_0(a) = \frac{1}{\pi} \lim_{y \to 0+} \int_a^b \Im m(x + iy)dx. \tag{1.9}$$

We'll continue the discussion on the spectral theory on canonical systems in Section 2.1.

1.2 de Branges chains

de Branges theory is naturally involved because for any Hamiltonian with a regular endpoint, the solution to the canonical equation (1.2) forms a chain of de Branges functions and consequently, a chain of de Branges spaces. On the other hand, once there exists a chain of de Branges spaces, we can always construct the Fourier transform using a corresponding chain of dB-functions, as shown in Theorem 1.28. However, the Fourier transform and the spectral measures exist regardless of whether the Hamiltonian of the chain of de Branges functions has a regular left endpoint or not.

In Section 1.2.1 we'll give the definitions of de Branges functions and de Branges spaces, and present some basic results on them. In Section 1.2.2 we introduce the Nevanlinna matrices and use them to define de Branges chains (dB-chains). In Section 1.2.3 we show how to construct a dB-chain for a given Hamiltonian with a regular left endpoint, and conversely, how a dB-chain uniquely determines the associated Hamiltonian. In Section 1.2.4 we give the definition of a spectral measure of a chain of dB-spaces, and investigate the structure of the chain based on whether $t \in I$ is H-ordinary or H-special. In Section 1.2.5 we define the Fourier transform from $L^2(H)$ to $L^2(\mu)$ explicitly for H associated with a dB-chain $\{\mathcal{B}(E_t)\}_{t\in I}$.

1.2.1 de Branges spaces and functions

For a function F defined on $\mathrm{dom}(F) \subseteq \mathbb{C}$, we define $F^{\#}(z) := \overline{F(\bar{z})}$ for z s.t. $\bar{z} \in \mathrm{dom}(F)$. A function F is called real entire if it's entire and $F^{\#} = F$. Namely, $F(x) \in \mathbb{R}$ for $x \in \mathbb{R}$. For a function F, we use $Z(F)$ to denote the set of zeros of F.

Definition 1.5. The *de Branges class dB* (a.k.a. *Hermite-Biehler class* $\mathcal{HB}$) of entire functions is defined as the set of entire functions E s.t. $|E(z)| > 0$ for $z \in \mathbb{C}_+$, and

$$|E^{\#}(z)| \leqslant |E(z)|, \quad \forall z \in \mathbb{C}_+. \tag{1.10}$$

$E \in dB$ is called a *de Branges function*, or a *dB-function* for short.

$E \in dB$ is called *strict* if it doesn't have real zeros.

$E \in dB$ is called *degenerate* if $|E^{\#}(z)| = |E(z)|$ for some (then all) $z \in \mathbb{C}_+$.

$E \in dB$ is called *normalized* if $E(0) = 1$.

Remark. We don't use strict inequality in (1.10) because we want dB to be closed, namely, if $E_n \in dB \to E$ locally uniformly and $E \not\equiv 0$, then $E \in dB$. More discussion on de Branges functions can be found in [Lev64, Chapter VII].

Through out this dissertation, $E \in dB$ is denoted by $E = A - iC$, where $A = \frac{E^{\#}+E}{2}$ and $C = \frac{E^{\#}-E}{2i}$ are both real entire. This is motivated by the Hermite-Biehler Theorem (cf. [Her05]), which states that for a polynomial $E = P - iQ$ where P, Q are polynomials with only real zeros, the following are equivalent:

(i) $E \in dB$,

(ii) $Z(E) \subseteq \mathbb{C}_-$,

(iii) Zeros of P and Q are simple and interlacing, and $Q'P - QP' > 0$ at some $x_0 \in \mathbb{R}$.

Example 1.6. (i) Any polynomial P with $Z(P) \subseteq \mathbb{C}_- \cup \mathbb{R}$ is a dB-function. In particular, P is non-degenerate if $Z(P) \subseteq \mathbb{C}_-$.

(ii) e^{-iaz} is a strict non-degenerate dB-function for $a > 0$.

Definition 1.7. A Hilbert space $(\mathcal{B}, (\cdot,\cdot))$ of entire functions is called a *de Branges space*, or a *dB-space* for short, if it satisfies the following axioms:

(H1) If $F \in \mathcal{B}$ and $F(w) = 0$ for some nonreal w, then $\frac{z-\bar{w}}{z-w}F(z) \in \mathcal{B}$ and

$$\left(\frac{z-\bar{w}}{z-w}F(z), \frac{z-\bar{w}}{z-w}G(z)\right) = (F, G), \quad \text{if } F, G \in \mathcal{B}, F(w) = G(w) = 0,$$

(H2) The point evaluation $F \mapsto F(w)$ is a continuous linear functional on $\mathcal{B}$, for all nonreal w,

(H3) If $F \in \mathcal{B}$, then $F^{\#} \in \mathcal{B}$ and

$$(F^{\#}, G^{\#}) = (G, F), \qquad \text{for } F, G \in \mathcal{B}.$$

A *nonzero* dB-space is a dB-space that contains at least one nonzero element.

Remark. For the inner product $(\cdot,\cdot)$, we assume it's linear in the first argument and conjugate linear in the second argument.

de Branges introduced such Hilbert spaces as he tried to generalize the identity

$$\int_{-\infty}^{+\infty} |F(t)|^2 dt = \frac{\pi}{a} \sum_{n=-\infty}^{+\infty} \left| F\left(\frac{n\pi}{a}\right) \right|$$

where F is an entire function of exponential type at most $a > 0$ that is also square integrable on the real line. de Branges proved that a similar summation formula, namely the sampling formula (2.25), holds for functions in a dB-space (cf. [dB68, Theorem 22]), which is much more general, as we shall seen later in Section 2.4.1.

dB-spaces are closely related to dB-functions and their relation is described by the following proposition (cf. [dB68, Theorem 23]):

Proposition 1.8. *(i) For a* non-degenerate *dB-function E, let*

$$\mathcal{B}(E) := \left\{ F \in \mathcal{A}(\mathbb{C}) : \frac{F}{E}, \frac{F^{\#}}{E} \in H^2(\mathbb{C}_+) \right\}$$

with inner product

$$(F, G)_{\mathcal{B}(E)} = \int_{-\infty}^{+\infty} \frac{F(t)\overline{G(t)}}{|E(t)|^2} dt,$$

then $\mathcal{B}(E)$ is a nonzero dB-space, and the reproducing kernel of $\mathcal{B}(E)$ at w is given by

$$K_w(z) = \frac{E(\bar{w})E^{\#}(z) - E(z)E^{\#}(\bar{w})}{2\pi i(z - \bar{w})} = \frac{\overline{A(w)}C(z) - A(z)\overline{C(w)}}{\pi(z - \bar{w})}. \tag{1.11}$$

(ii) For a non-degenerate dB-function E, a necessary and sufficient condition that an entire function F belongs to $\mathcal{B}(E)$ is that

$$\|F(t)\|_{\mathcal{B}(E)}^2 = \int_{-\infty}^{+\infty} \left| \frac{F(t)}{E(t)} \right|^2 dt < \infty$$

and that $|F(z)|^2 \leqslant \|F(t)\|_{\mathcal{B}(E)}^2 K_z(z)$ for all $z \in \mathbb{C}$.

(iii) For any nonzero dB-space $\mathcal{B}$, there exists a non-degenerate dB-function E s.t. $\mathcal{B} = \mathcal{B}(E)$.

Remark. Note that when we write $\mathcal{B}(E)$, we always assume E is a *non-degenerate* function.

We're mostly interested in strict non-degenerate dB-functions, as $\mathcal{B}(E)$ is only defined for non-degenerate dB-functions and any dB-space $\mathcal{B}(E)$ can be isometrically transformed to $\mathcal{B}(E_0)$ where E_0 is a *strict* dB-function, as stated in the following proposition.

Proposition 1.9. *If $\mathcal{B}(E)$ is a nonzero dB-space, then $E(z) = S(z)E_0(z)$ where $E_0(z)$ is strict and non-degenerate, and $S(z)$ is real entire. $F(z) \to S(z)F(z)$ is an isometric transformation of $\mathcal{B}(E_0)$ onto $\mathcal{B}(E)$.*

Proof. This can be easily verified by Proposition 1.8(i). ☐

For a given non-degenerate dB-function E, by definition it uniquely determines a dB-space $\mathcal{B}(E)$. On the other hand, however, it's possible to have two different non-degenerate dB-functions E_1 and E_2, s.t. $\mathcal{B}(E_1) = \mathcal{B}(E_2)$. Although $E_1 \neq E_2$, they're still closely related to each other.

Proposition 1.10. *Let $E_1 = A_1 - iC_1$, $E_2 = A_2 - iC_2$ be non-degenerate dB-functions, then the following are equivalent:*

(i) $\mathcal{B}(E_1) = \mathcal{B}(E_2)$,

(ii) $\exists V \in SL(2, \mathbb{R})$, s.t. $\binom{A_1}{C_1} = V \binom{A_2}{C_2}$.

Proof. See Section 1.A. □

Here are some examples of dB-spaces:

Example 1.11. (i) Let E_n be a polynomial of degree $n \geqslant 1$, $Z(E_n) \subseteq \mathbb{C}_-$, then E_n is a strict non-degenerate dB-function. Obviously the set of all polynomials of degree at most $n-1$ is contained in $\mathcal{B}(E_n)$. On the other hand, if $F \in \mathcal{B}(E_n)$, we want to show F is a polynomial of degree at most $n-1$.

First note that by (1.11), the reproducing kernel $K_w(z)$ is a polynomial of $\bar{w}$ and z which consists of terms $P_k(\bar{w})(z-\bar{w})^k$ where $k = 0, \cdots, n-1$ and $\deg P_k \leqslant 2n-1$. Let $w = \bar{z}$, then $K_z(z)$ is bounded by $M|z|^{2n-1}$ for $|z| > 1$, for some $M > 0$. Since $|F(z)| \leqslant \|F\|_{\mathcal{B}(E_n)} \sqrt{K_z(z)}$, we get $|F(z)| \leqslant \tilde{M}|z|^{n-\frac{1}{2}}$, for some $\tilde{M} > 0$. Then by Cauchy's inequality for analytic function we know F is a polynomial of degree at most $n-1$. Therefore $\mathcal{B}(E_n)$ is equal to the set of polynomials of degree at most $n-1$ as sets.

(ii) Paley-Wiener space PW_a $(a > 0)$ is the set of entire functions of exponential type at most a, with inner product of $L^2(dt)$. Paley-Wiener Theorem (cf. [PW34]) says PW_a is the Fourier transform of functions in $L^2(dt)$ which vanish outside of the interval $[-a, a]$. We claim $PW_a = \mathcal{B}(E_a)$, where $E_a = e^{-iaz}$, since the two spaces have the same reproducing kernel

$$K_w(z) = \frac{\sin(az - a\bar{w})}{\pi(z-\bar{w})} = \frac{\overline{\cos(aw)}\sin(az) - \cos(az)\overline{\sin(aw)}}{\pi(z-\bar{w})} = \frac{\overline{A_a(w)}C_a(z) - A_a(z)\overline{C_a(w)}}{\pi(z-\bar{w})},$$

where $A_a(z) := \cos(az)$ and $C_a(z) := \sin(az)$. For a complete proof please refer to [dB68, Theorem 16].

(iii) Let's consider $\mathcal{B}(E) = PW_1$. Since dB-space is invariant under multiplication by $SL(2,\mathbb{R})$ on $\binom{A}{C} = \binom{\cos z}{\sin z}$. For $k > 1$, let

$$\begin{pmatrix} A_k \\ C_k \end{pmatrix} = \begin{pmatrix} \frac{1}{k} & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} \frac{A}{k} \\ kC \end{pmatrix},$$

then $\forall F \in \mathcal{B}(E)$,

$$\|F\|^2_{\mathcal{B}(E)} = \|F\|^2_{\mathcal{B}(E_k)} = \int_{-\infty}^{+\infty} \frac{|F(t)|^2 dt}{\frac{\cos^2 t}{k^2} + k^2 \sin^2 t} = \int_{-\infty}^{+\infty} \frac{|F(t)|^2 k^2 dt}{\cos^2 t + k^4 \sin^2 t}. \tag{1.12}$$

Let $t_n = n\pi$ for some even n, then for small $\epsilon > 0$,

$$\begin{aligned}\int_{t_n-\epsilon}^{t_n+\epsilon} \frac{k^2 dt}{\cos^2 t + k^4 \sin^2 t} &= \int_{-\sin\epsilon}^{\sin\epsilon} \frac{k^2 dx}{(1+(k^4-1)x^2)\sqrt{1-x^2}} \quad (x = \sin t) \\ &= \frac{k^2}{\sqrt{k^4-1}} \int_{-\sqrt{k^4-1}\sin\epsilon}^{\sqrt{k^4-1}\sin\epsilon} \frac{ds}{1+s^2}(1+o(\epsilon)) \quad \left(s = \sqrt{k^4-1}x\right) \\ &\to \pi(1+o(\epsilon))\end{aligned}$$

as $k \to +\infty$. This holds for odd n as well. Since $\frac{k^2}{\cos^2 t + k^4 \sin^2 t}$ goes to 0 except at zeros of $\sin(t)$, the limit measure of $\frac{k^2 dt}{\cos^2 t + k^4 \sin^2 t}$ is $\pi \sum_{t_n \in Z(\sin)} \delta_{t_n}$ where δ_{t_n} is the Dirac measure at t_n. Then letting $k \to +\infty$ in (1.12) we can get a summation formula:

$$\int_{-\infty}^{+\infty} |F(t)|^2 dt = \|F\|^2_{\mathcal{B}(E)} = \pi \sum_{n=-\infty}^{+\infty} |F(n\pi)|^2.$$

If we use $\begin{pmatrix} k & 0 \\ 0 & \frac{1}{k} \end{pmatrix}$ instead we can get another summation formula:

$$\int_{-\infty}^{+\infty} |F(t)|^2 dt = \|F\|^2_{\mathcal{B}(E)} = \pi \sum_{n=-\infty}^{+\infty} \left| F\left(\frac{\pi}{2} + n\pi\right)\right|^2.$$

1.2.2 Chains of de Branges spaces and functions

One of the most intriguing result in de Branges theory is that for any dB-space, we are able to extrapolate a chain of dB-spaces, ordered by "almost isometric" inclusion (cf. [dB61b, Theorem III]), and the corresponding chain of generating dB-functions defines a canonical system via its Hamiltonian H. On the other hand, the solution to the canonical equation (1.2) actually forms a chain of dB-functions indexed by t, if t_- is a regular left endpoint of the Hamiltonian H and the spaces generated by the dB-functions are ordered by "almost isometric" inclusion. In this section we give the definitions of chains of de Branges functions and spaces. Moreover, de Branges' ordering theorem (cf. [dB68, Theorem 35]) is presented as well, which shows that a family of dB-spaces can be totally ordered under certain assumptions.

de Branges' ordering theorem uses the notion of functions of bounded type. An analytic function F defined on a simply connected region Λ is said to be *of bounded type*, or in the*Nevanlinna class* $\mathcal{N}(\Lambda)$, if $F(z) = \frac{P(z)}{Q(z)}$ for bounded analytic functions P and Q on Λ. An equivalent statement is $\log^+|F(z)|$ has a harmonic majorant on Ω. More information regarding functions of bounded type can be found in [dB68, Chapter 1] and [Gar07, Chapter II].

Theorem 1.12 (Ordering Theorem)**.** *Let $\mathcal{B}(E_a)$ and $\mathcal{B}(E_b)$ be nonzero dB-spaces which are contained isometrically in $L^2(\mu)$ for some positive measure μ on $\mathbb{R}$. If $\frac{E_b}{E_a} \in \mathcal{N}(\mathbb{C}_+)$ and $\frac{E_b}{E_a}$ has no real* *zeros or real poles, then either* $\mathcal{B}(E_a) \sqsubseteq \mathcal{B}(E_b)$ *or* $\mathcal{B}(E_b) \sqsubseteq \mathcal{B}(E_a)$.

The Ordering Theorem says if a family of dB-spaces sits isometrically in $L^2(\mu)$, then they're totally ordered under some technical assumptions. It's generalized from a theorem of Kreĭn [Kre53], where Kreĭn proved a special case of the ordering theorem for $\mathcal{B}(E)$ s.t. $E^{\#}(z) = E(-z)$. The ordering theorem is arguably one of the most important results of de Branges theory, which makes chains of dB-spaces of great importance. Before we give the formal definitions of chains of dB-functions/spaces, we first study the case when $\mathcal{B}(E_a)$ sits "almost isometrically" in $\mathcal{B}(E_b)$, where Nevanlinna matrices play an important role.

Definition 1.13. A matrix of real entire functions $M(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}$ is said to be a *Nevanlinna matrix* if $\det M(z) = 1$, $\forall z \in \mathbb{C}$, and

$$\frac{M^*(z)\Omega M(z) - \Omega}{z - \bar{z}} \geqslant 0, \quad \forall z \in \mathbb{C}. \tag{1.13}$$

A Nevanlinna matrix M is said to be *normalized* if $M(0) = I_2$, the 2×2 identity matrix.

Remark. Let $J := i\Omega$, we say a $M \in SL(2, \mathbb{C})$ is J*-expansive* (or J*-unitary*, J*-contractive*, respectively) if $M^*JM \geqslant J$ (or $M^*JM = J$, $M^*JM \leqslant J$, respectively). The definition of Nevanlinna matrix above is equivalent to say $M(z)$ is J-contractive for $z \in \mathbb{C}_+$, J-unitary for $z \in \mathbb{R}$ and J-expansive for $z \in \mathbb{C}_-$. Such matrices are also said to be J*-inner*. These properties have been studied extensively in Arov and Dym's monograph [AD08]. It's easy to verify

$$M \text{ is } J\text{−contractive} \Leftrightarrow M^{-1} \text{ is } J\text{−expansive} \Leftrightarrow \bar{M} \text{ is } J\text{−expansive} \Leftrightarrow M^T \text{ is } J\text{−expansive}. \tag{1.14}$$

For a normalized Nevanlinna matrix M, we define $t\,(M) = \operatorname{tr}(\Omega M'(0))$. We can get a useful inequality using $t(M)$, which leads to a normality condition for a family of Nevanlinna matrices.

Proposition 1.14. *Let M be a normalized Nevanlinna matrix, then $\Omega M'(0) \geqslant 0$, $t\,(M) \geqslant 0$, and*

$$1 + \|M(z) - I_2\|_F \leqslant e^{t(M)|z|}, \quad \forall z \in \mathbb{C},$$

where $\|\cdot\|_F$ is the Hilbert-Schmidt norm (Frobenious norm) of a matrix.

Proof. See Section 1.B. □

For a Nevanlinna matrix M, let $E := A - iC$, $\tilde{E} := B - iD$, then both E and $\tilde{E}$ are dB-functions, although they might be degenerate. Actually, E, $\tilde{E}$ are in a special class of dB-functions, namely the set of regular dB-functions (cf. Proposition 1.31) which will be discussed in detail in Section 1.3. From now on, when we refer to a Nevanlinna matrix M, we always denote it by $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, and define $E = A - iC$, $\tilde{E} = B - iD$. Moreover, just like dB-spaces can be constructed from dB-functions, we can construct Hilbert spaces using Nevanlinna matrices. The following theorems (cf. [dB68, Theorem 28,33,34]) were given by de Branges, where the *Bezout operator/Bezoutian* $T_{S,\alpha}$ is defined to be

$$\left(T_{S,\alpha}F\right)(z) = \frac{F(z)S(\alpha) - F(\alpha)S(z)}{z-\alpha} \tag{1.15}$$

for function S, F and complex numbers α and z.

Remark. We define

$$\mathrm{Assoc}\mathcal{B} := \left\{S \in \mathcal{A}(\mathbb{C}) : T_{S,\alpha}(F) \in \mathcal{B}, \quad \forall F \in \mathcal{B}, \quad \forall \alpha \in \mathbb{C}\right\}.$$

Namely, for $S \in \mathrm{Assoc}\mathcal{B}$, $T_{S,\alpha}$ acts on $\mathcal{B}$. Functions $S \in \mathrm{Assoc}\mathcal{B}$ are said to be *associated with* the dB-space $\mathcal{B}$ and has been studied extensively by Trutt and de Branges [dB68, Section 25-28]. We will occasionally use this notion and relevant results [dB68, Theorem 25-28] to simplify some of our proofs.

Theorem 1.15. *Let* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *be a Nevanlinna matrix s.t.* $E = A - iC \neq 0$ *on* $\mathbb{C}_+$, *then there exists a unique Hilbert space* $\mathcal{B}(M)$, *whose elements are pairs* $\binom{F_+}{F_-}$ *of entire functions, s.t.*

$$\left(\frac{M^*(w)\Omega M(z) - \Omega}{2\pi(z-\bar{w})}\right)^T \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{B}(M), \quad \forall u, v, w \in \mathbb{C} \tag{1.16}$$

and

$$\begin{pmatrix} u \\ v \end{pmatrix}^* \begin{pmatrix} F_+(w) \\ F_-(w) \end{pmatrix} = \left\langle \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix}, \left(\frac{M^*(w)\Omega M(t) - \Omega}{2\pi(t-\bar{w})}\right)^T \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{B}(M) \right\rangle, \quad \forall \begin{pmatrix} F_+ \\ F_- \end{pmatrix} \in \mathcal{B}(M)$$

and

$$\begin{aligned} 2\pi\left(F_+(\alpha), F_-(\alpha)\right)\Omega\overline{\begin{pmatrix} G_+(\beta) \\ G_-(\beta) \end{pmatrix}} &= \left\langle \begin{pmatrix} S(\alpha)F_+ \\ S(\alpha)F_- \end{pmatrix}, \begin{pmatrix} T_{S,\alpha}G_+ \\ T_{S,\alpha}G_- \end{pmatrix} \right\rangle - \left\langle \begin{pmatrix} T_{S,\alpha}F_+ \\ T_{S,\alpha}F_- \end{pmatrix}, \begin{pmatrix} S(\alpha)G_+ \\ S(\alpha)G_- \end{pmatrix} \right\rangle \\ &\quad + (\alpha - \bar{\beta})\left\langle \begin{pmatrix} T_{S,\alpha}F_+ \\ T_{S,\alpha}F_- \end{pmatrix}, \begin{pmatrix} T_{S,\alpha}G_+ \\ T_{S,\alpha}G_- \end{pmatrix} \right\rangle. \end{aligned}$$

Proposition 1.16. *Let* M *be a Nevanlinna matrix, then the following are equivalent:*

(i) $\mathcal{B}(M) = \{0\}$,

(ii) M *is a constant matrix,*

(iii) E *and* $\tilde{E}$ *are degenerate dB-functions.*

Proof. This comes from de Branges' construction of $\mathcal{B}(M)$ [dB68, Theorem 27,28]. ∎

Nevanlinna matrices enter de Branges theory because they're closely related to chain of dB-spaces and the Hamiltonian. Actually, they are the key to relating two dB-spaces if one sits inside the other one, as can be seen from the following theorem (cf. [dB68, Theorem 33,34]).

Theorem 1.17. *(i) Let $\mathcal{B}(E_a)$ and $\mathcal{B}(E_b)$ be two dB-spaces s.t. $\mathcal{B}(E_a) \sqsubseteq \mathcal{B}(E_b)$, and $\frac{E_a}{E_b}$ has no real zeros, then $\frac{E_a}{E_b}$ also has no real poles and there exists a unique Nevanlinna matrix $M_{a\to b}$ s.t.*

$$\begin{pmatrix} A_b \\ C_b \end{pmatrix} = M_{a\to b} \begin{pmatrix} A_a \\ C_a \end{pmatrix},$$

and the transformation

$$\begin{pmatrix} F_+ \\ F_- \end{pmatrix} \mapsto \sqrt{2}\,(A_a F_+ + C_a F_-)$$

takes $\mathcal{B}(M_{a\to b})$ isometrically onto $\mathcal{B}(E_b) \ominus \mathcal{B}(E_a)$.

(ii) Let $M_{a\to b}$ be a Nevanlinna matrix. Let $E_a = A_a - iC_a$ be a non-degenerate dB-function. If

$$\begin{pmatrix} A_b \\ C_b \end{pmatrix} = M_{a\to b} \begin{pmatrix} A_a \\ C_a \end{pmatrix},$$

then $E_b = A_b - iC_b$ is a non-degenerate dB-function and $\frac{E_a}{E_b}$ has no real zeros or poles. $\mathcal{B}(E_a) \subseteq \mathcal{B}(E_b)$ as sets and $\|F\|_{\mathcal{B}(E_a)} \geqslant \|F\|_{\mathcal{B}(E_b)}$, $\forall F \in \mathcal{B}(E_a)$. If there is no nonzero constant $\binom{u}{v}$ in $\mathcal{B}(M_{a\to b})$ s.t. $uA_a + vC_a \in \mathcal{B}(E_a)$, then $\mathcal{B}(E_a) \sqsubseteq \mathcal{B}(E_b)$ and the transformation

$$\begin{pmatrix} F_+ \\ F_- \end{pmatrix} \mapsto \sqrt{2}\,[A_a F_+ + C_a F_-]$$

takes $\mathcal{B}(M_{a\to b})$ isometrically onto $\mathcal{B}(E_b) \ominus \mathcal{B}(E_a)$.

Based on this theorem, we're now ready to give the definition of dB-chains.

Definition 1.18. (i) A family of non-degenerate dB-functions $\{E_t\}_{t\in I}$ ($I = (t_-, t_+)$, or $(t_-, t_+]$) is called a *chain of dB-functions* if there exist normalized non-constant Nevanlinna matrices $(M_{a\to b})_{t_-<a<b<t_+}$ s.t. $t(M_{a\to b})$ is continuous for $a, b \in I$, and

$$\begin{pmatrix} A_b \\ C_b \end{pmatrix} = M_{a\to b} \begin{pmatrix} A_a \\ C_a \end{pmatrix}, \quad \forall t_- < a < b < t_+, \tag{1.17}$$

and for $K_{t,z}$ defined by (1.11),

$$\lim_{t\to t_-} K_{t,z}(z) = 0, \quad \forall z \in \mathbb{C}. \tag{1.18}$$

Such unique matrices $M_{a\to b}$, $a < b \in I$ are called the *transition matrices of* the chain $\{E_t\}$. A chain $\{E_t\}_{t\in I}$ is called *normalized* if $t(M_{a\to b}) = b - a$, $\forall t_- < a < b < t_+$.

(ii) We say a dB-space $\mathcal{B}_a$ sits *almost isometrically*, or *a.i.* for short, in another dB-space $\mathcal{B}_b$, if $\mathcal{B}_a$ sits contractively in $\mathcal{B}_b$ and $\overline{\mathrm{dom}_{\mathcal{B}_a}(z)}$ sits isometrically in $\mathcal{B}_b$, where $\mathrm{dom}_{\mathcal{B}_a}(z)$ is the domain of multiplication by z in $\mathcal{B}_a$. We say $\mathcal{B}$ sits *almost isometrically* in $L^2(\mu)$ if $\overline{\mathrm{dom}_{\mathcal{B}}(z)}$ sits isometrically in $L^2(\mu)$ and $\mathrm{dom}_{\mathcal{B}}(z)^{\perp}$ sits contractively in $L^2(\mu)$.

(iii) A family of nonzero dB-spaces $\{\mathcal{B}_t\}$, $t \in I$, ($I = (t_-, t_+)$, or $(t_-, t_+]$) is called a *chain of dB-spaces* if:

- $\forall a < b \in I$, $\mathcal{B}_a \neq \mathcal{B}_b$ as dB-spaces, $\mathcal{B}_a$ sits a.i. in $\mathcal{B}_b$,
- $\|F\|_{\mathcal{B}_t}$ is a *continuous* function of $t \geqslant a$, for $F \in \mathcal{B}_a$, $\forall a \in I$,
- The reproducing kernels $K_{t,z}$ satisfy

$$\lim_{t \to t_-} K_{t,z}(z) = 0, \quad \forall z \in \mathbb{C}. \tag{1.19}$$

Remark. (i) The transition matrices $\{M_{a\to b}\}$ are unique by Theorem 1.17.

(ii) We assume $M_{a\to b}$ is non-constant in (i) and $\mathcal{B}_a \neq \mathcal{B}_b$, otherwise we could cut off the trivial part of the chain to satisfy the constraints. This is analogous to removing intervals where the Hamiltonian $H = 0$ almost everywhere.

(iii) The case $I = (t_+, t_-]$ means there's a largest (largest as sets) dB-space in the chain of dB-functions/dB-spaces. Since we can always add a tail to a chain, namely, we can define

$$\begin{pmatrix} A_t \\ C_t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ (t - t_+)\, z & 1 \end{pmatrix} \begin{pmatrix} A_{t_+} \\ C_{t_+} \end{pmatrix}, \quad \forall t > t_+,$$

then we can extend the chain beyond t_+. Therefore, we can always assume $I = (t_-, t_+)$ is an open interval unless otherwise stated.

(iv) The assumption (1.18) excludes the possibility that $E_{t_-} := \lim_{t\to t_-} E_t$ exists and is a non-degenerate dB-function. Therefore, the assumption (1.18) means the chain is "saturated" at its left endpoint. Namely, one can not extend the chain beyond the left endpoint t_-.

Actually, (1.18) can be implied by a seemingly weaker condition, that

$$\lim_{t \to t_-} K_{t,z}(z) = 0, \quad \text{for some } z \in \mathbb{C}\backslash\mathbb{R}.$$

The proof can be found in the proof of [dB68, Theorem 40]. In short, $K_z(z) = 0$ for some nonreal z means the dB-space is zero, and therefore $K_z(z) = 0$ for all $z \in \mathbb{C}$.

(v) The assumption $t(M_{a\to b})$ is continuous in a and b implies $E_t(z)$ is continuous in t for a given $z \in \mathbb{C}$. Similarly, for a chain of dB-spaces, we assume that $\|F\|_{\mathcal{B}_t}$ is continuous in $t \geqslant a$ for

$$\begin{tikzcd}
& \{M_{a\to b}\} \arrow[dl, dashed, shift left] \arrow[d, dashed, shift left] \arrow[dr, dashed, shift left, "V"] & \\
\{E_t\} \arrow[ur, Rightarrow] \arrow[r, Rightarrow, shift left] & \{\mathcal{B}(E_t)\} \arrow[l, dashed, shift left] \arrow[u, Rightarrow] \arrow[r, dashed, shift left] & \{\mathcal{B}_t\} \arrow[l, dashed, shift left, "V"] \arrow[ul, dashed, shift left]
\end{tikzcd}$$

Figure 1.1: Uniqueness diagram for chains of dB-functions/spaces

$F \in \mathcal{B}_a$ to exclude the possibility that $\mathcal{B}_t$ has a "jump." Without these two assumptions, we can "chop off" part of the chain and still get another chain. We add the assumptions here because, as mentioned above, we want the chain to be "saturated," namely any dB-subspace of $\mathcal{B}_t$ is equal to $\mathcal{B}_s$ for some $s \leqslant t$, as we will see in Theorem 1.27.

(vi) The structure of the chain of dB-spaces and the meaning of "saturated" chain will be further explained in Theorem 1.27 in Section 1.2.

By Theorem 1.17 we know if $\{E_t\}_{t\in I}$ is a chain of dB-functions, then $\{\mathcal{B}(E_t)\}_{t\in I}$ is a chain of dB-spaces. Conversely, for a chain of dB-spaces $\{\mathcal{B}(E_t)\}_{t\in I}$, we can choose E_t s.t. $\mathcal{B}_t = \mathcal{B}(E_t)$, $\forall t \in I$ and $\{E_t\}_{t\in I}$ is a chain of dB-functions.

Now we discuss the correspondence between chains of dB-functions and chains of dB-spaces. Let $\{E_t\}_{t\in I}$ be a chain of dB-functions, then by Theorem 1.17, $\{\mathcal{B}(E_t)\}_{t\in I}$ is a chain of dB-spaces and is uniquely determined by $\{E_t\}$. On the other hand, given a chain of dB-spaces, as by Proposition 1.10, the representation of a dB-space by dB-functions is only unique up to a $SL(2,\mathbb{R})$ transform on $\binom{A}{C}$. Then for $t_- < a < b < t_+$, we can find non-degenerate dB-functions E_a, E_b, and a Nevanlinna matrix $M_{a\to b}$ s.t.

$$\begin{pmatrix} A_b \\ C_b \end{pmatrix} = M_{a\to b}\begin{pmatrix} A_a \\ C_a \end{pmatrix},$$

where E_b and E_a are unique up to $SL(2,\mathbb{R})$ transforms V_b on $\binom{A_b}{C_b}$ and V_a on $\binom{A_a}{C_a}$, respectively. If we require $M_{a\to b}$ to be normalized, i.e., $M_{a\to b}(0) = I_2$, then $V_a = V_b$. Therefore, given a chain of dB-spaces, it determines a chain of dB-functions which is *unique up to a map*: $\binom{A_t}{C_t} \mapsto V\binom{A_t}{C_t}$ for $V \in SL(2,\mathbb{R})$. Consequently, for a chain of dB-spaces $\{\mathcal{B}_t\}$, the transition matrices of the chains of dB-functions $\{E_t\}$ s.t. $\mathcal{B}(E_t) = \mathcal{B}_t$, are *unique up to a map*: $M_{a\to b} \mapsto V^{-1}M_{a\to b}V$ for $V \in SL(2,\mathbb{R})$.

From now on, we'll use the term *de Branges chain* or *dB-chain* for short, denoted by $\{\mathcal{B}(E_t)\}_{t\in I}$, to denote a chain of dB-spaces formed by a prescribed chain of dB-functions $\{E_t\}_{t\in I}$. Two dB-chains are said to be equal if and only if they're formed by the same chain of dB-functions. On the other hand, as discussed earlier, two different dB-chains might be equal to each other *as chains of dB-spaces*.

Figure 1.1 summarizes the uniqueness relation between chains of dB-functions and spaces, where solid arrow means uniqueness and dashed arrow means non-uniqueness.

It is convenient to consider dB-chain $\{\mathcal{B}(E_t)\}_{t\in I}$ s.t. E_t is a normalized strict non-degenerate dB-function, as we can use Proposition 1.9 and Proposition 1.10 to transform any dB-chain so that E_t satisfies those requirements. We should point out that most results on dB-chain in later chapters assume E_t is normalized, strict, and non-degenerate. Moreover, if E_t is strict for *some* $t \in I$, then E_t is strict for *any* $t \in I$. This can be seen from Theorem 1.17(ii). Therefore, we can give the following definition.

Definition 1.19. A dB-chain $\{\mathcal{B}(E_t)\}_{t\in I}$ is said to be *strict* if E_t is strict, for some (hence for all) $t \in I$.

1.2.3 Hamiltonian of a dB-chain

In this section we'll focus on the relation between a Hamiltonian and a dB-chain. Firstly, given a Hamiltonian $H(t)$, $t \in I$ with a regular left endpoint, namely $\int_{t_-}^{c} H(t)dt$ has finite elements for $c \in I$, there always exists a dB-chain $\{\mathcal{B}(E_t)\}_{t\in I}$ having H as its Hamiltonian as defined below, and such a dB-chain is not unique (cf. Theorem 1.21). Secondly, given a dB-chain $\{\mathcal{B}(E_t)\}$, there exists a unique Hamiltonian $H(t)$ s.t. (1.20) holds (cf. Proposition 1.23). In Section 2.3 we'll give a sufficient and necessary condition for a Hamiltonian to be the Hamiltonian of some dB-chain.

Definition 1.20. We say a Hamiltonian $H(t)$, $t \in I$ is the *Hamiltonian of dB-chain* $\{\mathcal{B}(E_t)\}_{t\in I}$, or H is *associated with dB-chain* $\{\mathcal{B}(E_t)\}_{t\in I}$, or a dB-chain $\{\mathcal{B}(E_t)\}_{t\in I}$ *has Hamiltonian* H, if

$$\Omega\begin{pmatrix} A_b \\ C_b \end{pmatrix} - \Omega\begin{pmatrix} A_a \\ C_a \end{pmatrix} = z\int_a^b H(t)\begin{pmatrix} A_t \\ C_t \end{pmatrix}dt, \quad \forall t_- < a < b < t_+. \tag{1.20}$$

For a Hamiltonian $H(t)$, $t \in I$ with a regular left endpoint t_-, we can get two dB-chains from the matrix solution to the canonical equation.

Theorem 1.21. *Let $H = H(t)$, $t \in I$ be a Hamiltonian. If t_- is a regular left endpoint of H, then there exists a unique continuous matrix-valued function*

$$t \mapsto M_{t_-\to t}(z) := \begin{pmatrix} A_{t_-\to t}(z) & B_{t_-\to t}(z) \\ C_{t_-\to t}(z) & D_{t_-\to t}(z) \end{pmatrix}$$

s.t.

$$\Omega M_{t_-\to b} - \Omega = z\int_{t_-}^{b} H(t)M_{t_-\to t}dt, \quad \forall b \in I. \tag{1.21}$$

For $b \in I$, $M_{t_-\to b}$ is a non-constant normalized Nevanlinna matrix, and

$$M_{t_-\to b} = M_{a\to b}M_{t_-\to a}, \quad \forall t_- < a < b < t_+,$$

where $M_{a\to b}$ is also a non-constant normalized Nevanlinna matrix. Moreover, H is uniquely determined by the transition matrices $\{M_{a\to b}\}$.

Sketch of the Proof. The proof of this Theorem is quite straightforward. We use Picard's iterative method, namely we define $M_{0,t_-\to t} :\equiv I_2$ for $t \in I$ and $M_{n+1,t_-\to t} := \int_{t_-}^{t} H(s)M_{n,t_-\to s}ds$, then $M_{t_-\to t}(z) := \sum_{n=0}^{+\infty} M_{n,t_-\to t}z^n$ is convergent because of the inequality (1.16), and obviously it solves the canonical system (1.2). Uniqueness can be proved by taking the derivative of both sides of (1.21) w.r.t. z and evaluating them at 0. For a detailed proof please refer to [dB68, Theorem 38]. □

Remark. $M_{t_-\to b}$ and $M_{a\to b}$ are not constant matrices because we assume $H(t) \neq 0$ almost everywhere.

Such Nevanlinna matrices $\{M_{a\to b}\}_{t_-<a<b<t_+}$ are said to be *associated with* the Hamiltonian H. It is well defined for $t_- < a < b < t_+$ regardless of whether t_- is a regular left endpoint of H or not. The two columns are the solutions to the canonical equation (1.2) satisfying Dirichlet boundary condition and Neumann boundary condition at t_-, respectively. Let $E_t := A_{t_-\to t} - iC_{t_-\to t}$ and $\tilde{E}_t := B_{t_-\to t} - iD_{t_-\to t}$. Since $M_{t_-\to t} \to I_2$ as $t \to t_-$, condition (1.19) in the definition of a dB-chain is satisfied, and $\{\mathcal{B}(E_t)\}_{t\in I}$, $\{\mathcal{B}(\tilde{E}_t)\}_{t\in I}$ are two dB-chains satisfying the same canonical equation (1.2). Also from (1.2) we can see the reproducing kernels of the two dB-spaces $\mathcal{B}(E_t)$ and $\mathcal{B}\left(\tilde{E}_t\right)$ satisfy:

$$K_{t,0}(0) = \frac{C_t'(0)}{\pi} = \frac{1}{\pi}\int_{t_-}^{t} H_{11}(s)ds, \quad \tilde{K}_{t,0}(0) = -\frac{B_t'(0)}{\pi} = \frac{1}{\pi}\int_{t_-}^{t} H_{22}(s)ds,$$

and therefore $K_{t,0}(0)$ is not necessarily equal to $\tilde{K}_{t,0}(0)$, thus a Hamiltonian might be associated with more than one chain of dB-functions or chain of dB-spaces. However, under certain constraints, for example if E_t is *strict and normalized*, the dB-chain is unique up to multiplication by a zero-free real entire function S, as we will see in Section 2.4.4.

On the other hand, once we have a dB-chain $\{\mathcal{B}(E_t)\}_{t\in I}$, we can get at least one Hamiltonian associated with the chain (see Proposition 1.22 below). The uniqueness of such a Hamiltonian will be proved later, and we start with the following properties of the transition matrices $\{M_{a\to b}\}_{t\in I}$ of the dB-chain $\{\mathcal{B}(E_t)\}_{t\in I}$.

Proposition 1.22. *Let $\{M_{a\to b}\}$ be the transition matrices of the dB-chain $\{\mathcal{B}(E_t)\}_{t\in I}$, then*

(i) $M_{a\to c} = M_{b\to c}M_{a\to b}, \quad \forall t_- < a < b < c < t_+,$

(ii) For fixed $a \in I$, $\Omega M'_{a\to t}(0)$ is a strictly increasing matrix-valued function of t. $H(t) := \frac{d}{dt}\left(\Omega M'_{a\to t}(0)\right)$, $a < t$, exists for a.e. $t \in I$. $H(t)$ is independent of the choice of $a < t$, and

$$\Omega M_{a\to c}(z) - \Omega M_{a\to b}(z) = z\int_{b}^{c} H(t)M_{a\to t}(z)dt, \quad t_- < a < b < c < t_+, \tag{1.22}$$

and consequently,

$$\Omega\begin{pmatrix}A_b\\C_b\end{pmatrix}-\Omega\begin{pmatrix}A_a\\C_a\end{pmatrix}=z\int_a^b H(t)\begin{pmatrix}A_t\\C_t\end{pmatrix}dt. \tag{1.23}$$

Proof. See [dB68, Theorem 37,38]. □

For a given dB-chain $\{\mathcal{B}(E_t)\}_{t\in I}$, the Hamiltonian that satisfies (1.22) is unique. This can be seen by taking the derivative of both sides w.r.t. z and evaluating the equation at 0, as discussed earlier. Any Hamiltonian that satisfies (1.22) satisfies (1.23) as well. Actually, the Hamiltonian that satisfies (1.23) is also unique, as shown below.

Proposition 1.23. *Let $\{\mathcal{B}(E_t)\}_{t\in I}$ be a dB-chain, then there exists a unique Hamiltonian $H(t)$ s.t.*

$$\Omega\begin{pmatrix}A_b\\C_b\end{pmatrix}-\Omega\begin{pmatrix}A_a\\C_a\end{pmatrix}=z\int_a^b H(t)\begin{pmatrix}A_t\\C_t\end{pmatrix}dt.$$

Proof. See Section 1.C. □

In other words, there exists a unique Hamiltonian associated with given dB-chain. And from (1.22) we can see, if $M_{a\to b}$, $\forall a<b\in I$ are the transition matrices of the dB-chain $\{\mathcal{B}(E_t)\}_{t\in I}$ whose Hamiltonian is H, then $M_{a\to b}$ is associated with the Hamiltonian H.

A family of non-degenerate dB-functions $\{E_t\}_{t\in I}$ can form a dB-chain if the corresponding vector-valued functions $\binom{A_t}{C_t}$ solve the differential equation for some Hamiltonian H and satisfies the asymptotic constraint (1.18), as shown in the following proposition.

Proposition 1.24. *Let $\{E_t\}_{t\in I}$ be a family of dB-functions. If*

$$\Omega\begin{pmatrix}A_b\\C_b\end{pmatrix}-\Omega\begin{pmatrix}A_a\\C_a\end{pmatrix}=z\int_a^b H\begin{pmatrix}A_t\\C_t\end{pmatrix}dt,\qquad \forall t_-<a<b<t_+ \tag{1.24}$$

for some Hamiltonian H on I, then

$$\begin{pmatrix}A_b\\C_b\end{pmatrix}=M_{a\to b}\begin{pmatrix}A_a\\C_a\end{pmatrix},\qquad \forall t_-<a<b<t_+,$$

where $M_{a\to b}$ are the transition matrices associated with H.

Proof. See Section 1.D. □

Now we discuss the relation between chains of dB-spaces and Hamiltonian. We know a chain of dB-spaces determines a dB-chain which is unique up to a $SL(2,\mathbb{R})$ transform on $\binom{A}{C}$. From the discussion in Section 1.2.2 we know for a given chain of dB-spaces $\{\mathcal{B}_t\}$, its transition matrices are unique up to a map $M_{a\to b}\mapsto V^{-1}M_{a\to b}V$ for $V\in SL(2,\mathbb{R})$. As $H(t)=\frac{d}{dt}\left(\Omega M'_{a\to t}(0)\right)$, using the equality $V^*\Omega V=\Omega$ for $V\in SL(2,\mathbb{R})$, we can see the Hamiltonian associated with dB-chain $\{\mathcal{B}(E_t)\}$ s.t. $\mathcal{B}(E_t)=\mathcal{B}_t$ as dB-spaces for any $t\in I$ is unique up to a map: $H(t)\mapsto V^*H(t)V$.

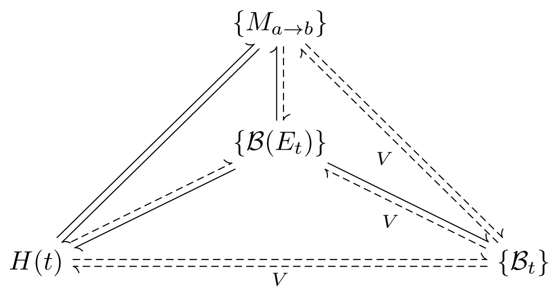

Figure 1.2: Uniqueness diagram for dB-chains and Hamiltonian

Figure 1.2 summarizes the uniqueness relation between dB-chains, Hamiltonian and the transition matrices. Again, solid arrow means uniqueness and dashed arrow means non-uniqueness.

Not all Hamiltonian H can be associated with a dB-chain. In Section 2.3 we will give a sufficient and necessary condition for H to be the Hamiltonian of a dB-chain, and here we give some simple necessary conditions for H to be the Hamiltonian of a dB-chain $\{\mathcal{B}(E_t)\}_{t\in I}$ s.t. E_t is normalized, $\forall t \in I$.

Actually, for such a Hamiltonian H, although H may not be locally integrable at t_-, the upper left element of H, namely H_{11}, is always locally integrable at t_-. Moreover, t_- is a point of growth of α, namely, $\alpha(t) > \alpha(t_-)$ for $t > t_-$.

Proposition 1.25. *Let $H = H(t)$ be the Hamiltonian of a dB-chain $\{\mathcal{B}(E_t)\}_{t\in I}$, $E_t(0) = 1, \forall t \in I$, then $0 < \int_{t_-}^{c} H_{11}(t)dt < \infty$ for $c \in I$. Or equivalently, let $h = h(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$ be an anti-derivative of H, then*

$$\begin{aligned} &\alpha(t_-) := \lim_{t\to t_-} \alpha(t) \text{ exists}, \\ &\alpha(t) > \alpha(t_-), \quad \forall t \in I. \end{aligned} \tag{1.25}$$

Proof. See Section 1.E. ☐

Note that this statement holds only because we assume $E_t(0) = 1$. Alternatively if we assume $E_t(0) = -i$, then the conditions become: $\gamma(t_-) := \lim_{t\to t_-} \gamma(t)$ exists and $\gamma(t) > \gamma(t_-)$ for $t \in I$.

Because H and h have such properties, from now on we *always assume* $\lim_{t\to t_-} \alpha(t) = 0$ for the anti-derivative $h(t)$ of the Hamiltonian $H(t)$ if the dB-chain satisfies $E_t(0) = 1$. Consequently, $\alpha(t) > 0$ for $t \in I$.

1.2.4 Spectral measures of a dB-chain

In this section we define spectral measures of a chain of dB-spaces, and classify the points in I into H-ordinary/H-special points. This classification enables us to further clarify the structure of chains of dB-spaces.

Definition 1.26. (i) A positive measure μ on $\mathbb{R}$ is said to be *associated with dB-space* $\mathcal{B}$, if $\mathcal{B}$ sits almost isometrically in $L^2(\mu)$, and $\overline{\mathrm{dom}_{\mathcal{B}}(z)} \sqsubseteq L^2(\mu)$.

(ii) A positive measure μ is said to be a *spectral measure of chain of dB-spaces* $\{\mathcal{B}_t\}_{t\in I}$, if μ is associated with $\mathcal{B}_t$, $\forall t \in I$.

Remark. The exact meaning of "almost isometric" inclusion is given in Section 1.2.2.

From Proposition 1.8(ii) it's easy to see for a strict dB-function E, $\frac{d\lambda}{|E(\lambda)|^2}$ is a measure associated with the dB-space $\mathcal{B}(E)$. In Section 2.4.1 we will see any sampling measure of $\mathcal{B}(E)$ is associated with $\mathcal{B}(E)$, hence a dB-space always has infinitely many associated measures as $\mathcal{B}(E)$ has infinitely many sampling measures.

A chain of dB-spaces $\{\mathcal{B}_t\}$ has at least one spectral measure μ. Moreover, under certain conditions (for example, $\Omega M'_{a\to t}(0)$ is unbounded as $t \to t_+$), the spectral measure of a chain of dB-spaces is unique. These results will be discussed and proved in Section 2.4.2 and Section 2.4.3.

From the definition of a spectral measure we can see for any dB-space $\mathcal{B}_t$ in the chain $\{\mathcal{B}_t\}_{t\in I}$, $\overline{\mathrm{dom}_{\mathcal{B}_t}(z)} \sqsubseteq L^2(\mu)$ always holds where μ is a spectral measure of the chain. However, there are two cases for $\mathrm{dom}_{\mathcal{B}_t}(z)^\perp$:

- $\mathrm{dom}_{\mathcal{B}_t}(z)^\perp$ sits *strictly contractively* in $L^2(\mu)$.
- $\mathrm{dom}_{\mathcal{B}_t}(z)^\perp$ sits *isometrically* in $L^2(\mu)$.

The two different cases are closely related to another concept, namely the H-ordinary/H-special points of the Hamiltonian $H(t)$. For a dB-chain or a chain of dB-functions, since it has a unique Hamiltonian H, we can classify points on I into H-ordinary/H-special points using this unique associated Hamiltonian. For a chain of dB-spaces, as discussed in Section 1.2.2, there are multiple dB-chains corresponding to it but they are equal up to a $SL(2,\mathbb{R})$ transform on $\binom{A}{C}$, therefore their Hamiltonian is unique up to transform $H(t) \mapsto V^*H(t)V$. From the remark following Definition 1.3 we can see the H-indivisible intervals are invariant under such a transform, therefore we can define H-ordinary/H-special points for a chain of dB-spaces.

The following result (cf. [dB68, Theorem 40]) gives more insight into the structure of a chain of dB-spaces and its relation with $L^2(\mu)$ where μ is a spectral measure of the chain.

Theorem 1.27. *Let $\{\mathcal{B}_t\}_{t\in I}$ be a chain of dB-spaces. Let μ be a spectral measure of the chain. Then:*

(i) For H-ordinary $a \in I$, $\mathcal{B}_a \sqsubseteq \mathcal{B}_b$ for $b \in (a, t_+)$, and $\mathcal{B}_a \sqsubseteq L^2(\mu)$.

(ii) For H-special $a \in I$, $\mathrm{dom}_{\mathcal{B}_a}(z)^\perp$ sits strictly contractively in $\mathcal{B}_b$ for $b \in (a, t_+)$ and $L^2(\mu)$.

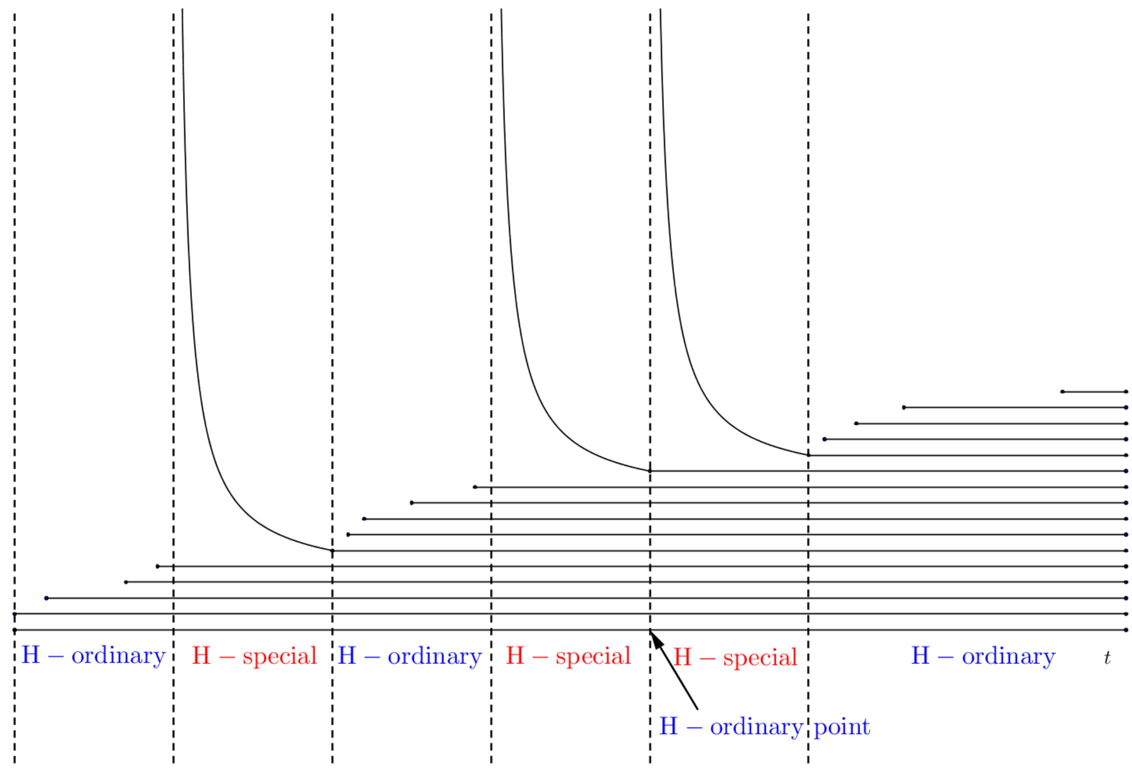

Figure 1.3: Structure of a chain of dB-spaces

(iii) For H-ordinary $b \in I$, if b is not a left endpoint of an H-indivisible interval, then

$$\mathcal{B}_b = \bigcap_{\substack{c>b \\ c\ H\text{-ordinary}}} \mathcal{B}_c.$$

(iv) For H-ordinary $b \in I$, if b is not a right endpoint of an H-indivisible interval, then

$$\mathcal{B}_b = \overline{\bigcup_{\substack{a<b \\ a\ H\text{-ordinary}}} \mathcal{B}_a}.$$

(v) For a maximal H-indivisible interval (a,c), $\mathcal{B}_c \ominus \mathcal{B}_a = \{uA_a + vC_a : u\bar{v} \in \mathbb{R}\}$, where $E_a = A_a - iC_a$ is any dB-function s.t. $\mathcal{B}_a = \mathcal{B}(E_a)$.

(vi) If $\mathcal{B}$ is a nonzero dB-space and $\mathcal{B} \sqsubseteq \mathcal{B}_b$ for H-ordinary b, then $\mathcal{B} = \mathcal{B}_a$ for some H-ordinary $a \in (t_-, b)$.

(vii) If $\mathcal{B}$ is a nonzero dB-space and $\mathcal{B}_b \sqsubseteq \mathcal{B} \sqsubseteq L^2(\mu)$ for H-ordinary b, then $\mathcal{B} = \mathcal{B}_c$ for some H-ordinary $c \in (b, t_+)$.

The structure of a chain of dB-spaces is illustrated figuratively in Figure 1.3. To summarize, $\mathcal{B}_t$ keeps expanding continuously as a dB-space as t increases on an H-ordinary interval (i.e., an interval that contains only H-ordinary points). Here the continuity means the continuity of the reproducing kernels of the dB-spaces. When t enters an H-indivisible interval (a, c), elements $uA_a + vC_a$, $u\bar{v} \in \mathbb{R}$ are added to the dB-space, and the norms of such elements are strictly decreasing (from ∞) for $t \in (a, c)$. When t reaches c, the norms of such elements are fixed as their norms in $\mathcal{B}_c$ and they sit isometrically in any $\mathcal{B}_t$ for $t > c$. And it's possible to have two adjacent maximal H-indivisible intervals, as long as they have different types (cf. Definition 1.3).

1.2.5 Generalized Fourier transform associated with a dB-chain

Once we have a dB-chain, we can define the generalized Fourier transform (a.k.a. the Titchmarsh-Weyl-Fourier transform, or the Fourier transform) accordingly (cf. [dB68, Theorem 43-45]). Since every dB-chain has at least one (scalar) spectral measure μ as we will see in Section 2.4.2, essentially we get a transform from $L^2(H)$ to $L^2(\mu)$.

Theorem 1.28. *Let $\{\mathcal{B}(E_t)\}_{t\in I}$ be a dB-chain with Hamiltonian $H(t)$. Let $h(t) := \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$ be one anti-derivative of H. Assume $\alpha(t) > 0$ for $t > t_-$, $\lim_{t\to t_-} \alpha(t) = 0$, and E_t is strict and normalized, then for any H-ordinary $c \in I$, $\chi_{(t_-,c]}(t)\binom{A_t(z)}{C_t(z)} \in L^2(H)$. Let $\mathcal{W}_{\mathcal{B}}$ be the map*

$$\mathcal{W}_{\mathcal{B}}: \quad \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \mapsto F(z) := \frac{1}{\sqrt{\pi}} \left(\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} \overline{A_t(z)} \\ \overline{C_t(z)} \end{pmatrix} \right)_{L^2(H)} = \frac{1}{\sqrt{\pi}} \int_I (f_1(t), f_2(t)) H(t) \begin{pmatrix} A_t(z) \\ C_t(z) \end{pmatrix} dt, \tag{1.26}$$

then $\mathcal{W}_{\mathcal{B}}$ maps $L^2(H; (t_-, c])$ isometrically onto $\mathcal{B}(E_c)$. Moreover, if $\binom{g_1}{g_2} \in L^2(H; (t_-, c])$ is orthogonal to $\binom{1}{0}$, then there exists $\binom{f_1}{f_2} \in L^2(H; (t_-, c])$, s.t.

$$\Omega \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}^{\bullet} = H \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}.$$

Let $F := \mathcal{W}_{\mathcal{B}}\binom{f_1}{f_2}$ and $G := \mathcal{W}_{\mathcal{B}}\binom{g_1}{g_2}$, then $G(z) = zF(z)$.

This theorem is explained in Figure 1.4,where μ is a spectral measure of the dB-chain $\{\mathcal{B}(E_t)\}$.

$$\begin{tikzcd}
L^2(H;(t_-,c]) \arrow[d, "\mathcal{W}_{\mathcal{B}}"] & \mathcal{D}\left(H^{-1}\Omega\frac{d}{dt}\right) \cap L^2(H;(t_-,c]) \arrow[l, "\sqsupseteq"'] \arrow[r, "H^{-1}\Omega\frac{d}{dt}"] \arrow[d, "\mathcal{W}_{\mathcal{B}}"] & L^2(H;(t_-,c]) \arrow[r, "\sqsubseteq"] \arrow[d, "\mathcal{W}_{\mathcal{B}}"] & L^2(H) \arrow[d, dashed, "\mathcal{W}_{\mathcal{B}}"] \\
\mathcal{B}_c & \mathrm{dom}_{\mathcal{B}_c}(z) \arrow[l, "\sqsupseteq"'] \arrow[r, "z"] & \mathcal{B}_c \arrow[r, "\sqsubseteq"] & L^2(\mu)
\end{tikzcd}$$

Figure 1.4: The generalized Fourier transform $\mathcal{W}_{\mathcal{B}}$

The significance of Theorem 1.28 is that it applies to a dB-chain regardless of whether it's regular or not. This enables us to extend the whole theory from the regular case to certain singular cases. Namely, the canonical system (the Hamiltonian) can have a singular left endpoint, but as long as there exists a dB-chain with H as its Hamiltonian, we can get a *scalar* spectral measure and the generalized Fourier transform $\mathcal{W}_{\mathcal{B}}$ as defined in Theorem 1.28. In Chapter 2 we show that for a given Hamiltonian H, there exists a dB-chain $\{\mathcal{B}(E_t)\}_{t\in I}$ with H as its Hamiltonian if and only if the Titchmarsh-Weyl m-function for the canonical system on $(t_-, c]$ has a meromorphic extension. In Chapter 3 and Chapter 4 we show how to construct a dB-chain for certain Hamiltonian (or equivalently, canonical systems) which satisfy the sufficient and necessary condition for the existence of a dB-chain given in Chapter 2.

1.3 Regular de Branges spaces

In this section we focus on regular de Branges spaces, which are closely related to Hamiltonian with regular left endpoints and regular measures. The main result of this section is the one-to-one correspondence between regular dB-chains, Hamiltonian with regular left endpoints, and regular measures. We will give the precise statements in Theorem 1.34.

For a Hamiltonian $H = H(t)$, recall that it has a *regular* left endpoint if H is locally integrable at t_-, i.e., $\int_{t_-}^{c} H(t)dt$ has finite elements for some $c \in I$. Note that this is equivalent to the condition $\lim_{t\to t_-} \alpha(t) + \gamma(t) > -\infty$, where $h(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$ is an anti-derivative of $H(t)$. A positive measure μ on $\mathbb{R}$ is said to be *regular* if

$$\int_{-\infty}^{+\infty} \frac{d\mu(\lambda)}{1+\lambda^2} < \infty.$$

In order to define regular dB-functions/spaces, we introduce the *Cartwright class* of entire functions. The Cartwright class Cart_a consists of entire functions F of exponential type at most a that satisfy a weaker integrability condition on $\mathbb{R}$:

$$\int_{-\infty}^{+\infty} \frac{\log^+ |F|}{1+t^2} dt < \infty,$$

and Cart$:= \cup_{a\geqslant 0}\mathrm{Cart}_a$. Cartwright functions are considered in detail in [Lev64], [BJ54], and [dB68]. The following *Kreĭn's theorem* will be used in later proofs: an entire function F belongs to the Cartwright class Cart if and only if it belongs to the Nevanlinna classes $\mathcal{N}(\mathbb{C}_+)$ and $\mathcal{N}(\mathbb{C}_-)$.

Definition 1.29. (i) A dB-function E is said to be *regular* if $E \in$ Cart and

$$\int_{-\infty}^{+\infty} \frac{d\lambda}{|E(\lambda)|^2(1+\lambda^2)} < \infty.$$

(ii) A dB-space $\mathcal{B}$ is said to be *regular* if

$$\frac{F(z)-F(w)}{z-w}\in\mathcal{B},\quad \forall F\in\mathcal{B},\quad \forall w\in\mathbb{C}.$$

Remark. From the definition we can see that regular dB-functions must be strict, i.e., zero-free on the real line.

The two definitions are equivalent in the following sense (cf. [dB60, Theorem III]):

Proposition 1.30. *Let E be a non-degenerate dB-function. Then E is a regular dB-function if and only if $\mathcal{B}(E)$ is a regular dB-space.*

Moreover, for a dB-chain $\{\mathcal{B}(E_t)\}_{t\in I}$, if $\mathcal{B}(E_t)$ is regular for any $t\in I$, then $\mathcal{B}(E_t)$ is regular for all $t\in I$. This can be proved using the transition matrices between any two dB-spaces in a dB-chain. The following proposition shows how the Nevanlinna matrix is related to regular dB-functions.

Proposition 1.31. *Let $M(z) := \begin{pmatrix} A(z) & B(z)\\ C(z) & D(z)\end{pmatrix}$ be a Nevanlinna matrix, then $E := A - iC$, $\tilde{E} := B - iD$ are regular dB-functions.*

Proof. See Section 1.F. $\square$

Remark. The dB-functions E or $\tilde{E}$ might be degenerate, even if $M(z)$ is a *non-constant* Nevanlinna matrix. For example,

$$M(z) = \begin{pmatrix} 1 & 0\\ z & 1\end{pmatrix}$$

is a non-constant Nevanlinna matrix where $\mathcal{B}(M) = \left\{\binom{w}{0} : w\in\mathbb{C}\right\} \neq \{0\}$, but $\tilde{E}(z)\equiv -i$ is a *degenerate* dB-function.

Based on this result, the following proposition establishes the regularity for a dB-chain in terms of the regularity of every individual dB-space.

Proposition 1.32. *Let $\{\mathcal{B}(E_t)\}_{t\in I}$ be a dB-chain. If $\mathcal{B}(E_t)$ is regular for some $t\in I$, then $\mathcal{B}(E_t)$ is regular for all $t\in I$.*

Proof. Assume $\mathcal{B}(E_b)$ is a regular dB-space for some $b\in I$. For $c\in(b,t_+)$, $\binom{A_c}{C_c} = M_{b\to c}\binom{A_b}{C_b}$. Since $E_b = A_b - iC_b$ is regular, then we can find $\binom{B_b}{D_b}$ s.t. $M_b := \begin{pmatrix} A_b & B_b\\ C_b & D_b\end{pmatrix}$ is a Nevanlinna matrix, according to [dB68, Theorem 27]. Then $M_c := M_{b\to c}M_b$ is a Nevanlinna matrix as well. Therefore $E_c = A_c - iC_c$, where $\binom{A_c}{C_c}$ is the first column of M_c, is regular.

On the other hand, for $a\in(t_-,b)$, note that

$$A_a = D_{a\to b}A_b - B_{a\to b}C_b.$$

It can be shown A_b, C_b, $D_{a\to b}$, $B_{a\to b}$ all belong to $\mathcal{P}_0$, the Pólya class of entire functions which will be discussed in Chapter 3. One of the most important properties for function $F \in \mathcal{P}_0$ is that for any fixed $x \in \mathbb{R}$, $|F(x+iy)|$ is a non-decreasing function for $y \geqslant 0$, therefore $\sup_{y\geqslant 1}\left|\frac{1}{A_a(iy)}\right| < \infty$. Similarly we have $\sup_{y\geqslant 1}\left|\frac{1}{C_a(iy)}\right| < \infty$, therefore $\sup_{y\geqslant 1}\left|\frac{1}{E_a(iy)}\right| < \infty$ as $E_a(0) \neq 0$. The rest follows from [dB68, Theorem 26], a criterion to determine whether $\mathcal{B}(E)$ is a regular dB-space or not based on an estimate of $\frac{1}{|E(z)|}$ on the imaginary axis.

□

Based on this proposition, we can define a regular dB-chain as below.

Definition 1.33. A dB-chain $\{\mathcal{B}(E_t)\}$ is called *regular* if any dB-space in the chain is regular.

Now we give the precise statements on the one-to-one correspondence between Hamiltonian with regular left endpoints, chains of regular dB-spaces, and regular measures. This is the most important result in the regular case, which gives a complete solution to the inverse spectral problem.

Theorem 1.34. *(i) Let $\{\mathcal{B}(E_t)\}_{t\in I}$ be a dB-chain with Hamiltonian H and a spectral measure μ. If $\{\mathcal{B}(E_t)\}$ is regular, then μ is regular, t_- is a regular left endpoint of H, and $\lim_{t\to t_-} E_t(z) \equiv w$ locally uniformly in z for some complex constant w which doesn't depend on z.*

(ii) Let $H = H(t)$, $t \in I$ be a Hamiltonian. If t_- is a regular left endpoint of H, then there exists a regular dB-chain $\{\mathcal{B}(E_t)\}_{t\in I}$ with H as its Hamiltonian. The chain is unique if we specify $E_t(0) = 1$ for $t \in I$.

(iii) Let μ be a regular positive measure on $\mathbb{R}$, then there exists a chain of regular dB-spaces $\{\mathcal{B}_t\}_{t\in I}$ s.t. μ is a spectral measure of $\{\mathcal{B}_t\}$. The chain is unique up to re-parametrization of t.

Proof. See Section 1.G. □

Remark. (i) From part (i) of Theorem 1.34 we know if $\{\mathcal{B}(E_t)\}_{t\in I}$ is a regular dB-chain, then $\lim_{t\to t_-} E_t$ exists and is a constant function. The converse, however, is not true. In Section 3.2.2, Theorem 3.13 shows that it's possible to construct a dB-chain s.t. $\lim_{t\to t_-} E_t \equiv 1$ ($\beta(t) \equiv 0$ in the setting of Theorem 3.13), and E_t is of Pólya class but is not regular.

(ii) For part (ii), if we do not assume regularity of the dB-chain, then there would be multiple chains sharing the same Hamiltonian H: for example, we can multiply any real entire function S to the regular chain to get another $\{\mathcal{B}(SE_t)\}_{t\in I}$, which would have the same Hamiltonian H as $\{\mathcal{B}(E_t)\}_{t\in I}$.

(iii) Similarly, for part (iii), let E be a regular dB-function. If $\mathcal{B}(E) \sqsubseteq L^2(\mu)$ for regular μ, the dB-space $\mathcal{B}(E)$ is not necessarily regular. For example, for regular $\mathcal{B}(E) \sqsubseteq L^2(\mu)$, μ is regular.

$$\begin{tikzcd}[column sep=huge] H \arrow[r, shift left, "\text{exists unique}"] & \mathcal{B}(E_t) \arrow[l, shift left, "\text{exists unique}"] \arrow[r, shift left, "\text{exists unique}"] & \mathcal{B}_t \arrow[l, dashed, shift left, "\text{exists unique up to transform } V\in SL(2,\mathbb{R})"] \arrow[r, dashed, shift left, "\text{exists unique if } K_{t,z}(z)\to+\infty"] & L^2(\mu) \arrow[l, shift left, "\text{exists unique up to reparametrization of } t"] \end{tikzcd}$$

Figure 1.5: Correspondence between H, $\{\mathcal{B}(E_t)\}$, $\{\mathcal{B}_t\}$, and μ in regular case

Let $\tilde{E} = e^{z^2}E$ and $d\tilde{\mu} = e^{-t^2}d\mu$, then $\mathcal{B}(\tilde{E}) \sqsubseteq L^2(\tilde{\mu})$, while E is not a regular dB-function as it's not of exponential type and $\tilde{\mu}$ is regular.

Moreover, by Theorem 1.34(iii) we know there exists one regular dB-chain whose spectral measure is $\tilde{\mu}$, then the dB-chains having $\tilde{\mu}$ as the spectral measure is not unique if we don't require the chain to be regular.

(iv) For part (iii), by the ordering theorem, loosely speaking, the uniqueness of the dB-chains sitting in $L^2(\mu)$ can be achieved in the Cartwright class. Actually, any dB-chain of Pólya class $\mathcal{P}_0$ sitting in $L^2(\mu)$ for regular μ, is a regular dB-chain if E_t doesn't have real zeros (cf. [dB68, Section 26]). The Pólya class $\mathcal{P}_0$ will be introduced in Section 3.2 and

$$\text{regular} \subsetneq dB \cap \text{Cart} \subsetneq dB \cap \text{Exp} \subsetneq \mathcal{P}_0.$$

Combining Theorem 1.34 with the unique correspondences discussed in Section 1.2 and the uniqueness of the spectral measures that is given by Theorem 2.12, we can get Figure 1.5, which illustrates the one-to-one correspondence between the Hamiltonian H, the dB-chain $\{\mathcal{B}(E_t)\}$, and the spectral measure μ.

We point out again Theorem 1.34 is the most important result in the theory of regular dB-spaces. In Chapter 3 and Chapter 4 we will see such an one-to-one correspondence does not hold for dB-spaces which are not necessarily regular. Partial results have been obtained for certain classes of dB-functions, among which the results on dB-functions of Pólya class and the first generalized Pólya class will be presented in Section 3.2 and Section 4.2 respectively.

1.4 Example: the classical Fourier transform

The simplest example is the Schrödinger equation with zero potential (i.e., $q \equiv 0$):

$$-\ddot{y} = zy, \quad t \in [0, +\infty), \tag{1.27}$$

or alternatively

$$\Omega\dot{Y} = zHY - QY, \quad t \in [0, +\infty), \tag{1.28}$$

where $H = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$. It's easy to see $y_1(t,z) = \cos(\sqrt{\lambda}t)$, $y_2(t,z) = -\frac{\sin(\sqrt{z}t)}{\sqrt{z}}$ are the solutions to (1.27) with initial conditions (boundary conditions) $y_1(0,z) = 1$ and $y_2(0,z) = 0$, respectively. In the special case where $z = 0$, $y_2(t,z)$ should be interpreted as the linear function $y_2(t,0) = t$. Equivalently,

$$Y(t,z) := (Y_1(t,z), Y_2(t,z)) = \begin{pmatrix} \cos(\sqrt{z}t) & -\frac{\sin(\sqrt{z}t)}{\sqrt{z}} \\ \sqrt{z}\sin(\sqrt{z}t) & \cos(\sqrt{z}t) \end{pmatrix}$$

is the solution to (1.28) s.t. $Y(0,z) = I_2$.

Let $V(t) := Y(t,0) = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}$, then the self-adjoint system (1.28) can be transformed to a canonical system

$$\Omega \dot{X} = z\tilde{H}X, \quad t \in [0, +\infty), \tag{1.29}$$

where $\tilde{H} = \tilde{H}(t) := V^*(t)HV(t) = \begin{pmatrix} 1 & -t \\ -t & t^2 \end{pmatrix}$ for $t \in [0, +\infty)$. It's easy to see $\tilde{H}$ doesn't have any H-special points.

As discussed in Section 1.2.2,

$$M_t(z) := \begin{pmatrix} A_t(z) & B_t(z) \\ C_t(z) & D_t(z) \end{pmatrix} := V^{-1}(t)\,(Y_1(t,z), Y_2(t,z))$$

is the solution to the canonical system (1.29) and it defines a dB-function $E_t := A_t - iC_t$. Theorem 1.28 then implies

$$\mathcal{W}_{\mathcal{B}} : \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \mapsto \frac{1}{\sqrt{\pi}} \left(\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \overline{\begin{pmatrix} A_t(z) \\ C_t(z) \end{pmatrix}} \right)_{L^2(\tilde{H})} \tag{1.30}$$

maps $L^2(\tilde{H}; [0,c])$ isometrically onto $\mathcal{B}(E_c)$, for any $c > 0$.

In particular, for any $f \in L^2(\mathbb{R}_+)$, obviously

$$\begin{pmatrix} f \\ 0 \end{pmatrix} \in L^2(H; [0, +\infty)),$$

then the map

$$
\begin{aligned}
f \mapsto \begin{pmatrix} f \\ 0 \end{pmatrix} \overset{\mathcal{W}_{\mathcal{B}}}{\longmapsto} & \int_0^c (f(t),0)\tilde{H}(t)\begin{pmatrix} A_t(z) \\ C_t(z) \end{pmatrix} dt \\
&= \int_0^c (f(t),0)V^* \underbrace{V^{-*}\tilde{H}(t)V(t)^{-1}}_{H(t)} \underbrace{V(t)\begin{pmatrix} A_t(z) \\ C_t(z) \end{pmatrix}}_{Y_1(t,z)} dt \\
&= \int_0^c (f(t),0)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} Y_1(t,z)dt \\
&= \int_0^c f(t)\cos(\sqrt{z}t)dt
\end{aligned}
$$

is an isometry from $L^2([0,c])$ to $\mathcal{B}(E_c)$.

Now let's calculate the Titchmarsh-Weyl m-function and the spectral measure for the canonical system (1.29). It's easy to see $m(z) := i\sqrt{z}$ satisfies $y_1(t,z) - m(z)y_2(t,z) = e^{-i\sqrt{z}} \in L^2(\mathbb{R}_+)$, therefore

$$
\begin{aligned}
& \int_0^{+\infty} (1,-m(z))M_t(z)^T\tilde{H}(t)\bar{M}_t(z)\begin{pmatrix} 1 \\ -m(z) \end{pmatrix} dt \\
= & \int_0^{+\infty} (1,-m(z))\begin{pmatrix} Y_1(t,z)^T \\ Y_2(t,z) \end{pmatrix} H(t)\left(\overline{Y_1(t,z)}, \overline{Y_2(t,z)}\right)\begin{pmatrix} 1 \\ -m(z) \end{pmatrix} dt \\
= & \int_0^{+\infty} |y_1(t,z) - m(z)y_2(t,z)|^2\, dt < \infty.
\end{aligned}
$$

Then

$$
\begin{pmatrix} A_t(z) \\ C_t(z) \end{pmatrix} - m(z)\begin{pmatrix} B_t(z) \\ D_t(z) \end{pmatrix} \in L^2(H;[0,+\infty)),
$$

and $m(z) = m_0(z)$ is the Titchmarsh-Weyl m-function for $\alpha = 0$ in (1.6). Note that the transform (1.30) uses $\binom{A_t(z)}{C_t(z)}$, therefore the corresponding Titchmarsh-Weyl m-function corresponds to $\alpha = \frac{3\pi}{2}$, and is given by

$$
m_{\frac{3\pi}{2}}(z) = -\frac{1}{m_0(z)} = \frac{i}{\sqrt{z}}.
$$

Then according to (1.9), the spectral measure is given by

$$
\mu(\lambda) = \begin{cases} \frac{2}{\pi}\sqrt{\lambda}, & \lambda > 0, \\ 0, & \lambda \leqslant 0. \end{cases}
$$

1.A Proof of Proposition 1.10

First assume $\mathcal{B}(E_1) = \mathcal{B}(E_2)$. Since $\mathcal{B}(E_1) = \mathcal{B}(E_2)$, they have the same reproducing kernel $K_w(z)$, i.e., $\forall z, w \in \mathbb{C}$,

$$\frac{C_1(z)\overline{A_1(w)} - A_1(z)\overline{C_1(w)}}{\pi(z-\bar{w})} = \frac{C_2(z)\overline{A_2(w)} - A_2(z)\overline{C_2(w)}}{\pi(z-\bar{w})}.$$

In particular, let $z \in \mathbb{C}_+$ and $w = x \in \mathbb{R}$ which is not a zero of A_1, C_1, A_2, C_2. Writing it in matrix form we get

$$(A_1(x), C_1(x)) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} A_1(z) \\ C_1(z) \end{pmatrix} = (A_2(x), C_2(x)) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} A_2(z) \\ C_2(z) \end{pmatrix}.$$

Since $\frac{A_1}{C_1}$ is not a constant, we can choose $y \in \mathbb{R}$ s.t. $\frac{A_1(y)}{C_1(y)} \neq \frac{A_1(x)}{C_1(x)}$, then

$$\begin{pmatrix} A_1(x) & C_1(x) \\ A_1(y) & C_1(y) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} A_1(z) \\ C_1(z) \end{pmatrix} = \begin{pmatrix} A_2(x) & C_2(x) \\ A_2(y) & C_2(y) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} A_2(z) \\ C_2(z) \end{pmatrix}.$$

Note that

$$\det \begin{pmatrix} A_1(x) & C_1(x) \\ A_1(y) & C_1(y) \end{pmatrix} = \pi(y-x)K_x(y) = \det \begin{pmatrix} A_2(x) & C_2(x) \\ A_2(y) & C_2(y) \end{pmatrix} \neq 0,$$

recall $\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then

$$V := \Omega^{-1} \begin{pmatrix} A_1(x) & C_1(x) \\ A_1(y) & C_1(y) \end{pmatrix}^{-1} \begin{pmatrix} A_2(x) & C_2(x) \\ A_2(y) & C_2(y) \end{pmatrix} \Omega,$$

$$\begin{pmatrix} A_1 \\ C_1 \end{pmatrix} = V \begin{pmatrix} A_2 \\ C_2 \end{pmatrix}.$$

And since A_1, C_1, A_2, C_2 are real entire, we know $V \in SL(2, \mathbb{R})$.

Now assume $\binom{A_1}{C_1} = V \binom{A_2}{C_2}$, then

$$\begin{aligned} (\overline{A_1(w)}, \overline{C_1(w)}) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} A_1(z) \\ B_1(z) \end{pmatrix} &= (\overline{A_2(w)}, \overline{B_2(w)}) V^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} V \begin{pmatrix} A_2(z) \\ B_2(z) \end{pmatrix} \\ &= (\overline{A_2(w)}, \overline{B_2(w)}) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} A_2(z) \\ B_2(z) \end{pmatrix}, \end{aligned}$$

hence $\mathcal{B}(E_1)$ and $\mathcal{B}(E_2)$ have the same reproducing kernels and then $\mathcal{B}(E_1)=\mathcal{B}(E_2)$.

1.B Proof of Proposition 1.14

Since $M(x)^*\Omega M(x)\equiv\Omega$ for $x\in\mathbb{R}$, we have

$$\begin{aligned}\frac{M^*(i\epsilon)\Omega M(i\epsilon)-\Omega}{2i\epsilon}&=\frac{(I_2-i\epsilon M'^*(x)+o(\epsilon))\,\Omega\,(I_2+i\epsilon M'(x)+o(\epsilon))-\Omega}{2i\epsilon}\\&\to-\frac{1}{2}(M'^*(0)\Omega-\Omega M'(0)),\qquad\text{as }\epsilon\to 0.\end{aligned}$$

Taking derivatives w.r.t. x in the equation $M(x)^*\Omega M(x)\equiv\Omega$ we can get

$$M^{*\prime}(x)\Omega M(x)+M^*(x)\Omega M'(x)=0.$$

Since elements of M are all real entire, then $M^{*\prime}(x)=M'^*(x)$, and therefore

$$M'^*(x)\Omega M(x)+M^*(x)\Omega M'(x)=0.$$

In particular, $M'^*(0)\Omega=-\Omega M'(0)$, then

$$\Omega M'(0)=-\frac{1}{2}(M'^*(0)\Omega-\Omega M'(0))=\lim_{z\to i0+}\frac{M^*(z)\Omega M(z)-\Omega}{z-\bar{z}}\geqslant 0,$$

and consequently $t(M)=\operatorname{tr}\left(\Omega M'(0)\right)\geqslant 0$.

Now pick any $c>0$, by [dB68, Theorem 37] we know there exist Nevanlinna matrices M_a for $a\in[0,c]$, s.t. $M=M_c$, $t(M_a)=\frac{a}{c}t(M)$, and

$$\Omega M_a(z)-\Omega=z\int_0^a dh(t)M_t(z),$$

where $h(t):=\Omega M_a'(0):=\begin{pmatrix}\alpha(a)&\beta(a)\\\beta(a)&\gamma(a)\end{pmatrix}$. Choosing $a=c$ and taking derivatives of both sides w.r.t. z and evaluate it at 0 we know

$$\left\|\Omega M_c^{(n)}(0)\right\|_F=n\left\|\int_0^a dh(t)M_t^{(n-1)}(z)\right\|_F.$$

For $n=1$, RHS becomes $\sqrt{\alpha(c)^2+2|\beta(c)|^2+\gamma(c)^2}\leqslant\alpha(c)+\gamma(c)$, and inductively one can show

$$\left\|M_c^{(n)}(0)\right\|_F\leqslant(\alpha(c)+\gamma(c))^n,\quad\forall n\in\mathbb{Z}_+,$$

then

$$1 + \|M(z) - I_2\|_F \leqslant e^{(\alpha(c)+\gamma(c))|z|} = e^{t(M)|z|}, \quad \forall z \in \mathbb{C}.$$

1.C Proof of Proposition 1.23

The existence of the Hamiltonian follows from Proposition 1.22. Now suppose there exists another Hamiltonian $\tilde{H}(t)$ that satisfies

$$\Omega\begin{pmatrix} A_b \\ C_b \end{pmatrix} - \Omega\begin{pmatrix} A_a \\ C_a \end{pmatrix} = z\int_a^b \tilde{H}(t)\begin{pmatrix} A_t \\ C_t \end{pmatrix} dt.$$

Let $\tilde{M}_{a\to b}$ be the Nevanlinna matrices associated with $\tilde{H}(t)$, then for fixed $a \in I$,

$$\Omega\tilde{M}_{a\to b} - \Omega = z\int_a^b \tilde{H}(t)\tilde{M}_{a\to t} dt,$$

$$\Omega\begin{pmatrix} \tilde{A}_b \\ \tilde{C}_b \end{pmatrix} - \Omega\begin{pmatrix} A_a \\ C_a \end{pmatrix} = z\int_a^b \tilde{H}(t)\begin{pmatrix} \tilde{A}_t \\ \tilde{C}_t \end{pmatrix} dt,$$

where

$$\begin{pmatrix} \tilde{A}_t \\ \tilde{C}_t \end{pmatrix} := \tilde{M}_{a\to t}\begin{pmatrix} A_a \\ C_a \end{pmatrix}.$$

Taking the difference of the canonical equations that $\binom{A_t}{C_t}$, $\binom{\tilde{A}_t}{\tilde{C}_t}$ satisfy, we can get

$$\Omega\begin{pmatrix} A_b - \tilde{A}_b \\ C_b - \tilde{C}_b \end{pmatrix} = z\int_a^b H(t)\begin{pmatrix} A_t - \tilde{A}_t \\ C_t - \tilde{C}_t \end{pmatrix} dt. \tag{1.31}$$

As $\binom{A_t(0)}{C_t(0)} = \binom{A_a(0)}{C_a(0)} = \binom{\tilde{A}_t(0)}{\tilde{C}_t(0)}$, by taking derivatives of both sides of (1.31) at 0 we can show $\binom{A_b^{(n)}(0)}{C_b^{(n)}(0)} = \binom{\tilde{A}_b^{(n)}(0)}{\tilde{C}_b^{(n)}(0)}$ for $n \geqslant 0$, hence $\binom{A_b}{C_b} \equiv \binom{\tilde{A}_b}{\tilde{C}_b}$. Then from Theorem 1.17 we know $\tilde{M}_{a\to b} = M_{a\to b}$, and therefore $\tilde{H}(t) = H(t)$ almost everywhere.

1.D Proof of Proposition 1.24

For each $a \in I$, let

$$\begin{pmatrix} \tilde{A}_t \\ \tilde{C}_t \end{pmatrix} := M_{a\to t}\begin{pmatrix} A_a \\ C_a \end{pmatrix},$$

then

$$\Omega\begin{pmatrix} \tilde{A}_b \\ \tilde{C}_b \end{pmatrix} - \Omega\begin{pmatrix} A_a \\ C_a \end{pmatrix} = z\int_a^b H\begin{pmatrix} A_t \\ C_t \end{pmatrix} dt, \quad \forall t_- < a < b < t_+.$$

Comparing this equation with (1.24) we can get

$$\Omega\begin{pmatrix} \tilde{A}_b \\ \tilde{C}_b \end{pmatrix} - \Omega\begin{pmatrix} A_b \\ C_b \end{pmatrix} = z\int_a^b H\begin{pmatrix} \tilde{A}_t - A_t \\ \tilde{C}_t - C_t \end{pmatrix} dt, \quad \forall t_- < a < b < t_+. \tag{1.32}$$

Evaluating this equation at $z = 0$ we know

$$\begin{pmatrix} \tilde{A}_b(0) \\ \tilde{C}_b(0) \end{pmatrix} = \begin{pmatrix} A_b(0) \\ C_b(0) \end{pmatrix}, \quad \forall t_- < a < b < t_+.$$

Taking the n-th derivative w.r.t. z for both sides of (1.32) and then evaluating the equation at $z = 0$ we can inductively show

$$\begin{pmatrix} \tilde{A}_b^{(n)}(0) \\ \tilde{C}_b^{(n)}(0) \end{pmatrix} = \begin{pmatrix} A_b^{(n)}(0) \\ C_b^{(n)}(0) \end{pmatrix}, \quad \forall t_- < a < b < t_+.$$

Therefore

$$\begin{pmatrix} A_b \\ C_b \end{pmatrix} = M_{a\to b} \begin{pmatrix} A_a \\ C_a \end{pmatrix}, \quad \forall t_- < a < b < t_+.$$

1.E Proof of Proposition 1.25

The first row of (1.23) gives

$$C_b(z) - C_a(z) = z \int_a^b (H_{11}(t), H_{12}(t)) \begin{pmatrix} A_t(z) \\ C_t(z) \end{pmatrix} dt, \quad \forall t_- < a < b < t_+.$$

Taking derivative w.r.t. z on both sides and letting $z = 0$, then as $A_t(0) = 1$ and $C_t(0) = 0$, we have

$$C_b'(0) - C_a'(0) = \int_a^b H_{11}(t)dt.$$

Since $C_a'(0) = \pi K_{a,0}(0) \geqslant 0$, we must have

$$\int_a^b H_{11}(t)dt < C_b'(0), \quad \forall t_- < a < b < t_+.$$

Hence $\int_{t_-}^b H_{11}(t)dt = \lim_{a\to t_-} \int_a^b H_{11}(t)dt$ is finite.

By definition of a dB-chain, $\lim_{a\to t_-} C_a'(0) = \lim_{a\to t_-} \pi K_{a,0}(0) = 0$, then

$$\int_{t_-}^b H_{11}(t)dt = C_b'(0) > 0, \quad \forall b \in I,$$

where the last inequality is strict because E_t is normalized and non-degenerate.

1.F Proof of Proposition 1.31

(We'll need some notations and results from Chapter 2.) Note that $\bar{M}$ is J-expansive for $z \in \mathbb{C}_+$, then $\tau_{\bar{M}}(\mathbb{C}_+) \subseteq \mathbb{C}_+$ by Proposition 2.10, where

$$\tau_{\bar{M}}(w) = \frac{\overline{A(z)}w + \overline{B(z)}}{\overline{C(z)}w + \overline{D(z)}}.$$

In particular $\Im \frac{A(\bar{z})}{C(\bar{z})} = \Im \tau_{\bar{M}}(\infty) \geqslant 0$, for $z \in \mathbb{C}_+$ where $C(z) \neq 0$. Since A and C are real entire, then we get $\Im \frac{A}{C} \leqslant 0$ for $z \in \mathbb{C}_+$ where $C(z) \neq 0$. Then $E = A - iC$ is a dB-function although it might be degenerate. By considering $\tau_{\bar{M}}(0)$ we can show $\tilde{E}$ is a dB-function as well. Both E and $\tilde{E}$ are regular by [dB68, Theorem 27].

1.G Proof of Theorem 1.34

(i) Fix $b \in I$. As E_b is regular, by [dB68, Theorem 27] we can find another regular dB-function $\tilde{E}_b = B_b - iD_b$ s.t. $M_b := \begin{pmatrix} A_b & B_b \\ C_b & D_b \end{pmatrix}$ is a Nevanlinna matrix. In order to show H has a regular left endpoint, it suffices to show $\lim_{a \to t_-} t(M_{a\to b}) < \infty$.

We know

$$\begin{pmatrix} A_b \\ C_b \end{pmatrix} = M_{a\to b} \begin{pmatrix} A_a \\ C_a \end{pmatrix} = M_b \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

As E_a is non-degenerate, there exists a Nevanlinna matrix M (cf. [dB68, Section 36]), s.t.

$$\begin{pmatrix} A_a \\ C_a \end{pmatrix} = M \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad M_b = M M_{a\to b}.$$

The second equation implies $t(M_{a\to b}) = t(M_b) - t(M) \leqslant t(M_b)$, hence $t(M_{a\to b})$ is uniformly bounded from above. Then $\int_a^b H(t)dt = \Omega M'_{a\to b}(0)$ is uniformly bounded by $\Omega M'_b(0)$ as well.

Let $M_{t_-\to b} := \lim_{a\to t_-} M_{a\to b}$, then $M_{t_-\to b}$ is a normalized Nevanlinna matrix. Note that

$$\begin{pmatrix} A_a \\ C_a \end{pmatrix} = M_{a\to b}^{-1} \begin{pmatrix} A_b \\ C_b \end{pmatrix} = \begin{pmatrix} D_{a\to b} & -B_{a\to b} \\ -C_{a\to b} & A_{a\to b} \end{pmatrix} \begin{pmatrix} A_b \\ C_b \end{pmatrix}.$$

Then

$$\begin{pmatrix} A_{t_-} \\ C_{t_-} \end{pmatrix} := \lim_{a\to t_-} \begin{pmatrix} A_a \\ C_a \end{pmatrix} = M_{t_-\to b}^{-1} \begin{pmatrix} A_b \\ C_b \end{pmatrix} \tag{1.33}$$

exists. From the definition of a dB-chain we know $E_{t_-} := A_{t_-} - iC_{t_-}$ must be a degenerate dB-function. We now show it's of Cartwright class and is zero-free, therefore it must be a constant. E_{t_-} is of Cartwright class by (1.33) and Kreĭn's theorem on Cartwright functions.

Suppose $E_{t_-}(z) = 0$, then $E_{t_-}(\bar{z}) = 0$ as E_{t_-} is degenerate, hence we can assume $z \in \mathbb{C}_+ \cup \mathbb{R}$. Obviously, A_{t_-} and C_{t_-} can't have same zeros, otherwise we get $A_b(z) = C_b(z) = 0$ for some $z \in \mathbb{C}_+ \cup \mathbb{R}$, a contradiction. As A_{t_-} and C_{t_-} are linearly dependent, we can find $c, d \in \mathbb{R}$ s.t. $cA_{t_-} + dC_{t_-} \equiv 0$. WLOG $c \neq 0$, then C_{t_-} must be zero-free by above arguments. As C_{t_-} is of Cartwright class, by Hadamard's factorization it's easy to see C_{t_-} is a constant, and so is A_{t_-}.

The spectral measure μ is regular by [dB68, Theorem 32].

(ii) The existence of a dB-chain comes from [dB68, Theorem 38]. The uniqueness comes from part (i), as E_t is uniquely determined by E_{t_-}:

$$\begin{pmatrix} A_t \\ C_t \end{pmatrix} = M_{t_- \to t} \begin{pmatrix} A_{t_-} \\ C_{t_-} \end{pmatrix}.$$

(iii) The existence of a chain of regular dB-spaces is given by [dB61a, Theorem XII]. It can be obtained via approximating μ weakly by finite discrete measures μ_n, for which it's easy to find a polynomial E_n s.t. $\mathcal{B}(E_n) \sqsubseteq L^2(\mu_n)$.

The uniqueness comes from the ordering theorem Theorem 1.12.

Chapter 2

Full-line problems: spectral matrices, Israel Kats' Theorem, and chains of non-regular de Branges spaces

Our main focuses in this chapter are the spectral theory of canonical system where both endpoints of $I = (t_-, t_+)$ are singular and the de Branges theory of non-regular dB-spaces. The case t_-, t_+ are singular is known as the full-line problem as one can normalize the Hamiltonian s.t. $\operatorname{tr} H(t) \equiv 1$, $\forall t \in I$, in which case an endpoint is singular if and only if it's infinite. The spectral matrix approach is introduced in Section 2.1. In Section 2.3 we give a necessary and sufficient condition on a Hamiltonian H s.t. there exists a dB-chain with H as its Hamiltonian, in which case one can always define a (scalar) spectral measure and corresponding generalized Fourier transform according to Theorem 1.28. In Section 2.4 we present some results on chains of dB-spaces which are not necessarily regular, to better illustrate the correspondence between dB-chains, spectral measures, and Hamiltonian. In particular, the nice one-to-one correspondence given by Theorem 1.34 for the regular case doesn't hold in the more general setting.

2.1 More on the spectral theory of canonical systems on the half-line

In this section we list some well known properties of the Weyl disk and the Weyl solution. These results will be used in the proof of the main theorem of this chapter, namely Theorem 2.3.

WLOG we assume the Hamiltonian is normalized and the interval I is $(-\infty, +\infty)$. Therefore

$$\lim_{t \to t+} \alpha(t) + \gamma(t) = \lim_{t \to +\infty} t = \infty,$$

$$\lim_{t \to t-} \alpha(t) + \gamma(t) = \lim_{t \to -\infty} t = -\infty,$$

where α, γ are the diagonal elements of the anti-derivative $h(t) := \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$ of H, s.t. $\alpha(t) + \gamma(t) = t$, $\forall t \in (-\infty, +\infty)$. Moreover if there's no *maximal H-ordinary* point, then by [dB68, Theorem 42], we're in the Weyl limit point case at the *right* endpoint since

$$\|Y_1(\cdot, z)\|_{L^2(H;[0,b])} \to +\infty, \quad \|Y_2(\cdot, z)\|_{L^2(H;[0,b])} \to +\infty,$$

as $b \to +\infty$ for nonreal z, where Y_1, Y_2 are defined as the solutions to the canonical system with boundary values

$$Y_1(0, z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad Y_2(0, z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

respectively. Similarly, if there's no *minimal H-ordinary* point, then we're in the Weyl limit point case at the *left* endpoint as

$$\|Y_1(\cdot, z)\|_{L^2(H;[a,0])} \to +\infty, \quad \|Y_2(\cdot, z)\|_{L^2(H;[a,0])} \to +\infty,$$

as $a \to -\infty$ for nonreal z.

Remark. Actually, [dB68, Theorem 42] only states that $\|Y_1(\cdot, z)\|_{L^2(H;[0,b])} \to +\infty$. By considering the Hamiltonian $\tilde{H} := \Omega H \Omega^*$, then $Y_2 = \tilde{Y}_1$ and it follows that $\|Y_2(\cdot, z)\|_{L^2(H;[0,b])} \to +\infty$ as well, as $\tilde{H}$ and H have the same ordinary/special points structure and same trace.

The Lagrange's identity is introduced here and will be used in the proof of following lemmas, and it also plays an important role when we determine the uniqueness of the dB-chains sharing the same Hamiltonian, as we will see in Section 2.4.4.

Lemma 2.1 (Lagrange's Identity). *Let $X_1(t, z)$, $X_2(t, w)$ be solutions to the canonical system*

$$\Omega \dot{X} = zHX, \quad t \in I, \tag{2.1}$$

then $\forall t_- < a < b < t_+$,

$$\begin{aligned} \langle \Omega X_1(b, z), X_2(b, w) \rangle - \langle \Omega X_1(a, z), X_2(a, w) \rangle &= (z - \bar{w}) \int_a^b \langle H(t) X_1(t, z), X_2(t, w) \rangle dt \\ &= (z - \bar{w}) \left\langle X_1(t, z), X_2(t, w) \right\rangle_{L^2(H;[a,b])}. \end{aligned} \tag{2.2}$$

Proof. From the canonical system (2.1) we know

$$
\begin{aligned}
\langle \Omega X_1(t,z), X_2(t,w)\rangle^{\bullet} &= \left\langle \Omega X_1(t,z)^{\bullet}, X_2(t,w)\right\rangle - \left\langle X_1(t,z), \Omega X_2(t,w)^{\bullet}\right\rangle \\
&= \langle zH(t)X_1(t,z), X_2(t,w)\rangle - \langle X_1(t,z), wH(t)X_2(t,w)\rangle \\
&= (z-\bar{w})\langle H(t)X_1(t,z), X_2(t,w)\rangle.
\end{aligned}
$$

□

Remark. Both Lagrange's identity and Lemma 2.2 are classical facts in the spectral theory of ordinary differential equations. The readers may refer to [Tit62, Chapter II], [CL55, Chapter 9], [LS75, Chapter 2], and [LS90, Chapter 2,8] for more details. Nevertheless we give the proofs in the appendix using notations that are consistent with this dissertation.

First let's take a closer look at the Weyl disk. Recall the definition

$$
l_b(z,w) = \frac{Y_{11}(b,z)w + Y_{12}(b,z)}{Y_{21}(b,z)w + Y_{22}(b,z)} = \frac{A_b(z)w + C_b(z)}{B_b(z)w + D_b(z)}.
$$

In particular, since $l_b\left(z, -\frac{D_b(z)}{B_b(z)}\right) = \infty$,

$$
l_b\left(z, -\frac{\overline{D_b(z)}}{\overline{B_b(z)}}\right) = \frac{A_b\overline{D_b} - \overline{B_b}C_b}{B_b\overline{D_b} - \overline{B_b}D_b}
$$

is the center of Weyl disk at $t=b$. Since $l_b(z,0)$ lies on the circle, we can get the radius of the Weyl disk:

$$
r_b(z) = \left|\frac{A_b\overline{D_b} - \overline{B_b}C_b}{B_b\overline{D_b} - \overline{B_b}D_b} - \frac{C_b}{D_b}\right| = \frac{1}{\left|B_b\overline{D_b} - \overline{B_b}D_b\right|}, \tag{2.3}
$$

which goes to 0 by [dB68, Theorem 42], for nonreal z.

Moreover, by Lagrange's identity (2.2), we have

$$
\begin{aligned}
B_b\overline{D_b} - \overline{B_b}D_b &= -\langle \Omega Y_2(b,z), Y_2(b,z)\rangle \\
&= -2i\Im z \int_0^b \langle H(t)Y_2(t,z), Y_2(t,z)\rangle dt \\
&= -2i\Im z \|Y_2\|^2_{L^2(H;[0,b])},
\end{aligned} \tag{2.4}
$$

hence the Weyl disk at $t=b$ has radius $\frac{1}{2\Im z\|\phi\|^2_{L^2(H;[0,b])}}$ for $z \in \mathbb{C}_+$.

Then let's look at the asymptotic behavior of the Weyl solution $\psi(t,z) = Y_1(t,z) - m(z)Y_2(t,z)$ as $t \to +\infty$.

Lemma 2.2. *Let $H(t)$, $t \in I := [0,+\infty)$ be a Hamiltonian with no maximal H-ordinary point, then for $z \in \mathbb{C}\backslash\mathbb{R}$, one has $\lim_{t\to+\infty}\langle \Omega\psi(t,z), \psi(t,z)\rangle = 0$ where ψ is the Weyl solution to the canonical* *system*

$$\Omega\dot{X} = zHX, \quad t \in [0, +\infty). \tag{2.5}$$

Proof. See Section 2.A. □

We should also point out that the Titchmarsh-Weyl m-function depends on the choice of the boundary values. In general, one may consider the following boundary value problem

$$\begin{aligned} &\Omega\dot{X} = zHX, \quad t \in I := [0, +\infty), \\ &X(0, z) = \begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix}, \end{aligned} \tag{2.6}$$

for $\alpha \in [0, 2\pi)$. Let $Y_{1,\alpha}$, $Y_{2,\alpha}$ be the solutions to the canonical system with boundary values

$$Y_{1,\alpha}(0, z) = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}, \quad Y_{2,\alpha}(0, z) = \begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix}, \tag{2.7}$$

and similarly we can define $m_\alpha(z)$ for $z \in \mathbb{C}\backslash\mathbb{R}$ s.t. $Y_{1,\alpha} - m_\alpha Y_{2,\alpha} \in L^2(H; [0, +\infty))$, and a measure μ_α on $\mathbb{R}$ s.t.

$$m_\alpha(z) = a_\alpha + b_\alpha z + \int_{-\infty}^{+\infty} \left(\frac{1}{z - \lambda} - \frac{\lambda}{1 + \lambda^2} \right) d\mu_\alpha(\lambda)$$

for some $a_\alpha \in \mathbb{R}$ and $b_\alpha > 0$.

From the definitions we can see that the previously defined Y_1, Y_2, m are actually $Y_{1,0}$, $Y_{2,0}$, m_0. In particular, as $\sin\left(\alpha + \frac{\pi}{2}\right) = \cos\alpha$ and $\cos\left(\alpha + \frac{\pi}{2}\right) = -\sin\alpha$, we then obtain

$$Y_{1,\alpha+\frac{\pi}{2}} = Y_{2,\alpha}, \quad Y_{2,\alpha+\frac{\pi}{2}} = -Y_{1,\alpha},$$

where $\alpha + \frac{\pi}{2}$ is interpreted as $\alpha + \frac{\pi}{2} \mod 2\pi$, and as a special case we have

$$m_0(z) = -\frac{1}{m_{\frac{3\pi}{2}}(z)}, \quad \forall z \in \mathbb{C}\backslash\mathbb{R}. \tag{2.8}$$

As mentioned in Section 1.1, the formal differential operator can be extended to a self-adjoint operator on a proper domain, as shown in [GK70] and [Kat07]. Moreover, the spectrum of the self-adjoint operator is discrete if and only if the spectral measure μ_α of the boundary value problem (2.6) has a discrete support which coincides with the spectrum of the self-adjoint operator. Note that the spectral measure μ_α has a discrete support if and only if the corresponding Titchmarsh-Weyl m-function m_α for (2.6) has a meromorphic extension, according to the relation (1.8) between m_α and μ_α. The poles of the Titchmarsh-Weyl m-function m_α, accordingly, are the spectrum of the boundary value problem (2.6).

There is an intuitive explanation for this correspondence. If $Y_{2,\alpha}(\cdot, z)$ is a $L^2(H)$ solution to the canonical system for some $z \in \mathbb{C}$, then $m_\alpha(z) = \infty$, as otherwise $Y_{1,\alpha}$ is a $L^2(H)$ solution as well since it's a linear combination of the Weyl solution, which is in $L^2(H)$, and $Y_{2,\alpha} \in L^2(H)$. Thus if $Y_1(\cdot, z) = Y_{2,\frac{3\pi}{2}}(\cdot, z)$ is a $L^2(H)$ solution to the canonical system, and $m_{\frac{3\pi}{2}}$ has a meromorphic extension, then m_0 has a meromorphic extension by (2.8), and $m_0(z) = 0$. We'll need this result to prove Theorem 2.3, the main theorem of this chapter.

2.2 The spectral matrix approach

In this section we introduce the full-line problem and the spectral matrix approach. The approach below for full-line problems is well known for the Schrödinger equation, Dirac system and Sturm-Liouville equations (see [Tit62, Chapter III], [CL55, Section 9.5], and [LS75, Chapter 2,3]). Here we state the classical results without giving proofs.

For the full-line problem

$$\Omega \dot{X} = zHX, \quad t \in I := (-\infty, +\infty), \tag{2.9}$$

where H is a normalized Hamiltonian on I. We can approximate I by expanding finite intervals $\Delta := [a, b]$ so that $a \to -\infty$ and $b \to +\infty$.

Let $[a, b]$ be an arbitrary finite interval, consider the boundary value problem:

$$\begin{aligned} &\Omega \dot{X} = zHX, \quad t \in I := \Delta, \\ &\left\langle X(b, z), \begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix} \right\rangle = 0, \\ &\left\langle X(a, z), \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} \right\rangle = 0, \end{aligned} \tag{2.10}$$

where $\alpha, \beta \in [0, \pi)$. It can be shown for boundary value problem (2.10), there are countably many eigenvalues $\{\lambda_n\}_{n=-\infty}^{+\infty}$ and a complete set of orthonormal vector-valued eigenfunctions $\{X_n\}_{n=-\infty}^{+\infty}$ s.t. X_n corresponds to eigenvalue λ_n.

Let $Y_1(t, z)$, $Y_2(t, z)$ be the solutions to the canonical system (2.9) s.t.

$$Y_1(0, z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad Y_2(0, z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

then there exist β_n, $\gamma_n \in \mathbb{R}$ s.t.

$$X_n(t) = \beta_n Y_1(t, \lambda_n) + \gamma_n Y_2(t, \lambda_n).$$

Let $\alpha_n := \|X_n\|_{L^2(H;\Delta)}$, we can use Parseval's identity to get: $\forall f = \binom{f_1}{f_2} \in L^2(H;\Delta)$,

$$\|f\|^2_{L^2(H;\Delta)} = \sum_{n=-\infty}^{+\infty} \frac{1}{\alpha_n^2} \left|(f, X_n)_{L^2(H;\Delta)}\right|^2 \tag{2.11}$$
$$= \sum_{n=-\infty}^{+\infty} \frac{1}{\alpha_n^2} (f, X_n)_{L^2(H;\Delta)} \left(\bar{f}, X_n\right)_{L^2(H;\Delta)} \tag{2.12}$$
$$= \sum_{n=-\infty}^{+\infty} \frac{\beta_n^2}{\alpha_n^2} (f, Y_1)_{L^2(H;\Delta)} \left(\bar{f}, Y_1\right)_{L^2(H;\Delta)} \tag{2.13}$$
$$+ \sum_{n=-\infty}^{+\infty} \frac{\beta_n\gamma_n}{\alpha_n^2} (f, Y_1)_{L^2(H;\Delta)} \left(\bar{f}, Y_2\right)_{L^2(H;\Delta)} \tag{2.14}$$
$$+ \sum_{n=-\infty}^{+\infty} \frac{\beta_n\gamma_n}{\alpha_n^2} \left(\bar{f}, Y_1\right)_{L^2(H;\Delta)} (f, Y_2)_{L^2(H;\Delta)} \tag{2.15}$$
$$+ \sum_{n=-\infty}^{+\infty} \frac{\gamma_n^2}{\alpha_n^2} (f, Y_2)_{L^2(H;\Delta)} \left(\bar{f}, Y_2\right)_{L^2(H;\Delta)}. \tag{2.16}$$

We introduce the notation

$$\xi_\Delta(\lambda) = \sum_{0\leqslant\lambda_n<\lambda} \frac{\beta_n^2}{\alpha_n^2} \text{ for } \lambda > 0, \qquad \xi_\Delta(\lambda) = \sum_{\lambda<\lambda_n\leqslant 0} \frac{\beta_n^2}{\alpha_n^2} \text{ for } \lambda \leqslant 0,$$
$$\eta_\Delta(\lambda) = \sum_{0\leqslant\lambda_n<\lambda} \frac{\beta_n\gamma_n}{\alpha_n^2} \text{ for } \lambda > 0, \qquad \eta_\Delta(\lambda) = \sum_{\lambda<\lambda_n\leqslant 0} \frac{\beta_n\gamma_n}{\alpha_n^2} \text{ for } \lambda \leqslant 0,$$
$$\zeta_\Delta(\lambda) = \sum_{0\leqslant\lambda_n<\lambda} \frac{\gamma_n^2}{\alpha_n^2} \text{ for } \lambda > 0, \qquad \zeta_\Delta(\lambda) = \sum_{\lambda<\lambda_n\leqslant 0} \frac{\gamma_n^2}{\alpha_n^2} \text{ for } \lambda \leqslant 0.$$

Let $F_\Delta(\lambda) := (f, Y_1)_{L^2(H;\Delta)}$, $G_\Delta(\lambda) := (f, Y_2)_{L^2(H;\Delta)}$, then the Parseval's identity (2.11) can be written as

$$\|f\|^2_{L^2(H;\Delta)} = \int_{-\infty}^{+\infty} (F_\Delta, G_\Delta) d \begin{pmatrix} \xi_\Delta & \eta_\Delta \\ \eta_\Delta & \zeta_\Delta \end{pmatrix} \begin{pmatrix} \overline{F_\Delta} \\ \overline{G_\Delta} \end{pmatrix}.$$

It can be shown the limits of F_Δ, G_Δ exist as $\Delta \to (-\infty, +\infty)$. We denote them by F, G respectively. Moreover, one can show $\xi_\Delta, \eta_\Delta, \zeta_\Delta$ have finite total variation on any bounded interval, and the upper bound is independent of Δ. Therefore, by Helly's selection theorem, there exist limit functions ξ, η and ζ, s.t.

$$\|f\|^2_{L^2(H;(-\infty,+\infty))} = \int_{-\infty}^{+\infty} (F, G) d \begin{pmatrix} \xi & \eta \\ \eta & \zeta \end{pmatrix} \begin{pmatrix} \overline{F} \\ \overline{G} \end{pmatrix}$$

for all $f \in L^2(H;(-\infty,+\infty))$. The matrix measure $\begin{pmatrix} \xi & \eta \\ \eta & \zeta \end{pmatrix}$ is known as the *spectral matrix* of the boundary value problem (2.10).

Similar to the half-line problem, the spectral matrix can be calculated using the Titchmarsh-Weyl m-functions. Let m_1, m_2 be the Titchmarsh-Weyl m-functions for the canonical system on $[0,+\infty)$, $(-\infty,0]$, respectively, then

$$\xi(\lambda) = \lim_{y\to 0} \frac{1}{\pi} \int_0^\lambda -\Im \frac{1}{m_1(x+iy) - m_2(x+iy)} dx, \tag{2.17}$$

$$\eta(\lambda) = \lim_{y\to 0} \frac{1}{\pi} \int_0^\lambda -\Im \frac{1}{2} \frac{m_1(x+iy) + m_2(x+iy)}{m_1(x+iy) - m_2(x+iy)} dx, \tag{2.18}$$

$$\zeta(\lambda) = \lim_{y\to 0} \frac{1}{\pi} \int_0^\lambda -\Im \frac{m_1(x+iy)m_2(x+iy)}{m_1(x+iy) - m_2(x+iy)} dx. \tag{2.19}$$

See [LS75, LS90] for more details.

2.3 Israel Kats' Theorem

In Section 1.3 we showed if Hamiltonian H has a regular left endpoint t_- (i.e., H integrable on a neighborhood of t_-), then there exists a regular dB-chain $\{\mathcal{B}(E_t)\}_{t\in I}$ with H as its Hamiltonian. In this section we discuss the necessary and sufficient conditions for the existence of a dB-chain, for a given Hamiltonian with a *singular* left endpoint. These results are summarized in the main theorem of this section, namely Theorem 2.3, which was announced by Kac in 1995 in [Kac95], and proved in 2007 in [Kat07]. Here the proof of the sufficiency part of Theorem 2.3 was given by Kats, and we present a simpler proof of the necessity part.

Theorem 2.3. *Let $H(t)$, $t \in I := (-\infty,+\infty)$ be a normalized Hamiltonian, s.t. there is no minimal H-ordinary point and $0 < \int_{-\infty}^0 H_{11}(t)dt < +\infty$, then the following are equivalent:*

(i) There exists a dB-chain $\{\mathcal{B}(E_t)\}_{t\in I}$ with H as its Hamiltonian, i.e.,

$$\Omega \begin{pmatrix} A_b \\ C_b \end{pmatrix} - \Omega \begin{pmatrix} A_a \\ C_a \end{pmatrix} = z \int_a^b H(t) \begin{pmatrix} A_t \\ C_t \end{pmatrix} dt, \quad \forall t_- < a < b < t_+,$$

(ii) The Titchmarsh-Weyl m-function $m(z)$ for the corresponding canonical system on interval $(-\infty,0]$ can be extended to a meromorphic function.

Remark. As discussed in Section 2.1, the spectrum of the boundary value problem

$$\begin{aligned} &\Omega \dot{X} = zHX, \quad t \in I := [0,+\infty), \\ &X(0,z) = \begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix}, \end{aligned} \tag{2.20}$$

is discrete if and only if the Titchmarsh-Weyl m-function $m_\alpha(z)$ has a meromorphic extension whose poles coincide with the spectrum. Recall that $m(z) = m_0(z)$, therefore we can replace the condition (ii) by: *the spectrum of the boundary value problem* (2.20) *for* $\alpha = 0$ *is discrete.*

The reason we only consider the case that H doesn't have a *minimal H-ordinary* point is that this condition is necessary for the existence of a dB-chain when t_- is a *singular* left endpoint of H.

Proposition 2.4. *Let $\{\mathcal{B}(E_t)\}_{t\in I}$ be a dB-chain with Hamiltonian H. If t_- is a singular left endpoint of H, then there exists no minimal H-ordinary point on I. Namely, there is no $t_0 \in I$ s.t. (t_-, t_0) is an H-indivisible interval w.r.t. H.*

Proof. WLOG we assume the Hamiltonian H is normalized and defined on $I := (-\infty, \infty)$. Suppose the minimal H-ordinary point t_0 exists, WLOG we assume $t_0 = 0$, then

$$H(t) = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix} = \begin{pmatrix} u^2 & uv \\ uv & v^2 \end{pmatrix}, \quad \forall t < 0.$$

Consequently,

$$M_{a\to b}(z) = \begin{pmatrix} 1-(b-a)uvz & -(b-a)v^2z \\ (b-a)u^2z & 1+(b-a)uvz \end{pmatrix}, \quad \forall t_- < a < b < t_+.$$

Note that for $t < 0$ and $z \in \mathbb{C}\backslash\mathbb{R}$,

$$\begin{aligned}
K_{t,z}(z) &= \frac{\overline{A_t(z)}C_t(z) - A_t(z)\overline{C_t(z)}}{\pi(z-\bar z)} \\
&= \frac{1}{\pi(z-\bar z)}\left\langle \Omega\begin{pmatrix} A_t(z) \\ C_t(z)\end{pmatrix}, \begin{pmatrix} A_t(z) \\ C_t(z)\end{pmatrix}\right\rangle \\
&= \frac{1}{\pi(z-\bar z)}\left\langle \Omega M_{t\to 0}^{-1}\begin{pmatrix} A_0(z) \\ C_0(z)\end{pmatrix}, M_{t\to 0}^{-1}\begin{pmatrix} A_0(z) \\ C_0(z)\end{pmatrix}\right\rangle \\
&= -\frac{1}{\pi(z-\bar z)}\left(A_0(z), C_0(z)\right)\begin{pmatrix} 1-tuvz & tu^2z \\ -tv^2z & 1+tuvz\end{pmatrix}\Omega\begin{pmatrix} 1-tuv\bar z & -tv^2\bar z \\ tu^2\bar z & 1+tuv\bar z\end{pmatrix}\begin{pmatrix} \overline{A_0(z)} \\ \overline{C_0(z)}\end{pmatrix} \\
&= \frac{1}{\pi(z-\bar z)}\left(A_0(z), C_0(z)\right)\left(-\Omega + \begin{pmatrix} tu^2(z-\bar z) & 0 \\ 0 & tv^2(z-\bar z)\end{pmatrix}\right)\begin{pmatrix} \overline{A_0(z)} \\ \overline{C_0(z)}\end{pmatrix} \\
&= K_{0,z}(z) + \frac{tu^2}{\pi}\left|A_0(z)\right|^2 + \frac{tv^2}{\pi}\left|C_0(z)\right|^2.
\end{aligned}$$

Then for $z \in \mathbb{C}_+$, as $|A_0(z)| > 0$ and $|C_0(z)| > 0$, we get $K_{t,z}(z) \to -\infty$ as $t \to -\infty$, a contradiction to the asymptotic assumption in the definition of a dB-chain. $\square$

First we consider the case $I = [0, +\infty)$ and $\lim_{t\to+\infty}\alpha(t) < \infty$. The main idea of the proof of the sufficiency part of Theorem 2.3 (i.e., (ii)$\Rightarrow$(i)) is to "flip" the interval $[0, +\infty)$ to get the existence of a dB-chain for $t \in (-\infty, 0]$. This will be explained in the proof of the sufficiency part of Theorem 2.3 below.

Lemma 2.5. *If the canonical system*

$$\Omega \dot{X} = zHX, \quad t \in I = [0, +\infty) \tag{2.21}$$

satisfies $\lim_{t\to+\infty} \alpha(t) < \infty$, *$H$ is normalized and doesn't have a maximal H-ordinary point, and if the Titchmarsh-Weyl m-function $m(z)$ has a meromorphic extension, then $m(0) = 0$, and there exists a vector function $\theta(t,z) = \binom{\theta_1(t,z)}{\theta_2(t,z)}$ s.t.*

(i) $\forall t \in I$, *$\theta_1(t,z)$ and $\theta_2(t,z)$ are real entire functions of z,*

(ii) $\theta(t,0) = \binom{1}{0}$, $\forall t \in I$,

(iii) $\forall z \in \mathbb{C}$, *$\theta(t,z)$ is a nonzero solution of the canonical equation s.t.*

$$\theta(0,z) = \begin{pmatrix} Q(z) \\ -P(z) \end{pmatrix},$$

where P, Q are real entire, have no nonreal zeros and no common real zeros, $P(0) = 0$ and $Q(0) = 1$,

(iv) $\forall z \in \mathbb{C}$,

$$\lim_{t\to+\infty} \langle \Omega\theta(t,z), \theta(t,z)\rangle = \lim_{t\to+\infty} \left(\overline{\theta_1(t,z)}\theta_2(t,z) - \theta_1(t,z)\overline{\theta_2(t,z)}\right) = 0.$$

Proof. By definition in Section 1.1, $Y_1(t,z)$ is a solution to (2.1) s.t. $Y_1(t,0) \equiv \binom{1}{0}$, $\forall t \in I$. Let $X_1 = X_2 = Y_1$ in the Lagrange's identity (2.2), then

$$\|Y_1(t,0)\|^2_{L^2(H;[0,+\infty))} = \int_0^{+\infty} (1,0)H(t)\begin{pmatrix}1\\0\end{pmatrix}dt = \int_0^{+\infty} H_{11}(t)dt < \infty.$$

Therefore 0 is in the spectrum of the boundary value problem

$$\Omega\dot{X} = zHX, \quad t \in [0,+\infty),$$
$$X(0,z) = \begin{pmatrix}1\\0\end{pmatrix}.$$

Then by the discussion at the end of Section 2.1 we know $m(0) = m_0(0) = 0$.

As m is meromorphic and $m^{\#}(z) = m(z)$ for $z \in \mathbb{C}\backslash\mathbb{R}$, we can find real entire functions P and Q s.t.

$$m(z) = \frac{P(z)}{Q(z)}, \quad P(0) = 0, \quad Q(0) = 1.$$

Since m doesn't have nonreal zeros or poles, we can choose P, Q not to have nonreal zeros or common real zeros.

Now let

$$\begin{pmatrix}\theta_1(t,z)\\ \theta_2(t,z)\end{pmatrix} := \theta(t,z) = Q(z)\psi(t,z) = Q(z)Y_1(t,z) - P(z)Y_2(t,z) = Q(z)\begin{pmatrix}A_t(z)\\ C_t(z)\end{pmatrix} - P(z)\begin{pmatrix}B_t(z)\\ D_t(z)\end{pmatrix},$$

then it's easy to see (i) and (ii) hold.

To prove (iii), note that for fixed z, $\theta(\cdot,z)$ is a linear combination of $Y_1(\cdot,z)$ and $Y_2(\cdot,z)$, and hence is a solution to the canonical equation (2.21), and

$$\theta(0,z) = Q(z)\psi(0,z) - P(z)\phi(0,z) = Q(z)\begin{pmatrix}1\\ 0\end{pmatrix} - P(z)\begin{pmatrix}0\\ 1\end{pmatrix} = \begin{pmatrix}Q(z)\\ -P(z)\end{pmatrix}.$$

And since P, Q don't have nonreal zeros and no common real zeros, $\theta(t,z)$ is a *nonzero* solution to the canonical system (2.21).

(iv) follows from the asymptotic behavior of the Weyl solution ψ,

$$\langle \Omega\theta(t,z), \theta(t,z)\rangle = |Q(z)|^2\langle \Omega\psi(t,z), \psi(t,z)\rangle \to 0, \quad \forall z \in \mathbb{C}\backslash\mathbb{R}$$

as t goes to infinity, by Lemma 2.2. The case $z \in \mathbb{R}$ is obvious as both $\theta_1(t,\cdot)$ and $\theta_2(t,\cdot)$ are real entire. $\square$

Based on Lemma 2.5, we then "flip" the Hamiltonian H to get the following theorem, which establishes the sufficiency of the condition that the Titchmarsh-Weyl m-function is meromorphic, for the existence of a dB-chain.

Proof of Theorem 2.3, the Sufficiency Part. Let $\tilde{H}(t) := H(-t), \forall t \in \tilde{I} = [0,+\infty)$. By Lemma 2.5 we can get $\theta(t,z)$, a solution to the canonical system corresponding to $\tilde{H}$ with spectral parameter z. Let $u(t,z) = \theta(-t,-z)$ for $t \in (-\infty, 0]$. Let $s = -t$, then

$$\begin{aligned}\Omega\frac{du(t,z)}{dt} &= \Omega\frac{d\theta(s,-z)}{ds}\frac{ds}{dt}\\ &= -(-z\tilde{H}(s)\theta(s,-z))\\ &= zH(t)u(t,z).\end{aligned}$$

Denote $u(t,z)$ by

$$u(t,z) = \begin{pmatrix}A_t(z)\\ C_t(z)\end{pmatrix},$$

then by Lemma 2.5(4),

$$\lim_{t\to-\infty}\left(A_t(z)\overline{C_t(z)}-\overline{A_t(z)}C_t(z)\right)=0,\quad \forall z\in\mathbb{C}. \tag{2.22}$$

Now we show $E_t := A_t - iC_t$ is a strict, non-degenerate dB-function for $t \in I$. By Lagrange's identity (2.2) and (2.22), we know

$$u_1(t_0,z)\overline{u_2(t_0,z)}-\overline{u_1(t_0,z)}u_2(t_0,z)=\langle\Omega u(t,z),u(t,z)\rangle=-2i\Im z\int_{-\infty}^{t}\langle H(t)u(t,z),u(t,z)\rangle dt,$$
$$\Im(u_1(t_0,z)\overline{u_2(t_0,z)}-\overline{u_1(t_0,z)}u_2(t_0,z))=-2\Im z\|u\|^2_{L^2(H;(-\infty,t_0])}<0$$

for $z \in \mathbb{C}_+$. Hence

$$\begin{aligned}
\overline{A_t(z)}C_t(z)-A_t(z)\overline{C_t(z)}&=\left\langle\Omega\begin{pmatrix}A_t\\C_t\end{pmatrix},\begin{pmatrix}A_t\\C_t\end{pmatrix}\right\rangle\\
&=\langle\Omega u(t,z),u(t,z)\rangle\\
&=2i\Im z\|u\|^2_{L^2(H;I)},\\
\Im\overline{A_t(z)}C_t(z)&=\Im z\|u\|^2_{L^2(H;I)}>0,\quad z\in\mathbb{C}_+.
\end{aligned}$$

The last inequality further implies A_t, C_t don't have nonreal zeros. Then for $z \in \mathbb{C}_+$,

$$\begin{aligned}
\Im\overline{A_t(z)}C_t(z)>0&\Rightarrow\Im\frac{C_t(z)}{A_t(z)}>0,\\
&\Rightarrow\left|\frac{C_t(z)}{A_t(z)}-(-i)\right|>\left|\frac{C_t(z)}{A_t(z)}-i\right|,\\
&\Rightarrow|C_t(z)+iA_t(z)|>|C_t(z)-iA_t(z)|,\\
&\Rightarrow|E_t(z)|>\left|E_t^{\#}(z)\right|,
\end{aligned}$$

therefore E_t is a non-degenerate dB-function. Then by Proposition 1.24 together with the asymptotic condition (2.22), $\{\mathcal{B}(E_t)\}_{t\in I}$ is a dB-chain with H as its Hamiltonian.

Moreover, as $\theta(t,z)$ is a nonzero solution to the canonical system, we can't have $A_t(z) = C_t(z) = 0$ for real z. Therefore $E_t(z) = A_t(z) - iC_t(z) \neq 0$ for real z. Namely, E_t is a strict non-degenerate dB-function. The conclusion $E_t(0) = 1$ comes from evaluating the equation

$$\Omega\begin{pmatrix}A_b(z)\\C_b(z)\end{pmatrix}-\Omega\begin{pmatrix}A_a(z)\\C_a(z)\end{pmatrix}=z\int_a^b H(t)\begin{pmatrix}A_t(z)\\C_t(z)\end{pmatrix}dt,$$

at $z = 0$, and the fact

$$\begin{pmatrix}A_0(0)\\C_0(0)\end{pmatrix}=u(0,0)=\theta(0,0)=\begin{pmatrix}Q(0)\\P(0)\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}.$$

Therefore $E_t(0) = 1$, $\forall t \in (-\infty, 0]$. $\square$

So far we have shown (ii)$\Rightarrow$(i) in Theorem 2.3. For the necessity of m being meromorphic, we show $m(z) = -\frac{C_0(z)}{A_0(z)}$ and hence $m(z)$ is meromorphic as it's the quotient of two entire functions.

Proof of Theorem 2.3, the Necessity Part. Let $Y_1(t,z)$, $Y_2(t,z)$ be the solutions to the canonical system

$$\Omega \dot{X} = zHX, \quad t \in (-\infty, 0], \tag{2.23}$$

with boundary values

$$Y_1(0,z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad Y_2(0,z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

respectively. Let

$$\begin{pmatrix} u_1(t,z) \\ u_2(t,z) \end{pmatrix} := u(t,z) := A_0(z)Y_1(t,z) + C_0(z)Y_2(t,z),$$

then $u(t,z)$ is a solution to the canonical system (2.23) and

$$u(0,z) = A_0(z)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_0(z)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} A_0(z) \\ C_0(z) \end{pmatrix}.$$

By uniqueness of the solution to the canonical system with given boundary values, we must have

$$u(t,z) = \begin{pmatrix} A_t(z) \\ C_t(z) \end{pmatrix}, \quad \forall t \leqslant 0.$$

For nonreal z, by Theorem 1.28, $\binom{A_t(z)}{C_t(z)} \in L^2(H; (-\infty, 0])$. Moreover, since $A_0(z) \neq 0$, we must have $Y_1 + \frac{C_0}{A_0} Y_2 \in L^2(H; (-\infty, 0])$. Since we're in the Weyl limit-point case, then $m(z) = -\frac{C_0(z)}{A_0(z)}$ is the Titchmarsh-Weyl m-function for $z \in \mathbb{C} \backslash \mathbb{R}$, and obviously it's meromorphic. $\square$

2.4 General de Branges spaces

In this section we introduce results on chains of dB-spaces which are not necessarily regular. In particular, in Section 2.4.1 we discuss the Bezout operator $T_{A,0}$ and its eigenfunctions. We discuss the existence and uniqueness of the spectral measures for a given chain of dB-spaces in Section 2.4.2 and Section 2.4.3, respectively. For a given Hamiltonian H, the uniqueness of the dB-chains $\{\mathcal{B}(E_t)\}_{t \in I}$ with H as its Hamiltonian is discussed in Section 2.4.4.

2.4.1 Bezout operators and their eigenfunctions

In this section we give an orthogonal set of elements (functions) in $\mathcal{B}(E)$. The orthogonal functions are actually also the eigenfunctions of some particular Bezout operator, namely $T_{A,0}$. We will use this property to show that the Bezout operator $T_{A,0}$ is compact in certain circumstances.

The *sampling formula*

$$\int_{-\infty}^{+\infty} |F(t)|^2 dt = \frac{\pi}{a} \sum_{n=-\infty}^{+\infty} \left| F\left(\frac{n\pi}{a}\right)\right|^2 \tag{2.24}$$

holds for $F \in PW_a$, and is a well-known result in Fourier analysis. One of de Branges' motivations to develop his theory of entire functions was that he found a similar "sampling formula" in a totally different way, and the result itself was meaningful even without any knowledge of Fourier analysis. (2.24) is just a special case of the more general sampling formula where $E = e^{-iaz}$, and $\mathcal{B}(E) = PW_a$.

Theorem 2.6 (Sampling Formula)**.** *Let $\mathcal{B}(E)$ be a nonzero dB-space. For a given $\alpha \in \mathbb{R}$, the functions $\left\{\frac{K_{t_n}(z)}{E(t_n)}\right\}$, where $\{t_n\} = \{t \in \mathbb{R} : e^{i\alpha}E(t) \in \mathbb{R}\}$, form an orthogonal set in $\mathcal{B}(E)$. The only elements of $\mathcal{B}(E)$ which are orthogonal to this set are constant multiples of $e^{i\alpha}E - e^{-i\alpha}E^{\#}$. If this function does not belong to $\mathcal{B}(E)$, then*

$$\int_{-\infty}^{+\infty} \left|\frac{F(t)}{E(t)}\right|^2 dt = \sum_n \frac{|F(t_n)|^2}{K_{t_n}(t_n)}, \tag{2.25}$$

and

$$\mu_\alpha := \sum_{t: e^{i\alpha}E(t)\in\mathbb{R}} \frac{\delta_t}{K_t(t)}$$

are called the sampling measures of $\mathcal{B}(E)$.

In the special case E is strict and normalized, if we choose $\alpha = \frac{\pi}{2}$, then $e^{i\alpha}E(t) \in \mathbb{R}$ if and only if $A(t) = 0$. Let t_ns be the (real) zeros of A, then

$$K_{t_n}(z) = \frac{A(t_n)C(z) - C(t_n)A(z)}{\pi(z - t_n)} = -\frac{C(t_n)}{\pi}\frac{A(z)}{z - t_n},$$

$$\frac{K_{t_n}(z)}{\|K_{t_n}\|_{\mathcal{B}(E)}} = \frac{-\frac{C(t_n)}{\pi}\frac{A(z)}{z-t_n}}{\sqrt{-\frac{C(t_n)}{\pi}A'(t_n)}} = \pm\sqrt{-\frac{C(t_n)}{\pi A'(t_n)}}\frac{A(z)}{z - t_n},$$

where the $\pm$ depends on whether $C(t_n) > 0$ or not. Therefore we can get an orthonormal basis

$$\left\{\sqrt{-\frac{C(t_n)}{\pi A'(t_n)}}\frac{A(z)}{z - t_n}\right\}_{t_n\in Z(A)} \quad \text{or} \quad \left\{\sqrt{-\frac{C(t_n)}{\pi A'(t_n)}}\frac{A(z)}{z - t_n}\right\}_{t_n\in Z(A)} \bigcup \left\{\frac{A}{\|A\|_{\mathcal{B}}}\right\},$$

of $\mathcal{B}(E)$, depending on whether $A \in \mathcal{B}(E)$ or not.

The constant multiples of $e^{i\alpha}E - e^{-i\alpha}E^{\#}$ are actually very special in $\mathcal{B}(E)$. de Branges [dB68, Theorem 29] proved that the only elements in $\mathcal{B}(E)$ that are orthogonal to $\mathrm{dom}_{\mathcal{B}(E)}(z)$ must be of this form:

Theorem 2.7. *For $F \in \mathcal{B}(E)$, the following are equivalent:*

(1) $F \perp \mathit{dom}_{\mathcal{B}(E)}(z)$,

(2) $F = uA + vC$ *for* u, v *s.t.* $u\bar{v} \in \mathbb{R}$*,*

(3) $F = c\left(e^{i\alpha}E - e^{-i\alpha}E^{\#}\right)$ *for some* $c \in \mathbb{C}$ *and* $\alpha \in \mathbb{R}$*.*

With this theorem we can have a better understanding of the structure of a dB-chain as discussed in Section 1.2.4. For any $\mathcal{B}(E_t)$ in a dB-chain $\{\mathcal{B}(E_t)\}_{t\in I}$ s.t. $e^{i\alpha}E_t - e^{-i\alpha}E_t^{\#}$ for some $\alpha \in \mathbb{R}$, t must be H-special or a right endpoint of an H-indivisible interval. If such elements are absent in $\mathcal{B}(E_t)$, then t must be H-ordinary except for being a right endpoint of an H-indivisible interval.

Now we consider the Bezout operator $T_{A,0}$ acting on dB-space $\mathcal{B}(E)$ where E is strict and normalized. By the definition of Bezout operator (1.15) in Section 1.2.2, we know

$$(T_{A,0}F)(z) = \frac{F(z) - A(z)F(0)}{z}.$$

From the discussion above we know there is a complete orthogonal set in $\mathcal{B}(E)$:

$$\left\{\frac{A(z)}{z - t_n}\right\}_{t_n \in Z(A)} \quad \text{or} \quad \left\{\frac{A(z)}{z - t_n}\right\}_{t_n \in Z(A)} \cup \{A\}.$$

Obviously, we have

$$T_{A,0}\frac{A}{z - t_n} = \frac{1}{t_n}\frac{A(z)}{z - t_n}, \quad T_{A,0}A = 0 \in \mathcal{B}(E). \tag{2.26}$$

Since $A(0) = 1$, $|t_n|$ has a positive lower bound. $T_{A,0}$ is a well-defined bounded operator which maps $\mathcal{B}(E)$ to $\overline{\operatorname{dom}_{\mathcal{B}(E)}(z)}$. Actually, the Bezout operator $T_{A,0}$ is the left inverse of multiplication by z at 0: it's easy to check $\forall F \in \operatorname{dom}_{\mathcal{B}(E)}(z)$, we have $T_{A,0}(zF) = F$.

By (2.26) it's easy to see that we can approximate $T_{A,0}$ (in the norm topology) by finite rank operators, therefore $T_{A,0}$ is compact. This fact echoes Theorem 2.3 and explains why the de Branges theory only applies if the formal differential operator $H^{-1}\Omega\frac{d}{dt}$ has a compact resolvent on $(-\infty, 0]$ (in the full-line problem). The case that $T_{A,0}$ belongs to the $(2k+2)$-th Schatten class is discussed later in Section 4.5, and we will show it's closely related to the assumption that the generating dB-function E is in the generalized Pólya class $\mathcal{P}_{\leqslant k}$.

2.4.2 Existence of scalar spectral measures

In this section we show that any strict dB-chain $\{\mathcal{B}(E_t)\}_{t\in I}$ has at least one spectral measure. The uniqueness of the spectral measures will be established under certain assumptions in Section 2.4.3.

In Section 1.2.4, we say a measure μ is associated with dB-space $\mathcal{B}$ if $\mathcal{B}$ sits almost isometrically in $L^2(\mu)$. In order to show the existence and uniqueness of the spectral measures, we give an equivalent statement for μ to be a spectral measure of a dB-chain, which is technical and less intuitive compared to the definition, but easier to work with in order to do calculations. The following theorem was proved by de Branges [dB68, Theorem 32].

Theorem 2.8. *Let $\mathcal{B}(E)$ be a nonzero dB-space. If a positive measure μ on $\mathbb{R}$ is associated with a dB-space $\mathcal{B}(E)$, then there exists $W \in \mathcal{A}(\mathbb{C}_+)$ that is bounded by 1 and $p \geqslant 0$, s.t.*

$$\Re \frac{E(z)+E^{\#}(z)W(z)}{E(z)-E^{\#}(z)W(z)} = py + \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{|E(\lambda)|^2 d\mu(\lambda)}{(\lambda-x)^2+y^2} \tag{2.27}$$

where $z = x+iy \in \mathbb{C}_+$. Moreover, $dom_{\mathcal{B}(E)}(z)^{\perp}$ sits isometrically in $L^2(\mu)$ if and only if $p = 0$.

On the other hand, for any analytic function bounded by 1 on $\mathbb{C}_+$, there exists $p \geqslant 0$ and a positive measure μ on $\mathbb{R}$ s.t. (2.27) *holds. Any measure μ satisfying* (2.27) *is associated with the dB-space $\mathcal{B}(E)$.*

Based on this theorem, the spectral measures of a dB-chain can be similarly characterized as well.

Theorem 2.9. *Let $\{\mathcal{B}(E_t)\}_{t\in I}$ be a dB-chain. A positive measure μ on $\mathbb{R}$ is a spectral measure of $\{\mathcal{B}(E_t)\}$ if and only if there exists $W_t \in \mathcal{A}(\mathbb{C}_+)$ which is bounded by 1 and $p_t \geqslant 0$ for $t \in I$, s.t.*

$$\Re \frac{E_t(z)+E_t^{\#}(z)W_t(z)}{E_t(z)-E_t^{\#}(z)W_t(z)} = p_t y + \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{|E_t(\lambda)|^2 d\mu(\lambda)}{(\lambda-x)^2+y^2} \tag{2.28}$$

where $z = x+iy \in \mathbb{C}_+$. Moreover, $t \in I$ is H-ordinary if and only if $p_t = 0$.

Based on Theorem 2.9, we can construct the functions W_t for $t \in I$ and then get a spectral measure for the dB-chain. Before we do that, we first give an important property of the Nevanlinna matrices. For a matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$, we define

$$\tau_M(w) = \frac{aw+b}{cw+d}, \quad w \in \mathbb{C}.$$

Then for Nevanlinna matrices, we have:

Proposition 2.10. *If M is a Nevanlinna matrix, then $\tau_{M^T}(\mathbb{C}_+) \subseteq \mathbb{C}_+$. Moreover, if B and D are linearly independent, then $\tau_{M^T}(\mathbb{C}_+)$ is a disk contained in $\mathbb{C}_+$ with radius $\frac{1}{|\overline{B(z)}D(z)-B(z)\overline{D(z)}|}$.*

Proof. For $\omega \in \mathbb{C}_+$, let $\binom{w_1}{w_2} = M^T\binom{w}{1}$, then by (1.14) we know M^* is J-contractive,

$$\begin{aligned} 2\Im w_1 \bar{w}_2 = -i(w_1\bar{w}_2 - \bar{w}_1 w_2) &= i\left\langle \Omega\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right\rangle \\ &= i\left\langle \Omega M^T\begin{pmatrix} w \\ 1 \end{pmatrix}, M^T\begin{pmatrix} w \\ 1 \end{pmatrix}\right\rangle \\ &= -(w,1)M\underbrace{i\Omega}_{J} M^*\begin{pmatrix} \bar{w} \\ 1 \end{pmatrix} \\ &\geqslant -i(w,1)\Omega\begin{pmatrix} \bar{w} \\ 1 \end{pmatrix} \\ &= 2\Im w > 0. \end{aligned}$$

Therefore $\Im \tau_{M^T} w = \Im \frac{w_1}{w_2} > 0$ for $w \in \mathbb{C}_+$. Proving $\tau_{M^T}(\mathbb{C}_+)$ is a disk is the same as proving the Weyl disk is a disk and the proof can be found in [CL55]. □

Remark. For $M \in SL(2,\mathbb{C})$, $\tau_M(\mathbb{C}_+) \subseteq \mathbb{C}_+$ is actually a necessary and sufficient condition for M to be J-expansive.

Theorem 2.11 (Existence of a Spectral Measure)**.** *Let $\{\mathcal{B}(E_t)\}_{t\in I}$ be a strict dB-chain, then there exists a family of analytic functions $W_t \in \mathcal{A}(\mathbb{C}_+)$ which is bounded by 1, $p_t \geqslant 0$ for $t \in I$, and tehre exists a positive measure μ on $\mathbb{R}$, s.t.*

$$i\frac{1-W_a}{1+W_a} = \tau_{M^T_{a\to b}}\left(i\frac{1-W_b}{1+W_b}\right), \quad \forall t_- < a < b < t_+,$$

where $M_{a\to b}$ are the transition matrices associated with the Hamiltonian H of the dB-chain, and

$$\Re\frac{E_t(z) + E_t^{\#}(z)W_t(z)}{E_t(z) - E_t^{\#}(z)W_t(z)} = p_t y + \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{|E_t(\lambda)|^2 d\mu(\lambda)}{(\lambda - x)^2 + y^2}.$$

Proof. First we show the existence of W_a, $\forall a \in I$. By Proposition 2.10,

$$\Im\frac{E_{a\to b}}{\tilde{E}_{a\to b}} = \Im\tau_{M^T_{a\to b}}(i) > 0, \quad \forall z \in \mathbb{C}_+.$$

Define

$$w_{a,b} = \frac{E_{a\to b}}{\tilde{E}_{a\to b}} \quad \left(= i\frac{1-W_{a,b}}{1+W_{a,b}}\right), \quad W_{a,b} = \frac{i - w_{a,b}}{i + w_{a,b}} = \frac{i - \frac{E_{a\to b}}{\tilde{E}_{a\to b}}}{i + \frac{E_{a\to b}}{\tilde{E}_{a\to b}}},$$

then $W_{a,b} \in \mathcal{A}(\mathbb{C}_+)$ and $|W_{a,b}(z)| < 1$ on $\mathbb{C}_+$. Since $M^T_{a\to c} = M^T_{a\to b}M^T_{b\to c}$, we have

$$\begin{pmatrix} E_{a\to c} \\ \tilde{E}_{a\to c} \end{pmatrix} = M^T_{a\to c}\begin{pmatrix} i \\ 1 \end{pmatrix} = M^T_{a\to b}M^T_{b\to c}\begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} A_{a\to b} & B_{a\to b} \\ C_{a\to b} & D_{a\to b} \end{pmatrix}\begin{pmatrix} E_{b\to c} \\ \tilde{E}_{b\to c} \end{pmatrix},$$
$$i\frac{1-W_{a,c}}{1+W_{a,c}} = w_{a,c} = \tau_{M^T_{a\to b}}(w_{b,c}) = \tau_{M^T_{a\to b}}\left(i\frac{1-W_{b,c}}{1+W_{b,c}}\right).$$

Since $W_{a,b}$ are bounded by 1, we can choose a sequence b_n s.t. W_{a,b_n} goes to $W_a \in \mathcal{A}(\mathbb{C}_+)$ locally uniformly for $z \in \mathbb{C}_+$, therefore

$$i\frac{1-W_a}{1+W_a} = \tau_{M^T_{a\to b}}\left(i\frac{1-W_b}{1+W_b}\right), \quad \forall t_- < a < b < t_+. \tag{2.29}$$

As for the second part, by [dB68, Theorem 32], for each $b \in I$, there exists a positive measure μ_b on $\mathbb{R}$ s.t.

$$\Re\frac{E_b + E_b^{\#}W_b}{E_b - E_b^{\#}W_b} = p_b y + \frac{y}{\pi}\int_{\mathbb{R}} \frac{d\mu_b(t)}{(t-x)^2+y^2} \tag{2.30}$$

for $y > 0$. Now we'll show $\left|\frac{E_a}{E_b}\right|^2 d\mu_b = d\mu_a$. Let

$$M_a(z) := \begin{pmatrix} A_a(z) & -C_a(z) \\ C_a(z) & A_a(z) \end{pmatrix},$$
$$M_b(z) := M_{a\to b}(z)M_a(z) := \begin{pmatrix} A_b(z) & B_b(z) \\ C_b(z) & D_b(z) \end{pmatrix},$$

then it's easy to check both M_a, M_b are dB-matrices (defined as in [dB68, Theorem 27]) with associated function $S = E_a$. By (2.29) it's straightforward to check that

$$\frac{E_a + E_a^{\#}W_a}{E_a - E_a^{\#}W_a} = i\frac{\tilde{E}_b - \tilde{E}_b^{\#}W_b}{E_b - E_b^{\#}W_b}. \tag{2.31}$$

We know for a given dB function E_b and associated function S, the dB pair function (defined as in [dB68, Theorem 27]) is unique up to adding a product of E_b and a linear function in z. In particular, by [dB68, Theorem 27] we can choose $\hat{E}$ s.t. $\lim_{y\to+\infty} \frac{\hat{E}_b(iy)}{iyE_b(iy)} = 0$. Note that such $\hat{E}$ is unique up to adding a real multiple of E. Now by [dB68, Theorem 32], we have

$$\Re i\frac{\hat{E}_b - \hat{E}_b^{\#}W_b}{E_b - E_b^{\#}W_b} = p(E_a, E_a)y + \frac{y}{\pi}\int_{\mathbb{R}} \frac{1}{(t-x)^2+y^2}\left|\frac{E_a}{E_b}\right|^2 d\mu_b(t). \tag{2.32}$$

Let $p = -\lim_{y\to+\infty} \frac{\tilde{E}_b(iy)}{iyE_b(iy)} \geqslant 0$, since both $-\frac{\hat{E}_b}{E_b}$ and $-\frac{\tilde{E}_b + pzE_b}{E_b}$ have nonnegative imaginary parts on $\mathbb{C}_+$ and the same limits on the positive imaginary axis, can be continuously extended to $\mathbb{R}$, and have the same imaginary parts on $\mathbb{R}$, then they differ by a real number. We can choose $\hat{E}_b$ s.t. the real number is 0, then $\hat{E}_b = \tilde{E}_b + pzE_b$. Plug this in (2.32), we get:

$$\Re i \frac{\tilde{E}_b - \tilde{E}_b^{\#} W_b}{E_b - E_b^{\#} W_b} - py = p(E_a, E_a)y + \frac{y}{\pi} \int_{\mathbb{R}} \frac{1}{(t-x)^2 + y^2} \left| \frac{E_a}{E_b} \right|^2 d\mu_b(t).$$

Combining this with (2.31) we get:

$$\Re \frac{E_a + E_a^{\#} W_a}{E_a - E_a^{\#} W_a} = (p(E_a, E_a) + p)y + \frac{y}{\pi} \int_{\mathbb{R}} \frac{1}{(t-x)^2 + y^2} \left| \frac{E_a}{E_b} \right|^2 d\mu_b(t).$$

On the other hand, by our definition of μ_a (see (2.30)), we have

$$\Re \frac{E_a + E_a^{\#} W_a}{E_a - E_a^{\#} W_a} = p_a y + \frac{y}{\pi} \int_{\mathbb{R}} \frac{d\mu_a(t)}{(t-x)^2 + y^2}.$$

Since such a representation is unique, we can conclude $p_a = p(E_a, E_a) + p$ and $d\mu_a = \left| \frac{E_a}{E_b} \right|^2 d\mu_b$. Let $d\mu = \frac{d\mu_b}{|E_b|^2}$, then (2.30) becomes

$$\Re \frac{E_b + E_b^{\#} W_b}{E_b - E_b^{\#} W_b} = p_b y + \frac{y}{\pi} \int_{\mathbb{R}} \frac{|E_b|^2 d\mu(t)}{(t-x)^2 + y^2}$$

and then the proof is complete. □

2.4.3 Discussion on the uniqueness of the spectral measures

In this section we give a necessary and sufficient condition for a strict dB-chain $\{\mathcal{B}(E_t)\}_{t \in I}$ to have a *unique* spectral measure. Let H be the Hamiltonian of $\{\mathcal{B}(E_t)\}$, then the uniqueness of the spectral measures depends on the properties of H near the right endpoint t_+. If there exists a maximal H-ordinary point $b \in I$, then it's easy to check any measure associated with $\mathcal{B}(E_b)$ is a spectral measure for the dB-chain $\{\mathcal{B}(E_t)\}$. Note that if t_+ is a *regular* right endpoint, namely $\int_c^{t_+} H(t)dt$ has finite elements for some $c \in I$, then there must exist a maximal H-ordinary point or we can extend the interval to be $(t_-, t_+]$ so that t_+ is the maximal H-ordinary point. The results are summarized in the following theorem.

Theorem 2.12 (Uniqueness of the Spectral Measures)**.** *Let $\{\mathcal{B}(E_t)\}_{t \in I}$ be a strict dB-chain with Hamiltonian H, and $h(t) := \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$ be an anti-derivative of H. If there's a maximal H-ordinary point $b \in I$, then any measure associated with $\mathcal{B}(E_b)$ is a spectral measure of $\{\mathcal{B}(E_t)\}$. Assume there's no maximal H-ordinary point, then:*

- *(i) If $\lim_{t \to t_+} \alpha(t) + \gamma(t) < \infty$, then $E_{t_+} := \lim_{t \to t_+} E_t$ exists and is strict non-degenerate, and any measure associated with $\mathcal{B}(E_{t_+})$ is a spectral measure of $\{\mathcal{B}(E_t)\}$.*

(ii) If $\lim_{t\to t_+}\alpha(t)+\gamma(t)=\infty$*, then the spectral measure* μ *of dB-chain* $\{\mathcal{B}(E_t)\}$ *is unique and*

$$\overline{\bigcup_{b\ H\text{-}ordinary}\mathcal{B}(E_b)}=L^2(\mu).$$

Proof. In the case $b\in I$ is a maximal H-ordinary point, it's easy to check any measure associated with $\mathcal{B}(E_b)$ is a spectral measure of the dB-chain $\{\mathcal{B}(E_t)\}$ by Theorem 2.8 and Theorem 2.9.

If there's no maximal H-ordinary point and $\lim_{t\to t_+}\alpha(t)+\gamma(t)<\infty$, then $\lim_{b\to t_+}M_{a\to b}=:M_{a\to t_+}$ exists and is a Nevanlinna matrix, and we can define

$$\begin{pmatrix}A_{t_+}\\ C_{t_+}\end{pmatrix}:=M_{a\to t_+}\begin{pmatrix}A_a\\ C_a\end{pmatrix},\quad E_{t_+}:=A_{t_+}-iC_{t_+}.$$

It's easy to see $\forall t\in I$, $\mathcal{B}(E_t)$ sits almost isometrically in $\mathcal{B}(E_{t_+})$, therefore any measure associated with $\mathcal{B}(E_{t_+})$ is a spectral measure of the dB-chain $\{\mathcal{B}(E_t)\}$.

If there's no maximal H-ordinary point and $\lim_{t\to t_+}\alpha(t)+\gamma(t)=\infty$, the uniqueness of the spectral measure is given by [dB68, Theorem 42], and the fact

$$\overline{\bigcup_{b\ H\text{-ordinary}}\mathcal{B}(E_b)}=L^2(\mu)$$

comes from [dB60, Theorem VIII]. $\square$

2.4.4 Discussion on the uniqueness of the dB-chains for a given Hamiltonian

In Section 1.3 we showed in the regular case, namely for Hamiltonian H with a regular left endpoint, there is a *unique* regular dB-chain $\{\mathcal{B}(E_t)\}_{t\in I}$ with H as its Hamiltonian s.t. $E_t(0)=1$, $\forall t\in I$. The proof uses the fact that the limit function $\lim_{t\to t_-}E_{t_-}$ exists and is a constant. Actually, a more general result holds for Hamiltonian with singular left endpoints. In this case, the uniqueness we get is up to a multiplication by a zero-free real entire function.

Theorem 2.13 (Uniqueness of the dB-chains for a given Hamiltonian)**.** *Let* $H(t)$ *be a Hamiltonian for* $t\in I$*. If* H *is associated with strict dB-chains* $\{\mathcal{B}(E_{+,t})\}_{t\in I}$ *and* $\{\mathcal{B}(E_{+,t})\}_{t\in I}$ *s.t.* $E_{+,t}(0)=E_{-,t}(0)=1$*, then there exists a zero-free real entire function* S*, s.t.* $E_{-,t}=SE_{+,t}$*.*

Proof. WLOG we assume $t_+=\infty$. If t_- is a regular left endpoint of H, WLOG we assume $t_-=0$. Let $\{\mathcal{B}(E_t)\}_{t\in I}$ be a dB-chain with H as its Hamiltonian, then $E_0:=\lim_{t\to 0}E_t$ exists as $M_{0\to b}:=\lim_{t\to 0}M_{t\to b}$ exists, for $b\in I$. Since $\lim_{t\to 0}K_{t,z}(z)=0$, let $E_0:=A_0-iC_0$, then A_0 and C_0 must be linearly dependent. Since $A_0(0)=1$ and $C_0(0)=0$, then $C_0\equiv 0$. Therefore A_0 must be zero free. Let $S(z) := \frac{A_{-,0}(z)}{A_{+,0}(z)}$, then from

$$\begin{pmatrix} A_{\pm,t}(z) \\ C_{\pm,t}(z) \end{pmatrix} = M_{0\to t} \begin{pmatrix} A_{\pm,0}(z) \\ 0 \end{pmatrix}$$

we can see $E_{-,t}(z) = S(z)E_{+,t}(z)$.

If t_- is a singular left endpoint of H, WLOG we assume $t_- = -\infty$. By Proposition 2.4, there is no minimal H-ordinary point. Let $\{\mathcal{B}(E_t)\}_{t\in I}$ be a dB-chain with H as its Hamiltonian, then similar to the proof of Theorem 2.3, we define $\tilde{H}(t) = H(-t)$ for $t \geqslant 0$. Since $\binom{A_t(z)}{C_t(z)}$ solves the canonical system for H and is contained in $L^2(H; (-\infty, 0])$ by Theorem 1.28, then $\binom{A_{-t}(-z)}{C_{-t}(-z)}$ is a solution to the canonical system for $\tilde{H}$ that is in $L^2(\tilde{H}; [0, +\infty))$. As $H(t)$ is not integrable on any neighborhood of $+\infty$ and there is no maximal H-ordinary point, we are in Weyl limit point case and the Titchmarsh-Weyl m-function is unique. It's easy to see $\frac{C_0(-z)}{A_0(-z)}$ is the Titchmarsh-Weyl m-function, therefore

$$\frac{C_{+,0}(-z)}{A_{+,0}(-z)} = \frac{C_{-,0}(-z)}{A_{-,0}(-z)}, \quad \forall z \in \mathbb{C}\backslash\mathbb{R}.$$

Let $w = \bar{z}$ in the Lagrange's identity (2.2), we can see

$$C_{+,b}(z)A_{-,b}(z) - C_{-,b}(z)A_{+,b}(z) = C_{+,a}(z)A_{-,a}(z) - C_{-,a}(z)A_{+,a}(z), \quad \forall -\infty < a < b < \infty. \tag{2.33}$$

Then we know for any $b \in I$,

$$C_{+,b}(z)A_{-,b}(z) = C_{-,b}(z)A_{+,b}(z).$$

Since $E_{+,b}$ is strict, $A_{+,b}$ and $C_{+,b}$ can not have common real zeros, and since $A_{+,b}(z)$ and $A_{-,b}(z)$ do not have nonreal zeros, we can conclude

$$S(z) := \frac{A_{-,b}(z)}{A_{+,b}(z)}$$

is a zero-free real entire function. By (2.33) we know $E_{-,z} = SE_{+,z}$, $\forall t \in I$. $\square$

Applying this theorem to chains of dB-functions of Cartwright class we get the following result.

Corollary 2.14. *Let $\{\mathcal{B}(E_{+,t})\}_{t\in I}$, $\{\mathcal{B}(E_{-,t})\}_{t\in I}$ be two strict dB-chains sharing the same Hamiltonian H, s.t. $E_{-,t}(0) = E_{+,t}(0) = 1$. If $E_{+,t}, E_{-,t} \in$ Cart for some $t \in I$, then $E_{-,t} = E_{+,t}$, $\forall t \in I$.*

Proof. By Theorem 2.13 we know $E_{-,t} = SE_{+,t}$ for some zero-free real entire function S. Since $E_{-,t}, E_{+,t} \in$ Cart, then $S \in$ Cart. Any zero-free real entire Cartwright function must be a constant according to its canonical factorization, and since $S(0) = 1$ we get $S \equiv 1$. $\square$

‘

2.A Proof of Lemma 2.2

In particular, $l_t(z,\infty) = \frac{A_t}{B_t}$ lies on the Weyl circle at t. Since $m(z)$ lies inside the (closed) Weyl disk at t, we have

$$|l(z) - l_t(z,\infty)| \leqslant 2r_t(z).$$

Since $A_t - l_t(z,\infty)B_t = A_t - \frac{A_t}{B_t}B_t = 0$, we have

$$\begin{aligned}
&\langle \Omega(Y_1(t,z) - l_t(z,\infty)Y_2(t,z)), Y_1(t,z) - l_t(z,\infty)Y_2(t,z)\rangle \\
=&\left\langle \Omega\begin{pmatrix} 0 \\ C_t - \frac{A_tD_t}{B_t}\end{pmatrix}, \begin{pmatrix} 0 \\ C_t - \frac{A_tD_t}{B_t}\end{pmatrix}\right\rangle \\
=&\left\langle \begin{pmatrix} C_t - \frac{A_tD_t}{B_t} \\ 0\end{pmatrix}, \begin{pmatrix} 0 \\ C_t - \frac{A_tD_t}{B_t}\end{pmatrix}\right\rangle \\
=&0.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\langle \Omega(\psi(t,z) - (l_t(z,\infty) - m(z))Y_2(t,z)), \psi(t,z) - (l_t(z,\infty) - m(z))Y_2(t,z)\rangle = 0, \\
&\langle \Omega\psi(t,z), \psi(t,z)\rangle \\
&\qquad = 2\Re\underbrace{\langle \Omega\psi(t,z), (l_t(z,\infty) - m(z))Y_2(t,z)\rangle}_{\mathrm{I}} - \underbrace{|l_t(z,\infty) - m(z)|^2\langle \Omega Y_2(t,z), Y_2(t,z)\rangle}_{\mathrm{II}}.
\end{aligned}$$

For I, by Lagrange's identity (2.2), we know

$$|\langle \Omega\psi(t,z), Y_2(t,z)\rangle - \langle \Omega\psi(0,z), Y_2(0,z)\rangle| = 2\,|\Im z|\left|(\psi(t,z), Y_2(t,z))_{L^2(H;[0,t])}\right|.$$

Then by the Cauchy-Schwarz inequality,

$$\begin{aligned}
|\langle \Omega\psi(t,z), Y_2(t,z)\rangle| &\leqslant O(1) + 2\,|\Im z|\,\|\psi(t,z)\|_{L^2(H;[0,t])}\|Y_2(t,z)\|_{L^2(H;[0,t])} \\
&= O\left(\|Y_2(t,z)\|_{L^2(H;[0,t])}\right).
\end{aligned}$$

Note that by the formula of radius r_t (2.3) and (2.4),

$$|l_t(z,\infty) - l(z)| \leqslant 2r_t(z) = \frac{1}{\Im z\|Y_2\|^2_{L^2(H;[0,t])}}.$$

Then

$$\mathrm{I} = O\left(\frac{1}{\|Y_2(t,z)\|_{L^2(H;[0,t])}}\right) \to 0, \quad \text{as } t \to +\infty.$$

For II, similarly, by Lagrange's identity (2.2) and Cauchy-Schwarz Inequality,

$$\langle \Omega Y_2(t,z), Y_2(t,z)\rangle = O(1) + 2\Im z\|Y_2(t,z)\|^2_{L^2(H;[0,t])}.$$

Hence

$$\mathrm{II} = O\left(\frac{1}{\|Y_2(t,z)\|^2_{L^2(H;[0,t])}}\right) \to 0, \quad \text{as } t \to +\infty,$$

which completes the proof of this lemma.

Chapter 3

Kreĭn's strings, entrance type condition, and de Branges' Theorem 41

In this chapter we review some known results in the spectral theory of canonical systems with singular left endpoints. We introduce Kreĭn's strings and treat them as a special type of canonical system, and we show the relation between Kreĭn's strings (or equivalently, diagonal Hamiltonian), symmetric (i.e., even) measures and dB-spaces which are symmetric about the origin. We briefly discuss Kotani's results [Kot75, Kot07, Kot13] on Kreĭn's strings in Section 3.1. In Section 3.2 we introduce de Branges' Theorem 41 in [dB68] on the existence of a dB-chain for a given Hamiltonian satisfying certain integrability condition near t_-. de Branges' Theorem 41 is more general than some of the results on Kreĭn's strings as it considers more general asymptotic condition on E_t as $t \to t_-$, instead of assuming E_t converges to an entire function as $t \to t_-$. We also prove the converse of de Branges' Theorem 41 and show the connection between the assumption that $\lim_{t\to t_-} E_t(z)e^{\beta(t)z}$ exists and the assumption that E_t is in the Pólya class $\mathcal{P}_0$.

3.1 Kreĭn's strings

Kreĭn's string is a special type of canonical system where the Hamiltonian H is diagonal. It was first studied in early 1950's by Kreĭn [Kre52] as a generalization of the classical theory of Stieltjes on the moment problem and continued fractions. Since then Kreĭn's strings have been studied extensively due to its comparative simplicity over the general canonical system. Kreĭn's strings also appear in other contexts, for instance in the one-dimensional diffusion processes [Fel57], in the prediction of stationary Gaussian processes [DM70, DM08]. In this section we formulate Kreĭn's strings in the framework of de Branges theory, and present partial results on the direct and inverse spectral problems for Kreĭn's strings with singular left endpoints.

3.1.1 Canonical systems with diagonal Hamiltonian

A Kreĭn's string is a second order ordinary differential equation

$$-\ddot{u} = z\rho u, \quad t \in I := (t_-, t_+),$$

where $\rho = \rho(t)$ is a locally integrable function which takes *positive* values almost everywhere, and is interpreted as the "density of the string." The formulation of Kreĭn's string we consider is slightly different. Indeed, we use the spectral parameter z^2, rather than z:

$$-\ddot{u} = z^2 \rho u, \quad t \in I := (t_-, t_+). \tag{3.1}$$

For $z \neq 0$, let $v = -\frac{\dot{u}}{z}$, then (3.1) can be written as

$$\Omega \dot{X} = z \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix} X, \quad t \in I$$

where the Hamiltonian $\begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}$ is diagonal. By re-parametrization, this is equivalent to the following form which is more general.

Definition 3.1. A *Kreĭn's string* is a canonical system

$$\Omega \dot{X} = zHX, \quad \forall t \in I, \tag{3.2}$$

where the Hamiltonian H is diagonal.

de Branges [dB62b] showed Kreĭn's strings correspond to a certain type of de Branges space which has certain "symmetry" property that $F(z) \in \mathcal{B}$ implies $F(-z) \in \mathcal{B}$. Such de Branges spaces are called *even* or *symmetric about the origin*, and the generating de Branges functions are also "symmetric" in the following sense (cf. [dB62b, Theorem I]).

Proposition 3.2. *Let $\mathcal{B}$ be a nonzero dB-space, then the following are equivalent:*

(i) $F(-z) \in \mathcal{B}$ *whenever* $F(z) \in \mathcal{B}$,

(ii) $\mathcal{B} = \mathcal{B}(E)$ *for a dB-function* E *s.t.* $E^{\#}(z) = E(-z)$.

If $\mathcal{B}$ contains an element F s.t. $F(0) \neq 0$, then E may be chosen to be normalized, and such E is unique.

Remark. Note that $E^{\#}(z) = E(-z)$ is equivalent to $A(z) = A(-z)$ and $C(z) = -C(-z)$.

If a dB-space $\mathcal{B}$ is symmetric about the origin, then any of its dB-subspaces is also symmetric about the origin, as shown in the following proposition. This result was stated by de Branges in [dB68, Section 47] as a problem without giving a proof. More results can be found in [dB68, Section 47], especially for dB-chain s.t. $E_t^{\#}(-z) = E_t(z)$, but that's beyond the scope of this dissertation and it's inevitable to include too many other preparatory results to give complete proofs.

Proposition 3.3. *(i) Let E be a strict non-degenerate dB-function s.t. $\frac{E^{\#}(-z)}{E(z)} \in \mathcal{N}(\mathbb{C}_+)$, $\mathcal{B}(E) \sqsubseteq L^2(\mu)$ for a symmetric measure μ, then $\mathcal{B}(E)$ is symmetric about the origin.*

(ii) Let E_a, E_b be two strict non-degenerate dB-functions. If $\mathcal{B}(E_a) \sqsubseteq \mathcal{B}(E_b)$ and $\mathcal{B}(E_b)$ is symmetric about the origin, then $\mathcal{B}(E_a)$ is also symmetric about the origin.

(iii) Let $\{\mathcal{B}(E_t)\}_{t\in I}$ be a strict dB-chain. If $E_t^{\#}(-z) = E_t(z)$, $\forall t \in I$, then its Hamiltonian H is diagonal.

Proof. (i) Firstly, $\tilde{E}(z) := E^{\#}(-z)$ is a strict non-degenerate dB-function as well as

$$\left|\tilde{E}(z)\right| = \left|E^{\#}(-z)\right| = |E(-\bar{z})| > |E(-z)| = \left|E^{\#}(-\bar{z})\right| = \left|\tilde{E}(\bar{z})\right|, \quad \forall z \in \mathbb{C}_+.$$

Then $\mathcal{B}(\tilde{E})$ is a nonzero dB-space, and it's easy to see $F(z) \in \mathcal{B}(E)$ if and only if $F(-z) \in \mathcal{B}(\tilde{E})$ by Proposition 1.8(ii). Since μ is symmetric, $\mathcal{B}(\tilde{E})$ sits in $L^2(\mu)$ isometrically. By the ordering theorem, namely Theorem 1.12, either $\mathcal{B}(\tilde{E}) \sqsubseteq \mathcal{B}(E)$ or $\mathcal{B}(E) \sqsubseteq \mathcal{B}(\tilde{E})$. We now show $\mathcal{B}(\tilde{E}) = \mathcal{B}(E)$. If $\mathcal{B}(E) \sqsubseteq \mathcal{B}(\tilde{E})$, let $G \in \mathcal{B}(\tilde{E}) \ominus \mathcal{B}(E)$, then

$$\int_{-\infty}^{+\infty} F(t)\overline{G(t)}d\mu(t) = 0, \quad \forall F \in \mathcal{B}(E).$$

Since μ is symmetric, we can get

$$\int_{-\infty}^{+\infty} F(-t)\overline{G(-t)}d\mu(t) = 0, \quad \forall F \in \mathcal{B}(E).$$

Since $F(z) \in \mathcal{B}(E)$ if and only if $F(-z) \in \mathcal{B}(\tilde{E})$, then $G(-z) \in \mathcal{B}(E)$ is orthogonal to $\mathcal{B}(\tilde{E})$. Therefore $G(-z) = 0$ and we can conclude $\mathcal{B}(\tilde{E}) = \mathcal{B}(E)$. Similarly if $\mathcal{B}(\tilde{E}) \sqsubseteq \mathcal{B}(E)$ we can show $\mathcal{B}(\tilde{E}) = \mathcal{B}(E)$ by the same arguments. Therefore $\mathcal{B}(E)$ is symmetric about the origin.

(ii) Since $\mathcal{B}(E_b)$ is symmetric about the origin, we have $\mathcal{B}(\tilde{E}_a) \sqsubseteq \mathcal{B}(E_b)$. Then by the ordering theorem again, either $\mathcal{B}(E_a) \sqsubseteq \mathcal{B}(\tilde{E}_a)$ or $\mathcal{B}(\tilde{E}_a) \sqsubseteq \mathcal{B}(E_a)$. Using the same arguments as in part (i) one can show $\mathcal{B}(\tilde{E}_a) = \mathcal{B}(E_a)$. The condition $\frac{E_a^{\#}(-z)}{E_a(z)} \in \mathcal{N}(\mathbb{C}_+)$ is satisfied because $\frac{E_a^{\#}(-z)}{E_b(z)} \in \mathcal{N}(\mathbb{C}_+)$ and $\frac{E_a(z)}{E_b(z)} \in \mathcal{N}(\mathbb{C}_+)$.

(iii) Since $E_t^{\#}(-z) = E_t(z)$, then $A_t(z) = A_t(-z)$ and $C_t(z) = -C_t(-z)$, $\forall t \in I$. In particular,

$A_t'(0) = 0$ and $C_t(0) = 0$. From the canonical system

$$\Omega\begin{pmatrix} A_b \\ C_b \end{pmatrix} - \Omega\begin{pmatrix} A_a \\ C_a \end{pmatrix} = z\int_a^b H(t)\begin{pmatrix} A_t \\ C_t \end{pmatrix} dt$$

it's easy to see $A_b(0) = A_a(0) = E_a(0) \neq 0$, and by taking derivatives w.r.t. z on both sides and evaluating the equation at 0, we can get

$$-A_b'(0) + A_a'(0) = A_t(0)\int_a^b H_{21}(t)dt.$$

Since $A_t'(0) = 0$, $\forall t \in I$, $H_{21} \equiv 0$ for a.e. $t \in I$, then H is diagonal.

□

On the other hand, for a diagonal Hamiltonian H, its transition matrices $M_{a\to b}$ also have some symmetry properties as shown below.

Lemma 3.4. *Let $H = H(t)$, $t \in I$ be a diagonal Hamiltonian with transition matrices $M_{a\to b}$, then*

$$\begin{aligned} A_{a\to b}(z) &= A_{a\to b}(-z), \qquad B_{a\to b}(z) = -B_{a\to b}(-z), \\ C_{a\to b}(z) &= -C_{a\to b}(-z), \quad D_{a\to b}(z) = D_{a\to b}(-z), \end{aligned} \tag{3.3}$$

i.e., $A_{a\to b}$, $D_{a\to b}$ are even, $B_{a\to b}$, $C_{a\to b}$ are odd, $E^{\#}_{a\to b}(z) = E_{a\to b}(z)$ and $\tilde{E}^{\#}_{a\to b}(-z) = -\tilde{E}_{a\to b}(z)$.

Proof. By [dB68, Theorem 38] the associated transition matrices $M_{a\to b}$ must be unique, and it's easy to check

$$\begin{pmatrix} A_{a\to b}(-z) & -B_{a\to b}(-z) \\ -C_{a\to b}(-z) & D_{a\to b}(-z) \end{pmatrix}$$

solves the same canonical system for strings, namely (4.26). □

Therefore, if t_- is a *regular* left endpoint of the diagonal Hamiltonian H, the unique dB-chain $\{\mathcal{B}(E_t)\}_{t\in I}$ with Hamiltonian H satisfies $E_t^{\#}(-z) = E_t(z)$. For a detailed introduction to the theory of Kreĭn's strings from the point of view of de Branges theory, we refer the readers to [Dym71]. A survey of known results on the spectral theory of Kreĭn's strings was presented by Kac [Kac95]. Later in Section 3.2 and Section 4.3 we will see such a "symmetric" dB-chain can also be constructed when t_- is a *singular* left endpoint of H if H satisfies certain integrability conditions near t_-.

3.1.2 Entrance type condition

We're interested in the case when a diagonal Hamiltonian H is the Hamiltonian of a dB-chain. From Theorem 2.3 we know for this purpose, the spectrum of the canonical system on $(t_-, c]$ must be discrete for some $c \in (t_-, t_+)$. The exact condition on H s.t. the spectrum is discrete is unknown for general canonical systems, but for Kreĭn's strings, Kac and Kreĭn [KK58] give a necessary and sufficient condition for the discreteness of the spectrum, hence the existence of a dB-chain with Hamiltonian H. By Proposition 1.25 we're mostly interested in the case

$$\begin{aligned} &\alpha(t_-) := \lim_{t \to t_-} \alpha(t) = 0, \\ &\alpha(t) > 0, \quad \forall t \in I, \end{aligned} \tag{3.4}$$

where $h(t) := \begin{pmatrix} \alpha(t) & 0 \\ 0 & \gamma(t) \end{pmatrix}$ is an anti-derivative of the Hamiltonian H. In this case the necessary and sufficient condition given by Kac and Kreĭn [KK58] has a simple expression.

Theorem 3.5. *Let $H(t)$, $t \in (t_-, t_+)$ be a Hamiltonian satisfying* (3.4)*, then the spectrum for the canonical system* (3.2) *on $(t_-, c]$ is discrete if and only if* $\lim_{t \to t_-} \alpha(t)\gamma(t) = 0$.

Kotani [Kot75] considered the entrance type condition (named by Feller [Fel66] in the context of diffusion processes)

$$\int_{t_-}^{c} \alpha(t) d\gamma(t) < \infty, \quad c \in (t_-, t_+), \tag{3.5}$$

which clearly implies $\lim_{t \to t_-} \alpha(t)\gamma(t) = 0$. Kotani proved if a Hamiltonian satisfies (3.4) and (3.5), then there exists a dB-chain $\{\mathcal{B}(E_t)\}_{t \in I}$ with H as its Hamiltonian. The proof is similar to the proof in the regular case, where one uses Picard's iteration method on (t_-, t) for $t \in I$ to get a solution and show it forms a chain of dB-functions. Alternatively, one can show $E_{a \to t}$ has a finite limit as $a \to t_-$, under the assumption (3.5).

The original formulation in [Kot75] of entrance type condition is different from but actually equivalent to (3.5). Later in Section 3.2 we will see this condition on H is both sufficient and necessary for the existence of a dB-chain $\{\mathcal{B}(E_t)\}_{t \in I}$ having Hamiltonian H in the general case (i.e., H is not necessarily diagonal), and the dB-functions E_t belong to the Pólya class $\mathcal{P}_0$ of entire functions, which will be discussed in detail in Section 3.2.1. For this reason, (3.5) is also known as the *Pólya condition*.

On the other hand, Kotani [Kot75, Theorem 4.4] showed a symmetric measure μ is a spectral measure of a dB-chain in the Pólya class (see Definition 3.9) under certain assumptions, as shown below.

Theorem 3.6. *Let μ be a symmetric measure on $\mathbb{R}$. Let $\mu = \mu_s + \mu_a$ be the decomposition where μ_s is singular w.r.t. the Lebesgue measure on $\mathbb{R}$ and μ_a is absolute continuous w.r.t. the Lebesgue measure, then if*

$$\int_{-\infty}^{+\infty} \frac{|\log \mu_a'(t)|}{1+t^2} dt < \infty$$

and either of the following holds:

(i) $\int_0^{+\infty} e^{-r(t)} d\mu_a < \infty$ *with a positive increasing* r *s.t.* $\int_1^{+\infty} \frac{r(t)}{t^2} dt < \infty$,

(ii) $\log \mu_a'(t)$ *is uniformly continuous in* $t \in [0, +\infty)$,

then there exists a dB-chain $\{\mathcal{B}(E_t)\}_{t\in I}$ *with* H *as its Hamiltonian, s.t.* $E_t \in \mathcal{P}_0$ *and* $E_t^{\#}(-z) = E_t(z)$, $\forall t \in I$.

In the simpler case that the singularity of H is of "polynomial type," Kotani [Kot07, Theorem 7] showed it corresponds to a "polynomial growth" condition on the corresponding spectral measure. Here the meaning of the spectral measure is described in [Kat94] without using dB-chain as intermediates.

Theorem 3.7. *For* $n \geqslant 1$*, if a diagonal Hamiltonian* H *satisfies*

$$\int_{t_-}^{c} \alpha(t)^{n-1} dx < \infty, \quad c \in (t_-, t_+), \tag{3.6}$$

then the spectral measure satisfies

$$\int_{-\infty}^{+\infty} \frac{d\mu(t)}{1+t^{2n}} < \infty. \tag{3.7}$$

Conversely, if a symmetric measure μ *on* $\mathbb{R}$ *satisfies* (3.7)*, then it's the spectral measure for a Kreĭn's string that satisfies* (3.6)*.*

The cases $n = 2, 3$ can also be found in [DM08, Section 6.12]. This result was later enhanced by Kotani in [Kot13].

A more recent result was obtained by Eckhardt [Eck13], where the one-to-one correspondence between discrete measures satisfying certain summability condition and diagonal Hamiltonian satisfying the entrance type condition at both endpoints is established.

Theorem 3.8. *Let* $H(t)$ *be a Hamiltonian on a finite interval* $I = (t_-, t_+)$ *and* $h(t) = \begin{pmatrix} \alpha(t) & 0 \\ 0 & \gamma(t) \end{pmatrix}$ *be its anti-derivative s.t.* $\alpha(t_-) := \lim_{t\to t_-} \alpha(t) = 0$*. If* $\alpha(t_+) := \lim_{t\to t_+} \alpha(t) < \infty$ *and*

$$\int_{t_-}^{c} \alpha(t) d\gamma(t) < \infty, \quad \int_{c}^{t_+} (\alpha(t_+) - \alpha(t))\, d\gamma(t) < \infty, \tag{3.8}$$

then the spectral measure μ *is discrete and*

$$\sum_{\lambda\in\sigma} \frac{1}{\lambda^2} < \infty, \tag{3.9}$$

where σ *is the support of* μ*. Conversely, if a symmetric measure* μ *is discrete and satisfies* (3.9)*, then it's the spectral measure of some Kreĭn's string that satisfies* (3.8)*.*

3.2 dB-functions in the Pólya class

In this section we introduce the Pólya class $\mathcal{P}_0$ of entire functions and de Branges' Theorem 41 in [dB68], which shows the existence of a dB-chain in $\mathcal{P}_0$ with Hamiltonian H s.t. $E_t(0) = 1$, if H satisfies conditions (3.14)—(3.16) at the left endpoint. Moreover, we'll show (3.14)—(3.16) are also necessary for the existence of a dB-chain in the Pólya class $\mathcal{P}_0$ s.t. $E_t(0) = 1$. The main result of this section is Theorem 3.15 for the Pólya case, which is analogous to Theorem 1.34 for the regular case, so before we prove that we introduce the Pólya class $\mathcal{P}_0$ first.

3.2.1 The Pólya class $\mathcal{P}_0$

The Pólya class first arose as people studied the limit functions of polynomials whose zeros lie in a given region, usually the real line, the lower half plane $\mathbb{C}_-$, and the sector $\theta_1 \leqslant \arg z \leqslant \theta_2$ where $|\theta_2 - \theta_1| < \pi$. The first results in this direction are due to Laguerre [Lag98] and Pólya [Pól13]. A summary of the theory of functions of Pólya class can be found in [dB68, Section 7,14] and [Lev64, Chapter VIII].

Definition 3.9. An entire function E having no zero on $\mathbb{C}_+$ is said to belong to the *Pólya class* $\mathcal{P}_0$ if there exist polynomials $\{P_n\}$ with no zeros on $\mathbb{C}_+$, s.t. P_n converges to E locally uniformly.

Remark. The definition excludes the possibility that $F \in \mathcal{P}_0$ and $F \equiv 0$.

From the definition we can see that $\mathcal{P}_0$ is closed in the following sense: if $E_n \in \mathcal{P}_0$ and $E_n \to E$ locally uniformly and $E \not\equiv 0$, then $E \in \mathcal{P}_0$.

Here are some sufficient and necessary conditions for $E \in \mathcal{P}_0$ that characterize $\mathcal{P}_0$ from different angles (cf. [dB68, Section 7], [Lev64, Theorem 4, Chapter VIII]). Recall that an entire function F is said to have *genus* k if it admits the following (unique) factorization

$$z^m e^{Q(z)} \prod_{n=1}^{N(F)} \left(1 - \frac{z}{z_n}\right) e^{\frac{z}{z_n} + \cdots + \frac{1}{\rho-1}\left(\frac{z}{z_n}\right)^{\rho-1}},$$

where Q is a polynomial and ρ is chosen to be the smallest integer s.t. $\sum_{n=1}^{N(F)} \frac{1}{|z_n|^{\rho+1}} < \infty$, and k is defined to be $\max\{\deg Q, \rho\}$.

Proposition 3.10. *Let E be an entire function, then the following are equivalent:*

(i) $E \in \mathcal{P}_0$,

(ii) $E \in dB$, *and its zeros* $\{z_n\}_{n=1}^{N(E)}$ *satisfies*

$$\sum_{n=1}^{N(E)} \frac{1}{|z_n|^2} < \infty,$$

and it admits the product representation

$$E(z) = cz^r e^{P(z)-ihz} \prod_{n=1}^{N(E)} \left(1 - \frac{z}{z_n}\right) e^{z\Re\frac{1}{z_n}}, \tag{3.10}$$

where $P(z) = p_2 z^2 + p_1 z$ *is a real polynomial,* $p_2 \leqslant 0$, $h \geqslant 0$, $c \in \mathbb{C}$ *and* $r \in \mathbb{N}_0$.

(iii) $E = e^{-az^2} E_0$, *where* $a \geqslant 0$ *and* E_0 *is a dB-function with genus at most 1,*

(iv) $E \in dB$ *and* $|E(x+iy)|$ *is a nondecreasing function of* $y > 0$ *for each fixed* x.

Condition (iii) above implies that $E \in \mathcal{P}_0$ has order at most 2, while on the other hand, not all dB-functions of order at most 2 belong to $\mathcal{P}_0$. Actually, the exceptions for the converse can be characterized precisely by Proposition 3.10(ii): let E be a dB-function of order at most 2, then $E \notin \mathcal{P}_0$ if and only if E has order 2, and in its canonical factorization

$$E(z) = cz^r e^{q_2 z^2 + q_1 z} \prod_{n=1}^{N(E)} \left(1 - \frac{z}{z_n}\right) e^{\frac{z}{z_n} + \frac{z^2}{z_n^2}}, \tag{3.11}$$

either $\sum_{n=1}^{N(E)} \frac{1}{|z_n|^2} = \infty$ or $\sum_{n=1}^{N(E)} \frac{1}{|z_n|^2} < \infty$ and $q_2 + \sum_{n=1}^{N(E)} \frac{1}{z_n^2} > 0$ (entire functions of order 2 s.t. $\sum_{n=1}^{N(E)} \frac{1}{|z_n|^2} < \infty$ are said to be of *convergence class*, see [BJ54, Definition 2.5.20]). Such a relation holds for the generalized Pólya classes $\mathcal{P}_k$, as we shall see in Section 4.1 (Proposition 4.5 in particular), where we show $\cup_{k\in\mathbb{N}_0}\mathcal{P}_k$ is a partition of the set of dB-functions of finite order, and the partition is closely related to the order of the dB-functions.

Condition (iii) also implies

$$\text{regular} \subsetneq dB \cap \text{Cart} \subsetneq dB \cap \text{Exp} \subsetneq \mathcal{P}_0.$$

Condition (iv) is interesting for its own sake, as it implies that for any $E \in \mathcal{P}_0$, we can "shift" E downwards and get another dB-function, namely, $\tilde{E}(z) := E(z+ic)$, $c > 0$. Moreover, for $E \in \mathcal{P}_0$, $|E(x+iy)|$ is a strictly increasing function of $y > 0$ for fixed x, unless $E(z) = E(0)e^{hz}$ for some real number h (cf. [dB68, Section 7]).

Here are some examples of functions of the Pólya class $\mathcal{P}_0$:

Example 3.11. (i) Any polynomial with no zeros in $\mathbb{C}_+$ belongs to $\mathcal{P}_0$.

(ii) $e^{-az^2} \in \mathcal{P}_0$ if and only if $a \geqslant 0$. Note that $e^{-az^2} = \lim_{n\to\infty}\left(1 - \frac{az^2}{n}\right)^n$, and $\left(1 - \frac{az^2}{n}\right)^n$ is a polynomial with no zeros on $\mathbb{C}_+$. Later in Section 4.1 we'll see $e^{-az^2} \in \mathcal{P}_1$ if $a < 0$.

(iii) $e^{-iaz} \in dB \cap \text{Exp} \subseteq \mathcal{P}_0$ for $a \geqslant 0$, $\sin z, \cos z \in dB \cap \text{Exp} \subseteq \mathcal{P}_0$.

(iv) $\frac{1}{\Gamma(z)} \in \mathcal{P}_0$ as $\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}\left(1 + \frac{z}{n}\right)e^{-\frac{z}{n}}$, where γ is the Euler-Mascheroni constant.

Despite the many interesting properties $E \in \mathcal{P}_0$ has, the main benefit we gain from using $\mathcal{P}_0$ rather than the set of dB-functions of order at most 2, is the normality condition that $\mathcal{P}_0$ has. Let $E \in \mathcal{P}_0$ s.t. $E(0) = 1$, then by factorization (3.11) and inequality (4.3),

$$\log \left| E(z)e^{-q_1(E)z} \right| \leqslant -q_2|z|^2 + \log \left| \prod_{n=1}^{N(E)} \left(1 - \frac{z}{z_n}\right) e^{\frac{z}{z_n}} \right| \leqslant \left(-q_2 + \sum_{n=1}^{N(E)} \frac{1}{|z_n|^2} \right) |z|^2. \tag{3.12}$$

Moreover, the factor $-q_2 + \sum_{n=1}^{N(E)} \frac{1}{|z_n|^2}$ in the RHS can be bounded above by the derivatives of E at 0, as shown by the following lemma.

Lemma 3.12. *Let $E = A - iC \in \mathcal{P}_0$ be normalized, i.e., $E(0) = 1$, then*

$$\log \left| E(z)e^{-q_1(E)z} \right| \leqslant \frac{1}{2} \left(A'(0)^2 - A''(0) + C'(0)^2 \right) |z|^2, \quad \forall z \in \mathbb{C}, \tag{3.13}$$

where $q_1(E)$ is the coefficient in the canonical factorization (3.11)*. Moreover, $A'(0) - A''(0) \geqslant 0$.*

Proof. The inequality comes from [dB61a, Lemma 5]. The last statement is actually a special case of a more general inequality given by Lemma 4.14. □

Now let $\mathcal{F} \subseteq \mathcal{P}_0$ be a family of *normalized* dB-functions. If $A'(0)^2 - A''(0) + C'(0)^2$ is uniformly bounded for $E := A - iC \in \mathcal{F}$, then $\{Ee^{-q_1(E)z} : E \in \mathcal{F}\}$ is a normal family of entire functions.

The inequality (3.13) is critical to prove de Branges' Theorem 41, where we use it to get a "limit" dB-function from the family $\left\{E_{a\to b}e^{-q_1(E_{a\to b})z} : a \in (t_-, b)\right\}$ for fixed b, as we shall see in Section 3.2.2.

3.2.2 Chains of dB-functions in the Pólya class $\mathcal{P}_0$

The main result of this subsection is Theorem 3.15, which establishes the one-to-one correspondence between a Hamiltonian that satisfies conditions (3.14)—(3.16) at its left endpoint, and a dB-chain in $\mathcal{P}_0$ that is unique up to a factor e^{az^2+bz} for $a, b \in \mathbb{R}$ if we assume $E_t(0) = 1$. In particular, if a Hamiltonian H has a regular left endpoint, then it satisfies (3.14)—(3.16), while the converse is not true.

de Branges' Theorem 41 below shows the existence of a dB-chain in $\mathcal{P}_0$ for a given Hamiltonian H that satisfies (3.14)—(3.16). As a byproduct, the chain always has a certain type of asymptotic behavior, and is unique if the asymptotic behavior is specified (e.g. (3.17)).

Theorem 3.13 (de Branges' Theorem 41)**.** *Let $H = H(t)$, $t \in I$ be a Hamiltonian and $h = h(t) =$*

$\begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$ *be its anti-derivative. Assume that*

$$\alpha(t_-) := \lim_{t \to t_-} \alpha(t) = 0, \tag{3.14}$$

$$\alpha(t) > 0 \text{ for } t \in I, \tag{3.15}$$

$$\int_{t_-}^{b} \alpha(t) d\gamma(t) < \infty \text{ for some (hence for all) } b \in I, \tag{3.16}$$

then there exists a unique dB-chain $\{\mathcal{B}(E_t)\}_{t \in I}$ *with* H *as its Hamiltonian, s.t.* $E_t \in \mathcal{P}_0$ *is normalized, strict, non-degenerate, and*

$$\lim_{t \to t_-} E_t(z) e^{\beta(t) z} = 1 \tag{3.17}$$

locally uniformly in z.

Remark. $E_t(0) = 1$ can actually be implied from the asymptotic condition (3.17).

de Branges' original proof can be found in [dB68, Section 41]. An alternative proof using theory of functions of Laguerre classes will be presented in Section 4.4.

Note that conditions (3.14), (3.15) are necessary because of Proposition 1.25 and the assumption $E_t(0) = 1$. Only the condition (3.16) is critical for E_t to belong to $\mathcal{P}_0$. Actually, (3.16) is a also necessary condition on H if H is the Hamiltonian of a dB-chain $\{\mathcal{B}(E_t)\}_{t \in I}$ s.t. $E_t(0) = 1$, $E_t \in \mathcal{P}_0$ or $\lim_{t \to t_-} E_t(z) e^{\beta(t) z} = S(z)$ locally uniformly in z, for some real entire function S.

Theorem 3.14. *Let* $\{\mathcal{B}(E_t)\}_{t \in I}$ *be a dB-chain with Hamiltonian* H*, s.t.* $E_t(0) = 1$ *for* $t \in I$*. Let* $h = h(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$ *be an anti-derivative of* H *s.t.* $\alpha(t_-) = 0$*. If* $E_t \in \mathcal{P}_0$ *for* $t \in I$*, or* $\lim_{t \to t_-} E_t(z) e^{\beta(t) z} = S(z)$ *locally uniformly in* z *where* S *is real entire, then* H *satisfies* (3.16) *as well.*

Proof. WLOG we assume $t_- = 0$. We can re-write the canonical equation as

$$\begin{aligned} A_b(z) - A_a(z) &= -z \int_a^b A_t(z) d\beta(t) - z \int_a^b C_t(z) d\gamma(t), \\ C_b(z) - C_a(z) &= z \int_a^b A_t(z) d\alpha(t) + z \int_a^b C_t(z) d\beta(t), \end{aligned} \tag{3.18}$$

then

$$C_b'(0) - C_a'(0) = \alpha(b) - \alpha(a).$$

Since $C_a'(0) = K_{a,0}(0) \geqslant 0$, $\lim_{a \to 0+} K_{a,0}(0) = 0$ by definition of a dB-chain, $\lim_{a \to 0+} \alpha(a) = 0$ by assumption, then $C'(b) = \alpha(b)$.

Moreover, by (3.18) again we know

$$\begin{aligned}
A_b'(0) - A_a'(0) &= -(\beta(b) - \beta(a)),\\
A_b''(0) - A_a''(0) &= -2\int_a^b A_t'(0)d\beta(t) - 2\int_a^b C_t'(0)d\gamma(t)\\
&= -2\int_a^b (A_t'(0) - A_a'(0))d\beta(t) - 2A_a'(0)(\beta(b) - \beta(a)) - 2\int_a^b C_t'(0)d\gamma(t)\\
&= (\beta(b) - \beta(a))^2 - 2A_a'(0)(\beta(b) - \beta(a)) - 2\int_a^b \alpha(t)d\gamma(t).
\end{aligned} \tag{3.19}$$

Now suppose $E_t \in \mathcal{P}_0$ for all $t > 0$, then by (3.19),

$$\begin{aligned}
A_b''(0) - A_b'(0)^2 =& A_a''(0) + (\beta(b) - \beta(a))^2 - 2A_a'(0)(\beta(b) - \beta(a)) - 2\int_a^b \alpha(t)d\gamma(t)\\
&- (A_a'(0) - (\beta(b) - \beta(a)))^2\\
=& A_a''(0) - A_a'(0)^2 - 2\int_a^b \alpha(t)d\gamma(t),\\
2\int_a^b \alpha(t)d\gamma(t) = (&\underbrace{A_a''(0) - A_a'(0)^2}_{\leqslant 0}) - (A_b''(0) - A_b'(0)^2) \leqslant -(A_b''(0) - A_b'(0)^2) < \infty,
\end{aligned}$$

where the first term is nonpositive by (3.12). Therefore $\int_0^b \alpha(t)d\gamma(t) \leqslant -\frac{1}{2}(A_b''(0) - A_b'(0)^2)$ is finite. On the other hand, suppose $\lim_{t\to 0+} E_t(z)e^{\beta(t)z} = S(z)$ locally uniformly in z and S is real entire, then by (3.19),

$$-2\int_a^b \alpha(t)d\gamma(t) = A_b''(0) - \beta(b)^2 - \left(\underbrace{A_a''(0) + 2A_a'(0)\beta(a) + \beta(a)^2}_{\to S''(0)}\right) + 2\beta(b)(\underbrace{A_a'(0) + \beta(a)}_{\to S'(0)}),$$

where the second and third terms have finite limits as $a \to 0+$ because the derivatives of $A_a(z)e^{\beta(a)z}$ converge to derivatives of $S(z)$ as $a \to 0+$. Thus the proof is complete. □

Combining Theorem 3.13, Theorem 3.14 and Theorem 2.13 we get a one-to-one correspondence between dB-chain in $\mathcal{P}_0$ up to a factor e^{az^2+bz}, $a, b \in \mathbb{R}$ if we assume $E_t(0) = 1$, and Hamiltonian H that satisfies (3.14)—(3.16). Actually, for a chain $\{\mathcal{B}(E_t)\}_{t\in I}$ to belong to the Pólya class $\mathcal{P}_0$, it suffices to have $E_t \in \mathcal{P}_0$ for one $t \in I$. This is a special case of Corollary 4.8, which states that if E_t belongs to the generalized Pólya class $\mathcal{P}_k$ for some $t \in I$, then $E_t \in \mathcal{P}_k$ for all $t \in I$.

Theorem 3.15. *(i) Let $H = H(t)$, $t \in I$ be a Hamiltonian. If H satisfies* (3.14)—(3.16), *then there exists a unique dB-chain with H as its Hamiltonian, s.t. E_t is strict, normalized, and $\lim_{t\to t_-} E_t(z)e^{\beta(t)z} = 1$ locally uniformly in z. For this unique dB-chain $\{\mathcal{B}(E_t)\}$, we also have $E_t \in \mathcal{P}_0$, $\forall t \in I$.*

(ii) Let $\{\mathcal{B}(E_t)\}_{t\in I}$ be a dB-chain with Hamiltonian $H = H(t)$ s.t. E_t is strict and normalized for some $t \in I$, then:

- *If $E_t \in \mathcal{P}_0$ for some $t \in I$, then $E_t \in \mathcal{P}_0$ for all $t \in I$, $H(t)$ satisfies (3.16) and $E_t(z)e^{\beta(t)z}$ converges to $S(z) := e^{az^2+bz}$ for some $a, b \in \mathbb{R}$ locally uniformly in z, as $t \to t_-$.*
- *If $S(z) := \lim_{t\to t_-} E_t(z)e^{\beta(t)z}$ exists and is real entire, then $H(t)$ satisfies (3.16), and $E_t = S\tilde{E}_t$ where $\tilde{E}_t \in \mathcal{P}_0$, $\forall t \in I$, and $\lim_{t\to t_-} \tilde{E}_t(z)e^{\beta(t)z} = 1$ locally uniformly in z.*

Proof. (i) The proof can be found in [dB68, Section 41]. An alternative proof using theory of Laguerre classes of entire functions is given in Section 4.4.

(ii) From Proposition 1.25 we know conditions (3.14)—(3.15) are satisfied. Assume $E_t \in \mathcal{P}_0$ for some $t \in I$, then by Theorem 2.13 we know $E_t \in \mathcal{P}_0$, $\forall t \in I$. From Theorem 3.14 we know the Hamiltonian of $\{\mathcal{B}(E_t)\}_{t\in I}$ also satisfies (3.16). Then from Theorem 3.13 we know there exists a dB-chain $\{\mathcal{B}(\tilde{E}_t)\}_{t\in I}$ with Hamiltonian H, $\tilde{E}_t \in \mathcal{P}_0$ and $\lim_{t\to t_-} \tilde{E}_t(z)e^{\beta(t)z} = 1$. Now we have two chains in $\mathcal{P}_0$ with the same Hamiltonian, then by Corollary 2.14, $\frac{E_t}{\tilde{E}_t}$ is of the form $S(z) := e^{az^2+bz}$ for some $a, b \in \mathbb{R}$, therefore $E_t(z) = S(z)\tilde{E}_t$ converges to $S(z)$ locally uniformly in z as $t \to t_-$.

On the other hand, if $S(z) := \lim_{t\to t_-} E_t(z)e^{\beta(t)z}$ exists and is real entire, then by the same arguments as above, we know $S = \frac{E_t}{\tilde{E}_t}$ where $\tilde{E}_t \in \mathcal{P}_0$ and $\lim_{t\to t_-} \tilde{E}_t(z)e^{\beta(t)z} = 1$ locally uniformly in z.

□

In Section 4.5 we'll see for any dB-space in the dB-chain given by Theorem 3.15(i), the Bezout operator $T_{A,0}$ is in the Hilbert-Schmidt class $\mathfrak{S}_2$. Conversely, for a strict normalized non-degenerate dB-function E, if the Bezout operator $T_{A,0}$ is in the Hilbert-Schmidt class $\mathfrak{S}_2$, then $E = SE_0$ where S is a zero-free real entire function and E_0 is a strict normalized non-degenerate dB-function in the Pólya class $\mathcal{P}_0$. We will see the correspondence actually holds for larger classes of Bezout operators (i.e., Schatten classes) and larger classes of dB-functions (i.e., generalized Pólya classes).

3.3 Example: Bessel functions and the Hankel transform

We take Bessel's equation as an example to illustrate some of the above results. Bessel's differential equation of order ν is given by

$$t^2\frac{d^2y}{dt} + t\frac{dy}{dt} + (zt^2 - \nu^2)y = 0, \quad t > 0. \tag{3.20}$$

By means of the substitution $u = y\sqrt{t}$ (3.20) reduces to the form

$$-\ddot{u} + \frac{\nu^2 - \frac{1}{4}}{t^2} u = zu, \quad t > 0. \tag{3.21}$$

Namely it becomes the Schrödinger equation with potential $q(t) := \frac{\nu^2 - \frac{1}{4}}{t^2}$. Similar to Example 1.2 we define $v = -\dot{u}$ and let $Y = \binom{u}{v}$, then (3.21) reduces to

$$\Omega \dot{Y} = z \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} Y - \begin{pmatrix} q & 0 \\ 0 & -1 \end{pmatrix} Y. \tag{3.22}$$

To reduce it further to a canonical system, let $z = 0$, and then

$$V(t) := \frac{1}{\sqrt{2\nu}} \begin{pmatrix} t^{\frac{1}{2}+\nu} & t^{\frac{1}{2}-\nu} \\ -\left(\frac{1}{2}+\nu\right) t^{-\frac{1}{2}+\nu} & -\left(\frac{1}{2}-\nu\right) t^{-\frac{1}{2}-\nu} \end{pmatrix} \in SL(2, \mathbb{R})$$

is a solution of (3.22) with determinant 1. Let $X := V^{-1} Y$, then X satisfies the following differential equation

$$\Omega \dot{X} = zHX, \quad t > 0, \tag{3.23}$$

where

$$H(t) = V^*(t) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} V(t) = \frac{1}{2\nu} \begin{pmatrix} t^{1+2\nu} & t \\ t & t^{1-2\nu} \end{pmatrix} \tag{3.24}$$

is a Hamiltonian which has no H-indivisible intervals.

For $\nu \geqslant 1$, obviously, H is not integrable near 0, then $t_- = 0$ is a *singular* left endpoint of H. We now show it satisfies conditions (3.14)—(3.16) in Theorem 3.13. Let

$$h(t) := \frac{1}{2\nu} \begin{pmatrix} \frac{t^{2+2\nu}}{2(1+\nu)} & \frac{t^2}{2} \\ \frac{t^2}{2} & \frac{t^{2-2\nu}}{2(1-\nu)} \end{pmatrix}$$

be the anti-derivative of H, then (3.14)—(3.15) follow directly. For (3.16),

$$\int_0^1 \alpha(t) d\gamma(t) = \frac{1}{4v^2} \frac{1}{2(1+\nu)} \int_0^1 t^{2+2\nu} t^{1-2\nu} dt = \frac{1}{32\nu^2(1+\nu)} < \infty.$$

Note that since $\lim_{t\to 0} \beta(t) = 0$, Theorem 3.13 then asserts that there exists a unique dB-chain $\{\mathcal{B}(E_t)\}_{t\in(0,+\infty)}$ s.t. $\lim_{t\to 0} E_t(z) = 1$ locally uniformly, and

$$\Omega \begin{pmatrix} A_b \\ C_b \end{pmatrix} - \Omega \begin{pmatrix} A_a \\ C_a \end{pmatrix} = z \int_a^b H(t) \begin{pmatrix} A_t \\ C_t \end{pmatrix} dt, \quad \forall b > a > 0.$$

Let $p = \frac{\nu}{2}$. It is well known that

$$\sqrt{t}J_\nu(t\sqrt{z}) = \sqrt{t}\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+2p)!}\left(\frac{t}{2}\right)^{2(n+p)} z^{n+p} \tag{3.25}$$

is a solution to (3.21), where J_ν is the Bessel function of the first kind. Since the radius of convergence is

$$r = \lim_{n\to+\infty}\left|\frac{\frac{1}{n!(n+2p)!}}{\left(\frac{t}{2}\right)^2\frac{1}{(n+1)!(n+2p+1)!}}\right| = +\infty,$$

then $\frac{1}{z^p}\sqrt{t}J_\nu(t\sqrt{z})$ is an entire function of z for $t > 0$. From the series expansion (3.25) we know

$$y(t,z) = \frac{1}{(2p)!}\frac{z^p}{2^{2p}}t^{\frac{1}{2}+2p} + o\left(t^{\frac{1}{2}+2p}\right).$$

Similarly we have

$$\dot{y}(t,z) = \frac{1}{(2p)!}\frac{z^p}{2^{2p}}\left(\frac{1}{2}+2p\right)t^{-\frac{1}{2}+2p} + o\left(t^{-\frac{1}{2}+2p}\right).$$

Let $y(t,z) := \sqrt{t}J_\nu(t\sqrt{z})$, then $Y(t,z) := \binom{y(t,z)}{-\dot{y}(t,z)}$ solves (3.22), and $X(t,z) := V(t)^{-1}Y(t,z)$ solves (3.23). Denote $X(t,z)$ by $\binom{x_1(t,z)}{x_2(t,z)}$, then by construction we know

$$\begin{aligned}
x_1(t,z) &= 2\sqrt{p}\left(-\left(\frac{1}{2}-2p\right)t^{-\frac{1}{2}-2p}y(t,z) + t^{\frac{1}{2}-2p}\dot{y}(t,z)\right)\\
&= 2\sqrt{p}\left(-\left(\frac{1}{2}-2p\right)\frac{1}{(2p)!}\frac{z^p}{2^{2p}} + \left(\frac{1}{2}+2p\right)\frac{1}{(2p)!}\frac{z^p}{2^{2p}} + o(1)\right)\\
&= \frac{p^{\frac{3}{2}}}{(2p)!2^{2p-3}}z^p + o(1),\\
x_2(t,z) &= 2\sqrt{p}\left(\left(\frac{1}{2}+2p\right)t^{-\frac{1}{2}+2p}y(t,z) + t^{\frac{1}{2}+2p}\dot{y}(t,z)\right)\\
&= 2\sqrt{p}\left(\left(\frac{1}{2}+2p\right)\frac{1}{(2p)!}\frac{z^p}{2^{2p}}t^{4p} + \left(\frac{1}{2}+2p\right)\frac{1}{(2p)!}\frac{z^p}{2^{2p}}t^{4p} + o(t^{4p})\right)\\
&= \left(\frac{1}{2}+2p\right)\frac{\sqrt{p}t^{4p}}{(2p)!2^{2p-2}}z^p + o(t^{4p}).
\end{aligned}$$

By the same arguments as in the proof of Theorem 2.3 we know $\{\frac{1}{z^p}\left(x_1(t,z) - ix_2(t,z)\right)\}_{t\in(0,+\infty)}$ is a chain of dB-functions, then by the uniqueness in Theorem 3.13, we must have

$$\begin{pmatrix}A_t(z)\\C_t(z)\end{pmatrix} = \frac{1}{C_p z^p}\begin{pmatrix}x_1(t,z)\\x_2(t,z)\end{pmatrix} = \frac{1}{C_p z^p}V(t)^{-1}\begin{pmatrix}y(t,z)\\-\dot{y}(t,z)\end{pmatrix},$$

where

$$C_p := \frac{p^{\frac{3}{2}}}{(2p)!2^{2p-3}}.$$

Let $f \in L^2(\mathbb{R})$, then $V^{-t}\binom{f}{0} \in L^2(H)$. Theorem 1.28 then gives the following Fourier transform from $L^2(H;(0,c])$ to $\mathcal{B}(E_c)$:

$$\begin{aligned} V^{-1}\begin{pmatrix} f \\ 0 \end{pmatrix} &\mapsto \int_0^c (f(t),0)V^{-T}H\begin{pmatrix} A_t(z) \\ C_t(z) \end{pmatrix} dt \\ &= \frac{1}{C_p z^p}\int_0^c (f(t),0)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} y(t,z) \\ -\dot{y}(t,z) \end{pmatrix} dt \\ &= \frac{1}{C_p z^p}\int_0^c f(t)\sqrt{t}J_\nu(t\sqrt{z})dt. \end{aligned}$$

To summarize, we get a “multiple” of the classical Hankel transfrom which maps $L^2(\mathbb{R})$ isometrically into $\mathcal{B}(E_c)$:

$$f \mapsto \frac{1}{C_p z^p}\int_0^c f(t)\sqrt{t}J_\nu(t\sqrt{z})dt.$$

For comparison, the Fourier transform using the associated spectral matrix can be found in [Tit62, Section 4.11]. See also [LS75, Section 5.2,5.3] to find a different approach without using de Branges theory, where the generalized Fourier transform is shown to be well-defined as $\sqrt{t}J_\nu(t\sqrt{z})$ is in $L^2\left((0,c]\right)$ according its asymptotic values as $t \to 0$.

Chapter 4

Generalized Pólya classes

In this chapter we use the theory of the generalized Pólya classes of entire functions to extend Theorem 3.13, to get the existence of a dB-chain in a broader class, namely the first generalized Pólya class when we impose weaker integrability assumptions on the Hamiltonian near its left endpoint. The generalized Pólya classes were introduced by Kaltenbäck and Woracek in [KW05], and we will present their main results in Section 4.1. In particular, we will present one special sub-class of generalized Pólya classes, namely the Laguerre classes which were defined and studied by de Branges in [dB68, Problem 305–313]. In Section 4.2 we show how to construct a dB-chain in the first generalized Pólya class $\mathcal{P}_{\leqslant 1}$ assuming given integrability conditions on the Hamiltonian near the left endpoint, which is analogous to Theorem 3.13 for the existence of a dB-chain in the Pólya class $\mathcal{P}_0$. In Section 4.3 we show how to apply the theory of the Laguerre classes of entire functions to extend Theorem 3.13 for Kreĭn's strings beyond the Pólya class $\mathcal{P}_0$, and in Section 4.4 we show how to prove de Branges' Theorem 41 using this theory. We continue our discussion on the Bezout operator in Section 4.5 by showing the connection between the assumption that E is in the generalized Pólya class $\mathcal{P}_{\leqslant k}$ and the assumption that the Bezout operator $T_{A,0}$ is in the $(2k+2)$-th Schatten class.

4.1 Generalized Pólya classes and Laguerre classes

From Proposition 3.10 we know $\mathcal{P}_0 = \{e^{-az^2}E_0 : a \geqslant 0, E_0 \in dB \text{ has genus at most } 1\}$. The corresponding normality condition (3.13) is crucial in the proof of Theorem 41. It's natural to consider functions of the form $e^{-az^{2k+2}}E_0$, where $a \geqslant 0$ and $E_0 \in dB$ has genus at most $2k+1$ for $k \in \mathbb{N}_0$, and try to get a similar normality condition. Such sets with $k \in \mathbb{N}_0$, called the generalized Pólya classes, actually form a partition of the set of dB-functions of finite order (see Proposition 4.5), and are the main subject of this section.

Generalized Pólya classes were first introduced by Kaltenbäck and Woracek in [KW05], as a natural extension of the Pólya class $\mathcal{P}_0$. de Branges proved for a normalized dB-function E, if we define $\log E$ continuously on $\mathbb{C}_+ \cup \{0\}$, then

$$E \in \mathcal{P}_0 \iff -\frac{\log E(z)}{z} \in \mathcal{N}_0,$$

where $\mathcal{N}_0$ is the set of *Nevanlinna functions* (a.k.a. *Herglotz/Pick/R functions*). Motivated by this relationship between $\mathcal{P}_0$ and $\mathcal{N}_0$, Kaltenbäck and Woracek defined $\mathcal{P}_k$ and proved for $E \in dB$ s.t. $E(0) = 1$ and $\log E$ is defined continuously on $\mathbb{C}_+ \cup \{0\}$, then

$$E \in \mathcal{P}_k \iff -\frac{\log E(z)}{z} \in \mathcal{N}_k,$$

where $\mathcal{N}_k$ are the generalized Nevanlinna classes whose definitions can be found in [KW05].

The formal definition of the generalized Pólya classes is given below.

Definition 4.1. An entire function E is said to belong to the class $\mathcal{P}_{\leqslant k}$ if it's a dB-function, and its zeros $\{z_n\}_{n=1}^{N(E)}$ satisfy

$$\sum_{n=1}^{N(E)} \frac{1}{|z_n|^{2k+2}} < \infty,$$

and it admits the product representation

$$E(z) = cz^r e^{P(z)-ihz} \prod_{n=1}^{N(E)} \left(1 - \frac{z}{z_n}\right) e^{z\Re\frac{1}{z_n} + \cdots + z^{2k+1}\Re\frac{1}{(2k+1)z_n^{2k+1}}}, \tag{4.1}$$

where $P(z) = p_{2k+2}z^{2k+2} + \cdots + p_1 z$ is a real polynomial with $p_{2k+2} \leqslant 0$, $h \geqslant 0$, $c \in \mathbb{C}$ and $r \in \mathbb{N}_0$.

For $k \geqslant 1$ we put $\mathcal{P}_k := \mathcal{P}_{\leqslant k} \backslash \mathcal{P}_{\leqslant k-1}$ and refer to $\mathcal{P}_k$ as the k-th generalized Pólya class, and denote $\mathcal{P}_{\leqslant 0}$ by $\mathcal{P}_0$, which is the Pólya class.

$E \in \mathcal{P}_{\leqslant k}$ is said to belong to $\mathcal{L}_{\leqslant k}$ if $E \in \mathcal{P}_{\leqslant k}$ has only real zeros, $E(0) = 1$, and $p_1 = \cdots = p_{2k+1} = h = 0$.

For $k \geqslant 1$ we put $\mathcal{L}_k := \{1\} \cup \mathcal{L}_{\leqslant k} \backslash \mathcal{L}_{\leqslant k-1}$ and refer to $\mathcal{L}_k$ as the k-th Laguerre class, and denote $\mathcal{L}_{\leqslant 0}$ by $\mathcal{L}_0$.

As mentioned in Section 3.2.1, the Pólya class first arose as people studied the limit functions of polynomials whose zeros lie in certain regions. The limit functions are of the Pólya class $\mathcal{P}_0$ if the region is the closed lower half plane; and if we choose the region to be the real line, then the limit functions are of the so-called *Laguerre-Pólya class*, which is a subset of $\mathcal{L}_0$. Actually, the 0-th Laguerre class $\mathcal{L}_0$ consists of functions F of the Laguerre-Pólya class, s.t. $F(0) = 1$ and $F'(0) = 0$.

From the definition of $\mathcal{L}_{\leqslant k}$ and $\mathcal{L}_k$ we can see

$$\mathcal{L}_k = \{1\} \cup \{F \in \mathcal{P}_k : F = F^{\#}, F(0) = 1, F'(0) = \cdots = F^{(2k+1)}(0) = 0\}. \tag{4.2}$$

We define $\mathcal{L}_k$ in this way so that $\mathcal{L}_k$ is closed when taking limits of entire functions (in the sense of Corollary 4.4). To prove this, we first give some equivalent conditions for $F \in \mathcal{L}_k$ in Proposition 4.3 below. In order to prove it, the following inequality for the *canonical factor* (a.k.a. *primary factor*) is needed.

Lemma 4.2.

$$\left|(1-z)\exp\left(z+\frac{z^2}{2}+\cdots+\frac{z^r}{r}\right)-1\right| \leqslant e^{|z|^{r+1}}-1, \quad \forall z \in \mathbb{C}, \quad \forall r \in \mathbb{N}. \tag{4.3}$$

Proof. See Section 4.A. □

Proposition 4.3. *Let F be a real entire function s.t. $F(0) = 1$ and $F(z) \neq 0$ for $z \in \mathbb{C}\backslash\mathbb{R}$. Then $F \in \mathcal{L}_k$ if and only if*

(i) $F'(z)$ has a zero of order at least $2k+1$ at the origin,

(ii) $\Re\frac{iF'(z)}{z^{2k}F(z)} \geqslant 0$ for $z \in \mathbb{C}_+$.

Proof. See Section 4.B. □

Remark. de Branges [dB68] defined k-th Laguerre class to be the set of real entire functions s.t. $F(0) = 1$, $F(z) \neq 0$ for $z \in \mathbb{C}\backslash\mathbb{R}$ and $\Re\frac{iF'(z)}{z^{2k}F(z)} \geqslant 0$ for $z \in \mathbb{C}_+$. However it seems necessary to include the assumption that $F'(z)$ has a zero of order at least $2k+1$ at the origin for the results on [dB68, Pages 288-292] to hold.

Corollary 4.4. *For $F_n \in \mathcal{L}_k$, $n \geqslant 1$. If F_n goes to F locally uniformly and $F \not\equiv 0$ then $F \in \mathcal{L}_k$.*

Proof. Firstly, we show F doesn't have any nonreal zeros. For any $z_0 \in \mathbb{C}\backslash\mathbb{R}$ s.t. F, we can choose small r s.t. $\overline{D_r(z_0)} \subseteq \mathbb{C}\backslash\mathbb{R}$ and F doesn't vanish on $\partial D_r(z_0)$, where $D_r(z_0)$ is the open disk centered at z_0 with radius r. Since $F_n(z) \neq 0$ for $z \in \mathbb{C}_+$, we have

$$\log|F_n(z_0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|F_n(z_0+re^{i\theta})|d\theta.$$

Let $n \to +\infty$, then the RHS has a finite limit, therefore $\log|F(z_0)|$ is finite as the limit of the LHS must be finite. Therefore F is a real entire function that has only real zeros, and $F(0) = 1$. The conditions (i) and (ii) in Proposition 4.3 are automatically satisfied since $F_n \to F$, therefore $F \in \mathcal{L}_k$. □

As mentioned earlier, $\{\mathcal{P}_k\}_{k\in\mathbb{N}_0}$ forms a partition of the set of dB-functions of finite order. The following statements are from [KW05].

Proposition 4.5. *We have*

$$\{E \in dB : E \text{ is of finite order}\} = \bigcup_{k\in\mathbb{N}_0} \mathcal{P}_k.$$

More exactly:

- *If $E \in \mathcal{P}_k$, then $E \in dB$ and the order ρ of E satisfies $\rho \in [2k, 2k+2]$.*
- *If $E \in dB$ and has order ρ, there are two subcases:*
  - *If ρ is not an even integer, then $E \in \mathcal{P}_k$ where k is the unique integer with $\rho \in (2k, 2k+2)$.*
  - *If ρ is an even integer, then $E \in \mathcal{P}_{\frac{\rho}{2}-1}$ if E is of convergence class and the coefficient of the power z^ρ in the polynomial Q in the Hadamard product*

$$E(z) = cz^r e^{Q(z)} \prod_{n=1}^{N(E)} \left(1 - \frac{z}{z_n}\right) e^{\frac{z}{z_n} + \cdots + \frac{1}{\rho-1}\left(\frac{z}{z_n}\right)^{\rho-1}}$$

is nonpositive. Otherwise $E \in \mathcal{P}_{\frac{\rho}{2}}$.

Remark. This proposition explains why $e^{-az^2} \in \mathcal{P}_0$ for $a \geqslant 0$ and $e^{-az^2} \in \mathcal{P}_1$ for $a < 0$.

The following proposition and corollaries imply that for dB-chain $\{\mathcal{B}(E_t)\}_{t\in I}$, if $E_t \in \mathcal{P}_k$ for some $t \in I$, then $E_t \in \mathcal{P}_k$ for all $t \in I$. This result is analogous to Proposition 1.32 for regular dB functions. Proposition 4.6 was proved by Kaltenbäck and Woracek (cf. [KW05, Theorem 1.3]).

Proposition 4.6. *Let E, F be dB functions, $E(0) = F(0) = 1$, and $\frac{E}{F} \in \mathcal{N}(\mathbb{C}_+)$. For $k \in \mathbb{N}_0$,*

$$E \in \mathcal{P}_k \iff F \in \mathcal{P}_k.$$

For a non-degenerate dB function $E = A - iC$, we have $\Im\frac{C}{A} \geqslant 0$ on $\mathbb{C}_+$, thus $\frac{C}{A} \in \mathcal{N}(\mathbb{C}_+)$, and $\frac{E}{A} = 1 - i\frac{C}{A} \in \mathcal{N}(\mathbb{C}_+)$. Then by Proposition 4.6 $A \in \mathcal{P}_k \iff E \in \mathcal{P}_k$. The case E is degenerate and normalized simply means $C \equiv 0$ and $E \equiv A$, therefore we have:

Corollary 4.7. *Let $E = A - iC$ be a normalized dB-function, then*

$$A \in \mathcal{P}_k \iff E \in \mathcal{P}_k,$$
$$A \in \mathcal{L}_k \Longrightarrow E \in \mathcal{P}_k \cup \{1 - ipz : p \geqslant 0\} \subseteq \mathcal{P}_{\leqslant k}.$$

Now for a dB-chain $\{\mathcal{B}(E_t)\}_{t\in I}$, for $t_- < a < b < t_+$, by definition of a dB-chain we have

$$\begin{pmatrix} A_b \\ C_b \end{pmatrix} = \begin{pmatrix} A_{a\to b} & B_{a\to b} \\ C_{a\to b} & D_{a\to b} \end{pmatrix} \begin{pmatrix} A_a \\ C_a \end{pmatrix},$$

then

$$\frac{A_b}{A_a} = A_{a\to b} + B_{a\to b}\frac{C_a}{A_a}.$$

By Proposition 1.31, $A_{a\to b}$, $B_{a\to b}$ are of Cartwright class, hence they belong to $\mathcal{N}(\mathbb{C}_+)$. As $\frac{C_a}{A_a} \in \mathcal{N}(\mathbb{C}_+)$ as well, we get $\frac{A_b}{A_a} \in \mathcal{N}(\mathbb{C}_+)$ for $t_- < a < b < t_+$. To summarize, we have:

Corollary 4.8. *Let $\{\mathcal{B}(E_t)\}_{t\in I}$ be a dB-chain. If $E_t \in \mathcal{P}_k$ for some $t \in I$, then $E_t \in \mathcal{P}_k$, $\forall t \in I$.*

Similar to functions in $\mathcal{P}_0$, functions in $\mathcal{L}_k$ have upper bounds which are closely related to their derivatives at 0. Actually, the inequality for functions in $\mathcal{L}_k$ is more elegant and easier to prove, as any function in $\mathcal{L}_k$ is just the product of a factor $e^{-az^{2k+2}}$ and a product of canonical factors. The bound for the product of canonical factors can be derived from (4.3). With this result, we can prove the following proposition for Laguerre classes which says functions of Laguerre classes can be uniformly bounded if the $(2k+2)$-th derivatives at 0 are uniformly bounded. This property is critical and we will use it in Section 4.3 to get a normal family of entire functions and then a convergent subsequence.

Proposition 4.9. *If $F(z)$ belongs to the k-th Laguerre class $\mathcal{L}_k$, let $\delta := -\lim_{z\to 0}\frac{F'(z)}{z^{2k+1}}$, then $\delta = -\frac{F^{(2k+2)}(0)}{(2k+1)!} \geqslant 0$ and*

$$\log(1 + |F(z) - 1|) \leqslant \delta |z|^{2k+2}. \tag{4.4}$$

Proof. Note that any $F \in \mathcal{L}_k$ admits the canonical factorization

$$F(z) = e^{p_{2k+2}z^{2k+2}} \prod_{n=1}^{N(F)} \left(1 - \frac{z}{z_n}\right) e^{\frac{z}{z_n} + \cdots + \frac{1}{2k+1}\left(\frac{z}{z_n}\right)^{2k+1}},$$

where $p_{2k+2} \leqslant 0$ and $z_n \in \mathbb{R}$ are the zeros of F. Let $\delta(F) := -\lim_{z\to 0}\frac{F'(z)}{z^{2k+1}}$ and if $F = GH$ s.t. $F(0) = G(0) = H(0) = 1$ and $\delta(G)$, $\delta(H)$ exist, then $\delta(F) = \delta(G) + \delta(H)$. Note that

$$1 + |ab - 1| \leqslant (1 + |a - 1|)(1 + |b - 1|), \quad \forall a, b \in \mathbb{C},$$

then it suffices to prove (4.4) for factors $e^{p_{2k+2}z^{2k+2}}$ and $\left(1 - \frac{z}{z_n}\right) e^{\frac{z}{z_n} + \cdots + \frac{1}{2k+1}\left(\frac{z}{z_n}\right)^{2k+1}}$. For the factor $e^{p_{2k+2}z^{2k+2}}$ the proof is straightforward and the inequality for the canonical factor of degree $2k + 1$ comes from Lemma 4.2. $\square$

For convenience we also compare the factorization (4.1) in the definition of the generalized Pólya classes and the canonical factorization. The following proposition can be proved via direct calculation.

Proposition 4.10. *Let $E \in \mathcal{P}_{\leqslant k}$, then the factorization* (4.1) *can also be written as the canonical* *factorization*

$$E(z) = cz^r e^{Q(z)} \prod_{n=1}^{N(E)} \left(1 - \frac{z}{z_n}\right) e^{\frac{z}{z_n} + \cdots + \frac{1}{(2k+1)}\left(\frac{z}{z_n}\right)^{2k+1}}, \tag{4.5}$$

where $\sum_{n=1}^{N(E)} \left|\Im \frac{1}{z_n^l}\right| < \infty$*, for* $l = 1, \cdots, N(E)$*, and* $Q(z) = P(z) - ihz - i\sum_{l=1}^{2k+1} \left(\frac{z^l}{l} \sum_{n=1}^{N(E)} \Im \frac{1}{z_n^l}\right)$.

4.2 Chains of dB-functions in the first generalized Pólya class $\mathcal{P}_{\leqslant 1}$

The main result of this section is Theorem 4.11, which extends Theorem 3.13 in the sense that for a given Hamiltonian H satisfying certain integrability conditions which are weaker than (3.14)—(3.16) in de Branges' Theorem 41, there exists a dB-chain $\{\mathcal{B}(E_t)\}_{t \in I}$ with H as its Hamiltonian, s.t. E_t is of the first generalized Pólya class $\mathcal{P}_{\leqslant 1}$, which by definition is larger than the Pólya class $\mathcal{P}_0$.

Here we briefly discuss the main idea of the proof. Note that the Hamiltonian is locally integrable at a, for any $a \in (t_-, t_+)$. Therefore, we can treat a as a regular left endpoint of H for $t \in (a, t_+)$. The corresponding de Branges functions are denoted by $E_{a \to t}$. Although $E_{a \to b}$ doesn't have a finite limit as $a \to t_-$ for fixed b, we can multiply some "controlling polynomials" $P_a(z)$ to it so that the product $E_{a \to b}(z) e^{P_a(z)}$ is uniformly bounded in a and thus forms a normal family for fixed b and $a \in (t_-, b)$. The "controlling polynomial" de Branges used in proving [dB68, Theorem 41] is $(\beta(b) - \beta(a))\, z$, a linear polynomial of z. This choice naturally leads a limit function $\tilde{E}_t := \lim_{a_n \to t_-} E_{a \to b}(z) e^{P_a(z)}$ in the Pólya class, which is intuitive as $E_{a \to b}(z) e^{P_a(z)}$ is a dB-function of order at most 1. We use a polynomial of degree at most 3 instead of a linear polynomial:

$$(\beta(b) - \beta(a_n))\, z + \left(\int_{a_n}^{b} (\alpha(t) - \alpha(a_n))\, d\gamma(t)\right) z^2 + 2\left(\int_{a_n}^{b} \int_{a_n}^{t} (\beta(t) - \beta(s)) d\alpha(s) d\gamma(t)\right) z^3,$$

and this choice leads to a limit function $\tilde{E}_t$ that is contained in the first generalized Pólya class $\mathcal{P}_{\leqslant 1}$, which is intuitive as $E_{a \to t}(z) e^{P_a(z)}$ is clearly a de Branges function or order at most 3. The normality of the family is established by replacing inequality (3.13), which plays a pivotal role in proving de Branges' Theorem 41, by a more general inequality as we will see in the proof below.

Theorem 4.11. *Let* $H = H(t)$*,* $t \in I$ *be a Hamiltonian and* $h = h(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$ *be its* *anti-derivative. Assume that*

$$\alpha(t_-) := \lim_{t\to t_-} \alpha(t) = 0, \tag{4.6}$$

$$\alpha(t) > 0 \text{ for } t \in I, \tag{4.7}$$

$$\int_{t_-}^{b}\int_{t_-}^{t} \alpha(s)^2 d\gamma(s) d\gamma(t) < \infty \text{ for some (hence for all) } b \in I, \tag{4.8}$$

$$\int_{t_-}^{b}\int_{t_-}^{t} (\beta(t)-\beta(s))^2\, d\alpha(s) d\gamma(t) < \infty \text{ for some (hence for all) } b \in I, \tag{4.9}$$

then there exists a unique dB-chain $\{\mathcal{B}(E_t)\}_{t\in I}$ *with* H *as its Hamiltonian, s.t.* $E_t \in \mathcal{P}_{\leqslant 1}$ *is normalized, strict, non-degenerate, and*

$$\lim_{t\to t_-} E_t(z) \exp\left(\beta(t)z - \left(\int_t^1 \alpha(s)d\gamma(s)\right) z^2 - 2\left(\int_t^1 \int_{t_-}^{s} (\beta(s)-\beta(u))\, d\alpha(u) d\gamma(s)\right) z^3\right) = 1 \tag{4.10}$$

locally uniformly in z.

The conditions (4.6), (4.7) are necessary because of Proposition 1.25. The condition (3.16), i.e., $\int_{t_-}^{b} \alpha(s)d\gamma(s) < \infty$, is stronger than (4.8) and (4.9) combined because

$$\int_c^b \int_c^t \alpha(s)^2 d\gamma(s)d\gamma(t) \leqslant \int_c^b \alpha(t) \int_c^t \alpha(s)d\gamma(s)d\gamma(t) \leqslant \left(\int_{t_-}^{b} \alpha(t)d\gamma(t)\right)^2,$$

$$\begin{aligned}\int_c^b \int_c^t (\beta(t)-\beta(s))^2\, d\alpha(s)d\gamma(t) &\leqslant \int_c^b \int_c^t (\alpha(t)-\alpha(s))(\gamma(t)-\gamma(s))\, d\alpha(s)d\gamma(t),\\ &\leqslant \int_c^b \alpha(t)\int_c^t (\alpha(s)-\alpha(c))\, d\gamma(s)d\gamma(t) \leqslant \left(\int_{t_-}^{b} \alpha(t)d\gamma(t)\right)^2.\end{aligned}$$

And by letting $c \to t_-$ we can get (4.8) and (4.9). On the other hand, (4.8) and (4.9) combined are strictly weaker than the condition $\int_{t_-}^{b} \alpha(s)d\gamma(s) < \infty$, and here's an example showing this:

Example 4.12. Let $I = (0, \epsilon)$ where ϵ is very small, $\alpha(t) = -\frac{t^2}{\log t}$, $\beta(t) = \log(-\log t)$, $\gamma(t) = -\frac{1}{2t^2}$ be defined on I. This defines a Hamiltonian as

$$\dot{\alpha}\dot{\gamma} = \left(-\frac{2t^2}{\log t} + \frac{t}{\log^2 t}\right)\frac{1}{t^3} \geqslant \frac{1}{t^2\log^2 t} = \dot{\beta}^2,$$

and

$$\int_0^b \alpha(t) d\gamma(t) = -\int_0^b \frac{1}{t\log t} dt = -\log(-\log(t))\Big|_0^b = \infty,$$
$$\int_0^b \alpha(t)^2 d\gamma(t) = \int_0^b \frac{t}{\log^2 t} dt \leqslant \frac{b^2}{2\log^2 b} < \infty,$$
$$\int_0^b \int_0^t \alpha(s)^2 d\gamma(s) d\gamma(t) \leqslant \int_0^b \frac{t^2}{2\log^2 t}\frac{dt}{t^3} = -\frac{1}{2}\frac{1}{\log t}\Big|_0^b = -\frac{1}{2\log b} < \infty,$$
$$\int_0^t (\beta(t)-\beta(s))^2 \, d\alpha(s) = 2\int_0^t \alpha(s)\left(\beta(t)-\beta(s)\right) d\beta(s) \leqslant \int_0^t \frac{2s}{\log^2 s}\log\left(-\log s\right) ds$$
$$\leqslant \int_0^t \frac{2s}{(-\log s)^{3/2}} ds \leqslant \frac{t^2}{(-\log t)^{3/2}} < \infty,$$
$$\int_0^b \int_0^t (\beta(t)-\beta(s))^2 \, d\alpha(s) d\gamma(t) \leqslant \int_0^b \frac{t^2}{(-\log t)^{3/2}}\frac{1}{t^3} dt = \int_{-\log b}^{+\infty} \frac{1}{u^{3/2}} du < \infty.$$

Before proving Theorem 4.11, we'll first prove some intermediate results. To simplify calculation, we'll introduce the notation $N_l(E)$ for an entire function E s.t. $E(0) = 1$. The reason we use $N_l(E)$ is to get rid of the first to the $(2k+1)$-th derivatives of E at the origin, which leads to a simpler inequality as shown in Lemma 4.14 below.

Lemma 4.13. *Let E be an entire function s.t. $E(0) = 1$, then for $k \geqslant 1$, there exists a unique polynomial P of degree at most k and $P(0) = 0$, s.t. $\left(Ee^P\right)'$ has a zero of order at least k at the origin. Moreover, if we define*

$$N_1(E) = E'(0),$$
$$N_{j+1}(E) = \left(E(z)\exp\left(-\sum_{i=1}^{j}\frac{N_i(E)}{i!}z^i\right)\right)^{(j+1)}(0), \quad j \geqslant 1,$$

then $P(z) = -\sum_{j=1}^{k}\frac{N_j(E)}{j!}z^j$.

Proof. Let $P(z) = -\sum_{j=1}^{k}\frac{N_j(E)}{j!}z^j$, and we'll show P satisfies the constraint inductively. For $k = 1$,

$$(E\exp(-N_1(E)z))'(0) = E'(0) - N_1(E) = 0.$$

Now for $k = j+1$, we know that $\left(E\exp\left(-\sum_{i=1}^{j}\frac{N_i(E)}{i!}z^i\right)\right)'$ has a zero of order at least j at the origin.

$$
\begin{aligned}
&\left(E\exp\left(-\sum_{i=1}^{j+1}\frac{N_i(E)}{i!}z^i\right)\right)^{(j+1)}\\
=&\sum_{l=0}^{j+1}\binom{j+1}{l}\left(E\exp\left(-\sum_{i=1}^{j}\frac{N_i(E)}{i!}z^i\right)\right)^{(l)}\left(\exp\left(-\frac{N_{j+1}(E)}{(j+1)!}z^{j+1}\right)\right)^{(j+1-l)}\\
=&-N_{j+1}(E)+\left(E\exp\left(-\sum_{i=1}^{j}\frac{N_i(E)}{i!}z^i\right)\right)^{(j+1)}\\
=&0.
\end{aligned}
$$

For the uniqueness, suppose there are two polynomials P and Q of degree at most k, s.t. $\left(Ee^P\right)'$ and $\left(Ee^Q\right)'$ have zeros of order at least k at the origin, then $E\left(e^P-e^Q\right)$ has a zero of order at least $k+1$ at the origin, and by taking the first k derivatives at the origin we can get $P=Q$. $\square$

From now on, for an entire function E of finite order, we use $q_l(E)$ to denote the coefficient q_l in the polynomial $Q(z)$ in its canonical factorization (4.11). Actually, for finite order entire function E, $N_l(E)$ and $q_l(E)$ are closely related to each other. Moreover, we prove inequality (4.13) which generalized (3.13). It is crucial to our proof and we use it to get the normality of the family of entire functions.

Lemma 4.14. *Let E be an entire function s.t. $E(0)=1$ and has factorization*

$$E(z)=e^{Q(z)}\prod_{n=1}^{N(E)}\left(1-\frac{z}{z_n}\right)e^{\frac{z}{z_n}+\cdots+\frac{1}{(2k+1)}\left(\frac{z}{z_n}\right)^{2k+1}}, \tag{4.11}$$

where $Q(z)=q_{2k+2}z^{2k+2}+\cdots+q_1z$, then $q_j=\frac{N_j(E)}{j!}$, $j=1,\cdots,2k+1$, and

$$E(z)\exp\left(-\sum_{j=1}^{2k+1}\frac{N_j(E)}{j!}z^j\right)=e^{q_{2k+2}z^{2k+2}}\prod_{n=1}^{N(E)}\left(1-\frac{z}{z_n}\right)e^{\frac{z}{z_n}+\cdots+\frac{1}{(2k+1)}\left(\frac{z}{z_n}\right)^{2k+1}}, \tag{4.12}$$

$$\log\left|E(z)\exp\left(-\sum_{j=1}^{2k+1}\frac{N_j(E)}{j!}z^j\right)\right|\leqslant\left(|q_{2k+2}|+\sum_{n=1}^{N(E)}\frac{1}{|z_n|^{2k+2}}\right)|z|^{2k+2}, \tag{4.13}$$

and $N_{2k+2}(E)=(2k+2)!\left(q_{2k+2}-\frac{1}{2k+2}\sum_{n=1}^{N(E)}\frac{1}{z_n^{2k+2}}\right)$. Moreover, if $A\in\mathcal{P}_{\leqslant k}$, $A(0)=1$ and $A^{\#}=A$, then $N_{2k+2}(A)\leqslant 0$.

Proof. It's easy to see the derivative of $E(z)\exp\left(-\sum_{j=1}^{2k+1}q_jz^j\right)$ has a zero of order at least $2k+1$ at the origin. Then by the uniqueness in Lemma 4.13 we know $q_j=\frac{N_j(E)}{j!}$ for $j=1,\cdots,2k+1$. The formula for $N_{2k+2}(E)$ comes from the definition of $N_{2k+2}(E)$. The rest comes from Lemma 4.2. $\square$

For $E\in\mathcal{P}_{\leqslant 1}$, $q_4(E)=p_4(E)\leqslant 0$, therefore we have:

Corollary 4.15. *Let $E \in \mathcal{P}_{\leqslant 1}$ be normalized, then*

$$-\frac{N_4(E)}{4!} = -q_4 + \frac{1}{4}\sum_{n=1}^{N(E)} \frac{1}{z_n^4}, \tag{4.14}$$

$$\log\left|E(z)e^{-q_3z^3-q_2z^2-q_1z}\right| \leqslant \left(-q_4 + \sum_{n=1}^{N(E)} \frac{1}{|z_n|^4}\right)|z|^4 \leqslant 4\left(-q_4 + \frac{1}{4}\sum_{n=1}^{N(E)} \frac{1}{|z_n|^4}\right)|z|^4. \tag{4.15}$$

The first few $N_j(F)$s are listed here for future reference:

$$\begin{aligned} N_1(E) &= E'(0), \\ N_2(E) &= E''(0) - E'(0)^2, \\ N_3(E) &= E^{(3)}(0) - 3E''(0)E'(0) + 2E'(0)^3 \\ &= E^{(3)}(0) - E'(0)^3 - 3E'(0)N_2(E), \\ N_4(E) &= E^{(4)}(0) - 4E^{(3)}(0)E'(0) - 3E''(0)^2 + 12E'(0)^2E''(0) - 6E'(0)^4 \\ &= E^{(4)}(0) - E'(0)^4 - 3N_2(E)^2 - 6E'(0)^2N_2(E) - 4E'(0)N_3(E). \end{aligned} \tag{4.16}$$

Now that we have the inequality (4.15) for $E \in \mathcal{P}_{\leqslant 1}$, and by (4.14) we know the factor $-q_4 + \frac{1}{4}\sum_{n=1}^{N(E)} \frac{1}{|z_n|^4}$ has an upper bound that is closely related to $N_4(E)$. The next step, is to relate the derivatives of $A_{a\to b}$, $C_{a\to b}$ and furthermore $N_l(A_{a\to b})$, $l \in \mathbb{N}$, with the associated Hamiltonian H.

Lemma 4.16. *For a Hamiltonian H with associated transition matrices* $M_{a\to b} = \begin{pmatrix} A_{a\to b} & B_{a\to b} \\ C_{a\to b} & D_{a\to b} \end{pmatrix}$,

$$\begin{aligned}
A'_{a\to b}(0) &= -(\beta(b) - \beta(a)), \\
C'_{a\to b}(0) &= \alpha(b) - \alpha(a), \\
A''_{a\to b}(0) &= (\beta(b) - \beta(a))^2 - 2\int_a^b (\alpha(t) - \alpha(a))d\gamma(t), \\
C''_{a\to b}(0) &= -2(\alpha(b) - \alpha(a))(\beta(b) - \beta(a)) + 4\int_a^b (\alpha(t) - \alpha(a))d\beta(t), \\
A^{(3)}_{a\to b}(0) &= -(\beta(b) - \beta(a))^3 + 6(\beta(b) - \beta(a))\int_a^b (\alpha(s) - \alpha(a))d\gamma(s) \\
&\quad - 12\int_a^b \int_a^t (\alpha(s) - \alpha(a))d\beta(s)d\gamma(t), \\
C^{(3)}_{a\to b}(0) &= 3(\beta(b) - \beta(a))^2(\alpha(b) - \alpha(a)) - 6\int_a^b (\alpha(b) - \alpha(s))(\alpha(s) - \alpha(a))d\gamma(s) \\
&\quad - 12(\beta(b) - \beta(a))\int_a^b (\alpha(t) - \alpha(a))d\beta(t) + 12\int_a^b (\beta(b) - \beta(t))^2 d\alpha(t), \\
N_1(A_{a\to b}) &= -(\beta(b) - \beta(a)), \\
N_2(A_{a\to b}) &= -2\int_a^b (\alpha(t) - \alpha(a))d\gamma(t), \\
N_3(A_{a\to b}) &= -12\int_a^b \int_a^t (\beta(t) - \beta(s))d\alpha(s)d\gamma(t), \\
N_4(A_{a\to b}) &= -24\left(\int_a^b \int_a^t (\alpha(s) - \alpha(a))^2 d\gamma(s)d\gamma(t) + 2\int_a^b \int_a^t (\beta(t) - \beta(s))^2 d\alpha(s)d\gamma(t)\right).
\end{aligned} \tag{4.17}$$

Proof. See Section 4.C. □

Following all these preparations, we can now state the proof of Theorem 4.11.

Proof of Theorem 4.11. The proof can be divided into three parts. Firstly, based on Corollary 4.15, we prove an inequality for $E_{a\to b}e^{-q_3z^3 - q_2z^2 - q_1z} = E_{a\to b}\exp\left(-\sum_{j=1}^3 \frac{N_j(E_{a\to b})}{j!}z^j\right)$, which is analogous to inequality (3.13) for $\mathcal{P}_0$. Secondly, using Lemma 4.16, we show the bounds in the inequalities are uniformly bounded above if conditions (4.6)—(4.9) hold, and thus we can find a convergent subsequence of $E_{a\to b}\exp\left(-\sum_{j=1}^3 \frac{N_j(E_{a\to b})}{j!}z^j\right)$ for fixed $b \in I$ and $a \to t_-$, and then construct the dB-functions E_t. Thirdly, we show the dB-chain is unique if we specify the asymptotic condition (4.10). WLOG we assume $t_- = 0$.

Part 1: Inequality and the Normality Condition

Let's consider $E_{a\to b} := A_{a\to b} - iC_{a\to b}$ for $b > a > 0$, where $\binom{A_{a\to b}}{C_{a\to b}}$ is the left column of $M_{a\to b}$. By Proposition 1.31, $A_{a\to b} - iC_{a\to b}$ is regular, hence $A_{a\to b}$, $C_{a\to b}$ are of exponential type and

$E_{a\to b} \in \mathrm{Exp} \cap dB \subseteq \mathcal{P}_0 \subseteq \mathcal{P}_{\leqslant 1}$. By Corollary 4.15 and the fact $q_4 = p_4$,

$$\log \left| E_{a\to b}(z) \exp\left(-\sum_{j=1}^{3} \frac{N_j(E_{a\to b})}{j!} z^j \right) \right| \leqslant 4 \left(-p_4 + \frac{1}{4} \sum_{n=1}^{N(E_{a\to b})} \frac{1}{|z_n|^4} \right) |z|^4. \tag{4.18}$$

Our goal is to find an upper bound for $-p_4 + \frac{1}{4}\sum_{n=1}^{N(E_{a\to b})} \frac{1}{|z_n|^4}$ which is uniform in a for each fixed b. We'll split it into a multiple of $\Re N_4(E_{a\to b})$ and some remainder. To simplify notation, we'll omit the subscripts for now and denote $E_{a\to b}$ by E, $A_{a\to b}$ by A, and $C_{a\to b}$ by C. Let $z_n = x_n - iy_n$, then $y_n \geqslant 0$ and

$$\begin{aligned}
-\frac{N_4(E)}{4!} &= -p_4 + \frac{1}{4} \sum_{n=1}^{N(E)} \frac{1}{z_n^4}, \\
-\frac{\Re N_4(E)}{4!} &= -p_4 + \frac{1}{4} \sum_{n=1}^{N(E)} \frac{x_n^4 + y_n^4 - 6x_n^2 y_n^2}{(x_n^2 + y_n^2)^4}, \\
-p_4 + \frac{1}{4} \sum_{n=1}^{N(E)} \frac{1}{|z_n|^4} &= -\frac{\Re N_4(E)}{4!} + \sum_{n=1}^{N(E)} \frac{2x_n^2 y_n^2}{(x_n^2 + y_n^2)^4}.
\end{aligned}$$

First, let's look at $\Re N_4(E)$. By (4.16) it's easy to get

$$\begin{aligned}
\Re N_4(E) =& N_4(A) + 36A'(0)^2 C'(0)^2 - 12A''(0)C'(0)^2 - 6C'(0)^4 \\
& - 24A'(0)C'(0)C''(0) + 3C''(0)^2 + 4C'(0)C^{(3)}(0).
\end{aligned}$$

On the other hand, by the relation between p_l and q_l described in Proposition 4.10,

$$\begin{aligned}
C'(0) &= -\Im q_1 = h + \sum_{n=1}^{N(E)} \Im \frac{1}{z_n} = h + \sum_{n=1}^{N(E)} \frac{y_n}{x_n^2 + y_n^2} \geqslant \sum_{n=1}^{N(E)} \frac{y_n}{x_n^2 + y_n^2} \geqslant 0, \\
C'(0)^3 &\geqslant \sum_{n=1}^{N(E)} \frac{y_n^3}{(x_n^2 + y_n^2)^3}, \\
\Im N_3(E) &= 3! \Im q_3 = -2 \sum_{n=1}^{N(E)} \Im \frac{1}{z_n^3} = -2 \sum_{n=1}^{N(E)} \frac{-3x_n^2 y_n + y_n^3}{(x_n^2 + y_n^2)^3} = \sum_{n=1}^{N(E)} \frac{6x_n^2 y_n - 2y_n^3}{(x_n^2 + y_n^2)^3}, \\
C'(0)\left(\Im N_3(E) + 2C'(0)^3\right) &\geqslant \left(\sum_{n=1}^{N(E)} \frac{y_n}{x_n^2 + y_n^2} \right) \left(\sum_{n=1}^{N(E)} \frac{6x_n^2 y_n}{(x_n^2 + y_n^2)^3} \right) \geqslant \sum_{n=1}^{N(E)} \frac{6x_n^2 y_n^2}{(x_n^2 + y_n^2)^4}.
\end{aligned}$$

By (4.16) again we can get

$$\Im N_3(E) = -6A'(0)^2 C'(0) + 3A''(0)C'(0) + 2C'(0)^3 + 3A'(0)C''(0) - C^{(3)}(0).$$

To summarize,

$$
\begin{aligned}
4!\left(-p_4+\frac{1}{4}\sum_{n=1}^{N(E)}\frac{1}{|z_n|^4}\right) =& -\Re N_4(E)+\sum_{n=1}^{N(E)}\frac{48x_n^2y_n^2}{(x_n^2+y_n^2)^4} \\
\leqslant& -\Re N_4(E)+8C'(0)\left(\Im N_3(E)+2C'(0)^3\right) \\
=& -N_4(A)+38C'(0)^4\underbrace{-36\left(\frac{C^{(3)}(0)}{3}-A''(0)C'(0)\right)C'(0)}_{\mathrm{I}} \\
& \underbrace{-3C''(0)^2+48A'(0)C'(0)C''(0)-84A'(0)^2C'(0)^2}_{\mathrm{II}}.
\end{aligned}
\tag{4.19}
$$

Part 2: Existence of a dB-chain

Now we show for fixed $b>0$, the terms in the RHS are bounded uniformly in $a\in(0,b)$ under assumptions (4.6)—(4.9).

Firstly, for $-N_4(A_{a\to b})$, by Lemma 4.16,

$$
\begin{aligned}
-N_4(A_{a\to b}) &= 24\left(\int_a^b\int_a^t(\alpha(s)-\alpha(a))^2d\gamma(s)d\gamma(t)+2\int_a^b\int_a^t(\beta(t)-\beta(s))^2d\alpha(s)d\gamma(t)\right) \\
&\leqslant 24\left(\int_0^b\int_0^t\alpha(s)^2d\gamma(s)d\gamma(t)+2\int_0^b\int_0^t(\beta(t)-\beta(s))^2d\alpha(s)d\gamma(t)\right)
\end{aligned}
\tag{4.20}
$$

is uniformly bounded above (uniform in $a\in(0,b)$).

Secondly, by Lemma 4.16 again,

$$
C'_{a\to b}(0)^4=(\alpha(b)-\alpha(a))^4\leqslant\alpha(b)^4. \tag{4.21}
$$

Thirdly, by Lemma 4.16, for term I in (4.19), we have:

$$
\begin{aligned}
\frac{C^{(3)}_{a\to b}(0)}{3} =&(\beta(b)-\beta(a))^2(\alpha(b)-\alpha(a))-2\int_a^b(\alpha(b)-\alpha(s))(\alpha(s)-\alpha(a))d\gamma(s) \\
&-4(\beta(b)-\beta(a))\int_a^b(\alpha(t)-\alpha(a))d\beta(t)+4\int_a^b(\beta(b)-\beta(t))^2d\alpha(t), \\
-A''_{a\to b}(0)C'_{a\to b}(0) =& -(\beta(b)-\beta(a))^2(\alpha(b)-\alpha(a)) \\
&+2\int_a^b(\alpha(b)-\alpha(a))(\alpha(s)-\alpha(a))d\gamma(s), \\
\frac{C^{(3)}_{a\to b}(0)}{3}-A''_{a\to b}(0)C'_{a\to b}(0) =&2\int_a^b(\alpha(s)-\alpha(a))^2d\gamma(s)-4(\beta(b)-\beta(a))\int_a^b(\alpha(t)-\alpha(a))d\beta(t) \\
&+4\int_a^b(\beta(b)-\beta(t))^2d\alpha(t).
\end{aligned}
$$

On the other hand, for term II in (4.19),

$$
\begin{aligned}
\mathrm{II} &= -3\left(C''_{a\to b}(0)^2 - 16A'_{a\to b}(0)C'_{a\to b}(0)C''_{a\to b}(0) + 28A'_{a\to b}(0)^2C'_{a\to b}(0)^2\right)\\
&= -3\left(C''_{a\to b}(0) - 2A'_{a\to b}(0)C'_{a\to b}(0)\right)\left(C''_{a\to b}(0) - 14A'_{a\to b}(0)C'_{a\to b}(0)\right)\\
&= -12\left(\int_a^b (\alpha(t)-\alpha(a))d\beta(t)\right)\left(4\int_a^b (\alpha(t)-\alpha(a))d\beta(t) + 12(\alpha(b)-\alpha(a))(\beta(b)-\beta(a))\right)\\
&= -48\left(\int_a^b (\alpha(t)-\alpha(a))d\beta(t)\right)^2 - 144(\alpha(b)-\alpha(a))(\beta(b)-\beta(a))\left(\int_a^b (\alpha(t)-\alpha(a))d\beta(t)\right).
\end{aligned}
$$

Combining the above calculation we know

$$
\begin{aligned}
\mathrm{I}+\mathrm{II} =& -72(\alpha(b)-\alpha(a))\left(\int_a^b (\alpha(s)-\alpha(a))^2 d\gamma(s) + 2\int_a^b (\beta(b)-\beta(t))^2 d\alpha(t)\right)\\
& - 48\left(\int_a^b (\alpha(t)-\alpha(a))d\beta(t)\right)^2.
\end{aligned}
\tag{4.22}
$$

From (4.8)—(4.9) we know both $\int_0^b \alpha(s)^2 d\gamma(s)$ and $\int_0^b (\beta(b)-\beta(t))^2 d\alpha(t)$ are finite, and by Hölder's inequality we have

$$
\left|\int_a^b (\alpha(t)-\alpha(a))d\beta(t)\right| = \left|\int_a^b (\beta(b)-\beta(t))d\alpha(t)\right| \leqslant \sqrt{\int_a^b (\beta(b)-\beta(t))^2\, d\alpha(t)}\sqrt{\alpha(b)-\alpha(a)},
$$

thus $\mathrm{I}+\mathrm{II}$ in (4.19) are uniformly bounded above, and moreover the RHS of (4.19) is uniformly bounded above.

To summarize, we get a uniform upper bound for $\left|E_{a\to b}(z)\exp\left(-\sum_{j=1}^3 \frac{N_j(E_{a\to b})}{j!}z^j\right)\right|$. Now we'll show for fixed b, $N_j(E_{a\to b}) - N_j(A_{a\to b})$ remains uniformly bounded for $j=1,2,3$:

$$
\begin{aligned}
N_1(E_{a\to b}) - N_1(A_{a\to b}) =& -iC'_{a\to b}(0) = -i\left(\alpha(b)-\alpha(a)\right),\\
N_2(E_{a\to b}) - N_2(A_{a\to b}) =& C'_{a\to b}(0) - i\left(C''_{a\to b}(0) - 2A'_{a\to b}(0)C'_{a\to b}(0)\right)\\
=& (\alpha(b)-\alpha(a)) - 4i\int_a^b (\alpha(t)-\alpha(a))d\beta(t),
\end{aligned}
\tag{4.23}
$$

$$
\begin{aligned}
N_3(E_{a\to b}) - N_3(A_{a\to b}) =& 3C'_{a\to b}(0)\left(C''_{a\to b}(0) - 2A'_{a\to b}(0)C'_{a\to b}(0)\right) \\
& - 3i\Bigg(\frac{C^{(3)}_{a\to b}(0)}{3} - C'_{a\to b}(0)A''_{a\to b}(0) \\
& \quad - A'_{a\to b}(0)\left(C''_{a\to b}(0) - 2A'_{a\to b}(0)C'_{a\to b}(0)\right)\Bigg) \\
=& 12\left(\alpha(b) - \alpha(a)\right)\int_a^b (\alpha(t) - \alpha(a))d\beta(t) \\
& - 6i\left(\int_a^b (\alpha(s) - \alpha(a))^2 d\gamma(s) + 2\int_a^b (\beta(b) - \beta(t))^2 d\alpha(t)\right),
\end{aligned}
\tag{4.24}
$$

and by the arguments above it's easy to see all terms above are uniformly bounded for $a \in (0, b)$, for any fixed b. Therefore, as $a \to 0$, we have a convergent subsequence

$$
\begin{aligned}
E_{a_n\to b}\exp\Bigg(&\underbrace{(\beta(b) - \beta(a_n))}_{-N_1(A_{a_n\to b})} z + \underbrace{\left(\int_{a_n}^b (\alpha(t) - \alpha(a_n))\, d\gamma(t)\right)}_{-\frac{N_2(A_{a_n\to b})}{2!}} z^2 \\
& + 2\underbrace{\left(\int_{a_n}^b \int_{a_n}^t (\beta(t) - \beta(s))d\alpha(s)d\gamma(t)\right)}_{-\frac{N_3(A_{a_n\to b})}{3!}} z^3\Bigg) \to \tilde{E}_b.
\end{aligned}
$$

Let $\tilde{E}_b = \tilde{A}_b - i\tilde{C}_b$, then $\tilde{E}_b$ is a normalized dB-function which might be non-degenerate. We also have $\tilde{A}_b = \lim_{n\to+\infty} A_{a_n\to b}\exp\left(-\sum_{l=1}^3 \frac{N_l(A_{a_n\to b})}{l!} z^l\right) \in \mathcal{L}_{\leqslant 1}$, since by Corollary 4.4 the Laguerre classes $\mathcal{L}_k$ are closed. Therefore $\tilde{E}_b \in \mathcal{P}_{\leqslant 1}$ according to Corollary 4.7.

Now let

$$
E_b = \tilde{E}_b \exp\left(-\beta(b)z + \left(\int_b^1 \alpha(t)d\gamma(t)\right) z^2 + 2\left(\int_b^1 \int_0^t (\beta(t) - \beta(s))\, d\alpha(s)d\gamma(t)\right) z^3\right),
$$

then $E_b \in \mathcal{P}_{\leqslant 1}$. Let $P_{a\to b}(z)$ be defined as

$$
P_{a\to b}(z) = \exp\left(-\beta(a)z + \int_a^b (\alpha(t) - \alpha(a))\, d\gamma(t)z^2 + 2\int_a^b \int_a^t (\beta(t) - \beta(s))\, d\alpha(s)d\gamma(t)z^3\right).
$$

Multiplying $P_{a_n\to 1}$ to the following equation

$$
\Omega\begin{pmatrix} A_{a_n\to b} \\ C_{a_n\to b}\end{pmatrix} - \Omega\begin{pmatrix} A_{a_n\to a} \\ C_{a_n\to a}\end{pmatrix} = z\int_a^b H(t)\begin{pmatrix} A_{a_n\to t} \\ C_{a_n\to t}\end{pmatrix} dt,
$$

and letting $n \to +\infty$, we can get

$$\Omega\begin{pmatrix} A_b \\ C_b \end{pmatrix} - \Omega\begin{pmatrix} A_a \\ C_a \end{pmatrix} = z\int_a^b H(t)\begin{pmatrix} A_t \\ C_t \end{pmatrix} dt.$$

Moreover, for $b > 0$,

$$\begin{aligned}
A_b(0) &= \tilde{A}_b(0) = \lim_{n\to+\infty} A_{a_n\to b}(0) = 1, \\
C_b(0) &= \tilde{C}_b(0) = \lim_{n\to+\infty} C_{a_n\to b}(0) = 0, \\
C_b'(0) &= \tilde{C}_b'(0) = \lim_{n\to+\infty} C_{a_n\to b}'(0) = \lim_{n\to+\infty} \alpha(b) - \alpha(a_n) = \alpha(b) > 0.
\end{aligned}$$

Then A_b and C_b must be linearly independent, E_b is non-degenerate, and therefore the dB-space $\mathcal{B}(E_b)$ exists.

As for the asymptotic condition, we have

$$\begin{aligned}
&\lim_{b\to 0+} E_b(z)\exp\left(\beta(b)z - \left(\int_b^1 \alpha(t)d\gamma(t)\right)z^2 - 2\left(\int_b^1\int_0^t (\beta(t)-\beta(s))\,d\alpha(s)d\gamma(t)\right)z^3\right) \\
&= \lim_{b\to 0+} \tilde{E}_b(z) = 1,
\end{aligned}$$

where the last equality comes from inequalities (4.18), (4.19), (4.20), (4.21), (4.22), (4.23), and (4.24).

The asymptotic condition implies

$$\lim_{t\to t_-} E_t(\overline{iy})E_t^{\#}(iy)e^{2\int_t^1 \alpha s d\gamma(s) y^2} = 1, \quad y > 0,$$

therefore $\lim_{t\to t_-} K_{t,iy}(iy) = 0$ for $y > 0$. Then $\{\mathcal{B}(E_t)\}$ is a dB-chain.

By Theorem 1.17, if $E_b(c) = 0$ for $c \in \mathbb{R}$, then $E_a(c) = 0$ for all $a \in (0, b)$, which contradicts the asymptotic condition (4.10). To summarize, we showed the existence of $\{\mathcal{B}(E_t)\}$ with H as its Hamiltonian, s.t. $E_t \in \mathcal{P}_{\leqslant 1}$ is strict and non-degenerate, and it satisfies the asymptotic condition (4.10).

Part 3: Uniqueness of the dB-chain

Suppose there are two chains satisfying the same asymptotic condition (4.10). Plug $w = \bar{z}$ in Lagrange's identity (2.2), then

$$G_a(z) := C_{+,a}(z)A_{-,a}(z) - A_{+,a}(z)C_{-,a}(z) = \frac{E_{+,a}^{\#}E_{-,a} - E_{+,a}E_{-,a}^{\#}}{2i}$$

is independent of a. On the imaginary axis,

$$\lim_{b\to 0+} E_{\pm,b}(iy)\exp\left(i\beta(b)y+\left(\int_b^1 \alpha(t)d\gamma(t)\right)y^2+2i\left(\int_b^1\int_0^t (\beta(t)-\beta(s))\,d\alpha(s)d\gamma(t)\right)y^3\right)=1,$$
$$\lim_{b\to 0+} E^{\#}_{\pm,b}(iy)\exp\left(-i\beta(b)y+\left(\int_b^1 \alpha(t)d\gamma(t)\right)y^2-2i\left(\int_b^1\int_0^t (\beta(t)-\beta(s))\,d\alpha(s)d\gamma(t)\right)y^3\right)=1,$$

then

$$\lim_{a\to 0+} G_a(iy)e^{2\int_a^1 \alpha(s)d\gamma(s)y^2}=0,\quad \forall y\in\mathbb{R}.$$

For $a<1$, $\int_a^1 \alpha(s)d\gamma(s)\geqslant 0$, then

$$\lim_{a\to 0+}|G_a(iy)|\leqslant \lim_{a\to 0+}|G_a(iy)|\,e^{2\int_a^1 \alpha(s)d\gamma(s)y^2}=0.$$

Thus $G_b(z)\equiv 0$ on the imaginary axis as $G_b(z)$ is independent of b. Therefore $G_b(z)\equiv 0$ for all $z\in\mathbb{C}$. By definition of $G_b(z)$, we now have

$$C_{+,b}(z)A_{-,b}(z)-A_{+,b}(z)C_{-,b}(z)\equiv 0,\quad \forall z\in\mathbb{C}.$$

By the same arguments as above we know both $E_{+,b}$ and $E_{-,b}$ are strict non-degenerate dB-functions given the asymptotic conditions, then $A_{+,b}$ and $C_{+,b}$, $A_{-,b}$ and $C_{-,b}$ can't have the same zeros, therefore

$$S_b:=\frac{A_{-,b}}{A_{+,b}}=\frac{C_{-,b}}{C_{+,b}}$$

is real entire and zero free, and

$$E_{-,b}=S_bE_{+,b}.$$

Since

$$\begin{pmatrix}A_{-,b}\\C_{-,b}\end{pmatrix}=S_b\begin{pmatrix}A_{+,b}\\C_{+,b}\end{pmatrix}=S_bM_{a\to b}\begin{pmatrix}A_{+,a}\\C_{+,a}\end{pmatrix}=\frac{S_b}{S_a}M_{a\to b}\begin{pmatrix}A_{-,a}\\C_{-,a}\end{pmatrix}=\frac{S_b}{S_a}\begin{pmatrix}A_{-,b}\\C_{-,b}\end{pmatrix},$$

we have $S_b=S_a$. As $E_{-,t}$ and $E_{+,t}$ satisfy the same asymptotic condition (4.10) as $t\to 0$, $S_b=\lim_{t\to 0+}S_t\equiv 1$, hence $E_{-,b}=E_{+,b}$, $\forall b>0$. To summarize, we proved if there are two dB-chains with the same associated Hamiltonian, which satisfy the same asymptotic condition (4.10), then they must be the same dB-chain. $\square$

To summarize, for a Hamiltonian H that satisfies (4.6)—(4.9) we can find a dB-chain $\{\mathcal{B}(E_t)\}_{t\in I}$ with H as its Hamiltonian, then by Theorem 1.28 the spectral measures and the Weyl-transform are well-defined. We should also point out that unlike the case $E_t\in\mathcal{P}_0$, the necessity conditions of (4.8) and (4.9) haven't been proved yet, and we expect this to be completed in the near future. Nevertheless, in next section we'll show that for Kreĭn's strings where $\beta(t)\equiv 0$, condition (4.8) is both sufficient and necessary for the existence of a dB-chain in $\mathcal{P}_{\leqslant 1}$ for a given Hamiltonian $H(t)$.

4.3 Kreĭn's strings: beyond the Pólya class

In this section we focus on Kreĭn's strings, namely the canonical systems with the Hamiltonian $H(t)$ being a diagonal matrix, then Theorem 3.13 and Theorem 4.11 together show the existence of a dB-chain $\{\mathcal{B}(E_t)\}_{E\in I}$ with H as its Hamiltonian under certain assumptions on H. Namely, if $\alpha(t_-)=0$ and $\alpha(t)>\alpha(t_-)$ for $t\in I$, then

$$\begin{aligned}
&\int_{t_-}^{c}\alpha(t)d\gamma(t)<\infty \Rightarrow \exists\{\mathcal{B}(E_t)\} \text{ with Hamiltonian } H \text{ s.t.} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \begin{cases} E_t^{\#}(-z)=E(z), \\ E_t\in\mathcal{P}_0, \\ \lim_{t\to t_-}E_t(z)=1. \end{cases} \\
&\int_{t_-}^{c}\int_{t_-}^{t}\alpha(s)^2 d\gamma(s)d\gamma(t)<\infty \Rightarrow \exists\{\mathcal{B}(E_t)\} \text{ with Hamiltonian } H \text{ s.t.} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \begin{cases} E_t^{\#}(-z)=E(z), \\ E_t\in\mathcal{P}_{\leqslant 1}, \\ \lim_{t\to t_-}E_t(z)e^{-\int_t^c\alpha(s)d\gamma(s)z^2}=1. \end{cases}
\end{aligned} \tag{4.25}$$

In this section we will give a different approach, which uses the theory of function of Laguerre classes (most importantly, Proposition 4.9), to prove the above results. Moreover, this approach further leads to possible generalization to $\mathcal{P}_{\leqslant 2}$ and a new proof of de Branges' Theorem 41. The latter will be presented in Section 4.4.

First we will discuss some basic results for Kreĭn's strings. For Kreĭn's strings, The canonical equation (1.23) becomes

$$\begin{pmatrix} A_{a\to b}(z)-1 \\ C_{a\to b}(z) \end{pmatrix} = z\begin{pmatrix} -\int_a^b C_{a\to t}(z)d\gamma(t) \\ \int_a^b A_{a\to t}(z)d\alpha(t) \end{pmatrix}. \tag{4.26}$$

Taking the derivatives w.r.t. z at 0 on both sides, then for $n\geqslant 1$,

$$\begin{aligned}
A_{a\to b}^{(n)}(0) &= -n\int_a^b C_{a\to t}^{(n-1)}(0)d\gamma(t), \\
C_{a\to b}^{n}(0) &= n\int_a^b A_{a\to t}^{(n-1)}(0)d\alpha(t).
\end{aligned} \tag{4.27}$$

This simple recursive relation between $A_{a\to b}^{(n)}$ and $C_{a\to b}^{(n)}$ makes the calculations of $N_l(A_{a\to b}(z))$ and

$N_l\left(\frac{C_{a\to b}(z)}{zC'_{a\to b}(0)}\right)$ feasible. Moreover, since the Hamiltonian H is diagonal, by Lemma 3.4 we know $A_{a\to b}$ is even and $C_{a\to b}$ is odd, which simplify the recursive relation and calculation even further.

Now we present the theorem on the existence of a dB-chain for diagonal Hamiltonian, which is a special case of Theorem 4.11. The key to the proof is the normality condition for functions in $\mathcal{L}_1$, namely (4.4) and the facts that $A_{a\to b}\exp\left(-\frac{N_2(A_{a\to b})}{2}z^2\right)\in\mathcal{L}_{\leqslant 1}$ and $F_{a\to b}\exp\left(-\frac{N_2(F_{a\to b})}{2}z^2\right)\in\mathcal{L}_{\leqslant 1}$, where $F_{a\to b}:=\frac{C_{a\to b}}{zC'_{a\to b}(0)}$ is an entire function s.t. $F_{a\to b}(0)=1$.

Theorem 4.17. *Let $H=H(t)$, $t\in I$ be a diagonal Hamiltonian and $h=h(t)=\begin{pmatrix}\alpha(t) & 0\\ 0 & \gamma(t)\end{pmatrix}$ be its anti-derivative. Assume that*

$$\alpha(t_-):=\lim_{t\to t_-}\alpha(t)=0, \tag{4.28}$$

$$\alpha(t)>0 \text{ for } t\in I, \tag{4.29}$$

$$\int_{t_-}^{b}\int_{t_-}^{t}\alpha(s)^2 d\gamma(s)d\gamma(t)<\infty \text{ for some (hence for all) } b\in I, \tag{4.30}$$

then there exists a unique dB-chain $\{\mathcal{B}(E_t)\}_{t\in I}$ with H as its Hamiltonian, s.t. $E_t\in\mathcal{P}_{\leqslant 1}$ is strict and non-degenerate, and

$$\lim_{t\to t_-}E_t(z)e^{-\left(\int_t^1\alpha(s)d\gamma(s)\right)z^2}=1 \tag{4.31}$$

locally uniformly in z.

Proof. The proof has three parts. First, we show

$$A_{a\to b}\exp\left(-\frac{N_2(A_{a\to b})}{2}z^2\right)\in\mathcal{L}_{\leqslant 1},\quad F_{a\to b}\exp\left(-\frac{N_2(F_{a\to b})}{2}z^2\right)\in\mathcal{L}_{\leqslant 1}$$

and get the normality condition accordingly. The second part is to show $\lim_{a\to t_-}N_2(F_{a\to b})-N_2(A_{a\to b})$ exists, therefore we can find a convergent subsequence for $E_{a\to b}\exp\left(-\frac{N_2(A_{a\to b})}{2}z^2\right)$ as $a\to t_-$. The third part is to construct the dB-chain from the limit function, which would be the same as the proof of Theorem 4.11. WLOG we assume $t_-=0$.

Part 1: Normality Condition

Let b, $A_{a\to b}$, $C_{a\to b}$ be as defined in the proof of Theorem 4.11. As $\alpha(b)>0$ and $\alpha(a)\to 0$ as $a\to 0+$, we have $C'_{a\to b}(0)=\alpha(b)-\alpha(a)>0$ for a small enough. Let $F_{a\to b}(z):=\frac{C_{a\to b}}{zC'_{a\to b}(0)}$. Because of the symmetry conditions (3.3), (4.16), and (4.17) can be simplified to

$$
\begin{aligned}
&N_1(A_{a\to b}) = N_1(F_{a\to b}) = N_3(A_{a\to b}) = N_3(F_{a\to b}) = 0,\\
&N_2(A_{a\to b}) = A''_{a\to b}(0) = -\frac{2}{\alpha(b)-\alpha(a)}\int_a^b (\alpha(t)-\alpha(a))d\gamma(t),\\
&N_2(F_{a\to b}) = F''_{a\to b}(0) = \frac{C''_{a\to b}(0)}{3C'_{a\to b}(0)} = -2\int_a^b (\alpha(b)-\alpha(s))(\alpha(s)-\alpha(a))d\gamma(s),\\
&N_4(A_{a\to b}) = A^{(4)}_{a\to b}(0) - 3A''_{a\to b}(0)^2 = -24\int_a^b\int_a^t (\alpha(s)-\alpha(a))^2 d\gamma(s)d\gamma(t),\\
&N_4(F_{a\to b}) = F^{(4)}_{a\to b}(0) - 3F''_{a\to b}(0)^2.
\end{aligned}
\tag{4.32}
$$

Basically, we want to show that for fixed $b>0$, both $\left\{A_{a\to b}\exp\left(-\frac{N_2(A_{a\to b})}{2}z^2\right) : a\in(0,b)\right\}$ and $\left\{F_{a\to b}\exp\left(-\frac{N_2(F_{a\to b})}{2}z^2\right) : a\in(0,b)\right\}$ are normal families of entire functions, and therefore we can find the limit functions and follow similar arguments as in the proof of Theorem 4.11.

As $A_{a\to b}$, $C_{a\to b}$ of exponential type, we know

$$A_{a\to b}\exp\left(-\frac{N_2(A_{a\to b})}{2}z^2\right)\in\mathcal{P}_{\leqslant 1},\quad F_{a\to b}\exp\left(-\frac{N_2(F_{a\to b})}{2}z^2\right)\in\mathcal{P}_{\leqslant 1}.$$

Furthermore, by definition of $N_l(A_{a\to b})$ and the facts that $A_{a\to b}$, $F_{a\to b}$ are real entire and each of them has value 1 at the origin,

$$A_{a\to b}\exp\left(-\frac{N_2(A_{a\to b})}{2}z^2\right)\in\mathcal{L}_{\leqslant 1},\quad F_{a\to b}\exp\left(-\frac{N_2(F_{a\to b})}{2}z^2\right)\in\mathcal{L}_{\leqslant 1}.$$

There are two cases here. Firstly, for fixed $b>0$, if $N_2(A_{a\to b})=0$ for all $a\in(0,b)$, then $\int_0^b \alpha(t)d\gamma(t)=0$ and we're reduced to the regular case, and the results obviously hold. Secondly, if $N_2(A_{a\to b})>0$ for some $a\in(0,b)$, then $N_2(A_{a\to b})>0$ for all a small enough, and therefore

$$A_{a\to b}\exp\left(-\frac{N_2(A_{a\to b})}{2}z^2\right)\in\mathcal{L}_1,\quad F_{a\to b}\exp\left(-\frac{N_2(F_{a\to b})}{2}z^2\right)\in\mathcal{L}_1$$

for a small enough.

By the inequality (4.4) for functions in $\mathcal{L}_1$ we know

$$
\begin{aligned}
\log\left|1-A_{a\to b}\exp\left(-\frac{N_2(A_{a\to b})}{2}z^2\right)\right| &\leqslant -\frac{N_4(A_{a\to b})}{3!}|z|^4,\\
\log\left|1-F_{a\to b}\exp\left(-\frac{N_2(F_{a\to b})}{2}z^2\right)\right| &\leqslant -\frac{N_4(F_{a\to b})}{3!}|z|^4.
\end{aligned}
$$

Because of (4.30),

$$-N_4(A_{a\to b}) = 24\int_a^b\int_a^t (\alpha(s)-\alpha(a))^2 d\gamma(s)d\gamma(t) \leqslant 24\int_0^b\int_0^t \alpha(s)^2 d\gamma(s)d\gamma(t) < \infty$$

is uniformly bounded for $a \in (0,b)$ and fixed $b>0$. Next we will show $-N_4(F_{a\to b})$ is uniformly bounded too.

By (4.26), and noting that $\frac{d\alpha(t)}{C'_{a\to b}(0)}$ is a probability measure, we can get

$$\begin{aligned} F''_{a\to b}(0) &= \frac{C^{(3)}_{a\to b}(0)}{3C'_{a\to b}(0)} = \int_a^b A''_{a\to t}(0)\frac{d\alpha(t)}{C'_{a\to b}(0)}, \\ F''_{a\to b}(0)^2 &\leqslant \int_a^b A''_{a\to t}(0)^2\frac{d\alpha(t)}{C'_{a\to b}(0)}, \\ F^{(4)}_{a\to b}(0) &= \frac{C^{(5)}_{a\to b}(0)}{5C'_{a\to b}(0)} = \int_a^b A^{(4)}_{a\to t}(0)\frac{d\alpha(t)}{C'_{a\to b}(0)}, \\ -N_4(F_{a\to b}) &= -F^{(4)}_{a\to b}(0) + 3F''_{a\to b}(0)^2 \leqslant -\int_a^b N_4(A_{a\to t})\frac{d\alpha(t)}{C'_{a\to b}(0)} \leqslant -N_4(A_{a\to b}), \end{aligned}$$

where we applied Jensen's inequality to get the inequality on the second row. Thus we get normality conditions for both $A_{a\to b}\exp\left(-\frac{N_2(A_{a\to b})}{2}z^2\right)$ and $F_{a\to b}\exp\left(-\frac{N_2(F_{a\to b})}{2}z^2\right)$.

Part 2: Convergent Subsequence and Limit Function

Now we show $N_2(F_{a\to b}) - N_2(A_{a\to b})$ is uniformly bounded as well. By (4.32),

$$N_2(F_{a\to b}) - N_2(A_{a\to b}) = 2\int_a^b (\alpha(s)-\alpha(a))^2\, d\gamma(s)$$

is uniformly bounded under the assumption (4.30). Therefore, there exists a sequence $\{a_n\}$ s.t. $a_n \to 0$ and

$$\begin{aligned} \lim_{n\to+\infty} A_{a_n\to b}\exp\left(-\frac{N_2(F_{a\to b})}{2}z^2\right) &= \tilde{A}_b, \\ \lim_{n\to+\infty} F_{a_n\to b}\exp\left(-\frac{N_2(F_{a\to b})}{2}z^2\right) &= \tilde{F}_b, \\ \lim_{n\to+\infty} C_{a_n\to b}\exp\left(-\frac{N_2(F_{a\to b})}{2}z^2\right) &= \tilde{C}_b = z\alpha(b)\alpha(b)\tilde{F}_b, \\ \lim_{n\to+\infty} E_{a_n\to b}\exp\left(-\frac{N_2(F_{a\to b})}{2}z^2\right) &= \tilde{E}_b = \tilde{A}_b - i\tilde{B}_b. \end{aligned}$$

Part 3: Existence and Uniqueness of the dB-chain

Let $E_b := \tilde{E}_b e^{\int_b^1 \alpha(t)d\gamma(t)z^2}$, and the rest is the same as the proof of Theorem 4.11. □

Remark. For Kreĭn's string, we may apply the same method to $\mathcal{P}_{\leqslant k}$ for $k \geqslant 2$. For example, for

$k = 2$ the condition becomes

$$-\frac{N_6(A_{a\to b})}{2\cdot 6!} = \int_0^b\int_0^t\int_0^s \alpha(s)\alpha(u)^2 d\gamma(u)d\gamma(s)d\gamma(t) < \infty.$$

This approach leads to possible generalization of Theorem 4.17.

On the other hand, the condition $\int_0^b\int_0^t \alpha(s)^2 d\gamma(s)d\gamma(t) < \infty$ is also necessary for the existence of a dB-chain in $\mathcal{P}_{\leqslant 1}$, for a given diagonal Hamiltonian $H(t)$. To prove this, we need the following lemma.

Lemma 4.18. *Let $E = A - iC$ be dB-function, $A(0) \neq 0$, then $\left(\frac{C(z)}{zA(z)}\right)''\Big|_{z=0} \geqslant 0$. Let $F(z) := \frac{C(z)}{z}$, then*

$$F''(0) - F(0)A''(0) - 2(F'(0) - F(0)A'(0)A'(0)) \geqslant 0. \tag{4.33}$$

Proof. See Section 4.D. $\square$

Remark. Inequality (4.33) can also be used in proving Theorem 3.13, to show $N_2(F_{a\to b}) = F''_{a\to b}(0) - F'_{a\to b}(0)^2$ is bounded below. See Section 4.4 for more details. Moreover, this inequality can also be used to simplify the proof of Theorem 4.11. To be more specific, the proof of boundedness of I+II can be simplified using (4.33).

Theorem 4.19. *Let $\{\mathcal{B}(E_t)\}_{t\in I}$ be a dB-chain with diagonal Hamiltonian H, and $E_t(0) = 1$ for $t \in I$. Let $h = h(t) = \begin{pmatrix}\alpha(t) & 0\\ 0 & \gamma(t)\end{pmatrix}$ be an anti-derivative of H s.t. $\alpha(t_-) = 0$. If $E_t \in \mathcal{P}_{\leqslant 1}$ for some $t \in I$ or $\lim_{t\to t_-} E_t(z)e^{-\left(\int_t^1 \alpha(s)d\gamma(s)\right)z^2} = S(z)$ locally uniformly in z where S is real entire,then H satisfies* (4.30) *as well.*

Proof. WLOG we assume $t_- = 0$. Since $H = H(t)$ is a diagonal matrix, the canonical equation (1.23) can be re-written as

$$\begin{aligned} A_b(z) - A_a(z) &= -z^2\int_a^b \frac{C_t(z)}{z}d\gamma(t),\\ \frac{C_b(z)}{z} - \frac{C_a(z)}{z} &= \int_a^b A_t(z)d\alpha(t), \end{aligned} \tag{4.34}$$

from which we can see that $A_b'(0) = A_a'(0) = \lambda \in \mathbb{R}$. WLOG we assume $\lambda = 0$, otherwise we can multiply $e^{-\lambda z}$ to the dB-chain. Let $F_t(z) := \frac{C_t(z)}{z}$, then $F_t(0) = C_t'(0) \to 0$ as $t \to 0+$, by definition of a dB-chain. Evaluating (4.34) at the origin, we know $F_b(0) - F_a(0) = \alpha(b) - \alpha(a)$, then as $\lim_{a\to 0+} F_a(0) = \lim_{a\to 0+}\alpha(a) = 0$, we get $F_b(0) = \alpha(b)$, $\forall b > 0$. Taking the derivatives of (4.34) and evaluating them at 0, we get:

$$A_b''(0) - A_a''(0) = -2\int_a^b \alpha(t)d\gamma(t),$$

$$F_b''(0) - F_a''(0) = \int_a^b A_t''(0)d\alpha(t) = A_a''(0)(\alpha(b)-\alpha(a)) - 2\int_a^b\int_a^t \alpha(s)d\gamma(s)d\alpha(t)$$

$$= A_a''(0)(\alpha(b)-\alpha(a)) - 2\int_a^b (\alpha(b)-\alpha(s))\alpha(s)d\gamma(s),$$

$$A_b^{(4)}(0) - A_a^{(4)}(0) = -12\int_a^b F_t''(0)d\gamma(t)$$

$$= -12F_a''(0)(\gamma(b)-\gamma(a)) - 12A_a''(0)\int_a^b(\alpha(t)-\alpha(a))d\gamma(t)$$

$$+ 24\int_a^b\int_a^t(\alpha(t)-\alpha(s))\alpha(s)d\gamma(s)d\gamma(t)$$

$$= -12F_a''(0)(\gamma(b)-\gamma(a)) - 12A_a''(0)\int_a^b(\alpha(t)-\alpha(a))d\gamma(t)$$

$$+ 12\left(\int_a^b \alpha(s)d\gamma(s)\right)^2 - 24\int_a^b\int_a^t \alpha(s)^2 d\gamma(s)d\gamma(t).$$

Therefore

$$N_4(A_b) = A_b^{(4)}(0) - 3A_b''(0)^2$$

$$= A_a^{(4)}(0) - 3A_a''(0)^2 - 12F_a''(0)(\gamma(b)-\gamma(a)) - 12A_a''(0)\int_a^b(\alpha(t)-\alpha(a))d\gamma(t)$$

$$+ 12A_a''(0)\int_a^b \alpha(t)d\gamma(t) - 24\int_a^b\int_a^t \alpha(s)^2 d\gamma(s)d\gamma(t)$$

$$= N_4(A_a) + 12\left(A_a''(0)F_a(0) - F_a''(0)\right)(\gamma(b)-\gamma(a)) - 24\int_a^b\int_a^t \alpha(s)^2 d\gamma(s)d\gamma(t).$$

We now show the second term is always nonnegative. By Lemma 4.18 and the assumption $A_b'(0) = 0$,

$$F_b''(0) - F_b(0)A_b''(0) \geqslant 0.$$

And because $\gamma(b) \geqslant \gamma(a)$, the second term is always nonnegative. Therefore

$$24\int_a^b\int_a^t \alpha(s)d\gamma(s)d\gamma(t) \leqslant N_4(A_a) - N_4(A_b). \tag{4.35}$$

If $E_t \in \mathcal{P}_{\leqslant 1}$, then $A_t \in \mathcal{P}_{\leqslant 1}$ by Corollary 4.7, and by (4.35) we have

$$24\int_a^b\int_a^t \alpha(s)^2 d\gamma(s)d\gamma(t) \leqslant -N_4(A_b)$$

is uniformly bounded for $a \in (0, b)$.

On the other hand, if $\lim_{a\to 0+} E_a(z)e^{\beta(a)z} = S(z)$, then as S is real entire,

$$A_a(z)e^{-\left(\int_t^1 \alpha(s)d\gamma(s)\right)z^2} \to S(z),$$
$$A_a''(0) - 2\int_a^1 \alpha(s)d\gamma(s) \to S''(0),$$
$$A_a''(0)^2 - 4A_a''(0)\int_a^1 \alpha(s)d\gamma(s) + 4\left(\int_a^1 \alpha(s)d\gamma(s)\right)^2 \to S''(0)^2,$$
$$A_a^{(4)}(0) - 12A_a''(0)\int_a^1 \alpha(s)d\gamma(s) + 12\left(\int_a^1 \alpha(s)d\gamma(s)\right)^2 \to S^{(4)}(0),$$

as $a \to 0+$. Therefore

$$N_4(A_a) = A_a^{(4)}(0) - 3A_a''(0)^2 \to S^{(4)}(0) - 3S''(0)^2 = N_4(S)$$

is finite.

Then in either case,

$$\int_0^b \int_0^t \alpha(s)^2 d\gamma(s)d\gamma(t) = \lim_{a\to 0+} \int_a^b \int_a^t \alpha(s)^2 d\gamma(s)d\gamma(t) < \infty.$$

□

Combining Theorem 4.17, Theorem 4.19 and Theorem 2.13 we get a one-to-one correspondence between a dB-chain in $\mathcal{P}_{\leqslant 1}$ up to a factor $\exp\left(\sum_{n=1}^4 a_n z^n\right)$, $a_n \in \mathbb{R}$, and a diagonal Hamiltonian H that satisfies (4.28)—(4.30). This is analogous to Theorem 1.34 for regular E_t and Theorem 3.15 for $E_t \in \mathcal{P}_0$.

Theorem 4.20. *(i) Let $H = H(t)$, $t \in I$ be a diagonal Hamiltonian. If H satisfies (4.28)—(4.30), then there exists a unique dB-chain $\{\mathcal{B}(E_t)\}_{t\in I}$ with H as its Hamiltonian, s.t.*

$$\lim_{t\to t_-} E_t(z)e^{-\left(\int_t^1 \alpha(s)d\gamma(s)\right)z^2} = 1$$

locally uniformly in z. For this unique dB-chain $\{\mathcal{B}(E_t)\}$, we also have $E_t \in \mathcal{P}_{\leqslant 1}$, $\forall t \in I$.

(ii) Let $\{\mathcal{B}(E_t)\}_{t\in I}$ be a dB-chain with diagonal Hamiltonian $H = H(t)$, and $E_t(0) = 1$ for $t \in I$.

- *If $E_t \in \mathcal{P}_{\leqslant 1}$ for some $t \in I$, then $E_t \in \mathcal{P}_{\leqslant 1}$ for all $t \in I$, $H(t)$ satisfies (4.30) and $E_t(z)e^{-\left(\int_t^1 \alpha(s)d\gamma(s)\right)z^2}$ converges to $S(z) := \exp\left(\sum_{n=1}^4 a_n z^n\right)$ for some $a_n \in \mathbb{R}$ locally uniformly in z, as $t \to t_-$.*
- *If $S(z) := \lim_{t\to t_-} E_t(z)e^{-\left(\int_t^1 \alpha(s)d\gamma(s)\right)z^2}$ exists and is real entire, then $H(t)$ satisfies (4.30), and $E_t = S\tilde{E}_t$ where $\tilde{E}_t \in \mathcal{P}_{\leqslant 1}$, $\forall t \in I$, and $\lim_{t\to t_-} \tilde{E}_t(z)e^{-\left(\int_t^1 \alpha(s)d\gamma(s)\right)z^2} = 1$ locally uniformly in z.*

Proof. The proof is essentially the same as the proof of Theorem 3.15. □

4.4 An alternative proof of de Branges' Theorem 41

The approach we used to prove Theorem 4.17 can also be used to prove de Branges' Theorem 41. Firstly, we show the uniform boundedness of $N_2(A_{a\to b})$ and $N_2(F_{a\to b})$. Secondly, we show that $N_1(A_{a\to b}) - N_1(F_{a\to b})$ is uniformly bounded, and therefore we can find a convergent subsequence of $\{E_{a\to b}e^{-N_1(A_{a\to b})z}\}$ as $a \to t_-$.

Proof of de Branges' Theorem 41. WLOG let $t_- = 0$. Let $A_{a\to b}$, $C_{a\to b}$, $F_{a\to b}$ be defined as usual. The canonical equation (1.23) becomes

$$\begin{aligned} A_{a\to b}(z) - 1 &= -z\int_a^b A_{a\to t}(z)d\beta(t) - z\int_a^b C_{a\to t}(z)d\gamma(t), \\ C_{a\to b}(z) &= z\int_a^b A_{a\to t}(z)d\alpha(t) + z\int_a^b C_{a\to t}(z)d\beta(t), \\ \frac{C_{a\to b}(z)}{z} &= \int_a^b A_{a\to t}(z)d\alpha(t) + \int_a^b C_{a\to t}(z)d\beta(t). \end{aligned} \tag{4.36}$$

As $A_{a\to b}(z), C_{a\to b}$ are of exponential type, and by definition of $\mathcal{L}_0$ and N_1 we know

$$A_{a\to b}e^{-N_1(A_{a\to b})z} \in \mathcal{L}_0, \quad F_{a\to b}e^{-N_1(F_{a\to b})z} \in \mathcal{L}_0,$$

By (4.17),

$$-N_2(A_{a\to b}) = 2\int_a^b (\alpha(t) - \alpha(a))\, d\gamma(t) \leqslant 2\int_0^b \alpha(t)d\gamma(t)$$

is uniformly bounded under assumption (3.16).

There are at least two ways to show $N_2(F_{a\to b})$ is uniformly bounded from below. Firstly, by Lemma 4.18,

$$F''_{a\to b}(0) - A''_{a\to b}(0) \geqslant 2\left(F'_{a\to b}(0) - A'_{a\to b}(0)\right) A'_{a\to b}(0).$$

Therefore

$$\begin{aligned} N_2(F_{a\to b}) &= F''_{a\to b}(0) - F'_{a\to b}(0)^2 \\ &\geqslant A''_{a\to b}(0) + 2\left(F'_{a\to b}(0) - A'_{a\to b}(0)\right) A'_{a\to b}(0) - F'_{a\to b}(0)^2 \\ &= N_2 A_{a\to b} - \left(F'_{a\to b}(0) - A'_{a\to b}(0)\right)^2. \end{aligned}$$

By (4.17) again,

$$F'_{a\to b}(0) = \frac{C''_{a\to b}(0)}{2C'_{a\to b}(0)} = A'_{a\to b}(0) + \frac{2}{C'_{a\to b}(0)}\int_a^b (\alpha(t) - \alpha(a))\, d\beta(t).$$

As

$$\begin{aligned}\left(\frac{1}{C'_{a\to b}(0)}\int_a^b (\alpha(t)-\alpha(a))\,d\beta(t)\right)^2 &= \left|\int_a^b (\beta(b)-\beta(t))\,\frac{d\alpha(t)}{C'_{a\to b}(0)}\right| \\ &\leqslant \int_a^b (\beta(b)-\beta(t))^2\,\frac{d\alpha(t)}{C'_{a\to b}(0)} \\ &\leqslant \int_a^b (\alpha(b)-\alpha(t))\,(\gamma(b)-\gamma(t))\,\frac{d\alpha(t)}{C'_{a\to b}(0)} \\ &\leqslant \int_a^b (\gamma(b)-\gamma(t))\,d\alpha(t) \\ &= \int_a^b (\alpha(t)-\alpha(a))\,d\gamma(t) \\ &\leqslant \int_0^b \alpha(t)d\gamma(t)\end{aligned}$$

is bounded uniformly, we can get

$$0 \geqslant N_2(F_{a\to b}) \geqslant N_2(A_{a\to b}) - \int_0^b \alpha(t)d\gamma(t) \geqslant -3\int_0^b \alpha(t)d\gamma(t) > -\infty,$$
$$|N_1(F_{a\to b}) - N_1(A_{a\to b})| \leqslant \sqrt{\int_0^b \alpha(t)d\gamma(t)} < \infty.$$

Secondly, we can verify by direct calculation that $N_2(F_{a\to b})$ is uniformly bounded from below. From (4.17) we know

$$\begin{aligned}&F''_{a\to b}(0) - F'_{a\to b}(0)^2 \\ =&\underbrace{\frac{1}{\alpha(b)-\alpha(a)}\int_a^b(\beta(t)-\beta(a))^2 d\alpha(t)}_{\text{I}}\underbrace{-\frac{2}{\alpha(b)-\alpha(a)}\int_a^b(\alpha(b)-\alpha(t))(\alpha(t)-\alpha(a))d\gamma(t)}_{\text{II}} \\ &\underbrace{-\frac{2}{\alpha(b)-\alpha(a)}\int_a^b(\alpha(t)-\alpha(a))(\beta(t)-\beta(a))d\beta(t)}_{\text{III}}+\underbrace{\frac{4}{\alpha(b)-\alpha(a)}\int_a^b\int_a^t(\alpha(s)-\alpha(a))d\beta(s)d\beta(t)}_{\text{IV}} \\ &\underbrace{-(\beta(b)-\beta(a))^2}_{\text{V}}+\underbrace{4\frac{\beta(b)-\beta(a)}{\alpha(b)-\alpha(a)}\int_a^b(\alpha(t)-\alpha(a))d\beta(t)}_{\text{VI}} \\ &\underbrace{-\frac{4}{(\alpha(b)-\alpha(a))^2}\left(\int_a^b(\alpha(t)-\alpha(a))d\beta(t)\right)^2}_{\text{VII}}.\end{aligned}$$

Using integration by parts, we can get

$$
\begin{aligned}
\mathrm{I}+\mathrm{V} &= -\frac{2}{\alpha(b)-\alpha(a)}\int_a^b(\alpha(t)-\alpha(a))(\beta(t)-\beta(a))d\beta(t),\\
\mathrm{I}+\mathrm{III}+\mathrm{V} &= -\frac{4}{\alpha(b)-\alpha(a)}\int_a^b(\alpha(t)-\alpha(a))(\beta(t)-\beta(a))d\beta(t),\\
\mathrm{I}+\mathrm{III}+\mathrm{V}+\mathrm{VI} &= \frac{4}{\alpha(b)-\alpha(a)}\int_a^b(\alpha(t)-\alpha(a))(\beta(b)-\beta(t))d\beta(t),\\
\mathrm{IV} &= \frac{4}{\alpha(b)-\alpha(a)}\int_a^b(\alpha(t)-\alpha(a))(\beta(b)-\beta(t))d\beta(t),\\
\mathrm{I}+\mathrm{III}+\mathrm{IV}+\mathrm{V}+\mathrm{VI} &= \frac{8}{\alpha(b)-\alpha(a)}\int_a^b(\alpha(t)-\alpha(a))(\beta(b)-\beta(t))d\beta(t)\\
&= \frac{4}{\alpha(b)-\alpha(a)}\int_a^b(\beta(b)-\beta(t))^2d\alpha(t),\\
\mathrm{VII} &= -\frac{4}{(\alpha(b)-\alpha(a))^2}\left(\int_a^b(\beta(b)-\beta(t))d\alpha(t)\right)^2\\
&\geqslant -\frac{4}{\alpha(b)-\alpha(a)}\int_a^b(\beta(b)-\beta(t))^2d\alpha(t),
\end{aligned}
$$

$$
\mathrm{I}+\mathrm{III}+\mathrm{IV}+\mathrm{V}+\mathrm{VI}+\mathrm{VII} \geqslant 0.
$$

Hence

$$
\begin{aligned}
F''(0)-F'(0)^2 &\geqslant \mathrm{II}\\
&\geqslant -2\int_a^b(\alpha(t)-\alpha(a))d\gamma(t)\\
&\geqslant -2\int_0^b\alpha(t)d\gamma(t).
\end{aligned}
$$

To summarize, for fixed $b>0$, we get local uniform boundedness of

$$
A_{a\to b}e^{(\beta(b)-\beta(a))z}, \quad \frac{C_{a\to b}}{zC'_{a\to b}(0)}e^{(\beta(b)-\beta(a))z}
$$

for all small enough $a>0$. The rest is similar to the proof of Theorem 4.11. □

Remark. Although the theory of Laguerre classes might lead to possible generalization of Theorem 4.17 for Kreĭn's strings, we should point out that the approach via direct calculation we presented here becomes intractable for $N_4(F_{a\to b})$ if H is *not diagonal*. It's probably because we treat $A_{a\to b}$ and $C_{a\to b}$ separately, while the other approach uses the interaction between $A_{a\to b}$ and $C_{a\to b}$, namely Lemma 4.18. To extend Theorem 4.11 even further, an inequality for higher order derivatives which is analogous to Lemma 4.18 is desired.

4.5 Bezout operators and Schatten classes

In Section 2.4.1 we showed for a *strict normalized non-degenerate* dB-function E, the Bezout operator $T_{A,0}$ is a compact self-adjoint operator acting on $\mathcal{B}(E)$. Moreover, the nonzero spectrum (eigenvalues) of $T_{A,0}$ consists of $\frac{1}{t_n}$ where t_ns are the zeros of A.

The p-th Schatten class $\mathfrak{S}_p$ consists of bounded linear operators T from a (separable) Hilbert space $\mathcal{H}_1$ to another (separable) Hilbert space $\mathcal{H}_2$ s.t. the p-th Schatten norm

$$\|T\|_p := (\operatorname{tr}|T|^p)^{\frac{1}{p}} = \left(\sum_{n\geqslant 1} s_n^p(T)\right)^{\frac{1}{p}}$$

is finite, where $s_1(T) \geqslant s_2(T) \geqslant \cdots \geqslant s_n(T) \geqslant \cdots \geqslant 0$ are the singular values of the operator T. Some basics of the Schatten classes can be found in [Wei80] and [Tes00]. More discussion on the Schatten classes as ideals of operators can be found in [GK69], [Sim05], and [Pie80].

Remark. Some authors use $\mathcal{C}_p$, $\mathfrak{C}_p$, $\mathcal{J}_p$, or $\mathcal{B}_p$ to denote the p-th Schatten class. We adopt Gohberg and Kreĭn's notation $\mathfrak{S}_p$ in [GK69].

For $A \in \mathcal{P}_{\leqslant k}$, by definition $\sum_{n=1}^{N(A)} \frac{1}{|t_n|^{2k+2}} < \infty$, therefore the Bezout operator $T_{A,0}$ is of $(2k+2)$-th Schatten class $\mathfrak{S}_{2k+2}$ (in our case $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{B}(E)$). Conversely, if $T_{A,0} \in \mathfrak{S}_{2k+2}$, we can find a zero-free real entire function S s.t. $E = SE_0$ where $E_0 = A_0 - iC_0$ for some $A_0 \in \mathcal{L}_{2k+2}$.

Theorem 4.21. *Let E be a strict normalized non-degenerate dB-function.*

(i) If $E \in \mathcal{P}_{\leqslant k}$, then $T_{A,0} \in \mathfrak{S}_{2k+2}$ and

$$\|T_{A,0}\|_{2k+2}^{2k+2} = \sum_{n=1}^{N(A)} \frac{1}{t_n^{2k+2}} < \infty,$$

where $\{t_n\}_{n=1}^{N(E)}$ are the zeros of A. Moreover if $A \in \mathcal{L}_k$, then

$$\|T_{A,0}\|_{2k+2}^{2k+2} \leqslant -\frac{A^{(2k+2)}(0)}{(2k+1)!} < \infty.$$

(ii) If $T_{A,0} \in \mathfrak{S}_{2k+2}$, then $E = SE_0$ where S is a zero-free real entire function and $E_0 \in \mathcal{P}_{\leqslant k}$ is a strict normalized non-degenerate dB-function s.t. $A_0 \in \mathcal{L}_k$.

Proof. (i) From the discussion above we know

$$\|T_{A,0}\|_{2k+2}^{2k+2} = \sum_{n=1}^{N(A)} \frac{1}{t_n^{2k+2}} < \infty$$

as $A \in \mathcal{P}_{2k+2}$. If $A \in \mathcal{L}_k$, then it admits the following factorization:

$$A(z) = e^{-pz^{2k+2}} \prod_{n=1}^{N(A)} \left(1 - \frac{z}{t_n}\right) e^{\frac{z}{t_n} + \cdots + \frac{1}{2k+1}\left(\frac{z}{t_n}\right)^{2k+1}},$$

where $p \geqslant 0$, and it's easy to see

$$\frac{A^{(2k+2)}(0)}{(2k+1)!} = \lim_{z \to 0} \frac{A'(0)}{z^{2k+1}} = -(2k+2)p - \sum_{n=1}^{N(A)} \frac{1}{t_n^{2k+2}},$$

and this proves the second inequality.

(ii) Let $\{t_n\}$ be the zeros of A, then $\sum_{n=1}^{+\infty} \frac{1}{t_n^{2k+2}} < \infty$. Let

$$A_0(z) := \prod_{n=1}^{N(A)} \left(1 - \frac{z}{t_n}\right) e^{\frac{z}{t_n} + \cdots + \frac{1}{2k+1}\left(\frac{z}{t_n}\right)^{2k+1}},$$

then A_0 belongs to k-th Laguerre class. Let $S = \frac{A}{A_0}$, it's real entire and zero-free by the definition of A_0, and $S(0) = 1$ as $A(0) = A_0(0) = 1$. Let $E_0 = \frac{E}{S}$, then E_0 is a strict normalized non-degenerate dB-function, as E is strict and non-degenerate. We get $E_0 \in \mathcal{P}_{\leqslant k}$ by Corollary 4.7.

□

Remark. We should point out that one can replace the assumption that $E(0) = 1$ by $A(0) = 1$, and the results still hold with minor modifications of the proof.

Therefore, dB-chains in the generalized Pólya class $\mathcal{P}_{\leqslant k}$ correspond to Bezout operators $T_{A,0}$ in the $(2k+2)$-th Schatten class in the sense of Theorem 4.21. In other words, the meaning of our extension of de Branges' Theorem 41, namely Theorem 4.11, is to obtain the Bezout operators in the the 4-th Schatten class $\mathfrak{S}_4$, which is larger than the Hilbert-Schmidt class $\mathfrak{S}_2$ that de Branges considered. Since in general the Bezout operator $T_{A,0}$ is always compact, we expect to employ this approach even further to obtain Bezout operators of p-th Schatten class for $p = 2n$, $n \geqslant 3$.

4.A Proof of Lemma 4.2

Let $F(z) := (1-z)\exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^r}{r}\right) - 1$, then $F(0) = 0$, and

$$\begin{aligned} F'(z) &= (-1 + (1-z)(1 + z + \cdots + z^{r-1}))\exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^r}{r}\right) \\ &= -z^r \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^r}{r}\right). \end{aligned}$$

We'll prove the following inequality first:

$$x + \frac{x^2}{2} + \cdots + \frac{x^r}{r} \leqslant x^{r+1} + \log(1+r), \quad \forall x > 0. \tag{4.37}$$

Denote LHS by $f(x)$ and RHS by $g(x)$ respectively. For $x = 1$, the inequality is well known:

$$f(1) \leqslant 1 + \sum_{k=2}^{r} \int_{k-1}^{k} \frac{1}{k} ds < 1 + \sum_{k=2}^{r} \int_{k-1}^{k} \frac{1}{s} ds = 1 + \int_1^r \frac{1}{s} ds = 1 + \log(r) < 1 + \log(1+r) = g(1).$$

For $x > 1$,

$$f(x) - f(1) = \int_1^x (1 + s + \cdots + s^{r-1}) ds < \int_1^x r s^r ds < \int_1^x (r+1) s^r ds = g(x) - g(1). \tag{4.38}$$

Now we show (4.37) holds for $x < 1$. First we'll show $f\left(\frac{r}{r+1}\right) < \log(1+r)$. This is because

$$f\left(\frac{r}{r+1}\right) = \int_0^{\frac{r}{r+1}} \frac{1-s^r}{1-s} ds < \int_0^{\frac{r}{r+1}} \frac{1}{1-s} ds = \int_{\frac{1}{r+1}}^1 \frac{1}{t} dt = \log(1+r).$$

Hence for $0 < x \leqslant \frac{r}{r+1}$, $f(x) < f\left(\frac{r}{r+1}\right) < \log(1+r) < g(x)$.

For $\frac{r}{r+1} < x < 1$, we claim that if $f'(x) \leqslant g'(x)$, then $f(x) < g(x)$. Suppose $f'(x) \leqslant g'(x)$, then $g(x) - \log(1+r) = \frac{x}{r+1} g'(x) \geqslant \frac{x}{r+1} f'(x)$. On the other hand, as f' is positive and monotone increasing, $f(x) - f\left(\frac{r}{r+1}\right) = \int_{\frac{r}{r+1}}^x f'(s) ds < \left(x - \frac{r}{r+1}\right) f'(x)$. As $x < 1$, $x\frac{r}{r+1} < \frac{r}{r+1}$, i.e., $x - \frac{r}{r+1} < \frac{x}{r+1}$, then

$$f(x) < f\left(\frac{r}{r+1}\right) + \left(x - \frac{r}{r+1}\right) f'(x) < \log(1+r) + \frac{x}{r+1} f'(x) \leqslant g(x).$$

Suppose there exists $x_0 \in \frac{r}{r+1} < x < 1$ s.t. $f(x_0) > g(x_0)$, then $f'(x_0) > g'(x_0)$. Let x be the smallest number in $(x_0, 1)$ s.t. $f'(x) = g'(x)$, then $f(x) < g(x)$. Such an x exists because $f'(1) = r < r + 1 = g'(1)$. But this is impossible as $f(x_0) > g(x_0)$ and $f'(s) > g'(s)$, $\forall s \in (x_0, x)$. Hence such x_0 doesn't exist, therefore $f(x) \leqslant g(x)$ for any $x > 0$.

Now

$$\begin{aligned}\left|F(z)-F(0)\right| &\leqslant \int_0^{|z|}\left|F'\left(t\frac{z}{|z|}\right)\right|dt\\ &\leqslant \int_0^{|z|} t^r\exp\left(t+\cdots+\frac{t^r}{r}\right)dt\\ &\leqslant \int_0^{|z|}(1+r)t^r\exp(t^{r+1})dt\\ &=\exp(|z|^{r+1})-1.\end{aligned}$$

4.B Proof of Proposition 4.3

Assume $F\in\mathcal{L}_k$, then by the factorization (4.1) we know

$$F(z)=e^{p_{2k+2}z^{2k+2}}\prod_{n=1}^{N(F)}\left(1-\frac{z}{z_n}\right)e^{\frac{z}{z_n}+\cdots+\frac{1}{2k+1}\left(\frac{z}{z_n}\right)^{2k+1}},$$

where $p_{2k+2}\leqslant 0$ and $z_n\in\mathbb{R}$ are the zeros of F. Taking its derivative we get

$$F'(z)=(2k+2)p_{2k+2}z^{2k+1}F(z)-F(z)\sum_{n=1}^{N(F)}\frac{1}{\left(1-\frac{z}{z_n}\right)e^{\frac{z}{z_n}+\cdots+\frac{1}{2k+1}\left(\frac{z}{z_n}\right)^{2k+1}}}\frac{z^{2k+1}}{z_n^{2k+2}}, \tag{4.39}$$

which clearly shows $F'(z)$ has a zero of order at least $2k+1$ at the origin.

For the last statement $\Re\tilde{F}(z)\geqslant 0$ for $z\in\mathbb{C}_+$ where

$$\tilde{F}(z):=\frac{iF'(z)}{z^{2k}F(z)}.$$

As $(F_1F_2)\tilde{}=\tilde{F}_1+\tilde{F}_2$, it suffices to show the condition holds for each factor of F. For $G(z):=e^{p_{2k+2}z^{2k+2}}$, clearly $\Re\tilde{G}(z)=\Re\frac{iG'(z)}{z^{2k}G(z)}=-(2k+1)p_{2k+2}\cdot y\geqslant 0$ for $z=x+iy\in\mathbb{C}_+$ as $p_{2k+2}\leqslant 0$. For $F_n(z):=\left(1-\frac{z}{z_n}\right)e^{\frac{z}{z_n}+\cdots+\frac{1}{2k+1}\left(\frac{z}{z_n}\right)^{2k+1}}$, one can show

$$\Re\tilde{F}_n(z)=\Re\frac{iF_n'(z)}{z^{2k}F_n(z)}=\Re\frac{\frac{-iz}{z_n^{2k+2}}}{1-\frac{z}{z_n}}=\frac{1}{z_n^{2k+2}}\Im\frac{z}{1-\frac{z}{z_n}}=\frac{1}{z_n^{2k+2}}\frac{\Im z}{\left|1-\frac{z}{z_n}\right|^2}>0,\quad\forall z\in\mathbb{C}_+.$$

On the other hand, assume F is real entire s.t. $F(0)=1$, $F(z)\neq 0$ for $z\in\mathbb{C}\backslash\mathbb{R}$, and $F'(z)$ has a zero of order at least $2k+1$ at the origin and $\Re\tilde{F}(z)=\Re\frac{iF'(z)}{z^{2k}F(z)}\geqslant 0$ for $z\in\mathbb{C}_+$. The proof is divided into two parts. Firstly, we show if F is zero-free, then it is of the form $e^{p_{2k+2}z^{2k+2}}$ for some $p_{2k+2}\leqslant 0$. Secondly, if F has a real zero z_1 with multiplicity M_1, let F_1 be the canonical factor of

$\frac{z}{z_n}$ of degree $2k+1$, namely

$$F_1(z) = \left(1 - \frac{z}{z_1}\right) e^{\frac{z}{z_1} + \cdots + \frac{1}{2k+1}\left(\frac{z}{z_1}\right)^{2k+1}}.$$

We show $G_1 := \frac{F}{F_1^{M_1}}$ satisfies the same conditions which F satisfies, namely, G_1 is real entire s.t. $G_1(0) = 1$, $G_1(z) \neq 0$ for $z \in \mathbb{C}\backslash\mathbb{R}$, and $G_1'(z)$ has a zero of order at least $2k+1$ at the origin and $\Re\tilde{G}_1(z) = \Re\frac{iG_1'(z)}{z^{2k}G_1(z)} \geqslant 0$ for $z \in \mathbb{C}_+$.

The proof of the first part is straightforward. By assumption $\tilde{F}(z)$ is a zero-free entire function. Since $\Re\tilde{F}(z) \geqslant 0$ for $z \in \mathbb{C}_+$, by the Poisson representation formula we know

$$\Re\tilde{F}(z) = py + \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{d\mu(x)}{(t-x)^2 + y^2}, \quad \forall z = x + iy \in \mathbb{C}_+, \tag{4.40}$$

where $p \geqslant 0$ and μ is a positive measure on $\mathbb{R}$. Since $\Re\tilde{F}(z) = 0$ for $z \in \mathbb{R}$, then $d\mu = 0$ and $\Re\tilde{F}(z) = py$, $\forall z = x + iy \in \mathbb{C}$. As $\tilde{F}(0) = 0$, we can conclude $\tilde{F}(z) = -ipz$, $\forall z \in \mathbb{C}$ and therefore $F(z) = e^{-pz^{2k+2}}$.

Now assume F has a real zero z_1, and let F_1, G_1 be defined as above. It suffices to check $\Re\tilde{G}_1 \geqslant 0$ for $z \in \mathbb{C}_+$ as other assumptions are automatically satisfied. Since $\Re\left(i\frac{1}{z^{2\rho}}\frac{F'}{F}\right) \geqslant 0$, by the Poisson representation formula again there exists $p \geqslant 0$ and a positive measure μ on $\mathbb{R}$ s.t.

$$\Re\tilde{F}(z) = py + \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{d\mu(x)}{(t-x)^2 + y^2}, \quad \forall z = x + iy \in \mathbb{C}_+.$$

Since $\Re\tilde{F} = 0$ on the real line except at zeros of F and $\mu(b) - \mu(a) = \lim_{y\to 0+}\int_a^b \Re\tilde{F}(x+iy)dx$ at points of continuity a, b of μ, then μ is a discrete measure and has jumps only at z_n. Now let's calculate jump of μ at z_1. Note that

$$\Re\tilde{F} = \Re\tilde{G_1} + M_1\Re\tilde{F}_1.$$

As for $\Re\tilde{F}_1$, by (4.40) we know

$$\Re\tilde{F}_1(z) = \frac{1}{z_1^{2k}}\frac{\Im z}{|z - z_1|^2}$$

$$\mu(z_1 + \epsilon) - \mu(z_1 - \epsilon) = M_1\frac{1}{z_1^{2k}}\lim_{y\to 0+}\int_{z_1-\epsilon}^{z_1+\epsilon} \frac{y}{(x - z_1)^2 + y^2}dx = \pi M_1\frac{1}{z_1^{2k}}$$

Hence

$$\Re\tilde{F}(z) = py + \sum_{n\geqslant 1} M_n \frac{1}{z_n^{2k}} \frac{y}{|z-z_k|^2},$$
$$\Re\tilde{G}_1(z) = \Re\tilde{F}(z) - M_1\Re\tilde{F}_1(z) = py + \sum_{n>1} M_n \frac{1}{z_n^{2k}} \frac{y}{|z-z_k|^2} \geqslant 0, \quad \forall z \in \mathbb{C}_+.$$

Now it suffices to show $\prod_{n=1}^{N} F_n^{M_n}$ converges to an entire function as $N \to \infty$. This is because

$$0 \leqslant \sum_{n=1}^{N} M_n \frac{1}{z_n^{2k+2}} \leqslant -p + \lim_{y\to 0+} \frac{\Re\tilde{F}(iy)}{y} = -p - \Re \lim_{z\to i0+} \frac{F'(z)}{z^{2k+1}} < +\infty,$$

where the upper bound is independent of N, then the limit function F_∞ exists by Lemma 4.2. Let $G := \frac{F}{F_\infty}$, then $G = e^{-pz^{2k+2}}$ for some $p \geqslant 0$, and $F \in \mathcal{L}_k$ since it admits the canonical factorization (4.39).

4.C Proof of Lemma 4.16

Note that the first column of the canonical equation (1.21) can be re-written as

$$\begin{aligned} A_{a\to b}(z) - 1 &= -z\int_a^b A_{a\to t}(z)d\beta(t) - z\int_a^b C_{a\to t}(z)d\gamma(t), \\ C_{a\to b}(z) &= z\int_a^b A_{a\to t}(z)d\alpha(t) + z\int_a^b C_{a\to t}(z)d\beta(t). \end{aligned} \tag{4.41}$$

Taking the first two derivatives w.r.t. z in (4.41) and evaluating them at $z = 0$, we can get

$$\begin{aligned} A'_{a\to b}(0) &= -(\beta(b) - \beta(a)), \\ C'_{a\to b}(0) &= \alpha(b) - \alpha(a), \\ N_1(A_{a\to b}) &= A'_{a\to b}(0) = -(\beta(b) - \beta(a)), \end{aligned}$$

$$\begin{aligned}
A''_{a\to b}(0) &= -2\int_a^b A'_{a\to t}(0)d\beta(t) - 2\int_a^b C'_{a\to t}(0)d\gamma(t)\\
&= (\beta(b)-\beta(a))^2 - 2\int_a^b (\alpha(t)-\alpha(a))d\gamma(t),\\
C''_{a\to b}(0) &= 2\int_a^b A'_{a\to t}(0)d\alpha(t) + 2\int_a^b C'_{a\to t}(0)d\beta(t)\\
&= -2\int_a^b (\beta(t)-\beta(a))d\alpha(t) + 2\int_a^b (\alpha(t)-\alpha(a))d\beta(t)\\
&= -2(\alpha(b)-\alpha(a))(\beta(b)-\beta(a)) + 4\int_a^b (\alpha(t)-\alpha(a))d\beta(t),\\
N_2(A_{a\to b}) &= A''_{a\to b}(0) - A'_{a\to b}(0)^2 = -2\int_a^b (\alpha(t)-\alpha(a))d\gamma(t).
\end{aligned}$$

Taking the third derivative w.r.t. z in (4.41) and evaluating them at $z=0$, we can get

$$\begin{aligned}
A^{(3)}_{a\to b}(0) =& -3\int_a^b A''_{a\to t}(0)d\beta(t) - 3\int_a^b C''_{a\to t}(0)d\gamma(t)\\
=& -3\int_a^b (\beta(t)-\beta(a))^2 d\beta(t) + 6\int_a^b\int_a^t (\alpha(s)-\alpha(a))d\gamma(s)d\beta(t)\\
&+ 6\int_a^b (\alpha(t)-\alpha(a))(\beta(t)-\beta(a))d\gamma(t) - 12\int_a^b\int_a^t (\alpha(s)-\alpha(a))d\beta(s)d\gamma(t)\\
=& -(\beta(b)-\beta(a))^3 + 6\int_a^b (\alpha(s)-\alpha(a))(\beta(b)-\beta(s))d\gamma(s)\\
&+ 6\int_a^b (\alpha(t)-\alpha(a))(\beta(t)-\beta(a))d\gamma(t) - 12\int_a^b\int_a^t (\alpha(s)-\alpha(a))d\beta(s)d\gamma(t)\\
=& -(\beta(b)-\beta(a))^3 + 6(\beta(b)-\beta(a))\int_a^b (\alpha(s)-\alpha(a))d\gamma(s)\\
&- 12\int_a^b\int_a^t (\alpha(s)-\alpha(a))d\beta(s)d\gamma(t),\\
C^{(3)}_{a\to b}(0) =& 3\int_a^b A''_{a\to t}(0)d\alpha(t) + 3\int_a^b C''_{a\to t}(0)d\beta(t)\\
=& 3\int_a^b (\beta(t)-\beta(a))^2 d\alpha(t) - 6\int_a^b\int_a^t (\alpha(s)-\alpha(a))d\gamma(s)d\alpha(t)\\
&- 6\int_a^b (\alpha(t)-\alpha(a))(\beta(t)-\beta(a))d\beta(t) + 12\int_a^b\int_a^t (\alpha(s)-\alpha(a))d\beta(s)d\beta(t)\\
=& 3(\beta(b)-\beta(a))^2(\alpha(b)-\alpha(a)) - 6\int_a^b\int_a^t (\alpha(s)-\alpha(a))d\gamma(s)d\alpha(t)\\
&- 12\int_a^b (\alpha(t)-\alpha(a))(\beta(t)-\beta(a))d\beta(t) + 12\int_a^b (\alpha(s)-\alpha(a))(\beta(b)-\beta(s))d\beta(s).
\end{aligned}$$

We can get a simple expression for $N_3(A_{a\to b})$:

$$\begin{aligned}
N_3(A_{a\to b}) =& A^{(3)}_{a\to b}(0) - 3A''_{a\to b}(0)A'_{a\to b}(0) + 2A'_{a\to b}(0)^3 \\
=& A^{(3)}_{a\to b}(0) - A'_{a\to b}(0)^3 - 3A'_{a\to b}(0)(A''_{a\to b}(0) - A'_{a\to b}(0)) \\
=& 6(\beta(b)-\beta(a))\int_a^b (\alpha(s)-\alpha(a))d\gamma(s) - 12\int_a^b\int_a^t (\alpha(s)-\alpha(a))d\beta(s)d\gamma(t) \\
& - 6(\beta(b)-\beta(a))\int_a^b (\alpha(t)-\alpha(a))d\gamma(t) \\
=& -12\int_a^b\int_a^t (\alpha(s)-\alpha(a))d\beta(s)d\gamma(t) \\
=& -12\int_a^b\int_a^t (\beta(t)-\beta(s))d\alpha(s)d\gamma(t).
\end{aligned}$$

Now let's look at $A^{(4)}_{a\to b}(0)$:

$$\begin{aligned}
\int_a^b A^{(3)}_{a\to t}(0)d\beta(t) =& -\int_a^b (\beta(t)-\beta(a))^3 d\beta(t) + 6\int_a^b\int_a^t (\beta(t)-\beta(a))(\alpha(s)-\alpha(a))d\gamma(s)d\beta(t) \\
& - 12\int_a^b\int_a^t\int_a^s (\alpha(u)-\alpha(a))d\beta(u)d\gamma(s)d\beta(t) \\
=& -\frac{1}{4}(\beta(b)-\beta(a))^4 + 6\int_a^b\int_a^t (\alpha(s)-\alpha(a))(\beta(t)-\beta(a))d\gamma(s)d\beta(t) \\
& - 12\int_a^b\int_a^t (\alpha(s)-\alpha(a))(\beta(b)-\beta(t))d\beta(s)d\gamma(t), \\
\int_a^b C^{(3)}_{a\to b}(0)d\gamma(t) =& 3\int_a^b (\beta(t)-\beta(a))^2(\alpha(t)-\alpha(a))d\gamma(t) \\
& - 12\int_a^b\int_a^t (\alpha(s)-\alpha(a))(\beta(s)-\beta(a))d\beta(s)d\gamma(t) \\
& - 6\int_a^b\int_a^t (\alpha(t)-\alpha(s))(\alpha(s)-\alpha(a))d\gamma(s)d\gamma(t) \\
& + 12\int_a^b\int_a^t (\alpha(s)-\alpha(a))(\beta(t)-\beta(s))d\beta(s)d\gamma(t) \\
=& 3\int_a^b (\beta(t)-\beta(a))^2(\alpha(t)-\alpha(a))d\gamma(t) \\
& - 12\int_a^b\int_a^t (\alpha(s)-\alpha(a))(\beta(s)-\beta(a))d\beta(s)d\gamma(t) \\
& + 6\int_a^b\int_a^t (\alpha(s)-\alpha(a))^2 d\gamma(s)d\gamma(t) - 3\left(\int_a^b (\alpha(t)-\alpha(a))d\gamma(t)\right)^2 \\
& + 12\int_a^b\int_a^t (\alpha(s)-\alpha(a))(\beta(t)-\beta(s))d\beta(s)d\gamma(t).
\end{aligned}$$

Now we can get a simple formula for $N_4(A_{a\to b})$,

$$\begin{aligned}
A^{(4)}_{a\to b}(0) - 3(\underbrace{A''_{a\to b}(0) - A'_{a\to b}(0)^2}_{N_2(A_{a\to b})}) - A'_{a\to b}(0)^4 = & \underbrace{-24\int_a^b\int_a^t(\alpha(s)-\alpha(a))(\beta(t)-\beta(a))d\gamma(s)d\beta(t)}_{\mathrm{I}} \\
& + 48\int_a^b\int_a^t(\alpha(s)-\alpha(a))(\beta(b)-\beta(t))d\beta(s)d\gamma(t) \\
& - 12\int_a^b(\beta(t)-\beta(a))^2(\alpha(t)-\alpha(a))d\gamma(t) \\
& + 48\int_a^b\int_a^t(\alpha(s)-\alpha(a))(\beta(s)-\beta(a))d\beta(s)d\gamma(t) \\
& - 24\int_a^b\int_a^t(\alpha(s)-\alpha(a))^2 d\gamma(s)d\gamma(t) \\
& - 48\int_a^b\int_a^t(\alpha(s)-\alpha(a))(\beta(t)-\beta(s))d\beta(s)d\gamma(t).
\end{aligned}$$

Notice that

$$\begin{aligned}
\mathrm{I} = & -24\int_a^b\int_b^s(\alpha(s)-\alpha(a))(\beta(t)-\beta(a))d\beta(t)d\gamma(s) \\
= & -12\int_a^b(\alpha(s)-\alpha(a))\left((\beta(b)-\beta(a))^2-(\beta(s)-\beta(a))^2\right)d\gamma(s) \\
= & -12(\beta(b)-\beta(a))^2\int_a^b(\alpha(s)-\alpha(a))d\gamma(s) + 12\int_a^b(\alpha(s)-\alpha(a))(\beta(s)-\beta(a))^2d\gamma(s) \\
= & 6A'_{a\to b}(0)^2(A''_{a\to b}(0) - A_{a\to b}(0)^2) + 12\int_a^b(\alpha(s)-\alpha(a))(\beta(s)-\beta(a))^2d\gamma(s),
\end{aligned}$$

hence

$$
\begin{aligned}
&A^{(4)}_{a\to b}(0) - 3(\underbrace{A''_{a\to b}(0) - A'_{a\to b}(0)^2}_{N_2(A_{a\to b})})^2 - A'_{a\to b}(0)^4 - 6A'_{a\to b}(0)^2(A''_{a\to b}(0) - A_{a\to b}(0)^2) \\
=& -24\int_a^b\int_a^t (\alpha(s)-\alpha(a))^2 d\gamma(s)d\gamma(t) \\
&+48\int_a^b\int_a^t (\alpha(s)-\alpha(a))(\beta(b)-\beta(t))d\beta(s)d\gamma(t) \\
&+48\int_a^b\int_a^t (\alpha(s)-\alpha(a))(\beta(s)-\beta(a))d\beta(s)d\gamma(t) \\
&-48\int_a^b\int_a^t (\alpha(s)-\alpha(a))(\beta(t)-\beta(s))d\beta(s)d\gamma(t) \\
=& -24\int_a^b\int_a^t (\alpha(s)-\alpha(a))^2 d\gamma(s)d\gamma(t) \\
&+48\int_a^b\int_a^t (\alpha(s)-\alpha(a))(\beta(b)-\beta(a)-2(\beta(t)-\beta(s)))d\beta(s)d\gamma(t) \\
=& -24\int_a^b\int_a^t (\alpha(s)-\alpha(a))^2 d\gamma(s)d\gamma(t) \\
&\underbrace{+48(\beta(b)-\beta(a))\int_a^b\int_a^t (\alpha(s)-\alpha(a))d\beta(s)d\gamma(t)}_{\mathrm{II}} \\
&-96\int_a^b\int_a^t (\alpha(s)-\alpha(a))(\beta(t)-\beta(s))d\beta(s)d\gamma(t).
\end{aligned}
$$

It's easy to see $\mathrm{II} = 4A'_{a\to b}(0)N_3(A_{a\to b})$, then

$$
\begin{aligned}
N_4(A_{a\to b}) =& A^{(4)}_{a\to b}(0) - 3(A''_{a\to b}(0) - A'_{a\to b}(0)^2)^2 - A'_{a\to b}(0)^4 - 6A'_{a\to b}(0)^2(A''_{a\to b}(0) - A_{a\to b}(0)^2) \\
&-4A'_{a\to b}(0)(A^{(3)}_{a\to b}(0) - 3A''_{a\to b}(0)A'_{a\to b}(0) + 2A'_{a\to b}(0)^3) \\
=& -24\int_a^b\int_a^t (\alpha(s)-\alpha(a))^2 d\gamma(s)d\gamma(t) \\
&-96\int_a^b\int_a^t (\alpha(s)-\alpha(a))(\beta(t)-\beta(s))d\beta(s)d\gamma(t).
\end{aligned}
$$

Note that

$$
\begin{aligned}
\int_a^t (\alpha(s)-\alpha(a))(\beta(t)-\beta(s))d\beta(s) &= -\int_a^t (\alpha(s)-\alpha(a))(\beta(t)-\beta(s))d(\beta(t)-\beta(s)) \\
&= -\int_a^t (\alpha(s)-\alpha(a))d\frac{(\beta(t)-\beta(s))^2}{2} \\
&= \int_a^t \frac{(\beta(t)-\beta(s))^2}{2}d\alpha(s),
\end{aligned}
$$

therefore we can get the final formula for $N_4(A_{a\to b})$:

$$N_4(A_{a\to b}) = -24\left(\int_a^b\int_a^t(\alpha(s)-\alpha(a))^2 d\gamma(s)d\gamma(t) + 2\int_a^b\int_a^t(\beta(t)-\beta(s))^2 d\alpha(s)d\gamma(t)\right).$$

4.D Proof of Lemma 4.18

Since E is dB and $\Im\frac{C}{A}\geqslant 0$ on $\mathbb{C}_+$, by the Poisson representation formula, we get

$$\Im\frac{C}{A}(x+iy) = py + \frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{d\mu(t)}{(t-x)^2+y^2}$$

for $y>0$, where $p\geqslant 0$, μ is non-decreasing and $\int_{-\infty}^{+\infty}\frac{d\mu(t)}{1+t^2}<\infty$. By the Stieltjes inversion formula, if a and b are points of continuity of μ, $a<b$, then

$$\mu(b)-\mu(a) = \lim_{y\to 0+}\int_a^b \Im\frac{C}{A}(x+iy)dx.$$

Since $\frac{C}{A}$ is well-defined in a neighborhood of the origin, say $(-\epsilon,\epsilon)$, and $\Im\frac{C}{A}(x)=0$ for $x\in(-\epsilon,\epsilon)$, μ is not supported on $(-\epsilon,\epsilon)$. Consequently, $\int_{-\infty}^{+\infty}\frac{d\mu(t)}{t^2}<\infty$. Now by dominated convergence theorem,

$$\begin{aligned}
\Re\frac{C(iy)}{iyA(iy)} &= p + \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{d\mu(t)}{t^2+y^2},\\
\frac{\partial}{\partial y}\Re\frac{C(iy)}{iyA(iy)} &= -\frac{2}{\pi}\int_{-\infty}^{+\infty}\frac{yd\mu(t)}{(t^2+y^2)^2},\\
\frac{\partial^2}{\partial y^2}\Re\frac{C(iy)}{iyA(iy)} &= -\frac{2}{\pi}\int_{-\infty}^{+\infty}\frac{t^2-3y^2}{(t^2+y^2)^3}d\mu(t),\\
\frac{\partial^2}{\partial y^2}\Re\frac{C(iy)}{iyA(iy)}\bigg|_{y=0} &= -\frac{2}{\pi}\int_{-\infty}^{+\infty}\frac{1}{t^4}d\mu(t)\leqslant 0.
\end{aligned}$$

On the other hand, let $\frac{C(z)}{zA(z)} = u+iv$ where u and v are real harmonic functions, then

$$\left(\frac{C(z)}{zA(z)}\right)''\bigg|_{z=0} = u_{xx}(0)+iv_{xx}(0) = u_{xx}(0) = -u_{yy}(0)\geqslant 0.$$

Let $F(z) := \frac{C(z)}{z}$, then

$$\begin{aligned}
\left(\frac{F}{A}\right)' &= \frac{F'A-FA'}{A^2},\\
\left(\frac{F}{A}\right)'' &= \frac{(F''A-FA'')A-2(F'A-FA')A'}{A^3},\\
\left(\frac{F}{A}\right)''(0) &= F''(0)-F(0)A''(0)-2(F'(0)-F(0)A'(0)A'(0))\geqslant 0.
\end{aligned}$$

List of Symbols

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