CONSERVED QUANTITIES AND THE FORMATION OF BLACK HOLES IN THE BRANS-DICKE THEORY OF GRAVITATION

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John Joseph Dykla

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ABSTRACT

In Part I, we construct a symmetric stress-energy-momentum pseudo-tensor for the gravitational fields of Brans-Dicke theory, and use this to establish rigorously conserved integral expressions for energy-momentum P^i and angular momentum J^{ik}. Application of the two-dimensional surface integrals to the exact static spherical vacuum solution of Brans leads to an identification of our conserved mass with the active gravitational mass. Application to the distant fields of an arbitrary stationary source reveals that P^i and J^{ik} have the same physical interpretation as in general relativity. For gravitational waves whose wavelength is small on the scale of the background radius of curvature, averaging over several wavelengths in the Brill-Hartle-Isaacson manner produces a stress-energy-momentum tensor for gravitational radiation which may be used to calculate the changes in P^i and J^{ik} of their source.

In Part II, we develop strong evidence in favor of a conjecture by Penrose--that, in the Brans-Dicke theory, relativistic gravitational collapse in three dimensions produce black holes identical to those of general relativity. After pointing out that any black hole solution of general relativity also satisfies Brans-Dicke theory, we establish the Schwarzschild and Kerr geometries as the only possible spherical and axially symmetric black hole exteriors, respectively. Also, we show that a Schwarzschild geometry is necessarily formed in the collapse of an uncharged sphere.

Appendices discuss relationships among relativistic gravity theories and an example of a theory in which black holes do not exist.

CONTENTS

INTRODUCTION

WHAT IS THE BRANS-DICKE THEORY OF GRAVITY AND WHAT ARE THE PROBLEMS DEALT WITH IN THIS THESIS?

With the acceptance of the special theory of relativity early in the twentieth century, physicists had to face the concept, apparently necessary if the observed phenomena of electromagnetism are to be intelligible, that no signal may be propagated at a speed greater than that of light in vacuum. Clearly, the Newtonian view of gravitation as an instantaneous action at a distance was no longer acceptable. Albert Einstein (1916) laid the foundations of "the general theory of relativity," which is still regarded by most theoreticians as the simplest theory of gravity consistent both with special relativity (and indeed as the most "natural" extension of it) and with the observed phenomena of gravitation. His theory soon had the support of three "crucial experiments":

(i) the advance of the perihelion of the planet Mercury by 43" of arc per century (Leverrier, 1859; Newcomb, 1897),

(ii) the deflection of light rays from distant stars in the gravitational field of the sun (Dyson, Eddington, and Davidson, 1920),

(iii) the shift towards the red end of the spectrum of spectral lines in the light coming to us from the surfaces of compact stars (Adams, 1925).

Other observations lent support both to some of the fundamental assumptions (such as the principle of equivalence (von Eötvös, Pekár, and Fekete, 1922)) as well as to some further predictions (such as the expansion of the universe (Hubble, 1929)) of general relativity theory (GRT).

But even as the evidence for and acceptance of GRT grew, some theoreticians considered modifications of the theory. As an outgrowth of attempts to create a unified field theory of gravity and electromagnetism, Pascual Jordan (1955, 1959) proposed a theory of gravitation involving a scalar field in addition to the (second-rank tensor) metric of GRT. This theory was open to the objection that, in general, it required violation of the law of conservation of energy-momentum. Motivated by the desire to bring relativistic gravity theory into closer accord with Mach's principle, Brans and Dicke (1961) proposed another version of scalar-tensor gravity theory, which will hereafter be referred to as BDT. It is easy to show (see Appendix B) that BDT is equivalent to the only special case of Jordan's theory which does not violate conservation laws.

Our own approach to the study of scalar-tensor gravity theory is different from the motivations of either Jordan or Brans and Dicke. As shown in Appendix A, BDT is, in a very natural sense, the generalization of GRT involving the least additional complication. Thus, if experiments should force us to a theory more general than GRT, BDT is the candidate which should receive the most serious consideration. Furthermore, a large amount of theoretical effort has been expended in calculating the astrophysical implications of GRT. It would be very desirable to know which of these predictions remain unchanged, or at least are not modified qualitatively, in BDT.

The first section of this thesis addresses itself to the problem of defining rigorously conserved quantities, for isolated systems, which can be reasonably interpreted as energy-momentum and angular momentum in BDT. Although the distribution of stress-energy-momentum in the gravitational fields cannot be described in a generally covariant manner, we show how to define a physically meaningful stress-energy-momentum tensor for gravitational waves in the high-frequency limit. Like the corresponding work in GRT (Landau and Lifshitz, 1962; Isaacson, 1968), the definition of conserved, physically interpretable quantities should prove most useful in the further understanding and applications of BDT.

The second section of this thesis addresses itself to the problem of gravitational collapse and the formation of black holes in BDT. Ever since the pioneering investigations of the collapse of a sphere of dust in GRT (Tolman, 1934; Oppenheimer and Snyder, 1939), the conjecture that compression of a mass M into a region with circumference $C \lesssim 4\pi GM/c^2$ in all three directions necessarily produces a black hole has received support from all further investigations of gravitational collapse. Now the scalar field of BDT, which has its source in the trace of the stress-energy-momentum tensor of matter, plays the role of reciprocal "gravitational constant." Thus, compression of matter weakens the gravitational attraction, so one might naturally ask whether this effect may not be enough to prevent the formation of black holes. We present evidence in support of Penrose's (1970) conjecture that this is <u>not</u> the case; that in fact black holes <u>identical</u> to those of GRT are formed in BDT. That is, after all the waves have flowed off to infinity and the "dust has settled", the hole's exterior will be the "charged-Kerr" solution to the Einstein-Maxwell field equations, or one of the special cases of this solution:

Schwarzschild, Reissner-Nordström, or uncharged-Kerr.

The black hole thus presents itself to us as a truly remarkable phenomenon: a theoretical object with identically the same properties in two different theories. I cannot resist the temptation to speculate that the "charged-Kerr" black hole exterior solution may be common to a large class of relativistic theories of gravitation. Finding simple criteria by which to judge whether or not a given theory is a member of this class would be a most interesting problem for further research. The fact that not all relativistic gravity theories imply black holes is established by a counterexample discussed in Appendix C.

PART I

CONSERVED INTEGRALS AND THE STRESS-ENERGY-MOMENTUM OF GRAVITATIONAL WAVES

Chapter 1

INTRODUCTION AND SUMMARY

The original conformal form of Brans-Dicke theory (BDT) (Brans and Dicke, 1961) may be derived from a variational principle in which the Lagrangian density for matter is independent of the scalar field ϕ . Therefore, the "equation of motion" for matter in a given gravitational field is the vanishing of the covariant divergence of the stress-energy-momentum tensor T^{ik} , exactly as in general relativity theory (GRT). As is well known, the components of this vanishing divergence are not the differential expression of global conservation laws for an isolated system, because terms involving Christoffel symbols prevent the application of Gauss' theorem to integrals over a volume of spacetime. This mathematical statement supports our intuitive notion that energy-momentum (and angular momentum) must be associated with the gravitational fields as well as with matter.

To determine conserved integrals for isolated systems, we construct a symmetric pseudo-tensor τ^{ik} for the gravitational fields such that the ordinary divergence of $\phi(-g)(T^{ik}+ \tau^{ik})$ vanishes. This allows us to define for an isolated system an energy-momentum vector

$$P^i = (c^3/16\pi\phi_o) \int [\phi^2(-g)(g^{ik}g^{\ell m}- g^{i\ell}g^{km})]_{,\ell m}\, dS_k \quad ,$$

and an angular momentum tensor

$$J^{ik} = (c^3/16\pi\phi_o)\int_k \{x^i[\phi^2(-g)(g^{k\ell}g^{mp} - g^{km}g^{\ell p})]_{,mp}$$

$$- x^k[\phi^2(-g)(g^{i\ell}g^{mp} - g^{im}g^{\ell p})]_{,mp}\}\, dS_\ell \quad ,$$

both of which are rigorously conserved in the full nonlinear Brans-Dicke theory. We also construct two-dimensional surface integrals for P^i and J^{ik}.

Application to the static, spherical vacuum solution of Brans (external field of a star) yields an identification of the "conserved mass" P^o/c with the active gravitational mass m which enters into Kepler's laws for the motions of planets in the weak gravitational fields far from a star. The Brans solution is also used to point up, even more forcefully than in GRT, the absurd concepts which can arise if one attempts to localize the mass in the gravitational fields.

Application of this formalism to weakly gravitating sources yields P^i and J^{ik} which are identical to the energy-momentum and angular momentum of special relativity: $P^i = \int T^{io}d^3x$, $J^{ik} = \int (x^iT^{ko} - x^kT^{io})d^3x$.

Application to the gravitational fields of a stationary source of arbitrary strength reveals that P^i and J^{ik} have the same physical interpretation as in general relativity: $(-P^iP_i)^{1/2}/c \equiv m$ is the active gravitational mass, which governs the Keplerian motion of distant planets. The four-vector $U^i \equiv P^i/m$ (a special relativistic vector in the flat spacetime far from the source) is the four-velocity of the source's distant, weak spherical field in the asymptotically flat spacetime. The four-vector $S^i = *(J^{ik} - L^{ik})U_k/c$

is the intrinsic angular momentum four-vector (here $*$ is the "dual operator"). It is orthogonal to the source's four-velocity, and it governs the precession of gyroscopes ("dragging of inertial frames") far from the source in the same way as in GRT, except that its effects are weaker[1] by the well-known (O'Connell, 1968) factor $(2\omega+3)/(2\omega+4)$. As in GRT, the tensor $L^{ik} = J^{ik} - \varepsilon^{ik\ell m}S_\ell U_m$ is the orbital angular momentum, which can be derived from the energy-momentum as $L^{ik} = X^iP^k - X^kP^i$, where $X^i = J^{ik}P_k/P^\ell P_\ell$ is the vector offset of the source's distant spherical field from being accurately centered on the Minkowskian coordinates at "infinity", i.e., X^i is the "location of the center of mass."

Whenever matter or (nongravitational) fields flow out of the source region to infinity, they decrease P^i and J^{ik} by precisely the (special relativistic) energy-momentum and angular momentum which the matter or fields carry. The decrease in P^i and J^{ik} produced by gravitational waves can be calculated, in the manner familiar in GRT, using the pseudotensor τ^{ik}.

For gravitational waves whose wavelength is small compared to the radius of curvature of the background spacetime, the pseudotensor τ^{ik} can be averaged over several wavelengths in the Brill-Hartle (1964)-Isaacson (1968) manner, to yield $T^{ik}_{(GW)} = (\varepsilon^2\phi_o c^4/32\pi)\langle f\, h^{\ell m|i}h_{\ell m}{}^{|k} + 2\omega/f(2\xi^{,i}\xi^{,k} - \gamma^{ik}\xi_{,\ell}\xi^{,\ell}) - 2\xi_{,\ell}(h^{\ell i|k} + h^{\ell k|i})\rangle$. Here γ_{ik} and f,

[1]Thus, the Lense-Thirring effect is in principle an experimental means of distinguishing between BDT and GRT.

both of order unity, are the fully relativistic but slowly changing background metric and scalar fields, h_{ik} and ξ are the gravitational waves in the corresponding fields, and ε = (wavelength) / (background curvature)$^{1/2} \ll 1$ is a small parameter embodying the high-frequency assumption. (Note that the background metric γ_{ik} is used to raise and lower indices and form covariant derivatives.) The object $T^{ik}_{(GW)}$, which is gauge invariant to the order $O(\varepsilon^2)$, is a stress-energy-momentum tensor (not merely pseudo-tensor) for gravitational waves. It has vanishing divergence with respect to the background spacetime, and can be used instead of τ^{ik} to calculate the change in P^i and J^{ik} of a source due to energy-momentum and angular momentum carried away by gravitational waves.

This paper is written rather concisely. Readers who find it too concise will find it helpful to review the treatments by Landau and Lifshitz (1962) and by Misner, Thorne, and Wheeler (1971) of conservation laws and pseudo-tensors in general relativity theory. The analysis given here is patterned after those treatments.

Chapter 2

CONSTRUCTION OF CONSERVED INTEGRALS

The field equations of the original formulation of BDT (Brans and Dicke, 1961) may be written in the form[2]

$$R^{ik} - \frac{1}{2} g^{ik} R = (8\pi\phi^{-1}/c^4) T^{ik} + (\omega/\phi^2)(\phi^{,i}\phi^{,k} - \frac{1}{2} g^{ik}\phi_{,\ell}\phi^{,\ell})$$

$$+ \phi^{-1}(\phi^{,i;k} - g^{ik}\phi^{,\ell}{}_{;\ell}) \qquad (1)$$

$$\phi^{,\ell}{}_{;\ell} = [8\pi/(2\omega+3)c^4]\, T \qquad (2)$$

Here, R^{ik} is the Ricci tensor, ϕ is the scalar field, and ω is a dimensionless coupling constant. As remarked above, the equations of motion for matter are

$$T^{ik}{}_{;k} = 0 \,. \qquad (3)$$

To establish global conservation laws, we would like to find some object $\mathcal{T}^{ik}$ whose ordinary divergence vanishes. It is clear that this will automatically be true if $\mathcal{T}^{ik}$ can be written in the form $\mathcal{T}^{ik} = \eta^{ik\ell}{}_{,\ell}$ for some "potential" $\eta^{ik\ell}$ which is antisymmetric on its last two indices. As the simplest heuristic way to introduce such a "potential", whose general usefulness will be established later, consider the evaluation of equation (3) at a particular (very special)

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[2] We use the conventions established by Landau and Lifshitz (1962), except that our line element is defined as $ds^2 = +g_{ik}dx^i dx^k$.

point where $\phi_{,i} = 0$. Further, choose the coordinate system so that $g^{ik}{}_{,\ell} = 0$ (locally inertial coordinates) there. Then, at this point, in the special coordinate system, equation (3) implies that $[(\phi/\phi_o)(-g)T^{ik}]_{,k} = 0$. Expressing T^{ik} in terms of g_{ik} and ϕ by means of equation (1), we can write $(\phi/\phi_o)(-g)T^{ik} = \eta^{ik\ell}{}_{,\ell}$ where the three-index "potential" is

$$\eta^{ik\ell} = (c^4/16\pi\phi_o)[\phi^2(-g)(g^{ik}g^{\ell m} - g^{i\ell}g^{km})]_{,m} \quad . \tag{4}$$

Here, the constant ϕ_o is the value of ϕ in the asymptotically flat spacetime far from any source, the so-called "cosmological value of ϕ." (For a true believer in the Machian motivation of BDT, the dimensional constant ϕ_o is in principle determined in accordance with equation (2) by the distribution and evolution of matter throughout the Universe--i.e., by the cosmological model (Brans and Dicke, 1961)). In practice, ϕ_o is determined in terms of Newton's gravitational constant G by requiring that at large distances from any source the expansion defining the active gravitational mass m

$$g_{oo} = -1 + 2Gm/c^2r + O(r^{-2}) \tag{5}$$

must agree with the weak field solution of linearized BDT. This yields the relationship $\phi_o = G^{-1}(2\omega+4)/(2\omega+3)$, as shown by Brans and Dicke (1961). Although the appearance of ordinary, rather than covariant, derivatives in equation (4) shows that $\eta^{ik\ell}$ is <u>not</u> a tensor, it does behave like a tensor under affine coordinate transformations of an arbitrary coordinate system. Thus, we are led to define the stress-energy-momentum pseudo-tensor τ^{ik} for the gravitational fields at an arbitrary point (where $\phi_{,i}$ may not vanish), and in coordinates which are arbitrary except for the requirement that they be Lorentzian "at infinity" (far from all sources), by writing

$$(\phi/\phi_o)(-g)(T^{ik} + \tau^{ik}) = \eta^{ik\ell}{}_{,\ell} \quad . \tag{6}$$

A glance at the defining equation (4) reveals the antisymmetry

$$\eta^{ik\ell} = -\eta^{i\ell k} \quad . \tag{7}$$

Like the tensor T^{ik}, the derivatives $\eta^{ik\ell}{}_{,\ell}$ are symmetric quantities, so that

$$\tau^{ik} = \tau^{ki} \quad . \tag{8}$$

Expressing T^{ik} in terms of g_{ik} and ϕ by means of the field equations (1), and using expression (4) for $\eta^{ik\ell}$, one obtains, after a rather tedious calculation, the following expression* for τ^{ik} :

$$\begin{aligned}\tau^{ik} = t^{ik} &+ (\phi^{-1}c^4/8\pi)[(\omega-1)\phi^{,i}\phi^{,k} - \tfrac{1}{2}(\omega-2)g^{ik}\,\phi_{,\ell}\phi^{,\ell}] \\ &+ (c^4/8\pi)\phi_{,n}\,[\Gamma^i{}_{\ell m}(g^{kn}g^{\ell m} - g^{\ell n}g^{km}) \\ &+ \Gamma^k{}_{\ell m}(g^{in}g^{\ell m} - g^{\ell n}g^{im}) + \Gamma^m{}_{\ell m}(2g^{ik}g^{\ell n} - g^{i\ell}g^{kn} - g^{in}g^{k\ell}) \\ &+ \Gamma^n{}_{\ell m}(g^{i\ell}g^{km} - g^{ik}g^{\ell m})] \quad ,\end{aligned} \tag{9}$$

where

$$t^{ik} = (\phi c^4/16\pi)[(2\Gamma^n{}_{\ell m}\Gamma^p{}_{np} - \Gamma^n{}_{\ell p}\Gamma^p{}_{mn} - \Gamma^n{}_{\ell n}\Gamma^p{}_{mp})(g^{i\ell}g^{km} - g^{ik}g^{\ell m}) +$$

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*After this was written (Spring 1971), it was pointed out to me that Nutku (Astrophys. J., 158, 991-996) had previously derived equation (9) for τ^{ik} . Other results of this chapter were not derived or discussed by him.

$$+ (\Gamma^k_{\ell p}\Gamma^p_{mn} + \Gamma^k_{mn}\Gamma^p_{\ell p} - \Gamma^k_{np}\Gamma^p_{\ell m} - \Gamma^k_{\ell m}\Gamma^p_{np}) g^{i\ell} g^{mn}$$

$$+ (\Gamma^i_{\ell p}\Gamma^p_{mn} + \Gamma^i_{mn}\Gamma^p_{\ell p} - \Gamma^i_{np}\Gamma^p_{\ell m} - \Gamma^i_{\ell m}\Gamma^p_{np}) g^{k\ell} g^{mn}$$

$$+ (\Gamma^i_{\ell n}\Gamma^k_{mp} - \Gamma^i_{\ell m}\Gamma^k_{np})\, g^{\ell m} g^{np}] \tag{10}$$

is formally identical to the well-known[3] Landau-Lifshitz pseudo-tensor of GRT, although $\phi \leftrightarrow G^{-1}$ is, of course, variable in BDT.

From the definition (6) and the antisymmetry equation (7), it follows that the equations

$$[(\phi/\phi_o)(-g)(T^{ik} + \tau^{ik})]_{,k} = 0 \tag{11}$$

are satisfied identically. This means there is a conservation law for the quantities ("total energy-momentum")

$$P^i = c^{-1} \int (\phi/\phi_o)(-g)(T^{ik} + \tau^{ik})dS_k \quad , \tag{12}$$

which transform as a vector (not merely pseudo-vector) in the asymptotically Lorentz coordinates. The integration in (12) can be taken over any infinite hypersurface. If we choose the hypersurface x^o = constant, then P^i can be written as a three-dimensional space integral:

$$P^i = c^{-1} \int (\phi/\phi_o)(-g)(T^{io} + \tau^{io})d^3x \quad . \tag{13}$$

The fact that the conserved energy-momentum of matter plus fields is expressible as the integral of a symmetric quantity implies that there

[3] Cf. Landau and Lifshitz (1962), p. 343, eq. (100.6).

is also a conservation law for the quantities

$$J^{ik} = \int (x^i dP^k - x^k dP^i)$$

$$= c^{-1} \int (\phi/\phi_o)(-g)\{x^i(T^{k\ell}+\tau^{k\ell}) - x^k(T^{i\ell}+\tau^{i\ell})\}dS_\ell \quad , \qquad (14)$$

which transform as a tensor (not merely pseudo-tensor) in the asymptotically Lorentz coordinates. Thus, for a closed system the total angular momentum is conserved and the "coordinates of the center of mass", given by

$$X^\alpha = \frac{\int (\phi/\phi_o) x^\alpha (T^{oo} + \tau^{oo})(-g) d^3x}{\int (\phi/\phi_o)(T^{oo} + \tau^{oo})(-g)\ d^3x} \quad , \qquad (15)$$

execute uniform motion as seen by a distant observer. The "coordinates of the center of mass" (15) have a physical interpretation only in the asymptotically flat spacetime far from the source. There, they are the offset, at Lorentz coordinate time t, of the source's distant spherical gravitational fields from being centered in the asymptotically Lorentz coordinates. The X^α can be made to vanish, at any fixed time in the asymptotically flat space, by a translation of the space coordinates ($x'^\alpha = x^\alpha - X^\alpha$). They can be made to vanish for all times by a suitable Lorentz transformation ($x'^i = \Lambda^{i'}{}_k x^k$, where $\Lambda^{i'}{}_k$ is a member of the Lorentz group).

By substituting equations (6) into the definition (12) and integrating by parts, we can write P^i as an integral over a two-dimensional surface:

$$P^i = (2c)^{-1} \oint \eta^{ik\ell} \, df^*_{k\ell} \quad . \tag{16}$$

If we choose the hypersurface in (12) to be x^o = constant, then the surface in (16) is a two-dimensional surface in ordinary three-dimensional space, enclosing an isolated system of matter and gravitational fields:

$$P^i = c^{-1} \oint \eta^{io\alpha} \, df_\alpha \quad . \tag{17}$$

Notice that $\eta^{ik\ell}$ can be derived from the four-index "super-potential" $\lambda^{ik\ell m}$, defined as

$$\lambda^{ik\ell m} = \frac{c^4}{16\pi\phi_o} \phi^2 (-g)(g^{ik} g^{\ell m} - g^{i\ell} g^{km}) \quad ; \tag{18}$$

it is the ordinary divergence

$$\eta^{ik\ell} = \lambda^{ik\ell m}{}_{,m} \, . \tag{19}$$

It is easy to see that the $\lambda^{ik\ell m}$ have the properties

$$\lambda^{ik\ell m} = - \lambda^{i\ell km} \, , \qquad \lambda^{ik\ell m} - \lambda^{\ell kim} = \lambda^{ikm\ell} \quad . \tag{20}$$

Substituting equations (6) and (19) into the definition (14), integrating by parts, and using equations (19) and (20), we find that the J^{ik} can also be expressed as a two-dimensional integral, namely

$$J^{ik} = (2c)^{-1} \oint (x^i \eta^{k\ell m} - x^k \eta^{i\ell m} + \lambda^{i\ell mk}) df^*_{\ell m} \quad . \tag{21}$$

If we choose the hypersurface in (14) to be x^o = constant, then the surface in (21) is a surface in ordinary space:

$$J^{ik} = c^{-1} \oint (x^i \eta^{ko\alpha} - x^k \eta^{io\alpha} + \lambda^{io\alpha k}) df_\alpha \quad . \tag{22}$$

Chapter 3

APPLICATION TO BRANS SOLUTION

Without loss of generality we can describe any static spherical configuration by a coordinate system in which the metric is diagonal and both metric and scalar fields are independent of coordinate time x^o. Then it is apparent from (17) and (22) that the spacelike components P^α and $J^{\beta\gamma}$ all vanish. If the $J^{\beta o}$ do not vanish, they merely indicate that the "center of mass" does not coincide with the spatial origin, but is located at $X^\beta = J^{\beta o}/P^o$. Such a clumsy choice can always be eliminated by a linear transformation in three-space which preserves the diagonal and time-independent features of the description. Thus, the only nontrivial conserved quantity is the conserved mass, given by

$$P^o/c = (\phi_o^{-1} c^2/16\pi) \oint (\phi^2(-g)g^{oo}g^{\alpha\beta})_{,\beta}\, df_\alpha \quad . \tag{23}$$

Assuming the isotropic line element

$$ds^2 = -e^{2\alpha}c^2dt^2 + e^{2\beta}[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)] \quad , \tag{24}$$

Brans has found (Brans and Dicke, 1961) that the most general solution of the field equations (1),(2) in vacuo can be written, for $\omega > -3/2$ (the physically acceptable range, with gravity always attractive), in the form

$$e^{2\alpha} = e^{2\alpha_o} \left(\frac{1 - B/r}{1 + B/r}\right)^{2/\lambda} \tag{25}$$

$$e^{2\beta} = e^{2\beta_o} \left(1 + \frac{B}{r}\right)^4 \left(\frac{1 - B/r}{1 + B/r}\right)^{2(\lambda - C - 1)/\lambda} \qquad (26)$$

$$\phi = \phi_o \left(\frac{1 - B/r}{1 + B/r}\right)^{C/\lambda} \qquad (27)$$

where

$$\lambda = [(C+1)^2 - C(1 - \tfrac{1}{2}\omega C)]^{1/2} \quad . \qquad (28)$$

Asymptotic flatness requires $\alpha_o = 0 = \beta_o$. By comparing with the weak field solution (Brans and Dicke, 1961), and by using the expansion (5) to define the active gravitational mass, one obtains for the remaining "arbitrary" constants:

$$C = \frac{-1}{\omega + 2} \quad , \qquad B = \frac{m}{2\phi_o c^2} \left(\frac{2\omega+4}{2\omega+3}\right)^{1/2} \quad , \qquad (29)$$

$$\phi_o = G^{-1}(2\omega+4)/(2\omega+3) \quad . \qquad (30)$$

Equations (28)-(30) imply that

$$\lambda = \left(\frac{2\omega+3}{2\omega+4}\right)^{1/2} \quad , \qquad \frac{B}{\lambda} = \frac{Gm}{2c^2} \quad . \qquad (31)$$

We will perform the integration (23) over the surface of a sphere of radius r , in the limit $r \to \infty$, so that all the fields are included. Thus, $df_\alpha = n_\alpha r^2 d\Omega$, where n_α is an outward-pointing unit normal and $\int d\Omega = 4\pi$. For any function $f(r)$ we have $[f(r)]_{,\beta} = f'(r) n_\beta$ and for the line element (24) we have $(-g) g^{oo} g^{\alpha\gamma} = -\delta^{\alpha\gamma} e^{4\beta}$. Combining these with equations (26) and (27) in evaluating equation (23), we find that

$$P^{o}/c = \lim_{r \to \infty} \left[\frac{-c^2}{4\phi_o} r^2 \frac{d}{dr} (\phi^2 e^{4\beta})\right]$$

$$= \lim_{r \to \infty} \left[m\left(1 - \frac{B}{\lambda r}\right)\left(1 - \frac{B}{r}\right)^{(3\lambda-2C-4)/\lambda}\left(1+\frac{B}{r}\right)^{(3\lambda+2C+4)/\lambda}\right] \tag{32}$$

That is

$$P^{o}/c = m \quad ; \tag{33}$$

the total conserved mass of matter and fields equals the active gravitational mass. This result is gratifying, but, as in GRT, any attempt to "localize" the mass in the gravitational fields leads to absurdity. Suppose we "define" the function $\tilde{P}^{o}(r)$ to be the expression whose limit is taken in equation (32):

$$P^{o} = \lim_{r \to \infty} \tilde{P}^{o}(r) \quad . \tag{34}$$

Then it may seem naively natural to ask the question: What contribution do infinitesimal shells of field-containing space make to P^{o} ; i.e., what is $d\tilde{P}^{o}/dr$? Using equation (32), we find

$$d\tilde{P}^{o}/dr = \frac{-mcB}{\lambda r^2}\left[(7+4C) - \frac{2B}{\lambda r}(4+3\lambda^2+2C) + \frac{7B^2}{r^2}\right]$$

$$\times \left(1 - \frac{B}{r}\right)^{2(\lambda-C-2)/\lambda}\left(1 + \frac{B}{r}\right)^{2(\lambda+C+2)/\lambda} \tag{35}$$

In the GRT limit, $\omega \to \infty$, this tends to

$$d\tilde{P}^{o}/dr \xrightarrow[(\omega\to\infty)]{} -\frac{7Gm^2}{2cr^2}\left(1 + \frac{Gm}{2c^2 r}\right)^6 \tag{36}$$

which at least has the "expected" property of being negative for all r : crudely speaking, the gravitational field is a potential energy "well". But, for arbitrary ω , equation (35) at large r approaches

$$d\tilde{P}^{o}/dr \xrightarrow[(r\to\infty)]{} -\frac{mcB}{\lambda r^{2}}(7+4C) \tag{37}$$

which, using (29), is <u>positive</u> for ω in the range

$$-\frac{3}{2} < \omega < -\frac{10}{7} \quad . \tag{38}$$

This bizarre result has absolutely no physical significance. The moral of the story is that no matter how peculiar and "unphysical" the behavior of the <u>integrands</u> in our expressions P^{i} and J^{ik}, their rigorously conserved integrals are reasonable, physically interpretable (see next section), numbers characterizing Brans-Dicke gravitational fields in terms of their sources.

Chapter 4

DISTANT FIELDS OF ANY STATIONARY SOURCE

Review of linearized BDT. In the obvious way, let

$$g_{ik} = \eta_{ik} + h_{ik} , \qquad |h_{ik}| << 1 \tag{39}$$

$$\phi = \phi_o + \xi , \qquad |\xi| << \phi_o \tag{40}$$

where η_{ik} is the Minkowski metric and ϕ_o is the constant value of ϕ far from the source. In the expansion of (1) and (2) we will keep only terms linear in h_{ik} and ξ ; i.e., we will neglect terms of $O(h^2, \xi^2, h\xi)$ and all their derivatives. Obviously, the linearized version of the scalar source equation (2) is

$$\xi^{,\ell}{}_{,\ell} = [8\pi/(2\omega+3)c^4]\, T \tag{41}$$

where $\xi^{,\ell}{}_{,\ell} = \underset{\sim}{\nabla}^2\xi - \ddot{\xi}$, the "d'Alembertian of ξ". If we introduce the "bar" notation

$$\overline{A}_{ik} = A_{ik} - \frac{1}{2}\eta_{ik} A \tag{42}$$

for any symmetric quantities A_{ik} , then $\overline{R}_{ik}$ is the linearized Einstein tensor, and the linearized form of the tensor field equation (1) is

$$\frac{1}{2}(\overline{h}_{i\ell,k}{}^{\ell} + \overline{h}_{k\ell,i}{}^{\ell} - \overline{h}_{ik}{}^{,\ell}{}_{\ell} - \eta_{ik}\overline{h}_{\ell m}{}^{,\ell m})$$

$$= (8\pi\phi_o^{-1}/c^4)T_{ik} + \phi_o^{-1}(\xi_{,ik} - \eta_{ik}\xi^{,\ell}{}_{,\ell}) \quad . \tag{43}$$

The third term on each side of (43) is a d'Alembertian; the term proportional to T_{ik} is the matter source; and the remaining terms serve merely to keep the equations gauge invariant; they can be eliminated by imposing the gauge condition

$$\bar{h}_{ik}{}^{,k} = \phi_o^{-1}\, \xi_{,i} \quad . \tag{44}$$

If (44) is satisfied and we make use of (41), then equations (43) can be written

$$\bar{h}_{ik}{}^{,\ell}{}_{\ell} = -\frac{16\pi}{\phi_o c^4}\left(T_{ik} - \frac{\eta_{ik}}{4\omega+6}\, T\right) \quad . \tag{45}$$

Symmetry arguments. The "retarded-time" solution of the linearized equations (45),(41) can be written

$$\bar{h}_{ik}(t,\underset{\sim}{x}) = \frac{4}{\phi_o c^4}\int \Big[T_{ik}\Big(t - \frac{|\underset{\sim}{x} - \underset{\sim}{x}'|}{c}, \underset{\sim}{x}'\Big) - \frac{\eta_{ik}}{4\omega+6}\, T\Big(t - \frac{|\underset{\sim}{x} - \underset{\sim}{x}'|}{c}, \underset{\sim}{x}'\Big)\Big]\frac{d^3x'}{|\underset{\sim}{x} - \underset{\sim}{x}'|} \tag{46a}$$

$$\xi(t,\underset{\sim}{x}) = \frac{-2}{(2\omega+3)c^4}\int T\Big(t - \frac{|\underset{\sim}{x} - \underset{\sim}{x}'|}{c}, \underset{\sim}{x}'\Big)\frac{d^3x'}{|\underset{\sim}{x} - \underset{\sim}{x}'|} \quad . \tag{46b}$$

If the source is localized ($T_{ik} = 0$ for $|\underset{\sim}{x}'| > R$), and stationary, then far outside it (for $r = |\underset{\sim}{x}| >> R$), the solution (46) has the form

$$\bar{h}_{oo} = \frac{A_o}{r} + \frac{B_\alpha n^\alpha}{r^2} + O(r^{-3}) \tag{47a}$$

$$\bar{h}_{o\alpha} = \frac{A_\alpha}{r} + \frac{B_{\alpha\beta} n^\beta}{r^2} + O(r^{-3}) \tag{47b}$$

$$\bar{h}_{\alpha\beta} = \frac{A_{\alpha\beta}}{r} + \frac{B_{\alpha\beta\gamma}n^{\gamma}}{r^2} + O(r^{-3}) \tag{47c}$$

$$\xi = \frac{a}{r} + \frac{b_{\alpha}n^{\alpha}}{r^2} + O(r^{-3}) \tag{47d}$$

where $n^{\alpha} = x^{\alpha}/r$, $A_{\alpha\beta} = A_{(\alpha\beta)}$, $B_{\alpha\beta\gamma} = B_{(\alpha\beta)\gamma}$, and round brackets on indices denote symmetrization. All the quantities A_{ik}, B_{ik} ,$\cdots$ and $a, b_i, \cdots$ are constants.

Applying the gauge condition (44) to the solution (47), we find that

$$A_{\alpha} = 0 , \qquad B_{\alpha\beta}(\delta^{\alpha\beta} - 3n^{\alpha}n^{\beta}) = 0 \tag{48}$$

$$A_{\alpha\beta} = \phi_o^{-1} a n_{\alpha} n_{\beta} , \qquad B_{(\alpha\beta)\gamma}(\delta^{\beta\gamma} - 3n^{\beta}n^{\gamma}) = \phi_o^{-1}(b_{\alpha} - 3n_{\alpha}b_{\beta}n^{\beta}) . \tag{49}$$

Equations (49) imply that

$$a = \phi_o A_{\alpha\beta} n^{\alpha} n^{\beta} , \quad b_{\alpha}n^{\alpha} = -\frac{1}{2}\phi_o B_{(\alpha\beta)\gamma} n^{\alpha}(\delta^{\beta\gamma} - 3n^{\beta}n^{\gamma}) . \tag{50}$$

Break $B_{\alpha\beta}$ up into its "irreducible tensorial parts"

$$B_{\alpha\beta} = B'\delta_{\alpha\beta} + S_{(\alpha\beta)} + \varepsilon_{\alpha\beta\gamma} F^{\gamma}, \quad \text{with} \quad S_{\alpha}{}^{\alpha} = 0 .$$

Then (48) implies that $S_{(\alpha\beta)} = 0$. Similarly, equations (49) tell us that if we break $B_{(\alpha\beta)\gamma}$ up into its "irreducible parts"

$$B_{(\alpha\beta)\gamma} = \delta_{\alpha\beta}C_{\gamma} + D_{(\alpha}\delta_{\beta)\gamma} + \varepsilon_{\mu\gamma(\alpha}E_{\beta)}{}^{\mu} + S_{(\alpha\beta\gamma)}$$

where $E^{\alpha\beta} = E^{(\alpha\beta)}$, $E^{\alpha}{}_{\alpha} = 0$, $S^{\alpha}{}_{\alpha\gamma} = 0$, then

$$C_{\alpha} + \frac{1}{2}D_{\alpha} = \phi_o^{-1} b_{\alpha} , \quad E_{(\alpha\beta)} = 0 , \quad S_{(\alpha\beta\gamma)} = 0 .$$

Finally, we can use coordinate transformations which maintain the gauge conditions (44) to fix any four constants arbitrarily. Clearly a choice which effects considerable simplification is $B' = 0 = D_\alpha$. (This is always possible, because equation (44) does not determine the coordinate system uniquely. Indeed, if these conditions are satisfied in some system of coordinates x^i, they will be satisfied in any infinitesimally transformed coordinates $x^{i'} = x^i + \chi^i$, provided that the generating functions χ^i of the transformation satisfy $\chi_i{}^{,\ell}{}_{,\ell} = \phi_o^{-1}\,\xi_{,i}$.) Using all these results, we can write the solution (47) in the form

$$\overline{h}_{oo} = \frac{A_o}{r} + \frac{B_\alpha n^\alpha}{r^2} + O(r^{-3})$$

$$\overline{h}_{o\alpha} = \frac{\varepsilon_{\alpha\beta\gamma} n^\beta F^\gamma}{r^2} + O(r^{-3})$$

$$\overline{h}_{\alpha\beta} = \frac{A_{\alpha\beta}}{r} + \frac{\delta_{\alpha\beta} C_\gamma n^\gamma}{r^2} + O(r^{-3})$$

$$\xi = \phi_o\left[\frac{A_{\alpha\beta} n^\alpha n^\beta}{r} + \frac{C_\alpha n^\alpha}{r^2} + O(r^{-3})\right] \quad .$$

Reverting to the "unbarred" metric perturbations, we find that the solution can be written in the form

$$h_{oo} = \frac{A_o'}{r} + \frac{B_\alpha' n^\alpha}{r^2} + O(r^{-3}) \tag{51a}$$

$$h_{o\alpha} = \frac{\varepsilon_{\alpha\beta\gamma} n^\beta F^\gamma}{r^2} + O(r^{-3}) \tag{51b}$$

$$h_{\alpha\beta} = \frac{A'_{\alpha\beta}}{r} + \frac{\delta_{\alpha\beta} C'_{\gamma} n^{\gamma}}{r^2} + O(r^{-3}) \qquad (51c)$$

$$\xi = \phi_o \{ r^{-1} [A'_{\alpha\beta} n^{\alpha\beta} + \frac{1}{2}(A'_o - A'^{\gamma}_{\gamma})]$$

$$+ \frac{1}{2} r^{-2} (B'_{\alpha} - C'_{\alpha}) n^{\alpha} + O(r^{-3}) \} \qquad (51d)$$

in a suitable coordinate system satisfying the gauge conditions(44). Here, the primed constants are related to the unprimed constants of the preceding discussion by

$$A'_o = \frac{1}{2}(A_o + A_{\alpha}{}^{\alpha}) , \qquad B'_{\alpha} = \frac{1}{2}(3C_{\alpha} + B_{\alpha})$$

$$A'_{\alpha\beta} = A_{\alpha\beta} + \frac{1}{2}\delta_{\alpha\beta}(A_o - A_{\gamma}{}^{\gamma}) , \qquad C'_{\alpha} = \frac{1}{2}(B_{\alpha} - C_{\alpha}) \quad .$$

(Notice that the property $\bar{\bar{h}}_{ik} = h_{ik}$ of the "bar" notation is reflected in the fact that

$$A_o = \frac{1}{2}(A'_o + A'^{\alpha}{}_{\alpha}) , \qquad B_{\alpha} = \frac{1}{2}(3C'_{\alpha} + B'_{\alpha})$$

$$A_{\alpha\beta} = A'_{\alpha\beta} + \frac{1}{2}\delta_{\alpha\beta}(A'_o - A'^{\gamma}{}_{\gamma}) , \qquad C_{\alpha} = \frac{1}{2}(B'_{\alpha} - C'_{\alpha})$$

are the inverse relations among these constants.)

Evaluating the constants. For a stationary configuration ($h_{ik,o} = 0 = \xi_{,o}$) in linear order the conserved integrals (17) and (22) reduce to

$$P^o = \frac{\phi_o c^3}{16\pi} \oint (h^{oo,\alpha} - 3h^{\alpha\beta}{}_{,\beta}) \, df_{\alpha} \qquad (52a)$$

$$P^{\beta} = \frac{\phi_o c^3}{16\pi} \oint (h^{o\beta,\alpha} - 3\delta^{\beta\alpha} h^{o\gamma}{}_{,\gamma})\, df_{\alpha} \tag{52b}$$

$$J^{\beta\gamma} = \frac{\phi_o c^3}{16\pi} \oint [x^{\beta}h^{o\gamma,\alpha} - x^{\gamma}h^{o\beta,\alpha} - 3(\delta^{\gamma\alpha}x^{\beta} - \delta^{\beta\alpha}x^{\gamma})h^{o\delta}{}_{,\delta} + \delta^{\gamma\alpha}h^{o\beta} - \delta^{\beta\alpha}h^{o\gamma}]\, df_{\alpha} \tag{52c}$$

$$J^{\beta o} = \frac{\phi_o c^3}{16\pi} \oint [x^{\beta}(h^{oo,\alpha} - 3h^{\alpha\gamma}{}_{,\gamma}) - x^{o}h^{o\beta,\alpha} + h^{\beta\alpha} + \delta^{\beta\alpha}(3x^{o}h^{o\gamma}{}_{,\gamma} + 1 + h - h^{oo} + \phi_o^{-1}\xi)]\, df_{\alpha} \; . \tag{52d}$$

Inserting equations (51) into (52) and performing the integrals on a sphere of radius $r \to \infty$ yields

$$P^{o} = \frac{1}{4}\phi_o c^2(-A'_o + 3A'_{\alpha\beta}n^{\alpha}n^{\beta}) \tag{53a}$$

$$P^{\beta} = 0 \tag{53b}$$

$$J^{\beta\gamma} = \frac{1}{2}\phi_o c^3 \varepsilon^{\beta\gamma\alpha}F_{\alpha} \tag{53c}$$

$$J^{\beta o} = \frac{1}{4}\phi_o c^3 n^{\beta}(-B'_{\gamma}n^{\gamma} + 3C'_{\gamma}n^{\gamma}) \quad . \tag{53d}$$

But g_{oo}, $g_{\alpha\beta}$, and ξ should include terms of order $O(G^2m^2/c^4r^2)$ which linearized BDT ignores. Since these are completely determined by m , they may be added ad hoc from a second order expansion of the Brans solution. When this is done, and the results combined with an inversion of equations (53), we find that the distant fields of an arbitrary stationary source have the expansions

$$ds^2 = -\left[1 - \frac{2Gm}{c^2 r} - \frac{2Gm\, X_\gamma x^\gamma}{c^2 r^3} + \frac{2G^2 m^2}{c^4 r^2}\right] c^2 dt^2$$

$$+ 4\left(\frac{2\omega+3}{2\omega+4}\right)\frac{G}{c^2}\frac{\varepsilon_{\alpha\beta\gamma} x^\beta S^\gamma}{r^3}\, dt\, dx^\alpha + \left[1 + 2\left(\frac{\omega+1}{\omega+2}\right)\frac{Gm}{c^2 r}\right.$$

$$\left. + 2\left(\frac{\omega+1}{\omega+2}\right)\frac{GmX_\gamma x^\gamma}{c^2 r^3} + \frac{6\omega^2 - 11\omega + 4}{4(\omega+2)^2}\frac{G^2 m^2}{c^4 r^2}\right]\delta_{\alpha\beta} dx^\alpha dx^\beta$$

$$+ O(r^{-3}) dx^i dx^k \qquad (54a)$$

$$\phi = G^{-1}\left(\frac{2\omega+4}{2\omega+3}\right)\left[1 - \frac{Gm}{(\omega+2)c^2 r} - \frac{GmX_\gamma x^\gamma}{(\omega+2)c^2 r^3} + \frac{G^2 m^2}{2(\omega+2)^2 c^4 r^2} + O(r^{-3})\right] \qquad (54b)$$

where

$$m = P^o/c, \quad P^\beta = 0, \quad J^{\beta o} = mcX^\beta, \quad J^{\beta\gamma} = \varepsilon^{\beta\gamma\alpha} S_\alpha , \qquad (55a)$$

and where, as can be seen by inspection of the constant multiplying the right-hand side of equation (54b), Newton's gravitational constant can be expressed in terms of the cosmological value ϕ_o of the scalar field by

$$G = \phi_o^{-1}(2\omega+4)/(2\omega+3) \quad . \qquad (55b)$$

Notice that these gravitational fields far from a completely general stationary source are determined uniquely by the source's total mass-energy m (measurable via Kepler's third law), by its intrinsic angular momentum S_α (measurable via the "dragging of inertial frames"), and by the offset X^α of the gravitational fields from being centered in the asymptotically Lorentzian coordinate system. A simple translation of the space coordinates ($x'^{\alpha} = x^{\alpha} - X^{\alpha}$) completely removes the X^{α} terms ("dipole terms") from both metric and scalar fields.

The results (54),(55) strictly speaking apply only for sources which are assumed to move on geodesics in the conformal form of BDT defined by equations (1),(2). Bodies with significant gravitational binding energy violate the equivalence principle in BDT. S. W. Hawking (1971, private communication through K. S. Thorne) has recently pointed out that black holes in BDT move on geodesics in a conformally transformed version of BDT (Dicke, 1962) in which the tensor field equations are Einstein's equations with the stress-energy-momentum tensor augmented by that for a scalar field, but test particles do not move on geodesics. Hawking splits the mass of the gravitational field into two parts, M_{tensor} and M_{scalar}. Keplerian orbits of black holes distant from a source measure M_t ; Keplerian orbits of distant test masses (planets) measure the sum $M_t + M_s$. Our pseudo-tensor τ^{ik} includes <u>all</u> gravitational stress-energy-momentum of the gravitational fields.

Chapter 5

GRAVITATIONAL WAVES IN THE HIGH-FREQUENCY LIMIT

We consider gravitational waves to be of "high frequency" whenever their wavelengths are small compared to the radius of curvature of the background geometry and the scale on which the background gravitational "constant" changes. Obviously, this is the case for such conventional weak-field sources as binary star systems or oscillating and rotating spheroids. Further, this situation holds for waves of optical frequency, such as the gravitational bremsstrahlung produced by the thermal motion of matter. Most important, all gravitational radiation from isolated systems is of high frequency when it gets far enough away from its source, since, assuming that the wavelength λ remains approximately constant, at increasing distance r from a source of active gravitational mass m, the ratio of wavelength to radius of curvature of space is of the order $\lambda(m/\phi c^2 r^3)^{1/2}$ and the ratio of wavelength to length scale for change in the background scalar field is of the order λ/r. At large r both of these ratios become negligible.

Our picture of high frequency gravitational waves is one of small ripples in the geometry of space-time and in the strength of the gravitational "constant" propagating through strong, fully relativistic, but slowly changing background fields. Although their amplitudes are small, the energy carried by the gravitational waves may be a major (if not the only) source for the background fields. Generalizing the approach used by Brill and Hartle (1964) and

Isaacson (1968) in GRT, we assume that the total metric g_{ik} and the total scalar field ϕ take the forms

$$g_{ik} = \gamma_{ik} + \varepsilon h_{ik}$$

$$\phi = \phi_o(f + \varepsilon\xi)$$

where γ_{ik} and f represent the background, h_{ik} and ξ (assumed of order unity) are the waves, and ε is a smallness parameter which guarantees that the geometry and the gravitational "constant" exhibit only microscopic fluctuations in the laboratory. If L and λ are characteristic lengths over which the background and the waves change significantly, the field derivatives are typically of the orders (suppressing indices)

$$\partial\gamma \sim \gamma/L \,, \qquad \partial h \sim h/\lambda$$

$$\partial f \sim f/L \,, \qquad \partial\xi \sim \xi/\lambda \quad .$$

The high-frequency assumption is that $L >> \lambda$. The effective energy density of the waves is of order $(\phi_o c^4)(\varepsilon/\lambda)^2$, while the curvature of the background is of order L^{-2} . In view of the field equations (1) and (2), $L^{-2} \gtrsim (\phi_o c^4)^{-1}\,(\phi_o c^4)(\varepsilon/\lambda)^2 = (\varepsilon/\lambda)^2$. That is, $\varepsilon \lesssim \lambda/L$. The most interesting case occurs when gravitational waves are the <u>only</u> source of the background fields, i.e., $T^{ik} = 0$. In the remainder of this chapter we will restrict our attention to this situation. (The generalization to the case where matter and nongravitational fields are present is straightforward.) Then, our two small dimensionless numbers are equal, $\varepsilon \sim \lambda/L << 1$, so that we may regard L as a constant, say of order unity, and λ may be replaced by ε , since $O(\lambda) = O(\varepsilon)$. Thus, we are studying the one-parameter class of background fields differing infinitesimally by high frequency wave sources. We will say that the background fields contain high-frequency waves if and only if there exists a family of coordinate systems (called "steady coordinates", following Isaacson's (1968) usage in GRT), related by infinitesimal coordinate transformations, in which the total fields take the forms[4]

$$g_{ik}(x^\ell) = \gamma_{ik}(x^\ell) + \varepsilon h_{ik}(x^\ell,\varepsilon) \quad , \tag{56a}$$

$$\varepsilon << 1 \ , \quad \gamma_{ik} = O(1) \ , \quad h_{ik} = O(1) \quad , \tag{56b}$$

$$\gamma_{ik,\ell} = O(1) \ , \qquad h_{ik,\ell} = O(\varepsilon^{-1}) \ , \tag{56c}$$

$$\gamma_{ik,\ell m} = O(1) \ , \qquad h_{ik,\ell m} = O(\varepsilon^{-2}) \ , \tag{56d}$$

$$\phi(x^\ell) = \phi_o[f(x^\ell) + \varepsilon\xi(x^\ell,\varepsilon)] \tag{57a}$$

$$\varepsilon << 1 \ , \quad f = O(1) \ , \quad \xi = O(1) \quad , \tag{57b}$$

$$f_{,\ell} = O(1) \ , \qquad \xi_{,\ell} = O(\varepsilon^{-1}) \ , \tag{57c}$$

$$f_{,\ell m} = O(1) \ , \qquad \xi_{,\ell m} = O(\varepsilon^{-2}) \quad . \tag{57d}$$

Equations (56) imply a highly curved background spacetime, since the Riemann tensor components $R^i{}_{k\ell m}$, whose typical terms are of the symbolic form (suppressing indices)

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[4]Here $F(x^\ell,\varepsilon) = O(\varepsilon^n)$ means that $F < N\varepsilon^n$ for some constant N , as $\varepsilon \to 0$.

$$\partial(g\partial g) = \partial[(\gamma + \varepsilon h)(\partial\gamma + \varepsilon\partial h)]$$

$$= \varepsilon\gamma\partial^2 h \qquad [O(\varepsilon^{-1})]$$

$$+ \gamma\partial^2\gamma + \varepsilon^2 h\partial^2 h + (\partial\gamma + \varepsilon\partial h)(\partial\gamma + \varepsilon\partial h) \qquad [O(1)]$$

$$+ \varepsilon h\partial^2\gamma \quad , \qquad [O(\varepsilon)]$$

is clearly of the order $O(\varepsilon^{-1})$. Similarly, although $\phi'^{\ell}{}_{;\ell}/\phi = 0$ by equation (2) applied in vacuum, equations (57) imply that this zero scalar field curvature is due to cancellation of terms of the form

$$(f + \varepsilon\xi)^{-1}\, \partial^2(f + \varepsilon\xi) = (f + \varepsilon\xi)^{-1}\, (\partial^2 f + \varepsilon\partial^2\xi)$$

$$= \varepsilon f^{-1}\partial^2\xi \qquad [O(\varepsilon^{-1})]$$

$$+ f^{-1}[\partial^2 f - \varepsilon^2(\xi/f)\partial^2\xi] \qquad [O(1)]$$

$$+ O(\varepsilon) \quad ,$$

and so of the order $O(\varepsilon^{-1})$ in steady coordinates.

Brill-Hartle Averaging and the Stress-Energy-Momentum Tensor. In discussing the energy and momentum carried by gravitational waves, we are not really interested in the fine details of the geometry's fluctuations. Thus, it is natural and advantageous to introduce an averaging process over regions containing many wavelengths. Following Isaacson (1968), we let the symbol $<\cdots>$ denote an average over a spacetime region with characteristic dimension d that (i) is small compared to the scale over which the background changes, but (ii) is independent of ε, and therefore is large compared to the wavelength of the radiation in the limit $\varepsilon \to 0$.

In order that the result be a tensor, we must construct an average over this region of spacetime by carrying all quantities through the background geometry to a common point and adding them there. To do this uniquely, we may use the bivector of geodesic parallel displacement (Synge, 1966) $g_i{}^k(x,x')$ for the background geometry. This object transforms as a vector with respect to coordinate transformations at either x or x' , and if these points are close enough to insure a unique geodesic of the metric γ_{ik} between them, then $A^k = g_{i'}{}^k A^{i'}$ is the unique vector at x obtained by parallel-transport of $A^{i'}$ from x' along the geodesic. If S^{ik} is a tensor which is the sum of (i) high-frequency components and (ii) background containing only low frequency components, then we define the Brill-Hartle (BH) average to be the tensor

$$\bar{S}^{ik}(x) = \int_{\substack{\text{all}\\ \text{spacetime}}} g_{\ell'}{}^i(x,x')\, g_{m'}{}^k(x,x') S^{\ell'm'}(x') w(x,x') d^4x' , \tag{58}$$

where $w(x,x')$ is a weighting function which falls smoothly to zero when x and x' are an interval d apart $(\lambda << d << L)$, and where

$$\int_{\substack{\text{all}\\ \text{spacetime}}} w(x,x')\, d^4x = 1 \quad . \tag{59}$$

Clearly, $\partial w \sim w/d$ and $\partial g \sim g/L$ so that the only rapidly varying quantity in the definition (58) is S , since $\partial S \sim S/\lambda = O(\varepsilon^{-1})$.

By writing (again suppressing indices)

$$gg(\partial S)w = \partial(ggSw) - (\partial g)gSw - g(\partial g)Sw - ggS(\partial w) \quad ,$$

we see that, when averaged by (58), divergences are smaller than one's <u>a priori</u> expectations by a factor ε . (Notice that the first term can be converted to a surface integral taken in the region where $w \to 0$.) As a trivial corollary, we may integrate by parts and neglect the integrated terms whenever we are interested in only the dominant contribution to the BH average.

Applying BH averaging (58),(59) to the field equations (1),(2) with $T^{ik} = 0$ and assuming the forms (56),(57) for the fields, we see that

$$R^{ik(0)} - \frac{1}{2}\gamma^{ik}R^{(0)} = (8\pi/\phi_o f c^4)\, T^{ik}{}_{(GW)} + (\omega/f^2)(f^{,i}f^{,k} - \frac{1}{2}\gamma^{ik} f_{,\ell}f^{,\ell}) + f^{-1}(f^{,i|k} - \gamma^{ik}f^{,\ell}{}_{|\ell} \tag{60}$$

$$f^{,\ell}{}_{|\ell} = [8\pi/(2\omega+3)\phi_o c^4]\, \gamma_{ik}T^{ik}{}_{(GW)} \tag{61}$$

Here $R^{ik(0)}$ is the Ricci tensor formed from the background metric γ_{ik}, $R^{(0)}$ is its trace, and the slash denotes covariant differentiation with respect to the background metric, which is used to raise and lower all indices.

The quantity $T^{ik}{}_{(GW)}$ is the BH average of the pseudotensor τ^{ik} ,

$$T^{ik}{}_{(GW)} = \langle \tau^{ik} \rangle \quad , \tag{62}$$

defined by (9),(10); we call $T^{ik}{}_{(GW)}$ the "stress-energy-momentum tensor for gravitational waves". It is the analogue of the Isaacson tensor in GRT, and like the Isaacson tensor it transforms as a tensor under background coordinate transformations, and it has vanishing covariant divergence with respect to the background metric γ_{ik}. The explicit form of the dominant terms,

$$T^{ik}{}_{(GW)} = (\varepsilon^2 \phi_o f c^4/32\pi) \ \langle h^{\ell m|i} \, h_{\ell m}{}^{|k} + (2\omega/f^2)(2\xi^{,i}\xi^{,k} - \gamma^{ik}\xi_{,\ell}\xi^{,\ell}) - (2/f) \ \xi_{,\ell}(h^{\ell i|k} + h^{\ell k|i}) \rangle \tag{63}$$

can be used instead of τ^{ik} to calculate the energy-momentum and angular momentum carried away from a source by gravitational radiation in the high-frequency limit. Although we made use of the gauge condition

$$(h^{ik} - \frac{1}{2}\gamma^{ik}\gamma^{\ell m}h_{\ell m})_{|k} = \phi_o^{-1} \, \xi^{,i} \tag{64}$$

in calculating equation (63), this result could easily be extended to an arbitrary gauge, as is done for GRT by Misner, Thorne, and Wheeler (1971, Chapter 35). Such an extension is unnecessary for our purposes, but would be useful for consideration of gravitational waves moving through matter or non-gravitational fields. In that case, the metric perturbations could not be made to satisfy equation (64) the analogue of the "transverse traceless" (or Lorentz) gauge in GRT.

PART II

GRAVITATIONAL COLLAPSE AND THE FORMATION OF BLACK HOLES

[Chapters 1, 2, and 3 are an extended version of the paper, "Black Holes in the Dicke-Brans-Jordan Theory of Gravity", co-authored with Kip S. Thorne; Astrophysical Journal Letters 166, L35 (June 1, 1971). The work discussed in Chapter 4 is due to the author alone.]

Chapter 1

INTRODUCTION AND SUMMARY

Compression of a body of mass M into a region of circumference $C \lesssim 4\pi GM/c^2$ in all directions (i.e., three-dimensional compression) necessarily produces a black hole. Moreover, after all matter and waves have flowed off toward infinity or down the hole, the hole's exterior will be the "charged-Kerr" solution to the Einstein-Maxwell field equations--or one of the three special cases of charged Kerr: Schwarzschild, Reissner-Nordström, or uncharged Kerr.

This slightly fuzzy conjecture is widely believed and is supported by extensive theoretical evidence within the framework of general relativity (GRT) [See, e.g., the review by Thorne (1971).], but the problem of gravitational collapse and the formation of black holes has not been so widely studied in other relativistic theories of gravity. However, Penrose (1970) has expressed the opinion that the conjecture is also true in the scalar-tensor theory[1] of Brans and Dicke (BDT) (1961). That this suggestion is not trivial is pointed up by the counterexample discussed in Appendix C: Here we have a relativistic theory of gravity (Nordström 1913) which has the appropriate correspondence limit with Newtonian theory, and whose spacetime metric is generated by matter according to a field equation derived

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[1] The Dicke-Brans theory is actually a special case, with change in outlook and motivation, of a wider class of theories due to Jordan (1955,1959). This relationship is carefully analyzed in Appendix B.

from an invariant action principle, and yet the formation of <u>any</u> type of black hole is <u>impossible</u> in this theory.

The investigations reported upon in this second part of the thesis have produced strong evidence in support of Penrose's conjecture for Brans-Dicke theory--that gravitational collapse in three dimensions produces black holes which are <u>identical</u> to those of general relativity.

First, we present two elementary results of exact analysis giving firm support to the conjecture: (i) The Schwarzschild, Reissner-Nordström, Kerr, and charged-Kerr solutions of the Einstein-Maxwell field equations are also exact solutions of the scalar-tensor field equations if the scalar-field ϕ is taken constant; and (ii) The Schwarzschild solution is the <u>only</u> static spherically symmetric vacuum solution of the scalar-tensor field equations which possesses a non-singular event horizon.

To establish further evidence, we consider an approximation to the scalar-tensor theory, constructed by expanding the full theory in powers of $1/\omega$ and keeping only zero- and first-order terms. (Recall that ω is the Dicke coupling constant, and that in the limit $\omega \to \infty$ the scalar-tensor theory agrees with general relativity.) In this "<u>large-ω version</u>" of the theory, any stationary, axially symmetric changes in the Kerr geometry of order $1/\omega$ would convert the Kerr horizon into a singularity. Furthermore, spherical gravitational collapse of an uncharged star in the large-ω version of the theory necessarily produces a Schwarzschild black hole.

Then, we argue that stationary, axially symmetric changes of the Kerr geometry of higher order in $1/\omega$ are also incompatible with a nonsingular Kerr horizon, and that higher order perturbations in the fields of an uncharged, nonrotating sphere are also radiated away as collapse proceeds to a Schwarzschild black hole.

Chapter 2

THE EVIDENCE OF EXACT RESULTS

Schwarzschild, Reissner-Nordström, Kerr, and Charged-Kerr Geometries. Recall that the field equations of the scalar-tensor theory of gravity, in the original form of Brans and Dicke (1961), can be written[2]

$$G_{ik} = (8\pi\phi^{-1}/c^4)\, T_{ik} + (\omega/\phi^2)(\phi_{,i}\phi_{,k} - \tfrac{1}{2} g_{ik}\phi_{,\ell}\phi^{,\ell}) + \phi^{-1}(\phi_{,i;k} - g_{ik}\phi^{,\ell}{}_{;\ell}) \tag{1}$$

$$\phi^{,\ell}{}_{;\ell} = [8\pi/(2\omega+3)c^4]\, T \quad . \tag{2}$$

Here T_{ik} is the stress-energy-momentum tensor for matter and non-gravitational fields, T is its trace, ϕ is the scalar field, g_{ik} is the metric, G_{ik} is the Einstein tensor, and c is the speed of light. Suppose that g_{ik} and f_{ik} are metric and electromagnetic fields, respectively, which satisfy the general relativistic, vacuum, Einstein-Maxwell field equations

$$G_{ik} = (8\pi G/c^4)\, T_{ik} \quad , \tag{3a}$$

$$T_{ik} = (1/4\pi)(f_{i\ell}f_k{}^{\ell} - \tfrac{1}{4} g_{ik}f_{\ell m}f^{\ell m}); \; T = T_\ell{}^\ell = 0 \; . \tag{3b}$$

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[2] As before, we use the conventions established by Landau and Lifshitz (1962) in GRT wherever applicable, except that our line element is defined as

$$ds^2 = +g_{ik}dx^i dx^k \; .$$

Then g_{ik} and f_{ik}, when augmented by the statement $\phi = 1/G =$ constant, are also a solution of the scalar-tensor field equations (1), (2). In particular, all the black-hole solutions of general relativity--Schwarzschild, Reissner-Nordström, Kerr, and charged-Kerr--are also valid in the scalar-tensor theory. Notice that there is no direct electromagnetic source for the scalar field, since the trace of the stress-energy-momentum tensor of an electromagnetic field vanishes identically.

Thus, in the work which follows we shall investigate the gravitational collapse of <u>uncharged</u> configurations, with the expectation that the generalization to black holes surrounded by electromagnetic fields would always be straightforward.

<u>Uniqueness of the Schwarzschild Black Hole.</u> Carl Brans (1962) has constructed all static, spherically symmetric solutions of the BDT field equations (1),(2) in vacuum. In the isotropic coordinates defined by a line element of the form

$$ds^2 = -e^{2\alpha}c^2dt^2 + e^{2\beta}[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)] \quad , \tag{4}$$

the four forms of the solution can be written:

$$e^{\alpha} = e^{\alpha_o} \left(\frac{1 - B/r}{1 + B/r}\right)^{1/\lambda} \tag{5a}$$

$$e^{\beta} = e^{\beta_o}(1 + B/r)^2 \left(\frac{1 - B/r}{1 + B/r}\right)^{(\lambda - C - 1)/\lambda} \tag{5b}$$

$$\phi = \phi_o\left(\frac{1 - B/r}{1 + B/r}\right)^{C/\lambda} \tag{5c}$$

where

$$\lambda^2 = (C+1)^2 - C(1 - \frac{1}{2}\omega C) > 0 \qquad ; \qquad (5d)$$

$$\alpha = \alpha_o + \frac{2}{\Lambda}\tan^{-1}(r/B) \qquad , \qquad (6a)$$

$$\beta = \beta_o - \frac{2(C+1)}{\Lambda}\tan^{-1}(r/B) - \log\frac{r^2}{r^2+B^2} \qquad , \qquad (6b)$$

$$\phi = \phi_o \exp[\frac{2C}{\Lambda}\tan^{-1}(r/B)] \qquad , \qquad (6c)$$

where

$$\Lambda^2 = C(1 - \frac{1}{2}\omega C) - (C+1)^2 > 0 \qquad ; \qquad (6d)$$

$$\alpha = \alpha_o - r/B \qquad , \qquad (7a)$$

$$\beta = \beta_o - 2\log(r/B) + (C+1)\, r/B \qquad , \qquad (7b)$$

$$\phi = \phi_o\, e^{-Cr/B} \qquad ; \qquad (7c)$$

$$\alpha = \alpha_o - 1/Br \qquad , \qquad (8a)$$

$$\beta = \beta_o + (C+1)/Br \qquad , \qquad (8b)$$

$$\phi = \phi_o\, e^{-C/Br} \qquad ; \qquad (8c)$$

where $C = \frac{-1 \pm (-2\omega-3)^{1/2}}{\omega + 2}$ for solutions (7) and (8). (9)

In each form of solution, α_o, β_o, and B are arbitrary constants. The constant $C \neq \frac{-1 \pm (-2\omega-3)^{1/2}}{\omega + 2}$ is arbitrary in (5) and (6). Since $(C+1)^2 - C(1 - \frac{1}{2}\omega C) = (\frac{1}{2}C+1)^2 + \frac{1}{2}(\omega + \frac{3}{2})C^2$, it is obvious that $\lambda^2 > 0$ and solution (5) applies for any real C provided that $\omega \geq -3/2$. But this is precisely the requirement which ω must satisfy in order that the contribution to ϕ from a local mass be positive, which is the usual assumption in applications of BDT. Nevertheless, for the sake of completeness, it is of interest to note the conditions under which the alternate forms of solution hold. Obviously, solution (6) can apply only if $\omega < -3/2$. Even so, it does not apply throughout this range, but only if one of the conditions

(i) $$\frac{-1-(-2\omega-3)^{1/2}}{\omega+2} < C < \frac{-1+(-2\omega-3)^{1/2}}{\omega+2} < 0$$

for $-2 < \omega < -3/2$,

(ii) $$C < \frac{-1+(-2\omega-3)^{1/2}}{\omega+2} < 0 \quad \underline{\text{or}} \quad C > \frac{-1-(-2\omega-3)^{1/2}}{\omega+2} > 0$$

for $\omega < -2$,

(iii) $C < -1$ for $\omega = -2$

is satisfied. If none of these conditions are satisfied, solution (5) is applicable even for $\omega < -3/2$, except at the singular values

$$C = \frac{-1 \pm (-2\omega-3)^{1/2}}{\omega+2} ,$$

where solutions (8) and (9) apply.

Examination of all four forms of the solution discloses the presence of naked singularities, <u>not</u> surrounded event horizons, unless

$C = 0$. But in that special case, $\lambda^2 = 1 > 0$, so that solution (5) is the only one which applies, and it reduces to the form

$$e^{\alpha} = e^{\alpha_o}\left(\frac{1 - B/r}{1 + B/r}\right) \tag{10a}$$

$$e^{\beta} = e^{\beta_o}(1 + B/r)^2 \tag{10b}$$

$$\phi = \phi_o \text{ , a constant } . \tag{10c}$$

Now, a glance at equations (1),(2) shows that if ϕ = constant the vacuum scalar-tensor equations reduce to the vacuum Einstein equations, whose general spherical solution is that of Schwarzschild. ("Birkhoff's theorem"). Thus, the Schwarzschild metric with ϕ = constant is the unique static spherical solution of the vacuum scalar-tensor field equations which possesses a nonsingular event horizon. Indeed, equations (10) are merely an expression of the Schwarzschild solution in the isotropic coordinates defined by equation (4), instead of in the more customary curvature coordinates.

Chapter 3

EVIDENCE IN AN APPROXIMATE SCALAR-TENSOR THEORY

The Large-ω Version of the Scalar-Tensor Theory. Consider an expansion of the stress-energy-momentum tensor, the metric, and the scalar field in powers of $1/\omega$

$$T_{ik} = T^{(E)}{}_{ik} + (1/\omega) t_{ik} + O(1/\omega^2) \tag{11a}$$

$$g_{ik} = g^{(E)}{}_{ik} + (1/\omega) h_{ik} + O(1/\omega^2) \tag{11b}$$

$$\phi = \phi_o + (1/\omega)\ \xi + O(1/\omega^2) \quad , \tag{11c}$$

$$\phi_o = \text{"cosmological } \phi\text{"; the value of } \phi \text{ far from sources.} \tag{11d}$$

Because the time scale (> 10 billion years) for changes in ϕ_o is very long compared to the time scale ($\lesssim$ 1 second) for gravitational collapse, we will assume that ϕ_o is a constant. Then using equations (11) in equations (1) and (2) leads to the following field equations:

$$G^{(E)}{}_{ik} = (8\pi/\phi_o c^4)\ T^{(E)}{}_{ik} \tag{12}$$

$$\begin{aligned} G^{(1)}{}_{ik} = {} & (8\pi/\phi_o c^4)[-(\xi/\phi_o)\ T^{(E)}{}_{ik} + t_{ik}] \\ & + (1/\phi_o^2)(\xi_{,i}\xi_{,k} - \tfrac{1}{2} g^{(E)}{}_{ik}\ \xi_{,\ell}\xi^{,\ell}) \\ & + (1/\phi_o)(\xi_{,i|k} - g^{(E)}{}_{ik}\xi^{,\ell}{}_{|\ell}) \end{aligned} \tag{13a}$$

$$\xi^{,\ell}{}_{|\ell} = (4\pi/c^4)\ T^{(E)} \tag{13b}$$

Here the zero-order "Einstein metric" $g^{(E)}{}_{ik}$ is used to raise and lower all indices; a slash denotes a covariant derivative with respect to $g^{(E)}{}_{ik}$; and $G^{(1)}{}_{ik}$ is the first-order perturbation in the Einstein tensor produced by the metric perturbation h_{ik} . These equations may be thought of in the following way: As shown by equation (12), the zero-order stress-energy-momentum $T^{(E)}{}_{ik}$ produces the "Einstein metric" $g^{(E)}{}_{ik}$ in the usual general relativistic manner. The perturbations t_{ik}, h_{ik}, and ξ "reside" and evolve in this "background" spacetime $g^{(E)}{}_{ik}$ in accordance with equations (13). One consequence of the field equations (1),(2) is $T_{ik}{}^{;k} = 0$. Calculated using equations (12) and (13) this becomes $T^{(E)}{}_{ik}{}^{|k} = 0$ and

$$t_{ik}{}^{|k} - (\phi_o c^4/8\pi)\, G^{(1)}{}_{ik}{}^{|k} = 0 \quad . \tag{13c}$$

Uniqueness of the Kerr Black Hole. Assuming that the background geometry is that of Kerr, consider solutions of the vacuum wave equation $\xi^{,\ell}{}_{|\ell} = 0$. These have been studied by Carter (1968) and by Fackerell and Ipser (1971) (but they were not aware of the application to the large-ω version of the scalar-tensor theory). One of the Fackerell-Ipser results is the following: "In a Kerr background metric with at least one real horizon (i.e., with $|a| \leq M$), the only solution of the vacuum wave equation $\xi^{,\ell}{}_{|\ell} = 0$ which is nonsingular at the horizon and at infinity is the trivial solution ξ = constant." Even this trivial solution is restricted by the requirement that $\phi \to \phi_o$ as $r \to \infty$. Thus, to retain a nonsingular horizon we must have $\xi = 0$, which implies that $G^{(1)}{}_{ik} = 0$. Carter (1971) has shown that all nontrivial, axially symmetric solutions h_{ik} of $G^{(1)}{}_{ik} = 0$ in the Kerr background with $|a| < M$, which are well-behaved at infinity, are singular at the horizon. Hence, both ξ and h_{ik} must vanish if the Kerr horizon is to remain nonsingular.

Uncharged Spherical Collapse Produces a Schwarzschild Black Hole. Our proof of this statement in the large-ω version of the scalar-tensor theory is based on the work of Price (1971a,b). His analysis of a monopole scalar field generated by a "scalar charge" in a general relativistic, spherically collapsing star applies directly to the "$1/\omega$" equation (13b) if one only replaces the gravitation constant G by $1/\phi_o$. It reveals that (i) the "$1/\omega$" field ξ is well-behaved outside and at the Schwarzschild radius; and (ii) ξ is radiated away completely as the collapse proceeds, with its decay at late times dominated by the "tail"

$$\xi(r,t) = \frac{F(r)}{t^2} \quad \text{for} \quad ct \gg \left| r + \frac{2M}{\phi_o c^2} \ln\left(\frac{\phi_o c^2 r}{2M} - 1\right) \right| , \qquad (14a)$$

$$F(r) = A \ln\left(1 - \frac{2M}{\phi_o c^2 r}\right) + B ; \quad A,B = \text{constants}. \qquad (14b)$$

We shall see that the metric perturbations decay in a similar manner, leaving behind a Schwarzschild black hole. To prove this, write the exact spacetime metric outside and at the star's surface in the form

$$ds^2 = -[1 - 2m(r,t)/\phi_o c^2 r]\, e^{\gamma(t)} c^2 dt^2 + \frac{dr^2}{1 - 2m(r,t)/\phi_o c^2 r} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \quad . \qquad (15)$$

(There exist coordinates in which any spherically symmetric system has a line element of this form; see, e.g., §VII-2 of Synge (1966).) The metric functions can be expressed in a purely geometrical manner in terms of the Einstein tensor, without any reference to the field equations (cf. §VII-3 of Synge (1966), but note differing sign conventions for G_{ik}):

$$m(r,t) = \text{const.} - \frac{1}{2}\phi_o c^2 \int^r r^2 G_t{}^t \, dr \tag{16a}$$

$$\gamma(r,t) = \text{const.} + \int^r \frac{r}{1 - 2m/\phi_o c^2 r} (G_r{}^r - G_t{}^t) \, dr. \tag{16b}$$

Now, in our large-ω version of the scalar-tensor theory the zero-order Schwarzschild metric satisfies $G^{(E)}{}_{ik} = 0$. Hence, equation (13a) outside and at the star's surface reads

$$G_i{}^k = \frac{1}{\omega} G^{(1)}{}_i{}^k = \frac{1}{\omega\phi_o{}^2} (\xi_{,i}\xi^{,k} - \frac{1}{2}\delta_i{}^k \xi_{,\ell}\xi^{,\ell} + \phi_o \xi_{,i}{}^{|k}) . \tag{17}$$

By using the zero-order Schwarzschild metric in equation (17) and combining with equations (14) and (16), we obtain

$$m(r,t) = M - \frac{M}{2\omega\phi_o}\xi(r,t) +$$

$$+ \frac{c^2}{4\omega\phi_o} \int_{r_1}^{r} \left[2\phi_o \xi_{,tt} + (\xi_{,t})^2 + \left(1 - \frac{2M}{\phi_o c^2 r}\right)^2 (\xi_{,r})^2 \right] \frac{r^2 dr}{1 - 2M/\phi_o c^2 r}$$

$$= M - \frac{M}{2\omega\phi_o}\frac{F(r)}{t^2} \quad \text{for} \quad ct \gg \left| r + \frac{2M}{\phi_o c^2} \ln\left(\frac{\phi_o c^2 r}{2M} - 1\right) \right| \quad , \tag{18a}$$

$$\gamma(r,t) = \frac{1}{\omega\phi_o}[r\xi_{,r} - \xi] + \frac{1}{\omega\phi_o^2}\int_{r_1}^{r}\left[(\xi_{,r})^2 + \frac{\phi_o\xi_{,tt} + (\xi_{,t})^2}{(1 - 2M/\phi_o c^2 r)^2}\right] r\,dr$$

$$= \frac{1}{\omega\phi_o}[rF'(r) - F(r)]\frac{1}{t^2} \text{ for } ct \gg \left| r + \frac{2M}{\phi_o c^2}\ell n\left(\frac{\phi_o c^2 r}{2M} - 1\right)\right| . \tag{18b}$$

Here $r_1 > 2M/\phi_o c^2$ is a fixed finite value of the radial coordinate. By choosing this lower limit of integration to be neither $r = 2M/\phi_o c^2$ nor $r = \infty$, one insures that the integrals vanish at any fixed r, $2M/\phi_o c^2 < r < \infty$, in the limit $t \to \infty$. The constant M is then the mass of the black hole as measured, after all the waves have passed a distant observer, using Keplerian orbits. Note that our approximate solutions (14) and (18) for the scalar field and metric perturbations in terms of $F(r)$ do not attempt to treat either (i) the region $r \sim ct$ in which the transients of the outgoing scalar waves reside, or (ii) the region $(\phi_o c^2 r/2M) - 1 \lesssim e^{-\phi_o c^3 t/2M}$ in which the collapsing star's surface and the transients of the ingoing scalar waves reside. It is sufficient to consider only the "tail" region, since at arbitrarily late times the boundaries of this region approach arbitrarily close to the Schwarzschild radius and to $r = \infty$.

Chapter 4

EVIDENCE IN HIGHER ORDER PERTURBATION THEORY

The Generalization to Second Order Expansion in $1/\omega$. A significant feature which Brans-Dicke theory shares with GRT is the nonlinearity of its field equations. Thus, any linearized perturbation theory runs the risk of yielding results which suffer from the failure to include some unforeseen, but essential, nonlinear effect. Although the first order $1/\omega$ expansion of BDT used in the preceding section was performed around a fully relativistic, and hence nonlinear, background metric, it included only the linear part of a variable scalar field. Because the time required for gravitational collapse is very brief on the cosmological time scale ($\lesssim 1/10^{17}$), it would be most unreasonable to suppose that taking into account the "variation" of the cosmological value of ϕ during collapse would have any appreciable effect. It is not intuitively obvious that we can make the same remark about variable parts of the scalar field of higher order in $1/\omega$. However, we shall show that the results of the preceding chapter carry over unchanged in quality in a second order $1/\omega$ expansion of BDT. Furthermore, we shall see that this generalization to second order is the key to inductive arguments about the uniqueness and necessary formation of black holes to all orders in a $1/\omega$ expansion.

To carry out the second order expansion, we assume that the stress-energy-momentum tensor, the metric, and the scalar field can be written in the forms

$$T_{ik} = T^{(E)}{}_{ik} + (1/\omega)t^{(1)}{}_{ik} + (1/\omega^2)t^{(2)}{}_{ik} + O(1/\omega^3) \quad , \qquad (19a)$$

$$g_{ik} = g^{(E)}{}_{ik} + (1/\omega)h^{(1)}{}_{ik} + (1/\omega^2)h^{(2)}{}_{ik} + O(1/\omega^3) \quad , \qquad (19b)$$

$$\phi = \phi_o + (1/\omega)\xi^{(1)} + (1/\omega^2)\xi^{(2)} + O(1/\omega^3) \quad , \qquad (19c)$$

$$\phi_o = \text{"cosmological value of } \phi\text{" , a constant.} \qquad (19d)$$

Using equations (19) in equations (1) and (2) is a tedious but straightforward exercise leading to field equations which can be written in the following form:

$$G^{(E)}{}_{ik} = (8\pi/\phi_o c^4)T^{(E)}{}_{ik} \quad , \qquad (20)$$

$$\xi^{(1),\ell}{}_{|\ell} = (4\pi/c^4)\, T^{(E)} \quad , \qquad (21a)$$

$$\begin{aligned} G^{(1)}{}_{ik} &= (8\pi/\phi_o c^4)[-(\xi/\phi_o)T^{(E)}{}_{ik} + t^{(1)}{}_{ik}] \\ &+ (1/\phi_o^2)(\xi^{(1)}{}_{,i}\xi^{(1)}{}_{,k} - \tfrac{1}{2}\, g^{(E)}{}_{ik}\xi^{(1)}{}_{,\ell}\xi^{(1),\ell} \\ &+ (1/\phi_o)(\xi^{(1)}{}_{,i|k} - g^{(E)}{}_{ik}\xi^{(1),\ell}{}_{|\ell}) \quad , \end{aligned} \qquad (21b)$$

$$\begin{aligned} \xi^{(2),\ell}{}_{|\ell} &= (4\pi/c^2)(-\tfrac{3}{2}\, T^{(E)} + t^{(1)} + h^{(1)ik}T^{(E)}{}_{ik}) \\ &+ \tfrac{1}{2}\, g^{(E)\ell m}[h^{(1)kp}(2g^{(E)}{}_{p\ell,m} - g^{(E)}{}_{\ell m,p}) \\ &+ g^{(E)kp}(2h^{(1)}{}_{p\ell,m} - h^{(1)}{}_{\ell m,p})]\xi^{(1)}{}_{,k} \\ &- h^{(1)\ell m}\xi^{(1)}{}_{,\ell|m} \quad , \end{aligned} \qquad (22a)$$

$$G^{(2)}{}_{ik} = (8\pi/\phi_o c^4)\ \{-[\xi^{(2)}/\phi_o - (\xi^{(1)}/\phi_o)^2]\ T^{(E)}{}_{ik}$$

$$- (\xi^{(1)}/\phi_o)t^{(1)}{}_{ik} + t^{(2)}{}_{ik}\}$$

$$+ (1/\phi_o{}^2)\{-2(\xi^{(1)}/\phi_o)\xi^{(1)}{}_{,i}\xi^{(1)}{}_{,k} + \xi^{(1)}{}_{,i}\xi^{(2)}{}_{,k} + \xi^{(2)}{}_{,i}\xi^{(1)}{}_{,k}$$

$$- \tfrac{1}{2}[-2(\xi^{(1)}/\phi_o)g^{(E)}{}_{ik}\xi^{(1)}{}_{,\ell}\xi^{(1),\ell} + 2g^{(E)}{}_{ik}\xi^{(1)}{}_{,\ell}\xi^{(2),\ell}$$

$$+ g^{(E)}{}_{ik}\xi^{(1)}{}_{,\ell}h^{(1)\ell m}\xi^{(1)}{}_{,m} + h^{(1)}{}_{ik}\xi^{(1)}{}_{,\ell}\xi^{(1),\ell}]\}$$

$$+ (1/\phi_o)\{\xi^{(2)}{}_{,i|k} - g^{(E)}{}_{ik}\xi^{(2),\ell}{}_{|\ell} - (\xi^{(1)}/\phi_o)\ [\xi^{(1)}{}_{,i|k}$$

$$- g^{(E)}{}_{ik}\ \xi^{(1),\ell}{}_{|\ell}] - h^{(1)}{}_{ik}\ \xi^{(1),\ell}{}_{|\ell}$$

$$- \frac{1}{2}\ \xi^{(1)}{}_{,\ell}[h^{(1)\ell m}(g^{(E)}{}_{mi,k} + g^{(E)}{}_{mk,i} - g^{(E)}{}_{ik,m})$$

$$+ g^{(E)\ell m}(h^{(1)}{}_{mi,k} + h^{(1)}{}_{mk,i} - h^{(1)}{}_{ik,m})]$$

$$+ \frac{1}{2}\ g^{(E)}{}_{ik}\ g^{(E)\ell m}\ \xi^{(1)}{}_{,r}[h^{(1)rp}(2g^{(E)}{}_{p\ell,m} - g^{(E)}{}_{\ell m,p})$$

$$+ g^{(E)rp}(2h^{(1)}{}_{p\ell,m} - h^{(1)}{}_{\ell m,p})] - g^{(E)}{}_{ik}\ h^{(1)\ell m}\ \xi^{(1)}{}_{,\ell|m}\} \qquad (22b)$$

As before, the zero-order "Einstein metric" $g^{(E)}{}_{ik}$ is used to raise and lower all indices, and a slash denotes a covariant derivative with respect to $g^{(E)}{}_{ik}$. The expansion of the metric (19b) implies that the Einstein tensor has the expansion

$$G_{ik} = G^{(E)}{}_{ik} + (1/\omega)\ G^{(1)}{}_{ik} + (1/\omega^2)\ G^{(2)}{}_{ik} + O(1/\omega^3) \quad .$$

We may think of these equations in the following way: As shown by equation (20), the zero-order stress-energy-momentum $T^{(E)}{}_{ik}$ produces the "Einstein metric" $g^{(E)}{}_{ik}$ in the usual general relativistic manner. The first-order perturbations $t^{(1)}{}_{ik}$, $h^{(1)}{}_{ik}$, and $\xi^{(1)}$ "reside" and evolve in this "background" spacetime $g^{(E)}{}_{ik}$ in accordance with equations (21). The second-order perturbations $t^{(2)}{}_{ik}$, $h^{(2)}{}_{ik}$, and $\xi^{(2)}$ also "reside" in the zero-order spacetime, but equations (22) show that their evolution is also influenced by the first-order perturbations.

One consequence of the field equations (1),(2) is $T_{ik}{}^{;k} = 0$. Calculated using equations (20),(21), and (22), this becomes $T^{(E)}{}_{ik}{}^{|k} = 0$ and

$$t^{(1)}{}_{ik}{}^{|k} = (\phi_o c^4/8\pi)\, G^{(1)}{}_{ik}{}^{|k} \quad , \tag{21c}$$

$$t^{(2)}{}_{ik}{}^{|k} = (\phi_o c^4/8\pi)\, [G^{(2)}{}_{ik}{}^{|k} + (\xi^{(1)}/\phi_o) G^{(1)}{}_{ik}{}^{|k}] \ . \tag{22c}$$

When applied to a vacuum ($T^{(E)}{}_{ik} = 0 = t^{(1)}{}_{ik} = t^{(2)}{}_{ik}$), equations (20)-(22) reduce to:

$$G^{(E)}{}_{ik} = 0 \quad , \tag{23}$$

$$\xi^{(1),\ell}{}_{|\ell} = 0 \quad , \tag{24a}$$

$$G^{(1)}{}_{ik} = (1/\phi_o{}^2)(\xi^{(1)}{}_{,i}\, \xi^{(1)}{}_{,k} - \tfrac{1}{2} g^{(E)}{}_{ik} \xi^{(1)}{}_{,\ell} \xi^{(1),\ell}) + (1/\phi_o)\xi^{(1)}{}_{,i|k} \quad , \tag{24b}$$

$$G^{(1)}{}_{ik}{}^{|k} = 0 \quad , \tag{24c}$$

$$\xi^{(2),\ell}{}_{|\ell} = \frac{1}{2}\, g^{(E)\ell m}[h^{(1)kp}(2g^{(E)}{}_{p\ell,m} - g^{(E)}{}_{\ell m,p})$$

$$+ g^{(E)kp}(2h^{(1)}{}_{p\ell,m} - h^{(1)}{}_{\ell m,p})]\xi^{(1)}{}_{,k} - h^{(1)\ell m}\xi^{(1)}{}_{,\ell|m}$$

(25a)

$$G^{(2)}{}_{ik} = (1/\phi_o{}^2)\{-2(\xi^{(1)}/\phi_o)\xi^{(1)}{}_{,i}\,\xi^{(1)}{}_{,k} + \xi^{(1)}{}_{,i}\,\xi^{(2)}{}_{,k}$$

$$+ \xi^{(2)}{}_{,i}\,\xi^{(1)}{}_{,k} - \frac{1}{2}[-2(\xi^{(1)}/\phi_o)g^{(E)}{}_{ik}\,\xi^{(1)}{}_{,\ell}\,\xi^{(1),\ell}$$

$$+ 2g^{(E)}{}_{ik}\,\xi^{(1)}{}_{,\ell}\,\xi^{(2),\ell} + g^{(E)}{}_{ik}h^{(1)\ell m}\,\xi^{(1)}{}_{,\ell}\,\xi^{(1)}{}_{,m}$$

$$+ h^{(1)}{}_{ik}\,\xi^{(1)}{}_{,\ell}\,\xi^{(1),\ell}]\} + (1/\phi_o)\{\xi^{(2)}{}_{,i|k} - (\xi^{(1)}/\phi_o)\xi^{(1)}{}_{,i|k}$$

$$- \frac{1}{2}\,\xi^{(1)}{}_{,\ell}[h^{(1)\ell m}(g^{(E)}{}_{mi,k} + g^{(E)}{}_{mk,i} - g^{(E)}{}_{ik,m})$$

$$+ g^{(E)\ell m}(h^{(1)}{}_{mi,k} + h^{(1)}{}_{mk,i} - h^{(1)}{}_{ik,m})]\}$$

(25b)

$$G^{(2)}{}_{ik}{}^{|k} = 0 \quad . \qquad (25c)$$

Consider the Kerr background metric, which is the solution of equation (23) for an uncharged rotating black hole in GRT. In the background, recall the Fackerell-Ipser (1971) theorem regarding solutions of equation (24a): "In a Kerr background metric with at least one real horizon (i.e., with $|a| \leq M$), the only solution which is nonsingular at the horizon and at infinity is the trivial solution ξ = constant." Since we require that $\phi \to \phi_o$ as $r \to \infty$, we must have $\xi = 0$. But, this implies that equation (24b) reduces to $G^{(1)}{}_{ik} = 0$.

Now, Carter's (1971) work on axially symmetric solutions of this equation in the Kerr background with $|a| < M$ shows that all nontrivial solutions which are well-behaved at infinity are singular at the horizon. Thus $h^{(1)}{}_{ik}$ also vanishes if the horizon[3] is to remain nonsingular. But then equation (25a) reduces to $\xi^{(2),\ell}{}_{|\ell} = 0$ and the Fackerell-Ipser theorem again applies, so that $\xi^{(2)} = 0$. The vanishing of $\xi^{(2)}$, $\xi^{(1)}$, and $h^{(1)}{}_{ik}$ is sufficient to reduce (25b) to the form $G^{(2)}{}_{ik} = 0$, and another application of Carter's result shows that $h^{(2)}{}_{ik} = 0$. To summarize: The only axially symmetric[3] solution of equations (24),(25) in a Kerr background metric with $|a| < M$ which is nonsingular at the horizon and at infinity, and in which the scalar field takes on its proper asymptotic value $\phi \to \phi_o$ as $r \to \infty$ is the trivial solution

$$\xi^{(1)} = \xi^{(2)} = 0 \tag{26a}$$

$$h^{(1)}{}_{ik} = h^{(2)}{}_{ik} = 0 \quad . \tag{26b}$$

It is obvious that the work of the preceding chapter on uncharged spherical collapse necessarily producing a Schwarzschild black hole could also be extended to the second-order $1/\omega$ expansion, since examination of equation (25b) shows that

$$\xi^{(1)}(r,t) = F(r)/t^2 \tag{27a}$$

$$h^{(1)}{}_{ik}(r,t) = H^{(1)}{}_{ik}(r)/t^2 \tag{27b}$$

[3] Recent work by Ipser (1971, private communication) shows that this result can be established without requiring axial symmetry.

implies that

$$\xi^{(2)}(r,t) = F^{(2)}(r)/t^2 \tag{28a}$$

$$h^{(2)}{}_{ik}(r,t) = H^{(2)}{}_{ik}(r)/t^2 \tag{28b}$$

for sufficiently late times, i.e., for

$$ct \gg \left| r + \frac{2M}{\phi_o c^2} \ln\left(\frac{\phi_o c^2 r}{2M} - 1\right) \right|$$

in the "tail" region of the gravitational waves.

Uniqueness of the Kerr Black Hole to All Orders in $1/\omega$. The "trick" which allowed us to write the second order expansion in the form (22) is the identity

$$\xi^{(2);\ell}{}_{;\ell} = \xi^{(2)|\ell}{}_{|\ell} - \frac{1}{2} g^{(E)\ell m}[h^{(1)kp}(g^{(E)}{}_{p\ell,m} + g^{(E)}{}_{pm,\ell}$$
$$- g^{(E)}{}_{\ell m,p}) + g^{(E)kp}(h^{(1)}{}_{p\ell,m} + h^{(1)}{}_{pm,\ell} - h^{(1)}{}_{\ell m,p})]\, \xi^{(1)}{}_{,k}$$
$$+ \frac{1}{\omega} h^{(1)\ell m}\, \xi^{(1)}{}_{,\ell|m} + O(1/\omega^2) \quad . \tag{29}$$

This split of the full covariant d'Alembertian into a covariant d'Alembertian taken with respect to the zero-order metric plus terms involving derivatives of all lower-order perturbations is always possible. Thus, the generalization to an arbitrary order of the results of the preceding section is straightforward. In the formal language of a proof by mathematical induction, the argument for the uniqueness of the Kerr black hole to all orders runs as follows:

(i) Expand the stress-energy-momentum tensor, the metric, and the scalar field in the forms

$$T_{ik} = T^{(E)}{}_{ik} + \sum_{n=1}^{\infty} t^{(n)}{}_{ik}/\omega^n \quad , \tag{30a}$$

$$g_{ik} = g^{(E)}{}_{ik} + \sum_{n=1}^{\infty} h^{(n)}{}_{ik}/\omega^n \quad , \tag{30b}$$

$$\phi = \phi_o + \sum_{n=1}^{\infty} \xi^{(n)}/\omega^n \quad , \tag{30c}$$

where

$$\phi_o = \text{"cosmological value of } \phi\text{", a constant.} \tag{30d}$$

Use this expansion in the field equations (1),(2) in vacuum. Let $g^{(E)}{}_{ik}$ be the Kerr solution of the zero-order equation (23).

(ii) If a scalar field perturbation satisfies

$$\xi^{(n)|\ell}{}_{|\ell} = 0 \tag{31a}$$

for any n, then the Fackerell-Ipser theorem and the asymptotic form of ϕ require that $\xi^{(n)}$ vanish if $|a| \leq M$ and both infinity and the event horizon are to be nonsingular.

(iii) If a metric perturbation satisfies

$$G^{(n)}{}_{ik} = 0 \tag{31b}$$

and if one has already proved $\xi^{(b)} = 0 = h^{(b)}{}_{ik}$ for $b=1,\cdots,n-1$, for any n, then Carter's theorem requires that $h^{(n)}{}_{ik}$ vanish if $|a| < M$, if the solution is assumed to be axially symmetric[4] and if both infinity and the event horizon are to be nonsingular.

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[4] Recall that Ipser's work shows that the requirement of axial symmetry may be relaxed.

(iv) The field equations derived from using (30) in (1),(2) guarantee that equations (31) are satisfied if

$$\xi^{(b)} = 0 = h^{(b)}_{ik} \quad \text{for} \quad b = 1,\cdots,n-1 \tag{32}$$

(v) But equations (31) are true if $n = 1$. Thus (ii), (iii), and (iv) guarantee that they are true for <u>all</u> n. Hence,

$$\xi^{(n)} = 0 = h^{(n)}_{ik} \quad \underline{\text{for all } n} \tag{33}$$

if $|a| < M$, if the solution is assumed to be axially symmetric[5], and if both infinity and the event horizon are to be nonsingular.

<u>Uncharged Spherical Collapse Produces a Schwarzschild Black Hole to All Orders in $1/\omega$.</u> The mathematical induction argument is an obvious generalization of remarks surrounding equations (27),(28):

(i) As before, use the expansion (30) in the field equations (1),(2). Then for the Schwarzschild background,

$$\xi^{(n)}(r,t) = F^{(n)}(r)/t^2 \tag{34a}$$

$$h^{(n)}_{ik}(r,t) = H^{(n)}_{(r)}/t^2 \tag{34b}$$

for any n at sufficiently late times

$$ct \gg \left| r + \frac{2M}{\phi_o c^2} \ln\left(\frac{\phi_o c^2 r}{2M} - 1\right) \right| \tag{34c}$$

provided that $\xi^{(b)}$ and $h^{(b)}_{ik}$ take on the forms (34) for all $b = 1,\cdots,n-1$.

[5] Ipser's (1971, private communication) work shows that the requirement of axial symmetry may be relaxed.

(ii) But equations (34) are satisfied in the "tail" region for $n = 1$.

(iii) Hence, (i) and (ii) guarantee that the scalar field and metric perturbations of the Schwarzschild background are of the forms (34) <u>for all n</u>. Since the boundaries of the "tail" region approach arbitrarily close to the Schwarzschild radius and to $r = \infty$ at sufficiently late times, we see that the dynamics of uncharged spherical collapse necessarily produces a Schwarzschild black hole.

APPENDICES

Appendix A

ON THE SIMPLICITY OF THE BRANS-DICKE THEORY OF GRAVITATION

It is well known (cf. Landau and Lifshitz, 1962, pp.313-315) that Einstein's field equations of general relativity theory (GRT)

$$R_{ik} - \frac{1}{2} g_{ik} R = (8\pi G/c^4)\, T_{ik} \tag{A1}$$

can be derived from the variational principle $\delta S = 0$, with

$$S = \frac{1}{c} \int \left(\frac{c^4}{16\pi\, G} R + L \right) \sqrt{-g}\, d^4x \quad . \tag{A2}$$

Here R_{ik} is the Ricci tensor with trace R formed from the metric tensor g_{ik} which has determinant g . The stress-energy-momentum tensor T_{ik} is derived from the Lagrangian density L for matter and nongravitational fields (e.g., electromagnetism) in the familiar manner

$$T_{ik} = \frac{2}{\sqrt{-g}} \left[\frac{\partial}{\partial x^{\ell}} \frac{\partial(\sqrt{-g}\, L)}{\partial g^{ik}_{\ \ ,\ell}} - \frac{\partial(\sqrt{-g}\, L)}{\partial g^{ik}} \right] \quad . \tag{A3}$$

As an alternative to the customary geometric interpretation we may take the view that GRT describes gravitation as a field derived from a (second-rank) tensor potential g_{ik} . From this perspective it is natural to ask whether potentials having other transformation properties may also contribute to the gravitational interaction, and if so what is the theory with the least complexity beyond GRT. We shall see that quite general assumptions and natural criteria of "simplicity" lead directly to the scalar-tensor theory of Brans and Dicke.

Since we seek to generalize GRT with the least complication, it will serve us well to list here a set of assumptions from which GRT may be derived. (See Misner, Thorne, and Wheeler (1971) for the explicit derivation from a set of assumptions completely equivalent to these, although worded somewhat differently.)

[1] The strong equivalence principle: In local inertial reference frames, special relativity holds. Thus there is a metric, free particles move along its geodesics, and clocks and rods measure proper times and proper lengths of the metric.

[2] The correspondence principle: Under Newtonian conditions the field equations of the theory reduce to $\nabla^2\Phi = 4\pi G\rho$.

[3] The field equations of the theory may be derived from an invariant action principle. This assumption is suggested by the requirements of quantum mechanics and the close relation between variational principles and conservation laws. Also, this assumption will guarantee that the theory has a well-posed initial-value problem.

[4] The field equations are as simple as possible:

(a) They are a set of second-order partial differential equations.

(b) They are linear in the second derivatives.

[5] The gravitational fields are generated by the stress-energy-momentum tensor T_{ik} of matter and nongravitational fields. If $T_{ik} = 0$, one solution of the field equations in a suitable coordinate frame is $g_{ik} = \eta_{ik}$.

[6] The only gravitational field is the (second-rank tensor) spacetime metric.

[7] The theory agrees with experiment. This requirement is necessary to rule out, e.g., Nordström's theory (see Appendix C), which

satisfies [1]-[6] but predicts <u>no</u> deflection for light in the gravitational field.

Of these assumptions, it is clear that [4] and [6] are the simplicity requirements. It is obvious that we tamper with [4] only at the peril of introducing monstrous complexity into the field equations, and so we shall accept assumption [4] unchanged. The modification of [6] introducing the minimum of new complexity is obviously:

[6'] The only gravitational fields are the spacetime metric and a scalar (zero-rank tensor) field.

The <u>simplest</u> source equation for the new field ϕ postulated in [6'], compatible with assumption [4], is

$$\phi^{,\ell}{}_{;\ell} = (\text{invariant derived from } L \text{ independent of } \phi) \qquad \text{(A4)}$$

Although assumption [4] does not <u>demand</u> that the right-hand-side of equation (A4) be independent of ϕ, this is clearly the simplest choice which we can make. It remains to be seen that equation (A4) is consistent with <u>all</u> of the assumptions [1]-[5], [6'], and [7]. In fact, we shall see that equation (A4) taken together with these assumptions leads to Brans-Dicke theory.

The most general form of action from which we can have any hope of deriving (A4) is

$$S = \int [f(\phi)R + k(\phi)\, L + h(\phi)\phi_{,i}\phi_{,k}\, g^{ik}]\, \sqrt{-g}\, d^4x \qquad \text{(A5)}$$

Carrying out the variation with respect to both metric and scalar fields is a straightforward exercise which leads to the field equations

$$R_{ik} - \frac{1}{2} g_{ik}R = (k/2f)T_{ik} + (f'/f)(\phi_{,i;k} - g_{ik}\phi^{,\ell}{}_{;\ell})$$
$$+ (f''/f)(\phi_{,i}\phi_{,k} - g_{ik}\phi_{,\ell}\phi^{,\ell})$$
$$-(h/f)(\phi_{,i}\phi_{,k} - \frac{1}{2} g_{ik}\phi_{,\ell}\phi^{,\ell})$$
$$+ \frac{k'}{f} \frac{\phi_{,\ell}}{\sqrt{-g}} \frac{\partial(\sqrt{-g}\, L)}{\partial g^{ik}{}_{,\ell}} \tag{A6}$$

$$2h\phi^{,\ell}{}_{;\ell} + h'\phi_{,\ell}\phi^{,\ell} = f'R + k'L \tag{A7}$$

Here, primes denote differentiation with respect to ϕ . Using the contracted form of equation (A6) to eliminate the scalar curvature in favor of matter as source in equation (A7) yields

$$(2h - 3f'^2/f)\phi^{,\ell}{}_{;\ell} + (h' + hf'/f - 3f'f''/f)\phi_{,\ell}\phi^{,\ell}$$
$$= - \frac{f'}{f} [\frac{1}{2} kT + k' \frac{\phi_{,\ell}}{\sqrt{-g}} g^{mp} \frac{\partial(\sqrt{-g}\, L)}{\partial g^{mp}{}_{,\ell}}] + k'L \tag{A8}$$

To identify this with equation (A4) we must satisfy the identities

(i) $f'k'/f = 0$,

(ii) $h' + hf'/f - 3f'f''/f = 0$,

(iii) $f'k/f = \alpha(2h - 3f'^2/f)$,

(iv) $k' = \beta(2h - 3f'^2/f)$,

where α and β are constants, not both zero. Comparing (i), (iii), and (iv) shows that at least one of α and β must be zero. Thus, we have two cases to consider.

<u>Case I. $\alpha = 0, \beta \neq 0$:</u> Then, since $k \neq 0$, (iii) implies $f' = 0$ [Without loss of generality (because of the arbitrariness of β) we can choose $f = 1.$], and (i) is automatically satisfied. Now (iv) implies $k' = 2\beta h$ and (ii) reduces to $h' = 0$, implying that h is a constant. Thus $k(\phi) = 2\beta h(\phi + \varepsilon)$, where ε is a constant of integration, so that equation (A8) becomes

$$\phi^{,\ell}{}_{;\ell} = \beta(\phi + \varepsilon)\, L \quad . \tag{A9}$$

However, since $k'(\phi) \neq 0$, free particles do <u>not</u> move along geodesics, so that this theory violates assumption [1] and will not be considered further.

<u>Case II. $\alpha \neq 0, \beta = 0$:</u> Then, (iv) implies k = constant and (i) is automatically satisfied. Now (iii) implies $h = \frac{1}{2}(3 + k/\alpha)(f'/f)$. As a matter of notational convenience, let us give the combination of constants in this equation a simple name:

$$\omega = -\frac{1}{2}(3 + k/\alpha) \quad . \tag{A10}$$

Thus $h = -\omega f'/f$ implies that $h' = -\omega f''/f + \omega f'^2/f^2$, so that (ii) reduces to $(\omega + 3f')f'' = 0$. Thus, f' = constant. Without loss of generality (because of the arbitrariness of ω) we can choose $f' = 1$. Thus $f(\phi) = \phi + \varepsilon$, $h(\phi) = \frac{-\omega}{\phi+\varepsilon}$, where ε is a constant of integration. Clearly ε merely sets the "zero-point" of ϕ , which should be established by boundary conditions on the field equations. Thus, without loss of generality we can set $\varepsilon = 0$. Then the action (A5) and the field equations (A6),(A8) reduce to

$$S = \int [\phi R + kL - (\omega/\phi)\phi_{,\ell}\phi^{,\ell}] \sqrt{-g}\, d^4x \quad , \tag{A11}$$

$$R_{ik} - \frac{1}{2} g_{ik}R = (k/2\phi)T_{ik} + (\omega/\phi^2)(\phi_{,i}\phi_{,k} - \frac{1}{2} g_{ik}\phi_{,\ell}\phi^{,\ell}) + (1/\phi)(\phi_{,i;k} - g_{ik}\phi^{,\ell}{}_{;\ell}) \quad , \tag{A12}$$

$$\phi^{,\ell}{}_{;\ell} = \frac{k}{2(2\omega+3)} T \quad , \tag{A13}$$

respectively. Imposing the correspondence principle (2) determines the constant $k = 16\pi/c^4$ and the correspondence $\phi \leftrightarrow 1/G$. This is indeed the Brans-Dicke scalar-tensor theory of gravitation, although our derivation of it has been along quite different lines from their original motivation (Brans and Dicke, 1961).

Incidentally, we may note that the failure of assumption [4] to uniquely impose a source equation for ϕ is reflected in the fact that we could use a transformed scalar field variable to arrive at a formally somewhat more complicated equation than (A13) in a completely equivalent scalar-tensor theory, such as the special case of Jordan's theory discussed in Appendix B.

Appendix B

BRANS-DICKE THEORY AS A SPECIAL CASE OF JORDAN'S THEORY

As an outgrowth of attempts to create a unified field theory of gravitation and electromagnetism, Pascual Jordan (1955,1959) proposed a scalar-tensor theory based on the variational principle $\delta S = 0$, with

$$S = \int \kappa^{\eta}[R + \frac{2\kappa}{c^2} L + \frac{\zeta}{\kappa^2} \kappa_{,i}\kappa_{,k} g^{ik}] \sqrt{-g}\, d^4x \quad . \tag{B1}$$

As usual, R_{ik} is the Ricci tensor, with trace R, formed from the metric tensor g_{ik}, which has determinant g. The scalar field is κ, while η and ζ are (as yet undetermined) constants. (If we set $\kappa = 8\pi G/c^2$, "Einstein's gravitational constant", the theory reduces to general relativity.) In Jordan's original formulation (1955), it was assumed that L was the Lagrangian density for electromagnetic fields, so that only the traceless stress-energy-momentum tensor

$$T_{ik} = (1/4\pi)(f_{i\ell}f_k{}^{\ell} - \frac{1}{4} g_{ik}f_{\ell m}f^{\ell m}) \tag{B2}$$

was considered. Later work in this theory assumed that L could represent any matter or nongravitational fields, with stress-energy-momentum derived in the familiar manner

$$T_{ik} = \frac{2}{\sqrt{-g}} [\frac{\partial}{\partial x^{\ell}} \frac{\partial(\sqrt{-g}\, L)}{\partial g^{ik}{}_{,\ell}} - \frac{\partial(\sqrt{-g}\, L)}{\partial g^{ik}}] \quad . \tag{B3}$$

When we speak of Jordan's theory we mean this latter form of it, including matter as well as electromagnetism.

Carrying out the variation of equation (B1) with respect to both metric and scalar fields leads to the field equations

$$R_{ik} - \frac{1}{2} g_{ik}R = (\kappa/c^2)T_{ik} + (\eta/\kappa)(\kappa_{,i;k} - g_{ik}\kappa^{,\ell}{}_{;\ell})$$
$$+ \frac{1}{\kappa^2} \{[\eta(\eta-1)- \zeta]\kappa_{,i}\kappa_{,k} - [\eta(\eta-1) - \tfrac{\zeta}{2}]\, g_{ik}\kappa_{,\ell}\kappa^{,\ell}\}$$
$$+ \frac{2}{c^2} (\eta+1) \frac{\kappa_{,\ell}}{\sqrt{-g}} \frac{\partial(\sqrt{-g}\, L)}{\partial g^{ik}{}_{,\ell}} \quad , \tag{B4}$$

$$\frac{2\zeta}{\kappa} \kappa^{,\ell}{}_{;\ell} + \frac{\zeta(\eta-2)}{\kappa^2} \kappa_{,\ell}\kappa^{,\ell} - \eta R = \frac{2\kappa}{c^2} (\eta+1)\, L \quad . \tag{B5}$$

Using the contracted form of equation (B4) to eliminate the scalar curvature in favor of matter as source in equation (B5) yields

$$(2\zeta - 3\eta^2)(\frac{1}{\kappa} \kappa^{,\ell}{}_{;\ell} + \frac{\eta-1}{\kappa^2} \kappa_{,\ell}\kappa^{,\ell}) = \frac{-\eta\kappa}{c^2}\, T$$
$$+ \frac{2}{c^2}(\eta+1)[\kappa L - \frac{\kappa_{,\ell}}{\sqrt{-g}}\, g^{mp} \frac{\partial(\sqrt{-g}\, L)}{\partial g^{mp}{}_{,\ell}}] \quad . \tag{B6}$$

The explicit appearance of the Lagrangian density on the right-hand-sides of equations (B4)-(B6) suggests that, in general, conservation of energy-momentum may be violated in this theory. To prove this, consider an infinitesimal transformation

$$x^{i'} = x^i + \chi^i \tag{B7}$$

of the spacetime coordinates, which induces the variations

$$\delta g^{ik} = \chi^{i;k} + \chi^{k;i} \tag{B8a}$$

$$\delta\kappa = \chi^i \kappa_{,i} \tag{B8b}$$

in the metric and scalar fields. When applied to (B1), this "trivial" variation of scalar and metric fields yields

$$(\kappa^{\eta+1} T_{ik})^{;k} = -(\eta+1)\kappa^{\eta}\kappa_{,i} L \tag{B9}$$

as the "equations of motion" for energy-momentum. The nonvanishing right-hand-side of equation (B9) shows that energy-momentum is created (or destroyed, depending on the value of η) unless

$$\eta = -1 \quad . \tag{B10}$$

Only in this special case do we have local conservation of energy momentum. Indeed, for the value $\eta=-1$, equation (B9) reduces to $T_{ik}{}^{;k} = 0$, the same equations of motion as in general relativity. If equation (B10) holds, the action (B1) and field equations (B4),(B6) become

$$S = \int [\kappa^{-1}R + (2/c^2)L + (\zeta/\kappa^3)\kappa_{,i}\kappa_{,k}g^{ik}]\sqrt{-g}\, d^4x \quad , \tag{B11}$$

$$R_{ik} - \frac{1}{2} g_{ik}R = (\kappa/c^2)\, T_{ik} - \kappa^{-1}(\kappa_{,i;k} - g_{ik}\kappa^{,\ell}{}_{;\ell})$$
$$+ \frac{1}{\kappa^2} [2-\zeta)\kappa_{,i}\kappa_{,k} - \frac{1}{2}(4-\zeta)g_{ik}\kappa_{,\ell}\kappa^{,\ell}] \quad , \tag{B12}$$

$$\frac{1}{\kappa}\kappa^{,\ell}{}_{;\ell} - \frac{2}{\kappa^2}\kappa_{,\ell}\kappa^{,\ell} = \frac{\kappa}{(2\zeta-3)c^2} T \quad . \tag{B13}$$

Now, if we redefine the scalar field coupling constant as

$$\omega = -\zeta \tag{B14}$$

and perform an elementary inverse transformation

$$\phi = (8\pi/c^2)\ \kappa^{-1} \tag{B15}$$

on the scalar field, the equations (B11)-(B13) become formally identical to the corresponding equations of Brans-Dicke theory. [That is, equations (A11)-(A13) of Appendix A.] Thus, the only special case of Jordan's theory which does not violate the law of conservation of energy-momentum is completely equivalent to the Brans-Dicke theory. The fact that BDT is a special case of Jordan's theory was known to the authors of BDT, as evidenced by a footnote in their original paper (Brans and Dicke, 1961). But they remarked on this without presenting proof or discussing the unique status of this special case with respect to conservation laws.

Appendix C

ON THE IMPOSSIBILITY OF BLACK HOLES IN NORDSTRÖM'S THEORY OF GRAVITY

Before Albert Einstein (1916) presented the foundations of the general theory of relativity, Gunnar Nordström (1912,1913) proposed a relativistic theory of gravity based upon a spacetime metric which, described in modern terminology, is conformally flat:

$$g_{ik} = \phi^2 \eta_{ik} \quad . \tag{C1}$$

Here, η_{ik} is the Minkowski metric of special relativity, and the scalar function ϕ satisfies the field equation

$$\phi^{-3} \Box \phi = -(4\pi G/c^4)\ T \ , \tag{C2}$$

where $\Box \phi = \underset{\sim}{\nabla}^2 \phi - \ddot{\phi}$ is the d'Alembertian of ϕ and T is the trace of the stress-energy-momentum tensor of matter.

It was quickly recognized (Einstein and Fokker, 1914) that Nordström's theory of gravity is a Riemann-geometric theory in which the field equation (C2) can be written as

$$R = (24\pi G/c^4)\ T \tag{C3}$$

if one assumes that the scalar curvature R is that derived from a metric of the special form (C1).

But the field equation (C3) can be derived from the variational principle $\delta S = 0$, with

$$S = c^{-1} \int [L - (c^4/48\pi G)\ R(\phi^2 \eta_{ik})]\ \phi^4 d^4x \quad . \tag{C4}$$

For the metric form (C1), the matter term in the action (C4) is of the same form as in GRT. Thus, it is immediately apparent that the "equations of motion" for matter are the familiar ones

$$T_{ik}{}^{;k} = 0 \quad , \tag{C5}$$

the vanishing of the covariant divergence of the stress-energy-momentum tensor.

Thus, it is clear that Nordström's theory satisfies the first six of the seven assumptions (listed in Appendix A) from which general relativity may be derived:

[1] The strong equivalence principle is satisfied because this is a Riemann-geometric theory and test particles move on geodesics, as shown by equation (C5).

[2] Equation (C2) reduces to the Newtonian field equation under the appropriate conditions if we make the identification

$$\phi = 1 + \Phi_{\text{Newtonian}} \quad .$$

[3] The field equations are derived from the invariant action (C4).

[4] The field equations (C2) are simple:

(a) They are second order differential equations.

(b) They are linear in the second derivatives.

[5] In the absence of matter or nongravitational fields, equations (C3) reduce to $R = 0$, which obviously have flat spacetime as one solution.

[6] The only gravitational field is the metric (C1). Unfortunately, this theory disagrees with experiment, since it is clear from

equation (C1) and the fact that light travels on null geodesics, that a ray of light will experience no gravitational deflection on passing near a massive body, such as the sun. [Even if one accepts a recent report (Sramek, 1971) indicating that the bending of light in the sun's gravitational field may be only $(90 \pm 5)\%$ of the value predicted by GRT, the fact that there is some relativistic deflection is very well established by experiments.]

However, as a question of principle it is interesting to inquire whether black holes are possible in this theory. When ϕ is static, as in the vacuum surrounding a possible black hole, the field equation (C2) reduces to Laplace's equation. The general spherically symmetric solution is $\phi = A + B/r$. Requirements of asymptotic flatness of the metric and correspondence with Newtonian theory at large distances fix the constants A and B . Thus,

$$\phi = 1 - \frac{Gm}{c^2 r} \tag{C6}$$

is the general vacuum solution for a spherically symmetric configuration with active gravitational mass m . From (C6) and (C1), it is clear that the metric component g_{oo} vanishes at

$$r_1 = Gm/c^2 \quad , \tag{C7}$$

which thus defines a "surface of infinite red shift." But for the solution (C6), the invariant

$$R^{ik\ell m} R_{ik\ell m} = \frac{8G^2 m^2}{c^4 r^6} \frac{1}{(1 - Gm/c^2 r)^6} \quad , \tag{C8}$$

constructed from the full Riemann curvature tensor, diverges as

$r \rightarrow r_1$. Thus, an observer falling toward $r = r_1$ would experience indefinitely increasing tidal gravitational forces, but would be able to send light signals to an arbitrarily distant receiver from any $r > r_1$. If a source is entirely contained within $r = r_1$, the "surface of infinite red shift" is not an event horizon surrounding a black hole, rather it is a naked singularity. Thus, a spherically symmetric black hole is impossible in Nordström's theory of gravity.

What of possible "nonspherical black holes"? Since light rays move on geodesics of a conformally flat metric, if they are able to enter any region of spacetime they are also able to leave. Thus, any conceivable "black hole" in Nordström's theory could neither have light sent into it, nor emit light to a distant observer. That is, the light cone structure becomes degenerate on the "surface of the black hole". This is a breakdown of the locally Minkowskian character of spacetime. Thus, if such a surface has finite proper area, we may say that it encloses an "excluded region of spacetime", a distributed (rather than point-like) naked singularity. But such a surface is quite unlike the "one-way membrane" around a GRT (or BDT) black hole, since such a surface in Nordström's theory would permit no communication in either direction.

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