# The Dimension of Spacetime

Dissertation by

**Philip J. Rosenthal**

In Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy

California Institute of Technology
Pasadena, California
1993
(Submitted May 21, 1993)

# Acknowledgments

It is utterly impossible to acknowledge all who deserve recognition for assisting with my research and making my years at Caltech rewarding. First, I must salute my parents for their cheerful endurance of and support for my decision to venture forth into the California wilderness for graduate school. My many friends also deserve special thanks, especially for never complaining as I virtually ignored them during my year-long sprint to the finish line. My friends in SEDS and the aerospace community must also be credited with introducing me to my future career. I would also like to thank Helen Tuck for moral support during the final months.

For assistance with my research, I give heartfelt thanks to my friend and collaborator, Jerry Cleaver. There are numerous others with whom I have had stimulating discussions, that have helped to shape my research. These include, Joe Polchinski, Mark Bowick, Cumrun Vafa, Henry Tye, Phil Argyres, Jaques Distler, Ramzi Khuri, Lance Dixon, Chris Pope, Jim Grochocinski, Stephen Chung, Costas Efthimiou, Rulin Xiu, and Ziad Massarani. I also owe a great deal to the many professors and students at Caltech with whom I have discussed this work. Of course, I am deeply grateful to my advisor, John Schwarz, for providing guidance and numerous suggestions about my research and for understanding when the time had come to move on.

# Abstract

The implications of string theory for the dimension of spacetime are investigated by two methods. First, a new potential class of string theories is studied, which have critical dimensions 3, 4 and 6. In particular, the partition functions of these theories are derived and interpreted using a generalized GSO projection. The possible uniqueness of field assignments, as well as the bosonization of the $K = 4$ model are also addressed. Second, using recent ideas in string cosmology, a new model is proposed to explain why three spatial dimensions grew large. Unlike the original work of Brandenberger and Vafa, this paradigm uses the theory of random walks. A simple computer model is developed to test the implications of this new approach. It is found that a four-dimensional spacetime can be explained by the proper choice of initial conditions.

# Table of Contents

**Acknowledgments** ..................... iii

**Abstract** ..................... iv

**Contents** ..................... v

**1. Introduction: Explaining the Dimension of Spacetime** ..................... 1

**2. Introduction to Conformal Field Theory** ..................... 2

2.1 Basic Definitions ..................... 2

2.2 Extended Algebras ..................... 5

2.3 Partition Functions ..................... 7

**3. The Fractional Superstring** ..................... 13

3.1 Introduction ..................... 13

3.2 A New Derivation of the Partition Functions ..................... 17

3.2.1 Preliminaries ..................... 17

3.2.2 The Affine Partition Function, "W" ..................... 20

3.2.3 The Theta Partition Function, "V," and the Twist Operator ... 23

3.3 A Generalized GSO Projection ..................... 31

3.4 The Uniqueness of $\epsilon$ and the Twist Field $\phi^{K/4}_{K/4}$ ..................... 39

3.5 Bosonization of the $K = 4$ Theory ..................... 46

3.6 Conclusions ..................... 50

**4. String Cosmology** ..................... 52

4.1 Introduction ..................... 52

4.1.1 Duality and the Initial Singularity ..................... 53

4.1.2 The Hagedorn Temperature ..................... 54

4.1.2a A Maximum Temperature? ..................... 56

4.1.2b Going Beyond the Hagedorn Temperature ..................... 58

4.1.3 The Original Paradigm of Brandenberger and Vafa ..................... 66

4.2 A New Cosmological Model That May Yield Four Dimensions ..................... 70

4.3 Testing the Model on a Computer ..................... 73

4.4 Results of the Test and Predictions of the Model ..................... 79

**References** .......... 85

**Appendix A: Figures** .......... 89

**Appendix B: Selected Results of the Cosmology Simulation** .......... 92

## 1. Introduction: Explaining the Dimension of Spacetime

The advent of string theory heralded a new era in particle physics. For the first time, there were good prospects to unify all the forces in a consistent quantum theory. Gravity, which had long resisted a quantum treatment, was naturally incorporated and even required by string theory. Furthermore, unlike standard particle theory, which is all but guaranteed to break down at energies near the Planck scale, string theory is likely to be valid at all energies.

String theory is also the first theory to address the dimension of spacetime. While all other theories are content to input this most fundamental quantity by hand, string theory promises to predict it. Indeed, string theory quickly produced a naive prediction for the dimension of spacetime. Demanding cancellation of the conformal anomaly for mathematical consistency, the bosonic and supersymmetric theories predicted critical dimensions of 26 and 10 respectively. Fortunately, the critical dimension need not be the observed dimension of spacetime. Anomaly cancellation only requires that the underlying conformal field theory have a particular central charge. Some of the fields contributing the charge can be internal degrees of freedom, rather than coordinate bosons. Furthermore, there may exist spacetime dimensions which are compactified near the Planck scale, where no terrestrial experiments have ever probed. Thus, one can construct a string theory with three large spatial dimensions, thereby maintaining the viability of the theory. However, the observed dimension of spacetime seems to be once again reduced to a parameter put in by hand.

This thesis is an attempt to explore how string theory can fulfill its original promise of predicting the dimension of spacetime. The first half of the dissertation examines recent proposals for variations of string theory that have different critical dimensions. In spite of the above discussion that the critical dimension is not required to be the dimension of spacetime, it is still the most "natural" dimension. A theory with a critical dimension of four would be an intriguing development. The second half of the thesis asks whether superstring theory, without modification, can predict three large dimensions. Modifying a previously proposed model of string cosmology, it tests whether such a model can be expected to reliably predict the spacetime we live in.

## 2. Introduction to Conformal Field Theory.

### 2.1 Basic Definitions.

As explained in the introduction, the physical motivation for the new string models studied in the next chapter is to achieve a more realistic critical dimension. The models also have a simple mathematical motivation, which is best understood in the context of Kač-Moody algebras in conformal field theory.

Conformal field theory (CFT) is quite fundamental for any string theory. The simplest string action,[1]

$$S = \frac{1}{2}\int d\sigma d\tau \sqrt{g} g^{ab} \partial_a X^\mu \partial_b X_\mu \tag{2.1.1}$$

leads us inexorably to a CFT. As desired, this action is invariant under general coordinate transformations. When this freedom is gauged away, a residual invariance remains, namely the Weyl rescaling of the metric. Indeed, eq. (2.1.1) is manifestly invariant under any transformation that rescales the metric $g$. In two dimensions, as is appropriate for the string worldsheet, the group of transformations that rescales the metric and thus the remaining symmetry group of the action, is the conformal group. As a result, the field theory appropriate for string theory is conformal field theory.[2]

In order to discuss CFT, one must introduce primary fields. Parameterizing the worldsheet by complex coordinates $z = \tau - i\sigma$ and $\bar{z} = \tau + i\sigma$, a conformal transformation is simply one where $z \to f(z)$ and $\bar{z} \to \bar{f}(\bar{z})$. Under a conformal transformation, a primary field, $\phi(z,\bar{z})$, is defined to transform as,

$$\phi(z,\bar{z}) \to \phi'(z,\bar{z}) = \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \phi(f(z),\bar{f}(\bar{z})) \tag{2.1.2}$$

where $(h,\bar{h})$ is the conformal weight of $\phi$. It is often instructive to consider an infinitesimal holomorphic transformation, [1]

$$z \longrightarrow z + \epsilon(z). \tag{2.1.3}$$

Then, a primary field $\phi(z)$ transforms as

$$\phi(z) \rightarrow \phi(z) + \delta_\epsilon \phi(z) \tag{2.1.4}$$

where[2]

$$\delta_\epsilon \phi(z) = [T_\epsilon, \phi(w)] = (h\partial\epsilon + \epsilon\partial)\,\phi(z). \tag{2.1.5}$$

Here, $T_\epsilon$ is the conserved charge associated with symmetry (2.1.3),

$$T_\epsilon = \oint \frac{dz}{2\pi i}\epsilon(z)T(z) \tag{2.1.6}$$

which also generates that symmetry, as required by Noether's theorem.

In the above expression for $T_\epsilon$, $T(z)$ is the $zz$ component of the energy-momentum tensor. Energy-momentum conservation and conformal invariance imply that $T_{z\bar{z}} = T_{\bar{z}z} = 0$, while $T_{zz}$ and $T_{\bar{z}\bar{z}}$ are conserved currents:

$$\bar{\partial} T_{zz} = 0 \tag{2.1.7a}$$

$$\partial T_{\bar{z}\bar{z}} = 0. \tag{2.1.7b}$$

Thus, $\epsilon(z)T_{zz}$ and $\bar{\epsilon}(\bar{z})T_{\bar{z}\bar{z}}$ form an infinite set of conserved currents, to which correspond the conserved charges, $\oint \frac{dz}{2\pi i}\epsilon(z)T(z)$. To find what transformations these generate, one can check how they act on $\partial X$.

$$\delta\,(\partial X) = [T_\epsilon, \partial X] = \oint \frac{dz}{2\pi i}\epsilon(z)T(z)\partial X(z). \tag{2.1.8}$$

Inserting the operator product expansion for $T(z)\partial X(w)$ and evaluating the integral, we find

$$\delta\,(\partial X) = (\partial\epsilon + \epsilon\partial)\,\partial X = \partial X\,(z+\epsilon) - \partial X\,(z)\,.$$

---

[1] The antiholomorphic case is similar. Simply take the complex conjugate of what follows and replace $h$ by $\bar{h}$.

[2] $\partial \equiv \frac{\partial}{\partial z}\ \bar{\partial} \equiv \frac{\partial}{\partial \bar{z}}$.

Thus, $T_\epsilon$ generates the conformal transformation, $z \longrightarrow z+\epsilon$, as claimed.

The energy-momentum tensor can be expanded in modes[3]

$$T(z) = \sum_n z^{-n-2} L_n \tag{2.1.9}$$

where the $L_n$ are the Virasoro generators. They generate conformal transformations with $\epsilon(z) = z^{n+1}$. The $L_n$ satisfy the Virasoro algebra,

$$[L_n, L_m] = \frac{c}{12} n(n-1)(n+1)\delta_{n+m} + (n-m)L_{n+m} \tag{2.1.10}$$

where $c$ is called the conformal anomaly.

Using eq. (2.1.2), we see that under a dilatation, $z \longrightarrow \lambda z$, $\phi \longrightarrow \lambda^{h+\bar{h}}\phi$. Since a dilatation by $\lambda$ is equivalent to a translation $\tau \longrightarrow \tau + \ln\lambda$, $L_0 + \bar{L}_0$ can be seen to play the role of the Hamiltonian. Indeed, we can write,

$$H = L_0 + \bar{L}_0 = \frac{1}{2}(p_L^2 + p_R^2) + N_L + N_R - 1 - 1. \tag{2.1.11}$$

Similarly, a rotation, $z \longrightarrow e^{-i\vartheta}z$ gives $\phi \longrightarrow e^{-i(h-\bar{h})\vartheta}\phi$, so that $h - \bar{h}$ is defined as conformal spin. Since the above transformation is also a $\sigma$ translation, we identify $L_0 - \bar{L}_0$ as the momentum.

With the Virasoro generators, we can describe physical states in our CFT. If the vacuum is $|0\rangle$, a state $|\phi\rangle$ is defined by $|\phi\rangle = \lim_{z\to 0}\phi(z)|0\rangle$. A state is called a highest weight state with weight $h$ if it satisfies,

$$L_0|\phi\rangle = h|\phi\rangle \tag{2.1.12a}$$

$$L_n|\phi\rangle = 0, \quad n > 0. \tag{2.1.12b}$$

In string theory, such a state with $h$ equal to the intercept is called a physical state. Since the $L_{-n}$ $(n > 0)$ act as raising operators, one can build "descendant states" by applying them to $|\phi\rangle$. $|\phi\rangle$ and the set of all its descendant states form a representation of the Virasoro algebra called the Verma module:

$$|\phi^{n_1......n_k}\rangle = L_{-n_1}......L_{-n_k}|\phi\rangle \;\; n_i > 0. \tag{2.1.13}$$

[3]The exponent of $z$ is $-n-2$ rather than $-n$ because we expand $T$ on a plane instead of a cylinder and the conformal weight of $T$ is 2.

### 2.2 Extended Algebras

The Virasoro algebra is by no means the only important algebra in string theory. To incorporate fermions into string theory, one introduces supersymmetry to the theory, arriving at the following action in light cone gauge:[1]

$$S = \frac{1}{2\pi} \int d^2\sigma \left\{ \partial_+ X \partial_- X + i(\psi_+ \partial_- \psi_+ + \psi_- \partial_+ \psi_-) \right\}. \qquad (2.2.1)$$

The fermion $\psi$ is the superpartner of the coordinate boson. Along with the energy momentum tensor, we now have a supercurrent, $T_F$. These take the following forms, respectively:

$$T_{\pm\pm} = \frac{1}{2}\partial_\pm X \partial_\pm X + \frac{i}{2}\psi_\pm \partial_\pm \psi \qquad (2.2.2a)$$

$$T_{F_\pm} = \frac{1}{2}\psi_\pm \partial_\pm X. \qquad (2.2.2b)$$

Taking $G_r$[4] to be the modes of $T_F$, (analogous to $L_n$), it is straightforward, though tedious, to derive the superconformal algebra:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - 2am)\delta_{m+n} \qquad (2.2.3a)$$

$$[L_m, G_r] = (\frac{m}{2} - r)G_{m+r} \qquad (2.2.3b)$$

$$[G_r, G_s] = 2L_{r+s} + \frac{c}{3}(r^2 - \frac{a}{2})\delta_{r+s}. \qquad (2.2.3c)$$

The fractional superstring is partly based on a generalization of this algebra.

Another algebra of profound importance to conformal field theory and string theory is the Kač-Moody (KM) algebra.[3] A classical KM algebra is easily constructed from its associated Lie algebra. Consider a Lie group $G$ with generators $T_a$ and Lie algebra, $[T_a, T_b] = if^{abc}T_c$. The modes of $T_a(z)$ are defined in the usual way, $T_a(z) = \sum_n T^a_n z^{-n}$, or

$$T^a_n = \oint \frac{dz}{2\pi i} z^{n-1} T^a(z). \qquad (2.2.4)$$

The KM algebra is simply the algebra of the modes $T^a_n$,

$$[T^a_m, T^b_n] = if^{abc}T^c_{m+n}. \qquad (2.2.5)$$

---

[4] $r \in \mathbb{Z} + a$ where $a$ is the intercept.

When dealing with quantum mechanical operators, rather than classical generators, one must include a central extension, $d^{ab}_{mnj}k^j$ on the right of eq. (2.2.5). For irreducible unitary representations of the KM algebra of a simple, compact group, the extension simplifies to yield,

$$[T^a_m, T^b_n] = if^{abc}T^c_{m+n} + Km\delta^{ab}\delta_{m+n} \tag{2.2.6}$$

where $K$ must be a positive integer.[5] $K$ is called the level of the algebra.

These algebras have physical relevance whenever there is a conserved current in the adjoint representation of a Lie group. Such a current can be expanded in modes, which satisfy a KM algebra. Furthermore, one can use purely algebraic means to determine the energy-momentum tensor and thus the $JJ$ and $TJ$ operator product expansions can be found as well. With an expression for $T$, one can calculate the OPE of $T$ with itself and find that the central charge is $c = \frac{K\mathrm{dim}G}{c_2/2+K}$ where $c_2 = 2(\frac{\mathrm{dim}G}{\mathrm{rank}G} - 1)$. It must be emphasized that the central charge is determined knowing only the symmetry group of the theory. The action need not be known. $G = SU(2)$ will be especially important in what follows, in which case $c = \frac{3K}{K+2}$.

One also often wants to consider coset representations. Given a group $G$, with subgroup $H$, and energy-momentum tensors $T_G$ and $T_H$ respectively, it can be shown that the coset $G/H$ has energy-momentum tensor $T_{G/H} = T_G - T_H$. Furthermore, $c_{G/H} = c_G - c_H$. The coset upon which the first half of this dissertation is based is $\frac{(SU(2))_K}{U(1)}$ with central charge $\frac{3K}{K+2} - 1 = \frac{2K-2}{K+2}$.

The coset $\frac{(SU(2))_2}{U(1)}$ deserves special attention. Its conformal charge of $\frac{1}{2}$ indicates that it contains a fermion as a primary field. Indeed the $(SU(2))_2$ primary fields can be factored as $\Phi^j_m(z) = \phi^j_m(z)\exp\left(\frac{im}{\sqrt{2}}\varphi(z)\right)$ where $\exp\left(\frac{im}{\sqrt{2}}\varphi(z)\right)$ is a bosonic representation of $U(1)$ with the boson, $\varphi$, compactified at radius $\sqrt{2}$. $\phi^1_0 \equiv \phi^0_1$ acting on the vacuum gives a fermion. For general $K$, the boson radius becomes $\sqrt{K}$ and a subset of the $\phi^j_m$ produce parafermions. Then $j$ and $m$ are restricted by $0 \le j$, $|m| \le \frac{K}{2}$, $j, m \in \frac{Z}{2}$ and $j - m = 0 \pmod 1$.

---

[5]excluding $G = U(1)$.

Note that the primary fields satisfy the identities:[4]

$$\phi^j_m = \phi^j_{m+K} = \phi^{\frac{K}{2}-j}_{m-\frac{K}{2}}. \tag{2.2.7}$$

The conformal dimension of these fields is, $h(\phi^j_m) = \frac{j(j+1)}{K+2} - \frac{m^2}{K}$. At a given level $K$ the fusion rules are,

$$\phi^{j_1}_{m_1} \otimes \phi^{j_2}_{m_2} = \sum_{j=|j_1-j_2|}^{r} \phi^j_{m_1+m_2} \tag{2.2.8}$$

where $r \equiv \min(j_1+j_2, K-j_1-j_2)$. Note that the subset of primary fields

$$\{\phi_i \equiv \phi^0_i \equiv \phi^{K/2}_{-K/2+i} \; ; \; 0 \leq i \leq K-1\} \tag{2.2.9}$$

($\phi^\dagger_i \equiv \phi_{K-i}$) with conformal dimensions $h(\phi_i) = \frac{i(K-i)}{K}$ form, under fusion, a closed subalgebra possessing a $Z_K$ Abelian symmetry:

$$\phi_i \otimes \phi_j = \phi_{i+j} \quad (\text{mod } K). \tag{2.2.10}$$

These fields are called $\mathbb{Z}_K$ parafermions.

### 2.3 Partition Functions.

When studying any theory, it is very natural to ask about the spectrum of the theory. We will find that the partition function can reveal a great deal of information about the spectrum of the fractional superstring, and is also relatively easy to determine. Thus, a few words should be said about general partition functions.

The most natural way to define the partition function is simply as a generating function that encodes the number of states at each level. For example, consider the Verma module for a primary field $\phi$ with "level" meaning conformal dimension (not to be confused with Kač-Moody level). The distinct states are listed in table 2.1:

Table 2.1: The Verma module of a primary field.

| level | dimension | field |
|---|---|---|
| 0 | $h$ | $\phi$ |
| 1 | $h+1$ | $L_{-1}\phi$ |
| 2 | $h+2$ | $L_{-2}\phi$, $L_{-1}^2\phi$ |
| 3 | $h+3$ | $L_{-3}\phi$, $L_{-1}L_{-2}\phi$, $L_{-1}^3\phi$ |
| ... | ... | ... |
| $n$ | $h+n$ | $P(n)$ fields |

The number of states at each level $n$ is the number of partitions of $n$ into positive integers. A proper generating function for $P(n)$ is $(\prod_{n=1}^{\infty}(1-q^n))^{-1} \equiv \sum_{n=0}^{\infty} P(n)q^n$. The partition function for a Verma module is called a character, since it corresponds to the trace of the corresponding representation matrix. More generally, we define the partition function to be

$$Z = \mathrm{tr} q^{L_{0\mathrm{cyl}}} \bar{q}^{\bar{L}_{0\mathrm{cyl}}} = q^{-\frac{c}{24}} \bar{q}^{-\frac{\bar{c}}{24}} \mathrm{tr} q^{L_0} \bar{q}^{\bar{L}_0} \tag{2.3.1}$$

where the latter expression is defined on the complex plane. Using the identifications of $H$ and $P$ made after eq. (2.1.10), we can write,

$$Z(\tau,\bar{\tau}) = \mathrm{tr} e^{2\pi i \tau_1 P} e^{-2\pi \tau_2 H} \tag{2.3.2}$$

where $q \equiv e^{2\pi i \tau}$ and $\tau = \tau_1 + i\tau_2$. Identifying imaginary time with the inverse temperature, $\beta$, we see how the partition function goes as $\mathrm{tr} e^{-\beta H}$ as one would expect. The term involving the momentum can be interpreted as twisting the cylinder before gluing its ends together to form a torus. This form leads one to suspect that the partition function in string theory plays a critical role in the string amplitude. Indeed, a careful computation of the one-loop vacuum amplitude gives,

$$A = \int_{\mathcal{F}} \frac{d^2\tau}{(\tau_2)^2} Z(\tau,\bar{\tau}). \tag{2.3.3}$$

$\mathcal{F}$ is the fundamental region given by the union of $\{-\frac{1}{2} \le \mathrm{Re}\,\tau \le 0,\ |\tau|^2 > 1\}$ and $\{0 < \mathrm{Re}\,\tau < \frac{1}{2},\ |\tau|^2 > 1\}$. $\mathcal{F}$ is shown in figure 2.1 of appendix A. The above relationship between the partition function and amplitude will put a number of constraints on the partition function, as will be seen below.

Real fermions provide an excellent example of the computation of characters. Majorana fermions only have two possible boundary conditions; they are either periodic (P) or antiperiodic (AP). On a cylinder,

$$\psi(w) = \sum_n \psi_n e^{-nw} \tag{2.3.4}$$

where $n \in \mathbb{Z} + \frac{1}{2}$ for AP fermions and $n \in \mathbb{Z}$ for P fermions. The one-loop partition function is calculated on a torus, which has two non-contractible loops. Thus, four independent sets of boundary conditions can be chosen. By convention, the "space" loop boundary condition determines the modding. Twisting in the "time" direction is implemented by the fermion number operator, $(-1)^F$, which anticommutes with the fermion. Thus, the associated characters for one chirality are listed in table 2.2:

Table 2.2: Fermion Characters

| | | | | |
|---|---|---|---|---|
| $\chi(AP, AP)$ | $=$ | $q^{-\frac{1}{48}}\prod_{n=0}^{\infty}(1+q^{n+1/2})$ | $=$ | $\sqrt{\frac{\vartheta_3}{\eta}}$ |
| $\chi(AP, P)$ | $=$ | $q^{-\frac{1}{48}}\prod_{n=0}^{\infty}(1-q^{n+1/2})$ | $=$ | $\sqrt{\frac{\vartheta_4}{\eta}}$ |
| $\chi(P, AP)$ | $=$ | $\frac{1}{\sqrt{2}}q^{\frac{1}{24}}\prod_{n=0}^{\infty}(1+q^{n})$ | $=$ | $\sqrt{\frac{\vartheta_2}{\eta}}$ |
| $\chi(P, P)$ | $=$ | $\frac{1}{\sqrt{2}}q^{\frac{1}{24}}\prod_{n=0}^{\infty}(1-q^{n})$ | $=$ | $\sqrt{\frac{\vartheta_1}{i\eta}}$ |

The last equality defines the Jacobi theta functions. $\eta$ is the Dedekind eta-function,

$$\eta = q^{1/24}\prod_{n=1}^{\infty}(1-q^n). \tag{2.3.5}$$

The partition function for a consistent theory must contain suitable combinations of all the above characters.

Similarly, the bosonic partition function is

$$Z = \frac{1}{\eta\bar{\eta}}\sum_{n,m=-\infty}^{\infty} q^{\frac{1}{2}(\frac{m}{2R}+nR)^2}\,\bar{q}^{\frac{1}{2}(\frac{m}{2R}-nR)^2}. \tag{2.3.6}$$

The characters for one chirality of $SU(2)_K$ primary fields, $\phi^j_m$ are expressed by[5]

$$Z(\tau) = \eta(\tau)c^{2j}_{2m}(\tau) \tag{2.3.7}$$

where $c^l_n(\tau)$ is a "string function"[6] defined by

$$c^{2j}_{2m} = \frac{1}{\eta^3(\tau)} \sum_{x,y} \mathrm{sign}(x) q^{x^2(K+2)-y^2 K} \tag{2.3.8a}$$

$$= q^{h^j_m + \frac{1}{4(K+2)}} \frac{1}{\eta^3} \sum_{r,s=0}^{\infty} (-1)^{r+s} q^{r(r+1)/2+s(s+1)/2+rs(K+1)} \times$$
$$\left\{ q^{r(j+m)+s(j-m)} - q^{K+1-2j+r(K+1-j-m)+s(K+1-j+m)} \right\} \tag{2.3.8b}$$

$$= q^{h^j_m - \frac{c(SU(2)_K)}{24}} (1 + \cdots) \tag{2.3.8c}$$

where in (2.3.8a) the conditions

1. $-|x| < y \leq |x|$,
2. either $x = \frac{l+1}{2(K+2)} \pmod 1$ or $(\frac{1}{2} - x) = \frac{l+1}{2(K+2)} \pmod 1$; and
3. either $y = \frac{n}{2K} \pmod 1$ or $(\frac{1}{2} + y) = \frac{n}{2K} \pmod 1$

must be met simultaneously.[7] $h^j_m \equiv h(\phi^j_m)$ and $c(SU(2)_K) = \frac{3K}{K+2}$, above. These string functions obey the same equivalences as their associated primary fields $\phi^j_m$:

$$c^{2j}_{2m} = c^{2j}_{2m+2K} = c^{K-2j}_{2m-K}\, . \tag{2.3.9a}$$

For the string functions, but not for the primary fields, one has also

$$c^{2j}_{2m} = c^{2j}_{-2m}\, . \tag{2.3.9b}$$

Since the partition function is essentially the integrand of the one-loop amplitude, it must satisfy stringent constraints. The amplitude is found by summing over all distinct Riemann surfaces, in this case parameterized by $\tau$. However, there is an infinite degeneracy, with each distinct surface corresponding to an infinite set of distinct $\tau$. Thus, the integral must be restricted to a "fundamental region." The conventional choice is shown in figure 2.1. Of course, restricting the range of integration is insufficient; the integrand (i.e., the partition function) must be invariant under changes in $\tau$ that do not change the underlying Riemann surface. Said differently, the partition function must be invariant under all conformal transformations, including those that are not continuously connected to the identity.

These considerations can be made more precise by studying the set of inequivalent tori. Note that a conformally distinct torus can be uniquely specified by a lattice of identified points, which without loss of generality can be rotated to be parallel to the Re $\tau$ axis. Since we have invariance under rescaling, the distance between adjacent points along the Re $\tau$ axis can be set to one. As shown in figure 2.2, the complex parameter $\tau$ now completely defines the lattice. It is easy to see that the transformation $T$: $\tau \rightarrow \tau + 1$ defines the same lattice and thus the same worldsheet. A less obvious symmetry transformation is $S$: $\tau \rightarrow -\frac{1}{\tau}$. These two transformations generate the modular group, $SL(2,\mathbb{Z})/\mathbb{Z}_2$ which has the general form,

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d} \tag{2.3.10}$$

where $a$, $b$, $c$ and $d \in \mathbb{Z}$ and $ad - bc = 1$.

It is easy to physically interpret the $S$ and $T$ transformations. To do this, observe that the torus has two non-contractible cycles. Imagine cutting one of them, twisting one end by $2\pi$ and gluing the torus back together. This operation, called a Dehn twist, clearly leaves the torus invariant and corresponds to a $T$ transformation. Another independent generator can be taken to correspond to cutting and twisting along the other cycle. However, it is more conventional to choose for the second independent generator the $S$ transformation, which corresponds to interchanging the two loops (i.e., relabeling what is considered worldsheet space and time). Clearly, the labeling is arbitrary, so this is a symmetry.[6]

The modular transformations of the boson, fermion and $SU(2)_K$ characters are well known. They are

$$S : \eta(\tau) \rightarrow \sqrt{-i\tau}\,\eta(\tau) \tag{2.3.11a}$$

$$T : \eta(\tau) \rightarrow e^{\frac{i\pi}{12}}\eta(\tau) \tag{2.3.11b}$$

$$\begin{aligned} S : \chi(AP,AP) &\rightarrow \chi(AP,AP) \\ \chi(AP,P) &\rightarrow \chi(P,AP) \\ \chi(P,AP) &\rightarrow \chi(AP,P) \end{aligned} \tag{2.3.12a}$$

[6]Note how the $S$ and $T$ transformation require that contributions from all fermion boundary conditions be included in the partition function.

$$\begin{aligned} T: \chi(AP,AP) &\to e^{\frac{i\pi}{24}}\chi(AP,P) \\ \chi(AP,P) &\to e^{-\frac{i\pi}{24}}\chi(AP,AP) \\ \chi(P,AP) &\to e^{\frac{i\pi}{12}}\chi(P,AP) \end{aligned} \tag{2.3.12b}$$

$$S: c^l_n \to \frac{1}{\sqrt{-i\tau}}\frac{1}{\sqrt{K(K+2)}}\sum_{l'=0}^{K}\sum_{\substack{n'=-K+1 \\ l'-n'\in 2\mathbb{Z}}}^{K} b(l,n,l',n')c^{l'}_{n'} \tag{2.3.13a}$$

$$T: c^l_n \to e^{2\pi i H^l_n}c^l_n \tag{2.3.13b}$$

where $c^l_n$ is the character for the $SU(2)_K$ primary field $\phi^{l/2}_{n/2}$, $H^l_n = \frac{l(l+2)}{4(K+2)} - \frac{n^2}{4K}$ and $b(l,n,l',n') = \exp\left\{\frac{i\pi nn'}{K}\right\}\sin\left\{\frac{\pi(l+1)(l'+1)}{K+2}\right\}$. The mixing of terms under S transformation makes the construction of modular invariant partition functions with $SU(2)_K$ fields highly non-trivial.

Everything discussed to this point assumed that the worldsheet was a smooth manifold. Often, it is preferable to consider manifolds which are modded out by discrete groups. The simplest such "orbifold" is $S^1/\mathbb{Z}_2$, where all the points are identified under reflection, $x \sim gx = -x$, $g \in \mathbb{Z}_2$. The effect of such an identification in the spectrum is twofold. First, states in the original $S^1$ spectrum that are not invariant under $\mathbb{Z}_2$ are projected out. Secondly, new sectors of states are added to the theory, which have twisted boundary conditions. For $S^1/\mathbb{Z}_2$ the resulting partition function is,

$$Z_{\text{orb}}(R) = \frac{1}{2}(Z_{\text{circ}}(R) + \frac{|\vartheta_3\vartheta_4|}{\eta\bar{\eta}} + \frac{|\vartheta_2\vartheta_3|}{\eta\bar{\eta}} + \frac{|\vartheta_2\vartheta_4|}{\eta\bar{\eta}}) \tag{2.3.14}$$

where $Z_{\text{circ}}$ is the bosonic partition function, eq. (2.3.6).

# 3. The Fractional Superstring.

## 3.1 Introduction.

At first glance, the fractional superstring[7] appears to be a modest generalization of the usual superstring. Rather than using $(SU(2))_2$ primary fields, these models employ $SU(2)$ primary fields of general level $K$. They replace the standard $c = \frac{1}{2}$ fermionic superpartner of the holomorphic (and likewise for antiholomorphic) worldsheet scalar $X(z)$ with the "energy operator," $\epsilon \equiv \phi_0^1$, [7] of the $Z_K$ parafermion conformal field theory. $\epsilon$ has conformal dimension (spin) $\frac{2}{K+2}$. Note that for $K \neq 2$, the conformal dimension (spin) is "fractional" (i.e., neither integral nor half-integral) which explains the name of these models. The central charge of each $\epsilon - X$ pair is $c = \frac{3K}{K+2}$.

The supercurrent must be modified in an analogous manner. The naive generalization to $\epsilon(z)\partial x$ proves to be inadequate. Instead, the proposed "fractional supercurrent" (FSC) is

$$J_{\rm FSC}(z) = \epsilon(z)\partial x(z) + :\epsilon\epsilon:(z) \tag{3.1.1}$$

where $:\epsilon\epsilon:$ denotes the first descendant of $\epsilon$ (which vanishes for $K = 2$.) $J_{\rm FSC}(z)$ suggests a local "fractional" worldsheet supersymmetry between $\epsilon(z)$ and $X(z)$, extending the Virasoro algebra of the stress-energy tensor $T(z)$. This local current of spin $h(J_{\rm FSC}) = 1 + \frac{2}{K+2}$ has fractional powers of $\frac{1}{(z-w)}$ in the OPE with itself, producing cuts on the worldsheet and implying a non-local worldsheet interaction. The corresponding "fractional superconformal algebra"[8] is,

$$T(z)T(w) = \frac{\frac{1}{2}c}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \ldots \tag{3.1.2a}$$

$$T(z)J_{\rm FSC}(w) = \frac{hJ_{\rm FSC}(w)}{(z-w)^2} + \frac{\partial J_{\rm FSC}(w)}{(z-w)} + \ldots \tag{3.1.2b}$$

---

[7] Note that $\epsilon$ is not in the $Z_K$ Abelian subgroup, and thus is not a $Z_K$ parafermion, except for the degenerate $K = 2$ superstring case where $\phi_0^1 \equiv \phi_1^0$.

$$J_{\rm FSC}(z)J_{\rm FSC}(w) = \frac{1}{(z-w)^{2h}} + \frac{\frac{2h}{c}T(w)}{(z-w)^{2h-2}} + \frac{\lambda_K(c_0)J_{\rm FSC}(w)}{(z-w)^h} + \frac{\frac{1}{2}\lambda_K(c_0)\partial J_{\rm FSC}(w)}{(z-w)^{h-1}} + \ldots \tag{3.1.2c}$$

where $h = \frac{K+4}{K+2}$, $c = Dc_0$, $D$ is the critical dimension and $c_0$ is the central charge for one dimension as above. Note that no mention has been made of an action. The worldsheet action for these models is simply unknown. However, a great deal of information about these theories can still be extracted. While it is impossible to find the partition function rigorously from first principles, a physically plausible partition function has been proposed, using the characters from eq. (2.3.7).

The one-loop partition function for the Type II fractional superstring can be found by requiring the existence of a massless graviton while excluding tachyons. Since each dimension corresponds to a boson and parafermion, which have characters $\frac{1}{\eta}$ and $\eta c^l_n$ respectively, each dimension contributes one factor of a string function, $c^l_n$. If we only consider the holomorphic part of the partition function, in light cone gauge, each term of the partition function must be made of $D-2$ string functions. Consider the lowest possible mass term, $(c^0_0)^{D-2}$, corresponding to $D-2$ factors of $\phi^0_0$. Using eq. (2.3.8b), it has leading order mass, $m^2 = \frac{-(D-2)K}{8(K+2)}$ and is tachyonic for $D > 2$. Thus, we exclude $(c^0_0)^{D-2}$ from the partition function. The next lowest acceptable[8] term is $(c^0_0)^{D-3}c^2_0$ with leading order mass, $m^2 = \frac{2}{K+2} - \frac{(D-2)K}{8(K+2)}$. We set the mass to zero so that it can produce a massless graviton. Solving the previous equation yields, for $K > 1$,

$$D = 2 + \frac{16}{K}. \tag{3.1.3}$$

We see that for an integral critical dimension, we can only consider $K = 2$, 4, 8 and 16, yielding critical dimensions 10, 6, 4, and 3, respectively. Requiring $(c^0_0)^{D-3}c^2_0$ to be in the partition function, one can try to add terms as needed to produce a modular invariant partition function. Computerized searches demonstrated that for each $K$ listed above, there is a unique partition function that satisfies the above constraints.[4] They are as follows:

---

[8]Acceptable means providing a massless spin 1 particle in the open string spectrum.[4] A better argument is given in the main body of the dissertation.

$D = 10 \qquad (K = 2): \qquad\qquad Z = |A_2|^2, \;\text{where}$

$$\begin{aligned} A_2 &= 8(c_0^0)^7(c_0^2) + 56(c_0^0)^5(c_0^2)^3 + 56(c_0^0)^3(c_0^2)^5 + 8(c_0^0)(c_0^2)^7 - 8(c_1^1)^8 \\ &= \frac{1}{2}\eta^{-12}(\vartheta_3^4 - \vartheta_4^4 - \vartheta_2^4) \end{aligned} \tag{3.1.4}$$

$D = 6 \qquad (K = 4): \qquad\qquad Z = |A_4|^2 + 3|B_4|^2, \;\text{where}$

$$A_4 = 4(c_0^0 + c_0^4)^3(c_0^2) - 4(c_0^2)^4 - 4(c_2^2)^4 + 32(c_2^2)(c_2^4)^3 \tag{3.1.5a}$$

$$B_4 = 8(c_0^0 + c_0^4)(c_0^2)(c_2^4)^2 + 4(c_0^0 + c_0^4)^2(c_2^2)(c_2^4) - 4(c_0^2)^2(c_2^2)^2 \tag{3.1.5b}$$

$D = 4 \qquad (K = 8): \qquad\qquad Z = |A_8|^2 + |B_8|^2 + 2|C_8|^2, \;\text{where}$

$$A_8 = 2(c_0^0 + c_0^8)(c_0^2 + c_0^6) - 2(c_0^4)^2 - 2(c_4^4)^2 + 8(c_4^6 c_4^8) \tag{3.1.6a}$$

$$B_8 = 4(c_0^0 + c_0^8)(c_4^6) + 4(c_0^2 + c_0^6)(c_4^8) - 4(c_0^4 c_4^4) \tag{3.1.6b}$$

$$C_8 = 4(c_2^2 + c_2^6)(c_2^8 + c_6^8) - 4(c_2^4)^2 \tag{3.1.6c}$$

$D = 3 \qquad (K = 16): \qquad\qquad Z = |A_{16}|^2 + |C_{16}|^2, \;\text{where}$

$$A_{16} = c_0^2 + c_0^{14} - c_0^8 - c_8^8 + 2c_8^{14} \tag{3.1.7a}$$

$$C_{16} = c_4^2 + c_4^{14} - c_4^8 \, . \tag{3.1.7b}$$

For convenience, a factor of $\tau_2^{-(D-2)/2}$ was omitted, where $\tau_2$ is the imaginary part of the modular parameter.

How to interpret these partition functions is not immediately obvious. The exception is, of course, the $K = 2$ superstring for which it is well known that the terms with $m = 0$ are due to spacetime bosons, while the term with $m = \frac{K}{2} = 1$ results from spacetime fermionic states. By analogy, it is argued[7] that for general $K$, terms with $m = 0$ are bosonic, while $m = \frac{K}{2}$ terms are fermionic. This gives an acceptable interpretation of the A sectors, which contain the graviton, the gravitino and indeed all the massless particles in these models. The B and C sectors have no analog in the superstring and are less transparent. Tye et al., claim[7] that they represent spin $\frac{1}{4}$ and $\frac{1}{8}$ particles respectively. Thus, if one wishes to maintain Lorentz invariance, compactification of one (two) dimension(s) is required for $D = 4$, (6) respectively. Thus, while the $K = 4$ and 8 models may have $D_{\text{crit}} = 6$ and 4, respectively, they have "natural" dimensions of 4 and 3. The $K = 16$ model has $D_{\text{crit}} = D_{\text{natural}} = 3$ since anyons are allowed in three dimensions.

It is also interesting to observe that all the sectors are individually numerically zero,

$$A_2 = A_4 = A_8 = A_{16} = B_4 = B_8 = C_8 = C_{16} = 0. \tag{3.1.8}$$

This is necessary for $N = 1$ SUSY, but is by no means a sufficient condition.

Another issue is whether the fractional superstring partition functions are modular invariant at all loops. One-loop modular invariance is equivalent to invariance under S and T transformations. However, modular invariance at higher orders requires an additional invariance under U transformations: Dehn twists mixing loops of neighboring tori of $g > 1$ Riemann surfaces. To date, U invariance has not been demonstrated, though it is expected to hold.[9]

The following five sections examine many aspects of the fractional superstring. The next section shows that the proposed partition functions have a simple origin and are not merely the arbitrary result of a computer search. Section three follows with an improved spacetime interpretation of the $K = 4$ partition function, based on a generalization of the GSO projection. Next, a more satisfactory argument is given for the critical dimension, and by similar means, the possibility of additional theories at other Kač-Moody levels is restricted. Section five considers the bosonization of the $K = 4$ theory and derives new identities, similar to theta function identities. The last section states some conclusions. Throughout the following chapters, a comparison of the superstring to the fractional superstring is made whenever appropriate, with the belief that much can be learned about the superstring in the current, more general context.

### 3.2 A New Derivation of the Partition Functions

#### 3.2.1 Preliminaries

As is often the case with computer generated results, the partition functions are not presented in the most natural or suggestive form. By using the string function identities, one can recast the partition function in many forms. To decide which to use, one notes that $S$ invariant expressions containing string functions with only even superscripts, as presented in the introduction, must only consist of symmetric combinations of the $c^l_n$, namely, $\frac{1}{2}(c^l_m + c^{k-l}_m)$. For $K \in 4\mathbb{Z}$ and $l$ even, we see that replacing $c^l_n$ by $\frac{1}{2}(c^l_m + c^{k-l}_m)$ maintains $T$ invariance. Thus one must be able to symmetrize the new partition functions while preserving modular invariance. Re-expressing the partition functions solely in terms of these symmetric combinations, one arrives at the following expressions:

$D = 10 \quad (K = 2)$: $\qquad Z = |A_2|^2$, where

$$
\begin{aligned}
A_2 &= \frac{1}{2}\left\{(c^0_0 + c^2_0)^8 - (c^0_0 - c^2_0)^8\right\}_{\text{boson}} - 8(c^1_1)^8_{\text{fermion}} \\
&= 8\left\{(c^0_0)^7 c^2_0 + 7(c^0_0)^5(c^2_0)^3 + 7(c^0_0)^3(c^2_0)^5 + c^0_0(c^2_0)^7\right\}_{\text{boson}} - 8(c^1_1)^8_{\text{fermion}}
\end{aligned}
\tag{3.2.1}
$$

$D = 6 \quad (K = 4)$: $\qquad Z = |A_4|^2 + 3|B_4|^2$, where

$$
\begin{aligned}
A_4 = \;& 4\left\{(c^0_0 + c^4_0)^3(c^2_0) - (c^2_0)^4\right\} \\
&+ 4\left\{(c^0_2 + c^4_2)^3(c^2_2) - (c^2_2)^4\right\}
\end{aligned}
\tag{3.2.2a}
$$

$$
\begin{aligned}
B_4 = \;& 4\left\{(c^0_0 + c^4_0)(c^2_0)(c^0_2 + c^4_2)^2 - (c^2_0)^2(c^2_2)^2\right\} \\
&+ 4\left\{(c^0_2 + c^4_2)(c^2_2)(c^0_0 + c^4_0)^2 - (c^2_2)^2(c^2_0)^2\right\}
\end{aligned}
\tag{3.2.2b}
$$

$D = 4 \qquad (K = 8): \qquad\qquad Z = |A_8|^2 + |B_8|^2 + 2|C_8|^2,$ where

$$\begin{aligned} A_8 = {} & 2\left\{(c_0^0 + c_0^8)(c_0^2 + c_0^6) - (c_0^4)^2\right\} \\ & + 2\left\{(c_4^0 + c_4^8)(c_4^2 + c_4^6) - (c_4^4)^2\right\} \end{aligned} \qquad (3.2.3a)$$

$$\begin{aligned} B_8 = {} & 2\left\{(c_0^0 + c_0^8)(c_4^2 + c_4^6) - (c_0^4 c_4^4)\right\} \\ & + 2\left\{(c_4^0 + c_4^8)(c_0^2 + c_0^6) - (c_4^4 c_0^4)\right\} \end{aligned} \qquad (3.2.3b)$$

$$\begin{aligned} C_8 = {} & 2\left\{(c_2^0 + c_2^8)(c_2^2 + c_2^6) - (c_2^4)^2\right\} \\ & + 2\left\{(c_2^0 + c_2^8)(c_2^2 + c_2^6) - (c_2^4)^2\right\} \end{aligned} \qquad (3.2.3c)$$

$D = 3 \qquad (K = 16): \qquad\qquad Z = |A_{16}|^2 + |C_{16}|^2,$ where

$$\begin{aligned} A_{16} = {} & \left\{(c_0^2 + c_0^{14}) - c_0^8\right\} \\ & + \left\{(c_8^2 + c_8^{14}) - c_8^8\right\} \end{aligned} \qquad (3.2.4a)$$

$$\begin{aligned} C_{16} = {} & \left\{(c_4^2 + c_4^{14}) - c_4^8\right\} \\ & + \left\{(c_4^2 + c_4^{14}) - c_4^8\right\} \; . \end{aligned} \qquad (3.2.4b)$$

The above form has many advantages. First, it reinforces the claim that these models have spacetime supersymmetry, if one assigns spacetime statistics in the $A$ sector according to subscripts on the string functions. For $K \geq 4$, the first bracket, with subscript zero, contains terms corresponding to spacetime bosons and the second bracket uniformly has the subscript $\frac{K}{2}$, which leads one to identify it with spacetime fermions. We see that the presumed bosonic and fermionic sectors have exactly the same form, with only the subscripts changed. This makes manifest a one-to-one cancellation of bosons with fermions, necessary to give SUSY.[9] Perhaps more

[9]Of course, this correspondence of bosonic and fermionic string functions in the partition function is neither necessary, nor sufficient for spacetime supersymmetry, as seen by the $K = 2$ case.

interestingly, the partition functions now hint that the $B$ and $C$ sectors may be simply related to the physically motivated $A$ sectors, rather than computer generated artifacts. In fact, if one removes the subscripts from every term in the partition function, we see that all sectors become identical, apart from an integer coefficient.

Indeed, the partition functions allow a simple analytic derivation. The key to this approach is the factorization theorem proved by Gepner and Qiu.[5] As we have already seen, the parafermion partition function for a single coordinate has the general form

$$Z = |\eta|^2 \sum N_{l,n,\bar{l},\bar{n}} c_n^l \bar{c}_{\bar{n}}^{\bar{l}} \,, \tag{3.2.5a}$$

Gepner and Qiu show that this can be written as,

$$Z = |\eta|^2 \sum \frac{1}{2} L_{l,\bar{l}} M_{n,\bar{n}} c_n^l \bar{c}_{\bar{n}}^{\bar{l}} \,, \tag{3.2.5b}$$

where $c_{n=2m}^{l=2j} = 0$ unless $l - n \in 2\mathbb{Z}$ since $\phi_m^j = 0$ for $j - m \notin \mathbb{Z}$. The $L_{l,\bar{l}}$ and $M_{n,\bar{n}}$ are defined as coefficients of the following partition functions:

$$W = \sum_{l,\bar{l}=0}^{K} L_{l,\bar{l}} \chi_l(\tau) \bar{\chi}_{\bar{l}}(\bar{\tau}) \tag{3.2.6}$$

and

$$V = \frac{1}{|\eta(\tau)|^2} \sum_{n,\bar{n}=-K+1}^{K} M_{n,\bar{n}} \vartheta_{n,K} \bar{\vartheta}_{\bar{n},K} . \tag{3.2.7}$$

$V$ is a $U(1)$ partition function written in terms of the Jacobi theta function, $\vartheta_{n,K}$, which is defined as

$$\vartheta_{n,K}(\tau) = \sum_{p \in \mathbb{Z} + \frac{n}{2K}} e^{2\pi i K p^2 \tau} . \tag{3.2.8}$$

$\chi_l(\tau)$ is the character of the spin-$\frac{1}{2}l$ representation of $SU(2)_K$, which can be written in terms of the theta functions,

$$\chi_l(\tau) = \frac{\vartheta_{l+1,K+2}(\tau) - \vartheta_{-l-1,K+2}(\tau)}{\vartheta_{1,2}(\tau) - \vartheta_{-1,2}(\tau)} . \tag{3.2.9}$$

Note that the $SU(2)_K$ characters and the theta functions provide the defining relation for the string functions,[10]

$$\chi_l(\tau) = \sum_{n=-K+1}^{K} c_n^l(\tau)\vartheta_{n,K}(\tau)\,. \tag{3.2.10}$$

With the preceding definitions in hand, one can state the factorization result simply as eq. (3.2.5a) is modular invariant if and only if

$$N_{l,n,\bar{l},\bar{n}} = \frac{1}{2}L_{l,\bar{l}}M_{m,\bar{m}}, \tag{3.2.11}$$

where $L_{l,\bar{l}}$ and $M_{m,\bar{m}}$ are chosen such that eqs. (3.2.6) and (3.2.7) are both modular invariant. Note that this result can be generalized to twisted tensor products of single factor partition functions. The corresponding tensors are denoted **L** and **M**.

### 3.2.2 The Affine Partition Function, "W"

In the last section, we discussed removing the subscripts from the string functions in the partition functions. This operation (which will be denoted by $\stackrel{\text{affine}}{\Longrightarrow}$) is equivalent to replacing each $c_n^l$ with the affine character $\chi_l$. As seen below, a unique affine partition function, ignoring overall factors, results for each $K$:

$D = 6 \qquad (K = 4)$:

$$A_4, B_4 \qquad \stackrel{\text{affine}}{\Longrightarrow} \qquad A_4^{\text{affine}} \equiv (\chi_0 + \chi_K)^3\chi_{K/2} - (\chi_{K/2})^4 \tag{3.2.12a}$$

$D = 4 \qquad (K = 8)$:

$$A_8, B_8, C_8 \qquad \stackrel{\text{affine}}{\Longrightarrow} \qquad A_8^{\text{affine}} \equiv (\chi_0 + \chi_K)(\chi_2 + \chi_{K-2}) - (\chi_{K/2})^2 \tag{3.2.12b}$$

---

[10] The associated relationship between the level-$K$ $SU(2)$ primary fields $\Phi^j$ and the parafermionic $\phi_m^j$ is

$$\Phi^j = \sum_{m=-j}^{j} \phi_m^j \; : \exp\left\{i\frac{m}{\sqrt{K}}\varphi\right\} :$$

where $\varphi$ is the $U(1)$ boson field of the $SU(2)$ theory.

$D = 3 \qquad (K = 16)$:

$$A_{16}, C_{16} \qquad \stackrel{\text{affine}}{\Longrightarrow} \qquad A_{16}^{\text{affine}} \equiv (\chi_2 + \chi_{K-2}) - \chi_{K/2} \ . \tag{3.2.12c}$$

If we denote the above affine terms as $A_K^{\text{affine}}$, then the affine factor in each parafermion partition function is:

$$Z^{\text{affine}}(K) = |A_K^{\text{affine}}|^2 \, , \tag{3.2.13}$$

where the general form of $A_K^{\text{affine}}$ is,[11]

$$A_K^{\text{affine}} \equiv (\chi_0 + \chi_K)^{D-3}(\chi_2 + \chi_{K-2}) - (\chi_{K/2})^{D-2} \ . \tag{3.2.14}$$

(Note that the modular invariance of $W$ requires that $A_K^{\text{affine}}$ transforms back into itself under $S$.)

The set of partition functions (3.2.13) is indeed modular invariant and satisfies special physical requirements. This is easiest to show for the $K = 16$ model. The $SU(2)_{16}$ MIPF's for $D = 3$ are trivial to classify since for this level the A–D–E classification forms a complete basis set of modular invariants. The only free parameters in $K = 16$ affine partition function $W(K = 16)$ are integers $a$, $b$, and $c$ where

$$W(K = 16) = a \times Z(\mathrm{A}_{17}) + b \times Z(\mathrm{D}_{10}) + c \times Z(\mathrm{E}_7) \ . \tag{3.2.15}$$

The specific choice of coefficients results from imposing physical constraints. Demanding that neither a left- nor a right-moving tachyonic state be in the Hilbert space of states of the $K = 16$ fractional superstring, when the intercept $v$, defined by

$$L_0|\text{physical}\rangle = v|\text{physical}\rangle \, , \tag{3.2.16}$$

is positive, removes these degrees of freedom and requires $a = -(b+c) = 0$, independent of the possible $(n, \bar{n})$ partition functions. These specific values for $a$, $b$, and $c$ give us (3.2.13) for this level:

$$W(K = 16) = Z(\mathrm{D}_{10}) - Z(\mathrm{E}_7) = |A_{16}^{\text{affine}}|^2 \, , \tag{3.2.17}$$

[11] For $K = 4$, only one factor of $\chi_2$ is intended.

which is exactly the form found by previous authors.[10]

Though not quite as straightforward a process, one can also derive the affine partition functions $W(K)$ for the remaining levels.[11] The affine factors in the $K=4$ and 8 partition functions involve twisting by a non-simple current.[12] These cases correspond to theories that are the difference between a $\bigotimes_{\substack{D-2\\ \text{factors}}} \mathrm{D}_{\frac{K}{2}+2}$ tensor product model and a $\bigotimes_{\substack{D-2\\ \text{factors}}} \mathrm{D}_{\frac{K}{2}+2}$ tensor product model twisted by the affine current

$$J^{\text{affine}}_{\text{non-simple,K}} = (\Phi^{\frac{K}{4}})^{D-2}\bar{\Phi}^{1}(\bar{\Phi}^{0})^{D-3} \ . \tag{3.2.20}$$

The equivalent parafermionic twist current is[13]

$$J^{\text{parafermion}}_{\text{non-simple,K}} = (\phi_0^{\frac{K}{4}})^{D-2}\bar{\phi}_0^{1}(\bar{\phi}_0^{0})^{D-3} \ . \tag{3.2.21}$$

---

[12] A simple current, $J_s$, is a primary field of a CFT which, when fused with any other primary field (including itself), $\Phi_l$, in the CFT produces only a single primary field as a product state:

$$J \otimes \Phi_l = \Phi_{l'} \ . \tag{3.2.18}$$

A non-simple current $J_{ns}$, when fused with at least one other primary field (possibly itself), produces more than one term:

$$J_{ns} \otimes \Phi_l = \sum_{l'} \Phi_{l'} \ . \tag{3.2.19}$$

[13] Spacetime indices are left off of most of the following currents and fields. All currents are written in light-cone gauge, so only indices for transverse modes are implied. The $D-2$ transverse dimensions are assigned indices in the range 1 to $D-2$ (and are generically represented by lowercase Greek superscripts.) When spacetime indices are suppressed, the fields and their corresponding partition functions acting along directions 1 to $D-2$ are ordered from left to right, respectively, for both the holomorphic and antiholomorphic sectors separately. Often, the notation will be more compressed and $r$ identical factors of $\phi^j_m$ along consecutive directions (when these directions are either all compactified or uncompactified) will be expressed as $(\phi^j_m)^r$. Thus, eq. (3.2.21) for $K=8$ should be read as

$$J^{K=8,\ \text{parafermion}}_{\text{non-simple}} \equiv (\phi_0^{K/4})^{\mu=1}(\phi_0^{K/4})^{\nu=2}(\bar{\phi}_0^{1})^{\bar{\mu}=1}(\bar{\phi}_0^{0})^{\bar{\nu}=2} \ .$$

### 3.2.3 The Theta Partition Function, "V," and the Twist Operator

Now consider the theta function factors of the fractional superstring partition functions which are labeled by the indices $(n, \bar{n})$, where $n$ represents $\vartheta_{n,K}$ defined in (3.2.8). Since all $A_K$, $B_K$, $C_K$ sectors in the level $K$ fractional superstring partition function and even the boson and fermion subsectors in $A_K$ contain the same affine factor, it is clearly the choice (or lack thereof after elimination of tachyons) of the theta function factor which determines the spacetime Lorentz properties as well as the level of spacetime supersymmetry of the fractional superstring theories. That is, the spacetime spin of particles in the Hilbert space of states depends upon the $M$'s that are allowed in tensored versions of eq. (3.2.7). In a situation when a single factor is sufficient, invariance of (3.2.7) under $S$ implies that the components $M_{n\bar{n}}$ must be related by

$$M_{n',\bar{n}'} = \frac{1}{2K} \sum_{n,\bar{n}=-K+1}^{K} M_{n,\bar{n}} e^{i\pi n n'/K} e^{i\pi \bar{n}\bar{n}'/K} \ . \tag{3.2.22a}$$

$T$ invariance provides a much simpler constraint,

$$\frac{n^2 - \bar{n}^2}{4K} \in \mathbb{Z} \ , \quad \text{if } M_{n,\bar{n}} \neq 0 \ . \tag{3.2.22b}$$

At every level $K$ there is a unique modular invariant function corresponding to each factorization[12], $\alpha\ \beta = K$, where $\alpha$, $\beta \in \mathbb{Z}$. Denoting the matrix elements of $\mathbf{M}^{\alpha,\beta}$ by $M^{\alpha,\beta}_{n,\bar{n}}$, they are given by[14]

$$M^{\alpha,\beta}_{n,\bar{n}} = \frac{1}{2} \sum_{\substack{x\in\mathbb{Z}_{2\beta} \\ y\in\mathbb{Z}_{2\alpha}}} \delta_{n,\alpha x+\beta y} \delta_{\bar{n},\alpha x-\beta y} \ . \tag{3.2.23}$$

Thus, for $K = 4$ the two distinct choices for the matrix $\mathbf{M}^{\alpha,\beta}$ are $\mathbf{M}^{1,4}$ and $\mathbf{M}^{2,2}$; for $K = 8$, we have $\mathbf{M}^{1,8}$ and $\mathbf{M}^{2,4}$; and for $K = 16$, the three alternatives are $\mathbf{M}^{1,16}$,

[14] By eq. (3.2.23), $M^{\alpha,\beta}_{n,\bar{n}} = M^{\beta,\alpha}_{n,-\bar{n}}$. Hence, choosing $\alpha > \beta$ or $\beta > \alpha$ results in equivalent fractional superstring partition functions. To avoid this redundancy, $\alpha \leq \beta$ is required.

Throughout this subsection $n$ is taken to represent, simultaneously, the holomorphic $\vartheta_{n,K}$ characters for $U(1)$ theories and, when appropriate, the holomorphic string functions, $c^0_n$, for parafermions. ($\bar{n}$ represents the antiholomorphic equivalent.) However, this does not imply that the string functions can actually be factored into $c^l_0 \otimes c^0_n = c^l_n$. In the case of the primary fields, $\phi^j_m$, the factorization is, indeed, valid: $\phi^j_0 \otimes \phi^0_m = \phi^j_m$ (for integer $j$, $m$).

$\mathbf{M}^{2,8}$, and $\mathbf{M}^{4,4}$. In general, $\mathbf{M}^{1,K}$ produces the level $K$ diagonal partition function, where $n = \bar{n}$. For $D > 3$ there is more than one transverse dimension, so that a single "$V$" partition function is insufficient. The most naive generalization is to simple tensor products. Unfortunately, such tensor products of these $\mathbf{M}^{\alpha,\beta}$ matrices do not produce fractional superstrings with spacetime SUSY. One can show that twisting by a special simple current is required to achieve this. Of the potential choices for the $U(1)$ modular invariant partition functions (MIPF's), $V$(level $K$), the following are the only ones that produce numerically zero fractional superstring partition functions:

$D = 6 \qquad (K = 4)$:

The $\mathbf{M} = \mathbf{M}^{2,2} * \mathbf{M}^{2,2} * \mathbf{M}^{2,2} * \mathbf{M}^{2,2}$ model twisted by the simple current[15]

$$J_4 \equiv \phi^0_{K/4}\phi^0_{K/4}\phi^0_{K/4}\phi^0_{K/4}\bar{\phi}^0_0\bar{\phi}^0_0\bar{\phi}^0_0\bar{\phi}^0_0 \tag{3.2.24}$$

results in the following $(n, \bar{n})$ partition function:[16]

$$\begin{aligned} V(K=4) = [&(0+4)^4(\bar{0}+\bar{4})^4 + (2+2)^4(\bar{2}+\bar{2})^4 \\ &+ (0+4)^2(2+2)^2(\bar{0}+\bar{4})^2(\bar{2}+\bar{2})^2 \\ &+ (2+2)^2(0+4)^2(\bar{2}+\bar{2})^2(\bar{0}+\bar{4})^2]_{\textbf{untwisted}} \end{aligned}$$

---

[15]Recall that the parafermion primary fields $\phi^0_m$ have simple fusion rules,

$$\phi^0_m \otimes \phi^0_{m'} = \phi^0_{m+m'} \quad (\text{mod } K)$$

and form a $Z_K$ closed subalgebra. This fusion rule, likewise, holds for the $U(1)$ fields $: \exp\{i\frac{m}{K}\varphi\} :$. This isomorphism makes it clear that any simple current, $J_K$, in this subsection that contains only integer $m$ can be expressed equivalently either in terms of these parafermion fields or in terms of $U(1)$ fields. Note that one must take $n = 2m$ even since $\phi^0_m = 0$ for half–integer $m$. In this section, all of the simple twist currents, $J_K$, will be defined in terms of the parafermion fields.

[16]The string function identities are used to simplify between (3.2.25a) and (3.2.25b) where in the first equation $n \sim c^0_n$. The same is done for the partition functions which follow.

$$+ [(2+2)^4(\bar{0}+\bar{4})^4 + (0+4)^4(\bar{2}+\bar{2})^4 \qquad (3.2.25a)$$
$$+ (2+2)^2(0+4)^2(\bar{0}+\bar{4})^2(\bar{2}+\bar{2})^2$$
$$+ (0+4)^2(2+2)^2(\bar{2}+\bar{2})^2(\bar{0}+\bar{4})^2]_{\text{twisted}}$$

$$\sim |(c_0^0)^4 + (c_2^0)^4|^2_{(A_4)} + |(c_0^0)^2(c_2^0)^2 + (c_2^0)^2(c_0^0)^2|^2_{(B_4)}. \qquad (3.2.25b)$$

$D = 4 \qquad (K = 8)$:

The $\mathbf{M} = \mathbf{M}^{4,2} * \mathbf{M}^{4,2}$ model twisted by the simple current

$$J_8 \equiv \phi^0_{K/4}\phi^0_{K/4}\bar{\phi}^0_0\bar{\phi}^0_0 \qquad (3.2.26)$$

results in:

$$V(K=8) = [(0+8)(\bar{0}+\bar{8}) + (4+4)(\bar{4}+\bar{4})]^2_{\text{untwisted}}$$
$$+ [(2+6)(\bar{2}+\bar{6}) + (2+6)(\bar{2}+\bar{6})]^2_{\text{untwisted}}$$

$$+ [(4+4)(\bar{0}+\bar{8}) + (0+8)(\bar{4}+\bar{4})]^2_{\text{twisted}} \qquad (3.2.27a)$$
$$+ [(6+2)(\bar{2}+\bar{6}) + (6+2)(\bar{2}+\bar{6})]^2_{\text{twisted}}$$

$$\sim |(c_0^0)^2 + (c_4^0)^2|^2_{(A_8)} + |(c_0^0)(c_4^0) + (c_4^0)(c_0^0)|^2_{(B_8)} + 4|(c_2^0)^2|^2_{(C_8)}. \qquad (3.2.27b)$$

$D = 3 \qquad (K = 16)$:

The $\mathbf{M} = \mathbf{M}^{4,4}$ model twisted by the simple current

$$J_{16} \equiv \phi^0_{K/4}\bar{\phi}^0_0 \qquad (3.2.28)$$

produces:

$$Z(K=16) = |(0+16) + (8+8)|^2 + |(4+12) + (4+12)|^2 \qquad (3.2.29a)$$

$$\sim |c_0^0 + c_8^0|^2_{(A_{16})} + 4|c_4^0|^2_{(C_{16})}. \qquad (3.2.29b)$$

In this case the twisting is trivial since $J$ is in the initial untwisted model.

The partition function for the standard $D = 10$ superstring can also be factored into affine and theta-function $(n, \bar{n})$ parts:

$D = 10 \quad K = 2$:

$$A_2 \overset{\text{affine}}{\Longrightarrow} \sum_{i \text{ odd } =1}^{7} \binom{8}{i} (\chi_0)^i (\chi_K)^{8-i} - (\chi_{K/2})^8. \tag{3.2.30a}$$

The accompanying $(n, \bar{n})$ factor is

$$V(K = 2) = |0^8 + 1^8 + 1^8 + 2^8|^2 \tag{3.2.30b}$$

which originates from the $\mathbf{M} = \mathbf{M}^{2,1} * \mathbf{M}^{2,1} * \mathbf{M}^{2,1} * \mathbf{M}^{2,1} * \mathbf{M}^{2,1} * \mathbf{M}^{2,1} * \mathbf{M}^{2,1} * \mathbf{M}^{2,1}$ model twisted by the (simple) current

$$J_2 \equiv (\vec{n} = 11111111,\ \vec{\bar{n}} = \bar{0}\bar{0}\bar{0}\bar{0}\bar{0}\bar{0}\bar{0}\bar{0}). \tag{3.2.31}$$

The difference between this factorization and those for $K > 2$ is that here we cannot define the $(n, \bar{n})$ twist current as $\left(\phi^0_{K/4}\right)^8$ since $\phi^0_{K/4} = 0$ for $K = 2$.

All of the above simple twist currents are of the general form

$$J_K = (\phi^0_{K/4})^{D-2} (\bar{\phi}^0_0)^{D-2} \text{ for } K > 2\ . \tag{3.2.32}$$

It is likely that this specific class of twist currents is the key to spacetime supersymmetry in the parafermion models. Its twisting effect is essential in producing numerically zero fractional superstring MIPF's in three, four, and six dimensions, which are necessary conditions for spacetime SUSY. This twisting also reveals much about the necessity of non-$A_K$ sectors. Terms from the twisted and untwisted sectors of these models become equally mixed in the $|A_K|^2$, $|B_K|^2$, and $|C_K|^2$ contributions to the level $K$ partition function. Further, this twisting keeps the string functions with $n \not\equiv 0, K/2 \pmod{K}$ from mixing with those with $n \equiv 0, K/2 \pmod{K}$. The importance of this will be seen later when the states in the $C_K$ sectors are interpreted as spacetime anyons (i.e., states with non-half-integral spin), while the other sectors contain only bosons and fermions. Mixing the sectors would require spacetime anyons, destroying Lorentz invariance for $D > 3$ and could yield spacetime fractional SUSY, rather than normal SUSY.

To understand the effect of the twist fields, one first observes that the new partition functions differ from the superstring partition function in a significant way. For the superstring, all terms corresponding to spacetime bosons enter with a positive sign, while all fermionic terms are negative. In contrast for $K > 2$, spacetime bosonic and fermionic fields contribute terms with both signs. Calling the fields that enter with the expected sign "common" and the fields that enter with unexpected sign "uncommon," one can divide the $A$ sectors into two classes. For example, consider $A_4$, which can be written[17]

$$A_4 \equiv A_4^{\text{boson}} - A_4^{\text{fermion}} \ , \tag{3.2.33a}$$

where

$$A_4^{\text{boson}} = 4\left\{(c_0^0 + c_0^4)^3(c_0^2) - (c_0^2)^4\right\} \tag{3.2.33b}$$

$$A_4^{\text{fermion}} = 4\left\{(c_2^2)^4 - (c_2^0 + c_2^4)^3(c_2^2)\right\} \ . \tag{3.2.33c}$$

Here, the bosonic common fields are $(\phi_0^0 + \phi_0^2)^3(\phi_0^1)$ and the bosonic uncommon field is $(\phi_0^1)^4$. Similarly, the fermionic common field is $(\phi_1^1)^4$, while the uncommon fields are $(\phi_1^0 + \phi_1^4)^3(\phi_1^1)$.

It has also been shown[10] that when $A_K$ sector is written as $A_K^{\text{boson}} - A_K^{\text{fermion}}$, as done above for $K = 4$, the $q$-expansions of both $A_K^{\text{boson}}$ and $A_K^{\text{fermion}}$ are all positive. With the uncommon terms added one arrives at,

$$\eta^{D-2} A_K^{\text{fermion}} = (D-2)\left(\frac{(\vartheta_2)^4}{16\eta^4}\right)^{\frac{D-2}{8}} \ . \tag{3.2.34a}$$

Eq. (3.2.34a) is the standard theta function expression for $D-2$ Ramond/Majorana-Weyl fermions on the worldsheet. Further,

$$\eta^{D-2} A_K^{\text{boson}} = (D-2)\left(\frac{(\vartheta_3)^4 - (\vartheta_4)^4}{16\eta^4}\right)^{\frac{D-2}{8}} \ . \tag{3.2.34b}$$

[17] Note that in the antiholomorphic sector $J_K$ acts as the identity, so that one need only consider the holomorphic sector in what follows.

With the preceding division in mind, one can state perhaps the most important property of the twist operator: it transforms bosonic (fermionic) common fields into fermionic (bosonic) uncommon fields and vice versa. As a result, it often permits one to separate terms that represent spacetime bosons and fermions. Consider how this may be used to understand the $B_K$ sectors. For $K = 4$ and 8 the operator $(\phi^0_{K/4})^{D-2}$ transforms the primary fields corresponding to the partition function terms in the first set of brackets on the RHS of eqs. (3.2.2b,3.2.3b) into the fields represented by the partition function terms in the second set. For example, in the $K = 4$ $(D = 6)$ case

$$(\phi^0_0 + \phi^2_0)(\phi^1_0)(\phi^0_1 + \phi^2_1)^2 \quad \overset{(\phi^0_{K/4})^{D-2}}{\Longleftrightarrow} \quad (\phi^2_1 + \phi^0_1)(\phi^1_1)(\phi^2_0 + \phi^0_0)^2 \tag{3.2.35a}$$

$$(\phi^1_0)^2(\phi^1_1)^2 \quad \overset{(\phi^0_{K/4})^{D-2}}{\Longleftrightarrow} \quad (\phi^1_1)^2(\phi^1_0)^2 \,. \tag{3.2.35b}$$

Making an analogy with what occurs in the $A_K$ sector, one can suggest that $(\phi^0_{K/4})^{D-2}$ transforms bosonic (fermionic) common fields into fermionic (bosonic) uncommon fields and vice-versa in the $B_K$ sector as well. Thus, use of the twist current $J_K$ allows for bosonic and fermionic interpretation of these fields:

$$B_4 \equiv B_4^{\text{boson}} - B_4^{\text{fermion}} \,, \tag{3.2.36a}$$

where

$$B_4^{\text{boson}} = 4\left\{(c^0_0 + c^4_0)(c^2_0)(c^0_2 + c^4_2)^2 - (c^2_0)^2(c^2_2)^2\right\} \tag{3.2.36b}$$

$$B_4^{\text{fermion}} = 4\left\{(c^2_2)^2(c^2_0)^2 - (c^0_2 + c^4_2)(c^2_2)(c^0_0 + c^4_0)^2\right\} \,. \tag{3.2.36c}$$

What appears as the uncommon term, $(c^2_0)^2(c^2_2)^2$, for the proposed bosonic part acts as the common term for the fermionic half when the subscripts are reversed. One interpretation is that this implies a compactification of two transverse dimensions.[18] The spin-statistics of the physical states of the $D = 6$ model, as observed in four-dimensional uncompactified spacetime, would then be determined by the (matching) $n$

[18]This was also suggested in ref. [4] by a different argument.

subscripts of the first two string functions[19] (corresponding to the two uncompactified transverse dimensions) in each term of four string functions, $c_n^{l_1} c_n^{l_2} c_{n'}^{l_3} c_{n'}^{l_4}$ . The $B_8$ terms can be interpreted similarly when one dimension is compactified.

Unfortunately, the $C_K$ sectors are harder to interpret. Under $(\phi^0_{K/4})^{D-2}$ twisting, string functions with $K/4$ subscripts are invariant. Thus, following the pattern of $A_K$ and $B_K$ we would end up writing, for example, $C_{16}$ as

$$C_{16} = C_{16}^a - C_{16}^b \tag{3.2.37a}$$

where,

$$C_{16}^a = (c_4^2 + c_4^{14}) - c_4^8 \tag{3.2.37b}$$

$$C_{16}^b = c_4^8 - (c_4^2 + c_4^{14}) \; . \tag{3.2.37c}$$

The transformations of the corresponding primary fields under the twist operator are not quite as trivial, though. $(\phi_2^1 + \phi_2^7)$ is transformed into its conjugate field $(\phi_{-2}^7 + \phi_{-2}^1)$ and likewise $\phi_2^4$ into $\phi_{-2}^4$, suggesting that $C_{16}^a$ and $C_{16}^b$ are the partition functions for conjugate fields. Remember, however, that $C_{16} = 0$. Even though we may interpret this sector as containing two conjugate spacetime fields, this means that the partition function for each is identically zero. This effect in the $C_K$ sector will be referred to as "self-cancellation." One interpretation is that there are no states in the $C_K$ sector of the Hilbert space that survive all of the internal projections. If this is correct, a question may arise as to the consistency of the $K = 8$ and 16 theories. Alternatively, perhaps anyon statistics allow two fields of either identical fractional spacetime spins $s_1 = s_2 = \frac{2m}{K}$, or spacetime spins related by $s_1 = \frac{2m}{K} = 1 - s_2$, where in both cases $0 < m < \frac{K}{2}$ (mod 1), to cancel each others contribution to the partition function.

In general, how can one assign spacetime spin to a state, simply by looking at the corresponding primary field? As discussed in the introduction, it appears that

[19] Using the subscripts $n'$ of last two string functions to define spin-statistics in $D = 4$ uncompactified spacetime corresponds to interchanging the definitions of $B_4^{\rm boson}$ and $B_4^{\rm fermion}$ and reversing the common and uncommon labels.

one wants to associate a distinct spacetime spin with each class of $m$ values of the primary field. Using the $\phi^j_m \equiv \phi^j_{m+K} \equiv \phi^{\frac{K}{2}-j}_{m-\frac{K}{2}}$ equivalences at level $K \in 4\mathbb{Z}$, a parafermion CFT with $m \in \mathbb{Z}$ has $K/2$ distinct classes of integer $m$ values. Assuming that $m$ and $-m$ yield identical assignments, since $(\phi^0_m)^\dagger = \phi^0_{-m}$, the number of spin classes is reduced to $\frac{K}{4}+1$. Since $m = 0$ ($m = \frac{K}{4}$) is associated with spacetime bosons (fermions), it is most natural to say that general $m$ correspond to particles of spacetime spin $\mathbb{Z}^+ \pm \frac{2m}{K}$. (Perhaps spin$(m) \in \mathbb{Z} + \frac{2m}{K}$ for $0 \le m \le K/4 \pmod{K/2}$ and spin$(m) \in \mathbb{Z}^+ - \frac{2m}{K}$ for $-K/4 < m < 0 \pmod{K/2}$.) This is one of the few spin assignment rules that maintains the equivalences of the fields $\phi^j_m$ under $(j, m) \to (\frac{k}{2} - j, m - \frac{K}{2}) \to (j, m + K)$ transformations. According to this rule, the fields in the $C_K$ sectors have quarter spins (statistics), which agrees with prior claims.[13,14,10]

In order for the previous assignments to apply to the $B_K$ sectors, some dimensions must be compactified. Otherwise, by the interpretation of $m$ values above, Lorentz invariance in uncompactified spacetime would be lost. In particular, Lorentz invariance requires that either all or none of the transverse modes in uncompactified spacetime be fermionic spinors. Further, $B$ sector particles cannot correspond to fractional spacetime spin particles for a consistent theory. Thus, the $D = 6\,(4)$ model must have two (one) of its dimensions compactified.[20]

Note that the $B_8$ sector of the $D = 4$ model may be necessary for the consistency of the theory. By the above spacetime spin assignments, this model suggests massive spin-quarter states or anyons in the $C$ sectors, which presumably cannot exist in $D > 3$ uncompactified dimensions. However, the $B$ sector, by forcing compactification to three dimensions where anyons are allowed, would save the model, making it self-consistent. Of course, anyons in the $K = 16$ theory with $D_{\text{crit}} = 3$ are physically acceptable. (Indeed, no $B$ sector is needed and none exists, which would otherwise reduce the theory to zero transverse dimensions.) Thus, $K = 8$ and $K = 16$ models are probably both allowed solutions for three uncompactified spacetime dimensional

[20] This implies the $D = 6$, 4 partition functions are incomplete. Momentum and winding factors for the two compactified dimensions would have to be added (with modular invariance maintained).

models. If this interpretation is correct then it is the $B$ sector for $K = 8$ which, rather than making the theory nonsensical, makes the theory consistent.

An alternative, less restrictive, assignment of spacetime spin is possible. Another view is that the $m$ quantum number is not fundamental for determining spacetime spin. Instead, the transformation of states under $\phi^j_{K/4}$ can be considered to be what divides the set of states into spacetime bosonic and fermionic classes. With this interpretation, compactification in the $B$ sector is no more necessary than in the $A$ sector. Unfortunately, it is not *a priori* obvious, in this approach, which group of states is bosonic, and which fermionic. In the $A$ sector, the assignment can be made phenomenologically. In the $B$ sector, we have no such guide. Of course, using the $m$ quantum number to determine spacetime spin does not truly tell us which states have bosonic or fermionic statistics either, since the result depends on the arbitrary choice of which of the two (one) transverse dimensions to compactify.

A final note of caution involves multiloop modular invariance. One-loop modular invariance amounts to invariance under $S$ and $T$ transformations. However modular invariance at higher orders requires an additional invariance under $U$ transformations: Dehn twists mixing loops of neighboring tori of $g > 1$ Riemann surfaces.[15] I believe neither this new method of generating the one-loop partition functions, nor the original method of Argyres *et al.*, firmly proves the multiloop modular invariance that is required for a truly consistent theory.

### 3.3 A Generalized GSO Projection

In the last chapter, we have seen how the twist operator completely determines which terms will appear in the partition function. Some combinations of fields have their corresponding characters included, while others are conspicuously absent. We know that the choices must be as they are in order to preserve spacetime supersymmety and modular invariance, yet have no fundamental understanding of why a particular state can be kept in the physical state space. From the superstring, we know an alternate way to impose modular invariance is for the state space to be subjected to the GSO projection. This chapter endeavors to achieve a similar prescription for accepting or rejecting states in the parafermion model.

The usual GSO projection[16] can be stated very simply. One introduces the G parity operator, $(-1)^F$, where $F$ is the fermion number. $(-1)^F$ commutes with Bose fields, $X^\mu$, and anticommutes with Fermi fields, $\psi^\mu$, on the string worldsheet. The GSO projection is then simply the requirement that $(-1)^F = +1$ for all physical states. States that are odd under $(-1)^F$ are projected out. All of the above assumes we are dealing with either periodic or antiperiodic worldsheet fermions,

$$\psi(\sigma = 2\pi) = \pm\psi(\sigma = 0) \ . \tag{3.3.1}$$

Of course, one can consider more general fermions with complex boundary conditions,

$$\psi(\sigma = 2\pi) = -e^{i\pi x}\psi(\sigma = 0) \ , \tag{3.3.2}$$

where $x$ is a rational number, denoted $x = \frac{a}{b}$, for $a$, $b \in \mathbb{Z}$. The set of distinct boundary conditions is the set of $a$, $b$ coprime with $-1 \leq \frac{a}{b} < 1$. In full generality, the GSO projection coming from a sector where the set of worldsheet fermions $\{\psi^i\}$ have boundary conditions,

$$\psi^i(2\pi) = -e^{i\pi x^i}\psi^i(0) \, , \tag{3.3.3a}$$

and acting on a physical state $|\text{phys}\rangle_{\vec{y}}$ in a sector where the same fermions have boundary conditions

$$\psi^i(2\pi) = -e^{i\pi y^i}\psi(0) \, , \tag{3.3.3b}$$

demands that,

$$\left\{ e^{i\pi\vec{x}\cdot\vec{F}_{\vec{y}}} = e^{\phi}(\vec{y},\vec{x}) \right\} |\text{phys}\rangle \tag{3.3.4}$$

be satisfied for states surviving the projection, where $\phi$ is a phase depending on whether there are periodic spacetime fermions in sectors $\vec{x}$ and $\vec{y}$, as well as on the coefficients of the partition function.[17] If a state fails (3.3.4) in any sector, it will be absent from the partition function. From (3.3.4), we see that Neveu-Schwarz fermions give $\mathbb{Z}_1$ projections and Ramond fermions give $\mathbb{Z}_2$ projections. If $x = \frac{a}{b}$, a $\mathbb{Z}_b$ projection results if $a$ is even and a $\mathbb{Z}_{2b}$ projection is obtained for $a$ odd.

The boundary condition information can also be encoded in (and indeed is derived from) the mode expansion of the fermion field, which has the following form for a general twist by $\frac{a}{b}$:

$$\psi^+(\sigma_1,\sigma_2) = \sum_{n=1}^{\infty}[\psi_{n-1/2-a/2b}\exp\{-i(n-1/2-a/2b)(\sigma_1+\sigma_2)\}$$
$$+\bar{\psi}_{1/2-n-a/2b}\exp\{-i(1/2-n-a/2b)(\sigma_1+\sigma_2)\}] \quad (3.3.5a)$$

$$\psi^-(\sigma_1,\sigma_2) = \sum_{n=1}^{\infty}[\psi_{1/2-n+a/2b}\exp\{-i(1/2-n+a/2b)(\sigma_1+\sigma_2)\}$$
$$+\bar{\psi}_{n-1/2+a/2b}\exp\{-i(n-1/2+a/2b)(\sigma_1+\sigma_2)\}] \quad (3.3.5b)$$

(where $\psi_r^\dagger = \psi_{-r}$ and $\bar{\psi}_r^\dagger = \bar{\psi}_{-r}$).

$$\psi^+(\sigma_1+2\pi) = e^{+i2\pi(1/2)}\,e^{i\pi a/b}\,\psi^+(\sigma_1) \quad (3.3.6a)$$
$$\psi^-(\sigma_1+2\pi) = e^{-i2\pi(1/2)}\,e^{-i\pi a/b}\,\psi^-(\sigma_1). \quad (3.3.6b)$$

The specification of the fields is completed by stating the commutation relation that the modes obey,

$$\{\psi_c^\dagger,\psi_d\} = \{\bar{\psi}_c^\dagger,\bar{\psi}_d\} = \delta_{c,d}. \quad (3.3.7)$$

A similar analysis can be done with the $\epsilon$ fields in the $K=4$ parafermion theory. The normal untwisted (*i.e.*, Neveu-Schwarz) modes of $\epsilon$ are $\epsilon^+_{-\frac{1}{3}-n}$ and $\epsilon^-_{\frac{1}{3}-n}$ where $n \in \mathbb{Z}$. That is, untwisted $\epsilon = \epsilon^+ + \epsilon^-$ has the following normal-mode expansions:

$$\epsilon^+(\sigma_1,\sigma_2) = \sum_{n=1}^{\infty}[\epsilon_{n-1/3}\exp\{-i(n-1/3)(\sigma_1+\sigma_2)\}$$
$$+\bar{\epsilon}_{2/3-n}\exp\{-i(2/3-n)(\sigma_1+\sigma_2)\}] \quad (3.3.8a)$$

$$\epsilon^-(\sigma_1,\sigma_2) = \sum_{n=1}^{\infty}[\epsilon_{1/3-n}\exp\{-i(1/3-n)(\sigma_1+\sigma_2)\}$$
$$+\bar{\epsilon}_{n-2/3}\exp\{-i(n-2/3)(\sigma_1+\sigma_2)\}] \quad (3.3.8b)$$

(where $\epsilon_r^\dagger = \epsilon_{-r}$ and $\bar{\epsilon}_r^\dagger = \bar{\epsilon}_{-r}$). As before one can read off the associated boundary conditions in this sector,

$$\epsilon^+(\sigma_1 + 2\pi) = e^{+i2\pi/3}\,\epsilon^+(\sigma_1) \tag{3.3.9a}$$

$$\epsilon^-(\sigma_1 + 2\pi) = e^{-i2\pi/3}\,\epsilon^-(\sigma_1)\ . \tag{3.3.9b}$$

After adding a twist of $\frac{a}{b}$, $\epsilon$ has the following mode expansion:

$$\epsilon^+(\sigma_1,\sigma_2) = \sum_{n=1}^{\infty}[\epsilon_{n-1/3-a/2b}\ \exp\{-i(n-1/3-a/2b)(\sigma_1+\sigma_2)\}$$

$$+\ \bar{\epsilon}_{2/3-n-a/2b}\ \exp\{-i(2/3-n-a/2b)(\sigma_1+\sigma_2)\}] \tag{3.3.10a}$$

$$\epsilon^-(\sigma_1,\sigma_2) = \sum_{n=1}^{\infty}[\epsilon_{1/3-n+a/2b}\ \exp\{-i(1/3-n+a/2b)(\sigma_1+\sigma_2)\}$$

$$+\ \bar{\epsilon}_{n-\frac{2}{3}+\frac{a}{2b}}\ \exp\{-i(n-2/3+a/2b)(\sigma_1+\sigma_2)\}]. \tag{3.3.10b}$$

The resulting boundary conditions are

$$\epsilon^+(\sigma_1 + 2\pi) = e^{+i2\pi(1/3)}\,e^{i\pi a/b}\,\epsilon^+(\sigma_1) \tag{3.3.11a}$$

$$\epsilon^-(\sigma_1 + 2\pi) = e^{-i2\pi(1/3)}\,e^{-i\pi a/b}\,\epsilon^-(\sigma_1). \tag{3.3.11b}$$

Unfortunately, parafermions do not have simple commutation relations. Zamolodchikov and Fateev[18] have derived relations that each involve an infinite number of modes. The relations are not instructive for our purposes and are thus relegated to the references.

Making an analogy with free-fermion models, one may suppose that in $K = 4$ parafermion models the presence of a sector containing twisted $\epsilon$ fields with boundary conditions (3.3.11a) or (3.3.11b) will result in $\mathbb{Z}_b$ or $\mathbb{Z}_{2b}$ GSO projections, depending on whether $a$ is even or odd respectively. (We assume $a$ and $b$ are relative primes and $-2/3 \le a/b < 4/3$.)

Zero modes, $\epsilon_n$, $n \in \mathbb{Z}$, are obtained from the particular twist with $a/b = -2/3$. One must conjecture that this twisted sector is not projected out, in order to have spacetime fermions. As discussed in the preceding paragraph, the existence of this sector should result in a generalized $\mathbb{Z}_3$ GSO projection. Likewise for the $K = 8$ and 16 models, one might expect $\mathbb{Z}_5$ and $\mathbb{Z}_9$ projections, respectively.

What could be the form of the new GSO projection? One might anticipate that it would depend on fermion number, as is the case for $K = 2$. However, this naive generalization proves to be insufficient. It is also necessary to consider the multiplicities of the other two "physically distinguished" fields, the twist field, $\phi_1^1$ and the field $\phi_1^0$, which increases the $m$ quantum number.

It is found that one can indeed uniquely select the surviving terms in the partition function by applying a $\mathbb{Z}_3$ projection to both the left-moving modes (LM) and right-moving modes (RM) independently. Survival of a physical state, $|\text{phys}\rangle$, in the Hilbert space under this $\mathbb{Z}_3$ projection requires[21]

$$\left\{ e^{\left\{ i\pi\frac{2}{3}\cdot\left[\vec{F}_{LM\,(RM)}(\phi_0^1)+\vec{F}_{LM\,(RM)}(\phi_{\pm 1}^1)\right]\right\}} = e^{i\pi\frac{2}{3}} \right\} |\text{phys}\rangle \ , \qquad (3.3.12a)$$

or equivalently

$$\left\{ Q_{3,LM\,(RM)} \equiv \sum_i F_{i,LM\,(RM)}(\phi_0^1) + \sum_i F_{i,LM\,(RM)}(\phi_{\pm 1}^1) = 1 \pmod 3 \right\} |\text{phys}\rangle \ , \qquad (3.3.12b)$$

where $F_i(\phi_m^j)_{LM\,(RM)}$ is the number operator for the field $\phi_m^j$ along the $i^{\text{th}}$ direction for left-moving (right-moving) modes. Note, this projection alone does not prevent mixing holomorphic $A$ sector and antiholomorphic $B$ sector terms or the complex conjugates. This is prevented by the standard requirement $M_{LM}^2 = M_{RM}^2$, i.e., $L_0 = \bar{L}_0$. Aside from this requirement, one can treat the right- and left-moving sectors independently, so only the left-moving sector will be discussed in detail.

Prior to projection by this extended GSO operator, one considers all physical states associated with the LM partition function terms in the expansion of $(c_0^0 + c_0^4 + c_0^2)^4$ or $(c_2^2+c_2^4)^4$ to be in the $A$ sector. Naturally, the RM physical states in the $A$ sector are those associated with the complex conjugates of these partition function terms. In the $B$ sector one begins with all the LM physical states associated with the partition

[21] Note that the GSO projection should be written in terms of its action on states, rather than it action on characters.

function terms in the expansion of $(c_2^2+c_2^4)^2(c_0^0+c_0^4+c_0^2)^2$ or $(c_0^0+c_0^4+c_0^2)^2(c_2^2+c_2^4)^2$. There is, however, a third class of states; let us call this the "$D$" class. This latter class would be present in the original Hilbert space if not for an additional $\mathbb{Z}_2$ GSO projection. Left-moving states in $D$ class, would have partition functions that are terms in the expansion of $(c_0^0+c_0^4+c_0^2)^3(c_2^2+c_2^4)$ or $(c_2^2+c_2^4)^3(c_0^0+c_0^4+c_0^2)$. The thirty-two $D$ terms in the expansions are likewise divisible into classes based on their associated $\mathbb{Z}_3$ charges, $Q_3$. Twelve have charge 0 (mod 3), twelve have charge 1 (mod 3) and eight have charge 2 (mod 3). Using only the previous $\mathbb{Z}_3$ projection it is impossible to keep just the correct terms in the $A$ and $B$ sectors, and also project away all of the $D$ sector terms. All $D$ terms can be eliminated, without further projections on the $A$ and $B$ terms, by the $\mathbb{Z}_2$ projection,

$$\left\{\sum_i F_{i,LM\,(RM)}(\phi_1^1)+\sum_i F_{i,LM\,(RM)}(\phi_{\pm 1}^0)=0 \bmod 2\right\}|\text{phys}\rangle\ . \tag{3.3.12c}$$

(Note that for $K=2$, $\phi_1^1$ is equivalent to the vacuum and $\phi_1^0$ is indistinguishable from the usual fermion, $\phi_0^1$. Thus for $K=2$ there is no additional $\mathbb{Z}_2$ GSO projection. Unfortunately, eq. (3.3.12b) fails to generalize to the $K=2$ case.)

There are many observations which should be made about the $\mathbb{Z}_3$ charge, which help us to understand its relation to spacetime supersymmetry. First, in all sectors, the charge $Q_3$ commutes with the twist operator, $(\phi_{K/4}^0)^{D-2}$, which transforms between common and uncommon states of opposite spacetime statistics in the $A$ and $B$ sectors. If this were not the case, the GSO projection would destroy SUSY, rather than enforce it. Indeed, the preceding construction of the partition function would then be fundamentally incompatible with the projection presented here. From Table 3.2, one also sees that the values of this charge are associated with specific $mass^2$ (mod 1) levels. Only for the $A$ and $B$ sector states does $mass^2$ (mod 1) commute with the twist operator $(\phi_{K/4}^0)^{D-2}$. This implies that only the $A$ and $B$ sectors can contribute to a supersymmetric theory. Just as in the $K=2$ case, the GSO projection acts to ensure the remaining theory is supersymmetric. Consider what would happen if the GSO projection were not applied. In that case, the $mass^2$ levels (mod 1) of states present would be $mass^2=0,\ \frac{1}{12},\ \frac{2}{12},\ \dots\frac{11}{12}$. When acting on the now allowed $D$ sector fields, $(\phi^0_{K/4})^{D-2}$ would transform $mass^2 = \frac{i}{12} \pmod 1$ states into $mass^2 = \frac{i+6}{12} \pmod 1$ states. Thus, states in the $D$ sector paired by supersymmetry, if the theory had supersymmetry, would be required to appear in different sectors (*i.e.*, different mod-squared terms) of the partition function, in order to preserve $T$ invariance. As a result, the paired contributions to the partition function cannot cancel, proving that we have a non-supersymmetric theory without the GSO projection. One can similarly understand the necessity of this specific projection in the $A$ and $B$ sectors. Although $mass^2 \pmod 1$ commutes with $(\phi^0_{K/4})^{D-2}$ in the $A(Q_3 = 0)$, $A(Q_3 = -1)$, $B(Q_3 = 0)$, and $B(Q_3 = -1)$ subsectors, in each of these there is either a single bosonic state or fermionic state of lowest mass (i.e., there is no possible superpartner of equal mass. See Table 3.1.) or the lowest mass states are tachyonic (Table 3.2). Thus, the specific GSO projection chosen in terms of the $\mathbb{Z}_3$ charge projection and the $\mathbb{Z}_2$ projection once again equates to demanding spacetime supersymmetry by restricting consideration to the $Q = 1$ sector where supersymmetric pairing can occur.

Table 3.1: Masses of $K = 4$ Primary States (represented by their associated characters). With $B$ sector states, it is not *a priori* obvious, when looking at the characters, whether a particular term, or some permutation of it, should be used. Following the convention in eqs. (3.2.36a- 3.2.36c), I place the squared string function to the right, when applicable. If there is no squared term or two squared terms, both permutations are possible and included. Note that since $\phi_0^0$ and $\phi_0^4$ have the same partner under twisting, terms in the expansion of $(\phi_0^0 + \phi_0^4)^2$ are considered to be squares.

| *A*-Sector | | | | Survives | *B*-Sector | | |
|---|---|---|---|---|---|---|---|
| Boson | $Mass^2$ | Fermion | $Q_3$ | GSO | Boson ? | $Mass^2$ | Fermion ? |
| $(c_0^4)^2(c_0^4)^2$ | $3\frac{2}{3}$ | | 0 | No | $(c_0^4)^2(c_2^4)^2$ | $3\frac{1}{6}$ | $(c_2^4)^2(c_0^4)^2$ |
| $c_0^2\ c_0^4\ (c_0^4)^2$ | 3 | | 1 | Yes | $c_0^2\ c_0^4\ (c_2^4)^2$ | $2\frac{1}{2}$ | $c_2^2\ c_2^4\ (c_0^4)^2$ |
| $c_0^0\ c_0^4\ (c_0^4)^2$ | $2\frac{2}{3}$ | $(c_2^4)^2(c_2^4)^2$ | 0 | No | $c_0^0\ c_0^4\ (c_2^4)^2$ | $2\frac{1}{6}$ | $(c_2^4)^2 c_0^0\ c_0^4$ |
| $(c_0^4)^2(c_0^2)^2$ | $2\frac{1}{3}$ | | $-1$ | No | $(c_0^4)^2(c_2^2)^2$ | $1\frac{5}{6}$ | $(c_2^4)^2(c_0^2)^2$ |
| $(c_0^2)^2(c_0^4)^2$ | | | | | $(c_0^2)^2(c_2^4)^2$ | | $(c_2^2)^2(c_0^4)^2$ |
| | | | | | $c_0^2\ c_0^4\ c_2^2\ c_2^4$ | | $c_2^2\ c_2^4\ c_0^2\ c_0^4$ |
| $c_0^2\ c_0^0\ (c_0^4)^2$ | 2 | $c_2^?\ c_2^4\ (c_2^4)^2$ | 1 | Yes | $c_0^2\ c_0^0\ (c_2^4)^2$ | $1\frac{1}{2}$ | $c_2^2\ c_2^4\ c_0^0\ c_0^4$ |
| $c_0^2\ c_0^4\ (c_0^2)^2$ | $1\frac{2}{3}$ | | 0 | No | $c_0^2\ c_0^4\ (c_2^2)^2$ | $1\frac{1}{6}$ | $c_2^2\ c_2^4\ (c_0^2)^2$ |
| $(c_0^0)^2(c_0^4)^2$ | | | | | $(c_0^0)^2(c_2^4)^2$ | | |
| $(c_0^4)^2(c_0^0)^2$ | | | | | | | $(c_2^4)^2(c_0^0)^2$ |
| $c_0^0\ c_0^4\ (c_0^2)^2$ | $1\frac{1}{3}$ | $(c_2^2)^2(c_2^4)^2$ | $-1$ | No | $c_0^0\ c_0^4\ (c_2^2)^2$ | $\frac{5}{6}$ | |
| | | $(c_2^4)^2(c_2^2)^2$ | | | $c_0^0\ c_0^2\ c_2^2\ c_2^4$ | | $c_2^2\ c_2^4\ c_0^0\ c_0^2$ |
| $(c_0^2)^2(c_0^2)^2$ | 1 | | 1 | Yes | $(c_0^2)^2(c_2^2)^2$ | $\frac{1}{2}$ | $(c_2^2)^2(c_0^2)^2$ |
| $c_0^2\ c_0^4\ (c_0^0)^2$ | | | | | | | $c_2^2\ c_2^4\ (c_0^0)^2$ |
| $c_0^2\ c_0^0\ (c_0^2)^2$ | $\frac{2}{3}$ | $c_2^2\ c_2^4\ (c_2^2)^2$ | 0 | No | $c_0^2\ c_0^0\ (c_2^2)^2$ | $\frac{1}{6}$ | |
| $c_0^0\ c_0^4\ (c_0^0)^2$ | | | | | | | |
| $(c_0^0)^2(c_0^2)^2$ | $\frac{1}{3}$ | | $-1$ | No | $(c_0^0)^2(c_2^2)^2$ | $-\frac{1}{6}$ | |
| $(c_0^2)^2(c_0^0)^2$ | | | | | | | $(c_2^2)^2(c_0^0)^2$ |
| $c_0^2\ c_0^0\ (c_0^0)^2$ | 0 | $(c_2^2)^2(c_2^2)^2$ | 1 | Yes | | | |
| $(c_0^0)^2(c_0^0)^2$ | $-\frac{1}{3}$ | | 0 | No | | | |

Table 3.2: Mass Sectors as Function of $\mathbb{Z}_3$ Charge

Columns one and seven give the lowest $mass^2$ of a state with center column $\mathbb{Z}_3$ charge in the appropriate sector. For the $D$ sector states, $mass^2$ values in column two transform into $mass^2$ values in column six of the same row and vice-versa, under $(\phi^0_{K/4})^{D-2}$ twistings.

| Lowest $M^2$ | $M^2$ mod 1 | Sector | $\mathbb{Z}_3$ Charge | Sector | $M^2$ mod 1 | Lowest $M^2$ |
|---|---|---|---|---|---|---|
| 0 | 0 | $A$ | $Q_3 = 1$ | $B$ | $\frac{6}{12}$ | $\frac{6}{12}$ |
| $-\frac{1}{12}$ | $\frac{11}{12}$ | $D$ | $Q_3 = 0$ | $D$ | $\frac{5}{12}$ | $\frac{5}{12}$ |
| $-\frac{2}{12}$ | $\frac{10}{12}$ | $B$ | $Q_3 = -1$ | A | $\frac{4}{12}$ | $\frac{4}{12}$ |
| $-\frac{3}{12}$ | $\frac{9}{12}$ | $D$ | $Q_3 = 1$ | $D$ | $\frac{3}{12}$ | $\frac{3}{12}$ |
| $-\frac{4}{12}$ | $\frac{8}{12}$ | $A$ | $Q_3 = 0$ | $B$ | $\frac{2}{12}$ | $\frac{2}{12}$ |
| $\frac{7}{12}$ | $\frac{7}{12}$ | $D$ | $Q_3 = -1$ | D | $\frac{1}{12}$ | $\frac{1}{12}$ |

In the $K = 4$ case, unlike $K = 2$, we find that the $\mathbb{Z}_3$ projection in the Ramond sector wipes out complete spinor fields, not just some of the modes within a given spin field. Of course, this type of projection need not occur in the Ramond sector of the superstring since there are no fermionic states with fractional $mass^2$ values in the $D = 10$ model.

### 3.4 The Uniqueness of $\epsilon$ and the Twist Field, $\phi^{K/4}_{K/4}$

Throughout the previous chapters, it has been assumed that the analogs of the fermion and twist operator of the superstring are $\epsilon = \phi^1_0$ and $\phi^{K/4}_{K/4}$ respectively. While both these fields reduce to the proper fields for $K = 2$, the possible uniqueness of this generalization has not been properly addressed. To begin with, we note that the choice of these fields has profound consequences. In particular, it determines the possible levels, $K$, and the critical dimensions of the theory. If one assumes that the twist operator $\phi_t^{\mu}$ acting on the (tachyonic) vacuum produces a massless spacetime spinor vacuum along the direction $\mu$, and $\epsilon^{\mu}$ produces a massless spin-one state, then for spacetime supersymmetry to be possible, (specifically $N = 2$ SUSY for fractional type II theories and $N = 1$ SUSY for fractional heterotic) $h(\epsilon)/h(\phi_t)$ must equal the number of transverse spin modes, i.e.,

$$h(\epsilon) = (D-2)h(\phi_t) \tag{3.4.1}$$

Thus we can demand,

$$h(\epsilon)/h(\phi_t) \in \mathbb{Z}^+ \ . \tag{3.4.2}$$

With the particular choice of $\epsilon$ and $\phi_t$ made previously, one arrives at,

$$\begin{aligned} h(\phi_0^1) &= (D-2)h(\phi_{K/4}^{K/4}) \\ \frac{2}{K+2} &= (D-2)\frac{K/8}{K+2}. \end{aligned} \tag{3.4.3}$$

Hence,

$$D = 2 + \frac{16}{K} \in \mathbb{Z} \ . \tag{3.4.4}$$

Thus, from this one assumption, the possible integer spacetime dimensions are determined along with the possible levels $K$. Perhaps not coincidentally, the allowed dimensions are precisely the ones in which minimal super Yang Mills theories are possible. This is clearly a complementary method to that in [10,14,13] for determining $D$ and is in ways simpler, since no knowledge of the intercept or conformal anomaly is required.

Demanding eq. (3.4.1), guarantees spin-1 and spin-1/2 partners at

$$m^2 = m^2(\text{vacuum}) + h(\phi_0^1) = m^2(\text{vacuum}) + (D-2) * h(\phi_{K/4}^{K/4}) \,, \tag{3.4.5}$$

as seen in fig. 3.1.

Figure 3.1

| $m^2$(spin 1) | $=$ | $m^2$(spin 1/2) |
|---|---|---|
| $h(\phi_0^1)\Uparrow$ | | $(D-2)\times h(\phi_{K/4}^{K/4})\Uparrow$ |

$m^2(vacuum)$

*A priori* simply demanding the ratio be integer is neither sufficient to guarantee spacetime supersymmetry, nor that the states be massless. Of course, in the models considered so far, we took $\phi_t = \phi_{K/4}^{K/4} \equiv \phi_{-K/4}^{K/4}$ and $\epsilon = \phi_0^1$ for $K = 4$, 8, and 16, as the generalization of $\phi_{1/2}^{1/2}$ at $K = 2$. We now are prepared to ask whether there any other pairs of parafermion operators at additional levels $K$ that could act on the tachyonic bosonic vacuum to yield a potential fermionic vacuum as well a spin-1 bosonic state of equal mass. Masslessness is not yet required.

We begin by looking for generalizations of $\phi_t$. Any potential level $K$ candidate $\phi_m^j$ must satisfy the condition that

$$\frac{K}{K+2}\left[j(j+1)-2\right] \leq m^2 \leq j^2 \leq K^2/4 \ . \tag{3.4.6}$$

The latter inequalities simply use $|m| \leq j \leq K/2$, which is required for any Kač-Moody fields at level $K$. The first half of the inequality, $\frac{K}{K+2}\left[j(j+1)-2\right] \leq m^2$, results from the weak requirement that the conformal dimension of the candidate field creating the fermion ground state along one spacetime direction cannot be greater than the conformal dimension of $\epsilon$, i.e., $h(\phi_t) \leq h(\epsilon)$.

Using equation (3.4.6), one can determine both the minimum and maximum values of $K$, for a given $j$, independent of the value of $m$. These limits are $K_{\rm min} = 2j$ and $K_{\rm max} = \text{int}\left(\frac{2j^2}{j-2}\right)$, where the latter applies only for $j > 2$. Thus, the number of different levels $K$ that can correspond to the field $\phi_m^j$ is $\text{int}\left(\frac{5j-2}{j-2}\right)$. This number decreases to six as $j$ increases to 10 and equals 5 for $j > 10$. Hence, for $j > 10$ we can express the possible levels $K$ under consideration by $K_i = 2j + i$, where $i = 0$ to 4.

Also, coupled with $K_{min} = 2j$, the weak constraint on $m$ implies that for $j > 1$ we need only consider $\phi^j_j$ fields.

For $j > 10$ the search reduces to finding fields $\phi^j_j$ whose conformal dimensions satisfy

$$\frac{h(\epsilon)}{h(\phi^j_j)} = \frac{\frac{2}{K_i+2}}{\frac{j(j+1)}{K_i+2} - \frac{j^2}{K_i}} \in \mathbb{Z}. \tag{3.4.7}$$

One finds that there are no solutions to for $i = 0$ to 4 and $j > 10$. Thus we have reduced our search for possible alternative sources of fermionic ground states to only $\phi^j_j$ with $0 < j \leq 10$. Within this range of $j$, a computer search reveals the following complete set of $\phi^j_m$ fields that obey equation (3.4.7), as shown in Table 3.3:

Table 3.3: Potential Alternatives to $\phi^{K/4}_{K/4}$ for Spin Fields (An asterisk marks the models discussed in the previous sections.)

| $j$ | $\pm m$ | $K$ | $i$ | $h(\epsilon)$ | $h(\phi^j_m)$ | $D$ |
|---|---|---|---|---|---|---|
| 1/2 | 1/2 | 2 | 1 | 1/2 | 1/16 | 10 * |
| | | 3 | 2 | 2/5 | 1/15 | 8 |
| | | 5 | 4 | 2/7 | 2/35 | 7 |
| 1 | 1 | 3 | 1 | 2/5 | 1/15 | 8 |
| | | 4 | 2 | 1/3 | 1/12 | 6 * |
| | | 6 | 4 | 1/4 | 1/12 | 5 |
| 3/2 | 3/2 | 9 | 6 | 2/11 | 1/11 | 4 |
| 2 | 2 | 5 | 1 | 2/7 | 2/35 | 7 |
| | | 6 | 2 | 1/4 | 1/12 | 5 |
| | | 8 | 4 | 1/5 | 1/10 | 4 * |
| 2.5 | 2.5 | 25 | 20 | 2/27 | 2/27 | 3 |
| 3 | 3 | 9 | 3 | 2/11 | 1/11 | 4 |
| | | 18 | 12 | 1/10 | 1/10 | 3 |
| 4 | 4 | 16 | 8 | 1/9 | 1/9 | 3 * |
| 6 | 6 | 18 | 6 | 1/10 | 1/10 | 3 |
| 10 | 10 | 25 | 5 | 2/27 | 2/27 | 3 |

The set of distinct, physically relevant fields can be further reduced because there is redundancy in the above list. For all but the standard $\phi^{K/4}_{K/4}$ solutions, there are two fields at each level, with distinct values of $j$. However, these pairs are related by the field equivalences (2.2.7):

$$\phi^{1/2}_{\pm 1/2} \equiv \phi^{1}_{\mp 1} \quad \text{at level} \quad K = 3 \tag{3.4.8a}$$

$$\phi^{1/2}_{\pm 1/2} \equiv \phi^{2}_{\mp 2} \quad \text{at level} \quad K = 5 \tag{3.4.8b}$$

$$\phi^{1}_{\pm 1} \equiv \phi^{2}_{\mp 2} \quad \text{at level} \quad K = 6 \tag{3.4.8c}$$

$$\phi^{3/2}_{\pm 3/2} \equiv \phi^{3}_{\mp 3} \quad \text{at level} \quad K = 9 \tag{3.4.8d}$$

$$\phi^{3}_{\pm 3} \equiv \phi^{6}_{\mp 6} \quad \text{at level} \quad K = 18 \tag{3.4.8e}$$

$$\phi^{5/2}_{\pm 5/2} \equiv \phi^{10}_{\mp 10} \quad \text{at level} \quad K = 25. \tag{3.4.8f}$$

Because $\phi^j_m$ and $\phi^j_{-m}$ have identical partition functions and $\phi^j_{-m} \equiv (\phi^j_m)^\dagger$ we can cut the number of possible alternative fields in half, down to six.[22]

If we want models with minimal super Yang-Mills Lagrangians it may be possible to reduce the number of new possibilities to investigate. Such theories exist classically only in $D_{SUSY} = 10, 6, 4, 3,$ (and 2) spacetime. With no compactification, one can consider only those $\phi^j_j$ in the above list that have integer conformal dimension ratios of $D_{SUSY} - 2 = h(\phi^1_0)/h(\phi^j_j) = 8, 4, 2,$ and 1. This would reduce the number of new fields to consider to just the three for $D = 4$, and 3 since there there are no new choices for $D = 10$ or 6. Of course, compactification may be desired or required, as previously discussed, which could allow more possibilities.

In fact, it is easy to see that demanding masslessness of the fermionic vacuum and the spin-1 boson implies the uniqueness of the twist field for $K < 500$, as the following argument demonstrates:

Assume we have a consistent closed fractional superstring theory at level $K$ with supersymmetry in $D$ dimensional spacetime, ($N = 2$ for fractional type II theories and $N = 1$ SUSY for fractional heterotic). Let the massless left- (right-) moving

[22]Note, we have not been distinguishing between $\pm$ on $m$ in any case.

spin-1 field be $(\phi_m^j)^\mu|\text{vacuum}>$. This requires that $\phi_m^j$ have conformal dimension

$$h(\phi_m^j) = c_{\rm tr}/24 = (D-2)\frac{K}{8(K+2)} \ , \tag{3.4.9}$$

where $c_{\rm tr}$ is the total central charge of the transverse dimensions. Thus, the twist field $\phi_t$ that produces the spinor vacuum along one of the $D-2$ transverse dimensions must have conformal dimension

$$h(\phi_t) = \frac{K}{8(K+2)} \ . \tag{3.4.10}$$

For $K < 500$ the only primary fields with dimension $\frac{K}{8(K+2)}$ are the series of $\phi_{K/4}^{K/4}$ for $K \in 2\mathbb{Z}$, and the accidental solutions $\phi_0^2$ for $K = 48$, $\phi_0^3$ for $K = 96$, and $\phi_{7/2}^{9/2}$ for $= 98$. The solutions with $m = 0$ clearly cannot be used to generate spacetime fermions. The $K = 98$ case could not be used because there is no candidate field at that level whose conformal dimension is an integer multiple of (3.4.10) and thus no replacement exists for $\epsilon \equiv \phi_0^1$.

The other alternatives previously considered must therefore not be consistent string theories. They lack the massless left- and right-moving spin-1 spacetime fields, whose tensor product is required to form a massless spin-2 graviton in closed string theory. The physical meaning of these theories without gravity is not clear.

The above assertion that $\phi_{K/4}^{K/4}$ must be the spin-$\frac{1}{2}$ operator, does not immediately lead one to conclude that $\phi_0^1$ is the only possible choice for producing massless boson fields. Table 3.4 shows alternative fields at new levels $K \neq 2$, 4, 8, or 16 whose conformal dimension is one, two, or four times the conformal dimension of $\phi_{K/4}^{K/4}$. (Note that successful alternatives to $\phi_0^1$ would lead to a relationship between level and spacetime dimension differing from eq. (3.4.4).) However, nearly all alternatives are of the form $\phi_0^{j>1}$ and one would expect that modular invariant models using $\phi_0^{j>1}$ to create massless bosons, would very likely include tachyonic $(\phi_0^i)^\mu|\text{vacuum}>$ states with $i < j$. Alternatively, we may conjecture that it is impossible to construct valid GSO projections which would project away these tachyons while simultaneously keeping the massless graviton and gravitino and preserving modular invariance. Further, the remaining fields on the list have $m \neq 0 \pmod{K}$. Each of these would not have the correct fusion rules with itself, nor with $\phi_{K/4}^{K/4}$, to be a spacetime boson.

Table 3.4: Fields $\phi^{j_1}_{m_1} \neq \phi^1_0$ with Conformal Dimensions in Integer Ratio with $h(\phi^{K/4}_{K/4})$.

| $K$ | $\phi^j_m$ | $h(\phi^j_m)/h(\phi^{K/4}_{K/4})$ |
|---|---|---|
| 12 | $\phi^2_0$ | 4 |
| 24 | $\phi^2_0$ | 2 |
| | $\phi^3_0$ | 4 |
| 36 | $\phi^7_6$ | 4 |
| 40 | $\phi^4_0$ | 4 |
| 48 | $\phi^2_0$ | 1 |
| | $\phi^3_0$ | 2 |
| 60 | $\phi^5_0$ | 4 |
| 80 | $\phi^4_0$ | 2 |
| 84 | $\phi^6_0$ | 4 |
| 96 | $\phi^3_0$ | 1 |
| 112 | $\phi^7_0$ | 4 |
| 120 | $\phi^5_0$ | 2 |
| $\vdots$ | $\vdots$ | $\vdots$ |

### 3.5 Bosonization of the $K = 4$ Theory.

A general parafermion theory can be bosonized using background charge. However, for $K = 4$, a much simpler bosonization prescription is possible. This is because the $\mathbb{Z}_4$ parafermion has a central charge of one, precisely the central charge of a free boson. For $c = 1$ the bosonization of CFT's has been completely classified.[19] The three acceptable solutions correspond to a boson propagating on either (1) a torus of radius R, (2) a $\mathbb{Z}_2$ orbifold of radius R, or (3) a discrete orbifold space defined on $SU(2)/\Gamma_i$, where $\Gamma_i$ are discrete subgroups of $SU(2)$. The $\mathbb{Z}_4$ parafermion CFT is equivalent to the $\mathbb{Z}_2$ orbifold at radius $R = \sqrt{6}/2$ (or $R = 1/\sqrt{6}$ by duality).

As discussed in the introduction, an $S^1/\mathbb{Z}_2$ orbifold at radius $R$ has the partition function,

$$Z_{\rm orb}(R) = \frac{1}{2}\{Z(R) + \frac{|\eta|}{|\vartheta_2|} + \frac{|\eta|}{|\vartheta_3|} + \frac{|\eta|}{|\vartheta_4|}\} \tag{3.5.1a}$$

$$= \frac{1}{2}\{Z(R) + \frac{|\vartheta_3\vartheta_4|}{|\eta|^2} + \frac{|\vartheta_2\vartheta_4|}{|\eta|^2} + \frac{|\vartheta_2\vartheta_3|}{|\eta|^2}\} \tag{3.5.1b}$$

where,

$$Z(R) = \frac{1}{\eta\bar{\eta}} \sum_{m,n=\infty}^{\infty} q^{(\frac{m}{2R}+nR)^2/2}\bar{q}^{(\frac{m}{2R}-nR)^2/2} \tag{3.5.1c}$$

is the partition function for a free scalar boson compactified on a circle of radius $R$.

For $R = \frac{\sqrt{6}}{2}$ the generalized momentum states $p = \frac{m}{\sqrt{6}} + \frac{n\sqrt{6}}{2}$ can be classified into four sectors based on the value of $\frac{p^2}{2}$ (mod 1). The classes are $\frac{p^2}{2} = 0, \frac{1}{12}, \frac{1}{3}$, and $\frac{3}{4}$ (mod 1). Also, it is easily seen that for a given $(m,n)$, $p = \frac{m}{\sqrt{6}} + \frac{n\sqrt{6}}{2}$ and $\bar{p} = \frac{m}{\sqrt{6}} - \frac{n\sqrt{6}}{2}$ belong to the same sector. That is,

$$\frac{1}{2}(p^2 - \bar{p}^2) \equiv 0 \bmod 1\,, \tag{3.5.2}$$

(as required by level matching). This will be significant below when we match bosonized and unbosonized terms of the partition functions.

The untwisted sector of the model corresponds to the first two terms on the right-hand side of eq. (3.5.1b) and the twisted sector the remaining two terms. The factor of $\frac{1}{2}$ results from a "GSO projection" requiring invariance of states under the orbifold operation, $g:\ X(z,\bar{z}) \to -X(z,\bar{z})$. In the untwisted sector, this invariance leads one to pair $|m,n\rangle$ and $|-m,-n\rangle$ into symmetric and antisymmetric combinations and reject half of them, depending on whether the net number of oscillations acting on them is even or odd. The second term in (3.5.1a) and (3.5.1b) corresponds to states antiperiodic along the "time" loop and thus can only be states built from net even numbers of $\alpha(z)$ and $\bar{\alpha}(\bar{z})$'s acting on $|m=n=0\rangle$. Similarly, the surviving twisted sector states have a net even number of $\alpha_r$ and $\bar{\alpha}_r$, $r \in \mathbb{Z}+\frac{1}{2}$, oscillations acting on the $|m=n=0\rangle$ twisted vacuum with $h = \bar{h} = \frac{1}{16}$. Thus the twisted states have conformal dimensions of the form $(h,\bar{h}) \in (\frac{1}{16}+\mathbb{Z},\ \frac{1}{16}+\mathbb{Z})$ or $(\frac{1}{16}+\mathbb{Z}+\frac{1}{2},\ \frac{1}{16}+\mathbb{Z}+\frac{1}{2})$.

It is instructive to consider the $\mathbb{Z}_4$ primary fields and their corresponding conformal dimensions and characters. These are shown in table 3.5:

Table 3.5: The $\mathbb{Z}_4$ primary fields, their associated conformal dimensions and characters.

| Primary Fields | Conformal Dimension h | Character |
|---|---|---|
| $\phi_0^0 \equiv \phi_0$ | $0$ | $\eta c_0^0$ |
| $\phi_{-1}^2 = \phi_1^0 \equiv \phi_1 \equiv \phi_3^\dagger$ | $\frac{3}{4}$ | $\eta c_2^4$ |
| $\phi_0^2 = \phi_2^0 \equiv \phi_2$ | $1$ | $\eta c_0^4$ |
| $\phi_1^2 = \phi_3^0 \equiv \phi_3 = \phi_1^\dagger$ | $\frac{3}{4}$ | $\eta c_2^4$ |
| $\phi_0^1 \equiv \epsilon$ | $\frac{1}{3}$ | $\eta c_0^2$ |
| $\phi_1^1 = \phi_{-1}^1$ | $\frac{1}{12}$ | $\eta c_2^2$ |
| | | |
| $\phi_{-1/2}^{1/2}$ | $\frac{1}{16}$ | $\eta c_{-1}^1$ |
| $\phi_{1/2}^{1/2}$ | $\frac{1}{16}$ | $\eta c_1^1$ |
| $\phi_{-1/2}^{3/2}$ | $\frac{9}{16}$ | $\eta c_{-1}^3$ |
| $\phi_{1/2}^{3/2}$ | $\frac{9}{16}$ | $\eta c_1^3$ |

The first six primary fields listed above have representations in the untwisted sector of the partition function and the latter four have representations in the twisted sector.[23] Considering the classes of conformal dimensions of these states, $\frac{p^2}{2} = 0,\ \frac{1}{12},\ \frac{1}{3},\ \frac{3}{4}\pmod 1$ one finds the following identities for string functions:

$$|\eta c_0^0|^2 + |\eta c_0^4|^2 = \frac{1}{2}\left\{\frac{1}{|\eta|^2}\sum_{\frac{p^2}{2}\equiv 0 \bmod 12} q^{(\frac{m}{2R}+nR)^2/2}\bar{q}^{(\frac{m}{2R}-nR)^2/2} + \frac{|\vartheta_3\vartheta_4|}{|\eta|^2}\right\} \tag{3.5.3a}$$

$$|\eta c_2^2|^2 = \frac{1}{2}\frac{1}{|\eta|^2}\sum_{\frac{p^2}{2}\equiv 1 \bmod 12} q^{(\frac{m}{2R}+nR)^2/2}\bar{q}^{(\frac{m}{2R}-nR)^2/2} \tag{3.5.3b}$$

$$|\eta c_0^2|^2 = \frac{1}{2}\frac{1}{|\eta|^2}\sum_{\frac{p^2}{2}\equiv 4 \bmod 12} q^{(\frac{m}{2R}+nR)^2/2}\bar{q}^{(\frac{m}{2R}-nR)^2/2} \tag{3.5.3c}$$

---

[23] We note that independent of the choice of the affine factor in the partition functions of section 3.2, the required $(n,\bar{n})$ partition functions effectively remove from the theory the primary fields with half-integer $j$, $m$. The only theory which uses such fields is the $K = 2$ superstring. This observation may prove significant in understanding what distinguishes the superstring from other models.

$$|\eta c^4_{2,-2}|^2 = \frac{1}{2}\frac{1}{|\eta|^2} \sum_{\frac{p^2}{2}\equiv 9 \bmod 12} q^{(\frac{m}{2R}+nR)^2/2}\bar{q}^{(\frac{m}{2R}-nR)^2/2} \tag{3.5.3d}$$

$$|\eta c^1_{1,-1}|^2 + |\eta c^3_{1,-1}|^2 = \frac{1}{2}\left\{\frac{|\vartheta_2\vartheta_4|}{|\eta|^2} + \frac{|\vartheta_2\vartheta_3|}{|\eta|^2}\right\}. \tag{3.5.3e}$$

Since the above identities involve $|\eta c^{2j}_{2m}|^2$ rather than just $\eta c^{2j}_{2m}$, they do not necessarily imply the exact equivalence, term by term, of the parafermion and orbifold models. However, more fundamental identities for the string functions do exist. Since none of the $\mathbb{Z}_4$ parafermion fields connected with the twisted orbifold sector appear in the $K = 4$ FSC model, we can look just at a left-moving (holomorphic) boson compactified at $R = \sqrt{6}$ with no twisting (i.e., we can drop the terms resulting from twisting by the orbifold).

$$Z(z, R = \sqrt{6}) = \frac{1}{\eta} \sum_{m=-\infty}^{\infty} q^{[\frac{m}{R}]^2/2} \tag{3.5.4a}$$

If we change the summation index, using $m = 6n + i$, $i = 0$ to 5, then the partition function can be split into[24]

$$Z(z, R = \sqrt{6}) = \frac{1}{\eta} \sum_{\substack{n=-\infty \\ i=0 \text{ to } 5}}^{\infty} q^{[\frac{(6n+i)}{R}]^2/2} \tag{3.5.4b}$$

which suggests the following more succinct identities:[25]

$$\eta c^2_2 = \frac{1}{\eta} q^{\frac{1}{12}} \sum_{n=-\infty}^{\infty} q^{3n^2+n} \tag{3.5.5a}$$

$$\eta c^2_0 = \frac{1}{\eta} q^{\frac{1}{3}} \sum_{n=-\infty}^{\infty} q^{3n^2+2n} \tag{3.5.5b}$$

$$\eta c^4_2 = \eta c^4_{-2} = \frac{1}{2\eta} q^{\frac{3}{4}} \sum_{n=-\infty}^{\infty} q^{3n^2+3n} \tag{3.5.5c}$$

$$\eta(c^0_0 + c^4_0) = \frac{1}{\eta} \sum_{n=-\infty}^{\infty} q^{3n^2}. \tag{3.5.5d}$$

---

[24]Note that $m = i \pmod 6$ terms are equivalent to $m = -i \pmod 6$ terms, so including a factor of two, we need only sum over $i = 0$ to 3.

[25]These were verified up to $q^{1300}$ using Mathematica.

### 3.6 Conclusions

A viable and consistent generalization of the superstring would be a significant development. It would add to the very small number of viable string theories and possibly shed light on the relationship between different string theories. For example, it has been conjectured, but not rigorously proven, that all string theories are special cases of the bosonic string.[20] Reparameterization invariance is the one symmetry required for all string theories, though other symmetries can be added to arrive at restricted subsets. This claim is hard to study as we now only have one additional symmetry, supersymmetry, that is known to give a consistent string theory. Work on fractional supersymmetry and W algebras may ultimately provide the framework for addressing claims about the relationships between different string theories.

This raises the obvious question whether the fractional string is viable. While this thesis does not provide a definitive answer, it does shown that the fractional superstring has many intriguing features that merit further study. The partition functions for these theories are found to have simple origins and the new massive sectors are shown to be related to the more typical $A_K$ sector. A possible GSO projection is found, as well as a spacetime interpretation of the fields in the $B_K$ sectors. A form of self-cancellation was also indicated in the $C_K$ sectors.

Nevertheless, fundamental questions remain about the ghost system and current algebra, which prevent a definitive statement about whether or not this is a truly consistent theory. The ghost system is not known, but there are indications that it is not free. More troubling are recent indications that the conformal charge of the matter is not $c = D \times 3K/(K+2)$ as one would expect in a tensor product theory. For example, when one considers the appearance of extra null states in the spectrum, it seems that the $K = 4$ theory has $c = 10$.[21] This would imply that the theory is fundamentally not a tensor product theory and that the algebra is not linear. This seems to contradict strong evidence given in this dissertation that the partition function is precisely what one would get from a simple tensor product. If indeed the theory turns out not to result from a tensor product, it seems much effort would be required to explain why its partition function is in tensor product form. As is often the case in research, answering some questions raises an entirely new set of problems. Even if fractional superstring theories are ultimately shown to be flawed, I believe such work will remain valuable in that it may provide new insight into the one case we know is consistent, the $K = 2$ superstring.

# 4. String Cosmology

## 4.1 Introduction

In spite of extraordinary successes, traditional cosmology has left unanswered a number of fundamental questions and been plagued by potential inconsistencies. Arguably the most troubling problem is the pointlike initial singularity at the time of the big bang. Almost equally distressing is the related prediction of infinite initial temperature. Frequently, one sidesteps the preceding problems by appealing to some future theory of quantum gravity. A classical theory, general relativity is expected to break down at small scales where quantum effects should dominate. Thus, the divergences predicted by the standard Friedman-Robertson-Walker cosmology are expected to be artifacts of using a classical theory in a quantum regime. One hopes that a proper theory of gravity would predict a small but non-singular universe at the time of the big bang, which would have no divergent physical quantities. Indeed, such an outcome can be seen as a test for any candidate theory of quantum gravity.

An equally compelling, albeit less common, open question in traditional cosmology is why we live in a four-dimensional universe. While many are content to insert the dimension of spacetime by hand, it would be more satisfying to explain its value.

One need no longer talk about quantum gravity as a distant dream; with the advent of string theory, we have a candidate theory of quantum gravity today and therefore an unrivaled potential tool for understanding cosmology. Conversely, cosmology provides a unique arena for testing string theory's performance as a theory of quantum gravity. Since, string theory may make qualitatively different predictions than point particle theories, one can hope that some of the consequences are observable and will lead to the first experimental (or at least observational) tests of string theory.

Indeed, as will be seen below, string theory completely resolves the problem of an initial pointlike singularity. Furthermore, it goes a long way towards providing a maximum finite temperature for the universe. Arguments are also being developed for why there are three "large" spatial dimensions in our universe, rather than nine or 25, etc.. Finally, string theory may suggest solutions to many other cosmological problems, as it naturally could provide "cosmic" strings, other sources of dark matter and ultimately a resolution to the cosmological constant problem.

### 4.1.1 Duality and the Initial Singularity

String theory has already resolved the problem of an initial singularity. Using the duality symmetry of string theory, one can immediately see that the radius of the universe has some non-zero minimum value. Duality is easily seen by considering the string coordinate expansion (using the bosonic string for simplicity) for a single circular dimension:[16]

$$X = x + (\frac{m}{2R} + nR)(\tau + \sigma) + (\frac{m}{2R} - nR)(\tau - \sigma) + \text{oscillators} \tag{4.1.1}$$

where $m$, $n \in \mathbb{Z}$. We see that the left- and right-moving momenta are,

$$(p_L, p_R) = (\frac{m}{2R} + nR, \frac{m}{2R} - nR). \tag{4.1.2}$$

The first term, $\frac{m}{2R}$, is interpreted as one half the center of mass momentum of the string, while the second term, $\pm nR$ is the winding mode "momentum." The corresponding string mass spectrum is,

$$\frac{1}{4}M^2 = N + \frac{1}{2}(\frac{m}{2R} - nR)^2 - 1 + \tilde{N} + \frac{1}{2}(\frac{m}{2R} + nR)^2 - 1. \tag{4.1.3}$$

If one lets $R \to \frac{1}{2R}$ with $m \leftrightarrow n$, the spectrum is preserved. Indeed, the scattering amplitudes also respect "$R \leftrightarrow \frac{1}{R}$ duality," and it has been shown that replacing $R$ with $\frac{1}{R}$ produces an isomorphic conformal field theory.[22] With duality, a pointlike universe is equivalent to one that is infinitely big. The smallest universe possible has $R = 1$ in units of the Planck length, which is exactly the minimum size one expected to be predicted by quantum gravity!

### 4.1.2 The Hagedorn Temperature

The very high temperature behavior of string theory remains a mystery. There is still little agreement in the literature about whether string theory even provides a maximum temperature. I would argue that duality gives stringy evidence for a finite maximum temperature. The temperature must respect duality, so it remains finite at $R = 0$ (i.e., at $R = \infty$). It could possibly diverge at $R = 1$, (see fig. 4.1a), but this would give the universe an infinite total energy. The resulting infinite energy density could not be reduced by any finite amount of expansion, so such a scenario seems untenable. Furthermore, as will be discussed shortly, the early universe likely oscillated about $R = 1$, so that a finite maximum temperature (see fig. 4.1b) is preferred.

The existence of a distinguished temperature in dual models has long been known. The basic idea dates back to the work of Hagedorn on hadronic "fireballs" in 1965.[23] Such a temperature results from the exponential dependence of the density of states on energy in the large energy limit. For an open string, the oscillator density of states can be easily calculated as the coefficient of $q^n$ in the generating function,[16]

$$G(q) = \mathrm{tr} q^N = \sum_{n=0}^{\infty} d(n) q^n, \tag{4.1.4}$$

where $N$ is the relevant sum of number operators. For the bosonic string,

$$N = \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n \tag{4.1.5a}$$

and for the superstring,

$$N = \sum_{n=1}^{\infty} (\alpha_{-n}\alpha_n + nS_{-n}S_n) \tag{4.1.5b}$$

where the $S$ are Majorana fermions. The bosonic string provides the simplest example:

$$\mathrm{tr} w^N = \prod_{n=1}^{\infty} \mathrm{tr} w^{\alpha_{-n}\alpha_n} = \left\{ \prod_{n=1}^{\infty} \left( \frac{1}{1-q^n} \right) \right\}^{-24} = |f(q)|^{-24} \tag{4.1.6}$$

where $f(q) = \prod_{n=1}^{\infty}(1-q^n)$ is the classical partition function. The degeneracy at a given level can be extracted by a contour integral over the generating function:

$$d_n = \frac{1}{2\pi i} \oint \frac{G(q)}{q^{n+1}} dq. \tag{4.1.7}$$

The high energy behavior of the density of states can be obtained by finding the limiting form of $f(q)$ as $q \to 1$ and doing a saddle point approximation.

For closed strings, one has left- and right-moving excitations which are only constrained by the level matching constraint:

$$n_R = n_L + \sum_i m_i n_i. \tag{4.1.8}$$

The corresponding density of states in the high energy limit is

$$d(n) = d_R(n) d_L(n) \sim (2n)^{-c} \exp(\beta_H \sqrt{\frac{4n}{\alpha'}}). \tag{4.1.9}$$

The values of $\beta_H$ and $c$ depend only on the dimension of spacetime, the worldsheet symmetry and the string mass scale.[24] In the appropriate critical dimensions, they are shown in table 4.1:[25]

Table 4.1: Parameters for the asymptotic density of states.

| | $c$ | $\beta_H$ |
|---|---|---|
| Bosonic string | $\frac{27}{2}$ | $4\pi\sqrt{\alpha'}$ |
| Type II superstring | $\frac{11}{2}$ | $\pi\sqrt{8\alpha'}$ |
| Heterotic string | $\frac{11}{2}$ | $(2+\sqrt{2})\pi\sqrt{\alpha'}$ |

Using $n = \frac{1}{4}\alpha' m^2$, the density of states goes as, $\rho \sim e^{\beta_H E}$ in all cases. One can see that the canonical partition function,

$$Z = \int dE \rho(E) e^{-\beta E} \tag{4.1.10}$$

will diverge unless $\beta \geq \beta_H$ or $T \leq T_H$, where $T_H$ is the Hagedorn temperature. The proper interpretation of this temperature is still a matter of debate. Troubled by the divergence of the above partition function and other physical quantities, some authors have claimed that the Hagedorn temperature is a true maximum physically attainable temperature. Others contend that the divergence of the partition function can be ignored, as the partition function is less fundamental than the microcanonical ensemble. The divergence then merely signals a phase transition. The nature of such an alleged phase transition is also unclear. Is the Hagedorn transition analogous to deconfinement in QCD, as its name suggests? Alternately, is some condensate formed above $T_H$, as the very topology of spacetime breaks down? While complete answers to these questions cannot be given, the possibilities can be elucidated.

#### 4.1.2a A Maximum Temperature?

The argument for a maximum temperature, using the canonical partition function, focuses on the finiteness of the energy and specific heat as one approaches the Hagedorn temperature from below. If both quantities are finite at $T_H$, it is hard to argue that this temperature cannot be exceeded. Conversely, if a divergence is found from below, where the partition function is still well defined, it is claimed that a maximum has been reached. The standard superstring partition function can be written,

$$Z = \prod_{k,\alpha} \left( \frac{1 + e^{-\beta\epsilon_{k,\alpha}}}{1 - e^{-\beta\epsilon_{k,\alpha}}} \right)^{d(n)} . \tag{4.1.11}$$

Above, $\epsilon_{k,\alpha}$ is the energy of a string characterized by quantum numbers $\alpha = n_i, m_i, N, \tilde{N}$ and uncompactified momentum $k$. $d(n)$ is the density of states at level $n$. Assuming we have a large $D-1$ dimensional box with sides of length $L$, the free energy is[25]

$$F = -\frac{1}{\beta}(\frac{L}{2\pi})^{D-1} \sum_{\alpha} d(n) \int d^{D-1}p(\ln(1 + e^{-\beta\epsilon_{k\alpha}}) - \ln(1 - e^{-\beta\epsilon_{k\alpha}})). \tag{4.1.12}$$

Using the asymptotic form of the density of states, valid above some cutoff $m_0$, and performing some standard manipulations, one can show that the free energy is,

$$-\frac{1}{\beta}^{\frac{D+1}{2}} \int_{m_0}^{\infty} \frac{dm}{m^{\frac{D+1}{2}}} e^{m(\beta_H - \beta)}. \tag{4.1.13}$$

The preceding expression shows that strings have what resembles a standard Boltzmann form, even at high temperatures, provided that $T < T_H$. More importantly, $< E >= -\frac{\partial}{\partial\beta} \ln Z \sim (\beta - \beta_H)^{\frac{D-3}{2}}$. Similarly, $C_V = \frac{\partial E}{\partial T} \sim (\beta - \beta_H)^{\frac{D-5}{2}}$. Thus, for $D > 5$, fundamental physical quantities are finite at the Hagedorn temperature, so $T_H$ can be surpassed, even though the partition function will diverge. Oddly, for $D = 4$, the energy density is finite, but the specific heat diverges. The above has led some authors[24,25] to conclude that for $D \leq 3$ the Hagedorn temperature is unsurpassable. Similar work has been done in open string models[26] to suggest that $T_H$ is maximal, even in six dimensions.

How do we interpret the divergence of the specific heat? With $S$ denoting the entropy[26] and the specific heat given by,

$$C_V = -\beta^2 (\frac{\partial^2 S}{\partial E^2})^{-1}$$

the partition function can be expanded about its maximum as

$$Z(\beta) = Z_0 \int \exp\left\{ -\frac{1}{2}(E - E_0)^2 \frac{1}{T^2}(C_V)^{-1} \right\} dE. \tag{4.1.14}$$

If $C_V$ diverges at $T_H$, fluctuations in energy about the mean energy can be of arbitrary size even if the partition function remains finite. As a result, one cannot confidently use the canonical formalism arbitrarily close to the Hagedorn temperature.[27] Also observe the importance of the sign of $C_V$ in eq. (4.1.14). Note, however, that a negative specific heat is not fatal to a thermodynamic treatment of a string gas.[28] Indeed, if conservation laws are included, the specific heat is always negative at sufficiently high energy densities.

Of course, finiteness of the partition function is not strictly a sufficient condition for its use. Indeed, if one assumes that there is a finite total energy in the universe, eq. (4.1.10) will be cut off at $E_{\text{tot}}$ and remain finite. Nevertheless, the canonical approach will break down, since it assumes all states that it describes have energies much less than the total energy in the system. In this limit, one can speak of an infinite reservoir of energy and integrate to infinity. For a finite system, integrating to the maximum energy in the total system explicitly violates the canonical assumption. However, the Boltzmann distribution may remain a valid description of relative probabilities, arbitrarily close to the Hagedorn temperature, provided that the energies of the states

[26]The entropy can be found in the canonical ensemble in the usual way in terms of the average energy, $\bar{E}$ by $S = \ln Z + \beta \bar{E}$.

considered are small compared to the total energy of the universe. After all, this is the only assumption used in expanding the microcanonical density of states as

$$\ln \Omega(E_{\text{tot}} - E) = \ln \Omega_{\text{res}}(E_{\text{tot}}) - \beta E. \tag{4.1.15}$$

Expressing relative probabilities as $e^{-\beta E}$ is then a purely microcanonical result, for $E << E_{\text{tot}}$.

Even with the use of the microcanonical ensemble, some authors have found the Hagedorn temperature to be maximal.[27,28] Brandenberger and Vafa derived the temperature as a function of radius of the universe and found a temperature plateau about $R = 1$ at the Hagedorn temperature. The origin of such a temperature plateau is easy to understand without doing the computation. Near $R = 1$, all string energy levels are equally spaced and thus equally accessible. In such a high energy regime, the exponential form of the oscillator density of states leads one to suspect that most of the energy will be in the oscillator modes. Since these modes have absolutely no radial dependence, one would not anticipate a significant change in the number of states accessible to the system when the radius is increased slightly. Thus we have a temperature plateau. However, after a sufficient increase in the radius, the momentum modes become lighter and more significant. Having explicit dependence on radius, these modes give the temperature a non-trivial dependence on the temperature.

#### 4.1.2b Going Beyond the Hagedorn Temperature

Many recent papers discuss ways of probing the phase that may exist above the Hagedorn temperature. The most straightforward way to study this regime is to directly construct the microcanonical density of states for a string gas. The first issue is finding the single string density of states,

$$f(\epsilon) \equiv \sum_a \delta(\epsilon - \epsilon_a). \tag{4.1.16}$$

Conserved charges, like momentum and winding number, can be accommodated by defining $f(\epsilon, q) \equiv \sum \delta(\epsilon - \epsilon_a)\delta_{q,q_a}$ as the number of single particle states with energy $\epsilon$ and charge $q$.[28] It is often more convenient to enforce conservation by defining[29]

$$f(\epsilon, \mu) = \text{tr}[e^{2\pi i \mu q}\delta(\epsilon - \epsilon_\alpha)]. \tag{4.1.17}$$

Total charge $Q$ can be imposed by multiplying the resulting total density of states by $e^{-2\pi i\mu Q}$ and integrating,

$$\Omega(E,Q)=\int_{-\frac{1}{2}}^{\frac{1}{2}} d\mu\Omega(E,\mu)e^{-2\pi i\mu Q}. \tag{4.1.18}$$

The starting point for deriving the single string density of states is the energy spectrum for one string given by equation (4.1.3). Using level matching one can write,[28,29]

$$N=\frac{1}{8}[\epsilon^2-k^2-\sum_i(\frac{m_i}{R_i}-2R_in_i)^2] \tag{4.1.19a}$$

$$\tilde{N}=\frac{1}{8}[\epsilon^2-k^2-\sum_i(\frac{m_i}{R_i}+2R_in_i)^2]. \tag{4.1.19b}$$

Then the single string density of states with uncompactified momentum between $k$ and $k+dk$, winding $n_i$ and internal momentum $m_i$ is

$$f(\epsilon,m,n,k)d\epsilon dk=d_Nd_{\tilde{N}}e^{2\pi i(\kappa\cdot k+\mu\cdot n+\nu\cdot m)}\frac{\partial n}{\partial\epsilon}. \tag{4.1.20}$$

The single string density of states is then finally obtained by integrating out everything but the energy to obtain,[29]

$$\omega(\epsilon,\kappa,\mu,\nu)d\epsilon\sim V_d\int\frac{d^dk}{(2\pi)^d}\sum_{m_i,n_i}\frac{\epsilon d\epsilon}{4}e^{2\pi i(\kappa k+\mu n+\nu m)}\frac{e^{4\pi\sqrt{N}}+e^{4\pi\sqrt{\tilde{N}}}}{(N\tilde{N})^{(d+1)/4}} \tag{4.1.21}$$

where $d$ is the number of uncompactified dimensions and $N$ and $\tilde{N}$ have complicated dependence on $\epsilon$, $k$, $n_i$, $m_i$ given by eqs. (4.1.19a, 4.1.19b).

The single-string density of states is clearly not easy to evaluate explicitly. However, in the high energy limit where $\epsilon >> R_i, \frac{1}{R_i}$ for all $i$, one can replace the sums in eq. (4.1.21) with integrals. In addition, one can expand the square roots and evaluate the $k$ integral by saddle point methods. With all radii equal, the result is[29]

$$\omega(\epsilon,\kappa,\mu,\nu)\sim\frac{1}{\epsilon^{d/2+1}}e^{\beta_H(\kappa,\mu,\nu)\epsilon} \tag{4.1.22}$$

where

$$\beta_H(\kappa,\mu,\nu)=\beta_H-\sqrt{2}\pi\left\{\frac{\kappa^2}{2}+(\frac{\nu}{4R}+\frac{\mu R}{2})^2+(\frac{\nu}{4R}-\frac{\mu R}{2})^2\right\} \tag{4.1.23}$$

and $\beta_H$ is the usual Hagedorn temperature.

Next the multi-string density of states, $\Omega(E) = \sum_\alpha \delta(E - E_\alpha)$, must be constructed from the single string density of states. To do so, imagine breaking up the set of states, $\alpha$, into subsets $\alpha_N$ which all have $N$ strings in them. Defining $\Omega_N(E) = \sum_{\alpha_N} \delta(E_N - E_{\alpha_N})$, we immediately can write $\Omega(E) = \sum_{N=1}^{\infty} \Omega_N(E)$. Now note that $\prod_{i=1}^{N} f(\epsilon_i)d\epsilon_i$ is the number of $N$ particle states where each $\epsilon_i$ is between $\epsilon_i$ and $\epsilon_i + d\epsilon_i$. For classical identical particles, one simply divides this by $N!$. $\Omega_N(E)$ is then the integral over all possible $\epsilon_i$ that satisfy the constraint that the total energy is $E$. Thus, one inserts a delta function to arrive at[28]

$$\Omega_N(E) = \frac{1}{N!} \int_0^E \prod_{i=1}^{N} d\epsilon_i f(\epsilon_i) \delta(E - \sum_{i=1}^{N} \epsilon_i). \tag{4.1.24}$$

The Maxwell-Boltzmann density of states is then

$$\Omega(E) = \sum_{N=1}^{\infty} \frac{1}{N!} \int_0^E \prod_{i=1}^{N} d\epsilon_i f(\epsilon_i) \delta(E - \sum_{i=1}^{N} \epsilon_i). \tag{4.1.25}$$

Using eq. (4.1.22), one can show that

$$\Omega(E, \kappa, \mu, \nu) \sim \frac{e^{\beta_H(\kappa,\mu,\nu)E}}{E^{\frac{d}{2}+1}} e^{2/dm_0^{d/2}} \quad \text{for } d > 0 \tag{4.1.26a}$$

$$\sim \frac{1}{m_0} e^{\beta_H(\kappa,\mu,\nu)E} \quad \text{for } d = 0. \tag{4.1.26b}$$

$m_0$ is a low energy cutoff introduced to avoid divergences. Its origin is in the departure from the asymptotic form of the density of states at low energy. As will be seen below, one can easily go beyond the preceding Maxwell-Boltzmann form to the full quantum expression. Also, note that in evaluating (4.1.24), one is led to integrating

$$\int_0^E \prod_{i=1}^{N} \frac{d\epsilon_i}{\epsilon_i^{d/2+1}} \delta(E - \sum_{i=1}^{N} \epsilon_i), \tag{4.1.27}$$

as an intermediate step. This clearly shows the tendency for one energetic string to dominate at high energies.

In addition to the density of states, energy and number distribution functions are of enormous value in understanding any statistical system. Define $D(\epsilon; E)d\epsilon$ as the average number of strings with energy between $\epsilon$ and $\epsilon + d\epsilon$ in a gas of energy $E$. To find $D(\epsilon; E)$, insert $\sum_{i=1}^{N} \delta(\epsilon - \epsilon_i) d\epsilon$ into the definition of $\Omega_N(E)$ to obtain $\sum_{\alpha_N} \delta(E - E_{\alpha_N}) \sum_{i=1}^{N} \delta(\epsilon - \epsilon_i) d\epsilon$. This expression counts the number of $N$ particle states that have particles with energy between $\epsilon$ and $\epsilon + d\epsilon$ and weights the result by the number of such particles. Thus,[28]

$$D(\epsilon; E) = \frac{1}{\Omega(E)} \sum_{N=1}^{\infty} \sum_{\alpha_N} \delta(E - E_{\alpha_N}) \sum_{i=1}^{N} \delta(\epsilon - \epsilon_i). \tag{4.1.28}$$

Now inserting $\sum_{i=1}^{N} \delta(\epsilon - \epsilon_i)$ in eq. (4.1.24) and performing the integral over $\epsilon_i$ yields $\sum_{\alpha_N} \delta(E - E_{\alpha_N}) \sum_{i=1}^{N} \delta(\epsilon - \epsilon_i) = f(\epsilon) \Omega_{N-1}(E - \epsilon)$. Combining the last two equations we see that

$$D(\epsilon; E) = f(\epsilon) \frac{\Omega(E - \epsilon)}{\Omega(E)}. \tag{4.1.29}$$

In the regime where (4.1.15) is valid, this clearly reduces to the usual canonical result, $D(\epsilon; E) = f(\epsilon) e^{-\beta E}$.

Ironically, an extremely powerful alternate method for investigating strings above the Hagedorn temperature uses the partition function. To see its role one begins with the microcanonical density of states:[29]

$$\begin{aligned}
\Omega(E) &= \mathrm{Tr}\delta(E - E_\alpha) \\
&= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \mathrm{Tr} e^{ik(E - E_\alpha)} \\
&= \int_{L-i\infty}^{L+i\infty} \frac{d\beta}{2\pi i} e^{\beta E} \mathrm{Tr} e^{-\beta E_\alpha} \\
&= \int_{L-i\infty}^{L+i\infty} \frac{d\beta}{2\pi i} e^{\beta E} Z(\beta)
\end{aligned} \tag{4.1.30}$$

where the Laplace transform can be inverted to obtain

$$Z(\beta) = \int_0^{\infty} dE \Omega(E) e^{-\beta E}. \tag{4.1.31}$$

$L$ is a convergence factor that is inserted to allow continuation of the partition function throughout the complex plane. Even though very close to, or above the Hagedorn temperature, $\beta$ loses physical meaning as an inverse temperature and $Z$ is no longer directly useful for computing thermodynamic properties, $Z(\beta)$ remains a well defined analytic function. Taking the log of eq. (4.1.11) one obtains,

$$\ln Z(\beta) = \sum_{r=1}^{\infty} \left( \frac{f_B(\beta r)}{r} - (-1)^r \frac{f_F(\beta r)}{r} \right) \tag{4.1.32}$$

where $f_B$ and $f_F$ are defined by

$$f_B(\beta) = \int_0^{\infty} d\epsilon e^{-\beta\epsilon} f_B(\epsilon) \tag{4.1.33a}$$

$$f_F(\beta) = \int_0^{\infty} d\epsilon e^{-\beta\epsilon} f_F(\epsilon). \tag{4.1.33b}$$

Inserting the above partition functions into eq. (4.1.30), we arrive at

$$\Omega(E) = \int_{L-i\infty}^{L+i\infty} \frac{d\beta}{2\pi i} e^{\beta E} \exp\left\{ \sum_{r=1}^{\infty} \left( \frac{f_B(\beta r)}{r} - (-1)^r \frac{f_F(\beta r)}{r} \right) \right\}. \tag{4.1.34}$$

Taking $r = 1$ with $f(\epsilon) = f_B(\epsilon) + f_F(\epsilon)$ reproduces eq.(4.1.25). Reference [30] shows that the Maxwell-Boltzmann form is, to good approximation, sufficient.

To investigate temperatures above the Hagedorn temperature, one must study the singularity structure of the partition function. An exhaustive analysis has been done[28] to obtain the microcanonical density of states, as well as the single string energy distribution $D(\epsilon, E)$ in a variety of different regimes.

If there are three or more uncompactified dimensions, $d$,

$$\Omega(E) \sim C \frac{e^{\beta_H E}}{(E - \sigma_H V)^{d/2+1}} \tag{4.1.35}$$

where $\sigma_H$ is Hagedorn energy density and $V$ is the volume of uncompactified space. From this one finds that

$$T = \frac{T_H}{1 - \frac{\nu}{\beta_H E}} \tag{4.1.36}$$

where $\nu = \frac{d}{2} + 1$. We see that the temperature goes *above* the Hagedorn temperature. High energy strings have the distribution,

$$D(\epsilon; E) \sim e^{-\beta_H (E-\epsilon)} \Omega(E - \epsilon)[(\frac{E}{E - \sigma_H V})(1 - \frac{E - \epsilon}{E})]^{-(d/2+1)}. \tag{4.1.37}$$

With one and two uncompactified dimensions, the situation is more complicated.

With all spatial dimensions compactified at the Planck scale,

$$\Omega(E) = \beta_0 e^{\beta_0 E + \lambda_0}(1 - g(E)e^{-\eta E}) \tag{4.1.38}$$

where $g(E) = \frac{1}{(2D-1)!}(\eta E)^{2D-1} e^{\lambda_1 - \lambda_0}$ and $\eta$ is the distance between the Hagedorn singularity, $\beta_0 \equiv \beta_H$, and the next nearest pole, $\beta_1$; i.e., $\eta = \beta_0 - \beta_1$. The $\lambda_i$ result from a power series expansion of the regular part of the partition function. For the heterotic string with $R$ in units of $\sqrt{\alpha'}$,

$$\eta(R) = \sqrt{2\pi^2\alpha'}\left(1 + \sqrt{2} - \sqrt{2 - \frac{1}{2R^2}} - \sqrt{1 - \frac{1}{2R^2}}\right)$$

with $R \geq 1$. From the full density of states, one can find the formal temperature to be

$$T \sim \frac{1}{\beta_0}(1 - \frac{\eta}{\beta_0} g(E) e^{-\eta E}). \tag{4.1.39}$$

First we see that for $R \ll E$, the correction term is very small and there is a constant temperature plateau at the Hagedorn temperature.[27] It is also interesting to note that the temperature never exceeds the Hagedorn temperature. In principle one could use this expression to determine at which radius the temperature plateau ends, as a function of the total energy of the universe. Unfortunately, the unknown dependence of $\lambda_0$ and $\lambda_1$ on $R$ prevents this. For a crude estimate, one might assume that $\lambda_1 \sim \lambda_0$ and expand $\eta$ for large $R$. In this case the temperature drops much too slowly for any reasonable value of $E$ so the approximation must fail.

The single string distribution function is found to have the simple form,[28]

$$D(\epsilon; E) \sim \frac{1}{\epsilon}. \tag{4.1.40}$$

Thus, the energy distribution function, $\epsilon D(\epsilon, E)$ is independent of energy. The energy is spread out over all scales, rather than concentrated in a single string. One can also find the number of strings in the universe (accounting for all but a few $m_{Pl}$ of the total energy) to be $\int D(\epsilon; E) d\epsilon \sim \ln E$.

---

[27] This agrees with the original result of Brandenberger and Vafa.

How can we account for the unusual behavior of a string gas? Mathematically, the heart of the matter is the exponential form of the asymptotic density of states. As a result, the entropy is linear with energy, to leading order. Thus, the entropy is extensive in the high energy limit, but all other thermodynamic quantities, which depend on derivatives of the entropy, are not extensive. Subleading terms, which are not extensive in either the string or particle cases, dominate the behavior of the string gas, producing effects that violate our intuition built upon extensive point particle gases.

Physically understanding these gases is more difficult. For the case where at least three dimensions are uncompactified, it is interesting to look at the energy distribution function $\epsilon D(\epsilon; E)$. It is found that most of the energy is deposited in extremely high energy strings. If one computes the average number of such strings from $D(\epsilon; E)$, it turns out to be of order one! Thus the original picture of Frautschi and Carlitz[31] re-emerges where one highly energetic string takes most of the energy. The picture one obtains is of a gas of strings at the Hagedorn temperature surrounding one extremely energetic string.[28,29] The Hagedorn temperature is then a physical maximum for the gas, since additional energy simply goes into the energetic string.

Atick and Witten[32] develop a similar picture, but reach markedly different conclusions. They show that the Hagedorn temperature can be interpreted as the temperature at which the first winding mode becomes massless. Compactifying the time coordinate (radius $R = \frac{\beta}{2\pi}$), they write,

$$\frac{1}{4}M^2 = N + \frac{1}{2}(\frac{m\pi}{\beta} + \frac{n\beta}{2\pi})^2 - 1 + \tilde{N} + \frac{1}{2}(\frac{m\pi}{\beta} - \frac{n\beta}{2\pi})^2 - 1 \tag{4.1.41}$$

for the mass shell conditions where $N$ and $\tilde{N}$ are the oscillator contributions for the right- and left-moving modes respectively. If one sets $N = \tilde{N} = m = 0$ and $n = \pm 1$ the result is $M^2 = -8 + \frac{\beta^2}{\pi^2} = -8 + \frac{1}{\pi^2 T^2}$. At $T_H = \frac{1}{2\sqrt{2}\pi}$ the first winding mode becomes massless. The same phenomena occurs for type II and heterotic strings, but at the proper Hagedorn temperatures for those theories. With a new massless channel opening up at $T_H$, one would expect the temperature to stop increasing as energy is deposited in the new levels. The question becomes whether the new levels can ever be filled; i.e., whether the phase transition can be completed. The single high energy string described above might seem to give an infinite latent heat to the phase transition.

Considering the preceding remarks and the appearance of a tachyon, even for the superstring, it would seem that a maximum temperature has been reached unless the fundamental degrees of freedom change above the Hagedorn temperature. Such a change is precisely what Atick and Witten claim. They argue that above the Hagedorn temperature one would expect a genus zero contribution to the free energy, which means that the Riemann surface interpretation of string theory is breaking down. They conjecture that the $n = \pm 1$ windings are creating tiny holes in the worldsheet so that it is not even simply connected locally. This is claimed to be analogous to the QCD deconfinement transition, where an effective description in terms of Riemann surfaces must be replaced by a simpler theory in terms of more fundamental degrees of freedom. Thus, the thermodynamic treatment described above which might imply that the Hagedorn temperature is a maximum temperature would then be fundamentally flawed, since it assumes the "effective" string spectrum. Further evidence of deconfinement is given by the vanishing of the string tension as $T \rightarrow T_H$.[33] However, note that even if the Hagedorn transition is completely analogous to QCD deconfinement, it still could be a limiting temperature, if it corresponds to the $N \rightarrow \infty$ limit.

The QCD analogy, while appealing, may also be misleading. At extremely high densities, it seems unnatural to expect a phase where the basic degrees of freedom become more free. Pushing the QCD analogy too far, one might expect the high temperature phase to be described by an underlying 10- or 26-dimensional field theory which would be badly divergent. Instead one might expect a ball of strings or string condensate to form.[30] One model explicitly makes an analogy with nucleation models of supersaturated gases.[34] Strings are assumed to decay, split and recombine much as droplets do. Breaking up the total energy into finite sized bins, the variation with energy density of the fraction of energy, $c_i$, in each bin $i$ is studied. $c_i$ is then fit to $i^u e^{vi}$, where $u$ and $v$ are determined by the fits. Good fits can be found, but they must be markedly different above and below the Hagedorn temperature. In addition, the total number of strings and the fluctuations in the total energy are tracked and found to abruptly change at the Hagedorn temperature. It is concluded that these phenomena indicate a first-order transition to a "liquid" string condensate phase. Since the preceding analysis employs perturbation theory above the Hagedorn temperature and neglects winding modes as well as gravitational effects, it is not conclusive. Models of strings that are allowed to break and rejoin[33] have led to similar conclusions. Other authors[24] have suggested that the Hagedorn temperature is a multicritical point, above which phases with different spacetime dimensions are coexisting. Dropping below $T_H$, our universe had to choose one particular phase or dimension Many more alternatives and physical analogs exist. Rather than providing an exhaustive enumeration, one must conclude as one expert in the field did[29], that "what happens when we try to increase the temperature past [the Hagedorn temperature] is, despite much effort, merely a speculation."

### 4.1.3 The Original Paradigm of Brandenberger and Vafa

The basic framework of string cosmology rests on a reversal of the usual compactification scenario. Rather than begin with $D_{\text{crit}}$ uncompactified (i.e., large) dimensions and posit the spontaneous compactification of $D_{\text{crit}} - 4$ of them, Brandenberger and Vafa[27] adopt a view more compatible with the cosmological idea of a small universe that expands. They assume that the universe began with all $D_{crit} - 1$ of its spatial dimensions compactified near the Planck radius. One then tries to explain why precisely three of the dimensions became very large ("uncompactified") in a stringy big bang. Since the universe has not expanded infinitely since the big bang, we anticipate that all of its spatial dimensions are still compactified today. Some simply have a larger radius of compactification than others. This view allows strings to have winding modes about all spatial dimensions.

In their seminal paper,[27] Brandenberger and Vafa suggested a tantalizing scenario in which winding modes, a purely stringy phenomena, could be used to explain the dimension of spacetime. They argued heuristically that winding modes exert a negative pressure on the universe, thereby slowing and ultimately reversing the expansion. One can simply see that since their energy is linear with the radius of compactification (i.e., the scale factor of the universe), there is a large energy cost to expanding with winding modes present. The question becomes "in how many dimensions can winding modes be expected to interact frequently enough to annihilate?" If the universe expands in a dimension where annihilation is incomplete, the windings will eventually force a recollapse of the universe to and possibly past $R = 1$, which by duality can be interpreted as another attempt at expansion. Note that in many ways the model is incomplete because no mechanism or even justification is given for expansion. One could imagine that the universe sits at the Planck scale forever. This makes the model very hard to test or constrain.

Furthermore, one might be troubled that there are many examples of cosmological features, like the cosmological constant and domain walls, which seem to require energy during expansion, but on closer analysis actually promote expansion. Indeed, Einstein's equations lead us to believe that all matter, by contributing to $\rho$, should accelerate expansion. Brandenberger and Vafa sidestepped the issue by proclaiming that the Einstein equations are invalid because they do not respect duality. Furthermore, it was implied that if ordinary matter accelerates expansion, windings (i.e., "dual matter") should stop expansion. Fortunately, reference [35] shows more convincingly that winding modes do indeed inhibit expansion by studying the low energy expansion of the tree level gravitational-dilaton effective action:

$$S_0 = -\int d^D x\sqrt{-G}e^{-2\phi}[R + 4(\partial\phi)^2]. \tag{4.1.42}$$

Here, $D$ is the total dimension of spacetime. Now consider a time dependent dilaton and a metric of the special form

$$ds^2 = -dt^2 + \sum_{i=1}^{D-1} a_i^2(t)dx_i^2 \tag{4.1.43}$$

in the presence of a string gas in thermal equilibrium at temperature $T = \frac{1}{\beta}$. Defining $d = D - 1$, $a_i(t) = e^{\lambda_i(t)}$ and $\varphi = 2\phi - \sum_{i=1}^{d} \lambda_i$, and truncating to zero energy modes, the full action becomes,

$$-\int dt\sqrt{-G_{00}}[e^{-\varphi}(-G^{00}\sum_{i=1}^{d}\dot{\lambda}_i^2 + G^{00}\dot{\varphi}^2) - F(\lambda_i, \beta\sqrt{-G_{00}})] \qquad (4.1.44)$$

where a term, $F$, has been added to include the free energy of matter. The equations of motion are found by varying with respect to $G_{00}$, $\lambda$ and $\varphi$. Rewritten in terms of the original dilaton $\phi$, the equations of motion are:

$$-\sum_{i=1}^{d}\dot{\lambda}_i^2 + (2\dot{\phi} - \sum_{i=1}^{d}\dot{\lambda}_i)^2 = e^{2\phi}\rho \qquad (4.1.45a)$$

$$\ddot{\lambda}_j - (2\dot{\phi} - \sum_{i=1}^{d}\dot{\lambda}_i)\dot{\lambda}_j = \frac{1}{2}e^{2\phi}p_j \qquad (4.1.45b)$$

$$2\ddot{\phi} - \sum_{i=1}^{d}\ddot{\lambda}_i - \sum_{i=1}^{d}\dot{\lambda}_i^2 = \frac{1}{2}e^{2\phi}\rho. \qquad (4.1.45c)$$

Above, $\rho = \frac{E}{V}$ and $p_i = \frac{P_i}{V}$ is the pressure where $V = \exp(\sum_i \lambda_i)$ and $E$ and $P_i$ are defined,

$$\begin{aligned} E &= F + \beta\frac{\partial F}{\partial \beta} \\ P_i &= -\frac{\partial F}{\partial \lambda_i} \end{aligned} . \qquad (4.1.46)$$

In order to solve these equations, we specialize to the isotropic case where all the $\lambda_i$ are taken to be equal and $P \equiv -\frac{1}{d}\frac{\partial E}{\partial \lambda}$. Then equations (4.1.45a- 4.1.45c) reduce to

$$-d\dot{\lambda}^2 + \dot{\varphi}^2 = e^{\varphi}E \qquad (4.1.47a)$$

$$\ddot{\lambda} - \dot{\varphi}\dot{\lambda} = \frac{1}{2}e^{\varphi}P \qquad (4.1.47b)$$

$$\ddot{\varphi} - d\dot{\lambda}^2 = \frac{1}{2}e^{\varphi}E. \qquad (4.1.47c)$$

These equations can now be solved once initial conditions are chosen, if $E(\lambda)$ is known. Unfortunately, this function is not well understood. However, a universe with much of its energy in windings will have $E \sim R$ so it is reasonable to assume that for relatively large $R$

$$E(\lambda) = e^{\alpha\lambda} \qquad (4.1.48)$$

where $\alpha$ is of order unity.

The solution is easiest when $\alpha = 0$, as is appropriate in the very early oscillator dominated regime. Then the dilaton and radius vary by

$$e^{-\varphi} = \frac{Et^2}{4} - \frac{dA^2}{E} \qquad (4.1.49a)$$

$$\lambda = \lambda_0 + \ln\left(\frac{t - 2\sqrt{d}A/E}{t + 2\sqrt{d}A/E}\right) \qquad (4.1.49b)$$

where $A$ is an integration constant. We see that the even in the extreme case of $\alpha = 0$, the expansion slows and ultimately is halted. When $\alpha > 0$ it has been shown that not only is the expansion stopped in finite time, but it is also reversed. Thus we see that winding strings must collide and annihilate for significant and continued expansion to occur.

In what number of spacetime dimensions can annihilation be expected? Clearly, annihilation is easier in fewer dimensions. For example, in $1+1$ dimensions, the windings must lie on top of each other. In $2+1$ dimensions, they can be separated by one coordinate, but would be expected to interact frequently. In a very large number of dimensions, one would expect the equilibrium between winding modes to most likely be lost, so that their number density need not fall drastically as their energy increases. What is the maximum spacetime dimension which would allow thermal equilibrium between winding modes and thus lead to their annihilation? Many[27,36,37] have argued that two 2-dimensional worldsheets should generically intersect in four or fewer spacetime dimensions because $2+2=4$. Even if four were a rigorously proven maximum, the question would remain why four dimensions are favored over a smaller number, which seems much more likely from the point of view of ease of interaction. It has been suggested that entropy considerations would favor four dimensions.[38] However, to my knowledge, no attempt has yet been made to demonstrate in detail how four spacetime dimensions are dynamically favored for winding mode annihilation.

The following section proposes a new model for understanding origins of the dimension of spacetime, that has its roots in the preceding discussion. The next section will argue using the theory of random surfaces and walks that four is indeed the maximum large dimension of spacetime, if the background spacetime is quasi-static and Euclidean. Possible modifications for a more realistic spacetime are also discussed. Next, a simple computer model is described that was used to explore qualitatively the implications of this new paradigm. The goal is to test the viability of the model by asking if it could yield sensible results. Detailed predictions must await a fuller understanding of string interactions and string thermodynamics at extreme temperatures. Nevertheless, approximate limits on the magnitude of the Hubble expansion are found which are consistent with theoretical estimates. Furthermore, it is found that the preferred dimension of spacetime need not be two, as might have been inferred from the original model of Brandenberger and Vafa.

### 4.2 A New Cosmological Model That May Yield Four Dimensions

In this section, I suggest a new argument that shows the maximum spacetime dimension in which windings can be expected to annihilate is four. The starting point is the observation that worldsheets are not simple planes, but instead have many complex bends and twists. As a result, one may question the classical intuition used above that found four to be a limiting dimension.[28] Instead, one may choose to model strings as random surfaces. This is analogous to what is commonly done with point particle theories. Using the field theory of point particles, it is rigorously known that correlation functions can often be bounded by the intersection properties of two random walks.[40] Whether or not two random walks will intersect depends of their Hausdorf dimension, $d_H$. Strictly, the Hausdorf dimension of a set is defined by considering a covering of the set by boxes of size $\epsilon$ and defining

$$l_d(\epsilon) = \inf \sum_i \epsilon_i^d \tag{4.2.1}$$

where the infimum is over all choices of $\epsilon_i$ with $\epsilon_i < \epsilon$. The Hausdorf dimension is then defined implicitly by[41]

$$\begin{aligned} l_d = lim_{\epsilon \to 0} l_d(\epsilon) &= 0 \quad \text{for } d > d_H \\ &= \infty \text{ for } d < d_H \end{aligned} . \tag{4.2.2}$$

---

[28]C. Vafa claims that winding self-interactions allow the previous interpretation.[39]

The Hausdorf dimension is extremely difficult to determine. Thus, one typically finds instead the capacity dimension which is defined as follows:

$$d_c = lim_{\epsilon \to 0} \frac{\ln N(\epsilon)}{\ln(\frac{1}{\epsilon})} \tag{4.2.3}$$

where $N(\epsilon)$ = the number of boxes of side $\epsilon$ needed to cover the set.[42] In most cases the capacity or fractal dimension are equal. Rigorously, what one knows is that $d_H \leq d_c$. [42] For our purposes, we can use a more intuitive definition of $d_H$ which is that $< X^2 > \sim N^{\frac{2}{d_H}}$ for a walk of $N$ steps. Thus, the Hausdorf dimension indicates how the size of the walk scales, as well as indicating how "space filling" it is. Clearly, the higher the Hausdorf dimension, the slower the size grows and the better space is filled. It can be proved that there will be a non-trivial intersection if $2d_H > D$ and no intersection if $2d_H < D$, for walks embedded in $\mathbb{R}^D$. For random walks, it is well known that the Hausdorf dimension is two, independent of the embedding space. This analysis leads to the triviality of $\phi^4$ theory in $D > 4$. It turns out that in the boundary case, $D = 4$, intersections just fail to occur, so that for random walks one demands $D < 4$.

It is expected that string interactions can be studied by considering the intersection properties of random surfaces.[40] Of course, this conjecture, though extremely plausible, cannot be proven without understanding string field theory. Nevertheless, one can reasonably believe that whether two free strings will interact in $D$ dimensions depends on the sum of the Hausdorf dimensions of two random surfaces. Unfortunately, the Hausdorf dimension of a random surface is very difficult to determine. Some authors have claimed that it is infinite, regardless of the dimension of the embedding space,[43] while others claim a range from four to infinity, depending on $D$. For embedding dimensions greater than one, the Hausdorf dimension should be at least 8.[44] Thus, it appears that two strings should interact in at least 16 dimensions. Ten is as good as four!

Can we believe this result? For non-winding strings, which have point particle analogs, it is reasonable. However, for strings with winding modes about a direction with a large or growing radius, I submit that a significant modification is necessary.

Unlike true random surfaces, winding strings cannot move freely in all directions; they must lie more or less parallel to the axis about which they wind. It is expected that their Hausdorf dimension is significantly less than that corresponding to a non-winding string.[45] Of course, if winding strings have a significant number of oscillations, they can bend far away from parallel to the winding axis. However, the central difficulty for expansion is the energy cost of expanding with windings present, so one would expect the oscillation energy to be minimized. As shown by Tseytlin and Vafa, matter energy tends to slow the expansion, so oscillations would not be favored in a regime where the winding energy was significant. Said more simply, expansion should stretch and smooth out the windings. Indeed, the exponential inflation considered in the next section would smooth out the oscillations in the same way that inflation smoothes density variations to solve the flatness problem. As a result, one should treat the windings as points doing a random walk in $D-1$ dimensions transverse to the winding axis. Now we can adopt the previous results concerning the intersection properties of random walks and see that interaction is expected if and only if $D-1<4$ or $D<5$! Without any fine tuning, we have a very phenomenologically pleasing result.

While intriguing, the preceding argument is too naive to directly apply to cosmology. This is because random walk analyses have typically been done on a static Euclidean background. The early universe is profoundly different. First, the existence of true winding modes presupposes that we begin with a topologically non-trivial universe, which for simplicity we assume to be $T^9 \times \mathbb{R}$. This will naturally greatly enhance the interaction rate. On the other hand, the universe is undergoing significant expansion, which has the opposite effect. A fully detailed model would also consider the radial dependence of the annihilation cross section for winding strings, as well as other winding interactions that might catalyze the annihilation process. The creation rate of windings in the early universe and the effects of forces between the windings could be added, as well as a host of other effects.

### 4.3 Testing the Model on a Computer

Treating all these effects analytically is prohibitively difficult, since we are forced to consider the universe near or even at the Planck scale. Without a well developed theory of quantum gravity, one may doubt the plausibility of analytically proving that exactly three spatial dimensions are expected to expand. Additionally, a proper treatment of the creation rate of windings requires a knowledge of string thermodynamics well beyond the current level of understanding. The first difficulty, discussed in the introduction, is that near the Hagedorn temperature, the microcanonical ensemble must be used, which ostensibly requires counting all the states in the universe. Progress has been made in limiting regimes, but general results for independent radii of any size have yet to be exhibited and are sure to be unwieldy at best. Furthermore, the inclusion of strong gravity, appropriate for the early universe, would lead one to question the validity of even using thermodynamics. Even if a careful thermodynamic treatment of winding creation in an equilibrium ensemble were possible, it would leave unanswered the most interesting question: how does the winding creation rate drop as equilibrium is lost? This would require understanding non-equilibrium statistical mechanics in the early universe. Another problem is that the stability of the very topology of spacetime has come into question at extreme temperatures. Above the Hagedorn temperature, the conservation of winding number cannot be guaranteed.[32,46]

In spite of the preceding difficulties, it is feasible to test whether this tantalizing model for the expansion of the universe can work. In other words, one can ask whether one can make reasonable phenomenological assumptions about various processes in the early universe which in this model would lead to a strong prediction that three dimensions expand. This would not prove that the paradigm of Brandenberger and Vafa does work, but that it can. Furthermore, one can turn the problem around, asking what must the early universe have been like in order to produce our four-dimensional spacetime. Ultimately, useful constraints may be placed on the expansion rate, the radius at which equilibrium is lost, the number of windings surviving at that radius, interaction rates, temperature and other quantities in the early universe by this procedure.

In this spirit, a computer model was developed for winding string collisions in the early universe. While one would like to follow the model from $t = 0$, it is expected to be much more reliable below $T_H$, where one can more confidently use a string description and assume that oscillations are suppressed. Thus, we begin evolving the model soon after the temperature has dropped below $T_H$, in an inflationary era. The primary goal is to better understand the evolution of the universe just after the equilibrium of winding strings is lost.

Based on the previous discussion, the windings about each dimension are represented by points in $D-2$ dimensions, where $D$ is the total dimension of spacetime.[29] For the Type II superstring, $D = 10$. For the heterotic string, one takes $D = 10$ as well, assuming that internal degrees of freedom, not extra compactified spatial coordinates, provide the extra central charge for the left moving sector. The bosonic case is not investigated, since we seek a phenomenologically realistic model. Dimensions other than the critical dimension are also considered to study the effect of dimension on various processes. Since the radius at which the temperature drops significantly below the Hagedorn temperature and the windings drop out of equilibrium is not well known, the appropriate starting radius for the model is not precisely known. Thus, the initial radii of the spatial dimensions is left variable, but is typically chosen to be a few Planck lengths. A large initial radius would invariably lead to only one large spatial dimension. The radii of compactification, which truly are quantum mechanical objects, can be allowed to fluctuate, typically up to $L_{Pl}$ per step. Since this effect is not considered to be very significant, it is only incorporated for some trials. For simplicity, the fluctuations are taken to be independent of position.

A certain number of windings are presumed to remain in this epoch, but the precise number is unknown, so the initial number of windings is also a free parameter. Since the total number of high energy strings in the early universe roughly equals the log of the energy, the number of windings about each direction should not

[29] "Total" means the total number of spacetime coordinates in the theory, including both those that stay small and those that become effectively "uncompactified."

be huge. If the primordial universe contained precisely the energy in our observable universe, assuming critical density, there would only be 135 energetic strings in the entire universe! Of course, this is an extreme lower bound. Nevertheless, since one does not expect all the strings in the universe to be winding strings, the number of windings about each direction can reasonably be assumed to be at most of order ten when equilibrium is lost. Only windings of $\pm 1$ about a single direction are considered. Since we are below the Hagedorn temperature, a string with higher winding excitations about a given direction can be expected to have decayed to strings with unit winding number, as required by Boltzmann suppression. Perhaps more important is the possibility of strings with single windings about more than one direction. While these may have significance, they are not tracked in this first modeling attempt, since although they may increase the overall interaction rate and thereby may alter the quantitative results, they should not change the kinds of qualitative effects we seek to study. Indeed, to create these strings and allow them to participate in annihilation interactions requires two separate interactions with a correspondingly reduced net probability. While it is true that multiple winding strings would execute walks in effectively fewer dimensions, the enhancement is not expected to be extreme. The net winding number is set to zero, in order to satisfy observational constraints on the isotropy of the universe.[47] The windings then execute a random walk, stepping up to $L_{Pl}$ in each time step.

During each step, the computer checks for annihilations of pairs of winding modes with opposite winding number. Unlike most work in this field that assumes an ideal gas, this analysis explicitly allows interactions. Naively, if two windings come within $L_{Pl}$, they can be expected to annihilate.[27,36] However, as the length of the string and thus its energy grow, oscillations cost less and less energy, compared to the total energy of the string so that the effective thickness of the string increases. Owing to quantum correlations, interactions are expected for windings that come within $\sqrt{c + \ln L}$[48]30 of each other, where $L$ is the length of the string and $c$ is a constant of order one or ten. Note that having the extra oscillations in no way violates our

30Unless otherwise specified, all quantities are expressed in Planckian units.

assumption regarding the straightness of the winding strings, since the scale of the oscillations grows slowly compared to the size of the string. The probability of interaction, given a collision, may also vary with radius. Using equations (4.1.49a, 4.1.49b) one can show that the coupling remains fairly constant for small radii in the somewhat unphysical case that $\alpha = 0$ and all the radii are equal. More precisely,

$$g^2 = e^{2\phi} = K\left(1 - (\frac{R}{R_0})^{\sqrt{d}}\right)^2 . \tag{4.3.1}$$

Above, $R_0$ is the radius at which expansion stops and $K$ is an unknown constant. We see that for $R \ll R_0$ the coupling remains constant to lowest order. For larger $R$, the coupling $e^{2\phi}$ decreases with radius, ultimately dropping to zero. This agrees with other studies that show that one expects only trivial scattering in the infinite radius limit.[37] One can use eq. (4.3.1) to get an indication of the importance the variation of the dilaton, even in the current context where the radii are all independent, if one replaces $(\frac{R}{R_0})^{\sqrt{d}}$ by $(\frac{V}{V_0})^{\frac{1}{\sqrt{d}}}$. At $R = 1$, the probability of annihilation, given a collision, is taken to be one, providing the normalization to the coupling constant. Runs are conducted assuming either a constant dilaton or a coupling varying by eq. (4.3.1). In any case, the decrease in the coupling is not expected to be significant, at least in ten dimensions, since the dramatic drop in the collision rate with increasing radius will dominate any effect of the dilaton.

The universe is also allowed to expand during each time step. The proper expansion equation can be found using equations (4.1.45a- 4.1.45c), provided that one knew how the energy of the matter varied with independent radii. The assumptions made by [35] that all the radii are equal and that $E \sim R^\alpha$ clearly do not hold here. Eq. (4.1.49b) is also inappropriate since it shows all radii tending to a fixed value. No dimensions effectively "decompactify." Even if $E(R_i)$ were well known, one would be forced to solve a system of 19 coupled differential equations to get a rigorous result. However, in order to understand the qualitative implications of the model, one only needs to use an expansion equation that has the correct features. The Brandenberger-Vafa framework, verified in special cases by [35], requires that the windings slow the expansion as the radius increases and can ultimately stop or reverse it. These essential features are captured by the following procedure: During each time step, each radius $R_i$ is rescaled by a function of the number of windings, $n_i$, about that dimension and the radius itself,

$$R_i(t+1) = R_i(t)(1 + \epsilon(n_i(t), R_i(t))). \tag{4.3.2}$$

If $\epsilon$ were independent of time, this would give an exponential expansion, which is a form predicted by some authors to result from string driven inflation.[47,49] This also would give a constant Hubble parameter, $H = \dot{R}/R$, which is appropriate for de Sitter inflation. In light of the previous discussion of a phase transition at the Hagedorn temperature, we will assume that in the absence of windings, $\epsilon$ is constant and that it decreases with increasing radius and increasing number of windings. The following form satisfies all requirements:

$$\epsilon(n_i(t), R_i(t)) = H(1 - \frac{n_i(t)R_i(t)}{2R_0}) \tag{4.3.3}$$

where $H$ is the maximum expansion rate, as well as the Hubble constant for de Sitter inflation and $R_0$ is the radius at which two windings will halt the expansion. $R_0$ appeared previously in eq. (4.3.1). Clearly, $R_0$ must be less than the radius at which GUT physics takes place, but is not otherwise well constrained. The importance and reasonable ranges of both parameters will be determined by studying how they effect the prediction for the dimension of spacetime. Finally, note that replacing $1+\epsilon$ with $\exp(\epsilon)$ so the latter physically motivated form is used in eq. (4.3.2).

This prescription yields an expansion rate that decreases to zero as $R$ increases with windings present and expands indefinitely about any dimension about which all the windings have been annihilated. Dimensions do not recontract if windings remain, instead staying compactified at $2R_0/n$. Though an expansion equation that allows contraction could be constructed, it would not be useful in the model. It is of course possible that in a given expansion attempt that no dimensions lose all their windings and begin to "decompactify." In that case, the dimensions are expected to recontract and ultimately begin expansion again. However, for inflation to occur a second time, the universe must have recontracted back into the phase above the Hagedorn temperature and returned to its original symmetrical vacuum state. We cannot follow the windings as they enter this phase. When the universe begins expansion again, one could begin modeling below the Hagedorn temperature as before. This would be essentially treated as an independent attempt at expansion. Thus, the model should be seen as following the evolution of the universe during its final and only successful attempt at expansion. The above reasoning also shows that if some dimensions lose all their windings, those that do not stay forever at the Planck scale. With some dimensions large, the temperature can no longer grow high enough to restore the universe to its original vacuum so that the inflation of the small dimensions cannot be repeated.

Causality raises some questions about how to implement the preceding prescription. These difficulties result from trying to incorporate the effects of a purely global concept like winding number into local physical effects like expansion. The number of windings about a given direction is globally defined, irrespective of position. However, the effect of those windings on local physics cannot change everywhere instantaneously. If a winding is annihilated at some spacetime point, $X^{\mu}$, one would expect the expansion rate far away to be unaffected initially to satisfy causality. Strictly, winding annihilations would lead to bubbles of spacetime, whose walls move at the speed of light, expanding at a faster rate. After a number of annihilations, each spacetime point would have expanded a different amount. This is very difficult to model. Instead, a retardation is introduced so that $n$ in eq. (4.3.3) counts windings that either have not annihilated or have annihilated too recently for most of the universe to know about it. Specifically, an annihilation is "counted" after a time equal to the effective radius of the universe, $R = \sqrt{\sum_{i=1}^{d} R_i^2}$, at the time of the annihilation. Runs are conducted both with and without the retardation to determine its importance.

The very early universe should be hot enough to create pairs of winding strings. As discussed previously, it is highly nontrivial to compute that rate. Following ref. [29], one could find both the number of winding states and the total number of states for a gas of strings in the high energy limit. One first integrates out the momenta and energy from eq. (4.1.20) to find the number of states with a particular winding and then integrates over winding number to get the total number of states for a single string. Then an analog of eq. (4.1.25) is used to find these quantities for the full system. The ratio of these numbers should constrain the creation rate in thermal equilibrium. Ideally, one would want to relax the equilibrium assumption to get more accurate results in the more interesting era when equilibrium is not present. Indeed, having an equilibrium description obviates the need for computer modeling, telling us exactly how many windings should exist at any given time. A naive argument can show that creation of windings must cease at a very small radius. If the expansion process is roughly adiabatic, $T \sim \frac{1}{R}$. Furthermore, the energy of windings is linear with $R$. Thus, the Boltzmann factor would go as $e^{-R^2}$. Even though deviations from adiabaticity may occur, the suppression is strong enough that one can believe the creation rate is negligible in the relevant regime. Thus, numerical trials are conducted after creation has ceased.

### 4.4 Results of the Test and Predictions of the Model

The central result of the computer simulation is that a two-dimensional universe need not be the most probable outcome of the model just presented. If one considers the full parameter space described in the last section, the vast majority of it corresponds to either a two- or a ten-dimensional spacetime. However, appropriate choices of parameters can be found to make any dimension, from two to ten, the most probable. Unfortunately, in cases when the most likely dimension is neither two nor ten, it becomes impossible to predict the outcome with reasonable certainty.

Two dimensions result when annihilations are extremely unlikely. All dimensions then have $n_i$ windings about them that survive so that each dimension can at most expand to $2R_0/n_i$. The simulation would then show a result of zero large spatial dimensions or a one-dimensional spacetime. However, we know that given sufficient time, annihilation must ultimately occur, since the space is no longer rapidly expanding. This time may have to be integrated over several expansion attempts if it is more likely that the universe will recollapse before such annihilations occur. In this case the entire scenario would be repeated, presumably with the same choice of initial parameters, since they are determined by the poorly understood physics of the Planck scale. Once the rare annihilation occurs that leaves a dimension without windings, this dimension will expand without bound, forever suppressing annihilations along other dimensions. More than one large dimension would require two rare annihilations to occur almost simultaneously. If the annihilations along different dimensions occurred at significantly different times, then the large spatial dimensions we observe today would have undergone vastly different amounts of expansion. This is probably ruled out by the isotropy of our universe.

A ten dimensional spacetime is achieved when the parameters are such that annihilation is extremely efficient once equilibrium is lost. Then all winding strings are destroyed almost immediately and all dimensions expand without constraint.

The more interesting situation occurs for a relatively narrow band of parameter space in which winding annihilation is moderately likely. The most important variable is the radius at which equilibrium is lost and the simulation begins. The importance of radius can be seen by examining how the collision rate falls with the radius of compactification in various dimensions. For walks in one spatial dimension, one would expect the number of steps required for collision to scale as the square of the radius of compactification. This is largely true, especially for large radii. Deviations result from the logarithmic growth of the size of the string with radius. At small radii, $\frac{\sqrt{\ln R}}{R}$ is not negligible, accounting for the greater deviation in small spaces. Fig. 1 in appendix B shows how the number of steps required for a pair of windings to annihilate with at least 98 percent probability varies with the radius of compactification. As expected, in more dimensions the collision rate drops dramatically. In nine dimensions, roughly $180,000$ steps are required to get a collision with 98 percent probability with a radius of only three. As a result, we see that the winding creation rate must drop effectively to zero at a very small radius, or the expansion rate must be small enough so that hundreds of thousands of time steps lead to negligible expansion. (An expansion rate of $10^{-4}$ would increase the radius by a factor of $6 \times 10^{7}$ in $180,000$ time steps.) If not, windings would be created at a radius where they had little chance of annihilating, leading to a two dimensional universe.

The above raises the question about how fast the universe can expand without preventing winding collision and annihilation. To answer this question, trials were conducted with the expansion rate taken to be a constant, independent of the number of windings present. One then checked to see how large the expansion rate could be such that two windings would collide with 98 percent probability before the radii of compactification were clearly too large for annihilation to occur ($R > 500$.) In three spatial dimensions, a very large Hubble parameter (of order one) is allowed if equilibrium is lost at the rather improbable value, $R = 1$. However, if the proper initial radius for the model is $R = 4$ then the maximum Hubble parameter is about $10^{-4}$ in Planck units (see fig 2.) With nine spatial dimensions, as is appropriate for the superstring, the largest possible Hubble parameter can be as low as $10^{-5}$ at $R = 1$, depending on the choice of other parameters. These constraints are not precise limits, since the actual expansion rate is not constant, as assumed above, but gets reduced in the presence of windings. Thus, the maximum expansion rate without windings could be larger. More complicated string processes than those treated here could also increase the annihilation rate and allow greater expansion. Nevertheless, the preceding analysis indicates that the expected magnitude of the maximum expansion is very small.

Are the above values reasonable? It has been shown that the size of the Hubble parameter is bounded by observations of the cosmic microwave background radiation. Abbott and Wise[50] found that a Hubble parameter during inflation greater than $10^{-4}$ would allow gravitational waves of sufficient amplitude to produce a microwave anisotropy greater than that observed by COBE. It is pleasing that the model described here produces similar bounds.

Of course, the most direct and revealing way to determine the effect of the radius of compactification and Hubble parameter on the expected dimension of spacetime is to simply run many (50) trials for various values of these parameters and compute the average dimension obtained. Typically, one finds $\langle D \rangle = 10$ up to some $R_1$ and then falls rapidly as a function of $R$ up until $R_2$, beyond which the expected dimension is two. Unless otherwise specified, all following runs use 10 windings about each direction, an $R_0$ (the maximum radius with two windings present) of 50 and a string width chosen equal to one at $R = 1$. If the Hubble parameter, $H$, is between .1 and .01, $R_1$ is equal to one and $R_2$ is a very small 1.5. Since it is hard to believe that the temperature could have dropped sufficiently below the Hagedorn temperature for the windings to have fallen out of equilibrium so near to the dual radius, we again conclude that a small expansion rate is necessary. For $H = .001$, the interesting range of radii has only gone up to 1.2 to 1.6. If $H = 10^{-4}$, $R_1$ and $R_2$ are 1.5 and 2.5 respectively. Plots of average dimension versus initial radius are given in figs. 3 to 6.

Another parameter upon which the final dimension of spacetime sensitively depends is the effective width of a string, determined by $c$ in the expression above eq. (4.3.1). Even with $R = 1$ and $H = .01$, one finds that $\langle D \rangle = 0$ for $c$ ranging from zero to 15. The expected dimension rises rapidly as $c$ increases from 15 to 30 (see fig 7). Of course, a wider string should act equivalently to a narrower string in a smaller space, so this behavior is not surprising.

The number of windings surviving when equilibrium is lost has a variable effect. If the initial radius is small, it has almost no effect. For example, with $R = 1.4$, and $H = .1$, 500 trials were conducted with either 2, 10, 50 or 100 windings about each dimension. Even with a sensitivity of .07 in the average dimension, no statistically significant change in the average dimension was observed when the number of windings ranged from 10 to 100. One could argue that since the initial volume of the transverse space was only about 15, ten or more windings completely filled the space, so that the total number is irrelevant. This might lead one to expect that almost all of them would annihilate, as they are guaranteed to be in close proximity. In reality, no dimensions got large, most of the time. In a larger space, the effect of the number of windings is very significant. When the initial radius is two and $H = .0001$, the average dimension of spacetime drops by over four when the number of windings increases from 10 to 25.

Other parameters are less significant. The radius at which two windings stop expansion, $R_0$, does not greatly affect the results. In many cases, varying $R_0$ from 5 to 100 has no effect, above error. If the initial radius is close enough to $R_0$, then this parameter can reduce the expected dimension of spacetime by about one (see fig. 8). This is to be expected since a larger $R_0$ allows faster expansion for a given number of windings, resulting in less collisions and a smaller dimension. The effect of the evolution of the dilaton was also considered. While it can sometimes drop the expected dimension by one or two sigma, the effect is insignificant compared to other uncertainties, so many trials are conducted with a constant dilaton. This result gives us confidence that deviations from the approximate dilaton evolution equation being used (4.3.1), will not significantly affect the results. The effect of radius fluctuations was also studied. For $R \gtrsim 1.3$ it had no effect whatsoever, so fluctuations were subsequently ignored for other runs.

Finally, the importance of a time delay to enforce causality was determined. In almost all cases the time delay had almost no statistically significant impact on the results. For some trials the time delay reduced the expected dimension by up to two sigma (i.e., by .4) for $D$ near five. The minimal effect of the time delay indicates that one need not be concerned with constructing a more realistic time delay algorithm.

The average dimension of spacetime is by no means the only quantity that should be studied. The width of the expected distribution of dimensions is also critically important. When the average dimension is one (ultimately two) or ten, the width can be arbitrarily small. However, for intermediate values, $\sigma$ is is roughly $1-1.5$ (see figs. 9-20). Thus, while it is possible to have an average dimension of spacetime of four (see fig. 20), one cannot rule out other alternatives based on the gross initial conditions of the universe. This lack of determinism is not pleasing.

The above analysis shows that there are a number of parameters that can be tuned to produce any desired average dimension of spacetime. The maximum expansion rate, the radius at which equilibrium is lost, the number of strings remaining at this time and the effective width of those strings are certainly the most important. Unfortunately, a firm prediction for the most probable dimension of spacetime is not possible from this model because of the number of free parameters and the omission of possibly important physical effects. Nevertheless, this work does demonstrate how string theory can be used to make such a prediction. A more complete model, properly incorporating as yet poorly understood physics, is clearly called for. Lastly, the the narrow range of parameters that give a four- dimensional universe should be seen as a blessing in disguise, rather than a fine tuning disaster. Once our knowledge of some of the relevant parameters improves, we can use the fact that we live in a four-dimensional world in analysis as done above to determine the values of the remaining parameters to good accuracy.

**References**

[1] M. Green, J. Schwarz, and E. Witten, Superstring Theory, V. I & II, (1987);
D. Lüst and S. Theisen, Lectures on String Theory, Chap. 4, Springer-Verlag (Lecture Notes in Physics), Berlin, (1989).

[2] There are many excellent introductions to CFT. They include:
A. Belavin, A. Polyakov and A. Zamolodchikov, Nucl. Phys. B241 (1984) 333;
P. Ginsparg, "Applied Conformal Field Theory," in E.Brézin and J. Zinn-Justin, eds., Fields, Strings and Critical Phenomena, Les Houches 1988, Elsevier (1989);
D. Lüst and S. Theisen, Chap. 4;
M. Kaku, Introduction to Superstrings, Chap. 4, Springer-Verlag (Graduate Texts in Contemporary Physics), New York, (1990);
M. Grisaru, "Conformal Field Theory: A primer," in L. Alverez-Gaume et al., eds., Superstrings '87, Proceedings of the Trieste Spring School 1987, World Scientific, Singapore, (1987);
A. Schellekens, "Conformal Field Theory for Four-Dimensional Strings," in Sixth Adriatic meeting on particle physics, Dubrovnik, 1989.

[3] D. Lüst and S. Theisen, chap. 11;
P. Goddard and D. Olive, "Kač Moody and Virasoro Algebras," in 'Advanced Series in Mathematical Physics," World Scientific, Singapore, (1988).

[4] P. Argyres, K. Dienes and S.-H. Tye, CLNS 91/1113

[5] D. Gepner and Z. Qiu, Nucl. Phys. B285 (1987) 423.

[6] V. Kač, Adv. Math. 35 (1980) 264;
V. Kač and D. Peterson, Bull. AMS 3 (1980) 1057; Adv. Math. 53 (1984) 125.

[7] P. Argyres, K. Dienes and S.-H. Tye, CLNS 91/1113;
P. Argyres, and S.-H. Tye, Phys. Rev. Lett. 67 (1991) 3339;
K. Dienes and S.-H Tye, Nucl. Phys. B376 (1992) 297.

[8] P. Argyres, J. Grochocinski and S.-H. Tye, CLNS 91/1059;
S.-H. Tye, private communication.

[9] P. Argyres, K. Dienes and S.-H. Tye, CLNS 91/1113;

H. Kawai, D. Lewellen, and S.-H. Tye, Nucl. Phys. B288 (1987) 1.;
I. Antoniadis, C. Bachas, and C. Kounnas, Nucl. Phys. B289 (1987) 87;
H.Kawai, D. Lewellen, J.Schwartz and S.-H. Tye, Nucl Phys. 299B (1988) 431;
L. Alvarez-Gaumé, G. Moore, and C. Vafa, Comm. Math. Phs. 106 (1986) 1.

[10] P. Argyres, K. Dienes and S.-H. Tye, CLNS 91/1113; McGill-91-37.

[11] G. Cleaver and D. Lewellen, Phys. Lett. B300 (1993) 354.

[12] D. Gepner and Z. Qiu, Nucl. Phys. B285 (1987) 423.

[13] K. Dienes and S. -H. Tye, Nucl. Phys. B376 (1992) 297.

[14] P. Argyres and S. -H. Tye, Phys. Rev. Lett. 67 (1991) 3339.

[15] H. Kawai, D. Lewellen, and S.-H. Tye, Nucl. Phys. B288 (1987) 1;
I. Antoniadis, C. Bachas, and C. Kounnas, Nucl. Phys. B289 (1987) 87;
I. Antoniadis and C. Bachas, Nucl. Phys. B298 (1987) 586;
L. Alvarez-Gaumé, G. Moore, and C. Vafa, Comm. Math. Phys. 106 (1986) 1.

[16] M. Green, J. Schwarz, and E. Witten, Superstring Theory, V. I & II, Cambridge University Press, Cambridge, (1987).

[17] I. Antoniadis, C. Bachas, and C. Kounnas, Nucl. Phys. B289 (1987) 87;
I. Antoniadis and C. Bachas, Nucl. Phys. B298 (1987) 586.

[18] A. Zamolodchikov and V. Fateev, Sov. Phys. JETP 62 (1985) 215; Teor. Mat. Phys. 71 (1987) 163;
D. Gepner and Z. Qiu, Nucl. Phys. B285 (1987) 423.

[19] P. Ginsparg, Nucl. Phys. B295 (1988) 153;
E. Kiritsis, Phys. Lett. B217 (1988) 427.

[20] A. Schellekens, "Conformal Field Theory for Four-Dimensional Strings," in Sixth Adriatic meeting on particle physics, Dubrovnik, 1989.

[21] P. Argyres, E. Lyman, and S.-H. Tye, "Low-Lying States of the Six-Dimensional Fractional Superstring," preprint CLNS 91/1121.

[22] K. Kikkawa and M. Yamasaki, Phys. Lett. B149 (1984) 357;
N. Sakai and I. Senda, Prog. Theo. Phys. 75 (1986) 692.

[23] R. Hagedorn, Nuovo Cim. 56A (1968) 1027.

[24] M. Axenides, S. Ellis and C. Kounnas, Phys. Rev. D37 (1988) 2964.

[25] G. Athanasiu and J. Atick, "Remarks on Thermodynamics of Strngs," preprint IASSNS/HEP-88/46

[26] L. Clavelli, et al., Int. J. Mod. Phys. A5 (1990) 175.

[27] R. Brandenberger and C. Vafa, Nuc. Phys. B136 (1989) 391.

[28] K. O'Brien and C. Tan, Phys. Rev. D36 (1987) 1184;
N. Deo, S. Jain and C. Tan, Phys. Lett. B220 (1989) 125;
N. Deo, S. Jain and C. Tan, Phys. Rev. D40 (1989) 2626;
N. Deo, S. Jain and C. Tan, "The Ideal Gas of Strings," in S. Das et al., eds., Proceedings of the International Colloquium on Modern Quantum Field Theory, World Scientfic, Singapore, (1990);
N. Deo, S. Jain and C. Tan, Phys. Rev. D45 (1992) 3641.

[29] M. Bowick, "Finite Temperature Strings," preprint SUHEP-4241-522.

[30] M. Bowick and S. Giddings, Nucl. Phys. B325 (1989) 631.

[31] S. Frautschi, Phys. Rev. D3 (1971) 2821;
R. Carlitz, Phys. Rev. D5 (1972) 3231.

[32] J. Atick and E. Witten, Nuc. Phys. B310 (1988) 291.

[33] P. Salomonson and B. Skagerstam, Nucl. Phys. B268 (1986) 349.

[34] F. Lizzi and I. Senda, Phys. Lett. B244 (1990) 27;
F. Lizzi and I. Senda, Nucl. Phys. B268 (1986) 349.

[35] A. Tseytlin and C. Vafa, Nucl. Phys. B372 (1992) 443.

[36] J. Polchinski, Phys. Lett. B209 (1988) 252.

[37] R. Khuri, "Classical Dynamics of Macroscopic Strings," preprint CTP/TAMU-80/92;
R. Khuri, "Veneziano Amplitude for Winding Strings," preprint CTP/TAMU-10/93.

[38] H. Cateau and K. Sumiyoshi, Phys. Rev. D46 (1992) 2366.

[39] C. Vafa, private communication.

[40] R. Fernández, J. Fröhlich and A. Sokal, Random Walks, Critical Phenomena, and Triviality in Quantum Field Theory, Springer-Verlag, Berlin, (1992).

[41] B. Mandelbrot, The Fractlal Geometry of Nature, Freeman, New York (1977).

[42] M. Barnsley, Fractals Everywhere, Academic Press, Boston, (1988).

[43] D. Gross, Phys. Lett. B138 (1984) 185.

[44] J. Distler, Z. Hlousek and H Kawai, Int. J. Mod. Phys. 5A (1990) 1093;
G. Parisi, Phys. Lett. B81 (1979) 357;
J.Distler, private communication.

[45] J.Distler, private communication

[46] E. Witten, "Phases of $N = 2$ Theories in Two Dimensions," preprint IASSNS-HEP-93-3.

[47] N. Turok, Phys. Rev. Lett. 60 (1988) 549.

[48] J. Polchinski, private communication.

[49] M. Gasperini, N. Sánchez and G. Veneziano, Int. J. Mod. Phys. A6 (1991) 3853;
M. Gasperini, N. Sánchez and G. Veneziano, Nuc. Phys. B364 (1991) 365.

[50] L. Abbott and M. Wise Nucl. Phys. B224 (1984) 541.

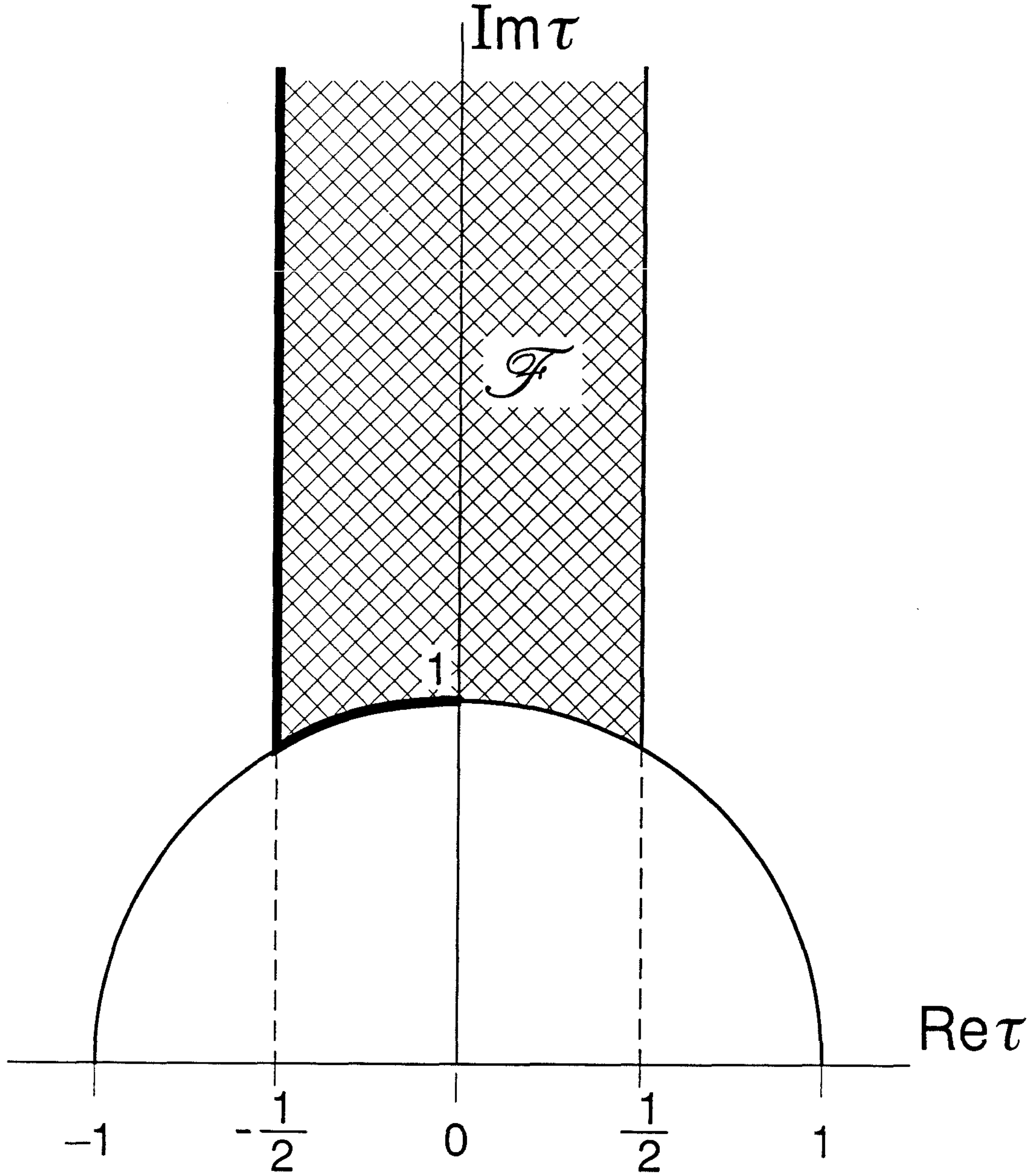


Fig. 2.1 The Fundamental Region.

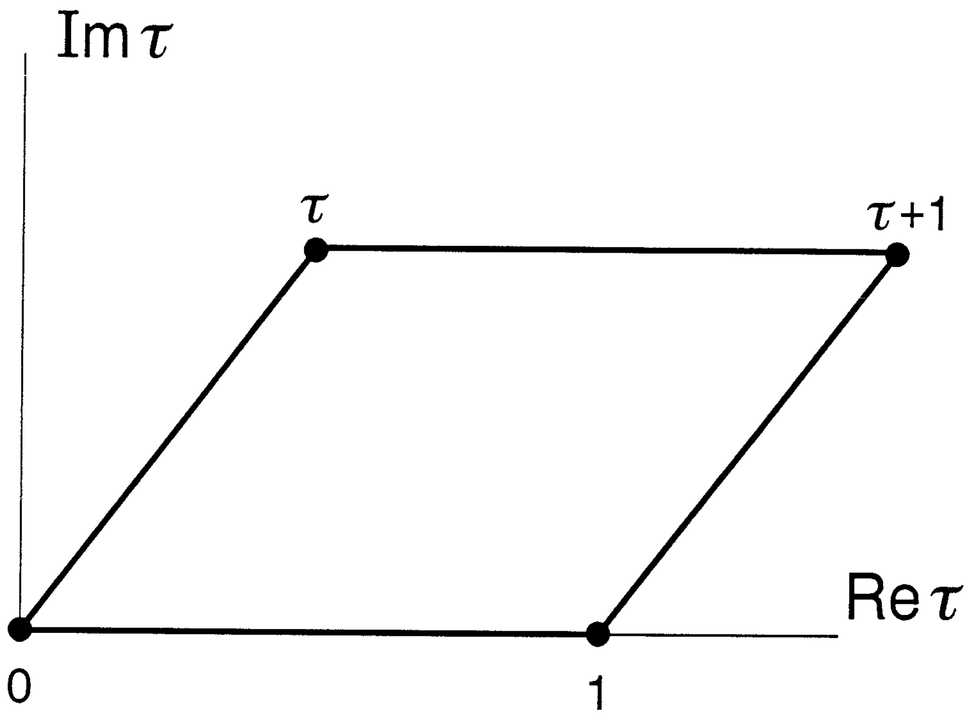


Fig. 2.2 Defining a two-dimensional torus with a lattice specified by $\tau$.

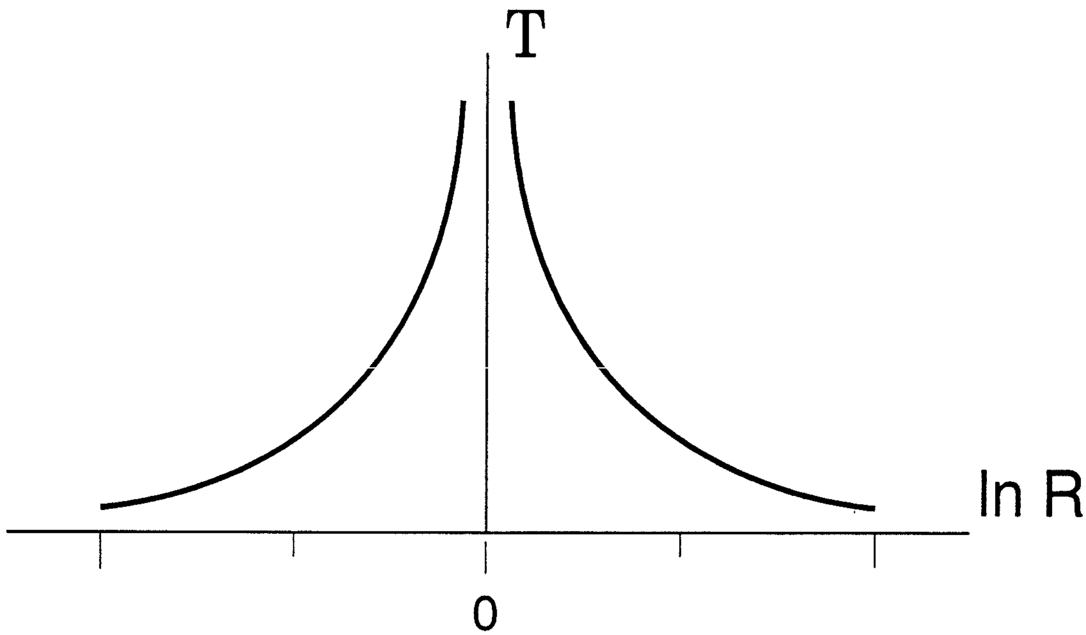


Fig. 4.1a An infinite temperature at R=1.

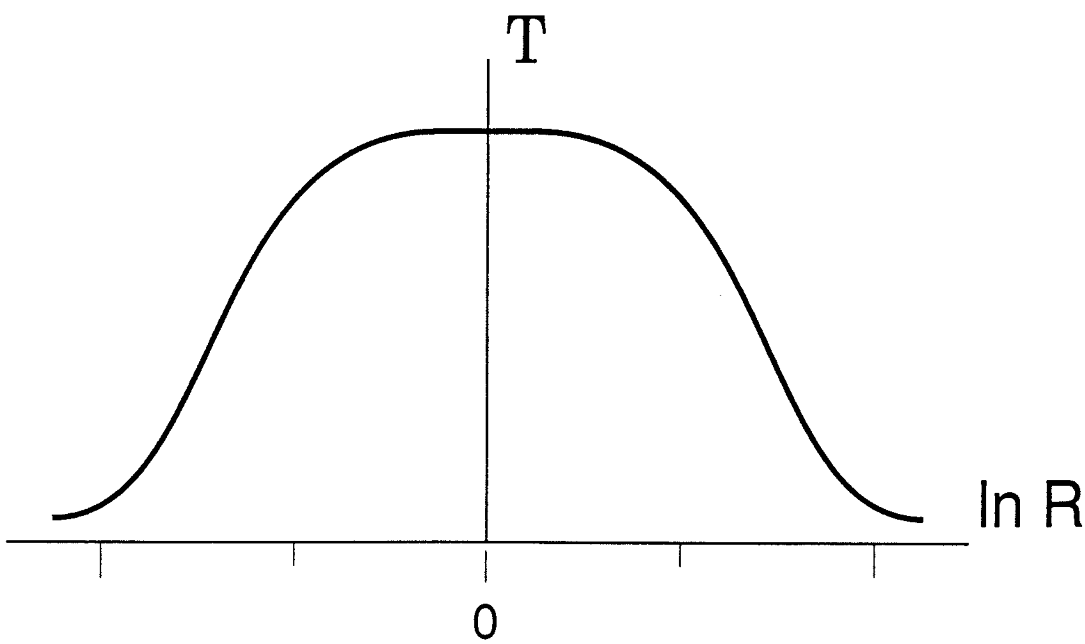


Fig. 4.1b A finite maximum temperature at R=1.

## Figure Captions for Appendix B

Unless otherwise noted, the effective string width is $\sqrt{37.64 + \ln L}$, $R_0 = 50.0$ and ten windings are initially placed about each of nine spatial dimensions.

Figure 1: Number of steps required for collision of two windings in three spacetime dimensions vs. initial radius, with at least 98 percent probability for collision.

Figure 2: Maximum expansion rate allowing collision of two windings in four spacetime dimensions vs. initial radius, with at least 98 percent probability for collision.

Figure 3: Average dimension of spacetime vs. initial radius with $H = .1$.

Figure 4: Average dimension of spacetime vs. initial radius with $H = .01$.

Figure 5: Average dimension of spacetime vs. initial radius with $H = .001$.

Figure 6: Average dimension of spacetime vs. initial radius with $H = .0001$.

Figure 7: Average dimension of spacetime vs. $c$ where the effective string width is $\sqrt{c + \ln L}$, $H = .01$ and the initial radius is 1.0.

Figure 8: Average dimension of spacetime vs. $R_0$ where $H = .001$ and the initial radius is 1.5.

Figure 9: Histogram of the dimension of spacetime for 50 trials with an initial radius of 1.0 and $H = .001$.

Figure 10: Histogram of the dimension of spacetime for 50 trials with an initial radius of 1.1 and $H = .001$.

Figure 11: Histogram of the dimension of spacetime for 50 trials with an initial radius of 1.2 and $H = .001$.

Figure 12: Histogram of the dimension of spacetime for 50 trials with an initial radius of 1.3 and $H = .001$.

Figure 13: Histogram of the dimension of spacetime for 50 trials with an initial radius of 1.4 and $H = .001$.

Figure 14: Histogram of the dimension of spacetime for 50 trials with an initial radius of 1.5 and $H = .001$.

Figure 15: Histogram of the dimension of spacetime for 50 trials with an initial radius of 1.6 and $H = .001$.

Figure 16: Histogram of the dimension of spacetime for 50 trials with an initial radius of 1.7 and $H = .001$.

Figure 17: Histogram of the dimension of spacetime for 50 trials with an initial radius of 1.8 and $H = .001$.

Figure 18: Histogram of the dimension of spacetime for 50 trials with an initial radius of 1.9 and $H = .001$.

Figure 19: Histogram of the dimension of spacetime for 50 trials with an initial radius of 2.0 and $H = .001$.

Figure 20: Histogram of the dimension of spacetime for 50 trials with an initial radius of 1.19 and $H = .1$. A four-dimensional spacetime is then the most probable outcome.

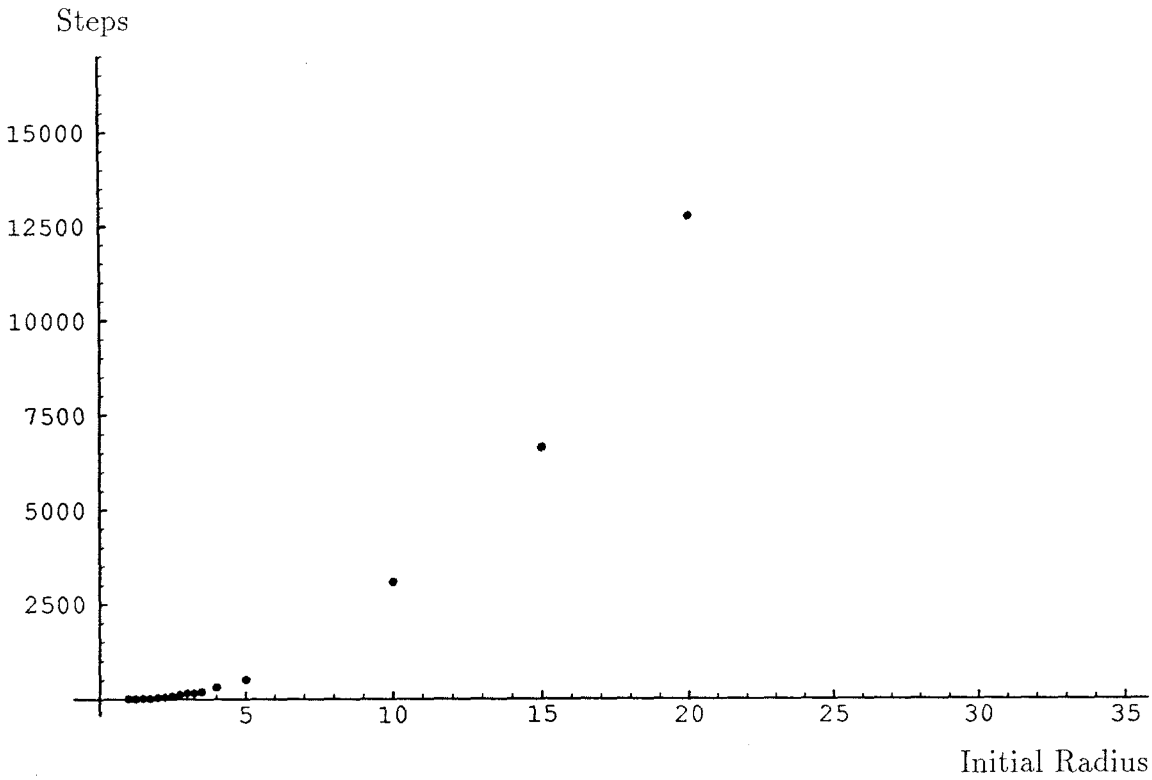


Fig. 1. Steps for Collision vs. Initial Radius

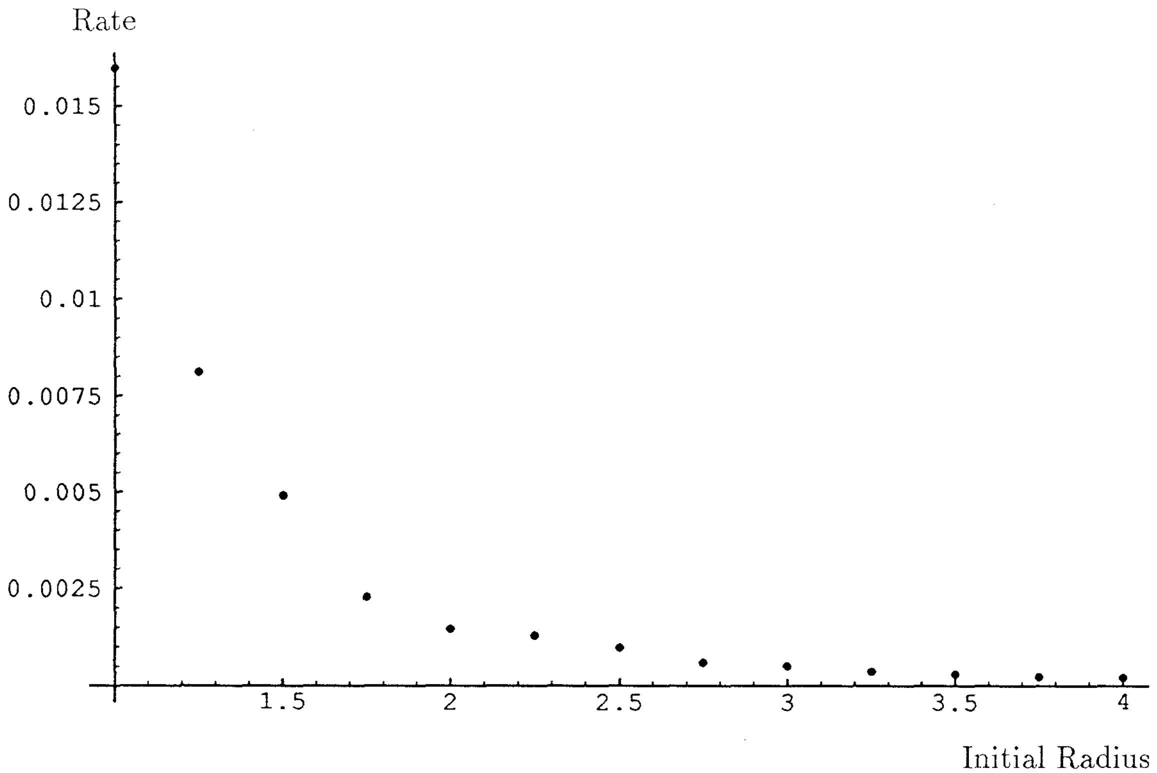


Fig. 2. Maximum Expansion Rate vs. Initial Radius

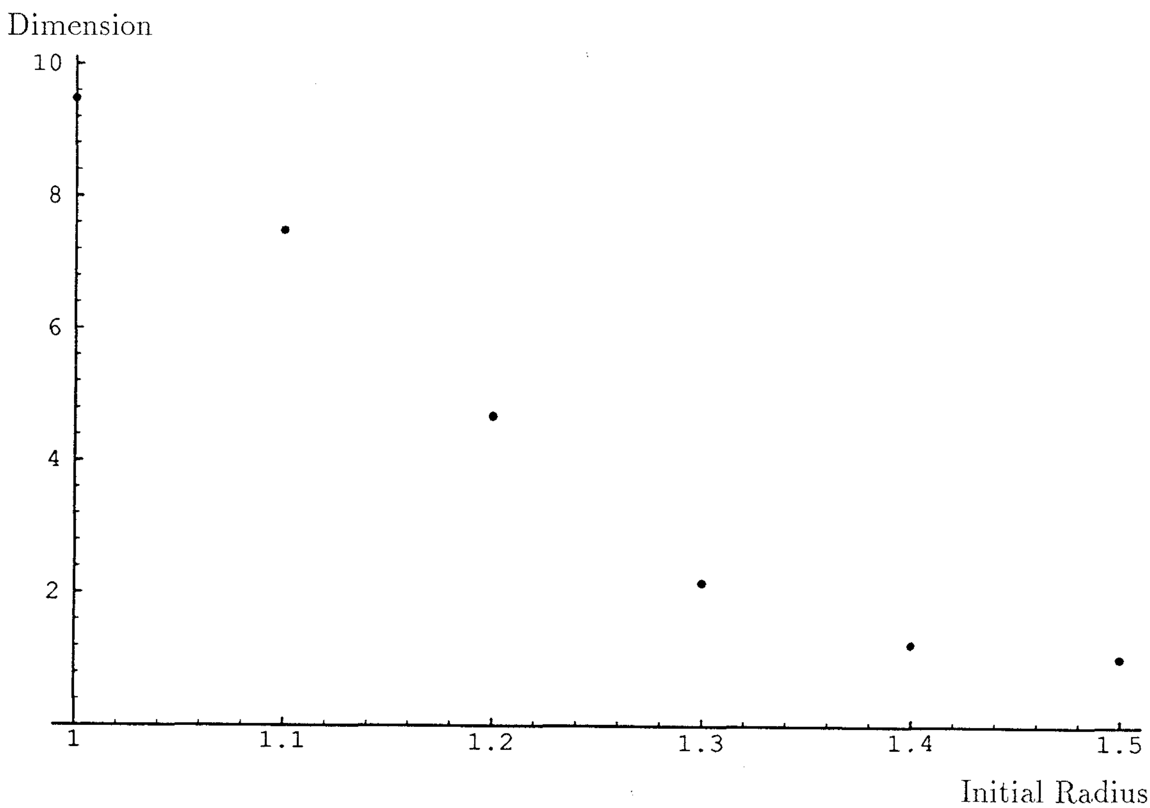


Fig. 3. Dimension vs. Initial Radius with $H = .1$

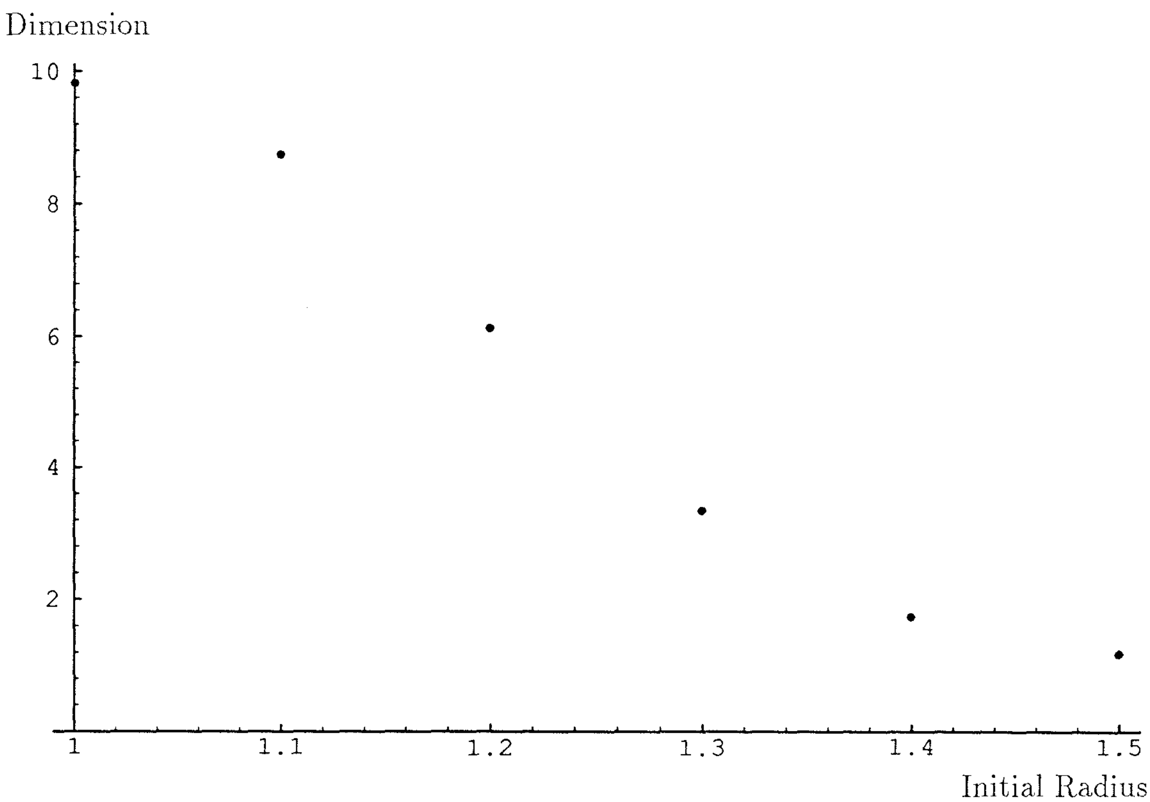


Fig. 4. Dimension vs. Initial Radius with $H = .01$

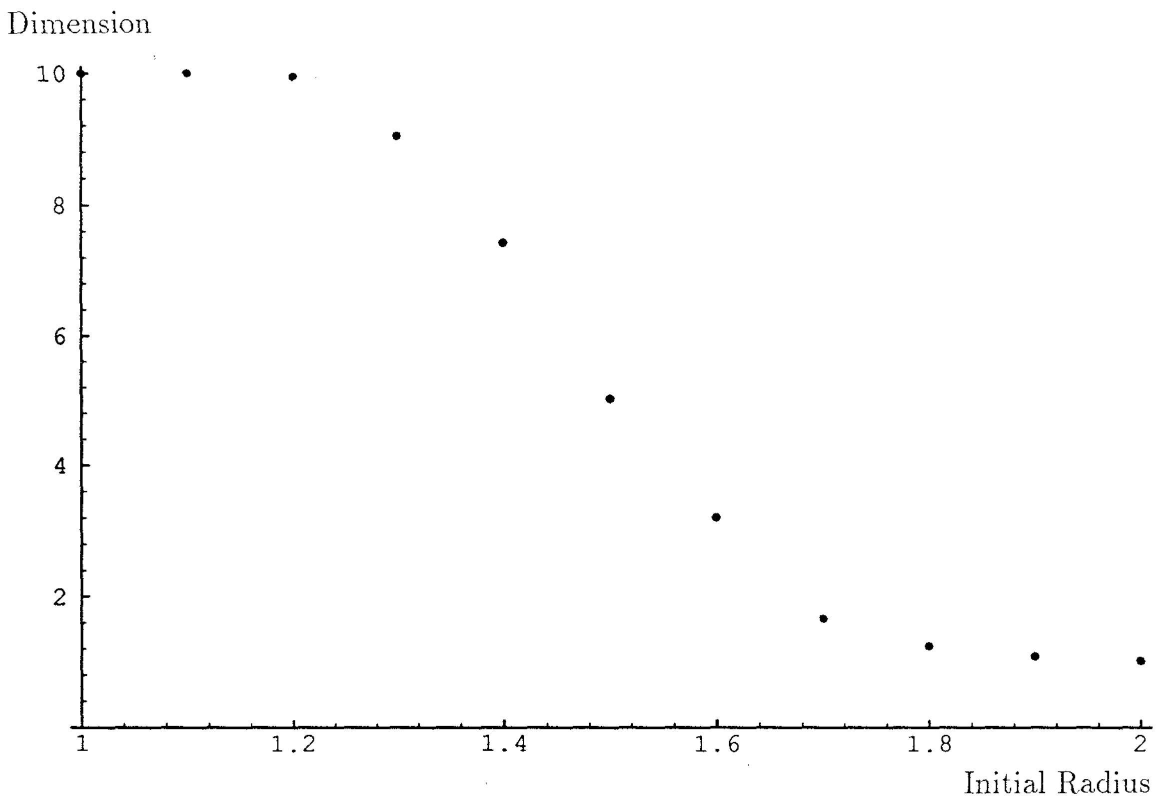


Fig. 5. Dimension vs. Initial Radius with $H = .001$

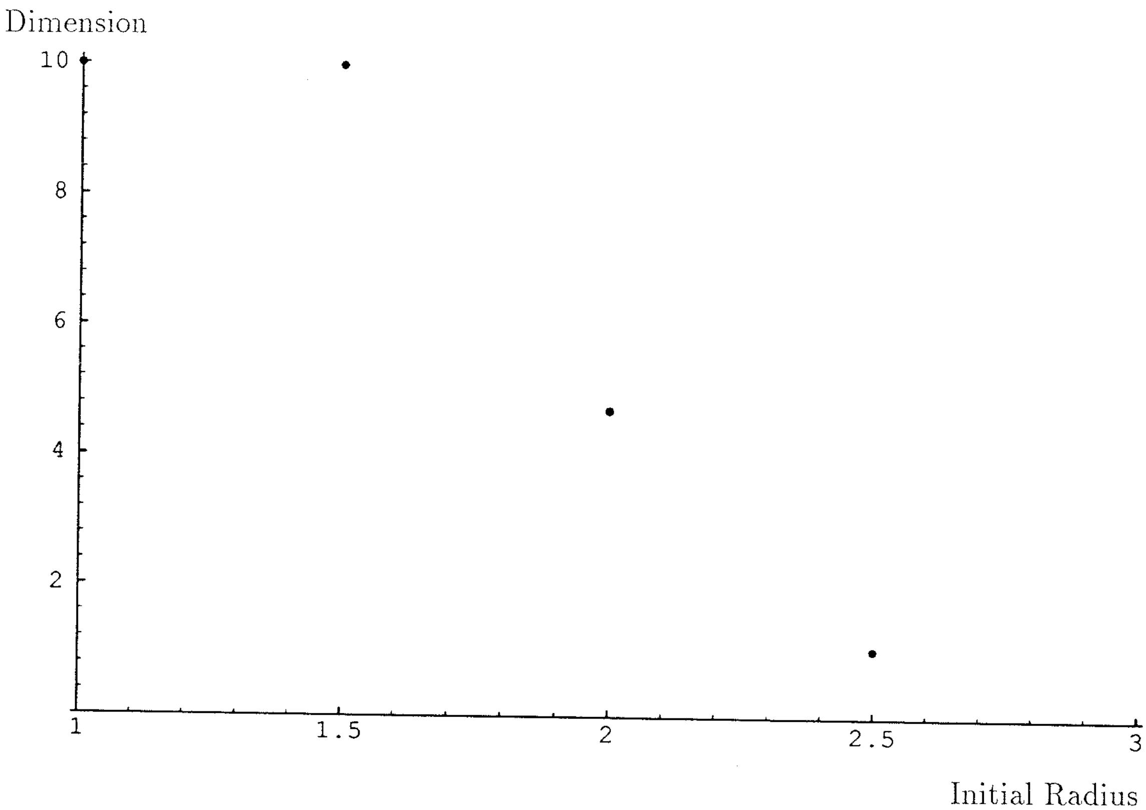


Fig. 6. Dimension vs. Initial Radius with $H = .0001$

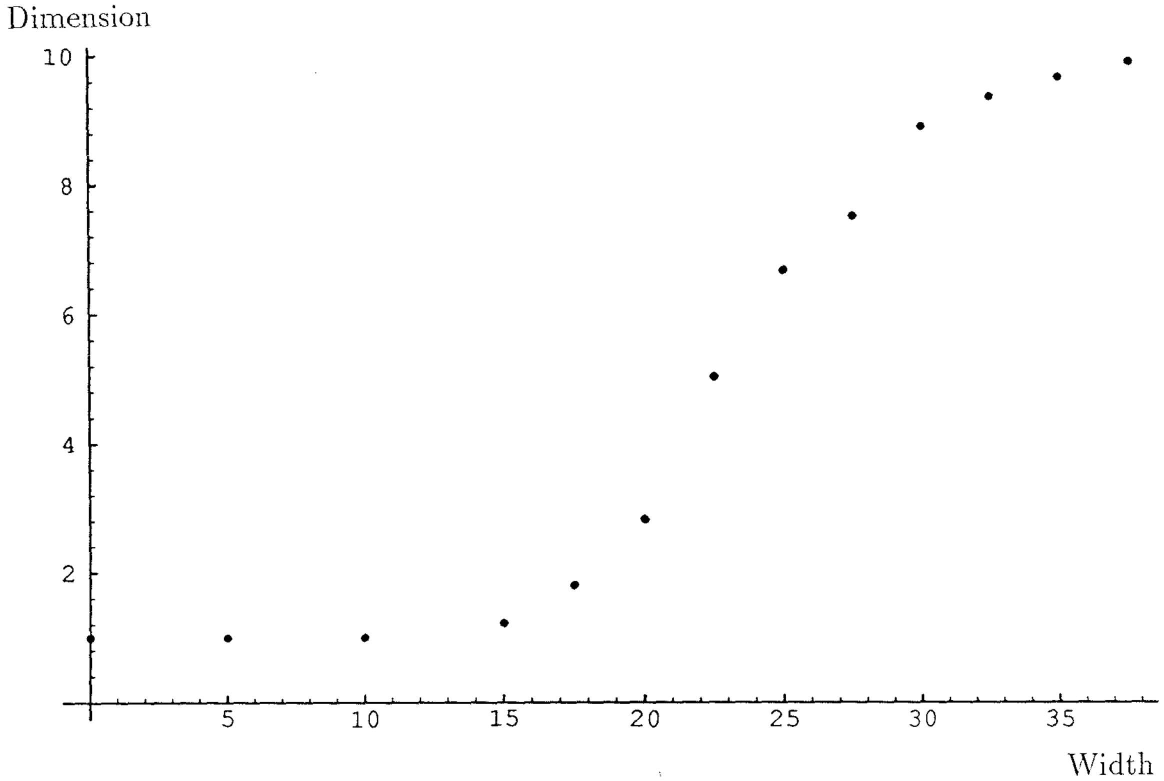


Fig. 7. Dimension vs. Width

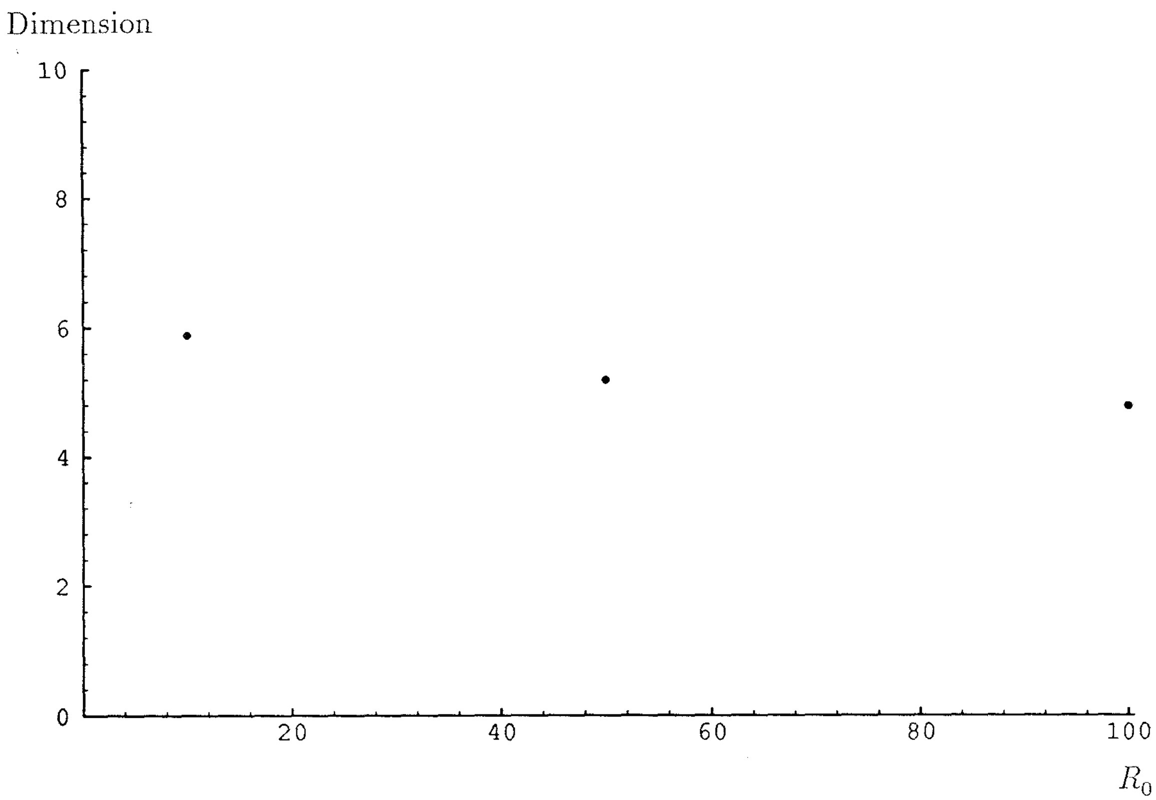


Fig. 8. Dimension vs. $R_0$

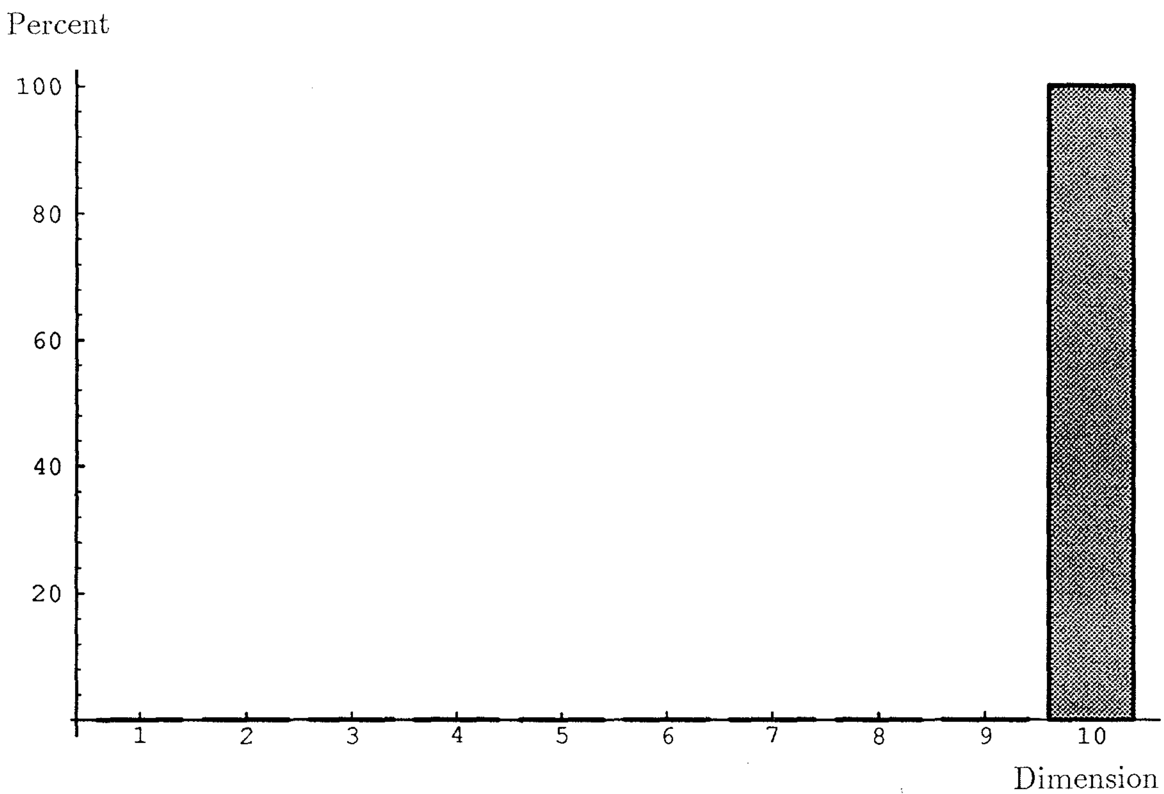


Fig. 9. Histogram of Dimension, $R = 1.0$

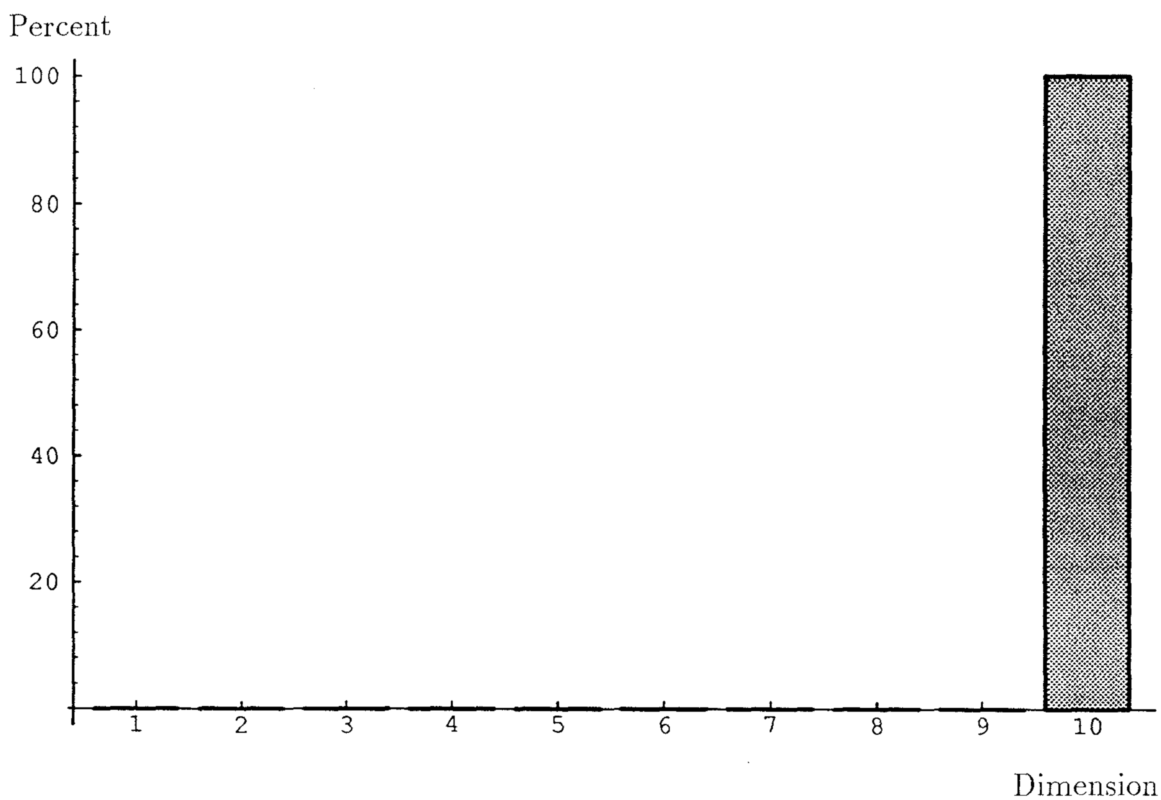


Fig. 10. Histogram of Dimension, $R = 1.1$

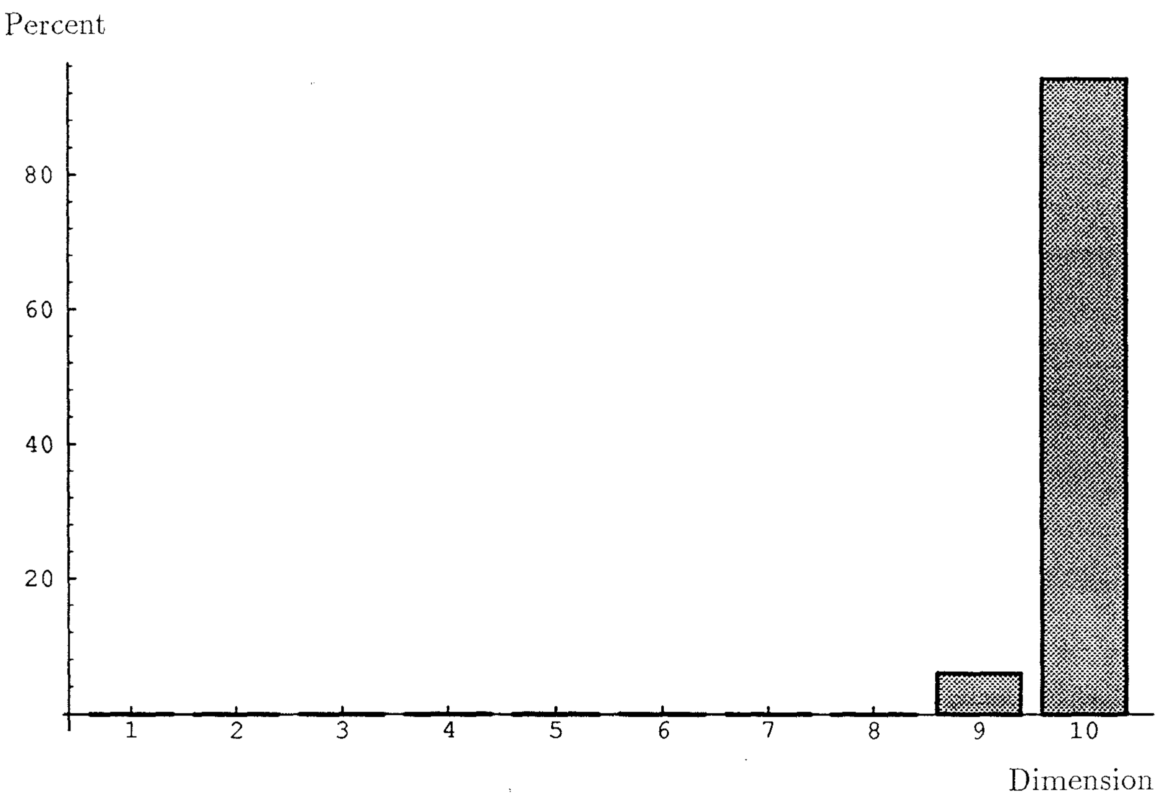


Fig. 11. Histogram of Dimension, $R = 1.2$

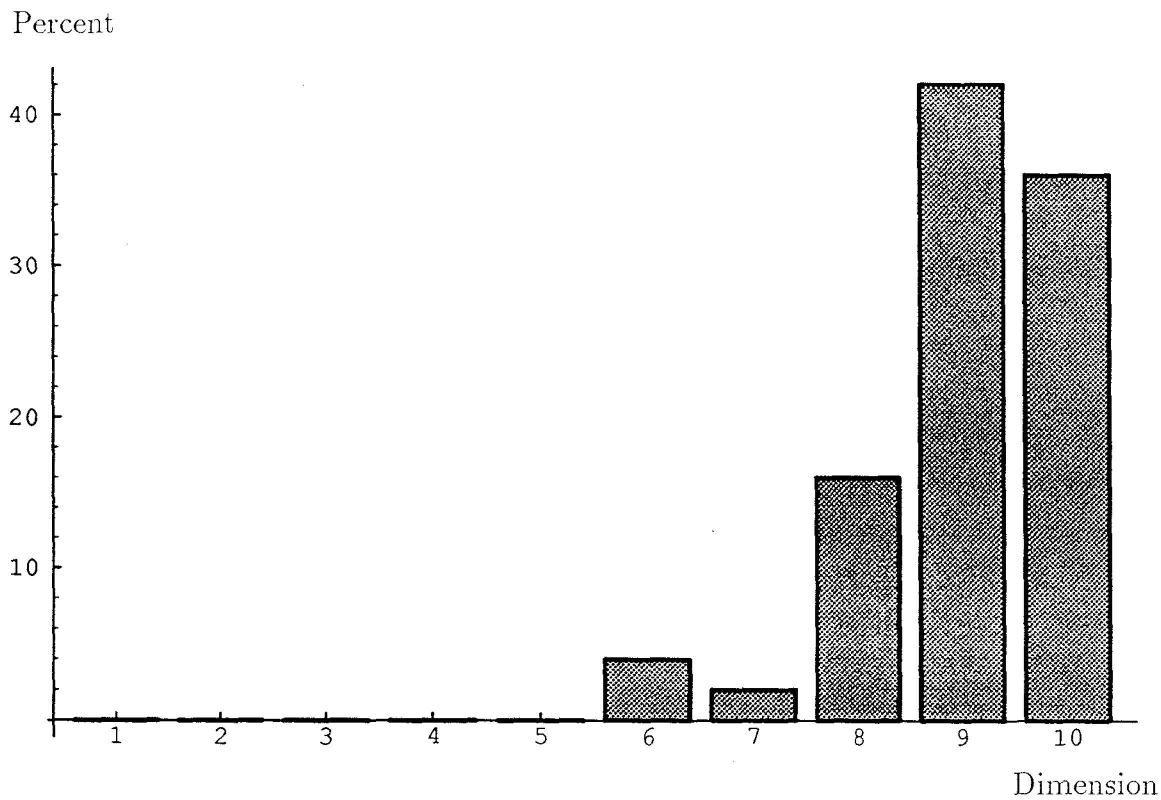


Fig. 12. Histogram of Dimension, $R = 1.3$

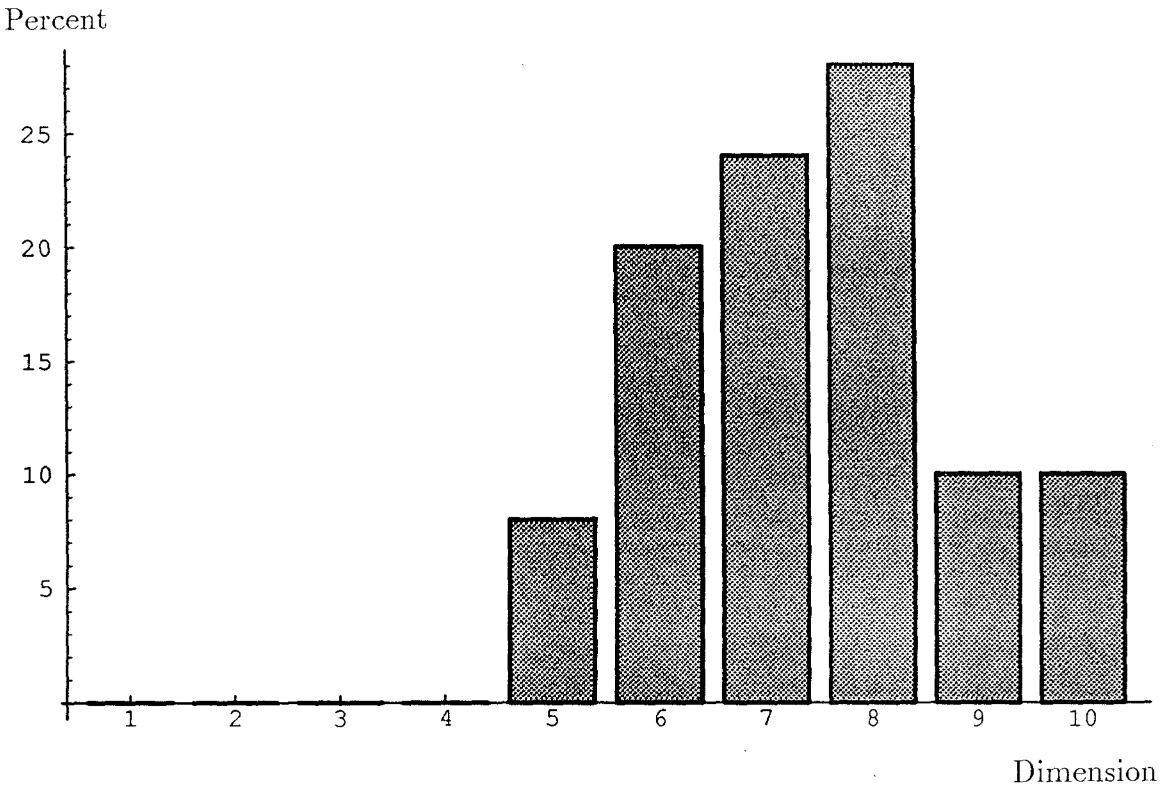


Fig. 13. Histogram of Dimension, $R = 1.4$

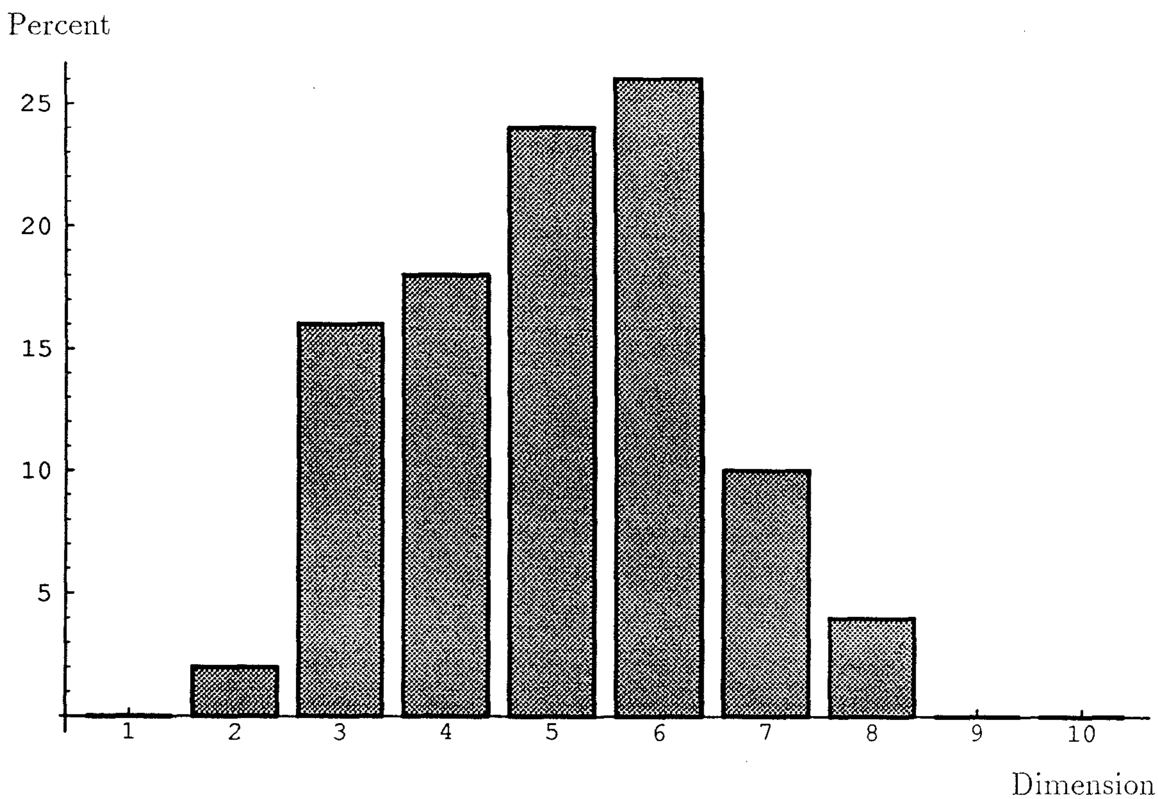


Fig. 14. Histogram of Dimension, $R = 1.5$

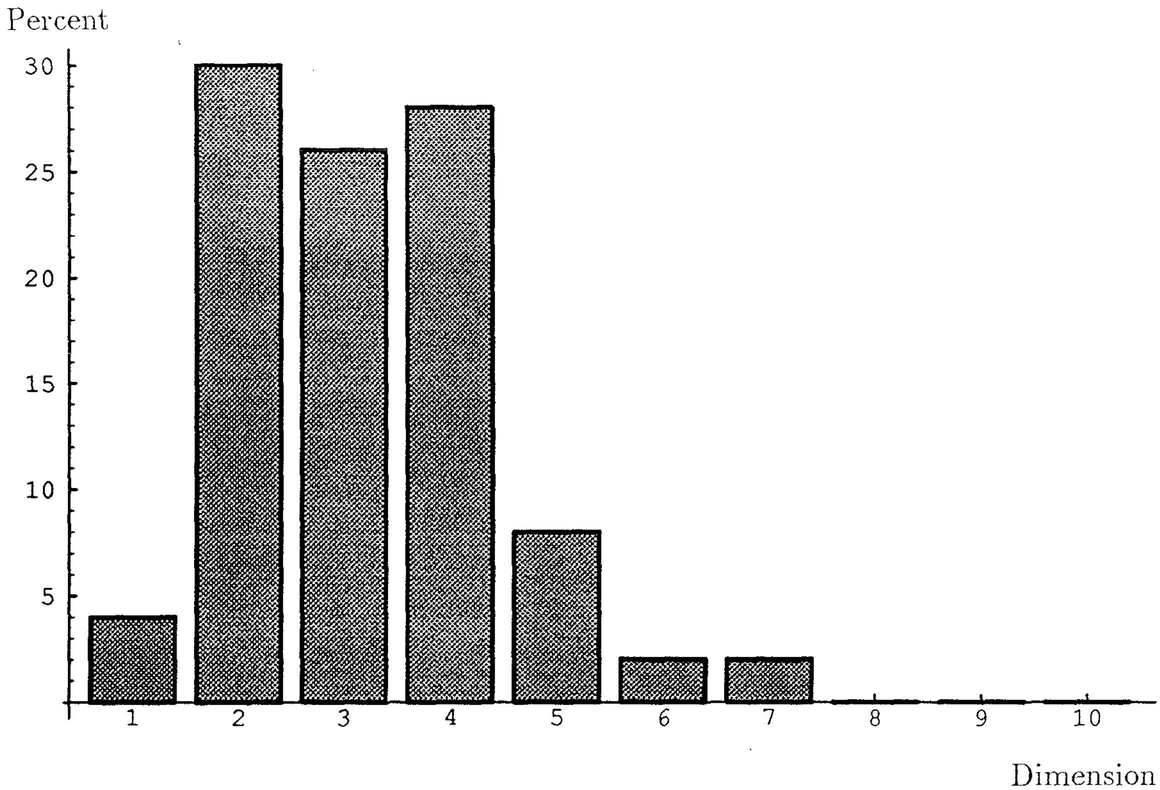


Fig. 15. Histogram of Dimension, $R = 1.6$

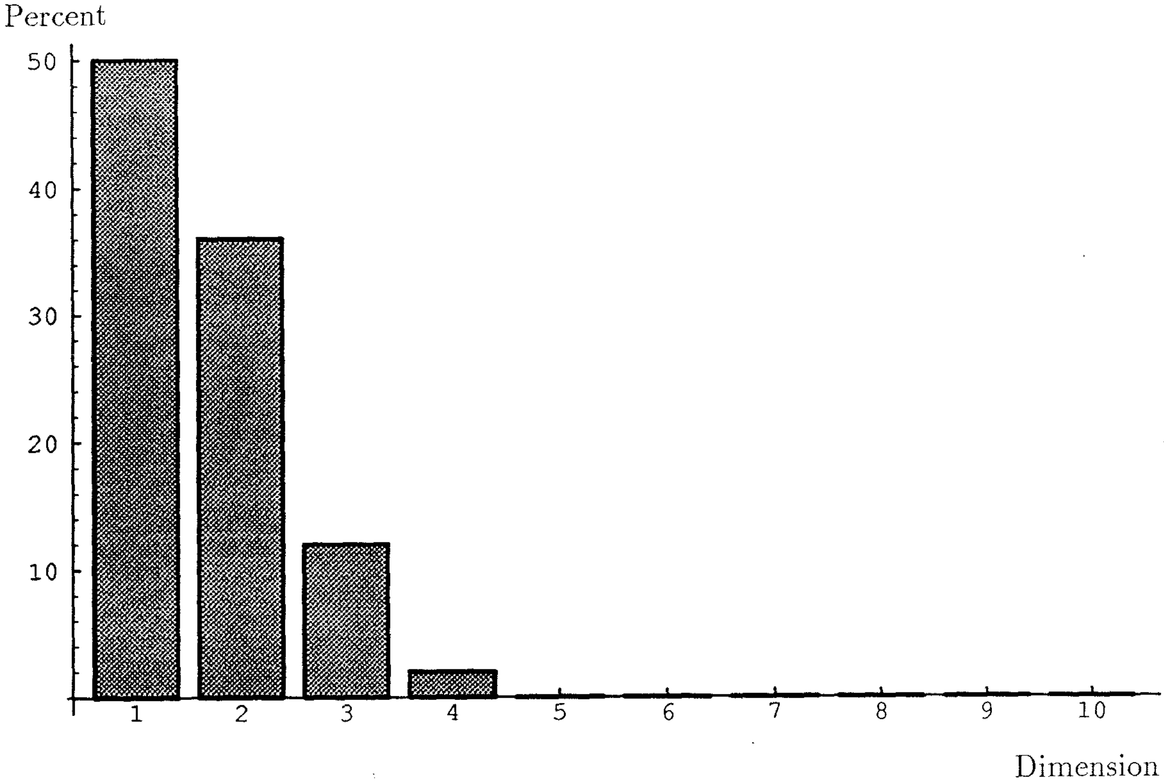


Fig. 16. Histogram of Dimension, $R = 1.7$

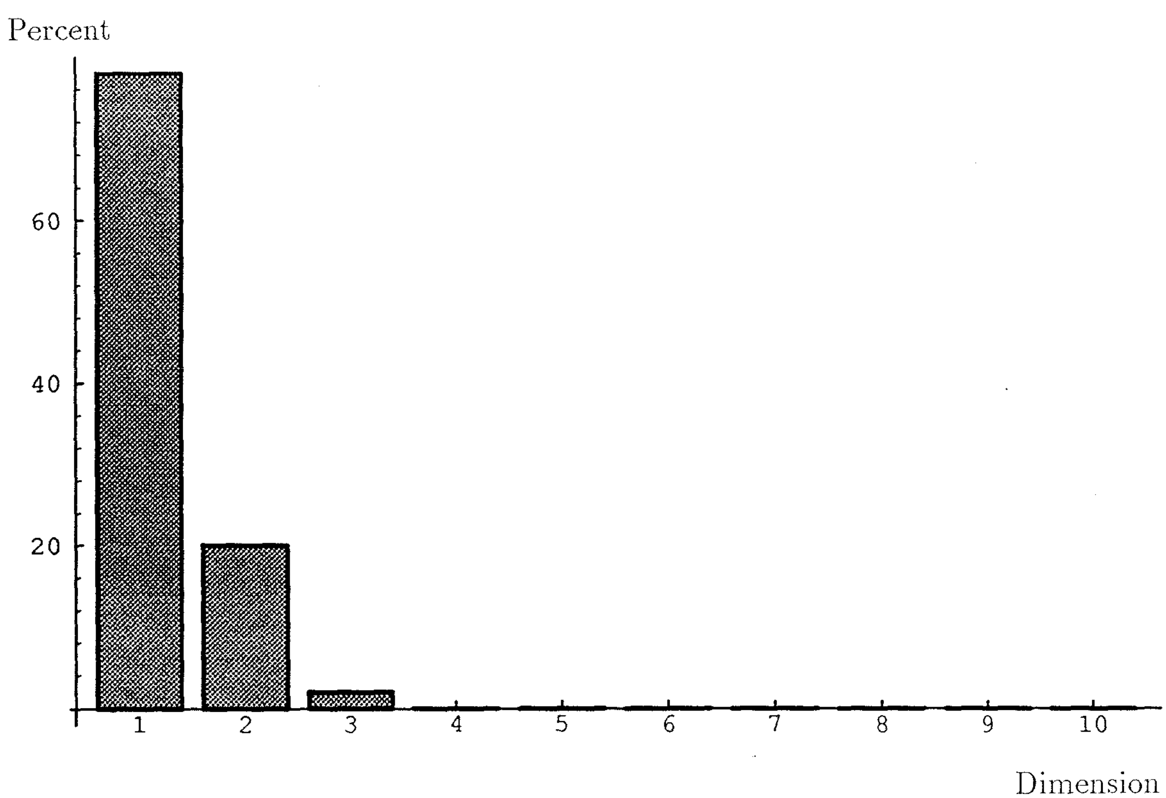


Fig. 17. Histogram of Dimension, $R = 1.8$

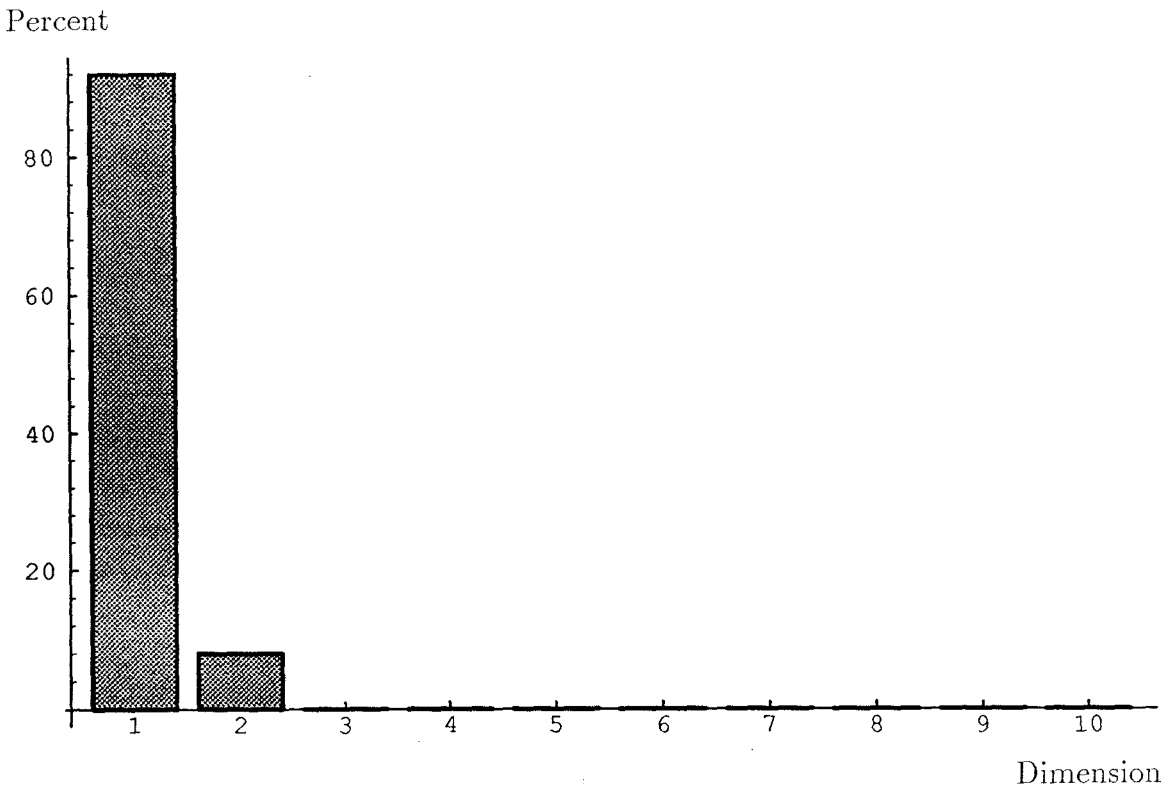


Fig. 18. Histogram of Dimension, $R = 1.9$

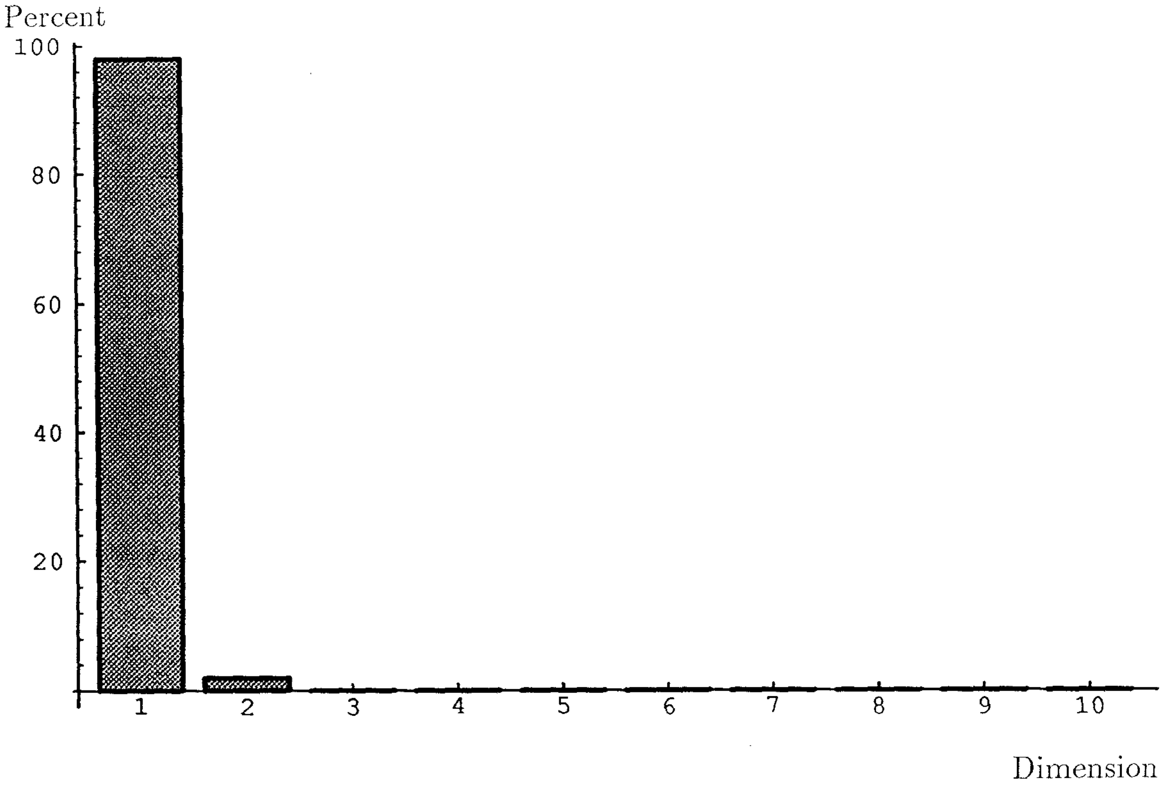


Fig. 19. Histogram of Dimension, $R = 2.0$

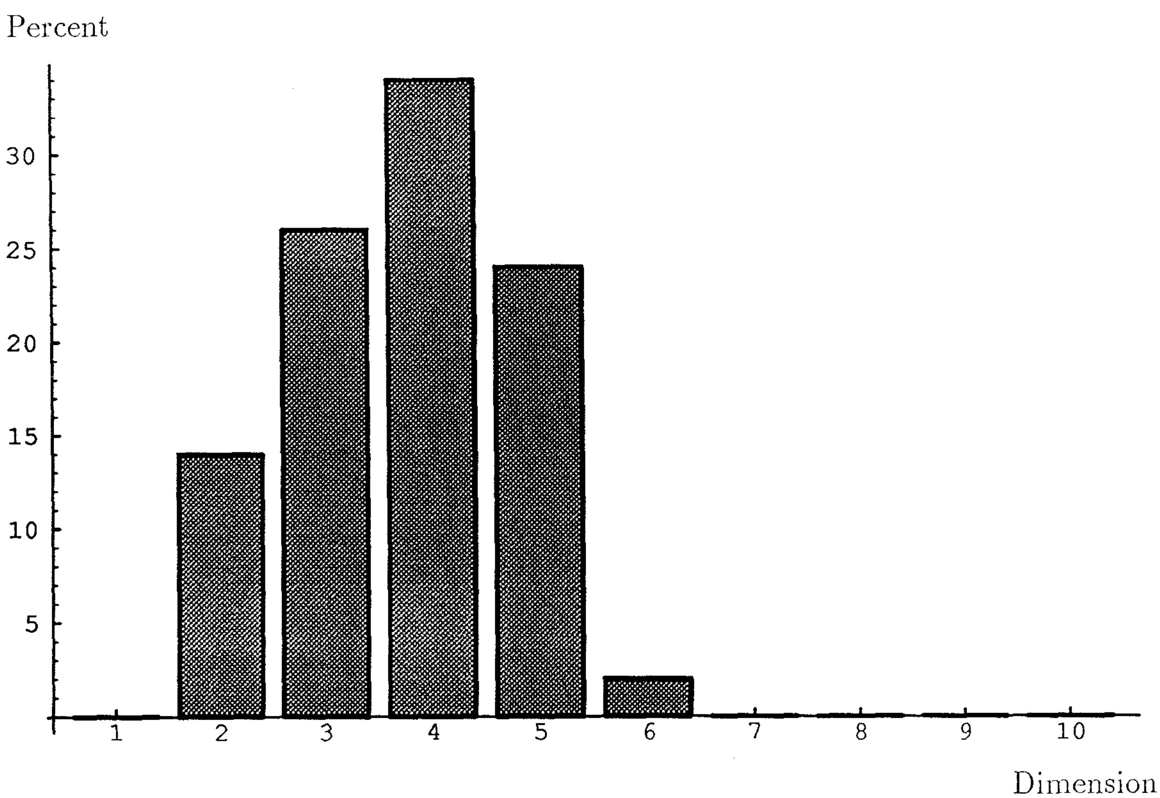


Fig. 20. A Maximum at Four Dimensions!