Topological Strings, Double Affine Hecke Algebras, and Exceptional Knot Homology

Thesis by

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For

Gram and Gunky,

in loving memory.

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Abstract

This thesis is primarily based on the publications [CE, EG], which respectively study refined, composite invariants and exceptional knot homologies of torus knots. The main technical tools are double affine Hecke algebras ("DAHA") and various insights from topological string theory.

In [CE], we define and study the composite DAHA-superpolynomials of torus knots, which depend on pairs of Young diagrams and generalize the composite HOMFLY-PT polynomials from the full HOMFLY-PT skein of the annulus. We establish the correct notions of stabilization, duality, and evaluation in the DAHA theory, as well as the connection to the HOMFLY-PT theory via a composite analogue of the Rosso-Jones formula. At the end, we construct two DAHA-hyperpolynomials which are closely related to the Deligne-Gross exceptional series of root systems.

In [EG], we describe a rich structure of differentials that act on homological knot invariants for exceptional groups. These follow from the physics of BPS states and the adjacencies/spectra of singularities associated with Landau-Ginzburg potentials. When combined with the DAHA-Jones theory, these differentials uniquely determine hyperpolynomials (with positive coefficients), which are Poincaré polynomials for exceptional $(\mathfrak{e}_6, \mathbf{27})$ torus knot homologies.

We also provide new results connecting DAHA-Jones polynomials to quantum torus knot invariants for Cartan types A and D, as well as the first appearance of quantum $(\mathfrak{e}_6, \mathbf{27})$ knot invariants in the literature. These results are previously unpublished.

Contents

Chapter 1

Introduction

Our brains are hard-wired for topology. Indeed, life would be very confusing if we had so strict a notion of "sameness" that we could not read each other's handwriting or agree that Earth's surface is a sphere. Only by dividing the world into equivalence classes, and then studying the relationships between those classes, can we begin to understand it as a whole. Otherwise, we are hopelessly lost and overwhelmed, trying to classify all of the "beautiful and unique snowflakes" out there.

While our brains are inclined to *interpret* topologically, they are also inclined to *compute* algebraically. *Topological quantum field theory* ("TQFT") provides a general framework for re-expressing topological problems in an algebraic setting. Therefore, TQFT is extraordinarily useful for making progress where our topological intuitions fail. As its name suggests, TQFT has its origins in physics, though by now it belongs equally to mathematics.

In this thesis, we will use algebra to study topology via the relation provided by physics (TQFT). In particular, we consider some aspects of *knot theory*, a subject in topology whose basic goal is to determine when two closed, knotted loops of string are equivalent or *isotopic*, i.e., when one may be deformed (without cutting) to match the other. The framework for this study will be a related set of TQFT's which produce the (algebraic) *quantum knot invariants.*

Basic concepts

Quantum knot invariants have many different but equivalent definitions throughout the mathematical and physical literature. However, in every case, they are essentially assignments:

$$(K;\mathfrak{g},V) \rightsquigarrow P^{\mathfrak{g},V}(K;q) \in \mathbb{Z}[q^{\pm 1}], \tag{1.1}$$

where $K \subset \mathbf{S}^3$ is a knot, $\mathfrak{g}$ is a complex, simple Lie algebra, and V is a representation of the Hopf algebra $\mathcal{U}_q(\mathfrak{g})$. From these three pieces of data, one obtains an integer Laurent polynomial $P^{\mathfrak{g},V}(K;q)$, which is well-defined on the isotopy class of K.

A more recent theme in this subject is the *categorification* of quantum knot invariants via doubly-graded homology theories $\mathcal{H}^{\mathfrak{g},V}_{i,j}(K)$ for which

$$P^{\mathfrak{g},V}(K;q) = \sum_{i,j} (-1)^j q^i \dim \mathcal{H}^{\mathfrak{g},V}_{i,j}(K). \tag{1.2}$$

These so-called "knot homologies" have many advantages over quantum knot invariants. Most notably, they possess a category-theoretic (TQFT) structure, which, among other things, provides a window into 4-dimensional topology.

One can view quantum invariants and their categorifications differently depending on his or her interests. At various points in this thesis, it will be unclear whether we are studying topology, representation theory, or physics. In fact, we are studying all three!

Problems considered

The original motivation for the work in this thesis was to understand "exceptional knot homologies," i.e., $\mathcal{H}^{\mathfrak{g},V}_{*}(K)$ for which $\mathfrak{g} \in \{\mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8\}$ is an exceptional Lie algebra. For a variety of reasons, both practical and theoretical, this presents a significant challenge. Using insights from topological strings and double affine Hecke algebras ("DAHA"), we are able to make several concrete predictions about the structure of such a homology theory [EG]. Let us briefly describe these two approaches.

Topological strings

From the viewpoint of topological strings [GSV], knot homologies are realized as Hilbert spaces of BPS states, which are refined (graded) by their charges:

$$\mathcal{H}_{\text{knot}}(K) \;\cong\; \mathcal{H}_{\text{BPS}}(K). \tag{1.3}$$

Geometrically, $\mathcal{H}_{\text{BPS}}(K)$ may be described by the enumerative geometry of a certain Calabi-Yau space X called the resolved conifold.

In [DGR, GW, GS, GGS], the authors use (1.3) to make predictions about $\mathcal{H}_{\text{knot}}(K)$. In particular, they conjecture triply-graded (or quadruply-graded) homology theories $\mathcal{H}_*(K)$ with rigid structures of *differentials* $d_{\mathfrak{g},V}$ such that

$$H_*(\mathcal{H}_*, d_{\mathfrak{g},V}) \cong \mathcal{H}^{\mathfrak{g},V}_*, \tag{1.4}$$

i.e., $\mathcal{H}_*$ unifies the homology theories associated to different $(\mathfrak{g}, V)$. The individual theories may be recovered by taking homology with respect to the differential $d_{\mathfrak{g},V}$. One can use this structure to predict explicit Poincaré polynomials, a.k.a *superpolynomials*, for knot homologies.

Double affine Hecke algebras

In another line of development, the authors of [AS] use physical arguments to refine quantum $(\mathfrak{sl}_N, \lambda)$-invariants of torus knots in terms of an additional parameter t. For $\lambda = \square$, the N-stable version of this refinement was conjecturally related to the superpolynomials of [DGR]. In [C5, C6], the author used DAHA to formalize their construction and generalize it to arbitrary $(\mathfrak{g}, V)$.

In particular, to any torus knot $T^{\mathsf{r},\mathsf{s}}$, root system R, and dominant weight $b \in P_+$ for R, one associates the *DAHA-Jones polynomial* $\widetilde{JD}^{R}_{\mathsf{r},\mathsf{s}}(b; q, t) \in \mathbb{Z}[q, t]$. Conjecturally,

$$\widetilde{JD}^{R}_{\mathsf{r},\mathsf{s}}(b; q, t \mapsto q) = P^{\mathfrak{g}, V_b}(T^{\mathsf{r},\mathsf{s}}; q), \tag{1.5}$$

where R is the root system for the Lie algebra $\mathfrak{g}$, and V_b is the representation with highest weight b. One may also define the *DAHA-superpolynomials* by the (infinitely many) specializations

$$HD_{\mathsf{r},\mathsf{s}}(\lambda; q, t, a \mapsto -t^{n+1}) = \widetilde{JD}^{A_n}_{\mathsf{r},\mathsf{s}}(\lambda; q, t), \tag{1.6}$$

where, for n sufficiently large, λ is interpreted as a Young diagram for A_n. For rectangular λ, the DAHA-superpolynomial $HD_{\mathsf{r},\mathsf{s}}(\lambda; q, t, a)$ has positive coefficients and is related to the superpolynomials in [DGR, GS].

Exceptional knot homologies

Each approach has its advantages and disadvantages. In [EG] we used *both* approaches to make explicit predictions for *hyperpolynomials*, i.e., Poincaré polynomials, for $\mathcal{H}^{\mathfrak{e}_6, \mathbf{27}}(T^{\mathsf{r},\mathsf{s}})$. Specifically, we combined the t-refinement from the DAHA theory with a rigid structure of differentials, from which we could uniquely restore positive coefficients to $\widetilde{JD}^{E_6}_{\mathsf{r},\mathsf{s}}(\omega_1; q, t)$ in three convincing examples. The differentials were justified by a detailed analysis of the critical structure of the corresponding Landau-Ginzburg potential $W_{E_6, 27}$; see [GW].

While considering other exceptional knot homologies, the author R.E. observed that the DAHA-Jones polynomials exhibited regularities within the Deligne-Gross "exceptional series" [DG]:

$$e \subset A_1 \subset A_2 \subset G_2 \subset D_4 \subset F_4 \subset E_6 \subset E_7 \subset E_8. \tag{1.7}$$

In [CE], we construct two hyperpolynomials, defined by specializations,

$$HD^{\mathfrak{ad}}_{\mathsf{r},\mathsf{s}}(q, t, a = -t^{\nu(G)}) = \widetilde{JD}^{G}_{\mathsf{r},\mathsf{s}}(\mathfrak{ad}; q, t), \text{ excluding } G_2, F_4, \tag{1.8}$$

to the DAHA-Jones polynomials for $T^{3,2}$, $T^{4,3}$ "colored" by the adjoint representation for the algebraic groups of type ADE in this series. The parameter $\nu(G)$ is used in [DG].

The role of the exceptional series in [CE] was mainly as inspiration. That is, consideration of the exceptional series required, in particular, consideration of the adjoint representation. For A_n, the adjoint representation has highest weight $\omega_1 + \omega_n$. Observe that this weight depends on the rank, so stabilization is a more subtle issue there.

Weights of this sort are called *composite weights* and were studied in [K]. They depend on pairs $[\lambda, \mu]$ of Young diagrams, where λ is the portion that encodes the n dependence, e.g.,

$$[\square, \square] = \omega_1 + \omega_n. \tag{1.9}$$

The main subjects of [CE] were establishing the existence and properties of DAHA-superpolynomials for composite weights, as well as their connection to composite HOMFLY-PT polynomials. The latter have a topological interpretation in [HM] via skein theory in the annulus, as well as physical interpretations in [GJKS, Ma3, PBR].

Outline

The original work in this thesis is based mostly on the author's contributions to [CE] and [EG]. One notable addition is Theorem 4.3.4, which establishes the coincidence (1.5) between DAHA-Jones polynomials and quantum knot invariants in the cases that R is of type A or D. Also, Appendix D contains expressions for $P^{\mathfrak{e}_6,\mathbf{27}}(K;q)$ for many knots, which have not previously appeared in the literature (for *any* knots).

This thesis has the following structure:

- In Chapter 2, we provide both mathematical and physical accounts of TQFT's and how they give rise to quantum knot invariants. We briefly describe the ideas of categorification and knot homologies from the mathematics perspective.

- In Chapter 3, we give a detailed account of the physical realization of knot homologies as spaces of BPS states in topological string theory. We also introduce the specific approaches via differentials in [DGR] and refined Chern-Simons theory in [AS].

- In Chapter 4, we define the DAHA-Jones polynomials and explain their relationship to torus knot polynomials and homologies. We also prove Theorem 4.3.4, which establishes the connection to quantum knot invariants in the cases of A and D.

- In Chapter 5, we recount the results from Sections 1-3 of [CE], which study the composite HOMFLY-PT polynomials, composite DAHA-superpolynomials, and various examples and confirmations. The main results are Proposition 5.2.1, Theorem 5.3.1, and Theorem 5.3.4.

- In Chapter 6, we recount the results on exceptional knot homologies from [EG] as well as their justification using the classical theory of singularities. We also discuss the stabilization of DAHA-Jones polynomials in the Deligne-Gross exceptional series from Section 4 of [CE].
  - Appendix A contains DAHA-Jones formulas, which support Section 6.2.
  - Appendix B contains diagrams which depict our proposals in Section 6.2.
  - Appendix C contains the adjacency tree and spectra, which support Section 6.3.
  - Appendix D contains new formulas for $P^{\mathfrak{e}_6,\mathbf{27}}(K;q)$.

Chapter 2

TQFT, Quantum Knot Invariants, and Knot Homologies

We introduce topological quantum field theories in Section 2.1, as a unifying theme for the ideas considered in this thesis. They provide a general framework for studying topological problems algebraically. These constructs are of mathematical and physical interest, and we give an even-handed account of both viewpoints.

Of particular interest to us are topological quantum field theories that give rise to isotopy invariants of knots and links in 3-manifolds. We provide both physical and mathematical definitions of these *quantum link invariants* in Sections 2.2 and 2.3, respectively. As we will see, they also depend on the representation theory of Lie algebras, underscoring the interdisciplinary nature of this field.

Finally, in Section 2.4, we introduce the idea of categorification and how we can apply it to quantum knot invariants to obtain "knot homologies." Only the mathematical viewpoint is sketched there, as we give a more detailed account of the physical constructions in Chapter 3.

2.1 Topological quantum field theory

Mathematics definition

Mathematically, as proposed in [At1, At2, At3], a *d-dimensional topological quantum field theory* ("TQFT") is a functor:

$$Z : d\mathrm{Cob} \to \mathrm{Vect}. \tag{2.1}$$

That is, Z assigns to every closed, oriented d-dimensional manifold Σ a finite-dimensional, complex vector space $Z(\Sigma) = \mathcal{H}_\Sigma$ and to every compact, oriented $(d+1)$-dimensional manifold (cobordism) X a vector $Z(X) \in \mathcal{H}_{\partial X}$. Furthermore, such a functor should satisfy the axioms

1. $\mathcal{H}_{-\Sigma} = \mathcal{H}_\Sigma^*$, where "$-$" reverses orientation, and "$*$" denotes the dual vector space,

2. $\mathcal{H}_{\Sigma_1 \sqcup \Sigma_2} = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2}$,

3. $Z(X_1 \sqcup_{\Sigma_2} X_2) = Z(X_2) \circ Z(X_1)$ for $(d+1)$-manifolds X_i such that $\partial X_i = -\Sigma_i \sqcup \Sigma_{i+1}$,

4. $\mathcal{H}_\varnothing = \mathbb{C}$,

5. $Z(\Sigma \times I) = \mathrm{id}_{\mathcal{H}_\Sigma}$.

In light of (1) and (2), we see that if X^{d+1} is such that $\partial X = -\Sigma \sqcup \Sigma'$, then $Z(X) \in \mathcal{H}_\Sigma^* \otimes \mathcal{H}_{\Sigma'}$. Therefore, we may identify $Z(X)$ with a linear map $\mathcal{H}_\Sigma \to \mathcal{H}_{\Sigma'}$. This justifies our notation in axioms (3) and (5). Observe that (4) implies $Z(X) \in \mathbb{C}$ for a *closed* $(d+1)$-manifold X.

Physics definitions

Physically, there are two equivalent definitions of a TQFT, which respectively correspond to the path integral and operator formalism approaches to quantum field theory ("QFT"). In the path integral approach, a TQFT is a QFT whose action (Lagrangian) is invariant with respect to changes in the background metric. Consequently, the partition function and observables are topological invariants of the background geometry.

Recall that in the operator formalism, vectors in the *quantum Hilbert space* $\mathcal{H}_\Sigma$ represent physical states in the quantum theory on Σ. The dynamics of the system are then summarized by *propagators* $U_t : \mathcal{H}_\Sigma \to \mathcal{H}_\Sigma$, i.e., unitary operators that describe the evolution over time t of physical states of the theory on Σ. Propagators may be written in the form $U_t = e^{iHt/\hbar}$, where H is the *Hamiltonian* of the system.

This approach better resembles the mathematical axioms for a TQFT: for a cobordism X_t such that $\partial X_t = -\Sigma_0 \sqcup \Sigma_t$, the linear operator $Z(X_t) : \mathcal{H}_{\Sigma_0} \to \mathcal{H}_{\Sigma_t}$ describes a *particular* time-evolution from the theory on Σ_0 to the theory on Σ_t. Which time-evolution $Z(X_t)$ describes depends entirely on the topology of X_t. For example, axiom (5) indicates that the evolution from Σ_0 to Σ_t along X_t is nontrivial only if it includes a topology change. Equivalently, $H = 0$ defines a TQFT.

2.2 Knot invariants via Chern-Simons theory

Here we describe the physics approaches to quantum link invariants via the Chern-Simons TQFT. Good general references for the material in this section are [At3, Gua, GSa, Koh, Ma1, Ma2, Oh, W1].

Path integral

For M a closed, oriented 3-manifold and G a compact, connected Lie group, let $\mathcal{A} = \Omega^1(M; \mathfrak{g})$ be the space of ($\mathfrak{g}$-valued) connection 1-forms on the trivial principal G-bundle $P \to M$. The integral of the Chern-Simons form [CS] is the *Chern-Simons action functional* $CS : \mathcal{A} \to \mathbb{R}$, defined by

$$CS(A) = \frac{1}{8\pi^2} \int_M \operatorname{tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right), \text{ for } A \in \mathcal{A}. \tag{2.2}$$

The *gauge group* $\mathcal{G}$ consists of automorphisms of $P \to M$ (equiv. sections), and its action on $\mathcal{A}$ is

$$g^*A \coloneqq g^{-1}Ag + g^{-1}dg, \text{ for } g \in \mathcal{G} \text{ and } A \in \mathcal{A}. \tag{2.3}$$

One can normalize the trace tr so that $CS(g^*A) - CS(A) \in \mathbb{Z}$, and the Chern-Simons functional then descends to a well-defined map $CS : \mathcal{A}/\mathcal{G} \to \mathbb{R}/\mathbb{Z}$.

Following [W1], the *partition function* of the corresponding quantum theory at *level* $k \in \mathbb{Z}$ is given by the path integral:

$$Z^k_{CS}(M) \coloneqq \int_{\mathcal{A}/\mathcal{G}} [\mathcal{D}A] e^{2\pi i k CS(A)} \in \mathbb{C}. \tag{2.4}$$

Let $L \subset M$ be a link, whose components $K_1 \sqcup \cdots \sqcup K_\ell$ are colored by representations $\vec{V} = (V_1, \ldots, V_\ell)$ of $\mathfrak{g}$. One can consider the product of the holonomies of A along K_i traced in V_i, yielding the gauge-invariant *Wilson loop operator*:

$$\mathbf{W}^L_{\vec{V}}(A) \coloneqq \prod_{i=1}^{\ell} \operatorname{tr}_{V_i}\big(\operatorname{Hol}_{K_i}(A)\big). \tag{2.5}$$

The Wilson loops are a complete set of observables for the Chern-Simons QFT. Expanding the *correlation function* of a Wilson loop,

$$\left\langle \mathbf{W}^L_{\vec{V}} \right\rangle_M \coloneqq \frac{1}{Z^k_{CS}(M)} \int_{\mathcal{A}/\mathcal{G}} [\mathcal{D}A] e^{iS(A)} \mathbf{W}^L_{\vec{V}}(A), \tag{2.6}$$

in $q \coloneqq e^{\frac{2\pi i}{k+h^\vee}}$ produces an integer Laurent polynomial $W_{\mathfrak{g},\vec{V}}(M, L; q) \in \mathbb{Z}[q^{\pm 1}]$, which we will (tentatively[1]) call the *quantum* $(\mathfrak{g}, \vec{V})$*-invariant of* $L \subset M$.

Observe that $CS(A)$ is invariant with respect to changes in the metric on M. Therefore, the Chern-Simons QFT is a TQFT according to the physical definition. We are led to conclude that $Z^k_{CS}(M)$ is a topological invariant of M, and $W_{\mathfrak{g},\vec{V}}$ are isotopy invariants of $L \subset M$.

However, a mathematically-oriented reader will observe that the space $\mathcal{A}/\mathcal{G}$ is infinite-dimensional, so the measure $[\mathcal{D}A]$—whence the integrals $Z^k_{CS}(M)$ and $\langle \mathbf{W}^L_{\vec{V}} \rangle_M$—is not well-defined. Physicists typically circumvent this difficulty by a formal analogy to oscillatory integrals $\int_{\mathbb{R}^n} \mathrm{e}^{ikf(\vec{x})} d^n\vec{x}$, which (as $k \to \infty$) are determined by contributions from the critical points of the function $f(\vec{x})$.

[1] The definition used in other sections will be (2.39). Conjecturally, the two definitions are equivalent.

We do not describe this "perturbative expansion" of Chern-Simons theory here, but we do remark that the critical points of the action functional CS are *flat connections*:

$$F_A \coloneqq dA + A \wedge A = 0, \tag{2.7}$$

which are also independent of the metric on M. Thus, even though the path integral is not mathematically well-defined, it still suggests a mathematically rigorous recipe for producing topological invariants of 3-manifolds M and links $L \subset M$.

Operator formalism

Recall that every closed, oriented 3-manifold has a *Heegaard splitting*, i.e., a decomposition $M = H_1 \sqcup_f H_2$ into handlebodies, H_i, glued along their common boundary, $\partial H_i = \Sigma$, via an orientation-reversing homeomorphism $f : \partial H_1 \to \partial H_2$. The topology of the resulting manifold M depends only on the isotopy class of f, i.e., we need only specify an element of the mapping class group $\mathrm{MCG}(\Sigma)$.

We would like to use a Heegaard splitting to evaluate the partition function $Z^k_{CS}(M)$. Schematically, this means that we associate a quantum Hilbert space $\mathcal{H}_\Sigma$ to Σ, vectors $|\Psi_i\rangle \in \mathcal{H}_\Sigma$ to the handlebodies H_i, and a unitary operator $U_f : \mathcal{H}_\Sigma \to \mathcal{H}_\Sigma$ to $[f] \in \mathrm{MCG}(\Sigma)$. Then we evaluate

$$Z^k_{CS}(M) = \langle \Psi_2 | U_f | \Psi_1 \rangle \,, \tag{2.8}$$

where we have used Dirac's "bra-ket" notation for the inner product on $\mathcal{H}_\Sigma$. As we will see, observables may be evaluated similarly.

This approach requires a definition of the Chern-Simons path integral for 3-manifolds with non-empty boundary. That is, we need a recipe for obtaining quantum Hilbert spaces and vectors in those Hilbert spaces from the Chern-Simons field theory. Such a recipe was obtained by Witten in [W1], where he identified $\mathcal{H}_\Sigma$ with the space of conformal blocks of the WZW model on Σ.

Conformal blocks on the torus

The space of conformal blocks of the WZW model may be rigorously formulated for arbitrary Riemann surfaces (potentially with marked points), e.g., see [DMS, Koh]. Here, we focus on the torus T^2 and provide an explicit description of $\mathcal{H}_{T^2}$, which we will later use for concrete computations. Our exposition follows [Ma1, Ma2].

The Hilbert space $\mathcal{H}_{T^2}$ of conformal blocks for the torus is the finite-dimensional, complex vector space spanned by the integrable representations at level k of the affine Lie algebra corresponding to

$\mathfrak{g}$. In the large k limit, an orthonormal basis for this space is

$$\big\{|\rho+\lambda\rangle \,:\, \lambda \in P_+\big\}, \tag{2.9}$$

where $P_+ := \mathrm{span}_{\mathbb{Z}_{\geq 0}}\{\omega_i\}$ is the set of dominant weights and $\rho := \sum \omega_i$ is the Weyl vector, relative to the set $\{\omega_i\}$ of fundamental weights for $\mathfrak{g}$.

The state $|\rho+\lambda\rangle \in \mathcal{H}_{T^2}$ corresponds to the Wilson loop operator $\mathbf{W}^U_{V_\lambda}$. Here, U is an unknot inserted along the noncontractible cycle in the solid torus $\mathbf{S}^1 \times D^2$, and V_λ is the representation of $\mathfrak{g}$ whose highest weight is $\lambda \in P_+$. In particular, $|\rho\rangle \in \mathcal{H}_{T^2}$ corresponds to the "empty" solid torus, i.e., with no unknot inserted.

Mapping class group of the torus

Recall that, in the homotopy category, the mapping class group of the torus is $\mathrm{MCG}(T^2) \cong \mathrm{SL}(2,\mathbb{Z})$. A presentation is

$$\mathrm{SL}(2,\mathbb{Z}) = \langle \sigma, \tau \,:\, \sigma^4 = 1,\, (\sigma\tau)^3 = \sigma^2\rangle, \tag{2.10}$$

and a representation on $H_1(T^2;\mathbb{Z}) \cong \mathbb{Z}^2$ is given by

$$\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{2.11}$$

where the action is by left multiplication, and we have chosen a basis in which $(1,0)^{tr}$ corresponds to a longitude of T^2 and $(0,1)^{tr}$ corresponds to a meridian of T^2.

The action of $SL(2,\mathbb{Z})$ on $H_1(T^2;\mathbb{Z})$ induces an action on $\mathcal{H}_{T^2}$. We will write this action in terms of matrices $S = (S_{\lambda\mu})$, $T = (T_{\lambda\mu})$, where $\lambda, \mu \in P_+$ and, e.g., λ corresponds to the basis element $|\rho+\lambda\rangle \in \mathcal{H}_{T^2}$. We then have

$$S_{\lambda\mu} = C_{\mathfrak{g},k} \sum_{w \in W} (-1)^{\mathrm{sgn}(w)} q^{-(\rho+\lambda, w(\rho+\mu))} \quad \text{and} \quad T_{\lambda\mu} = \delta_{\lambda,\mu} \mathrm{e}^{2\pi i (h_\lambda - c/24)}, \tag{2.12}$$

where $q = e^{\frac{2\pi i}{k+h^\vee}}$ as defined above, W is the Weyl group for $\mathfrak{g}$, and $C_{\mathfrak{g},k} \in \mathbb{C}$ is a constant which depends on $\mathfrak{g}$ and k. We will not use T directly but remark that $h_\lambda := \frac{(2\rho+\lambda,\lambda)}{2(k+h^\vee)}$ is the *conformal weight* of the primary field associated to $\rho+\lambda$, and c is the central charge of the WZW model.

Partition function and unknot observables

We now have enough information to evaluate the partition functions for and unknot observables in 3-manifolds which have genus-1 Heegaard splittings $M = H_1 \sqcup_f H_1'$, i.e., lens spaces. Of particular interest to us is the case $M = \mathbf{S}^3$, which corresponds to $[f] = \sigma \in \mathrm{MCG}(T^2)$.

Using the operator formalism, we evaluate

$$Z^k_{CS}(\mathbf{S}^3) = \langle \rho | S | \rho \rangle = S_{00} = C_{g,k} \sum_{w \in W} (-1)^{\mathrm{sgn}(w)} q^{-(\rho, w(\rho))}, \tag{2.13}$$

for the partition function on $\mathbf{S}^3$ at level k. The quantum $(\mathfrak{g}, V_\lambda)$-invariant of the unknot $U \subset \mathbf{S}^3$ is

$$W_{\mathfrak{g},V_\lambda}(\mathbf{S}^3, U; q) = \frac{S_{0\lambda}}{S_{00}} = \frac{\sum_{w \in W} (-1)^{\mathrm{sgn}(w)} q^{-(\rho, w(\rho+\lambda))}}{\sum_{w \in W} (-1)^{\mathrm{sgn}(w)} q^{-(\rho, w(\rho))}}, \tag{2.14}$$

which may be identified with the Weyl character ch_λ evaluated at $q^{-\rho}$ or, via Weyl's denominator formula, with the quantum dimension :

$$\dim_q V_\lambda := \prod_{\alpha \in R_+} \frac{q^{(\alpha,\lambda+\rho)/2} - q^{-(\alpha,\lambda+\rho)/2}}{q^{(\alpha,\rho)/2} - q^{-(\alpha,\rho)/2}}. \tag{2.15}$$

Compare to the formula (4.58) and those in Section 5.2 below.

Torus knots

Torus links are links which may be embedded in T^2 and are labeled by pairs (r, s) of integers, where the corresponding torus link $T^{\mathsf{r},\mathsf{s}}$ has $\gcd(\mathsf{r}, \mathsf{s})$ components and represents the homology class $(\mathsf{r}, \mathsf{s}) \in H_1(T^2; \mathbb{Z})$. A *torus knot* is a torus link with $\gcd(\mathsf{r}, \mathsf{s}) = 1$.

Consider the following algorithm to "create" a torus knot: perform a transformation $\kappa^{-1} \in \mathrm{SL}(2, \mathbb{Z})$ on T^2, insert an unknot along the longitude $(1, 0)^{tr}$, and then perform the inverse transformation κ on T^2. Under this procedure, the torus T^2 remains unchanged, but the unknot becomes a torus knot representing $\kappa \cdot (1, 0)^{tr}$.

In [LLR], the authors defined *knot operators* $\mathbf{W}^{(\mathsf{r},\mathsf{s})}_\lambda$ corresponding to torus knots $T^{\mathsf{r},\mathsf{s}}$ colored by V_λ in the boundary T^2 . Using the procedure, they observed that

$$K \mathbf{W}^{(1,0)}_\lambda K^{-1} = \mathbf{W}^{(\mathsf{r},\mathsf{s})}_\lambda, \text{ for } K \text{ lifting } \kappa = \left(\begin{smallmatrix} \mathsf{r} & * \\ \mathsf{s} & * \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}), \tag{2.16}$$

if $\gcd(\mathsf{r}, \mathsf{s}) = 1$. The entries labeled $*$ can be any integers for which $\det(\kappa) = 1$. In [St], the action of the torus knot operators on $\mathcal{H}_{T^2}$ was described explicitly:

$$\mathbf{W}^{(\mathsf{r},\mathsf{s})}_\lambda \left| \rho \right\rangle = \sum_{\mu \in P_+} c^\mu_{\lambda;\mathsf{r}} \mathrm{e}^{2\pi i \frac{\mathsf{s}}{\mathsf{r}} h_\mu} \left| \rho + \mu \right\rangle, \tag{2.17}$$

where the coefficients $c^\mu_{\lambda;\mathsf{r}}$ are determined by the r-Adams operation; see (4.60) and (5.23) below.

We now have

$$W_{\mathfrak{g},V_\lambda}(\mathbf{S}^3, T^{\mathrm{r,s}}; q) = \frac{1}{S_{00}} \left\langle \rho \middle| \mathbf{W}_\lambda^{(\mathrm{r,s})} S \middle| \rho \right\rangle = \sum_{\mu \in P_+} c_{\lambda;\mathrm{r}}^{\mu} \mathrm{e}^{2\pi i \frac{\mathrm{s}}{\mathrm{r}} h_\mu} \dim_q(V_\mu), \tag{2.18}$$

for the (framed, unreduced) quantum $(\mathfrak{g}, V_\lambda)$-invariant of a torus knot.

In [St], the author generalized equation (2.18) to produce, in the framework of Chern-Simons theory, formulas for the colored HOMFLY and Kauffman polynomials for torus links. These formulas agree with those produced in the framework of quantum groups by [LZ] and [CC], respectively.

2.3 Knot invariants via quantum groups

Here we describe the mathematics approach to quantum link invariants via quantum groups. Good general references for the material in this section are [CP, Dr1, Ja, Ka, Oh, Tu2].

Conventions

It is important to note that the variable q used here in Section 2.3 differs from that in every other part of this thesis. The relationship is

$$q \text{ here } = q^{\frac{1}{2}} \text{ everywhere else.} \tag{2.19}$$

However, q here is equal to s in Section 5.2, e.g., see equation (5.10). The historical significance of these different conventions with overlapping notation is discussed in [Sa].

Algebraic considerations

Quantum groups

For a complex, finite-dimensional, simple Lie algebra $\mathfrak{g}$, we denote by $\mathcal{U}_q(\mathfrak{g})$ its quantized universal enveloping algebra, which we will call a *quantum group.* Suppose that $\mathfrak{g}$ has Cartan matrix $(a_{ij})_{1\leq i,j\leq n}$, and let $q \in \mathbb{C}$ be a nonzero complex number. Then, as defined in [Dr1, Ji], the quantum group $\mathcal{U}_q(\mathfrak{g})$ is the $\mathbb{Q}(q)$-algebra generated by $\{K_i^{\pm 1}, E_i, F_i\}_{1\leq i,j\leq n}$, subject to relations

$$[K_i, K_j] = 0, \quad K_i K_i^{-1} = 1 = K_i^{-1} K_i, \tag{2.20}$$

$$K_i E_j = q_i^{a_{ij}} E_j K_i, \quad [E_i, F_j] = \delta_{i,j} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \quad K_i F_j = q_i^{-a_{ij}} F_j K_i, \tag{2.21}$$

$$\sum_{\ell=0}^{1-a_{ij}} (-1)^\ell \begin{bmatrix} 1 - a_{ij} \\ \ell \end{bmatrix}_{q_i} E_i^{1-a_{ij}-\ell} E_j E_i^\ell = 0, \text{ for } i \neq j, \tag{2.22}$$

$$\sum_{\ell=0}^{1-a_{ij}} (-1)^{\ell} \begin{bmatrix} 1-a_{ij} \\ \ell \end{bmatrix}_{q_i} F_i^{1-a_{ij}-\ell} F_j F_i^{\ell} = 0, \text{ for } i \neq j, \tag{2.23}$$

where $q_i := q^{d_i}$ for a set $\{\alpha_i\}_{1 \leq i \leq n}$ of simple roots for $\mathfrak{g}$ with $d_i := \frac{1}{2}(\alpha_i, \alpha_i)$, and

$$\begin{bmatrix} m \\ n \end{bmatrix}_q := \frac{[m]_q!}{[m-n]_q![n]_q!}, \text{ where, e.g., } [m]_q! := \prod_{k=1}^{m} [k]_q \text{ for } [k]_q := \frac{q^k - q^{-k}}{q - q^{-1}}. \tag{2.24}$$

The quantum group $\mathcal{U}_q(\mathfrak{g})$ is a Hopf algebra, for the coproduct Δ_q, counit ϵ_q, and (invertible) antipode S_q defined by:

$$\Delta_q(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta_q(K_i) = K_i \otimes K_i, \quad \Delta_q(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \tag{2.25}$$

$$\epsilon_q(E_i) = 0, \quad \epsilon_q(K_i) = 1, \quad \epsilon_q(F_i) = 0, \tag{2.26}$$

$$S_q(E_i) = -K_i^{-1} E_i, \quad S_q(K_i) = K_i^{-1}, \quad S_q(F_i) = -F_i K_i, \tag{2.27}$$

and extended uniquely to maps on the entire $\mathcal{U}_q(\mathfrak{g})$.

Universal R-matrices

A Hopf algebra H is *quasitriangular* if there is an invertible element $R \in H \otimes H$ such that

1. $R\Delta(x) = \Delta^{op}(x)R$, for all $x \in H$,
2. $(\Delta \otimes \mathrm{id}_H)(R) = R_{13}R_{23}$,
3. $(\mathrm{id}_H \otimes \Delta)(R) = R_{13}R_{12}$,

where, if $R = \sum_i \alpha_i \otimes \beta_i$, then using Sweedler notation

$$R_{12} := \sum_i \alpha_i \otimes \beta_i \otimes 1, \quad R_{13} := \sum_i \alpha_i \otimes 1 \otimes \beta_i, \quad R_{23} := \sum_i 1 \otimes \alpha_i \otimes \beta_i. \tag{2.28}$$

Such an R is called a *universal R-matrix.* If one exists, it satisfies the *Yang-Baxter equation*

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{2.29}$$

The Hopf algebra $\mathcal{U}_q(\mathfrak{g})$ is *not* quasitriangular. However, we may consider the $\mathbb{C}[[h]]$-algebra $\mathcal{U}_h(\mathfrak{g})$, which is generated ($h$-adically) by $\{H_i, E_i, F_i\}_{1 \leq i,j \leq n}$ with the relations and Hopf algebra structure for $\mathcal{U}_q(\mathfrak{g})$, but modified by replacing

$$q \mapsto \mathrm{e}^{-\frac{h}{2}} \text{ and } K_i \mapsto \mathrm{e}^{-\frac{h}{2} d_i H_i}. \tag{2.30}$$

$\mathcal{U}_h(\mathfrak{g})$ *is* a quasitriangular Hopf algebra: a universal R-matrix R_h was obtained for $\mathcal{U}_h(\mathfrak{g})$ in [Dr1]. In particular, let $\mathcal{U}_h(\mathfrak{b}_+)$ and $\mathcal{U}_h(\mathfrak{b}_-)$ be the subalgebras of $\mathcal{U}_h(\mathfrak{g})$ generated by $\{H_i, E_i\}$ and $\{H_i, F_i\}$, respectively. If we let $\{e^k\}$ and $\{e_k\}$ be dual bases for $\mathcal{U}_h(\mathfrak{b}_+)$ and $\mathcal{U}_h(\mathfrak{b}_-)$, then

$$R_h := \sum_k e_k \otimes e^k \in \mathcal{U}_h(\mathfrak{g}) \otimes \mathcal{U}_h(\mathfrak{g}) \tag{2.31}$$

is a universal R-matrix for $\mathcal{U}_h(\mathfrak{g})$. Explicit formulas for R_h were obtained in [KiR, LS].

Representation theory

Assuming that q is not a root of unity, the finite-dimensional, irreducible representations of the quantum group $\mathcal{U}_q(\mathfrak{g})$ are labeled by dominant weights for $\mathfrak{g}$. In particular, given a weight $\lambda \in P_+(\mathfrak{g})$, the corresponding representation V_λ of $\mathcal{U}_q(\mathfrak{g})$ has highest weight λ [Lu, Ro].

Even though $\mathcal{U}_q(\mathfrak{g})$ does not have a universal R-matrix, we can adapt R_h to the situation of *finite-dimensional* representations. Suppose $\rho_i : \mathcal{U}_q(\mathfrak{g}) \to \mathrm{End}(V_i)$ are finite-dimensional representations and that $R_h = \sum_k \alpha_k \otimes \beta_k$. Then we may define

$$R_{V_1,V_2} := (\rho_1 \otimes \rho_2)(R_h) = \sum_k \rho_1(\alpha_k) \otimes \rho_2(\beta_k). \tag{2.32}$$

This *quantum R-matrix* $R_{V_1,V_2} \in \mathrm{End}(V_1 \otimes V_2)$ inherits the defining properties of the universal R-matrix. However, quantum R-matrices are restricted to this setting of finite-dimensional representations and so are not "universal."

Quantum link invariants

Braids and links

The *braid group* B_n on n strands is generated by $\{\sigma_i\}_{1 \le i < n}$, subject to relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \ |i-j| \ge 2 \quad \text{and} \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}. \tag{2.33}$$

Generators σ_i (resp. σ_i^{-1}) may be depicted by planar diagrams of n ordered, upward-oriented "strands" with a positive (resp. negative) crossing between the i-th and $(i+1)$-th strands and no crossings between any other strands. The diagram for $\sigma_i \sigma_j$ is formed by placing the diagram for σ_j atop the diagram for σ_i.

The *closure* $\hat{\sigma}$ of a braid σ is the knot or link diagram obtained by identifying the i-th strand at the top of the braid diagram for σ with the i-th strand at the bottom, without introducing additional crossings. By Alexander's theorem, every link in $\mathbf{S}^3$ can be represented as the closure of a braid.

However, such a representative is not unique. Different braids have isotopic closures if and only if they are related by a sequence of *Markov moves*, an equivalence relation $\sim$ on $\bigsqcup B_n$ defined by

$$\alpha\beta \sim \beta\alpha \quad \text{and} \quad \alpha \sim \alpha\sigma_n^{\pm 1}, \quad \text{for} \quad \alpha, \beta \in B_n. \tag{2.34}$$

That is, if $\sigma, \tau \in \bigsqcup B_n$, then $\hat{\sigma}$ is isotopic to $\hat{\tau}$ iff $\sigma \sim \tau$.

Toward link invariants

We would like to construct a sort of "0-dimensional TQFT" which produces link invariants. That is, to an (oriented) point we associate a vector space V, and to a braid, $\sigma \in B_n$, we associate a vector, $\rho(\sigma) \in V^{\otimes n} \otimes V^{*\otimes n}$, which depends only on the isotopy class of the corresponding braid diagram. This assignment is equivalent to a representation $\rho : B_n \to \mathrm{End}(V^{\otimes n})$.

To obtain a corresponding link invariant I_ρ, it is natural in the context of TQFT to consider traces, i.e., $I_\rho(\hat{\sigma}) \coloneqq \mathrm{tr}_\rho(\sigma)$. However, to ensure that I_ρ is an isotopy invariant, we must ensure that

1. $\mathrm{tr}_\rho(\alpha\beta) = \mathrm{tr}_\rho(\beta\alpha)$ for $\alpha, \beta \in B_n$,
2. $\mathrm{tr}_\rho(\alpha) = \mathrm{tr}_\rho(\alpha\sigma_n^{\pm 1})$ for $\alpha \in B_n$,

i.e., tr_ρ is invariant under Markov moves. Such a tr_ρ is called a *Markov trace.*

Observe that (1) is a general property of traces. However, (2) requires a bit more work and, in particular, compatibility with the inclusions $B_n \subset B_{n+1}$. This method for constructing link invariants can generally be realized in the setting of a ribbon category.

Ribbon category $\mathbf{Rep}(\mathcal{U}_q(\mathfrak{g}))$

For the general definition of a ribbon category see [Tu2]. Here, we will only describe the ribbon category structure on $\mathrm{Rep}(\mathcal{U}_q(\mathfrak{g}))$, the finite-dimensional complex representations of $\mathcal{U}_q(\mathfrak{g})$.

For any $U, V, W \in \mathrm{Rep}(\mathcal{U}_q(\mathfrak{g}))$, let $P_{U,V} : U \otimes V \to V \otimes U$ be the "flip" isomorphism which sends $u \otimes v \mapsto v \otimes u$ and extends by linearity. Define the *braiding* $c_{U,V} \coloneqq P_{U,V}R_{U,V}$, a natural isomorphism $U \otimes V \to V \otimes U$ satisfying

1. $c_{U\otimes V,W} = (c_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes c_{V,W})$,
2. $c_{U,V\otimes W} = (\mathrm{id}_V \otimes c_{U,W})(c_{U,V} \otimes \mathrm{id}_W)$.

A consequence of (1), (2) is that the $c_{U,V}$ satisfy the *Yang-Baxter identity*

$$(c_{V,W} \otimes \mathrm{id}_U)(\mathrm{id}_V \otimes c_{U,W})(c_{U,V} \otimes \mathrm{id}_W) = (\mathrm{id}_W \otimes c_{U,V})(c_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes c_{V,W}). \tag{2.35}$$

We also define the *twist*,

$$\theta_V : V \to V, \tag{2.36}$$

to be the natural isomorphism induced by multiplication by a certain element $K_{-2\rho} \in \mathcal{U}_q(\mathfrak{g})$. As shown in [Dr2], $K_{-2\rho}$ acts by the scalar $q^{-(2\rho+\lambda,\lambda)}$ on V_λ. Finally, we will mention but not define the *duality*, a necessary feature of a ribbon category.

Quantum link invariants

Let $\sigma \in B_n$ be a braid whose i-th strand is colored by a representation $V_i \in \mathrm{Rep}(\mathcal{U}_q(\mathfrak{g}))$ in a way that is consistent on the components of the closure $\hat{\sigma}$. Define

$$R_k^{\pm 1} \coloneqq \mathrm{id}_{V_1} \otimes \cdots \otimes \mathrm{id}_{V_{k-1}} \otimes c^{\pm 1}_{V_k, V_{k+1}} \otimes \mathrm{id}_{V_{k+2}} \otimes \cdots \otimes \mathrm{id}_{V_n} \in \mathrm{End}\left(\bigotimes_{i=1}^{n} V_i\right), \tag{2.37}$$

and observe that $R_j R_k = R_k R_j$ for $|j-k| \geq 2$, and that (2.35) implies $R_k R_{k+1} R_k = R_{k+1} R_k R_{k+1}$. Thus, $\rho_R : \sigma_k^{\pm 1} \mapsto R_k^{\pm 1}$ is a representation of the braid group on $\bigotimes_i V_i$.

For any $X \in \mathrm{End}_{\mathcal{U}_q(\mathfrak{g})}(V \otimes W)$, define the *partial quantum trace* by

$$\mathrm{tr}_W(X) \coloneqq \mathrm{tr}_2(X \cdot \mathrm{id}_V \otimes \rho_W(K_{-2\rho})) \in \mathrm{End}_{\mathcal{U}_q(\mathfrak{g})}(V). \tag{2.38}$$

Then, in particular, we have that $\mathrm{tr}_V(c^{\pm 1}_{V,V}) = \theta_V^{\pm 1}$. Now, as defined in, e.g., [Re, Tu1],

$$P^{\mathfrak{g},\hat{V}}(L;q) \coloneqq \theta_1^{-\mathrm{wr}(K_1)} \cdots \theta_\ell^{-\mathrm{wr}(K_\ell)} \mathrm{tr}\left(\rho_i^{\otimes n}(K_{-2\rho}) \cdot \rho_R(\sigma)\right), \tag{2.39}$$

where L is the closure of σ with components $K_1 \sqcup \cdots \sqcup K_\ell$ colored by $\hat{V} \coloneqq (\hat{V}_1, \ldots, \hat{V}_\ell)$; $\mathrm{wr}(K_i)$ is the writhe of the component K_i; the twist $\theta_i \coloneqq \theta_{V_j}$ if $\hat{V}_i = V_j$; and ρ_i corresponds to the representation on the $\mathcal{U}_q(\mathfrak{g})$-module V_i.

It follows almost immediately from $\mathrm{tr}_V(c^{\pm 1}_{V,V}) = \theta_V^{\pm 1}$ that $\sigma \mapsto \mathrm{tr}\left(\rho_i^{\otimes n}(K_{-2\rho}) \cdot \rho_R(\sigma)\right)$ is a Markov trace, and the factors $\theta_i^{-\mathrm{wr}(K_i)}$ simply adjust for the framing. Thus, the polynomial $P^{\mathfrak{g},\hat{V}}(L;q)$ is the invariant of unframed, oriented links, which we will henceforth call the *quantum* $(\mathfrak{g},\hat{V})$*-invariant*.

Torus links

Torus links $T^{\mathsf{r},\mathsf{s}}$ have a uniform expression as the closure of the braid

$$\sigma_{\mathsf{r},\mathsf{s}} \coloneqq (\sigma_1 \cdots \sigma_{\mathsf{r}-1})^{\mathsf{s}} \in B_{\mathsf{r}}. \tag{2.40}$$

Observe that $(\sigma_1 \cdots \sigma_{n-1})^n$ is in the center of B_n. This fact was used in [RJ] to produce a sort of "eigendecomposition" for (2.39), commonly called the *Rosso-Jones formula*,

$$P^{\mathfrak{g},V_\lambda}(\hat{\sigma}_{\mathsf{r},\mathsf{s}};q) = \theta_\lambda^{-\mathsf{rs}} \sum_{\substack{\mu \in P(\lambda,\mathsf{r}) \\ Y \in \mathcal{Y}}} \theta^{\frac{\mathsf{s}}{\mathsf{r}}} \chi_Y(\bar{\sigma}_{\mathsf{r},\mathsf{s}})[Y:\mu] \dim_q(V_\mu). \tag{2.41}$$

It is a general expression for the quantum invariant of any torus link for any $\mathfrak{g}$ colored by any V_λ. The coefficients in this expansion belong to classical invariant theory and are generally not very computable without appealing to some specific facts about $\mathfrak{g}$.

In [LZ, CC], the authors do just that for the classical groups (HOMFLY, Kauffman) and produce (4.69) below, which we use in our proof of Theorem 4.3.4. Their method is via a “cabling-projection rule” and results in an expansion in terms of the r-Adams operation which matches [St].

We also mention Proposition 4 of [ZGB], which is computable for any $(\mathfrak{g}, V_\lambda)$ such that $V_\lambda^{\otimes 2}$ is multiplicity free. However, it is restricted to $T^{2,\mathsf{s}}$ torus links. This is because the coefficient corresponding to μ in the expansion is just ± 1 (or 0) according to whether V_μ occurs symmetrically/antisymmetrically (or not at all) in $V_\lambda^{\otimes 2}$. In fact, this is identically the coefficient $c^\mu_{\lambda;2}$ from the 2-Adams operation.

2.4 Knot homologies

Categorification

“Categorification” is an informal process or program by which one realizes set-theoretic objects as naturally coming from category-theoretic ones. The prototype for categorification in low-dimensional topology is the relationship between the Euler characteristic and singular homology of a space:

$$\chi(M) = \sum_{k\geq 0} (-1)^k H_k(M) \in \mathbb{Z}. \tag{2.42}$$

Homology is a better invariant of spaces than the Euler characteristic (consider closed M of odd dimension). It also has the structure of a graded abelian group vs. an integer, which is useful in determining more refined properties of spaces.

Most notably, homology is a category-theoretic object, i.e., it is functorial:

$$H_* : \{\text{spaces, cts. maps}\} \to \{\text{abelian groups, homomorphisms}\}, \tag{2.43}$$

which suggests an analogy between homology and TQFT. In fact, such an analogy between the Eilenberg-Steenrod axioms for homology and Atiyah’s axioms for a TQFT was explored in [PS].

Knot homologies

Using (2.42) as a model, mathematicians have constructed several doubly-graded homology theories which categorify polynomial knot invariants. For example, the *graded Euler characteristic* of the knot Floer homology [OS1, Ras1] is the Alexander polynomial:

$$\Delta_K = \sum_{i,j} (-1)^j t^i \dim \widehat{HFK}_j(K, i). \tag{2.44}$$

Similarly, Khovanov homology [Kh] categorifies the Jones polynomial:

$$J(K) = \sum_{i,j} (-1)^j q^i \dim Kh_{i,j}(K), \tag{2.45}$$

and Khovanov-Rozansky homology [KhR1, KhR2] categorifies the quantum $(\mathfrak{sl}_N, \square)$-invariant:

$$P^{\mathfrak{sl}_N, \square}(K) = \sum_{i,j} (-1)^j q^i \dim HKR^N_{i,j}(K). \tag{2.46}$$

By now there are many more examples of "knot homologies." Notably, [Web] categorifies any $P^{\mathfrak{g},V}(K)$, though the construction is rather abstract and computable only in the simplest cases.

TQFT structure

Knot homologies are better invariants of knots than quantum knot invariants, (e.g., $Kh_{i,j}$ vs. J in [BN]). Also, that they are doubly-graded vector spaces vs. polynomials is useful in determining more refined properties of knots, (e.g., fibredness [Ni], Seifert genus [OS2]).

Most notably, many knot homologies are functorial:

$$\mathcal{H}_{\text{knot}} : \{\text{knots, cobordisms}\} \to \{\text{graded vector spaces, homomorphisms}\}, \tag{2.47}$$

and so provide examples of TQFT's. Then Atiyah's axioms imply that $\mathcal{H}_{\text{knot}}$ should provide invariants of closed surfaces in 4-manifolds and concordance invariants of knots and links.

In [Ras2], the author utilizes this structure to define a concordance invariant $s(K) \in \mathbb{Z}$ from $Kh_{i,j}(K)$. He subsequently uses this invariant to provide a simple proof of the Milnor conjecture, which concerns the 4-ball genus of torus knots.

Chapter 3

Physical Constructions of Knot Homologies

While formal algebraic constructions of $\mathcal{H}^{\mathfrak{g},V}(K)$ exist for arbitrary $(\mathfrak{g}, V)$ [Web], they are not, in general, amenable to calculations. However, the physical arguments of [GSV, G, W2] suggest alternative, geometric constructions following from the identification of vector spaces:

$$\mathcal{H}_{\text{knot}}(K) \ = \ \mathcal{H}_{\text{BPS}}(K), \tag{3.1}$$

where $\mathcal{H}_{\text{BPS}}(K)$ is the Hilbert space of BPS states in topological string theory or, more generally, M-theory. This geometric / physical realization reveals structural properties of knot homologies which have aided greatly in their calculation.

In Section 3.1, we describe how the knot invariants defined in Section 2.2 are reformulated in terms of BPS degeneracies in a certain topological string theory, defined on a space X called the resolved conifold [OV]. Mathematically speaking, this motivates a conjecture relating quantum knot invariants to the enumerative geometry of X.

In Section 3.2, we describe the fundamental result of [GSV] which leads to the conjecture (3.1). The M-theory version of this conjecture lends many different and apparently unrelated points of view to the story of knot homologies. Two such points of view were initiated in [DGR] and [AS]. They are essential to this thesis and so are recounted in Sections 3.3 and 3.4, respectively.

3.1 Knot invariants and topological strings

Here we describe how to realize the knot invariants in equation (2.6) in the context of topological strings. Good general references for the material in this section are [AK, H, Ma1, Ma2, OV].

Large-N Duality

Let M be a closed 3-manifold and consider the open string theory described by the topological A-model on the cotangent bundle T^*M with N D-branes wrapping the Lagrangian $M \subset T^*M$, and coupling constant:

$$g_s := \frac{2\pi i}{k+N}. \tag{3.2}$$

When $G = SU(N)$, it was shown in [W3] that the $\frac{1}{N}$ expansion of the Chern-Simons free energy $F(M) = \log Z^k_{CS}(M)$ is naturally identified with with the contribution to free energy of the degenerate instantons in this topological string setup.

Instantons there are generally described by holomorphic maps of Riemann surfaces with Lagrangian boundary conditions:

$$(\Sigma, \partial\Sigma) \hookrightarrow (T^*M, M). \tag{3.3}$$

However, an easy consequence of Witten's "vanishing theorem" is that the *only* such maps are the degenerate (constant) ones. Therefore, one identifies:

$$Z^k_{\mathrm{CS}}(M) = Z^{\mathrm{open}}_{\mathrm{string}}(T^*M), \tag{3.4}$$

the partition functions for Chern-Simons theory on M and open topological string theory on T^*M.

In the special case of $M = \mathbf{S}^3$, it was conjectured [GV] that at large N, this open string setup undergoes a geometric transition which produces a (physically equivalent) closed string theory. This *conifold transition* shrinks the 3-cycle of the *deformed conifold* $T^*\mathbf{S}^3$ to a point and resolves the resulting conical singularity with a small blow-up. The resulting space X is the *resolved conifold*, i.e., the total space of the $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ bundle over $\mathbb{C}\mathbf{P}^1$.

Observe that the conifold transition eliminates the N branes wrapping $\mathbf{S}^3$, producing a closed string theory on X. In the worldsheet description of this theory, based on the genus-g topological sigma model coupled to 2-dimensional gravity, the free energy is:

$$F_g(t) = \sum_{Q \in H_2(X)} N_{g,Q} e^{-tQ}, \tag{3.5}$$

where the parameter t is the Kähler modulus for the Calabi-Yau space X:

$$t = \frac{2\pi i N}{k+N} = \mathrm{vol}(\mathbb{C}\mathbf{P}^1), \tag{3.6}$$

and $N_{g,Q}$ is the Gromov-Witten invariant "counting" holomorphic maps of genus g representing the integral 2-homology class Q.

The numbers $N_{g,Q}$ are rational, in general. However, as shown in [GV], this model also admits a target space description in which the all-genus free energy is naturally described in terms of integer invariants $n_Q^s \in \mathbb{Z}$:

$$F(g_s, t) = \sum_{g=0}^{\infty} g_s^{2g-2} F_g(t) = \sum_{\substack{Q \in H_2(X), \\ s \geq 0}} n_Q^s \left[\sum_{m \geq 1} \frac{1}{m} \left(2 \sin \frac{m g_s}{2} \right)^{2s-2} e^{-mtQ} \right], \tag{3.7}$$

which encode degeneracies of so-called "BPS states."

BPS states

In a general supersymmetric quantum theory, a *BPS state* is one whose mass is equal to the central charge of the supersymmetry algebra. In the case at hand, a state is a D2-brane wrapping $\mathbb{C}\mathbf{P}^1$, and the BPS condition means that it is supported on a calibrated 2-submanifold of the Calabi-Yau X, (i.e., on a holomorphic curve in X).

Thus, a minimally embedded surface representing $Q \in H_2(X; \mathbb{Z})$ gives rise to a component of the Hilbert space $\mathcal{H}_{BPS}$, i.e., a projective unitary representation of the spatial rotation group,

$$SO(4) \sim SU(2)_L \times SU(2)_R, \tag{3.8}$$

of $\mathbb{R}^4$ obtained upon compactification from M-theory. This representation can be specified by two half-integer charges $j_L, j_R \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, which are the weights of the respective $SU(2)$ representations.

One might be tempted to introduce integers $n_Q^{(j_L, j_R)}$ counting these states. However, as one deforms the theory, BPS states can combine into non-BPS states, so these numbers are not invariant. On the other hand, the index:

$$n_Q^{j_L} := \sum_{j_R} (-1)^{2j_R} (2j_R + 1) n_Q^{(j_L, j_R)}, \tag{3.9}$$

is well-defined on the moduli of X. The integers n_Q^s are then related by a change of basis for the representation ring of $SU(2)$.

Incorporating Wilson loops

As explained in [OV], when $G = SU(N)$, Wilson loops can be incorporated in the open string on the deformed conifold by introducing $L_K \subset T^*\mathbf{S}^3$, the *conormal bundle* to $K \subset \mathbf{S}^3$. In particular, L_K is a Lagrangian submanifold of $T^*\mathbf{S}^3$, which is topologically $\mathbf{S}^1 \times \mathbb{R}^2$ and with $L_K \cap \mathbf{S}^3 = K$. Wrapping M "probe" branes on L_K produces a theory with three kinds of strings:

1. both ends on $\mathbf{S}^3 \rightsquigarrow SU(N)$ Chern-Simons theory on $\mathbf{S}^3$,

2. both ends on $L_K \rightsquigarrow SU(M)$ Chern-Simons theory on L_K,

3. one end on each $\mathbf{S}^3$ and $L_K \rightsquigarrow$ complex $SU(N) \otimes SU(M)$ scalar field on K.

Let U, V be the holonomies around K of gauge fields A, A' in (1),(2), respectively. Then the last kind of string (3) contributes to the overall action by:

$$S(U,V) := \sum_{n=1}^{\infty} \frac{1}{n} \mathrm{tr} U^n \mathrm{tr} V^{-n} = \log \left[\sum_{\lambda} \mathrm{tr}_\lambda U \mathrm{tr}_\lambda V^{-1} \right]. \tag{3.10}$$

In turn, the effective action for the theory on $\mathbf{S}^3$ is

$$S(A;K) := CS(A;\mathbf{S}^3) + S(U,V), \tag{3.11}$$

and integrating A out of the overall theory involves evaluating

$$\langle S(U,V) \rangle_{\mathbf{S}^3} = \frac{1}{Z^k_{CS}(\mathbf{S}^3)} \int_{\mathcal{A}} [\mathcal{D}A] e^{iS(A;K)} = \sum_{\lambda} \langle \mathbf{W}^K_\lambda \rangle (\mathrm{tr}_\lambda V^{-1}), \tag{3.12}$$

for fixed V, which produces a generating functional for all Wilson loops associated to $K \subset \mathbf{S}^3$, (i.e., for all Young diagrams λ).

If one follows the Lagrangian $L_K \subset T^*\mathbf{S}^3$ through the conifold transition, the result is another Lagrangian $L'_K \subset X$, where the M branes will still reside. In the resulting open string theory, the worldsheet perspective again "counts," in an appropriate sense, holomorphic maps of Riemann surfaces with Lagrangian boundary conditions:

$$(\Sigma, \partial\Sigma) \hookrightarrow (X, L'_K), \tag{3.13}$$

described by the open Gromov-Witten theory.

From the target space perspective, states correspond to configurations in which D2-branes wrap relative cycles $Q \in H_2(X, L'_K; \mathbb{Z})$ and end on D4-branes which wrap L'_K. BPS states are then minimally-embedded surfaces $\Sigma \subset X$ with boundaries $\partial\Sigma \subset L'_K$.

In [OV], the authors also showed that the generating functional for Wilson loops has an interpretation in terms of BPS degeneracies:

$$\langle S(U,V) \rangle_{\mathbf{S}^3} = i \sum_{\lambda,Q,s} N_{\lambda,Q,s} \left[\sum_{m \geq 1} \frac{e^{m(-tQ+isg_s)}}{2m \sin\left(\frac{mg_s}{2}\right)} \mathrm{tr}_\lambda V^m \right], \tag{3.14}$$

where $N_{\lambda,Q,s} \in \mathbb{Z}$ are certain modifications of n_Q^s. One can then express the quantum invariant $P^{\mathfrak{sl}_N,\lambda}(K;q)$ directly in these terms. For example, if $\lambda = \Box$ we have:

$$P_N(K;q) = \frac{1}{q-q^{-1}} \sum_{Q,s} N_{\Box,Q,s} q^{NQ+s}, \tag{3.15}$$

directly relating quantum knot invariants to the enumerative geometry of X.

3.2 Knot homologies and refined BPS states

In light of the mathematical development of homology theories categorifying quantum knot invariants, one might ask whether they also admit physical descriptions in the contexts outlined above. This program was initiated in [GSV], where the authors refined the BPS degeneracies:

$$N_{\Box,Q,s}(K) = \sum_r (-1)^r D_{Q,s,r}(K), \tag{3.16}$$

introducing *non-negative* integers $D_{Q,s,r} \in \mathbb{Z}_{\geq 0}$, which also reflect the charge r of $U(1)_R \in SU(2)_R$. If the Calabi-Yau X is rigid, these numbers are invariant under complex structure deformations.[1]

This led to a conjecture relating the knot homology categorifying $P^{\mathfrak{sl}_N,\Box}(K;q)$ to refined BPS degeneracies:

$$(q-q^{-1})KhR_N(K;q,t) = \sum_{Q,s,r} D_{Q,s,r}(K) q^{NQ+s} t^r, \tag{3.17}$$

for sufficiently large N, where $KhR_N(K;q,t)$ is the Poincaré polynomial for the Khovanov-Rozansky homology [KhR1, KhR2].

More generally, one might view the charges Q, s, r as gradings on the Hilbert space $\mathcal{H}_{\text{BPS}}(K)$ and conjecture an isomorphism of graded vector spaces:

$$\bigoplus_{i,j} \mathcal{H}_{i,j}(K) = \mathcal{H}_{\text{knot}}(K) \cong \mathcal{H}_{\text{BPS}}(K) = \bigoplus_{Q,s,r} \mathcal{H}_{Q,s,r}(K), \tag{3.18}$$

with $\dim \mathcal{H}_{Q,s,r}(K) = D_{Q,s,r}(K)$. This new perspective has revealed hidden structures of knot homologies that are manifest in the context of BPS states. In particular, $\mathcal{H}_{\text{knot}}(K)$ should:

- stabilize in dimension for sufficiently large N,
- be triply-graded, the additional Q-grading encoding N-dependence of the homology theory,
- "jump" in dimension as one varies N, corresponding to wall-crossing behavior of $\mathcal{H}_{\text{BPS}}(K)$,

[1] As mentioned in (3.6), the Kähler modulus of X is related to the rank of the underlying root system via $q^N = e^t = \exp(\text{vol}(\mathbb{C}\mathbf{P}^1))$, so that changes in the BPS spectrum as one varies the Kähler parameter t (a.k.a. the 'stability parameter') reflect changes of homological knot invariants at different values of N. See [GS] for details.

and, in fact, all of these structures were realized in [DGR], where the authors proposed a triply-graded homology theory categorifying the HOMFLY polynomial. Furthermore, they were able to construct explicit Poincaré polynomials for this homology theory ("superpolynomials") based on a rigid structure of differentials, which was later formalized in [Ras3] (see Section 3.3). Similar constructions for other choices of $(\mathfrak{g}, R)$ were proposed in [GW, GS, GGS].

M-Theory Descriptions

M-theory on an eleven-dimensional space-time incorporates the various (equivalent) versions of string/gauge theory and the dualities between them. The individual theories can then be recovered by integrating out the dependence of M-theory on some portion of the background geometry.

Naturally, this framework can offer several equivalent but nontrivially different points of view on the same object. In the case of knot homologies, we are looking for new descriptions of:

$$\mathcal{H}_{\text{knot}}(K) \cong \mathcal{H}_{\text{BPS}}(K), \tag{3.19}$$

and so promote the topological string setups described above.

In particular, the five-brane configuration relevant to the physical description of the $(\mathfrak{sl}_N, \lambda)$ knot homologies on the deformed conifold is:

$$\begin{aligned} \text{space-time}: \quad & \mathbb{R} \times T^*\mathbf{S}^3 \times M_4, \\ N \text{ M5-branes}: \quad & \mathbb{R} \times \mathbf{S}^3 \times D, \\ |\lambda| \text{ M5-branes}: \quad & \mathbb{R} \times L_K \times D, \end{aligned} \tag{3.20}$$

and the equivalent (large-N dual) configuration on the resolved conifold is:

$$\begin{aligned} \text{space-time}: \quad & \mathbb{R} \times X \times M_4, \\ |\lambda| \text{ M5-branes}: \quad & \mathbb{R} \times L'_K \times D, \end{aligned} \tag{3.21}$$

where states correspond to configurations in which M2-branes wrap relative cycles $Q \in H_2(X, L'_K; \mathbb{Z})$, fill $D \subset M_4$, and end on the M5-branes.

The precise form of the 4-manifold M_4 and the surface $D \subset M_4$ is not important (in most applications $D \cong \mathbb{R}^2$ and $M_4 \cong \mathbb{R}^4$), as long as they enjoy a $U(1)_F \times U(1)_P$ symmetry action, corresponding to the charges that comprise the s,r-gradings. In fact, these charges are more easily seen in this context: the first (resp. second) factor is a rotation symmetry of the normal (resp. tangent) bundle of $D \subset M_4$. Following [W2], let us denote the corresponding quantum numbers by F and P. These quantum numbers were denoted, respectively, by $2S_1$ and $2(S_1 - S_2)$ in [AS] and by $2j_3$ and n in [GS].

This description of $\mathcal{H}_{\text{BPS}}(K)$ in the M-theory framework led to a number of developments which shed light on various aspects of knot homologies and yield powerful computation techniques. Some examples include

- [W2] formulates the relevant space of BPS states within (3.21);
- [AS] refines torus knot invariants directly within Chern-Simons based on its relationship with (3.21) discovered in [W3];
- [DGH] takes the perspective of M_4 on which the BPS invariants are expressed via equivariant instanton counting.

3.3 Superpolynomials

Here we motivate and describe the approach to knot homologies in the seminal paper [DGR], which we will use to approach exceptional knot homologies in Section 6.

HOMFLY-PT polynomial

Recall the skein theoretic definition of the (normalized) *HOMFLY-PT polynomial* of a knot K:

$a\mathbf{P}\Big($  $\Big) - a^{-1}\mathbf{P}\Big($  $\Big) = (q - q^{-1})\mathbf{P}\Big($  $\Big),$ (3.22)

$\mathbf{P}\Big($  $\Big) = 1.$ (3.23)

One applies this relation recursively to a diagram for K to obtain an integer Laurent polynomial $\mathbf{P}(K) \in \mathbb{Z}[q^{\pm 1}, a^{\pm 1}]$, which is an invariant of K.

Additionally, the HOMFLY-PT polynomial satisfies the specializations

$$\mathbf{P}(K; q, a \mapsto q^N) = P^{\mathfrak{sl}_N, \square}(K; q), \tag{3.24}$$

$$\mathbf{P}(K; q, a \mapsto 1) = \Delta_K(q), \tag{3.25}$$

to the quantum $(\mathfrak{sl}_N, \square)$-invariants and the Alexander polynomial of K, respectively.

HOMFLY-PT homology?

We have now seen the unification of various polynomial knot invariants into the HOMFLY-PT polynomial. We have also seen the categorifications of each of these invariants separately in Section 2.4.

Question: Does there exist a triply-graded "HOMFLY-PT homology" theory $\mathcal{H}_{i,j,k}$ which simultaneously categorifies the unifications and unifies the categorifications?

For $P_N := P^{\mathfrak{sl}_N,\square}$, this question is represented schematically by the commutative diagram:

$$\begin{CD} \mathcal{H}_{i,j,k} @>{\chi_j}>> \mathbf{P} \\ @V{?}VV @VV{a=q^N}V \\ HKR^N_{i,j} @>{\chi_j}>> P_N \end{CD} \tag{3.26}$$

Clearly, $\mathbf{P}$ is just the graded Euler characteristic of $\mathcal{H}_{i,j,k}$ with respect to j, but the relationship between $\mathcal{H}_{i,j,k}$ and $HKR^N_{i,j}$ is not as straightforward. *A priori*, we know that such a relationship should somehow "categorify" the specialization $a = q^N$ if the diagram is to commute. However, we do not know the right categorical analogue of specialization.

Differentials

In [DGR], the authors resolve this ambiguity by first introducing Poincaré polynomials:

$$\mathscr{P} := \sum_{i,j,k} q^i t^j a^k \dim \mathcal{H}_{i,j,k}, \tag{3.27}$$

$$\mathscr{P}_N := \sum_{i,j} q^i t^j \dim HKR^N_{i,j}, \tag{3.28}$$

which, using (3.18), may be interpreted as generating functions for refined BPS states. Recall that as we vary N, which corresonds to the Kähler modulus of X, the dimension of $\mathcal{H}_{\text{BPS}}$ can "jump." The authors use this observation to infer the behavior of $\mathcal{H}_{i,j,k}$ with respect to changes in N.

In particular, they predict that this wall-crossing behavior (see [GS]) translates to a *differential* d_N in the language of homological algebra and conjecture that taking homology,

$$H_*(\mathcal{H}_{i,j,k}, d_N) \cong HKR^N_{i,j}, \tag{3.29}$$

provides the right categorical analogue of the specialization $a = q^N$. These differentials were later formalized in [Ras3].

Superpolynomials

Our commutative diagram (3.26) is now complete:

$$\begin{array}{ccccc} \mathcal{H}_{i,j,k} & \xrightarrow{qta-\dim} & \mathscr{P} & \xrightarrow{t=-1} & \mathbf{P} \\ \downarrow{\scriptstyle d_N} & & & & \downarrow{\scriptstyle a=q^N} \\ HKR^N_{i,j} & \xrightarrow{qt-\dim} & \mathscr{P}_N & \xrightarrow{t=-1} & P_N \end{array} \tag{3.30}$$

Beyond simply observing this structure of differentials, the authors of [DGR] used them to compute $\mathscr{P}$ for knots with up to 10 crossings, a feat that was impossible from the formal definitions in [KhR1, KhR2]. These *superpolynomials* have been widely studied and generalized since then; see, e.g., [GS, GW, GGS]. In [EG], which is the subject of Section 6.2, we extend this story to the exceptional Lie algebra $\mathfrak{e}_6$.

3.4 Refined Chern-Simons theory

Something special happens when $K = T^{\mathrm{r,s}}$ is a torus knot: the five-brane theory in (3.20) has an extra $U(1)$-symmetry that acts on $\mathbf{S}^3$ leaving the knot $K = T^{\mathrm{r,s}}$, hence the Lagrangian $L_K \subset T^*\mathbf{S}^3$, invariant. In particular

$$\mathbf{S}^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_1|^2 = 1\}, \tag{3.31}$$

$$T^{\mathrm{r,s}} = \{(z_1, z_2) \in \mathbf{S}^3 : z_1^r = z_2^s\}, \tag{3.32}$$

and the $U(1)$-action is given by

$$U(1) : \mathbf{S}^3 \longrightarrow \mathbf{S}^3, \tag{3.33}$$

$$\zeta : (z_1, z_2) \mapsto (\zeta^s z_1, \zeta^r z_2). \tag{3.34}$$

Furthermore, this action is *semi-free*, meaning that the actions of finite subgroups of $U(1)$ may have fixed points. For example $(z_1, 0)$ is fixed by $\mathbb{Z}/s\mathbb{Z} \subset U(1)$.

In [AS], the authors use this semi-free action to refine the partition function of and torus knot observables in (3.20) by a new parameter t, where

$$q := \mathrm{e}^{\frac{2\pi i}{k+\beta N}} \quad \text{and} \quad t := q^\beta = \mathrm{e}^{\frac{2\pi i\beta}{k+\beta N}}, \quad \text{for} \quad \beta \in \mathbb{N}, \tag{3.35}$$

and upon $\beta \to 1$, (i.e., $t \mapsto q$), the usual theory is recovered. Then, using the relationship [W3], they formulate and solve the *refined Chern-Simons theory* for $G = SU(N)$.

Their solution is a t-refinement of the operator formalism using the type-A Macdonald polynomials P_λ for partitions $\lambda = (\lambda_i)$ (see Section 4.1). In particular, the refined Hilbert space $\tilde{\mathcal{H}}_{T^2}$ has the same basis as $\mathcal{H}_{T^2}$, but the refined matrix elements corresponding to S and T are

$$\tilde{S}_{\lambda\mu} = C_N(q,t) P_\mu(t^\rho q^\lambda) P_\lambda(t^\rho), \quad \tilde{T}_{\lambda\mu} = \delta_{\lambda,\mu} C'_N(q,t) q^{\frac{1}{2}\sum_i \lambda_i(\lambda_i - 1)} t^{\sum_i \lambda_i(i-1)}, \tag{3.36}$$

which, as explained in [GN], is exactly the $SL(2,\mathbb{Z})$-action defined in [Ki1]. Upon $t \mapsto q$, type-A Macdonald polynomials become Schur polynomials, and this coincides with the ordinary $SU(N)$ Chern-Simons theory.

One can then refine the knot operators (2.16) to obtain refined torus knot invariants:

$$P^{\mathsf{r},\mathsf{s}}_{[\mathrm{AS}]}(\lambda; q, t) := \frac{1}{\tilde{S}_{00}} \left\langle \rho \left| \widetilde{\mathbf{W}}^{(\mathsf{r},\mathsf{s})}_\lambda \tilde{S} \right| \rho \right\rangle. \tag{3.37}$$

These stabilize for sufficiently large N, and the N-dependence can be described by a polynomial $\mathcal{P}^{\mathsf{r},\mathsf{s}}_{[\mathrm{AS}]}(\lambda; q, t, a)$ with additional variable a, such that

$$\mathcal{P}^{\mathsf{r},\mathsf{s}}_{[\mathrm{AS}]}(\lambda; q, t, a \mapsto t^N t^{\frac{1}{2}} q^{-\frac{1}{2}}) = P^{\mathsf{r},\mathsf{s}}_{[\mathrm{AS}]}(\lambda; q, t). \tag{3.38}$$

To make a connection to [DGR] and Section 3.3, perform the change of variables:

$$\bar{q} = \sqrt{t}, \qquad \bar{t} = -\sqrt{q/t}, \qquad \bar{a} = \sqrt{a}, \tag{3.39}$$

Then, as conjectured in [AS], for the fundamental representation $\Box$ of $SU(N)$,

$$\mathcal{P}^{\mathsf{r},\mathsf{s}}_{[\mathrm{AS}]}(\Box; \bar{q}, \bar{t}, \bar{a}) = \mathscr{P}^{\mathsf{r},\mathsf{s}}_{[\mathrm{DGR}]}(\Box; \bar{q}, \bar{t}, \bar{a}). \tag{3.40}$$

By now this conjecture includes all rectangular Young diagrams (c.f. (4.78) and Conjecture 4.4.2).

Chapter 4

DAHA and Refined Torus Knot Invariants

Given the (r,s)-torus knot, a root system R, and a weight b, the corresponding *DAHA-Jones polynomial* was defined in [C5, C6] by the simple formula:

$$JD^R_{\mathsf{r},\mathsf{s}}(b;q,t) \coloneqq \{\hat{\gamma}_{\mathsf{r},\mathsf{s}}(P_b)/P_b(q^{-\rho_k})\}_{ev}. \tag{4.1}$$

We will briefly explain the meaning of this expression and then describe its properties and relations to torus knot polynomials and homologies.

In Section 4.1, we will recall the relevant facts on affine root systems / Weyl groups, double affine Hecke algebras, and Macdonald polynomials needed to define the DAHA-Jones polynomials, which we will do in Section 4.2.

Section 4.3 is dedicated to the proof of Theorem 4.3.4, a new result which establishes the connection between DAHA-Jones polynomials upon $t \mapsto q$ and quantum torus knot invariants for root systems of types A, D. The approach follows a suggestion of [C5], where this coincidence is proved for A_1. The treatment of arbitrary root systems is relegated to future work.

Finally, in Section 4.4, we define the *DAHA-superpolynomials* and describe the conjectures which relate them to [DGR] and [AS], the subjects of Sections 3.3 and 3.4, respectively.

4.1 Double affine Hecke algebras

Good general references for the material in this section are [C4, Ha, Hu, Ki2, M1, M5] as well as the original papers [C1, C2, C3, M2, M3]. Our conventions for root systems will be from [B].

Hecke algebras

Let R be a (crystallographic) root system of rank n with respect to the Euclidean inner product $(-,-)$ on $\mathbb{R}^n$, and let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be any set of simple roots. The *Weyl group* W for R is generated by the simple reflections:

$$s_i : \beta \mapsto \beta - \frac{2(\beta, \alpha_i)}{(\alpha_i, \alpha_i)}\alpha_i \text{ for } 1 \le i \le n, \ \ \beta \in R, \tag{4.2}$$

subject to the Coxeter relations $(s_i s_j)^{m_{ij}} = 1$. The numbers m_{ij} are 2,3,4,6 when the corresponding nodes in the Dynkin diagram for R are joined by 0,1,2,3 edges, respectively.

Then the (nonaffine) *Hecke algebra* H for R is generated over $\mathbb{C}(t_1^{\frac{1}{2}}, \ldots, t_n^{\frac{1}{2}})$ by elements $\{T_1, \ldots, T_n\}$, subject to relations:

$$(T_i - t_i^{\frac{1}{2}})(T_i + t_i^{-\frac{1}{2}}) \text{ for } 1 \le i \le n, \tag{4.3}$$

$$T_iT_jT_i \ldots = T_jT_iT_j \ldots \text{ with } m_{ij} \text{ terms on each side,} \tag{4.4}$$

where the number of distinct t_i is equal to the number the orbits of W acting on R, so at most 2 in the nonaffine case. That is, we normalize the form by $(\alpha, \alpha) = 2$ for short roots $\alpha \in R$ and set $\nu_\beta := \frac{(\beta,\beta)}{2}$ for $\beta \in R$. Then $t_i := t_{\nu_{\alpha_i}}$ for each simple root $\alpha_i \in \Delta$.

Twisted affine root systems

Before defining an affine root system, we recall the identification $\mathbb{R}^{n+1} \cong \text{Aff}(\mathbb{R}^n)$. That is, we interpret a vector $[\vec{u}, c] \in \mathbb{R}^n \times \mathbb{R}$ as an affine linear function on $\mathbb{R}^n$:

$$[\vec{u}, c] : \vec{v} \mapsto (\vec{u}, \vec{v}) - c, \tag{4.5}$$

whose zero set $[\vec{u}, c]^{-1}(0)$ is an affine hyperplane in $\mathbb{R}^n$, $H_{[\vec{u},c]} := \{\vec{v} \in \mathbb{R}^n : (\vec{u}, \vec{v}) = c\}$. Observe that $H_{[\vec{u},c]} = H_{[\vec{u},0]} + \frac{c}{2}\vec{u}^\vee$, where $\vec{u}^\vee := \frac{\vec{u}}{\nu_{\vec{u}}}$.

The reflection of $\mathbb{R}^n$ through $H_{[\vec{u},c]}$ is

$$s_{[\vec{u},c]} : \vec{v} \mapsto \vec{v} - [(\vec{u}, \vec{v}) - c]\, \vec{u}^\vee, \tag{4.6}$$

which fixes $H_{[\vec{u},c]}$ and maps 0 to $c\vec{u}^\vee$. We can extend the domain of affine reflections to act on $\text{Aff}(\mathbb{R}^n) \cong \mathbb{R}^n \times \mathbb{R}$ by

$$s_{[\vec{u},c]}([\vec{v}, k]) := [\vec{v}, k] \circ s_{[\vec{u},c]} = [\vec{v}, k] - (\vec{v}, \vec{u}^\vee)[\vec{u}, c]. \tag{4.7}$$

Alternatively, we could describe $s_{[\vec{u},c]}$ as a reflection in $H_{[\vec{u},0]}$ with a subsequent translation by $c\vec{u}^\vee$, where "translations" are

$$s_{[\pm\vec{u},c]}s_{[\vec{u},0]} = s_{[\vec{u},0]}s_{[\mp\vec{u},c]} : \begin{cases} \vec{v} \mapsto \vec{v} \pm c\vec{u}^\vee, \\ [\vec{v}, k] \mapsto [\vec{v}, k \pm (\vec{v}, \vec{u}^\vee)c], \end{cases} \tag{4.8}$$

and we will often confuse $c\vec{u}^\vee \in \mathbb{R}^n$ with this action below.

Define the *(twisted) affine root system* $R \subset \widetilde{R}$ by:

$$\widetilde{R} = \{[\alpha, k\nu_\alpha] : \alpha \in R, k \in \mathbb{Z}\}, \tag{4.9}$$

with $R = \{[\alpha, 0]\}$. The simple roots for $\widetilde{R}$ are $\widetilde{\Delta} := \{\alpha_0 = [-\vartheta, 1]\} \cup \Delta$, where $\vartheta \in R$ is the highest *short* root with respect to Δ.

Affine Weyl groups

The *affine Weyl group* $\widetilde{W}$ is generated by $s_i := s_{\alpha_i}$, $0 \leq i \leq n$ subject to relations $s_i^2 = 1$ and

$$s_i s_j s_i \ldots = s_j s_i s_j \ldots \text{ with } m_{ij} \text{ terms on each side}, \tag{4.10}$$

where m_{ij} correspond, as above, to the affine Dynkin diagram.

We saw that $s_{[\alpha,k\nu_\alpha]}$ admits a description as a reflection $s_\alpha \in W$ composed with a translation by $k\nu_\alpha\alpha^\vee = k\alpha \in Q$, where Q is the root lattice for R, i.e., the $\mathbb{Z}$-span of Δ. Therefore, one easily concludes that

$$\widetilde{W} = W \ltimes Q, \tag{4.11}$$

where Q acts by "translations" as described above.

If we enlarge the group Q to include translations by the weight lattice,

$$Q \subset P := \bigoplus_{i=1}^{n} \mathbb{Z}\omega_i, \tag{4.12}$$

where $\{\omega_i\}$ are fundamental weights, we obtain the *extended affine Weyl group*,

$$\widehat{W} := W \ltimes P = \widetilde{W} \ltimes \Pi, \tag{4.13}$$

where $\Pi := P/Q$ in the semidirect product decomposition relative to $\widetilde{W}$.

To describe the subgroup $\Pi \lhd \widehat{W}$ more explicitly, we can introduce a length function l on $\widehat{W}$:

$$l(\hat{w}) := \left| \widetilde{R}_+ \cap \hat{w}^{-1}(-\widetilde{R}_+) \right|, \tag{4.14}$$

where $\widehat{R}_+$ is the set of positive roots with respect to $\widetilde{\Delta}$. Then $\Pi = \{\hat{w} \in \widehat{W} : l(\hat{w}) = 0\}$. Geometrically, these these are the elements of $\widehat{W}$ which permute $\widetilde{\Delta}$, and we can label an element $\pi_r \in \Pi$ by its action $\pi_r(\alpha_0) = \alpha_r$.

Alternatively, define the set of indices of minuscule weights:

$$O' \coloneqq \{r : 0 \leq (\omega_r, \alpha^\vee) \leq 1, \text{ , for all } \alpha \in R_+\} \subset \{1, \ldots, n\}. \tag{4.15}$$

Then $O = \{0\} \cup O'$ is a system of representatives for P/Q in the sense that every $b \in P$ can be written uniquely as $b = \omega_r + \alpha$ for some $r \in O$, $\alpha \in Q$, where $\omega_0 = 0$. For $r \in O$ let $u_r \in W$ be the shortest element such that $u_r(\omega_r) \in -P_+$. We can define

$$\Pi = \{\pi_r : \omega_r = \pi_r u_r,\ r \in O\}, \tag{4.16}$$

and observe that $\pi_0 = \text{id}$.

The affine Weyl group $\widetilde{W}$ (or, to be more precise, its group algebra) has a simple physical interpretation [GWi] as the algebra of line operators in four dimensional gauge theory on $M_4 \cong \mathbf{S}^1 \times \mathbb{R}^3$ in the presence of ramification along $D \cong \mathbf{S}^1 \times \mathbb{R}$. (In physics, ramification is often called a *surface operator*.)

Affine Hecke algebras

The *affine Hecke algebra* $\mathcal{H}$ for $R \subset \widetilde{R}$ is generated over $\mathbb{C}(t_0^{\frac{1}{2}}, t_1^{\frac{1}{2}}, \ldots, t_n^{\frac{1}{2}})$. It admits two equivalent descriptions, each emphasizing one of the two equivalent descriptions of the extended affine Weyl group $\widehat{W}$:

- For $\widehat{W} = \widetilde{W} \ltimes \Pi$, $\mathcal{H}$ is generated by elements $\{T_0, T_1, \ldots, T_n\}$ and $\pi_r \in \Pi$, subject to relations:
  1. $(T_i - t_i^{\frac{1}{2}})(T_i + t_i^{-\frac{1}{2}})$ for $0 \leq i \leq n$,
  2. $T_i T_j T_i \ldots = T_j T_i T_j \ldots$ with m_{ij} terms on each side,
  3. $\pi_r T_i \pi_r^{-1} = T_j$ if $\pi_r(\alpha_i) = \alpha_j$.
- For $\widehat{W} = W \ltimes P$, $\mathcal{H}$ is generated by $\{T_1, \ldots, T_n\}$ and $\{Y_b : b \in P\}$, subject to relations:
  1. $(T_i - t_i^{\frac{1}{2}})(T_i + t_i^{-\frac{1}{2}})$ for $1 \leq i \leq n$,
  2. $T_i T_j T_i \ldots = T_j T_i T_j \ldots$ with m_{ij} terms on each side,
  3. $Y_{b+c} = Y_b Y_c$ for $b, c \in P$,
  4. $T_i Y_b = Y_b Y_{\alpha_i}^{-1} T_i^{-1}$ if $(b, \alpha_i^\vee) = 1$ for $0 \leq i \leq n$,
  5. $T_i Y_b = Y_b T_i$ if $(b, \alpha_i^\vee) = 0$ for $0 \leq i \leq n$.

To translate from the first to the second description, one can define pairwise-commuting elements:

$$Y_b := \prod_{i=1}^{n} Y_i^{l_i} \text{ for } b = \sum_{i=1}^{n} l_i \omega_i \in P, \tag{4.17}$$

where $Y_i := T_{\omega_i}$ for $\omega_i \in \widehat{W}$. That is, if $l = l(\tilde{w})$ so that $\tilde{w} = s_{i_l} \cdots s_{i_1} \in \widetilde{W}$ is a reduced decomposition, then $T_{\pi_r \tilde{w}} := \pi_r T_{i_l} \cdots T_{i_1}$. For example, $Y_\vartheta = T_0 T_{s_\vartheta}$.

Much like the affine Weyl group, the affine Hecke algebra $\mathcal{H}$ can also be interpreted as the algebra of line operators in 4d gauge theory on M_4 with a ramification (surface operator) along $D \subset M_4$. The only difference is that now one has to introduce a so-called Ω-background in the normal bundle of D. (See [G] for a review.)

Double affine Hecke algebras

Let m be the least natural number satisfying $(P, P) \subset \frac{1}{\mathsf{m}}\mathbb{Z}$. Suppose that $\tilde{b} = [b, j]$ with $b = \sum_{i=1}^{n} l_i \omega_i \in P$ and $j \in \frac{1}{\mathsf{m}}\mathbb{Z}$. Then for $\{X_1, \ldots, X_n : [X_i, X_j] = 0\}$ we define elements:

$$X_{\tilde{b}} := \prod_{i=1}^{n} X_i^{l_i} q^j, \tag{4.18}$$

and an action of $\hat{w} \in \widehat{W}$ by $\hat{w}(X_{\tilde{b}}) := X_{\hat{w}(\tilde{b})}$. Observe that $X_0 := X_{\alpha_0} = q X_\vartheta^{-1}$.

The *double affine Hecke algebra* ("DAHA") $\mathcal{H\!\!H}$ for $R \subset \widetilde{R}$ is generated over $\mathbb{Z}_{q,t} := \mathbb{Z}[q^{\pm\frac{1}{\mathsf{m}}}, t_\nu^{\pm\frac{1}{2}}]$ by elements $\{T_i, X_b, \pi_r : 0 \le i \le n,\ b \in P,\ r \in O\}$ subject to relations:

1. $(T_i - t_i^{\frac{1}{2}})(T_i + t_i^{-\frac{1}{2}})$ for $0 \le i \le n$,
2. $T_i T_j T_i \ldots = T_j T_i T_j \ldots$ with m_{ij} terms on each side,
3. $\pi_r T_i \pi_r^{-1} = T_j$ if $\pi_r(\alpha_i) = \alpha_j$,
4. $T_i X_b = X_b X_{\alpha_i}^{-1} T_i^{-1}$ if $(b, \alpha_i^\vee) = 1$ for $0 \le i \le n$,
5. $T_i X_b = X_b T_i$ if $(b, \alpha_i^\vee) = 0$ for $0 \le i \le n$,
6. $\pi_r X_b \pi_r^{-1} = X_{\pi_r(b)} = X_{u_r^{-1}(b)} q^{(\omega_{\iota(r)}, b)}$ for $r \in O'$,

where in (6) we have used the involution $\iota : O' \to O'$ defined by $\pi_r^{-1} = \pi_{\iota(i)}$.

Observe that $\mathcal{H\!\!H}$ contains two subalgebras isomorphic to the affine Hecke algebra $\mathcal{H}$ for $R \subset \widetilde{R}$:

$$\mathcal{H}_1 := \langle \pi_r, T_0, \ldots, T_n \rangle \subset \mathcal{H\!\!H}, \tag{4.19}$$

$$\mathcal{H}_2 := \langle T_1, \ldots, T_n, X_b \rangle \subset \mathcal{H\!\!H}. \tag{4.20}$$

One can make $\mathcal{H}_1$ look more like $\mathcal{H}_2$ by defining pairwise-commuting elements Y_b as in (4.17). Then we have that

$$\mathcal{H}_1 = \langle T_1, \ldots, T_n, Y_b \rangle. \tag{4.21}$$

In fact, $\mathcal{HH}$ is also generated by elements $\{X_a, T_w, Y_b : a, b \in P,\ w \in W\}$. While relations between these generators are more complicated, this presentation has some nice properties that will be useful in our definitions of Macdonald and DAHA-Jones polynomials below. In particular, we have the PBW theorem for DAHA.

Theorem 4.1.1. *(PBW Theorem) Any $h \in \mathcal{HH}$ can be written uniquely in the form*

$$h = \sum_{a,w,b} c_{a,w,b} X_a T_w Y_b, \tag{4.22}$$

for $c_{a,w,b} \in \mathbb{Z}_{q,t}$. The similar statement holds for each ordering of $\{X_a, T_w, Y_b\}$.

Macdonald polynomials

Polynomial representation

To define the Macdonald polynomials using DAHA, we need the *polynomial representation*

$$\varrho : \mathcal{HH} \to \mathcal{V}, \tag{4.23}$$

where $\mathcal{V} := \mathrm{End}(\mathbb{Z}_{q,t}[X])$. In generators $\{X_b, \pi_r, T_i\}$ its action is given by

$$\varrho : \begin{cases} X_b \cdot g = X_b g \\ \pi_r \cdot g = \pi_r g \pi_r^{-1}, \text{ where, e.g., } \pi_r \cdot X_b = X_{\pi_r(b)} \\ T_i \cdot g = \widehat{T}_i g \end{cases}, \tag{4.24}$$

for $g \in \mathbb{Z}_{q,t}[X]$. The action of T_i is by the *Demazure-Lusztig* operators:

$$\widehat{T}_i := t_i^{\frac{1}{2}} s_i + (t_i^{\frac{1}{2}} - t_i^{\frac{-1}{2}}) \frac{s_i - 1}{X_{\alpha_i} - 1}, \tag{4.25}$$

where, again, $s_i X_b = X_{s_i(b)}$. Observe that if $g \in \mathbb{Z}_{q,t}[X]^W$ is *any* symmetric polynomial, then $\widehat{T}_i g = t_i^{\frac{1}{2}} g$. Remarkably, ϱ is a faithful representation.

Symmetric Macdonald polynomials

The symmetric Macdonald polynomials $P_b \in \mathbb{Z}_{q,t}[X]$ for $b \in P_+$ were introduced in [M2, M3]. They form a basis for the symmetric (W-invariant) polynomials $\mathbb{Z}_{q,t}[X]^W$. DAHA provides a uniform construction of P_b for any root system as the simultaneous eigenfunctions for a commuting family of W-invariant operators L_f for $f \in \mathbb{Z}_{q,t}[Y]^W = Z(\mathcal{H}_1)$; see [C1].

Now for $f \in \mathbb{Z}_{q,t}[Y]^W \subset \mathcal{H}\!\!\!\mathcal{H}$, we can use the polynomial representation to write an operator $L_f \coloneqq \varrho(f)$ on $\mathbb{Z}_{q,t}[X]$. The *symmetric Macdonald polynomials* are uniquely defined by

$$L_f(P_b) = f(q^{\rho_k+b})P_b, \tag{4.26}$$

as simultaneous eigenfunctions of the pairwise-commuting W-invariant operators L_f for all $f \in \mathbb{Z}_{q,t}[Y]^W$. In fact, $P_b \in \mathbb{Q}(q, t_\nu)[X]^W$.

In expressing P_b as an eigenfunction, we used the notation

$$\rho_k \coloneqq \frac{1}{2}\sum_{\alpha\in R_+} k_\alpha \alpha = k_{\mathrm{sht}}\rho_{\mathrm{sht}} + k_{\mathrm{lng}}\rho_{\mathrm{lng}}, \quad \text{where, e.g.,} \quad \rho_{\mathrm{sht(lng)}} \coloneqq \frac{1}{2}\sum_{\substack{\alpha \text{ short} \\ \text{(long)}}} k_\alpha \alpha, \tag{4.27}$$

for the Weyl vector weighted by a function $k_\alpha = k_{\nu_\alpha}$ which is invariant on W-orbits. We also use the notation $X_b(q^a) \coloneqq q^{(b,a)}$, and in particular, $X_b(q^{\rho_k}) = q^{(b,\rho_k)} = t_{\mathrm{sht}}^{(b,\rho_{\mathrm{sht}})} t_{\mathrm{lng}}^{(b,\rho_{\mathrm{lng}})}$. Following [C2], we have the duality and evaluation formulas:

$$P_b(q^{c-\rho_k})P_c(q^{-\rho_k}) = P_c(q^{b-\rho_k})P_b(q^{-\rho_k}) \quad \text{for} \quad b, c \in P_-, \tag{4.28}$$

$$P_b(q^{-\rho_k}) = q^{-(\rho_k,b)} \prod_{\alpha\in R_+} \prod_{j=0}^{(\alpha^\vee,b)-1} \left(\frac{1 - q_\alpha^j t_\alpha X_\alpha(q^{\rho_k})}{1 - q_\alpha^j X_\alpha(q^{\rho_k})} \right). \tag{4.29}$$

The corresponding *spherical polynomial* is $P_b^\circ \coloneqq P_b/P_b(q^{-\rho_k})$.

Nonsymmetric Macdonald polynomials

The *nonsymmetric Macdonald polynomials* $E_b \in \mathbb{Z}_{q,t}[X]$ for $b \in P_+$ were introduced in [M4, Op]. Following [C3], DAHA provides a uniform construction for E_b for any root system as simultaneous eigenfunctions of the pairwise-commuting Y-operators (*Dunkl operators*),

$$Y_a(E_b) = q^{-(a,b+w_b(\rho_k))}E_b, \quad \text{for} \quad a \in P, \tag{4.30}$$

which act in the polynomial representation. Here $w_b \in W$ is the element of maximal length in the centralizer of b, and E_b is normalized by the condition that the coefficient of X_b is 1. We have the evaluation formula:

$$E_b(q^{-\rho_k}) = q^{-(\rho_h,b)} \prod_{\alpha\in R_+} \prod_{j=1}^{(\alpha^\vee,b)-1} \left(\frac{1 - q_\alpha^j t_\alpha X_\alpha(q^{\rho_k})}{1 - q_\alpha^j X_\alpha(q^{\rho_k})} \right), \tag{4.31}$$

and the corresponding spherical polynomial is $E_b^\circ := E_b/E_b(q^{-\rho_k})$.

The spherical counterparts of symmetric and nonsymmetric Macdonald polynomials are related by the *t-symmetrizer*

$$\delta^+ := \frac{\sum_{w\in W} t_{\mathrm{sht}}^{l_{\mathrm{sht}}(w)/2} t_{\mathrm{lng}}^{l_{\mathrm{lng}}(w)/2} T_w}{\sum_{w\in W} t_{\mathrm{sht}}^{l_{\mathrm{sht}}(w)} t_{\mathrm{lng}}^{l_{\mathrm{lng}}(w)}}, \tag{4.32}$$

where $l_\nu(w) := \#\{s_i : w = s_{l(w)}\cdots s_1, \nu_{\alpha_i} = \nu\}$. Then we have that

$$P_b^\circ = \delta^+ E_{b'}^\circ, \quad \text{for any} \quad b \in P_+, \, b' \in W(b). \tag{4.33}$$

As we will see, this relationship will prove very useful in calculating DAHA-Jones polynomials.

4.2 DAHA-Jones Polynomials

Here we provide an efficient definition of the DAHA-Jones polynomials, which were originally defined in [C5, C6] for torus knots and extended to iterated torus knots in [CD]. We also state their main (algebraic) properties, which were conjectured in [C5] and mostly proved in [C6, GN].

$PSL_2^\wedge(\mathbb{Z})$-action

Define a central idempotent:

$$e := \frac{1}{|W|}\sum_{w\in W} w, \tag{4.34}$$

in the group algebra of W. Then the *spherical DAHA* is $\mathcal{SH} := e\mathcal{H}e \subset \mathcal{H}$. In particular, $P_b^\circ, E_b^\circ \in \mathcal{SH}$. Further, define the *projective* $PSL_2(\mathbb{Z})$ by

$$PSL_2^\wedge(\mathbb{Z}) := \langle \tau_\pm : \tau_+\tau_-^{-1}\tau_+ = \tau_-^{-1}\tau_+\tau_-^{-1}\rangle, \tag{4.35}$$

as a group whose action $\mathcal{H}$ is represented by

$$\tau_+ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tau_- = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \text{ where } \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{cases} X_\lambda \mapsto X_\lambda^a Y_\lambda^c \\ T_i \mapsto T_i \\ Y_\lambda \mapsto X_\lambda^b Y_\lambda^d \end{cases}, \tag{4.36}$$

for $\lambda \in P$, $i > 0$ and extends to an action on all of $\mathcal{H}$, which restricts to an action on $\mathcal{SH} \subset \mathcal{H}$.

Evaluation coinvariant

We define a functional $\{\cdot\}_{ev} : \mathcal{H\!H} \to \mathbb{Z}_{q,t}$ called the *evaluation coinvariant*, which first writes $h \in \mathcal{H\!H}$,

$$h = \sum_{a,w,b} c_{a,w,b} X_a T_w Y_b, \tag{4.37}$$

in the unique form guaranteed by the PBW Theorem 4.1.1 and then substitutes

$$X_a \mapsto q^{-(\rho_k,a)}, \quad T_i \mapsto t_i^{\frac{1}{2}}, \quad Y_b \mapsto q^{(\rho_k,b)}. \tag{4.38}$$

This process factors through the polynomial representation, which allows one to avoid making direct use of the PBW theorem (which can be rather complicated to implement). In other words, $\{\cdot\}_{ev}$ is equivalent to projection onto the polynomial representation followed by the substitution (4.38). See [CM].

Main definition

Corresponding to the (r,s)-torus knot, choose an element $\hat{\gamma}_{\mathsf{r,s}} \in PSL_2^{\wedge}(\mathbb{Z})$, which is *any* word in $\tau_\pm$ that can be represented by

$$\gamma_{\mathsf{r,s}} = \begin{pmatrix} \mathsf{r} & * \\ \mathsf{s} & * \end{pmatrix}, \tag{4.39}$$

where the $*$ entries do not matter, since $\hat{\gamma}_{\mathsf{r,s}}$ will act on a polynomial in X_i, see (4.36). For any root system R and dominant weight $b \in P_+$, let

$$JD^R_{\mathsf{r,s}}(b; q, t) \coloneqq \{\hat{\gamma}_{\mathsf{r,s}}(P_b)/P_b(q^{-\rho_k})\}_{ev}, \tag{4.40}$$

$$q^\bullet t^\bullet \widetilde{JD}^R_{\mathsf{r,s}}(b; q, t) \coloneqq JD^R_{\mathsf{r,s}}(b; q, t), \tag{4.41}$$

where $q^\bullet t^\bullet$ is the lowest q,t-monomial in $JD^R_{\mathsf{r,s}}(b; q, t)$, if it is well-defined. Then $\widetilde{JD}^R_{\mathsf{r,s}}(b; q, t) \in \mathbb{Z}[q, t]$ is the (reduced, tilde-normalized) *DAHA-Jones polynomial.*

Alternative definition

We may also define DAHA-Jones polynomials in terms of the nonsymmetric Macdonald polynomials.

Theorem 4.2.1. *([C6]) We have that* $JD^R_{r,s}(b; q, t) = \{\hat{\gamma}_{r,s}(E_{b'})/E_{b'}(q^{-\rho_k})\}_{ev}$ *for* $b \in P_+$, $b' \in W(b)$.

Proof. Combining (4.33) and (4.40),

$$JD^R_{\mathsf{r,s}}(b; q, t) = \{\hat{\gamma}_{\mathsf{r,s}}(P_b)/P_b(q^{-\rho_k})\}_{ev} = \{\hat{\gamma}_{\mathsf{r,s}}(\delta_+ E^\circ_{b'})\}_{ev}, \text{ for } b' \in W(b). \tag{4.42}$$

Observe from (4.36) that the action of $PSL_2^{\wedge}(\mathbb{Z})$ commutes with the action of T_i for $i > 0$. Therefore,

$$= \{\delta_+ \hat{\gamma}_{\mathsf{r},\mathsf{s}}(E^{\circ}_{b'})\}_{ev} = (\delta_+|_{T_i \mapsto t_i^{1/2}})\{\hat{\gamma}_{\mathsf{r},\mathsf{s}}(E^{\circ}_{b'})\}_{ev}, \tag{4.43}$$

by the general formula $\{T_i(f)\}_{ev} = t_i^{1/2}\{f\}_{ev}$. Then (4.32) implies that $\delta_+|_{T_i \mapsto t_i^{1/2}} = 1$. □

Since the nonsymmetric Macdonald polynomials are generally much simpler than the symmetric ones, this definition is extremely useful for computations.

Example: E_8 and $\vartheta = \omega_8$

We compute the DAHA-Jones polynomial corresponding to the E_8 root system and its highest short root $\vartheta = \omega_8$, which is the highest weight of the (248-dimensional) adjoint representation of $\mathfrak{e}_8$.

From formula (4.1) of [C5], the nonsymmetric Macdonald polynomial is

$$E_\vartheta = X_\vartheta + \frac{q(1-t)}{1 - qt^{(\vartheta,\rho_k)}}, \quad E_\vartheta(q^{-\rho_k}) = q^{(\vartheta,\rho_k)} \frac{1 - qt^{(\vartheta,\rho_k)+1}}{1 - qt^{(\vartheta,\rho_k)}}. \tag{4.44}$$

Observe that since E_8 is simply-laced, we have a single t. Furthermore, $(\vartheta, \rho_k) = h^\vee - 1 = 29$ with $k = 1$. Now we want to apply $\tau_+\tau_-^m = \begin{pmatrix} m+1 & 1 \\ m & 1 \end{pmatrix} \in PSL_2^{\wedge}(\mathbb{Z})$, which represents the torus knot $T^{m+1,m}$ and may be done inductively:

$$\tau_+\tau_-^m(E_\vartheta) = \big(X_\vartheta Y_\vartheta - (t^{\frac{1}{2}} - t^{-\frac{1}{2}})X_\vartheta T_{s_\vartheta}\big)\big(\tau_+\tau_-^{m-1}(E_\vartheta)\big) + (1 - t^{-1})X_\vartheta. \tag{4.45}$$

Observe that $Y_\vartheta = T_0 T_{s_\vartheta}$, where

$$s_\vartheta = s_{8,7,6,5,4,3,2,4,5,6,7,1,3,4,5,6,2,4,5,3,4,1,3,2,4,5,6,7,8,7,6,5,4,2,3,1,4,3,5,4,2,6,5,4,3,1,7,6,5,4,2,3,4,5,6,7,8} \tag{4.46}$$

is a reduced expression, and T_0 acts in the polynomial representation by:

$$\hat{T}_0 = t^{\frac{1}{2}} s_0 + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \frac{s_0 - 1}{qX_\vartheta^{-1} - 1}, \tag{4.47}$$

the Demazure-Lusztig operator (4.25) for $s_0(X_b) \coloneqq s_{[-\vartheta,1]}(X_b) = s_\vartheta(X_b)q^{(b,\vartheta)}$.

We now have enough information to project onto the polynomial representation and then evaluate at $q^{-\rho_k}$. Carrying this out on a computer algebra program for $m = 2$, we obtain

$$JD^{E_8}_{3,2}(\omega_8; q, t) = \{\tau_+\tau_-^2(E^{\circ}_\vartheta)\}_{ev} = \tag{4.48}$$

$\frac{1}{q^6 t^{116}}\big(1 + q(t + t^6 + t^{10} - t^{20} - t^{24} - t^{29}) + q^2 t^{12}(1 + t^4 + t^8 - t^{14} + t^{17} - 3t^{18} - t^{22} - t^{23} - t^{27} + t^{32} + t^{37} + t^{41}) - q^3 t^{29}(-1 - t^6 + t^7 - t^{10} + t^{11} + t^{12} + t^{16} + 2t^{20} - t^{21} + t^{24} - t^{25} - t^{26} + t^{29} - 2t^{30} - t^{34} + t^{44}) - q^4 t^{58}(-1 +$

$t + t^6 - t^7 + t^{10} - t^{11} - t^{20} + t^{21} - t^{24} + t^{25}) + q^5(-t^{87} + t^{88})),$

which, after multiplying by $q^6 t^{116}$, agrees with the tilde-normalized $\widetilde{JD}^{E_8}_{3,2}(\omega_8; q, t)$ obtained in Section 4.5.3 of [C5].

Properties of DAHA-Jones Polynomials

Here we recall some important properties of DAHA-Jones polynomials, which were conjectured in [C5] and proved in Theorem 1.2 of [C6]. First, we remark that the tilde-normalized DAHA-Jones polynomials are, in fact, polynomials:

$$\widetilde{JD}^{R}_{\mathsf{r},\mathsf{s}}(b; q, t) \in \mathbb{Z}[q, t]. \tag{4.49}$$

Then, in anticipation of a connection to quantum knot invariants, we expect that DAHA-Jones polynomials should satisfy the usual topological properties with respect to the torus knot $T^{\mathsf{r},\mathsf{s}}$:

1. *(well-defined)* $\widetilde{JD}^{R}_{\mathsf{r},\mathsf{s}}(b; q, t)$ does not depend on the choice of $\hat{\gamma}_{\mathsf{r},\mathsf{s}} \in PSL_2^{\wedge}(\mathbb{Z})$,
2. *(unknot)* $\widetilde{JD}^{R}_{\mathsf{r},1}(b; q, t) = 1$,
3. *(r, s-symmetry)* $\widetilde{JD}^{R}_{\mathsf{r},\mathsf{s}}(b; q, t) = \widetilde{JD}^{R}_{\mathsf{s},\mathsf{r}}(b; q, t)$,
4. *(orientation)* $\widetilde{JD}^{R}_{\mathsf{r},\mathsf{s}}(b; q, t) = \widetilde{JD}^{R}_{-\mathsf{r},-\mathsf{s}}(b; q, t)$,
5. *(mirror image)* $JD^{R}_{\mathsf{r},-\mathsf{s}}(b; q, t) = JD^{R}_{\mathsf{r},\mathsf{s}}(b; q^{-1}, t^{-1})$.

Finally, the following *evaluation* is a property of the refinement which reflects "exponential growth" in the number of terms in $\widetilde{JD}^{R}_{\mathsf{r},\mathsf{s}}(b; q, t)$ with respect to $|b|$:

$$\widetilde{JD}^{R}_{\mathsf{r},\mathsf{s}}(\sum_{i=1}^{n} b_i\omega_i; q = 1, t) = \prod_{i=1}^{n} \widetilde{JD}^{R}_{\mathsf{r},\mathsf{s}}(\omega_i; q = 1, t)^{b_i}. \tag{4.50}$$

It is related to the fact that $P_{b+c} = P_b P_c$ upon $q \to 1$. We do not discuss the *color exchange*, which is also part of Theorem 1.2 and corresponds to generalized level-rank duality.

4.3 Connection to quantum groups: types A and D

We are going to establish the relationship between DAHA-Jones polynomials and quantum invariants of torus knots that was Conjecture 2.1 of [C5]. Presently, we focus on the cases of A and D, using the approach suggested there, and a general proof will be relegated to future work. This relationship was already demonstrated for A_1 in [C5] for torus knots and in [CD] for iterated torus knots.

Difference shift operators

Recall the function $k_\alpha = k_{\nu_\alpha}$, which is invariant on W-orbits. For simplicity, we restrict to the simply-laced root systems (types ADE) and $k_\alpha = k \in \{0,1\}$. The *difference shift operator*,

$$\chi_k = \prod_{\alpha\in R_+}\prod_{j=0}^{k-1}\big((q^jX_\alpha)^{\frac{1}{2}} - (q^jX_\alpha)^{-\frac{1}{2}}\big), \tag{4.51}$$

was defined in [C2] and used to prove the Macdonald constant term and duality conjectures. As we will see, it may be used to perform the induction $k \mapsto k+1$ for integral k.

Let $\mathcal{H}^{(k)}$ be the DAHA with structural parameters q and $t = q^k$, and define the projections

$$\begin{aligned} \mathcal{H} &\longrightarrow \mathcal{H}^{(k)} \xrightarrow{\varrho_k} \mathcal{V}^{(k)}, \\ H &\longmapsto H^{(k)} \xmapsto{\varrho_k} \hat{H}^{(k)}. \end{aligned} \tag{4.52}$$

We will also need the automorphisms $\tau_\pm^{(k)}$ of $\mathcal{H}^{(k)}$, i.e., $(\tau_\pm H)^{(k)} = \tau_\pm^{(k)}H^{(k)}$.

Of particular interest for us is the case $k=0$, for which we will use $\circ$. Then for $b\in P$ and $w\in W$, the operators $\hat{Y}_b^\circ, \hat{T}_w^\circ \in \mathcal{V}^\circ$ are the difference operator b^{-1} and w, respectively. We also have

$$\tau_+^\circ(Y_b) = q^{-(b,b)/2}X_bY_b, \quad \tau_-^\circ(X_b) = q^{(b,b)/2}Y_bX_b. \tag{4.53}$$

Now we are ready to state Lemma 2.2 of [C5], the main property of shift operators that we will use.

Lemma 4.3.1. Suppose that $H^{(k)}$ is an algebraic expression with $\mathbb{C}$-coefficients in terms of W-invariant polynomials $\mathbb{C}_{q,q^k}[X]^W$ and $\mathbb{C}_{q,q^k}[Y]^W$. In this case, $\hat{H}^{(k)} \in \mathcal{V}^{(k)}$ restricts to an operator $\hat{H}^{(k)}_{\mathrm{sym}} \in (\mathcal{V}^{(k)})^W$, i.e., on the subspace of W-invariant elements of $\mathcal{V}^{(k)}$. Then

$$\hat{H}^{(k)}_{\mathrm{sym}} = \chi_k^{-1}\hat{H}^\circ_{\mathrm{sym}}\chi_k \quad \text{and} \quad (\widehat{\tau_\pm^{(k)}H^{(k)}})_{\mathrm{sym}} = \chi_k^{-1}(\widehat{\tau_\pm^\circ H^\circ})_{\mathrm{sym}}\chi_k. \tag{4.54}$$

Free $SL(2,\mathbb{Z})$-action

Following the discussion on pages 15-16 of [C5], consider the $\mathcal{H}^{(0)}$-module $\tilde{\mathcal{V}}$ spanned by elements $\{X_\lambda q^{zx^2/2} : \lambda\in P, z\in\mathbb{C}\}$, where $X_\lambda = q^{(\lambda,x)}$ and $x^2 = (x,x)$ for a formal variable x. Then $\tau_\pm^\circ$ act as inner automorphisms on $\tilde{\mathcal{V}}$ and may be extended to the *free* $SL(2,\mathbb{Z})$*-action*

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})\colon \begin{cases} \gamma^\circ(X_\lambda q^{\frac{zx^2}{2}}) = \frac{1}{(cz+d)^{\frac{1}{2}}} q^{\frac{-\lambda^2c}{2(cz+d)}} X_{\frac{\lambda}{cz+d}} q^{\frac{az+b}{cz+d}x^2/2}, \\ (\gamma^\circ)^{-1}(X_\lambda q^{\frac{zx^2}{2}}) = \frac{1}{(-cz+a)^{\frac{1}{2}}} q^{\frac{\lambda^2c}{2(-cz+a)}} X_{\frac{\lambda}{-cz+a}} q^{\frac{dz-b}{-cz+a}x^2/2}, \end{cases} \tag{4.55}$$

for generic $z\in\mathbb{C}$. We remark that $\tilde{\gamma}^{(k)}$ acts in $\tilde{\mathcal{V}}^W$, so the conditions of Lemma 4.3.1 are satisfied. Thus, we have $\tilde{\gamma}^{(k)} = \chi_k^{-1}\gamma^\circ\chi_k$, which leads to Proposition 2.3 of [C5]:

Proposition 4.3.2. *For $k\in\{0,1\}$, $\lambda\in P_+$, and any $\gamma = \left(\begin{smallmatrix} r & u \\ s & v\end{smallmatrix}\right) \in SL(2,\mathbb{Z})$, define*

$$R^{k,\lambda}_{r,s} := \chi_k^{-1}\gamma^\circ\big((P^{(k)}_\lambda / P^{(k)}_\lambda(q^{-\rho_k})(\gamma^\circ)^{-1}(\chi_k)\big). \tag{4.56}$$

Then the DAHA-Jones polynomial may be computed by

$$JD^R_{r,s}(\lambda; q, t) = \{R^{k,\lambda}_{r,s}\}_k = R^{k,\lambda}_{r,s}(q^{-\rho_k}). \tag{4.57}$$

Main result

First, we remark that the Macdonald polynomial $P^{(1)}_\lambda$ for $\lambda \in P_+$ is just the Weyl character of the corresponding finite-dimensional irreducible representation:

$$P^{(1)}_\lambda(X) = \frac{\sum_{w\in W}(-1)^{\mathrm{sgn}(w)}X_{w(\rho+\lambda)}}{\sum_{w\in W}(-1)^{\mathrm{sgn}(w)}X_{w(\rho)}}, \tag{4.58}$$

and the difference shift operator χ_1 is just the Weyl denominator:

$$\chi_1(X) = \sum_{w\in W}(-1)^{\mathrm{sgn}(w)}X_{w(\rho)}. \tag{4.59}$$

Also, the *r-Adams operation* is generally defined by the coefficients $c^\mu_{\lambda;\mathsf{r}} \in \mathbb{Z}$ in the expansion

$$P^{(1)}_\lambda(X^{\mathsf{r}}) = \sum_{\mu\in P(\lambda,\mathsf{r})} c^\mu_{\lambda;\mathsf{r}} P^{(1)}_\mu(X), \tag{4.60}$$

where $P(\lambda,\mathsf{r})$ is the set of highest weights of the irreducible summands in $V_\lambda^{\otimes\mathsf{r}}$. The existence and uniqueness of $c^\mu_{\lambda;\mathsf{r}}$ follow from the fact that the Weyl characters are a $\mathbb{Z}$-basis for the Weyl character ring. See Section 5.2 for a more detailed discussion of Adams operations.

Lemma 4.3.3. For $R \in \{A_n, D_n, E_6, E_7, E_8\}$ and any $b \in P_+$,

$$JD^R_{\mathsf{r},\mathsf{s}}(\lambda; q, t \mapsto q) = \frac{1}{\dim_q(V_\lambda)} \sum_\mu c^\mu_{\lambda;\mathsf{r}} q^{\frac{-(2\rho+\mu,\mu)\mathsf{s}}{2\mathsf{r}}} \dim_q(V_\mu). \tag{4.61}$$

Proof. Observe that $t \mapsto q$ corresponds to the case $k = 1$. Thus, by Proposition 4.3.2, we would like to evaluate

$$JD_{\mathsf{r},\mathsf{s}}(\lambda; q, t \mapsto q) = \{R^{1,\lambda}_{\mathsf{r},\mathsf{s}}\}_1 = R^{1,\lambda}_{\mathsf{r},\mathsf{s}}(q^{-\rho}), \text{ where} \tag{4.62}$$

$$R^{1,\lambda}_{\mathsf{r},\mathsf{s}} = \chi_1^{-1}\gamma^\circ((P^{(1)}_\lambda/P^{(1)}_\lambda(q^{-\rho_1})(\gamma^\circ)^{-1}(\chi_1))\ \text{, for}\quad \gamma = \left(\begin{smallmatrix}\mathsf{r} & u\\ \mathsf{s} & v\end{smallmatrix}\right) \in SL(2,\mathbb{Z}), \tag{4.63}$$

which we do in stages. First, using (4.55) and (4.59),

$$(\gamma^\circ)^{-1}(\chi_1) = \frac{1}{\mathsf{r}^{1/2}} q^{\frac{\rho^2 s - ux^2}{2\mathsf{r}}} \sum_{w\in W}(-1)^{\mathrm{sgn}(w)}X_{\frac{w(\rho)}{\mathsf{r}}}. \tag{4.64}$$

Observe that, upon sending $X_b \mapsto X_{\frac{b}{\mathsf{r}}} = X_b^{\frac{1}{\mathsf{r}}}$, (4.60) becomes

$$P^{(1)}_\lambda(X) = \sum_\mu c^\mu_{\lambda;\mathsf{r}} P^{(1)}_\mu(X^{\frac{1}{\mathsf{r}}}). \tag{4.65}$$

Then combining (4.64) and (4.65),

$$P_\lambda^{(1)}(\gamma^\circ)^{-1}(\chi_1) = \frac{1}{\mathsf{r}^{1/2}} q^{\frac{\rho^2 s - ux^2}{2\mathsf{r}}} \sum_\mu c^\mu_{\lambda;\mathsf{r}} \sum_{w\in W} (-1)^{\mathrm{sgn}(w)} X_{\frac{w(\rho+\mu)}{\mathsf{r}}}. \tag{4.66}$$

Applying (4.55) again,

$$\gamma^\circ(P_\lambda^{(1)}(\gamma^\circ)^{-1}(\chi_1)) = \sum_\mu c^\mu_{\lambda;\mathsf{r}} q^{\frac{-(2\rho+\mu,\mu)s}{2\mathsf{r}}} \sum_{w\in W} (-1)^{\mathrm{sgn}(w)} X_{w(\rho+\mu)}. \tag{4.67}$$

Finally, we have

$$\chi_1^{-1}\gamma^\circ((P_\lambda^{(1)}/P_\lambda^{(1)}(q^{-\rho_1})(\gamma^\circ)^{-1}(\chi_1)) = \frac{1}{P_\lambda^{(1)}(q^{-\rho_1})} \sum_\mu c^\mu_{\lambda;\mathsf{r}} q^{\frac{-(2\rho+\mu,\mu)s}{2\mathsf{r}}} P_\mu^{(1)}(X). \tag{4.68}$$

Evaluating at $q^{-\rho_1}$ and observing that $P_\mu^{(1)}(q^{-\rho_1}) = \dim_q(V_\mu)$ implies the result. □

In Theorem 5.1 of [LZ], a "cabling-projection rule" was used to produce a formula for the colored HOMFLY-PT polynomials of torus links. In Theorem 3.6 of [CC], the same technique was employed to produce a formula for the colored Kauffman polynomials of torus links. Both formulas are combinatorial and emphasize stabilization within the classical series of Lie algebras.

However, our interest in them here is for their use of the r-Adams operation (4.60). We will not need to use their combinatorial nature or the stabilization. Thus, we present a generalization of both formulas, which is not combinatorial and does not emphasize stabilization. Let $T^{\mathsf{r},\mathsf{s}}$ be a torus knot colored by V_λ for $\lambda \in P_+(\mathfrak{g})$, where $\mathfrak{g}$ is any *classical* Lie algebra (type ABCD). Then

$$P^{\mathfrak{g},V_\lambda}(T^{\mathsf{r},\mathsf{s}};q) = \theta_\lambda^{-\mathsf{rs}} \sum_{\mu\in P(\lambda,\mathsf{r})} c^\mu_{\lambda;\mathsf{r}} \theta_\mu^{\frac{\mathsf{s}}{\mathsf{r}}} \dim_q(V_\mu) \tag{4.69}$$

is the (unreduced) quantum $(\mathfrak{g}, V_\lambda)$ torus knot invariant. Recall that θ_μ is the twist (2.36), i.e., the scalar $q^{-\frac{(2\rho+\lambda,\lambda)}{2}}$ by which $K_{-2\rho}$ acts on V_μ [Dr2]. Now we are ready to prove the main result.

Theorem 4.3.4. *Let $\mathfrak{g}$ be a complex, simple Lie algebra of type A or D, and let R be the root system corresponding to $\mathfrak{g}$. Then*

$$JD^R_{r,s}(\lambda; q, t\mapsto q) = \frac{q^{\frac{(2\rho+\lambda,\lambda)rs}{2}}}{\dim_q(V_\lambda)} P^{\mathfrak{g},V_\lambda}(T^{r,s};q), \tag{4.70}$$

for any dominant weight $\lambda \in P_+$ and representation V_λ of $\mathfrak{g}$ with highest weight λ.

Proof. This follows immediately from Lemma 4.3.3, which applies to root systems of type ADE, and equation (4.69), which applies to Lie algebras of type ABCD. □

4.4 Relation to Knot Homologies

DAHA-superpolynomials

Here we restrict to type-A root systems and present the "three super-conjectures" from Section 2.2 of [C5], which are now theorems due to [C6, GN].

Theorem 4.4.1. *For any $n \geq m-1$, we may naturally interpret $\lambda \in P_+(A_m)$ as a weight for A_n.*

1. **(Stabilization)** *There exists a unique polynomial $HD_{r,s}(\lambda; q, t, a) \in \mathbb{Z}[q, t^{\pm 1}, a]$, defined by the (infinitely many) specializations*

$$HD_{r,s}(\lambda; q, t, a \mapsto -t^{n+1}) = \widetilde{JD}^{A_n}_{r,s}(\lambda; q, t), \text{ for } n \geq m-1. \tag{4.71}$$

We will call $HD_{r,s}(\lambda; q, t, a)$ the DAHA-superpolynomial.

2. **(Duality)** *Let $q^A t^B$ be the greatest q, t-monomial in $HD_{r,s}(\lambda; q, t, a)$ whose a-degree is 0. Then*

$$HD_{r,s}(\lambda^{tr}; q, t, a) = t^A q^B HD_{r,s}(\lambda; t^{-1}, q^{-1}, a), \tag{4.72}$$

where λ^{tr} indicates the transposed Young diagram for λ.

3. **(Evaluation)** *It immediately follows from (4.50) that*

$$HD_{r,s}\Big(\sum_{i=1}^{m} \lambda_i \omega_i; 1, t, a\Big) = \prod_{i=1}^{m} \Big(HD_{r,s}(\omega_i; 1, t, a)\Big)^{\lambda_i}. \tag{4.73}$$

When combined with the duality, this implies

$$HD_{r,s}\Big(\sum_{i=1}^{m} \lambda_i \omega_i; q, 1, a\Big) = \prod_{i=1}^{m} \Big(HD_{r,s}(\omega_i; q, 1, a)\Big)^{\lambda_i}. \tag{4.74}$$

Currently, the latter has no direct interpretation in terms of Macdonald polynomials or the DAHA-Jones construction.

We can generally make contact with the conventions used in the literature on superpolynomials, e.g., [DGR], by a transformation DAHA $\mapsto$ DGR:

$$t \mapsto q^2\,, \qquad q \mapsto q^2 t^2\,, \qquad a \mapsto a^2 t. \tag{4.75}$$

Then we have the following conjecture, which extends the conjecture (3.40) from [AS].

Conjecture 4.4.2. For a *rectangular* Young diagram $i \times j$, i.e., a weight $j\omega_i \in P_+$, the coefficients of $HD_{\mathrm{r,s}}(j\omega_i; q, t, a)$ are positive integers. In this case, upon the transformation (4.75), one recovers the superpolynomials (3.27) from [DGR, GS, GGS].

In light of Conjecture 4.4.2, one can attribute the duality to the "mirror symmetry" and the evaluation to the "refined exponential growth" of [GS, GGS].

DAHA-Jones and refined Chern-Simons invariants

In Lemma 2.8 of [GN] the authors demonstrate that for any $H \in \mathcal{SH}$,

$$\tilde{S} H \tilde{S}^{-1} = \sigma(H) \quad \text{and} \quad \tilde{T} H \tilde{T}^{-1} = \tau(H), \tag{4.76}$$

where $\tilde{S}, \tilde{T}$ are from (3.36) and σ, τ from (2.11) act by extending the $PSL_2^\wedge(\mathbb{Z})$-action from (4.36) to a $SL_2(\mathbb{Z})$-action. Therefore, if $\tilde{K}$ is a lift of $\gamma_{\mathsf{r},\mathsf{s}} = \left(\begin{smallmatrix} \mathsf{r} & * \\ \mathsf{s} & * \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ to the refined Hilbert space $\tilde{H}_{T^2}$,

$$\frac{1}{\tilde{S}_{00}} \widetilde{\mathbf{W}}_\lambda^{\mathsf{r},\mathsf{s}} = \tilde{K} P_\lambda^\circ \tilde{K}^{-1} = \hat{\gamma}_{\mathsf{r},\mathsf{s}}(P_\lambda^\circ). \tag{4.77}$$

Furthermore, in Corollary 2.9 they establish that (in the case R is of type A)

$$P_{[\mathrm{AS}]}^{\mathsf{r},\mathsf{s},n}(\lambda; q, t) = \frac{1}{\tilde{S}_{00}} \left\langle \rho \left| \widetilde{\mathbf{W}}_\lambda^{\mathsf{r},\mathsf{s}} \tilde{S} \right| \rho \right\rangle = \{\hat{\gamma}_{\mathsf{r},\mathsf{s}}(P_\lambda^\circ)\}_{ev} = \widetilde{JD}_{\mathsf{r},\mathsf{s}}^{A_{n-1}}(\lambda; q, t). \tag{4.78}$$

Therefore, the DAHA-Jones polynomials are a proper (formal) generalization—to any root system and weight—of the refined torus knot invariants of [AS].

Chapter 5

Refined Composite Invariants

Here we recount the results of [CE], where we introduced and studied the *composite DAHA-superpolynomials* for torus knots and arbitrary *composite weights* [K], i.e., pairs $[\lambda, \mu]$ of Young diagrams. They depend on a, q, t and unify the corresponding n-series of (refined) DAHA-Jones q, t-polynomials of type A_n; all symmetries of superpolynomials from Theorem 4.4.1 hold for them.

When $t \mapsto q$ and $a \mapsto -a$, we establish their relation to the *composite HOMFLY-PT polynomials*, which may be computed using the procedure developed in Proposition 5.2.1. Topologically, these are based on the *full HOMFLY-PT skein* of the annulus [HM], an algebra generated by link diagrams drawn there. The physical significance is discussed in [GJKS, Ma3, PBR].

The simplest composite weight $[\square, \square]$ corresponds to the adjoint representation, which connects these results with two examples of adjoint DAHA-hyperpolynomials for the Deligne-Gross exceptional series of root systems considered in Section 6.4.

5.1 Composite representations

An irreducible, finite-dimensional $\mathfrak{sl}_N(\mathbb{C})$-module V is uniquely specified by its *highest weight*:

$$b = \sum_{i=1}^{N-1} b_i \omega_i \in P_+ := \bigoplus_{i=1}^{N-1} \mathbb{Z}_+ \omega_i, \quad \mathbb{Z}_+ := \mathbb{Z}_{\geq 0}, \tag{5.1}$$

where $\{\omega_i\}$ are fundamental, dominant weights for A_{N-1}.

Equivalently, we may encode b (and V) in a *partition* or its corresponding *Young diagram* $\lambda = \lambda_1 \geq \lambda_2 \ldots \lambda_{N-1} \geq \lambda_N = 0$ with at most $N-1$ nonempty rows and k_{th} row of length $\lambda_k := b_k + \cdots + b_{N-1}$. The highest weight b is recovered from λ by taking $b_i = \lambda_i - \lambda_{i+1}$, i.e., b_i is the number of columns of λ of height i.

The dual representation V^* has highest weight $b^* := \iota(b)$, where $\iota : \omega_i \mapsto \omega_{N-i}$. Alternatively, the Young diagram λ^* has rows of length $\lambda^*_k = \lambda_1 - \lambda_{N+1-k}$ (this operation depends on N).

A weight $b \in P_+$ for $\mathfrak{sl}_N(\mathbb{C})$ is interpreted for $\mathfrak{sl}_M(\mathbb{C})$ by setting $b_i = 0$ for $i \geq \min\{M, N\}$.

Accordingly, we may interpret the corresponding Young diagram λ as a dominant weight for $\mathfrak{sl}_M(\mathbb{C})$ by removing any columns of height $\geq M$. It is precisely this sort of "packaging" of representations for all ranks that leads to the HOMFLY-PT polynomial and its generalizations.

One can generalize this procedure to any number of Young diagrams by "placing" them in the Dynkin diagram of type A_{N-1} with breaks in between. The *composite representations* are labeled by pairs of partitions (or Young diagrams) "placed" at the ends of the Dynkin diagram. Namely, for Young diagrams λ and μ with $\ell(\lambda)$ and $\ell(\mu)$ rows, $N \geq \ell(\lambda) + \ell(\mu)$ (always assumed), and P_+ of type A_{N-1}, let

$$[\lambda, \mu]_N = b^* + c \in P_+ = P_+^{A_{N-1}}, \text{ for } \; b, c \text{ associated with } \lambda, \mu. \tag{5.2}$$

We call the pair $[\lambda, \mu]$ a *composite diagram/partition* and will constantly identify dominant weights $[\lambda, \mu]_N$ and the corresponding Young diagrams (with no greater than $N - 1$ rows).

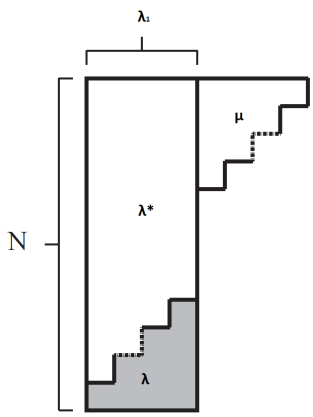


Figure 5.1: The Young diagram $[\lambda, \mu]_N$

Schur functions

In what follows, we will require some basic facts about Schur functions and their generalization to composite representations in [K].

Let $\Lambda_n := \mathbb{Z}[x_1, \ldots, x_n]^{S_n}$ denote the *ring of symmetric functions in n variables*, where the action of S_n is permutation of the variables (indices). For any $m \geq n$, the map which sends $x_i \mapsto 0$ for $i > n$, and $x_i \mapsto x_i$ otherwise, is the restriction homomorphism $\Lambda_m \to \Lambda_n$. Then the *ring of symmetric functions* is

$$\Lambda_x := \varprojlim_n \Lambda_n, \tag{5.3}$$

where the projective limit is taken with respect to the restriction homomorphisms.

If λ is a partition with length at most n, one can define the corresponding *Schur function* $s_\lambda(x_1, \ldots, x_n) \in \Lambda_n$. The set of Schur functions for all such partitions is a $\mathbb{Z}$-basis for Λ_n. We may naturally interpret a given $s_\lambda(x_1,\dots,x_n)$ as having infinitely-many variables, for which we write $s_\lambda(\vec{x}) \in \Lambda_x$. The set of all $s_\lambda(\vec{x})$ is a $\mathbb{Z}$-basis for Λ_x.

The Schur functions satisfy many interesting properties. For our purposes, we will interpret $s_\lambda(\vec{x}) \in \Lambda_x$ as a character for the irreducible polynomial representation V_λ. Consequently, the *Littlewood-Richardson rule*, that is,

$$s_\lambda(\vec{x})s_\mu(\vec{x}) = \sum_\nu N^\nu_{\lambda,\mu} s_\nu(\vec{x}), \tag{5.4}$$

shows that the multiplicity of an irreducible summand V_ν in the tensor product decomposition of $V_\lambda \otimes V_\mu$ is equal to the *Littlewood-Richardson coefficient* $N^\nu_{\lambda,\mu}$.

The composite case

In [K], the author introduces $s_{[\lambda,\mu]}(\vec{x},\vec{y}) \in \Lambda_x \otimes \Lambda_y$, which generalize the Schur functions and provide characters for irreducible representations $V_{[\lambda,\mu]}$ corresponding to composite partitions. Their natural projection onto the character ring for $\mathfrak{sl}_N$ is the (ordinary) Schur function $s_{[\lambda,\mu]_N}(x_1,\dots,x_{N-1}) \in \Lambda_{N-1}$. Recall that we always assume that $N \geq \ell(\lambda)+\ell(\mu)$ for the length $\ell(\lambda)$ of λ; see (5.2).

The following formulas, proved in [K], will be used as definitions:

$$s_{[\lambda,\mu]}(\vec{x},\vec{y}) := \sum_{\tau,\nu,\xi} (-1)^{|\tau|} N^\lambda_{\nu,\tau} N^\mu_{\tau,\xi} s_\nu(\vec{x}) s_\xi(\vec{y}), \tag{5.5}$$

$$\text{where } \; s_\eta(\vec{x})s_\delta(\vec{y}) = \sum_{\alpha,\beta,\delta} N^\eta_{\beta,\alpha} N^\delta_{\gamma,\alpha} s_{[\beta,\gamma]}(\vec{x},\vec{y}); \tag{5.6}$$

the sums here are over arbitrary triples of Young diagrams.

5.2 Composite HOMFLY-PT polynomials

Skein theory in the annulus

The *colored HOMFLY-PT polynomial* for a knot K and a partition λ is the integer Laurent polynomial $\mathbf{P}_\lambda(K;q,a) \in \mathbb{Z}[q^{\pm 1}, a^{\pm 1}]$ satisfying $\mathbf{P}_\lambda(K;q;q^N) = P^{\mathfrak{sl}_N,\lambda}(K;q)$ to the corresponding quantum knot invariant for $\mathfrak{sl}_N$ and partition (dominant weight) λ.

The *composite HOMFLY-PT polynomial* for $[\lambda,\mu]$ is defined similarly via the specializations $\mathbf{P}_{[\lambda,\mu]}(K;q,q^N) = P^{\mathfrak{sl}_N,[\lambda,\mu]_N}(K;q)$ for all sufficiently large N. In particular, $\mathbf{P}_{[\varnothing,\mu]}(K) = \mathbf{P}_\mu(K)$. Recall that the composite diagram $[\lambda,\mu]_N$ is from (5.2).

The HOMFLY-PT polynomial has two normalizations. For connection with DAHA, as in Theorem 5.3.4, we will be interested in the *normalized* polynomial $\mathbf{P}$. However, for many of our intermediate calculations, we will also need the *unnormalized* HOMFLY-PT polynomial $\bar{\mathbf{P}}$. These are generally defined and related by:

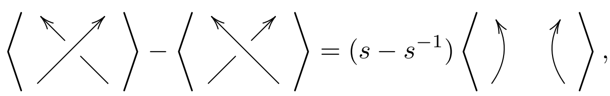
$$\bar{\mathbf{P}}(K) = \bar{\mathbf{P}}(U)\mathbf{P}(K) \ , \ \bar{\mathbf{P}}(U) = \dim_{q,a}(V), \tag{5.7}$$

where K is any knot, U is the unknot, and $\dim_{q,a}$ is defined in Section 5.2 for $V = V_{[\lambda,\mu]}$. Observe that with this definition, $\mathbf{P}(U) = 1$. In the specializations described earlier in this section, the normalized (resp. unnormalized) HOMFLY-PT polynomials coincide with the reduced (resp. unreduced) quantum knot invariants.

We will briefly recall the approach to composite HOMFLY-PT polynomials from [HM]. The *full HOMFLY-PT skein algebra* $\mathcal{C}$ is a commutative algebra over the coefficient ring $\Upsilon = \mathbb{Z}[v^{\pm 1}, s^{\pm 1}](\{s^k - s^{-k}\}_{k\geq 1})^{-1}$. It consists of Υ-linear combinations of oriented link diagrams in $\mathbf{S}^1 \times I$.

The *product* of two diagrams in $\mathcal{C}$ is the diagram obtained by identifying the outer circle of one annulus with the inner circle of the other; the identity with respect to this product is the empty diagram (with coefficient 1).

The relations in $\mathcal{C}$ are the (framed) HOMFLY-PT skein relation



(5.8)

together with the relation that accompanies a type-I Reidemeister move on a positively (resp. negatively) oriented loop with multiplication by a factor of v^{-1} (resp. v). As a consequence, observe

$$\left\langle K \sqcup \circlearrowleft \right\rangle = \left(\frac{v^{-1} - v}{s - s^{-1}}\right) \langle K \rangle \, . \tag{5.9}$$

Furthermore, for a given diagram $D = D(K)$ of a knot K,

$$\langle D \rangle = a^{\frac{1}{2}\mathrm{wr}(D)} \bar{\mathbf{P}}(K; q, a) \ \text{ under } s \mapsto q^{\frac{1}{2}}, v \mapsto a^{-\frac{1}{2}}, \tag{5.10}$$

tying the variables s, v used in [HM] to the variables q, a used elsewhere in this thesis; $\mathrm{wr}(D)$ is the *writhe* of D (see there).

The meridian maps

Let $\varphi : \mathcal{C} \to \mathcal{C}$ be the *meridian map* induced by adding a single oriented, unknotted meridian to any diagram in $\mathbf{S}^1 \times I$ and extending linearly to $\mathcal{C}$. Let $\bar{\varphi}$ be the map induced by adding a meridian with an orientation opposite that of φ. Then, $\varphi, \bar{\varphi}$ are diagonal in their common eigenbasis $\{Q_{\lambda,\mu}\} \subset \mathcal{C}$ indexed by pairs λ, μ of partitions.

The subalgebras of $\mathcal{C}$ spanned by $\{Q_{\lambda,\varnothing}\}$ and $\{Q_{\varnothing,\mu}\}$ are each isomorphic to the ring of symmetric functions in infinitely many variables. Under these isomorphisms, these bases are identified with the basis of Schur polynomials. Accordingly, the full basis $\{Q_{\lambda,\mu}\}$ is the skein-theoretic analog of the characters for composite partitions in [K] that we discussed in Section 5.1.

Now to a diagram D of a knot K and a composite partition $[\lambda,\mu]$, associate the satellite link $D \star Q_{\lambda,\mu}$, whose companion is D and whose pattern is $Q_{\lambda,\mu}$. We then have that

$$\bar{\mathbf{P}}_{[\lambda,\mu]}(K) = v^{\mathrm{wr}(D)}\langle D \star Q_{\lambda,\mu}\rangle, \ \mathrm{wr}(D) = \text{writhe of } D, \tag{5.11}$$

i.e., the corresponding composite, unnormalized HOMFLY-PT polynomial for K is equal to the framed, uncolored HOMFLY-PT polynomial for $D \star Q_{\lambda,\mu}$.

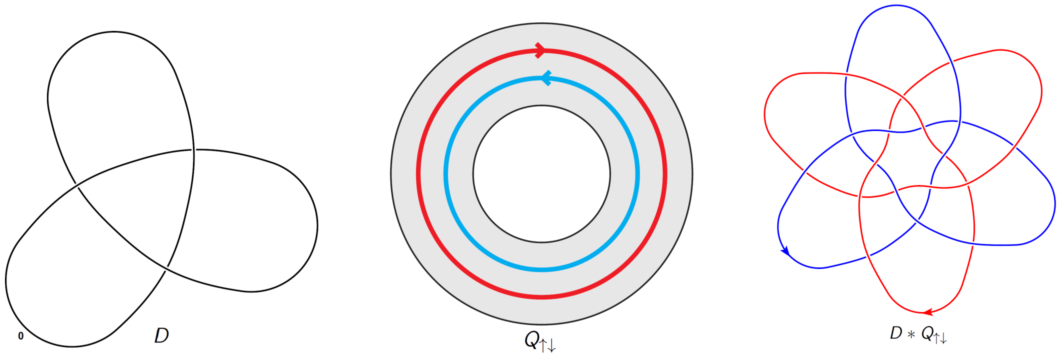


Figure 5.2: 0-framed trefoil cabled by the reverse parallel $Q_{\uparrow\downarrow}$ to form the satellite link $D * Q_{\uparrow\downarrow}$.

The pattern $Q_{\lambda,\mu}$ can be computed explicitly as the determinant of a matrix whose entries are certain idempotents $\{h_i, h_i^*\} \subset \mathcal{C}$. For the convenience of the reader, some patterns for $[\lambda,\mu]$ considered in Section 5.4 are included in the table below.

$[\lambda,\mu]$	$Q_{\lambda,\mu}$
$[\square,\square]$	$h_1h_1^* - 1$
$[\square,\begin{smallmatrix}\square\\\square\end{smallmatrix}]$	$h_1h_1^*h_1^* - h_1h_2^* - h_1^*$
$[\square\square,\square]$	$h_2h_1^* - h_1$
$[\square,\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}]$	$h_1h_1^*h_1^*h_1^* + h_1h_3^* + h_2^* - h_1h_1^*h_1^* - h_1h_1^*h_2^* - h_1^*h_1^*$
$[\begin{smallmatrix}\square\square\\\square\end{smallmatrix},\square]$	$h_1h_2h_1^* - h_1h_1 - h_3h_1^*$

(5.12)

The idempotents h_i are closures of linear combinations of upward-oriented braids $b_i \in \Upsilon[B_i]$:

$$b_1 = 1 = \uparrow \in \Upsilon[B_1], \ \ b_2 = \frac{1}{s[2]}(1 + s\sigma_1) \in \Upsilon[B_2], \tag{5.13}$$

$$b_3 = \frac{1}{s^3[2][3]}(1 + s\sigma_1)(1 + s\sigma_2 + s^2\sigma_2\sigma_1) \in \Upsilon[B_3], \tag{5.14}$$

in the annulus by homotopically nontrivial, counterclockwise-oriented strands. Here B_i is the ordinary braid group on i strands, and the quantum integers are denoted by $[k] := \frac{s^k - s^{-k}}{s - s^{-1}}$ (only in this section). The elements h_i^* are then obtained by rotating the diagrams for h_i about their horizontal axes. That is, h_i^* are linear combinations of closures of downward-oriented braids by clockwise-oriented strands.

In fact, the pattern $Q_{\lambda,\mu}$ for a *composite* partition $[\lambda,\mu]$ is distinguished by the fact that, in general, it contains strands oriented in both directions (clockwise and counterclockwise) around $\mathbf{S}^1 \times I$. On the other hand, the pattern $Q_\lambda = Q_{[\lambda,\varnothing]}$ for an ordinary partition will consist in strands oriented all in the same direction.

Let $K_{[\lambda,\mu]} \coloneqq \frac{\langle K \star Q_{[\lambda,\mu]}\rangle}{\langle Q_{[\lambda,\mu]}\rangle}$, which is well-defined on diagrams for K up to a framing coefficient, i.e., power of v. In [HM] the authors compute

$$K_{[\square,\square]}(z,v) = v^2 - 4v^4 + 4v^6 + z^2(1+2v^2-7v^4+4v^6) + z^4(v^2-2v^4+v^6), \tag{5.15}$$

for $K = T^{3,2}$ in terms of variables v and $z \coloneqq s - s^{-1}$. The relation to a, q that we use elsewhere is $v = a^{-\frac{1}{2}}$ and $z = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$; see below.

Composite Rosso-Jones formula

The usual theory

The *Rosso-Jones formula* [RJ] and its variants, e.g., [GMV, LZ, St, MM], expand the HOMFLY-PT polynomial for the (r,s)-torus knot and a partition $\lambda \vdash n$ in terms of the quantum dimensions of certain irreducible representations:

$$\theta_\lambda^{\mathsf{rs}} \bar{\mathbf{P}}_\lambda(T^{\mathsf{r},\mathsf{s}}) = \sum_{\mu \vdash \mathsf{r}n} c^{\mu}_{\lambda;\mathsf{r}} \theta_\mu^{\frac{\mathsf{s}}{\mathsf{r}}} \dim_{q,a}(V_\mu). \tag{5.16}$$

The formulas for $\theta_\lambda, \theta_\mu$ and the coefficients $c^{\mu}_{\lambda;\mathsf{r}}$ are provided below in (5.19), (5.24); $c^{\mu}_{\lambda;\mathsf{r}}$ is nonzero only if V_μ is an irreducible summand of $V_\lambda^{\otimes \mathsf{r}}$. Here $\theta_\lambda^{\mathsf{rs}}, \theta_\mu^{\frac{\mathsf{s}}{\mathsf{r}}}$ are powers, fractional for the latter. Note that (5.16) gives the *unnormalized* polynomial as defined in (5.7).

The composite theory

We are going to generalize the Rosso-Jones formula to the case of composite partitions $[\lambda,\mu]$. The stabilization of the corresponding expansion is not *a priori* clear. We will use the results of [K] described in Section 5.1. The following proposition matches formula (C.6) from [GJKS], independently obtained in the context of topological strings.

Proposition 5.2.1. *For any torus knot $T^{r,s}$ and composite partition $[\lambda,\mu]$ the corresponding (unnormalized) HOMFLY-PT polynomial admits an expansion:*

$$\theta_{[\lambda,\mu]}^{rs} \bar{\mathbf{P}}_{[\lambda,\mu]}(T^{r,s}) = \sum_{[\beta,\gamma]} c^{[\beta,\gamma]}_{[\lambda,\mu];r} \theta_{[\beta,\gamma]}^{\frac{s}{r}} dim_{q,a}(V_{[\beta,\gamma]}), \tag{5.17}$$

into finitely many terms for which the $c^{[\beta,\gamma]}_{[\lambda,\mu];r}$ are nonzero. Here $\theta_{[\lambda,\mu]}$, $\theta_{[\beta,\gamma]}$, and the coefficients $c^{[\beta,\gamma]}_{[\lambda,\mu];r}$ are provided in (5.22) and (5.27).

Proof. First of all, it is clear from (5.27) that $c_{[\lambda,\mu];\mathsf{r}}^{[\beta,\gamma]}$ is nonzero for only finitely many $[\beta,\gamma]$. Then, by construction, the resulting expansion (5.17) will satisfy the (infinitely many) specializations:

$$\mathbf{P}_{[\lambda,\mu]}(T^{\mathsf{r},\mathsf{s}};q,q^N) = \mathbf{P}_{[\lambda,\mu]_N}(T^{\mathsf{r},\mathsf{s}};q,q^N) = P^{\mathfrak{sl}_N,[\lambda,\mu]_N}(T^{\mathsf{r},\mathsf{s}};q), \tag{5.18}$$

which (uniquely) define the corresponding composite HOMFLY-PT polynomial.

We will divide the proof of (5.17) into several intermediate steps. In what follows, any occurrences of q^N will be replaced by a; all fractional exponents of N will cancel in the final formula.

Braiding eigenvalues

The constants $\theta_\lambda \in \mathbb{Z}[q^{\pm 1}, a^{\pm 1}]$ in (5.16) are *braiding eigenvalues* and correspond to the "twist" from (2.36) for $\mathfrak{sl}_N$. Explicitly, as computed in [AM], they are

$$\theta_\lambda = q^{-(\kappa_\lambda + nN - \frac{n^2}{N})/2} \text{ for } \kappa_\lambda := \sum_{x\in\lambda} 2c(x), \tag{5.19}$$

where the *content* of the box $x \in \lambda$ in the i_{th} row and j_{th} column is $c(x) := j - i$.

Now, for a composite partition $[\lambda,\mu]$ such that $\lambda \vdash m$ and $\mu \vdash n$, observe that $[\lambda,\mu]_N \vdash c := (n - m + \lambda_1 N)$. We would like to construct a $\kappa_{[\lambda,\mu]}$ such that

$$\kappa_{[\lambda,\mu]}|_{N=k} = \kappa_{[\lambda,\mu]_k}, \text{ for any } k. \tag{5.20}$$

To this end, we divide the Young diagram for $[\lambda,\mu]_N$ into two natural parts and count their individual contributions to $\kappa_{[\lambda,\mu]_N}$. Namely,

1. μ contributes $\kappa_\mu + 2\lambda_1|\mu|$ to $\kappa_{[\lambda,\mu]_N}$ and
2. λ^* contributes $\kappa_{\lambda^*} = \kappa_\lambda + N\lambda_1(\lambda_1+1) - \lambda_1 N(N+1) - 2|\lambda|(\lambda_1 - N)$.

Thus, we can set

$$\kappa_{[\lambda,\mu]} := \kappa_\lambda + \kappa_\mu + N\lambda_1(\lambda_1+1) - \lambda_1 N(N+1) + 2\lambda_1|\mu| - 2|\lambda|(\lambda_1 - N), \tag{5.21}$$

which satisfies (5.20), as desired. Furthermore we define the composite braiding eigenvalues:

$$\theta_{[\lambda,\mu]} := q^{-(\kappa_{[\lambda,\mu]} + cN - \frac{c^2}{N})/2}. \tag{5.22}$$

One has that $\theta_{[\lambda,\mu]} \overset{a\mapsto q^N}{=\!=} \theta_{[\lambda,\mu]_N}$ by construction.

The following is the key part of the proof of Proposition 5.2.1.

Adams operation

We will use Section 5.1, where we explained that the Schur functions $s_\lambda(\vec{x}) \in \Lambda_x$ are characters for the irreducible polynomial representations V_λ and described some of their properties. For applications to the Rosso-Jones formula we need to understand the r-*Adams operation* ψ_r on s_λ; see [GMV, MM].

Let $p_\mathsf{r} := \sum_i x_i^\mathsf{r} \in \Lambda_x$ be the *degree-*r *power sum symmetric function.* Then the r-Adams operation on s_λ may be defined formally by the plethysm $\psi_\mathsf{r}(s_\lambda) := p_\mathsf{r} \circ s_\lambda$. This means that $\psi_\mathsf{r}(s_\lambda)$ is determined by the coefficients $c^\nu_{\lambda;r} \in \mathbb{Z}$ in the expansion

$$s_\lambda(\vec{x}^\mathsf{r}) = \sum_\nu c^\nu_{\lambda;r} s_\nu(\vec{x}), \tag{5.23}$$

where $\vec{x}^\mathsf{r} := (x_1^\mathsf{r}, x_2^\mathsf{r}, x_3^\mathsf{r}, \ldots)$. The coefficients here are given an explicit description in [LZ]:

$$c^\nu_{\lambda;r} = \sum_\mu \frac{|C_\mu| \chi^\lambda(C_\mu) \chi^\nu(C_{r\mu})}{|\mu|}, \tag{5.24}$$

where χ^λ is the character of the symmetric group corresponding to λ, and C_μ is the conjugacy class corresponding to μ.

We need an analog of ψ_r for composite partitions $[\lambda, \mu]$, which must agree with the ordinary Adams operation upon specification of N. Thus, we need to switch from (5.23) to the expansion

$$s_{[\lambda,\mu]}(\vec{x}^r, \vec{y}^r) = \sum_\nu c^{[\beta,\gamma]}_{[\lambda,\mu];r} s_{[\beta,\gamma]}(\vec{x}, \vec{y}), \tag{5.25}$$

where $s_{[\lambda,\mu]}(\vec{x}, \vec{y}) \in \Lambda_x \otimes \Lambda_y$ is the universal character of [K], described in Section 5.1. Applying here the natural projection onto Λ_{N-1}, one recovers the following specialization of (5.23):

$$s_{[\lambda,\mu]_N}(x_1^r, \ldots, x_{N-1}^r) = \sum_\nu c^{[\beta,\gamma]_N}_{[\lambda,\mu]_N;r} s_{[\beta,\gamma]_N}(x_1, \ldots, x_{N-1}). \tag{5.26}$$

This demonstrates that $c^{[\beta,\gamma]}_{[\lambda,\mu];r}$ from (5.25) are exactly what we need, i.e., this formula agrees with (5.23) upon specification of N and therefore can be used for the proof of Proposition 5.2.1.

Now using (5.5), (5.6), and (5.23) we obtain an explicit expression for these coefficients:

$$c^{[\beta,\gamma]}_{[\lambda,\mu];r} = \sum_{\tau,\nu,\xi,\eta,\delta,\alpha} (-1)^{|\tau|} N^\lambda_{\nu,\tau} N^\mu_{\tau,\xi} c^\eta_{\nu;r} c^\delta_{\xi;r} N^\eta_{\beta,\alpha} N^\delta_{\gamma,\alpha}, \tag{5.27}$$

where the sum is over arbitrary sextuples of Young diagrams. Recall that $N^\lambda_{\nu,\tau}$, are the Littlewood-Richardson coefficients from (5.4).

Although this formula appears rather complicated, observe that the terms are only nonzero for relatively few (and finitely many) choices of $(\tau, \nu, \xi, \eta, \delta, \alpha)$. In light of (5.24) and the combinatorial nature of the Littlewood-Richardson rule, these formula provides a completely combinatorial description of $c^{[\beta,\gamma]}_{[\lambda,\mu];r}$. The following is the last step of the proof.

Quantum dimensions

We define the *q, a-integer* by

$$[uN+v]_{q,a} := \frac{a^{\frac{u}{2}}q^{\frac{v}{2}} - a^{-\frac{u}{2}}q^{-\frac{v}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}, \tag{5.28}$$

for $u, v \in \mathbb{Z}$, where N is "generic," i.e., it is treated here as a formal variable. Setting here $a = q^N$ for $N \in \mathbb{N}$, we obtain the ordinary quantum integer $[uN+v]_q$. We will suppress the subscript "q, a" in this and the next subsection, simply writing $[\,\cdot\,]$.

For an irreducible representation V_μ, its *stable quantum dimension* is given by the quantum Weyl dimension formula

$$\dim_{q,a}(V_\mu) = \prod_{\alpha \in A^+_{N-1}} \frac{[(\mu+\rho,\alpha)]}{[(\rho,\alpha)]}, \tag{5.29}$$

where the Young diagram μ is interpreted in the usual way as a weight for $\mathfrak{sl}_N$ for generic N and $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$ for A_{N-1}.

Then it only depends on the diagram μ, which includes the actual number of factors due to the cancelations. We note that such a stabilization holds in the theory of Macdonald polynomials of type A_{N-1} as well.

The stable quantum dimension for a composite partition $[\beta, \gamma]$ is defined as follows:

$$\dim_{q,a}(V_{[\beta,\gamma]}) := \prod_{\alpha \in A^+_{N-1}} \frac{[([\beta,\gamma]_N+\rho,\alpha)]}{[(\rho,\alpha)]}. \tag{5.30}$$

Similarly to (5.29), we claim that there is no actual dependence of N in this formula (including the actual number of factors). However, the justification is somewhat more involved because the weight,

$$[\beta,\gamma]_N = \sum_{j=1}^{\ell(\gamma)}(\gamma_i - \gamma_{i+1})\omega_i + \sum_{j=1}^{\ell(\beta)}(\beta_j - \beta_{j+1})\omega_{N-j}, \tag{5.31}$$

depends on N (in contrast to the case of one diagram). We will omit a straightforward justification; see table (5.35) below and the general formula (C.3) from [GJKS] (a calculation of normalized open-string stretched annulus amplitudes). Finally, the relation $\dim_{q,a}(V_{[\beta,\gamma]})|_{a\mapsto q^N} = \dim_q(V_{[\beta,\gamma]_N})$ concludes the proof of Proposition 5.2.1. □

Formula (5.17) provides a purely combinatorial and computationally effective way of producing HOMFLY-PT polynomials for arbitrary torus knots and composite representations. See examples in Section 5.4 below and also Section C from [GJKS].

Simplest examples

First, we evaluate the (ordinary) Rosso-Jones formula (5.16) for the trefoil $T^{3,2}$ and $\lambda = \square$. The necessary values are contained in table (5.32):

μ	θ_μ	$c^\mu_{\square;2}$	$\dim_{q,a}(V_\mu)$
$\square$	$a^{-\frac{1}{2}}q^{\frac{1}{2N}}$	0	$[N]$
$\square\square$	$a^{-1}q^{\frac{2}{N}-1}$	1	$\frac{[N][N+1]}{[2]}$
$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$a^{-1}q^{\frac{2}{N}+1}$	-1	$\frac{[N-1][N]}{[2]}$

(5.32)

Inserting the components of (5.32) into formula (5.16), we obtain the familiar expression:

$$\mathbf{P}_\square(T^{3,2};q,a) = \frac{\theta_\square^{-6}(\theta_{\square\square}^{\frac{3}{2}}\dim_{q,a}(V_{\square\square}) - \theta_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}^{\frac{3}{2}}\dim_{q,a}(V_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}))}{\dim_{q,a}(V_\square)} \tag{5.33}$$

$$= aq^{-1} - a^2 + aq, \tag{5.34}$$

the normalized HOMFLY-PT polynomial of $T^{3,2}$. Note that although $\square$ appears with coefficient 0 in the expansion (5.16), we include it in table (5.32) since both $\theta_\square$ and $\dim_{q,a}(V_\square)$ are needed to give the final, normalized polynomial, as defined in (5.7).

Similarly, we evaluate our composite Rosso-Jones formula (5.17) for the trefoil $T^{3,2}$ and $[\square,\square]$:

$[\beta,\gamma]$	$\theta_{[\beta,\gamma]}$	$c^{[\beta,\gamma]}_{[\square,\square];2}$	$\dim_{q,a}(V_{[\beta,\gamma]})$
$[\square,\square]$	a^{-1}	0	$[N-1][N+1]$
$[\square\square,\square\square]$	$q^{-2}a^{-2}$	1	$\frac{[N-1][N]^2[N+3]}{[2][2]}$
$[\square\square,\begin{smallmatrix}\square\\\square\end{smallmatrix}]$	a^{-2}	-1	$\frac{[N-2][N-1][N+1][N+2]}{[2][2]}$
$[\begin{smallmatrix}\square\\\square\end{smallmatrix},\square\square]$	a^{-2}	-1	$\frac{[N-2][N-1][N+1][N+2]}{[2][2]}$
$[\begin{smallmatrix}\square\\\square\end{smallmatrix},\begin{smallmatrix}\square\\\square\end{smallmatrix}]$	q^2a^{-2}	1	$\frac{[N-3][N]^2[N+1]}{[2][2]}$
$[\varnothing,\varnothing]$	1	1	1

(5.35)

Inserting the components of (5.35) into formula (5.17), we obtain

$$\mathbf{P}_{[\square,\square]}(T^{3,2};q,a) = a^2(q^{-2}+q^2+2) + a^3(-2q^{-2}+q^{-1}+q-2q^2-2) \\ + a^4(q^{-2}-2q^{-1}-2q+q^2+3) + a^5(q^{-1}+q-2), \tag{5.36}$$

where we include $[\square,\square]$ in table (5.35) for the same reason that we included $\square$ in table (5.32).

Observe that we can touch base with formula (5.15) from [HM] by

$$a^5 T^{3,2}_{[\square,\square]}(q^{\frac{1}{2}} - q^{-\frac{1}{2}}, a^{-\frac{1}{2}}) = \mathbf{P}_{[\square,\square]}(T^{3,2};q,a). \tag{5.37}$$

Our expression for $\mathbf{P}_{[\square,\square]}(T^{3,2};q,a)$ agrees with that obtained in [PBR]. See also examples (C.8-16) from [GJKS], obtained there via Chern-Simons theory (open-string amplitudes); they match ours.

5.3 Refined Composite Invariants via DAHA

Here we recount the results of Section 2.3 of [CE]. These establish the existence and properties of the composite DAHA-superpolynomials, as well as their relationship to HOMFLY-PT polynomials.

Composite DAHA-superpolynomials

The following is Theorem 2.3-2.4 of [CE]. It generalizes Theorem 4.4.1 above to the composite case.

Theorem 5.3.1. *([CE]) We switch to from $\mathfrak{sl}_N(\mathbb{C})$ to $A_n := A_{N-1}$ and set $t = q^k$. Let λ, μ be the Young diagrams corresponding to $b, c \in P_+^n := P_+^{A_n}$ (with no greater than n rows). Recall that $[\lambda, \mu]_N \in P_+^{N-1}$ is $b^* + c$, where $N \geq \ell(\lambda) + \ell(\mu)$ and $(\omega_i)^* = \omega_{N-i}$; see (5.2).*

1. **(Stabilization)** *There exists a unique polynomial $HD_{r,s}([\lambda, \mu]\,;\, q, t, a) \in \mathbb{Z}[q, t^{\pm 1}, a]$ such that*

$$HD_{r,s}([\lambda,\mu]\,;q,t,a\mapsto -t^N) = \widetilde{JD}_{r,s}^{A_{N-1}}(b^*+c\,;q,t) \text{ for any } N>n. \tag{5.38}$$

   This polynomial does not depend on the ordering of λ, μ.

2. **(Duality)** *Up to a power of q and t,*

$$HD_{r,s}([\lambda,\mu]\,;q,t,a) = q^{\bullet}t^{\bullet} HD_{r,s}([\lambda^{tr},\mu^{tr}]\,;t^{-1},q^{-1},a), \tag{5.39}$$

   where, e.g., λ^{tr} is the transposed Young diagram.

3. **(Evaluation)** *Setting $HD_{r,s}(\lambda) := HD_{r,s}([\varnothing, \lambda])$ and $q \mapsto 1$,*

$$\begin{aligned} HD_{r,s}([\lambda,\mu]\,;\,1,t,a) &= HD_{r,s}(\lambda\,;\,1,t,a)\,HD_{r,s}(\mu\,;\,1,t,a), \text{ where} \\ HD_{r,s}(\lambda\,;\,1,t,a) &= \prod_{i=1}^{n} HD_{r,s}(\omega_i\,;\,1,t,a)^{b_i} \quad \text{for } b=\sum_{i=1}^{n} b_i\omega_i, \end{aligned} \tag{5.40}$$

   b corresponds to λ and ω_i means the column with i boxes. We also have for $t \mapsto 1$

$$HD_{r,s}([\lambda,\mu]\,;\,q,1,a) = HD_{r,s}(\lambda\,;\,q,1,a)\,HD_{r,s}(\mu\,;\,q,1,a), \tag{5.41}$$

   which follows, as in (4.74), from combining the duality (5.39) and the evaluation (5.40).

Proof. (Sketch)
(1) This follows almost verbatim from the arguments in Section 3 of [GN], where they first argue that the evaluation $\{\cdot\}_k$ is compatible with the stabilization. Using the results in [SV1, SV2], the action of $PSL_2^{\wedge}(\mathbb{Z})$ and the formulas for Dunkl operators Y_ω are compatible, as well. Then, according to the definition of DAHA-Jones polynomials in Theorem 4.2.1 above, we only need to show that the nonsymmetric Macdonald polynomials E_b are also stable.

One way to see this is using the definition (4.30) in terms of intertwining operators:

$$Y_\omega(E_{b^*+c}) = q^{-(\omega\,,\,b^*+c+w_{b^*+c}(\rho_k))}E_{b^*+c}, \quad \text{for} \quad \omega \in P. \tag{5.42}$$

The results in [SV1, SV2] imply that this definition is stable, i.e., since the Y_ω are. That stability was used to deduce E_ϑ from just the single relation $Y_\vartheta(E_\vartheta) = q^{-2}q^{-(\vartheta,\rho_k)}E_\vartheta$. The result (4.44) is valid for the composite weight $[\omega_1, \omega_1]_N$ of A_n simultaneously for every n, as well as for ϑ in *every* root system R. One can also use the combinatorial formula for E_b in [HHL].

(2) According to the remark after the super-duality formula (1.44) from Section 1.6 of [C6], the standard (one-diagram) type-A duality is equivalent to $q^\bullet$-proportionality between $\widetilde{JD}^{A_n}_{\mathsf{r},\mathsf{s}}(\lambda\,; q, t)$ and $\widetilde{JD}^{A_m}_{\mathsf{r},\mathsf{s}}(\lambda^{tr}\,; t^{-1}, q^{-1})$ for $t = q^{-(m+1)/(n+1)}$, (i.e., for $k = -\frac{m+1}{n+1}$) and all possible relatively prime $m+1, n+1 \in \mathbb{N}$. This is directly connected with the generalized *level-rank duality*. Noting that q, n, m are essentially arbitrary, we conclude that these proportionality conditions (all of them considered simultaneously) are equivalent to the duality. The latter was proved in [GN]; the above argument (and the theory of perfect DAHA modules at roots of unity from [C4]) can be used for the justification of the standard super-duality as well (unpublished).

This reformulation of the super-duality in terms of DAHA-Jones polynomials, (i.e., without a) gives the composite super-duality upon considering diagrams in the form $[\lambda, \mu]_N$.

(3) This follows from (4.73) and the observation that $\widetilde{JD}^{A_n}_{\mathsf{r},\mathsf{s}}(b; q, t) = \widetilde{JD}^{A_n}_{\mathsf{r},\mathsf{s}}(b^*; q, t)$. ☐

The following theorem is Theorem 2.5 from [CE]. It is a special case of the more general color exchange theorem from [C6], which is discussed in detail in sections 1.6 and 1.7 there.

Theorem 5.3.2. (Color Exchange) *Let $t = q^k$ for $k \in -\mathbb{Q}_+$. For λ, μ as above, suppose that permutations $v, w \in S_n$ exist, which satisfy the following conditions. Setting $\lambda = \{l_1 \geq \ldots \geq l_n \geq 0\}$,*

$$\lambda' = \{l'_1, \ldots, l'_n\} := \{l_{v(i)} + k(i - v(i)),\ i = 1, 2, \ldots, n\}. \tag{5.43}$$

Suppose that λ' is a diagram, i.e., we require that $l'_i \geq l'_{i+1}$ and $l'_i \in \mathbb{Z}_+$. Similarly, suppose that μ' defined by μ, w (for the same k) is also a Young diagram. Then $HD_{\mathsf{r},\,\mathsf{s}}([\lambda, \mu]\,; q, t, a) = HD_{\mathsf{r},\,\mathsf{s}}([\lambda', \mu']\,; q, t, a)$ for such q, t and any r, s.

Let us provide an example for $t = q^{-\kappa}$, $\kappa \in \mathbb{N}$ (see [C6], formula (1.47) for details). For any $p > 0$ and $i \in \{1, 2\} \ni j$, one has:

$$\begin{aligned} &HD_{\mathsf{r},\mathsf{s}}(\,[\kappa b^{(i)}, \kappa b^{(j)}]\,; q, q^{-\kappa}, a) = q^\bullet\, HD_{\mathsf{r},\mathsf{s}}(\,[\kappa c^{(i)}, \kappa c^{(j)}]\,; q, q^{-\kappa}, a) \ \text{ for} \\ &b^{(1)} = \omega_{p+1}, c^{(1)} = (p+1)\omega_1 \ \text{ and } \ b^{(2)} = p\omega_{p+1}, c^{(2)} = (p+1)\omega_p\,, \end{aligned} \tag{5.44}$$

where the weights are identified with the corresponding diagrams. If $\kappa = 1$, then $t = q^{-1}$ and these relations are a special case of the duality: columns and rows.

The following conjecture from [CE] was based on the numerical evidence from Section 5.4 below and on a generalization of the construction from [GN] to the composite case.

Conjecture 5.3.3. Assuming that $\mathsf{r} > \mathsf{s}$, we conjecture that

$$\deg_a HD_{\mathsf{r},\,\mathsf{s}}([\lambda, \mu]\,; q, t, a) \;=\; \mathsf{s}(|\lambda| + |\mu|) - |\lambda \vee \mu|, \tag{5.45}$$

where the *join* $\lambda \vee \mu$ is the smallest Young diagram containing them, $|\lambda|$ is the number of boxes in λ.

Obtaining HOMFLY-PT polynomials

The following is Theorem 2.6 of [CE].

Theorem 5.3.4. (Connection) *For r, s and λ, μ as above,*

$$HD_{r,s}([\lambda,\mu]; q, t \mapsto q, a \mapsto -a) = \mathbf{P}_{[\lambda,\mu]}(T^{r,s}; q, a), \tag{5.46}$$

where $\mathbf{P}_{[\lambda,\mu]}(T^{r,s}; q, a)$ is the composite HOMFLY-PT polynomial for $[\lambda, \mu]$ normalized by the condition $\mathbf{P}(U) = 1$ for the unknot U.

Proof. This results from the coincidence of the $\widetilde{JD}$-polynomials in type A with the corresponding (reduced) quantum group invariants for torus knots. Both polynomials are uniquely determined by their (infinitely many) specializations, and by Theorem 4.3.4 above, these specializations coincide. Thus, the composite DAHA-superpolynomials and HOMFLY-PT polynomials also coincide. $\square$

5.4 Examples and Confirmations

We provide here examples of the composite DAHA-superpolynomials and discuss their symmetries. The first 5 composite representations considered below are contained in the following table.

$[b,c]$	$[\omega_1,\omega_1]$	$[\omega_1,\omega_2]$	$[2\omega_1,\omega_1]$	$[\omega_{\mathbf{1}},\omega_{\mathbf{3}}]$	$[\omega_1+\omega_2,\omega_1]$
$[\lambda,\mu]$	$[\square,\square]$	$[\square,(1,1)]$	$[(2),\square]$	$[\square,(1,1,1)]$	$[(2,1),\square]$
l	2	3	2	4	3
A_1	(2)	—	(3)	—	—
A_2	$(2,1)$	$(2,2)$	$(3,2)$	—	$(3,1)$
A_3	$(2,1,1)$	$(2,2,1)$	$(3,2,2)$	$(2,2,2)$	$(3,2,1)$
A_4	$(2,1,1,1)$	$(2,2,1,1)$	$(3,2,2,2)$	$(2,2,2,1)$	$(3,3,2,1)$
A_5	$(2,1,1,1,1)$	$(2,2,1,1,1)$	$(3,2,2,2,2)$	$(2,2,2,1,1)$	$(3,3,3,2,1)$
A_6	$(2,1,1,1,1,1)$	$(2,2,1,1,1,1)$	$(3,2,2,2,2,2)$	$(2,2,2,1,1,1)$	$(3,3,3,3,2,1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

(5.47)

Adjoint representation

The adjoint representation has the weight $\omega_1 + \omega_n$ and is represented in our notation by the pair $[\omega_1, \omega_1] = [\square, \square]$. We consider this representation for two knots.

Trefoil

The adjoint DAHA superpolynomial for the trefoil is given by the formula

$$HD_{3,2}([\omega_1,\omega_1];q,t,a) = \tag{5.48}$$

$$1+2qt+q^2t^2+a(3q^2-q^3+2qt^{-1}-q^2t^{-1}-q^3t^{-1}+2q^3t)+a^2(q^4+q^2t^{-2}-2q^3t^{-2}+q^4t^{-2}+2q^3t^{-1}-2q^4t^{-1})+a^3(-q^4t^{-3}+q^5t^{-3}+q^4t^{-2}-q^5t^{-2}).$$

Recall that it is defined by the relations

$$HD_{\mathsf{r,s}}([\lambda,\mu];q,t,a\mapsto -t^{n+1}) = \widetilde{JD}_{\mathsf{r,s}}^{A_n}(\lambda^*+\mu;q,t) \tag{5.49}$$

for $\lambda=\omega_1,\mu=\omega_1$ and all $n\geq 1$.

The corresponding normalized adjoint HOMFLY-PT polynomial for the unframed trefoil is given by formula (2.17) from [PBR]; see also (5.36) above. One has:

$$\mathbf{P}_{[\square,\square]}(T^{3,2};q,a) \;= \tag{5.50}$$

$$a^2(q^{-2}+q^2+2)+a^3(-2q^{-2}+q^{-1}+q-2q^2-2)+a^4(q^{-2}-2q^{-1}-2q+q^2+3)+a^5(q^{-1}+q-2),$$

and we have the following confirmation of Theorem 5.3.4:

$$a^2q^{-2}HD_{3,2}([\omega_1,\omega_1]\,;\,q,t\mapsto q,a\mapsto -a) = \mathbf{P}_{[\square,\square]}(T^{3,2}). \tag{5.51}$$

The super-duality from (5.39) in this case is as follows:

$$t^{-2}HD_{3,2}([\omega_1,\omega_1];q,t,a) = q^2HD_{3,2}([\omega_1,\omega_1];t^{-1},q^{-1},a). \tag{5.52}$$

The evaluation formula (5.41) reads

$$HD_{3,2}([\omega_1,\omega_1];q,1,a) \;=\; \big(1+q+aq\big)^2 =\; HD_{3,2}(\omega_1;q,1,a)^2. \tag{5.53}$$

$(4,3)$-torus knot

The adjoint DAHA-superpolynomial for the $(4,3)$-torus knot $T^{4,3}$ is given by the formula

$$HD_{4,3}([\omega_1,\omega_1]\,;\,q,t,a) \;= \tag{5.54}$$

$$1+2qt+2q^2t+3q^2t^2+2q^3t^2+q^4t^2+4q^3t^3+2q^4t^3+3q^4t^4+2q^5t^4+2q^5t^5+q^6t^6+a(5q^2+5q^3-q^4-3q^5-2q^6+2qt^{-1}+q^2t^{-1}-q^3t^{-1}-q^4t^{-1}-q^5t^{-1}+8q^3t+7q^4t+q^5t-3q^6t-q^7t+9q^4t^2+7q^5t^2-q^6t^2-q^7t^2+8q^5t^3+5q^6t^3-q^7t^3+5q^6t^4+q^7t^4+2q^7t^5)+a^2(7q^4+9q^5-2q^6-8q^7+q^2t^{-2}+2q^3t^{-2}-2q^4t^{-2}-3q^5t^{-2}+$$

$q^7t^{-2}+q^8t^{-2}+4q^3t^{-1}+5q^4t^{-1}-2q^5t^{-1}-8q^6t^{-1}+q^8t^{-1}+8q^5t+9q^6t-2q^7t-3q^8t+7q^6t^2+5q^7t^2-2q^8t^2+4q^7t^3+2q^8t^3+q^8t^4)+a^3(3q^6+5q^7-q^8-3q^9+q^4t^{-3}-q^5t^{-3}-3q^6t^{-3}+3q^7t^{-3}+q^8t^{-3}-q^9t^{-3}+q^4t^{-2}+3q^5t^{-2}-q^6t^{-2}-8q^7t^{-2}+4q^8t^{-2}+q^9t^{-2}+2q^5t^{-1}+5q^6t^{-1}-2q^7t^{-1}-8q^8t^{-1}+3q^9t^{-1}+2q^7t+3q^8t-q^9t+q^8t^2+q^9t^2)+a^4(q^9-q^7t^{-4}+2q^8t^{-4}-q^9t^{-4}+q^6t^{-3}-4q^8t^{-3}+4q^9t^{-3}-q^{10}t^{-3}+q^7t^{-2}+q^8t^{-2}-4q^9t^{-2}+2q^{10}t^{-2}+q^8t^{-1}-q^{10}t^{-1})+a^5(-q^{10}t^{-5}+q^{11}t^{-5}-q^9t^{-4}+2q^{10}t^{-4}-q^{11}t^{-4}+q^9t^{-3}-q^{10}t^{-3}),$

defined by (5.49) as for the trefoil. Computed using (5.17), the corresponding normalized HOMFLY-PT polynomial is

$$\mathbf{P}_{[\square,\square]}(T^{4,3}) \;= \tag{5.55}$$

$q^{-6}\big(a^6(q^{12}+2q^{10}+2q^9+3q^8+2q^7+5q^6+2q^5+3q^4+2q^3+2q^2+1)+a^7(-2q^{12}-q^{11}-4q^{10}-4q^9-6q^8-4q^7-8q^6-4q^5-6q^4-4q^3-4q^2-q-2)+a^8(q^{12}+2q^{11}+2q^{10}+2q^9+5q^8+2q^7+7q^6+2q^5+5q^4+2q^3+2q^2+2q+1)+a^9(-q^{11}-4q^8+2q^7-4q^6+2q^5-4q^4-q)+a^{10}(2q^8-4q^7+5q^6-4q^5+2q^4)+a^{11}(2q^7-4q^6+2q^5)\big)$

We have the *connection formula*

$$a^6q^{-6}HD_{4,3}([\omega_1,\omega_1]\,;\,q,t\mapsto q,a\mapsto -a)=\mathbf{P}_{[\square,\square]}(T^{4,3}). \tag{5.56}$$

The *super-duality* reads

$$t^{-6}HD_{4,3}([\omega_1,\omega_1];q,t,a)=q^6HD_{4,3}([\omega_1,\omega_1];t^{-1},q^{-1},a), \tag{5.57}$$

and the evaluation at $t=1$ is as follows:

$$\begin{aligned} HD_{4,3}([\omega_1,\omega_1];q,1,a) &= \big(1+q+2q^2+q^3+a(q+2q^2+2q^3)+a^2q^3\big)^2 \\ &= HD_{4,3}(\omega_1;q,1,a)^2. \end{aligned} \tag{5.58}$$

Column/row and a box

Rows and boxes correspond to the symmetric and wedge powers of the fundamental representation.

Two-row and a box: $[2\omega_1,\omega_1]=[\square\square,\square]$

The composite DAHA-superpolynomial for the trefoil is

$$HD_{3,2}([2\omega_1,\omega_1];q,t,a) \;= \tag{5.59}$$

$1+qt+q^2t+q^3t+q^3t^2+2q^4t^2+q^5t^3+a(3q^3+3q^4-2q^6-q^7+qt^{-1}+q^2t^{-1}-q^4t^{-1}-q^5t^{-1}+q^4t+4q^5t+2q^6t-q^7t+q^6t^2+2q^7t^2)+a^2(2q^6+4q^7-q^8-2q^9+q^3t^{-2}-q^5t^{-2}-q^6t^{-2}+q^8t^{-2}+q^4t^{-1}+3q^5t^{-1}+q^6t^{-1}-4q^7t^{-1}-2q^8t^{-1}+q^9t^{-1}+2q^8t+q^9t)+a^3(q^{10}-q^7t^{-3}+q^9t^{-3}+q^6t^{-2}+q^7t^{-2}-2q^8t^{-2}-2q^9t^{-2}+q^{10}t^{-2}+q^{11}t^{-2}+2q^8t^{-1}+q^9t^{-1}-2q^{10}t^{-1}-q^{11}t^{-1})+a^4(-q^{10}t^{-3}+q^{12}t^{-3}+q^{10}t^{-2}-q^{12}t^{-2}),$

defined by (5.49) for $\lambda = 2\omega_1, \mu = \omega_1$ and all $n \geq 1$.

The corresponding normalized HOMFLY-PT polynomial is given by formula (A.1) from [PBR], as well as computed using (5.17). It is

$$\mathbf{P}_{[\square\square,\square]}(T^{3,2}) \;=\; \tag{5.60}$$

$$q^{-3}\big(a^3(q^8+2q^6+q^5+q^4+q^3+q^2+1)+a^4(-1-q-2q^3-2q^4-q^5-2q^6-q^7-2q^9)+a^5(q+2q^4+q^5-q^6+2q^7+q^{10})+a^6(-q^5+q^6-2q^8+q^9)+a^7(-q^7+q^8+q^9-q^{10})\big),$$

and we have the relationship

$$a^3q^{-3}HD_{3,2}([2\omega_1,\omega_1]\,;\,q,t\mapsto q,a\mapsto -a) = \mathbf{P}_{[\square\square,\square]}(T^{3,2}), \tag{5.61}$$

confirming Theorem 5.3.4. The super-duality here requires $[\omega_1, \omega_2]$, which will be considered next. The evaluation at $t = 1$ reads

$$\begin{aligned} HD_{3,2}([2\omega_1,\omega_1]\,;\,q,1,a) &= \big(1+q+aq\big)\times\big(1+q^2+q^3+q^4+a(q^2+q^3+q^4+q^5)+a^2q^5\big)\\ &= HD_{3,2}(\omega_1;q,1,a)\times HD_{3,2}(2\omega_1;q,1,a). \end{aligned} \tag{5.62}$$

Two-column and a box: $[\omega_1, \omega_2] = [\square, \begin{smallmatrix}\square\\\square\end{smallmatrix}]$

The DAHA-superpolynomial for the trefoil reads

$$HD_{3,2}([\omega_1,\omega_2]\,;\,q,t,a) \;=\; \tag{5.63}$$

$$1+2qt+qt^2+q^2t^2+q^2t^3+q^2t^4+q^3t^5+a(4q^2-q^4+2qt^{-2}-q^2t^{-2}-q^3t^{-2}+qt^{-1}+2q^2t^{-1}-2q^3t^{-1}+q^2t+3q^3t-q^4t+3q^3t^2+q^4t^3+q^4t^4)+a^2(3q^4-q^5+q^2t^{-4}-2q^3t^{-4}+q^4t^{-4}+2q^2t^{-3}-q^3t^{-3}-2q^4t^{-3}+q^5t^{-3}+4q^3t^{-2}-4q^4t^{-2}+2q^3t^{-1}+q^4t^{-1}-q^5t^{-1}+q^4t+q^5t^2)+a^3(-q^4t^{-6}+q^5t^{-6}+q^3t^{-5}-2q^4t^{-5}+q^5t^{-5}+q^4t^{-4}-2q^5t^{-4}+q^6t^{-4}+2q^4t^{-3}-2q^5t^{-3}+q^5t^{-2}-q^6t^{-2}+q^5t^{-1})+a^4(-q^5t^{-7}+q^6t^{-7}+q^5t^{-5}-q^6t^{-5}),$$

where the specialization relations for all $n \geq 2$ are

$$HD_{3,2}([\omega_1,\omega_2];q,t,a\mapsto -t^{n+1}) = \widetilde{JD}^{A_n}_{3,2}(\omega_2+\omega_n;q,t). \tag{5.64}$$

The corresponding normalized HOMFLY-PT polynomial is given by formula (A.4) from [PBR], as well as computed using (5.17):

$$\mathbf{P}_{[\square,\begin{smallmatrix}\square\\\square\end{smallmatrix}]}(T^{3,2}) \;=\; \tag{5.65}$$

$$q^{-7}\big(a^3(q^2+2q^4+q^5+q^6+q^7+q^8+q^{10})+a^4(-2q-q^3-2q^4-q^5-2q^6-2q^7-q^9-q^{10})+a^5(1+2q^3-q^4+q^5+2q^6+q^9)+a^6(q-2q^2+q^4-q^5)+a^7(-1+q+q^2-q^3)\big),$$

and we have the *connection formula*

$$a^3 q^{-5} HD_{3,2}([\omega_1,\omega_2]; q, t \mapsto q, a \mapsto -a) = \mathbf{P}_{[\square,\begin{smallmatrix}\square\\\square\end{smallmatrix}]}(T^{3,2}). \tag{5.66}$$

The *super-duality* and *evaluation* are as follows:

$$t^{-3} HD_{4,3}([2\omega_1,\omega_1]; q,t,a) = q^5 HD_{4,3}([\omega_1,\omega_2]; t^{-1}, q^{-1}, a), \tag{5.67}$$

$$\begin{aligned} HD_{3,2}([\omega_1,\omega_2]; q,1,a) &= \big(1+q+aq\big)\times\big(1+q+aq\big)^2 \qquad (5.68)\\ &= HD_{3,2}(\omega_1; q,1,a)\times HD_{3,2}(\omega_2; q,1,a). \end{aligned}$$

The corresponding standard superpolynomials are

$$HD_{3,2}(\omega_1; q,t,a) = 1 + qt + aq, \tag{5.69}$$

$$HD_{3,2}(\omega_2; q,t,a) = 1 + \frac{a^2q^2}{t} + qt + qt^2 + q^2t^4 + a\left(q + \frac{q}{t} + q^2t + q^2t^2\right). \tag{5.70}$$

See, e.g., [C6] and references therein.

Three-column and a box: $[\omega_1,\omega_3] = [\square,\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}]$

This example is of $\deg_a = 5$, which matches our conjecture. The corresponding DAHA-superpolynomial for the trefoil is as follows:

$$HD_{3,2}([\omega_1,\omega_3]; q,t,a) \;= \tag{5.71}$$

$$1+2qt+qt^2+q^2t^2+qt^3+q^2t^3+2q^2t^4+q^2t^5+q^3t^5+q^2t^6+q^3t^6+q^3t^7+q^3t^9+q^4t^{10}+a(5q^2+q^3-2q^4+2qt^{-3}-$$
$$q^2t^{-3}-q^3t^{-3}+qt^{-2}+2q^2t^{-2}-2q^3t^{-2}+qt^{-1}+3q^2t^{-1}-q^3t^{-1}-q^4t^{-1}+2q^2t+5q^3t-2q^4t+q^2t^2+4q^3t^2+4q^3t^3+$$
$$q^4t^3-q^5t^3+q^3t^4+3q^4t^4-q^5t^4+q^3t^5+2q^4t^5+3q^4t^6+q^5t^7+q^5t^9)+a^2(q^3+6q^4-3q^5+q^2t^{-6}-2q^3t^{-6}+q^4t^{-6}+$$
$$2q^2t^{-5}-q^3t^{-5}-2q^4t^{-5}+q^5t^{-5}+2q^2t^{-4}+q^3t^{-4}-4q^4t^{-4}+q^5t^{-4}+q^2t^{-3}+5q^3t^{-3}-5q^4t^{-3}+5q^3t^{-2}-3q^5t^{-2}+$$
$$q^6t^{-2}+3q^3t^{-1}+4q^4t^{-1}-4q^5t^{-1}+4q^4t+q^5t-q^6t+2q^4t^2+2q^5t^2+3q^5t^3-q^6t^3+q^5t^4+q^5t^5+q^6t^6)+a^3(q^5+$$
$$q^6-q^7-q^4t^{-9}+q^5t^{-9}+q^3t^{-8}-2q^4t^{-8}+q^5t^{-8}+q^3t^{-7}-2q^4t^{-7}+q^6t^{-7}+2q^3t^{-6}-4q^5t^{-6}+2q^6t^{-6}+4q^4t^{-5}-$$
$$5q^5t^{-5}+q^6t^{-5}+3q^4t^{-4}-2q^5t^{-4}-q^6t^{-4}+2q^4t^{-3}+2q^5t^{-3}-3q^6t^{-3}+q^7t^{-3}+4q^5t^{-2}-2q^6t^{-2}+2q^5t^{-1}-q^6t^{-1}+$$
$$q^6t+q^6t^2)+a^4(-q^5t^{-11}+q^6t^{-11}-q^5t^{-10}+q^6t^{-10}+q^4t^{-9}-2q^5t^{-9}+q^6t^{-9}+q^5t^{-8}-2q^6t^{-8}+q^7t^{-8}+q^5t^{-7}-$$
$$2q^6t^{-7}+q^7t^{-7}+2q^5t^{-6}-2q^6t^{-6}+q^6t^{-5}-q^7t^{-5}+q^6t^{-4}-q^7t^{-4}+q^6t^{-3})+a^5(-q^6t^{-12}+q^7t^{-12}+q^6t^{-9}-q^7t^{-9}),$$

which is defined by (5.49) for all $n \geq 3$ and $\lambda = 2\omega_1, \mu = \omega_3$:

$$HD_{3,2}([\omega_1,\omega_3]; q,t,a \mapsto -t^{n+1}) = \widetilde{JD}^{A_n}_{3,2}(\omega_3+\omega_n; q,t). \tag{5.72}$$

The corresponding normalized HOMFLY-PT polynomial is

$$\mathbf{P}_{[\square,\,\substack{\square\\\square}]}(T^{3,2}) \;= \tag{5.73}$$

$$q^{-16}\big(a^4(q^{20}+q^{18}+q^{16}+q^{15}+2q^{14}+q^{13}+2q^{12}+q^{11}+2q^{10}+q^{9}+2q^{8}+q^{6})+a^5(-q^{20}-q^{18}-3q^{16}-q^{15}-3q^{14}-2q^{13}-4q^{12}-2q^{11}-4q^{10}-2q^{9}-4q^{8}-q^{7}-2q^{6}-2q^{4})+a^6(q^{18}+q^{16}+3q^{14}+q^{13}+3q^{12}+q^{11}+3q^{10}+2q^{9}+3q^{8}+q^{7}+2q^{6}+2q^{4}+q^{2})+a^7(-q^{3}-q^{5}-q^{7}-q^{8}-q^{9}-q^{10}-q^{12}-q^{14})+a^8(q^{7}-q^{6}+q^{5}-q^{4}+2q^{3}-q^{2}+q-1)+a^9(q^{4}-q^{3}-q+1)\big).$$

The connection and evaluation formulas are

$$a^4q^{-10}HD_{3,2}([\omega_1,\omega_3];q,t\mapsto q,a\mapsto -a)=\mathbf{P}_{[\square,\,\substack{\square\\\square}]}(T^{3,2}), \tag{5.74}$$

$$\begin{aligned} HD_{3,2}([\omega_1,\omega_3]\,;\,q,1,a) &= \big(1+q+aq\big)\times\big(1+q+aq\big)^3 \\ &= HD_{3,2}(\omega_1;q,1,a)\times HD_{3,2}(\omega_3;q,1,a). \end{aligned} \tag{5.75}$$

Three-hook and a box: $[\omega_1+\omega_2,\omega_1]=[\substack{\square\square\\\square},\square]$

The corresponding DAHA-superpolynomial for the trefoil is

$$HD_{3,2}([\omega_1+\omega_2,\omega_1]\,;\,q,t,a) \;= \tag{5.76}$$

$$1+3qt-qt^2+4q^2t^2+q^3t^2-2q^2t^3+4q^3t^3+q^4t^3-2q^3t^4+4q^4t^4-q^4t^5+3q^5t^5+q^6t^6+a(-2q^2+12q^3-4q^5-q^6+2qt^{-2}-q^2t^{-2}-q^3t^{-2}+6q^2t^{-1}-2q^3t^{-1}-2q^4t^{-1}-q^5t^{-1}-4q^3t+16q^4t+q^5t-4q^6t-q^7t-6q^4t^2+16q^5t^2-2q^7t^2-4q^5t^3+12q^6t^3-2q^7t^3-q^8t^3-2q^6t^4+6q^7t^4-q^8t^4+2q^8t^5)+a^2(-6q^5+26q^6-8q^7-6q^8+q^9+q^2t^{-4}-2q^3t^{-4}+q^4t^{-4}+5q^3t^{-3}-5q^4t^{-3}-2q^5t^{-3}+q^6t^{-3}+q^7t^{-3}-q^3t^{-2}+13q^4t^{-2}-8q^5t^{-2}-6q^6t^{-2}+q^7t^{-2}+q^8t^{-2}-4q^4t^{-1}+22q^5t^{-1}-8q^6t^{-1}-9q^7t^{-1}+q^8t^{-1}+q^9t^{-1}-6q^6t+22q^7t-8q^8t-2q^9t-4q^7t^2+13q^8t^2-5q^9t^2+q^{10}t^2-q^8t^3+5q^9t^3-2q^{10}t^3+q^{10}t^4)+a^3(-3q^8+14q^9-9q^{10}+q^{11}-q^4t^{-6}+q^5t^{-6}+2q^4t^{-5}-4q^5t^{-5}+q^6t^{-5}+q^7t^{-5}+6q^5t^{-4}-9q^6t^{-4}+3q^8t^{-4}-2q^5t^{-3}+14q^6t^{-3}-14q^7t^{-3}-3q^8t^{-3}+5q^9t^{-3}-3q^6t^{-2}+21q^7t^{-2}-18q^8t^{-2}-3q^9t^{-2}+3q^{10}t^{-2}-5q^7t^{-1}+21q^8t^{-1}-14q^9t^{-1}+q^{11}t^{-1}-2q^9t+6q^{10}t-4q^{11}t+q^{12}t+2q^{11}t^2-q^{12}t^2)+a^4(q^{12}-q^{13}-q^6t^{-7}+q^7t^{-7}+q^6t^{-6}-3q^7t^{-6}+q^8t^{-6}+q^9t^{-6}+4q^7t^{-5}-7q^8t^{-5}+2q^9t^{-5}+2q^{10}t^{-5}-q^{11}t^{-5}-q^7t^{-4}+8q^8t^{-4}-11q^9t^{-4}+2q^{10}t^{-4}+2q^{11}t^{-4}-2q^8t^{-3}+10q^9t^{-3}-11q^{10}t^{-3}+2q^{11}t^{-3}+q^{12}t^{-3}-2q^9t^{-2}+8q^{10}t^{-2}-7q^{11}t^{-2}+q^{12}t^{-2}-q^{10}t^{-1}+4q^{11}t^{-1}-3q^{12}t^{-1}+q^{13}t^{-1})+a^5(-q^9t^{-7}+q^{10}t^{-7}+q^9t^{-6}-2q^{10}t^{-6}+q^{11}t^{-6}+2q^{10}t^{-5}-3q^{11}t^{-5}+q^{12}t^{-5}-q^{10}t^{-4}+2q^{11}t^{-4}-2q^{12}t^{-4}+q^{13}t^{-4}+q^{12}t^{-3}-q^{13}t^{-3}),$$

defined by (5.49) for $\lambda=\omega_1+\omega_2,\mu=\omega_1$ and all $n\geq 2$. The corresponding normalized HOMFLY-PT polynomial is

$$\mathbf{P}_{[\omega_1+\omega_2,\,\square]}(T^{3,2}) \;= \tag{5.77}$$

$$q^{-8}\big(a^4(q^{14}+3q^{12}-q^{11}+4q^{10}-q^9+4q^8-q^7+4q^6-q^5+3q^4+q^2)+a^5(-2q^{15}+q^{14}-5q^{13}+4q^{12}-10q^{11}+5q^{10}-12q^9+6q^8-12q^7+5q^6-10q^5+4q^4-5q^3+q^2-2q)+a^6(q^{16}-2q^{15}+6q^{14}-6q^{13}+11q^{12}-11q^{11}+17q^{10}-13q^9+18q^8-13q^7+17q^6-11q^5+11q^4-6q^3+6q^2-2q+1)+a^7(q^{16}-3q^{15}+4q^{14}-7q^{13}+10q^{12}-14q^{11}+14q^{10}-18q^9+18q^8-18q^7+14q^6-14q^5+10q^4-7q^3+4q^2-3q+1)+a^8(-q^{15}+2q^{14}-3q^{13}+5q^{12}-7q^{11}+10q^{10}-11q^9+11q^8-11q^7+10q^6-7q^5+5q^4-3q^3+2q^2-q)+a^9(q^{12}-2q^{11}+2q^{10}-3q^9+4q^8-3q^7+2q^6-2q^5+q^4)\big),$$

which reduces to the HOMFLY-PT polynomial as follows:

$$a^4q^{-6}HD_{3,2}([\omega_1+\omega_2,\omega_1];q,t\mapsto q,a\mapsto -a)=\mathbf{P}_{[\omega_1+\omega_2,\,\square]}(T^{3,2}). \tag{5.78}$$

The exact super-duality identity from (5.39) is

$$t^{-6}HD_{3,2}([\omega_1+\omega_2,\omega_1];q,t,a)=q^6HD_{3,2}([\omega_1+\omega_2,\omega_1];t^{-1},q^{-1},a). \tag{5.79}$$

The evaluation at $t=1$ from (5.41) reads

$$\begin{aligned} HD_{3,2}([\omega_1+\omega_2,\omega_1]\,;\,q,1,a) \;&=\; \big(1+q+aq\big)\\ \times\,(1+q+aq)\,&\big(1+q^2+q^3+q^4+a^2q^5+a\,\big(q^2+q^3+q^4+q^5\big)\big)\\ =HD_{3,2}(\omega_1;q,1,a)&\times HD_{3,2}(\omega_1+\omega_2;q,1,a). \end{aligned} \tag{5.80}$$

Two-rows and two-columns

One of λ,μ in the previous examples was always $\square$. Let us discuss the cases when $\square\!\square$ and $\boxminus$ diagrams are combined. They match well our conjectural formula (5.45) for $\deg_a$; we also checked directly the super-duality and other properties provided by the theorems above.

Two two-columns: $[\omega_2,\omega_2]=[\boxminus,\boxminus]$

$$HD_{3,2}([\omega_2,\omega_2]\,;\,q,t,a)\;= \tag{5.81}$$

$$1+a^6\big(\tfrac{q^8}{t^{14}}-\tfrac{q^9}{t^{14}}-\tfrac{q^8}{t^{13}}+\tfrac{q^{10}}{t^{13}}-\tfrac{q^8}{t^{12}}+\tfrac{2q^9}{t^{12}}-\tfrac{q^{10}}{t^{12}}+\tfrac{q^8}{t^{11}}-\tfrac{q^{10}}{t^{11}}-\tfrac{q^9}{t^{10}}+\tfrac{q^{10}}{t^{10}}\big)+a^5\big(-\tfrac{q^6}{t^{13}}+\tfrac{q^7}{t^{13}}-\tfrac{q^6}{t^{12}}+\tfrac{3q^7}{t^{12}}-\tfrac{2q^8}{t^{12}}+\tfrac{q^6}{t^{11}}-\tfrac{q^7}{t^{11}}-\tfrac{q^8}{t^{11}}+\tfrac{q^9}{t^{11}}+\tfrac{q^6}{t^{10}}-\tfrac{5q^7}{t^{10}}+\tfrac{5q^8}{t^{10}}-\tfrac{q^9}{t^{10}}+\tfrac{2q^8}{t^9}-\tfrac{2q^9}{t^9}+\tfrac{2q^7}{t^8}-\tfrac{4q^8}{t^8}+\tfrac{2q^9}{t^8}-\tfrac{q^8}{t^7}+\tfrac{q^9}{t^7}+\tfrac{q^8}{t^6}-\tfrac{q^9}{t^6}\big)+a^4\big(-\tfrac{2q^5}{t^{11}}+\tfrac{2q^6}{t^{11}}+\tfrac{q^4}{t^{10}}-\tfrac{4q^5}{t^{10}}+\tfrac{4q^6}{t^{10}}-\tfrac{q^7}{t^{10}}+\tfrac{q^6}{t^9}-\tfrac{q^8}{t^9}+\tfrac{4q^5}{t^8}-\tfrac{10q^6}{t^8}+\tfrac{7q^7}{t^8}-\tfrac{q^8}{t^8}+\tfrac{2q^5}{t^7}-\tfrac{6q^6}{t^7}+\tfrac{4q^7}{t^7}+\tfrac{6q^6}{t^6}-\tfrac{9q^7}{t^6}+\tfrac{3q^8}{t^6}+\tfrac{3q^6}{t^5}-\tfrac{6q^7}{t^5}+\tfrac{3q^8}{t^5}+\tfrac{3q^7}{t^4}-\tfrac{3q^8}{t^4}+\tfrac{2q^7}{t^3}-\tfrac{2q^8}{t^3}+\tfrac{q^8}{t^2}\big)+2qt+2qt^2+q^2t^2+2q^2t^3+3q^2t^4+2q^3t^5+2q^3t^6+q^4t^8+a^3\big(q^6+q^7-\tfrac{q^4}{t^9}+\tfrac{q^5}{t^9}+\tfrac{2q^3}{t^8}-\tfrac{5q^4}{t^8}+\tfrac{3q^5}{t^8}+\tfrac{2q^3}{t^7}-\tfrac{4q^4}{t^7}+\tfrac{3q^6}{t^7}-\tfrac{q^7}{t^7}+\tfrac{6q^4}{t^6}-\tfrac{12q^5}{t^6}+\tfrac{7q^6}{t^6}-\tfrac{q^7}{t^6}+\tfrac{7q^4}{t^5}-\tfrac{11q^5}{t^5}+\tfrac{2q^6}{t^5}+\tfrac{2q^7}{t^5}+\tfrac{q^4}{t^4}+\tfrac{7q^5}{t^4}-\tfrac{12q^6}{t^4}+\tfrac{4q^7}{t^4}+\tfrac{10q^5}{t^3}-\tfrac{12q^6}{t^3}+\tfrac{2q^7}{t^3}+\tfrac{2q^5}{t^2}+\tfrac{4q^6}{t^2}-\tfrac{4q^7}{t^2}+\tfrac{7q^6}{t}-\tfrac{5q^7}{t}+2q^7t\big)+a^2\big(5q^4+3q^5-3q^6+\tfrac{q^2}{t^6}-\tfrac{2q^3}{t^6}+\tfrac{q^4}{t^6}+\tfrac{4q^2}{t^5}-\tfrac{4q^3}{t^5}-\tfrac{2q^4}{t^5}+\tfrac{2q^5}{t^5}+\tfrac{q^2}{t^4}+\tfrac{4q^3}{t^4}-\tfrac{9q^4}{t^4}+\tfrac{3q^5}{t^4}+\tfrac{q^6}{t^4}+\tfrac{10q^3}{t^3}-\tfrac{9q^4}{t^3}-\tfrac{2q^5}{t^3}+\tfrac{q^6}{t^3}+\tfrac{4q^3}{t^2}+\tfrac{5q^4}{t^2}-\tfrac{10q^5}{t^2}+\tfrac{2q^6}{t^2}+\tfrac{15q^4}{t}-\tfrac{10q^5}{t}-\tfrac{q^6}{t}+10q^5t-4q^6t+4q^5t^2-q^6t^2+4q^6t^3+q^6t^4\big)+a\big(5q^2+3q^3-4q^4+\tfrac{2q}{t^3}-\tfrac{q^2}{t^3}-\tfrac{q^3}{t^3}+$$

$$\tfrac{2q}{t^2}+\tfrac{q^2}{t^2}-\tfrac{3q^3}{t^2}+\tfrac{7q^2}{t}-\tfrac{3q^3}{t}-\tfrac{2q^4}{t}+10q^3t-3q^4t-q^5t+6q^3t^2+q^4t^2-q^5t^2+7q^4t^3-q^5t^3+5q^4t^4-q^5t^4+2q^5t^5+2q^5t^6\big).$$

Note that the specializations $a=-t^{n+1}$ to A_n begins here with $A_{n=3}$. We omit the formula for $HD_{3,2}([2\omega_1,2\omega_1]\,;\,q,t,a)$, since it can be obtained via the super-duality (checked numerically). Also,

$$HD_{3,2}([\omega_2,\omega_2]\,;\,q,t{=}1,a)=(1+q+qa)^4. \tag{5.82}$$

Two-column and two-row: $[\omega_2,2\omega_1]=[\begin{smallmatrix}\square\\\square\end{smallmatrix},\square\square]$

Note that the a-degree is 5 in this example vs. 6 in the previous one; $\lambda\vee\mu$ contains now 3 boxes (it is a 3-hook) in (conjectural) formula (5.45). This formula is self-dual with respect to $q\mapsto t^{-1}, t\mapsto q^{-1}, a\mapsto a$ (up to $q^{\bullet}t^{\bullet}$).

$$HD_{3,2}([\omega_2,2\omega_1]\,;\,q,t,a)\;= \tag{5.83}$$

$$1+a^5\big(-\tfrac{q^{11}}{t^7}+\tfrac{q^{13}}{t^7}+\tfrac{q^{11}}{t^5}-\tfrac{q^{13}}{t^5}\big)+a^4\big(-\tfrac{q^8}{t^7}+\tfrac{q^{10}}{t^7}-\tfrac{q^9}{t^6}-\tfrac{q^{10}}{t^6}+\tfrac{q^{11}}{t^6}+\tfrac{q^{12}}{t^6}+\tfrac{q^7}{t^5}+\tfrac{q^8}{t^5}-\tfrac{q^9}{t^5}-\tfrac{2q^{10}}{t^5}+\tfrac{q^{12}}{t^5}+\tfrac{q^9}{t^4}+\tfrac{q^{10}}{t^4}-\tfrac{2q^{11}}{t^4}-\tfrac{q^{12}}{t^4}+\tfrac{q^{13}}{t^4}+\tfrac{q^9}{t^3}+\tfrac{q^{10}}{t^3}-\tfrac{q^{11}}{t^3}-\tfrac{q^{12}}{t^3}+\tfrac{q^{11}}{t^2}-\tfrac{q^{13}}{t^2}+\tfrac{q^{11}}{t}\big)+qt+q^2t+q^3t+qt^2+q^3t^2+2q^4t^2+q^3t^3+q^4t^3+q^5t^3+q^2t^4+q^5t^4+q^4t^5+q^5t^5+q^6t^6+a^3\big(2q^9+2q^{10}-q^{11}-\tfrac{q^7}{t^6}+\tfrac{q^9}{t^6}+\tfrac{q^4}{t^5}-\tfrac{q^6}{t^5}-\tfrac{q^7}{t^5}-\tfrac{q^8}{t^5}+\tfrac{q^9}{t^5}+\tfrac{q^{10}}{t^5}+\tfrac{2q^6}{t^4}+\tfrac{q^7}{t^4}-\tfrac{4q^8}{t^4}-\tfrac{2q^9}{t^4}+\tfrac{2q^{10}}{t^4}+\tfrac{q^{11}}{t^4}+\tfrac{q^5}{t^3}+\tfrac{2q^6}{t^3}+\tfrac{q^7}{t^3}-\tfrac{q^8}{t^3}-\tfrac{3q^9}{t^3}-\tfrac{2q^{10}}{t^3}+\tfrac{q^{11}}{t^3}+\tfrac{q^{12}}{t^3}+\tfrac{2q^7}{t^2}+\tfrac{4q^8}{t^2}-\tfrac{q^9}{t^2}-\tfrac{4q^{10}}{t^2}-\tfrac{q^{11}}{t^2}+\tfrac{q^7}{t}+\tfrac{2q^8}{t}+\tfrac{q^9}{t}+\tfrac{q^{10}}{t}-\tfrac{q^{11}}{t}-\tfrac{q^{12}}{t}+q^9t+q^{11}t^2\big)+a^2\big(q^5+4q^6+4q^7-q^9-q^{10}+\tfrac{q^3}{t^4}-\tfrac{q^5}{t^4}-\tfrac{q^6}{t^4}+\tfrac{q^8}{t^4}+\tfrac{q^2}{t^3}+\tfrac{q^3}{t^3}-\tfrac{q^6}{t^3}-\tfrac{3q^7}{t^3}+\tfrac{2q^9}{t^3}+\tfrac{3q^4}{t^2}+\tfrac{4q^5}{t^2}-\tfrac{4q^7}{t^2}-\tfrac{4q^8}{t^2}+\tfrac{q^{10}}{t^2}+\tfrac{2q^4}{t}+\tfrac{q^5}{t}+\tfrac{4q^6}{t}+\tfrac{3q^7}{t}-\tfrac{4q^8}{t}-\tfrac{3q^9}{t}+q^5t+q^6t+q^7t+4q^8t-q^{10}t+2q^7t^2+3q^8t^2+q^9t^3+q^{10}t^3+q^9t^4\big)+a\big(q^2+3q^3+2q^4+2q^5-q^6-2q^7+\tfrac{q}{t^2}+\tfrac{q^2}{t^2}-\tfrac{q^4}{t^2}-\tfrac{q^5}{t^2}+\tfrac{q}{t}+\tfrac{2q^3}{t}+\tfrac{2q^4}{t}-\tfrac{q^5}{t}-\tfrac{2q^6}{t}-\tfrac{q^7}{t}+q^2t+2q^4t+5q^5t+2q^6t-q^7t-q^8t+3q^4t^2+2q^5t^2+2q^6t^2+2q^7t^2-q^8t^2+3q^6t^3+2q^7t^3+q^5t^4+q^6t^4+q^8t^4+q^7t^5+q^8t^5\big).$$

The evaluation at $t=1$ from formula (5.40) now reads as follows:

$$HD_{3,2}([\omega_2,2\omega_1]\,;\,q,1,a)=(1+q+aq)^2(1+q^2+q^3+q^4+a(q^2+q^3+q^4+q^5)+a^2q^5), \tag{5.84}$$

where the standard superpolynomial for $2\omega_1$ is

$$HD_{3,2}(2\omega_1\,;\,q,t,a)=1+a^2q^5+q^2t+q^3t+q^4t^2+a\left(q^2+q^3+q^4t+q^5t\right). \tag{5.85}$$

Here and above we omit the formulas for the composite HOMFLY-PT polynomials; they do satisfy the Connection Theorem 5.3.4.

Chapter 6

Exceptional Knot Homology

Here we recount the results of [EG], which extend the story of knot homologies and superpolynomials to the case of the exceptional Lie algebra $\mathfrak{e}_6$ and its fundamental, 27-dimensional representation.

This problem poses a number of unique challenges. Section 6.1 describes how we address them by combining the differentials from [DGR, GW, GS, GGS] with the refinements of [C5, C6, AS]. In Section 6.2, we provide a proposal for and three compelling examples of the structures which define our so-called *hyperpolynomials*. Finally, in Section 6.3, we describe the elements of classical singularity theory which justify our differentials; see also Appendix C.

Section 6.4 recounts the results of Section 4 of [CE], where we study the stabilization of DAHA-Jones polynomials within the Deligne-Gross exceptional series of root systems [DG].

The themes studied throughout Chapter 6 are (necessarily) of experimental nature. However, the examples we produce are convincing and merit further study.

6.1 Approach: DAHA + BPS

In [DGR] the authors introduce the superpolynomial for knot homologies, as a generating function of the refined BPS invariants on the one hand and as the Poincaré polynomial of the HOMFLY homology on the other (see Section 3.3). Analogous constructions for colored HOMFLY and Kauffman homologies were developed in [GS] and [GW], respectively. Here, we incorporate the exceptional Lie algebra $\mathfrak{e}_6$ and its 27-dimensional representation with (minuscule) highest weight ω_1.

Exceptional Lie algebras pose a number of unique challenges. For one, they are singular in the sense that they do not belong to infinite families in any obvious way. Thus, we are missing a natural notion of "stabilization," which helps the identification of gradings/differentials in the classical cases.

In [CE] and in Section 6.4 below, we consider stabilization for the Deligne-Gross "exceptional series." However, this is a fundamentally different phenomenon than considered here, as those examples contain negative coefficients. It is an interesting question, relegated to future research, of whether the approach in [CE] is compatible with the approach here.

We also face a more technical/computatational challenge. Even the ordinary (quantum group) knot invariants for $\mathfrak{e}_6$ have not been explicitly computed in the literature. The author R.E. has computed them for the cases considered here (unpublished) and verified their coincidence with the DAHA-Jones polynomials upon $t \mapsto q$. Furthermore, no corresponding homology theory has been formally defined (other than [Web]).

We manage to overcome these obstacles by applying the technique of differentials from [DGR, Ras3] to the DAHA-Jones polynomials, q, t-counterparts of quantum knot invariants defined in [C5]. This combination is sufficiently powerful to overcome all obstacles. Here, we propose so-called *hyperpolynomials* for $\mathfrak{e}_6, \mathbf{27}$ torus knot homologies, as well as produce some explicit examples.

Notation and conventions

We will use two sets of conventions: the standard DAHA conventions and conventions used in the literature on quantum group invariants ("QG"). While our calculations are performed in DAHA conventions (q, t, a), we are ultimately interested in QG conventions (q, t, u). To change DAHA $\to$ QG, we apply the "grading change" isomorphism:

$$a \mapsto ut^{-1}\,, \qquad q \mapsto qt^2\,, \qquad t \mapsto q. \tag{6.1}$$

Even though q, t are used in both sets of conventions, whether we are referring to DAHA or QG will be contextually clear.

Furthermore, for a given knot, polynomials in QG conventions are usually associated to a Lie algebra $\mathfrak{g}$ and a representation ($\mathfrak{g}$-module) V. Polynomials in DAHA conventions are (equivalently) associated to a root system R and a (dominant) weight $b \in P_+$. The correspondence between $\mathfrak{g}$ and R is via the classification of complex, semisimple Lie algebras, and b is the highest weight for V, as labeled in [B].

Now, in QG-conventions, our hyperpolynomials are Poincaré polynomials for a (hypothetical) triply-graded vector space:

$$\mathscr{H}^{\mathfrak{e}_6,\mathbf{27}}(K; q, t, u) \;:=\; \sum_{i,j,k} q^i t^j u^k \dim \mathcal{H}^{\mathfrak{e}_6,\mathbf{27}}_{i,j,k}(K). \tag{6.2}$$

The usual two-variable Poincaré polynomials are returned upon setting $u = 1$:

$$\mathscr{P}^{\mathfrak{e}_6,\mathbf{27}}(K; q, t) \;:=\; \mathscr{H}^{\mathfrak{e}_6,\mathbf{27}}(K; q, t, 1), \tag{6.3}$$

and we have, upon taking the graded Euler characteristic with respect to t,

$$\mathscr{P}^{\mathfrak{e}_6,\mathbf{27}}(K; q, -1) \;=\; P^{\mathfrak{e}_6,\mathbf{27}}(K; q), \tag{6.4}$$

i.e., these "categorify" the quantum knot invariants (2.39) for $\mathfrak{e}_6, \mathbf{27}$.

This story may be translated into DAHA conventions. In light of (6.1), we may also write the hyperpolynomials in DAHA conventions:

$$HD^{E_6}_{\mathsf{r},\mathsf{s}}(\omega_1; q, t, a) \;\coloneqq\; \sum_{i,j,k} q^{\frac{j+k}{2}} t^{\frac{2i-j+k}{2}} a^k \dim \mathcal{H}^{\mathfrak{e}_6,\mathbf{27}}_{i,j,k}(T^{\mathsf{r},\mathsf{s}}), \tag{6.5}$$

for the *same* vector space as in (6.2). Though we do not consider a DAHA analog of $\mathscr{P}^{\mathfrak{e}_6,\mathbf{27}}$ here, we may obtain the DAHA-Jones polynomial by taking the graded Euler characteristic with respect to a:

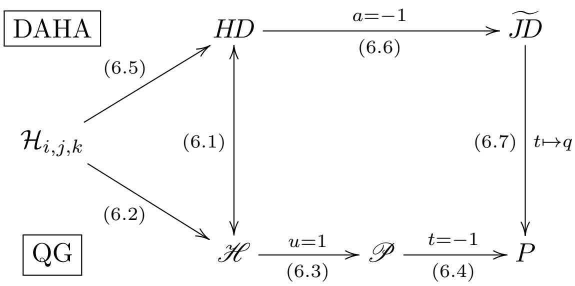
$$HD^{E_6}_{\mathsf{r},\mathsf{s}}(\omega_1; q, t, -1) \;=\; \widetilde{JD}^{E_6}_{\mathsf{r},\mathsf{s}}(\omega_1; q, t). \tag{6.6}$$

Recall that the DAHA-Jones polynomials are t-refinements of the QG knot invariants. They are (conjecturally) related by setting $t \mapsto q$:

$$\widetilde{JD}^{E_6}_{\mathsf{r},\mathsf{s}}(\omega_1; q, q) \;=\; P^{\mathfrak{e}_6,\mathbf{27}}(T^{\mathsf{r},\mathsf{s}}; q). \tag{6.7}$$

Thus, we come full circle and make contact with the QG conventions at the level of polynomials.

For the convenience of the reader, our conventions and notations are summarized in the following commutative diagram:



(6.8)

Our approach

Torus knots

Presently, our approach is confined to the torus knots and links for which the DAHA-Jones polynomials are defined. The reason for this limitation is algebraic from the DAHA point of view. The geometric and physical reasons were discussed in Section 3.4.

In either case, the origin of the extra grading (resp. variable u) has nothing to do with the choice of homology (Khovanov, colored HOMFLY, or other); it simply comes from a very special choice of the knot (link) and exists only for torus knots and links.

As a result, what for a generic knot K might be a doubly-graded homology $\mathcal{H}^{\mathfrak{g},V}_{i,j}(K)$ becomes a triply-graded homology $\mathcal{H}^{\mathfrak{g},V}_{i,j,k}(K)$ for a torus knot, with an extra u-grading. Likewise, what normally would be a triply-graded (say, HOMFLY or Kauffman) homology, for a torus knot $K = T^{\mathsf{r},\mathsf{s}}$ becomes a quadruply-graded homology $\mathcal{H}^{\mathfrak{g},V}_{i,j,k,\ell}(T^{\mathsf{r},\mathsf{s}})$, c.f. [GGS].

Hyper-lift

We wish to elevate the two-variable DAHA-Jones polynomial $\widetilde{JD}^{E_6}_{\mathsf{r},\mathsf{s}}(\omega_1; q, t)$, which in general has both positive and negative coefficients, to a three-variable *hyperpolynomial* $HD^{\mathsf{r},\mathsf{s}}_{E_6}(\omega_1; q, t, a)$ with only positive coefficients.

As in (6.5), this "upgraded" polynomial will be the Poincaré polynomial of a triply-graded vector space $\mathcal{H}^{\mathfrak{e}_6,\mathbf{27}}_{i,j,k}(T^{\mathsf{r},\mathsf{s}})$, accounting for its positive coefficients. As in (6.6), it is related to $\widetilde{JD}^{E_6}_{\mathsf{r},\mathsf{s}}$ by taking the graded Euler characteristic with respect to the k-grading (resp. variable a):

$$HD^{\mathsf{r},\mathsf{s}}_{E_6}(\omega_1; q, t, -1) = \widetilde{JD}^{E_6}_{\mathsf{r},\mathsf{s}}(\omega_1; q, t). \tag{6.9}$$

Note that we are here constructing the polynomial, $HD^{\mathsf{r},\mathsf{s}}_{E_6}(\omega_1)$, whose constituent monomials encode the graded dimensions of the irreducible components of the vector space $\mathcal{H}^{E_6,\mathsf{r},\mathsf{s}}_{i,j,k}$. We are not constructing this vector space itself.

Of course, there will be many polynomials $HD^{\mathsf{r},\mathsf{s}}_{E_6}(\omega_1)$ that satisfy only the aforementioned properties. We will define ours intelligently so that it is uniquely determined and so that like the HOMFLY-PT ("superpolynomial") and Kauffman homologies–which respectively unify $\mathfrak{sl}_N$ and $\mathfrak{so}_N$ invariants–our "hyperpolynomial" will unify the $(\mathfrak{e}_6, \mathbf{27})$-invariant with invariants associated to "smaller" algebras and representations $(\mathfrak{g}, V)$.

Differentials and specializations

This unification with other $(\mathfrak{g}, V)$ is effected using a certain (hypothetical) spectral sequence on $\mathcal{H}^{\mathfrak{e}_6,\mathbf{27}}_{*}$ induced by deformations of the potential $W_{E_6,27} \rightsquigarrow W_{\mathfrak{g},V}$, which are studied in Section 6.3. With the additional assumption that these spectral sequences converge on their second pages, such a deformation gives rise to a differential $d_{\mathfrak{g},V}$ such that the homology:

$$H_*(\mathcal{H}^{\mathfrak{e}_6,\mathbf{27}}_{*}, d_{\mathfrak{g},V}) \cong \mathcal{H}^{\mathfrak{g},V}_{*}. \tag{6.10}$$

Practically speaking, suppose that such a differential $d_{\mathfrak{g},V}$ exists ($= d_{R,b}$ in DAHA conventions), and that its (q, t, a)-degree is (α, β, γ). Then each monomial term in $HD^{\mathsf{r},\mathsf{s}}_{E_6}(\omega_1)$ will participate in exactly one of two types of direct summands in the chain complex $(\mathcal{H}^{\mathfrak{e}_6,\mathbf{27}}_{*}, d_{R,b})$:

$$0 \xrightarrow{d} q^i t^j a^k \xrightarrow{d} 0, \tag{6.11}$$

$$0 \xrightarrow{d} q^i t^j a^k \xrightarrow{\cong} q^{i+\alpha} t^{j+\beta} a^{k+\gamma} \xrightarrow{d} 0. \tag{6.12}$$

Observe that we can re-express this as a decomposition:

$$HD^{\mathsf{r,s}}_{E_6}(\omega_1) = \widetilde{HD}_R(b) + (1+q^\alpha t^\beta a^\gamma)\mathcal{Q}(q,t,a), \tag{6.13}$$

where $\widetilde{HD}_R(b)$ is related to $\widetilde{JD}^R(b)$ by the specialization

$$HD^{\mathsf{r,s}}_{E_6}(\omega_1; a = -q^{-\frac{\alpha}{\gamma}}t^{-\frac{\beta}{\gamma}}) = \widetilde{HD}_R(b; a = -q^{-\frac{\alpha}{\gamma}}t^{-\frac{\beta}{\gamma}}) = \widetilde{JD}^R(b), \tag{6.14}$$

which subsumes the differential $d_{R,b}$, realized by setting $(1+q^\alpha t^\beta a^\gamma) = 0$. Note that since these polynomials always have integer exponents (corresponding to integer gradings of a vector space), we will always be able to define the a-grading in such a way that γ divides α and β.

To restore the a-grading to $\widetilde{JD}^{E_6}(\omega_1)$, we must play this game in reverse. On the q,t level, we have a decomposition:

$$\widetilde{JD}^{E_6}(\omega_1) = \widetilde{JD}^R(b) + (1 \pm q^\alpha t^\beta)Q(q,t). \tag{6.15}$$

Since many of the polynomials $\widetilde{JD}^R(b)$ are known, we can hope to use this structure to recover the a-gradings of specific generators as well as the a-degrees of the $d_{R,b}$. If we can do this for sufficiently many (R,b), we will obtain enough constraints (specializations) to uniquely define the (relative) a-grading in $HD^{\mathsf{r,s}}_{E_6}(\omega_1)$.

Uniqueness

Suppose that we have defined HD by some (possibly infinite) set of differentials/specializations $S := \{(R,b,\alpha,\beta,\gamma)\}$, each of the form (6.13) with the *same* $\widetilde{HD}_R(b)$. If two hyperpolynomials HD_1, HD_2 each satisfy all of the specializations S, then evidently $HD_1 - HD_2 \in I_S$, where

$$I_S := \left(\prod_S 1 + q^\alpha t^\beta a^\gamma \right) \tag{6.16}$$

is an ideal in $\Gamma := \mathbb{Z}[[q,t,a]]$. Then HD corresponds to a unique coset $[HD] \in \Gamma/I_S$.

If S is infinite, then we may choose a distinguished representative of $[HD]$, i.e., the only one with finitely many terms. This is precisely the situation when considering superpolynomials and hyperpolynomials for the classical series of Lie algebras.

When S is finite, there is also a distinguished representative. Since HD is required to have *positive* coefficients, we may simply require that it is minimal in $[HD]$ with respect to that property, i.e., it has the minimum number of terms.

Indeed, suppose $HD_1 \neq HD_2$ are minimal, and write $HD_1 - HD_2 = F \cdot \prod_S(1+q^\alpha t^\beta a^\gamma) \in I_S$ for some $F \in R$. Since the HD_i both have positive coefficients, we may write $F = F_1 - F_2$, where each F_i has only positive coefficients. Then clearly the monomials in $F_i \cdot \prod_S(1+q^\alpha t^\beta a^\gamma)$ are all monomials in HD_i, and since these belong to I_S, they cancel in every specialization in S. Then

$$HD_i' \coloneqq HD_i - F_i \cdot \prod_S (1 + q^\alpha t^\beta a^\gamma) \tag{6.17}$$

is a new polynomial with positive coefficients and fewer terms, and which satisfies all of the specializations S. This contradicts the assumed minimality of HD_i.

Restricting ourselves to these distinguished representatives, the uniqueness of our HD depends on the uniqueness of the $\widetilde{HD}_R(b)$ chosen simultaneously for $\{(R,b)\} \subset S$. As we will see below, this is manifest in all cases considered.

6.2 E_6-Hyperpolynomials

In the standard knot theory (QG) conventions, our main proposal for $\mathscr{H}^{\mathfrak{e}_6,\mathbf{27}}$ is based on the following (finite) set of differentials/specializations:

$\mathfrak{g}, V$	$\mathscr{H}^{\mathfrak{e}_6,\mathbf{27}}(u=1, t=?) = P^{\mathfrak{g},V}$	$\deg(d_{\mathfrak{g},V})$
$\mathfrak{e}_6, \mathbf{27}$	-1	$(0,-1,1)$
$\mathfrak{d}_5, \mathbf{10}$	$-q^4$	$(4,-1,1)$
$\mathfrak{a}_6, \mathbf{7}$	$-q^5$	$(5,-1,1)$
canceling	$-q^8$	$(8,-1,1)$
canceling	$-q^{-13}$	$(13,1,1)$

(6.18)

which we will take as a definition for our hyperpolynomial. By a "canceling" differential, we mean that the corresponding homology is one dimensional. In other words, $\widetilde{HD}_R(b)$ in (6.13)—so also its equivalent in QG conventions—is a single monomial.

We construct three explicit examples, for $T^{3,2}, T^{5,2}, T^{4,3}$ torus knots, which are also known as the $\mathbf{3}_1, \mathbf{5}_1, \mathbf{8}_{19}$ knots, respectively. The result looks as follows:

$$\mathscr{H}^{\mathfrak{e}_6,\mathbf{27}}(\mathbf{3}_1) = 1 + q^2t^2 + q^5t^2 + q^{10}tu + q^{13}tu + q^{10}t^4 + q^{15}t^3u + q^{18}t^3u + q^{23}t^2u^2 \tag{6.19}$$

$$\mathscr{H}^{\mathfrak{e}_6,\mathbf{27}}(\mathbf{5}_1) \ = \tag{6.20}$$

$$1 + q^2t^2 + q^5t^2 + q^{10}tu + q^{13}tu + q^4t^4 + q^7t^4 + q^{10}t^4 + q^{12}t^3u + 2q^{15}t^3u + q^{18}t^3u + q^{23}t^2u^2 + q^{12}t^6 + q^{15}t^6 + q^{17}t^5u + 2q^{20}t^5u + q^{23}t^5u + q^{25}t^4u^2 + q^{28}t^4u^2 + q^{20}t^8 + q^{25}t^7u + q^{28}t^7u + q^{33}t^6u^2,$$

$$\mathscr{H}^{\mathfrak{e}_6,\mathbf{27}}(\mathbf{8}_{19}) \ = \tag{6.21}$$

$$1+q^2t^2+q^5t^2+q^{10}tu+q^{13}tu+q^3t^4+q^4t^4+q^6t^4+q^7t^4+q^{10}t^4+q^{11}t^3u+q^{12}t^3u+q^{14}t^3u+2q^{15}t^3u+q^{18}t^3u+ q^{23}t^2u^2+q^6t^6+q^8t^6+q^9t^6+q^{11}t^6+q^{12}t^6+q^{13}t^5u+q^{14}t^5u+q^{15}t^6+3q^{16}t^5u+2q^{17}t^5u+2q^{19}t^5u+2q^{20}t^5u+$$

$q^{21}t^4u^2 + q^{23}t^5u + 2q^{24}t^4u^2 + q^{25}t^4u^2 + q^{27}t^4u^2 + q^{28}t^4u^2 + q^{12}t^8 + q^{13}t^8 + q^{14}t^8 + q^{16}t^8 + q^{17}t^8 + q^{17}t^7u + q^{18}t^7u + q^{19}t^7u + q^{20}t^8 + q^{20}t^7u + 3q^{21}t^7u + 2q^{22}t^7u + 2q^{24}t^7u + 2q^{25}t^7u + q^{25}t^6u^2 + 2q^{26}t^6u^2 + q^{27}t^6u^2 + q^{28}t^7u + 3q^{29}t^6u^2 + q^{30}t^6u^2 + q^{32}t^6u^2 + q^{33}t^6u^2 + q^{34}t^5u^3 + q^{37}t^5u^3 + q^{18}t^{10} + q^{21}t^{10} + q^{22}t^{10} + q^{22}t^9u + q^{23}t^9u + q^{25}t^{10} + q^{25}t^9u + 3q^{26}t^9u + q^{27}t^9u + q^{27}t^8u^2 + 2q^{29}t^9u + 2q^{30}t^9u + 2q^{30}t^8u^2 + 2q^{31}t^8u^2 + q^{33}t^9u + q^{33}t^8u^2 + 3q^{34}t^8u^2 + q^{35}t^8u^2 + q^{35}t^7u^3 + q^{37}t^8u^2 + q^{38}t^8u^2 + q^{38}t^7u^3 + q^{39}t^7u^3 + q^{42}t^7u^3 + q^{30}t^{12} + q^{31}t^{11}u + q^{34}t^{11}u + q^{35}t^{11}u + q^{35}t^{10}u^2 + q^{36}t^{10}u^2 + q^{38}t^{11}u + 2q^{39}t^{10}u^2 + q^{40}t^9u^3 + q^{42}t^{10}u^2 + q^{43}t^{10}u^2 + q^{43}t^9u^3 + q^{44}t^9u^3 + q^{47}t^9u^3 + q^{48}t^8u^4.$

Spectral sequence diagrams, which reveal the structure of the proposed differentials, are included for these examples in Appendix B.

Computations with DAHA-Jones polynomials

Here we demonstrate explicitly how the DAHA-Jones polynomials are combined with the theory of differentials to produce our examples. First, we rewrite our proposal in DAHA conventions:

R, b	$HD^{E_6}_{\mathrm{r,s}}(\omega_1; a=?) = \widetilde{JD}^R(b)$	$\deg(d_{R,b})$
E_6, ω_1	-1	$(0,0,1)$
D_5, ω_1	$-t^{-4}$	$(0,4,1)$
A_6, ω_1	$-t^{-5}$	$(0,5,1)$
canceling	$-t^{-8}$	$(0,8,1)$
canceling	$-q^{-1}t^{-12}$	$(1,12,1)$

(6.22)

This is identically our proposal (6.18) for $\mathscr{H}^{\mathfrak{e}_6,\mathbf{27}}$ before the transformation (6.1). Now we consider each of our three examples individually.

The Trefoil $T^{3,2}$

The DAHA-Jones (E_6, ω_1) polynomial for the trefoil is

$$\widetilde{JD}^{E_6}_{3,2}(\omega_1; q, t) = 1 + qt + qt^4 - qt^9 - qt^{12} + q^2t^8 - q^2t^{13} - q^2t^{16} + q^2t^{21}. \tag{6.23}$$

To elevate this to a Poincaré polynomial with positive coefficients, we introduce an extra a-grading. For now this will only be a $\mathbb{Z}/2\mathbb{Z}$-grading ($a^{\underline{0}}$ or $a^{\underline{1}}$) compatible with the specialization $a = -1$:

$$\underline{HD}^{E_6}_{3,2}(\omega_1) = a^{\underline{0}} + qta^{\underline{0}} + qt^4a^{\underline{0}} + qt^9a^{\underline{1}} + qt^{12}a^{\underline{1}} + q^2t^8a^{\underline{0}} + q^2t^{13}a^{\underline{1}} + q^2t^{16}a^{\underline{1}} + q^2t^{21}a^{\underline{0}}. \tag{6.24}$$

Now we would like to lift this $\mathbb{Z}/2\mathbb{Z}$-grading to a genuine $\mathbb{Z}$-grading, for which we use the differential structure outlined above. Fortunately, this case is resolved rather easily by considering the

(D_n, ω_1) DAHA-Jones polynomial:

$$\widetilde{JD}^{D_n}_{3,2}(\omega_1; q, t) = 1 + qt + qt^{n-1} - qt^n - qt^{2n-2} + q^2t^{2n-2} - q^2t^{2n-1} - q^2t^{3n-3} + q^2t^{3n-2}, \quad (6.25)$$

which has the same dimension as $\widetilde{JD}^{E_6}_{3,2}$, so we can completely restore the a-grading by understanding just a single differential to some (D_n, ω_1), if one exists.

Indeed, such a differential to (D_5, ω_1) is indicated by the expression:

$$\underline{HD}^{E_6}_{3,2}(\omega_1) = \quad (6.26)$$

$$a^{\underline{0}} + qta^{\underline{0}} + qt^4a^{\underline{0}} + qt^5a^{\underline{1}} + qt^8a^{\underline{1}} + q^2t^8a^{\underline{0}} + q^2t^9a^{\underline{1}} + q^2t^{12}a^{\underline{1}} + q^2t^{13}a^{\underline{0}} + (1 + t^4a^{\underline{1}})(qt^5a^{\underline{0}} + qt^8a^{\underline{0}} + q^2t^9a^{\underline{0}} + q^2t^{12}a^{\underline{0}} + q^2t^{13}a^{\underline{1}} + q^2t^{17}a^{\underline{1}}).$$

Observe that the a-grading of this differential must be 1 if the corresponding specialization is to contain only integer powers of t. Thus, the a-grading of a generator corresponds to the number of canceling pairs of terms required to fit that generator into the expression above. For example, the generator $qt^9a^{\underline{1}}$ is realized in (6.26) as:

$$qt^9a^{\underline{1}} = qt^5a^{\underline{1}} + (1 + t^4a^{\underline{1}})qt^5a^{\underline{0}}, \quad (6.27)$$

so its a-grading is 1. However, the generator $q^2t^{21}a^{\underline{0}}$ is realized in (6.26) as:

$$q^2t^{21}a^{\underline{0}} = q^2t^{13}a^{\underline{0}} + (1 + t^4a^{\underline{1}})(q^2t^{13}a^{\underline{1}} + q^2t^{17}a^{\underline{1}}), \quad (6.28)$$

so its a-grading is 2. Overall, we restore the a-grading as a $\mathbb{Z}$-grading:

$$HD^{E_6}_{3,2}(\omega_1) = 1 + qt + qt^4 + qt^9a + qt^{12}a + q^2t^8 + q^2t^{13}a + q^2t^{16}a + q^2t^{21}a^2. \quad (6.29)$$

Observe that, as desired, we so far have the following specializations which determine the a-grading:

$$HD^{E_6}_{3,2}(\omega_1; a = -1) = \widetilde{JD}^{E_6}_{3,2}(\omega_1), \quad (6.30)$$
$$HD^{E_6}_{3,2}(\omega_1; a = -t^{-4}) = \widetilde{JD}^{D_5}_{3,2}(\omega_1). \quad (6.31)$$

We also find the two canceling differentials:

$$HD^{E_6}_{3,2}(\omega_1; a = -t^{-8}) = 1, \quad (6.32)$$
$$HD^{E_6}_{3,2}(\omega_1; a = -q^{-1}t^{-12}) = q^2t^8, \quad (6.33)$$

as well as the differential to (A_6, ω_1):

$$HD^{E_6}_{3,2}(\omega_1; a = -t^{-5}) = \widetilde{JD}^{A_6}_{3,2}(\omega_1). \quad (6.34)$$

The Torus Knot $T^{5,2}$

We repeat the above construction for $T^{5,2}$ and restore the a-grading to $\widetilde{JD}^{E_6}_{5,2}(\omega_1)$ in a way that includes all of the same structure. We have the DAHA-Jones (E_6, ω_1) polynomial for $T^{5,2}$:

$$\widetilde{JD}^{E_6}_{5,2}(\omega_1; q, t) \;= \tag{6.35}$$

$$1 + qt + qt^4 - qt^9 - qt^{12} + q^2t^2 + q^2t^5 + q^2t^8 - q^2t^{10} - 2q^2t^{13} - q^2t^{16} + q^2t^{21} + q^3t^9 + q^3t^{12} - q^3t^{14} - 2q^3t^{17} - q^3t^{20} + q^3t^{22} + q^3t^{25} + q^4t^{16} - q^4t^{21} - q^4t^{24} + q^4t^{29}.$$

As above, we introduce a mod-2 grading compatible with the specialization $a = -1$:

$$\underline{HD}^{E_6}_{5,2}(\omega_1) \;= \tag{6.36}$$

$$a^{\underline{0}} + qta^{\underline{0}} + qt^4a^{\underline{0}} + qt^9a^{\underline{1}} + qt^{12}a^{\underline{1}} + q^2t^2a^{\underline{0}} + q^2t^5a^{\underline{0}} + q^2t^8a^{\underline{0}} + q^2t^{10}a^{\underline{1}} + 2q^2t^{13}a^{\underline{1}} + q^2t^{16}a^{\underline{1}} + q^2t^{21}a^{\underline{0}} + q^3t^9a^{\underline{0}} + q^3t^{12}a^{\underline{0}} + q^3t^{14}a^{\underline{1}} + 2q^3t^{17}a^{\underline{1}} + q^3t^{20}a^{\underline{1}} + q^3t^{22}a^{\underline{0}} + q^3t^{25}a^{\underline{0}} + q^4t^{16}a^{\underline{0}} + q^4t^{21}a^{\underline{1}} + q^4t^{24}a^{\underline{1}} + q^4t^{29}a^{\underline{0}}.$$

The D_5 DAHA-Jones is:

$$\widetilde{JD}^{D_5}_{5,2}(\omega_1; q, t) \;= \tag{6.37}$$

$$1 + qt + qt^4 - qt^5 - qt^8 + q^2t^2 + q^2t^5 - q^2t^6 + q^2t^8 - 2q^2t^9 - q^2t^{12} + q^2t^{13} + q^3t^9 - q^3t^{10} + q^3t^{12} - 2q^3t^{13} + q^3t^{14} - q^3t^{16} + q^3t^{17} + q^4t^{16} - q^4t^{17} - q^4t^{20} + q^4t^{21},$$

which again has the same dimension as $\widetilde{JD}^{E_6}_{5,2}$, so we can restore the a-grading in the same manner:

$$HD^{E_6}_{5,2}(\omega_1) \;= \tag{6.38}$$

$$1 + qt + qt^4 + qt^9a + qt^{12}a + q^2t^2 + q^2t^5 + q^2t^8 + q^2t^{10}a + 2q^2t^{13}a + q^2t^{16}a + q^2t^{21}a^2 + q^3t^9 + q^3t^{12} + q^3t^{14}a + 2q^3t^{17}a + q^3t^{20}a + q^3t^{22}a^2 + q^3t^{25}a^2 + q^4t^{16} + q^4t^{21}a + q^4t^{24}a + q^4t^{29}a^2.$$

Observe that, as with the trefoil, we have specializations:

$$HD^{E_6}_{5,2}(\omega_1; a = -1) = \widetilde{JD}^{E_6}_{5,2}(\omega_1), \tag{6.39}$$

$$HD^{E_6}_{5,2}(\omega_1; a = -t^{-4}) = \widetilde{JD}^{D_5}_{5,2}(\omega_1), \tag{6.40}$$

$$HD^{E_6}_{5,2}(\omega_1; a = -t^{-8}) = 1, \tag{6.41}$$

$$HD^{E_6}_{5,2}(\omega_1; a = -q^{-1}t^{-12}) = q^4t^{16}, \tag{6.42}$$

$$HD^{E_6}_{5,2}(\omega_1; a = -t^{-5}) = \widetilde{JD}^{A_6}_{5,2}(\omega_1). \tag{6.43}$$

The Torus Knot $T^{4,3}$

We have the DAHA-Jones (E_6, ω_1) polynomial for $T^{4,3}$:

$$\widetilde{JD}^{E_6}_{4,3}(\omega_1; q, t) = \tag{6.44}$$

$$1 + qt + qt^4 - qt^9 - qt^{12} + q^2t + q^2t^2 + q^2t^4 + q^2t^5 + q^2t^8 - q^2t^9 - q^2t^{10} - q^2t^{12} - 2q^2t^{13} - q^2t^{16} + q^2t^{21} + q^3t^3 + q^3t^5 + q^3t^6 + q^3t^8 + q^3t^9 - q^3t^{10} - q^3t^{11} + q^3t^{12} - 3q^3t^{13} - 2q^3t^{14} - 2q^3t^{16} - 2q^3t^{17} + q^3t^{18} - q^3t^{20} + 2q^3t^{21} + q^3t^{22} + q^3t^{24} + q^3t^{25} + q^4t^8 + q^4t^9 + q^4t^{10} + q^4t^{12} - q^4t^{14} - q^4t^{15} - 3q^4t^{17} - 2q^4t^{18} - 2q^4t^{20} - q^4t^{21} + 2q^4t^{22} + q^4t^{23} - q^4t^{24} + 3q^4t^{25} + q^4t^{26} + q^4t^{28} + q^4t^{29} - q^4t^{30} - q^4t^{33} + q^5t^{13} + q^5t^{16} - q^5t^{18} - 3q^5t^{21} - 2q^5t^{24} + 2q^5t^{26} + 3q^5t^{29} + q^5t^{32} - q^5t^{34} - q^5t^{37} + q^6t^{24} - q^6t^{25} - q^6t^{28} + q^6t^{30} - q^6t^{32} + 2q^6t^{33} - q^6t^{34} + q^6t^{36} - q^6t^{38} - q^6t^{41} + q^6t^{42},$$

and the D_5 DAHA-Jones is:

$$\widetilde{JD}^{D_5}_{4,3}(\omega_1; q, t) \ = \tag{6.45}$$

$$1 + qt + qt^4 - qt^5 - qt^8 + q^2t + q^2t^2 + q^2t^4 - q^2t^6 - 2q^2t^9 - q^2t^{12} + q^2t^{13} + q^3t^3 + q^3t^5 - q^3t^7 + q^3t^8 - 2q^3t^9 - q^3t^{10} - q^3t^{12} + q^3t^{14} + q^3t^{17} + q^4t^8 - q^4t^{11} - q^4t^{13} + q^4t^{15} - q^4t^{16} + q^4t^{17}.$$

From the outset it is apparent that these do not have the same dimension, so the same approach will be less effective. However, we can try to assign a monomial in $\widetilde{JD}^{E_6}_{4,3}(\omega_1)$ to each monomial in $\widetilde{JD}^{D_5}_{4,3}(\omega_1)$ so that they coincide in the specialization $a = -t^{-4}$. That is, we consider the following subset of $\underline{HD}_{E_6}(\omega_1)$:

$$\underline{HD}_{D_5/E_6} \ = \tag{6.46}$$

$$a^{\underline{0}} + qta^{\underline{0}} + qt^4a^{\underline{0}} + qt^9a^{\underline{1}} + qt^{12}a^{\underline{1}} + q^2ta^{\underline{0}} + q^2t^2a^{\underline{0}} + q^2t^4a^{\underline{0}} + q^2t^{10}a^{\underline{1}} + 2q^2t^{13}a^{\underline{1}} + q^2t^{16}a^{\underline{1}} + q^2t^{21}a^{\underline{0}} + q^3t^3a^{\underline{0}} + q^3t^5a^{\underline{0}} + q^3t^{11}a^{\underline{1}} + q^3t^8a^{\underline{0}} + 2q^3t^{13}a^{\underline{1}} + q^3t^{14}a^{\underline{1}} + q^3t^{16}a^{\underline{1}} + q^3t^{22}a^{\underline{0}} + q^3t^{25}a^{\underline{0}} + q^4t^8a^{\underline{0}} + q^4t^{15}a^{\underline{1}} + q^4t^{17}a^{\underline{1}} + q^4t^{23}a^{\underline{0}} + q^4t^{20}a^{\underline{1}} + q^4t^{25}a^{\underline{0}},$$

which should specialize to $\widetilde{JD}^{D_5}_{4,3}$, and thus lifts to:

$$HD_{D_5/E_6} \ = \tag{6.47}$$

$$1 + qt + qt^4 + qt^9a + qt^{12}a + q^2t + q^2t^2 + q^2t^4 + q^2t^{10}a + 2q^2t^{13}a + q^2t^{16}a + q^2t^{21}a^2 + q^3t^3 + q^3t^5 + q^3t^{11}a + q^3t^8 + 2q^3t^{13}a + q^3t^{14}a + q^3t^{16}a + q^3t^{22}a^2 + q^3t^{25}a^2 + q^4t^8 + q^4t^{15}a + q^4t^{17}a + q^4t^{23}a^2 + q^4t^{20}a + q^4t^{25}a^2.$$

Now we turn our eye to the complementary subset:

$$\underline{HD}_{E_6 \setminus D_5} \ = \tag{6.48}$$

$$q^2t^5a^{\underline{0}} + q^2t^8a^{\underline{0}} + q^2t^9a^{\underline{1}} + q^2t^{12}a^{\underline{1}} + q^3t^6a^{\underline{0}} + q^3t^9a^{\underline{0}} + q^3t^{10}a^{\underline{1}} + q^3t^{12}a^{\underline{0}} + q^3t^{13}a^{\underline{1}} + q^3t^{14}a^{\underline{1}} + q^3t^{16}a^{\underline{1}} +$$

$2q^3t^{17}a^{\underline{1}}+q^3t^{18}a^{\underline{0}}+q^3t^{20}a^{\underline{1}}+2q^3t^{21}a^{\underline{0}}+q^3t^{24}a^{\underline{0}}+q^4t^9a^{\underline{0}}+q^4t^{10}a^{\underline{0}}+q^4t^{12}a^{\underline{0}}+q^4t^{14}a^{\underline{1}}+2q^4t^{17}a^{\underline{1}}+2q^4t^{18}a^{\underline{1}}+$
$q^4t^{20}a^{\underline{1}}+q^4t^{21}a^{\underline{1}}+2q^4t^{22}a^{\underline{0}}+q^4t^{24}a^{\underline{1}}+2q^4t^{25}a^{\underline{0}}+q^4t^{26}a^{\underline{0}}+q^4t^{28}a^{\underline{0}}+q^4t^{29}a^{\underline{0}}+q^4t^{30}a^{\underline{1}}+q^4t^{33}a^{\underline{1}}+q^5t^{13}a^{\underline{0}}+$
$q^5t^{16}a^{\underline{0}}+q^5t^{18}a^{\underline{1}}+3q^5t^{21}a^{\underline{1}}+2q^5t^{24}a^{\underline{1}}+2q^5t^{26}a^{\underline{0}}+3q^5t^{29}a^{\underline{0}}+q^5t^{32}a^{\underline{0}}+q^5t^{34}a^{\underline{1}}+q^5t^{37}a^{\underline{1}}+q^6t^{24}a^{\underline{0}}+q^6t^{25}a^{\underline{1}}+$
$q^6t^{28}a^{\underline{1}}+q^6t^{30}a^{\underline{0}}+q^6t^{32}a^{\underline{1}}+2q^6t^{33}a^{\underline{0}}+q^6t^{34}a^{\underline{1}}+q^6t^{36}a^{\underline{0}}+q^6t^{38}a^{\underline{1}}+q^6t^{41}a^{\underline{1}}+q^6t^{42}a^{\underline{0}}.$

We can use the degrees of the differentials (now known) to restore the a-grading on these generators. For example, $q^2t \in HD_{D_5/E_6}$ and $q^2t^9a^{\underline{1}} \in \underline{HD}_{E_6\backslash D_5}$ should cancel in the differential of degree $(0,8,1)$, so we restore the a-degree q^2t^9a on that generator. Carrying this out fully, we obtain:

$$HD_{E_6\backslash D_5} \;=\; \tag{6.49}$$

$$q^2t^5+q^2t^8+q^2t^9a+q^2t^{12}a+q^3t^6+q^3t^9+q^3t^{10}a+q^3t^{12}+q^3t^{13}a+q^3t^{14}a+q^3t^{16}a+2q^3t^{17}a+q^3t^{18}a^2+$$
$$q^3t^{20}a+2q^3t^{21}a^2+q^3t^{24}a^2+q^4t^9+q^4t^{10}+q^4t^{12}+q^4t^{14}a+2q^4t^{17}a+2q^4t^{18}a+q^4t^{20}a+q^4t^{21}a+2q^4t^{22}a^2+$$
$$q^4t^{24}a+2q^4t^{25}a^2+q^4t^{26}a^2+q^4t^{28}a^2+q^4t^{29}a^2+q^4t^{30}a^3+q^4t^{33}a^3+q^5t^{13}+q^5t^{16}+q^5t^{18}a+3q^5t^{21}a+2q^5t^{24}a+$$
$$2q^5t^{26}a^2+3q^5t^{29}a^2+q^5t^{32}a^2+q^5t^{34}a^3+q^5t^{37}a^3+q^6t^{24}+q^6t^{25}a+q^6t^{28}a+q^6t^{30}a^2+q^6t^{32}a+2q^6t^{33}a^2+$$
$$q^6t^{34}a^3+q^6t^{36}a^2+q^6t^{38}a^3+q^6t^{41}a^3+q^6t^{42}a^4.$$

Finally, observe that some generators that should cancel in certain specializations do not. For example, q^4t^9 should cancel in the differential of degree $(0,4,1)$, but there is no $q^4t^{13}a$. Taking all differentials into account, we add the generators:

$$\begin{gathered}\{q^4t^{13}, q^4t^{13}a, q^4t^{16}a, q^4t^{21}a^2, q^5t^{17}a, q^4t^{16}, q^5t^{17}, 2q^5t^{25}a^2, q^5t^{20},\\ q^5t^{28}a^2, q^5t^{22}a^2, q^5t^{20}a, q^4t^{21}a, q^5t^{22}a, q^5t^{30}a^3, 2q^5t^{25}a, q^5t^{33}a^3,\\ q^6t^{29}a^2, q^5t^{28}aq^6t^{29}a, q^6t^{37}a^3, q^5t^{30}a^2, q^5t^{33}a^2, q^6t^{37}a^2\},\end{gathered} \tag{6.50}$$

and take the sum $HD_{D_5/E_6} + HD_{E_6\backslash D_5}+$ (6.50) to obtain:

$$HD^{E_6}_{4,3}(\omega_1) \;=\; \tag{6.51}$$

$$1+qt+qt^4+qt^9a+qt^{12}a+q^2t+q^2t^2+q^2t^4+q^2t^5+q^2t^8+q^2t^9a+q^2t^{10}a+q^2t^{12}a+2q^2t^{13}a+q^2t^{16}a+$$
$$q^2t^{21}a^2+q^3t^3+q^3t^5+q^3t^6+q^3t^8+q^3t^9+q^3t^{10}a+q^3t^{11}a+q^3t^{12}+3q^3t^{13}a+2q^3t^{14}a+2q^3t^{16}a+2q^3t^{17}a+$$
$$q^3t^{18}a^2+q^3t^{20}a+2q^3t^{21}a^2+q^3t^{22}a^2+q^3t^{24}a^2+q^3t^{25}a^2+q^4t^8+q^4t^9+q^4t^{10}+q^4t^{12}+q^4t^{13}+q^4t^{13}a+q^4t^{14}a+$$
$$q^4t^{15}a+q^4t^{16}+q^4t^{16}a+3q^4t^{17}a+2q^4t^{18}a+2q^4t^{20}a+2q^4t^{21}a+q^4t^{21}a^2+2q^4t^{22}a^2+q^4t^{23}a^2+q^4t^{24}a+$$
$$3q^4t^{25}a^2+q^4t^{26}a^2+q^4t^{28}a^2+q^4t^{29}a^2+q^4t^{30}a^3+q^4t^{33}a^3+q^5t^{13}+q^5t^{16}+q^5t^{17}+q^5t^{17}a+q^5t^{18}a+q^5t^{20}+$$
$$q^5t^{20}a+3q^5t^{21}a+q^5t^{22}a+q^5t^{22}a^2+2q^5t^{24}a+2q^5t^{25}a+2q^5t^{25}a^2+2q^5t^{26}a^2+q^5t^{28}a+q^5t^{28}a^2+3q^5t^{29}a^2+$$
$$q^5t^{30}a^2+q^5t^{30}a^3+q^5t^{32}a^2+q^5t^{33}a^2+q^5t^{33}a^3+q^5t^{34}a^3+q^5t^{37}a^3+q^6t^{24}+q^6t^{25}a+q^6t^{28}a+q^6t^{29}a+q^6t^{29}a^2+$$
$$q^6t^{30}a^2+q^6t^{32}a+2q^6t^{33}a^2+q^6t^{34}a^3+q^6t^{36}a^2+q^6t^{37}a^2+q^6t^{37}a^3+q^6t^{38}a^3+q^6t^{41}a^3+q^6t^{42}a^4,$$

and verify that it satisfies:

$$HD^{E_6}_{4,3}(\omega_1; a=-1) = \widetilde{JD}^{E_6}_{4,3}(\omega_1), \tag{6.52}$$

$$HD^{E_6}_{4,3}(\omega_1; a=-t^{-4}) = \widetilde{JD}^{D_5}_{4,3}(\omega_1), \tag{6.53}$$

$$HD^{E_6}_{4,3}(\omega_1; a=-t^{-8}) = 1, \tag{6.54}$$

$$HD^{E_6}_{4,3}(\omega_1; a=-q^{-1}t^{-12}) = q^6 t^{24}, \tag{6.55}$$

$$HD^{E_6}_{4,3}(\omega_1; a=-t^{-5}) = \widetilde{JD}^{A_6}_{4,3}(\omega_1). \tag{6.56}$$

Further properties

We observe that our hyperpolynomials exhibit a number of potentially meaningful structures beyond their defining specializations/differentials.

Dimensions

First, observe that

$$HD^{E_6}_{\mathsf{r,s}}(\omega_1; q, \pm 1, a) = HD^{A}_{\mathsf{r,s}}(\omega_2; q, \pm 1, a) \tag{6.57}$$

in all examples considered, in spite of the fact that the weight ω_2 for A_n is non-minuscule. These relations generalize the special evaluations at $t = 1$ of DAHA-Jones polynomials and DAHA-superpolynomials. In particular, using the evaluation and super-duality theorems from [C5], equation (6.57) implies that

$$HD^{E_6}_{\mathsf{r,s}}(\omega_1; q, 1, a) = \left(HD^{A}_{\mathsf{r,s}}(\omega_1; q, 1, a)\right)^2. \tag{6.58}$$

In turn, we see that the *dimensions*

$$\dim HD^{E_6}_{\mathsf{r,s}} := HD^{E_6}_{\mathsf{r,s}}(\omega_1; 1, 1, 1) \tag{6.59}$$

are perfect squares. The dimensions for our examples $T^{3,2}$, $T^{5,2}$, and $T^{4,3}$ are 9, 25, and 121, respectively. These properties are analogues of the *refined exponential growth* [GS, GGS] for the exceptional groups.

Hat symmetry

We also have a "hat symmetry" corresponding to the involution of the Dynkin diagram for E_6 which sends $\omega_1 \mapsto \omega_6$. We define

$$\widehat{HD}^{E_6}_{\mathsf{r,s}}(\omega_1; q, t, a) := HD^{E_6}_{\mathsf{r,s}}(\omega_1; q \mapsto qt^4, t, a \mapsto at^{-4}), \tag{6.60}$$

which satisfies the specializations

$$\widehat{HD}^{E_6}_{\mathsf{r,s}}(\omega_1; q, t, -1) = \widetilde{JD}^{E_6}_{\mathsf{r,s}}(\omega_6; q, t), \tag{6.61}$$

$$\widehat{HD}^{E_6}_{\mathsf{r,s}}(\omega_1; q, t, -t^{-1}) = \widetilde{JD}^{A_6}_{\mathsf{r,s}}(\omega_1; qt^4, t), \tag{6.62}$$

$$\widehat{HD}^{E_6}_{\mathsf{r,s}}(\omega_1; q, t, -t^{-4}) = 1, \tag{6.63}$$

$$\widehat{HD}^{E_6}_{\mathsf{r,s}}(\omega_1; q, t, -q^{-1}t^{-12}) = q^{\alpha}t^{\beta}. \tag{6.64}$$

Other evaluations

We also have another potentially meaningful specialization of our hyperpolynomials at $a = q^{-1}t^{-9}$:

$$HD^{E_6}_{3,2}(\omega_1; a = -q^{-1}t^{-9}) = qt - q^2t^7 + q^2t^8, \tag{6.65}$$

$$HD^{E_6}_{5,2}(\omega_1; a = -q^{-1}t^{-9}) = q^2t^2 - q^2t^8 + q^3t^9 - q^3t^{15} + q^4t^{16}, \tag{6.66}$$

$$\begin{aligned} HD^{E_6}_{4,3}(\omega_1; a = -q^{-1}t^{-9}) = {} & q^3t^3 - q^3t^9 + q^4t^{10} - q^4t^{16} + q^5t^{13} - q^5t^{19} + q^5t^{17} \\ & -q^5t^{23} - q^4t^{12} + q^6t^{24}. \end{aligned} \tag{6.67}$$

We do not recognize the resulting polynomials. However, observe the significant reduction in the number of terms, as well as their regularity.

6.3 Singularities and differentials

In the previous section, we encountered several "exceptional" differentials that relate homological invariants of knots colored by representations of exceptional groups to knot homologies associated with classical groups. In this section we explain the origin of such differentials.

There are two general ways to predict *a priori* the structure of the differentials, both of which are rooted in physics. One approach [GS] involves analysis of the spectrum (3.18) of BPS states (a.k.a. Q-cohomology) and how it changes when one varies stability parameters, such as the Kähler modulus (3.6). The second approach [Go] is based on deformations of the Landau-Ginzburg potential, which for the 27-dimensional representation of $\mathfrak{g} = \mathfrak{e}_6$ has the form [GW]

$$W_{E_6,27} \;=\; z_1^{13} - \frac{25}{169} z_1 z_4^3 + z_4 z_1^9. \tag{6.68}$$

In general (and in every physics-based approach to knot homology), homology of the unknot can be represented as a Q-cohomology, i.e., the space of Q-closed but not Q-exact states (called BPS states) in a two-dimensional theory on a cylinder, $\mathbb{R} \times (\text{unknot}) = \mathbb{R} \times \mathbf{S}^1$. In some cases, this two-dimensional theory admits a Landau-Ginzburg description, which for certain Lie algebras $\mathfrak{g}$ and representations V has been identified in [GW]. In this approach, spectral sequences and differentials correspond to relevant deformations and RG flows of the two-dimensional "unknot theory" which, in the Landau-Ginzburg description, simply manifest as deformations of the potential.

Therefore, in our present problem we need to explore deformations of the potential (6.68) which correspond to the adjacencies of the singularity $Z_{3,0}$. Additionally, we perform a nontrivial verification of our calculations using the adjacency of the spectra of singularities. A good general reference for material in this section is [AGV].

Adjacency tree of the corank-2 singularity $Z_{3,0}$

Singularities and Adjacency

A singularity is an analytic apparatus that captures the local geometry of a holomorphic (smooth) function at a critical point. For our purposes, we will consider functions $f : \mathbb{C}^n \to \mathbb{C}$ and without loss of generality, critical points at $0 \in \mathbb{C}^n$.

Let $\mathcal{O}_n$ be the space of all germs at $0 \in \mathbb{C}^n$ of holomorphic functions $f : \mathbb{C}^n \to \mathbb{C}$. Then the group of germs of diffeomorphisms (biholomorphic maps) $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ acts on $\mathcal{O}_n$ by $g \cdot f = f \circ g^{-1}$. The orbits of this action define equivalence classes in $\mathcal{O}_n$, and those classes for which 0 is a critical point are called *singularities.* Consider a class L as a subspace of $\mathcal{O}_n$. An *l-parameter deformation* of $f \in L \subset \mathcal{O}_n$ with *base* $\Lambda = \mathbb{C}^l$ is the germ of a smooth map $F : \Lambda \to \mathcal{O}_n$ such that $F(0) = f$.

If L is contained in the closure of some other subspace, $L \subset \bar{K} \subset \mathcal{O}_n$, then an infinitesimal neighborhood of every $f \in L \subset \mathcal{O}_n$ intersects K nontrivially. This geometric notion can be reformulated equivalently in terms of deformations and gives rise to the concept of adjacency. That is, suppose that every function $f \in L$ can be transformed to a function in the class K by an arbitrarily small deformation. Here the "size" of a deformation is a restriction on $\lambda \in \Lambda$, induced by the standard metric on $\mathbb{C}^l$. In this case, we say that the singularity classes L, K are *adjacent*, written $L \to K$.

Versal deformations

Here we aim to find the adjacencies to the specific class $Z_{3,0}$, that is the classes K such that $Z_{3,0} \to K$. We go about this by considering a specific type of deformation.

A deformation $F : \Lambda \to \mathcal{O}_n$ of f is *versal* if every deformation of f is equivalent to one induced (by change of base Λ) from F. If, in addition, Λ has the smallest possible dimension, F is said to be *miniversal*, i.e., "minimal and universal."

We can construct an explicit miniversal deformation of $f \in L$ as follows. Let g_t be a path of diffeomorphisms of $(\mathbb{C}^n, 0)$ such that g_0 is the identity. Then the tangent space $T_f L$ consists of elements of the form

$$\frac{\partial}{\partial t}(f \circ g_t)|_{t=0} = \sum_{i=1}^{n} \frac{\partial f}{\partial z_i} \cdot \frac{\partial g_i}{\partial t}\bigg|_{t=0} . \tag{6.69}$$

In other words, the partial derivatives of f form an $\mathcal{O}_n$-linear basis for $T_f L$, motivating the following important invariants.

Let $I_{\nabla f} \subset \mathcal{O}_n$ be the gradient ideal, generated by the partial derivatives of f. Then we define the *local algebra* $A_f := \mathcal{O}_n / I_{\nabla f}$ and its *multiplicity* or *Milnor number* $\mu := \dim A_f$, which are both invariants of the singularity L.

Then if $\{\varphi_k\}$ is a monomial basis for A_f, we can define a miniversal deformation:

$$F(\lambda) = f + \sum_{k=1}^{\mu} \lambda_k \varphi_k. \tag{6.70}$$

Indeed, the graph of this deformation is a linear subspace of $\mathcal{O}_n$ which is centered at the germ $f \in L$ and is transversal to its orbit. In particular, this subspace will necessarily intersect every class adjacent to L. To determine these adjacent classes, we restrict to arbitrarily small $\epsilon \in \Lambda$ and use Arnold's algorithm [Ar1] to classify the possible $F(\epsilon)$.

Spectrum of a singularity

Here we define the vanishing cohomology of a singularity and the spectrum of its associated mixed Hodge structure. The latter is a highly nontrivial verification and refinement of adjacency.

Nonsingular fibers and monodromy

Let $f : \mathbb{C}^n \to \mathbb{C}$ be a germ with (isolated) critical point at $0 \in \mathbb{C}^n$ of multiplicity μ and critical value $f(0) = 0$. Let U be a small ball about $0 \in \mathbb{C}^n$ and B be a small ball about $0 \in \mathbb{C}$. If the radii of these balls are sufficiently small, the following holds [Mi].

Theorem 6.3.1. *For $b \in B' := B \backslash \{0\}$, the level set $X_b = f^{-1}(b) \cap U$ is a nonsingular hypersurface, homotopy equivalent to $\vee^{\mu} S^{n-1}$. The level set $X_0 = f^{-1}(0) \cap U$ is nonsingular away from 0.*

Then $f : X' \to B'$ (where $X' := f^{-1}(B') \cap U$) is a locally trivial fibration with fiber $X_b \simeq \vee^{\mu} S^{n-1}$. Suppose $b_0 \in \partial B$ is a noncritical value of f, and let $[\gamma] \in \pi_1(B', b_0) \cong \mathbb{Z}$. Then $\gamma(t)$ lifts to a continuous family of maps $h_t : X_{b_0} \to X_t$ which can be chosen so that h_0 is the identity on X_{b_0} and $h = h_1$ is the identity on $\partial X_{b_0} = f^{-1}(b_0) \cap \partial U$.

The map $h : X_{b_0} \to X_{b_0}$ is the *monodromy* of γ. The induced map on homology,

$$h_* : H_{n-1}(X_{b_0}) \to H_{n-1}(X_{b_0}), \tag{6.71}$$

is the corresponding *monodromy operator*, which is well-defined on the class $[\gamma]$. If, in addition, $[\gamma] \in \pi_1(B', b_0)$ is a counterclockwise generator, h_* is called the *classical-monodromy operator*.

Vanishing cohomology

Observe that the (reduced) integral [co]homology is nonzero only in dimension $n-1$, where $H_{n-1}(X_b) \cong \mathbb{Z}^{\mu}$. We construct a distinguished basis for this homology group by first considering the simple case where f has a nondegenerate critical point of multiplicity $\mu = 1$.

The Morse lemma tells us that in some neighborhood of $0 \in \mathbb{C}^n$, there is a coordinate system in which $f(\vec{z}) = z_1^2 + \cdots + z_n^2$. In this coordinate system, let $S^{n-1} = \{\vec{z} : \|\vec{z}\|^2 = 1, \mathrm{Im}(z_i) = 0\}$ and let $\varphi : [0,1] \to B$ be a path with $\varphi(0) = b_0$ and $\varphi(1) = 0$. Then the family of spheres,

$$S_t = \sqrt{\varphi(t)} S^{n-1} \subset X_{\varphi(t)}, \tag{6.72}$$

depends continuously on the parameter t and vanishes to the singular point $S_1 = 0 \in X_0$. The sphere $S_0 = \sqrt{b_0} S^{n-1}$ corresponds to a homology class $\Delta \in H_{n-1}(X_{b_0})$, called a *vanishing cycle.*

In the more general case that f has a degenerate critical point of arbitrary multiplicity μ, one can slightly perturb f into a function $f_\epsilon = f + \epsilon g$ with μ nondegenerate critical points in a small neighborhood of $0 \in \mathbb{C}^n$, having distinct critical values a_i. Now consider a system of paths $\varphi_1, \ldots, \varphi_\mu$ with $\varphi_i(0) = b_0$ and $\varphi_i(1) = a_i$. Suppose that these paths satisfy the following conditions:

1. The loops formed by traversing φ_i, followed by a small counterclockwise loop around a_i, followed by φ_i^{-1} generate $\pi_1(B', b_0)$;
2. The paths φ_i do not intersect themselves and intersect each other only at b_0 for $t = 0$;
3. The paths are indexed clockwise in $\arg \varphi_i(\epsilon)$.

Then, as above, each path φ_i determines a distinct vanishing cycle $\Delta_i \in H_{n-1}(X_{b_0})$, and the set $\{\Delta_1, \ldots, \Delta_\mu\}$ form a *distinguished basis of vanishing cycles* for the homology $H_{n-1}(X_{b_0}) \cong \mathbb{Z}^\mu$.

Mixed Hodge structure

For $f : X' \to B'$ as in Theorem 6.3.1, the μ-dimensional complex vector bundle $\pi_f^* : \mathcal{H}_f^* \to B'$, whose fibers are the complex [co]homology groups $H^{n-1}(X_b; \mathbb{C})$, is called the *vanishing [co]homology bundle* of the singularity f. There is a natural connection ∇ in the vanishing [co]homology bundle, called the *Gauss-Manin connection*, which is defined by covariant derivation ∇_b along the holomorphic vector field $\frac{\partial}{\partial b}$ on the base B'.

We would like to define a mixed Hodge structure in the vanishing cohomology bundle and so review the relevant definitions. Suppose we have an integer lattice $H_\mathbb{Z}$ in a real vector space $H_\mathbb{R} = H_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{R}$. Let $H = H_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{C}$ be its complexification. Then for $k \in \mathbb{Z}$, a *pure Hodge structure of weight* k on H is a decomposition:

$$H = \bigoplus_{p+q=k} H^{p,q}, \tag{6.73}$$

into complex subspaces satisfying $H^{p,q} = \overline{H^{q,p}}$, where the bar denotes complex conjugation in $\mathbb{C}$.

Equivalently, we may specify a Hodge structure by a *Hodge filtration*: a finite, decreasing filtration F^p on H satisfying $F^p \oplus \overline{F^{p+1}} = H$. Indeed, from a Hodge filtration, one can recover a Hodge structure by $H^{p,q} = F^p \cap \overline{F^q}$, and from a Hodge structure, one can recover a Hodge filtration by $F^p = \oplus_{i \geq p} H^{i,k-i}$. We generalize these notions to a *mixed Hodge structure* on H, specified by

1. A *weight filtration*: a finite, increasing filtration W_k on H which is the complexification of an increasing filtration on $H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$,

2. A *Hodge filtration*: a finite, decreasing filtration F^p on H,

such that for each k, the filtration,

$$F^p gr_k^W H := (F^p \cap W_k + W_{k-1})/W_{k-1}, \tag{6.74}$$

satisfies $F^p gr_k^W H \oplus \overline{F^{k-p+1} gr_k^W H} = gr_k^W H$. That is, $F^p gr_k^W H$ induces a pure Hodge structure of weight k on $gr_k^W H := W_k/W_{k-1}$.

The vanishing cohomology is obtained as the complexification of the integral cohomology of the nonsingular fibers X_b. So to define a mixed Hodge structure in the vanishing cohomology, it remains to specify the relevant weight and Hodge filtrations there. We follow the construction of [V1, V2, V3].

Hodge filtration

To obtain a Hodge filtration, first consider a holomorphic $(n-1)$-form ω defined in a neighborhood of $0 \in \mathbb{C}^n$. Since X_b is a complex $(n-1)$-manifold, the restriction $\omega_b = \omega|_{X_b}$ represents a cohomology class $[\omega_b] \in H^{n-1}(X_b, \mathbb{C})$ for all $b \in B'$. That is, ω defines a global section $s_\omega : B' \to \mathcal{H}_f^*$, $b \mapsto [\omega_b]$ of the vanishing cohomology bundle.

In the neighborhood of every nonsingular manifold X_b, there exists a holomorphic $(n-1)$-form ω/df, with the property that $\omega = df \wedge \omega/df$ in that neighborhood. As above, the restriction $\omega/df_b = \omega/df|_{X_b}$, called the *residue form*, represents a cohomology class $[\omega/df_b] \in H^{n-1}(X_b, \mathbb{C})$ and defines a global section $\sigma_\omega : B' \to \mathcal{H}_f^*$, $b \mapsto [\omega/df_b]$ of the vanishing cohomology bundle.

The section σ_ω is called a *geometric section.* For a set of μ forms that do not satisfy a complex analytic relation, the set of their geometric sections trivalizes the vanishing cohomology bundle, i.e., the corresponding residue forms are a basis in each fiber.

The above sections are holomorphic, meaning that if δ_b is a cycle in the (integer) homology of the fiber which depends continuously on b, (i.e., is covariantly constant via the Gauss-Manin connection), then the map $\sigma_\omega \delta : b \mapsto \int_{\delta_b} \sigma_\omega(b)$ is a holomorphic map $B' \to \mathbb{C}$. We consider an asymptotic expansion of such a map around zero.

For example, in the simple case (6.72) of a nondegenerate critical point, one can easily see that

$$s_\omega S(b) = \int_{S_b} s_\omega(b) = \int_{B_b} ds_\omega(b) = cb^{n/2} + \cdots . \tag{6.75}$$

S_b is the vanishing sphere, B_b is its interior, and the expansion is proportional to $\mathrm{vol} B_b$ and $d\omega|_0$.

For general f with a (possibly degenerate) critical point at 0, we can take a set of forms $\omega_1, \dots, \omega_\mu$ such that their geometric sections trivialize the vanishing cohomology bundle. Then analysis of the Picard-Fuchs equations of these geometric sections yields the following theorem:

Theorem 6.3.2. *Let δ_b be a continuous family of vanishing cycles over the sector $\theta_0 < arg b < \theta_1$ in B'. Let σ_ω be a section of the vanishing cohomology. Then the corresponding integral admits an asymptotic expansion:*

$$\sigma_\omega \delta(b) = \int_{\delta_b} \omega/df_b = \sum_{k,\alpha} \frac{T_{k,\alpha} b^\alpha (\log b)^k}{k!}, \tag{6.76}$$

which converges for b sufficiently close to 0. The numbers $e^{2\pi i\alpha}$ are the eigenvalues of the classical monodromy operator.

If we fix ω, the coefficients $T_{k,\alpha}$ do not depend on b, but they do depend linearly on the section δ, and so determine sections $\tau^\omega_{k,\alpha}$ of the vanishing cohomology bundle via the pairing $\langle \tau^\omega_{k,\alpha}, \delta\rangle = T_{k,\alpha}$ between homology and cohomology. Thus, we can rewrite the asymptotic expansion (6.76) as a series expansion of the geometric section:

$$\sigma_\omega = \sum_{k,\alpha} \frac{\tau^\omega_{k,\alpha} b^\alpha (\ln b)^k}{k!}. \tag{6.77}$$

This expansion induces a filtration of the vanishing cohomology as follows. Define

$$\alpha(\omega) \coloneqq \min\{\alpha : \exists k \geq 0 \text{ such that } \tau^\omega_{k,\alpha} \neq 0\}. \tag{6.78}$$

Given a geometric section σ_ω, the number $\alpha(\omega)$ is its *order*, and the corresponding expansion,

$$\Sigma_\omega \coloneqq \sum_{k} \frac{\tau^\omega_{k,\alpha(\omega)} b^{\alpha(\omega)} (\ln b)^k}{k!}, \tag{6.79}$$

is its *principal part*. Now define a finite, decreasing filtration of F^p_b of the fiber $H^{n-1}(X_b;\mathbb{C})$ by

$$F^p_b \coloneqq \langle \Sigma_{\omega,b} : \alpha(\omega) \leq n-p-1\rangle \subseteq H^{n-1}(X_b;\mathbb{C}), \tag{6.80}$$

and the *asymptotic Hodge filtration* filtration of the vanishing cohomology bundle by

$$F^p \coloneqq \bigcup_b F^p_b. \tag{6.81}$$

Weight filtration

Suppose we have a nilpotent operator N acting on a finite-dimensional vector space H. Then there is exactly one finite, increasing filtration W_k on H which satisfies:

1. $N(W_k) \subset W_{k-2}$,
2. $N^k : W_{r+k}/W_{r+k-1} \cong W_{r-k}/W_{r-k-1}$ for all k,

called the *weight filtration of index* r of N.

We obtain a weight filtration in the vanishing cohomology bundle using this construction and the classical-monodromy operator M. As is true for any invertible linear operator, M has a Jordan-

Chevalley decomposition $M = M_u M_s$ into commuting unipotent and semisimple parts. Define a nilpotent operator N to be the logarithm of the unipotent part:

$$N := \sum_i \frac{(-1)^{i+1}(M_u - I)^i}{i}. \tag{6.82}$$

Now for each eigenvalue λ of the monodromy operator on $H^{n-1}(X_b;\mathbb{C})$, let $H_{\lambda,b}$ be the corresponding root subspace. Define a filtration $W_{k,b,\lambda}$ according to the following rules:

1. If $\lambda = 1$, let $W_{k,b,\lambda}$ be the weight filtration of index n of N on $H_{\lambda,s}$,
2. If $\lambda \neq 1$, let $W_{k,b,\lambda}$ be the weight filtration of index $n-1$ of N on $H_{\lambda,s}$.

Now define a filtration $W_{k,b}$ of the fiber $H^{n-1}(X_b;\mathbb{C})$ by

$$W_{k,b} := \bigoplus_\lambda W_{k,b,\lambda}, \tag{6.83}$$

and a filtration W_k of the vanishing cohomology bundle by

$$W_k := \bigcup_b W_{k,b}. \tag{6.84}$$

The subbundle W_k is the weight filtration in the vanishing cohomology bundle. Now we may state the following theorem from [V3].

Theorem 6.3.3. *For all k and p, the filtrations W_k and F^p are analytic subbundles of the vanishing cohomology bundle, which are invariant under the action of the semisimple part of the monodromy operator. Furthermore, they specify a mixed Hodge structure in the vanishing cohomology bundle:*

$$gr_k^W H = \bigoplus_{p+q=k} H^{p,q}, \tag{6.85}$$

where $H^{p,q} := F^p \cap W_k / F^{p+1} \cap W_k + W_{k-1}$.

Spectrum

In light of Theorem 6.3.3, we are now in a position to define the spectrum of a singularity $f \in K$.

Let $f \in K$ be a singularity. If λ is an eigenvalue of the semisimple part of the classical-monodromy operator on $H^{p,q}$, one can associate to f the set of μ rational numbers:

$$\{n - 1 - l_p\lambda\}, \quad \text{where} \quad l_p\lambda := \log(\lambda/2\pi i) \quad \text{and} \quad \mathrm{Re}(l_p\lambda) = p. \tag{6.86}$$

This (unordered) set of numbers is the *spectrum* of the singularity K.

To see what the spectrum of a singularity f has to do with adjacencies to f, we first construct a fibration, analagous to the fibration $f : X' \to B'$. Choose a miniversal deformation,

$$F(z, \lambda) = f(z) + \sum_{i=0}^{\mu-1} \lambda_i \varphi_i(z), \tag{6.87}$$

where $\lambda \in \mathbb{C}^\mu$ and $\varphi_0 = 1$. As before, we consider sufficiently small ball U about $0 \in \mathbb{C}^n$ and a another small ball, this time Λ about $0 \in \mathbb{C}^\mu$.

For $\lambda \in \Lambda$, define the level set $V_\lambda := \{z \in U : F(z, \lambda) = 0\}$ and the hypersurface $V := \{(z, \lambda) \in U \times \Lambda : F(z, \lambda) = 0\}$. Let $\Sigma \subset \Lambda$ be the set of values of λ for which V_λ is singular, called the *level bifurcation set*. Let $\pi_\Lambda : V \to \Lambda$ be the restriction of the canonical projection, called the *Whitney map*. Finally, let $\Lambda' := \Lambda \backslash \Sigma$ and $V' := \pi_\Lambda^{-1}(\Lambda')$. The locally trivial fibration $\pi_\Lambda : V' \to \Lambda'$ with fiber V_λ over $\lambda \in \Lambda'$ is the *Milnor fibration* of f.

Observe that the fibration $f : X' \to B'$ can be embedded in the Milnor fibration by identifying B' with the λ_0-axis in the base Λ' (recall that $\varphi_0 = 1$). Furthermore, we can repeat the constructions outlined above for the Milnor fibration and then ask how the spectrum varies as we vary the deformation parameter λ in an infinitesimal neighborhood of 0. This leads to observations on the *semicontinuity* of the spectrum, including the following [Ar2].

Theorem 6.3.4. *Suppose that a critical point of type L has (ordered) spectrum $\alpha_1 \leq \cdots \leq \alpha_\mu$ and a critical point of type L' has spectrum $\alpha'_1 \leq \cdots \leq \alpha'_{\mu'}$ where $\mu' \leq \mu$. Then a necessary condition for the adjacency $L \to L'$ is that the spectra be adjacent in the sense that $\alpha_i \leq \alpha'_i$.*

6.4 Exceptional series

General procedure

Here we consider the "exceptional series":

$$e \subset A_1 \subset A_2 \subset G_2 \subset D_4 \subset F_4 \subset E_6 \subset E_7 \subset E_8, \tag{6.88}$$

discussed in [DG]. This is actually the bottom row of the triangle considered in that paper.

Recall that the algebraic groups G in this series are given a parameter ν in that paper as follows:

$$\nu(G) = \frac{h^\vee}{6}, \tag{6.89}$$

where $h^\vee$ is the dual Coxeter number of G. This very quantity provides the specializations of our hyperpolynomials.

The *E-hyperpolynomials* we will construct below unify the DAHA-Jones polynomials (also called refined polynomials) for $T^{3,2}$, $T^{4,3}$ "colored" by the adjoint representation for the groups of type ADE in this series. The root systems G_2 and F_4 play an important role in the exceptional series, but we cannot incorporate them so far (see also the end of this section).

As with the (colored) superpolynomial and hyperpolynomials of [C5, C6] and in Chapter 5, this unification works by packaging the corresponding DAHA-Jones polynomials into a single polynomial, denoted by $HD^{\mathfrak{ad}}_{\mathsf{r,s}}(q,t,a)$, with an additional parameter a, where the individual polynomials are recovered via the following specializations:

$$HD^{\mathfrak{ad}}_{\mathsf{r,s}}(q,t,a=-t^{\nu(G)}) = \widetilde{JD}^{G}_{\mathsf{r,s}}(\mathfrak{ad};q,t), \text{ excluding } G_2, F_4. \tag{6.90}$$

Thus a is associated with the (dual) Coxeter number, rather than with the rank. Relations (6.90) appeared sufficient to determine $HD^{\mathfrak{ad}}$ for $T^{3,2}$ and $T^{4,3}$, but this cannot be expected for arbitrary torus knots.

In general, such polynomials cannot be uniquely determined via these specializations for sufficiently complicated torus knots; one needs an infinite family of root systems in (6.90) to restore a for any knots. Practically speaking, however, only two specializations to E_8 and E_7 are enough for the trefoil. We will demonstrate this in detail below. Even more convincingly, the three specializations to E_8, E_7, and E_6 were enough for $T^{4,3}$; the resulting polynomial has hundreds of terms.

Here we construct $HD^{\mathfrak{ad}}_{\mathsf{r,s}}$ for two knots, the trefoil $T^{3,2}$ and $T^{4,3}$. We will call this polynomial the *adjoint exceptional hyperpolynomial*, since we consider only the adjoint representations. As in [C5], we use the name "hyperpolynomial," since "superpolynomial" is commonly reserved for the root systems of type A.

For the trefoil we will show explicitly how $HD^{\mathfrak{ad}}_{3,2}$ is obtained from the relevant DAHA-Jones polynomials for E_8, E_7 and the adjoint representation $\mathfrak{ad}$ whose highest weight is the highest short root ϑ.

For $T^{4,3}$, we obtain $HD^{\mathfrak{ad}}_{4,3}$ using the same procedure, though E_6 is also required to find some coefficients. Since the DAHA-Jones polynomials in these cases are rather long, we do not include them and instead refer the reader to [C5] where they are posted.

Both $HD^{\mathfrak{ad}}_{3,2}$ and $HD^{\mathfrak{ad}}_{4,3}$ will satisfy all six of the defining specializations from (6.90), even though they are only constructed using two and three of these specializations, respectively. This is a convincing confirmation that the formulas we found are meaningful. See Section 6.4, where we discuss this relations and some further interesting symmetries.

E-type hyperpolynomials

Trefoil

Here we will demonstrate how $HD^{\mathfrak{ad}}_{3,2}(q,t,a)$ is obtained from only the specializations (6.90) for G of types E_8, E_7. The relevant DAHA-Jones polynomial for E_8 from [C6] is

$$\widetilde{JD}^{E_8}_{3,2}(\omega_8;q,t) \ = \tag{6.91}$$

$$1+q(t+t^6+t^{10}-t^{20}-t^{24}-t^{29})+q^2(t^{12}+t^{16}+t^{20}-t^{26}+t^{29}-3t^{30}-t^{34}-t^{35}-t^{39}+t^{44}+t^{49}+t^{53})+ q^3(t^{29}+t^{35}-t^{36}+t^{39}-t^{40}-t^{41}-t^{45}-2t^{49}+t^{50}-t^{53}+t^{54}+t^{55}-t^{58}+2t^{59}+t^{63}-t^{73})+q^4(t^{58}-t^{59}- t^{64}+t^{65}-t^{68}+t^{69}+t^{78}-t^{79}+t^{82}-t^{83})+q^5(-t^{87}+t^{88}),$$

and the relevant DAHA-Jones polynomial for E_7 is

$$\widetilde{JD}^{E_7}_{3,2}(\omega_1;q,t) \ = \tag{6.92}$$

$$1+q(t+t^4+t^6-t^{12}-t^{14}-t^{17})+q^2(t^8+t^{10}+t^{12}-t^{16}+t^{17}-3t^{18}-t^{20}-t^{21}-t^{23}+t^{26}+t^{29}+t^{31})+ q^3(t^{17}+t^{21}-t^{22}+t^{23}-t^{24}-t^{25}-t^{27}-2t^{29}+t^{30}-t^{31}+t^{32}+t^{33}-t^{34}+2t^{35}+t^{37}-t^{43})+q^4(t^{34}-t^{35}- t^{38}+t^{39}-t^{40}+t^{41}+t^{46}-t^{47}+t^{48}-t^{49})+q^5(-t^{51}+t^{52}).$$

The (lexicographic) order in which these two polynomials are printed gives a perfect, one-to-one correspondence between their terms, which respects the signs $\pm$ of these terms.

For example, in this correspondence $-q^2t^{39}$ in the E_8 polynomial is paired with $-q^2t^{23}$ in the E_7 polynomial. Determining the common exponent x of a that satisfies the right specializations from (6.90) readily reduces to finding a solution to $39-5x=23-3x$, since $\nu(E_8)=5$ and $\nu(E_7)=3$. Evidently, this solution is $x=8$, and the corresponding term in $HD^{\mathfrak{ad}}_{3,2}$ will then be $-q^2t^{-1}a^8$.

Applying this procedure to every pair of terms in these two polynomials, the *adjoint exceptional hyperpolynomial* for the trefoil is

$$HD^{\mathfrak{ad}}_{3,2}(q,t,a) \ = \tag{6.93}$$

$$1+q(t-ta+a^2-a^4+t^{-1}a^5-t^{-1}a^6)+q^2(t^2a^2-ta^3+a^4+ta^5+t^{-1}a^6-3a^6+t^{-1}a^7+a^7-t^{-1}a^8- t^{-1}a^9+t^{-1}a^{10}-t^{-2}a^{11})+q^3(t^{-1}a^6-a^7+ta^7+t^{-1}a^8-a^8-ta^8+a^9-2t^{-1}a^{10}+a^{10}+t^{-2}a^{11}-t^{-1}a^{11}- a^{11}-t^{-2}a^{12}+2t^{-1}a^{12}-t^{-2}a^{13}+t^{-2}a^{15})+q^4(t^{-2}a^{12}-t^{-1}a^{12}+t^{-1}a^{13}-a^{13}-t^{-2}a^{14}+t^{-1}a^{14}+t^{-2}a^{16}- t^{-1}a^{16}-t^{-3}a^{17}+t^{-2}a^{17})+q^5(-t^{-3}a^{18}+t^{-2}a^{18}).$$

$(4,3)$-torus knot

As was mentioned above, we will not provide the corresponding formulas for DAHA-Jones polynomials for $E_{6,7,8}$ from [C6] here, since they are long. The adjoint exceptional hyperpolynomial for the torus knot $T^{4,3}$ can be constructed using essentially the same method as that for the trefoil. However, since the DAHA-Jones polynomials $\widetilde{JD}^{E_8}_{4,3}$ and $\widetilde{JD}^{E_7}_{4,3}$ have different numbers of terms, their lexicographic orderings are (for some powers of q) insufficient to determine a correspondence between their respective monomials. These few ambiguities are resolved by also considering $\widetilde{JD}^{E_6}_{4,3}$.

Once such a correspondence between triples of monomials is established, the a-degrees are uniquely restored using the relevant specializations from (6.90), as for the trefoil. The resulting hyperpolynomial is long, but we think that the formula must be provided, since it has various symmetries beyond those discussed here and we expect that further relations will be found. For instance, its connection to the root systems F_4, G_2 is an open problem. One has:

$$HD^{\mathfrak{ad}}_{4,3}(q,t,a) \;=\; \tag{6.94}$$

$$1+q(-t^{-1}a^6+t^{-1}a^5-a^4+a^2-ta+t)+q^2(-t^{-2}a^{11}+t^{-1}a^{10}-t^{-1}a^9-t^{-1}a^8+t^{-1}a^7+a^7-4a^6+ta^5+t^{-1}a^5+$$
$$a^5-ta^4-ta^3+t^2a^2+ta^2+a^2-t^2a-ta+t^2+t)+q^3(t^{-2}a^{15}-t^{-2}a^{13}+3t^{-1}a^{12}-3t^{-1}a^{11}-t^{-2}a^{11}-a^{11}+t^{-2}a^{10}+$$
$$3a^{10}-t^{-1}a^9-ta^8-t^{-1}a^8-3a^8+4ta^7+2t^{-1}a^7+2a^7-t^2a^6-4ta^6+t^{-1}a^6-4a^6+t^2a^5+2ta^5+a^5+a^4-t^3a^3-$$
$$t^2a^3-2ta^3+t^3a^2+2t^2a^2+ta^2-t^3a-t^2a+t^3)+q^4(2t^{-2}a^{17}-2t^{-1}a^{16}-t^{-3}a^{16}+2t^{-1}a^{15}+t^{-2}a^{15}+2t^{-1}a^{14}-$$
$$t^{-2}a^{14}-2t^{-1}a^{13}-2t^{-2}a^{13}-3a^{13}+ta^{12}+3t^{-1}a^{12}+6a^{12}-3ta^{11}-5t^{-1}a^{11}+t^{-2}a^{11}-4a^{11}+2ta^{10}-2t^{-1}a^{10}+$$
$$2a^{10}+t^2a^9+2ta^9+t^{-1}a^9+2a^9-4t^2a^8-5ta^8+t^{-1}a^8-6a^8+t^3a^7+4t^2a^7+7ta^7+t^{-1}a^7-t^3a^6-4t^2a^6-2ta^6+$$
$$t^{-1}a^6-ta^5+t^4a^4+t^3a^4+3t^2a^4+ta^4+a^4-t^4a^3-2t^3a^3-t^2a^3-ta^3+t^4a^2+t^3a^2+t^2a^2)+q^5(-t^{-2}a^{21}+t^{-3}a^{20}+$$
$$t^{-2}a^{19}-3t^{-1}a^{18}-t^{-3}a^{18}+3t^{-1}a^{17}+4t^{-2}a^{17}-t^{-3}a^{17}+2a^{17}-2t^{-1}a^{16}-t^{-2}a^{16}-3a^{16}+2t^{-1}a^{15}-2t^{-2}a^{15}+$$
$$3ta^{14}+5t^{-1}a^{14}-t^{-2}a^{14}+4a^{14}-t^2a^{13}-6ta^{13}-4t^{-1}a^{13}-8a^{13}+3t^2a^{12}+6ta^{12}-3t^{-1}a^{12}+9a^{12}-2t^2a^{11}-4ta^{11}-$$
$$t^{-1}a^{11}+2t^{-2}a^{11}+a^{11}-t^3a^{10}-2t^2a^{10}-4ta^{10}-2t^{-1}a^{10}-3a^{10}+4t^3a^9+5t^2a^9+7ta^9+t^{-1}a^9+a^9-t^4a^8-4t^3a^8-$$
$$7t^2a^8-2ta^8+2t^{-1}a^8-2a^8+t^4a^7+2t^3a^7+2t^2a^7+ta^7-2a^7+t^4a^6+t^2a^6+ta^6+t^{-1}a^6+a^6-t^5a^5-t^4a^5-2t^3a^5-$$
$$t^2a^5-ta^5+t^5a^4+t^4a^4+t^3a^4+t^2a^4-t^4a^3)+q^6(-t^{-2}a^{23}+t^{-1}a^{22}+2t^{-3}a^{22}-t^{-1}a^{21}-2t^{-2}a^{21}+t^{-3}a^{21}-t^{-1}a^{20}+$$
$$t^{-2}a^{20}+2t^{-1}a^{19}+4t^{-2}a^{19}-t^{-3}a^{19}+3a^{19}-2ta^{18}-7t^{-1}a^{18}+2t^{-2}a^{18}-t^{-3}a^{18}-4a^{18}+3ta^{17}+4t^{-1}a^{17}-t^{-2}a^{17}-$$
$$2t^{-3}a^{17}+5a^{17}+4t^{-1}a^{16}+t^{-3}a^{16}-2a^{16}-3t^2a^{15}-4ta^{15}-t^{-1}a^{15}-3t^{-2}a^{15}-6a^{15}+t^3a^{14}+6t^2a^{14}+8ta^{14}-$$
$$2t^{-2}a^{14}+6a^{14}-3t^3a^{13}-4t^2a^{13}-9ta^{13}+3t^{-2}a^{13}+a^{13}+2t^2a^{12}-ta^{12}-7t^{-1}a^{12}+t^{-2}a^{12}-a^{12}+t^4a^{11}+3t^3a^{11}+$$
$$2t^2a^{11}+3ta^{11}+t^{-1}a^{11}+t^{-2}a^{11}+3a^{11}-4t^4a^{10}-2t^3a^{10}-6t^2a^{10}-ta^{10}+t^{-1}a^{10}-2a^{10}+t^5a^9+t^4a^9+4t^3a^9+2ta^9-$$
$$3a^9-t^3a^8+2ta^8+t^{-1}a^8+a^8-t^5a^7-t^3a^7-ta^7-a^7+t^6a^6+t^4a^6+t^2a^6+a^6)+q^7(-t^{-3}a^{26}+t^{-1}a^{24}+t^{-3}a^{24}-$$
$$t^{-1}a^{23}-4t^{-2}a^{23}+2t^{-3}a^{23}-a^{23}+3t^{-1}a^{22}-t^{-2}a^{22}+t^{-3}a^{22}-t^{-4}a^{22}+a^{22}-t^{-1}a^{21}+2t^{-2}a^{21}+2t^{-3}a^{21}+a^{21}-$$
$$3ta^{20}-5t^{-1}a^{20}+2t^{-2}a^{20}-t^{-3}a^{20}-2a^{20}+2t^2a^{19}+3ta^{19}+t^{-2}a^{19}-3t^{-3}a^{19}+7a^{19}-2t^2a^{18}-2ta^{18}+3t^{-1}a^{18}+$$
$$5t^{-2}a^{18}-t^{-3}a^{18}-4a^{18}-2t^2a^{17}+2ta^{17}-t^{-1}a^{17}-6t^{-2}a^{17}+t^{-3}a^{17}-4a^{17}+3t^3a^{16}+2t^2a^{16}+5ta^{16}+5t^{-1}a^{16}-$$
$$3t^{-2}a^{16}+a^{16}-t^4a^{15}-3t^3a^{15}-3t^2a^{15}-4ta^{15}+2t^{-1}a^{15}+t^{-2}a^{15}+t^4a^{14}+5t^2a^{14}-3ta^{14}-7t^{-1}a^{14}+t^{-2}a^{14}+$$
$$t^4a^{13}-t^3a^{13}+2t^2a^{13}-ta^{13}-t^{-1}a^{13}+t^{-2}a^{13}+7a^{13}-t^5a^{12}-t^4a^{12}-t^3a^{12}-t^2a^{12}-2ta^{12}-t^{-1}a^{12}+t^{-2}a^{12}-a^{12}+$$
$$t^5a^{11}+2t^3a^{11}-t^2a^{11}+ta^{11}-a^{11}-t^4a^{10}+t^3a^{10}-t^2a^{10}+2ta^{10}+t^{-1}a^{10}-t^2a^9-a^9+ta^8)+q^8(-t^{-3}a^{28}+t^{-2}a^{27}-$$
$$t^{-3}a^{27}+t^{-4}a^{26}-2t^{-2}a^{25}+t^{-3}a^{25}-a^{25}+ta^{24}+4t^{-1}a^{24}-3t^{-2}a^{24}+t^{-3}a^{24}-t^{-4}a^{24}+t^{-1}a^{23}+5t^{-3}a^{23}-2t^{-4}a^{23}-$$
$$2a^{23}-ta^{22}-2t^{-1}a^{22}-2t^{-2}a^{22}+t^{-3}a^{22}+a^{22}+2t^2a^{21}-2t^{-1}a^{21}+3t^{-2}a^{21}-3t^{-3}a^{21}+4a^{21}-t^3a^{20}-4ta^{20}-$$
$$t^{-1}a^{20}+6t^{-2}a^{20}-2t^{-3}a^{20}+2a^{20}+t^2a^{19}+ta^{19}-5t^{-1}a^{19}-t^{-2}a^{19}+t^{-3}a^{19}-a^{19}+t^3a^{18}-t^2a^{18}+2ta^{18}+6t^{-1}a^{18}-$$
$$5t^{-2}a^{18}-t^4a^{17}-2t^2a^{17}+ta^{17}+3t^{-1}a^{17}-t^{-2}a^{17}+t^{-3}a^{17}-3a^{17}+t^3a^{16}-t^{-1}a^{16}-2a^{16}-t^3a^{15}+t^2a^{15}-t^{-1}a^{15}+$$
$$3a^{15}-2ta^{14}-t^{-1}a^{14}+t^{-2}a^{14}+a^{14}-t^{-1}a^{13}+a^{13}+t^{-2}a^{12})+q^9(t^{-2}a^{29}-t^{-3}a^{29}-t^{-1}a^{28}+t^{-2}a^{28}-t^{-3}a^{28}+$$
$$t^{-4}a^{28}-2t^{-3}a^{27}+2t^{-4}a^{27}+t^{-1}a^{26}-t^{-2}a^{26}+2t^{-1}a^{25}-t^{-2}a^{25}+3t^{-3}a^{25}-2t^{-4}a^{25}-2a^{25}+ta^{24}+t^{-1}a^{24}-$$
$$5t^{-2}a^{24}+5t^{-3}a^{24}-t^{-4}a^{24}-a^{24}+2t^{-1}a^{23}-t^{-2}a^{23}-t^{-3}a^{23}-ta^{22}-t^{-1}a^{22}+3t^{-2}a^{22}-2t^{-3}a^{22}+a^{22}+t^2a^{21}-$$

$ta^{21}-3t^{-1}a^{21}+2t^{-2}a^{21}+a^{21}-t^{-1}a^{20}-t^{-2}a^{20}+2a^{20}+t^{-1}a^{19}-a^{19}-t^{-2}a^{18}+t^{-3}a^{18}\big)+q^{10}\big(-t^{-3}a^{29}+2t^{-4}a^{29}-t^{-5}a^{29}+t^{-2}a^{28}-2t^{-3}a^{28}+t^{-4}a^{28}-t^{-2}a^{26}+2t^{-3}a^{26}-t^{-4}a^{26}+t^{-1}a^{25}-2t^{-2}a^{25}+t^{-3}a^{25}\big)+q^{11}\big(t^{-4}a^{30}-t^{-5}a^{30}\big).$

Specializations

For $\{\mathsf{r},\mathsf{s}\}\in\{\{3,2\},\{4,3\}\}$, the following specializations, which are special cases of (6.90), are easily verified:

$$HD^{\mathfrak{ad}}_{\mathsf{r},\mathsf{s}}(q,t,a=-t^5)=\widetilde{JD}^{E_8}_{\mathsf{r},\mathsf{s}}(\omega_8;q,t), \tag{6.95}$$

$$HD^{\mathfrak{ad}}_{\mathsf{r},\mathsf{s}}(q,t,a=-t^3)=\widetilde{JD}^{E_7}_{\mathsf{r},\mathsf{s}}(\omega_1;q,t), \tag{6.96}$$

$$HD^{\mathfrak{ad}}_{\mathsf{r},\mathsf{s}}(q,t,a=-t^2)=\widetilde{JD}^{E_6}_{\mathsf{r},\mathsf{s}}(\omega_2;q,t), \tag{6.97}$$

$$HD^{\mathfrak{ad}}_{\mathsf{r},\mathsf{s}}(q,t,a=-t^1)=\widetilde{JD}^{D_4}_{\mathsf{r},\mathsf{s}}(\omega_2;q,t), \tag{6.98}$$

$$HD^{\mathfrak{ad}}_{\mathsf{r},\mathsf{s}}(q,t,a=-t^{\frac{1}{2}})=\widetilde{JD}^{A_2}_{\mathsf{r},\mathsf{s}}(\omega_1+\omega_2;q,t), \tag{6.99}$$

$$HD^{\mathfrak{ad}}_{\mathsf{r},\mathsf{s}}(q,t,a=-t^{\frac{1}{3}})=\widetilde{JD}^{A_1}_{\mathsf{r},\mathsf{s}}(2\omega_1;q,t). \tag{6.100}$$

The DAHA-Jones polynomials for the first four specializations may be found in [C5]. The last two DAHA-Jones polynomials are specializations of the DAHA-superpolynomials from Section 5.4.

In addition to these defining specializations, the expressions for $HD^{\mathfrak{ad}}_{\mathsf{r},\mathsf{s}}$ possess two structures that resemble the "canceling differentials" from [DGR] and other papers. On the level of polynomials, these canceling differentials correspond to specializations of the parameters with respect to which $HD^{\mathfrak{ad}}_{\mathsf{r},\mathsf{s}}$ becomes a single monomial.

The simplest such specialization corresponds to the evaluation at $t=1$ of DAHA-Jones polynomials. On the level of hyperpolynomials, we set $a\mapsto -t^{\nu}=-1$, which readily results in the relation

$$HD^{\mathfrak{ad}}_{\mathsf{r},\mathsf{s}}(q,t=1,a=-1)=1. \tag{6.101}$$

The following example of a "canceling differential" is more interesting. We set $t=qa^6$. Then

$$HD^{\mathfrak{ad}}_{3,2}(q,t,a)=q^3t^{-1}a^6+(1-qt^{-1}a^6)\mathcal{Q}_{3,2}(q,t,a), \tag{6.102}$$

$$HD^{\mathfrak{ad}}_{4,3}(q,t,a)=q^7t^{-1}a^6+(1-qt^{-1}a^6)\mathcal{Q}_{4,3}(q,t,a), \tag{6.103}$$

for some polynomials $\mathcal{Q}_{\mathsf{r},\mathsf{s}}(q,t,a)$. Observe that $qt^{-1}a^6\mapsto -qt^{h^\vee-1}$ in the specialization $a\mapsto -t^{\nu}$. Upon this specialization, the above relations reflect the $PSL^c_2(\mathbb{Z})$-invariance of the image of non-symmetric Macdonald polynomials E_ϑ in the quotient of the polynomial representation of the corresponding DAHA under the relation $qt^{h^\vee-1}=-1$ by its radical. However we did not check all details.

Let us also mention potential links of our hyperpolynomials evaluated at $a=-t^{-1}$ and $a=-1$ to the root systems D_6 and A_3, respectively, which we are going to investigate elsewhere.

Finally, let us touch upon the root systems G_2, F_4 in the Deligne-Gross series. For $\nu(G_2)=\frac{2}{3}$ and for $\nu(F_4)=\frac{3}{2}$, the corresponding specializations of $HD^{\mathfrak{ad}}_{\mathsf{r,s}}$ resemble the polynomials $\widetilde{JD}^{G_2}_{3,2}(\omega_1;q,r,t)$ and $\widetilde{JD}^{F_4}_{3,2}(\omega_1;q,r,t)$ from [C5] at $r=t$, but do not coincide with them. Hopefully, these specializations are connected with the *untwisted* variants of these two DAHA-Jones polynomials, but they are known so far only in the twisted setting.

Appendix A

DAHA-Jones formulas

Type A

The formulas for $\widetilde{JD}^{A_n}(b)$, can be readily obtained from the following well-known type-A super-polynomials $HD^A_{\rm r,s}(b;q,t,a)$ upon the substitution $a=-t^{n+1}$. We will need only A_6 here, which corresponds to $a=-t^7$:

$$HD^A_{3,2}(\omega_1) = 1+aq+qt, \tag{A.1}$$

$$HD^A_{5,2}(\omega_1) = 1+qt+q^2t^2+a\big(q+q^2t\big), \tag{A.2}$$

$$HD^A_{4,3}(\omega_1) = 1+a^2q^3+qt+q^2t+q^2t^2+q^3t^3+a\big(q+q^2+q^2t+q^3t+q^3t^2\big). \tag{A.3}$$

The simplest colored formulas for the super-polynomials of type A, defined for ω_2, are known from [GS, FGS] and [C5]. They play an important role for the super-polynomials of the pair (E_6,ω_1), in spite of the fact that this weight is non-minuscule.

$$HD^A_{3,2}(\omega_2;q,t,a) \;=\; \tag{A.4}$$
$$1+\frac{a^2q^2}{t}+qt+qt^2+q^2t^4+a\big(q+\frac{q}{t}+q^2t+q^2t^2\big),$$

$$HD^A_{5,2}(\omega_2;q,t,a) \;=\; \tag{A.5}$$
$$1+qt+qt^2+q^2t^2+q^2t^3+q^2t^4+q^3t^5+q^3t^6+q^4t^8+a^2\big(q^3+\tfrac{q^2}{t}+q^3t+q^4t^3\big)+a\big(q+q^2+\tfrac{q}{t}+2q^2t+q^2t^2+q^3t^2+2q^3t^3+q^3t^4+q^4t^5+q^4t^6\big),$$

$$HD^A_{4,3}(\omega_2;q,t,a) \;=\; \tag{A.6}$$
$$1+\tfrac{a^4q^6}{t^2}+qt+q^2t+qt^2+2q^2t^2+q^2t^3+2q^3t^3+q^2t^4+2q^3t^4+q^4t^4+q^3t^5+q^4t^5+q^3t^6+2q^4t^6+q^4t^7+q^5t^7+q^4t^8+q^5t^8+q^5t^9+q^5t^{10}+q^6t^{12}+a^3\big(q^5+q^6+\tfrac{q^4}{t^2}+\tfrac{q^5}{t^2}+\tfrac{q^4}{t}+\tfrac{q^5}{t}+q^5t+q^6t+q^6t^2+q^6t^3\big)+a^2\big(2q^3+2q^4+q^5+\tfrac{q^3}{t^2}+\tfrac{q^2}{t}+\tfrac{2q^3}{t}+\tfrac{q^4}{t}+q^3t+4q^4t+2q^5t+2q^4t^2+3q^5t^2+q^4t^3+3q^5t^3+q^6t^3+2q^5t^4+q^6t^4+q^5t^5+$$

$2q^6t^5 + q^6t^6 + q^6t^7\big) + a\big(q + 2q^2 + q^3 + \frac{q}{t} + \frac{q^2}{t} + 2q^2t + 4q^3t + q^4t + q^2t^2 + 4q^3t^2 + 2q^4t^2 + 2q^3t^3 + 4q^4t^3 + q^5t^3 + q^3t^4 + 4q^4t^4 + 2q^5t^4 + 2q^4t^5 + 3q^5t^5 + q^4t^6 + 3q^5t^6 + 2q^5t^7 + q^6t^7 + q^5t^8 + q^6t^8 + q^6t^9 + q^6t^{10}\big).$

More specifically, we will need the values of these super-polynomials at $t = 1$:

$$HD^A_{3,2}(\omega_2; q, t = 1, a) \;=\; (1 + q + aq)^2, \tag{A.7}$$

$$HD^A_{5,2}(\omega_2; q, t = 1, a) \;=\; (1 + q + aq + q^2 + aq^2)^2, \tag{A.8}$$

$$HD^A_{4,3}(\omega_2; q, t = 1, a) = (1 + q + aq + 2q^2 + 2aq^2 + q^3 + 2aq^3 + a^2q^3)^2. \tag{A.9}$$

For instance, the corresponding *dimensions* $HD^A(q = 1, t = 1, a = 1)$ are $9, 25, 121$.

Type D

We will need the following DAHA-Jones polynomials of type D_5 for ω_1 (which is minuscule):

$$\widetilde{JD}^{D_5}_{3,2}(\omega_1; q, t) \;= \tag{A.10}$$
$$1 + qt + qt^4 - qt^5 - qt^8 + q^2t^8 - q^2t^9 - q^2t^{12} + q^2t^{13},$$

$$\widetilde{JD}^{D_5}_{5,2}(\omega_1; q, t) \;= \tag{A.11}$$

$1 + qt + q^2t^2 + qt^4 - qt^5 + q^2t^5 - q^2t^6 - qt^8 + q^2t^8 - 2q^2t^9 + q^3t^9 - q^3t^{10} - q^2t^{12} + q^3t^{12} + q^2t^{13} - 2q^3t^{13} + q^3t^{14} - q^3t^{16} + q^4t^{16} + q^3t^{17} - q^4t^{17} - q^4t^{20} + q^4t^{21},$

$$\widetilde{JD}^{D_5}_{4,3}(\omega_1; q, t) \;= \tag{A.12}$$

$1 + qt + q^2t^2 + qt^4 - qt^5 + q^2t^5 - q^2t^6 - qt^8 + q^2t^8 - 2q^2t^9 + q^3t^9 - q^3t^{10} - q^2t^{12} + q^3t^{12} + q^2t^{13} - 2q^3t^{13} + q^3t^{14} - q^3t^{16} + q^4t^{16} + q^3t^{17} - q^4t^{17} - q^4t^{20} + q^4t^{21}.$

We will also need the super-polynomials for the case when the *last* fundamental weight is taken for $D_n (n \geq 4)$:

$$\widehat{HD}^{D}_{3,2}(\omega_n) = 1 + aqt^6 + qt^3, \tag{A.13}$$

$$\widehat{HD}^{D}_{5,2}(\omega_n) = 1 + qt^3 + q^2t^6 + a\big(qt^6 + q^2t^9\big), \tag{A.14}$$

$$\widehat{HD}^{D}_{4,3}(\omega_n) = 1 + a^2q^3t^{14} + qt^3 + q^2t^5 + q^2t^6 + a\big(qt^6 + q^2 + q^2t^8 + q^2t^9 + q^3t^{11} + q^3t^{12}\big), \tag{A.15}$$

where the relevant specializations are

$$\widehat{HD}^{D}(q, t, a \mapsto -t^{n-4}) = \widetilde{JD}^{D_n}(\omega_n\, ;\, q, t). \tag{A.16}$$

The DAHA-superpolynomials and DAHA-Jones polynomials for ω_{n-1} are identical to those for ω_n.

Interestingly, these super-polynomials are related to those for (A, ω_1):

$$\widehat{HD}^{D}_{\mathrm{r,s}}(\omega_n; q, t, a) = HD^{A}_{\mathrm{r,s}}(\omega_1; q \mapsto tq^2, t, a \mapsto at^4), \tag{A.17}$$

so we have essentially similar "stable theories" for the pairs (A_{n-1}, ω_1) and (D_n, ω_n).

Type E_6

We will need the DAHA-Jones polynomials for the minuscule weight ω_1:

$$\widetilde{JD}^{E_6}_{3,2}(\omega_1; q, t) = \tag{A.18}$$
$$1 + q\big(t + t^4 - t^9 - t^{12}\big) + q^2\big(t^8 - t^{13} - t^{16} + t^{21}\big),$$

$$\widetilde{JD}^{E_6}_{5,2}(\omega_1; q, t) = \tag{A.19}$$

$1 + q\big(t + t^4 - t^9 - t^{12}\big) + q^2\big(t^2 + t^5 + t^8 - t^{10} - 2t^{13} - t^{16} + t^{21}\big) + q^3\big(t^9 + t^{12} - t^{14} - 2t^{17} - t^{20} + t^{22} + t^{25}\big) + q^4\big(t^{16} - t^{21} - t^{24} + t^{29}\big),$

$$\widetilde{JD}^{E_6}_{4,3}(\omega_1; q, t) = \tag{A.20}$$

$1 + q\big(t + t^4 - t^9 - t^{12}\big) + q^2\big(t + t^2 + t^4 + t^5 + t^8 - t^9 - t^{10} - t^{12} - 2t^{13} - t^{16} + t^{21}\big) + q^3\big(t^3 + t^5 + t^6 + t^8 + t^9 - t^{10} - t^{11} + t^{12} - 3t^{13} - 2t^{14} - 2t^{16} - 2t^{17} + t^{18} - t^{20} + 2t^{21} + t^{22} + t^{24} + t^{25}\big) + q^4\big(t^8 + t^9 + t^{10} + t^{12} - t^{14} - t^{15} - 3t^{17} - 2t^{18} - 2t^{20} - t^{21} + 2t^{22} + t^{23} - t^{24} + 3t^{25} + t^{26} + t^{28} + t^{29} - t^{30} - t^{33}\big) + q^5\big(t^{13} + t^{16} - t^{18} - 3t^{21} - 2t^{24} + 2t^{26} + 3t^{29} + t^{32} - t^{34} - t^{37}\big) + q^6\big(t^{24} - t^{25} - t^{28} + t^{30} - t^{32} + 2t^{33} - t^{34} + t^{36} - t^{38} - t^{41} + t^{42}\big).$

The next series of DAHA-Jones polynomials will be for ω_6 (minuscule):

$$\widetilde{JD}^{E_6}_{3,2}(\omega_6; q, t) = \tag{A.21}$$
$$1 + q(t^5 + t^8 - t^9 - t^{12}) + q^2(t^{16} - t^{17} - t^{20} + t^{21}),$$

$$\widetilde{JD}^{E_6}_{5,2}(\omega_6; q, t) = \tag{A.22}$$

$1 + q\big(t^5 + t^8 - t^9 - t^{12}\big) + q^2\big(t^{10} + t^{13} - t^{14} + t^{16} - 2t^{17} - t^{20} + t^{21}\big) + q^3\big(t^{21} - t^{22} + t^{24} - 2t^{25} + t^{26} - t^{28} + t^{29}\big) + q^4\big(t^{32} - t^{33} - t^{36} + t^{37}\big),$

$$\widetilde{JD}^{E_6}_{4,3}(\omega_6; q, t) = \tag{A.23}$$

$1 + q\big(t^5 + t^8 - t^9 - t^{12}\big) + q^2\big(t^9 + t^{10} + t^{12} - t^{14} - 2t^{17} - t^{20} + t^{21}\big) + q^3\big(t^{15} + t^{17} - t^{19} + t^{20} - 2t^{21} - t^{22} - t^{24} + t^{26} + t^{29}\big) + q^4\big(t^{24} - t^{27} - t^{29} + t^{31} - t^{32} + t^{33}\big).$

Appendix B

Figures

This appendix contains diagrams which depict our proposals for $\mathscr{H}^{\mathfrak{e}_6,\mathbf{27}}$ in Section 6.2. We use QG-conventions; see (6.8). In particular, Figure B.1 corresponds to our proposal for $T^{3,2}$, figure B.2 corresponds to our proposal for $T^{5,2}$, and figure B.2 corresponds to our proposal for $T^{4,3}$.

In each figure, a monomial $q^i t^j u^k$ corresponds to the number k placed on the diagram in position (i, j), i.e., with x-coordinate i and y-coordinate j. The differentials are depicted by line segments connecting pairs of monomials, color-coded as follows.

$\mathfrak{g}, V$	color	$\deg(d_{\mathfrak{g},V})$
$\mathfrak{e}_6, \mathbf{27}$	–	$(0, -1, 1)$
$\mathfrak{d}_5, \mathbf{10}$	Red	$(4, -1, 1)$
$\mathfrak{a}_6, \mathbf{7}$	Yellow	$(5, -1, 1)$
canceling	Green	$(8, -1, 1)$
canceling	Blue	$(13, 1, 1)$

(B.1)

Observe that while the differential corresponding to $(\mathfrak{d}_5, \mathbf{10})$ only appears in the diagram for $\mathscr{H}^{\mathfrak{e}_6,\mathbf{27}}(T^{4,3})$, that structure still exists as a *specialization* in the other two cases; see Section 6.2.

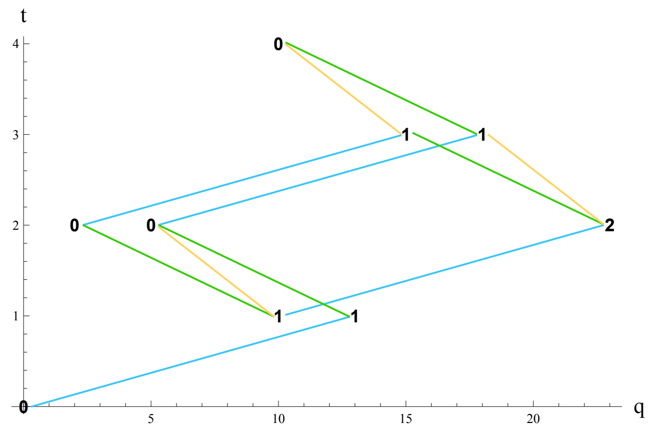


Figure B.1: Differentials for $T^{3,2}$

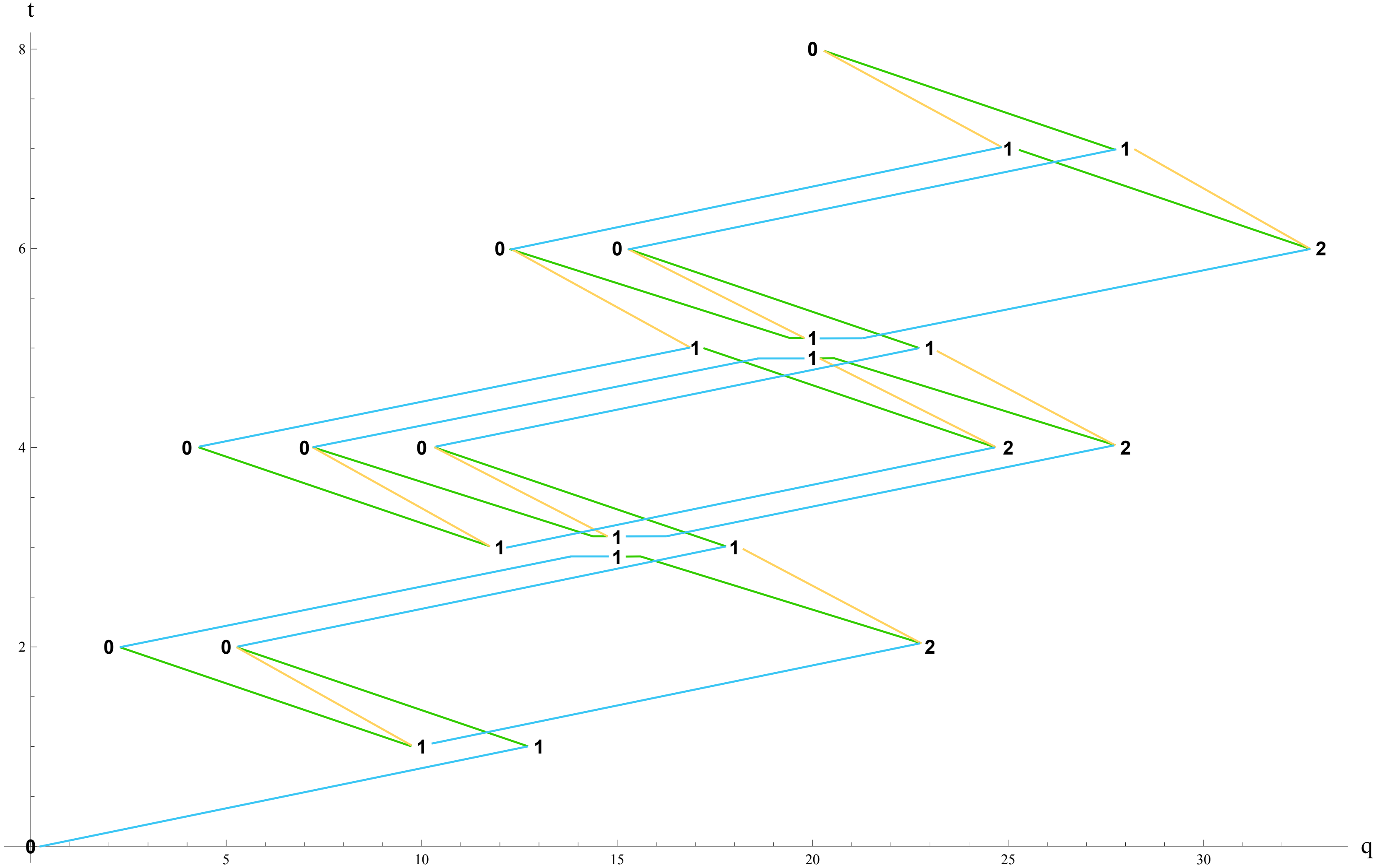


Figure B.2: Differentials for $T^{5,2}$

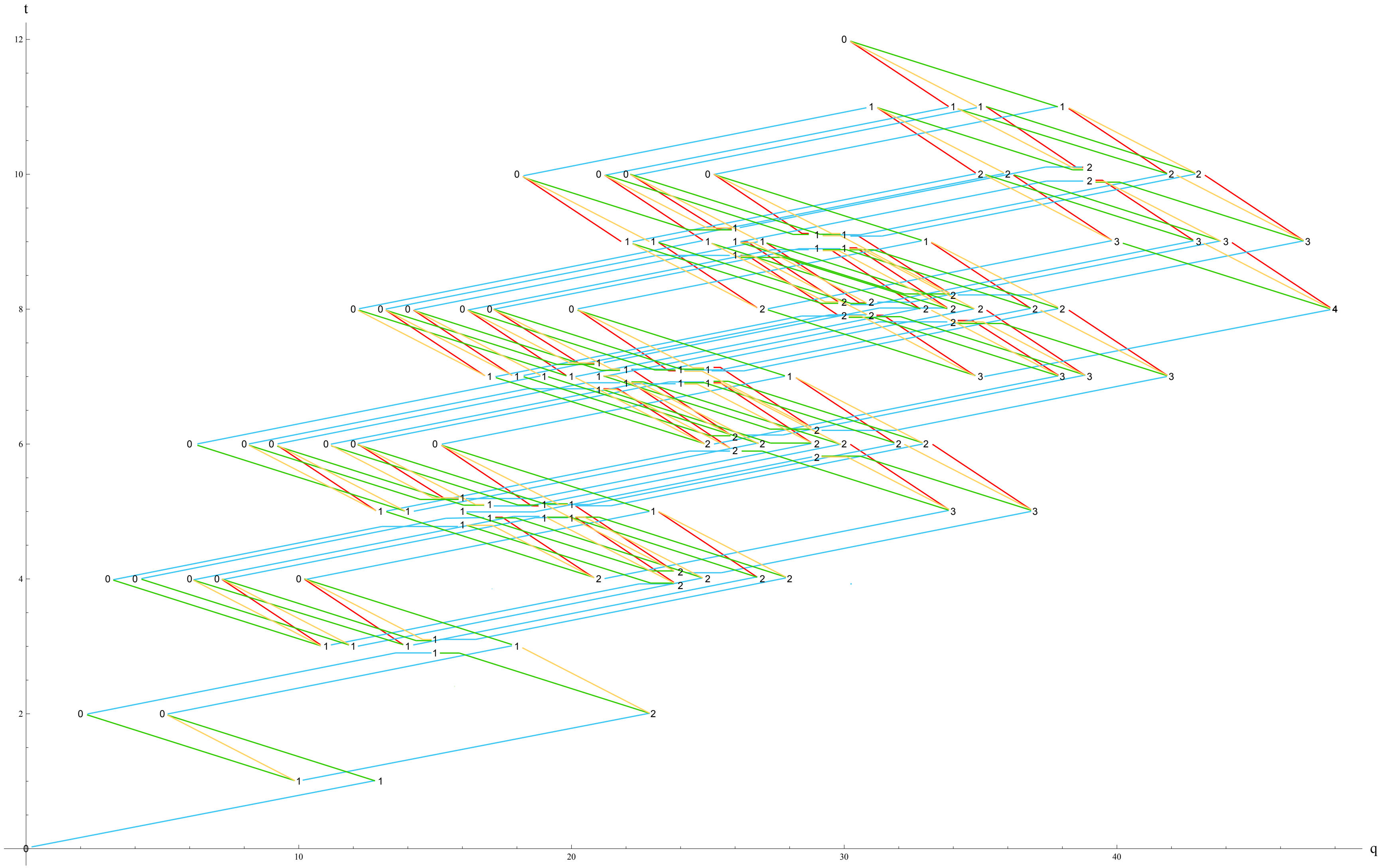


Figure B.3: Differentials for $T^{4,3}$

Appendix C

Adjacencies and spectra

This appendix contains the adjacency tree to $Z_{3,0}$, computed as outlined in section 6.3. This tree displays only those adjacencies ("arrows") that arise in the classification of singularities by their jets [Ar1], though there are other internal adjacencies. Observe that as a direct consequence of the definition of adjacency, this tree is transitive in that $A \to B \to C$ implies $A \to C$.

One can check this list using the adjacency of the spectra, which are also listed. There are many ways to compute the spectrum of a singularity, and we will outline one method here. Suppose $f \in \mathcal{O}_n$ with Taylor expansion $f = \sum a_{\mathbf{k}} \mathbf{z}^{\mathbf{k}}$. Then we can take the set,

$$\mathrm{supp} f = \{\mathbf{k} \in \mathbb{N}^n_{\geq 0} : a_{\mathbf{k}} \neq 0\}. \tag{C.1}$$

Now we let define a subset of $\mathbb{R}^n_+$ by

$$G(f) = \bigcup_{\mathbf{k} \in \mathrm{supp} f} \{\mathbf{k} + \mathbb{R}^n_+\}. \tag{C.2}$$

The convex hull of $G(f)$ constitutes the *Newton polyhedron* of f, and the union of the compact faces of the Newton polyhedron is the *Newton diagram* $\Gamma(f)$ of f.

A Newton diagram induces a decreasing filtration on power series as follows. If we assume that any monomial contained in the Newton diagram is quasihomogeneous of degree 1, then each face $e_i \in \Gamma(f)$ determines a set of weights ν_i such that $\langle \mathbf{j}, \nu_i \rangle = 1$ for all $\mathbf{z}^{\mathbf{j}} \in e_i$. We can then define the Newton degree of an arbitrary monomial by:

$$\deg \mathbf{z}^{\mathbf{k}} = \min_i \langle \mathbf{k}, \nu_i \rangle. \tag{C.3}$$

Then if every monomial in a power series has Newton degree greater than or equal to d, that power series belongs to the d^{th} subspace of the Newton filtration.

The Newton filtration also descends to forms, e.g., the Newton order of the form $\mathbf{z}^{\mathbf{k}} dz_1 \wedge \cdots \wedge dz_n$ coincides with the Newton order of the monomial $\mathbf{z}^{\mathbf{k}} z_1 \cdots z_n$. Furthermore, the Newton filtration on forms coincides with the Hodge filtration after a shift of indices, and one can show that for an appropriate set of monomials (ones whose corresponding forms trivialize the vanishing cohomology bundle), the spectrum coincides with the set of numbers:

$$\min_i \langle \mathbf{k}+\mathbf{1}, \boldsymbol{\nu}_i \rangle - 1, \tag{C.4}$$

for those monomials, which can often be taken to be a basis for the local algebra or, using the symmetry of the spectrum about $\frac{n}{2}-1$, a set of *subdiagrammatic monomials*–those $\mathbf{z}^{\mathbf{k}}$ for which $\mathbf{k}+\mathbf{1}$ does not belong to the interior of the Newton polyhedron. The following table lists the singularities adjacent to $Z_{3,0}$ and their normal forms, relevant deformations, and Milnor numbers.

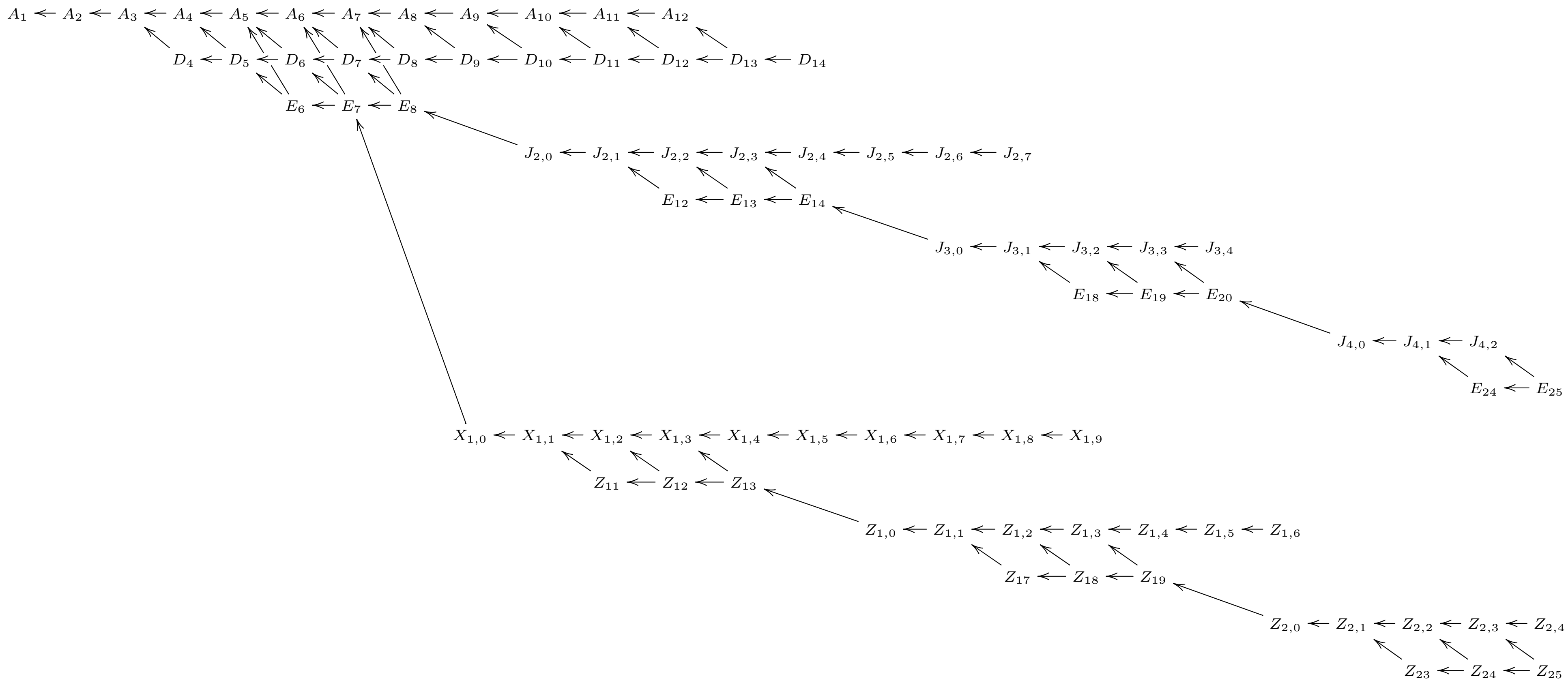
Singularity	**Normal Form**[1]	$\mathbf{\Delta W}/\epsilon$	μ
$Z_{3,0}$	$x^3y+dx^2y^5+\mathbf{a}_4xy^{10}+y^{13}$	$--$	27
A_k, $1\le k\le 12$	x^{k+1}	x^2+y^{k+1}	k
D_k, $4\le k\le 14$	x^2y+y^{k-1}	x^2y+y^{k-1}	k
E_6	x^3+y^4	x^3+y^4	6
E_7	x^3+xy^3	x^3+xy^3	7
E_8	x^3+y^5	x^3+y^5	8
$J_{2,0}$	$x^3+bx^2y^2+y^6$	$x^3+x^2y^2+y^6$	10
$J_{2,p}$, $1\le p\le 7$	$x^3+x^2y^2+ay^{6+p}$	$x^3+x^2y^2+y^{6+p}$	$10+p$
E_{12}	$x^3+y^7+\mathbf{a}_2xy^5$	x^3+y^7	12
E_{13}	$x^3+xy^5+\mathbf{a}_2y^8$	x^3+xy^5	13
E_{14}	$x^3+y^8+\mathbf{a}_2xy^6$	x^3+y^8	14
$J_{3,0}$	$x^3+bx^2y^3+y^9+axy^7$	$x^3+x^2y^3+y^9$	16
$J_{3,p}$, $1\le p\le 4$	$x^3+x^2y^3+\mathbf{a}_3y^{9+p}$	$x^3+x^2y^3+y^{9+p}$	$16+p$
E_{18}	$x^3+y^{10}+\mathbf{a}_3xy^7$	x^3+y^{10}	18
E_{19}	$x^3+xy^7+\mathbf{a}_3y^{11}$	x^3+xy^7	19
E_{20}	$x^3+y^{11}+\mathbf{a}_3xy^8$	x^3+y^{11}	20
$J_{4,0}$	$x^3+bx^2y^4+y^{12}+\mathbf{a}_3xy^9$	$x^3+x^2y^4+y^{12}$	22
$J_{4,1}$	$x^3+x^2y^4+\mathbf{a}_4y^{13}$	$x^3+x^2y^4$	23
$J_{4,2}$	$x^3+x^2y^4+\mathbf{a}_4y^{14}$	$x^3+xy^9+x^2y^4$	24
E_{24}	$x^3+y^{13}+\mathbf{a}_4xy^9$	x^3+y^{10}	24
E_{25}	$x^3+xy^9+\mathbf{a}_4y^{14}$	x^3+xy^9	25
$X_{1,0}$	$x^4+ax^2y^2+y^4$, $a\ne 4$	$y^4+xy^3+x^2y^2$	9
$X_{1,p}$, $1\le p\le 9$	$x^4+x^2y^2+ay^{4+p}$, $a\ne 0$	$x^2y^2+y^{4+p}$	$9+p$
Z_{11}	$x^3y+y^5+axy^4$	y^5	11
Z_{12}	$x^3y+xy^4+ax^2y^3$	xy^4	12
Z_{13}	$x^3y+y^6+axy^5$	y^6	13
$Z_{1,0}$	$x^3y+dx^2y^3+axy^6+y^7$	$x^2y^3+y^7$	15
$Z_{1,p}$, $1\le p\le 6$	$x^3y+x^2y^3+\mathbf{a}_3y^{7+p}$	$x^2y^3+y^{7+p}$	$15+p$
Z_{17}	$x^3y+y^8+\mathbf{a}_3xy^6$	y^8	17
Z_{18}	$x^3y+xy^6+\mathbf{a}_3y^9$	xy^6	18
Z_{19}	$x^3y+y^9+\mathbf{a}_3xy^7$	y^9	19
$Z_{2,0}$	$x^3y+dx^2y^4+\mathbf{a}_3xy^8+y^{10}$	$x^2y^4+y^{10}$	21
$Z_{2,p}$, $1\le p\le 4$	$x^3y+x^2y^4+\mathbf{a}_4y^{9+p}$	$x^2y^4+y^{9+p}$	$21+p$
Z_{23}	$x^3y+y^{11}+\mathbf{a}_4xy^8$	y^{11}	23
Z_{24}	$x^3y+xy^8+\mathbf{a}_4y^{12}$	xy^8	24
Z_{25}	$x^3y+y^{12}+\mathbf{a}_4xy^9$	y^{12}	25

[1] Here we have that $\mathbf{a}_k := a_0+\cdots+a_{k-2}y^{k-2}, \mathbf{a}_1 := 0$.

The following table lists the spectra of the singularities which are adjacent to $Z_{3,0}$. Observe that, by Theorem 6.3.4, it supports our list of adjacencies.

	Spectrum
$Z_{3,0}$	$\frac{-3}{26}, \frac{-1}{26}, \frac{1}{26}, \frac{3}{26}, \frac{5}{26}, \frac{5}{26}, \frac{7}{26}, \frac{7}{26}, \frac{9}{26}, \frac{9}{26}, \frac{11}{26}, \frac{11}{26}, \frac{13}{26}, \frac{\mathbf{13}}{\mathbf{26}}, \frac{13}{26}, \frac{15}{26}, \frac{15}{26}, \frac{17}{26}, \frac{17}{26}, \frac{19}{26}, \frac{19}{26}, \frac{21}{26}, \frac{21}{26}, \frac{23}{26}, \frac{25}{26}, \frac{27}{26}, \frac{29}{26}$
A_k	$\frac{2}{2k+2}, \frac{4}{2k+2}, \cdots, \frac{2k}{2k+2}$
D_k	$\left\{\frac{1}{2k-2}, \frac{3}{2k-2}, \cdots, \frac{2k-3}{2k-2}\right\} \cup \left\{\frac{k-1}{2k-2}\right\}$
E_6	$\frac{1}{12}, \frac{4}{12}, \frac{5}{12}, \frac{7}{12}, \frac{8}{12}, \frac{11}{12}$
E_7	$\frac{1}{18}, \frac{5}{18}, \frac{7}{18}, \frac{\mathbf{9}}{\mathbf{18}}, \frac{11}{18}, \frac{13}{18}, \frac{17}{18}$
E_8	$\frac{1}{30}, \frac{7}{30}, \frac{11}{30}, \frac{13}{30}, \frac{17}{30}, \frac{19}{30}, \frac{23}{30}, \frac{29}{30}$
$J_{2,p}$	$\left\{\frac{0}{6(p+6)}, \frac{6}{6(p+6)}, \cdots, \frac{6(p+6)}{6(p+6)}\right\} \cup \left\{\frac{2(p+6)}{6(p+6)}, \frac{3(p+6)}{6(p+6)}, \frac{4(p+6)}{6(p+6)}\right\}$
E_{12}	$\frac{-1}{42}, \frac{5}{42}, \frac{11}{42}, \frac{13}{42}, \frac{17}{42}, \frac{19}{42}, \frac{23}{42}, \frac{25}{42}, \frac{29}{42}, \frac{31}{42}, \frac{37}{42}, \frac{43}{42}$
E_{13}	$\frac{-1}{30}, \frac{3}{30}, \frac{7}{30}, \frac{9}{30}, \frac{11}{30}, \frac{13}{30}, \frac{\mathbf{15}}{\mathbf{30}}, \frac{17}{30}, \frac{19}{30}, \frac{21}{30}, \frac{23}{30}, \frac{27}{30}, \frac{31}{30}$
E_{14}	$\frac{-1}{24}, \frac{2}{24}, \frac{5}{24}, \frac{7}{24}, \frac{8}{24}, \frac{10}{24}, \frac{11}{24}, \frac{13}{24}, \frac{14}{24}, \frac{16}{24}, \frac{17}{24}, \frac{19}{24}, \frac{22}{24}, \frac{25}{24}$
$J_{3,p}$	$\left\{\frac{9}{18(p+9)}, \frac{27}{18(p+9)}, \cdots, \frac{9(2p+17)}{18(p+9)}\right\} \cup \left\{\frac{-(p+9)}{18(p+9)}, \frac{5(p+9)}{18(p+9)}, \frac{7(p+9)}{18(p+9)}, \frac{9(p+9)}{18(p+9)}, \frac{11(p+9)}{18(p+9)}, \frac{13(p+9)}{18(p+9)}, \frac{19(p+9)}{18(p+9)}\right\}$
E_{18}	$\frac{-2}{30}, \frac{1}{30}, \frac{4}{30}, \frac{7}{30}, \frac{8}{30}, \frac{10}{30}, \frac{11}{30}, \frac{13}{30}, \frac{14}{30}, \frac{16}{30}, \frac{17}{30}, \frac{19}{30}, \frac{20}{30}, \frac{22}{30}, \frac{23}{30}, \frac{26}{30}, \frac{29}{30}, \frac{32}{30}$
E_{19}	$\frac{-3}{42}, \frac{1}{42}, \frac{5}{42}, \frac{9}{42}, \frac{11}{42}, \frac{13}{42}, \frac{15}{42}, \frac{17}{42}, \frac{19}{42}, \frac{\mathbf{21}}{\mathbf{42}}, \frac{23}{42}, \frac{25}{42}, \frac{27}{42}, \frac{29}{42}, \frac{31}{42}, \frac{33}{42}, \frac{37}{42}, \frac{41}{42}, \frac{45}{42}$
E_{20}	$\frac{-5}{66}, \frac{1}{66}, \frac{7}{66}, \frac{13}{66}, \frac{17}{66}, \frac{19}{66}, \frac{23}{66}, \frac{25}{66}, \frac{29}{66}, \frac{31}{66}, \frac{35}{66}, \frac{37}{66}, \frac{41}{66}, \frac{43}{66}, \frac{47}{66}, \frac{49}{66}, \frac{53}{66}, \frac{59}{66}, \frac{65}{66}, \frac{71}{66}$
$J_{4,p}$	$\left\{\frac{12}{12(p+12)}, \frac{24}{12(p+12)}, \cdots, \frac{12(p+11)}{12(p+12)}\right\} \cup$ $\left\{\frac{-(p+12)}{12(p+12)}, \frac{0}{12(p+12)}, \frac{3(p+12)}{12(p+12)}, \frac{4(p+12)}{12(p+12)}, \frac{5(p+12)}{12(p+12)}, \frac{6(p+12)}{12(p+12)}, \frac{7(p+12)}{12(p+12)}, \frac{8(p+12)}{12(p+12)}, \frac{9(p+12)}{12(p+12)}, \frac{12(p+12)}{12(p+12)}, \frac{13(p+12)}{12(p+12)}\right\}$
E_{24}	$\frac{-7}{78}, \frac{-1}{78}, \frac{5}{78}, \frac{11}{78}, \frac{17}{78}, \frac{19}{78}, \frac{23}{78}, \frac{25}{78}, \frac{29}{78}, \frac{31}{78}, \frac{35}{78}, \frac{37}{78}, \frac{41}{78}, \frac{43}{78}, \frac{47}{78}, \frac{49}{78}, \frac{53}{78}, \frac{55}{78}, \frac{59}{78}, \frac{61}{78}, \frac{67}{78}, \frac{73}{78}, \frac{79}{78}, \frac{85}{78}$
E_{25}	$\frac{-5}{54}, \frac{-1}{54}, \frac{3}{54}, \frac{7}{54}, \frac{11}{54}, \frac{13}{54}, \frac{15}{54}, \frac{17}{54}, \frac{19}{54}, \frac{21}{54}, \frac{23}{54}, \frac{25}{54}, \frac{\mathbf{27}}{\mathbf{54}}, \frac{29}{54}, \frac{31}{54}, \frac{33}{54}, \frac{35}{54}, \frac{37}{54}, \frac{39}{54}, \frac{41}{54}, \frac{43}{54}, \frac{47}{54}, \frac{51}{54}, \frac{55}{54}, \frac{59}{54}$
$X_{1,p}$	$\left\{\frac{0}{4(p+4)}, \frac{4}{4(p+4)}, \cdots, \frac{4(p+4)}{4(p+4)}\right\} \cup \left\{\frac{p+4}{4(p+4)}, \frac{2(p+4)}{4(p+4)}, \frac{2(p+4)}{4(p+4)}, \frac{3(p+4)}{4(p+4)}\right\}$
Z_{11}	$\frac{-1}{30}, \frac{5}{30}, \frac{7}{30}, \frac{11}{30}, \frac{13}{30}, \frac{\mathbf{15}}{\mathbf{30}}, \frac{17}{30}, \frac{19}{30}, \frac{23}{30}, \frac{25}{30}, \frac{31}{30}$
Z_{12}	$\frac{-1}{22}, \frac{3}{22}, \frac{5}{22}, \frac{7}{22}, \frac{9}{22}, \frac{11}{22}, \frac{11}{22}, \frac{13}{22}, \frac{15}{22}, \frac{17}{22}, \frac{19}{22}, \frac{23}{22}$
Z_{13}	$\frac{-1}{18}, \frac{2}{18}, \frac{4}{18}, \frac{5}{18}, \frac{7}{18}, \frac{8}{18}, \frac{\mathbf{9}}{\mathbf{18}}, \frac{10}{18}, \frac{11}{18}, \frac{13}{18}, \frac{14}{18}, \frac{16}{18}, \frac{19}{18}$
$Z_{1,p}$	$\left\{\frac{7}{14(p+7)}, \frac{14}{14(p+7)}, \cdots, \frac{7(2p+13)}{14(p+7)}\right\} \cup \left\{\frac{-(p+7)}{14(p+7)}, \frac{3(p+7)}{14(p+7)}, \frac{5(p+7)}{14(p+7)}, \frac{7(p+7)}{14(p+7)}, \frac{7(p+7)}{14(p+7)}, \frac{9(p+7)}{14(p+7)}, \frac{11(p+7)}{14(p+7)}, \frac{15(p+7)}{14(p+7)}\right\}$
Z_{17}	$\frac{-2}{24}, \frac{1}{24}, \frac{4}{24}, \frac{5}{24}, \frac{7}{24}, \frac{8}{24}, \frac{10}{24}, \frac{11}{24}, \frac{\mathbf{12}}{\mathbf{24}}, \frac{13}{24}, \frac{14}{24}, \frac{16}{24}, \frac{17}{24}, \frac{19}{24}, \frac{20}{24}, \frac{23}{24}, \frac{26}{24}$
Z_{18}	$\frac{-3}{34}, \frac{1}{34}, \frac{5}{34}, \frac{7}{34}, \frac{9}{34}, \frac{11}{34}, \frac{13}{34}, \frac{15}{34}, \frac{17}{34}, \frac{17}{34}, \frac{19}{34}, \frac{21}{34}, \frac{23}{34}, \frac{25}{34}, \frac{27}{34}, \frac{29}{34}, \frac{33}{34}, \frac{37}{34}$
Z_{19}	$\frac{-5}{54}, \frac{1}{54}, \frac{7}{54}, \frac{11}{54}, \frac{13}{54}, \frac{17}{54}, \frac{19}{54}, \frac{23}{54}, \frac{25}{54}, \frac{\mathbf{27}}{\mathbf{54}}, \frac{29}{54}, \frac{31}{54}, \frac{35}{54}, \frac{37}{54}, \frac{41}{54}, \frac{43}{54}, \frac{47}{54}, \frac{53}{54}, \frac{59}{54}$
$Z_{2,p}$	$\left\{\frac{0}{10(p+10)}, \frac{10}{10(p+10)}, \cdots, \frac{10(p+10)}{10(p+10)}\right\} \cup$ $\left\{\frac{-(p+10)}{10(p+10)}, \frac{2(p+10)}{10(p+10)}, \frac{3(p+10)}{10(p+10)}, \frac{4(p+10)}{10(p+10)}, \frac{5(p+10)}{10(p+10)}, \frac{5(p+10)}{10(p+10)}, \frac{6(p+10)}{10(p+10)}, \frac{7(p+10)}{10(p+10)}, \frac{8(p+10)}{10(p+10)}, \frac{11(p+10)}{10(p+10)}\right\}$
Z_{23}	$\frac{-7}{66}, \frac{-1}{66}, \frac{5}{66}, \frac{11}{66}, \frac{13}{66}, \frac{17}{66}, \frac{19}{66}, \frac{23}{66}, \frac{25}{66}, \frac{29}{66}, \frac{31}{66}, \frac{\mathbf{33}}{\mathbf{66}}, \frac{35}{66}, \frac{37}{66}, \frac{41}{66}, \frac{43}{66}, \frac{47}{66}, \frac{49}{66}, \frac{53}{66}, \frac{55}{66}, \frac{61}{66}, \frac{67}{66}, \frac{73}{66}$
Z_{24}	$\frac{-5}{46}, \frac{-1}{46}, \frac{3}{46}, \frac{7}{46}, \frac{9}{46}, \frac{11}{46}, \frac{13}{46}, \frac{15}{46}, \frac{17}{46}, \frac{19}{46}, \frac{21}{46}, \frac{23}{46}, \frac{23}{46}, \frac{25}{46}, \frac{27}{46}, \frac{29}{46}, \frac{31}{46}, \frac{33}{46}, \frac{35}{46}, \frac{37}{46}, \frac{39}{46}, \frac{43}{46}, \frac{47}{46}, \frac{51}{46}$
Z_{25}	$\frac{-4}{36}, \frac{-1}{36}, \frac{2}{36}, \frac{5}{36}, \frac{7}{36}, \frac{8}{36}, \frac{10}{36}, \frac{11}{36}, \frac{13}{36}, \frac{14}{36}, \frac{16}{36}, \frac{17}{36}, \frac{\mathbf{18}}{\mathbf{36}}, \frac{19}{36}, \frac{20}{36}, \frac{22}{36}, \frac{23}{36}, \frac{25}{36}, \frac{26}{36}, \frac{28}{36}, \frac{29}{36}, \frac{31}{36}, \frac{34}{36}, \frac{37}{36}, \frac{40}{36}$

Adjacency tree to $Z_{3,0}$:



Appendix D

Quantum $(\mathfrak{e}_6, \mathbf{27})$ knot invariants

Here we include expressions for the exceptional quantum invariants $P^{\mathfrak{e}_6,\mathbf{27}}(K;q)$ for many knots K. They are computed directly from the definition (2.39), using quantum R-matrices obtained from the GAP package QuaGroup [GAP, Qua]. I am grateful to W.A. de Graaf for explaining his package.

As these matrices are $27^2 = 729$-dimensional, we needed to use the `SparseArray` function in Mathematica in order to make these computations feasible. Even then, we were confined to considering knots with braid index ≤ 3.

We compute normalized/reduced, framing-independent invariants of knots. The quantum dimension of this representation is

$$\dim_q(\mathbf{27}) = \frac{[12][9]}{[4]}. \tag{D.1}$$

We also make the following connections with DAHA-Jones polynomials:

$$q^{16}\widetilde{JD}^{E_6}_{3,2}(\omega_1; q, t \mapsto q) = P^{\mathfrak{e}_6,\mathbf{27}}(\mathbf{3}_1; q), \tag{D.2}$$

$$q^{32}\widetilde{JD}^{E_6}_{5,2}(\omega_1; q, t \mapsto q) = P^{\mathfrak{e}_6,\mathbf{27}}(\mathbf{5}_1; q), \tag{D.3}$$

$$q^{48}\widetilde{JD}^{E_6}_{4,3}(\omega_1; q, t \mapsto q) = P^{\mathfrak{e}_6,\mathbf{27}}(\mathbf{8}_{19}; q). \tag{D.4}$$

The following table contains $P^{\mathfrak{e}_6,\mathbf{27}}(K;q)$ for knots with braid index ≤ 3 and crossing number ≤ 8.

K	σ	$P^{\mathfrak{e}_6,\mathbf{27}}(K;q)$
$\mathbf{0}_1$	$\mathrm{id} \in B_1$	1
$\mathbf{3}_1$	σ_1^3	$q^{16}+q^{18}+q^{21}-q^{29}-q^{31}-q^{34}+q^{39}$
$\mathbf{4}_1$	$\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$	$\frac{1}{q^{18}}-\frac{1}{q^{13}}+\frac{1}{q^{12}}-\frac{1}{q^{10}}+\frac{1}{q^9}-\frac{1}{q^8}-\frac{1}{q^4}+\frac{1}{q^3}-\frac{1}{q}+3-q+q^3-q^4-q^8+q^9-q^{10}+q^{12}-q^{13}+q^{18}$
$\mathbf{5}_1$	σ_1^5	$q^{32}+q^{34}+q^{36}+q^{37}+q^{39}-q^{45}-q^{47}-q^{49}-q^{50}-q^{52}+q^{65}$
$\mathbf{5}_2$	$\sigma_1^3\sigma_2\sigma_1^{-1}\sigma_2$	$q^{16}-q^{17}+q^{18}-q^{20}+2q^{21}-q^{22}-q^{26}+q^{27}+q^{28}-2q^{29}+3q^{30}-2q^{31}+2q^{33}-q^{34}+q^{36}-q^{38}+3q^{39}-q^{40}-q^{41}+q^{42}-2q^{43}-q^{46}-q^{47}+q^{48}-q^{49}+q^{51}-q^{52}+q^{57}$
$\mathbf{6}_2$	$\sigma_1^3\sigma_2^{-1}\sigma_1\sigma_2^{-1}$	$\frac{1}{q^2}+1+q^4-q^5+q^7-2q^8+2q^9-q^{10}-q^{11}-q^{13}-q^{14}+2q^{16}-3q^{17}+3q^{18}-q^{19}-2q^{20}+4q^{21}-3q^{22}+q^{23}+2q^{27}-2q^{29}+3q^{30}-3q^{31}+q^{32}+q^{33}-2q^{34}+q^{35}+q^{39}-q^{40}-q^{43}+q^{44}$
$\mathbf{6}_3$	$\sigma_1^2\sigma_2^{-1}\sigma_1\sigma_2^{-2}$	$\frac{1}{q^{18}}+\frac{1}{q^{17}}+\frac{2}{q^{16}}+\frac{3}{q^{15}}+\frac{5}{q^{14}}+\frac{6}{q^{13}}+\frac{10}{q^{12}}+\frac{12}{q^{11}}+\frac{15}{q^{10}}+\frac{19}{q^9}+\frac{22}{q^8}+\frac{26}{q^7}+\frac{30}{q^6}+\frac{33}{q^5}+\frac{36}{q^4}+\frac{39}{q^3}+\frac{41}{q^2}+\frac{41}{q}+45+41q+41q^2+39q^3+36q^4+33q^5+30q^6+26q^7+22q^8+19q^9+15q^{10}+12q^{11}+10q^{12}+6q^{13}+5q^{14}+3q^{15}+2q^{16}+q^{17}+q^{18}$

K	σ	$P^{\mathfrak{e}_6,\mathbf{27}}(K;q)$
$\mathbf{7}_1$	σ_1^7	$q^{48}+q^{50}+q^{52}+q^{53}+q^{54}+q^{55}+q^{57}-q^{61}-q^{63}-q^{65}-q^{66}-q^{67}-q^{68}-q^{70}+q^{91}$
$\mathbf{7}_3$	$\sigma_1^5\sigma_2\sigma_1^{-1}\sigma_2$	$q^{32}-q^{33}+q^{34}-q^{35}+2q^{37}-2q^{38}+2q^{39}-q^{40}-q^{45}+3q^{46}-3q^{47}+3q^{48}-2q^{50}+3q^{51}-2q^{52}+q^{53}+q^{54}+q^{55}+q^{57}-2q^{59}+3q^{60}-3q^{61}-2q^{64}+q^{65}-q^{67}-q^{68}-2q^{70}-q^{73}+q^{74}+q^{77}+q^{83}$
$\mathbf{7}_5$	$\sigma_1^4\sigma_2\sigma_1^{-1}\sigma_2^2$	$q^{32}-q^{33}+2q^{34}-q^{35}+3q^{37}-3q^{38}+3q^{39}-q^{40}-q^{41}+q^{42}-2q^{45}+4q^{46}-6q^{47}+5q^{48}-5q^{50}+6q^{51}-5q^{52}+q^{53}+2q^{54}+q^{55}-2q^{56}+3q^{57}-5q^{59}+7q^{60}-6q^{61}+2q^{63}-4q^{64}+2q^{65}+q^{66}-q^{67}-2q^{68}+3q^{69}-4q^{70}+q^{71}+2q^{72}-3q^{73}+2q^{74}+q^{77}+q^{78}-q^{79}-q^{82}+q^{83}$
$\mathbf{8}_2$	$\sigma_1^5\sigma_2^{-1}\sigma_1\sigma_2^{-1}$	$q^{14}+q^{16}+q^{18}+q^{20}-q^{24}+2q^{25}-2q^{26}+q^{27}-q^{28}-q^{29}-2q^{31}+q^{32}-2q^{33}+2q^{34}-2q^{35}+q^{36}+q^{37}-3q^{38}+3q^{39}-3q^{40}+q^{41}+q^{44}-q^{45}+q^{46}-q^{47}+2q^{48}-q^{49}+q^{51}-q^{52}+2q^{53}+q^{56}-q^{57}-q^{60}+q^{65}-q^{66}-q^{69}+q^{70}$
$\mathbf{8}_5$	$\sigma_1^3\sigma_2^{-1}\sigma_1^3\sigma_2^{-1}$	$q^{14}+2q^{16}+q^{18}+q^{19}+q^{20}+q^{22}-q^{24}+3q^{25}-3q^{26}+q^{27}-q^{28}-4q^{29}-3q^{31}-q^{32}-3q^{33}+q^{34}-4q^{35}+q^{36}+2q^{37}-5q^{38}+6q^{39}-4q^{40}+2q^{41}+4q^{42}+3q^{44}+q^{45}+2q^{46}-q^{47}+4q^{48}-2q^{49}-q^{50}+2q^{51}-4q^{52}+3q^{53}-q^{54}-2q^{55}+q^{56}-2q^{57}-q^{58}-q^{61}+q^{62}+2q^{65}-q^{66}-q^{69}+q^{70}$
$\mathbf{8}_7$	$\sigma_1^{-4}\sigma_2\sigma_1^{-1}\sigma_2^2$	$\frac{1}{q^{49}}-\frac{1}{q^{48}}+\frac{1}{q^{47}}-\frac{1}{q^{45}}+\frac{2}{q^{44}}-\frac{1}{q^{43}}-\frac{1}{q^{40}}+\frac{1}{q^{38}}-\frac{2}{q^{36}}+\frac{3}{q^{35}}-\frac{4}{q^{34}}+\frac{2}{q^{32}}-\frac{6}{q^{31}}+\frac{4}{q^{30}}-\frac{3}{q^{29}}+\frac{2}{q^{27}}-\frac{1}{q^{24}}+\frac{3}{q^{23}}-\frac{6}{q^{22}}+\frac{7}{q^{21}}-\frac{2}{q^{20}}-\frac{2}{q^{19}}+\frac{7}{q^{18}}-\frac{6}{q^{17}}+\frac{6}{q^{16}}-\frac{2}{q^{10}}+\frac{5}{q^{9}}-\frac{5}{q^{8}}+\frac{3}{q^{7}}-\frac{3}{q^{5}}+\frac{3}{q^{4}}-\frac{3}{q^{3}}-\frac{1}{q}+1-q+q^2-q^4+q^5-q^6+q^7$
$\mathbf{8}_9$	$\sigma_1^3\sigma_2^{-1}\sigma_1\sigma_2^{-3}$	$\frac{1}{q^{28}}-\frac{1}{q^{27}}+\frac{1}{q^{26}}-\frac{1}{q^{25}}-\frac{1}{q^{24}}+\frac{2}{q^{23}}-\frac{2}{q^{22}}+\frac{1}{q^{21}}+\frac{2}{q^{17}}-\frac{1}{q^{16}}-\frac{2}{q^{15}}+\frac{5}{q^{14}}-\frac{7}{q^{13}}+\frac{5}{q^{12}}+\frac{1}{q^{11}}-\frac{4}{q^{10}}+\frac{6}{q^{9}}-\frac{4}{q^{8}}+\frac{1}{q^{7}}-\frac{1}{q^{6}}+\frac{3}{q^{5}}-\frac{6}{q^{4}}+\frac{3}{q^{3}}+\frac{1}{q^{2}}-\frac{8}{q}+13-8q+q^2+3q^3-6q^4+3q^5-q^6+q^7-4q^8+6q^9-4q^{10}+q^{11}+5q^{12}-7q^{13}+5q^{14}-2q^{15}-q^{16}+2q^{17}+q^{21}-2q^{22}+2q^{23}-q^{24}-q^{25}+q^{26}-q^{27}+q^{28}$
$\mathbf{8}_{10}$	$\sigma_1^{-3}\sigma_2\sigma_1^{-2}\sigma_2^2$	$\frac{1}{q^{49}}-\frac{1}{q^{48}}+\frac{1}{q^{47}}-\frac{1}{q^{45}}+\frac{3}{q^{44}}-\frac{2}{q^{43}}+\frac{1}{q^{41}}-\frac{2}{q^{40}}+\frac{2}{q^{39}}+\frac{1}{q^{38}}-\frac{2}{q^{36}}+\frac{4}{q^{35}}-\frac{7}{q^{34}}+\frac{1}{q^{33}}+\frac{3}{q^{32}}-\frac{11}{q^{31}}+\frac{7}{q^{30}}-\frac{6}{q^{29}}-\frac{2}{q^{28}}+\frac{3}{q^{27}}-\frac{1}{q^{26}}-\frac{2}{q^{25}}+\frac{4}{q^{23}}-\frac{9}{q^{22}}+\frac{13}{q^{21}}-\frac{4}{q^{20}}-\frac{2}{q^{19}}+\frac{13}{q^{18}}-\frac{8}{q^{17}}+\frac{9}{q^{16}}+\frac{2}{q^{15}}-\frac{1}{q^{13}}+\frac{3}{q^{12}}-\frac{2}{q^{11}}-\frac{3}{q^{10}}+\frac{7}{q^{9}}-\frac{9}{q^{8}}+\frac{4}{q^{7}}-\frac{1}{q^{6}}-\frac{5}{q^{5}}+\frac{3}{q^{4}}-\frac{3}{q^{3}}-\frac{1}{q^{2}}-\frac{1}{q}+1-2q+2q^2-q^4+2q^5-q^6+q^7$
$\mathbf{8}_{16}$	$(\sigma_1^{-2}\sigma_2)^2\sigma_1^{-1}\sigma_2$	$\frac{1}{q^{49}}-\frac{2}{q^{48}}+\frac{1}{q^{47}}-\frac{2}{q^{45}}+\frac{5}{q^{44}}-\frac{4}{q^{43}}+\frac{2}{q^{41}}-\frac{4}{q^{40}}+\frac{3}{q^{39}}+\frac{2}{q^{38}}-\frac{2}{q^{36}}+\frac{8}{q^{35}}-\frac{10}{q^{34}}+\frac{3}{q^{33}}+\frac{7}{q^{32}}-\frac{17}{q^{31}}+\frac{14}{q^{30}}-\frac{8}{q^{29}}-\frac{3}{q^{28}}+\frac{7}{q^{27}}-\frac{2}{q^{26}}-\frac{4}{q^{25}}+\frac{1}{q^{24}}+\frac{6}{q^{23}}-\frac{18}{q^{22}}+\frac{21}{q^{21}}-\frac{10}{q^{20}}-\frac{6}{q^{19}}+\frac{19}{q^{18}}-\frac{16}{q^{17}}+\frac{11}{q^{16}}+\frac{2}{q^{15}}-\frac{2}{q^{14}}-\frac{3}{q^{13}}+\frac{6}{q^{12}}-\frac{5}{q^{11}}-\frac{4}{q^{10}}+\frac{14}{q^{9}}-\frac{15}{q^{8}}+\frac{9}{q^{7}}-\frac{7}{q^{5}}+\frac{6}{q^{4}}-\frac{3}{q^{3}}-\frac{1}{q^{2}}-\frac{1}{q}+3-4q+4q^2-q^3-2q^4+3q^5-2q^6+q^7$
$\mathbf{8}_{17}$	$\sigma_1^2(\sigma_2^{-1}\sigma_1)^2\sigma_2^{-2}$	$\frac{1}{q^{28}}-\frac{2}{q^{27}}+\frac{2}{q^{26}}-\frac{1}{q^{25}}-\frac{2}{q^{24}}+\frac{5}{q^{23}}-\frac{6}{q^{22}}+\frac{3}{q^{21}}+\frac{1}{q^{20}}-\frac{3}{q^{19}}+\frac{2}{q^{18}}+\frac{3}{q^{17}}-\frac{3}{q^{16}}-\frac{4}{q^{15}}+\frac{11}{q^{14}}-\frac{17}{q^{13}}+\frac{13}{q^{12}}+\frac{3}{q^{11}}-\frac{13}{q^{10}}+\frac{18}{q^{9}}-\frac{10}{q^{8}}+\frac{2}{q^{6}}+\frac{5}{q^{5}}-\frac{15}{q^{4}}+\frac{11}{q^{3}}+\frac{2}{q^{2}}-\frac{21}{q}+31-21q+2q^2+11q^3-15q^4+5q^5+2q^6-10q^8+18q^9-13q^{10}+3q^{11}+13q^{12}-17q^{13}+11q^{14}-4q^{15}-3q^{16}+3q^{17}+2q^{18}-3q^{19}+q^{20}+3q^{21}-6q^{22}+5q^{23}-2q^{24}-q^{25}+2q^{26}-2q^{27}+q^{28}$
$\mathbf{8}_{18}$	$(\sigma_1\sigma_2^{-1})^4$	$\frac{1}{q^{28}}-\frac{3}{q^{27}}+\frac{3}{q^{26}}-\frac{1}{q^{25}}-\frac{3}{q^{24}}+\frac{8}{q^{23}}-\frac{9}{q^{22}}+\frac{5}{q^{21}}+\frac{2}{q^{20}}-\frac{5}{q^{19}}+\frac{3}{q^{18}}+\frac{4}{q^{17}}-\frac{5}{q^{16}}-\frac{6}{q^{15}}+\frac{16}{q^{14}}-\frac{26}{q^{13}}+\frac{18}{q^{12}}+\frac{4}{q^{11}}-\frac{21}{q^{10}}+\frac{26}{q^{9}}-\frac{15}{q^{8}}-\frac{1}{q^{7}}+\frac{4}{q^{6}}+\frac{7}{q^{5}}-\frac{21}{q^{4}}+\frac{18}{q^{3}}+\frac{4}{q^{2}}-\frac{30}{q}+47-30q+4q^2+18q^3-21q^4+7q^5+4q^6-q^7-15q^8+26q^9-21q^{10}+4q^{11}+18q^{12}-26q^{13}+16q^{14}-6q^{15}-5q^{16}+4q^{17}+3q^{18}-5q^{19}+2q^{20}+5q^{21}-9q^{22}+8q^{23}-3q^{24}-q^{25}+3q^{26}-3q^{27}+q^{28}$
$\mathbf{8}_{19}$	$(\sigma_1\sigma_2)^4$	$q^{48}+q^{50}+q^{51}+q^{52}+q^{53}+2q^{54}+q^{55}+q^{56}+q^{57}+q^{60}-q^{61}-q^{62}-q^{63}-2q^{64}-2q^{65}-q^{66}-3q^{67}-2q^{68}-q^{69}-2q^{70}-q^{71}-q^{74}+2q^{75}+q^{77}+2q^{78}+q^{79}+q^{80}+q^{81}+q^{82}+q^{84}-q^{86}+q^{87}-q^{88}-q^{92}-q^{95}+q^{96}$
$\mathbf{8}_{20}$	$(\sigma_1^3\sigma_2^{-1})^2$	$q^{14}+2q^{16}+q^{18}+q^{19}+q^{20}+q^{22}-q^{24}+3q^{25}-3q^{26}+q^{27}-q^{28}-4q^{29}-3q^{31}-q^{32}-3q^{33}+q^{34}-4q^{35}+q^{36}+2q^{37}-5q^{38}+6q^{39}-4q^{40}+2q^{41}+4q^{42}+3q^{44}+q^{45}+2q^{46}-q^{47}+4q^{48}-2q^{49}-q^{50}+2q^{51}-4q^{52}+3q^{53}-q^{54}-2q^{55}+q^{56}-2q^{57}-q^{58}-q^{61}+q^{62}+2q^{65}-q^{66}-q^{69}+q^{70}$
$\mathbf{8}_{21}$	$\sigma_1^3\sigma_2\sigma_1^{-2}\sigma_2^2$	$3q^{16}-2q^{17}+2q^{18}+q^{19}-2q^{20}+4q^{21}-q^{22}+q^{25}-3q^{26}+2q^{27}+2q^{28}-7q^{29}+6q^{30}-5q^{31}-3q^{32}+3q^{33}-4q^{34}-2q^{35}+q^{36}-4q^{38}+8q^{39}-3q^{40}-q^{41}+6q^{42}-4q^{43}+2q^{44}+3q^{45}-q^{46}-q^{47}+4q^{48}-2q^{49}+3q^{51}-4q^{52}+q^{53}-q^{54}-2q^{55}+q^{57}-q^{58}+q^{60}-q^{61}+q^{62}$

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