Logarithmic Potential Theory on Riemann Surfaces

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This thesis is dedicated to Cy Skinner (1920-2014).

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Abstract

We develop a logarithmic potential theory on Riemann surfaces which generalizes logarithmic potential theory on the complex plane. We show the existence of an equilibrium measure and examine its structure. This leads to a formula for the structure of the equilibrium measure which is new even in the plane. We then use our results to study quadrature domains, Laplacian growth, and Coulomb gas ensembles on Riemann surfaces. We prove that the complement of the support of the equilibrium measure satisfies a quadrature identity. Furthermore, our setup allows us to naturally realize weak solutions of Laplacian growth (for a general time-dependent source) as an evolution of the support of equilibrium measures. When applied to the Riemann sphere this approach unifies the known methods for generating interior and exterior Laplacian growth. We later narrow our focus to a special class of quadrature domains which we call Algebraic Quadrature Domains. We show that many of the properties of quadrature domains generalize to this setting. In particular, the boundary of an Algebraic Quadrature Domain is the inverse image of a planar algebraic curve under a meromorphic function. This makes the study of the topology of Algebraic Quadrature Domains an interesting problem. We briefly investigate this problem and then narrow our focus to the study of the topology of classical quadrature domains. We extend the results of Lee and Makarov [24] and prove (for $n \geq 3$)

$$c \leq 5n - 5,$$

where c and n denote the connectivity and degree of a (classical) quadrature domain. At the same time we obtain a new upper bound on the number of isolated points of the algebraic curve corresponding to the boundary and thus a new upper bound on the number of special points. In the final chapter we study Coulomb gas ensembles on Riemann surfaces.

0.1 Notation Index

Unless otherwise specified:

dA	"A" represents the Lebesgue measure on $\mathbb{C}$.
$AL^1(\Omega, \mu)$	The set of μ-integrable analytic functions on Ω.
$Bal(\mu, \kappa)$	The partial balayage of the measure μ onto the measure κ.
$\hat{\mathbb{C}}$	The Riemann sphere.
C^*	$\mathbb{C}\backslash\{0\}$.
$C_b(D)$	The set of bounded continuous functions on D.
$f^*\omega$	The pullback of the form ω by f.
$f_*\mu$	The pushforward of the measure μ by f.
g	The Riemannian metric.
G_g	The Green's function of Laplace-Beltrami operator for the metric g.
G_ν	The interaction kernel with background measure ν (defined in chapter 3).
$HL^1(\Omega, \mu)$	The set of μ-integrable harmonic functions on Ω.
$\hat{\mu}$	The balayage of μ onto a specified boundary.
μ_{eq}	The equilibrium measure.
ν	The background measure (defined in chapter 3).
$\mu^{\otimes n}$	The product measure $\mu \otimes \cdots \otimes \mu$ (n-times).
M	A compact Riemann surface.
$\mathbb{P}(M)$	The set of probability measures on M.
Q	The external field.
$Q_{\nu,g}$	The modified external field (see chapter 3).
R	A meromorphic function on the compact Riemann surface, M.
S	The support of the equilibrium measure μ_{eq}.
$SL^1(\Omega, \mu)$	The set of μ-integrable subharmonic functions on Ω.
U_ν^μ	The potential generated by μ with respect to ν (defined in chapter 3).
V	The energy of the equilibrium measure.
vol_g	The measure induced by the volume form for the metric g.
Δ_g	The Laplace-Beltrami operator for the metric g.
Δ	The diagonal of a product set or the Laplacian in $\mathbb{C}$.
1_Ω	The characteristic function of Ω.
$\lvert\Omega\rvert_g$	The volume of Ω with respect to vol_g.
χ	The Euler characteristic.

Contents

Chapter 1

Introduction and Summary of Results

1.1 Introduction

The origins of logarithmic potential theory (with external fields) can be traced back to the work of C. F. Gauss in what is known as the Gauss variation problem. There has been significant progress since then and logarithmic potential theory has proved invaluable in the study of orthogonal polynomials, polynomial approximation, 2-D Laplacian growth (Hele-Shaw flow), and quadrature domains. More recently, there have been important applications to random matrix theory and conformal field theory in the plane.

Some of these areas of application have well-developed or burgeoning Riemann surface generalizations. Etingof and Varchenko [28] sketched the beginnings of the theory of Hele-Shaw flow on surfaces in $\mathbb{R}^3$ and Hedenmalm and Shimorin [14] studied Hele-Shaw flow on simply connected surfaces with negative curvature. At the same time, there has been interest in developing a theory of quadrature domains on Riemann surfaces (see [5] section 3). Conformal field theory is richest when studied on general Riemann surfaces and while there is not an obvious Riemann surface generalization of random matrix theory, there are generalizations of the eigenvalue distributions which are motivated by conformal field theory. This leads to Coulomb gas ensembles on Riemann surfaces and related ensembles motivated by the recent study of the Fractional Quantum Hall effect on curved space [29]. Logarithmic potential theory is a theory in the plane and as such it is not applicable to problems in these areas. This thesis begins by developing a generalization of logarithmic potential theory on Riemann surfaces which is suitable for application to these areas. We then explore its applications to quadrature domains, Laplacian growth, and Coulomb gas ensembles. Along the way we prove new results in the classical setting of logarithmic potential and quadrature domains. In the following section we precisely state our main results while providing an overview of the thesis.

1.2 Overview and Main Results

In this section we summarize the main results of this thesis. Some terminology (admissible external field, weak solutions of Hele-Shaw flow, balayage, etc.) will be defined later. To distinguish between the existing theories on the plane and Riemann surface generalizations, we call the former the "classical setting" (although it is a very rich and active area of study).

1.2.1 Logarithmic Potential Theory on Riemann Surfaces

We begin by developing a generalization of logarithmic potential theory on compact Riemann surfaces. Throughout this thesis M will always denote a compact Riemann surface. Although we sometimes endow M with a Riemannian metric this is only a matter of convenience and all results in this section depend only on the complex structure of M. We consider a finite positive measure, ν, on M, which we call the background measure. The typical cases of interest are when ν is linear combination of point masses and when ν is the measure induced by a volume form. We show that corresponding to such a ν, there exists, up to an additive constant, a unique symmetric kernel G_ν satisfying

$$\int_M G_\nu(z,w)\partial\bar{\partial}f(w) = \frac{1}{2i}\Big[\int_M f d\nu - \nu(M)f(z)\Big],$$

for all $f \in C^\infty(M)$. When $M = \hat{\mathbb{C}}$ and $\nu = 2\pi\delta_\infty$, G_ν is the logarithmic kernel, $\log\frac{1}{|z-w|}$. and so we recover logarithmic potential theory on the plane. Consider the cylinder $\mathbb{C}/\mathbb{N}$. We can compactify $\mathbb{C}/\mathbb{N}$ by adding two points p and q at either end to obtain the Riemann sphere. If we choose $\nu = \pi\delta_p + \pi\delta_q$ we obtain

$$G_\nu(z,w) = \log\frac{1}{|\sin(\pi(z-w))|},$$

which is the natural potential on $\mathbb{C}/\mathbb{N}$ obtained from the logarithmic potential by lifting to $\mathbb{C}$ (see example 3.2.5). If M is a compact Riemann surface endowed with an automorphic metric, g, and ν is the measure induced by the corresponding volume form, then G_ν is conformally invariant (see Prop 6.5.5 for a stronger statement). We note that these particular kernels appear in Conformal Field Theory; in the aforementioned examples, $G_\nu(z,w)$ is the two-point correlation function, $\mathbb{E}[\Phi(z)\Phi(w)]$, where Φ is a free Bosonic field, on the plane, the cylinder, and M respectively. Connections with Conformal Field Theory are discussed further in chapter 6.

In the presence of an external field, Q, the energy of a charge distribution $\mu \in \mathbb{P}(M)$, is defined to be

$$I[\mu] \equiv \int_{M\times M} G_\nu d\mu^{\otimes 2} + 2\int_M Q d\mu.$$

We prove the existence of an equilibrium measure $\mu_{eq} \in \mathbb{P}(M)$ satisfying

$$I[\mu_{eq}] = \inf_{\mu\in\mathbb{P}(M)} I[\mu],$$

for a certain class of external fields called admissible external fields. We prove an analogue of Frostman's Theorem in this setting (Lemma 3.4.10 and Lemma 3.4.11). In general it is difficult to explicitly describe the equilibrium measure. The main theorems of this chapter show under mild hypotheses that the equilibrium measure has an a priori structure. To determine the equilibrium measure, it suffices only to know its support (which we denote by S). Before stating the main theorems of this section, we take an opportunity to clarify some notation. Let $f \in C^2(M)$; $\partial\bar{\partial} f$ is a (1-1)-form defined in local coordinates, $z = x + iy$ by $\partial_z \partial_{\bar{z}} f dz \wedge d\bar{z}$, where $\partial_z = \frac{1}{2}[\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}]$ and $\partial_{\bar{z}} = \frac{1}{2}[\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}]$. From now on we identify (1-1)-forms with measures. That is if ω is a (1-1)-form then we identify it with the induced measure, ω, defined by

$$\omega(A) \equiv \int_A \omega, \tag{1.1}$$

for all measurable sets A. A measure, μ, is simply said to be absolutely continuous (without reference to another measure) if μ is absolutely continuous with respect to the measure induced by some (and thus any) volume form. Similarly, when we speak of properties of the Radon-Nikodym of a measure μ, we implicitly mean the Radon-Nikodym derivative taken with respect to the measure induced by a volume form. As such, we only consider properties which are metric independent. The following are the main theorems of this chapter.

Theorem 3.5.1. *(Structure Theorem 1). Let Q be an admissible potential which is C^2 in a neighborhood of S. Then*

$$\mu_{eq} = \frac{1}{\nu(M)} 1_S[\nu + 2i\partial\bar{\partial} Q + \kappa],$$

where κ is the balayage of the measure $1_{S^c}[\nu + 2i\partial\bar{\partial}\tilde{Q}]$ onto ∂S, $\tilde{Q} \in C^2(M)$ is any C^2-smooth extension of $Q|_U$, where U is a neighborhood of S.

Under stricter hypotheses on the background measure we obtain the following structure theorem:

Theorem 3.5.2. *(Structure Theorem 2). Let Q be an admissible external field which is C^2 in a neighborhood U of S. Suppose further that $\nu|_U$ is absolutely continuous and its Radon-Nikodym derivative is in $C_b(U)$, then*

$$\mu_{eq} = \frac{1}{\nu(M)} 1_S[\nu + 2i\partial\bar{\partial} Q].$$

As an almost immediate consequence of these structure theorems we recover the well-known formula for the structure of the equilibrium measure in the classical setting :

$$\mu_{eq} = \frac{1}{2\pi} 1_S \Delta Q dA,$$

(under the original hypotheses that Q is C^2 in a neighborhood of the support). Using the first structure theorem, we can strengthen this result. In the classical setting, it follows from Frostman's theorem that Q must be subharmonic on S. Using the first structure theorem we prove a formula for the structure of the equilibrium measure when Q is subharmonic in a neighborhood of S, but no other regularity is assumed.

If Q is subharmonic in a (bounded) neighborhood U of S, it follows from the Riesz Decomposition theorem, that (for $z \in \mathbb{C}$):

$$Q(z) = \int_M \log \frac{1}{|z-w|} d\tau(w) + h(z), \tag{1.2}$$

where τ is a *negative* compactly supported measure, and h is harmonic on U.

Theorem 3.5.4. *Let Q be a (classically) admissible external field in the plane. Suppose that Q is subharmonic in a neighborhood U of S. Choose τ and h satisfying 1.2. Then*

$$\mu_{eq} = \frac{1}{\tau(\mathbb{C})} 1_S [\tau - \omega], \tag{1.3}$$

where ω is defined as follows. Let $\tilde{h}$ denote a C^2 extension of h from a neighborhood of S to $\hat{\mathbb{C}}$. ω is defined as the balayage of $1_{S^c}[-\tau + \Delta\tilde{h}dA]$ onto ∂S.

The structure theorems show that μ_{eq} has an a priori structure and thus to determine μ_{eq} is suffices to determine S. In section 3.6 we show that S is related to a certain obstacle problem.

1.2.2 Quadrature Domains and Laplacian Growth on Surfaces

2-D Laplacian growth (formerly known as Hele-Shaw flow) is a planar growth process where the evolution of a domain at any stage of the process is governed by its harmonic measure at a fixed point (see section 2.3 for a precise definition). It is remarkable that several seemingly disparate physical processes turn out to be examples of 2-D Laplacian growth. Some of the many examples are: viscous fingering, diffusion limited aggregation, slow freezing of fluids, fluid flow in Hele-Shaw cells, and crystal growth (see [29] for a more complete list along with references). Some of these processes have analogues on surfaces. For example Laplacian growth is a mathematical model for the evolution of a fluid injected into a thin mould created by two almost parallel surfaces separated by a small distance (see [28] chapter 7 for more details). On the other hand, even on $\mathbb{C}$, the introduction of a metric is very interesting. In the example of fluid dynamics, the metric determines the permeability of the medium. Laplacian growth in this setting is known as Elliptic Growth and has applications in fluid dynamics, electro-deposition, and cellular growth. [20] provides a nice introduction to the subject. Etingof and Varchenko [28] sketched the beginnings of a theory of Laplacian growth on Riemann surfaces endowed with a Riemannian metric. Later Hedenmalm and Shimorin [14] studied Laplacian growth on simply connected surfaces with negative curvature.

If $\{\Omega_t\}_{t=0}^{t_*}$ is the evolution of Ω_0 under 2-D Laplacian growth with source $b \in \Omega_0$ then by definition the normal velocity, $v_n(t)$, of the evolution of domains at time t satisfies

$$v_n(t) = -\frac{1}{2\pi} \partial_n G_{\Omega_t}(\cdot, b), \tag{1.4}$$

where G_{Ω_t} is the Green's function of Ω_t, and ∂_n is the outward normal derivative with respect to $\partial\Omega_t$. If a Riemann surface, M, is endowed with a Riemannian metric, g, then we can define Laplacian growth on M analogously. Both the normal velocity and the normal derivative depend on the choice of the metric, but Equation 1.4 is not invariant under a conformal change of metric. We introduce a weighted version of Laplacian growth on Riemann surfaces which depends only on the complex structure and the weight. Given a weight $F : M \to \mathbb{R}$, we define F-weighted Laplacian growth by the equation

$$\Delta_g F v_n(t) = \frac{1}{2\pi}\partial_n G_{\Omega_t}(\cdot, b), \tag{1.5}$$

where g is any metric compatible with the complex structure of M. Equation 1.5 does not depend on the choice of metric compatible with the complex structure and thus only depends on the complex structure of M. We discuss an interesting connection between logarithmic potential theory on Riemann surfaces and Q-weighted Laplacian growth (where Q is the external field). In the classical case, this unifies the known methods for producing interior and exterior 2-D Laplacian growth. In chapter 5 section 3 we study $|R|^2$-weighted Laplacian growth where R is a meromorphic function on M (when $M = \hat{\mathbb{C}}$ and $R(z) = z$ we recover 2-D Laplacian growth).

There has been less work on quadrature domains on Riemann surfaces. In the plane there is a beautiful relation between quadrature domains and operator theory given by the exponential transform. In [30] Xia provides a definition of quadrature domains on Riemann surfaces and studies an analogue of the exponential transform. For our purposes Xia's definition is not general enough to capture the interplay between Laplacian growth and quadrature domains nor is it specific enough to capture the algebraic properties of the boundary of a quadrature domain. We instead provide a more general definition of quadrature domain which we study in chapter four, and later in chapter five focus on a special case which we call Algebraic Quadrature Domains. For a measure, μ, and an open set, Ω, we let $AL^1(\Omega,\mu)$, $HL^1(\Omega,\mu)$, and $SL^1(\Omega,\mu)$ denote the sets of μ-integrable function on Ω which are analytic, harmonic, and subharmonic, respectively. We begin by defining quadrature domains on Riemann surfaces. Let $\Omega \subset M$ be an open set (despite the name, it is convenient to not require that Ω be connected) and let μ be an absolutely continuous measure on Ω with continuous and bounded Radon-Nikodym derivative. Let $\Psi : AL^1(\Omega,\mu) \to \mathbb{C}$ be a linear functional. We call an open set $\Omega \subset M$ an analytic quadrature domain with data (μ, Ψ) if

$$\int_\Omega f d\mu = \Psi[f] \tag{1.6}$$

for every $f \in AL^1(\Omega, \mu)$. We define harmonic quadrature domains analogously by replacing $AL^1(\Omega, \mu)$ in the above definition with $HL^1(\Omega, \mu)$. We call Ω a subharmonic quadrature domain if

$$\int_{\Omega} h d\mu \geq \Psi[h] \tag{1.7}$$

for every $h \in SL^1(\Omega, \mu)$ where $\Psi : SL^1(\Omega, \mu) \to \mathbb{C}$ is a linear functional. We recover classical analytic quadrature domains by choosing $M = \hat{\mathbb{C}}$, μ the Lebesgue measure, and Ψ the functional defined by

$$\Psi[f] = \sum_{i=1}^{n} \sum_{j=0}^{n_i - 1} c_{i,j} f^j(a_i).$$

We begin by studying the simplest type of quadrature domains, those where

$$\Psi = c\delta_b,$$

where $c \in \mathbb{R}_+$ and $b \in \Omega$. Following the terminology in the classical setting, we call such domains one-point quadrature domains. In the classical setting, the discs are one-point (harmonic and analytic) quadrature domains. This classical result is the well-known (area) mean value theorem for harmonic functions. It is also easy to show that the discs are also one-point subharmonic quadrature domains. It is significantly more difficult to show that discs are the *only* one-point quadrature domains. There has been much work on this problem see ([6], [7], [20], [1]). Using the Schottky Double construction, Gustaffson [9] provided another proof that the discs are the only one-point analytic quadrature domains.
We prove a hyperbolic analogue of the result above. Effectively, we consider the description of one-point quadrature domains on the Poincar disc $(\mathbb{D}, \mu_h)$, where μ_h denotes the measure induced by the volume form for the hyperbolic metric. We prove that the bounded one-point harmonic and subharmonic quadrature domains on the Poincar disc are precisely the hyperbolic discs. The proof relies on techniques different from those used in the proof of the analogue on $\mathbb{C}$, but it is relatively elementary.

Theorem 4.2.1. *Let M be a hyperbolic Riemann surface (which is not necessarily compact). Let $\Omega \subset M$ be an open set which is bounded with respect to the hyperbolic metric and does not contain a homology cycle of M. Let ω_h be the measure induced by the volume form corresponding to the hyperbolic metric on M. Let $c \in \mathbb{R}_+$, and let $b \in \Omega$. Then Ω is a harmonic quadrature domain with data $(\omega_h, c\delta_b)$ only if Ω is the hyperbolic disc centered at b with area c. Moreover, hyperbolic discs not containing cycles are subharmonic quadrature domains.*

More interesting examples of quadrature domains are obtained as a consequence of the structure theorems. Namely, the complement of the support of the equilibrium measure is a quadrature domain. More precisely:

Theorem 4.2.3. *Let Q be C^2 in a neighborhood of S and suppose* $\operatorname{supp}(\nu) \subset (\overline{S})^c$. *Let $\tilde{Q} \in C^2(M)$ denote a C^2*

extension of $1_U Q$ *where* U *is a neighborhood of* S. *Then* S^c *is a Quadrature Domain with data* $(\frac{2}{i}\partial\bar{\partial}\tilde{Q}, \nu, H)$. *That is:*

$$\frac{2}{i}\int_{S^c} h\partial\bar{\partial}\tilde{Q} = \int_M h d\nu,$$

for every $h \in HL^1(S^c, 2i\partial\bar{\partial}Q)$.

1.2.3 Algebraic Quadrature Domains

It was shown in [9] that the boundary of a quadrature domain is a subset of an algebraic curve. We introduce Algebraic Quadrature Domains (abbreviated AQD) to capture an analogue of this property on Riemann surfaces. Let R be a meromorphic function on M. We call a domain $D \subsetneq M$ with piecewise C^1 boundary an AQD for R if $\overline{D}$ does not contain any poles of R and there exists a meromorphic function, $S_{D,R}$, on D that extends continuously to ∂D and satisfies

$$S_{D,R}|_{\partial D} = \bar{R}|_{\partial D}.$$

$S_{D,R}$ is called the (generalized) Schwarz function. We show that these domains turn out to be analytic quadrature domains with data $(2i\partial\bar{\partial}|R|^2, \Psi)$ where

$$\Psi[f] \equiv 2i\int_{\partial D} f S_{D,R}\partial R, \tag{1.8}$$

for all $f \in AL^1(D, 2i\partial\bar{\partial}|R|^2)$. We prove that if D is an AQD for R then ∂D is a subset of the inverse image of a planar algebraic curve under R which motivates the name. In the classical setting $M = \hat{\mathbb{C}}$ and $R(z) = z$, and so we recover the well-known result that the boundary of a classical quadrature domain is a subset of a planar algebraic curve. Moreover, the boundary of an algebraic quadrature domain for a rational function is also a planar algebraic curve. This makes the study of the topology of algebraic quadrature domains an interesting problem.

We narrow our focus to the study of the topology of classical quadrature domains. Let Ω be a (classical) quadrature domain of degree n. The connectivity, c, of Ω is defined to be the number of components in $\hat{\mathbb{C}}\backslash\Omega$. It is well known that $\partial\Omega$ is subset of an algebraic curve, Γ, of degree $2n$. Moreover $\Gamma\backslash\partial\Omega$ consists of a finite number of points called special points. We denote the number of special points of Ω by s. We provide an upper bound on the connectivity of a quadrature domain in terms of its degree (the degree of a classical quadrature domain is defined in chapter 2 section 2). At the same time we prove a new upper bound on the number of special points of a quadrature domain as a function of its degree. As quadrature domains of degree one and two are completely classified, we restrict our attention to the case of $n \geq 3$. In a beautiful paper, Lee and Makarov [24] proved using qausiconformal surgery and complex dynamics the following upper bound when $\Omega = int(\overline{\Omega})$ (and $n \geq 3$):

$$c \leq 2n - 4. \tag{1.9}$$

Moreover they constructed examples showing that 1.9 is sharp. In this section we eliminate the hypothesis $\Omega = int(\overline{\Omega})$ and prove the following:

Theorem 5.3.3. *Let Ω be a quadrature domain with degree $n \geq 3$. Let c denote the connectivity of Ω. Then*

$$c \leq 5n - 5. \tag{1.10}$$

At the same time we provide a new upper bound on the number of special points. Special points are interesting for a number of reasons. It turns out that $z \in \Omega$ is a special point iff $\Omega \backslash \{z\}$ remains an analytic quadrature domain. In other words, Ω continues to satisfy the quadrature identity for the new test function $f(z) \equiv \frac{1}{z-a}$. More interestingly, special points are isolated points of the algebraic curve which defines the boundary of Ω. They have been well studied. In particular, there has been interest in providing upper and lower bounds on the number of special points. For estimates of special points, it makes sense (and we lose nothing) to assume $\Omega = int(\overline{\Omega})$ (so that there exists nontrivial lower bounds on s). We will do so from now on. Let b denote the number of cusps on $\partial\Omega$, c denote the connectivity of Ω, and d denote the number of double points of $\partial\Omega$. Gustafsson [13] proved:

$$s \leq (n-1)^2 + 1 - c - b - 2d \tag{1.11}$$

and Sakai [27] proved

$$s \geq n - 2 + c - b, \tag{1.12}$$

see also McCarthy and Yang [25]. We prove:

Theorem 5.3.2. *Let Ω be a quadrature domain of degree $n \geq 3$ satisfying $\Omega = int(\overline{\Omega})$. Let s denote the number of special points and c denote the connectivity of Ω. Then*

$$s \leq 5n - 5 - c.$$

Our proof makes use of Lemma 4.3 from [24]. The new ingredient is a recent result of D. Khavinson and G. Neumann [21] which resolved an open problem in gravitational lensing.

1.2.4 Coulomb Gas Ensembles and CFT

Let $Q : \mathbb{C} \to \mathbb{R} \cup \{\infty\}$ and $\beta > 0$. The following sequence of probability measures

$$\Pi_n \equiv \frac{1}{Z_n} \int_{\mathbb{C}^n} e^{-\beta H_n} dA^{\otimes n}, \tag{1.13}$$

where

$$H_n(z_1, ..., z_n) \equiv \sum_{1 \leq i < j \leq n} \log \frac{1}{|z_i - z_j|} + (n-1) \sum_{i=1}^{n} Q(z_i),$$

and Z_n is a normalizing constant is called a Coulomb gas ensemble or β-ensemble. Coulomb gas ensembles occur frequently in random matrix theory. For example, when $\beta = 2$ they are the eigenvalue distributions of certain classes of random normal matrices. There is also a connection to conformal field theory. The internal energy component of the integrand of 1.13 is related to the vacuum expectation of vertex operators. Coulomb gas ensembles generalize nicely to Riemann surfaces. Let

$$H_n^{\nu,Q}(z_1, ..., z_n) \equiv \sum_{1 \leq i < j \leq n} [G_\nu(z_i, z_j) + Q(z_i) + Q(z_j)].$$

Physically, H can be interpreted as the energy of configuration of n unit point charges placed at $\{z_1, ..., z_n\}$ ($z_i \neq z_j$), in the presence of an external field $(n-1)Q$. There is a discrete analogue of the energy problem. For every such configuration of n particles there is an associated probability measure

$$\mu_z \equiv \frac{1}{n} \sum_{i=1}^{n} \delta_{z_i}, \tag{1.14}$$

where $z = (z_1, ..., z_n) \in M^n$ and $z_i \neq z_j$. We denote the class of such probability measures by $\mathbb{P}_n(M)$. All measures in $\mathbb{P}_n(M)$ have infinite energy, so we introduce a regularized energy energy functional:

$$I^*[\mu] \equiv \int_{(M \times M) \setminus \Delta} K_\nu^Q d\mu^{\otimes 2}. \tag{1.15}$$

If Q is admissible then there exists a $\mu_n \in \mathbb{P}_n(M)$ of minimal energy. In analogy with the classical setting the associated $(z_1^{(n)}, ..., z_n^{(n)})$ are called Fekete points. We call μ_n the Fekete measure (at level n) and we prove:

Theorem 6.2.3. *Let Q be an admissible external field. Then*

$$\mu_n \rightharpoonup \mu_{eq} \tag{1.16}$$

and

$$I^*[\mu_n] \to I[\mu_{eq}] \tag{1.17}$$

as $n \to \infty$.

Theorem 6.2.3 extends the classical result of Fekete to this setting.
We define the partition function

$$Z_n^{Q,\nu,\beta} \equiv \int_{M^n} e^{-\beta H_n} dvol_g^{\otimes n}.$$

In anology with statistical mechanics, the Boltzmann Gibbs measure corresponding to the energy function H is defined to be

$$\Pi_n \equiv \frac{1}{Z_n} e^{-\beta H_n} dvol_g^{\otimes n}. \tag{1.18}$$

For certain ν, $G_\nu(z,w) = \mathbb{E}[\Phi(z)\Phi(w)]$ where Φ is a Free Bosonic Field on M. For such ν the internal energy component of the integrand 1.14 is related to the vacuum expectation of vertex operators. Let $k \leq n$, and let $A \subset M^k$. The k-th marginal measure of Π_n is defined by

$$\Pi_{n,k}(A) \equiv \Pi_n(A \times M^{n-k}).$$

The main result of this section is:

Theorem 6.3.3. *Let Q be an admissible potential. If $Q_{\nu,g}$ is continuous then*

$$\Pi_{n,k} \rightharpoonup \mu_{eq}^{\otimes k}$$

as $n \to \infty$.

Johansson [18] proved this result on $\mathbb{R}$ and Hedenmalm and Makarov [15] later proved the analogue in the complex plane. Our proof follows the general structure of Johansson's although some portions are more technical in this setting (e.g. Lemma 6.3.1).

Chapter 2

Background

This background section provides a very brief survey of some fundamental results in the study of logarithmic potential theory and quadrature domains. [26], [28], and [12] provide more thorough introductions to the respective subjects and our presentation is influenced by them. Nothing more than basic complex analysis is needed to understand the background material. This thesis studies generalizations of these subjects to compact Riemann surfaces (sometimes with punctures), and as such will use results from Riemann surface theory. We only assume knowledge of the basics of Riemann surface theory (e.g. the uniformization theorem, covering spaces, differential forms, Riemann-Hurwitz theorem etc.). We further assume that the reader is familiar with the basics of Riemannian Geometry (e.g. volume forms, Stoke's Theorem, the Laplace-Beltrami operator, Sobolev spaces on manifolds, etc.).

2.1 Logarithmic Potential Theory

2.1.1 Overview

The central object of logarithmic potential theory is the energy functional $I : \mathbb{P}(\mathbb{C}) \to \mathbb{R} \cup \{\infty\}$ defined by

$$I[\mu] \equiv \int_{\mathbb{C}\times\mathbb{C}} \log \frac{1}{|z-w|} d\mu(z)d\mu(w) + \int_{\mathbb{C}} Q d\mu, \tag{2.1}$$

where $Q : \mathbb{C} \to \mathbb{R} \cup \{\infty\}$. If Q is an admissible external field (see def) then 2.1 is well-defined and there exists a unique compactly supported probability measure, μ_{eq}, called the equilibrium measure satisfying:

$$I[\mu_{eq}] = \inf_{\mu\in\mathbb{P}(\mathbb{C})} I[\mu]. \tag{2.2}$$

There is a helpful physical intuition coming from electrostatics which motivates the terminology. Probability measures can be thought of as positive charge distributions on the conductor $\mathbb{C}$ with total charge one. Each charge distribution generates an electric potential defined by

$$U^{\mu} \equiv \int_{\mathbb{C}} \log \frac{1}{|\cdot - w|} d\mu(w), \tag{2.3}$$

which we call the logarithmic potential of μ. The external field, Q, is the electric potential on $\mathbb{C}$ in the absence of a charge distribution. The first term on the right side of 2.1 represents the internal energy - the potential energy of the charge distribution from interacting with its own electric field. The second term is the external energy arising from the interaction of the charge distribution with the external field Q. The energy functional, I, is then the total potential energy of the charge configuration in the presence of the external field. The external field is necessary. If a charge distribution, μ, were placed on $\mathbb{C}$ in the absence of an external field, it would not reach an equilibrium configuration. Rather, it would continue spreading out —continually reducing its potential energy. The growth condition on the external field at ∞ is just enough to hold back the charge. More precisely, it ensures a lower bound on the potential energy. μ_{eq} represents the charge configuration of minimal potential energy. The name equilibrium comes from the fact that if the charge distribution μ_{eq} were placed statically on $\mathbb{C}$ it would remain in static equilibrium - any motion would further reduce its potential energy, contradicting the definition.

One of the central concerns of logarithmic potential theory is the precise understanding of μ_{eq}. The following sections provide some prerequisites for understanding μ_{eq}.

2.1.2 Logarithmic Potentials

Definition 2.1.1. *The logarithmic kernel.*
We call the function $K : \mathbb{C} \times \mathbb{C} \to \mathbb{R} \cup \{\infty\}$ defined by

$$K(z, w) \equiv \log \frac{1}{|z - w|}$$

the logarithmic kernel.

Definition 2.1.2. *Logarithmic potential.*
Let μ be a finite measure. The logarithmic potential for μ is defined to be

$$U^{\mu} \equiv \int_{\mathbb{C}} \log \frac{1}{|\cdot - w|} d\mu(w). \tag{2.4}$$

The logarithmic potential is in general not continuous. However, for the typical class of measures that we will consider - Borel measures that are finite, positive, and compactly supported - U^μ satisfies a weaker notion of continuity called lower semi-continuity.

Definition 2.1.3. *Lower semi-continuous functions.*
Let D be a topological space, and let $f : D \to \mathbb{R} \cup \{\infty\}$. f is called lower semi-continuous if for every $\alpha \in \mathbb{R}$ the set

$$\{f > \alpha\}$$

is open.

A function f is called upper semi-continuous if $-f$ is lower semi-continuous. The "semi" comes from the fact that f is continuous if and only if f is both lower semi-continuous and upper semi-continuous. The following is an alternative characterization of lower semi-continuity which is used several times throughout this thesis.

Proposition 2.1.4. *f is a lower semi-continuous function on a compact set K if and only if f is the pointwise limit of an increasing sequence of continuous functions on K.*

The following is a useful property of lower semi-continuous functions.

Proposition 2.1.5. *If f is lower semi-continuous on a compact set K, then f attains its minimum on K.*

Under the hypotheses on μ given above, U^μ, satisfies a stronger property called superharmonicity. Before, defining this we review some properties of harmonic functions.

Definition 2.1.6. *Let $D \subset \mathbb{C}$ be a domain. $f \in C^2(D)$, is called harmonic if*

$$\Delta f = 0. \tag{2.5}$$

It is remarkable that Equation 2.5 actually implies that f is real-analytic. In fact, we can even reduce the a priori regularity hypothesis on f to continuity and the existence of the second order partial derivatives f_{xx} and f_{yy}. Harmonic functions satisfy the mean-value property. That is, if f is harmonic in D, then

$$f(z) = \frac{1}{2\pi r} \int_{\partial B(z,r)} f ds, \tag{2.6}$$

for every $B(z, r)$ which is compactly contained in D. If $f \in C(D)$, the converse holds true.

Definition 2.1.7. *Superharmonic functions on $\mathbb{C}$.*
Let $D \subset \mathbb{C}$ be a domain. A lower semi-continuous function $f : D \to \mathbb{R} \cup \{\infty\}$ is called superharmonic if for every $z \in D$ and every $r > 0$ such that $\overline{B(z, r)} \subset D$:

$$f(z) \geq \frac{1}{2\pi r} \int_{\partial B(z,r)} f ds. \tag{2.7}$$

Note that 2.7 is well-defined since by Proposition 5.41, f is bounded from below on the compact set $\partial B(z, r)$. We say a function f is subharmonic if $-f$ is superharmonic. It follows from the remark above that f is harmonic if and only if f is both superharmonic and subharmonic. Unlike harmonicity, superharmonicity does not provide any a posteriori regularity. However, if a superharmonic function is in C^2, there is an alternative characterization.

Proposition 2.1.8. *Let $D \subset \mathbb{C}$ be a domain. If $f \in C^2(D)$, then f is superharmonic if and only if $\Delta f \leq 0$.*

It is not difficult to check that for fixed w, $K(\cdot, w)$ is superharmonic. The following proposition is then a consequence of Fubini's theorem:

Proposition 2.1.9. *Let μ be a finite positive compactly supported Borel measure. Then U^μ is superharmonic.*

Moreover, since $K(\cdot, w)$ is harmonic on $\mathbb{C}\backslash\{w\}$, it follows that U^μ is harmonic on $\operatorname{supp}(\mu)^c$. The following is an important partial converse to proposition 2.1.9.

Theorem 2.1.10. *(Riesz Decomposition Theorem)*
If f is a superharmonic function on a domain Ω, then there exists a positive measure μ on Ω such that on every subdomain $D \subset \Omega$ for which $\overline{D} \subset \Omega$ we have the decomposition:

$$f(z) = u_D(z) + \int_D \log \frac{1}{|z - w|} d\mu(w), \tag{2.8}$$

for $z \in D$, where u_D is a harmonic.

2.1.3 Energy Problem

A set $A \subset \mathbb{C}$ is said to have zero capacity if every $\mu \in \mathbb{P}(\mathbb{C})$ supported in A has infinite energy logarithmic energy i.e.

$$\int_{\mathbb{C}\times\mathbb{C}} K d\mu^{\otimes 2} = \infty.$$

If a property holds everywhere except on a set with zero capacity the property is said to hold quasi-everywhere. If a set doesn't have zero capacity it is said to have positive capacity.

Definition 2.1.11. *Admissible external field.*
We call a function $Q : \mathbb{C} \to \mathbb{R} \cup \{\infty\}$ an admissible external field if the following three conditions hold:

i *Q is lower semi-continuous.*

ii $\lim_{z\to\infty} Q(z) - \log|z| = \infty$.

iii $\{Q \leq \infty\}$ *has positive capacity.*

Recall that the energy functional $I : \mathbb{P}(\mathbb{C}) \to \mathbb{R}_+$ is defined by

$$I[\mu] \equiv \int_{\mathbb{C}\times\mathbb{C}} \log \frac{1}{|z-w|} d\mu(z) d\mu(w) + \int_{\mathbb{C}} Q d\mu. \tag{2.9}$$

Define

$$V \equiv \inf_{\mu\in\mathbb{P}(\mathbb{C})} I[\mu].$$

The following theorem is fundamental.

Theorem 2.1.12. *(Frostman's Theorem)*

If Q is an admissible external field then the following holds:

i *V is finite.*

ii *There exists a unique element $\mu_{eq} \in \mathbb{P}(\mathbb{C})$ satisfying $I[\mu_{eq}] = V$.*

iii $\int_{\mathbb{C}\times\mathbb{C}} \log \frac{1}{|z-w|} d\mu_{eq}(z) d\mu_{eq}(w)$ *is finite.*

iv $S \equiv \mathrm{supp}(\mu_{eq})$ *is compact.*

v *Let $F \equiv V - \int_{\mathbb{C}} Q d\mu_{eq}$ then*

$$U^{\mu_{eq}} + Q \geq F$$

holds quasi-everywhere.

vi

$$(U^{\mu_{eq}} + Q)|_S \leq F.$$

2.2 Quadrature Domains

Let f be analytic and integrable on $a + r\mathbb{D}$, then it is well-known that f satisfies the mean-value property:

$$\int_{a+r\mathbb{D}} f dA = \pi r^2 f(a). \tag{2.10}$$

We shift the viewpoint and view Equation 2.11 as a property of the disc (where the test functions f are understood to be those that are analytic and integrable). Fix $\{a, b\} \subset \mathbb{C}$ and $\{c_1, c_2\} \in \mathbb{R}_+$. It is natural to ask if there is an open set Ω satisfying

$$\int_{\Omega} f dA = c_1 f(a) + c_2 f(b) \tag{2.11}$$

for all function f which are analytic and integrable on Ω. Of course, if $|a - b| \geq \sqrt{\frac{c_1}{\pi}} + \sqrt{\frac{c_2}{\pi}}$, then the discs $(\sqrt{\frac{c_1}{\pi}}\mathbb{D} + a)$ and $(\sqrt{\frac{c_2}{\pi}}\mathbb{D} + b)$ are disjoint and it follows from Equation 2.11 that $\Omega \equiv (\sqrt{\frac{c_1}{\pi}}\mathbb{D} + a) \sqcup (\sqrt{\frac{c_2}{\pi}}\mathbb{D} + b)$ satisfies 2.2. But what if $|a - b| < \sqrt{\frac{c_1}{\pi}} + \sqrt{\frac{c_2}{\pi}}$?

Example 2.2.1. *Fix $r > 1$ and let Ω be the domain whose boundary is the following algebraic curve:*

$$(x^2 + y^2)^2 - 2r^2(x^2 + y^2) - 2(x^2 - y^2) = 0.$$

Then Ω satisfies the following property

$$\int_{\Omega} f dA = \pi r^2 f(1) + \pi r^2 f(-1),$$

for all f analytic and integrable on Ω.

Example 2.2.1 provides an explicit description of a domain satisfying 2.11. Remarkably sets satisfying 2.11 are completely understood —even under the more general hypothesis that $c_1, c_2 \in \mathbb{R}$. These are the simplest non-trivial examples of quadrature domains. It will be useful to introduce some notation. Let Ω be a domain and let μ be a measure, we define $AL^1(\Omega, \mu)$, $HL^1(\Omega, \mu)$, and $SL^1(\Omega, \mu)$, to be the sets of functions which are integrable on Ω with respect to μ and which are respectively analytic, harmonic, and subharmonic on Ω.

Definition 2.2.2. *Classical (analytic) quadrature domains*
We say that a bounded open set, $\Omega \subset \mathbb{C}$, is a classical quadrature domain if there exists $\{a_1\}_{i=1}^n \subset \Omega$ and $c_{i,j} \in \mathbb{C}$ such that

$$\int_{\Omega} f dA = \sum_{i=1}^{n} \sum_{j=0}^{n_i - 1} c_{i,j} f^j(a_i) \tag{2.12}$$

for all $f \in AL^1(\Omega, dA)$. *Equation 2.12 is a called a quadrature identity and* $d \equiv \sum_{i=1}^{n} n_i$ *is called the degree of the quadrature domain. We call* $\{a_1, ..., a_n\}$ *and* $\{c_{i,j}\}$ *the quadrature data.*

Example 2.2.3. *The cardioids are a simple example of quadrature domains. In particular let*

$$\Omega \equiv \{z + \frac{z^2}{2} : |z| < 1\}.$$

Then

$$\int_\Omega f dA = \frac{3\pi}{2} f(0) + \frac{\pi}{2} f^{'}(0), \tag{2.13}$$

for all $f \in AL^1(\Omega, dA)$.

Despite much work, there are still many unsolved problems. In particular, it is not even known if two simply connected domains can have the same quadrature data. A lot is known however. The following is a (in no way complete) summary of some main results

- The boundary of a quadrature domain is subset of an algebraic curve [1].
- If Ω is a bounded simply connected domain, then Ω is a quadrature domain iff every uniformization $\phi : \mathbb{D} \to \Omega$, is a rational function ([1], [18]).
- More generally if Ω is a bounded finitely connected domain, then Ω is a quadrature domain iff Ω is a subset of a compact symmetric Riemann surface M (this will be defined later) and the identity map on Ω extends to a meromorphic function on M.
- If Ω is a connected quadrature domain, with connectivity n, then there exists an $n-1$-parameter family of quadrature domains with the same quadrature data as Ω.
- Let Ω be a quadrature domain with connectivity c and degree $n \geq 3$ satisfying $int(\overline{\Omega}) = \Omega$. Then

$$c \leq 2n - 4. \tag{2.14}$$

  Moreover 2.14 is sharp [24].

We now introduce two related classes of quadrature domains called harmonic and subharmonic quadrature domains.

Definition 2.2.4. *Classical harmonic quadrature domains*
We say that a bounded open set, $\Omega \subset \mathbb{C}$, *is a classical harmonic quadrature domain if there exists* $\{a_i\}_{i=1}^{n} \subset \Omega$

and $\{c_i\}_{i=1}^n \subset \mathbb{R}$ *such that*

$$\int_\Omega f dA = \sum_{i=1}^n c_i f(a_i) \tag{2.15}$$

for all $f \in HL^1(\Omega, dA)$.

Definition 2.2.5. *Classical subharmonic quadrature domains*
We say that a bounded open set, $\Omega \subset \mathbb{C}$*, is a classical subharmonic quadrature domain if there exists* $\{a_i\}_{i=1}^n \subset \Omega$ *and* $\{c_i\}_{i=1}^n \subset \mathbb{R}_+$ *such that*

$$\int_\Omega f dA \geq \sum_{i=1}^n c_i f(a_i) \tag{2.16}$$

for all $f \in SL^1(\Omega, dA)$.

The restriction of $\{c_i\}$ to the positive reals in Definition 2.2.5 is not really a restriction, as 2.16 cannot hold if $c_i < 0$. Indeed choosing $f = -K(\cdot, a_i)$ would yield a contradiction. Unlike analytic and harmonic quadrature domains, classical subharmonic quadrature domains (essentially) uniquely depend on the quadrature data.

Proposition 2.2.6. *Let* Ω_1 *and* Ω_2 *be a classical subharmonic quadrature domain with data* $\{a_i\}_{i=1}^n$ *and* $\{c_i\}_{i=1}^n$*. Then* $\Omega_1 = \Omega_2$ *up to null sets.*

2.2.1 Subharmonic Quadrature Domains and Partial Balayage

Let's look at 2.12 from another view. Imagine placing the two discs, $(\sqrt{\frac{c_1}{\pi}}\mathbb{D} + a)$ and $(\sqrt{\frac{c_2}{\pi}}\mathbb{D} + b)$, on top of each other. A modified version of 2.12 for the union of the discs holds if the area measure is doubled on the intersection. There is a beautiful technique called partial balayage which allows us to "sweep" the excess measure off the intersection to form a subharmonic quadrature domain Ω satisfying

$$\int_\Omega f dA \geq c_1 f(a) + c_2 f(b) \tag{2.17}$$

for all $f \in SL^1(\Omega, dA)$.
An open set, D, is called saturated if for every ball $B(z, r)$ satisfying $|D \backslash B(z, r)| = 0$, it follows that $B(z, r) \subset D$.

Theorem 2.2.7. *Let* μ *be a measure satisfying*

$$\int_{\text{supp}(\mu)} f d\mu \geq \int_{\text{supp}(\mu)} f dA, \tag{2.18}$$

for all positive $f \in C(\mathbb{C})$. Then there exists a unique bounded saturated open set D containing the $\mathrm{supp}(\mu)$ *and satisfying*

$$\int_D f dA \geq \int_{\mathbb{C}} f d\mu, \tag{2.19}$$

for all $f \in SL^1(D, dA)$.

We denote the domain, D, in theorem 2.28 by $Bal(\mu, dA)$. $Bal(\cdot, \cdot\cdot)$ is called the partial balayage operator and is defined under a weaker hypothesis than 2.19. A large class of quadrature domains can be constructed in this way as the following corollary shows.

Corollary 2.2.8. *Let $\{a_i\}_{i=1}^n \subset \mathbb{C}$ and $\{c_i\}_{i=1}^n \in \mathbb{R}_+$. Then there exists a unique saturated open set Ω satisfying*

$$\int_\Omega f dA \geq \sum_{i=1}^n c_i f(a_i),$$

for all $f \in SL^1(\Omega, dA)$.

By corollary 2.2.8 subharmonic quadrature domains with data $\{a_i\}_{i=1}^n \subset \mathbb{C}$ and $\{c_i\}_{i=1}^n \in \mathbb{R}_+$ exist iff $Bal(\sum_{i=1}^n c_i\delta_{a_i}, dA)$ is a domain.

Chapter 3

Potential Theory on Riemann Surfaces

3.1 Introduction

We begin by developing a generalization of logarithmic potential theory on compact Riemann surfaces. Throughout this thesis M will always denote a compact Riemann surface. Although we sometimes endow M with a Riemannian metric this is only a matter of convenience and all results in this section depend only on the complex structure of M. We consider a finite positive measure, ν, on M, which we call the background measure. The typical cases of interest are when ν is a linear combination of point masses and when ν is the measure induced by a volume form. We show that corresponding to such a ν, there exists, up to an additive constant, a unique symmetric kernel G_ν satisfying

$$\int_M G_\nu(z,w)\partial\bar{\partial}f(w) = \frac{1}{2i}[\int_M f d\nu - \nu(M)f(z)],$$

for all $f \in C^\infty(M)$. When $M = \hat{\mathbb{C}}$ and $\nu = 2\pi\delta_\infty$, G_ν is the logarithmic kernel, $\log\frac{1}{|z-w|}$. and so we recover logarithmic potential theory on the plane. Consider the cylinder $\mathbb{C}/\mathbb{N}$. We can compactify $\mathbb{C}/\mathbb{N}$ by adding two points p and q at either end to obtain the Riemann sphere. If we choose $\nu = \pi\delta_p + \pi\delta_q$ we obtain

$$G_\nu(z,w) = \log\frac{1}{|\sin(\pi(z-w))|}$$

which is the natural potential on $\mathbb{C}/\mathbb{N}$ obtained from the logarithmic potential by lifting to $\mathbb{C}$ (see example 3.2.5). If M is a compact Riemann surface endowed with an automorphic metric, g, and ν is the measure induced by the corresponding volume form, then G_ν is conformally invariant (see Prop 6.5.5 for a stronger statement). We note that these particular kernels appear in Conformal Field Theory; in the aforementioned examples, $G_\nu(z,w)$ is the two-point correlation function, $\mathbb{E}[\Phi(z)\Phi(w)]$, where Φ is a free Bosonic field, on the plane, the cylinder, and M respectively. Connections with Conformal Field Theory are discussed further in chapter 6.

In the presence of an external field, Q, the energy of a charge distribution $\mu \in \mathbb{P}(M)$, is defined to be

$$I[\mu] \equiv \int_{M\times M} G_\nu d\mu^{\otimes 2} + 2\int_M Q d\mu.$$

We prove the existence of an equilibrium measure $\mu_{eq} \in \mathbb{P}(M)$ satisfying

$$I[\mu_{eq}] = \inf_{\mu \in \mathbb{P}(M)} I[\mu],$$

for a certain class of external fields called admissible external fields. We prove an analogue of Frostman's theorem in this setting (Lemma 3.4.10 and Lemma 3.4.11). In general it is difficult to explicitly describe the equilibrium measure. The main theorems of this chapter show under mild hypotheses that the equilibrium measure has an a priori structure. To determine the equilibrium measure, it suffices only to know its support (which we denote by S). Before stating the main theorems of this section, we take an opportunity to clarify some notation. Let $f \in C^2(M)$; $\partial\bar{\partial} f$ is a (1-1)-form defined in local coordinates, $z = x + iy$ by $\partial_z \partial_{\bar{z}} f dz \wedge d\bar{z}$, where $\partial_z = \frac{1}{2}[\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}]$ and $\partial_{\bar{z}} = \frac{1}{2}[\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}]$. From now on we identify (1-1)-forms with measures. That is, if ω is a (1-1)-form then we identify it with the induced measure, ω, defined by

$$\omega(A) \equiv \int_A \omega, \tag{3.1}$$

for all measurable sets A. A measure, μ, is simply said to be absolutely continuous (without reference to another measure) if μ is absolutely continuous with respect to the measure induced by some (and thus any) volume form. Similarly, when we speak of properties of the Radon-Nikodym of a measure μ, we implicitly mean the Radon-Nikodym derivative taken with respect to the measure induced by a volume form. As such, we only consider properties which are metric independent. The following are the main theorems of this chapter.

Theorem 3.5.1. *(Structure Theorem 1). Let Q be an admissible potential which is C^2 in a neighborhood of S. Then*

$$\mu_{eq} = \frac{1}{\nu(M)} 1_S [\nu + 2i\partial\bar{\partial} Q + \kappa],$$

where κ is the balayage of the measure $1_{S^c}[\nu + 2i\partial\bar{\partial}\tilde{Q}]$ onto ∂S, $\tilde{Q} \in C^2(M)$ is any C^2-smooth extension of $Q|_U$, where U is a neighborhood of S.

Under stricter hypotheses on the background measure we obtain the following structure theorem:

Theorem 3.5.2. *(Structure Theorem 2). Let Q be an admissible external field which is C^2 in a neighborhood U of S. Suppose further that $\nu|_U$ is absolutely continuous and its Radon-Nikodym derivative is in $C_b(U)$, then*

$$\mu_{eq} = \frac{1}{\nu(M)} 1_S [\nu + 2i\partial\bar{\partial} Q].$$

As an almost immediate consequence of these structure theorems we recover the well-known formula for the structure of the equilibrium measure in the classical setting :

$$\mu_{eq} = \frac{1}{2\pi} 1_S \Delta Q dA$$

(under the original hypotheses that Q is C^2 in a neighborhood of the support). Using the first structure theorem, we can strengthen this result. In the classical setting, it follows from Frostman's theorem that Q must be subharmonic on S. Using the first structure theorem we prove a formula for the structure of the equilibrium measure when Q is subharmonic in a neighborhood of S, but no other regularity is assumed.
If Q is subharmonic in a (bounded) neighborhood U of S, it follows from the Riesz Decomposition theorem (see section 2.1), that (for $z \in \mathbb{C}$):

$$Q(z) = \int_M \log \frac{1}{|z-w|} d\tau(w) + h(z), \tag{3.2}$$

where τ is a *negative* compactly supported measure, and h is harmonic on U.

Theorem 3.5.4. *Let Q be a (classically) admissible external field in the plane. Suppose that Q is subharmonic in a neighborhood U of S. Choose τ and h satisfying 1.2. Then*

$$\mu_{eq} = \frac{1}{\tau(\mathbb{C})} 1_S [\tau - \omega], \tag{3.3}$$

where ω is defined as follows. Let $\tilde{h}$ denote a C^2 extension of h from a neighborhood of S to $\hat{\mathbb{C}}$. ω is defined as the balayage of $1_{S^c}[-\tau + \Delta\tilde{h} dA]$ onto ∂S.

The structure theorems show that μ_{eq} has an a priori structure and thus to determine μ_{eq} is suffices to determine S. In section 3.6 we show that S is related to a certain obstacle problem.

3.2 Interaction Kernel

The purpose of this section is to find a generalization of the logarithmic kernel on compact Riemann surfaces. The motivation for our choice of generalization comes not just from the desire to develop a logarithmic potential theory on Riemann surfaces, but also to preserve the connections between logarithmic potential theory, Hele-Shaw flow, and Quadrature Domains that exist in the planar setting. For convenience let

$$K(z, w) \equiv \log \frac{1}{|z-w|}$$

denote the logarithmic kernel. K has the well-known property that

$$\int_{\mathbb{C}} K(z,w)\Delta f(w)dA(w) = -2\pi f(z),$$

for any $f \in C_c^2(\mathbb{C})$. This is more commonly written as the distributional equation:

$$\Delta K(z,\cdot) = -2\pi\delta_z,$$

where Δ acts on the second variable. We will generally avoid this notation, because it suppresses the dependence of the left side on the area measure which will change throughout this section. K is defined on $(\mathbb{C}\times\mathbb{C})\backslash\Delta$ but naturally extends to an extended real valued function on $(\hat{\mathbb{C}}\times\hat{\mathbb{C}})\backslash\Delta$ by defining $K(\infty,w) = K(z,\infty) = -\infty$. We will show that the extended K satisfies a similar distributional equation. We begin by choosing a metric, g, on $\hat{\mathbb{C}}$ compatible with its complex structure. This allows us to define the (positive) Laplace Beltrami operator, Δ_g, the natural generalization of the Laplacian (or rather the negative of the Laplacian). For the new space, $C^2(\hat{\mathbb{C}})$, of test functions, K satisfies a new distributional equation.

Lemma 3.2.1. *Let K be defined above. Then for all $f \in C^2(\hat{\mathbb{C}})$,*

$$\int_{\hat{\mathbb{C}}} K(z,w)\Delta_g f(w)dvol_g(w) = 2\pi(f(z) - f(\infty)). \tag{3.4}$$

Before we prove this, we remark that since $\Delta_g f vol_g = -2i\partial\bar{\partial}f$, the left side depends only on the complex structure. In fact, our formulation of logarithmic potential theory depends only on the complex structure of M. Yet, it is sometimes convenient to introduce a metric to prove statements which are metrically independent, and so we prefer the notation from Riemannian geometry except in the statements of the main results. We now prove 3.2.1

Proof. From the note above, it suffices to verify Equation 3.4 for the spherical metric. We prove this first for $z = 0$ and then use the translational symmetry of K to obtain the general result.

$$\begin{aligned}
\int_{\hat{\mathbb{C}}} K(0,w)\Delta_g f(w)dvol_g(w) &= \int_{\mathbb{C}} \log\frac{1}{|w|}(-(1+|w|^2)^2\Delta)f(w)\frac{1}{(1+|w|^2)^2}dA(w)\\
&= -\int_{\mathbb{C}} \log\frac{1}{|w|}\Delta f(w)dA(w)\\
&= -\int_{\mathbb{D}^c} \log\frac{1}{|w|}\Delta f(w)dA(w) - \int_{\mathbb{D}} \log\frac{1}{|w|}\Delta f(w)dA(w)\\
&= -\int_{\mathbb{D}} \log|w|(\Delta f)(\frac{1}{w})\frac{1}{|w|^4}dA(w) - \int_{\mathbb{D}} \log\frac{1}{|w|}\Delta f(w)dA(w)\\
&= -\int_{\mathbb{D}} \log|w|\Delta(f(\frac{1}{w}))dA(w) + \int_{\mathbb{D}} \log|w|\Delta f(w)dA(w)
\end{aligned}$$

$$= -2\pi f(\infty) - \int_{\partial\mathbb{D}}(\log|w|\partial_n f(\frac{1}{w}) - f(\frac{1}{w})\partial_n\log|w|))ds(w) + 2\pi f(0) + \int_{\partial\mathbb{D}}(\log|w|\partial_n f(w) - f(w)\partial_n\log(|w|))ds(w)$$

$$= 2\pi(f(0) - f(\infty)),$$

where we have used Green's third formula in the second equality and the fact that the boundary terms cancel in the last equality. We then have for any $z \in \mathbb{C}$

$$\begin{aligned}\int_{\hat{\mathbb{C}}} K(z,w)\Delta_g f(w)dvol_g(w) &= -\int_{\mathbb{C}} \log\frac{1}{|z-w|}(1+|w|^2)^2\Delta f(w)\frac{1}{(1+|w|^2)^2}dA(w)\\ &= -\int_{\mathbb{C}} \log\frac{1}{|w|}\Delta f(z+w)dA(w) = 2\pi(f(z)-f(\infty)).\end{aligned}$$

□

3.2.1 Existence, Uniqueness, and Properties of the Interaction Kernel

We saw from Lemma 3.2.1 that the logarithmic kernel distinguishes the point $\infty \in \hat{\mathbb{C}}$. Indeed, $K(z,\cdot)$ is a Bipolar Green's function on $\hat{\mathbb{C}}$ with poles at z and ∞. In the sense of distributions we say that K has a "background measure" at ∞. We are interested in symmetric kernels on compact Riemann surfaces with more general background measures. The reason is twofold. Firstly, because we do not want to unnecessarily provide preference to any point or measure, and more importantly, as we will later show in chapter four, there are interesting connections between the background measure, quadrature domains, and Laplacian growth.

Definition 3.2.2. *The Interaction Kernel.*
Let M be a compact Riemann surface with metric g compatible with its complex structure. Let ν be a finite positive Borel measure which we call the background measure. We define the Interaction Kernel, G_ν to be a symmetric function in $L^2(M \times M, vol_g \otimes vol_g)$, satisfying

$$\int_M G_\nu(z,w)\Delta_g f(w)dvol_g(w) = \nu(M)f(z) - \int_M f d\nu, \tag{3.5}$$

for all $f \in C^\infty(M)$.

Lemma 3.2.3. *The interaction kernel with background measure ν is unique up to an additive constant.*

The choice of the additive constant will be unimportant in what follows. By an abuse of terminology, for a fixed background measure ν, we refer to a function satisfying Definition 3.2.2 as *the* interaction kernel with background measure ν.

Proof. Fix $z \in M$. Let G be another interaction kernel with background measure ν. Let $\{\psi_i\}_{i\in\mathbb{N}}$ be an eigenbasis of the Δ_g, with corresponding eigenvalues $\{\lambda_i\}$ in increasing order. Recall that ϕ_0 is constant and thus $\phi_0 = \sqrt{\frac{1}{|M|_g}}$. Since G_ν and G are in $L^2(M \times M, vol_g \otimes vol_g)$, by hypothesis,

$$(G_\nu - G)(z,w) = \sum_{i,j} c_{i,j}\phi_i(z)\phi_j(w),$$

where $c_{i,j} \in \mathbb{C}$. For $i \geq 1$, let $f = \frac{1}{\lambda_i}\phi_i$, then

$$\begin{aligned} c_{i,j} &= \int_{M\times M} (G_\nu - G)(z,w)\phi_i(w)\phi_j dvol_g(w)dvol_g(z) \\ &= \int_M [\int_M (G_\nu - G)(z,w)\Delta_g f dvol_g(w)]\phi_j dvol_g(z) = 0, \end{aligned}$$

where the third equality follows from the fact that G and G_ν both satisfy Equation 3.5 and thus the expression in the inner bracket is zero. We thus have $c_{i,j} = 0$ for $i > 0$. By the assumed symmetry of G and G_ν, it follows that $c_{i,j} = c_{j,i}$ and thus $c_{i,j} = 0$ if $i > 0$ or $j > 0$. Since $\phi_0 = \sqrt{\frac{1}{|M|_g}}$, we have thus shown that

$$(G_\nu - G) = \frac{c_{0,0}}{|M|_g}.$$

We have thus shown that the interaction kernel is unique up to an additive constant. □

Before considering examples, we remark about the relationship between the interaction kernel and Bipolar Green's functions. Let $G_{p,q}(\cdot)$ denote the Bipolar Green's function with logarithmic pole at p and negative logarithmic pole at q. It is well-known that $G_{p,q}$ satisfies, the distributional identity,

$$\int_M G_{p,q}(z)\Delta_g f(z)dvol_g(z) = 2\pi(f(p) - f(q))$$

for any $f \in C^2(M)$. It follows from Fubini's theorem that

$$\int_M [\frac{1}{2\pi}\int_M G_{p,q}(z)d\nu(q)]\Delta_g f(z)dvol_g(z) = \nu(M)f(p) - \int_M f d\nu.$$

So $\frac{1}{2\pi}\int_M G_{p,q}(z)d\nu(q)$ satisfies the same distributional equation as $G_\nu(p,\cdot)$. The problem is that $G_{p,q}(\cdot)$ is uniquely determined up to an additive constant for each pair (p,q), and it is not immediately obvious how to choose a constant for each (p,q) such that $\frac{1}{2\pi}\int_M G_{p,q}(z)d\nu(q)$ is symmetric —that is $\frac{1}{2\pi}\int_M G_{p,q}(z)d\nu(q) = \frac{1}{2\pi}\int_M G_{z,q}(p)d\nu(q)$. We prefer to take a different approach to constructing G_ν. The following are some important examples of interaction kernels.

Example 3.2.4. *$M = \hat{\mathbb{C}}$ and $\nu = 2\pi\delta_\infty$.*
K is clearly symmetric, and by Lemma 3.2.1

$$\begin{aligned} \int_{\hat{\mathbb{C}}} K(z,w)\Delta_g f(w)dvol_g(w) &= 2\pi(f(z) - f(\infty)) \\ &= \nu(\hat{\mathbb{C}})f(z) - \int_M f d\nu, \end{aligned}$$

so K is the interaction kernel on $\hat{\mathbb{C}}$ with background measure $2\pi\delta_\infty$.

Example 3.2.5. *The Cylinder.*
In this example we show that the potential on the cylinder $\mathbb{C}/\mathbb{N}$ obtained from the logarithmic potential by lifting to $\mathbb{C}$ can be identified with the interaction kernel on the Riemann Sphere with background measure $\pi\delta_0 + \pi\delta_\infty$.

To determine the potential at a point z in the cylinder $\mathbb{C}/\mathbb{N}$ in the presence of a unit point charge at $w \in \mathbb{C}/\mathbb{N}$, we lift to the universal cover, $\mathbb{C}$, of the cylinder. The preimage of the unit point charge at w under the covering map is $\{\tilde{w} + n\}_{n\in\mathbb{N}}$, where $\tilde{w}$ is any lift of w. Let $\tilde{z}$ be a lift of z. The potential at $\tilde{z}$ in the presence of unit point charges at $\{\tilde{w} + n\}_{n\in\mathbb{N}}$ is formally the divergent series

$$\sum_{n\in\mathbb{N}} \log \frac{1}{|\tilde{z} - \tilde{w} + n|}. \tag{3.6}$$

However, we can regularize 3.6 by considering instead the limit

$$\lim_{N\to\infty} \sum_{|n|<N} \log \frac{1}{|\tilde{z} - \tilde{w} + n|} - \sum_{0<|n|<N} \log \frac{1}{|n|} - \log(\pi).$$

Observe that

$$\begin{aligned}
\lim_{N\to\infty} \sum_{|n|<N} \log \frac{1}{|\tilde{z} - \tilde{w} + n|} - \sum_{0<|n|<N} \log \frac{1}{|n|} - \log(\pi) &= -\lim_{N\to\infty} \log|\pi(\tilde{z} - \tilde{w})| - \sum_{0<|n|<N} \log|\frac{\tilde{z} - \tilde{w}}{n} + 1| \\
&= -\log|\pi(\tilde{z} - \tilde{w}) \prod_{n=1}^{\infty}(1 - \frac{(\tilde{z} - \tilde{w})^2}{n})| \\
&= -\log|\sin(\pi(\tilde{z} - \tilde{w}))|.
\end{aligned}$$

So the regularized potential is $\log \frac{1}{|\sin(\pi(\tilde{z}-\tilde{w}))|}$. This expression is independent of the choice of lifts $\tilde{w}$ and $\tilde{z}$ and so descends to a function on $((\mathbb{C}/\mathbb{N}) \times (\mathbb{C}/\mathbb{N}))\backslash\Delta$ given by

$$\log \frac{1}{|\sin(\pi(z - w))|},$$

which is our desired potential on the cylinder. Let $\phi : \mathbb{C}^* \to \mathbb{C}/\mathbb{N}$ be defined by $\phi(z) \equiv \frac{1}{2\pi i} \log z$. ϕ is a *conformal diffeomorphism so the potential on $\mathbb{C}^*$ is*

$$\begin{aligned}\log \frac{1}{|\sin(\pi(\phi(z)-\phi(w)))|} &= \log \frac{2}{|e^{\pi i(\phi(z)-\phi(w))} - e^{-\pi i(\phi(z)-\phi(w))}|}\\ &= \log \frac{2|e^{\pi i \phi(z)+\phi(w)}|}{|e^{2\pi i\phi(z)} - e^{2\pi i \phi(w)}|}\\ &= \log \frac{2|\sqrt{zw}|}{|z-w|}\\ &= \log \frac{1}{|z-w|} - \frac{1}{2}\log\frac{1}{|z|} - \frac{1}{2}\log\frac{1}{|w|} - \log 2.\end{aligned}$$

Let $G(z,w) \equiv \log \frac{1}{|z-w|} - \frac{1}{2}\log\frac{1}{|z|} - \frac{1}{2}\log\frac{1}{|w|} - \log 2$. We claim that G is equal to the interaction kernel G_ν on $\hat{\mathbb{C}}$ with background measure $\nu = \pi\delta_0 + \pi\delta_\infty$. Since G is symmetric, we only need to check that it satisfies the distributional property. Let $f \in C^\infty(\hat{\mathbb{C}})$. Then

$$\begin{aligned}&\int_{\hat{\mathbb{C}}} G(z,w)\Delta_g f(w) dvol_g(w)\\ &= \int_{\hat{\mathbb{C}}} \log\frac{1}{|z-w|}\Delta_g f(w)dvol_g(w) - \frac{1}{2}\int_{\hat{\mathbb{C}}}\log\frac{1}{|w|}\Delta_g f(w) dvol_g(w) - \frac{1}{2}\int_{\hat{\mathbb{C}}}(\log\frac{1}{|z|} - \log(2))\Delta_g f(w)dvol_g(w)\\ &= 2\pi(f(z)-f(\infty)) - \pi(f(0)-f(\infty)) - \frac{1}{2}(\log\frac{1}{|z|} - \log(2))\int_{\hat{\mathbb{C}}}\Delta_g f(w)dvol_g(w)\\ &= 2\pi f(z) - \pi f(\infty) - \pi f(0)\\ &= \nu(\hat{\mathbb{C}})f(z) - \int_{\hat{\mathbb{C}}} f d\nu\end{aligned}$$

where we used Lemma 3.2.1 twice in the second equality and the fact that

$$\int_{\hat{\mathbb{C}}} \Delta_g f(w) dvol_g(w) = 0$$

in the third. We have thus shown that $G = G_\nu$.

Example 3.2.6. *The torus with background measure induced by the flat metric.*
In this example we compute the interaction kernel for the torus with background measure equal to the volume measure on $\mathbb{T}^2$ induced by the flat metric, g. Let $\mathbb{T}^2 = \mathbb{C}/\Lambda$ where $\Lambda = \langle 1, \tau\rangle$ and $\tau \in \mathbb{H}$. Let ν be the volume measure induced by g. Note that $\nu(\mathbb{T}^2) = \Im(\tau)$. Let $\theta(z;\tau)$ denote the Jacobi theta function. The Jacobi theta function satisfies the following identities:

$$\begin{aligned}\theta(z+1,\tau) &= \theta(z;\tau),\\ \theta(z+\tau;\tau) &= e^{-\pi i\tau}e^{-2\pi i z}\theta(z;\tau).\end{aligned}$$

We begin by defining a function G on $(\mathbb{C}\times\mathbb{C})\backslash\Delta$ and show that it is doubly periodic in each variable. Let

$$G(z,w) = \frac{1}{2\pi}[\log\frac{1}{|\theta(z-w)|} + \frac{1}{\Im(\tau)}(\Im(z-w))^2].$$

Using the identities for the Jacobi theta function above, it is not difficult to verify that G is doubly periodic. Therefore G descends to a function $\tilde{G}$ on $(\mathbb{T}^2\times\mathbb{T}^2)\backslash\Delta$ defined by $\tilde{G}(\pi(z),\pi(w)) = G(z,w)$ where $\pi:\mathbb{C}\to\mathbb{T}^2$ is the canonical projection. Notice that G is symmetric and thus so is $\tilde{G}$. Also since $G(z,\cdot)$ has a logarithmic pole at z, $\tilde{G}(\pi(z),\cdot)$ has a logarithmic pole at $\pi(z)$. Moreover, off the diagonal

$$\begin{aligned}\Delta G(z,\cdot) &= \frac{1}{2\Im(\tau)}\Delta(\Im(z-\cdot))^2\\ &= \frac{2}{2\Im\tau}\\ &= \frac{1}{\Im\tau}.\end{aligned}$$

Notice that off the diagonal G_ν also satisfies $\Delta_g G_\nu(z,\cdot)vol_g = \nu$. It thus follows that $G_\nu(z,\cdot)-\tilde{G}(z,\cdot)$ is harmonic and thus constant. By the symmetry of G_ν and $\tilde{G}$, it follows that $G_\nu = \tilde{G}+c$ where c is a constant.

Let g be a Riemannian metric on M. The special case of interaction kernel corresponding to $\nu = \frac{1}{|M|_g}vol_g$ has been well studied and is called the Green's function for the Laplace-Beltrami operator, Δ_g. Let $\{\phi_i\}_{i=0}^\infty$ be a real orthonormal basis of eigenfunctions for Δ_g and let $\{\lambda_i\}_{i=0}^\infty$ be the corresponding eigenvalues in non-decreasing order. It is known that the kernel of Δ_g is precisely the constants and so $\phi_0 = \sqrt{\frac{1}{|M|_g}}$, $\lambda_0 = 0$, and $\lambda_i > 0$ for $i > 0$ since Δ_g is a nonnegative. Let

$$G_g(z,w) \equiv \sum_{i=1}^\infty \frac{1}{\lambda_i}\phi_i(z)\phi_i(w).$$

By Weyl's lemma $\lambda_i \asymp i$ as $i\to\infty$ and so the series converges in $L^2(M\times M)$.
The following properties of G_g are well-known

i $\int_M G_g(z,w)\Delta_g\psi(w)dvol_g(w) = \psi(z) - \frac{1}{|M|_g}\int_M \psi dvol_g$, for any $\psi\in C^\infty(M)$.

ii $G_g(z,w)$ is smooth outside the diagonal $\Delta\subset M\times M$.

iii Let (U,ϕ) be a coordinate chart then $G_g(z,w)+\frac{1}{2\pi}\log|\phi(z)-\phi(w)|$ is smooth on $U\times U$.

iv For each $z\in M$, $||G_g(z,\cdot)||_{L^1(M,g)}$ is uniformly bounded.

v $\int_M G_g(z,w)dvol_g = 0$ for each $z\in M$.

vi For $z\neq w$, $\Delta_g G(z,w) = -\frac{1}{|M|_g}$, where Δ_g acts on the first variable (or the second).

The Green's function is an important example of an interaction kernel and we will show in Proposition 3.2.8 that it can be used to construct general interaction kernels. Before we can do this we need to prove two elementary lemmas.

Lemma 3.2.7. *Let M and let ν be a finite positive Borel measure on M. Then for $p > 0$*

$$G_g \in L^p(M \times M, \nu \otimes vol_g).$$

Proof. Since G_g is smooth off the diagonal, it suffices to show that G_g is in L^p in a neighborhood of the diagonal. Since M is compact, there is covering

$$M = \bigcup_{i=1}^{n} D_i,$$

where $D_i = \phi_i^{-1}(\mathbb{D})$ and $\phi_i : \mathbb{D} \to D_i$ is a conformal map which extends conformally to a neighborhood of $\mathbb{D}$. Since

$$\bigcup_{i=1}^{n} D_i \times D_i \supset \Delta$$

is an open covering of the diagonal, it suffices to show that $G_g \in L^p(D_i \times D_i, \nu \otimes vol_g)$. Indeed, recall by property (iii), $G_g(\phi(z), \phi(w)) = \frac{1}{2\pi} \log|z - w| + \Psi(z, w)$ where Ψ is smooth. The result follows since

$$\int_{D_i \times D_i} |G_g(z, w)|^p d\nu(z) dvol_g(w) = \int_{\mathbb{D}\times\mathbb{D}} |\frac{1}{2\pi} \log|z - w| + \Psi(z, w)|^p d(\phi_*^{-1}\nu)(z) d(\phi_*^{-1} vol_g)(w) < \infty,$$

where the last inequality follows since $\phi_*^{-1}\nu$ is a finite and $\phi_*^{-1} vol_g$ is absolutely continuous with bounded Radon-Nikodym derivative with respect to the area measure on $\mathbb{C}$. □

Lemma 3.2.8. *Let M be a Riemann surface with metric g. Let ν be a background measure. Let $f \in C^\infty(M)$. Then*

$$\int_M [\int_M G_g(w, s) d\nu(s)] \Delta_g f(w) dvol_g(w) = \int_M f d\nu - \frac{\nu(M)}{|M|_g} \int_M f dvol_g.$$

Proof. By Lemma 3.2.7, $G_g \in L^1(M \times M, vol_g \otimes \nu)$. Moreover, since $\Delta_g f$ is bounded on M, $G_g \in L^1(M \times M, \Delta_g f vol_g \otimes \nu)$. By Fubini's theorem the order of integration in the expression on the left side of the above equation can be interchanged. Using this and the distributional property (i) of G_g we have

$$\int_M [\int_M G_g(w, s) d\nu(s)] \Delta_g f(w) dvol_g(w) = \int_M [\int_M G_g(w, s) \Delta_g f(w) dvol_g(w)] d\nu(s)$$
$$= \int_M [f(s) - \frac{1}{|M|_g} \int_M f dvol_g] d\nu(s) = \int_M f d\nu - \frac{\nu(M)}{|M|_g} \int_M f dvol_g.$$

□

The following proposition allows us to construct G_ν from G_g.

Proposition 3.2.9. *Let M be a Riemann surface with metric g and let ν be a background measure. Then (up to an additive constant)*

$$G_\nu(z,w) = \nu(M)G_g(z,w) - \int_M G_g(z,s)d\nu(s) - \int_M G_g(w,s)d\nu(s). \tag{3.7}$$

Moreover the right side of 3.7 has mean zero with respect to vol_g.

Proof. Let

$$H(z,w) \equiv \nu(M)G_g(z,w) - \int_M G_g(z,s)d\nu(s) - \int_M G_g(w,s)d\nu(s).$$

Since G_g is symmetric, so is H. By Lemma 3.2.7, $H \in L^2(M \times M, vol_g \otimes vol_g)$. All that remains is to show that H satisfies the required distributional property. Let $f \in C^2(M)$, then

$$\begin{aligned}
&\int_M H(z,w)\Delta_g f(w)dvol_g(w)\\
&= \nu(M)\int_M G_g(z,w)\Delta_g f(w)dvol_g(w) - \int_M[\int_M G_g(z,s)d\nu(s)]\Delta_g f(w)dvol_g(w)\\
&\quad - \int_M[\int_M G_g(w,s)d\nu(s)]\Delta_g f(w)dvol_g(w)\\
&= \nu(M)(f(z) - \frac{1}{|M|_g}\int_M f dvol_g) - [\int_M G_g(z,s)d\nu(s)]\int_M \Delta_g f(w)dvol_g(w)\\
&\quad - \int_M f d\nu + \frac{\nu(M)}{|M|_g}\int_M f dvol_g\\
&= \nu(M)f(z) - \frac{\nu(M)}{|M|_g}\int_M f dvol_g + 0 - \int_M f d\nu + \frac{\nu(M)}{|M|_g}\int_M f dvol_g\\
&= \nu(M)f(z) - \int_M f d\nu.
\end{aligned}$$

We have used Lemma 3.2.8 in the second equality and that $\int_M \Delta_g f(w)dvol_g(w) = 0$ in the third equality. We have thus shown that $H = G_\nu$. Next we show that $\int_M H(z,w)dvol_g(w) = 0$. Indeed,

$$\begin{aligned}
\int_M H(z,w)dvol_g(w) &= \nu(M)\int_M G_g(z,w)dvol_g(w) - \int_M[\int_M G_g(z,s)d\nu(s)]dvol_g(z)\\
&\quad - \int_M[\int_M G_g(w,s)d\nu(s)]dvol_g(w)\\
&= 0 - \int_M[\int_M G_g(z,s)dvol_g(z)]d\nu(s) - \int_M[\int_M G_g(w,s)dvol_g(w)]d\nu(s) = 0,
\end{aligned}$$

where we have used Fubini's theorem (which is justified in Lemma 3.2.7), and the fact that

$$\int_M G_g(z,w)dvol_g(w) = 0,$$

which is property (v) of G_g.

□

3.3 Physical Interpretation

Electrostatics provides a useful intuition, if not motivation, for the remaining sections. We outline the analogy with electrostatics in this brief section. In chapter four we explain the connection with fluid dynamics when we discuss Hele-Shaw flow on surfaces.
Let μ be a compactly supported charge distribution (in $\mathbb{R}^3$) with smooth density ρ. Let E denote the electric field generated by μ and let V denote the potential energy. According to Maxwell's equations,

$$\nabla \cdot E = \frac{\rho}{\epsilon_0}, \tag{3.8}$$

where E is the electric field, and ϵ_0 is the permittivity of free space. The force acting on a point charge with charge q placed at x is $qE(x)$. The potential V is the gradient of this force. Therefore

$$\Delta V = \nabla \cdot (\nabla V) = \nabla \cdot (qE) = \frac{q\rho}{\epsilon_0}.$$

So in the sense of distributions

$$\Delta V = \frac{q}{\epsilon_0}\mu. \tag{3.9}$$

Equation 3.5 even holds for more general μ. We now compare this with the interaction kernel. By definition the interaction kernel satisfies

$$\Delta_g G_\nu(z,\cdot) = \nu(M)(\nu - \delta_z)$$

in the sense of distributions (where Δ_g acts on the second variable). Thus the interaction kernel, $G_\nu(z,w)$, can be viewed as the potential at w generated by a charge distribution ν and a point charge at z with charge $-\nu(M)$. This partly motivates the name background measure for ν. "Background" comes from the fact that we assume that it does not interact with any other charges or itself. We will make this clearer in a moment. The symmetry of G_ν implies that the potential at z generated by a point charge on w is the same as the potential at w generated by a point charge at z. Unlike in $\mathbb{R}^3$, the analogous statement for forces is not true. The magnitude of the force on a point charge at z induced by a the point charge at w may be different from the magnitude of the force on the charge at w induced by the charge at z. In special cases, like when M is the torus and the background measure ν is proportional to the flat metric, the symmetry at the level of forces persists.

Given a charge distribution μ in $\mathbb{R}^3$, the energy of configuration of μ is $\int_{\mathbb{R}^3} V_\mu d\mu$ where V_μ is the potential generated by μ. In the absence of an external field, a charge distribution μ in $\mathbb{R}^3$ can never be in static equilibrium.

$$\int_M (V_\mu + Q)d\mu.$$

In what follows we will be interested in positive charge distributions with minimal energy of configuration for a given total charge. If the external field is properly chosen, there will exist a corresponding charge distribution μ_{eq} of positive density and total charge one such that

$$\int_M (V_{\mu_{eq}} + Q)d\mu_{eq} \leq \int_M (V_\mu + Q)d\mu$$

for every positive charge distribution μ with total charge one. It can be shown mathematically that μ_{eq} is in static equilibrium. Physically, any motion would conserve the total energy and thus reduce potential energy contradicting the fact that the charge distribution given by μ_{eq} has minimal potential energy.

We now return to the situation on M. Let μ be a probability measure on M. By our assumption that the background measure does not interact with other charges or itself, the potential U^μ generated by the charge distribution μ is $\int_M G_\nu(z,w)d\mu(w)$. Therefore the energy of configuration of μ in the presence of the external field $2Q$ is

$$\int_M (U^\mu + 2Q)d\mu = \int_M U^\mu d\mu + 2\int_M Q d\mu$$

$$= \int_M \int_M G_\nu(z,w)d\mu(z)d\mu(w) + \int_M Q(z)d\mu(z)d\mu(w) + \int_M Q(w)d\mu(z)d\mu(w)$$

$$= \int_M \int_M [G_\nu(z,w) + Q(z) + Q(w)]d\mu(z)d\mu(w).$$

In analogy with electrostatics, we call the functional I, defined on the space of probability measures on M, and defined as

$$I[\mu] \equiv \int_M \int_M [G_\nu(z,w) + Q(z) + Q(w)]d\mu(z)d\mu(w),$$

the energy functional, where Q is a function on M called the external field. In what follows we will be interested in finding a probability measure with minimal energy for a given Q. Since physically such measures must be in static equilibrium, such a measure is called an equilibrium measure. We note that unlike the situation in $\mathbb{R}^3$, for certain background measures there exist equilibrium measures even if $Q = 0$.

3.4 The Equilibrium Measure

In section 3.2.1 we showed that associated to any background measure there is an interaction kernel which can be viewed as the interaction between electrons on a Riemann surface in the presence of a charge distribution corresponding to the background measure. We will additionally assume that there is an external field Q present. In the previous section we showed that the energy of a unit charge distribution μ is given by

$$I[\mu] = \int_{M\times M} K_\nu^Q d\mu^{\otimes 2},$$

where $K_\nu^Q(z,w) \equiv G_\nu(z,w) + Q(z) + Q(w)$. We note that there is not a one-one correspondence between (Q,ν) and K_ν^Q. The following lemma clarifies the dependency.

Lemma 3.4.1. *Let g be a metric on a compact Riemann surface M. Let Q be an external field and let ν and $\tilde{\nu}$ be background measures. Then (up to an additive constant)*

$$K_\nu^Q = \frac{\nu(M)}{\tilde{\nu}(M)} K_{\tilde{\nu}}^{\tilde{Q}},$$

where

$$\tilde{Q} \equiv \frac{\tilde{\nu}(M)}{\nu(M)}[Q - \int_M G_g(\cdot,s)d\nu(s)] + \int_M G_g(\cdot,s)d\tilde{\nu}(s).$$

Proof. This follows easily from Proposition 3.2.9, by writing G_ν and $G_{\tilde{\nu}}$ in terms of G_g and substituting these expressions into K_ν^Q and $K_{\tilde{\nu}}^{\tilde{Q}}$. □

The apparent metric dependence of $\tilde{Q}$ is illusory. Let $\tilde{g}$ be another metric. As we have shown earlier, for the background measure $\nu = \frac{1}{|M|_{\tilde{g}}} vol_{\tilde{g}}$, $G_\nu = G_{\tilde{g}}$. We can then apply Proposition 3.2.9 to $G_{\tilde{g}}$ to show that the expressions for $\tilde{Q}$ obtained from g and $\tilde{g}$ differ by a constant. Lemma 3.4.1 shows that the effect of modifying the interaction of charges by changing the background measure can be replicated by instead modifying the external field. The converse does not hold for the simple reason that not every external field is of the form $\int_M G_\nu(\cdot,w)d\mu(w)$ for some finite measure μ. Despite this, ν and Q naturally play different roles. For example as we will show later in the case of Hele-Shaw flow, ν is the accumulation of the source, and the permeability of the medium is given by $\Delta_g Q$.

The following definition formalizes the notion of sets which are negligible in the potential theoretic sense.

Definition 3.4.2. *Zero Capacity Sets*
Let A be a Borel set and let Λ_A denote the set of probability measures supported on A. We say that A has zero capacity if for some (and thus any) metric g,

$$\inf_{\mu \in \Lambda_A} \int_{M \times M} G_g d\mu^{\otimes 2} = \infty.$$

A Borel set that does not have zero capacity is said to have positive capacity. If a property holds except perhaps on a set with zero capacity we say that the property holds quasi-everywhere which we abbreviate by q.e.

The fact that the definition of zero capacity sets is independent of the metric follows from property (*iii*) of G_g. It is also not difficult to see that a countable union of zero capacity sets has zero capacity —a fact we will use later. We now introduce a class of external fields whose importance will be seen in a moment.

Definition 3.4.3. *Admissible External Field*
Let Q be an extended real valued function on M. We call Q an admissible external field if for some (and thus any) metric g on M,

$$Q_{\nu,g}(z) \equiv Q(z) - \int_M G_g(z,s) d\nu(s)$$

is lower semi-continuous and $M \backslash \{(Q_{\nu,g})^{-1}(\infty)\}$ has positive capacity.

We stress that whether a given external field is admissible depends only on the background measure and not on the choice of metric. Indeed, if $\tilde{g}$ is another metric then $G_{\tilde{g}}(z,s) - G_g(z,s)$ is smooth on $M \times M$ and so $\int_M (G_{\tilde{g}}(z,s) - G_g(z,s)) d\nu(s)$ is continuous and thus bounded on M, and it is easy to see that the two defining properties of an admissible external field are stable under addition of a continuous (bounded) function. The importance of the notion of admissibility is that if Q is admissible then K_ν^Q is lower semi-continuous. Indeed, by Proposition 3.2.9

$$K^Q(z,w) \equiv G_\nu(z,w) + Q(z) + Q(w) \tag{3.10}$$

$$= \nu(M) G_g(z,w) + Q_{\nu,g}(z) + Q_{\nu,g}(w). \tag{3.11}$$

Since Q is admissible, $Q_{\nu,g}$ is lower semi-continuous and thus expression 3.11 is also lower semi-continuous since it is a sum of lower semi-continuous functions.
We define the energy functional, I, by

$$I[\mu] \equiv \int_{M \times M} K_\nu^Q d\mu^{\otimes 2}$$

for $\mu \in \mathbb{P}(M)$. A probability measure μ satisfying $I[\mu] < \infty$ is said to have finite energy. Let

$$V \equiv \inf_{\mu \in \mathbb{P}(M)} I[\mu]. \tag{3.12}$$

V of course depends on ν and Q, but we suppress this dependence in the notation. We now show that if Q is admissible then V is finite. This property is so crucial that we will from now on restrict our consideration only to admissible external fields.

Lemma 3.4.4. *Let Q be an admissible external field. Then V is finite.*

Proof. Since K_ν^Q is lower semi-continuous and $M \times M$ is compact, K_ν^Q is bounded from below on $M \times M$, and so $V > -\infty$. We now show that $V < \infty$. By hypothesis $M \backslash \{(Q_{\nu,g})^{-1}(\infty)\}$ has positive capacity. Suppose for the sake of contradiction that for every $n \in \mathbb{N}$,

$$M \backslash \{Q_{\nu,g}^{-1}([n, \infty])\}$$

has zero capacity. Then

$$M \backslash \{(Q_{\nu,g})^{-1}(\infty)\} = \bigcup_{n \in \mathbb{N}} (M \backslash \{(Q_{\nu,g})^{-1}([n, \infty])\}),$$

is a countable union of zero capacity sets and thus has zero capacity —a contradiction. Therefore there exists an $n \in \mathbb{N}$ such that $M \backslash \{(Q_{\nu,g})^{-1}([n, \infty])\}$ has positive capacity. Therefore there exists a finite energy probability measure μ with support in $M \backslash \{(Q_{\nu,g})^{-1}([n, \infty])\}$. Then

$$\begin{aligned} \int_{M \times M} K_\nu^Q d\mu^{\otimes 2} &= \int_{M \times M} G_g d\mu^{\otimes 2} + 2 \int_M Q_{\nu,g} d\mu \\ &\leq \int_{M \times M} G_g d\mu^{\otimes 2} + 2n < \infty. \end{aligned}$$

□

We call a minimizer of the energy functional an equilibrium measure. We will soon show that it is unique. Later in sections 3.5 and 3.6 we analyze the dependence of the equilibrium measure on the background measure and the external field. We will now show that for an admissible potential the equilibrium measure exists.

Lemma 3.4.5. *Let Q be an admissible external field. There exists a $\mu \in \mathbb{P}(M)$ such that $I[\mu] = V$.*

Proof. Let $\{\mu_n\}_{n \in \mathbb{N}} \subset \mathbb{P}(M)$ satisfy $\lim_{n \to \infty} I[\mu_n] = V$. Then by Prokhorov's theorem (as M is compact), there exists a subsequence which converges weakly to a measure μ. For clarity, we relabel this subsequence by $\{\mu_n\}_{n \in \mathbb{N}}$. We will now show that $I[\mu] = V$. As K_ν^Q is lower semi-continuous there exist continuous functions $K_n \nearrow K_\nu^Q$. As a result we have:

$$
\begin{aligned}
I[\mu] &\equiv \int_{M\times M} K_\nu^Q d\mu^{\otimes 2} \\
&= \int_{M\times M} \lim_{n\to\infty} K_n d\mu^{\otimes 2} \\
&\leq \liminf_{n\to\infty} \int_{M\times M} K_n d\mu^{\otimes 2} \\
&= \liminf_{n\to\infty} \lim_{m\to\infty} \int_{M\times M} K_n^Q d\mu_m^{\otimes 2} \\
&\leq \liminf_{n\to\infty} \lim_{m\to\infty} \int_{M\times M} K_\nu^Q d\mu_m^{\otimes 2} \\
&= \lim_{m\to\infty} \int_{M\times M} K_\nu^Q d\mu_m^{\otimes 2} \\
&= V.
\end{aligned}
$$

As trivially $I[\mu] \geq V$, we have $I[\mu] = V$. □

Recall in the background section that in the classical setting, an external field, Q, was called admissible if Q is lower semi-continuous, $\lim_{z\to\infty} Q(z) - \log|z| = \infty$, and $\mathbb{C}\backslash Q^{-1}(\infty)$ has positive capacity. The following example clarifies the relationship between this definition of admissibility and Definition 3.4.3.

Example 3.4.6. *Admissibility for $M = \hat{\mathbb{C}}$ and $\nu = 2\pi\delta_\infty$ versus classical admissibility.*
There are two conditions given in Definition 3.4.3 for an external field to be admissible. From the first condition, Q is admissible only if $Q(z) - 2\pi G_g(z,\infty)$ is lower semi-continuous. By the properties of G_g, $2\pi G_g(z,\infty)$ is smooth except at ∞ and for any chart (U,ϕ) containing ∞, $2\pi G_g(z,w) + \log|\phi(z) - \phi(w)|$ is smooth on $U \times U$. Thus the lower semi-continuity of $Q(z) - 2\pi G_g(z,\infty)$ is equivalent to the fact that $\limsup_{z\to\infty} Q(z) - \log|z| > -\infty$. This condition is just strong enough to ensure that an equilibrium measure exists. If additionally we want the support of the equilibrium measure to be contained in $\mathbb{C}$, the growth condition for Q must be improved to $\lim_{z\to\infty} Q(z) - \log|z| = \infty$, which is one of the criteria for an admissible potential in logarithmic potential theory. We can thus view logarithmic potential theory on $\mathbb{C}$ as a special case of logarithmic potential theory on $\hat{\mathbb{C}}$ by forcing the external field to be sufficiently large in a neighborhood of ∞. More generally we obtain logarithmic potential theory on a compact surface with punctures by forcing the external field to be sufficiently large in a neighborhood of the punctures, to disallow the accumulation of charge at the punctures.

We next prove the uniqueness of the equilibrium measure, but first we need a technical lemma.

Lemma 3.4.7. *Let g be a metric on M and let μ_1, μ_2 be probability measures on M with finite energy, then*

$$\int_{M\times M} G_g d(\mu_1 - \mu_2)^{\otimes 2} \geq 0,$$

with equality only if $\mu_1 = \mu_2$.

Proof. Let $\{\phi_i\}_{i\in\mathbb{N}}$ be an eigenbasis of Δ_g with corresponding eigenvalues $\{\lambda_i\}_{i\in\mathbb{N}}$ in non-decreasing order. We will show that

$$\int_{M\times M} G_g d(\mu_1-\mu_2)^{\otimes 2} = \sum_{j=1}^{\infty}\frac{1}{\lambda_j}(\int_M \phi_j d(\mu_1-\mu_2))^2. \tag{3.13}$$

Since the spectrum of Δ_g is nonnegative, the right side of Equation 3.13 is clearly nonnegative. Moreover, if the right side equals zero, then $\int_M \phi_j d(\mu_1-\mu_2) = 0$ for all $j > 0$, and since $\mu_1-\mu_2$ has zero mass and ϕ_0 is constant, we have $\int_M \phi_j d(\mu_1-\mu_2) = 0$ for all $j \geq 0$. This implies that $\mu_1 = \mu_2$. So it suffices to prove 3.13. We introduce the heat kernel $e(t,z,w) = \sum_{j=0}^{\infty} e^{-\lambda_j t}\phi_j(z)\phi_j(w)$. Recall that by Weyl's theorem, $\lambda_n \asymp n$ and by Hormander's theorem, $\|\phi_n\|_\infty \leq c\lambda_n^{\frac{1}{4}}$ for a constant c independent of n. It follows that $e(t,z,w)$ converges uniformly in the spatial variables and thus

$$\int_{M\times M} e(t,z,w)d(\mu_1-\mu_2)(z)d(\mu_1-\mu_2)(w) = \sum_{j=0}^{\infty} e^{-\lambda_j t}(\int_M \phi_j d(\mu_1-\mu_2))^2. \tag{3.14}$$

It is also well-known that

$$G_g(z,w) = \int_0^\infty (e(t,z,w) - \frac{1}{|M|_g^2})dt$$

and

$$\int_0^n (e(t,\cdot,\cdot\cdot) - \frac{1}{|M|_g^2})dt \rightrightarrows G_g$$

as $n\to\infty$. Using these facts we have

$$\int_{M\times M} G_g d(\mu_1-\mu_2)^{\otimes 2} \tag{3.15}$$
$$= \int_{M\times M}\lim_{n\to\infty}[\int_0^n (e(t,z,w)-\frac{1}{|M|_g^2})dt]d(\mu_1-\mu_2)(z)d(\mu_1-\mu_2)(w) \tag{3.16}$$
$$= \lim_{n\to\infty}\int_{M\times M}[\int_0^n (e(t,z,w)-\frac{1}{|M|_g^2})dt]d(\mu_1-\mu_2)(z)d(\mu_1-\mu_2)(w) \tag{3.17}$$
$$= \lim_{n\to\infty}\int_{M\times M}[\int_0^n e(t,z,w)dt]d(\mu_1-\mu_2)(z)d(\mu_1-\mu_2)(w) \tag{3.18}$$
$$= \lim_{n\to\infty}\int_0^n[\int_{M\times M} e(t,z,w)d(\mu_1-\mu_2)(z)d(\mu_1-\mu_2)(w)]dt \tag{3.19}$$
$$= \int_0^\infty[\sum_{j=0}^{\infty} e^{-\lambda_j t}(\int_M \phi_j d(\mu_1-\mu_2))^2]dt \tag{3.20}$$
$$= \sum_{j=0}^{\infty}\int_0^\infty e^{-\lambda_j t}(\int_M \phi_j d(\mu_1-\mu_2))^2 dt \tag{3.21}$$
$$= \sum_{j=1}^{\infty}\frac{1}{\lambda_j}(\int_M \phi_j d(\mu_1-\mu_2))^2. \tag{3.22}$$

The interchange of integration in 3.19 is justified since $e(t,z,w) \geq 0$. Equation 3.20 follows from 3.14. The interchange of summation and integration in 3.21 is justified since the summands are nonnegative. The proof is complete.

□

The following proposition shows that the equilibrium measure is unique.

Proposition 3.4.8. *Let Q be an admissible potential and ν be a background measure. Then there exists precisely one measure μ_{eq} which minimizes the energy functional I.*

Proof. We have already shown that for an admissible external field, Q, an equilibrium measure exits, so all that remains is to show that it is unique. Assume for the sake of contradiction that μ_1 and μ_2 are two different equilibrium measures. Let g be a metric compatible with the complex structure of M. From the previous lemma we have

$$\begin{aligned} I[\frac{1}{2}(\mu_1 - \mu_2)] &= \int_{M\times M} K^{\nu,Q} d(\frac{1}{2}(\mu_1 - \mu_2))^{\otimes 2} \\ &= \nu(M)\int_{M\times M} G_g d(\frac{1}{2}(\mu_1 - \mu_2))^{\otimes 2} > 0. \end{aligned}$$

Note that the last term is independent of the choice of metric. The last equality uses Equation 3.7 as well as the fact that

$$\begin{aligned} &\int_{M\times M} Q_{\nu,g}(z) d(\frac{1}{2}(\mu_1 - \mu_2))(z) d(\frac{1}{2}(\mu_1 - \mu_2))(w) \\ &= (\frac{1}{2}(\mu_1 - \mu_2))(M)\int_{M\times M} Q_{\nu,g}(z) d(\frac{1}{2}(\mu_1 - \mu_2))(z) = 0. \end{aligned}$$

We now compute the energy of the probability measure $\frac{1}{2}(\mu_1 + \mu_2)$.

$$\begin{aligned} I[\frac{1}{2}(\mu_1 + \mu_2)] &< I[\frac{1}{2}(\mu_1 + \mu_2)] + I[\frac{1}{2}(\mu_1 - \mu_2)] \\ &= \frac{1}{2}(I[\mu_1] + I[\mu_2]) = \frac{1}{2}(V + V) = V. \end{aligned}$$

This contradicts the definition of V. This completes the proof. □

From now on, we denote the equilibrium measure by μ_{eq} and its support by S. Unlike in the classical setting, for certain background measures, there can exist equilibrium measures even in the absence of an external field (precisely the background measures for which $Q = 0$ is admissible). If a unit drop of charge were placed on M in the presence of a unit static charge distribution $-\nu$, our physical intuition suggests that in the absence of an external field the droplet would arrange itself to neutralize the $-\nu$. The follow example rigorously proves this when $\nu = \frac{1}{|M|_g} vol_g$. We note that a more general result follows from Theorem 3.5.2.

Example 3.4.9. *The equilibrium measure for $\nu = \frac{1}{|M|_g} vol_g$ and $Q = 0$.*
If $\nu = \frac{1}{|M|_g} vol_g$, then $\int_M G_g(\cdot, w) d\nu(w)$ is continuous and so $Q = 0$ is an admissible potential. By Proposition

3.4.8, there exists a corresponding equilibrium measure μ_{eq}. It follows directly from property (v) of G_g, that

$$\int_M G_g(\cdot, w)d\nu(w) = 0. \tag{3.23}$$

We then have

$$
\begin{aligned}
I[\mu_{eq}] &= \int_{M\times M} G_\nu d\mu_{eq}^{\otimes 2} &(3.24)\\
&= \int_{M\times M} [\nu(M)G_g(z,w) - \int_M G_g(z,s)d\nu(s) - \int_M G_g(s,w)d\nu(s)]d\mu_{eq}(z)d\mu_{eq}(w) &(3.25)\\
&= \int_{M\times M} G_g(z,w)d\mu_{eq}(z)d\mu_{eq}(w) - \int_{M\times M} G_g(z,w)d\mu_{eq}(z)d\nu(w) - \int_{M\times M} G_g(z,w)d\nu(z)d\mu_{eq}(w) &(3.26)\\
&= \int_{M\times M} G_g(z,w)d\mu_{eq}(z)d\mu_{eq}(w) - \int_{M\times M} G_g(z,w)d\mu_{eq}(z)d\nu(w) - \int_{M\times M} G_g(z,w)d\nu(z)d\mu_{eq}(w) &(3.27)\\
&\quad + \int_{M\times M} G_g(z,w)d\nu(z)d\nu(w) &(3.28)\\
&= \int_{M\times M} G_g d(\nu - \mu_{eq})^{\otimes 2} &(3.29)\\
&\geq 0, &(3.30)
\end{aligned}
$$

where 3.25 follows from Proposition 3.2.9, 3.26 follows since $\nu(M) = 1$. 3.27 and 3.28 follow from 3.23, and 3.30 follows from Proposition 3.4.7. On the other hand

$$I[\nu] = \int_{M\times M} G_g d\nu^{\otimes 2} = 0,$$

so by the uniqueness of the equilibrium measure, $\mu_{eq} = \nu \equiv \frac{1}{|M|_g} vol_g$.

Let μ be a probability measure, we define the potential to be

$$U_\nu^\mu \equiv \int_M G_\nu(\cdot, s)d\mu(s).$$

Let $F \equiv V - \int_M Q d\mu_{eq}$. The analogue of F in the classical setting is called the modified Robin's constant. We suppress the dependence of F on Q and ν. Indeed, since V depends on our choice of G_ν (recall G_ν is unique up to an additive constant) so does F. The following two lemmas are generalizations of Frostman's theorem to this setting. For the proof, it will be convenient to extend the definition of the energy functional to the bilinear functional:

$$I[\mu_1, \mu_2] \equiv \int_{M\times M} K_\nu^Q(z,w)d\mu_1(z)d\mu_2(w).$$

Lemma 3.4.10. *$(U_\nu^{\mu_{eq}} + Q) \geq F$, q.e.*

Proof. Let τ be a probability measure with finite energy. Then for $\delta \in (0, 1]$ we have by the extremality of

μ_{eq}:

$$I[\mu_{eq}] \leq I[\frac{\mu_{eq} + \delta\tau}{1+\delta}] = (1+\delta)^{-2}(I[\mu_{eq}] + 2\delta I[\mu_{eq}, \tau] + \delta^2 I[\tau]).$$

Rearranging the inequality yields:

$$\delta I[\mu_{eq}] \leq \delta I[\mu_{eq}, \tau] + O(\delta^2).$$

Dividing by δ and letting $\delta \to 0^+$ gives

$$I[\mu_{eq}] \leq I[\mu_{eq}, \tau].$$

Equivalently,

$$\int_M (U_\nu^{\mu_{eq}} + Q) d\tau \geq F.$$

If $U_\nu^{\mu_{eq}} + Q$ is less than F on a set of positive capacity, then choosing τ to be any finite energy measure supported on this set contradicts the inequality above. Therefore $U_\nu^{\mu_{eq}} + Q \geq F$ q.e. □

Lemma 3.4.11. $(U_\nu^{\mu_{eq}} + Q)|_S \leq F$.

Proof. Let $\Gamma \equiv \{U_\nu^{\mu_{eq}} + Q > F\}$. Since Q is admissible, K_ν^Q is lower semi-continuous, and thus

$$U_\nu^{\mu_{eq}} + Q = \int_M K_\nu^Q(\cdot, w) d\mu_{eq}(w) - \int_M Q d\mu_{eq}$$

is also lower semi-continuous since μ_{eq} is a positive measure. The lower semi-continuity of $U_\nu^{\mu_{eq}} + Q$ implies that Γ is open. If $\Gamma \cap S \neq \emptyset$, then $\mu_{eq}(\Gamma \cap S) > 0$. By Lemma 3.4.10, we have $U_\nu^{\mu_{eq}} + Q \geq F$ q.e. and thus μ_{eq}-a.e., it follows that

$$\begin{aligned} F &= \int_\Gamma F d\mu_{eq} + \int_{M\setminus\Gamma} F d\mu_{eq} \\ &< \int_\Gamma (U_\nu^{\mu_{eq}} + Q) d\mu_{eq} + \int_{M\setminus\Gamma} (U_\nu^{\mu_{eq}} + Q) d\mu_{eq} \\ &= \int_M (U_\nu^{\mu_{eq}} + Q) d\mu_{eq} \\ &= \int_{M\times M} K_\nu^Q d\mu_{eq}^{\otimes 2} - \int_M Q d\mu_{eq} \\ &= V - \int_M Q d\mu_{eq} \\ &= F. \end{aligned}$$

This contradiction forces $\Gamma \cap S = \emptyset$ as desired.

□

For the proof of the following proposition it will be convenient to restate Lemmas 3.4.10 and 3.4.11 using a different notation. Let

$$T \equiv \nu(M)\int_M G_g(\cdot, s)d\mu_{eq}(s) - \int_M G_g(t, s)d\nu(t)d\mu_{eq}(s).$$

It is easy to check that T is lower semi-continuous and using Proposition 3.2.9 we have

$$T + Q_{\nu,g} = U_\nu^{\mu_{eq}} + Q.$$

The following lemma is simply a restatement of Lemmas 3.4.10 and 3.4.11 with this new notation.

Lemma 3.4.12. *Let Q be an admissible external field. Then*

$$T + Q_{\nu,g} \geq F, \quad q.e.$$

and

$$(T + Q_{\nu,g})|_S \leq F.$$

Recall that if Q is admissible then $Q_{\nu,g}$ is lower semi-continuous. In the event that $Q_{\nu,g}$ is continuous in the neighborhood of the support we can strengthen Lemmas 3.4.10 and 3.4.11.

Proposition 3.4.13. *Let $Q_{\nu,g}$ be continuous in a neighborhood B of S. Then*

$$(U_\nu^{\mu_{eq}} + Q)|_S = F, \tag{3.31}$$

$$(U_\nu^{\mu_{eq}} + Q)|_B \geq F. \tag{3.32}$$

Proof. Since Q is admissible it follows that $T + Q_{\nu,g}$ is lower semi-continuous and thus attains its minimum on the compact set S. Let m denote the minimum and let $z_* \in S$ be a point in S where the minimum is attained. Let (U, ϕ) be a coordinate chart containing z_* and compactly contained in B. Let $y \in B\backslash U$ and note that by property (ii) of G_g, $\nu(M)G_g(\cdot, y)$ is smooth on U. Let τ be the pushforward of the area measure on $\phi(U) \subset \mathbb{C}$ under ϕ^{-1}. Fix $\epsilon > 0$. Since $Q_{\nu,g} + \nu(M)G_g(\cdot, y)$ is continuous, there exists a conformal disc $D \Subset U$ centered at z_* sufficiently small such that

$$\frac{1}{\tau(D)}\int_D (Q_{\nu,g}(z) + \nu(M)G_g(z, y))d\tau(z) < Q_{\nu,g}(z_*) + \nu(M)G_g(z_*, y) + \epsilon. \tag{3.33}$$

It follows from properties (vi) and (iii) of G_g that for any $w \in U$, $G_g(\cdot, w) - G_g(\cdot, y)$ is superharmonic on $\mathbb{D}$ and thus

$$T - \nu(M)G_g(\cdot, y) = \nu(M)\int_M (G_g(\cdot, w) - G_g(\cdot, y))d\mu_{eq}(w) - \int_{M\times M} G_g(s,t)d\mu_{eq}(s)d\nu(t) \tag{3.34}$$

is also superharmonic. We thus have

$$\frac{1}{\tau(D)}\int_D (T(z) - \nu(M)G_g(z,y))d\tau(z) \leq T(z_*) - \nu(M)G_g(z_*, y). \tag{3.35}$$

Adding Inequalities 3.35 and 3.33 yields:

$$\frac{1}{\tau(D)}\int_D (T + Q_{\nu,g})d\tau < (T + Q_{\nu,g})(z_*) + \epsilon = m + \epsilon. \tag{3.36}$$

By Lemma 3.4.12,

$$T + Q_{\nu,g} \geq F \ q.e. \tag{3.37}$$

Since vol_g has finite energy, so does τ since it is seen from that definition of τ that $\tau < Cvol_g|_U$ for some constant C. Since Inequality 3.37 holds quasi-everywhere and τ has finite energy, 3.37 also holds τ-a.e. We thus have

$$F \leq \frac{1}{\tau(D)}\int_D (T + Q_{\nu,g})d\tau \leq m + \epsilon$$

and thus $F \leq m$. Now by Lemma 3.4.12, we also have

$$(T + Q_{\nu,g})|_S \leq F \leq m.$$

Now since m is the minimum of $T + Q_{\nu,g}$ on S we must have

$$(T + Q_{\nu,g})|_S = F, \tag{3.38}$$

and thus

$$(U_\nu^{\mu_{eq}} + Q)|_S = F.$$

The proof of the first statement of the proposition is complete. We now prove the second statement.

Let $\Gamma \equiv \{T + Q_{\nu,g} < F\} \cap B$. From 3.38, $S \cap \Gamma = \emptyset$. Since both T and $Q_{\nu,g}$ are continuous on S^c, it follows that Γ is open. By Lemma 3.4.12, Γ has zero capacity and thus zero volume measure. Therefore Γ does not contain a non-empty open set. We must have $\Gamma = \emptyset$ which completes the proof.

□

3.5 Structure of the Equilibrium Measure

3.5.1 Structure Theorems

In the previous section we showed that given an admissible external field there exists a unique equilibrium measure, μ_{eq}. In general it is difficult to explicitly determine the structure of μ_{eq}. The purpose of this section is to provide, under mild hypotheses, formulae for μ_{eq} in terms of S, Q, and ν. As we will see in the next chapter, these formulae provide a connection between potential theory, Hele-Shaw flow, and Quadrature Domains. In Section 3.6 we will explore a method for determining S. This combined with the following structure theorems, in principle, allows one to compute μ_{eq}.

Before proving the structure theorems we need to define the balayage of a measure. Given a measure μ supported on a domain D with boundary of positive capacity, the balayage of μ onto ∂D, denoted $\hat{\mu}$, is a measure supported on ∂D defined by

$$\hat{\mu} = \int_D \omega_z^D d\mu(z) \tag{3.39}$$

where ω_z^D is the harmonic measure with respect to the point z. If ∂D is C^1, then for $w \in \partial D$,

$$\frac{d\hat{\mu}}{ds}(w) = -\frac{1}{2\pi}\int_D \partial_n G_D(w,z)d\mu(z)$$

where G_D is the Green's function for D, and ∂_n is the normal derivative taken in the first variable. Let $f \in HL^1(D,\mu)$; if f extends continuously to ∂D then

$$\int_{\partial D} f d\hat{\mu} = \int_{\partial D} f(s) d\int_D \omega_z^D(s)d\mu(z) = \int_D [\int_{\partial D} f(s)d\omega_z^D(s)]d\mu(z) = \int_D f d\mu. \tag{3.40}$$

Theorem 3.5.1. *(Structure Theorem 1). Let Q be an admissible potential which is C^2 in a neighborhood of S. Then*

$$\mu_{eq} = \frac{1}{\nu(M)} 1_S[\nu + 2i\partial\bar{\partial}Q + \kappa],$$

where κ is the balayage of the measure $1_{S^c}[\nu + 2i\partial\bar{\partial}\tilde{Q}]$ onto ∂S, $\tilde{Q} \in C^2(M)$ is any C^2-smooth extension of

$Q|_U$, and U is a neighborhood of S.

Proof. In this proof there are several unimportant constant terms. We denote the ith such constant by c_i. Since by hypothesis Q is C^2-smooth in a neighborhood of S, there exists a $\tilde{Q} \in C^2(M)$, and a neighborhood $U \supset S$, such that $\tilde{Q}|_U = Q|_U$. We first observe that by Proposition 3.2.9

$$U_\nu^{\mu_{eq}}(z) \equiv \int_M G_\nu(z,w)d\mu_{eq}(w) = \int_M G_g(z,w)d[\nu(M)\mu_{eq} - \nu](w) + c_1. \tag{3.41}$$

Also note that by property (i) of G_g, we have

$$\int_M G_g(z,w)\Delta_g\tilde{Q}(w)dvol_g(w)|_U = (\tilde{Q}(z) - \int_M \tilde{Q}dvol_g)|_U = (Q - c_2)|_U. \tag{3.42}$$

Since ν is positive and $G_g(\cdot,w)$ is lower semi-continuous, $\int_M G_g(\cdot,w)d\nu(w)$ is lower semi-continuous. It follows that $Q_{\nu,g} \equiv Q(\cdot) - \int_M G_g(\cdot,w)d\nu(w)$ is upper semi-continuous on U, since $Q \in C^2(U)$. On the other hand, since Q is admissible, by definition $Q_{\nu,g}$ is lower semi-continuous. So $Q_{\nu,g}$ is continuous and we can apply Proposition 3.4.13 to obtain:

$$(U_\nu^{\mu_{eq}} + Q)|_S = F. \tag{3.43}$$

Substituting equations 3.41 and 3.42 into equation 3.43 (noting that $S \subset U$), yields

$$\int_M G_g(\cdot,w)d[\nu(M)\mu_{eq} - \nu + \Delta_g\tilde{Q}vol_g](w)|_S = c_3. \tag{3.44}$$

Note that since $\int_M \Delta_g\tilde{Q}dvol_g = 0$ and $\int_M(\nu(M)d\mu_{eq} - d\nu) = \nu(M)\mu_{eq}(M) - \nu(M) = 0$, the mass of the measure $[\nu(M)\mu_{eq} - \nu + \Delta_g\tilde{Q}vol_g]$ is zero. This fact will be used repeatedly throughout the proof. We now fix $p \in S$. From Prop 3.2.9

$$G_{\delta_p}(z,w) = G_g(z,w) - G_g(z,p) - G_g(p,w). \tag{3.45}$$

It follows from property (vi) of G_g that G_{δ_p} is harmonic separately in each variable on $(M\backslash\{p\} \times M\backslash\{p\})\backslash\Delta$. Observe that

$$\begin{aligned}
&\int_M G_{\delta_p}(\cdot, w) d[\nu(M)\mu_{eq} - \nu + \Delta_g \tilde{Q} vol_g](w)|_S \\
= &\int_M [G_g(\cdot, w) - G_g(\cdot, p) - G_g(p, w)] d[\nu(M)\mu_{eq} - \nu + \Delta_g \tilde{Q} vol_g](w)|_S \\
= &\int_M [G_g(\cdot, w) - G_g(p, w)] d[\nu(M)\mu_{eq} - \nu + \Delta_g \tilde{Q} vol_g](w)|_S \\
= &\int_M G_g(\cdot, w) d[\nu(M)\mu_{eq} - \nu + \Delta_g \tilde{Q} vol_g](w)|_S + c_4 \\
= &\ c_5.
\end{aligned}$$

We used Equation 3.45 in the first equality, the fact that the mass of $\nu(M)\mu_{eq} - \nu + \Delta_g \tilde{Q} vol_g$ is zero in the second equality, Equation 3.44 in the third equality (to show that $|\int_M G_g(p, w)] d[\nu(M)\mu_{eq} - \nu + \Delta_g \tilde{Q} vol_g](w)| < \infty$), and Equation 3.44 again in the fourth equality. As in the statement of the theorem, we define κ to be the balayage of $1_{S^c}[\nu - \Delta_g \tilde{Q} vol_g]$ onto ∂S. For $z \in S$, $G_{\delta_p}(z, \cdot)$ is harmonic on S^c, and so by the properties of balayage it follows

$$\int_{S^c} G_{\delta_p}(\cdot, w) d[\nu(M)\mu_{eq} - \nu + \Delta_g \tilde{Q} vol_g](w)|_S \tag{3.46}$$

$$= -\int_{S^c} G_{\delta_p}(\cdot, w) d[\nu - \Delta_g \tilde{Q} vol_g](w)|_S \tag{3.47}$$

$$= -\int_{\partial S} G_{\delta_p}(\cdot, w) d\kappa(w)|_S \tag{3.48}$$

everywhere on $S \setminus \partial S$, and quasi-everywhere on ∂S. We showed above that

$$\int_M G_{\delta_p}(\cdot, w) d[\nu(M)\mu_{eq} - \nu + \Delta_g \tilde{Q} vol_g](w)|_S = c_5. \tag{3.49}$$

Splitting the integral in 3.49 into integrals over S and S^c and using equation 3.48 yields that quasi-everywhere on ∂S and everywhere on $S \setminus \partial S$:

$$c_5 = \int_M G_{\delta_p}(\cdot, w) d[\nu(M)\mu_{eq} - \nu + \Delta_g \tilde{Q} vol_g](w)|_S \tag{3.50}$$

$$= \int_S G_{\delta_p}(\cdot, w) d[\nu(M)\mu_{eq} - \nu + \Delta_g \tilde{Q} vol_g](w)|_S + \int_{S^c} G_{\delta_p}(\cdot, w) d[\nu(M)\mu_{eq} - \nu + \Delta_g \tilde{Q} vol_g](w)|_S \tag{3.51}$$

$$= \int_S G_{\delta_p}(\cdot, w) d[\nu(M)\mu_{eq} - \nu + \Delta_g \tilde{Q} vol_g](w)|_S - \int_{\partial S} G_{\delta_p}(\cdot, w) d\kappa(w)|_S \tag{3.52}$$

$$= \int_S G_{\delta_p}(\cdot, w) d[\nu(M)\mu_{eq} - \nu + \Delta_g \tilde{Q} vol_g - \kappa](w)|_S \tag{3.53}$$

$$\equiv \int_S G_{\delta_p}(\cdot, w) d\tau(w)|_S, \tag{3.54}$$

where we let $\tau \equiv 1_S[\nu(M)\mu_{eq} - \nu + \Delta_g Q vol_g - \kappa]$ for convenience. Note that since the balayage operation preserves the mass of a measure, τ has mass zero. Note that in 3.53 we have used the fact that $\partial S \subset S$ since S is closed. We also have that $\text{supp}(\tau) \subset S$, since S is closed, and so

$$\int_M G_{\delta_p}(\cdot, w) d\tau(w)$$

is harmonic on S^c since for $w \in S$, $G_{\delta_p}(\cdot, w)$ is harmonic on S^c. We showed in 3.54 that

$$\int_M G_{\delta_p}(\cdot, w) d\tau(w)|_S = c_5 \tag{3.55}$$

quasi-everywhere on ∂S and everywhere on $S \backslash \partial S$. It follows from the generalized maximum principle applied to the harmonic functions $\int_M G_{\delta_p}(\cdot, w) d\tau(w)|_{S^c}$ and $-\int_M G_{\delta_p}(\cdot, w) d\tau(w)|_{S^c}$ that

$$\int_M G_{\delta_p}(\cdot, w) d\tau(w) = c_5 \tag{3.56}$$

where the equality above holds everywhere on $M \backslash \partial S$ and quasi-everywhere on ∂S. We then have

$$\begin{aligned} 0 &= \int_M c_5 d\tau(z) \\ &= \int_M [\int_M G_{\delta_p}(z, w) d\tau(w)] d\tau(z) \\ &= \int_{M\times M} [G_g(z, w) - G_g(z, p) - G_g(p, w)] d\tau(z) d\tau(w) \\ &= \int_M G_g d\tau^{\otimes 2}, \end{aligned}$$

where we used the fact that τ has zero mass in the first and fourth equalities. For the second equality we use 3.56 and the fact that since τ is a finite energy measure, $\tau(A) = 0$ if A has zero capacity. We now use Lemma 3.4.7 to conclude that $\tau = 0$. That is $1_S [\nu(M)\mu_{eq} - \nu + \Delta_g Q vol_g - \kappa] = 0$. Rearranging this equality and noting that $\text{supp}(\mu_{eq}) = S$, yields

$$\mu_{eq} = \frac{1}{\nu(M)} 1_S [\nu - \Delta_g Q vol_g + \kappa] \tag{3.57}$$

$$= \frac{1}{\nu(M)} 1_S [\nu + 2i\partial\bar{\partial} Q + \kappa]. \tag{3.58}$$

This completes the proof.

□

Under the further assumption that ν is absolutely continuous with a continuous Radon-Nikodym derivative (with respect to vol_g), we have a refinement of the first Structure Theorem.

Theorem 3.5.2. *(Structure Theorem 2). Let Q be an admissible external field which is C^2 in a neighborhood U of S. Suppose further that $\nu|_U$ is absolutely continuous and its Radon-Nikodym derivative is in $C_b(U)$, then*

$$\mu_{eq} = \frac{1}{\nu(M)} 1_S [\nu + 2i\partial\bar{\partial}Q].$$

Proof. We begin by showing that $U_\nu^{\mu_{eq}} \in W^{2,\infty}(U)$. Let $f \equiv \frac{d\nu|_U}{dvol_g}$. Then by hypothesis $f \in C_b(U)$ and so $\int_U G_g(\cdot, s) f(s) dvol_g(s)$ is in $W^{2,\infty}(U)$. We claim that $\int_M G_g(\cdot, s) d\nu(s)$ is also in $W^{2,\infty}(U)$. Indeed

$$\int_M G_g(\cdot, s) d\nu(s) = \int_U G_g(\cdot, s) f(s) dvol_g(s) + \int_{U^c} G_g(\cdot, w) d\nu(w),$$

where the first term on the right side is in $W^{2,\infty}(U)$ and the second term on the right side is smooth on U. By Proposition 3.2.9

$$G_\nu(z, w) = \nu(M) G_g(z, w) - \int_M G_g(z, s) d\nu(s) - \int_M G_g(w, s) d\nu(s), \tag{3.59}$$

for $w \in S$, $G_g(\cdot, w) \in W^{2,\infty}(U \backslash S)$. Since we showed above that $\int_M G_g(\cdot, s) d\nu(s) \in W^{2,\infty}(U)$, it follows from 3.59 that for $w \in S$, $G_\nu(\cdot, w) \in W^{2,\infty}(U \backslash S)$. Since $S \equiv \text{supp}(\mu_{eq})$, it follows that $U_\nu^{\mu_{eq}} \equiv \int_M G_\nu(\cdot, w) d\mu_{eq}(w)$ is in $W^{2,\infty}(U \backslash S)$.

Since ν is positive and $G_g(\cdot, w)$ is lower semi-continuous, $\int_M G_g(\cdot, w) d\nu(w)$ is lower semi-continuous. It follows that $Q_{\nu,g} \equiv Q(\cdot) - \int_M G_g(\cdot, w) d\nu(w)$ is upper semi-continuous on U, since $Q \in C^2(U)$. On the other hand since Q is admissible, by definition $Q_{\nu,g}$ is lower semi-continuous. So $Q_{\nu,g}$ is continuous and we can apply Proposition 3.4.13 to obtain:

$$(U_\nu^{\mu_{eq}} + Q)|_S = F, \tag{3.60}$$

$$(U_\nu^{\mu_{eq}} + Q)|_U \geq F. \tag{3.61}$$

We will now show that $U_\nu^{\mu_{eq}}$ is in $W^{2,\infty}(U)$. Fix an arbitrary $z_* \in S$. It suffices to show that $U_\nu^{\mu_{eq}}$ is in $W^2(V)$ where $V \subset U$ is a simply connected neighborhood of z_*. Let $\phi : \mathbb{D} \to V$ be a uniformization of V. By replacing V with $\phi((1-\epsilon)\mathbb{D})$ if necessary, we may assume that ϕ extends to a univalent map in a neighborhood, O, of $\mathbb{D}$. By equations 3.60 and 3.61, we have

$$(U_\nu^{\mu_{eq}} \circ \phi + Q \circ \phi)|_{\phi^{-1}(S)} = F, \tag{3.62}$$

$$(U_\nu^{\mu_{eq}} \circ \phi + Q \circ \phi)|_{\mathbb{D}} \geq F. \tag{3.63}$$

To show $U_\nu^{\mu_{eq}} \in W^{2,\infty}(U)$ it suffices to show that $U_\nu^{\mu_{eq}} \circ \phi \in W^{2,\infty}(\mathbb{D})$ since z_* was arbitrarily chosen. For convenience let $\Psi \equiv F - U_\nu^{\mu_{eq}} \circ \phi$ and let $R \equiv Q \circ \phi$. Note that $R \in C^2(\mathbb{D})$. Let $w \in \mathbb{D}$ and let $z \in \phi^{-1}(S)$. For t, satisfying $|t| < 1 - |z|$, we have $z + tw, z - tw \in \mathbb{D}$. We then have

$$\Psi(z+tw) + \Psi(z-tw) - 2\Psi(z) = \Psi(z+tw) + \Psi(z-tw) - 2R(z) \tag{3.64}$$

$$\leq R(z+tw) + R(z-tw) - 2R(z), \tag{3.65}$$

where 3.64 follows from 3.62 since $z \in \phi^{-1}(S)$, and 3.65 follows from 3.63. Dividing the expressions 3.64 and 3.65 by t^2 and then taking the limit $t \to 0$, yields

$$\partial_w^2 \Psi(z) \leq \partial_w^2 R(z) \leq M, \tag{3.66}$$

where ∂_w^2 denotes the second order directional derivative in the direction of w and $M \equiv \sup_{w \in \mathbb{D}, z \in \phi^{-1}(S)} \partial_w^2 Q(z)$ ($M < \infty$ since $R \in C^2(\mathbb{D})$). In particular, for $w = 1$ and $w = i$, we get

$$\partial_x^2 \Psi(z) \leq \partial_x^2 R(z) \leq M, \tag{3.67}$$

$$\partial_y^2 \Psi(z) \leq \partial_y^2 R(z) \leq M. \tag{3.68}$$

We now show that the partial derivatives (in the sense of distributions) of Ψ are bounded from below. In what follows c_1 and c_2 represent constants. We assumed that ϕ extends to a univalent map in a neighborhood, O, of $\mathbb{D}$. Fix $q \in \phi(O) \backslash V$. Notice that

$$\Psi \equiv F - U_\nu^{\mu_{eq}} \circ \phi(\cdot) \tag{3.69}$$

$$= F - \int_M G_\nu(\phi(\cdot), w) d\mu_{eq}(w) \tag{3.70}$$

$$= F - [\int_M G_g(\phi(\cdot), w) d[\nu(M)\mu_{eq} - \nu](w) + c_1] \tag{3.71}$$

$$= \int_M G_g(\phi(\cdot), w) d\nu(w) - \nu(M) \int_M G_g(\phi(\cdot), w) d\mu_{eq}(w) + c_2 \tag{3.72}$$

$$= \int_M G_g(\phi(\cdot), w) d\nu(w) + G_g(\phi(\cdot), q) - \nu(M) \int_M [G_g(\phi(\cdot), w) - G_g(\phi(\cdot), q)] d\mu_{eq}(w) + c_2, \tag{3.73}$$

where 3.71 follows from Proposition 3.2.9. We showed above that $\int_M G_g(\cdot, w) d\nu(w) \in W^{2,\infty}(U)$. It follows that the first term of 3.73 is in $W^{2,\infty}(\mathbb{D})$. Since $q \in \phi(O)\backslash\mathbb{D}$, the second term in 3.73 is smooth in a neighborhood of $\mathbb{D}$ and thus in $W^{2,\infty}(\mathbb{D})$. By property (vi) of G_g, it follows that $[G_g(\cdot, w) - G_g(\cdot, q)]$ is superharmonic on $M\backslash\{q\}$. Since ϕ is holomorphic, it follows that $[G_g(\phi(\cdot), w) - G_g(\phi(\cdot), q)]$ is superharmonic on $\mathbb{D}$. It then follows that the third term in 3.73 is subharmonic on $\mathbb{D}$. Putting this together it follows that the Laplacian of 3.73 is bounded from below in the sense of distributions. Therefore there exists $N \in \mathbb{R}$ such that

$$\Delta\Psi \geq N. \tag{3.74}$$

Combing this with inequalities 3.67 and 3.68 we have

$$M \geq \partial_x^2\Psi \geq \partial_x^2\Psi + \partial_y^2\Psi - M \tag{3.75}$$

$$= \Delta\Psi - M \tag{3.76}$$

$$\geq N - M, \tag{3.77}$$

and similarly for $\partial_y^2\Psi$. We have thus shown that the second order partials of Ψ are bounded and thus $\Psi \in W^{2,\infty}(\mathbb{D})$. We showed earlier that this implies that $U_\nu^{\mu_{eq}} \in W^{2,\infty}(U)$, as desired. It follows from [23], p. 53, and Equation 3.62, that the partial derivatives of $(U_\nu^{\mu_{eq}} \circ \phi + Q \circ \phi)|_{\phi^{-1}(S)} - F$ up to order two are equal to zero almost everywhere on $\phi^{-1}(S)$. Since $z_* \in S$ was arbitrary, it follows that, with respect to any chart, the partial derivatives up to order two of $(U_\nu^{\mu_{eq}} + Q)$ equal zero almost everywhere on S. We thus have

$$\Delta_g U_\nu^{\mu_{eq}}|_S = -\Delta_g Q|_S \ \ a.e. \tag{3.78}$$

Let $f \in C(M)$. By considering convolutions of $f 1_S$ with a mollifier, there exists a sequence $\{f_n\}_{n=1}^\infty \subset C^\infty(M)$, such that $\mathrm{supp}(f_n) \subset U$, $f_n \to f 1_S$ a.e., and $\|f_n\|_\infty \leq \|f\|_\infty$. We then have

$$
\begin{aligned}
-\int_S f \Delta_g Q dvol_g{}_g &= \int_S f \Delta_g U_\nu^{\mu_{eq}} dvol_g && (3.79)\\
&= \int_M f 1_S \Delta_g U_\nu^{\mu_{eq}} dvol_g && (3.80)\\
&= \lim_{n\to\infty} \int_M f_n \Delta_g U_\nu^{\mu_{eq}} dvol_g && (3.81)\\
&= \lim_{n\to\infty} \int_M U_\nu^{\mu_{eq}} \Delta_g f_n dvol_g && (3.82)\\
&= \lim_{n\to\infty} \int_M f_n d[\nu(M)\mu_{eq} - \nu] && (3.83)\\
&= \int_M f 1_S d[\nu(M)\mu_{eq} - \nu], && (3.84)\\
&= \int_S f d[\nu(M)\mu_{eq} - \nu], && (3.85)
\end{aligned}
$$

where we used equation 3.78 in 3.79, the bounded convergence theorem in 3.81, the distributional property of $U_\nu^{\mu_{eq}}$ in 3.83, and the bounded convergence theorem again in 3.84. We thus have shown that

$$[\nu(M)\mu_{eq} - \nu]1_S = -\Delta_g Q vol_g 1_S.$$

Noting that $1_S \mu_{eq} = \mu_{eq}$ we have

$$\mu_{eq} = \frac{1}{\nu(M)} 1_S [\nu - \Delta_g Q vol_g] = \frac{1}{\nu(M)} 1_S [\nu + 2i\partial\bar{\partial} Q].$$

This completes the proof.

□

3.5.2 The Structure of the Equilibrium Measure on $\mathbb{C}$

In this section we study the structure of the equilibrium measure in the (classical) setting of logarithmic potential theory. From here on we assume that Q is admissible in the classical sense that is Q is lower semi-continuous,

$$\lim_{|z|\to\infty} Q(z) - \log|z| = \infty,$$

and $M \backslash Q^{-1}(\infty)$ has positive capacity. It is a fundamental result of logarithmic potential theory that for a (classically) admissible potential there exists an equilibrium measure, μ, which is compactly supported in $\mathbb{C}$. As we showed in Example 3.2.4 when $M = \hat{\mathbb{C}}$ and $\nu = 2\pi\delta_\infty$, G_ν is the logarithmic kernel. Moreover, from

Example 3.4.6 we saw that if an external field Q is admissible in the classical sense, then by extending Q to $\hat{\mathbb{C}}$ by defining $Q(\infty) = \infty$ we obtain an admissible potential in the sense of Definition 3.4.3. It is not hard to see that $\mu = \mu_{eq}$ - the equilibrium measure in the classical sense agrees with the equilibrium measure on $\hat{\mathbb{C}}$ corresponding to the background measure $\frac{1}{2\pi}\delta_\infty$. We can then apply the structure theorems to obtain results about the structure of the equilibrium measure on the plane. We begin by showing that a well-known result on the structure of the equilibrium measure in the plane follows directly from the second structure theorem. After that we use the first structure theorem to prove a new result on the structure of the equilibrium measure in a very general setting.

Corollary 3.5.3. *Let Q be a classically admissible external field on the plane and let μ_{eq} denote the equilibrium measure. If Q is C^2-smooth in a neighborhood of* $\mathrm{supp}(\mu_{eq})$*, then*

$$\mu_{eq} = \frac{1}{2\pi} 1_S \Delta Q dA. \tag{3.86}$$

Proof. By the second structure theorem, we have

$$\begin{aligned} \mu_{eq} &= \frac{1}{\nu(M)} 1_S [\nu + 2i\partial\bar{\partial}Q] \\ &= \frac{1}{2\pi} 1_S [2\pi\delta_\infty + 2i\partial\bar{\partial}Q] \\ &= \frac{1}{2\pi} 1_S 2i\partial\bar{\partial}Q \\ &= \frac{1}{2\pi} 1_S \Delta Q dA, \end{aligned}$$

where the third equality follows from the fact that $S \subset \mathbb{C}$.

□

In the classical setting, if an admissible external field, Q, is C^2 in a neighborhood of S, then since the equilibrium measure is positive, it is easily seen by Corollary 3.5.3 that Q must be subharmonic on S. In fact even without additional regularity hypotheses, it follows from Frostman's theorem that Q must be subharmonic on S. Using the first structure theorem we prove a formula for the structure of the equilibrium measure when Q is subharmonic in a neighborhood of S, but no other regularity is assumed.

If Q is subharmonic in a (bounded) neighborhood U of S, it follows from the Riesz Decomposition theorem (see section 2.1), that (for $z \in \mathbb{C}$):

$$Q(z) = \int_M \log \frac{1}{|z-w|} d\tau(w) + h(z), \tag{3.87}$$

where τ is a finite *negative* compactly supported measure, and h is harmonic on U.

Theorem 3.5.4. *Let Q be a classically admissible external field in the plane. Suppose that Q is subharmonic in a neighborhood U of S. Choose τ and h satisfying 3.87. Then*

$$\mu_{eq} = \frac{1}{\tau(\mathbb{C})} 1_S [\tau - \omega], \tag{3.88}$$

where ω is defined as follows. Let $\tilde{h}$ denote a C^2 extension of h from a neighborhood of S to $\hat{\mathbb{C}}$. ω is defined as the Balayage of $1_{S^c}[\tau + \Delta \tilde{h} dA]$ onto ∂S.

Proof. It can be checked directly that

$$G_{-\tau}(z, w) = \tau(\mathbb{C}) \log \frac{1}{|z - w|} + \int_{\mathbb{C}} \log \frac{1}{|z - s|} d\tau(s) + \int_{\mathbb{C}} \log \frac{1}{|w - s|} d\tau(s). \tag{3.89}$$

Indeed, the right side of 3.89 is manifestly symmetric. Moreover it follows directly from the distributional property of the logarithmic kernel that the right side of 3.89 satisfies the same distributional property as $G_{-\tau}$. From Equations 3.87 and 3.89 we then have

$$\begin{aligned} K^Q_{2\pi\delta_\infty}(z, w) &\equiv \log \frac{1}{|z - w|} + Q(z) + Q(w) && (3.90) \\ &= G_{-\tau}(z, w) + h(z) + h(w) && (3.91) \\ &\equiv K^h_{-\tau}. && (3.92) \end{aligned}$$

The admissibility of h follows directly from the admissibility of Q. Since h is harmonic on U it is in particular C^2 on U and we can apply Structure Theorem 1, to obtain

$$\mu_{eq} = \frac{1}{-\tau(\hat{\mathbb{C}})} 1_S [-\tau + 2i\partial\bar{\partial} h + \kappa] \tag{3.93}$$

where κ is the balayage of the measure $1_{S^c} \frac{1}{-\tau(\hat{\mathbb{C}})}[-\tau + 2i\partial\bar{\partial}\tilde{h}]$ onto ∂S. Noting that $2i\partial\bar{\partial}\tilde{h} = \Delta h dA$, we have $\kappa = \omega$. So

$$\begin{aligned} \mu_{eq} &= \frac{1}{-\tau(\hat{\mathbb{C}})} 1_S [-\tau + 2i\partial\bar{\partial} h + \kappa] \\ &= \frac{1}{-\tau(\hat{\mathbb{C}})} 1_S [-\tau + 2i\partial\bar{\partial} h + \omega] \\ &= \frac{1}{-\tau(\hat{\mathbb{C}})} 1_S [-\tau + \omega] \\ &= \frac{1}{\tau(\mathbb{C})} 1_S [\tau - \omega], \end{aligned}$$

where the third equality uses the fact that h is harmonic on $U \supset S$. The proof is complete. □

3.6 Obstacle Problem

In the previous section we provided, under mild hypotheses, formulae for the equilibrium measure in terms of Q, ν, and S. To determine the equilibrium measure, it thus suffices to determine S. We show in this section that S can be realized as the coincidence set of a certain obstacle problem. Throughout this section we assume the hypotheses of the first structure theorem —that Q is an admissible potential and moreover, $Q \in C^2(M)$.

Let

$$\Gamma \equiv \{h \in L^2(M) \mid \Delta_g h vol_g \leq \nu,\ h \leq Q\},$$

where the first inequality is in the sense of distributions. This means that if $f \in C^\infty(M)$ is nonnegative and $h \in \Gamma$, then $\int_M h\Delta_g f dvol_g \leq \int_M f d\nu$. Let $\tilde{Q} \equiv \sup_{h\in\Gamma} h$. For convenience, let $\hat{Q} \equiv F - U_\nu^{\mu_{eq}}$. Q can be seen as an obstacle bounding $\tilde{Q}$. The obstacle problem is to determine the set $S^* \equiv \{\tilde{Q} = Q\}$. S^* is called the coincidence set.

Proposition 3.6.1. $\hat{Q} = \tilde{Q}$.

Proof. We begin by showing that $\hat{Q} \in \Gamma$. It follows from an argument analogous to that in Lemma 3.2.7, that $\hat{Q} \in L^2(M)$. Let $h \in C^2(M)$ be positive, then

$$\int_M \hat{Q}\Delta_g h dvol_g \tag{3.94}$$

$$= \int_M (F - U_\nu^{\mu_{eq}})\Delta_g h dvol_g \tag{3.95}$$

$$= -\int_M U_\nu^{\mu_{eq}} \Delta_g h dvol_g \tag{3.96}$$

$$= \int_M h d\nu - \nu(M)\int_M h d\mu_{eq} \tag{3.97}$$

$$\leq \int_M h d\nu. \tag{3.98}$$

The previous observations together show that $\hat{Q} \in \Gamma$ and so

$$\hat{Q} \leq \tilde{Q}, \tag{3.99}$$

Note that since ν is positive, $\int_M G_g(\cdot, w)d\nu(w)$ is lower semi-continuous and so $Q - \int_M G_g(\cdot, w)d\nu(w)$ is upper semi-continuous. Since Q is admissible, $Q - \int_M G_g(\cdot, w)d\nu(w)$ is also lower semi-continuous, and thus it is continuous. We can then apply Proposition 3.4.13 to conclude

$$Q|_S = \hat{Q}|_S.$$

Suppose that $\phi \in \Gamma$ then

$$\phi|_S \leq Q|_S = \hat{Q}|_S. \tag{3.100}$$

We now show that $\phi \leq \hat{Q}$. By 3.65, $\Delta_g \hat{Q} vol_g|_{S^c} = \nu|_{S^c}$ and $\Delta_g \phi vol_g|_{S^c} \leq \nu|_{S^c}$ in the sense of distributions.

$$\Delta_g(\hat{Q} - \phi) vol_g|_S \geq 0.$$

This shows that $(\hat{Q} - \phi)$ is superharmonic on S^c (recall that Δ_g is the positive Laplacian). By 3.68,

$$(\hat{Q} - \phi)|_{\partial S} \geq 0,$$

and so by the minimum principle $\hat{Q}|_{S^c} \geq \phi|_{S^c}$ and by using 3.68 again we obtain $\hat{Q} \geq \phi$. Since $\phi \in \Gamma$ was arbitrarily chosen, we have $\tilde{Q} \leq \hat{Q}$. Thus from 3.67 we have

$$\tilde{Q} = \hat{Q}$$

as desired. □

Earlier we introduced the coincidence set, $S^* \equiv \{\tilde{Q} = Q\}$. It follows directly from Proposition 3.6.1, that

$$S^* = \{\hat{Q} = Q\}.$$

By Proposition 3.4.13

$$(Q - \hat{Q})|_S, \tag{3.101}$$

$$(Q - \hat{Q}) \geq 0. \tag{3.102}$$

Equation 3.69 shows that

$$S \subset S^*. \tag{3.103}$$

Chapter 4

Quadrature Domains and Laplacian Growth on Surfaces

4.1 Introduction

2-D Laplacian growth (formerly known as Hele-Shaw flow) is a planar growth process where the evolution of a domain at any stage of the process is governed by its harmonic measure at a fixed point (see section 2.3 for a precise definition). It is remarkable that several seemingly disparate physical processes turn out to be examples of 2-D Laplacian growth. Some of the many examples are: viscous fingering, diffusion limited aggregation, slow freezing of fluids, fluid flow in Hele-Shaw cells, and crystal growth (see [29] for a more complete list along with references). Some of these processes have analogues on surfaces. For example Laplacian growth is a mathematical model for the evolution of a fluid injected into a thin mold created by two almost parallel surfaces separated by a small distance (see [28] chapter 7 for more details). On the other hand, even on $\mathbb{C}$, the introduction of a metric is very interesting. In the example of fluid dynamics, the metric determines the permeability of the medium. Laplacian growth in this setting is known as Elliptic Growth and has applications in fluid dynamics, electro-deposition, and cellular growth. [20] provides a nice introduction to the subject. Etingof and Varchenko [28] sketched the beginnings of a theory of Laplacian growth on Riemann surfaces endowed with a Riemannian metric. Later Hedenmalm and Shimorin [14] studied Laplacian growth on simply connected surfaces with negative curvature.

If $\{\Omega_t\}_{t=0}^{t_*}$ is the evolution of Ω_0 under 2-D Laplacian growth with source $b \in \Omega_0$ then by definition the normal velocity of the evolution of domains at time t satisfies

$$v_n(t) = -\frac{1}{2\pi}\partial_n G_{\Omega_t}(\cdot, b), \tag{4.1}$$

where G_{Ω_t} is the Green's function of Ω_t, and ∂_n is the outward normal derivative. If a Riemann surface, M, is endowed with a Riemannian metric, g, then we can define Laplacian growth on M analogously. Notice however, that equation 4.1 is not conformally invariant. We introduce a weighted version of Laplacian growth on Riemann surfaces which depends only on the complex structure and the weight. Given a weight $F : M \to \mathbb{R}$, we define F-weighted Laplacian growth by the equation

$$\Delta_g F v_n(t) = \frac{1}{2\pi}\partial_n G_{\Omega_t}(\cdot, b), \tag{4.2}$$

where g is any metric compatible with the complex structure of M. Equation 4.2 does not depend on the choice of metric compatible with the complex structure and thus only depends on the complex structure of M. We discuss an interesting connection between logarithmic potential theory on Riemann surfaces and Q-weighted Laplacian growth (where Q is the external field). In the classical case, this unifies the known methods for producing interior and exterior 2-D Laplacian growth. In chapter 5 section 3 we study $|R|^2$-weighted Laplacian growth where R is a meromorphic function on M (when $M = \hat{\mathbb{C}}$ and $R(z) = z$ we recover 2-D Laplacian growth).

There has been less work on quadrature domains on Riemann surfaces. In the plane there is a beautiful relation between quadrature domains and operator theory given by the exponential transform. In [30] Xia provides a definition of quadrature domains on Riemann surfaces and studies an analogue of the exponential transform. For our purposes Xia's definition is not general enough to capture the interplay between Laplacian growth and quadrature domains nor is it specific enough to capture the algebraic properties of the boundary of a quadrature domain. We instead provide a more general definition of quadrature domain which we study in Chapter 4, and focus on a special case which we call Algebraic Quadrature Domains in Chapter 5.

4.2 Quadrature Domains on Surfaces

We begin by defining quadrature domains on Riemann surfaces. Let $\Omega \subset M$ be an open set and let μ be an absolutely continuous measure on Ω. Let $\Psi : AL^1(\Omega, \mu) \to \mathbb{C}$ be a linear functional. We call an open set $\Omega \subset M$ an analytic quadrature domain with data (μ, Ψ) if

$$\int_\Omega f d\mu = \Psi[f] \tag{4.3}$$

for every $f \in AL^1(\Omega, \mu)$. We define harmonic quadrature domains analogously by replacing $AL^1(\Omega, \mu)$ in the above definition with $HL^1(\Omega, \mu)$. We call Ω a subharmonic quadrature domain if

$$\int_\Omega h d\mu \geq \Psi[h] \tag{4.4}$$

for every $h \in SL^1(\Omega,\mu)$ where $\Psi : SL^1(\Omega,\mu) \to \mathbb{C}$ is a linear functional. We recover the classical case by choosing $M = \hat{\mathbb{C}}$, μ the Lebesgue measure, and Ψ the functional defined by

$$\Psi[f] = \sum_{i=1}^{n} \sum_{j=0}^{n_i-1} c_{i,j} f^j(a_i).$$

Of course if Ω is a harmonic quadrature domain with data (μ,Ψ), it is also an analytic quadrature domain with the same data. Though, we saw in the classical setting, the converse does not. It easily follows from the definitions that if Ω is a subharmonic quadrature domain with data (μ,Ψ), it is also a harmonic quadrature domain with the same data. Indeed, let $h \in HL^1(\Omega,\mu)$, then in particular, $h \in SL^1(\Omega,\mu)$ and $-h \in SL^1(\Omega,\mu)$. Applying inequality 4.4 to h and $-h$ yields equation 4.3, and so Ω is also a harmonic quadrature domain. We have thus shown that for a fixed μ and Ψ, subharmonic quadrature domain $\implies$ harmonic quadrature domain $\implies$ analytic quadrature domain.

We begin by exploring the simplest examples of quadrature domains, those where

$$\Psi = c\delta_b,$$

where $c \in \mathbb{R}_+$ and $b \in \Omega$. Following the terminology in the classical setting, we call such Ω one-point quadrature domains. In the classical setting, the discs were long known to be one-point (harmonic and analytic) quadrature domains. This classical result is known as the (area) mean value theorem. It is also easy to show that the discs are also one-point subharmonic quadrature domains. It is significantly more difficult to show that the discs are the *only* one-point quadrature domains. There was much work on this problem see ([6], [7], [20], [1]). Using the Schottky Double construction, Gustaffson [9] provided another proof that the discs are the only one-point analytic quadrature domains.

It seems difficult to even describe one-point quadrature domains for all but the simplest μ. In what follows we prove a hyperbolic analogue of the result above. Effectively, we consider the description of one-point quadrature domains on the Poincar disc $(\mathbb{D},\mu_h)$, where μ_h denotes the volume measure induced by the hyperbolic metric. That is

$$d\mu_h(z) = \frac{1}{(1-|z|^2)^2} dA(z). \tag{4.5}$$

We prove that the bounded one-point harmonic and subharmonic quadrature domains on the Poincar disc are precisely the hyperbolic discs. The proof relies on techniques different from those used in the proof of the analogue on $\mathbb{C}$. The results on the Poincar disc extend to hyperbolic Riemann surfaces, if it is assumed that the quadrature domain does not contain a homology cycle of the Riemann surface. Although we are not aware of a proof, we suspect it is known that hyperbolic discs not containing cycles satisfy a mean value property for harmonic functions. The the proof of the converse is slightly more difficult.

Theorem 4.2.1. *Let M be a hyperbolic Riemann surface (which is not necessarily compact). Let $\Omega \subset M$ be an open set which is bounded with respect to the hyperbolic metric and does not contain a homology cycle of M. Let ω_h be the measure induced by the volume form corresponding to the hyperbolic metric on M. Let $c \in \mathbb{R}_+$, and let $b \in \Omega$. Then Ω is a harmonic quadrature domain with data $(\omega_h, c\delta_b)$ only if Ω is a hyperbolic disc centered at b with area c. Moreover, hyperbolic discs not containing cycles are subharmonic quadrature domains.*

Proof. We begin by reducing our consideration to quadrature domains on $\mathbb{D}$ with the hyperbolic metric. Let $\pi : \mathbb{D} \to M$, be the covering map. Since π is locally univalent and Ω does not contain a cycle, it follows from the Monodromy theorem that there exists a univalent map $\tau : \Omega \to \tau(\Omega) \subset \mathbb{D}$ whose inverse is $\pi|_{\tau(\Omega)}$. Recall that μ_h is defined in 4.5 as the measure induced by the volume form corresponding to the hyperbolic metric on $\mathbb{D}$. By definition of the hyperbolic metric, we have

$$(\pi|_{\tau(\Omega)})_*(\mu_h|_{\tau(\Omega)}) = \omega_h|_\Omega. \tag{4.6}$$

It then follows that

$$\begin{aligned}
\mu_h|_{\tau(\Omega)} &= (id|_{\tau(\Omega)})_*(\mu_h|_{\tau(\Omega)}) && (4.7)\\
&= (\tau \circ \pi|_{\tau(\Omega)})_*\mu_h && (4.8)\\
&= \tau_*(\pi|_{\tau(\Omega)})_*\mu_h && (4.9)\\
&= \tau_*(\omega_h|_\Omega). && (4.10)
\end{aligned}$$

Suppose Ω is a harmonic quadrature domain. Let $f \in HL^1(\tau(\Omega), \omega_h)$ then

$$\begin{aligned}
\int_{\tau(\Omega)} f d\mu_h &= \int_{\tau(\Omega)} f d\tau_*(\omega_h|_\Omega) && (4.11)\\
&= \int_\Omega f \circ \tau d\omega_h && (4.12)\\
&= c(f \circ \tau)(b) && (4.13)\\
&= cf(\tau(b)), && (4.14)
\end{aligned}$$

where 4.13 follows from the fact that $f \circ \tau \in HL^1(\Omega, w_h)$. Therefore $\tau(\Omega) \subset \mathbb{D}$ is a harmonic quadrature domain with data $(\mu_h, c\delta_{\tau(b)})$. It suffices to show that $\tau(\Omega)$ is a hyperbolic disc in $\mathbb{D}$ since this would imply that Ω is a hyperbolic disc, since τ is an isometry.

The reduction for the second assertion of the theorem is similar. Suppose we have shown that the hyperbolic disc in $\mathbb{D}$ with center b and hyperbolic area c is a subharmonic quadrature domain with data $(\mu_h, c\delta_b)$. If Ω is a hyperbolic disc not containing a cycle then $\tau(\Omega)$ is a hyperbolic disc in $\mathbb{D}$ and it follows from our hypothesis that if $s \in SL^1(\Omega, \omega_h)$ then

$$\int_\Omega s d\omega_h = \int_{\tau(\Omega)} s \circ \pi d\mu_h \geq c(s \circ \pi)(b) = cs(\pi(b)),$$

and thus Ω is a subharmonic quadrature domain with data $(\omega_h, c\delta_{\pi(b)})$ (we have used the fact that $s \circ \pi \in SL^1(\tau(\Omega)), \mu_h)$. In light of the reductions above, from now on we assume that $\Omega \subset \mathbb{D}$.

We next reduce to the case where $b = 0$. Let $\phi \in Aut(\mathbb{D})$ satisfy $\phi(b) = 0$. ϕ is an isometry and thus maps hyperbolic discs to hyperbolic discs. If Ω is a harmonic quadrature domain data $(\mu_h, c\delta_b)$, then $\phi(\Omega)$ is a harmonic quadrature domain with data $(\mu_h, c\delta_0)$. Indeed, if $f \in HL^1(\phi(\Omega), \mu_h)$, then

$$\int_{\phi(\Omega)} f d\mu_h = \int_\Omega f \circ \phi d(\phi^{-1})_* \mu_h = \int_\Omega f \circ \phi d\mu_h = c(f \circ \phi)(b) = cf(0).$$

So if we prove, under the hypotheses, that harmonic quadrature domain with data $(\mu_h, c\delta_0)$ are hyperbolic discs centered at 0 with area c, then it follows that $\Omega = \phi^{-1}(\phi(\Omega))$ is a hyperbolic disc centered at b with area c, since ϕ is an isometry. Similarly, it suffices to show that hyperbolic discs centered at zero are subharmonic quadrature domains.

We begin by showing that hyperbolic discs centered at 0 with area c are subharmonic quadrature domains with data $(c\delta_0, \mu_h)$. Let D be a hyperbolic disc centered at 0 with area c. By the radial symmetry of the hyperbolic metric at the origin, D is a Euclidean disc centered at 0. Let r satisfy

$$2\pi \int_0^r \frac{t}{(1-t^2)^2} dt = c. \tag{4.15}$$

Then $D = r\mathbb{D}$. Let $s \in SL^1(D, \mu_h)$. Then

$$\int_D s d\mu_h = \int_{r\mathbb{D}} s d\mu_h \tag{4.16}$$

$$= \int_0^r \frac{t}{(1-t^2)^2} \int_0^{2\pi} s(te^{i\theta}) d\theta dt \tag{4.17}$$

$$\geq s(0) 2\pi \int_0^r \frac{t}{(1-t^2)^2} dt \tag{4.18}$$

$$= cs(0). \tag{4.19}$$

Thus D is a subharmonic quadrature domain with data $(\mu_h, c\delta_0)$. We have thus shown that the hyperbolic discs not containing cycles are subharmonic quadrature domains.

We now prove the first assertion of the theorem. By the reduction, we only need to prove that bounded one-point harmonic quadrature domains in the Poincare disc with data $(\mu_h, c\delta_0)$ are hyperbolic discs. Let $s > 0$. Since the logarithmic kernel is superharmonic we have $\log \frac{1}{|z|} \geq \frac{1}{2\pi} \int_0^{2\pi} \log \frac{1}{|z-se^{i\theta}|} d\theta$ with strict inequality if $|z| < s$. Let r be defined as in 4.15. Then

$$\begin{aligned} U^{c\delta_0}(z) = c \log \frac{1}{|z|} &= 2\pi \int_0^r \frac{\tilde{r}}{(1-\tilde{r}^2)^2} d\tilde{r} \log \frac{1}{|z|} \\ &\geq \int_0^r \int_0^{2\pi} \frac{\tilde{r}}{(1-\tilde{r}^2)^2} \log \frac{1}{|z-\tilde{r}e^{i\theta}|} d\theta d\tilde{r} \\ &= \int_{r\mathbb{D}} \log \frac{1}{|z-w|} d\mu_h(w) \\ &= U^{1_{r\mathbb{D}}\mu_h}(z), \end{aligned}$$

with strict inequality if $|z| < r$. Since Ω is a harmonic quadrature domain with data $(\mu_h, c\delta_0)$ and $\log|z - \cdot|$ is harmonic on Ω for $z \in \Omega^c$, we have

$$U^{1_\Omega \mu_h}|_{\Omega^c} = U^{c\delta_0}|_{\Omega^c} \geq U^{1_{r\mathbb{D}}\mu_h}|_{\Omega^c}. \tag{4.20}$$

We now subtract $U^{1_{\Omega \cap r\mathbb{D}}\mu_h}$ from both sides of 4.20 to obtain

$$\begin{aligned} U^{1_{\Omega\backslash r\mathbb{D}}\mu_h}|_{\Omega^c} &= U^{1_\Omega\mu_h}|_{\Omega^c} - U^{1_{\Omega\cap r\mathbb{D}}\mu_h}|_{\Omega^c} \\ &\geq U^{1_{r\mathbb{D}}\mu_h}|_{\Omega^c} - U^{1_{\Omega\cap r\mathbb{D}}\mu_h}|_{\Omega^c} \\ &= U^{1_{r\mathbb{D}\backslash\Omega}\mu_h}|_{\Omega^c}. \end{aligned}$$

So in particular $U^{1_{\Omega\backslash r\mathbb{D}}\mu_h}|_{\Omega^c}$ dominates $U^{1_{r\mathbb{D}\backslash\Omega}\mu_h}|_{\Omega^c}$ on $r\mathbb{D}\backslash\Omega = \text{supp}(1_{r\mathbb{D}\backslash\Omega}\mu_h)$. Applying the quadrature identity to the harmonic function 1, shows

$$\begin{aligned} \int_{\mathbb{D}} 1_{\Omega\backslash r\mathbb{D}} d\mu_h &= \int_\Omega d\mu_h - \int_{\Omega\cap r\mathbb{D}} d\mu_h = c - \int_{\Omega\cap r\mathbb{D}} d\mu_h \\ &= \int_{r\mathbb{D}} d\mu_h - \int_{\Omega\cap r\mathbb{D}} d\mu_h = \int_{\mathbb{D}} 1_{r\mathbb{D}\backslash\Omega} d\mu_h, \end{aligned}$$

so the measures $1_{\Omega\backslash r\mathbb{D}}\mu_h$ and $1_{r\mathbb{D}\backslash\Omega}\mu_h$ have the same mass. Also note that both $1_{\Omega\backslash r\mathbb{D}}\mu_h$ and $1_{r\mathbb{D}\backslash\Omega}\mu_h$ have finite logarithmic energy (this is where we used the fact that Ω is bounded): we can apply the principle of domination ([26] page 43) to conclude that

$$U^{1_{\Omega\backslash r\mathbb{D}}\mu_h} \geq U^{1_{r\mathbb{D}\backslash\Omega}\mu_h} \tag{4.21}$$

on all of $\mathbb{C}$. By adding $U^{1_{\Omega\cap r\mathbb{D}}\mu_h}$ to both sides of 4.21 we obtain

$$U^{1_\Omega\mu_h} \geq U^{1_{r\mathbb{D}}\mu_h}. \tag{4.22}$$

We now show this implies that $\partial\Omega \subset \overline{r\mathbb{D}}$. Suppose not. Let $z_* \in \partial\Omega\backslash\overline{r\mathbb{D}}$. We then have

$$(U^{1_\Omega\mu_h} - U^{1_{r\mathbb{D}}\mu_h})(z_*) = 0.$$

Since we have showed that $(U^{1_\Omega\mu_h} - U^{1_{r\mathbb{D}}\mu_h}) \geq 0$, it follows that $(U^{1_\Omega\mu_h} - U^{1_{r\mathbb{D}}\mu_h})$ is subharmonic at z_*. On the other hand, since $z_* \in (\overline{r\mathbb{D}})^c$, $U^{1_{r\mathbb{D}}\mu_h}$ is harmonic at z_*. $U^{1_\Omega\mu_h}$ is superharmonic everywhere and strictly superharmonic at z_* (since z_* cannot be an isolated point since Ω is open) it follows that $(U^{1_\Omega\mu_h} - U^{1_{r\mathbb{D}}\mu_h})$ is strictly superharmonic at z_* which is a contradiction. We thus have shown that $\partial\Omega \subset \overline{r\mathbb{D}}$. It thus follows that $\Omega \subset \overline{r\mathbb{D}}$ since $\overline{r\mathbb{D}}$ is simply connected, and furthermore that $\Omega \subset r\mathbb{D}$ since Ω is open. Since Ω and $r\mathbb{D}$ both have the same hyperbolic area, if follows that $\int_{r\mathbb{D}\backslash\Omega} d\mu_h = 0$. It then follows that $U^{1_\Omega\mu_h} = U^{1_{r\mathbb{D}}\mu_h}$. Suppose for the sake of contradiction that $\hat{z} \in r\mathbb{D}\backslash\Omega$. Then the function $\log \frac{1}{|\hat{z}-w|}$ is harmonic on Ω. However we would then have

$$
\begin{aligned}
c \log \frac{1}{|\hat{z}|} &= U^{1_\Omega \mu_h}(\hat{z}) \\
&= U^{1_{r\mathbb{D}} \mu_h}(\hat{z}) \\
&< U^{c\delta_0}(\hat{z}) \\
&= c \log \frac{1}{|\hat{z}|}.
\end{aligned}
$$

It thus follows that $r\mathbb{D} \backslash \Omega = \emptyset$ and thus since $\Omega \subset r\mathbb{D}$ we have $\Omega = r\mathbb{D}$. This completes the proof.

□

A Riemann surface is called parabolic if its universal cover is $\mathbb{C}$. The parabolic Riemann surfaces inherit the Euclidean metric from $\mathbb{C}$ which is referred to as the flat metric. Let μ_f denote the measure induced by the volume form for the flat metric. The parabolic Riemann surfaces are a very small class and consist only of the complex plane, the torus, and the cylinder. Using the classification of one-point quadrature domains in the plane and an analogous argument to the one used in the beginning of the proof of Theorem 4.2.1, we easily obtain an analogous result for the class of parabolic Riemann surfaces.

Proposition 4.2.2. *(Description of Parabolic One-Point Quadrature Domains)*
Let M be a parabolic Riemann surface. Let $\Omega \subset M$ be an open set which is bounded with respect to the parabolic metric and does not contain a homology cycle of M. Let $c \in \mathbb{R}_+$, and let $b \in \Omega$. Then Ω is a quadrature domain with data $(\mu_f, c\delta_b)$ iff Ω is a disc (with respect to the flat metric) centered at b with area c.

It would be very interesting to provide a complete classification of one-point quadrature domains by removing the cycle condition. We expect that there should be continuous families of analytic quadrature domains with the same data. We have not yet considered the classification of one-point quadrature domains on the sphere with the spherical metric and it would be interesting to do so. Together this would provide a complete classification of one-point quadrature domains with respect to the natural geometries. For a general geometry, it appears difficult to explicitly describe all but the simplest quadrature domains.

The following theorem illustrates a relationship between quadrature domains and logarithmic potential theory on surfaces, and allows us to construct more interesting examples of quadrature domains.

Theorem 4.2.3. *Let Q be an admissible potential which is C^2 in a neighborhood of the support of the equilibrium measure, S, and suppose* $\operatorname{supp}(\nu) \subset (\overline{S})^c$. *Let $\tilde{Q} \in C^2(M)$ denote a C^2 extension of $1_U Q$ where U is a neighborhood of S. Then S^c is a harmonic quadrature domain with data $(\frac{2}{i}\partial\bar{\partial}\tilde{Q}, \nu)$. That is:*

$$\frac{2}{i}\int_{S^c} h\partial\bar{\partial}\tilde{Q} = \int_M h d\nu$$

for every $h \in HL^1(S^c, \frac{2}{i}\partial\bar{\partial}\tilde{Q})$.

The result follows easily from the two structure theorems.

Proof. From the hypotheses on Q and ν we can apply the first and second structure theorems (3.5.1 and 3.5.2) to obtain two representations for μ_{eq}. This yields the following identity:

$$\frac{1}{\nu(M)} 1_S[\nu + 2i\partial\bar{\partial}Q + \kappa] = \mu_{eq} = \frac{1}{\nu(M)} 1_S[\nu + 2i\partial\bar{\partial}Q]. \tag{4.23}$$

where

$$\kappa \equiv 1_{S^c}[\widehat{\nu + 2i\partial\bar{\partial}\tilde{Q}}]. \tag{4.24}$$

The hat indicates the balayage of the measure from S^c onto ∂S (recall that we identify the form in 4.24 with its associated measure). Equation 4.23 indicates that $\kappa = 0$. It follows from the definition of κ and the linearity of balayage that

$$\begin{aligned} \hat{\nu} &= \widehat{1_{S^c}\nu} &(4.25)\\ &= -(\widehat{1_{S^c}2i\partial\bar{\partial}\tilde{Q}}). &(4.26)\end{aligned}$$

Let $h \in HL^1(S^c, \frac{2}{i}\partial\bar{\partial}\tilde{Q})$ have a continuous extension to ∂S. Using equations 4.25-4.26 and a property of balayage we have

$$\begin{aligned} \int_{S^c} h(-2i\partial\bar{\partial}\tilde{Q}) &= -\int_{\partial S} hd(\widehat{1_{S^c}2i\partial\bar{\partial}\tilde{Q}}) &(4.27)\\ &= \int_{\partial S} hd\hat{\nu} &(4.28)\\ &= \int_{S^c} hd\nu. &(4.29)\end{aligned}$$

This completes the proof. □

4.3 Laplacian Growth on Surfaces

In this section we consider two formulations of Laplacian growth which we call the weighted and unweighted cases. The unweighted case involves the specification of a metric and is not conformally invariant, whereas in the weighted case we obtain a conformally invariant flow. Let M be a compact Riemann surface with metric g and let $t_* > 0$. We will consider the evolution of domains on M. In this section, $\{\Omega_t\}_{t\in[0,t_*]}$, will always be an increasing family of domains such that $\{\partial\Omega_t\}_{t\in[0,t_*]}$ is a C^1-smooth lamination. The normal velocity is then defined and we denote it by v_n. Let $\{\tau_t\}_{t\in[0,t_*]}$ be a family of finite positive Borel measures supported on $M\backslash\Omega_0$. $\{\tau_t\}_{t\in\in[0,t_*]}$ will be referred to as the source. We say that the source is integrable (smooth, C^1 etc.) if for any $f \in C(M)$, $\tau_t(f)$ is an integrable (smooth, C^1, etc.) on $(0, t_*)$. We will always assume that the source is integrable. Given a measure μ supported on a domain D with boundary of positive capacity, the balayage of μ onto ∂D, denoted $\hat{\mu}$, is a measure supported on ∂D defined by

$$\hat{\mu} = \int_D \omega_z^D d\mu(z) \tag{4.30}$$

where ω_z^D is the harmonic measure with respect to the point z. If ∂D is C^1, then for $w \in \partial D$,

$$\frac{d\hat{\mu}}{ds}(w) = -\frac{1}{2\pi}\int_D \partial_n G_D(w,z)d\mu(z)$$

where G_D is a the Green's function for D, and ∂_n is the outward normal derivative taken in the first variable. Let $f \in HL^1(D,\mu)$ if f extends continuously to ∂D then

$$\int_{\partial D} f d\hat{\mu} = \int_{\partial D} f(s)d\int_D \omega_z^D(s)d\mu(z) = \int_D [\int_{\partial D} f(s)d\omega_z^D(s)]d\mu(z) = \int_D f d\mu. \tag{4.31}$$

Definition 4.3.1. *Laplacian Growth*
Let $\{\Omega_t\}_{t\in[0,t_*]}$ satisfy the properties above and let $\{\tau_t\}_{t\in[0,t_*]}$ be a C^1-source for $\{\Omega_t\}_{t\in[0,t_*]}$. We call evolution of domains $\{\Omega_t\}_{t\in[0,t_*]}$ Laplacian growth with source $\{\tau_t\}_{t\in[0,t_*]}$ and initial domain Ω_0.

$$v_n(t) = \frac{d\hat{\tau}_t}{ds}, \tag{4.32}$$

where $\hat{\tau}_t$ is the balayage of τ_t onto $\partial\Omega(t)$.

We note that it is possible for $\{\Omega_t\}_{t\in[0,t_*]}$ to be Laplacian growth for different sources. For example, on $\hat{\mathbb{C}}$, $\Omega_t = B(0, \sqrt{\frac{2+t}{\pi}})$ is Laplacian growth for the sources $\tau_t = \delta_0$ and $\tau_t = \frac{1}{\pi}\chi_{B(0,1)}A$, where A is the area measure on $\mathbb{C}$.

Definition 4.3.2. *Weighted Laplacian Growth*

Let $\{\Omega_t\}_{t\in[0,t_*]}$ satisfy the properties above. Let F be C^2 and subharmonic in a neighborhood of Ω_{t_*}. We call $\{\Omega_t\}_{t\in[0,t_*]}$ F-weighted Laplacian growth with source $\{\tau_t\}_{t\in[0,t_*]}$ if:

$$-\Delta_g F v_n(t) = \frac{d\hat{\tau}_t}{ds_g}. \tag{4.33}$$

Notice that under a conformal change of metric $g \to e^{2f}g$. We have $\Delta_g F \to e^{-2f}\Delta_g F$, $\frac{d\hat{\tau}_t}{ds} \to e^{-f}\frac{d\hat{\tau}_t}{ds}$, and $v_n(t) \to e^f v_n(t)$. Therefore equation 4.33 also holds for the metric $e^f g$ and thus F-weighted Laplacian growth is conformally invariant. We will consider the weighted case from now on, and as weighted Laplacian growth depends on the complex structure but not the metric structure of M, we will drop the subscript g when we can. The following proposition is a generalization of Richardson's theorem in this setting.

Proposition 4.3.3. *Suppose $\{\Omega_t\}_{t\in[0,t_*]}$ is the solution of the F-weighted Laplacian growth with source $\{\tau_t\}_{t\in[0,t_*]}$. Let $U \subset S_{r-\epsilon}$ be open. Let f be continuous on $S_r\backslash U$. Then*

$$\frac{d}{dt}\int_{\Omega(t)\backslash U} f dd^c Q\Big|_{t=r} = \int_M \tilde{f} d\tau_r,$$

where $\tilde{f}$ is the harmonic extension of f on $M\backslash S_r$ and $d^c = -i(\partial - \bar{\partial})$.

Proof.

$$\frac{d}{dt}\int_{\Omega(t)\backslash U} f dd^c Q\Big|_{t=r} = \frac{d}{dt}\int_{\Omega(t)\backslash U} f\Delta_g Q dvol_g\Big|_{t=r}$$

$$= \int_{\partial S_r} f\Delta_g Q v_n(r) ds_g = \int_{\partial S_r} f d\hat{\tau}_r = \int_{M\backslash S_r} \tilde{f} d\tau_r = \int_M \tilde{f} d\tau_r.$$

The last equality follows from the fact that the support of τ_r is contained in the complement of S_r. □

Corollary 4.3.4. *In addition to the hypotheses of the proposition above, assume that f is subharmonic on $M\backslash S_r$. Then*

$$\frac{d}{dt}\int_{\Omega(t)\backslash U} f dd^c Q\Big|_{t=r} \geq \int_M f d\tau_r.$$

Proof. Let $\tilde{f}$ denote the harmonic extension of f to $M\backslash S_r$, then, since f is subharmonic, $f \leq \tilde{f}$ on $M\backslash S_r$. From the proof of the previous proposition, we have

$$\frac{d}{dt}\int_{\Omega(t)\backslash U} f dd^c Q\Big|_{t=r} = \int_{M\backslash S_r} \tilde{f} d\tau_r \geq \int_{M\backslash S_r} f d\tau_r = \int_M f d\tau_r.$$

□

Chapter 5

Algebraic Quadrature Domains

5.1 Introduction

It was shown in [9] that the boundary of a quadrature domain is a subset of an algebraic curve. We introduce Algebraic Quadrature Domains (abbreviated AQD) to capture an analogue of this property on Riemann surfaces. Let R be a meromorphic function on M. We call a domain $D \subsetneq M$ with piecewise C^1 boundary an AQD for R if $\overline{D}$ does not contain any poles of R and there exists a meromorphic function, $S_{D,R}$, on D that extends continuously to ∂D and satisfies

$$S_{D,R}|_{\partial D} = \bar{R}|_{\partial D}.$$

S is called the (generalized) Schwarz function. We show that these domains turn out to be analytic quadrature domains with data $(2i\partial\bar{\partial}|R|^2, \Psi)$ where

$$\Psi[f] \equiv 2i \int_{\partial D} f S_{D,R} \partial R, \tag{5.1}$$

for all $f \in AL^1(D, 2i\partial\bar{\partial}|R|^2)$. We prove that if D is an AQD for R then ∂D is a subset of the inverse image of a planar algebraic curve under R which motivates the name.

We later narrow our focus to the study of the topology of classical quadrature domains. Let Ω be a quadrature domain of degree n. The connectivity, c, of Ω is defined to be the number of components in $\hat{\mathbb{C}}\backslash\Omega$. Recall that $\partial\Omega$ is subset of an algebraic curve, Γ, of degree $2n$. Moreover $\Gamma\backslash\partial\Omega$ consists of a finite number of points called special points. We denote the number of special points of Ω by s. We provide an upper bound on the connectivity of a quadrature domain in terms of its degree. At the same time we prove a new upper bound on the number of special points of a quadrature domain as a function of its degree. As quadrature domains of degrees one and two are completely classified, we restrict our attention to the case of $n \geq 3$. In a beautiful paper, Lee and Makarov [24] proved using qausiconformal surgery and complex dynamics the following upper bound when $\Omega = int(\overline{\Omega})$ (and $n \geq 3$):

$$c \leq 2n - 4. \tag{5.2}$$

Moreover they constructed examples showing that 5.2 is sharp. In this section we eliminate the hypothesis $\Omega = int(\overline{\Omega})$ and prove the following:

Theorem 5.3.3. *Let Ω be a quadrature domain with degree $n \geq 3$. Let c denote the connectivity of Ω. Then*

$$c \leq 5n - 5. \tag{5.3}$$

At the same time we provide a new upper bound on the number of special points. Special points are interesting for a number of reasons. It turns out that $z \in \Omega$ is a special point iff $\Omega\backslash\{z\}$ remains an analytic quadrature domain. In other words, Ω continues to satisfy the quadrature identity for the new test function $f(z) \equiv \frac{1}{z-a}$. More interestingly, special points are isolated points of the algebraic curve which defines the boundary of Ω. They have been well studied. In particular, there has been interest in providing upper and lower bounds on the number of special points. For estimates of special points, it makes sense (and we lose nothing) to assume $\Omega = int(\overline{\Omega})$ (so that there exists nontrivial lower bounds on s). We will do so from now on. Let b denote the number of cusps on $\partial\Omega$, c denote the connectivity of Ω, and d denote the number of double points of $\partial\Omega$. Gustafsson [13] proved:

$$s \leq (n-1)^2 + 1 - c - b - 2d \tag{5.4}$$

and Sakai [27] proved

$$s \geq n - 2 + c - b, \tag{5.5}$$

see also McCarthy and Yang [25]. We prove:

Theorem 5.3.2. *Let Ω be a quadrature domain of degree $n \geq 3$ satisfying $\Omega = int(\overline{\Omega})$. Let s denote the number of special points and c denote the connectivity of Ω. Then*

$$s \leq 5n - 5 - c.$$

Our proof makes use of Lemma 4.3 from [24]. The new ingredient is a recent result of D. Khavinson and G.

Neumann [21] which resolved an open problem in gravitational lensing.

5.2 Algebraic Quadrature Domains

We begin by introducing a generalization of the Schwarz function, which is the central object of this section.

Definition 5.2.1. *Schwarz function*
Let $D \subsetneq M$ be a domain with piecewise C^1 boundary, and let R be a meromorphic function on M. We say S is a Schwarz function for (D, R) iff S is meromorphic on D and S extends continuously to ∂D, and satisfies

$$S|_{\partial D} = \bar{R}|_{\partial D}. \tag{5.6}$$

It is easy to show that the Schwarz function for (D, R) is unique if it exists.

Lemma 5.2.2. *Let $D \subsetneq M$ be a domain wtih piecewise C^1 boundary, and let R be a meromorphic function on M. Then there is at most one Schwarz function for the pair (D, R).*

Proof. Suppose S_1 and S_2 are Schawrz functions for (D, R). Then $S_1 - S_2$ is a meromorphic function and $(S_1 - S_2)|_{\partial D} = 0$. Since D is an open Riemann surface, there exists a holomorphic function f on D whose zeros with multiplicity are precisely the poles of $S_1 - S_2$. Moreover, by the hypothesis on the boundary, f can be chosen to extend continuously to the boundary. Then $f(S_1 - S_2)$ is holomorphic and zero on ∂D. It follows from the maximum principle that $f(S_1 - S_2) = 0$. Since the zeros of f and $S_1 - S_2$ are discrete unless they are identically zero, it follows that $S_1 = S_2$. □

We will denote the Schwarz function for the pair (D, R), by $S_{D,R}$.

Definition 5.2.3. *Algebraic Quadrature Domain (AQD)*
Let $D \subsetneq M$ be a domain with piecewise C^1 boundary and let R be a meromorphic function on M. We call D an Algebraic Quadrature Domain if the Schwarz function for (D, R) exists.

The following lemma shows that AQD's are analytic quadrature domains.

Proposition 5.2.4. *Let D be an AQD for R. Then D is an analytic quadrature domain with data $(2i\partial\bar{\partial}|R|^2, \Psi)$ where Ψ is the linear functional defined by*

$$\Psi[f] \equiv 2i \int_{\partial D} f S_{D,R} \partial R \tag{5.7}$$

for $f \in AL^1(D, 2i\partial\bar{\partial}|R|^2)$.

Proof. We only need to show that D satisfies the required quadrature identity. Let $f \in AL^1(\Omega, 2i\partial\bar{\partial}|R|^2)$. This follows from the following computation:

$$\int_D f 2i\bar{\partial}\partial|R|^2 = 2i\int_D \bar{\partial}(f\partial|R|^2) \tag{5.8}$$

$$= 2i\int_D \bar{\partial}(f\bar{R}\partial R) \tag{5.9}$$

$$= 2i\int_D (\partial+\bar{\partial})(f\bar{R}\partial R) \tag{5.10}$$

$$= 2i\int_D d(f\bar{R}\partial R) \tag{5.11}$$

$$= 2i\int_{\partial D} f\bar{R}\partial R \tag{5.12}$$

$$= 2i\int_{\partial D} f S_{D,R}\partial R \tag{5.13}$$

$$= \Psi[f], \tag{5.14}$$

where 5.8 follows since f is holomorphic, 5.9 follows since $\bar{R}$ is antiholomorphic on D, 5.10 follows since $f\overline{R}\partial R$ is a $(1,0)$-form and thus $\partial(f\overline{R}\partial R)$ is zero, 5.12 is Stoke's theorem, and 5.13 follows since $S_{D,R}|_{\partial D} = \overline{R}|_{\partial D}$.

□

An important example of AQD's for R are given by the class of harmonic quadrature domains with data $(2i\partial\bar{\partial}|R|^2, \Psi)$, where

$$\Psi \equiv \sum_{i=1}^{n} c_i \delta_{a_i}, \tag{5.15}$$

where $\{a_i\}_{i=1}^n, \{c_i\}_{i=1}^n \subset \mathbb{C}$. We prove this below. Of particular importance is the case where $\Psi = t\delta_a$. This corresponds to Laplacian growth with an empty initial condition.

Proposition 5.2.5. *Let D be a bounded harmonic quadrature domain with piecewise C^1 boundary and data $(2i\partial\bar{\partial}|R|^2, \Psi)$, where Ψ is given by 5.15; then D is an AQD for R and*

$$S_{D,R} = \frac{1}{\partial R}\partial[\int_D G_{\delta_p}(\cdot, w)2i\partial\bar{\partial}|R|^2 - \sum_{i=1}^{n} c_i G_{\delta_p}(\cdot, a_i) + |R|^2], \tag{5.16}$$

where p is any point in D^c.

Proof. Let $p \in D^c$. We claim that

$$S_{D,R} = \frac{1}{\partial R}\partial[\int_D G_{\delta_p}(\cdot, w) 2i\partial\bar{\partial}|R|^2 - \sum_{i=1}^{n} c_i G_{\delta_p}(\cdot, a_i) + |R|^2], \tag{5.17}$$

is a Schwarz function for (D, R). Let S denote the right side of 5.17. We begin by showing

$$S|_{\partial D} = \bar{R}|_{\partial D}.$$

Let $z \in (\overline{D})^c$, then $G_{\delta_p}(z, \cdot)$ is harmonic on D. By the quadrature identity for D we have

$$\int_D G_{\delta_p}(z, w) 2i\partial\bar{\partial}|R|^2(w) = \Psi[G_{\delta_p}(z, \cdot)] \tag{5.18}$$

$$= \sum_{i=1}^{n} c_i G_{\delta_p}(z, a_i). \tag{5.19}$$

It then follows that

$$\partial \int_D G_{\delta_p}(z, w) 2i\partial\bar{\partial}|R|^2(w) = \partial \sum_{i=1}^{n} c_i G_{\delta_p}(z, a_i), \tag{5.20}$$

where $z \in (\overline{D})^c$ and ∂ acts on the first variable. By continuity and by hypothesis on the boundary it follows that 5.20 continues to hold for $z \in \partial D$. It then follows that

$$\begin{aligned} S|_{\partial D} &= \frac{1}{\partial R}\partial|R|^2|_{\partial D} \\ &= \frac{1}{\partial R}\bar{R}\partial R|_{\partial D} \\ &= \bar{R}|_{\partial D}. \end{aligned}$$

We now show that S is meromorphic on D. Since ∂R is a holomorphic one-form on D, from 5.17 it suffices to show that

$$\partial[\int_D G_{\delta_p}(\cdot, w) 2i\partial\bar{\partial}|R|^2(w) - \sum_{i=1}^{n} c_i G_{\delta_p}(\cdot, a_i) + |R|^2]$$

is a meromorphic one-form on D. To show this it suffices to show that

$$\int_D G_{\delta_p}(\cdot, w) 2i\partial\bar{\partial}|R|^2(w) - \sum_{i=1}^{n} c_i G_{\delta_p}(\cdot, a_i) + |R|^2 \tag{5.21}$$

is harmonic on D except at a discrete set of points where it has logarithmic poles. Since

$$\sum_{i=1}^{n} c_i G_{\delta_p}(\cdot, a_i)$$

is harmonic on D except at $\{a_i\}$ where it has logarithmic poles, it suffices to show that

$$\int_D G_{\delta_p}(\cdot, w) 2i\partial\bar{\partial}|R|^2(w) + |R|^2 \tag{5.22}$$

is harmonic on D. Let $\tilde{R} \in C^2(M)$ be an extension of $R|_D$. Then

$$\begin{aligned}
\int_D G_{\delta_p}(\cdot, w) 2i\partial\bar{\partial}|R|^2(w)|_D + |R|^2|_D &= \int_D G_{\delta_p}(\cdot, w) 2i\partial\bar{\partial}|\tilde{R}|^2(w)|_D + |\tilde{R}|^2|_D && (5.23)\\
&= \int_M G_{\delta_p}(\cdot, w) 2i\partial\bar{\partial}|\tilde{R}|^2(w)|_D - \int_{D^c} G_{\delta_p}(\cdot, w) 2i\partial\bar{\partial}|\tilde{R}|^2(w)|_D + |\tilde{R}|^2|_D && (5.24)\\
&= (|\tilde{R}|^2(p) - |\tilde{R}|^2)|_D - \int_{D^c} G_{\delta_p}(\cdot, w) 2i\partial\bar{\partial}|\tilde{R}|^2(w)|_D + |\tilde{R}|^2|_D && (5.25)\\
&= |\tilde{R}|^2(p) - \int_{D^c} G_{\delta_p}(\cdot, w) 2i\partial\bar{\partial}|\tilde{R}|^2(w)|_D, && (5.26)
\end{aligned}$$

where we used the distributional property of G_{δ_p} in 5.25. Since for $w \in D^c$, $G_{\delta_p}(\cdot, w)$ is harmonic on D, 5.26 is easily seen to be harmonic on D. This completes the proof.

□

The following lemma allows us to generate AQDs from classical quadrature domains.

Lemma 5.2.6. *Let $\Omega \subset \mathbb{C}$ be a classical analytic quadrature domain with linear functional Ψ. Let R be a meromorphic function on a compact Riemann surface M. Let D be a connected component of $R^{-1}(\Omega)$, then D is an AQD for R.*

Proof. Since Ω is a classical analytic quadrature domain, the domain Ω has a Schwarz function, S_Ω. We claim that $S_{D,R} = S_\Omega \circ R$. First observe that $S_\Omega \circ R$ is meromorphic, so it suffices to show that

$$S_\Omega \circ R|_{\partial D} = \overline{R}|_{\partial D}, \tag{5.27}$$

Let $z_* \in \partial D$, then $z_* \notin D$ and thus $R(z_*) \notin \Omega$. Also, $z_* \in \overline{D}$, and thus $R(z_*) \in \overline{R(D)}$ by the continuity of R on $\overline{D}$. So $R(z_*) \in \overline{D} \backslash D \equiv \partial D$. Identity 5.27 then follows immediately:

$$S_\Omega \circ R(z_*) = S_\Omega(R(z_*)) = \overline{R(z_*)}. \tag{5.28}$$

□

The importance of the previous lemma is that it provides an easy way to construct examples of AQDs since classical quadrature domains have been well studied. However, most AQDs cannot be generated in this fashion and the subject seems much richer than the classical setting.

If D possesses a Schwarz function, it actually possesses an infinite dimensional family of Schwarz functions as the following simple lemma shows.

Lemma 5.2.7. *Let D be an AQD for R and let T be a rational function with poles off $\overline{R(D)}$. Then D is an AQD for $T \circ R$ where the Schwarz function is given by*

$$S_{D,T\circ R} = \overline{T \circ \overline{S_{D,R}}}.$$

Proof. The function $\overline{T \circ \overline{S_{D,R}}}$ is clearly meromorphic, and

$$\overline{T \circ \overline{S_{D,R}}}|_{\partial D} = \overline{T \circ \overline{\overline{R}}}|_{\partial D} = \overline{T \circ R}|_{\partial D}.$$

Therefore,

$$S_{D,T\circ R} = \overline{T \circ \overline{S_{D,R}}}.$$

□

In the next proposition we provide a correspondence between *AQDs* and a special type of meromorphic functions on compact Riemann surfaces with real structure. The benefit of this correspondence is that there is no direct reference to the quadrature domain D —one can find AQDs of a connectivity c simply by finding such a meromorphic function on a Riemann surface N of genus $c - 1$ with real structure.

Proposition 5.2.8. *Let N be a compact Riemann surface with real structure given by an antiholomorphic involution τ. Let Γ denote the set of fixed points of τ. Assume that Γ divides N into two components and let E be a connected component of $N \backslash \Gamma$. Suppose there exists a meromorphic function ψ on N satisfying*

$$\psi|_{\overline{E}} = R \circ \phi|_{\overline{E}},$$

where $\phi : \overline{E} \to M$ is continuous, injective, and univalent on E and R is a meromorphic function on M with poles off $\overline{E}$. Then $\phi(E)$ is an AQD with for R. Conversely, if D is an AQD for R then there exists a compact Riemann surface N satisfying the hypotheses above and a meromorphic function ψ on N satisfying

$$\psi|_{\overline{E}} = R \circ \phi|_{\overline{E}},$$

where $\phi : \overline{E} \to \overline{D}$ is continuous, injective, and univalent on E.

Proof. Suppose we are given such a ψ, then the function

$$S \equiv \overline{\psi \circ \tau \circ \phi^{-1}},$$

is meromorphic on $\phi(E)$. If $z_* \in \partial\phi(E)$, then by the hypotheses on ϕ, $\phi^{-1}(z_*) \in \partial E$, and so

$$\begin{aligned} S(z_*) &= \overline{\psi \circ \tau \circ \phi^{-1}}(z_*) \\ &= \overline{\psi \circ \phi^{-1}}(z_*) \\ &= \overline{(R \circ \phi) \circ \phi^{-1}}(z_*) \\ &= \bar{R}(z_*). \end{aligned}$$

So $S|_{\partial\phi(E)} = \bar{R}|_{\partial\phi(E)}$. Therefore S is a Schwarz function for $\phi(E)$, and so $\phi(E)$ is an AQD for R.

We now prove the converse. Suppose D is an AQD for R. From the definition of an *AQD*, there exists a univalent map $\kappa : D \to E$, where E is a finite bordered Riemann surface which extends to a continuous injection from $\overline{D} \to \overline{E}$. Denote the inverse of κ by ϕ. Let $\tilde{E}$ be a copy of E and let $N \equiv E \sqcup \partial E \sqcup \tilde{E}$ be the Schottky double, then N satisfies the hypotheses in the proposition and

$$\psi(z) \equiv \begin{cases} R \circ \phi(z) & : z \in E \\ \overline{S_{D,R}} \circ \phi(z) & : z \in \tilde{E} \end{cases}$$

is a meromorphic function on N which satisfies

$$\psi|_{\overline{E}} = R \circ \phi,$$

and $\phi : \overline{E} \to \overline{D}$ is continuous, injective, and is univalent on D.

□

Proposition 5.2.9. *Let $D \subset M$ be an AQD for R. Let c denote the connectivity of D and let g denote the* *genus of M. Then there exists a compact Riemann surface N of genus gc and a meromorphic function T on N such that*

$$T \circ \phi = S_{D,R},$$

where $\phi : \overline{D} \to N$ is continuous, injective and univalent on D.

Proof. Since D is an AQD, there exists a finite bordered Riemann surface, E, and a continuous and injective map $\phi : \overline{D} \to \overline{E}$, which is univalent on D. The connectivity of E is also c. Let $\{B_i\}_{i=1}^{c}$, denote the connected components of $M\backslash E$. Consider the collection of open Riemann surfaces $\{M\backslash B_i\}$ with opposite conformal structure. Let N be the Riemann surface obtained by gluing each $M\backslash B_i$ to E along the boundary ∂E_i. Let N be the resulting compact surface with genus gc. Since the boundary of E is analytic, the complex structure of E extends in a neighborhood of the boundary which is consistent with the complex structures on each $\{M\backslash B_i\}$. This induces a complex structure on N. Let

$$T(z) \equiv \begin{cases} S_{D,R} \circ \phi^{-1} & : z \in E \\ \overline{R} \circ \phi^{-1} & : z \in M\backslash B_i,\ 1 \le i \le c. \end{cases}$$

Then T is meromorphic and $T \circ \phi = S_{D,R}$.

□

Corollary 5.2.10. *Let $D \subset \hat{\mathbb{C}}$ be an AQD for R. Then there exists a rational function T such that*

$$T \circ \phi = S_{D,R},$$

where $\phi : D \to \hat{\mathbb{C}}$ is univalent.

We use proposition 5.2.8 to characterize the boundaries of AQDs.

Proposition 5.2.11. *Let D be an AQD for R. Then there exists an irreducible polynomial $P \in \mathbb{C}[x, y]$, such that*

$$\partial D \subset \{P(R(z), \overline{R}(z)) = 0\}$$

and thus $R(\partial D)$ is a subset of an algebraic curve in $\mathbb{C}$.

Since D is an AQD, there exists a univalent map $\kappa : D \to E$, where E is a finite bordered Riemann surface which extends to a continuous injection from $\overline{D} \to \overline{E}$. Denote the inverse of κ by ϕ. Let $\tilde{E}$ be a copy of E and let $N \equiv E \sqcup \partial E \sqcup \tilde{E}$ be the Schottky double. On N we have the following functions:

Proof.

$$\psi(z) \equiv \begin{cases} R \circ \phi(z) & : z \in E \\ \overline{S_{D,R}} \circ \phi(z) & : z \in \tilde{E} \end{cases}$$

and

$$\tilde{\psi}(z) \equiv \begin{cases} S_{D,R} \circ \phi(z) & : z \in E \\ \bar{R} \circ \phi(z) & : z \in \tilde{E}. \end{cases}$$

We saw in prop 5.2.8 that ψ is meromorphic on N. An analogous argument shows that $\tilde{\psi}$ is meromorphic. Recall that the transcendence degree of the field of meromorphic functions on a compact Riemann Surface is one. It follows that there exists a polynomial P such that $P(\psi, \tilde{\psi}) = 0$. In particular, for $z \in \partial D = \phi(\partial E)$, we have $P(R(z), \bar{R}(z)) = 0$. It then follows that $R(\partial D) \subset \{P(z, \bar{z}) = 0\}$ and thus $R(\partial D)$ is a subset of an algebraic curve in $\mathbb{C}$.

□

5.3 Topology of Quadrature Domains

In this section we prove an upper bound on the connectivity of quadrature domains in terms of their degree. At the same time we prove a new upper bound on the number of special points of a quadrature domain as a function of its degree. In this section Ω will always denote a classical quadrature domain and n will denote its degree. As quadrature domains of degree one and two are completely classified, we restrict our attention to the case where $n \geq 3$. Recall that $\partial\Omega$ is subset of an algebraic curve, Γ, of degree $2n$. Moreover $\Gamma \backslash \partial\Omega$ consists of a finite number of points called special points. We denote the number of special points of Ω by s. The connectivity of Ω is defined to be the number of components in $\hat{\mathbb{C}} \backslash \Omega$. We denote the connectivity of Ω by c. In a beautiful paper, Lee and Makarov [24] proved, using quasiconformal surgery and complex dynamics, the following upper bound when $\Omega = int(\overline{\Omega})$ (and $n \geq 3$):

$$c \leq 2n - 4. \tag{5.29}$$

Moreover they constructed examples to show that 5.29 is sharp. In this section we eliminate the hypothesis $\Omega = int(\overline{\Omega})$ and prove (for $n \geq 3$) that

$$c \leq 5n - 5. \tag{5.30}$$

Special points are interesting for a number of reasons. It turns out that $z \in \Omega$ is a special point iff $\Omega \backslash \{z\}$ remains an analytic quadrature domain. That is Ω continues to satisfy the analytic quadrature identity for the new test function $f(z) \equiv \frac{1}{z-a}$. More interestingly, the special points are precisely the isolated points of the algebraic curve, Γ, which are contained in Ω. They have been well studied. In particular, there has been interest in providing upper and lower bounds. For estimates of special points, it makes sense (and we lose nothing) to assume $\Omega = int(\overline{\Omega})$ (so that there exist nontrivial lower bounds on s). We will do so from now on (we will eliminate the assumption later when we prove 5.30). Let b denote the number of cusps on $\partial\Omega$, c denote the connectivity of Ω, and d denote the number of double points of $\partial\Omega$. Gustafsson [13] proved

$$s \leq (n-1)^2 + 1 - c - b - 2d \tag{5.31}$$

and Sakai [27] proved

$$s \geq n - 2 + c - b, \tag{5.32}$$

see also McCarthy and Yang [25]. We prove

$$s \leq 5n - 5 - c. \tag{5.33}$$

Our proof makes use of Lemma 4.3 from [24]. The new ingredient is a result of D. Khavinson and G. Neumann [21]. They resolved an open problem in gravitational lensing by proving that for a rational function R of degree ≥ 3

$$|\{z \in \mathbb{C} \mid R(z) = \bar{z}\}| \leq 5n - 5, \tag{5.34}$$

where $|\cdot|$ denotes the cardinality. Before stating Lemma 4.3, we first need to define an important set $\hat{A} \subset \Omega$ used in the lemma. Let S denote the Schwarz function of Ω. Further assume that $\bar{S}$ has no critical value on $\partial\Omega$ (we will remove this assumption later). Let $K \equiv \hat{\mathbb{C}} \backslash \Omega$. Let

$$K = \bigsqcup_{i=1}^{c} K_i \tag{5.35}$$

be the decomposition of K into connectivity components. By the hypothesis on $\bar{S}$, each K_i is a closed Jordan domain (recall that we are assuming for now that $\Omega = int(\overline{\Omega})$). Let

$$\bar{S}^{-1}(K) \equiv \{z \in \overline{\Omega} : \bar{S}(z) \in K\}. \tag{5.36}$$

Then

$$\bar{S}^{-1}K = \bigsqcup_{i=1}^{c} \bar{S}^{-1}K_i. \tag{5.37}$$

We denote by A_i the component of $\bar{S}^{-1}K_i$ that contains ∂K_i. This component contains a set homeomorphic to an annulus such that ∂K_i is one of the boundary components of the annulus (see [24]). The set $\hat{A}_i$ is defined by filling in the holes of the annulus, that is

$$\hat{A}_i \equiv A_i \cup K_i. \tag{5.38}$$

$\hat{A}$ is defined to be the set

$$\hat{A} \equiv \bigcup_{i=1}^{c} \hat{A}_i. \tag{5.39}$$

In the case where $\bar{S}$ has critical values on $\partial\Omega$, $\hat{A}$ is defined similarly but with a small modification (see [24] section 4.3 for details). The following Lemma is from [24]:

Lemma 5.3.1. *(Lee-Makarov) There exists a branched covering map $G : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of degree n such that*

i $G = \bar{S}$ *on* $\hat{\mathbb{C}}\backslash\hat{A}$.

ii G is quasi-conformally equivalent to a rational map, i.e.

$$G = \Phi^{-1} \circ \bar{R} \circ \Phi, \tag{5.40}$$

for some rational map R and some orientation-preserving quasi-conformal homeomorphism $\Phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.

iii Each component, K_i of K contains a fixed point of G which attracts the orbits of all points of K_i.

Theorem 5.3.2. *Let Ω be a quadrature domain of degree $n \geq 3$ satisfying $\Omega = int(\overline{\Omega})$. Let s denote the number of special points and c denote the connectivity of Ω. Then*

$$s \leq 5n - 5 - c.$$

Proof. By Lemma 4.3 in [24], $\bar{S}|_{\hat{\mathbb{C}}\backslash\hat{A}} = \Phi^{-1} \circ \bar{R} \circ \Phi|_{\hat{\mathbb{C}}\backslash\hat{A}}$ for some rational map R and some orientation-preserving homeomorphism $\Phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. We claim that $\bar{S}$ has no fixed points on $\hat{A} \cap \Omega$. Indeed it suffices to show that $\bar{S}$ has no fixed points on $\hat{A}_i \cap \Omega$. Note that $\hat{A}_i \cap \Omega = A_i \backslash \partial K_i$, and by definition $\bar{S}(A_i) = K_i$ and thus

$\bar{S}(A_i \backslash \partial K_i) \subset K_i$. Now since $A_i \backslash \partial K_i \subset \Omega$ and $K_i \subset \Omega^c$, we have shown that $\bar{S}$ has no fixed points on $\hat{A}_i \cap \Omega$ and thus $\bar{S}$ has no fixed points on $\hat{A} \cap \Omega$.

By property (iii) from Lemma 5.3.1, $\Phi^{-1} \circ \bar{R} \circ \Phi$ has at least c fixed points on $\hat{A}$. We then have

$$\begin{aligned} s &= |\{z \in \Omega | S(z) = \bar{z}\}| && (5.41)\\ &= |\{z \in \Omega | \bar{S}(z) = z\}| && (5.42)\\ &= |\{z \in \hat{\mathbb{C}} \backslash \hat{A} | \bar{S}(z) = z\}| && (5.43)\\ &= |\{z \in \hat{\mathbb{C}} \backslash \hat{A} | \Phi^{-1} \circ \bar{R} \circ \Phi(z) = z\}| && (5.44)\\ &\leq |\{z \in \hat{\mathbb{C}} | \Phi^{-1} \circ \bar{R} \circ \Phi(z) = z\}| - c && (5.45)\\ &= |\{z \in \hat{\mathbb{C}} | \bar{R} = z\}| - c && (5.46)\\ &\leq 5n - 5 - c, && (5.47) \end{aligned}$$

where 5.43 follows since $\bar{S}$ has no fixed points on $\hat{A} \cap \Omega$ and $\hat{\mathbb{C}} \backslash \hat{A} = \Omega \backslash \hat{A}$. 5.45 follows since $\Phi^{-1} \circ \bar{R} \circ \Phi$ has at least c fixed points on $\hat{A}$. 5.46 follows since Φ is a bijection. 5.47 follows from the inequality 5.34 (Theorem 3 in [21]). □

We are now ready to prove 5.30.

Theorem 5.3.3. *Let Ω be a quadrature domain with degree $n \geq 3$. Let c denote the connectivity of Ω. Then*

$$c \leq 5n - 5. \tag{5.48}$$

Proof. Let $\Lambda \equiv int(\overline{\Omega})$, let $\tilde{c}$ denote the connectivity of Λ, and let $\tilde{s}$ denote the number of special points in Λ. Then $int(\overline{\Lambda}) = \Lambda$ and so by Theorem 5.3.2,

$$\tilde{s} \leq 5n - 5 - \tilde{c}. \tag{5.49}$$

Recall that $\partial\Omega$ is subset of an algebraic curve, Γ, of degree $2n$. Moreover $\Gamma \backslash \partial\Omega$ is the set of special points of Ω. Let S_Ω, S_Λ denote the sets of special points in Ω and Λ respectively. Note that $S_\Omega \subset S_\Lambda$ and that

$$|S_\Lambda \backslash S_\Omega| = |\partial\Omega \backslash \partial\Lambda|. \tag{5.50}$$

By hypothesis on Λ, $\partial\Lambda$ does not contain any special points. It then follows that

$$|\partial\Omega \backslash \partial\Lambda| \leq |\Gamma \backslash \partial\Lambda| = \tilde{s}. \tag{5.51}$$

Note that Ω is obtained from Λ by removing from Λ the set of special points in $\Lambda \backslash \Omega$. That is

$$\Omega = \Lambda \backslash \{S_\Lambda \backslash S_\Omega\}. \tag{5.52}$$

Since removing a point from Ω increases the connectivity by one, it follows that $c - \tilde{c} = |\{S_\Lambda \backslash S_\Omega\}|$. Putting everything together we have:

$$\begin{aligned} c &= |\{S_\Lambda \backslash S_\Omega\}| + \tilde{c} && (5.53) \\ &= |\partial\Omega \backslash \partial\Lambda| + \tilde{c} && (5.54) \\ &\leq \tilde{s} + \tilde{c} && (5.55) \\ &\leq 5n - 5, && (5.56) \end{aligned}$$

where 5.54 follows from 5.50, 5.55 follows from 5.51, and 5.56 follows from Theorem 5.3.2. $\square$

Chapter 6

Coulomb Gas Ensembles and CFT

6.1 Introduction

Let $Q : \mathbb{C} \to \mathbb{R} \cup \{\infty\}$ and $\beta > 0$. The following sequence of probability measures

$$\Pi_n \equiv \frac{1}{Z_n} \int_{\mathbb{C}^n} e^{-\beta H_n} dA^{\otimes n}, \tag{6.1}$$

where

$$H_n(z_1, ..., z_n) \equiv \sum_{1 \leq i < j \leq n} \log \frac{1}{|z_i - z_j|} + (n-1) \sum_{i=1}^{n} Q(z_i),$$

and Z_n is a normalizing constant is called a Coulomb gas ensemble or β-ensemble. Coulomb gas ensembles occur frequently in random matrix theory. For example, when $\beta = 2$ they are the eigenvalue distributions of certain classes of random normal matrices. There is also a connection to conformal field theory. The internal energy component of the integrand of 6.1 is related to the vacuum expectation of vertex operators. Coulomb gas ensembles generalize nicely to Riemann surfaces. Let

$$H_n^{\nu,Q}(z_1, ..., z_n) \equiv \sum_{1 \leq i < j \leq n} [G_\nu(z_i, z_j) + Q(z_i) + Q(z_j)].$$

Physically, H can be interpreted as the energy of configuration of n unit point charges placed at $\{z_1, ..., z_n\}$, in the presence of an external field $(n-1)Q$. We define the partition function

$$Z_n^{Q,\nu,\beta} \equiv \int_{M^n} e^{-\beta H_n} dvol_g^{\otimes n}.$$

In analogy with statistical mechanics, the Boltzmann Gibbs measure corresponding to the energy function H

is defined to be

$$\Pi_n \equiv \frac{1}{Z_n} e^{-\beta H_n} dvol_g^{\otimes n}. \tag{6.2}$$

For certain ν, $G_\nu(z,w) = \mathbb{E}[\Phi(z)\Phi(w)]$, where Φ is a Free Bosonic Field on M. Let $k \leq n$, and let $A \subset M^k$. The k-th marginal measure of Π_n is defined by

$$\Pi_{n,k}(A) \equiv \Pi_n(A \times M^{n-k}).$$

The main result of this section is:

Theorem 6.3.3. *Let Q be an admissible potential. If $Q_{\nu,g}$ is continuous then*

$$\Pi_{n,k} \rightharpoonup \mu_{eq}^{\otimes k}$$

as $n \to \infty$.

Johansson [18] proved this result on $\mathbb{R}$ and Hedenmalm and Makarov [15] later proved the analogue in the complex plane. Our proof follows the general structure of Johansson's although some portions are more technical in this setting (e.g. Lemma 6.3.1). The proof follows the structure of Johansson's [18] proof of the convergence of the marginal measures for β-ensembles on $\mathbb{R}$. Hedenmalm and Makarov [15] later proved the analogue in the plane.

We then show that the Bosonic free field on the cylinder can be realized as a limit of Fluctuations of a Coulomb gas ensemble on the cylinder.

6.2 Fekete Points

In this section we consider a discrete analogue of the setting in chapter three. More precisely, we concern ourselves only with probability measures of the form $\frac{1}{n}\sum_{j=1}^{n}\delta_{z_i}$ where $\{z_i\}_{i=1}^{n} \subset M$ are distinct points. Letting $z = (z_1, ..., z_n)$, we define

$$\mu_z \equiv \frac{1}{n}\sum_{i=1}^{n}\delta_{z_i}. \tag{6.3}$$

The set of probability measures given in 6.3 is denoted by $\mathbb{P}_n(M)$. All measures in $\mathbb{P}_n(M)$ have infinite energy, so we introduce a regularized energy energy functional:

$$I^*[\mu] \equiv \int_{(M\times M)\setminus\Delta} K_\nu^Q d\mu^{\otimes 2}. \tag{6.4}$$

This removes the portion of the measure $\mu_z^{\otimes 2}$ on the diagonal where G_ν diverges. It will be convenient to introduce the function

$$H_n^{\nu,Q}(z_1,...,z_n) \equiv \sum_{1\le i<j\le n} K^{\nu,Q}(z_i,z_j) = \sum_{1\le i<j\le n} G_\nu(z_i,z_j) + (n-1)\sum_{j=1}^{n} Q(z_j).$$

In what follows we suppress in the notation the dependence of H on ν and Q. From Equation 6.4 we have

$$H_n(z_1,...,z_n) = \frac{n^2}{2}\int_{(M\times M)\setminus\Delta} K^{\nu,Q} d\mu_z^{\otimes 2} = \frac{n^2}{2} I^*[\mu_z]. \tag{6.5}$$

Recall that the admissibility of Q implies that

$$K^{\nu,Q}(z,w) \equiv G_\nu(z,w) + Q(z) + Q(w)$$

is lower semi-continuous on $M\times M$. Therefore it follows that H_n is lower semi-continuous on M^n. Also note that since G_ν is symmetric, so is H.

In an earlier chapter we studied the equilibrium measure —the measure in $\mathbb{P}(M)$ minimizing the energy functional I. In this section, we consider the problem of minimizing the energy functional, I^*, over $\mathbb{P}_n(M) \subset \mathbb{P}(M)$. By Equation 6.5, the function $I^*[\mu_z]$ is lower semi-continuous on the compact set M^n. There is thus a global minimum $p_n = (z_1^{(n)},...,z_n^{(n)}) \in M^n$. $\{z_1^{(n)},...,z_n^{(n)}\}$ are called Fekete points. For convenience we define $\mu_n \equiv \mu_{p_n}$, and we call

$$\{\mu_n\}_{n\in\mathbb{N}},$$

the Fekete measures. Note that even though $\mathbb{P}_n(M) \subset \mathbb{P}(M)$, it does not follow that $I^*[\mu_n] \ge I^*[\mu_{eq}]$, since μ_{eq} does not minimize energy over all probability measures if self energy is excluded —the equilibrium measure in that case would simply be a unit point mass at any point in M. In fact the following lemma shows that the opposite is true:

Lemma 6.2.1. *Let $\mu \in \mathbb{P}(M)$. Then for any $n \in \mathbb{Z}_+$*

$$I[\mu] \ge \frac{n}{n-1} I^*[\mu_n].$$

Proof.

$$I[\mu] = \frac{2}{n(n-1)} \int_{M^n} H_n d\mu^{\otimes n} \tag{6.6}$$

$$\geq \frac{2}{n(n-1)} H(p_n)\mu^{\otimes n}(M^n) \tag{6.7}$$

$$= \frac{n}{(n-1)} I^*[\mu_n], \tag{6.8}$$

where 6.8 is due to equation 6.5, and the fact that $\mu^{\otimes n}(M^n) = (\mu(M))^n = 1$, since $\mu \in \mathbb{P}(M)$. □

In the next lemma, we will show that $\lim_{n\to\infty} I[\mu_n] = V$. In what follows $\Delta \subset M^n$ denotes the set

$$\Delta \equiv \{(z_1, ..., z_n) \in M^n \mid z_i \neq z_j \, \forall i \neq j\}.$$

Lemma 6.2.2. *If $n > m > 1$, and $\mu \in P_n(M)$, then*

$$I^*[\mu] \geq \frac{m(n-1)}{n(m-1)} I^*[\mu_m].$$

Proof. First observe that

$$\int_{M^n\setminus\Delta} H_m d\mu^{\otimes n} = m(m-1) \int_{M^n\setminus\Delta} K^{\nu,Q}(z_1, z_2) d\mu(z_1) d\mu(z_2) d\mu^{\otimes n-2} \tag{6.9}$$

$$= m(m-1) \frac{(n-2)!}{(n-m)!} n^{2-m} \int_{M^2\setminus\Delta} K^{\nu,Q} d\mu^{\otimes 2} \tag{6.10}$$

$$= m(m-1) \frac{(n-2)!}{(n-m)!} n^{2-m} I^*[\mu], \tag{6.11}$$

where 6.9 is due to the symmetry of the domain and integration measure and 6.10 is an easy combinatorial argument. By another easy combinatorial argument we have

$$\mu^{\otimes m}(M^m\setminus\Delta) = \frac{n!}{(n-m)!} n^{-m}. \tag{6.12}$$

Putting this together yields:

$$I^*[\mu] = (m(m-1) \frac{(n-2)!}{(n-m)!} n^{2-m})^{-1} \int_{M^n\setminus\Delta} H_m d\mu^{\otimes n} \tag{6.13}$$

$$\geq (m(m-1) \frac{(n-2)!}{(n-m)!} n^{2-m})^{-1} [H_m(p_m)\mu^{\otimes n}(M^n\setminus\Delta)] \tag{6.14}$$

$$= \frac{m(n-1)}{n(m-1)} I^*[\mu_m], \tag{6.15}$$

where 6.13 follows from 6.9-6.11, and 6.15 follows from 6.12. □

It is interesting, but perhaps not surprising, that the Fekete measures converge to the equilibrium measure.

Theorem 6.2.3. *Let Q be an admissible external field. Then*

$$\mu_n \rightharpoonup \mu_{eq} \tag{6.16}$$

and

$$I^*[\mu_n] \to I[\mu_{eq}], \tag{6.17}$$

as $n \to \infty$.

Proof. Let $\{\mu_{n_i}\}_{i\in\mathbb{N}}$ be an arbitrary subsequence of Fekete measures. Since M is compact, by Prokhorov's theorem there exists a weakly convergent subsequence of $\{\mu_{n_i}\}_{i\in\mathbb{N}}$ which converges to a probability measure μ. By relabeling if necessary, we denote this weakly convergent subsequence by $\{\mu_{n_i}\}_{i\in\mathbb{N}}$. Observe that to prove 6.16, it suffices to show that $\mu = \mu_{eq}$. By Lemma 6.2.1

$$I[\mu_{eq}] \geq \frac{n}{(n-1)} I^*[\mu_n].$$

Fix $N \in \mathbb{N}$ and let $\wedge$ denote the minimum. Observe the following series of inequalities (the reason for using lim inf will be apparent in a moment):

$$\begin{aligned}
I[\mu_{eq}] &\geq \limsup_i \frac{n_i}{(n_i-1)} I^*[\mu_{n_i}] && (6.18)\\
&= \limsup_i I^*[\mu_{n_i}] && (6.19)\\
&\geq \liminf_i I^*[\mu_{n_i}] && (6.20)\\
&\geq \liminf_i \int_{(M\times M)\setminus\Delta} K^{\nu,Q} \wedge N d\mu_{n_i}^{\otimes 2} && (6.21)\\
&= \liminf_i \int_{M\times M} K^{\nu,Q} \wedge N d\mu_{n_i}^{\otimes 2} - \frac{N}{n_i} && (6.22)\\
&= \liminf_i \int_{M\times M} K^{\nu,Q} \wedge N d\mu_{n_i}^{\otimes 2} && (6.23)\\
&= \int_{M\times M} K^{\nu,Q} \wedge N d\mu^{\otimes 2} && (6.24)
\end{aligned}$$

where 6.18 follows from Lemma 6.2.1 and 6.22 follows from the fact that $K_\nu^Q|_\Delta = \infty$. We then have

$$
\begin{aligned}
I[\mu_{eq}] &\geq \limsup_i I^*[\mu_{n_i}] && (6.25)\\
&\geq \liminf_i I^*[\mu_{n_i}] && (6.26)\\
&\geq \limsup_N \int_{M\times M} K_\nu^Q \wedge N d\mu^{\otimes 2} && (6.27)\\
&\geq \int_{M\times M} \liminf_N K_\nu^Q \wedge N d\mu^{\otimes 2} && (6.28)\\
&= \int_{M\times M} K_\nu^Q d\mu^{\otimes 2} && (6.29)\\
&\equiv I[\mu] && (6.30)
\end{aligned}
$$

where 6.27 follows from 6.18-6.24. Inequality 6.28 follows from Fatou's lemma (noting that $K_\nu^Q \wedge N$ is bounded from below uniformly in N). It then follows from 6.30 and the definition of the equilibrium measure that $I[\mu_{eq}] = I[\mu]$ and $\mu = \mu_{eq}$ which proves 6.16. Moreover, we have shown that

$$
\begin{aligned}
I[\mu_{eq}] &\geq \limsup_i I^*[\mu_{n_i}]\\
&\geq \liminf_i I^*[\mu_{n_i}]\\
&\geq I[\mu]\\
&\geq I[\mu_{eq}],
\end{aligned}
$$

hence $\lim_i I^*[\mu_{n_i}] = I[\mu_{eq}]$. Since $\{\mu_{n_i}\}_{i\in\mathbb{N}}$ is an arbitrary subsequence of Fekete measures we have proven 6.17.

□

6.3 The Boltzmann Gibbs Distribution

We introduce a probability measure on M^n called the Boltzmann Gibbs distribution which is fundamental in statistical mechanics. These distributions are parameterized by a positive real number β called the inverse temperature which will remain fixed throughout our discussion. We define the partition function

$$
Z_n^{Q,\nu,\beta} \equiv \int_{M^n} e^{-\beta H_n^{\nu,Q}} dvol_g^{\otimes n}.
$$

The Boltzmann Gibbs measure is then defined to be

$$
\Pi_n^{Q,\nu,\beta} \equiv \frac{1}{Z_n} e^{-\beta H_n^{\nu,Q}} vol_g^{\otimes n}.
$$

In what follows we will suppress the β,ν and Q dependence in the notation for Z and Π unless it is needed. Before proceeding, we prove a technical lemma which will be used later.

Lemma 6.3.1. *Let μ be a finite positive Borel measure with finite energy. Then there exists a family of absolutely continuous finite positive Borel measures $\{\mu_\delta\}_{\delta>0}$ such that the Radon-Nikodym derivative of μ_δ with respect to vol_g is continuous, $\mu_\delta(M) = \mu(M)$,*

$$\mu_\delta \rightharpoonup \mu. \tag{6.31}$$

Moreover suppose that $Q_{\nu,g}$ is continuous in a neighborhood of the support of μ. Then

$$I[\mu_\delta] \to I[\mu] \tag{6.32}$$

as $\delta \to 0$.

Proof. For notational convenience we will suppress the metric dependence and denote the geodesic ball centered at x with radius δ simply as $B(x,\delta)$. We will denote its volume with respect to the vol_g by $|B(x,\delta)|$. Let

$$\Psi_\delta(z) \equiv \int_{B(z,\delta)} (|B(w,\delta)|)^{-1} d\mu(w).$$

We note that Ψ is clearly continuous. We define

$$\mu_\delta \equiv \Psi_\delta vol_g.$$

We note that μ_δ has finite energy since $\mu_\delta < C_\delta vol_g$ and vol_g has finite energy. Note that

$$\begin{aligned}
\mu_\delta(M) &= \int_M \Psi_\delta vol_g && (6.33)\\
&= \int_M [\int_{B(z,\delta)} (|B(w,\delta)|)^{-1} d\mu(w)] dvol_g(z) && (6.34)\\
&= \int_M [\int_M 1_{B(z,\delta)}(w)(|B(w,\delta)|)^{-1} d\mu(w)] dvol_g(z) && (6.35)\\
&= \int_M [\int_M 1_{B(w,\delta)}(z)(|B(w,\delta)|)^{-1} d\mu(w)] dvol_g(z) && (6.36)\\
&= \int_M [\int_M 1_{B(w,\delta)}(z)(|B(w,\delta)|)^{-1} dvol_g(z)] d\mu(w) && (6.37)\\
&= \int_M d\mu(w) && (6.38)\\
&= \mu(M), && (6.39)
\end{aligned}$$

where equation 6.36 used the fact the simple identity $1_{B(z,\delta)}(w) = 1_{B(w,\delta)}(z)$, and 6.38 follows since the inner bracket is identically one. We now show

$$\mu_\delta \rightharpoonup \mu, \tag{6.40}$$

as $\delta \to 0$. Let $f \in C(M)$. Fix an $\epsilon > 0$, by uniform continuity, there exists a $\delta_0 > 0$ such that for all $w, z \in M$, $d(w,z) < \delta_0 \implies |f(w) - f(z)| < \epsilon$. For $\delta < \delta_0$,

$$|\mu_\delta(f) - \mu(f)| = |\int_M [\int_{B(z,\delta)} (|B(w,\delta)|)^{-1} d\mu(w)] f(z) dvol_g(z) - \int_M f(w) d\mu(w)| \tag{6.41}$$

$$= |\int_M [\int_M f(z) 1_{B(z,\delta)}(w)(|B(w,\delta)|)^{-1} dvol_g(z)] d\mu(w) - \int_M f(w) d\mu(w)| \tag{6.42}$$

$$= |\int_M \int_M (f(z) - f(w)) 1_{B(z,\delta)}(w)(|B(w,\delta)|)^{-1} dvol_g(z) d\mu(w)| \tag{6.43}$$

$$\leq \int_M \int_M |f(z) - f(w)| 1_{B(z,\delta)}(w)(|B(w,\delta)|)^{-1} dvol_g(z) d\mu(w) \tag{6.44}$$

$$\leq \epsilon \int_M [\int_M 1_{B(z,\delta)}(w)(|B(w,\delta)|)^{-1} dvol_g(z)] d\mu(w) \tag{6.45}$$

$$= \epsilon \mu(M), \tag{6.46}$$

where to obtain 6.42, we used $1_{B(z,\delta)}(w) = 1_{B(w,\delta)}(z)$ to bring f inside the inner integral. This completes the proof of 6.31. We are now ready to show 6.32.

$$I[\mu_\delta] - I[\mu] = \int_{M\times M} K_\nu^Q d\mu_\delta^{\otimes 2} - \int_{M\times M} K_\nu^Q d\mu^{\otimes 2} \tag{6.47}$$

$$\int_{M\times M} [G_g(z,w) + Q_{\nu,g}(z) + Q_{\nu,g}(w)] d\mu_\delta(z) d\mu_\delta(w) - \int_{M\times M} [G_g(z,w) + Q_{nu,g}(z) + Q_{\nu,g}(w)] d\mu(z) d\mu(w) \tag{6.48}$$

$$= [\int_{M\times M} G_g(z,w) d\mu_\delta(z) d\mu_\delta(w) - \int_{M\times M} G_g(z,w) d\mu(z) d\mu(w)] + 2\mu_\delta(M) \int_M Q_{\nu,g} d\mu_\delta - 2\mu(M) \int_M Q_{\nu,g} d\mu \tag{6.49}$$

$$= [\int_{M\times M} G_g d\mu_\delta^{\otimes 2} - \int_{M\times M} G_g d\mu^{\otimes 2}] + 2\mu(M)[\int_M Q_{\nu,g} d\mu_\delta - \int_M Q_{\nu,g} d\mu], \tag{6.50}$$

where 6.50 uses the fact that $\mu_\delta(M) = \mu(M)$. Note that since μ and μ_δ have finite energy we can split the integrand in 6.49. As we have shown that $\mu_\delta \rightharpoonup \mu$ as $\delta \to 0^+$, we have

$$\lim_{\delta\to 0^+} 2\mu(M)[\int_M Q_{v,g} d\mu_\delta - \int_M Q_{v,g} d\mu] = 0,$$

thus it suffices to prove that

$$\lim_{\delta\to 0^+} \int_{M\times M} G_g d\mu_\delta^{\otimes 2} - \int_{M\times M} G_g d\mu^{\otimes 2} = 0. \tag{6.51}$$

By an argument similar to that used in equations 6.41-6.46 we have

$$\int_{M\times M} G_g d\mu_\delta^{\otimes 2} - \int_{M\times M} G_g d\mu^{\otimes 2} \tag{6.52}$$

$$= \int_{M\times M} [\int_{M\times M} (G_g(z,w) - G_g(x,y)) \frac{1_{B(z,\delta)}(x)}{|B(x,\delta)|} \frac{1_{B(w,\delta)}(y)}{|B(y,\delta)|} dvol_g(z) dvol_g(w)] d\mu(x) d\mu(y). \tag{6.53}$$

Let

$$\Phi_\delta(x,y) = \int_{M\times M} (G_g(z,w) - G_g(x,y)) \frac{1_{B(z,\delta)}(x)}{|B(x,\delta)|} \frac{1_{B(w,\delta)}(y)}{|B(y,\delta)|} dvol_g(z) dvol_g(w) \tag{6.54}$$

denote the integrand in 6.54. Notice that

$$\frac{\Phi_\delta}{|G_g| + 1}$$

is uniformly bounded. Moreover, $\lim_\delta \Phi_\delta = 0$. Since μ has finite energy, $|G|_g + 1$ is integrable, and thus by the dominated convergence theorem

$$\lim_{\delta\to 0^+} \int_{M\times M} G_g d\mu_\delta^{\otimes 2} - \int_{M\times M} G_g d\mu^{\otimes 2}$$
$$= \lim_{\delta\to 0^+} \int_{M\times M} \Phi_\delta d\mu^{\otimes 2} = 0.$$

The proof is complete.

□

Proposition 6.3.2. *Existence of free energy on Riemann surfaces*
Let Q be an admissible potential and let $Q_{v,g}$ be continuous in a neighborhood of S. Then

$$\lim_{n\to\infty} -\frac{1}{n^2} \log Z_n^\beta = \frac{\beta}{2} V.$$

Proof. Since Q is admissible, the equilibrium measure μ_{eq} exists. Since by hypothesis $Q_{v,g}$ is continuous in a neighborhood of S, we can apply Lemma 6.3.1 to obtain a family of measures $\{\mu_\delta\}_{\delta>0}$ satisfying 6.31 and 6.32 where $\mu_\delta = \Psi_\delta vol_g$. Let $E_\delta \equiv \{\Psi_\delta > 0\}$.

$$Z_n^\beta \equiv \int_{M^n} e^{-\beta H_n} dvol_g^{\otimes n} \tag{6.55}$$

$$\geq \int_{E_\delta^n} e^{-\beta H_n} dvol_g^{\otimes n} \tag{6.56}$$

$$= \int_{E_\delta^n} e^{-\beta H_n(z) - \sum_{i=1}^n \log(\Psi_\delta(z_i))} \prod_{i=1}^n \Psi_\delta(z_i) dvol_g(z_1)...dvol_g(z_n). \tag{6.57}$$

By Jenson's inequality we have

$$\log Z_n^\beta \geq \int_{E_\delta^n} [-\beta H_n(z) - \sum_{i=1}^n \log(\Psi_\delta(z_i))] \prod_{i=1}^n \Psi_\delta(z_i) dvol_g(z_1)...dvol_g(z_n)$$

$$= -\frac{n(n-1)}{2}\beta \int_{E_\delta^2} K_v^Q(z,w)\Psi_\delta(z) dvol_g(z)\Psi_\delta(w) dvol_g(w) - n \int_{E_\delta} \log(\Psi_\delta(z))\Psi_\delta(z) dvol_g(z).$$

We then have

$$\limsup_n \frac{-1}{n^2} \log Z_n^\beta \leq \frac{\beta}{2} I[\Psi_\delta vol_g]. \tag{6.58}$$

Since by hypothesis

$$\lim_{\delta\to 0^+} I[\Psi_\delta vol_g] = I[\mu_{eq}]$$

we obtain:

$$\limsup_n \frac{-1}{n^2} \log Z_n^\beta \leq \lim_{\delta\to 0^+} \frac{\beta}{2} I[\Psi_\delta vol_g] \tag{6.59}$$

$$= \frac{\beta}{2} I[\mu_{eq}] \tag{6.60}$$

$$\equiv \frac{\beta}{2} V. \tag{6.61}$$

On the other hand,

$$Z_n^\beta \leq |M|_g^n \sup_{z\in M^n} e^{-\beta H_n(z)} = |M|_g^n exp(-\beta \frac{n^2}{2} I^*[\mu_n]),$$

We then have

$$\liminf_n \frac{-1}{n^2} \log Z_n^\beta \geq \frac{\beta}{2} \lim_{n\to\infty} I^*[\mu_n] = \frac{\beta}{2} V, \tag{6.62}$$

where the equality above is due to Proposition 6.2.3. By 6.61 and 6.62,

$$\lim_{n\to\infty} \frac{-1}{n^2} \log Z_n^{\beta} = \frac{\beta}{2} V. \tag{6.63}$$

The proof is complete.

□

Recall that the k-th marginal measure of Π_n is defined by

$$\Pi_{n,k}(A) = \Pi_n(A \times M^{n-k}).$$

In Proposition 6.2.3 we showed that the Fekete measures converge weakly to the equilibrium measure. We now prove that the k-th marginal measures converge weakly to the the $\mu_{eq}^{\otimes k}$. This is an analogue of the well-known theorem by Johansson in $\mathbb{R}$ ([18]).

Theorem 6.3.3. *Johansson's marginal measure theorem on Riemann surfaces*
Let Q be admissible and let $Q_{v,g}$ be continuous in a neighborhood of S. Then

$$\Pi_{n,k} \rightharpoonup \mu_{eq}^{\otimes k},$$

as $n \to \infty$.

Proof. Let $A_{n,\eta} \equiv \{z \in M^n | \frac{2}{n^2} H_n(z) \leq V + \frac{\eta}{n}\}$ for $\eta > 0$. As H_n is lower semi-continuous $A_{n,\eta}$ is closed, and thus compact.
By Proposition 6.3.2 , for a fixed $\eta > 0$, there exists N such that $n > N$ implies

$$Z_n^{\beta} \leq e^{-n^2 \frac{\beta}{2}(V - \frac{\eta}{n})}. \tag{6.64}$$

Then using 6.64 and the definition of $A_{n,\eta}$ we have:

$$\Pi_n[A_{n,\eta}^c] \equiv \frac{1}{Z_n^{\beta}} \int_{A_{n,\eta}^c} e^{-\beta H_n} dvol_g^{\otimes n} \leq e^{n^2 \frac{\beta}{2}(V - \frac{\eta}{n})} e^{-n^2 \frac{\beta}{2}(V + \frac{\eta}{n})} \int_{R^n} dvol_g^{\otimes n} = e^{-n\beta\eta} |M|_g^n. \tag{6.65}$$

Let $f \in C(M^k)$. We define the n-symmetrization of f, $Sym_{n,k}[f] : M^n \to \mathbb{R}$, by

$$Sym_{n,k}[f](z_1, ..., z_n) = \frac{1}{n^k} \sum_{1 \leq i_1,..,i_k \leq n} f(z_{i_1}, ..., z_{i_k}).$$

By the symmetry of the measure Π_n, we have

$$\Pi_{n,k}[f] = \Pi_n[Sym_{n,k}(f)]. \tag{6.66}$$

Choose $\eta \geq \frac{1}{\beta}(\log |M|_g + 1)$, then from 6.65:

$$\Pi_n[A^c_{n,\eta}] \leq e^{-n}. \tag{6.67}$$

It is easy to verify that

$$\|Sym_{n,k}[f]\|_\infty = \|f\|_\infty. \tag{6.68}$$

It follows from 6.67 and 6.68, that

$$\lim_{n\to\infty} \Pi_n[Sym_{n,k}(f)1_{A^c_{n,\eta}}] = 0, \tag{6.69}$$

so from 6.66 and 6.69, the proof is complete once we show:

$$\lim_{n\to\infty} \Pi_n[Sym_{n,k}(f)1_{A_{n,\eta}}] = \int_{M^k} f d\mu_{eq}^{\otimes k}. \tag{6.70}$$

The remaining portion of the proof is dedicated to proving 6.70. We argued above that $A_{n,\eta}$ is compact, and thus so is $A^n_{n,\eta}$. Since $Sym_{n,k}(f)$ is continuous, it attains its maximum and minimum on $A^n_{n,\eta}$. Let $a_{n,\eta}$ and $b_{n,\eta}$ denote such points. Recall for a $z \in M^n$, we defined μ_z to be:

$$\mu_z \equiv \frac{1}{n}\sum_{i=1}^{n} \delta_{z_i}. \tag{6.71}$$

Let

$$\mu^{(n,\eta)}_{max} \equiv \mu_{a_{n,\eta}}, \tag{6.72}$$

$$\mu^{(n,\eta)}_{min} \equiv \mu_{b_{n,\eta}}. \tag{6.73}$$

From the definitions of $\mu^{(n,\eta)}_{max}$ and $\mu^{(n,\eta)}_{min}$ and the symmetry of $Sym_{n,k}(f)$ we have:

$$\int_{M^k} f d(\mu^{n,\eta}_{min})^{\otimes k} = \int_{M^n} Sym_{n,k}(f) d(\mu^{n,\eta}_{min})^{\otimes n} \leq \Pi_n[Sym_{n,k}(f)1_{A_{n,\eta}}] \leq \int_{M^n} Sym_{n,k}(f) d(\mu^{n,\eta}_{max})^{\otimes n} = \int_{M^n} f d(\mu^{n,\eta}_{max})^{\otimes k}. \tag{6.74}$$

It thus suffices to show

$$(\mu_{max}^{(n,\eta)})^{\otimes k} \rightharpoonup \mu_{eq}^{\otimes k} \tag{6.75}$$

as $n \to \infty$ and similarly for $(\mu_{min}^{(n,\eta)})^{\otimes k}$. The proof for $\mu_{\min}$ is identical, so we proceed to show 6.75. To show 6.75, it suffices to simply show that $\mu_{max}^{(n,\eta)} \rightharpoonup \mu_{eq}$ as $n \to \infty$. It then clearly suffices to show that for any sequence of measures $\{\mu_{z_n}\}_{n\in\mathbb{N}}$ where $z_n \in A_{n,\eta}$ we have $\mu_{z_n} \rightharpoonup \mu_{eq}$ as $n \to \infty$ (since $\mu_{\max}^{n,\eta} \in \mathbb{P}_n(A_{n,\eta})$).
Let $\{\mu_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of measures with $\mu_n \in \mathbb{P}(A_{n,\eta})$. It suffices to show, that for any subsequence $\{\mu_{n_i}\}_{i\in\mathbb{N}}$, there exists a further subsequence which converges weakly to μ_{eq}. As M is compact, by Prokhorov's theorem there exists a convergent subsequence of $\{\mu_{n_i}\}_{i\in\mathbb{N}}$ converging to a probability measure μ. By relabeling if necessary, we denote this further subsequence simply by $\{\mu_{n_i}\}_{i\in\mathbb{N}}$. We will show that $\mu = \mu_{eq}$. Let $N \in \mathbb{Z}_+$. Now as $\mu_{n_i} \rightharpoonup \mu$ as $i \to \infty$ and

$$\int_{M\times M} K_\nu^Q \wedge N d\mu^{\otimes 2} = \lim_{i\to\infty} \int_{M\times M} K_\nu^Q \wedge N d\mu_{n_i}^{\otimes 2} \tag{6.76}$$

$$= \lim_{i\to\infty} \int_{M\times M\setminus\Delta} K_\nu^Q \wedge N d\mu_{n_i}^{\otimes 2} + \frac{N}{n_i} \tag{6.77}$$

$$= \lim_{i\to\infty} \frac{1}{n_i^2} H_{n_i}(z_{n_i}) + \frac{N}{n_i} \tag{6.78}$$

$$\leq \lim_{i\to\infty} V + \frac{1}{n_i} + \frac{N}{n_i} = V, \tag{6.79}$$

by the monotone convergence theorem

$$\int_{M\times M} K_\nu^Q d\mu^{\otimes 2} = \lim_{N\to\infty} \int_{M\times M} K_\nu^Q \wedge N d\mu^{\otimes 2} \leq V \tag{6.80}$$

which shows that $I[\mu] \leq V$ and thus $\mu = \mu_{eq}$, completing the proof.

□

6.4 The Free Bosonic Field

Let M be a closed orientable surface with a fixed differentiable structure. For each metric, g on M, we associate an action functional

$$S_g : L^2(N, g) \to \mathbb{R}.$$

In what follows M will remain fixed as a reference space but we will not fix a Riemannian structure on M. When M is endowed with a Riemannian structure by specifying a metric g, we will sometimes denote this by the pair (M, g).

6.4.1 Conformal Structure

Given metric g, we define the angle between $v_1, v_2 \in TM$ as

$$\cos^{-1}\left(\frac{g(v_1, v_2)}{|v_1|_g |v_2|_g}\right)$$

where the branch of $\cos^{-1}$ is chosen in the obvious way depending on the orientation of $\{v_1, v_2\}$. A metric $\tilde{g}$ is called conformally equivalent to g if it gives rise to the same angles, that is,

$$\cos^{-1}\left(\frac{\tilde{g}(v_1, v_2)}{|v_1|_{\tilde{g}} |v_2|_{\tilde{g}}}\right) = \cos^{-1}\left(\frac{g(v_1, v_2)}{|v_1|_g |v_2|_g}\right)$$

for every $v_1, v_2 \in TM$. Two inner products generate the same angles iff they are proportional, and so $\tilde{g}$ is conformally equivalent to g iff $\tilde{g} = e^f g$ for some $f \in C(M)$. Given a metric g, the conformal class $[g]_c$ is defined to be the set of all metrics conformally equivalent to g. The pair $(M, [g]_c)$ is called a conformal structure. A map C^1 map $F : (M, g) \to (M, h)$ is conformal if $dF : T(M, g) \to T(M, h)$ preserves angles. Equivalently, F is conformal if F^*h is conformally equivalent to g where F^*h denotes the pullback of h by F. Let $\tilde{g} \in [g]_c$ and $\tilde{h} \in [h]_c$, then it is easy to see that $F : (M, g) \to (M, h)$ is conformal iff $F : (M, \tilde{g}) \to (M, \tilde{h})$ is conformal. This shows that the notion of conformal map is defined as long as M is given a conformal structure. That is $F : (M, [g]_c) \to (M, [h]_c)$ is conformal iff for some (and thus any) choice of $g \in [g]_c$ and $h \in [h]_c$, $F : (M, g) \to (M, h)$ is conformal. It is easy to see that the set of all conformal diffeomorphisms $F : (M, [g]_c) \to (M, [g]_c)$ forms a group which we denote by $Aut(M, [g]_c)$. The group $Aut[M, [g]_c)$ is called the automorphism group and its elements are called conformal automorphisms.

Definition 6.4.1. *We say that the action (or rather the family of actions), is invariant under conformal change of metric if S_g depends only on $[g]_c$.*

Definition 6.4.2. *We say that an action (or rather the family of actions), is conformally invariant if*

$$S_g[\phi \circ \tau] = S_g[\phi]$$

for any $\tau \in Aut(M, [g]_c)$.

We note that there is no implication in either direction between these two conditions.

6.4.2 Complex Structure

Recall that a bundle-endomorphism $J : TM \to TM$ is called an almost complex structure iff $J^2 = -I$. A metric g gives rise to an almost complex structure on M in the following way. Let $p \in M$ and $v_p \in TM_p$. Since TM_p is two-dimensional, there is a unique $w_p \in TM_p$ such that $||w_p||_g = ||v_p||_g$, $g(v_p, w_p) = 0$, and

$\{v_p, w_p\}$ is positively oriented. Let J_g be the linear map sending v_p to w_p for each $p \in M$. Then it is easy to check that J_g is an almost complex structure. Also observe that $\tilde{g}$ is conformally equivalent to g iff $J_{\tilde{g}} = J_g$ and so J depends only on $[g]_c$. The Newlander-Nirenberg theorem states that any almost complex structure J satisfying a certain integrability condition arises from a unique complex structure. This integrability condition is trivially satisfied on surfaces and so from now on we will refer to J as a complex structure. Thus there is a one-one correspondence between conformal classes $[g]_c$, and complex structures $J_{[g]_c}$ on M. We will thus sometimes simply call $[g]_c$ a complex structure and we will denote by $(M, [g]_c)$ the surface M endowed with the complex structure $[g]_c$. An orientation-preserving C^1 map $F : (M, [g]_c) \to (M, [h]_c)$ is called holomorphic if

$$J_{[g]_c} \circ dF = dF \circ J_{[h]_c}.$$

If F is instead orientation reversing it is called antiholomorphic. It is not hard to check that F is conformal iff it is holomorphic or antiholomorphic. It is important to note that while the notions of conformal maps and holomorphic and antiholomorphic maps are equivalent for surfaces, they generalize differently in higher dimensions.

6.4.3 Moduli Space

Let $F : N \to N$ be an orientation-preserving diffeomorphism. F acts on the space of metrics by pullback which we denote by F^*. It is easy to see that if $\tilde{g}$ is conformally equivalent to g iff $F^*\tilde{g}$ is conformally equivalent to F^*g, and so the pullback descends to a map between conformal classes. Two complex structures $J_{[g]_c}$, $J_{[h]_c}$ are called equivalent iff there exists a orientation-preserving diffeomorphism $F : (M, h) \to (M, g)$, such that

$$J_{[g]_c} = dF \circ J_{[h]_c} \circ (dF)^{-1}.$$

It is easy to see that this is true iff $F^*[g]_c = [h]_c$ and equivalently F is a conformal diffeomorphism. The moduli space of M is the space of conformal classes of metrics modulo modular equivalence. We denote elements of the moduli space by $[g]_m$ where g is a representative metric.

The most basic bosonic action in Conformal Field Theory is that of a Free (Massless) Boson.

Example 6.4.3. *Free (Massless) Bosonic Action*
This action is

$$S_g[\phi] \equiv \frac{1}{4} \int_M |\nabla_g \phi|_g^2 dvol_g,$$

for $\phi \in W^{1,1}(M)$*. Now since* $\nabla_{e^{2f}g} = e^{-2f}\nabla_g$ *and* $|\cdot|_{e^{2f}g} = e^f|\cdot|_g$, $dvol_{e^{2f}g} = e^{2f}dvol_g$, S_g *is invariant under conformal change of metric and so* S *is conformally invariant. This indicates that there should be a way to* *rewrite the action in the notation of complex geometry and indeed*

$$\begin{aligned} S_g[\phi] &\equiv \frac{1}{4}\int_M |\nabla_g \phi|_g^2 dvol_g \\ &= -\frac{1}{4}\int_M \phi \Delta_g \phi dvol_g \\ &= \frac{1}{2i}\int_M \phi \partial\bar{\partial}\phi \\ &= \frac{i}{2}\int_M \partial\phi \wedge \bar{\partial}\phi, \end{aligned}$$

where we suppress in the notation the dependence of ∂ on the complex structure $[g]_c$. We claim further that S is conformally invariant. Let $\tau \in Aut(M, [g]_c)$. Without loss of generality, assume that τ is holomorphic (with respect to $(M, [g]_c)$). Since τ is holomorphic, $\tau^*\partial\phi = \partial(\phi \circ \tau)$ and so

$$\begin{aligned} S_g[\phi] = \int_M \partial\phi \wedge \bar{\partial}\phi = \int_M \tau^*(\partial\phi \wedge \bar{\partial}\phi) \\ = \int_M \tau^*\partial\phi \wedge \tau^*\bar{\partial}\phi \\ = \int_M \partial(\phi \circ \tau) \wedge \bar{\partial}(\phi \circ \tau) \\ = S_g[\phi \circ \tau]. \end{aligned}$$

6.4.4 Curvature

By the Korn-Lichtenstein theorem every metric is locally conformally flat. That is, there is a local holomorphic coordinate chart, (U, z), (with respect to the complex structure $[g]_c$) such that locally $g = \lambda dz \otimes d\bar{z}$. The curvature 2-form is defined locally as $\omega_g \equiv \frac{1}{2}\partial_z\partial_{\bar{z}} \log \lambda dz \wedge d\bar{z}$. It is an easy exercise to check that ω_g is independent of the choice of holomorphic coordinate and so ω_g extends to a (1-1)-form on M. Since both g and ω transform the same way under a holomorphic coordinate change, the quotient $s \equiv \frac{\partial_z\partial_{\bar{z}} \log \lambda}{2\lambda}$ extends to a well-defined function on M which is called the scalar curvature. The same argument which shows that the scalar curvature two-form is independent under local holomorphic coordinate change shows the curvature is a conformal invariant, that is

$$s \circ \tau = \tau$$

for all $\tau \in Aut(M, [g]_c)$.

Example 6.4.4. *Free Bosonic action coupled to curvature*

$$S_g[\phi] \equiv \frac{1}{2i}\int_M \partial\phi \wedge \bar{\partial}\phi + \omega_g\phi.$$

This action is not weakly conformally invariant but it transforms simply under a conformal change of metric:

$$S_{e^f g}[\phi] = S_g[\phi] + \int_M \phi\partial\bar{\partial}f.$$

In particular, by choosing f to be a constant, we see that the action is (globally) "scale invariant."

We now examine the condition for conformal invariance of S. Let τ be a conformal automorphism of (M, g), then

$$\begin{aligned} S_g[\phi \circ \tau] &= \int_M \partial(\phi \circ \tau) \wedge \bar{\partial}(\phi \circ \tau) + \omega_g \phi \circ \tau \\ &= \int_M \partial\phi \wedge \bar{\partial}\phi + \int_M \omega_g \phi \circ \tau \\ &= \int_M \partial\phi \wedge \bar{\partial}\phi + \int_M \tau^* \omega_g \phi. \end{aligned}$$

We thus see that a sufficient condition for conformal invariance is $\tau^*\omega_g = \omega_g$ for all $\tau \in Aut(M, [g]_c)$ ($H(M)$ will always be large enough so that this is also a necessary condition). For surface of genus greater than one, forms which transform as above are called automorphic and such forms always exist (for example the curvature form of the hyperbolic metric). More precisely, this shows that given a metric g, there is a metric $\tilde{g} \in [g]_c$ such that $S_{\tilde{g}}$ is conformally invariant. Similarly taking the flat metric yields an analogous result for the genus one case.

6.4.5 The Classical View

The action, S, will act as the starting point for defining a physical system. In the classical case we will take $H(M) \equiv C^\infty(M)$ and from now on we will assume that S has a unique minimum at f_0. Elements of $H(M)$ can be viewed as representing "trajectories" of the physical system. In the classical setting, f_0 is the only physically realizable trajectory. We can thus trivially view S as inducing a probability measure on $H(M)$, namely δ_{f_0}. In other words, given a physical system with action S, we expect with probability one to observe f_0.

6.4.6 The Quantum View

From the point of view of Quantum Field Theory, the action, S, defines a probability measure μ on $H(M)$ given by the formal expression

$$\mu \equiv \frac{1}{Z} e^{-S[\phi]}[d\phi].$$

$[d\phi]$ is formally a sigma-finite positive translation invariant measure on $H(M)$ called the functional integration measure (ϕ is just an index). The normalizing constant Z is called the partition function. We will be more precise in a moment. The important difference in the quantum view is that unlike in the classical view, when we observe a physical system with action S, we are not guaranteed to get an a priori fixed outcome. All trajectories are physical; the physical system only determines the probability distribution on the trajectories observed when a system is prepared with action S. From a physical perspective, this statement would be essentially meaningless if not for the fact that multiple systems can be prepared with the same action S and thus the repeatability allows for experimental confirmation of the probability distribution. Also note that since $[d\phi]$ is (formally) translation invariant on $H(M)$ and S is continuous with minimum at f_0, μ is "concentrated" near f_0, and in this sense the classical trajectory is the most likely.

We now focus on making sense of μ for the case of the Free Bosonic Action and Free Bosonic Action coupled to curvature. If $H(M)$ were finite dimensional, then choosing a linear isomorphism

$$h : \mathbb{R}^{\dim H(M)} \to H(M),$$

we define $[d\phi]_h$ to be the pushforward of the Lebesgue measure, V, by h. Let $\tilde{h}$ be another choice of linear isomorphism. Then

$$[d\phi]_{\tilde{h}} \equiv \tilde{h}_* V = h_*(h^{-1} \circ \tilde{h})_* V = h_* c\phi = c[d\phi]_h,$$

where c is the absolute value of the determinant of $h^{-1} \circ \tilde{h}$. Since Z is a normalizing constant, the measure μ does not depend on the choice of h and is thus canonically defined.

We will exclusively consider the case where $H(M)$ is infinite dimensional and this presents an immediate problem —there do not exist sigma-finite translation invariant measures on infinite dimensional vector spaces. We can still make sense of μ though, but it will depend on the metric g and not just the complex structure $[g]_c$. Even if the action is invariant under conformal change of metric, μ_g will in general not be. This is referred to as conformal anomaly —a symmetry at the classical level that is broken after quantization.

To begin the construction, we take $H(M) = L^2(M, \mathbb{R})$ where the measure for L^2 is given by the volume form for the metric g. Since M is compact, $H(M)$, as a set, is independent of the choice of g. It is well-known that the Laplace-Beltrami operator, Δ_g, has a orthonormal basis of smooth eigenvectors. Let $\{\phi_j\}_{j=0}^{\infty} \subset H(M)$ denote this set, ordered with respect to increasing eigenvalues $\{\lambda_i\}_{i=0}^{\infty}$. For our purposes, it will suffice to construct μ_g on the space

$$H_g(M) \equiv \prod_{j=1}^{\infty} \mathbb{R}\phi_j.$$

Note that we have started the indexing at 1 and thus excluded ϕ_0 which is the constant eigenvector (the kernel of Δ_g is one-dimensional). The reason for this exclusion will become clear later and is related to the conformal anomaly. Let

$$H_n \equiv \prod_{j=1}^{n} \mathbb{R}\phi_j.$$

Since H_n is finite-dimensional, the measure $\mu_n \equiv \frac{1}{Z_n} e^{-S}[d\phi]$ is canonically defined on H_n as shown above. Notice that for the Free Bosonic action

$$S_g[\sum_{j=1}^{n} c_j\phi_j] = \sum_{j=1}^{n} c_j^2\lambda_j.$$

For the Free Bosonic Action coupled to curvature, we have

$$S_g[\sum_{j=1}^{n} c_j\phi_j] = \sum_{j=1}^{n} c_j^2\lambda_j + \sum_{j=0}^{n} c_j \int_N \omega_g\phi_j.$$

The above evaluation of the action shows that μ_n splits into a product measure. It is thus easy to see that for $m > n$, and any Borel set $U \subset H_n$,

$$\mu_m(\mathbb{R}^{m-n} \times U) = \mu_n(U),$$

and so by Kolmogorov's extension theorem, there is a probability measure μ_g on $H_g(N)$ with marginal measures given by μ_n. For more general actions, we will not have a "splitting" of the action and thus the family of measures $\{\mu_n\}_{n=1}^{\infty}$, will not in general be consistent and thus may not give rise to a measure on H_g. Given a functional X on H_g, we define the expectation, variance, etc. of X (when it exists) by restricting X to H_n and computing it there, and then taking the limit as $n \to \infty$.

6.5 Bosonic Conformal Fields

Fix an action S_g. Consider the evaluation functional

$$\Phi : M \times C^{\infty}(M) \to \mathbb{R}$$

defined by $\Phi(x)[\phi] = \phi(x)$. If μ_g exists, we would like to extend $\Phi(x)$ to the measure space (H_g, μ_g) to obtain a random distribution, but $\Phi(x)$ is not well-defined on all of H_g. To remedy this, and to allow for cases where μ_g doesn't exist, we instead restrict $\Phi(x)$ to the measure space (H_n, μ_n). We can compute correlations of these finite dimensional approximations and take the limit as $n \to \infty$ to obtain the correlations for Φ. We call Φ the Bosonic field for the action S_g (although we will typically suppress in the notation the dependence on the action and metric). The action and metric manifest themselves only in the measure spaces (H_n, μ_n). If the action is the Free Bosonic action we will call Φ the free Bosonic field and if the action is the Free Bosonic action coupled to curvature, we will call Φ the free Bosonic field coupled to curvature.

We call the restriction of Φ to $M \times (H_n, \mu_n)$, the n-th truncation of Φ and denote it by Φ_n. For $x \in M$, $\Phi_n(x)$ is now a well-defined random variable on H_n and we can compute their correlations. The product of truncated fields

$$\Phi_n(\cdot)\Phi_n(\cdot) : M \times M \times (H_n(M), \mu_n) \to \mathbb{R}$$

is defined by

$$\Phi_n(x)\Phi_n(y)[\phi] = \phi(x)\phi(y).$$

The definition of products of Φ is analogous and restricts to the product of the truncations on H_n. Let F be the $C^\infty(M)$-algebra generated by Φ. The multiplication of fields is easily seen to be commutative and thus F is a commutative algebra.

Example 6.5.1. *Truncations of the Free Bosonic Field and the Gaussian Free Field*
Let $\{\zeta_1, ..., \zeta_n\}$ be independent centered Gaussian random variables with variance one on the probability space $(\Omega, F, \mathbb{P})$. Then the random function

$$\tilde{\Phi}_n \equiv \sum_{j=1}^{n} \Big(\frac{2\pi}{\lambda_j}\Big)^{\frac{1}{2}} \phi_j \zeta_j$$

has values in H_n and

$$\tilde{\Phi}_n^* \mathbb{P} = \mu_n.$$

$\tilde{\Phi}_n$ does not converge pointwise to a random function as $n \to \infty$. We can view $\tilde{\Phi}_n$ as a random distribution on $L^2(M, g)$ in the obvious way:

$$\tilde{\Phi}_n[f] \equiv \int_M \tilde{\Phi}_n f dvol_g.$$

$\tilde{\Phi}_n$ converges pointwise, as $n \to \infty$, to a random distribution $\tilde{\Phi}$, on $L^2(M)$, which is called the Gaussian Free Field.

Example 6.5.2. *Correlation Functions for the Free Bosonic Field*
We begin by calculating Z_n.

$$\begin{aligned} Z_n \equiv \int_{H_n} d\mu_n &= \int_{\mathbb{R}^n} e^{-S[\sum_{j=1}^n c_j \phi_j]} dc_1 ... dc_n \\ &= \int_{\mathbb{R}^n} e^{-\sum_{j=1}^n c_j^2 \lambda_j} dc_1 ... dc_n \\ &= \prod_{j=1}^{n} \int_{\mathbb{R}} e^{-c^2 \lambda_j} dc \\ &= \sqrt{\frac{\pi^n}{det(\Delta_g\big|_{H_n})}}. \end{aligned}$$

We can use this to formally compute the partition function $Z \equiv \lim_{n\to\infty} Z_n$" = " $\sqrt{\frac{\pi^n}{det\Delta_g}}$ where $det\Delta_g$ denotes *the ζ-regularized determinant of Δ_g. We now compute the expection of $\Phi(x)$.*

$$\mathbb{E}[\Phi(x)] \equiv \lim_{n\to\infty}\int_{H_n}\Phi(x)d\mu_n$$
$$= \lim_{n\to\infty}\int_{\mathbb{R}^n}\Phi(x)[\sum_{j=1}^{n}c_j\phi_j]\frac{1}{Z_n}e^{-S[\sum_{j=1}^{n}c_j\phi_j]}dc_1...dc_n$$
$$= \lim_{n\to\infty}\int_{\mathbb{R}^n}[\sum_{j=1}^{n}c_j\phi_j(x)]\frac{1}{Z_n}e^{-\sum_{j=1}^{n}c_j^2\lambda_j}dc_1...dc_n$$
$$\lim_{n\to\infty}\frac{1}{Z_n}\sum_{j=1}^{n}\phi_j(x)\int_{\mathbb{R}^n}c_je^{-\sum_{j=1}^{n}c_j^2\lambda_j}dc_1...dc_n$$
$$= 0$$

since $\int_{\mathbb{R}} ce^{-c^2\lambda}dc = 0$. We now compute $\mathbb{E}[\Phi(x)\Phi(y)]$ which is known as the correlation function or two-point function for Φ.

$$\mathbb{E}[\Phi(x)\Phi(y)] = \lim_{n\to\infty}\int_{\mathbb{R}^n}[\sum_{j=1}^{n}c_j\phi_j(x)] \tag{6.81}$$
$$[\sum_{i=1}^{n}c_i\phi_i(y)]\frac{1}{Z_n}e^{-\sum_{j=1}^{n}c_j^2\lambda_j}dc_1...dc_n \tag{6.82}$$
$$= \sum_{j=1}^{n}\sum_{i=1}^{n}\phi_j(x)\phi_i(y)\int_{\mathbb{R}^n}c_jc_ie^{-\sum_{j=1}^{n}c_j^2\lambda_j}dc_1...dc_n \tag{6.83}$$
$$= \sum_{j=1}^{n}\phi_j(x)\phi_j(y)\int_{\mathbb{R}^n}c_j^2e^{-\sum_{j=1}^{n}c_j^2\lambda_j}dc_1...dc_n \tag{6.84}$$
$$= \sum_{j=1}^{\infty}\frac{1}{\lambda_j}\phi_j(x)\phi_j(y) \tag{6.85}$$
$$= G_g(x,y) \tag{6.86}$$

where $G_g(x,y)$ is the Green's function for Δ_g defined earlier. Notice that the integral in 6.84 does not converge for $j = 0$, which is why we had to exclude the constant eigenvector. Continuing in this way we can compute the n-point functions for Φ:

$$\mathbb{E}[\Phi(x_1)\cdots\Phi(x_n)] = 0, \tag{6.87}$$

for n odd, and

$$\mathbb{E}[\Phi(x_1)\cdots\Phi(x_n)] = \frac{1}{2^n n!}\sum_{\lambda\in\Lambda}\prod_{k=1}^{\frac{n}{2}}G_g(\lambda(k)), \tag{6.88}$$

for n even, where each element of Λ consists of a partition of $\{x_1, ..., x_n\}$ into ordered pairs.

Example 6.5.3. *Correlations for the Free Bosonic Field coupled to curvature.*
We now repeat the previous computation for the Free Bosonic action coupled to curvature. Let $\tilde{\mathbb{E}}$, (resp. $\tilde{Z}_n$) denote the expectation (resp. normalizing constant) with respect to the measure given by the Free Bosonic action coupled to curvature and let $\mathbb{E}$, (resp. Z_n) denote the expectation (resp. normalizing constant) with respect to the Free Bosonic action. We can proceed as above, but it is more convenient to relate the correlation functions of the two actions. Observe

$$\begin{aligned}
\tilde{\mathbb{E}}[\Phi(x_1)\cdots\Phi(x_n)] &= \lim_{n\to\infty}\int_{\mathbb{R}^n}[\sum_{j=1}^{n} c_j\phi_j(x_1)]\cdots[\sum_{i=1}^{n} c_i\phi_i(x_n)]\frac{1}{\tilde{Z}_n}e^{-\sum_{j=1}^n c_j^2\lambda_j-\sum_{j=1}^n c_j\int_N \omega_g\phi_j}dc_1...dc_n \\
&= \lim_{n\to\infty}\int_{\mathbb{R}^n}[\sum_{j=1}^{n} c_j\phi_j(x_1)]\cdots[\sum_{i=1}^{n} c_i\phi_i(x_n)]\frac{1}{\tilde{Z}_n}e^{-\sum_{j=1}^n \lambda_j(c_j+(2\lambda_j)^{-1}\int_N\omega_g\phi_j)^2+\sum_{j=1}^n(4\lambda_j)^{-1}(\int_N\omega_g\phi_j)^2}dc_1...dc_n \\
&= \lim_{n\to\infty}\int_{\mathbb{R}^n}[\sum_{j=1}^{n}(c_j-(2\lambda_j)^{-1}\int_N\omega_g\phi_j)\phi_j(x_1)]\cdots[\sum_{i=1}^{n}(c_i-(2\lambda_j)^{-1}\int_N\omega_g\phi_i)\phi_i(x_n)]\frac{1}{\tilde{Z}_n}e^{-\sum_{j=1}^n c_j^2\lambda_j+\sum_{j=1}^n(4\lambda_j)^{-1}(\int_N\omega_g\phi_j)^2}dc_1...dc_n \\
&= \lim_{n\to\infty}\int_{\mathbb{R}^n}[\sum_{j=1}^{n}(c_j-(2\lambda_j)^{-1}\int_N\omega_g\phi_j)\phi_j(x_1)]\cdots[\sum_{i=1}^{n}(c_i-(2\lambda_j)^{-1}\int_N\omega_g\phi_i)\phi_i(x_n)]\frac{1}{Z_n}e^{-\sum_{j=1}^n c_j^2\lambda_j}dc_1...dc_n \\
&= \mathbb{E}[(\Phi(x_1)-\sum_{j=1}^{\infty}(2\lambda_j)^{-1}\int_N\omega_g\phi_j\phi_j(x_1))\cdots(\Phi(x_n)-\sum_{j=1}^{\infty}(2\lambda_j)^{-1}\int_N\omega_g\phi_j\phi_j(x_n))] \\
&= \mathbb{E}[(\Phi(x_1)-\frac{1}{2}\int_N G_g(x_1,y)\omega_g(y))\cdots(\Phi(x_n)-\frac{1}{2}\int_N G_g(x_n,y)\omega_g(y))].
\end{aligned}$$

This last expression can easily be computed by expanding the product inside the expectation and using the identities:

$$\mathbb{E}[\Phi(x_1)\cdots\Phi(x_n)] = 0, \tag{6.89}$$

for n odd, and

$$\mathbb{E}[\Phi(x_1)\cdots\Phi(x_n)] = \frac{1}{2^n n!}\sum_{\lambda\in\Lambda}\prod_{k=1}^{\frac{n}{2}} G_g(\lambda(k)), \tag{6.90}$$

for n even. In particular, $\tilde{\mathbb{E}}[\Phi(x)] = -\frac{1}{2}\int_N G_g(x,z)\omega_g(z)$ and

$$\tilde{\mathbb{E}}[\Phi(x)\Phi(y)] = G_g(x,y) + \frac{1}{4}\int_N G_g(x,z)\omega_g(z)\int_N G_g(y,z)\omega_g(z).$$

6.5.1 Correlation Functions

In conformal field theory we are primarily interested in fields which have conformally invariant correlation functions. Because of the conformal anomaly, even conformally invariant actions do not in general give rise to conformally invariant fields. However, there are choices of metrics for which conformal invariance remains. If $\tau \in Aut(M, [g]_c)$ and F is a function on the product space M^k, recall that the pullback of F by τ is defined as $\tau^* F(x_1, ..., x_n) \equiv F(\tau(x_1), ..., \tau(x_n))$.

Definition 6.5.4. *We call a Bosonic field conformally invariant if*

$$\tau^* \mathbb{E}[\Phi(x_1) \cdots \Phi(x_n)] = \mathbb{E}[\Phi(x_1) \cdots \Phi(x_n)]$$

for all $\tau \in Aut(M, [g]_c)$ and all $x_1, ..., x_n \in M$. A field that is conformally invariant will be simply called a conformal field.

The reason for using the pullback notation will become clear later on when we consider correlation functions which are actually differential forms. We now set out to determine the set of metrics for which G_g is conformally invariant.

Proposition 6.5.5. *The Free Bosonic Field is conformal iff the metric g is automorphic. That is $\tau^* g = g$ for all $\tau \in Aut(M, [g]_c)$.*

Proof. Since

$$\mathbb{E}[\Phi(x_1) \cdots \Phi(x_n)] = 0, \tag{6.91}$$

for n odd, and

$$\mathbb{E}[\Phi(x_1) \cdots \Phi(x_n)] = \frac{1}{2^n n!} \sum_{\lambda \in \Lambda} \prod_{k=1}^{\frac{n}{2}} G_g(\lambda(k)), \tag{6.92}$$

for n even, it follows that Φ is conformal iff $\tau^* G_g = G_g$ for all $\tau \in Aut(M, [g]_c)$. We now proceed to determine which metrics this holds for. Let $\tau \in Aut(M, [g]_c)$, then $\tau^* g = e^f g$ for some $f \in C(M)$. By the functoriality of the pullback, τ^{-1} is also conformal and $(\tau^{-1})^* g = e^{-f \circ \tau^{-1}} g$. We now obtain a distributional relation for $\tau^* G_g$.

Let $h \in C^\infty(M)$.

$$\begin{aligned}
\int_M \tau^* G_g(x,y)\Delta_g h(y) dvol_g(y) &= \int_M G_g(\tau(x),\tau(y))\Delta_g h(y) dvol_g(y) \\
&= \int_M G_g(\tau(x),y)(\Delta_g h)(\tau^{-1}(y)) d(\tau^{-1})^* vol_g(y) \\
&= \int_M G_g(\tau(x),y)\Delta_{(\tau^{-1})^* g}(h\circ\tau^{-1})(y) dvol_{(\tau^{-1})^* g}(y) \\
&= \int_M G_g(\tau(x),y) e^{f\circ\tau^{-1}(y)}\Delta_g(h\circ\tau^{-1})(y) e^{-f\circ\tau^{-1}(y)} dvol_g(y) \\
&= \int_M G_g(\tau(x),y)\Delta_g(h\circ\tau^{-1})(y) dvol_g(y) \\
&= (h\circ\tau^{-1})(\tau(x)) - \frac{1}{|M|_g}\int_M h\circ\tau^{-1} dvol_g \\
&= h(x) - \frac{1}{|M|_g}\int_M h d(\tau^{-1})^* vol_g \\
&= h(x) - \frac{1}{|M|_g}\int_M h dvol_{(\tau^{-1})^* g} \\
&= h(x) - \frac{1}{|M|_g}\int_M h e^{-f\circ\tau^{-1}} dvol_g.
\end{aligned}$$

Since G_g is symmetric and has mean zero, so is $\tau^* G_g - G_g$. Using these properties, it is not difficult to verify that $\int_M [\tau^* G_g(x,y) - G_g(x,y)]\Delta_g h(y) dvol_g(y) = 0$ for all $h \in C^\infty(M)$ iff $[\tau^* G_g(x,y) - G_g(x,y)] = 0$ a.e. Now since G is continuous off the diagonal and G_g is ∞ on the diagonal, this last condition is equivalent to $\tau^* G_g = G_g$ everywhere. We showed above that

$$\int_M \tau^* G_g(x,y)\Delta_g h(y) dvol_g(y) = h(x) - \frac{1}{|M|_g}\int_M h e^{-f\circ\tau^{-1}} dvol_g.$$

Now as

$$\int_M G_g(x,y)\Delta_g h(y) dvol_g(y) = h(x) - \frac{1}{|M|_g}\int_M h dvol_g,$$

we have

$$\int_M [G_g(\tau(x),\tau(y)) - G_g(x,y)]\Delta_g h(y) dvol_g(y) = \frac{1}{|M|_g}\int_M h(1 - e^{-f\circ\tau^{-1}}) dvol_g. \tag{6.93}$$

From the discussion in the previous paragraph and 6.93, it follows that $\tau^* G_g = G_g$ iff $\frac{1}{|M|_g}\int_M h(1-e^{-f\circ\tau^{-1}}) dvol_g = 0$ for all $h \in C^\infty(M)$. This last condition is easily seen to hold iff $f = 0$, that is $\tau^* g = g$. This completes the proof.

□

Proposition 6.5.6. *Let Φ be a Free Bosonic Action coupled to curvature. Then Φ is conformal iff g is automorphic.*

Proof. Recall that in example 6.5.3 we showed,

$$\tilde{\mathbb{E}}[\Phi(x_1)\cdots\Phi(x_n)] = \mathbb{E}[(\Phi(x_1) - \frac{1}{2}\int_M G_g(x_1,y)\omega_g(y))\cdots(\Phi(x_n) - \frac{1}{2}\int_M G_g(x_n,y)\omega_g(y))].$$

In particular, since $\tilde{\mathbb{E}}[\Phi(x)] = -\frac{1}{2}\int_N G_g(x,y)\omega_g(y)$, a necessary condition for Φ to be conformal is that

$$-\frac{1}{2}\int_M G_g(\tau(x),y)\omega_g(y) = -\frac{1}{2}\int_M G_g(x,y)\omega_g(y),$$

for all $\tau \in Aut(M,[g]_c)$. We then have

$$\begin{aligned}\tau^*\tilde{\mathbb{E}}[\Phi(x_1)\cdots\Phi(x_n)] &= \mathbb{E}[(\Phi(\tau(x_1)) - \frac{1}{2}\int_M G_g(\tau(x_1),y)\omega_g(y))\cdots(\Phi(\tau(x_n)) - \frac{1}{2}\int_M G_g(\tau(x_n),y)\omega_g(y))]\\ &= \mathbb{E}[(\Phi(\tau(x_1)) - \frac{1}{2}\int_M G_g(x_1,y)\omega_g(y))\cdots(\Phi(\tau(x_n)) - \frac{1}{2}\int_M G_g(x_n,y)\omega_g(y))],\end{aligned}$$

and thus by expanding the product inside the expectation and using the linearity of the expectation, it is easy to see by induction that the condition

$$\tau^*\tilde{\mathbb{E}}[\Phi(x_1)\cdots\Phi(x_n)] = \tilde{\mathbb{E}}[\Phi(x_1)\cdots\Phi(x_n)]$$

for all $n \in \mathbb{M}$ iff

$$\tau^*\mathbb{E}[\Phi(x_1)\cdots\Phi(x_n)] = \mathbb{E}[\Phi(x_1)\cdots\Phi(x_n)]$$

for all $n \in \mathbb{M}$ and

$$\tau^*\tilde{\mathbb{E}}[\Phi(x)] = \tilde{\mathbb{E}}[\Phi(x)].$$

In other words, if the Free Bosonic field coupled to curvature is conformal, then so is the Free Bosonic field. By the previous proposition, we must have that g is automorphic. So a necessary condition for the Free Bosonic field coupled to curvature to be conformal is that g is automorphic. We will next show that this is sufficient.

By the previous paragraph, it suffices to show that if g is automorphic then both $\tilde{\mathbb{E}}[\Phi(x)]$ and the Free Bosonic Field are conformal invariant. The latter is proven in Proposition 6.5.5, so we now proceed to show the former. We begin by showing that the automorphicity of g implies the automorphicity of ω_g. Let (U,z) be a local coordinate chart and let $g = \lambda dz \otimes d\bar{z}$ in (U,z). We define a local (1-1)-form, ι, called the fundamental form locally by $\iota = \lambda dz \wedge d\bar{z}$. It is easy to see that ι is invariant under a holomorphic coordinate change and thus extends to a globally defined (1-1)-form. It is also easy to see that the automorphicity of g implies the automorphicity of ι. Observe that $s\iota = \omega_g$, where s denotes the scalar curvature. Since g is automorphic, s is conformally invariant. Therefore for all $\tau \in Aut(M, [g]_c)$,

$$\tau^* \omega_g = \tau^*(s\iota) = (s \circ \iota)\tau^* \iota = s\iota = \omega_g, \tag{6.94}$$

thus ω_g is automorphic as desired. Using this property we then have

$$\begin{aligned}
\tau^* \tilde{\mathbb{E}}[\Phi(x)] &= -\frac{1}{2} \int_M G_g(\tau(x), y)\omega_g(y) \\
&= -\frac{1}{2} \int_M \tau^* G_g(x, \tau^{-1}(y))\omega_g(y) = -\frac{1}{2} \int_M G_g(x, \tau^{-1}(y))\omega_g(y) \\
&= -\frac{1}{2} \int_M G_g(x, y)\tau^* \omega_g(y) = -\frac{1}{2} \int_M G_g(x, y)\omega_g(y) \\
&= \tilde{\mathbb{E}}[\Phi(x)].
\end{aligned}$$

This completes the proof.

□

6.6 Next Steps: Linear Statistics on Coulomb Gas Ensembles and CFT

In section 6.3 the Boltzmann Gibbs measure was defined as:

$$\Pi_n^{Q,\nu,\beta} \equiv \frac{1}{Z_n} e^{-\beta H_n^{\nu,Q}} vol_g^{\otimes n}, \tag{6.95}$$

where

$$H_n^{\nu,Q}(z_1, ..., z_n) \equiv \sum_{1 \le i < j \le n} K^{\nu,Q}(z_i, z_j) = \sum_{1 \le i < j \le n} G_\nu(z_i, z_j) + (n-1) \sum_{j=1}^{n} Q(z_j). \tag{6.96}$$

and Z_n is the normalizing constant:

$$Z_n^{Q,\nu,\beta} \equiv \int_{M^n} e^{-\beta H_n^{\nu,Q}} dvol_g^{\otimes n}. \tag{6.97}$$

For $f \in C(M)$, we associate a random variable, $Tr_n[f]$, on $(M^n, \Pi_n^{Q,\nu,\beta})$ defined by:

$$Tr_n[f](z) \equiv \sum_{i=1}^{n} f(z_i). \tag{6.98}$$

Let $\mathbb{E}_n$ denote the expectation operator on the probability space $(M^n, \Pi_n^{Q,\nu,\beta})$. It follows that:

$$\mathbb{E}_n[Tr_n[f]] = n\int_M f d\Pi_{n,1}, \tag{6.99}$$

where $\Pi_{n,1}$ is the first marginal measure of $\Pi_n^{Q,\nu,\beta}$. Suppose Q is admissible and $Q_{\nu,g}$ is continuous in a neighborhood of S. Then as a result of Theorem 6.3.3,

$$\lim_{n\to\infty}\frac{1}{n}\mathbb{E}_n[Tr_n[f]] = \int_M f d\mu^Q. \tag{6.100}$$

$Tr_n[f]$ is an example of a linear statistic on $(M^n, \Pi_n^{Q,\nu,\beta})$. There are two other linear statistics of interest:

$$Fluct_n[f] \equiv Tr_n[f] - n\int_M f d\mu_{eq} \tag{6.101}$$

and

$$\Phi_n[f] \equiv Tr_n[f] - \mathbb{E}_n[Tr_n[f]]. \tag{6.102}$$

The study of these linear statistics in the special case where $M = \hat{C}$, $\nu = 2\pi\delta_\infty$, and $\beta = 2$, are of much interest and arise from the study of the distribution of eigenvalues of random normal matrix ensembles. In this setting and under additional assumptions on Q, Ameur, Hedenmalm, and Makarov proved the following (see [2] for the precise assumptions on Q):

Theorem 6.6.1. *(Ameur, Hedenmalm, and Makarov) Let $f \in C_0^\infty(\mathbb{C})$ then*

$$\Phi_n[f] \rightharpoonup N(0, \frac{1}{2\pi}\int_{\mathbb{C}} |\nabla f^S|^2 dA) \tag{6.103}$$

as

$$n \to \infty.$$

In the statement of the theorem, $\rightharpoonup$ denotes weak convergence, N denotes the normal distribution, and f^S denotes the function defined on S by $f|_S$ and defined on S^c as the harmonic extension of $f|_{\partial S}$ to S^c.

Ameur, Hedenmalm, and Makarov also proved a version of Theorem 6.6.1 for the linear statistic $Fluct_n$ (see [2]). Theorem 6.6.1 shows that Φ_n can be viewed as an approximation to a Gaussian free field. If M is a general compact Riemann surface and the external field is chosen such that the support of the equilibrium measure is all of M, an analogue of Theorem 6.6.1 would realize Bosonic fields on compact Riemann surfaces as weak limits of linear statistics on Coulomb Gas Ensembles.

Bibliography

[1] D. Aharonov and H. Shapiro, *Domains on which Analytic Functions Satisfy Quadrature Identities*, J. Analyse Math., 30 (1976), 39-73.

[2] Y. Ameur, H. Hedenmalm, and N. Makarov, *Random Normal Matrices and Ward Identities*, 2011 preprint. arXiv:1109.5941

[3] T. Can, M. Laskin, and P. Wiegmann, *Fractional Quantum Hall Effect in a Curved Space: Gravitational Anomaly and Electromagnetic Response*, 2014 preprint. arXiv:1402.1531v1

[4] P. J. Davis, *The Schwarz Function and its Applications*, Carus Math. Monographs No.17, Math. Assoc, Amer., 1974.

[5] P. Ebenfelt, B. Gustafsson, D. Kavinson, and M. Putinar, *Quadrature Domains and Their Applications*, Operator Theory Advances and Applications Vol. 156 Birkhauser 2005.

[6] B. Epstein, *On the Mean-Value Property of Harmonic Functions*, Proc. Amer. Math. Soc, 13 (1962), 830

[7] B. Epstein and M. M. Schiffer, *On the Mean-Value Property of Harmonic Functions*, J. Analyse Math., 14 (1965), 109-111.

[8] M. Goldstein and W. H. Ow *On the Mean-Value Property of Harmonic Functions*, Proc. Amer. Math. Soc., (1971), 341-344.

[9] B. Gustafsson, *Quadrature Identities and the Schottky Double*, Acta Applicandae Mathematicae **1** (1983) 209-240

[10] B. Gustafsson and A. Vasil'ev, *Conformal and Potential Analysis in Hele-Shaw Cells* Birkhauser Verlag 2006.

[11] B. Gustafsson, *On Quadrature Domains and an Inverse Problem in Potential Theory*, J. Analyse Math. **55** (1990), 172-216.

[12] B. Gustafsson and H. Shapiro, *What is a Quadrature Domain?*, Operator Theory: Advances and Aplications **156** (2004), 1-25.

[13] B. Gustafsson, *Singular and Special Points on Quadrature Domains from an Algebraic Geometric Point of View*, J. d'Analyse Math. bf 51 1 (1988), 91-117.

[14] H. Hedenmalm and S.Shimorin, *Hele-Shaw Flow on Hyperbolic Surfaces*, Journal De Mathjematiques Pures Appl. **91** (2002) 187-222.

[15] H. Hedenmalm and N. Makarov, *Coulomb Gas Ensembles and Laplacian Growth*. Proc. Lond. Math. Soc. 106 (2013), 859-907.

[16] H. Hedenmalm and N. Makarov, *Quantum Hele-Shaw Flow*. 2004 preprint. arXiv:math/0411437v1.

[17] V. Isakov, *Inverse Source Problems*, AMS Math. Surveys and Monographs **34**, Providence Rhode Island, 1990.

[18] K. Johansson, *On Fluctuations of Eigenvalues of Random Hermitian Matrices*. Duke Math. J. 91 (1998), 151-204.

[19] N. Kang and N. Makarov, *Gaussian Free Field and Conformal Field Theory*. 2013 preprint. arXiv:1101.1024v3.

[20] D. Khavinson, M. Mineev-Weinstein, and Mihai Putinar, *Planar Elliptic Growth* 2009 preprint. arXiv:0901.3126v1.

[21] D. Khavinson and G. Neumann, *From the Fundamental Theorem of Algebra to Astrophysics: A "Harmonious" Path*, Notices of the AMS 55, number 6 (2008).

[22] U. Kuran, *On the Mean-Value Property of Harmonic Functions*, Bull. London Math. Soc. 4 (1972), 311-312.

[23] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and their Applications*. Pure and Applied Mathematics, 88. Academic Press 1980.

[24] S. Lee and N. Makarov, *Topology of Quadrature Domains* 2013 preprint. arXiv:1307.0487v1

[25] J.E. McCarthy, Liming Yang, *Subnormal Operators and Quadrature Domains*, Adv. Math. 127 (1997), 52-72.

[26] E. B. Saff and V. Totik, *Logarithmic Potentials with External Fields*, Springer 1997.

[27] M. Sakai, *An Index Theorem on Singular Points and Cusps of Quadrature Domains*, vol. Holomorphic functions and moduli, Vol. 1 (1986) 119-131, Math. Sci. Rs. Inst Publ. 10, Springer, New York, 1988.

[28] A.N. Varchenko and P.I. Etingof, *Why the Boundary of a Round Drop Becomes a Curve of Order Four* AMS University Lecture Series, Volume **3**, Providence, Rhode Island 1992.

[29] M. Mineev-Weinstein, M. Putinar, and R. Teodorescu, *Random Matrices in 2-D, Laplacian Growth, and Operator Theory*, 2008 preprint v2. arXiv:0805.0049.

[30] D. Xia, *Trace Formulas for Some Operators Related To Quadrature Domains in Riemann Surfaces*, Integral Equations and Operator Theory **47** (2003) 123-130