ESSAYS IN SOCIAL AND ECONOMIC NETWORKS

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Abstract

This thesis consists of three chapters, and they concern the formation of social and economic networks. In particular, this thesis investigates the solution concepts of Nash equilibrium and pairwise stability in models of strategic network formation. While the first chapter studies the robustness property of *Nash equilibrium* in network formation games, the second and third chapters investigate the testable implication of *pairwise stability* in networks.

The first chapter of my thesis is titled "The Robustness of Network Formation Games". In this chapter, I propose a notion of equilibrium robustness, and analyze the robustness of Nash equilibria in a class of well-studied network formation games that suffers from multiplicity of equilibria. Under this notion of robustness, efficiency is also achieved. A Nash equilibrium is k-robust if k is the smallest integer such that the Nash equilibrium network can be perturbed by adding some k number of links. This chapter shows that acyclic networks are particularly fragile: with the exception of the periphery-sponsored star, all Nash equilibrium networks without cycles are 1-robust, or minimally robust. The main result of this paper then proves that for all Nash equilibria, cyclic or acyclic, the periphery-sponsored star is the most robust Nash equilibrium. Moreover the periphery-sponsored star is by far the most robust in the sense that asymptotically in large network, it must be at least twice as

robust as any other Nash equilibria.

The second chapter of my thesis is titled "On the Consistency of Network Data with Pairwise Stability: Theory". In this chapter, I characterize the consistency of social network data with pairwise stability, which is a solution concept that in a pairwise stable network, no agents prefer to deviate by forming or dissolving links. I take preferences as unobserved and nonparametric, and seek to characterize the networks that are consistent with pairwise stability. Specifically, given data on a single network, I provide a necessary and sufficient condition for the existence of some preferences that would induce this observed network as pairwise stable. When such preferences exist, I say that the observed network is rationalizable as pairwise stable. Without any restriction on preferences, any network can be rationalized as pairwise stable. Under one assumption that agents who are observed to be similar in the network have similar preferences, I show that an observed network is rationalizable as pairwise stable if and only if it satisfies the Weak Axiom of Revealed Pairwise Stability (WARPS). This result is generalized to include any arbitrary notion of similarity.

The third chapter of my thesis is titled "On the Consistency of Network Data with Pairwise Stability: Application". In this chapter, I investigate the extent to which real-world networks are consistent with WARPS. In particular, using the network data collected by Banerjee et al. (2013), I explore how consistency with WARPS is empirically associated with economic outcomes and social characteristics. The main empirical finding is that targeting of nodes that have central positions in social networks to increase the spread of information is more effective when the underlying networks are also more consistent with WARPS.

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Chapter 1

Robustness in Network Formation Games

1.1. Introduction

In this chapter, I propose a notion of equilibrium robustness, and analyze the robustness of Nash equilibria in a class of well-studied network formation games that suffers from multiplicity of equilibria. The idea is the following: in any Nash equilibrium networks, no agents have incentives to deviate, however for some equilibria, agents have incentives to free-ride (and deviate) following a slight perturbation of the equilibrium networks. The idea is then to select for equilibria that are more robust to such *off-path incentives to free-ride*. The goal of the chapter is to formalize this idea, and operationalize it as a measure of equilibrium robustness.

I show that a particular kind of network, the periphery-sponsored star, is the most robust Nash equilibrium. This result complements the extant literature[1] in that it uncovers another stability property of the remarkable periphery-sponsored star – it is the equilibrium network most robust to off-path incentive to free-ride. *Moreover this*

[1] Feri (2007); Hojman and Szeidl (2008)

proposed notion of robustness allows us to further quantify and compare the degrees of robustness, and this gives rise to new conclusions, while remaining consistent with the existing ones. For example, I can say there is an intuitive manner in which acyclic equilibrium networks are particularly fragile. Moreover, the periphery-sponsored star is significantly more robust than any equilibria, in the sense that asymptotically, it must at least twice as robust as any other equilibria. Due to its' tractability, this notion of considering off-path incentive to free-ride could see potential application in other games with positive externalities.

The model of network formation considered in this chapter is known as the distance-based network formation game (Bala and Goyal (2000)). An agent i in this game receives benefits from being connected to other agents in the network, but the benefit from being connected to an agent j decays as the network distance between i and j increases. In addition, an agent can pay a fixed cost to form a link, and thus faces a trade-off between minimizing costly link formation and maximizing connectedness. Multiplicity of Nash equilibria is a feature in this game.

Another important feature of this game is that a link in the network imposes positive externalities on other agents. Since links are costly, agents have incentives to free-ride on links that are formed by other agents, in order to better connect to other agents in the network. This paper shows that for some Nash equilibria, adding a link to the equilibrium network increases the incentives of some agents to free-ride to the extent that these agents would then subsequently deviate from their Nash equilibrium strategy. I say that these equilibrium networks are not robust to off-path incentive to free-ride.

This paper asks: which equilibrium network is the most impervious to off-path incentives to free-ride? Formally building on this intuition, *I define a Nash equilibrium to be k-robust if k is the smallest integer such that the Nash equilibrium network can be perturbed by adding some k number of links.* In another words, for a k-robust Nash equilibrium, it is not possible to perturb the equilibrium network with fewer than k links, and there exists a way of perturbing the Nash equilibrium by adding k links.

The **first** main result of the paper is as follows: I prove that Nash networks without cycles are especially fragile. With the exception of the periphery-sponsored star, all acyclic Nash equilibria are 1-robust, or minimally robust. Adding just one costless link is enough to perturb the equilibria. Although acyclic networks can be stable under Nash equilibria, this result shows that such equilibria must conceal rampant off-path incentives to free-ride. Thus this paper provides a new way to reconcile the fact that Nash equilibrium often predicts acyclic networks, and the fact that acyclic networks are not commonly observed in the real world.

The intuition is that adding a link to an acyclic equilibrium *always* creates a cycle in the network[2], the presence of a cycle greatly increases incentives to free-ride because a cycle creates multiple paths for agents to connect to other agents. As a result, some agents would have incentive to **not** pay for their links, and free-ride on the alternative paths in the cycle. This intuition however, does not carry over to the case where the equilibrium network itself has cycles, where some agents find it profitable to maintain cycles. Nonetheless we have the following result.

The **second** main result is: under the standard assumption that the benefit function satisfies decreasing marginal benefits, the periphery-sponsored star is in fact the most

[2] In a connected acyclic network, the addition of a link always creates a cycle, and I show that all Nash equilibria must either be connected or empty.

robust Nash equilibrium. It is more robust than any cyclic and acyclic Nash equilibria. Convexity of the benefit function is satisfied by well-known payoff functions in this class of network formation games.

In this class of network formation games, the periphery-sponsored star is also the most efficient equilibrium.[3] In addition, I show that in the limit as the network grows arbitrarily large, the periphery-sponsored star is at least two times more robust than any other Nash equilibria. In that sense, it is by far the most robust Nash equilibria.

The assumptions needed to obtain the result in this paper are quite minimal. In particular, the robustness result here is applicable to other models of network formation games in the economics and computer science literature. In economics, we can apply the result here to gain additional insights into Bala and Goyal (2000)'s model of network formation with decay. In computer science, the robustness of equilibria in Fabrikant et al. (2003)'s model of network formation can be obtained as corollaries.

1.1.1. Example

Figure 1.1 below shows a center-sponsored star network, where the center agent pays for all the links. This network is known to be stable under Nash equilibrium (see Proposition 1). Another type of star network, the periphery-sponsored star, is de-

[3] Moreover the center agent obtains the highest payoff, and everyone else (the periphery agents) receive less payoff. There is then a positive correlation between network centrality and payoffs. Many economic and social networks exhibit a core-periphery structure: A small number of central agents or hubs gather a disproportionate amount of connections, while most other agents have much fewer links. This leads to the network having a relatively small average and maximum distance, this small-world phenomenon has been documented extensively, both empirically and theoretically (see Newman (2010)).

picted on the right in Figure 1.1. It is just the star network where the periphery agents pay for links to the center instead.

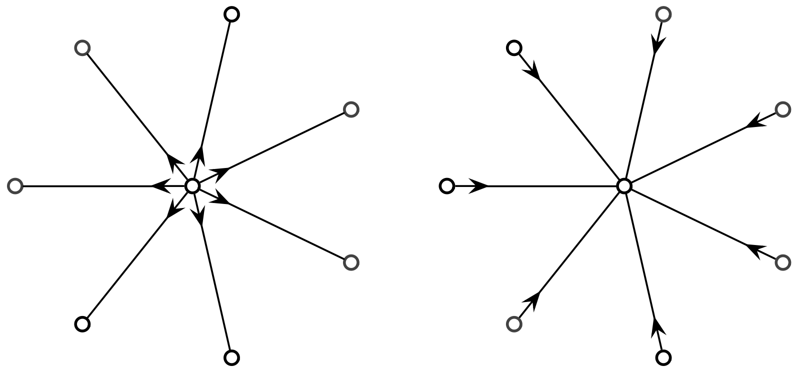

Figure 1.1: Left: center-sponsored star. Right: periphery-sponsored star.

The notion of robustness propounded in this paper makes the following stark statement: while the periphery sponsored star is the most robust Nash equilibrium, all other star networks are minimally robust Nash equilibria (adding one costless link is enough to perturb the equilibrium).

The center agent is maintaining a link that allows him to connect to just one agent. On the other hand, the periphery paying a link to the center that connects him to the center agent, *and* indirectly to all other agents. Therefore the periphery-to-center link provides a much higher marginal benefit than the center-to-periphery link. Nash equilibrium does not discriminate between these two configurations, even though there is an intuitive sense in which the periphery-to-center link serves a greater purpose, and hence should be the stronger and more robust type of link.

What heuristics allows us to say that the periphery-to-center link with higher marginal benefit should be formed? Consider adding a costless link to the center-sponsored star network, as depicted in Figure 1.2. A costless link is formed between agents i

and j where none of the agents have to pay for it.[4]

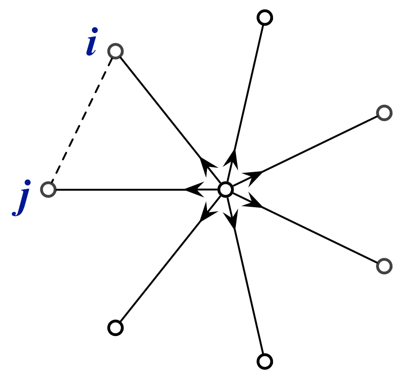

Figure 1.2: Off-path incentive to free-ride in center-sponsored star.

The effect of this new link is that it increases the incentive of the center agent to free-ride. The center agent's marginal benefit from having a link to the periphery agent i is now so low that he prefers to deviate from equilibrium strategy by severing the link to i or j. In this off-path network, he can access agent i through the costless link between i, j, and his link to j.

In general, by perturbing the equilibrium network with a new link, the center-to-periphery link now has a marginal benefit that is lower than the cost. The center agent can then free-ride on this new link, as he severs one of his link to the periphery. On the other hand, the periphery-to-center link provides such significant marginal benefit that even when perturbed with a new link, the periphery agent has no incentive to free-ride on this new link – if he severs his link to the center, he will be further away from the center agent which provides direct links to other periphery agents. Hence, the periphery-to-center link is robust and there is no off-path incentive to free-ride.

[4]For instance, they just happen to bond with each other at random. Many real-world networks have environments where strategic link formation is not the only driving force behind network formation (Jackson and Rogers (2007)).

1.1.2. Related Literature

Following the pioneering work of Jackson and Wolinsky (1996) and Bala and Goyal (2000), this paper builds on the literature of strategic network formation in economic theory. See Jackson (2008) and Goyal (2009) for surveys. A parallel strand of game-theoretic models can also be found in the theoretical computer science literature, following from the pioneering work of Fabrikant et al. (2003) and Corbo and Parkes (2005). The issue of equilibrium multiplicity is a challenge in analysing game-theoretic models of network formation. This motivates the study of the Price of Anarchy in computer science as how bad Nash equilibrium performs in the worst case relative to the socially optimal outcome.[5]

In the economics literature, the two most closely related papers that deal with the problem of multiple equilibria are Hojman and Szeidl (2008) and Feri (2007). By placing additional restrictions on the benefit function and the effect of decay, Hojman and Szeidl (2008) are able to fully characterize the set of Nash equilibria, and show that it is a periphery-sponsored star. Feri (2007) also characterizes periphery-sponsored stars as the unique stochastically stable equilibria of his model. Their approach is substantially different, and one notable difference is that I quantify and compare the degrees of robustness, and this gives rise to new conclusion, while remaining consistent with the existing conclusion.

[5]For example, Fabrikant et al. (2003) attempts to quantify the price of anarchy. The price of anarchy here is the ratio of the largest total cost generated by any Nash equilibrium, compared to the cost of the efficient network. It quantifies the degree of inefficiency in the system due to self-interested individuals not being able to internalize the positive externalities caused by their own actions. A few papers followed suit in providing incrementally better bounds on the worst-case Nash equilibrium, and the lack of a tight bound on the price of anarchy is an indication of how intractable it is to analytically characterize the set of Nash equilibria in a general class of network formation games. Chapter 19 of Vazirani et al. (2007) provides a concise survey of this strand of literature in computer science.

Stochastic stability is a useful tool for refinement and robustness analysis in models with multiple equilibria. Goyal and Vega-Redondo (2005); Jackson and Watts (2002a,b) study applications of stochastic stability to models of network formation beyond the unilateral class considered here. Finally, Bloch and Dutta (2009) contains references to other strategic models of network formation where multiple equilibria is an issue.[6]

[6] They remarked that: "However, Nash equilibrium has little predictive power in their model, and even when they resort to the refinement of strict Nash equilibria, they are unable to obtain a complete characterization in the two-way flow model with decay (see Bala and Goyal (2000)).

1.2. Setup

The setup is a distance-based (unilateral) network formation game. A network formation game has agents $N = \{1, 2, \dots n\}$, which we will also refer to as nodes. The action available to an agent i is as follows: agent i can choose any subset of agents, $s_i \in 2^{N \setminus \{i\}}$, and form a link to each agent $j \in s_i$. The strategy space of agent i is then given by the set $S_i = 2^{N \setminus \{i\}}$. A strategy profile $s = s_1, \dots, s_n \in S_1 \times \cdots \times S_n$ induces a directed network (N, g) where $g_i = \{(i, j) : j \in s_i\}$, and $g = \cup_{i=1}^{n} g_i$. When it is clear, we will refer to a network (N, g) by just g.

A few definitions are in order before I explain the payoffs of agents. Given a directed network g, let us define the corresponding undirected network by $\hat{g}$, where $(i, j) \in g \implies \{i, j\} \in \hat{g}$. Denote $d(i, j; \hat{g})$ as the distance between nodes i and j in the network $\hat{g}$, and $d(i, j; \hat{g})$ is defined to be the length of the shortest path from i to j in the undirected network $\hat{g}$.

Finally, let $b : \{1, \dots, n-1\} \to \mathbb{R}$ be the benefit function. The function b models the benefit that an agent receives from another agent as a function of the distance between them. Hence the benefit that agent i receives from agent j in the network g is $b[d(i, j; \hat{g})]$.

When the strategy profile is s, the payoff of agent i is given by:

$$u_i(s) = \sum_{\substack{j \neq i \\ j \in N}} b[d(i, j; \hat{g}_s)] - c|s_i| \tag{1.2.1}$$

g_s is the directed network induced by the strategy profile $s = s_1, \dots, s_n \in S_1 \times \cdots \times S_n$, and $\hat{g}_s$ is the corresponding undirected network. This payoff function is standard, and is exactly the one assumed in section 11.3.2 of Jackson (2008)).

The first term of Equation (1.2.1) is the sum of all benefits that agent i receives in the network g_s, while the second term is the total cost paid for by agent i in forming $|s_i|$ number of links, and where each link costs $c > 0$.

We will assume that an agent receives zero benefit from agents he has no connection to, i.e. $b(\infty) = 0$. Moreover, we will assume monotonicity of the benefit function, i.e. $b(k) \geq b(k+1)$ for any k and $c \geq 0$. These inequalities encapsulate the idea that an agent sees higher benefits for having a shorter distance to other agents.

Given the strategy profile s, the directed network g_s encodes all the payoff-relevant information. For convenience, we can write $u_i(g)$ to denote the utility that agent i derives from the network g, that is, $u_i(g) = \sum_{j \neq i} b[d(i,j;\hat{g})] - c \cdot d_{out}(g)$, where $d_{out}(g)$ is the out-degree of agent i in g, and $\hat{g}$ the undirected network of g.

1.2.1. remark

In the pioneering work of Bala and Goyal (2000), the benefit function is set to a decreasing, convex function: $b(x) = \delta^x$ for some $\delta \in (0,1]$. Agents exponentially discount benefits received from agents who are further away in the network. The smaller is δ, the more agents discount indirect connections. When δ is close to one, benefit does not decay with network distance, and agents only care how many direct and indirect connections they have.

This strategic network formation model captures the idea that agents obtain utility from their direct connections and also from their indirect connections, and the utility deteriorates with the distance between individuals. The utility function captures the competing incentives of the player to minimise its distance to other agents, while forming as few links as possible.

Although link formation is unilateral (i.e. the cost of forming a link is borne solely by one side, and a link can be formed without the consent of the other side), this model assumes that the benefit of any one link flows both ways. That is, once the links are installed, they are regarded as undirected and can be used in both directions.

1.2.2. Nash equilibrium

A (pure) **Nash equilibrium** of the game is an $s = (s_i, s_{-i})$ such that, for each player i, $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$ for all $s_i' \in S_i$. A Nash strategy profile s induces a network g, and for convenience, we will often say that g is a Nash network, although it should be clear that Nash equilibrium refers to the underlying strategy profile. In a Nash equilibrium, each player maximizes his or her own utility, without taking into account the positive externality that own's action has on other agents.

In contrast, a socially optimal or **efficient** network is one that maximizes the overall welfare and total payoffs in the network, giving equal weight to each of the player's utility. Formally, A strategy profile s and its induced network g is efficient relative to a profile of utility functions $(u_1, \ldots, u_n)$ if $\sum_i^n u_i(s) \geq \sum_i^n u_i(s')$ for all $s' \in S$.

Definition 1.2.1. A **periphery-sponsored star** network on n agents is a configuration where $n-1$ agents (the periphery nodes) each pays for a costly link to one node (the center node). These $n-1$ links are the only links in the network. Similarly, in a **center-sponsored star** network, the center node pays for costly links to each of the $n-1$ agents.

Proposition 1 and Lemma 1 below characterize the Nash equilibrium of the game. It is a more thorough characterization of equilibrium networks than found in Proposition 11.3 of Jackson (2008). In particular, Lemma 1 says that in any Nash equilibrium network, either there is a no link (empty network), or all agents are connected in the sense that there is a path between any two agents in the network. In another words, either there is exactly n components, or there is exactly one component in any Nash network. This result is rather striking and to the best of my knowledge has not been mentioned elsewhere. This lemma will be used in Lemma (3), and hence the main theorem of the paper.

Proposition 1 (Nash Equilibrium)**.**

(i) *For* $b(1) - b(2) > c$*, the unique Nash equilibrium of the game is the complete network.*

(ii) *For* $b(1) - b(2) < c < b(1)$*, any star is a Nash equilibrium but there are other equilibria.*

(iii) *For* $b(1) < c < b(1) + b(2)(n-2)$*, the empty network is a Nash equilibrium. The periphery-sponsored star is the only type of star that can be sustained in a Nash equilibrium. There are also other possible non-star Nash equilibria.*

(iv) *For* $b(1) + b(2)(n-2) < c$*, the empty network is the unique Nash equilibrium.*

Proof. (i) For $b(1) - b(2) > c$, the unique Nash equilibrium of the game is the complete network, if the distance between some two agents is more than 1, either player would be willing to pay c to form link so that the distance between them is 1.

(ii) For $b(1) - b(2) < c < b(1)$, any star is a Nash equilibrium but there can also be many other equilibria. The star is an equilibrium because if any player deletes his current link, his utility decreases by at least $b(1) - c$, which is a positive amount. There is no incentive to form additional link because all nodes are at most two links apart, and the marginal benefit of any additional link is just $b(1) - b(2)$, which is less than the marginal cost c.

(iii) For $b(1) < c < b(1) + b(2)(n-2)$, the empty network is certainly a Nash equilibrium. In an empty network, the marginal benefit from any link is $b(1)$, which is less than c. The periphery-sponsored star is a Nash equilibrium because if any of the periphery player severs the link to the center, he would lose $b(1) + b(2)(n-2)$ but only gain c, so overall utility still decreases. All nodes are at most two links apart, and the marginal benefit of any additional link is just $b(1) - b(2)$, which is less than the marginal cost c.

(iv) For $c > b(1) + b(2)(n-2)$, the empty network is the unique Nash equilibrium. The marginal benefit from any one link is at most $b(1) + b(2)(n-2)$, so for $c > b(1) + b(2)(n-2)$, there will be no incentive to form any link.

□

Lemma 1. *If a Nash equilibrium is nonempty, then it must be connected*[7]*, i.e. there is exactly one component in the undirected network induced by the Nash equilibrium.*

Proof. If a Nash equilibrium is nonempty, then it must be connected, i.e. there is exactly one component. Denote the Nash equilibrium strategy profile by s, and the

[7]An undirected network g is connected if there is a path between any two agents in the network, i.e. for any pair of agents i, j, there is a sequence of agents $a_1, \ldots, a_m$ such that $\{i, a_1\}$ is a link in g, $\{a_k, a_{k+1}\}$ is a link in g for $k = 1, \ldots, m-1$, and $\{a_m, j\}$ is a link in g.

corresponding network by g. Because the equilibrium is nonempty, we can find one component C_1 such that there is a player $i \in C_1$ with $|s_i| \geq 1$. That is, agent i pays for some links. Now suppose that agent i is linked to agent j, his marginal benefit from the link (i, j) is given by Equation (1.2.2) below, and it is positive because g is a Nash equilibrium.

$$
\begin{aligned}
&u_i(g) - u_i(g - ij) = \\
&b(1) - b(d(i, j; g - ij)) + \sum_{k \in L(i,j;g)} \big[b(d(i, k; g)) - b(d(i, k; g - ik)) \big] - c > 0 \qquad (1.2.2)
\end{aligned}
$$

Here, $L(i, j; g)$ is the set of nodes for which all the shortest paths from i to $l \in L(i, j; g)$ contains the node j in the undirected network of g. As a matter of notation, $g - ij$ is the network obtained from g by removing the directed link between agents i, j; and $g + ij$ is the network obtained from g by adding the directed link (i, j).

Suppose by contradiction that the network consists of more than one component, and let us pick an agent i' from this other component C_2. We will now show that $u_{i'}(g + i'j) - u_{i'}(g) > 0$, and hence agent i' has incentive to deviate from the Nash strategy by forming a link to agent j across component. Note that $d(i, k; g) = d(i', k; g + i'j)$ for all $k \in L(i, j; g)$

$$
\begin{aligned}
&u_{i'}(g + i'j) - u_{i'}(g) \\
&> b(1) - 0 + \sum_{k \in L(i,j;g) \subset C_1} \big[b(d(i', k; g + i'j)) - 0 \big] - c \\
&> u_i(g) - u_i(g - ij) \\
&> 0
\end{aligned}
$$

This contradicts the assumption that the strategy profile is a Nash equilibrium. □

Proposition 2 (Efficient Network)**.**

(i) For $2(b(1) - b(2)) > c$*, the unique efficient outcome is the complete network.*

(ii) For $2(b(1) - b(2)) < c < 2b(1) + b(2)(n - 2)$*, the star network is the unique efficient outcome*

(iii) For $2b(1) + b(2)(n - 2) < c$*, the empty network is the unique efficient outcome.*

Proposition 2 is given as Proposition 11.3 in Jackson (2008). Note that this network formation game is different from another strand of network formation games in which link formation is bilateral, and where both sides of the link must agree to bear the cost of the link. In a unilateral link formation game, Nash equilibrium is often used as an equilibrium concept (Bala and Goyal (2000)), whereas in bilateral link formation games, the notion of pairwise stability is used as an equilibrium concept (Jackson and Wolinsky (1996)).

1.3. Notion of Robustness

In this section, I will discuss the issue of multiple equilibria and introduce a new notion of equilibrium robustness.

1.3.1. Multiplicity of Nash equilibrium

In Proposition 1, we saw that when the cost of maintaining a link lies in an intermediate region, multiple networks arise as Nash equilibria.

Recall that the complete network and the empty network are the unique Nash equilibrium for $c < b(1) - b(2)$ and $c > b(1) + (n-2)(b(2) - b(1))$ respectively. Since there is no issue with equilibrium selection in this region, we will restrict ourselves to the intermediate range of c, i.e. $b(1) - b(2) < c < b(1) + (n-2)(b(2) - b(1))$, where the issue of multiplicity arises.

To starkly illustrate the problem of multiple equilibria, consider the region of cost high enough that the empty network is a Nash equilibrium. Specifically consider the region $H = [c : b(1) < c < b(1) + b(2)(n-2)]$. It turns out that even in the region $c \in H$, both the empty network as well as super-connected networks can arise as Nash equilibria. A super-connected network is one where the network remains connected upon the deletion of any one link. For example, the super-connected cycle of length 5 in Figure 1.3 is a Nash equilibrium for $b(1) - b(2) < c < b(1) - b(4) + b(2) - b(3)$. This interval overlaps with with the region H when the benefit function b declines rapidly enough such that $b(2) - b(3) - b(4) > 0$.

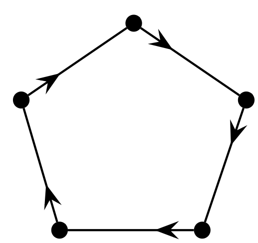

Figure 1.3: An equilibrium that is super-connected

1.3.2. Robustness

Our notion of robustness is based on perturbing an equilibrium network with **costless links**. More formally, define $g \cup \{i, j\}$ as adding an undirected link $\{i, j\}$ to the directed network g. The undirected link $\{i, j\}$ models a costless link, where no agent is paying for the link, but all agents derive some benefits from the link.

I now define the robustness value of an equilibrium network as follows: *what is the minimum number of costless links required to perturb the Nash equilibrium? More concretely, a Nash equilibrium is K-robust if (i) there exists a way of adding K or more costless links such that some agents now have the incentives to deviate from their Nash equilibrium strategy, and (ii) for all $k < K$, there does* **not** *exist a way of adding k costless links such that some agents now prefer to deviate from their Nash strategy.*

The formal definition of K-robust Nash equilibrium is given below, but first we need a notation. A network g' is **obtainable** from a network g by player i if the only changes between networks g' and g involve links that are directed from i to other agents. Formally, a network g' is obtainable from a network g by player i if $[(k, j) \notin g'$ and $(k, j) \in g]$ or $[(k, j) \in g'$ and $(k, j) \notin g]$ implies that $k = i$.

Definition 1.3.1. A Nash equilibrium s^* and its corresponding network g is a **K-robust** Nash equilibrium if and only if

(i) There exists K pairs of nodes $\{\{i_t, j_t\}_{t=1}^K\}$, such that when $\{\{i_t, j_t\}_{t=1}^K\}$ are added as costless links to (N, g), there exists a player i who has incentive to deviate from the Nash equilibrium strategy s^*. Formally, there exists $i \in N$ such that,

$$u_i(g^i \cup \{i_t, j_t\}_{t=1}^K) > u_i(g \cup \{i_t, j_t\}_{t=1}^K)$$

where g^i is obtainable from g by player i.

(ii) For all $k \in \{1, 2, \dots, K-1\}$, there does not exist $\{\{i_t, j_t\}_{t=1}^k\}$ such that

$$u_i(g^i \cup \{i_t, j_t\}_{t=1}^k) > u_i(g \cup \{i_t, j_t\}_{t=1}^k)$$

for some $i \in N$, where g^i is obtainable from g by player i.

In words, a Nash equilibrium is K-robust if K is the **minimum** number of costless links needed to perturb the Nash equilibrium. We will not be able to perturb the given Nash equilibrium with fewer than K costless links. If $K = 1$, then a network is K-robust means that the addition of **one** costless link between some two nodes will some agent to deviate from her Nash equilibrium strategy. I will also call such equilibrium minimally robust. It is possible that no such K exists, in that case, the equilibrium is maximally robust.

1.3.3. Robustness of periphery-sponsored star

Lemma 2. *The periphery-sponsored star is a $\lceil m \rceil$-robust Nash equilibrium, where*

$$m = \max\left\{\frac{b(1) - b(2) + (b(2) - b(3))(n-2) - c}{b(2) - b(3)}, 1\right\}$$

Proof. In a periphery-sponsored star network on n agents, the $n-1$ periphery agents each pay cost of c for a link to the center agent. Now consider perturbing this equilibrium network by adding a costless link between some two periphery agents (the center agent is already linked to everyone else). No agent will respond to this perturbation by paying for more links, nor will agents respond by switching link from the center agent to another agent. The only possible response of a periphery agent is to sever his existing link to the center. This argument holds true when more than one costless links are added to perturb the periphery-sponsored star.

A periphery player would have incentive to sever his link to the center when enough costless links are formed between him and other periphery agents. Consider adding t links between $\{1, \ldots, t\}$ periphery nodes and one particular periphery node i. Agent i's marginal benefit from the link to the center is now $b(1)-b(2)+(n-2-t)(b(3)-b(2))$. Without the link to the center, agent i would experience an increase in distance of 1 to each of the $(n-2-t)$ periphery node that he does not have a link to, which corresponds to a decrease in utility of $(n-2-t)(b(3)-b(2))$; as well as an increase in distance of 1 to the center (decrease in utility of $b(1)-b(2)$).

So the perturbed network is no longer a Nash equilibrium if the marginal benefit of i's link to the center is now less than c, that is, $b(1)-b(2)+(n-2-t)(b(3)-b(2)) \leq c$, or after rearranging, $t \geq \frac{b(1)-b(2)+(b(2)-b(3))(n-2)-c}{b(2)-b(3)}$. This is the most "efficient" way to perturb the periphery-sponsored star, in the sense that adding links any other ways would require more links to perturb the periphery-sponsored star. Therefore, $\lceil \frac{b(1)-b(2)+(b(2)-b(3))(n-2)-c}{b(2)-b(3)} \rceil$ is the minimum number of links we would need, and if this number is less than 1, then it takes only 1 link to perturb the equilibrium.

□

1.3.4. Example

To grasp the magnitude of the robustness value in Proposition 2, let us consider a special case of the distance-based utility model in which the benefit function is linear in distance. That is, consider the following objective function obtained from substituting $b(x) = -x$ into Equation (1.2.1):

$$u_i(s) = \sum_{j \neq i}^{n} -d(i, j; \hat{g}) - c|s_i| \tag{1.3.1}$$

where $d(i, j; \hat{g})$ is the shortest path length between i and j in the undirected network $\hat{g}$ induced by the strategy profile $s = (s_1, \ldots, s_n)$. Since the utilities are expressed so that they are always negative, we can think of the objective as minimizing the cost of communication. This class of unilateral network formation games was first studied in Fabrikant et al. (2003), where the intend was to quantify the price of anarchy.

Corollary 1: Consider the network formation game of Fabrikant et al. (2003), where the payoff function is given by Equation (1.3.1). The periphery-sponsored star is a $\lceil m \rceil$-robust Nash equilibrium, where $m = \max\{n - c - 1, 1\}$

As another example, we will consider the formulation of network formation game in Bala and Goyal (2000), where the benefit function is set to a decreasing, convex function: $b(x) = \delta^x$ for some $\delta \in (0, 1]$. The payoff of agent i is given by Equation (1.3.2) below.

$$u_i(s) = \sum_{j \neq i}^{n} \delta^{d(i,j;\hat{g})} - c|s_i| \tag{1.3.2}$$

Corollary 2: Consider the network formation game of Bala and Goyal (2000), where the payoff function given by Equation (1.3.2). The periphery-sponsored star is a $\lceil m \rceil$-robust Nash equilibrium, where $m = \max\{n + f(c, \delta) - 1, 1\}$ and $f(c, \delta) = \frac{1}{\delta} - 1 - \frac{c}{\delta(1-\delta)}$. The function $f(c, \delta)$ (hence the robustness value) decreases in c and decreases in δ for

$\delta > 0.5$.[8]

1.3.5. Robustness of acyclic equilibria

The following lemma shows rather strikingly that with the exception of the periphery-sponsored star, all acyclic equilibria are minimally robust. An acyclic equilibrium is one where the Nash strategy profile induces an undirected acyclic network. An undirected network is **acyclic** if and only if there does not exist a sequence of nodes $(a_1, \ldots, a_k)$ such that there is a link between nodes a_i and a_{i+1} for all $i = 1, \ldots, k-1$, and there is a link between nodes a_k and a_1.

Acyclic equilibria are especially fragile in general. The addition of just one costless link is sufficient to perturb an acyclic Nash equilibrium (other than the periphery-sponsored star).

Lemma 3. *Among all Nash equilibria whose networks are acyclic, the periphery-sponsored star is the most robust Nash equilibrium. Moreover all other acyclic equilibria are minimally robust.*

Proof. Take any path of length 2 in an **acyclic**, equilibrium network g, and denote the agents as i, j, k, and suppose that agent j pays for the link to i, as depicted in Figure 1.4. The key observation is the following equation (recall that the notation $g + ik$ denotes the network g with the addition of the directed link (i, k), and the network $g - ij$ denotes the network g with the directed link (j, i) removed):

$$u_k(g + ki) - u_k(g) = u_j(g + ki) - u_j(g + ki - ji)$$

[8]More thoroughly, $f(c, \delta)$ is decreasing in δ if either $\delta > 0.5$, or if $c < \frac{(1-\delta)^2}{1-2\delta}$.

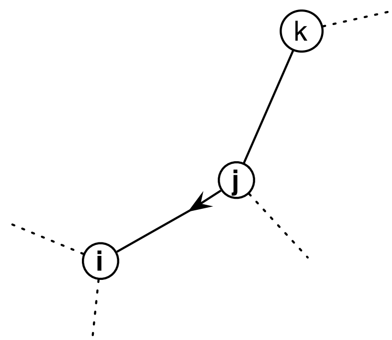

Figure 1.4: Acyclic equilibrium

Which says that the marginal benefit of the link (k, i) to k is the same as the marginal benefit of the link (j, i) to j when the link (k, i) is added. Since g is a Nash equilibrium, we have $u_k(g+ki) - u_k(g) < 0$, which implies $u_j(g+ki) - u_j(g+ki-ji) < 0$. Therefore when g is perturbed by adding a costless link between agents i and k, the best response of agent j is to sever the link to i.

As a result, all acyclic, equilibrium network with the structure depicted in Figure 1.4 must be minimally robust. For the network to be more than minimally robust, any path of length 2 must have the configuration depicted in Figure 1.5. That is, take any path ijk, for the equilibrium to be robust, we must have node i and node k paying for the link to the "middle" node j.

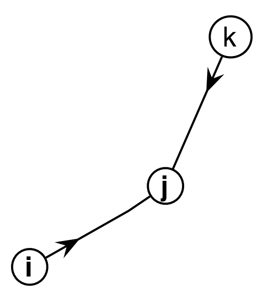

Figure 1.5: Acyclic equilibrium

We know from Lemma 1 that an equilibrium network is either connected or empty.

The only connected, acyclic network with this property is the periphery-sponsored star. Hence, all acyclic equilibria with the exception of the periphery-sponsored star are minimally robust. Although from Lemma 2, the periphery-sponsored star itself is minimally robust when the cost of link formation is high enough, i.e. when $c > b(1) - b(2) + (b(2) - b(3))(n-3)$, in which case, all acyclic equilibria are minimally robust. □

1.4. Main Theorem

In this section, the main result (Theorem 1) is presented. It states that not only is the periphery-sponsored star the most robust equilibrium in the class of acyclic equilibria (as presented in the previous section), but under one standard assumption on the benefit function, it is in fact more robust than any other equilibrium, with or without cycles. This assumption is the convexity of the benefit function. In Section 1.4.1, I further prove that the periphery-sponsored star is at least twice as robust as any other Nash equilibrium in large networks.

Assumption 1. The benefit function is **convex**, i.e. the function b satisfies $b(k) - b(k-1) \geq b(k') - b(k'-1)$ for $k' \geq k$. That is, the marginal benefit from being closer to an agent in the network decreases as the distance increases.

Convexity of the benefit function is a natural assumption, as it says that the marginal benefit from being closer to an agent in the network decreases as the distance increases. Agents care less about shortening his distance by 1 unit to an agent who is already far apart. Convexity of the benefit function is satisfied by existing models in the literature. In particular, the benefit functions $b(x) = \delta^x$ of Bala and Goyal (2000) and $b(x) = -x$ of Fabrikant et al. (2003) are convex.

Theorem 1. *For cost within the non-trivial range of* $b(1)-b(2) < c < b(1)+b(2)(n-2)$*, the periphery-sponsored star network is the unique most robust Nash equilibria. For* $c > b(1) + b(2)(n-2)$*, the empty network is the only Nash equilibrium. For* $c < b(1) - b(2)$*, the complete network is the unique Nash equilibrium.*

This main theorem follows from Lemma 2 and 3, as well as Lemmas 4 and 5 that we will state below. The proofs of Lemmas 4 and 5 are provided in Section 1.5. The role of Lemmas 4 and Lemma 5 are to provide upper bounds for the robustness of all Nash equilibria with (undirected) cycles[9].

From Lemmas 4 and Lemma 5 then, we conclude that all Nash equilibria containing some cycles have robustness value bounded above by the quantity $\max\left\{\frac{b(1)-b(2)+(b(2)-b(3))\lfloor n/2\rfloor - c}{b(2)-b(3)}, 1\right\}$. On the other hand, Lemmas 2 and 3 state that all acyclic equilibria are minimally robust, with the exception of the periphery-sponsored star, which has robustness value of $\lceil\theta\rceil$, where $\theta = \max\left\{\frac{b(1)-b(2)+(b(2)-b(3))(n-2)-c}{b(2)-b(3)}, 1\right\}$. Since this value is greater than the upper bound robustness value of all other Nash equilibria, cyclic or acyclic, it follows that the periphery-sponsored star is the most robust equilibrium.

Lemma 4. *Any Nash equilibrium whose network contains a cycle of length 3 is at most* $\lceil\theta\rceil$*-robust.*

$$\theta = \max\left\{\frac{b(1) - b(2) + (b(2) - b(3))\lfloor\frac{n-3}{2}\rfloor - c}{b(2) - b(3)}, 1\right\}$$

[9]A Nash equilibrium has a cycle of length K whenever the Nash strategy profile induces an **undirected** network g such that there exists a sequence of agents $(a_1, \ldots, a_K)$, and there is a link between agents a_k, a_{k+1} in g for $k = 1, \ldots, K-1$ and there a link between agents a_1 and a_K in g.

Corollary 3: Consider the network formation game with the payoff function given by Equation (1.3.1), i.e., the benefit function is $b(x) = -x$ per Fabrikant et al. (2003). Any Nash equilibrium whose network contains a cycle of length 3 is at most $\lceil\theta\rceil$-robust, where $\theta = \max\left\{\lfloor\frac{n-1}{2}\rfloor - c, 1\right\}$

Lemma 5. *Any Nash equilibrium whose network contains a cycle of length 4 or above is at most $\lceil\theta\rceil$-robust, where*

$$\theta = \max\left\{\frac{b(1) - b(2) + (b(2) - b(3))\lfloor n/2\rfloor - c}{b(2) - b(3)}, 1\right\}$$

Corollary 4: Consider the network formation game with the payoff function given by Equation (1.3.1), i.e., the benefit function is $b(x) = -x$ per Fabrikant et al. (2003). Any Nash equilibrium whose network contains a cycle of length 4 or above is at most $\lceil\theta\rceil$-robust, where $\theta = \max\left\{\lfloor\frac{n}{2}\rfloor - c, 1\right\}$.

1.4.1. Large networks

Our final result is to say something about how much more robust is the periphery-sponsored star compared to other Nash equilibria. The results from previous sections allow us to quantify and say that the periphery-sponsored star is by far the most robust Nash equilibrium. To allow for tractable comparison of robustness, assume that the number of agents n grows arbitrarily large (and other parameters stay constant).

Note that the qualitative comparison can still be made without resorting to asymptotics, but as evidenced by Corollary 5, simple comparisons of robustness can be made for large networks.

Corollary 5: In the limit as $n \to \infty$,

(i) the periphery-sponsored star is at least two times more robust than any other Nash equilibrium.

(ii) the periphery-sponsored star is at least $\frac{n}{2}$ more robust than any other Nash equilibrium.

Collecting all our previous results, this corollary becomes a matter of algebraic subtraction. Recall that the periphery-sponsored network is $\lceil m \rceil$-robust, where $m = \max\left\{\frac{b(1)-b(2)+(b(2)-b(3))(n-2)-c}{b(2)-b(3)}, 1\right\}$, and from Lemmas 4 and 5, all other equilibrium networks are at most $\lceil \theta \rceil$-robust, where $\theta = \max\{\frac{b(1)-b(2)+(b(2)-b(3))\lfloor n/2 \rfloor - c}{b(2)-b(3)}, 1\}$.

1.5. Main Proofs

Definition 1.5.1. For the proof, we would need the following definitions.

(i) A path in a network (N, g) between nodes i and j is a sequence of links or links $i_1 i_2 \ldots i_{K-1} i_K$ such that $(i_k, i_{k+1}) \in g$, for all $k = 1, \ldots K-1$, with $i_1 = i$ and $i_K = j$. The length of the path is K.

(ii) $L(i, j; g) = \{l_1, \ldots, l_L\}$ is the set of nodes for which all the shortest paths from i to $l \in L(i, j; g)$ contains the node j in the undirected network of g. When nodes i and j are linked, then the set $L(i, j; g)$ is exactly (and no more) those nodes in which the distance from i to them will increase when the link ij is removed from g.

1.5.1. Proof of Lemma 4

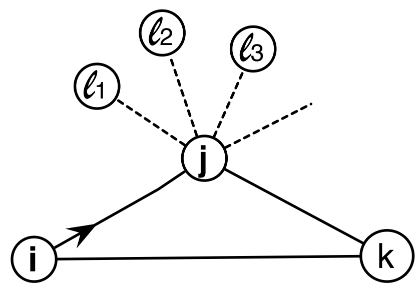

Figure 1.6: Equilibrium with a cycle of length 3

Proof. Denote the Nash equilibrium strategy and network by s and g respectively. Consider one such cycle in g with the nodes $\{i, j, k\}$, as shown in Figure 1.6. Without loss of generality, assume that the link ij is paid for by i. Note that i would strictly prefer not to form this link to j if there is no other node besides k connected to j, because then the marginal benefit of the link ij to i is $b(1) - b(2)$, which is less than c by assumption. Moreover, we can say that i prefers to form this link to j if and only if $|L(i, j; g)|$ is nonempty. That is, in a Nash equilibrium, i pays to form the link ij if and only if there exists a set of nodes $\{l_1, l_2, \ldots l_{L_1}\}$, such that all shortest paths between i and l contained j, for all $l \in \{l_1, l_2, \ldots l_{L_1}\}$. We will let $L_1 = |L(i, j; g)|$. Note that $L_1 > c - 1$.

Consider adding t links between i and some t nodes from the set $L(i, j; g) = \{l_1, l_2, \ldots l_{L_1}\}$. By concavity of b, we have $b(k) - b(k-1) \geq b(k') - b(k'-1)$ for $k' \geq k$, it then follows that player i's marginal benefit of the link ij in this perturbed network is bounded above by $b(1) + (b(2) - b(3))(L_1 - t)$. When $c > b(1) + (b(2) - b(3))(L_1 - t)$, player i's best-response is to delete the link to j. Solving for t, we obtain $t > \frac{b(1)+(b(2)-b(3))L_1-c}{b(2)-b(3)}$. Therefore, with $\lceil \frac{b(1)+(b(2)-b(3))L_1-c}{b(2)-b(3)} \rceil$ costless links, we can perturb the equilibrium network g.

The proof technique is to express the possible range of L_1 in terms of parameters of the model, n and c. A very crude upper bound for L_1 is just $n-3 \geq L_1$, because there are n nodes and $L(i,j;g) \cap \{i,j,k\} = \emptyset$. We can do better:

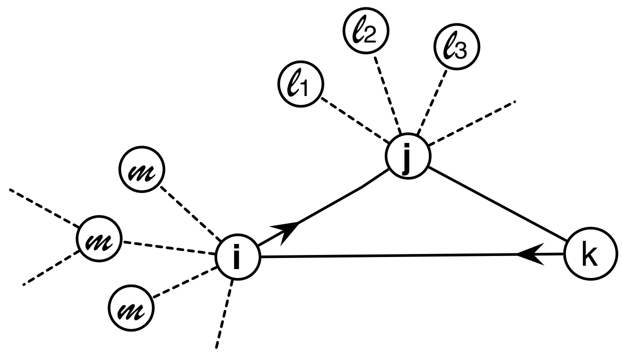

Figure 1.7: Cyclic equilibrium

Now without loss of generality, assume that the link between i and k is paid by k in equilibrium, as in Figure 1.7. The proof remains the same if we assume that i pays for the link ik. Now consider the set $L(k,i;g)$. If $|L(k,i;g)| = 0$, then k would not form link to i in equilibrium, so let $L(k,i;g) = \{m_1, \ldots, m_{L_2}\}$ with $|L(k,i;g)| = L_2 > 0$. Using the same reasoning as before, if $\lceil \frac{b(1)+(b(2)-b(3))L_2-c}{b(2)-b(3)} \rceil$ costless links are formed between k and some $\{m_1, \ldots, m_{L_2}\}$, then by deleting the link ki, player k could strictly increase his payoff.

Furthermore, we must have $L(i,j;g)$ and $L(k,i;g)$ disjoint, otherwise, if $x \in L(i,j;g) \cap L(k,i;g)$, then $ij\ldots x$ and $ki\ldots x$ are both shortest paths respectively. This implies that $kij\ldots x$ is the shortest path between k and x, but clearly $kj\ldots x$ is a shorter path, which gives a contradiction.

We now know it does not take more than $\min\{\lceil \frac{b(1)+(b(2)-b(3))L_1-c}{b(2)-b(3)} \rceil, \lceil \frac{b(1)+(b(2)-b(3))L_2-c}{b(2)-b(3)} \rceil\}$ costless links to disrupt the equilibrium, where $n-3 \geq L_1 + L_2$, with $L_1, L_2 > 0$. To obtain an upper bound for the robustness, we maximize $\min\{\lceil \frac{b(1)+(b(2)-b(3))L_1-c}{b(2)-b(3)} \rceil,$

$\lceil \frac{b(1)+(b(2)-b(3))L_2-c}{b(2)-b(3)} \rceil\}$ subject to $n-3 \geq L_1 + L_2$. The maximum occurs at $L_1 = L_2 = \lfloor \frac{n-3}{2} \rfloor$. Therefore, any Nash equilibrium that has a cycle of length 3 is less than $\lceil \theta \rceil$-robust, where

$$\theta = \max \left\{ \frac{b(1) - b(2) + (b(2) - b(3)) \lfloor \frac{n-3}{2} \rfloor - c}{b(2) - b(3)}, 1 \right\}$$

□

1.5.2. Proof of Lemma 5

Proof. Consider a network induced by some Nash equilibrium. There are two distinct cases, either this network has a cycle of length 3, in which case we can simply invoke Lemma 4, or the network is triangle-free and has no cycle of length 3. We will then only consider a triangle-free network. Now let us consider a cycle of length 4 or above in this triangle-free network. The cycle is depicted in Figure 1.8. Without loss of generality, let node i pays for the link ij in equilibrium. Now if we add a costless link between i and k, then a cycle of length 3 is obtained, and the problem is essentially transformed into the one encountered previously. From Lemma 4, it suffices to add $\lceil \frac{b(1)+(b(2)-b(3))L_1-c}{b(2)-b(3)} \rceil$ costless links, where $L_1 = |L(i,j)|$, in addition to the one costless link between i and k, to cause player i to sever the link ij to player j. In another words, we have $\lceil \frac{b(1)+(b(2)-b(3))L_1-c}{b(2)-b(3)} \rceil + 1 = \lceil \frac{b(1)+(b(2)-b(3))(L_1+1)-c}{b(2)-b(3)} \rceil$ as an upper bound.

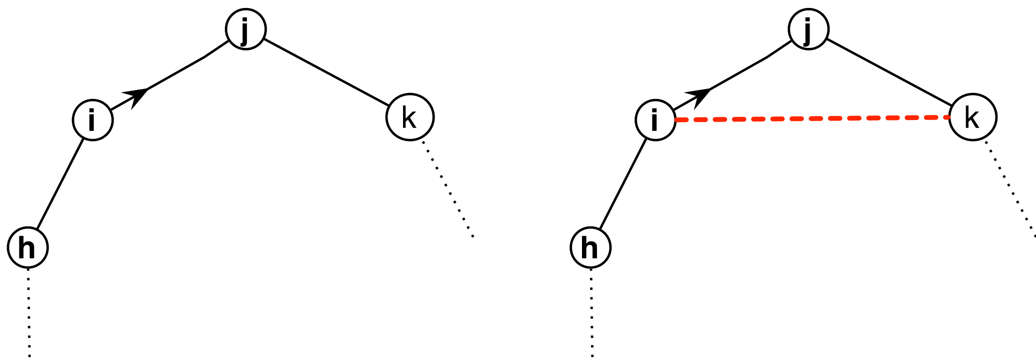

Figure 1.8: Equilibrium with cycle of length greater than 3

To improve on the upper bound, consider the link between h and i. Consider first the case where i is paying for the link to h, as shown in Figure 1.9. Adding a costless link between h and k as depicted in Figure 1.9 would produce a cycle of length 3, and we immediately deduce that $\lceil \frac{b(1)+(b(2)-b(3))(L_2+1)-c}{b(2)-b(3)} \rceil$ is another upper bound, where $L_2 = |L(i,h)|$. Since the sets $L(i,h)$ and $L(i,j)$ are disjoint, the worst case upper bound occurs when $L_1 = L_2 = \lfloor \frac{n-4}{2} \rfloor$.

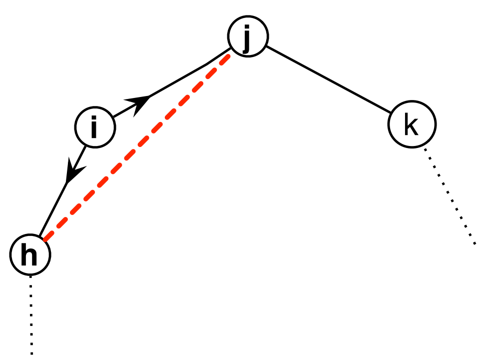

Figure 1.9: Equilibrium with cycle of length greater than 3

Finally, consider the second case where the node h is paying for the link to i. Similar to the preceding reasoning, adding a costless link between h and k as depicted in

Figure 1.10 would produce a cycle of length 3, and we can immediately deduce that $\lceil \frac{b(1)+(b(2)-b(3))(L_3+1)-c}{b(2)-b(3)} \rceil$ is another upper bound, where $L_3 = |L(h,i)|$. However, unlike before, the sets $L(h,i)$ and $L(i,j)$ are not necessarily disjoint. It turns out that the worst case occurs when $L(h,i)$ and $L(i,j)$ are disjoint. Consider a node $x \in L(h,i) \cap L(i,j)$, because of the link ih, the shortest distance between h and all such x would not increase after h deletes the link to i. The marginal benefit of the link to player i is determined by the cardinality $|L(h,i) \setminus L(i,j)|$. Explicitly, the marginal benefit of the link to i is bounded above by $(b(2)-b(3))|L(h,i) \setminus L(i,j)|$. Accordingly, the worst case upper bound occurs when $L_1 = L_3 = \lfloor \frac{n-4}{2} \rfloor$.

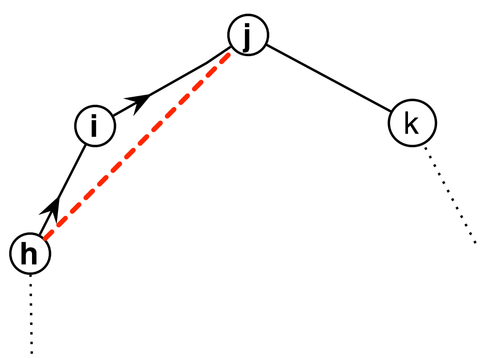

Figure 1.10: Equilibrium with cycle of length greater than 3.

Putting the steps together, it takes at most $\max\left\{\frac{b(1)-b(2)+(b(2)-b(3))\lfloor n/2 \rfloor - c}{b(2)-b(3)}, 1\right\}$ of costless links to disrupt the equilibrium, i.e. we have obtained a worst case upper bound for the robustness value of all equilibria with a cycle of length 4 or above. This concludes the proof.

$\square$

Chapter 2

On the Consistency of Network Data with Pairwise Stability: Theory

2.1. Introduction

This chapter examines theoretically and empirically, the consistency of social and economic networks with pairwise stability, which is a solution concept that in a pairwise stable network, no pair of agents prefer to deviate from the existing network configuration by unilaterally dissolving links, or bilaterally forming links (Jackson and Wolinsky (1996)).

More concretely, suppose we observe a single network of social relationships (who is friends with whom), or a network of financial relationships (who borrows money from whom, and who lends to whom). Now to say that the aforementioned network is pairwise stable, one has to check that agents do not *prefer* to deviate from this network. This requires knowing the preferences of everyone in the network.

In the spirit of classical revealed preference analysis, I assume that preferences are *unobserved* and *nonparametric*[1], and therefore, consistency of an observed network with pairwise stability requires that there exists some preferences that would induce this network as pairwise stable. This is the first paper to provide a revealed preference characterization of the set of networks that are *rationalizable* as pairwise stable. A network is said to be rationalizable as pairwise stable if and only if there exists preferences that induce the network as pairwise stable.

Without any restriction on preferences, any network is trivially rationalizable as pairwise stable, since we can just say that every agent prefers the observed network to any other networks. However it turns out, under the assumption that agents' preferences are not arbitrarily different from each other, some networks cannot be rationalized as pairwise stable.

More concretely, I impose only one assumption on preferences, which is that agents that are observed to be similar in the network have similar preferences. Now the notion of similarity or "which agents are similar to whom" is formalized as a similarity or equivalence relation. Under this restricted heterogeneity assumption, the Weak Axiom of Revealed Pairwise Stability (WARPS) is obtained which characterizes when an observed network is rationalizable as pairwise stable.

The result is then generalized: for any notion of similarity, and under the assumption that agents who are similar have similar preferences, I obtain a corresponding revealed preference axiom that characterizes the consistency of social network data with pairwise stability.

Our axiom resembles GARP (Generalized Axiom of Revealed Preference), which characterizes the consistency of consumption data set with utility maximization (Afriat

[1] That is, this paper never imposes parametric assumption on preferences

(1967); Samuelson (1938); Varian (1982, 2006)). Just as GARPS is tantamount to an algorithm checking for cycles in revealed preference relations, WARPS here is equivalent to the acyclicity of an appropriately modified profile of revealed preference relations.

For a given notion of similarity, WARPS can be understood as an algorithm that rules out the following revealed preference cycle: take a pair of agents who reveal preferred to form some links (not necessarily with each other), but there exists another pair of similar agents who reveal preferred not to form a link with each other.

One important departure from classical revealed preference theory arises because in networks and other two-sided settings, when a link is not observed between two agents, we are unsure as to which of the two sides blocked and chose not to form a link. Therefore the data do not reveal preferences directly, but induce a multiplicity of possible revealed preference relations.[2]

2.1.1. Related Literature

This is the first paper to characterize the empirical restriction of pairwise stability in networks. It belongs to the broader agenda of revealed preference in economics, championed by Afriat (1967); Richter (1966); Samuelson (1947, 1938); Varian (1982). They show that when preferences are *unobserved*, the empirical restriction of utility maximization is captured by GARP. Their works led to extensive empirical applications using GARP as a basis for measuring rationality (i.e. consistency with utility

[2]More concretely, preferences can be thought of as being revealed from network data as follows. If a link is observed between two agents, say that both agents reveal preferred to form this link, conditional on the rest of the network. However when a link is *not* observed between two agents, we can only say that **either** one **or** both of them reveal preferred not to form this link, conditional on the rest of the network.

maximization). Notably, Andreoni and Miller (2002); Blundell et al. (2003); Choi et al. (2014); Echenique et al. (2011); Harbaugh et al. (2001). For surveys on the subject, see Crawford and De Rock (2014); Varian (2006).

Beyond the setting of individual decision-making, the revealed preference approach has also been applied to other multi-agents (strategic or otherwise) settings. This paper extends to the network setting, although the desideratum is microfounding measures based on revealed preference characterizations, which is not the objective in these papers. For example, Brown and Matzkin (1996) give a revealed preference characterization of Walrasian equilibria; Carvajal et al. (2013) for Cournot equilibria; Echenique (2008); Echenique et al. (2013) for stability of two-sided matching models; Chambers and Echenique (2014) for bargaining equilibria; Cherchye et al. (2007, 2011); Chiappori (1988); Sprumont (2000) for collective rationality.

2.2. Illustrative examples

Before I formally introduce the general setup, the examples in this section help illustrate the heart of the matter. Consider the social network g depicted in Figure 2.1 below. From observing a link (friendship) between agents a and b, we say that agents a and b both *reveal preferred* the network in Figure 2.1 to the network in Figure 2.2. Notationally, the statement "agent a reveals preferred the network g to $g - ab$" is represented by $g \succ_a g - ab$. Now we can repeat the same step and define revealed preference this way for all pairs of agents who are linked in Figure 2.1.

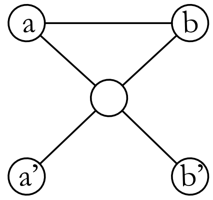

Figure 2.1: Network g

$\succ_a$

and

$\succ_b$

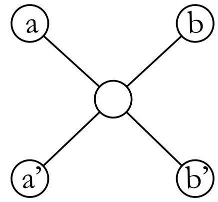

Figure 2.2: Network $g - ab$

Now from observing no link between agents a' and b', we say that **either** agent a' **or** b' (or both) reveal preferred the network in Figure 2.3 to the network in Figure 2.4. While forming a link requires the consent of both sides (bilateral), dissolving a link is unilateral. Therefore when we do not observe a link, we are unsure as to which side prefers not to form the link.

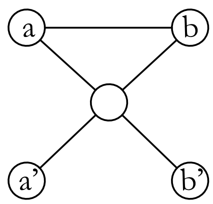

Figure 2.3: Network g

$\succ_{a'}$

or

$\succ_{b'}$

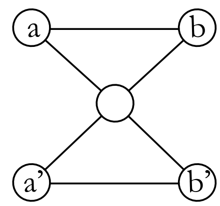

Figure 2.4: Network $g + a'b'$

The question we are interested in is: does there exist some underlying preferences (complete and transitive binary relations) that could induce these revealed preferences? If we can find such preferences, then we say the network g in Figure 2.1 is rationalizable as pairwise stable.

2.2.1. Network g is not rationalizable as pairwise stable

The network g in Figure 2.1 (in fact any network) is rationalizable as pairwise stable.[3] But under the following structure on preferences, it is not rationalizable as pairwise stable. Let an agent's degree in g be her number of links (friends) in the network g.

For this example (and this example only), assume that preferences are as follows: there is a ranking of degrees $d_0, d_1, d_2, \ldots$, such that everyone agrees having d_0 friends is better than d_1 friends, and better than d_2 friends and so on.[4]

From Figures 2.1 and 2.2, the degrees of agent a in the networks g and $g - ab$ are 2 and 1 respectively. Therefore the revealed preference $g \succ_a g - ab$ implies that agent a reveals preferred to have 2 friends to 1. On the other hand, we see in Figures 2.3 and 2.4 that the degrees of agent a' in the network g and $g + a'b'$ are 1 and 2 respectively. However we cannot be sure if agent a' reveals preferred to have 1 friend than 2 (recall that link deletion is unilateral, and it could have been agent b' who blocks the formation of the link with a').

Looking again at Figure 2.3 and 2.4, the degrees of agent b' in the network g and $g + a'b'$ are also 1 and 2 respectively, just like agent a'. Therefore even when revealed preferences of agents a' and b' are indeterminate, we can still say that there is some agent (either of a' or b') who reveals preferred 1 friend to 2; yet from the previous paragraph, we found an agent a who reveals preferred 2 friends to 1. Since agents

[3] To see this, the revealed preference relation we constructed is acyclic, there is therefore an underlying preferences (complete and transitive) that extend or induce them.

[4] For instance, this assumption is satisfied when the utility that agent i derives from network g is $u_i(g) = d_i(g) - cd_i^2(g)$, where $d_i(g)$ is the degree of i in g, and c is the cost parameter that is homogeneous among all agents.

agree on a preference ranking over degrees, we obtained a **cycle of length 2 in the revealed preference relation** – which implies that network g in Figure 2.1 is not rationalizable as pairwise stable.

2.2.2. Rationalizable networks

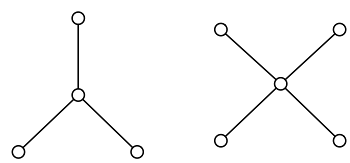

Figure 2.5: All star networks are rationalizable as pairwise stable.

Given the restrictive assumption on preferences in this example, one can still find networks that are rationalizable as pairwise stable. For example, all star networks with more than 3 agents (Figure 2.5) are rationalizable as pairwise stable. A preference that rationalizes star networks is as follows: agents prefer fewer friends than more friends below a threshold degree; and agents prefer more friends above a threshold degree. Moreover agents also prefer to have at least one friend to none.

2.2.3. Sufficient condition

Thus far, I have illustrated the intuition behind the necessary condition for a network to be rationalizable as pairwise stable:– check for cycle of length 2 in the revealed preference relations. Is this also sufficient? In particular, one can imagine the possibility of longer cycles, for example, when some agent (reveals) preferred d_0 degrees to d_1, another prefers d_1 to d_2, and another prefers d_2 to d_0.

However a simple argument shows that no cycles of length greater than 2 can exist without nesting a cycle of length 2. Therefore we can *ignore cycle of length greater* *than 2.* To see this, suppose there is a cycle of length 3, and without loss of generality, say that agent i (reveals) preferred d_0 number of friends to $d_0 + 1$ friends. If another agent j prefers d_0+1 to d_0 number of friends, then we obtain a cycle of length 2 in the revealed preference relation. If instead agent j prefers $d_0 + 1$ to $d_0 + 2$ friends, then there can be no agent who reveals preferred d_0+2 to d_0. This is an artifact of pairwise stability, which is a conservative solution concept that requires stable network to be robust to only one-link deviation. As a result, preferences over two networks that differ in more than one link are never revealed.

2.2.4. Modus Operandi

I will demonstrate the main result by applying Theorem 2 to this example, where preference heterogeneity is restricted to degrees (agents that have the same degrees have the same preference). The theorem is applicable generally to other (less restrictive) assumption on preference heterogeneity.

Let $d_i(g)$ be the degree (or the number of friends) of agent i in network g. *The network g is rationalizable as pairwise stable if and only if for all pairs of agents i, j, there does not exist a pair of agents i', j' that are not linked in network g, such that $d_i(g) = d_{i'}(g+i'j')$ and $d_j(g) = d_{j'}(g+i'j')$.* This condition is essentially an algorithm that decides if a given network data is rationalizable as pairwise stable under the particular restriction on preference heterogeneity of this example. To explain this result further:

Suppose for network g, the condition above is violated for the pair of agents i', j', and the pair of agents i, j. Denote d_i for $d_i(g) = d_{i'}(g + i'j') \geq 1$ and d_j for $d_j(g) = d_{j'}(g + i'j') \geq 1$. Since we observe no link between agents i' and j', **either** agent i'

reveals preferred $d_i - 1$ to d_i number of friends, **or** agent j' reveals preferred $d_j - 1$ to d_j number of friends. Otherwise they could have formed a link with each other to achieve d_i and d_j number of friends respectively. But we also know that node i reveals preferred d_i to $d_i - 1$ number of friends, otherwise he can unilaterally sever one of his links. Likewise node j reveals preferred d_j to $d_j - 1$ number of friends.

Therefore regardless of the indeterminacy in whether it was agent i' or j' who blocked the formation of the link between them, at least one of the following length-2 cycle must be present: (i) agents reveal preferred d_i to $d_i - 1$ number of friends, **yet** $d_i - 1$ to d_i number of friends, or (ii) they prefer d_j to $d_j - 1$ number of friends, **yet** $d_j - 1$ to d_j number of friends.

2.3. Setup

The primitives consist of (i) data, (ii) preferences, and (iii) the notion of pairwise stability.

2.3.1. Data

A network g is an ordered tuple $g = (N, E, X)$ where $N = \{1, 2, \ldots, n\}$ is the **set of agents** who can form links with each other. The set E is the **set of links**. It is a set listing who interacts with whom, also known as an adjacency list. For instance, $\{i, j\} \in E$ indicates there is a link between agents i and j. Implicit is the assumption that the network g is **undirected**, i.e., if i is linked to j, then j is also linked to i.

$X = (x_1, \ldots, x_n)$ is the vector of agents' characteristics, where $x_i \in \mathcal{X}_i$ is the characteristics of agent i. These characteristics are exogenously given and fixed across networks. For instance, x_i does not depend on who agent i is linked to, or the network that agent i is in. As a simplified example of a network, let $N = \{1, 2, 3\}$, $E = \{\{1, 2\}, \{1, 3\}\}$, and $X = (Black, White, Black)$. The network $g = (N, E, X)$ then describes a star (or line) network depicted in Figure 2.6.

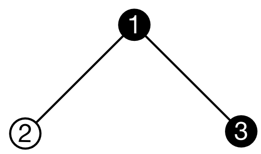

Figure 2.6: $g = (N, E, X)$

Let $\mathcal{G}(N, X)$ be the **set of all networks** with N as the set of nodes, and X as the profile of types. Unless stated otherwise, the sets N and X are fixed, and we will suppress the dependence of $\mathcal{G}(N, X)$ on N and X.

For ease of exposition, the notation $ij \in g$ refers to the undirected link $\{i, j\} \in E$ in the network $g = (N, E, X)$. A **non-link** $ij \notin g$ refers to a pair of agents $\{i, j\}$ that do not have a link between them in the network $g = (N, E, X)$, that is, $\{i, j\} \notin E$. In addition, I will use $g + ij$ to denote the network obtained by adding an undirected link between i and j in g, i.e., $g + ij$ is the network $(N, E \cup \{i, j\}, X)$. Similarly $g - ij$ is the network obtained by removing $ij \in g$.

2.3.2. Preferences

Agents have preferences over networks. For example, an agent i might prefer the network g to $g - ij$, denoted as $g \succeq_i g - ij$. When $g \succeq_i g - ij$, then we also say he prefers to form a link with agent j in the network g. In similar veins, $g \succeq_i g + ij$

means that agent i prefers not to form a link with agent j in the network g.

Recall that $\mathcal{G}$ is the set of all networks. Now I define agent i's **preference space** $\mathcal{G}_i$ as follows: $\mathcal{G}_i$ consists of pairs of networks (g, g') such that g and g' are one-link adjacent for agent i. That is, $\mathcal{G}_i = \{(g, g') \in \mathcal{G}^2 : \exists j \in N,\ g = g' + ij \text{ or } g = g' - ij\}$. That is, $(g, g') \in \mathcal{G}_i$ if and only if g, g' differs in one link that involves agent i.

Agent i's preference relation, denoted as $\succeq_i$, is formally a subset of $\mathcal{G}_i$. It is a binary relation $\succeq_i \subset \mathcal{G}_i$ that is complete and transitive. For instance, the statement $(g, g') \in \succeq_i$ is taken to mean $g \succeq_i g'$, which says that agent i weakly prefers the network g to the network g'. Moreover by definition of the preference space $\mathcal{G}_i$, we can always write $g \succeq_i g'$ as $g \succeq_i g \pm ij$, for some j. Finally let $\succeq = (\succeq_1, \ldots, \succeq_n)$ be a **profile** of preferences.

2.3.3. Pairwise Stability

In Definition 2.3.1 below, I introduce pairwise stability, which is the equilibrium condition that no agent wants to dissolve an existing relationship, and no pair of agents want to form a link between them. Pairwise stability was first introduced in Jackson and Wolinsky (1996), and is a weak requirement that individuals have a tendency to form relationships that are mutually beneficial and to drop relationships that are not. Other more restrictive equilibrium concepts exist, for example pairwise Nash equilibrium.[5] Pairwise stability is often considered as a necessary condition for any equilibrium concept in strategic network formation (Calvó-Armengol and İlkılıç (2009)).

[5] Intuitively, pairwise stable networks are robust to one-link deviations, while pairwise-Nash networks are robust to many-links deletion and single-link creation. See Goyal and Joshi (2003), Goyal (2009) for more.

Definition 2.3.1. The network $g \in \mathcal{G}$ is *pairwise stable* with respect to a profile of preferences $\succeq$ if

1.) For all $ij \in g$, $g \succ_i g - ij$ AND $g \succ_j g - ij$.

2.) For all $ij \notin g$, $g \succ_i g + ij$ OR $g \succ_j g + ij$

Implicit in this definition is the idea that while forming links is a collective and bilateral effort of two parties, it only takes one side to dissolve a link.

Remark 1: Definition 2.3.1 is the strict version of pairwise stability as defined in Jackson and Wolinsky (1996). In their definition, a link or a non-link can be explained by agents being indifferent to forming a link. This stability concept is too weak: by assuming that agents are always indifferent, any observed networks can be rationalizable as pairwise stable.

2.3.4. Statement of Problem

We can now formally define the statement of problem: what is the necessary and sufficient condition on network g such that g is rationalizable as pairwise stable? The notion of rationalizability is more carefully stated below.

Definition 2.3.2. The network data $g \in \mathcal{G}$ is **rationalizable as pairwise stable** if and only if there exists a profile of preferences $\succeq = (\succeq_1, \ldots, \succeq_n)$ such that g is pairwise stable with respect to $\succeq$.

Without additional assumption on preferences, any network is rationalizable as pairwise stable according to Definition 2.3.2 – we can just say that agents strictly prefer the observed network to any other networks. The following sections will introduce an assumption that restricts preference heterogeneity: agents that are observed to be similar in the network data have similar preferences. It turns out that under this one assumption, the problem stated here has a rich answer.

2.3.5. Similarity

In order to properly define the restricted heterogeneity assumption, we need a notion of similarity on networks. Only then, we can say that agents who are observed to be similar have similar preferences. Similarity is formalized as an equivalence relation on N (the set of agents) and $\mathcal{G}$ (the set of all networks on N). The idea that agent i in network g is similar to agent i' in network g' can be denoted by $(i,g) \sim (i',g')$.

Definition 2.3.3 (Similarity)**.** A similarity relation $\sim$, is an equivalence relation on $N \times \mathcal{G}$.

For instance, we can define the similarity relation $\sim$ to be such that $(i,g) \sim (i',g')$ if and only if $\theta_i(g) = \theta_{i'}(g')$, where $\theta_i(g)$ is a vector of observed characteristics of agent i in network g. The vector θ can consist of both exogenous characteristics of agent i (such as his gender, age, wealth), as well as characteristics that depend on the particular network that agent i is in. For instance, $\theta_i(g)$ could tell us the clustering coefficient and network centrality of agent i in network g.

2.3.6. Restriction on preference heterogeneity

The final primitive I introduce is the notion of heterogeneity restriction on preferences. Restricted heterogeneity says that agents that are observed to be similar have similar preferences. This assumption on preferences is the only restriction that we will impose on preferences.

Without this assumption, agents' preferences are allowed to be fully heterogeneous, and crucially, heterogeneous in ways that are unobserved to the analyst, so then there always exists some preferences that would rationalize any network data. For the question of what networks can be rationalized as pairwise stable to be meaningful, we need to impose some notion of bounded preference heterogeneity.

Restriction on preference heterogeneity is defined with respect to a given similarity relation, where a similarity relation (Definition 2.3.3) captures which agents are similar to whom from the analyst's perspective. Therefore, an analyst can propose a notion of similarity, and then heterogeneity of preferences is defined with respect to that.

Definition 2.3.5 below states the notion of preference heterogeneity restriction. Intuitively, it says that agents that are observed to be similar are deemed to have the same preferences. We will always specify similarity in terms of observable characteristics of the agents, therefore this assumption is tantamount to assuming that there is *no unobserved heterogeneity in preferences.* In particular, names or identities of the agents do not matter.

The restricted heterogeneity assumption we impose on preferences is weak in the sense that it does not restrict preferences for networks that are off-path. Off-path networks are those that are more than one-link different from the observed network. Therefore we are completely agnostic about preferences of agents when they are off-path from the observed network.

Definition 2.3.4. Let g be an observed network. The set of **on-path networks** with respect to g, $\mathcal{G}(g)$, consists of networks that are one-link adjacent to g. That is, $\mathcal{G}(g) = \{g' \in \mathcal{G} : \exists ij \in g \text{ or } \exists i'j' \notin g \text{ such that } g' = g - ij \text{ or } g' = g + i'j'\}$. In words, the set of on-path networks, $\mathcal{G}(g)$, consists of networks that are obtainable from g through a single deletion or addition of link. Therefore when observing the network g, pairwise stability only reveals preferences for on-path networks with respect to g, i.e. those networks in $\mathcal{G}(g)$.

Definition 2.3.5. Let g^0 be the observed network. The preference profile $\succeq$ is **heterogeneous with respect to the similarity relation** $\sim$ at g^0 if and only for all on-path networks: g, $g - ij$ and g', $g' - i'j'$:
$(i, g) \sim (i', g')$ *and* $(i, g - ij) \sim (i', g' - i'j')$ *implies that agent* i *must rank the networks* g *and* $g - ij$ *the same way as agent* i' *ranks the networks* g' *and* $g' - i'j'$.[6]

Imagine an agent i choosing between two alternatives, (i, g) and $(i, g - ij)$, that is, he chooses whether to form a link with agent j in network g. Now agent i' is also choosing between two alternatives, (i', g') and $(i', g' - i'j')$. Heterogeneity with respect to $\sim$ says that because agents i, i' are facing choice problems that are observationally similar according to $\sim$, they must have the same preference (ordinal ranking) in these two parallel choice problems. Whenever agent i prefers to form a link with j in g, then agent i' prefers to form a link with j' in g'.

In fact, the extensive example discussed in the last section imposes the restricted heterogeneity assumption where $\sim$ is defined to be $(i, g) \sim (i', g')$ whenever $d_i(g) = d_{i'}(g')$, the number of links (degree) of i in g is denoted by $d_i(g)$. Note that in this example, $(i, g) \sim (i', g') \iff (i, g - ij) \sim (i', g' - i'j')$

[6] More precisely, either (i) $g \succ_i g - ij$ and $g' \succ_{i'} g' - i'j'$, or (ii) $g \prec_i g - ij$ and $g' \prec_{i'} g' - i'j'$.

2.4. Main result

This section presents testable conditions that characterize what networks are rationalizable as pairwise stable. Only one restriction is imposed on preferences, that is preferences are heterogeneous with respect to a given similarity relation $\sim$, i.e. agents that are similar according to the similarity relation $\sim$ have similar preferences.

The following axiom, the Weak Axiom of Revealed Pairwise Stability, characterizes the testable implication of pairwise stability (under the assumption of restricted heterogeneity).

Definition 2.4.1 (WARPS)**.** Let $\sim$ be an equivalence relation on $N \times \mathcal{G}$. The network g satisfies the Weak Axiom of Revealed Pairwise Stability (WARPS) for $\sim$ if and only if for all pairs of links $(i,x),(j,y) \in g$, there does not exist a non-link $(i',j') \notin g$ such that:

(i) $(i,g) \sim (i', g + i'j')$ and $(i, g - ix) \sim (i', g)$

(ii) $(j,g) \sim (j', g + i'j')$ and $(j, g - jy) \sim (j', g)$

Theorem 2. Let g be an observed network, and $\sim$ be an equivalence relation on $N \times \mathcal{G}$. Assume that preferences are heterogeneous with respect to the similarity relation $\sim$ at g. *An observed network g is rationalizable as pairwise stable if and only if g satisfies WARPS for $\sim$ (Definition 2.4.1).*

Violation of WARPS by the network g implies that there is a link that should be formed (say between agents i' and j'), but it is not observed in the data g. As a consequence, there does not exist preferences that would rationalize g as pairwise stable. How do we know that the a link should be formed between agents i' and j'? From observing the revealed preference of other similar agents (say i and j). The choices of these similar agents reveal that they would like to form links ((i, x) and (j, y) respectively) in g, which implies that agents i', j' should also form the non-link $(i', j') \notin g$. Preference heterogeneity with respect to $\sim$ allows us to identify that agents i', j' have similar preferences to i, j' respectively.

A formal argument based on the above paragraph is used to prove that WARPS is necessary (Section 2.4.1 below). From this result, we can see that when too few agents are similar according to $\sim$, then WARPS is always satisfied by all networks. In particular, when $\sim$ is defined such that $(i, g) \sim (i', g')$ only if the networks g and g' have the same number of links, then WARPS is always satisfied, and all networks are rationalizable as pairwise stable. When WARPS has no power, we would need to impose a weaker heterogeneity restriction via $\sim$.

WARPS here is tantamount to an algorithm checking for cycles in an appropriately constructed profile of revealed preference relations. Existence of transitive and complete preference relations can then be found by extending these acyclic revealed preference relations. This is the technique used to prove the sufficiency of WARPS, which is the more difficult direction. The adjective ‘weak’ in WARPS is used because WARPS checks for cycles of length 2, which is reminiscent of the Weak Axiom of Revealed Preference (WARP) in consumer theory.

2.4.1. Outline of Proof

Instead of working with the similarity relation $\sim$ as in Definition 2.3.3, it will be more convenient to work with the following induced equivalence relation $\sim^*$.

Definition 2.4.2. Let $\sim^*$ be an equivalence relation on $N \times \mathcal{G}^2$ induced by the similarity relation $\sim$ such that $(i, g, g - ij) \sim^* (i', g', g' - i'j')$ if and only if $(i, g) \sim (i', g')$ and $(i, g - ij) \sim (i', g' - i'j')$

Following this new definition of similarity relation, a preference profile $\succeq$ is heterogeneous with respect to the similarity relation $\sim^*$ if and only if $(i, g, g - ij) \sim^* (i', g', g' - i'j')$ implies that agent i must rank the networks g and $g - ij$ the same way as agent i' ranks the networks g' and $g' - i'j'$.

2.4.1.1. Necessity

We will now argue that WARPS is necessary. So suppose that WARPS is violated for a network g, but there exists some preferences that rationalize g as pairwise stable. The next two paragraphs will demonstrate a contradiction. Begin with the fact that there is no link between agents i' and j'. From pairwise stability, we can only infer that *either or both* of i' and j' prefers g to $g + i'j'$.

Consider the first possibility that it is agent i' who prefers not to form the link $i'j'$. Now from observing the link (i, x) in the data, it must be that $g \succ_i g - ix$. Moreover (i, x) is such that $(i, g, g - ix) \sim^* (i', g + i'j', g)$. We can then infer that agents i, i' have similar preference, and in particular, $g \succ_i g - ix$ implies that $g + i'j' \succ_{i'} g$. This contradicts the assumption that agent i' prefers g to $g + i'j'$.

Consider the second possibility that it is agent j' who prefers not to form the link $i'j'$. Now from observing the link (j, y) in the data, it must be that $g \succ_j g - jy$. Moreover (j, y) is such that $(j, g, g - jy) \sim (j', g + i'j', g)$. We can then deduce that agents j, j' have similar preference, and in particular, $g \succ_j g - jy$ implies that $g + i'j' \succ_{j'} g$.

Again this contradicts the assumption that agent j' prefers g to $g + i'j'$.

2.4.1.2. Sufficiency

To show that WARPS is sufficient, we will utilize Lemma 6, which says if we can elicit a profile of revealed preference relations from the observed network g such that it is acyclic when extended according to the similarity relation $\sim$, then there exist complete and transitive preferences that rationalize g as pairwise stable (and satisfy restricted heterogeneity). The main machinery is Algorithms 1 and 2, which constructively demonstrate a profile of revealed preference relation whose extension according to $\sim$ is acyclic.

First, we will define how to elicit revealed preference relations from the data. If a link between agents i and j is observed in the network g, then we say that agents i and j both *reveal preferred* to have this link to not having this link in g. On the other hand, if we do not observe a link between agents i and j in the network g, then it is revealed that *either* agent i *or* agent j prefers not to form this link in g. More formally,

Definition 2.4.3. For a given network g, let $\succ^R = \prod_{i \in N} \succ_i^R$ be a profile of **revealed preference relations** defined as follows:

1.) For all $ij \in g$, $g \succ_i^R g - ij \quad \text{AND} \quad g \succ_j^R g - ij$.

2.) For all $ij \notin g$, $g \succ_i^R g + ij \quad \text{OR} \quad g \succ_j^R g + ij$

This definition does not pin down a unique revealed preference relation (due to part 2.) of the definition). Now let $\mathcal{R}(g)$ be the set of all such profiles of revealed preference relations derived via Definition 2.4.3 above. The set $\mathcal{R}(g)$ contains all the possible revealed preference relations that are induced by the data g.

Definition 2.4.4. The network $g \in \mathcal{G}$ is **rationalizable** as pairwise stable if and only if there exists a profile of strict preferences that extends some profile of revealed preference relations in $\mathcal{R}(g)$.

In classical revealed preference theory, the revealed preference relation is uniquely defined. In our setup, rationalization only requires that we can find just one revealed preference relation in $\mathcal{R}(g)$ that can be extended into a complete and transitive relations.

Definition 2.4.5. Let $\succ = \prod_{i\in N} \succ_i$ be a profile of binary relations. The extension of $\succ^R$ according to the similarity relation $\sim$ is a profile of binary relations $R = (R_1, \ldots, R_n)$ such that if $g \succ_i g - ij$, $(i, g) \sim (i', g')$ and $(i, g - ij) \sim (i', g' - i'j')$, then $g' R_i \, g' - i'j'$

Definition 2.4.6. The profile of strict (asymmetric) binary relation $\succ^R = (\succ_1^R, \ldots, \succ_n^R$) is acyclic if for all $i \in N$, there is no sequence $x_1, x_2, \ldots, x_L$ such that $x_1 \succ_i^R x_2 \succ_i^R \ldots x_L$ and $x_L \succ_i^R x_1$.

Lemma 6. *Assume that preferences are heterogeneous with respect to the similarity relation $\sim$. The network g is rationalizable as pairwise stable if and only if there exists a profile of revealed preference relations $\succ^R \in \mathcal{R}(g)$ such that the extension of $\succ^R$ according to $\sim$ is acyclic.*

Lemma 6 is an application of a version of *Szpilrajn extension theorem.* If a revealed preference relation is acyclic, then we can extend it to a complete and transitive preference relation. Since we also want the complete and transitive preferences to satisfy heterogeneity with respect to $\sim$, the revealed preference relations are first extended according to $\sim$, and it is the acyclicity of this $\sim$-extended relations that matter.

The profile of revealed preference relations itself is always acyclic by definition, but its' $\sim$-extension might not be. In fact, we now show that when WARPS for $\sim$ is satisfied, there exists a profile of revealed preference relation whose $\sim$-extension is acyclic, where $\sim$ is some similarity relation. Throughout the rest of this section, it will be maintained that (a) g denotes the observed network, (b) g satisfies WARPS for $\sim$, and (c) $\sim$ is some similarity relation.

The application of Algorithms 1 and 2 successively will demonstrate existence constructively. The input to the first algorithm is a special profile of relations that are elicited from g by assuming that link deletion is bilateral. This revealed preference relation is obtained by inferring that when a link is not observed between two individuals, *both* sides do not want to form this link. See Definition 2.4.7 below.

Definition 2.4.7. For an observed network g, let $\succ^0 = \prod_{i \in N} \succ_i^0$ be the unique profile of revealed preference relations defined as follows:

1.) For all $ij \in g$, $g \succ_i^0 g - ij$ AND $g \succ_j^0 g - ij$

2.) For all $ij \notin g$, $g \succ_i^0 g + ij$ AND $g \succ_j^0 g + ij$

Let R be an $\sim$-extension of some $\succ \in \mathcal{R}(g)$. To understand the following algorithms, observe that all possible cycles in R must take the following form: either ($g \, R_i \, g - ix$ and $g - ix \, R_i \, g$), or ($g \, R_i \, g + ix$ and $g + ix \, R_i \, g$). This is because in Definition 2.3.5, restricted heterogeneity is defined only locally on g, it only restricts preferences for on-path networks, hence restricted heterogeneity only causes cycles in R that involve on-path networks.

Algorithm 1: Eliminate cycles I

```
input  : ≻ as the profile of revealed preference relations constructed in
         Definition 2.4.7
output: ≻^1, a profile of binary relations

while there is a cycle in R of the form (g R_i g − ix and g − ix R_i g), where R is
the extension of ≻ according to ∼, do
    identify a cycle of the form (g R_i g − ix and g − ix R_i g). The relation
    g − ix R_i g is true if and only if there is a sequence of agents
    (i_0, i_2, ..., i_{M−1}) and a sequence of non-links (i_0, j_0), ..., (i_{M−1}, j_{M−1}) ∉ g
    such that for all m = 1, ..., M − 1, (i, g, g − ix) ∼* (i_m, g + i_m j_m, g) and
    g ≻_{i_m} g + i_m j_m;
    for m = 1, ..., M do
        replace (g ≻_{i_m} g + i_m j_m) with (g ≺_{i_m} g + i_m j_m);
        for each (i', j') ∉ g such that (i', g, g + i'j') ∼* (i_m, g, g + i_m j_m) do
            replace (g ≻_{i'} g + i'j') with (g ≺_{i'} g + i'j');
        end
    end
end
```

In Lemma 7 below, we will show that Algorithm 1 terminates in finite steps, and crucially, we will claim that the output $\succ^1$ from the algorithm belongs to the set $\mathcal{R}(g)$, i.e. it is a profile of revealed preference relations.

Lemma 7. *Algorithm 1 terminates in finite steps. The output of Algorithm 1 is a profile of revealed preference relations, i.e.* $\succ^1 \in \mathcal{R}(g)$.

Algorithm 2: Eliminate cycles II

input : $\succ$ as $\succ^1$, the profile of revealed preference relations from Algorithm 1;

output: $\succ^2$, a profile of binary relations;

while *there is a cycle in* R *of the form* $(g\, R_i\, g + ix$ *and* $g + ix\, R_i\, g)$, *where* R *be the extension of* $\succ$ *according to* $\sim$, **do**

  identify a cycle of the form $(g\, R_i\, g + ix$ and $g + ix\, R_i\, g)$;

  if $(g \succ_i g + ix)$ *is true* **then**

    replace $(g \succ_i g + ix)$ with $(g \prec_i g + ix)$;

    for *each* $(i', j') \notin g$ *such that* $(i', g, g + i'j') \sim^* (i, g, g + ix)$, *and* $(g \succ_{i'} g + i'j')$ **do**

      replace $(g \succ_{i'} g + i'j')$ with $(g \prec_{i'} g + i'j')$;

    end

  end

end

Lemma 8. *Algorithm 2 terminates in finite steps. The output* $\succ^2$ *is a profile of revealed preference relations, i.e.* $\succ^2 \in \mathcal{R}(g)$. *Moreover, the extension of* $\succ^2$ *according to* $\sim$ *is acyclic.*

What we have shown is whenever WARPS for $\sim$ is satisfied, we can construct (through Algorithms 1 and 2), a profile of revealed preference relations from the observed network, such that the extension according to $\sim$ is acyclic. By Lemma 6, this extended profile can be further extended into a complete and transitive preference relations that satisfy heterogeneity with respect to $\sim$.

2.5. Unobserved Heterogeneity

In this section, agents are allowed to have unobserved characteristics. When there is unbounded unobserved heterogeneity, every agent can have different preferences, and as a result, we can rationalize any network as pairwise stable (WARPS is always satisfied). However we can ask if under limited but reasonable amount of unobserved heterogeneity, a network would be rationalizable as pairwise stable.

The main result of this section is the formulation of WARPS with randomness (Proposition 3), which characterizes the likelihood that a network is rationalizable as pairwise stable, given some heterogeneity restriction on *both* observed and unobserved characteristics. The source of randomness is driven by the distribution of unobserved characteristics. The second result then derives the expected minimum number of types of agents that are needed to rationalize the network data as pairwise stable. Here, types are endogenous partition of the agents' characteristics (both observed and unobserved).[7]

The analysis with unobserved characteristics is based on the previous sections, the main difference is that the similarity relation, $\sim$, between agents is now probabilistic, to reflect our uncertainty about the unobserved characteristics agents have. As a result, whether or not an observed network satisfies WARPS is now a random variable, whose realization depends on the draw from the distribution of unobserved characteristics. This random version of WARPS is what characterizes the likelihood that

[7] A paper that is similar in spirit is Crawford and Pendakur (2013), which asks how many types of consumers are there that would rationalize a consumption data set.

a network is rationalizable as pairwise stable. For instance, we can then ask what is the probability that WARPS is rejected.

Let ϵ_i be a vector of (exogenous) unobserved characteristics of agent i. Agent i's true vector of characteristics is $\tau_i(g)$, which can depend on the network g, that agent i is in. We will assume that $\tau_i(g) = \theta_i(g) + \epsilon_i$, where $\theta_i(g)$ is the vector of observed characteristics of agent i in network g. The vector $\theta_i(g)$ can contain both exogenous characteristics of i, and the characteristics of i that is endogenous to the network g. For instance, $\theta_i(g)$ could have contain rows indicating the age and wealth of agent i, and it could contain rows indicating the network centrality and the degree of agent i in network g

Definition 2.5.1 (Similarity relation with unobserved characteristics)**.** Let $\sim$ be an equivalence relation on $N\times\mathcal{G}$ such that $(i,g)\sim(i',g')$ if and only if $\left|\tau_i(g)-\tau_{i'}(g)\right|<\delta$, where $\delta\in\mathbb{R}^L_+$ and $\tau_i(g)=\theta_i(g)+\epsilon_i$. Now $\theta: N\times\mathcal{G}\to\mathbb{R}^L$ is an L-dimensional vector of observed characteristics. For all $i\in N$, ϵ_i is an L-dimensional vector of random variable. $(\epsilon_1,\dots,\epsilon_n)$ is jointly distributed according to the distribution $F(\cdot)$. For simplicity, we will further assume that $(\epsilon_i)_{i\in N}$ are i.i.d.

Proposition 3 (Random WARPS)**.** *Let $\sim$ be a random similarity relation defined above. Let g be the observed network and assume that preferences are heterogeneous with respect to $\sim$ at g. The likelihood that a network g is rationalizable as pairwise stable is given by $\Pr[T(g,\delta)=1]$ in Equation (2.5.1) below, where the expectation is* *taken with respect to F, the joint probability distribution of $\epsilon = (\epsilon_1, \dots, \epsilon_n)$.*

$$\mathbb{E}\left[\max_{(i,x),(j,y)\in g,\,(i',j')\notin g}\left\{\mathbb{1}\left[\left|\tau_i(g)-\tau_{i'}(g+i'j')\right|<\delta\right]\cdot\mathbb{1}\left[\left|\tau_i(g-ix)-\tau_{i'}(g)\right|<\delta\right]\right.\right. \tag{2.5.1}$$
$$\left.\left.\cdots\mathbb{1}\left[\left|\tau_j(g)-\tau_{j'}(g+i'j')\right|<\delta\right]\cdot\mathbb{1}\left[\left|\tau_j(g-jy)-\tau_{j'}(g)\right|<\delta\right]\right\}\right]$$

Proof. The idea is to rewrite WARPS and use Theorem 2. WARPS in Definition (2.4.1) can represented by an indicator function $T(g,\sim)\in\{0,1\}$. When $T(g,\sim)=0$, WARPS is satisfied for g, and when $T(g,\sim)=1$, WARPS is rejected for g.

$$T(g,\sim)=\max_{(i,x),(j,y)\in g,\,(i',j')\notin g}\left\{\mathbb{1}\left[(i,g)\sim(i',g+i'j')\right]\cdot\mathbb{1}\left[(i,g-ix)\sim(i',g)\right]\right.$$
$$\left.\cdots\mathbb{1}\left[(j,g)\sim(j',g+i'j')\right]\cdot\mathbb{1}\left[(j,g-jy)\sim(j',g)\right]\right\}$$

Theorem 2 says that when there is no unobserved characteristics, a network g is rationalizable as pairwise stable (with preferences that are heterogeneous with respect to $\sim$) if and only if $T(g,\sim)=1$.

When there is unobserved characteristics, $T(g,\sim)$ becomes a random variable. For each draw of unobserved heterogeneity $(\epsilon_1,\dots,\epsilon_n)$ from $F(\cdot)$, a value of $T(g,\sim)\in\{0,1\}$ is realized. In particular, the likelihood that g is rationalizable as pairwise stable is the probability that $\Pr[T(g,\sim)=1]$, which is just $\mathbb{E}[T(g,\sim)]=\int T(g,\sim)dF$.

□

Provided that we specify what the threshold δ is, the quantity of interest, $\Pr[T(g,\delta)=1]$ in Equation (2.5.1) can be computed by simulation. For instance, assuming $(\epsilon_i)_{i\in N}$ are independent, we can set $\delta=0.1\ \mathrm{diag}(\Sigma)$ where $\mathrm{Var}(\epsilon_i-\epsilon_{i'})=\Sigma$, and $\mathrm{diag}(\Sigma)$ is the vector of the diagonals of Σ.

We can also ask what is the largest δ such that the data g is rationalizable at least $1-\alpha\%$ of the time. When δ is small enough, WARPS is never rejected and any network is rationalizable. We can think of δ as inversely proportional to how much heterogeneity there is among agents. If we need very large δ to achieve $1-\mathbb{E}[T(g,\delta)] \geq 1-\alpha$, or $\alpha \geq \mathbb{E}[T(g,\delta)]$, then it means that we even under limited heterogeneity, the network g can be rationalized as pairwise stable.

To elaborate on this, let δ^* as defined in Equation (2.5.2) below. Equation (2.5.2) says that we want to minimize the objective function, $\Pr[|\tau_i(g) - \tau_{i'}(g)| < \delta]$, which quantifies how likely are two agents similar to each other under some δ. The constraint is that WARPS is rejected less than $\alpha\%$ of the time. Therefore, N^* in Equation (2.5.3) can be interpreted as follows: we need at least N^* number of types in order to confidently rationalize the data g as pairwise stable at the $\alpha\%$ level.

Definition 2.5.2. The expected minimum number of types of agents needed to rationalize the network g as pairwise stable at least $1-\alpha\%$ of the time is N^*, which solves the following two equations. The function $\mathbb{E}[T(g,\delta)]$ depends on δ and is given in Equation (2.5.1).

$$\delta^* = \underset{\delta \in \mathbb{R}^L_+}{\operatorname{argmax}} \left\{ \Pr[|\tau_i(g) - \tau_{i'}(g)| < \delta] \right\} \text{ subject to } \alpha \geq \mathbb{E}[T(g,\delta)] \tag{2.5.2}$$

$$N^* = \frac{1}{\Pr[|\tau_i(g) - \tau_{i'}(g)| < \delta^*]} \tag{2.5.3}$$

The number N^* informs us the difficulty of rationalizing a network as pairwise stable. It tells us how much preference heterogeneity we need to rationalize a network as pairwise stable. If N^* is large relative to the number of agents in the network, then rationalizing the network as pairwise stable comes at a substantial cost of relying on the assumption that agents have myriad of different preferences. All else the same, we desire a smaller N^* relative to the size of the network, because then the data is rationalizable with a simpler model of preferences.

2.6. Additional Proofs

2.6.1. Proof of Lemma 7

Proof of Lemma 7. We want to show that the output of the algorithm is a profile of revealed preference relations. This consists of showing that WARPS implies the algorithm does not result in a profile of relations where there exists a pair of non-link $i'j' \notin g$ such that $(g+i'j' \succ_{i'} g)$ and $(g+i'j' \succ_{j'} g)$. That is, this would imply that both agents i', j' prefer to form the non-link $i'j' \notin g$, which is not consistent with Definition 2.4.3. If for some $(i', j') \notin g$ the algorithm changes $(g \succ_{i'} g + i'j')$ to $(g + i'j' \succ_{i'} g)$, and $(g \succ_{j'} g + i'j')$ to $(g + i'j' \succ_{j'} g)$, then there must exist links $(i, x), (j, y) \in g$ such that $(i, g, g - ix) \sim^* (i', g + i'j', g)$ and $(j, g, g - jy) \sim^* (j', g + i'j', g)$. This is precisely ruled out by WARPS.

Moreover the algorithm must terminate in finite steps because at each iteration of the while-loop, the relation $g - ix\, R_i\, g$ is reversed, and hence one cycle of the form $(g\, R_i\, g + ix$ and $g + ix\, R_i\, g)$ is eliminated, and no new cycle of that form is introduced. Therefore each iteration of the while-loop strictly decreases the number of cycles in R, and the algorithm must terminate in finite steps. $\square$

2.6.2. Proof of Lemma 8

Proof of Lemma 8. We will first prove by induction that whenever there is a cycle of the form ($g\,R_i\,g{+}ix$ and $g{+}ix\,R_i\,g$), there is only one possibility: $g \succ_i g{+}ix$, but there is a link $(p,q) \in g$ such that $(p, g, g-pq) \sim^* (i, g+ix, g)$. The two other possibilities are (i) $g \succ_i g+ix$, but there is a non-link $(i', j') \notin g$ such that $g + i'j' \succ_{i'} g$, and $(i', g+i'j', g) \sim^* (i, g+ix, g)$, or (ii) $g+ix \succ_i g$, but there is a non-link $(i', j') \notin g$ such that $g \succ_{i'} g+i'j'$, and $(i', g+i'j', g) \sim^* (i, g+ix, g)$.

We assume that this hypothesis holds for the 1st to t-th iterations of the while loop. Assume we are at the $t+1$-th iteration. Cases (i) and (ii) are directly ruled out by the for-loop in Lines (5) and (5) of Algorithms 1 and 2 respectively (similarly for base case of the induction).

At each iteration of the while loop, the cycle ($g\,R_i\,g+ix$ and $g+ix\,R_i\,g$) is destroyed because $g \succ_i g+ix$ is switched to $g \prec_i g+ix$. Moreover, no new cycle will be introduced. If there is a new cycle, then it would be because Case (i) or (ii) arises. Therefore the algorithm must terminate.

Finally we prove that the output of the algorithm $\succ^2$ is a profile of revealed preference relations, i.e. it belongs to $\mathcal{R}(g)$. The proof consists of showing WARPS implies that the algorithm does not result in a profile of relations such that there exists a pair of non-link $i'j' \notin g$ such that $(g+i'j' \succ_{i'} g)$ and $(g+i'j' \succ_{j'} g)$. That is, both agents i', j' prefer to form the non-link $i'j' \notin g$, which is not consistent with Definition 2.4.3. If the algorithm changes $(g \succ_{i'} g+i'j')$ to $(g+i'j' \succ_{i'} g)$, and $(g \succ_{j'} g+i'j')$ to $(g+i'j' \succ_{j'} g)$, then there exists links (i,x), (j,y) such that $(i, g, g-ix) \sim^* (i', g+i'j', g)$ and $(j, g, g-jy) \sim^* (j', g+i'j', g)$. This is precisely ruled out by WARPS. □

Chapter 3

On the Consistency of Network Data with Pairwise Stability: Application

3.1. Introduction

This chapter is presents an empirical application, the goal is to investigate the extent to which real-world networks are consistent with pairwise stability through consistency with WARPS. In particular, I consider a special implementation of WARPS where similarity is defined using graph isomorphism. That is, similarity relation is defined such that agents are similar when their respective network neighborhoods are isomorphic. This is a conservative notion of similarity that says agents have similar preferences only when their respective network structures are similar. The use of graph isomorphism allows preferences to depend freely on the network structures freely specifying how.

WARPS as developed in the theoretical section is either passed or violated by an observed network. For the application, I develop a continuous measure of how consistent an observed network is to WARPS. Moreover, this numerical consistency score is benchmarked with the consistency of a suitably calibrated random (Erdös-Rényi) network with WARPS.[1]

Using these measures, and I demonstrate that the consistency of real-world networks with WARPS matters for economic outcomes. To show this, I use the data set of Banerjee et al. (2013), which consists of a cross-section of 75 villages in rural India. For each village, we observe (i) the social network of the village, (ii) the nodes who were first informed of a new microfinance loan program, as well as (iii) the eventual adoption rate of microfinance.

Banerjee et al. (2013) show that the network centrality of the first-informed agents (injection points) is significantly and positively related to the adoption rate of microfinance. I find that this relationship is significantly more positive when the underlying network is more consistent with WARPS. In showing this, I maintain their exact specification and controls, incorporating only my measures of consistency. Therefore, the consistency of WARPS is new network statistics that has potentially useful application: when a network is more consistent with WARPS, targeting of agents that have central positions in the network to increase the diffusion of information is more effective.

Besides using different measures of consistency as a robustness check, a placebo test is also performed in which a host of other network measures are used in place of our measure of consistency. Like WARPS, these network measures are calculated using only information about the network structure, but unlike WARPS, they were not significant in explaining diffusion. Finally, the result is robust to using a specific

[1] This measure is in line with the notion of predictive success used in the empirical revealed preference literature (Beatty and Crawford (2011); Selten (1991)).

network type, the advice network, as the social network of each village (instead of an aggregate of different network types).

To get a sense of magnitudes, in their findings, one standard deviation increase in the diffusion centrality (a measure how central individuals are in their social network with regard to spreading information) of injection points would lead to an increase in the eventual adoption rate of 3.9 percentage points. I find that this result depends significantly at how consistent to WARPS the underlying network is. For instance, at the 90th, 75th, and 25th percentile consistency, the corresponding magnitudes are 7.2-8.1, 5.9-6.1, and 1.3-2.0 percentage points increase in the adoption rate.

The intuition for the result is as follows. Networks that are less consistent with WARPS are more difficult to rationalize as pairwise stable. There are only two reasons, either we do not expect pairwise stability to hold, or that there is unobserved heterogeneity not capture by the network structure. For either of these explanations, centrality of injection points *with respect to* the network structure does not correspond to the actual effectiveness of these agents in diffusion. Therefore, the relationship between network centrality and adoption rate is weaker as the underlying network violates WARPS more severely. This is precisely what we see in the data.

3.2. Graph theoretic definitions

Definition 3.2.1 (Isomorphism)**.** Consider networks $g, g' \in \mathcal{G}$, g and g' are isomorphic if and only if there is a bijection $\sigma : N \to N$ such that $ij \in g \iff \sigma(i)\sigma(j) \in g'$.

Two networks are isomorphic if they have the same network structure. That is, we can relabel and permute the identities of the nodes in one network to obtain another

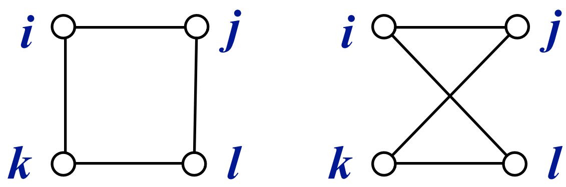

Figure 3.1: The left network: $g = \{ij, ik, jl, kl\}$, and the right network: $g' = \{ij, il, jk, kl\}$ are isomorphic. Both networks represent the same network structure.

isomorphic network. For example, consider the network $g = \{ij, ik, jl, kl\}$ depicted in the left side of Figure 3.1 below. Consider a permutation σ on the labels of the nodes such that $\sigma(k) = l$, $\sigma(l) = k$, $\sigma(i) = i$ and $\sigma(j) = j$. Then, we can see that $\sigma(g) = \{ij, il, jk, lk\} = g'$.

Definition 3.2.2 (k-neighborhoods)**.** The **neighborhood** of node i in network g is the set of nodes that i is linked to. While the k-neighborhood of node i in network g is the set of all nodes that are distance no more than k from i. Now the **k-neighborhood network** of node i in network g is the subgraph induced by the nodes in the k-neighborhood of i. For simplicity, k-neighborhood will be understood as the k-neighborhood *network*.

More formally, $N_i(g) = \{i\} \cup \{j : ij \in g\}$ is the neighborhood of node i in g. The k-neighborhood is recursively defined as $N_i^k(g) = N_i(g) \cup \left(\bigcup_{j \in N_i(g)} N_j^{k-1}(g)\right)$. The k-neighborhood graph of node i in g, denoted as $g(N_i^k(g))$, is the *subgraph* of g with $N_i^k(g)$ as the set of nodes such that $i'j' \in g(N_i^k(g))$ if and only $i', j' \in N_i^k(g)$ and $i'j' \in g$.

3.3. k-WARPS

A specific form of the Weak Axiom of Revealed Pairwise Stability (WARPS) is provided here. The main result of the previous section, Theorem 2, is applied to obtain Corollary 6 here.

Definition 3.3.1 (k-similar)**.** Let $\sim_k$ be an equivalence relation on $N \times \mathcal{G}$ such that $(i, g) \sim_k (i', g')$ (read as agent i in network g is **k-similar** to agent i' in network g') if and only if there exists an isomorphism σ from the k-neighborhood network of i to that of i' such that $\sigma(i) = i'$.

When the similarity relation $\sim_k$ is as in Definition 3.3.1, we can simplify the statement of WARPS through the following observation: whenever $(i, g) \sim_k (i', g + i'j')$, then there is a link $ix \in g$ such that $(i, g - ix) \sim_k (i', g)$.

Definition 3.3.2 (k-WARPS)**.** Let $\sim_k$ be a similarity relation from Definition 3.3.1. The network g satisfies k-WARPS if and only if for all pairs of agents (i, j), there does not exist a non-link $i'j' \notin g$ such that:

(i) $(i, g) \sim_k (i', g + i'j')$

(ii) $(j, g) \sim_k (j', g + i'j')$

Corollary 6: Let g be the observed network, and let $\sim_k$ be a similarity relation in Definition 3.3.1. Assume that preferences are heterogeneous with respect to $\sim_k$ at g. The network g is rationalizable as pairwise stable if and only if g satisfies k-WARPS.

3.3.1. Illustration

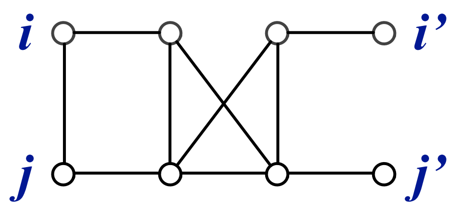

Figure 3.2: Network g

The network g depicted in Figure 3.2 fails to satisfy k-WARPS with $k = 2$. In particular, we will show that $(i, g) \sim_2 (i', g + i'j')$ and $(j, g) \sim_2 (j', g + i'j')$. The network in Figure 3.3 is the neighborhood of agent i in network g, while the network in Figure 3.4 shows the 2-neighborhood of agent i' in network $g + i'j'$. Figure 3.4 on the right is obtained by first adding a link between i' and j' (indicated by the color purple), and then removing the greyed out nodes and links.

Agents i and i' occupy similar positions in the respective neighborhoods, formally, there is an isomorphism σ of between the two neighborhood networks such that $\sigma(i) = i'$. Therefore we conclude that $(i, g) \sim_2 (i', g + i'j')$. Note that the network in Figure 3.4 is never observed in the data, but we postulate through $\sim$ that agent i' preference in this network $g + i'j'$, is similar to agent i's preference in the observed network g. Also observe that agents i, i' are never similar in the observed network g, as agent i has two links while i' has only one link in g.

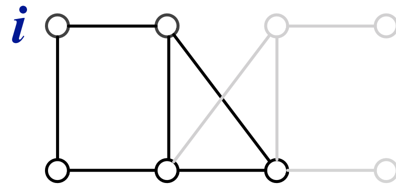

Figure 3.3: (i, g)

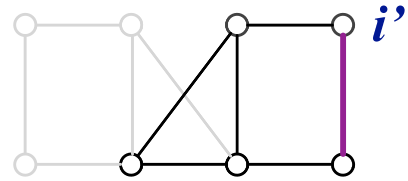

Figure 3.4: $(i, g + i'j')$

The same reasoning shows that $(j, g) \sim_2 (j', g+i'j')$. Therefore the network g violates k-WARPS with $k = 2$, as we have found a pair of agents i, j and a non-link $(i', j') \notin g$ such that $(i, g) \sim_2 (i', g + i'j')$ and $(j, g) \sim_2 (j', g + i'j')$.

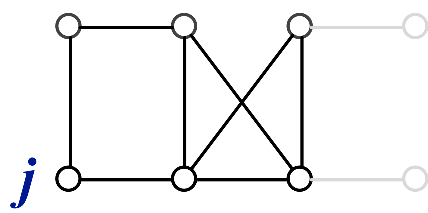

Figure 3.5: (j, g)

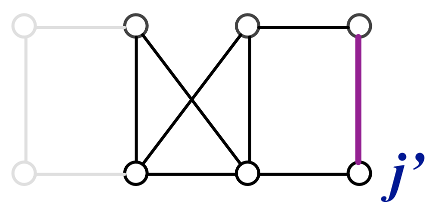

Figure 3.6: $(j', g + i'j')$

3.3.2. Measure of Consistency with WARPS

To what extent is a network consistent with k-WARPS (Definition 3.3.1)? This section proposes a continuous measure of how consistent an observed network is to satisfying k-WARPS. When a network g violates WARPS, the interpretation is that there are links that should be dissolved or formed under pairwise stability. One way to quantify the inconsistency with WARPS is to look at the fraction of such agents who should dissolve some links under pairwise stability.

In Definition 3.3.3 below, we propose **k-stability** as a continuous measure of consistency with k-WARPS. The k-stability of a network is a real number between 0 and

1, and when k-stability of a network g is 1, the network g is fully consistent with k-WARPS. It has the intuitive appeal of decomposing consistency of a network with WARPS into agents whose behaviors are consistent with WARPS.

Definition 3.3.3 (k-stability)**.** The k-stability of a network g, is the fraction of agents in g that are k-stable. An agent i is k-stable if and only if there does not exist another agent j and a non-link $i'j' \notin g$ such that $(i, g) \sim_k (i', g + i'j')$ and $(j, g) \sim_k (j', g + i'j')$

When an agent i is not k-stable, then inconsistency with pairwise stability can be resolved by having agent i dissolve a link. Therefore the measure of consistency with WARPS defined here intuitively measures the fraction of agents that should dissolve links.

3.3.2.1. Precision and power of WARPS

One criticism of the measure in Definition 3.3.3 is that it does not take into account the precision or the power of WARPS. For some class of data, it may be that WARPS never rejects. For example, the empty and the complete networks always satisfy WARPS. Therefore, WARPS may have little precision or power for very sparse and dense networks. Indeed Section 3.5 shows that the average consistency with WARPS of a random (Erdös-Rényi) network varies considerably depending on the parameters n, p, where n is the number of nodes and p is the network density.

The second measure of consistency we will use is the **excess k-stability**. As described in Definition 3.3.4, it is the difference between the consistency of an observed network with WARPS, and the average consistency with WARPS of a suitably calibrated random network. This measure is the notion of predictive success proposed by Selten (1991), and used in the empirical revealed preference literature (Beatty and Crawford (2011)).[2]

Definition 3.3.4 (Excess k-stability)**.** The **excess k-stability** of a network g is calculated by subtracting the expected k-stability of an Erdös-Rényi random network[3] with the same parameters as g, from the k-stability of g.

3.3.3. Discussion

This section discusses why this particular similarity relation is proposed (Definition 3.3.1). Firstly, it is a conservative notion of similarity that says agents have similar preferences only when their respective network structures are similar. The use of graph isomorphism allows preferences to depend freely on the network structures freely specifying how. More formally by definition of graph isomorphism, whenever $(i, g) \sim_k (i', g')$, we must have $\theta_i(g) = \theta_{i'}(g')$, as $k \to \infty$, and where $\theta : N \times \mathcal{G} \to \mathbb{R}$ is any possible network characteristics.[4]

The definition of k-WARPS made it clear that as $k \to \infty$, k-WARPS will never be rejected. For large k, pairwise stability has very little testable implication. In the empirical application that follows, we will set $k = 2$. That is, agents only care about

[2]Selten's index of predictive success is defined as the difference between the relative frequency of correct predictions (the 'hit rate') and the relative size of the set of predicted outcomes (the 'precision'). An attractive feature of Selten's predictive success is that it is uniquely characterized by a set of axioms.

[3]Specifically the $G(n, p)$ random network where there are n nodes and each link is formed independently with probability p

[4]Formally, a network characteristics is defined to be a property that is preserved under types-preserving isomorphisms of networks. In other words, it is a structural property of the network itself, not of a specific drawing or labeling of the network. For instance, $\theta_i(g)$ can be the network centrality of i in g, or the density of g.

network consisting of their friends, and their friends of friends. This is in line with the literature (de Paula et al. (2014); Mele (2013); Sheng (2014)).

In general, when we allow agents to have more heterogeneous preferences (through a less restrictive similarity relation $\sim$[5]), WARPS has less power. In Section 3.5, the power of WARPS is discussed, and it is defined as the probability that a random network would reject WARPS. We will see in Section 3.5, that the form of WARPS implemented in the empirical section (with $k = 2$), has just enough power for our purpose.

3.4. Empirical Application

3.4.1. Data

In this section, I present an empirical application of k-stability, which measures consistency with k-WARPS (Definition 3.3.3). The data set in this section is taken from Banerjee et al. (2013). They collected detailed network data by surveying households about a wide range of interactions in 75 rural villages in India. This information was then used to create different network graphs for each village.[6] I will consider 4 different types of networked relationships. Note that agents in this context refers to households.

Financial network with respect to money: there is a financial relationship with regards to money between agents i and j if and only if agent i reports borrowing money from

[5] More formally, $\sim$ is less restrictive than $\sim'$ if $\sim$ is a finer partition of $N \times \mathcal{G}$ than $\sim$

[6] More precisely, individuals are surveyed and a relationship between households exists if any household members indicated a relationship with members from the other household. Although surveyed individuals could name nonsurveyed individuals as friends, relatives, etc., such links are omitted unless both individuals were surveyed.

(or lending to) agent j, and agent j reports lending money to (or borrowing from) i. Similarly, I construct networks of financial relationships with regards to borrowing and lending rice or kerosene. *Advice network:* there is an advice relationship between agents i and j if and only if agent i reports receiving advice from (or giving to) agent j, and agent j reports giving advice to (or receiving from) i. *Social networks* are constructed in similar manner.[7]

Figure 3.7 shows the financial network of a particular village. This network does not satisfy k-WARPS with $k = 2$, and the degree to which it is consistent with 2-WARPS is measured at 80.8%, which is the 2-stability of the network (Definition 3.3.3). This is calculated as follows: 14 out of 73 households violate 2-WARPS agents (households), and these are the nodes whose links are bolded in the figure. Moreover, a (Erdös-Rényi) random network with the same parameter is expected to have a 2-stability score of 83.4%. The expected 2-stability of this network is then 80.8 minus 83.4.

3.4.2. Descriptive result

There is considerable variation in k-stability across villages for a given network type. For example, Figure 3.8 plots the histogram of k-stability with $k = 2$ for the cross section of 75 village networks. Figure 3.9 shows how excess 2-stability varies across villages and network types. Figures 3.8 and 3.9 also highlight how the empirical distribution of k-stability can be very different from the empirical distribution of the corresponding excess k-stability.

[7] *Social network:* there is a social relationship between agents i and j if and only if agent i reports visiting the house of (or receiving visits from) agent j, and agent j reports receiving visits from (or visiting the house of) agent i.

[8] There is a link between agents i and j if and only if agent i reports borrowing money from (or lending to) agent j, and agent j reports lending money to (or borrowing from) i. Non-surveyed households are not included. Graph is produced using the spring algorithm.

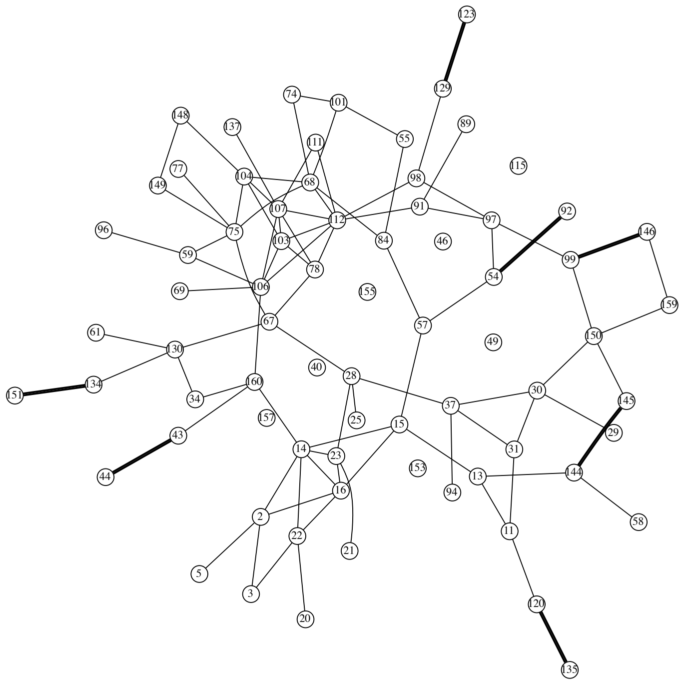

Figure 3.7: Network of borrowing and lending money[8] in village number 27 of Banerjee et al. (2013).

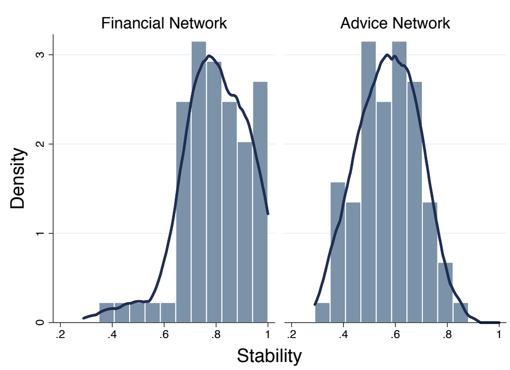

Figure 3.8: The histograms and kernel densities of 2-stability across 75 villages, for two main types of network: financial (money) and advice networks. The correlation between the two is 0.232. The histogram for 3-stability looks similar, but with mass shifted towards 1.

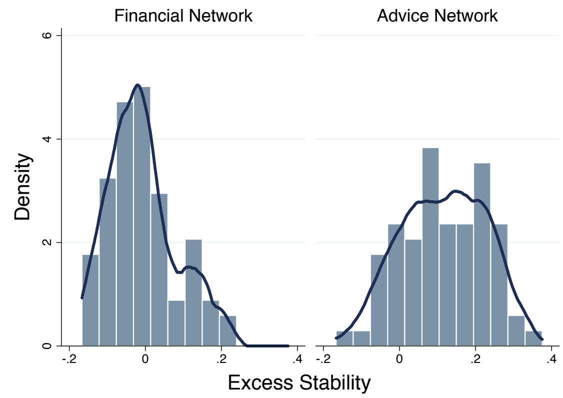

Figure 3.9: Comparing the excess 2-stability of various networks types across the 75 villages. The correlation between the two is -0.067.

3.4.3. Correlates of k-stability

This section examines the village level predictors of k-stability. Specifically, I will look at how the consistency of a village network with WARPS is related to the average characteristics of the village. In Table 3.1, I provide results for an ordinary least squares (OLS) regression of consistency with WARPS as a function of various village characteristics. The measure of consistency with WARPS here is the excess 2-stability (Definition 3.3.4), averaged across financial, social and advice networks.

The main village characteristics that explain consistency with WARPS is the number of households and the fraction of individuals in the village who travel outside the village for work. In particular, they are both significantly negatively related to consistency with WARPS. While the first variable is readily available in the data set provided by Banerjee et al. (2013), the "Work Outside Village" variable is not. I construct this variable by calculating the fraction of respondents in a village who answer 'Yes' to the question 'Do you travel outside the village for work?'. This empirical relationship suggests that (i) larger village tends to be less consistent with WARPS, and (ii) village where individuals spend less time in the village tends to also be less consistent with WARPS.

It is plausible that in a larger village, mutually beneficial relationships are not formed because agents have limited attention (Masatlioglu et al. (2012)) and do not consider all such beneficial relationships. This could be due to the geographical spread of the village as village size increases.

It is also possible that the variation in consistency with WARPS is driven by unequal measurement and sampling error across villages, where links are misreported with higher frequency in some villages.[9] Although households can name others not in the survey, I omit those links involving non-surveyed households (similar to Jackson et al. (2012)). Moreover, I construct the variable "Recip" that measures the proportion of reports that are reciprocated in a village, for instance, when household i reports borrowing from household j, how frequent does j also reports lending to i. This variable can be a proxy for measurement error in constructing village-level undirected network from survey data, but including it did not change the regression result in Table 3.1. Finally, the consistency of financial network with WARPS has negligible correlation (even negative) with the stability of social network, even though the measures are calculated based on the same sampled nodes.

[9] Chandrasekhar and Lewis (2011) examines the biases that would arise when sampled (and missing) network data are used in regression analysis.

	Excess 2-Stability (Consistency with WARPS)				
Number of Households (per 1000)	-0.508** (0.216)	-0.530** (0.226)		-0.563** (0.219)	
Work Outside Village	-0.125* (0.0674)	-0.260** (0.120)		-0.292** (0.136)	-0.310** (0.134)
Average Education (per 100)		-1.227 (0.824)	-0.437 (0.703)	-1.453 (0.990)	-1.446 (1.001)
Average Age (per 100)		-0.912 (0.620)	-0.139 (0.523)	-1.168 (0.737)	-0.975 (0.728)
Fraction GM			0.0413 (0.0471)	-0.0103 (0.0400)	-0.0239 (0.0426)
Fraction Nonnatives			-0.0888 (0.0975)	-0.117 (0.119)	-0.102 (0.0987)
Average No. Rooms				0.0239 (0.0284)	0.0170 (0.0310)
Constant	0.0961*** (0.0226)	0.544* (0.282)	0.110 (0.208)	0.668** (0.326)	0.563* (0.319)
N	75	75	75	75	75
R^2	0.092	0.144	0.021	0.163	0.101

Robust standard errors in parentheses

* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Table 3.1: The relationship between consistency with WARPS and village characteristics. The consistency of village networks with WARPS is negatively correlated with the number of households and the fraction of villagers who travel outside the village for work. Excess stability takes into account the random variation in consistency with WARPS due to variation in network density and number of households. Other village-level variables I control for are the average education level; average age of the villagers; fraction belonging to the upper two castes; fraction who are not natives of the village; and the average number of rooms owned by each person (wealth indicator).

3.4.4. Relation of k-stability to diffusion of microfinance

In this section, I show how the consistency with WARPS is associated with economic outcomes, specifically, the diffusion of microfinance.[10]

For each village, in addition to the network data, we also observe the nodes who were first informed of a new microfinance loan program, as well as the eventual adoption rate of microfinance. However, the full information is only available for 43 of the original 75 surveyed villages. The first-informed agents are also known as the injection point of the network, they are typically leaders in the village.

Banerjee et al. (2013) show that when the injection points occupy more central positions in the underlying social networks, then the eventual adoption rates of microfinance are greater. Using Banerjee et al. (2013)'s specification, I incorporate our measures of consistency with WARPS as additional variables. The main empirical finding is that when the underlying network is more consistent with WARPS, the positive relationship between network centrality of injection points and adoption rate is even more significant and stronger now.

Therefore, the consistency of WARPS is new network statistics that has potentially useful application: when a network is more consistent with WARPS, targeting of agents that have central positions in the network to increase the diffusion of information is more effective. Moreover the result is driven primarily by the consistency of advice network with WARPS. One expects the advice network to be the network type more salient for households passing on information about a new loan program.

[10] Potentially, the result obtained here is applicable for diffusion of information on networks in a broader sense. Galeotti et al. (2010); Jackson and Yariv (2007, 2010).

The intuition for the result is as follows. Networks that are less consistent with WARPS are more difficult to rationalize as pairwise stable. There are only two reasons, either we do not expect pairwise stability to hold, or that there is unobserved heterogeneity not capture by the network structure. For either of these explanations, centrality of injection points *with respect to* the network structure does not correspond to the actual effectiveness of these agents in diffusion. Therefore, the relationship between network centrality and adoption rate is weaker as the underlying network violates WARPS more severely. This is precisely what we see in the data.

3.4.4.1. Details of regression result

The first column of Table 3.2 replicates the regression result of Banerjee et al. (2013). The dependent variable is the microfinance take-up rate of non-leader households. The main explanatory variables introduced in Banerjee et al. (2013) are the communication and diffusion centrality of the injection points, which are measures of how central individuals are in their social network with regard to spreading information. We can see in Table 3.2 that the diffusion centrality of injection points helps significantly in predicting eventual adoption of microfinance. The controls include Number of households, Savings, SHG participation, Fraction GM, and Fraction Leaders.[11]

In Column (2) and (3) of Table 3.2, I introduced different measures of consistency with WARPS to the Banerjee et al. (2013)'s specification. Specifically, in Column (2) of Table 3.2, a village's consistency with WARPS is defined to be the 2-stability

[11]Savings: Fraction of households engaging in formal savings; SHG participation: Fraction of households participating in a self-help group; Fraction GM: Fraction of households that are not from the scheduled castes and scheduled tribes: groups that historically have been relatively disadvantaged; Fraction Leaders: Fraction of households that are leaders

(see Section 3.3.3), averaged across all the different network types: financial (money), financial (rice and kerosene), advice, and social networks. Analogously in Column (3), the measure of consistency is the excess 2-stability[12], averaged across different network types.

As seen in Table 3.2, our measures of consistency with WARPS added considerable explanatory power to the original model of diffusion. In all specifications, the interaction between consistency with WARPS and diffusion centrality is significantly positive. The main empirical finding is that when underlying networks are more consistent with WARPS, network centrality of injection points matters more for diffusion.

To get a sense of the magnitudes, using Column (3) of Table 3.2, one standard deviation increase in diffusion centrality is associated with 6.7, 4.6, 2.1 percentage points increase in the dependent variable at 90th, 75th, 25th percentile of excess 2-stability. Magnitudes using Column (2) are similar. In comparison to Banerjee et al. (2013), one standard deviation increase in diffusion centrality is associated with an increase of 3.9 percentage points (the average participation rate is 18.5%)

Interestingly the result also suggests that network that violates WARPS more has higher diffusion. Consistency with WARPS by itself seems to have small but negative effect on adoption rate. One explanation is that random and non-network-based meetings of agents is important for the spread of information, and networks that are stable have less random meetings of agents. Therefore our result suggests that for villages with higher stability, network-based diffusion is more prevalent; while for villages with lower stability, random non-network meetings drive diffusion.

[12]The excess k-stability score is the k-stability score in excess of a randomly generated network with the same network parameters (see Definition 3.3.4).

3.4.4.2. Robustness

In Table 3.3, I repeat the same exercise but focusing instead on advice networks. Table 3.3 verifies that the result is not sensitive to different ways of defining consistency with WARPS. The interaction between consistency with WARPS and diffusion centrality is significantly positive in all 4 columns representing different ways of defining consistency with WARPS.

In Table 3.4, I present a placebo test of the main specification (Table 3.2) using different network measures in place of consistency with WARPS. The network measures used are (1): Number of nodes (households); (2): Average degrees; (3) Average distance; (4) Average clustering; (5): Largest eigenvalue of the adjacency matrix. Column (6) shows that although average clustering has the same effect on the dependent variable as stability, this effect is not robust when communication centrality of injection points is used instead. Communication centrality is another measure of centrality that is similar to diffusion centrality (see Banerjee et al. (2013)).

Table 3.5 shows the correlation between our main measure of consistency and other commonly used network measures. Just like consistency with WARPS, these network measures are calculated using only information about the network structure, but nonetheless Table 3.4 shows that, with the exception of the number of nodes (households), they were not significant in explaining diffusion.

	(1)	(2)	(3)
Microfinance participation rate			
Diffusion Centrality	0.0222***	-0.0886*	0.0165***
	(0.00690)	(0.0470)	(0.00594)
Stability		-0.897***	
		(0.323)	
Stability × Diffusion Centrality		**0.143****	
		(0.0583)	
Excess Stability			-1.275***
			(0.391)
Excess Stability × Diffusion Centrality			**0.230*****
			(0.0734)
Number of Households	-0.540***	-0.472***	-0.507***
(per 1000)	(0.163)	(0.169)	(0.157)
Savings	-0.244*	-0.205*	-0.252*
	(0.138)	(0.118)	(0.138)
SHG Participation	-0.228	-0.179	-0.165
	(0.200)	(0.180)	(0.199)
Fraction GM	-0.0354	-0.0280	-0.0410
	(0.0396)	(0.0395)	(0.0404)
Fraction Leaders	0.270	0.301	0.222
	(0.457)	(0.442)	(0.425)
N	43	43	43
R^2	0.442	0.511	0.542

Robust standard errors in parentheses

* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Table 3.2: The dependent variable is the microfinance take-up rate of non-leader households. The first column reproduces exactly the result of Banerjee et al. (2013). Two measures of consistency with WARPS are used: (excess) stability is the average (excess) 2-stability across all network types. Excess k-stability is the k-stability of a network in excess of a randomly generated network with the same parameter. Using Column (2), one standard deviation increase in diffusion centrality is associated with 6.8, 5.6, 3.0 percentage points increase in the dependent variable at 90th, 75th, 25th percentile of stability scores. Using Column (3), one standard deviation increase in diffusion centrality is associated with 6.7, 4.6, 2.1 percentage points increase in the dependent variable at 90th, 75th, 25th percentile of stability scores.

	(1)	(2)	(3)	(4)
Diffusion Centrality	-0.0527	-0.133***	0.00370	0.0175***
	(0.0367)	(0.0282)	(0.00795)	(0.00548)
Stability	-0.760**	-0.954***		
	(0.299)	(0.190)		
Stability × Diffusion Centrality	0.128*	0.200***		
	(0.0657)	(0.0405)		
Excess Stability			-0.890***	-0.822***
			(0.272)	(0.295)
Excess Stability × Diffusion Centrality			0.155**	0.188**
			(0.0577)	(0.0707)
Constant	1.107***	1.191***	0.723**	0.846***
	(0.316)	(0.281)	(0.280)	(0.275)
N	43	43	43	43
R^2	0.529	0.583	0.572	0.550

Robust standard errors in parentheses

* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Table 3.3: The dependent variable is the microfinance take-up rate of non-leader households. Only Advice Networks are used. Demographic controls included as in Table 3.2. In columns (1) and (2), stability is defined as the 2 and 3-stability of advice networks. In Columns (3) and (4), excess stability is defined as the excess 2 and 3-stability of advice networks. Magnitudes are comparable with Table 3.2. For column (1), one standard deviation increase in the diffusion centrality is associated with 6.6, 5.3 and 1.8 percentage points increase in the dependent variable at 90th, 75th, 25th percentile of stability scores. For column (2), the numbers are 8.3, 5.8, and 1.2; for column (3): 7.2, 5.9, 1.3; for column (4): 8.1, 6.1, and 1.7

	(1)	(2)	(3)	(4)	(5)	(6)	(7)
	Microfinance participation rate						
Diffusion Centrality	0.0516***	0.00486	0.0318	-0.0560***	0.0506		
	(0.0142)	(0.0515)	(0.0724)	(0.0162)	(0.0429)		
Network Measures × Diffusion Centrality	-.000349*	0.00623	-0.00219	0.601***	-0.00699		
	(.000174)	(0.0199)	(0.0156)	(0.131)	(0.00971)		
Comm. Centrality						-0.695	0.449**
						(0.825)	(0.204)
Network Measures × Comm. Centrality						11.26	12.17***
						(6.725)	(2.536)
Network Measures	.000556	-0.0739	0.0480	-3.206***	-0.000738	-0.671	-0.847***
	(.000878)	(0.101)	(0.0879)	(0.685)	(0.0524)	(0.406)	(0.266)
Controls	Yes	Yes	Yes	Yes	Yes	Yes	Yes
N	43	43	43	43	43	43	43
R^2	0.492	0.470	0.468	0.582	0.495	0.443	0.574

Robust standard errors in parentheses

* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Table 3.4: Placebo test of the main specification (Table 3.2) using different network measures in place of consistency with WARPS. The network measures used are (1): Number of nodes (households); (2): Average degrees; (3) Average distance; (4) Average clustering; (5): Largest eigenvalue of the adjacency matrix. Column (6) shows that although average clustering has the same effect on the dependent variable as stability, this effect is not robust when communication centrality of injection points is used instead. Communication centrality is another measure of centrality that is similar to diffusion centrality (see Banerjee et al. (2013)). Finally, Column (7) shows that our new network measure, excess 2-stability, is robust to using communication centrality in place of diffusion centrality.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)
	Excess Stability						
No. of Households (per 1000)	-0.549**						0.0564
	(0.217)						(0.464)
Average degrees		0.0161					-0.0494
		(0.0217)					(0.0372)
Average distance			-0.0252*				-0.0187
			(0.0146)				(0.0207)
Average clustering				0.415***			0.568***
				(0.151)			(0.199)
1st Eigenvalue					0.00718		0.00926
					(0.0124)		(0.0249)
2nd Eig. Stoch. Adj						-0.620	-0.958
						(0.610)	(0.644)
Constant	0.0732***	-0.0203	0.140**	-0.0426*	-0.00969	0.634	1.050
	(0.0202)	(0.0581)	(0.0659)	(0.0246)	(0.0560)	(0.601)	(0.697)
N	75	75	75	75	75	75	75
R^2	0.062	0.012	0.052	0.121	0.005	0.016	0.192

Standard errors in parentheses

* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Table 3.5: Correlation of excess 2-stability with other network measures, at the village-level. All measures are averaged over the four types of networks. These network measures are the ones used in Banerjee et al. (2013)

3.5. Consistency of a random network with WARPS

In this section, I investigate the consistency with WARPS of a random network. The likelihood that a random network violates WARPS will be our measure of **power**. Moreover, the result obtained here will be used as a benchmark for gauging how severe real-world networks violate WARPS (see Excess k-stability of Definition 3.3.4).

Using k-stability (Definition 3.3.3) as our measure of consistency with WARPS, I will examine the k-stability of the canonical Erdös-Rényi random graph model. Here, I consider the $G(n, p, \lambda)$ random graph model, where a network with n number of nodes is randomly generated as follows: each link is included in the network with probability p (independently). With probability λ, a node is assigned as type 0, and with probability $1 - \lambda$, a node is assigned as type 1.

We are interested in the **expected stability function** $E(n, p, \lambda; k)$, which tells us on average, what is the k-stability for a network randomly drawn from the random graph model, $G(n, p, \lambda)$. For instance, if $E(n_0, p_0, \lambda_0) = 0.4$, then we expect a network randomly generated from $G(n_0, p_0, \lambda_0)$ to have a k-stability score of 0.4. Therefore the score of 0.4 is the appropriate benchmark for an observed network with n_0 number of nodes, density of p_0, and λ_0 proportion of type-0. In this section, we will focus on k-stability with $k = 2, 3$, but the same analysis can be carried out for different k-stability scores.

Definition 3.5.1. Let $\mathcal{S}(g; k)$ be the k-stability of the network g as defined in 3.3.2. Let n, p, λ be the number of nodes, the density, and the proportion of type-0 in the network g respectively. The **excess k-stability** of a network g is $\mathcal{S}(g; k) - E(n, p, \lambda; k)$

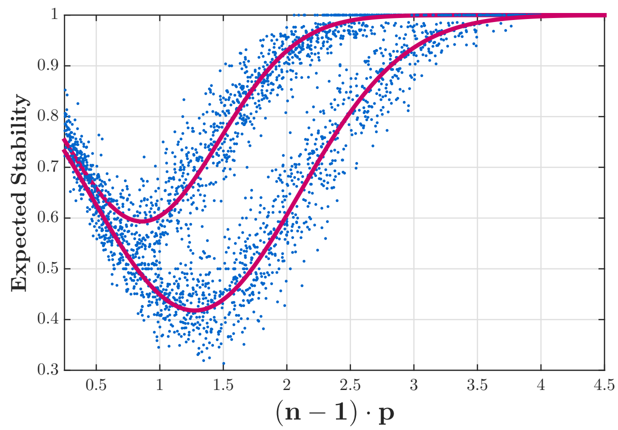

Figure 3.10: The simulated expected 2 and 3-stability can be accurately described by an inverted Gaussian function that depends only on the average degree $(n-1)p$. That is, the curves above are obtained by fitting $1-\alpha\cdot\exp\left[-(\frac{(n-1)p-\beta}{\gamma})^2\right]$ to the simulation points. The curve for expected 3-stability lies strictly on top.

First I fix $\lambda = 0$ (or equivalently $\lambda = 1$), that is, assuming that nodes do not have different types or characteristics. Although I am not able to analytically characterize the expected stability function, Figure 3.10 shows simulation evidence that the expected k-stability of a random network depends only on the product $(n-1)p$, which is the average degree of the network, i.e. how many links a node has on average. It is known that $(n-1)p$ characterized certain phase transitions of Erdös-Rényi random networks.[13] Moreover as can be seen in Figure 3.10, the simulated expected stability can be described by an inverted Gaussian function remarkably well. That is, $E(n,p) = 1-\alpha\cdot\exp\left[-\left(\frac{(n-1)p-\beta}{\gamma}\right)^2\right]$.

Ultimately we wish to know the general shape of the expected stability function, with $\lambda \in (0, 0.5]$. In Figure 3.11, I plot a non-parametric smoothed surface over the simulated expected stability. This is obtained by first simulating the expected stability over a range of values for n, p, and λ; and then secondly, computing the

[13]For instance, when the average degree is less than one, the network consists of many components that are small relative to the number of nodes, and when the average degree is greater than one, there is a unique giant component.

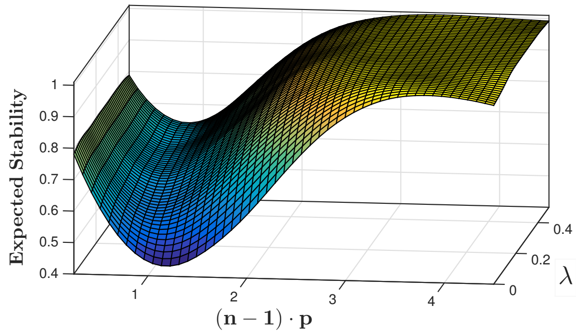

Figure 3.11: Non-parametric smoothed surface (Loess Curve) of the simulated expected stability function $E(n, p, \lambda)$. The function $E(n, p, \lambda)$ is the expected 2-stability score of a network randomly drawn from $G(n, p, \lambda)$. Stability is expected to be lowest around $(n-1)p = 1$, which corresponds to the well-known phase transition for the rise of a giant component in Erdös-Rényi random networks. As λ increases, stability is expected to increase.

local, non-parametric regression known as the LOESS Curve.

Again the non-parametric plot of the simulated expected stability in Figure 3.11 is suggestive of an inverted multivariate Gaussian function. That is, I obtain an R^2 of 0.96 when I fit to the simulated data, the inverted Gaussian function of the form $E(n, p, \lambda) = 1 - \alpha \exp\left[-\left(\frac{(n-1)p-\beta}{\gamma}\right)^2\right]$, where the height and location of the peak (α and β respectively) is a linear function of λ.[14]

Figures 3.10 is obtained by first drawing 1000 pairs of (n, p) uniformly from $n \in \{40, \ldots, 70\}$, and p such that $np \in [0, 4.5]$, then for each parameter (n, p), 10 random networks are drawn from $G(n, p)$ for a total of 10,000 networks. The 2 and 3-stability scores are then computed for each of these networks. To obtain Figure 3.11, I simulate the expected stability function at 10,000 distinct values of (n, p, λ) drawn uniformly from $n \in \{40, \ldots, 70\}$, $\lambda \in [0, 0.5]$, and p such that $np \in [0, 4.5]$. For each (n, p, λ),

[14]There is however a caveat: the Gaussian function appears only to match the non-parametric surface when λ is not too close to the boundary of 0 and 0.5, and $(n-1)p$ not too close to the boundary of 0.

I draw 10 random networks from $G(n, p, \lambda)$, and I then compute the 2-stability score for each randomly drawn networks (100,000 in total).

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