FREE ALGEBRAS IN VON NEUMANN-BERNAYS-GÖDEL SET THEORY AND POSITIVE ELEMENTARY INDUCTIONS IN REASONABLE STRUCTURES

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Abstract

This thesis consists of two independent chapters. The first chapter deals with universal algebra. It is shown, in von Neumann-Bernays-Gödel set theory, that free images of partial algebras exist in arbitrary varieties. It follows from this, as set-complete Boolean algebras form a variety, that there exist free set-complete Boolean algebras on any class of generators. This appears to contradict a well-known result of A. Hales and H. Gaifman, stating that there is no complete Boolean algebra on any infinite set of generators. However, it does not, as the algebras constructed in this chapter are allowed to be proper classes. The second chapter deals with positive elementary inductions. It is shown that, in any reasonable structure $\mathfrak{M}$, the inductive closure ordinal of $\mathfrak{M}$ is admissible, by showing it is equal to an ordinal measuring the saturation of $\mathfrak{M}$. This is also used to show that non-recursively saturated models of the theories ACF, RCF, and DCF have inductive closure ordinals greater than ω.

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CHAPTER 1 Free Algebras in Von Neumann-Bernays-Gödel Set Theory

In 1962, A. Hales and H. Gaifman independently proved the following theorem:

Theorem 1 If κ is a regular cardinal, there is no free set (κ,∞)-distributive complete Boolean algebra on a set of κ generators.

In view of this result, an existence theorem along these lines can only be obtained by allowing Boolean algebras which are proper classes. In fact, the following theorem can be established without much difficulty in a strong enough set theory, for example Morse-Kelley, which freely allows quantification over classes.

Theorem 2 Let E be a class of equations in the type of set-complete Boolean algebras. If the Boolean algebra $\mathbf{2}$ satisfies the equations E, then for any class X, there is a class X′ equipollent with X such that there is a free E-set-complete Boolean algebra on X′.

It will be shown in this chapter how results of this kind can be obtained in the weaker set theory of von Neumann-Bernays-Gödel (NBG).

In fact, the following general result is proved in NBG which easily gives Theorem 2 as a corollary.

Theorem 3 Let τ be a class similarity type. Let E be a class of equations in type τ. Then, if $\mathbb{A}$ is a partial algebra of type τ, there exists a $\mathcal{V}(E)$-free image of $\mathbb{A}$.

This theorem partially generalizes the following known result from universal algebra [Pierce].

Theorem 4 Let τ be a set similarity type. Let $\mathfrak{A}$ be a class of set partial algebras, closed under set products and subalgebras. Then, if $\mathbb{A}$ is a set partial algebra of type τ, there is an '$\mathfrak{A}$'-free image of $\mathbb{A}$.

§1. Free images of partial algebras

In this chapter, we will work in the axiom system NBGA (von Neumann-Bernays-Gödel with atoms) [see Rubin] + PR, where PR is:

There is a limited formula φ, a class Z such that for any nonempty class Y, there is a unique set x such that $\emptyset \neq x \subseteq Y$, and $\varphi(x,Y,Z)$. (See [Rubin & Rubin] for a weaker form of PR, and some of its uses.)

Uses of PR will usually be as follows: we will have classes Y_{ab}, for $\langle a,b\rangle$ in some index class I, and we will say that we construct sets x_{ab}, such that $\emptyset \neq x_{ab} \subseteq Y_{ab}$, by defining x_{ab} as the unique x such that $\varphi(x,Y_{ab},Z)$ holds.

Because of the restrictions of this set theory, and the extension of the concept of 'algebra' we will use, we must redefine 'algebra'. First, we note that we can define an indexed collection of classes $\langle\langle A_\beta\rangle\rangle_{\beta\in B}$, as in [Rubin], so that we can recover B and each A_β from $\langle\langle A_\beta\rangle\rangle_{\beta\in B}$, even if they are proper classes.

Definition 1.1 A class τ is a similarity type if τ is a function.

Definition 1.2 If τ is a similarity type, a partial algebra of type τ is a system $\mathbb{A} = \langle\langle A, \langle\langle F_j\rangle\rangle_{j\in\mathfrak{D}(\tau)}\rangle\rangle$ (abbreviated $\langle\langle A; F_j\rangle\rangle_{j\in\mathfrak{D}(\tau)}$) such that each F_j is a function from some subclass of $A^{\tau(j)}$ to A. If each $\mathfrak{D}(F_j) = A^{\tau(j)}$, A is called an abstract algebra of type τ.

Definition 1.3 If τ is a similarity type, and X is a class, then $W_{\tau X}$ (the words of τ with variables X) is the smallest class W such that

1) If $u \in X$, then $\langle 0,u \rangle$ (read x_u) $\in W$

2) If $j \in \mathfrak{D}(\tau)$, and $w \in W^{\tau(j)}$, then $\langle\langle 1,j \rangle, w\rangle$ (read $\Phi_j(w)$) $\in W$.

Note: This definition is not allowed in NBGA, as we cannot prove that $W_{\tau X}$ exists directly, but it can easily be modified to work properly as follows: Fix τ,X as above. For $\sigma \subseteq \tau$, $x \subseteq X$, σ and x sets, let $w_{\sigma x}$ be the smallest set w satisfying

1′) If $u \in x$, $\langle 0,u \rangle \in w$.

and

2′) If $j \in \mathfrak{D}(\sigma)$, and $t \in w^{\tau(j)}$, then $\langle\langle 1,j \rangle, t\rangle \in w$.

Let $W_{\tau X} = \bigcup_{\substack{\sigma \subseteq \tau \\ x \subseteq X}} w_{\sigma x}$.

Definition 1.4 Let τ,X be as in definition 1.3. We can then define operations on $W_{\tau X}$ by letting $F_j(w) = \Phi_j(w)$ if $\langle j,y \rangle \in \tau$ and $w \in (W_{\tau X})^y$. The algebra $\langle\langle W_{\tau X}; F_j \rangle\rangle_{j \in \mathfrak{D}(\tau)}$ is called the word algebra $W_{\tau X}$.

For $\mathbb{A} = \langle\langle A; F_j \rangle\rangle_{j \in \mathfrak{D}(\tau)}$ a (partial) algebra of type τ, $w \in W_{\tau X}$, and v a function from X to A, we define:

Definition 1.5 The value of w at v in $\mathbb{A}$, written $Val_{\mathbb{A}}(w,v)$ by set induction:

i) $Val_{\mathbb{A}}(x_u, v) = v(u)$

ii) $Val_{\mathbb{A}}(\Phi_j(w), v) = F_j(Val_{\mathbb{A}}(\cdot, v) \circ w)$ (if defined).

(See note following definition 1.3)

For future reference, we will also need some standard concepts in universal algebra.

<u>Definition 1.6</u> Let $\mathbb{A} = \langle\langle A; F_j \rangle\rangle_{j \in \mathcal{D}(\tau)}$ be a partial algebra of type τ.

A) If $B \subseteq A$, we define $\mathbb{A} \upharpoonright B$ (the <u>restriction</u> of $\mathbb{A}$ to B) = $\langle\langle B; F_j \cap (B^{\tau(j)} \times B) \rangle\rangle$.

B) If $\sigma \subseteq \tau$, we define $\mathbb{A} \downarrow \sigma$ (the <u>reduct</u> of $\mathbb{A}$ to σ) = $\langle\langle A; F_j \rangle\rangle_{j \in \mathcal{D}(\sigma)}$ as a partial algebra of type σ.

C) If $\tau \subseteq \sigma$, and σ is a similarity type, we define $A \uparrow \sigma$, the <u>expansion</u> of $\mathbb{A}$ to σ, as a partial algebra $\langle\langle A; F_j^* \rangle\rangle_{j \in \mathcal{D}(\sigma)}$ of type σ, so that

$$F_j^* = \begin{cases} F_j & j \in \mathcal{D}(\tau) \\ \emptyset & j \in \mathcal{D}(\sigma) \sim \mathcal{D}(\tau). \end{cases}$$

Note that, if all concepts are defined, if $\mathbb{A}$ is a partial algebra of type τ, then $(\mathbb{A} \uparrow \sigma) \downarrow \tau = \mathbb{A}$.

Most algebraic concepts can be defined in the conventional manner; however, we must redefine $[X]_{\mathbb{A}}$, the subalgebra of $\mathbb{A}$ generated by X.

<u>Definition 1.7</u> Let $\mathbb{A}$ be a partial algebra of type τ, $X \subseteq A$. Then $[X]_{\mathbb{A}}$, the <u>subalgebra of</u> $\mathbb{A}$ <u>generated by</u> X, is

$$\{\mathrm{Val}_{\mathbb{A}}(w, v) : v \text{ is a set, } \mathcal{R}(v) \subseteq X \text{ and } w \in W_{\tau\mathcal{D}(v)}\}.$$

We can now define the main concept used in this chapter.

<u>Definition 1.8</u> Let τ be a similarity type. Let $\mathbb{A}$ be a partial algebra of type τ. Let $\mathbb{P}$ be a property of partial algebras of type τ. A pair $\langle\langle \mathbb{B}, \sigma\rangle\rangle$ is called a $\mathbb{P}$-<u>free image</u> of $\mathbb{A}$ if the following four conditions hold:

1) $\mathbb{P}$ ($\mathbb{B}$) holds i.e., $\mathbb{B}$ has property $\mathbb{P}$. In particular, $\mathbb{B}$ is a partial algebra of type τ.

2) σ is a homomorphism from $\mathbb{A}$ to $\mathbb{B}$.

3) $[\sigma \text{"}A]_{\mathbb{B}} = B$.

4) If $\mathbb{C}$ has the property $\mathbb{P}$, and φ is a homomorphism from $\mathbb{A}$ to $\mathbb{C}$, there is a homomorphism ψ from $\mathbb{B}$ to $\mathbb{C}$ such that $\psi \circ \sigma = \varphi$.

<u>Definition 1.9</u> If, in definition 1.8, σ is id $\restriction A$, then $\mathbb{B}$ is a $\mathbb{P}$-<u>free extension of</u> $\mathbb{A}$.

<u>Definition 1.10.</u> If, in definition 1.8, σ is 1-1, then $\langle\langle \mathbb{B}, \sigma\rangle\rangle$ is a $\mathbb{P}$-<u>free quasi-extension of</u> $\mathbb{A}$.

<u>Definition 1.11</u> Let X be a class, and let $\mathbb{X} = \langle\langle X \,;\rangle\rangle$ be an algebra of type $\emptyset$. If $\mathbb{B}$ is a $\mathbb{P}$-free extension of $\mathbb{X} \uparrow \tau$, $\mathbb{B}$ is a <u>$\mathbb{P}$-free algebra on X.</u>

I will now give some examples of the above concepts.

<u>Example 1</u> Suppose we have groups G_1, G_2, considered as algebras of type $\{\langle \cdot, 2\rangle, \langle ^{-1}, 1\rangle\}$, with a common subgroup $H = G_1 \cap G_2$.
Let $\mathbb{A} = \langle\langle G_1 \cup G_2; \cdot_{G_1} \cup \cdot_{G_2}, {}^{-1}{}_{G_1} \cup {}^{-1}{}_{G_2}\rangle\rangle$. Then a group-free extension of $\mathbb{A}$ is the free product $G_1 *_H G_2$.

<u>Example 2</u> The free group on a set X is the group-free algebra on X.

Example 3 Suppose we have groups G_1, G_2 with $G_1 \cap G_2 = \emptyset$, and an isomorphism φ from $H \subseteq G_1$ into G_2. Let $\tau = \{\langle \cdot, 2\rangle, \langle {}^{-1}, 1\rangle, \langle i, 1\rangle\}$. Let $\mathbb{P}$ be the property of algebras $\langle\langle G; \cdot, {}^{-1}, i\rangle\rangle$ of type τ that

i) $\langle\langle G; \cdot, {}^{-1}\rangle\rangle$ is a group, and

ii) $i = id \upharpoonright G$.

Let $\mathbb{A} = \langle\langle G_1 \cup G_2; \cdot_{G_1} \cup \cdot_{G_2}, {}^{-1}_{G_1} \cup {}^{-1}_{G_2}, \varphi\rangle\rangle$, a partial algebra of type τ. Then $\langle\langle G_1 *_H G_2, \sigma_1 \cup \sigma_2\rangle\rangle$ is a $\mathbb{P}$-free image of $\mathbb{A}$, where σ_i is the natural map from G_i into $G_1 *_H G_2$.

We will now derive some simple properties of free images.

Lemma 1 Let $\langle\langle \mathbb{B}, \sigma\rangle\rangle$ be a $\mathbb{P}$-free image of $\mathbb{A}$, and let $\mathbb{C}, \varphi$ be as in condition 4 of definition 1.8. Then the ψ of that condition is unique.

Proof: Suppose ψ, ψ' satisfy condition 4. Fix a $b \in B$. By condition 3 of 1.8, $b = Val_{\mathbb{B}}(w, v)$ for some may v into $\sigma''A$.

Claim: $\psi \circ v = \psi' \circ v$.

Proof: Fix $t \in \mathcal{D}(v)$. $v(t) = \sigma(a)$ for some $a \in A$. Hence,

$\psi \circ v(t) = \psi(v(t)) = \psi(\sigma(a)) = \varphi(a) = \psi'(\sigma(a)) = \psi'(v(t)) = \psi' \circ v(t)$. ⊣

$$
\begin{aligned}
\text{So, } \psi(b) &= \psi(Val_{\mathbb{B}}(w,v)) \\
&= Val_{\mathbb{C}}(w, \psi \circ v) && (\psi \text{ is a homomorphism}) \\
&= Val_{\mathbb{C}}(w, \psi' \circ v) && (\text{claim}) \\
&= \psi'(Val_{\mathbb{B}}(w,v)) && (\psi' \text{ is a homomorphism}) \\
&= \psi'(b)
\end{aligned}
$$

⊣

<u>Lemma 2</u> If $\langle\langle \mathbb{B},\sigma\rangle\rangle$, $\langle\langle \mathbb{B}',\sigma'\rangle\rangle$ are $\mathbb{P}$-free images of $\mathbb{A}$, then there is an isomorphism $\psi : \mathbb{B} \to \mathbb{B}'$ such that $\psi \circ \sigma = \sigma'$.

<u>Proof</u>: As $\langle\langle \mathbb{B},\sigma\rangle\rangle$ is a $\mathbb{P}$-free image of $\mathbb{A}$, and σ' is a homomorphism from $\mathbb{A}$ to an algebra $\mathbb{B}'$ satisfying $\mathbb{P}$, there is a homomorphism ψ from $\mathbb{B}$ to $\mathbb{B}'$ such that $\psi \circ \sigma = \sigma'$. Similarly, there is a homomorphism ψ' from $\mathbb{B}'$ to $\mathbb{B}$ such that $\psi' \circ \sigma' = \sigma$. So, $\psi' \circ \psi \circ \sigma = \mathrm{id} \circ \sigma$. From lemma 1, with $\mathbb{C} = \mathbb{B}$, $\varphi = \sigma$, we get $\psi' \circ \psi = \mathrm{id} \restriction B$. Similarly $\psi \circ \psi' = \mathrm{id} \restriction B'$, so that ψ is an isomorphism from $\mathbb{B}$ to $\mathbb{B}'$. ⊣

§2. <u>Pierce's Theorem</u>

<u>Theorem 2.1</u> [Lemma 4.1.5 of [Pierce]]. Let τ be a similarity type which is a set. Let $\mathfrak{A}$ be a class of set partial algebras of type τ, closed under set products and subalgebras. Then, if $\mathbb{A}$ is a set partial algebra of type τ, and $\mathbb{P}$ is the property of belonging to $\mathfrak{A}$, then we can construct a $\mathbb{P}$-free image of $\mathbb{A}$.

The reasons I will prove this theorem here, are that I am not using the axiom of choice here, as Pierce does, and that this proof, but not Pierce's can be modified into a proof of the main theorem in Morse-Kelley set theory.

<u>Proof</u>: The main step of the proof, as in [Pierce], consists of showing that there is a (possibly empty) <u>set</u> of pairs $\{\langle \mathbb{A}_i, \varphi_i\rangle : i \in I\}$ such that

(i) $\mathbb{A}_i \in \mathfrak{A}$

(ii) φ_i is homomorphism from $\mathbb{A}$ to $\mathbb{A}_i$, and φ_i"A generates A_i in $\mathbb{A}_i$.

(iii) If $\mathbb{C} \in \mathfrak{A}$, and φ is a homomorphism from $\mathbb{A}$ to $\mathbb{C}$, there is $i \in I$, and a monomorphism θ from A_i to C, such that $\varphi = \theta \circ \varphi_i$.

Let Γ be the congruence relation on $\mathbb{W}_{\tau A}$ generated by $\Phi_j((x_{a_i})^{i\in\tau(j)}) \equiv x_b$ whenever $F_j((a_i)^{i\in\tau(j)}) = b$ in $\mathbb{A}$. (The notation $(x_i)^{i\in I}$ indicates the function defined on I which for each $i \in I$, takes the value x_i.) Let $I = \{\Delta : \Delta$ is a congruence relation on $\mathbb{W}_{\tau A}$, $\Delta \supseteq \Gamma$, and $\mathbb{W}_{\tau A}/\Delta \in \mathfrak{U}\}$. For $\Delta \in I$, let $\mathbb{A}_\Delta = \mathbb{W}_{\tau A}/\Delta$, and let φ_Δ be defined so that $\varphi_\Delta(a) = [x_a]_\Delta$.

Property i) above is clear. Property ii) follows from the fact that φ_Δ is a homomorphism from $\mathbb{A}$ to $\mathbb{A}_\Delta$ by construction. To show property iii), suppose $\mathbb{C} = \langle\langle C; H_j\rangle\rangle_{j\in\mathcal{D}(\tau)}$ is a $\mathfrak{U}$, and φ is a homomorphism from $\mathbb{A}$ to $\mathbb{C}$. Let $\Delta = \{\langle w_1, w_2\rangle : w_1, w_2 \in W_{\tau A}$ and $\mathrm{Val}_{\mathbb{C}}(w_1, \varphi) = \mathrm{Val}_{\mathbb{C}}(w_2, \varphi)\}$. $\Delta \supseteq \Gamma$, as φ is a homomorphism, and $\theta = \mathrm{Val}_{\mathbb{C}}(\cdot, \varphi)/\Delta$ clearly satisfies the condition of property iii).

To complete the proof, let $\mathbb{D} = \prod_{\Delta\in I}\mathbb{A}_\Delta$. Let σ be defined from $\mathbb{A}$ to $\mathbb{D}$ so that $\sigma(a)(\Delta) = \varphi_\Delta(a)$. Define $B = [\sigma''A]_{\mathbb{D}}$, and $\mathbb{B} = \mathbb{D} \restriction B$. I claim that $\langle\langle \mathbb{B}, \sigma\rangle\rangle$ is a $\mathbb{P}$-free image of $\mathbb{A}$. Property 1) holds because $\mathbb{B} \in \mathbf{S}\mathcal{P}\mathfrak{U} = \mathfrak{U}$. Properties 2) and 3) hold by construction. Property 4) holds, as follows: Suppose $\mathbb{C} \in \mathfrak{U}$, and φ is a homomorphism from $\mathbb{A}$ to $\mathbb{C}$. By property iii) there is a $\Delta \in I$, and a monomorphism θ from $\mathbb{A}_\Delta$ to $\mathbb{C}$, such that $\varphi = \theta \circ \varphi_\Delta$. Let ψ be defined on B by $\psi(b) = \theta(b(\Delta))$. Then, for $a \in A$, $\psi(\sigma(a)) = \theta(\sigma(a)(\Delta)) = \theta(\varphi_\Delta(a)) = \varphi(a)$. ⊣

§3. The main result

Theorem 3.1 Let τ be a similarity type. Let E be a class of equations in $W_\tau V$. Let $\mathbb{A}$ be a partial algebra of type τ. Then there exists a

$\gamma(E)$-free image of $\mathbb{A}$.

Proof: We will define a directed class of partial algebras $\mathbb{B}_{x,s}$ for sets $x \subseteq A$, $s \subseteq \tau$ and maps $\psi_{xs;yt} : \mathbb{B}_{x,s} \to \mathbb{B}_{y,t}$ such that $\varinjlim \mathbb{B}_{x,s} = \mathbb{B}$ is the $\gamma(E)$-free image of $\mathbb{A}$.

For $x \subseteq A$, $s \subseteq \tau$, let $\mathbb{A}_{xs} = A \downarrow s \upharpoonright x$, and let $\langle\langle \mathbb{B}_{xs}, \varphi_{xs} \rangle\rangle$ be the $\gamma(E \cap W_s \gamma)$-free image of $\mathbb{A}_{xs}$, as described in Theorem 2.1. For $x \subseteq y \subseteq A$, $s \subseteq t \subseteq \tau$, consider the map $\psi : x \to \mathbb{B}_{yt}$ defined by $\psi(a) = \varphi_{yt}(a)$. It is easy to see that ψ is a homomorphism from $\mathbb{A}_{xs}$ to $\mathbb{B}_{yt} \downarrow s$, and hence, from the fact that $\langle\langle \mathbb{B}_{xs}, \varphi_{xs} \rangle\rangle$ is a $\gamma(E \cap W_x \gamma)$-free image of $\mathbb{A}_{xs}$, there is a unique homomorphism $\psi_{xs;yt} : \mathbb{B}_{xs} \to \mathbb{B}_{yt} \downarrow s$ such that $\psi_{xs;yt} \circ \varphi_{xs} = \psi$.

Now we consider the direct limit:

If $a \in B_{xs}$, $b \in B_{yt}$, $z \subseteq A$, and $u \subseteq \tau$, we say

$$a \equiv_{zu} b \text{ if } x,y \subseteq z;\ s,t \subseteq u;\ \text{and, } \psi_{xs;zu}(a) = \psi_{yt;zu}(b).$$

We say $a \equiv b$ if there exist z and u such that $a \equiv_{zu} b$. Let $B^* = \bigcup_{x \subseteq A, s \subseteq \tau} B_{xs}$.

Claim 1: $\equiv$ is an equivalence relation on B^*.

Proof: $\equiv$ is reflexive on B^*, because if $a \in B_{xs}$, then $\psi_{xs;xs}(a) = \psi_{xs;xs}(a)$, so $a \equiv_{xs} a$.

$\equiv$ is clearly symmetric, so we must show that $\equiv$ is transitive. Suppose $a \in B_{xs}$, $b \in B_{x's'}$, $c \in B_{x''s''}$, $a \equiv_{yt} b$, and $b \equiv_{zu} c$. We claim $a \equiv_{y \cup z, t \cup u} c$ as follows:

$$\psi_{xs;y\cup z,t\cup u}(a) = \psi_{yt;y\cup z,t\cup u}(\psi_{xs;yt}(a))$$

$$= \psi_{yt;y\cup z,t\cup u}(\psi_{x's';yt}(b))$$

$$= \psi_{x's';y\cup z,t\cup u}(b)$$

$$= \psi_{zu;y\cup z,t\cup u}(\psi_{x's';zu}(b))$$

$$= \psi_{zu;y\cup z,t\cup u}(\psi_{x''s'';zu}(c))$$

$$= \psi_{x''s'';y\cup z,t\cup u}(c). \qquad \dashv$$

We would like to use $B = B^*/\equiv$, but we can't because equivalence classes are proper classes. However, using PR we can select from each equivalence class a canonical element, and B will then consist of those elements.

<u>Claim 2:</u> If $a \equiv b$, we can construct, from a and b, $z \subseteq A$ and $u \subseteq \tau$ such that $a \equiv_{zu} b$.

<u>Proof</u>: Let $W = \{\langle z,u\rangle : a \equiv_{zu} b\}$. By PR we can construct a non-empty set $w \subseteq W$. Let $z = \bigcup \mathfrak{D}(w)$, and $u = \bigcup \mathfrak{R}(w)$. It is now clear that $a \equiv_{zu} b$.

<u>Claim 3:</u> From $a \in B^*$ we can construct an $a^* \in B^*$, with the following properties:

(i) $a \equiv a^*$

(ii) If $a \equiv b$, then $a^* = b^*$.

Proof: For a fixed $a \in B^*$, let $[a] = \{b \in B^* : a \equiv b\}$. Using PR, we can construct a non-empty set $w \subseteq [a]$. By claim 1, for each $b, b' \in w$, $b \equiv a \equiv b'$, so $b \equiv b'$. Hence, by claim 2, we can construct $z_{bb'}$, $u_{bb'}$ such that $b \equiv_{z_{bb'}, u_{bb'}} b'$. Let $y = \bigcup_{b,b' \in w} z_{bb'}$, $t = \bigcup_{b,b' \in w} u_{bb'}$. It follows as in the proof of transitivity in claim 1, that, for $b, b' \in w$, $b \equiv_{yt} b'$. Hence if $b \in B_{xs}$, $b' \in B_{x's'}$, then $\phi_{xs;yt}(b) = \phi_{x's':yt}(b')$. Denote this common value by a^*.

Property (i) above follows from claim 1. Property (ii) follows from the fact that a^* was constructed from $[a]$; hence if $a \equiv b$, $[a] = [b]$, so $a^* = b^*$. ⊣

We can finally define B : $B = \{a^* : a \in B^*\}$.

We will now define the operations of τ on B^*:

Definition 3.1 Suppose $\langle F, x\rangle \in \tau$, and $b \in (B^*)^x$. For $y \in x$, suppose $b_y \in B_{x_y, u_y}$. Let $w = \bigcup_{y \in x} z_y$, and $t = \{\langle F, x\rangle\} \cup \bigcup_{y \in x} u_y$. Let b' be the (coordinate-wise) image of b in B_{wt} (using the ϕ's). Define $F_{\mathbb{B}^*}(b) = F_{\mathbb{B}_{wt}}(b')$.

Claim 4: If $\langle F, x\rangle \in \tau$, $b, c \in (B^*)^x$, and b and c are coordinate-wise $\equiv$, then $F_{\mathbb{B}^*}(b) \equiv F_{\mathbb{B}^*}(c)$.

Proof: By claim 2, for $y \in x$ we can construct $q_y \subseteq A$, $v_y \subseteq \tau$ such that $b_y \equiv_{q_y, v_y} c_y$. Let (similar to definition 3.1) $w' = \bigcup_{y \in x} q_y$, and $t' = \{\langle F, x\rangle\} \cup \bigcup_{y \in x} v_y$. Let d be the common (coordinate-wise) image of b and c in $(B_{w't'})^x$.

Subclaim 4.1 $F_{\mathbb{B}^*}(b) \equiv F_{\mathbb{B}^*}(d)$.

Proof: Let w, t, b' be as defined in definition 3.1. It is then clear that $w \subseteq w'$ and $t \subseteq t'$. From definition 3.1, $F_{\mathbb{B}^*}(b) = F_{\mathbb{B}_{wt}}(b')$ and $F_{\mathbb{B}^*}(d) = F_{\mathbb{B}_{w't'}}(d)$. It follows from properties of the ψ's, that $\psi_{wt;w't'} \circ b' = d$. Hence, as $\psi_{wt;w't'}$ is a homomorphism from $\mathbb{B}_{wt}$ to $\mathbb{B}_{w't'} \downarrow t$, $\psi_{wt;w't'}(F_{\mathbb{B}^*}(b)) = F_{\mathbb{B}^*}(d)$, and so $F_{\mathbb{B}^*}(b) \equiv_{w't'} F_{\mathbb{B}^*}(d)$ ⊣

Similarly, $F_{\mathbb{B}^*}(c) \equiv F_{\mathbb{B}^*}(d)$, so, by claim 1, $F_{\mathbb{B}^*}(b) \equiv F_{\mathbb{B}^*}(c)$ ⊣

In view of claim 4, we can define operation on B.

Definition 3.2 If $\langle F, x \rangle \in \tau$, $b \in (B)^x$, define $F_{\mathbb{B}}(b) = (F_{\mathbb{B}^*}(b))$.

Definition 3.3 If $a \in A$, $\varphi(a) = (\varphi_{\{a\},\emptyset}(a))^*$.

Claim 5: $\langle\langle \mathbb{B}, \varphi \rangle\rangle$ is a $\mathcal{V}(E)$-free image of $\mathbb{A}$.

Proof: There are four conditions in definition 1.8 to be verified.

(1) $P(\mathbb{B})$ holds, i.e., $\mathbb{B}$ satisfies the equations in E.

Suppose the equation $e =$ '$w_1 = w_2$' is in E; let the set t consist of the operation symbols occuring in e, and let x be the set of variable symbols occuring in e. Suppose f is a function from x to B. We must show:

(3.1) $Val_{\mathbb{B}}(w_1, f) = Val_{\mathbb{B}}(w_2, f)$.

Let g and h be defined on x so that $f(v) \in B_{g(v),h(v)}$ for all $v \in x$.

Let $y = \bigcup_{v \in x} g(v)$, and $u = \tau \restriction t \cup \bigcup_{v \in x} h(v)$. Let f' be the image of f in $\mathbb{B}_{yu}$.

It follows from claim 4 (and definition 1.2, and a simple induction argument) that

$$(3.2)\quad Val_{\mathbb{B}^*}(w_1,f) = Val_{\mathbb{B}^*}(w_1,f') \equiv Val_{\mathbb{B}^*_{yu}}(w_1,f'),$$

and similarly for w_2. But, as $\mathbb{B}_{yu}$ was defined as a free image, its property 1 implies that

$$(3.3)\quad Val_{\mathbb{B}_{yu}}(w_1,f') = Val_{\mathbb{B}_{yu}}(w_2,f'),$$

and so

$$(3.4)\quad Val_{\mathbb{B}^*}(w_1,f') \equiv Val_{\mathbb{B}_{yu}}(w_1,f') \qquad (3.2)$$

$$\equiv Val_{\mathbb{B}_{yu}}(w_2,f') \qquad (3.3)$$

$$\equiv Val_{\mathbb{B}^*}(w_2,f') \qquad (3.2)$$

It can be shown, using definition 3.2 and claim 4, that

$$(3.5)\quad Val_{\mathbb{B}}(w,f) \equiv Val_{\mathbb{B}^*}(w,f).$$

Hence, using (3.4) and (3.5) we obtain

(3.6) $Val_{\mathbb{B}}(w_1,f) \equiv Val_{\mathbb{B}}(w_2,f)$, which implies (3.1), as B selected a representative of each equivalence class of B^*. ⊣ (property 1)

(2) φ <u>is a homomorphism from $\mathbb{A}$ to $\mathbb{B}$.</u>

We must show that for any $\langle F,x\rangle \in \tau$, $f \in A^x$, if $F_{\mathbb{A}}(f)$ is defined, then

$$(3.7)\quad \varphi(F_{\mathbb{A}}(f)) = F_{\mathbb{B}}(\varphi \circ f).$$

Let $y = \mathcal{R}(f) \cup \{F_{\mathbb{A}}(f)\}$, and $t = \{\langle F, x\rangle\}$. Then:

$$
\begin{aligned}
F_{\mathbb{B}}(\varphi \circ f) &\equiv F_{\mathbb{B}^*}(\varphi \circ f) && \text{(definition 3.2)}\\
&\equiv F_{\mathbb{B}^*}(\varphi_{yt} \circ f) && \text{(claim 4)}\\
&= F_{\mathbb{B}_{yt}}(\varphi_{yt} \circ f) && \text{(definition 3.1)}\\
&= \varphi_{yt}(F_{\mathbb{A}}(f)) && \text{(property 2 of definition 3.8 for } \langle\langle \mathbb{B}_{yt}, \varphi_{yt}\rangle\rangle)\\
&\equiv \varphi(F_{\mathbb{A}}(f)) && \text{(definition 3.3)}
\end{aligned}
$$

This shows equivalence of the sides of (3.7), and hence equality.

⊣ (property 2)

(3) <u>φ"A generates $\mathbb{B}$.</u>

Suppose $b \in \mathbb{B}$, then $B \in \mathbb{B}_{xs}$ for some $x \subseteq A$, $s \subseteq \tau$. From property 3 for $\mathbb{B}_{xs}$, we have φ_{xs}" x generated $\mathbb{B}_{xs}$. Hence, there is a word $w \in W_{sx}$, such that $b = \mathrm{Val}_{\mathbb{B}_{xs}}(w, \varphi_{xs})$. But then:

$$
\begin{aligned}
b &= \mathrm{Val}_{\mathbb{B}_{xs}}(w, \varphi_{xs})\\
&\equiv \mathrm{Val}_{\mathbb{B}^*}(w, \varphi \restriction x) && \text{(claim 4)}\\
&\equiv \mathrm{Val}_{\mathbb{B}}(w, \varphi \restriction x),
\end{aligned}
$$

and as before equality must hold. ⊣ (property 3)

(4) If $\mathbb{C}$ satisfies E, and γ is a homomorphism from $\mathbb{A}$ to $\mathbb{C}$, there is a homomorphism ψ from $\mathbb{B}$ to $\mathbb{C}$ such that $\psi \circ \varphi = \gamma$.

Fix sets $x \subseteq A$, and $s \subseteq \tau$. Define homomorphisms γ_{xs} from $\mathbb{A}_{xs} \to \mathbb{C} \downarrow s$ by

(3.8) $\gamma_{xs} = \gamma \upharpoonright x$.

Since $\mathbb{C}$ satisfies $E \cap W_s V$, by property 4 of definition 1.8 there is a homomorphism $\psi_{xs} : \mathbb{B}_{xs} \to \mathbb{C} \downarrow s$ such that

(3.9) $\psi_{xs} \circ \varphi_{xs} = \gamma_{xs}$.

Now define $\psi^* : \mathbb{B}^* \to \mathbb{C}$ (not necessarily a homomorphism) by

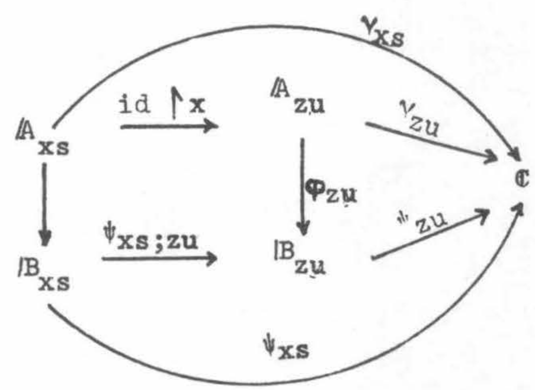
(3.10) $\psi^* = \bigcup_{x,s} \psi_{xs}$.

We now need an auxiliary result:

<u>Subclaim 3.1</u> If b, c $\in B^*$, and $b \equiv c$, then $\psi^*(b) = \psi^*(c)$.

<u>Proof</u>: Suppose $b \in B_{xs}$, $c \in B_{yt}$, and $b \equiv_{zu} c$. Let $d = \psi_{xs;zu}(b) = \psi_{yt;zu}(c)$.

It is sufficient to show that $\psi^*(b) = \psi^*(d)$, as the proof that $\psi^*(c) = \psi^*(d))$ is similar. We need to show that the adjoining diagram commutes. We already know that all circuits except the bottom $\mathbb{B}_{xs}, \mathbb{B}_{zu}, \mathbb{C}$ loop commute. But that is the desired result.



Define:

(3.11) $\psi'_{xs} = \psi_{zu} \circ \psi_{xs;zu}$.

To show $\psi'_{xs} = \psi_{xs}$, we will show that ψ'_{xs} satisfies property 4 of definition 1.8. Since that ψ is unique by lemma 1 of §1, we have $\psi'_{xs} = \psi_{xs}$, and $\psi^*(b) = \psi_{xs}(b) = \psi'_{xs}(b) = \psi_{zu}(d) = \psi^*(d)$.

We have:

$$\psi'_{xs} \circ \varphi_{xs} = \psi_{zu} \circ \psi_{xs;zu} \circ \varphi_{xs} \qquad (3.11)$$

$$= \psi_{zu} \circ (\varphi_{zu} \restriction x) \qquad (\text{definition } \psi_{xs;zu})$$

$$= \gamma_{zu} \restriction x \qquad (3.9)$$

$$= \gamma_{xs} \qquad (3.8)$$

Hence ψ'_{xs} satisfies the defining property (4) of ψ_{xs}, hence, by Lemma 1 of §1, $\psi'_{xs} = \psi_{xs}$.

Hence, $\psi^*(b) = \psi_{xs}(b) = \psi'_{xs}(b) = \psi_{zu}(d) = \psi^*(d) =$ (similarily) $\psi^*(c)$ ⊣ Subclaim 5.1

We now define

(3.12) $\psi = \psi^* \restriction B$.

We must show that ψ is a homomorphism from $\mathbb{B}$ to $\mathbb{C}$. Fix $\langle F,x\rangle \in \tau$, and $b \in B^x$. Let w, t, b′ be as in definition 3.1. Then:

$$\psi(F_{\mathbb{B}}(b)) = \psi^*(F_{\mathbb{B}^*}(b) \qquad (\text{Subclaim 5.1, definition 3.2})$$

$$= \psi^*(F_{\mathbb{B}_{wf}}(b')) \qquad (\text{definition 3.1})$$

$$= F_{\mathbb{C}}(\psi_{wf} \circ b') \qquad (\text{definition } \psi_{wt})$$

$$= F_{\mathbb{C}}(\psi^{*} \circ b') \qquad (3.10)$$

$$= F_{\mathbb{C}}(\psi^{*} \circ b) \qquad (\text{Subclaim } 5.1)$$

$$= F_{\mathbb{C}}(\psi \circ b) \qquad (3.12)$$

Hence ψ is a homomorphism, as asserted. All that remains to be shown is that $\psi \circ \varphi = \gamma$. To do this, note that for $a \in A$,

$$\psi(\varphi(a)) = \psi^{*}(\varphi_{\{a\},\emptyset}(a)) \qquad (\text{Subclaim } 5.1,\ (3.12),\ \text{definition } 3.3)$$

$$= \gamma_{\{a\},\emptyset}(a) \qquad (3.9)$$

$$= \gamma(a) \qquad (3.8)$$

⊣ (property 4)

⊣ claim 5

⊣ Theorem

§4 Applications

To use Theorem 3.1 to get results involving more familiar algebras, we need the following lemma:

Lemma 1 Let τ be the type $\{\langle '\vee', 2\rangle\} \cup \{\langle \bigvee_z, z\rangle : z \text{ a set}\}$. Let E consist of the following equations in $W_{\tau\infty}$:

(E 1z) $x \vee \bigvee_z (x)^{t\in z} = x$, z a set

(E 2z) $y \vee \bigvee_z (x_t)^{t\in z} = y \vee \bigvee_z (y \vee x_t)^{t\in z}$, z a set

(E 3zy) $x_y \vee \bigvee_z (x_t)^{t\in z} = \bigvee_z (x_t)^{t\in z}$, $y \in z$ sets

(E 4) $x \vee x = x$

(E 5) $x \vee y = y \vee x$

(E 6) $(x \vee y) \vee z = x \vee (y \vee z)$.

Then there is an exact correspondence between set-complete join semi-lattices $\mathbb{A} = \langle\langle A;\vee\rangle\rangle$ and algebras of type τ satisfying the equations E; by $\mathbb{A}^* = \langle\langle A;\vee, \bigvee_z\rangle\rangle$ defined by $\bigvee_z(a_t)^{t\epsilon z} = \bigvee\{a_t : t\epsilon z\}$.

Proof: We must show the following:

(1) If $\mathbb{A}$ is a set-complete join semi-lattice, then $\mathbb{A}^* \models E$

(2) If $\mathbb{B} \models E$, $\mathbb{B}$ of type τ, then $\mathbb{B} \downarrow \{\langle '\vee', 2\rangle\}$ is a set-complete join semi-lattice

(3) If $\mathbb{B} \models E$, then $\mathbb{B} = (\mathbb{B} \downarrow \{\langle '\vee', 2\rangle\})^*$

(4) If $\mathbb{A}$ is a set-complete join semi-lattice, then $\mathbb{A} = \mathbb{A}^* \downarrow \{\langle '\vee', 2\rangle\}$.

In order to show (1) suppose A be a set-complete join semi-lattice.

Equations E 1-3 for $\mathbb{A}^*$ follow from properties of the $\bigvee$ operation on A, while E 4-6 are the defining equations of a semi-lattice.

In order to show (2) suppose $\mathbb{B} \models E$. Clearly $\mathbb{B}' = \mathbb{B} \downarrow \{\langle\vee;2\rangle\}$ is a semi-lattice, as $\mathbb{B} \models$ E4 - E6. Suppose $z \subseteq B$. I claim

(4.1) $a = (\bigvee z)_{\mathbb{B}}(id \restriction z)$ is $\bigvee z$ in $\mathbb{B}'$. If $b \in z$, then

$$
\begin{aligned}
b \vee a &= b \vee \textstyle\bigvee_z (t)^{t\epsilon z} && (4.1)\\
&= \textstyle\bigvee_z (t)^{t\epsilon z} && (E\ 3zb)\\
&= a && (4.1),
\end{aligned}
$$

so $b \leq a$. If c is such that

(4.2) $b \leq c$ for all $b \in z$, then

$$\begin{aligned} c \vee a &= c \vee \bigvee_z (t)^{t \in z} && (4.1) \\ &= c \vee \bigvee_z (c \vee t)^{t \in z} && (E\ 2z) \\ &= c \vee \bigvee_z (c)^{t \in z} && (4.2) \\ &= c && (E\ 1z), \end{aligned}$$

so $a \leq c$. Hence $\bigvee(z) = a$ in B', as was to be shown.

In order to show (3) suppose $\mathbb{B} \models E$. Let $\mathbb{B}'$ be as in (2). We must show that, for z a set, and f a function from z to B, $(\bigvee_z)_{\mathbb{B}'}(f) = (\bigvee_z)_{\mathbb{B}}(f)$. However, a similar argument to that in (2) above shows that $\bigvee_{\mathbb{B}'}\{f(t) : t \in z\} = (\bigvee_z)_{\mathbb{B}}(f)$. Hence

$$(\bigvee_z)_{\mathbb{B}'^*}(f) = \bigvee_{\mathbb{B}'}\{f(t) : t \in z\} = (\bigvee_z)_{\mathbb{B}}(f).$$

Property (4) is clear. ⊣

Using this lemma we can describe a set-complete Boolean algebra as an algebraic system as follows:

Let $\tau = \{\langle \wedge, 2\rangle, \langle \vee, 2\rangle, \langle ', 1\rangle, \langle 0, 0\rangle, \langle 1, 0\rangle\} \cup \{\langle \bigwedge_z, z\rangle, \langle \bigvee_z, z\rangle\}^{(z \text{ a set})}$. A partial algebra of type τ will be called a <u>partial set-complete Boolean algebra</u> (p s-cBa). An algebra $\mathbb{B} = \langle\langle B; \wedge, \vee, ', 0, 1, \bigwedge_z, \bigvee_z\rangle\rangle_{z \text{ a set}}$ of type τ is a set-complete Boolean algebra (s-cBa) if

(1) $\langle\langle B; \wedge, \vee, ', 0, 1\rangle\rangle$ is a Boolean algebra.

(2) $\mathbb{B}$ satisfies the equations of lemma 1 for both $\wedge$ and $\vee$.

<u>Theorem 4.1</u> Let $\mathbb{A}$ be a p s-cBa, and suppose $\mathbb{A}' = \langle\langle A ; \wedge, \vee, ', 0, 1\rangle\rangle$ is a

Boolean algebra, and the partial operations $\wedge_z$, $\vee_z$ of $\mathbb{A}$ are preserved in $\mathbb{A}'$, i.e., if $b = \wedge_z(a_t)_{t\in z}$ is defined in $\mathbb{A}$, then $b = \wedge\{a_t : t \in z\}$ in $\mathbb{A}'$. Then there is a (s-cBa)-free quasi-extension of $\mathbb{A}$.

Proof: Using theorem 3.1, we get a (s-cBa)-free image $\langle\langle\mathbb{B},\sigma\rangle\rangle$ of $\mathbb{A}$. However, let $\mathbb{C}$ be the normal completion of $\mathbb{A}'$, and let φ be the induced map from A to C. Clearly $\mathbb{C}$ is a s-cBa, and by the preservation above, φ is a 1-1 homomorphism from $\mathbb{A}$ to $\mathbb{C}$. Hence, from property 4 of definition 1.8 applied to $\langle\langle\mathbb{B},\sigma\rangle\rangle$, there is a homomorphism ψ from $\mathbb{B}$ to $\mathbb{C}$ such that $\psi \circ \sigma = \varphi$. Hence σ is 1-1. $\dashv$

Theorem 4.2 Let E be a class of equations in the type of a s-cBa. If $\mathbf{2}$ (considered as a s-cBa) satisfies the equations E, then for any class X, there is a class X′ such that X is equipollent with X′, and there is a (E - s-cBa)-free algebra on X′.

Proof: Apply theorem 3.1 with the class of equations consisting of E and the s-cBa equations, and $\mathbb{A} = \langle\langle X ; \rangle\rangle \uparrow \tau$ of the type of s-cBa's to get $\langle\langle\mathbb{B},\sigma\rangle\rangle$ which is a free (E - s-cBa)-free image of $\mathbb{A}$. If we can show that σ is 1-1, we are done as then $\mathbb{B}$ is an (E - s-cBa)-free algebra on $\sigma''X$. Suppose u, v $\in$ X, u $\neq$ v. Let $\varphi : X \to 2$ be defined by

$$(4.3) \quad \varphi(x) = \begin{cases} 1 & x = u \\ 0 & x \neq u \end{cases} .$$

By property 4 of definition 1.8 for $\langle\langle\mathbb{B},\sigma\rangle\rangle$, there is a $\psi : \mathbb{B} \to \mathbf{2}$ such that $\psi \circ \sigma = \varphi$. But $\varphi(u) \neq \varphi(v)$, so $\sigma(u) \neq \sigma(v)$. $\dashv$

Contrast this with the Gaifman-Hales Theorem, which in our terminology reads:

<u>Theorem 4.3</u> If $\aleph$ is a regular cardinal, there is no (set $(\aleph,\infty)$-distributive cBa)-free algebra on a set of $\aleph$ generators.

The reason theorem 4.2 does not contradict this, is that Gaifman and Hales assumed that all of the Boolean algebras in question were sets; re-working the proof in the notation of this chapter would merely show that any such algebra is a proper class.

Here is one last theorem proved exactly as theorem 4.1.

<u>Theorem 4.4</u> Let A be a partial set-complete lattice, and suppose $\mathbb{A}' = \langle\langle A; \wedge, \vee\rangle\rangle$ is a normally distributive lattice. Suppose also the partial operations of $\mathbb{A}$ are preserved in $\mathbb{A}'$. Then there is a (distributive set-complete lattice)-free quasi-extension of $\mathbb{A}$.

CHAPTER 2 Positive Elementary Inductions in Reasonable Structures

Consider the following well-known result, from [Barwise 1] and [Moschovakis]:

Theorem 1 If $\mathfrak{M}$ is an almost acceptable structure, then $\kappa^{\mathfrak{M}}$ (the inductive closure ordinal of $\mathfrak{M}$) is equal to $o(\mathfrak{M})$ (the ordinal of the smallest admissible set above $\mathfrak{M}$, or equivalently, the recursive saturation ordinal of $\mathfrak{M}$). Hence $\kappa^{\mathfrak{M}}$ is admissible.

Moschovakis has shown that for any $\mathfrak{M}$, $\kappa^{\mathfrak{M}}$ is admissible or the limit of a countable sequence of admissible ordinals.

In 1977, Barwise [Barwise 2] proved the following theorem:

Theorem 2 If κ is a limit of a sequence of admissible ordinals, then there is a structure $\mathfrak{M}$ such that $\kappa^{\mathfrak{M}} = \kappa$.

In this chapter, I introduce saturation ordinals $\rho^{\mathfrak{M}}$ and $\rho_*^{\mathfrak{M}}$, to show:

Theorem 3 1) For any $\mathfrak{M}$, $\kappa^{\mathfrak{M}} \leq \rho_*^{\mathfrak{M}} \leq \rho^{\mathfrak{M}}$. 2) If $\mathfrak{M}$ is reasonable, $\kappa^{\mathfrak{M}} = \rho_*^{\mathfrak{M}} = \rho^{\mathfrak{M}}$, hence, $\kappa^{\mathfrak{M}}$ is admissible.

As a corollary we have:

Theorem 4 If $\mathrm{Th}(\mathfrak{M})$ admits recursive elimination of quantifiers, $o(\mathfrak{M}) > \omega$, and $\mathfrak{M}$ has an inductive definition with closure ordinal $\geq \omega$, then $\kappa^{\mathfrak{M}} > \omega$. In particular, if $\mathfrak{M}$ is a non-recursively saturated model of ACF, RCF or DCF, then $\kappa^{\mathfrak{M}} > \omega$.

§1. Saturation ordinals

Let $\mathfrak{M} = \langle M; R_1, \ldots, R_\ell \rangle$ be an infinite structure with finitely many relations, and <u>no</u> functions or constants. As in [Moschovakis], let $\kappa^{\mathfrak{M}}$ be the closure ordinal of the structure $\mathfrak{M}$; i.e., the supremum of the closure ordinals of the positive first-order operators on $\mathfrak{M}$. Let also $o(\mathfrak{M})$ be the ordinal of the smallest admissible set above $\mathfrak{M}$ as in [Barwise 1]. We shall introduce below some ordinals associated with $\mathfrak{M}$ which measure in a sense its "saturation". We need to explain some new notation first.

Let α be a limit ordinal and let $P' = \{\varphi_\beta(x,\bar{y}) : \beta < \alpha\}$ be an α-sequence of formulas of $\mathcal{L}_{\infty\omega}$ (where $\mathcal{L}$ is the language of the structure $\mathfrak{M}$) containing the same list of free variables $x, \bar{y}$ ($\bar{y} = y_1, \ldots, y_k$). Let $\bar{a} = (a_1, \ldots, a_k) \in M^k$ and put $p = \{\varphi_\beta(x,\bar{a}) : \beta < \alpha\}$. We say p is an (n,α)-type if the following conditions are satisfied:

i) For each $\beta < \alpha$, $\varphi_\beta \in \mathcal{L}_\alpha = L(\alpha) \cap \mathcal{L}_{\infty\omega}$, where $L(\alpha)$ is the α^{th} stage of Gödel's constructible hierarchy.

ii) $\beta \leq \beta' < \alpha \Rightarrow \mathfrak{M} \models (\forall x)[\varphi_{\beta'}(x,\bar{a}) \Rightarrow \varphi_\beta(x,\bar{a})]$.

iii) There is a list of exactly n variables x, $\bar{y}$, $\bar{z}$, such that all the free and bound variables of each $\varphi_\beta(x,\bar{y})$ are among x, $\bar{y}$, $\bar{z}$.

iv) The function $\beta \to \varphi_\beta$ is Σ_1-definable in $L(\alpha)$.

As we will see in §2, the complements of the stages of a positive first-order inductive definition correspond to some typical examples of (n,α)-types.

Let us call $\mathfrak{M}$ (n,α)-recursively saturated if for any (n,α)-type $p = \{\varphi_\beta(x,\bar{a}) : \beta < \alpha\}$, if $(\forall\ \beta < \alpha)(\mathfrak{M} \models (\exists x)\varphi_\beta\ (x,\bar{a}))$, then $\mathfrak{M} \models (\exists x) \bigwedge_{\beta<\alpha} \varphi_\beta(x,\bar{a})$. Define $\rho_n^{\mathfrak{M}}$ to be the least α such that $\mathfrak{M}$ is (n,α)-recursively saturated, $\rho_*^{\mathfrak{M}} = \sup_n \rho_n^{\mathfrak{M}}$, and $\rho^{\mathfrak{M}}$ to be the least α such that for all n, $\mathfrak{M}$ is (n,α)-recursively saturated. It is reasonable to call $\rho_*^{\mathfrak{M}}$ the weak saturation ordinal of $\mathfrak{M}$, and $\rho^{\mathfrak{M}}$ the saturation ordinal of $\mathfrak{M}$. It is clear that $\rho_*^{\mathfrak{M}} \leq \rho^{\mathfrak{M}}$.

Fact: For any $\mathfrak{M}$, $n \in \omega$, $\rho_n^{\mathfrak{M}}$ is admissible, $\rho^{\mathfrak{M}}$ is admissible, and $\rho_n^{\mathfrak{M}}$ is admissible or a limit of admissibles.

Proof: Suppose $\alpha = \rho^{\mathfrak{M}}$ or $\rho_n^{\mathfrak{M}}$ is not admissible. Then there is a $\beta < \alpha$ and a Σ_1 in $L(\alpha)$ function f such that $f : \beta \xrightarrow{\text{cof}} \alpha$. Consider the least such β. If there were an (n,β)-type P contradicting the definition of (n,β)-recursive saturation of $\mathfrak{M}$, with $p = \{\varphi_\gamma(x,\bar{a}) : \gamma < \beta\}$, then $p^* = \{\psi_\delta(x,\bar{a}) : \delta < \alpha\}$, with $\psi_\delta = \varphi_{(\text{the least } \gamma \text{ such that } f(\gamma) > \delta)}$, contradicts the definition of (n,α)-recursive saturation of $\mathfrak{M}$. It follows that $\rho^{\mathfrak{M}}$, $\rho_n^{\mathfrak{M}}$ are admissible, and clearly then that $\rho_*^{\mathfrak{M}}$ is admissible or a limit of admissibles.

§2. Comparing the closure and saturation ordinals.

The next theorem, which is the key result of this chapter, shows that the closure and both saturation ordinals coincide for many natural structures $\mathfrak{M}$. To make this precise, let us call a structure $\mathfrak{M}$ reasonable if there is an inductive relation which is not hyperelementary, or equivalently, if there is a positive first-order operator on $\mathfrak{M}$ with closure ordinal $\kappa^{\mathfrak{M}}$.

Theorem 2.1 i) For any structure $\mathfrak{M}$, $\kappa^{\mathfrak{M}} \leq \rho_*^{\mathfrak{M}}$.

ii) For any reasonable structure $\mathfrak{M}$, $\mathfrak{M}$ is $(n,\kappa^{\mathfrak{M}})$-saturated for all $n \in \omega$, so that $\rho^{\mathfrak{M}} \leq \kappa^{\mathfrak{M}}$. Hence $\rho^{\mathfrak{M}} = \rho_*^{\mathfrak{M}} = \kappa^{\mathfrak{M}}$, and so $\kappa^{\mathfrak{M}}$ is admissible.

Note: Part ii) of this theorem partially answers a question of Moschovakis about the possible values of $\kappa^{\mathfrak{M}}$. Barwise [2] has shown that there are structures $\mathfrak{M}$ such that $\kappa^{\mathfrak{M}}$ is any countable limit of admissibles, so that some assumption for part ii) is necessary.

Proof: i) This proof is similar to a conventional proof that $\kappa^{\mathfrak{M}} \leq o(\mathfrak{M})$. Let $\varphi(\bar{x},S)$ be a first-order positive operator. From theorem 4B.1 of [Moschovakis], there is an operator

$$\psi(\bar{x},S) \leftrightarrow (\bar{Q}\ \bar{z})(\forall\ \bar{u})(\theta(\bar{x},\ \bar{z},\ \bar{u}) \vee S(\bar{u}))$$ (each Q denotes $\exists$ or $\forall$),

with θ quantifier-free, such that for any α, $I^{\alpha}_{\varphi} = I^{\alpha}_{\psi}$.

Define, by transifinite recursion, formulas

$$\psi^{\beta}(x) = (\bar{Q}\ \bar{z})(\forall\ \bar{u})(\theta(\bar{x},\ \bar{z},\ \bar{u}) \vee (\exists\ \bar{x})(\bar{x} = \bar{u} \wedge \bigvee_{\gamma<\beta} \psi^{\gamma}(\bar{x}))).$$

It is clear that for any α, $\bar{x} \in I^{\alpha}_{\psi} \leftrightarrow \mathfrak{M} \models \psi^{\alpha}(\bar{x})$, and, in addition, the map $\beta \mapsto \psi^{\beta}$ is Σ_1 in $L(\alpha)$ for any admissible α. Let n be the number of variables occuring (free and bound) in the formulas ψ^{β}. Let $\alpha = \rho^{\mathfrak{M}}_{n}$. We wish to show that $\|\varphi\| \leq \alpha$; i.e., to show that there is no $x \in I^{\alpha}_{\varphi} \sim I^{<\alpha}_{\varphi}$.

We know that $x \in I^{\alpha}_{\varphi}$ iff

$$\bar{Q}\ \bar{z}\ \bar{\forall}\ \bar{u}\ [\theta(\bar{x},\ \bar{z},\ \bar{u}) \vee (\exists\ \bar{x})(\bar{x} = \bar{u} \wedge \bigvee_{\beta<\alpha} \psi^{\beta}(\bar{x}))]$$

iff

$$(*) \qquad \bar{Q}\ \bar{z}\ \bar{\forall}\ \bar{u} \bigvee_{\beta<\alpha} [\bigvee_{\gamma<\beta} [\theta(\bar{x},\ \bar{z},\ \bar{u}) \vee (\exists\ x)(x = u \wedge \psi^{\gamma}(\bar{x}))]]$$

Using the $(n\alpha)$-recursive saturation of $\mathfrak{M}$, we show that (*) holds iff

$$(**) \qquad \bigvee_{\beta<\alpha} \bar{Q}\ \bar{z}\ \bar{\forall}\ \bar{u}\ [\bigvee_{\gamma<\beta} [\theta(\bar{x},\ \bar{z},\ \bar{u}) \vee (\exists\ \bar{x})(\bar{x} = \bar{u} \wedge \psi^{\gamma}(\bar{x}))]],$$

which will give us our result since (**) asserts that $x \in I^{<\alpha}_{\varphi}$.

So we have left only to move the quantifiers, one by one, past the disjunction. The idea is essentially due to Keisler [Keisler]. We show inductively that if $\{\theta_\beta : \beta < \alpha\}$ is an (n,α)-type, and $\mathfrak{M}$ is (n,α)-recursively saturated, and $Q_1, \ldots, Q_k$ is any string of quantifiers, then for any $m \leq k - 1$,

$$\text{(A)} \quad \mathfrak{M} \models Q_1 v_1 \ \cdots \ Q_m v_m \bigvee_{\beta<\alpha} Q_{m+1} v_{m+1} \ \cdots \ Q_k v_k \bigvee_{\gamma<\beta} \neg\, \theta_\gamma(\bar{u},\bar{v})$$

iff

$$\text{(B)} \quad \mathfrak{M} \models Q_1 v_1 \ \cdots \ Q_m v_m\, Q_{m+1} v_{m+1} \bigvee_{\beta<\alpha} Q_{m+2} v_{m+2} \cdots Q_k v_k \bigvee_{\gamma<\beta} \neg\, \theta_\gamma(\bar{u},\bar{v}).$$

(This is what we need since the formula inside the brackets of (*) (for $\beta < \alpha$) form a (n,α)-cotype, i.e., their negations form an (n,α)-type.)

For $Q_{m+1} = \exists$ the result is trivial. For $Q_{m+1} = \forall$ we use Keisler's trick: It is trivial that (A) $\Rightarrow$ (B), so we only have to prove (B) $\Rightarrow$ (A).

Treat $v_1, \ldots, v_m$ as parameters: Suppose

$$\mathfrak{M} \not\models \bigvee_{\beta<\alpha} \forall\, v_{m+1} \cdots Q_k v_k \bigvee_{\gamma<\beta} \neg\, \theta_\gamma (\bar{u},\bar{v}).$$

Then

$$\mathfrak{M} \models \bigwedge_{\beta<\alpha} \exists\, v_{m+1}\, \breve{Q}_{m+2} v_{m+2} \cdots \breve{Q}_k v_k \bigwedge_{\gamma<\beta} \theta_\gamma(\bar{u},\bar{v}). \quad \text{[Where } \breve{\forall} = \exists,\ \breve{\exists} = \forall\text{]}.$$

As $\left\{\breve{Q}_{m+2} v_{m+2} \ \cdots \ \breve{Q}_k v_k \bigwedge_{\gamma<\beta} \theta_\gamma(\bar{u},\bar{v}) : \beta < \alpha\right\}$ form an (n,α)-type, by (n,α)-recursive saturation we have:

$$\mathfrak{M} \models \exists\, v_{m+1} \bigwedge_{\beta<\alpha} \breve{Q}_{m+2} v_{m+2} \ \cdots \ \breve{Q}_k v_k \bigwedge_{\gamma<\beta} \theta_\gamma(\bar{u},\bar{v}), \text{ so}$$

$$\mathfrak{M} \not\models \forall v_{m+1} \bigvee_{\beta<\alpha} Q_{m+2}\, v_{m+2}, \ldots, Q_k\, v_k \bigvee_{\gamma<\beta} \neg\, \theta_\gamma(\bar{u},\bar{v}).$$

Hence $\neg(A) \Rightarrow \neg(B)$, so $(A) \Leftrightarrow (B)$.

This gives us $\|\varphi\|^{\mathfrak{M}} \leq \rho_n^{\mathfrak{M}} \leq \rho_*^{\mathfrak{M}}$ for an arbitrary first order positive operator φ, so $\varkappa^{\mathfrak{M}} \leq \rho_*^{\mathfrak{M}}$.

ii) Let $\mathfrak{M}$ be a reasonable structure. There is then an inductive definition $\varphi(\bar{x},S)$ with closure ordinal $\varkappa^{\mathfrak{M}}$. To complete the proof we 'simulate' $L(\varkappa^{\mathfrak{M}})$ within $\mathfrak{M}$ and use the construction to show that $\mathfrak{M}$ is $(n,\varkappa^{\mathfrak{M}})$-saturated for all n. We know ([Moschovakis]) that $\varkappa^{\mathfrak{M}}$ is admissible or a limit of admissibles; this fact will be needed for the following.

Lemma: There is a function F from $\varkappa^{\mathfrak{M}}$ onto $L(\varkappa^{\mathfrak{M}})$ such that if $F(\beta) \in F(\alpha)$, there is a $\gamma < \alpha$ such that $F(\beta) = F(\gamma)$, and such that the following relations are inductive: (Putting $\varkappa = \varkappa^{\mathfrak{M}}$).

a) $T_n(\bar{u},m_1, \ldots, m_n) \leftrightarrow \bar{u} \in I_\varphi^\varkappa$ and $F(|\bar{u}|_\varphi)$ is a formula $\ulcorner\psi\urcorner$ of $\mathcal{L}_\varkappa$ with only n variables, free and bound, $v_1, \ldots, v_n$ and $\mathfrak{M} \models \psi(m_1, \ldots, m_n)$

b) $N_n(\bar{u}, m_1, \ldots, m_n) \leftrightarrow \bar{u} \in I_\varphi^\varkappa$ and $F(|\bar{u}|_\varphi)$ is a formula $\ulcorner\psi\urcorner$ of $\mathcal{L}_\varkappa$ with only n variables, free and bound, $v_1, \ldots, v_n$ and $\mathfrak{M} \not\models \psi(m_1, \ldots, m_n)$

c) $T^*(\bar{u},\bar{s}) \leftrightarrow \bar{u},\bar{s} \in I_\varphi^\varkappa$, and $F(|\bar{u}|_\varphi)$ is a Δ_0 formula $\ulcorner\psi\urcorner$ of $\mathcal{L}_\omega^*$, where $L^* = \{'\varepsilon'\}$, and $F(|\bar{s}|_\varphi)$ is a finite sequence, and $L(\varkappa) \models \psi(F(|\bar{s}|_\varphi))$.

d) $N^*(\bar{u},\bar{s}) \leftrightarrow \bar{u},\bar{s} \in I^{\kappa}_{\varphi}$ and $F(|\bar{u}|_{\varphi})$ is a Δ_0 formula $\ulcorner\psi\urcorner$ of $\mathcal{L}^*_{\omega}$ and $F(|\bar{s}|_{\varphi})$ is a finite sequence, and $L(\kappa) \not\models \psi(F(|\bar{s}|_{\varphi}))$.

The proof of the lemma will be given in an appendix. Given the lemma, we proceed:

Let $\underline{p} = \{\psi_\beta(x,\bar{a}) : \beta < \kappa^{\mathfrak{M}}\}$ be an $(n,\kappa^{\mathfrak{M}})$-type, defined in $L(\kappa^{\mathfrak{M}})$ by a formula $t = \ulcorner\psi_\beta\urcorner \Leftrightarrow L(\kappa^{\mathfrak{M}}) \models (\exists\, \bar{u})(\theta(\beta,\ t,\ \bar{u},\ z_1,\ \ldots,\ z_q)$, where θ is a Δ_0 formula and $z_1,\ \ldots,\ z_q \in L(\kappa^{\mathfrak{M}})$. Then, using the lemma, we can define an inductive relation

$$M(x,y) \leftrightarrow \mathfrak{M} \models \neg\, \psi_{|y|_{\varphi}}(x,\bar{a}) \quad \text{(supressing parameters).}$$

Let p, q, and r be distinct elements of $\mathfrak{M}$. As in [Moschovakis], we create a single inductive definition $\chi(x,\ \bar{y},\ \bar{z},\ \bar{u},\ S)$ such that:

i) $(x,\ \bar{y},\ p,\ \bar{u}) \in I^{\alpha}_{\chi} \leftrightarrow \bar{y} \in I^{\alpha}_{\varphi}$

ii) There is a $\bar{u}_0$ such that

$$(x,\ \bar{y},\ q,\ \bar{u}_0) \in I^{\infty}_{\chi} \leftrightarrow \mathfrak{M} \models \neg\, \psi_{|\bar{y}|_{\varphi}}(x,\bar{a})$$

iii) $(x,\ \bar{y},\ r,\ \bar{u}) \in I^{\alpha}_{\chi} \leftrightarrow (\forall x)(\exists \bar{y})\Big[(x,\ \bar{y},\ p,\ \bar{u}) \in I^{<\alpha}_{\chi}$

$$(x,\ \bar{y},\ q,\ \bar{u}) \in I^{<\alpha}_{\chi}\Big].$$

Suppose now that $\underline{p}$ were such that $(\forall\ \beta < \kappa)\ \mathfrak{M} \models (\exists x)\psi_\beta(x,\bar{a})$, but $\mathfrak{M} \not\models \exists x \bigwedge_{\beta<\kappa} \psi_\beta(x,\bar{a})$. I claim then that $(r,\bar{r},r,\ \bar{u}_0) \in I^{\infty}_{\chi}$, but $(r,\bar{r},r,\bar{u}_0) \notin I^{<\kappa}_{\chi}$, so that $|\chi|^{\mathfrak{M}} > \kappa^{\mathfrak{M}}$, a contradiction; hence we would be done. So now,

$$(r,\bar{r},r,\bar{u}_0) \in I^{\infty}_{\chi} \leftrightarrow (\forall x)(\exists \bar{y})((x,\bar{y},p,\bar{u}_0) \in I^{\infty}_{\chi} \;\&\; (x,\bar{y},q,\bar{u}_0) \in I^{\infty}_{\chi})$$

$$\leftrightarrow (\forall x)(\exists \bar{y})(\bar{y} \in I^{\infty}_{\varphi} \;\&\; \mathfrak{M} \models \neg\, \psi_{|\bar{y}|_{\varphi}}(x,\bar{a}))$$

$$\leftrightarrow (\forall x)(\exists \beta < \varkappa)(\mathfrak{M} \not\models \psi_{\beta}(x,\bar{a}))$$

$$\leftrightarrow \mathfrak{M} \not\models (\exists x) \bigwedge_{\beta<\varkappa} \psi_{\beta}(x,\bar{a}), \text{ which was assumed.}$$

$$(r,r,r,\bar{u}_0) \in I^{<\varkappa}_{\chi} \leftrightarrow (\exists \beta < \varkappa)((r,\bar{r},r,\bar{u}_0) \in I^{\beta+1}_{\chi})$$

$$\leftrightarrow (\exists \beta < \varkappa)(\forall x)(\exists \bar{y})\Big[(x,\bar{y},p,\bar{u}_0) \in I^{\beta}_{\chi} \;\&\; (x,\bar{y},q,\bar{u}_0) \in I^{\beta}_{\chi}\Big]$$

$$\rightarrow (\exists \beta < \varkappa)(\forall x)(\exists \bar{y})\Big[\bar{y} \in I^{\beta}_{\varphi} \;\&\; (x,\bar{y},q,\bar{u}_0) \in I^{\infty}_{\chi}\Big]$$

$$\leftrightarrow (\exists \beta < \varkappa)(\forall x)(\exists \bar{y})\Big[\bar{y} \in I^{\beta}_{\varphi} \;\&\; \mathfrak{M} \models \neg\, \psi_{|\bar{y}|_{\varphi}}(x,\bar{a})\Big]$$

$$\leftrightarrow (\exists \beta < \varkappa)(\forall x)(\exists \gamma \le \beta)\Big[\mathfrak{M} \not\models \psi_{\gamma}(x,\bar{a})\Big]$$

$$\leftrightarrow (\exists \beta < \varkappa)\; \mathfrak{M} \not\models (\exists x) \bigwedge_{\gamma\le\beta} \psi_{\gamma}(x,\bar{a})$$

$$\leftrightarrow \neg\, (\forall \beta < \varkappa)\; \mathfrak{M} \models (\exists x)\; \psi_{\beta}(x,\bar{a}), \text{ which was assumed false.} \quad \dashv$$

By the methods of the next section, this result can be extended to models $\mathfrak{M}$ which have function symbols.

§3 <u>The closure ordinal of non-recursively saturated structures.</u>

In this section, $\mathfrak{M}$ can have function symbols.

<u>Lemma</u> 1 If $\mathfrak{M}$ is such that $Th(\mathfrak{M})$ admits recursive elimination of

quantifiers, and $\mathfrak{M}$ is (n,ω)-recursively saturated for every $n \in \omega$, then $\mathfrak{M}$ is recursively saturated. (See [Schlipf] for definitions of recursively saturated structures.)

Proof The idea of this proof, is to find, for any recursive ω-type (a type satisfying conditions i) and iv) of an (n,ω)-type), an $m \in \omega$ and an (m,ω)-type which fails to satisfy the conditions for (m,ω)-recursive saturation, if the original type fails to satisfy the conditions for recursive saturation.

Let $\Gamma = \{\varphi_m(x,\bar{a}) : m \in \omega\}$ be a recursive ω-type. Eliminate quantifiers to find $\{\psi_m(x,\bar{a}) : m \in \omega\}$ such that $\mathfrak{M} \models \varphi_m \leftrightarrow \psi_m$, and each ψ_m is quantifier-free.

The next phase of the proof is to eliminate function symbols, replacing them by the appropriate relations, as the proof in §2 was only correct in the absence of function symbols. Find, for each k-ary relation R of $\mathfrak{M}$, k new variable symbols $v_1^R, \ldots, v_k^R$; for each k-ary function f of $\mathfrak{M}$, 2k new variable symbols $u_1^f, \ldots, u_k^f, v_1^f, \ldots, v_k^f$; and a new variable $v^=$, none of which occur in any ψ_m. We will assign to any formula ψ, using only the variables of the ψ_m's, a formula χ with variables only among $v_1^R, u_1^f, v_1^f, v^=$, and the variables of ψ, such that $\models \psi \leftrightarrow \chi$, and the function symbols of $\mathfrak{M}$ only appear in χ as $v = f(v_1, \ldots, v_k)$, where $v, v_1, \ldots, v_k$ are variables. This allows us to replace $v = f(v_1, \ldots, v_k)$ by $R_f\,(v,v_1, \ldots, v_k)$, and we can treat $\mathfrak{M}$ as a relational system.

By replacing each

$R(t_1, \ldots, t_k)$ by $(\exists v_1^R) \cdots (\exists v_k^R)(R(v_1^R, \ldots, v_k^R)$ &

$v_1^R = t_1 \ \& \ v_2^R = t_2 \ \& \cdots v_k^R = t_k)$, and each

$t_1 = t_2$ by $(\exists v^{=})(v^{=} = t_1 \ \& \ v^{=} = t_2))$, we can reduce ψ to an equivalent ψ' where functions only occur in atomic formulas of the form $v = t$, where v is a variable and t is a term. (The construction following is essentially due to Moschovakis.) We now replace each formula $v = t$ by an equivalent formula using more variables (the u_1^f, v_1^f), but containing function symbols only as $v = f(v_1, \ldots, v_n)$ where the v's are <u>variables</u>. This is done by a complicated recursion.

Let $F(v{=}t) = v{=}t$ if t is a variable symbol.

Let $F(v{=}f(t_1, \ldots, t_k)) = (\exists v_1^f) \cdots (\exists v_k^f)(v = f(v_1^f, \ldots, v_k^f) \ \&$

$$(\exists u_1^f) \cdots (\exists u_k^f)(u_1^f = v_1^f \ \& \cdots \& \ u_k^f = v_k^f \ \& \ F(u_1^f = t_1) \ \& \cdots$$

$$\& \ F(u_k^f = t_k))).$$

Let $F(\neg \varphi) = \neg F(\varphi)$, and similarly for all connectives and quantifiers. Let then $\chi = F(\psi)$.

This is a fairly complicated method, so let us consider an example. Suppose $\mathfrak{M}$ has a binary relation R, a unary function f, and a binary function g, and let ψ by $R(f(f(v_0)), g(g(v_0,v_1), g(v_2,v_3))$. Then ψ' is $(\exists v_1^R)(\exists v_2^R)(R(v_1^R, v_2^R) \ \& \ v_1^R = f(f(v_0)) \ \& \ v_2^R = g(g(v_0,v_1), g(v_2,v_3))$. χ then is:

$$(\exists v_1^R)(\exists v_2^R)(R(v_1^R,v_2^R) \ \&$$

$$(\exists v_1^f)(v_1^R = f(v_1^f) \ \& \ (\exists u_1^f)(u_1^f = v_1^f \ \& \ (\exists v_1^f)(u_1^f = f(v_1^f) \ \&$$

$$(\exists u_1^f)(u_1^f = v_1^f \ \& \ u_1^f = v_0)))) \ \&$$

$$(\exists v_1^g)(\exists v_2^g)(v_2^R = g(v_1^g, v_2^g) \ \& \ (\exists u_1^g)(\exists u_2^g)(u_1^g = v_1^g \ \& \ u_2^g = v_2^g \ \&$$

$$(\exists v_1^g)(\exists v_2^g)(u_1^g = g(v_1^g, v_2^g) \;\&\; (\exists u_1^g)(\exists u_2^g)(u_1^g = v_1^g \;\&\; u_2^g = v_2^g \;\&$$

$$u_1^g = v_0 \;\&\; u_2^g = v_1)) \;\&$$

$$(\exists v_1^g)(\exists v_2^g)(u_2^g = g(v_1^g, v_2^g) \;\&\; (\exists u_1^g)(\exists u_2^g)(u_1^g = v_1^g \;\&\; u_2^g = v_2^g \;\&$$

$$u_1^g = v_2 \;\&\; u_2^g = v_3))))),$$

which is clearly equivalent to ψ.

We now associate by this procedure a χ_m to each ψ_m such that $\vdash \psi_m \leftrightarrow \chi_m$ and χ_m has all its function symbols in the form $v = f(v_1, \ldots, v_k)$. Let $\mathfrak{M}^*$ be $\mathfrak{M}$ with all k-ary functions replaced by (k+1)-ary relations, and construct χ_m^* in $\mathcal{L}(\mathfrak{M}^*)$ by the obvious replacement. Let $\theta_k = \bigwedge_{\ell<k} \chi_\ell^*$.

Then $\{\theta_k : k \in \omega\}$ is an (n,ω)-type for som n, which satisfies the defining property of (n,ω)-recursive saturation if and only if $\{\varphi_m : m \in \omega\}$ satisfies the defining property for recursive saturation. ⊣

<u>Corollary 1</u> If Th($\mathfrak{M}$) admits recursive elimination of quantifiers, $\mathfrak{M}$ is not recursively saturated, and $\mathfrak{M}$ has an inductive definition with closure ordinal $\geq \omega$, then $\kappa^{\mathfrak{M}} > \omega$.

<u>Proof</u>: Suppose the hypothesis, and $\kappa^{\mathfrak{M}} = \omega$, then, clearly, $\mathfrak{M}$ is reasonable, so $\kappa^{\mathfrak{M}} = \rho^{\mathfrak{M}}$ from the theorem, but, by the above lemma, as $\rho^{\mathfrak{M}} = \omega$, $\mathfrak{M}$ is recursively saturated, a contradiction. ⊣

<u>Corollary 2</u> If $\mathfrak{M}$ is a non-recursively saturated model of ACF, RCF or DCF, then $\kappa^{\mathfrak{M}} > \omega$.

<u>Proof</u>: All the assumptions of Corollary 1 are clear except the inductive definition with closure ordinal $\geq \omega$. For ACF_0, RCF, or DCF_0, we could take

$\varphi(x,R) \leftrightarrow x = 0$ or $(\exists y)(x = y + 1 \;\&\; Ry)$, with closure ordinal ω.

However, for ACF_p, DCF_p, we need a more complicated formula:

$$\varphi(x,y,R) \leftrightarrow \{y = 0 \;\&\; [x = 1 \text{ or } (\exists z,w)(R(z,0) \;\&\; R(w,0) \;\&\; (x = z + w \text{ or }$$

$$x = z - w \text{ or } x = z\,w \text{ or } (w \neq 0 \;\&\; x\,w = z))]\}$$

$$\text{or } y = 1 \text{ or } (\exists z,w)(R(x,z) \;\&\; R(w,0) \;\&\; y = x\,z + w)$$

It is apparent that

$(x,y) \in I_\varphi^\infty \leftrightarrow y$ is the value of a monic polynomial with algebraic coefficients at x;hence

$(x,0) \in I_\varphi^\infty \leftrightarrow x$ is algebraic. This induction clearly cannot terminate in at a finite stage; hence $|\varphi|^{\mathfrak{M}} \geq \omega$. ⊣

We can use the example of Kunen ([Moschovakis], p. 159, exercise 5) to show that the condition in Corollary 1 of the existence of an inductive definition with closure ordinal $\geq \omega$, is necessary. Let $\mathfrak{M} = \langle M;E \rangle$, where E is an equivalence relation on M with exactly one equivalence class of each finite cardinality, and no infinite equivalence classes. It can be shown that any first order positive inductive definition closes after a finite number of stages, so that $\kappa^{\mathfrak{M}} = \omega$. But, the type $\{\varphi_n(x) : n \in \omega\}$, where $\varphi_n(x)$ states that there are at least n elements equivalent to x, violates the definition of recursive saturation for $\mathfrak{M}$.

<u>Appendix</u>

<u>Lemma</u> There is a function $F : \kappa^{\mathfrak{M}} \xrightarrow{\text{onto}} L(\kappa^{\mathfrak{M}})$ such that $F(\beta) \in F(\alpha) \to (\exists\ \gamma < \alpha)\ F(\beta) = F(\gamma)$ and such that the following relations are inductive:

a) $T_n(\bar{u}, m_1, \ldots, m_n) \leftrightarrow \bar{u} \in I^{\infty}_{\varphi}$ and $F(|\bar{u}|_{\varphi}$ is a formula $\ulcorner\psi\urcorner$ of $\mathcal{L}_{\kappa}$ with only n variables, free and bound, $v_1, \ldots, v_n$ and $\mathfrak{M} \models \psi(m_1, \ldots, m_n)$

b) $F_n(\bar{u}, m_1, \ldots, m_n) \leftrightarrow \bar{u} \in I^{\infty}_{\varphi}$ and $F(|\bar{u}|_{\varphi})$ is a formula $\ulcorner\psi\urcorner$ of $\mathcal{L}_{\kappa}$ with only n variables, free and bound, $v_1, \ldots, v_n$ and $\mathfrak{M} \not\models \psi(m_1, \ldots, m_n)$

c) $T^*(\bar{u}, \bar{s}) \leftrightarrow \bar{u}, \bar{s} \in I^{\infty}_{\varphi}$, $F(|\bar{u}|_{\varphi})$ is a Δ_0 formula $\ulcorner\psi\urcorner$ of $\mathcal{L}^*_{\omega}$, $F(|\bar{s}|_{\varphi})$ is a finite sequence and $L(\kappa^{\mathfrak{M}}) \models \psi(F(|\bar{s}|_{\varphi}))$, $[L^* = \{\varepsilon\}]$.

d) $N^*(\bar{u}, \bar{s}) \leftrightarrow \bar{u}, \bar{s} \in I^{\infty}_{\varphi}$, $F(|\bar{u}|_{\varphi})$ is a Δ_0 formula $\ulcorner\psi\urcorner$ of $\mathcal{L}^*_{\omega}$, $F(|\bar{s}|_{\varphi})$ is a finite sequence and $L(\kappa^{\mathfrak{M}}) \not\models \psi(F(|s|_{\varphi}))$.

<u>Proof</u> We define $\mathcal{J}_1$ operators in a similar manner [Barwise 1], p. 63 ff, as follows

$$\mathcal{J}_1(x, y, z) = \{x, y\}$$

$$\mathcal{J}_2(x, y, z) = \{x\}$$

$$\mathcal{F}_3(x, y, z) = \langle x,y \rangle$$

$$\mathcal{F}_4(x, y, z) = \{z\}$$

$$\mathcal{F}_5(x, y, z) = \{z, \langle x,y \rangle\}$$

$$\mathcal{F}_6(x, y, z) = \langle z,x,y \rangle$$

$$\mathcal{F}_7(x, y, z) = \bigcup x$$

$$\mathcal{F}_8(x, y, z) = x-y$$

$$\mathcal{F}_9(x, y, z) = x \times y$$

$$\mathcal{F}_{10}(x, y, z) = \mathcal{D}(x)$$

$$\mathcal{F}_{11}(x, y, z) = \mathcal{R}(x)$$

$$\mathcal{F}_{12}(x, y, z) = \{\langle u,v,w \rangle : \langle w,u,v \rangle \in x \cap y \cap z\}$$

$$\mathcal{F}_{13}(x, y, z) = \{\langle u,w,v \rangle : \langle w,u,v \rangle \in x \cap y \cap z\}$$

$$\mathcal{F}_{14}(x, y, z) = \{\langle u,v \rangle : u \in x \;\&\; v \in y \;\&\; u = v\}$$

$$\mathcal{F}_{15}(x, y, z) = \{\langle u,v \rangle : u \in x \;\&\; v \in y \;\&\; u \in v\}$$

We define a well-ordering on $16 \times \varkappa \times \varkappa \times \varkappa$ by:

$$\begin{aligned} \langle k,\alpha_1,\alpha_2,\alpha_3 \rangle \ll \langle k',\alpha_1',\alpha_2',\alpha_3' \rangle \text{ iff } & \max(\alpha_1,\alpha_2,\alpha_3) < \max(\alpha_1',\alpha_2',\alpha_3') \\ & \text{or } (\max(\alpha_1,\alpha_2,\alpha_3) = \max(\alpha_1',\alpha_2',\alpha_3') \\ & \text{and } (\alpha_3 < \alpha_3' \text{ or } (\alpha_3 = \alpha_3' \text{ and} \\ & \quad (\alpha_2 < \alpha_2' \text{ or} \\ & \quad (\alpha_2 = \alpha_2' \text{ and } (\alpha_1 < \alpha_1' \text{ or} \\ & \quad (\alpha_1 = \alpha_1' \text{ and } k < k') \ldots) \end{aligned}$$

Since $\varkappa$ is a limit of admissibles, it is easily seen as in [Rubin], that the order type of $16 \times \aleph \times \aleph \times \aleph$ under $\ll$ is $\aleph$, hence we can define a 1-1 order-preserving map $J : \aleph \to 16 \times \aleph \times \aleph \times \aleph$. Now, we can finally define F by transfinite recursion:

$$F(\alpha) = \begin{cases} F''\alpha & \text{if} \quad (J(\alpha))_0 = 0 \\ \mathcal{F}_k(F(\alpha_1), F(\alpha_2), F(\alpha_3)) & \text{if} \quad J(\alpha) = \langle k, \alpha_1, \alpha_2, \alpha_3 \rangle. \end{cases}$$

It can be shown (with a large amount of work as in [Barwise 1] or [Rubin]) that F satisfies the conditions of the lemma. Now, we go on to construct inductive definitions. First, we note that the relations

$$\bar{x} <^*_\varphi \bar{y} \to \bar{x} \in I^\infty_\varphi \text{ and } (\bar{y} \notin I^\infty_\varphi \text{ or } |\bar{x}|_\varphi < |\bar{y}|_\varphi)$$

and

$$\bar{x} \leq^*_\varphi \bar{y} \to \bar{x} \in I^\infty_\varphi \text{ and } (\bar{y} \notin I^\infty_\varphi \text{ or } |\bar{x}|_\varphi \leq |\bar{y}|_\varphi)$$

are inductive, as shown in [Moschovakis] theorem 2A.2. It can easily be seen that, in a similar manner, the relations $\ll^*$ and $\underline{\ll}^*$ can be shown to be inductive. Hence, using theorem 2A.2 of [Moschovakis] again, we can define the function J inductively. We will now define relations E_1, $\bar{E}_1$, E, $\bar{E}$, I, and $\bar{I}$ such that

1) $\bar{x}\, E\bar{y} \leftrightarrow F(|\bar{x}|_\varphi) \in F(|\bar{y}|_\varphi)$

2) $\bar{x}\, \bar{E}\bar{y} \leftrightarrow F(|\bar{x}|_\varphi) \notin F(|\bar{y}|_\varphi)$

3) $\bar{x}\, I\bar{y} \leftrightarrow F(|\bar{x}|_\varphi) = F(|\bar{y}|_\varphi)$

4) $\bar{x}\, \bar{I}\bar{y} \leftrightarrow F(|\bar{x}|_\varphi) \neq F(|\bar{y}|_\varphi)$

5) $\bar{x}\, E_1\bar{y} \to \bar{x}\, E\bar{y}$

6) $\bar{x}\ \overline{Ey} \to \bar{x}\ \bar{E}_1\bar{y}$

7) $\bar{x}\ \bar{E}_1\bar{y} \to \bar{x} \overset{*}{<}_{\varphi} \bar{y}$

8) $\bar{y} \overset{*}{\leq}_{\varphi} \bar{x} \to \bar{x}\ \bar{E}_1\bar{y}$

9) $\bar{x}\ E\bar{y} \to (\exists\bar{z})(\bar{z}\ E_1\bar{y}\ \&\ \bar{z}\ I\bar{z})$

10) $(\forall\bar{z})(\bar{z}\ \bar{E}_1\bar{y}$ or $\bar{z}\ \overline{Ix}) \to \bar{x}\ \overline{Ey}$.

I will give the inductive definitions for E_1, $\bar{E}$, E, I and $\bar{I}$ here.

$\bar{x}\ E_1\bar{y} \leftrightarrow \bar{x} \overset{*}{<}_{\varphi} \bar{y}\ \&\ (\exists k)(\exists\bar{y}_1)(\exists\bar{y}_2)(\exists\bar{y}_3)(J(\bar{y}, k, \bar{y}_1, \bar{y}_2, \bar{y}_3)$ and

$[k = 0]$ or

$[k = 1\ \&\ \bar{x} = {}^{*}\bar{y}_1]$ or

$[k = 2\ \&\ (\bar{x} = {}^{*}\bar{y}_1$ or $\bar{x} = {}^{*}\bar{y}_2)]$ or

$[k = 3\ \&\ (J(\bar{x}, 1, \bar{y}_1, \bar{y}_2, \bar{y}_3)$ or $J(\bar{x}, 2, \bar{y}_1, \bar{y}_2, \bar{y}_3)]$

or $[k = 4\ \&\ \bar{x} = {}^{*}\bar{y}_3]$ or

$[k = 5\ \&\ \bar{x} = {}^{*}\bar{y}_3$ or $J(\bar{x}, 3, \bar{y}_1, \bar{y}_2, \bar{y}_3))]$ or

$[k = 6\ \&\ (J(\bar{x}, 4, \bar{y}_1, \bar{y}_2, \bar{y}_3)$ or $J(\bar{x}, 5, \bar{y}_1, \bar{y}_2, \bar{y}_3))]$

$[k = 7\ \&\ (\exists\bar{z})(\bar{x}\ E_1\ \bar{z}\ \&\ \bar{z}\ E_1\ \bar{y}_1)$

$[k = 8\ \&\ \bar{x}\ E_1\bar{y}_1\ \&\ \bar{x}\ \overline{E\bar{y}_2}]$ or

$[k = 9\ \&\ (\exists\bar{u})(\exists\bar{v})(J(\bar{x}, 3, \bar{u}, \bar{v}, 0)\ \&\ \bar{u}\ E_1\bar{y}_1\ \&\ \bar{v}E_1\bar{y}_2)]$

or $[k = 10\ \&\ (\exists\bar{v})(\exists\bar{w})(J(\bar{w}, 3, \bar{x}, \bar{v}, 0)\ \&\ \bar{w}\ E\bar{y}_1]$ or

$[k = 11\ \&\ (\exists\bar{v})(\exists\bar{w})(J(\bar{w}, 3, \bar{v}, \bar{x}, 0)\ \&\ \bar{w}\ E\bar{y}_1]$ or

$$[k = 12 \mathbin{\&} (\exists \bar{u})(\exists \bar{v})(\exists \bar{w})(\exists \bar{t})(J(\bar{x}, 6, \bar{v}, \bar{w}, \bar{u}) \mathbin{\&}$$

$$J(\bar{t}, 6, \bar{u}, \bar{v}, \bar{w}) \mathbin{\&} \bar{t}\ E\bar{y}_1 \mathbin{\&} \bar{t}\ E\bar{y}_2 \mathbin{\&} \bar{t}\ E\bar{y}_3] \text{ or}$$

$$[k = 13 \mathbin{\&} (\exists \bar{u})(\exists \bar{v})(\exists \bar{w})(\exists \bar{t})(J(\bar{x}, 6, \bar{w}, \bar{v}, \bar{u}) \mathbin{\&}$$

$$J(\bar{t}, 6, \bar{u}, \bar{v}, \bar{w}) \mathbin{\&} \bar{t}\ E\bar{y}_1 \mathbin{\&} \bar{t}\ E\bar{y}_2 \mathbin{\&} \bar{t}\ E\bar{y}_3] \text{ or}$$

$$[k = 14 \mathbin{\&} (\exists \bar{u})(\exists \bar{v})(\bar{u}\ E\bar{y}_1 \mathbin{\&} \bar{v}\ E\bar{y}_2 \mathbin{\&} \bar{u}I\bar{v} \mathbin{\&} J(\bar{x},3,\bar{u},\bar{v},0))]$$

or

$$[k = 15 \mathbin{\&} (\exists \bar{u})(\exists \bar{v})(\bar{u}\ E\bar{y}_1 \mathbin{\&} \bar{v}\ E\bar{y}_2 \mathbin{\&} \bar{u}\ E\bar{v} \mathbin{\&}$$

$$J(\bar{x}, 3, \bar{u}, \bar{v}, 0)])$$

$$\bar{x}\ E\bar{y} \leftrightarrow (\exists z)(zE_1\bar{y} \mathbin{\&} \bar{x}\ Iz)$$

$$\bar{x}\ \overline{E\bar{y}} \leftrightarrow (\forall z)(\overline{zE_1\bar{y}} \text{ or } \bar{x}\ \overline{Iz})$$

$$\bar{x}\ I\bar{y} \leftrightarrow (\forall z)(\bar{z}\ \bar{E}_1\ \bar{x} \text{ or } \bar{z}\ E\bar{y}) \mathbin{\&} (\forall z)(\bar{z}\ \bar{E}_1\bar{y} \text{ or } \bar{z}\ E\bar{x})$$

$$\bar{x}\ \overline{I\bar{y}} \leftrightarrow (\exists z)(\bar{z}\ E_1\ \bar{x} \mathbin{\&} \bar{z}\ \overline{E\bar{y}}) \text{ or } (\exists z)(\bar{z}\ E_1\bar{y} \mathbin{\&} \bar{z}\ \overline{E\bar{x}}).$$

Using these definitions, it is easy to see that for any Σ_1 formula $\psi(x_1, \ldots, x_n)$, the relation $L(\aleph^{\mathfrak{M}}) \models \psi(|\bar{y}_1|_\varphi, |\bar{y}_2|_\varphi, \ldots |\bar{y}_n|_\varphi)$ is inductive. This completes the proof as the desired relations are Σ_1 as shown in the proposition V.1.6 of [Barwise 1]. ⊣

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