# On the link Floer homology of $L$-space links

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# Abstract

We will prove that, for a 2 or 3 component $L$-space link, $HFL^{-}$ is completely determined by the multi-variable Alexander polynomial of all the sub-links of $L$, as well as the pairwise linking numbers of all the components of $L$. We will also give some restrictions on the multi-variable Alexander polynomial of an $L$-space link. Finally, we use the methods in this paper to prove a conjecture of Yajing Liu classifying all 2-bridge $L$-space links.

# Contents

# Chapter 1

# Introduction

In [18] and [23], knot and link Floer homology were defined as a part of Ozsváth and Szabó's Heegaard Floer theory (introduced in [20]). These give rise to graded homology groups which are invariants of isotopy classes of knots and links embedded in $S^3$. Carefully examining these groups has yielded a wealth of topological insights (see [14], [15], [16], [25], [26] and [30]). The Euler characteristic of link (knot) Floer homology is the multi-variate (single variable) Alexander polynomial[a].

Throughout this paper, we will work over the field $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$, and $L = L_1 \sqcup L_2 \sqcup \ldots \sqcup L_l$ will always be an $l$ component link inside $S^3$ unless otherwise specified. We will focus on links all of whose large positive surgeries yield $L$-spaces.

$L$-spaces are rational homology spheres whose Heegaard Floer homology is the simplest possible. More specifically, recall that for any rational homology 3-sphere $Y$ we must have $\dim(\widehat{HF}(Y)) \geq |H_1(Y)|$, and so we define an $L$-space as:

**Definition 1.1** $Y$ a $\mathbb{Q}HS^3$ is an $L$-space if $\dim(\widehat{HF}(Y)) = |H_1(Y)|$.

Lens spaces are the simplest examples of $L$-spaces. Further examples include any connected sums of 3-manifolds with elliptic geometry [21], as well as double branched covers of quasi-alternating links [22]. It was shown in Theorem 1.4 of [20] that such manifolds do not admit co-orientable $C^2$-taut foliations.

We will define an $L$-space link as follows:

**Definition 1.2** $L \subset S^3$ is an $L$-space link if the 3-manifolds $S^3_{n_1,\ldots,n_l}(L)$ obtained by surgery on $L$ are all $L$-spaces when all of the $n_i$ are sufficiently large.[b].

$L$-space links were first studied in [6], where it was shown that any link arising as the embedded link of a complex plane curve singularity (i.e. algebraic link) is an $L$-space link (note that this

[a] This is "almost true", we will make it precise in Definition 1.4.

[b] Note that this definition does not depend on the orientation of the components of $L$.

includes all torus links). The general study of properties and examples of $L$-space links was initiated in [10] see also [5]. $L$-space knots were first examined in [21]. In that paper it was shown that for an $L$-space knot, the knot Floer homology is completely determined by its Euler Characteristic (i.e. the Alexander polynomial). In this paper, we give a generalization of this statement to 2 and 3 component $L$-space links inside $S^3$. First, we recall some standard facts and notation.

**Definition 1.3** Let $\mathbb{H}(L)_i$ denote the affine lattice over $\mathbb{Z}$ given by $\mathrm{lk}(L_i, L\backslash L_i)/2 + \mathbb{Z}$. We define:

$$\mathbb{H}(L) := \bigoplus_{i=1}^{l} \mathbb{H}(L)_i.$$

We can think of every element of $\mathbb{H}(L)$ as an element of the set of relative Spin$^c$ structures of $L \subset S^3$ via the identification $\mathbb{H}(L) \to \underline{\mathrm{Spin}}^c(S^3, L)$ given in section 8.1 of [23]. Note that $\mathbb{H}(L)$ is an affine lattice over $H_1(S^3 - L) \cong \mathbb{Z}^l$.

Both $HFL^-$ and $\widehat{HFL}$ for a link $L$ inside $S^3$ split into direct summands indexed by pairs $(d, \mathbf{s})$, where $d \in \mathbb{Z}$ (the homological grading) and $\mathbf{s} \in \mathbb{H}(L)$. We will write these summands as $HFL^-_d(L, \mathbf{s})$ and $\widehat{HFL}_d(L, \mathbf{s})$.

Now, if $\mathbf{s} = (s_1, s_2, \ldots, s_l) \in \mathbb{H}(L)$, we denote by $u^{\mathbf{s}}$ the monomial $u_1^{s_1} \ldots u_l^{s_l}$.

**Definition 1.4** In this paper, we define the symmetric multi-variable Alexander polynomial $\Delta_L(u_1, u_2, \ldots, u_l)$ for $L$ so that the following equality[c] holds:

$$\sum_{\mathbf{s} \in \mathbb{H}(L)} \chi(\widehat{HFL}_*(L, \mathbf{s})) u^{\mathbf{s}} = \prod_{i=1}^{l} \left( u_i^{1/2} - u_i^{-1/2} \right) \Delta_L(u_1, u_2, \ldots, u_l).$$

**Theorem 1.5** *Let $L \subset S^3$ be a 2 or 3 component $L$-space link and let $\mathbf{s} \in \mathbb{H}(L)$. Then $HFL^-(L, \mathbf{s})$ is completely determined by the symmetric multi-variable Alexander polynomials $\pm\Delta_M$ for every sub-link $M \subset L$, as well as the pairwise linking numbers of components of $L$.*

In [21], it was shown that being an $L$-space knot forces strong restrictions on the Alexander polynomial, and we will generalize this to links. Our restrictions will depend on the Alexander polynomial of the link $L$, as well as the Alexander polynomial of all its sub-links after a shift depending on various linking numbers.

**Definition 1.6** Given a proper subset $S = \{i_1, i_2, \ldots, i_k\} \subsetneq \{1, \ldots, l\}$, we let $\{j_1, j_2, \ldots, j_{l-k}\} = \{1, \ldots, l\} \backslash S$ where $j_a < j_b$ when $a < b$. Let $L_S \subset L$ be the sub-link $L_{i_1} \sqcup L_{i_2} \sqcup \ldots \sqcup L_{i_k}$. The

[c] In proposition 9.1 of [23], the above equality was only shown to hold up to sign. So our sign convention for $\Delta_L$ here may not be standard, but it will make the statement of some of our Theorems easier. For our main Theorem, we only need to know $\Delta_L$ up to sign.

polynomial $P^L_{L_S}$ is defined as follows:

When $S = \emptyset$ we have,

$$P^L_\emptyset = \left(\prod_{i=1}^{l} u_i^{1/2}\right) \Delta_L(u_1, \dots, u_l);$$

When $l - k > 1$ we have,

$$P^L_{L_S}(u_{j_1}, u_{j_2}, \dots, u_{j_{l-k}}) = \left(\prod_{p=1}^{l-k} u_{j_i}^{1/2+\mathrm{lk}(L_{j_i}, L_S)/2}\right) \Delta_{L \setminus L_S}(u_{j_1}, \dots, u_{j_{(l-k)}});$$

And finally when $l - k = 1$ we have,

$$P^L_{L_S}(u_{j_1}) = u_{j_1}^{\frac{\mathrm{lk}\left(L_{j_1}, L_S\right)}{2}} \left(\sum_{i \geq 0} u_{j_1}^{-i}\right) \Delta_{L \setminus L_S}(u_{j_1}).$$

Now, fix some $\mathbf{s} = (s_1, s_2, \dots, s_l) \in \mathbb{H}(L)$ and $r \in \{1, \dots, l\}$ so that $r \notin S$. Then, define

$$R_{\substack{\mathbf{s}' \geq \mathbf{s} \\ s'_r = s_r}}(P^L_{L_S})$$

to be the sum of all the coefficients of monomials $u_{j_1}^{s'_1} \dots u_{j_{l-k}}^{s'_{j_{l-k}}}$ of $P^L_{L_S}$ that satisfy $s'_r = s_r$ and $s'_{j_p} \geq s_{j_p}$ for $j_p \neq r$.

**Example 1.7** Consider the 2-bridge link $L = b(20, -3)$ (see Section 5 for definitions and notation). Then;

$$\begin{aligned}
\Delta_L(u_1, u_2) =& u_1^{1/2}u_2^{3/2} + u_1^{3/2}u_2^{1/2} + u_1^{1/2}u_2^{-1/2} + u_1^{-1/2}u_2^{1/2} + u_1^{-3/2}u_2^{-1/2} + u_1^{-1/2}u_2^{-3/2} - u_1^{3/2}u_2^{3/2} \\
&- u_1^{1/2}u_2^{1/2} - u_1^{-1/2}u_2^{-1/2} - u_1^{-3/2}u_2^{-3/2}. \\
P^L_\emptyset(u_1, u_2) =& u_1u_2^2 + u_1^2u_2 + u_1 + u_2 + \frac{1}{u_1} + \frac{1}{u_2} - u_1^2u_2^2 - u_1u_2 - 1 - \frac{1}{u_1u_2}.
\end{aligned}$$

$L = L_1 \sqcup L_2$ is a 2 component link with both components unknots. The linking number of the 2 components is 2 so;

$$P^L_{L_1}(u_2) = u_2 \left(\sum_{i \geq 0} u_2^{-i}\right) \text{ and } P^L_{L_2}(u_1) = u_1 \left(\sum_{i \geq 0} u_1^{-i}\right).$$

**Theorem 1.8** *If $L$ is an $L$-space link, then for any $\mathbf{s} \in \mathbb{H}(L)$ and $r \in \{1, 2, \dots, l\}$:*

$$\sum_{\substack{S \subset \{1,\dots,l\} \\ r \notin S}} (-1)^{l-1-|S|} R_{\substack{\mathbf{s}' \geq \mathbf{s} \\ s'_r = s_r}} (P^L_{L_S}) = 0 \text{ or } 1.$$

**Remark 1.9** When $l = 1$, this says that the coefficients of $P^L_\emptyset$ are all 1 or 0, which follows from the work in [21].

Given any 2 variable polynomial $F(u_1, u_2)$, we define $F|_{(i,j)}$, where $i = 1$ or 2, to be the polynomial obtained from $F$ by discarding all monomials where the exponent of $u_i$ is not equal to $j$. Then the above Theorem, when restricted to the $l = 2$ case, reads as follows:

**Corollary 1.10** *Suppose that $L = L_1 \sqcup L_2$ is an $L$-space link. Then the nonzero coefficients of $P^L_\emptyset$ are all $\pm 1$. The nonzero coefficients of $P^L_\emptyset|_{(r,s'_r)}$ for $r = 1$ or 2 and any $s'_r \in \mathbb{H}(L)_r$, alternate in sign. The first nonzero coefficient of $P^L_\emptyset|_{(r,s'_r)}$ is $-1$ if the coefficient of $u_r^{s'_r}$ in $P^L_{L_{3-r}}$ is 0; and the first nonzero coefficient of $P^L_\emptyset|_{(r,s'_r)}$ is 1 if the coefficient of $u_r^{s'_r}$ in $P^L_{L_{3-r}}$ is 1.*

**Proof:** As in Theorem 1.8, fix $\mathbf{s}' = (s'_1, s'_2)$. Suppose without loss of generality that $r = 1$. We denote by $a_{s_1,s_2}$ the coefficient of $u_1^{s_1} u_2^{s_2}$ in $P^L_\emptyset(u_1, u_2)$, and $a_{s_1}$ the coefficient of $u_1^{s_1}$ in $P^L_{L_2}(u_1)$. Then according to Theorem 1.8:

$$a_{s'_1} - \sum_{s_2 \geq s'_2} a_{s'_1,s_2} = 0 \text{ or } 1. \tag{1.1}$$

Similarly;

$$a_{s'_1} - \sum_{s_2 \geq s'_2+1} a_{s'_1,s_2} = 0 \text{ or } 1. \tag{1.2}$$

Subtracting 1.1 from 1.2 gives $a_{s'_1,s'_2} = -1, 0$ or 1. We have thus shown that all the coefficients of $P^L_\emptyset$ or $-1, 0$ or $-1$. We know that $a_{s'_1}$ must be either 1 or 0 (see Remark 1.9). Combining this with equation 1.1 gives that $\sum_{s_2 \geq s'_2} a_{s'_1,s_2} = 0$ or 1 if $a_{s'_1} = 1$, and $\sum_{s_2 \geq s'_2} a_{s'_1,s_2} = 0$ or $-1$ if $a_{s'_1} = 0$. The rest of the corollary now immediately follows. □

Part of the above corollary was already shown directly in Theorem 1.15 of [10]. Additionally in [10], it was shown that when $q$ and $k$ are odd positive integers $b(qk-1, -k)$ is an $L$-space link. This was conjectured to be a complete list of 2-bridge $L$-space links, which is correct.

**Theorem 1.11** *If $L$ is a 2-bridge $L$-space link, then, after possibly reversing the orientation of one of the components, $L$ is equivalent to $b(qk-1, -k)$ for some positive odd integers $q$ and $k$.*

The organization of this paper is as follows. Section 2 consists of some homological algebra needed to compute $HFL^-(L)$ from its Euler characteristic when $L$ is a 2 or 3 component $L$-space link. Section 3 generalizes the arguments in [23] to work on links. In Section 4 Theorem 1.5 is proved, as well the the restrictions on the Alexander polynomials of $L$-spaces. In Section 5 we prove the classification of 2 bridge $L$-space links.

# Chapter 2

# Homological Preliminaries

**Definition 2.1** Let $E_n = \{0,1,2\}^n \subset \mathbb{R}^n$ where $n \geq 1$. We will denote $(0,0,\ldots,0)$, $(1,1,\ldots 1)$ and $(2,2,\ldots,2)$ by **0**,**1** and **2** respectively. For any $\varepsilon \in E_n$, we denote by $\varepsilon_j$ the $j$th coordinate of $\varepsilon$ and by $e_j$ the $j$th elementary coordinate vector. We define an **n-dimensional short exact cube of chain complexes**, $\boldsymbol{C}$ (or **short exact cube** for short), as follows:

**1** For every $\varepsilon \in E_n$ there is a chain complex $\boldsymbol{C}_\varepsilon$ over $\mathbb{F}$.

**2** Suppose that $\varepsilon', \varepsilon$ and $\varepsilon''$ are in $E_n$ and only differ in the $j$th coordinate with $\varepsilon'_j = 0, \varepsilon_j = 1$ and $\varepsilon''_j = 2$. Then there is a short exact sequence

$$0 \longrightarrow \boldsymbol{C}_{\varepsilon'} \xrightarrow{i_{\varepsilon'\varepsilon}} \boldsymbol{C}_\varepsilon \xrightarrow{j_{\varepsilon\varepsilon''}} \boldsymbol{C}_{\varepsilon''} \longrightarrow 0.$$

**3** The diagram made by all of the complexes $\boldsymbol{C}_\varepsilon$ and maps $i_{\varepsilon'\varepsilon}, j_{\varepsilon\varepsilon''}$ is commutative.

We will denote $\boldsymbol{C}_{(2,2,\ldots,2)}$ as $\overline{\boldsymbol{C}}$ for short. We define the **cube of inclusions**, $\boldsymbol{C}^{\boldsymbol{I}}$, to be the sub-diagram consisting of all the chain complexes $\boldsymbol{C}_\varepsilon$ with $\varepsilon \in \{0,1\}^n$ and the corresponding inclusion maps. We call a short exact cube **basic** if the following additional properties hold:

**4** For $\varepsilon \in \{0,1\}^n$, $H_*(\boldsymbol{C}_\varepsilon) \cong \mathbb{F}[U]$ where multiplication by $U$ drops homological grading by 2. We do not specify what the top grading for $\mathbb{F}[U]$ is, but we do require that it is even.

**5** All of the maps $(i_{\varepsilon'\varepsilon})_*$, induced by homology in the cube of inclusions are either isomorphisms in all degrees, or $(i_{\varepsilon'\varepsilon})_*$ is injective in all degrees and the top degree supported in $H_*(C_\varepsilon)$ is 2 higher than the top degree supported in $H_*(C_{\varepsilon'})$. In other words $UH_*(\boldsymbol{C}_\varepsilon) \cong H_*(\boldsymbol{C}_{\varepsilon'})$

When the top grading for $\mathbb{F}[U]$ is $d$, we will write it as $\mathbb{F}_{(d)}[U]$. Similarly, $\mathbb{F}_{(d)}$ will be used to denote $\mathbb{F}$ supported in degree $d$.

Given an $n$ dimensional basic short exact cube $\boldsymbol{C}$, if we restrict to the commutative diagram coming from the subset of $E_n$ with $j$th coordinate $i$ where $i = 0, 1$ or $2$, this can be thought of as an $n-1$ dimensional short exact cube of chain complexes which we will denote by ${}^{j}_{i}\boldsymbol{C}$. For any $j \in \{1, 2, \ldots, n\}$, $\overline{{}^{j}_{2}\boldsymbol{C}}$ is the same as $\overline{\boldsymbol{C}}$; and ${}^{j}_{0}\boldsymbol{C}$ and ${}^{j}_{1}\boldsymbol{C}$ are basic.

**Lemma 2.2** *Suppose* $\boldsymbol{C}$ *is a basic short exact cube of chain complexes. Also let* $\varepsilon \in E_n$ *have some coordinate equal to* 2*. Then,* $H_*(\boldsymbol{C}_\varepsilon)$ *is finite dimensional.*

**Proof:** In the $n = 1$ case, $H_*(\overline{\boldsymbol{C}})$ is either $\mathbb{F}$ or 0 by property 5 of basic short exact cubes. Thus, for any $n$-dimensional basic short exact cube $\boldsymbol{C}$, the homologies of the complexes in ${}^{j_1}_{2}\boldsymbol{C}^{\boldsymbol{I}}$ are only either $\mathbb{F}$ or 0 for any $j_1$. From here we can conclude that the homologies of the complexes in ${}^{j_2}_{2}{}^{j_1}_{2}\boldsymbol{C}^{\boldsymbol{I}}$ are finite and continuing with this argument proves the claim. □

**Definition 2.3** If $\boldsymbol{C}$ is a basic short exact cube, then we define the **hypercube graph of** $\boldsymbol{C}$, $HC(\boldsymbol{C})$, as a directed graph with labeled edges as follows:

- The vertices correspond to the elements of the set $\{0, 1\}^n$.
- There is a directed edge from $\varepsilon'$ to $\varepsilon$ if the two agree in all coordinates except the $j$th for some $1 \le j \le n$ and $\varepsilon'_j = 0, \varepsilon_j = 1$. We will denote the edge from $\varepsilon'$ to $\varepsilon$ by $e_{\varepsilon'\varepsilon}$.
- An edge $e_{\varepsilon'\varepsilon}$ is labeled with 0 if $(i_{\varepsilon'\varepsilon})_*$ is an isomorphism in all degrees and 1 otherwise. We will denote the label of an edge $e$ by $l_{\boldsymbol{C}}(e_{\varepsilon'\varepsilon})$ or $l(e_{\varepsilon'\varepsilon})$ when $\boldsymbol{C}$ is clear from context.

We will denote by $\widetilde{HC}(\boldsymbol{C})$ the subgraph of $HC(\boldsymbol{C})$ induced by all the vertices except the origin and we will refer to $\widetilde{HC}(\boldsymbol{C})$ as the **hypercube subgraph of** $\boldsymbol{C}$.

**Remark 2.4** Note that, since $\boldsymbol{C}^{\boldsymbol{I}}$ is a commutative diagram, for any two directed paths between vertices the sum of the edge labels must be the same in $HC(\boldsymbol{C})$. If we are given a directed hypercube graph $G$ (directed as in definition 2.3) with edge labels 0 and 1 that satisfies the property that the sum of the edge labels along any two directed paths between vertices is the same, we can easily construct a basic short exact cube with $G$ as its hypercube graph. Also note that $\chi(H_*(\overline{\boldsymbol{C}}))$ is completely determined by $HC(\boldsymbol{C})$.

**Lemma 2.5** *Suppose that* $\boldsymbol{C}$ *is a basic short exact cube. There are only two mutually exclusive possibilities:*

**1** *If* $\boldsymbol{C'}$ *is another basic short exact cube then* $\widetilde{HC}(\boldsymbol{C'}) = \widetilde{HC}(\boldsymbol{C}) \Rightarrow HC(\boldsymbol{C'}) = HC(\boldsymbol{C})$*.*

**2** *Either all of the edges in* $HC(\boldsymbol{C}) \backslash \widetilde{HC}(\boldsymbol{C})$ *(i.e. all the edges emerging from* $\boldsymbol{0}$ *)are labeled with* 0 *or they are all labeled with* 1*.*

**Proof:** Note first that, if possibility 1 is satisfied, possibility 2 cannot also be satisfied since if all the edges emerging from $\mathbf{0}$ are labeled with $i$ (where $i$ is 0 or 1) then we can get another valid labeling by simply replacing all the $i$'s emerging from the origin with $(1-i)$s (see Remark 2.4).

Suppose that $\boldsymbol{C}$ and $\boldsymbol{C}'$ satisfy $\widetilde{HC}(\boldsymbol{C'}) = \widetilde{HC}(\boldsymbol{C})$, but $HC(\boldsymbol{C}') \neq HC(\boldsymbol{C})$. Then there must be some $\varepsilon'$ connected to the origin such that the edge from $\mathbf{0}$ to $\varepsilon'$ is labeled differently in $HC(\boldsymbol{C}')$ and $HC(\boldsymbol{C})$. Assume without loss of generality that $l_{\boldsymbol{C}}(e_{\mathbf{0}\varepsilon'}) = 1$ and $l_{\boldsymbol{C}'}(e_{\mathbf{0}\varepsilon'}) = 0$. Consider any other vertex $\varepsilon$ connected to the origin and consider $l_{\boldsymbol{C}}(e_{\mathbf{0}\varepsilon})$. We claim that $l_{\boldsymbol{C}}(e_{\mathbf{0}\varepsilon})$ must be 1. To see this, consider the square subgraph induced by the vertices $\mathbf{0}, \varepsilon, \varepsilon'$ and $\delta = \varepsilon + \varepsilon'$. If $l_{\boldsymbol{C}}(e_{\mathbf{0}\varepsilon}) = 0$ then since $l_{\boldsymbol{C}}(e_{\mathbf{0}\varepsilon'}) = 1$ this forces $l_{\boldsymbol{C}}(e_{\varepsilon\delta}) = 1 = l_{\boldsymbol{C}'}(e_{\varepsilon\delta})$ and $l_{\boldsymbol{C}}(e_{\varepsilon'\delta}) = 0 = l_{\boldsymbol{C}'}(e_{\varepsilon'\delta})$ (see the Remark 2.4). However this is impossible because we know $l_{\boldsymbol{C}'}(e_{\mathbf{0}\varepsilon'}) = 0$ and if $0 = l_{\boldsymbol{C}'}(e_{\varepsilon'\delta}), 1 = l_{\boldsymbol{C}'}(e_{\varepsilon\delta})$ there is no label that works for $e_{\mathbf{0}\varepsilon'}$. So we get that in $\boldsymbol{C}$ every edge emerging from the origin must be labeled 1 if one of them is. By the same argument, we can show that every edge emerging from the origin must be labeled 0 if one of them is. This proves that the two cases stated in the Lemma are exhaustive and mutually exclusive.

□

**Lemma 2.6** *Suppose that $\boldsymbol{A}$ and $\boldsymbol{B}$ are two basic short exact cubes satisfying $\widetilde{HC}(\boldsymbol{A}) = \widetilde{HC}(\boldsymbol{B})$, every edge in $HC(\boldsymbol{A}) \backslash \widetilde{HC}(\boldsymbol{A})$ is labeled with 0, and every edge in $HC(\boldsymbol{B}) \backslash \widetilde{HC}(\boldsymbol{B})$ is labeled with 1. Then,*

$$\chi(H_*(\overline{\boldsymbol{A}})) = \chi(H_*(\overline{\boldsymbol{B}})) + (-1)^n.$$

**Proof:** We will prove this inductively. For the $n = 1$ case using the fact that both $H_*(\boldsymbol{A}_0)$ and $H_*(\boldsymbol{B}_0)$ have even top grading we directly compute that $\chi(H_*(\overline{\boldsymbol{A}})) = 0$ and $\chi(H_*(\overline{\boldsymbol{B}})) = 1$. Now we can proceed with the induction. Note that we have:

$$\chi(\overline{\boldsymbol{A}}) = \chi(\overline{{}^1_1\boldsymbol{A}}) - \chi(\overline{{}^1_0\boldsymbol{A}}) \text{ and } \chi(\overline{\boldsymbol{B}}) = \chi(\overline{{}^1_1\boldsymbol{B}}) - \chi(\overline{{}^1_0\boldsymbol{B}}).$$

$\chi(\overline{{}^1_1\boldsymbol{A}}) = \chi(\overline{{}^1_1\boldsymbol{B}})$ since they are both completely determined by the hypercube subgraph $\widetilde{HC}$ and also $\chi(\overline{{}^1_0\boldsymbol{A}}) = \chi(\overline{{}^1_0\boldsymbol{B}}) + (-1)^{n-1}$ by induction. □

**Lemma 2.7** *Suppose that $\boldsymbol{C}$ is a 1, 2 or 3-dimensional basic cube of chain complexes. Then we can compute $H_*(\overline{\boldsymbol{C}})$ as a graded vector space if we know $H_*(\boldsymbol{C}_\varepsilon)$ for any $\boldsymbol{C}_\varepsilon$ in the cube of inclusions $\boldsymbol{C}^I$, as well as all the maps $(i_{\varepsilon'\varepsilon})_*$ induced by homology in the cube of inclusions $\boldsymbol{C}^I$.*

**Proof:** When $n = 1$, we have a short exact sequence:

$$0 \longrightarrow \boldsymbol{C}_0 \xrightarrow{i_{01}} \boldsymbol{C}_1 \xrightarrow{j_{12}} \overline{\boldsymbol{C}} \longrightarrow 0.$$

Thus if $(i_{01})_*$ is an isomorphism, we get that $H_*(\overline{\boldsymbol{C}}) \cong 0$; and if not, then $H_*(\overline{\boldsymbol{C}}) \cong \mathbb{F}$. For the $n = 2$ case we show all possibilities for $HC$ in Figure 2.1.

(1) (2) (3) (4) (5) (6)

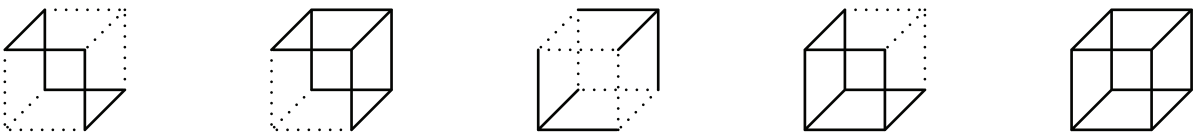
Figure 2.1: All possible hypercube graphs in the $n = 2$ case ((0,0) is on the bottom left). The dotted lines denote edges labeled with 0 and the solid lines are edges labeled with 1

If we assume that $H_*(\boldsymbol{C_0}) \cong \mathbb{F}_{(0)}[U]$, then $H_*(\overline{\boldsymbol{C}})$ is $0, 0, 0, \mathbb{F}_{(3)}, \mathbb{F}_{(2)}, \mathbb{F}_{(4)} \oplus \mathbb{F}_{(3)}$ for the 6 possibilities shown in Figure 2.1, respectively.

In the $n = 3$ case we only need to consider those $HC$ which do not have a facet equal to $(1), (2)$ or $(3)$ in Figure 2.1 , as otherwise we would have for some $j = 1, 2$ or 3, $H_*(\overline{{}^{j}_{0}\boldsymbol{C}}) = 0$ or $H_*(\overline{{}^{j}_{1}\boldsymbol{C}}) = 0$. This would allow us to compute $H_*(\overline{\boldsymbol{C}})$ from the long exact sequence for the short exact sequence:

$$0 \longrightarrow \overline{{}^{j}_{0}\boldsymbol{C}} \longrightarrow \overline{{}^{j}_{1}\boldsymbol{C}} \longrightarrow \overline{\boldsymbol{C}} \longrightarrow 0.$$

We show all the possibilities for $HC$ when $n = 3$ and none of the facets are as $(1), (2)$ or $(3)$ of Figure 2.1 in Figure 2.2.



(1) (2) (3) (4) (5)

Figure 2.2: Some hypercube graphs in the $n = 3$ case. Once again the $(0, 0, 0)$ is on the bottom left and the dotted lines denote edges labeled with 0 and the solid lines are edges labeled with 1

If we assume that $H_*(\boldsymbol{C_0}) \cong \mathbb{F}_{(0)}[U]$, then $H_*(\overline{\boldsymbol{C}})$ is $\mathbb{F}^2_{(3)}, \mathbb{F}_{(4)} \oplus \mathbb{F}^2_{(3)}, \mathbb{F}^2_{(4)}, \mathbb{F}^2_{(5)} \oplus \mathbb{F}_{(4)}, \mathbb{F}_{(6)} \oplus \mathbb{F}^2_{(5)} \oplus \mathbb{F}_{(4)}$ for the five cases shown, respectively. $\square$

**Remark 2.8** The above Lemma does not hold when $n \geq 4$. Consider the basic 4 dimensional short exact cube $\boldsymbol{C}$ where every edge of $HC(\boldsymbol{C})$ is labeled with 1 and $H_*(\boldsymbol{C_0}) \cong \mathbb{F}_{(0)}[U]$. For any $j_1, j_2 \in \{1, 2, 3, 4\}$ we have $H_*({}^{j_1 j_2}_{2\ 2}\boldsymbol{C}_{00}) \cong \mathbb{F}_4 \oplus \mathbb{F}_3, H_*({}^{j_1 j_2}_{2\ 2}\boldsymbol{C}_{10}) \cong H_*({}^{j_1 j_2}_{2\ 2}\boldsymbol{C}_{01}) \cong \mathbb{F}_6 \oplus \mathbb{F}_5$ and $H_*({}^{j_1 j_2}_{2\ 2}\boldsymbol{C}_{11}) \cong \mathbb{F}_8 \oplus \mathbb{F}_7$. It follows that all maps on homology in the cube of inclusions for ${}^{j_1 j_2}_{2\ 2}\boldsymbol{C}$ are trivial. So for all $j$ the map from $H_*(\overline{{}^{j}_{0}\boldsymbol{C}}) \cong \mathbb{F}_{(6)} \oplus \mathbb{F}^2_{(5)} \oplus \mathbb{F}_{(4)}$ to $H_*(\overline{{}^{j}_{1}\boldsymbol{C}}) \cong \mathbb{F}_{(8)} \oplus \mathbb{F}^2_{(7)} \oplus \mathbb{F}_{(6)}$ may be of rank 0 or 1 without violating commutativity. Thus $H_*(\overline{C})$ may be either $\mathbb{F}_{(8)} \oplus \mathbb{F}^3_{(7)} \oplus \mathbb{F}^3_{(6)} \oplus \mathbb{F}_{(5)}$ or $\mathbb{F}_{(8)} \oplus \mathbb{F}^2_{(7)} \oplus \mathbb{F}^2_{(6)} \oplus \mathbb{F}_{(5)}$. See also Theorem 1.5.1.d in [7].

# Chapter 3

# The Chain Complex

For a complete overview of Heegaard Floer homology, admissible multi-pointed Heegaard diagrams for knots and links, the definition of $L$-spaces and their relationship with the Heegaard Floer complex, see [20],[19], [18], [23], [21], [24], [25] and [12]. Suppose that $L \subset S^3$ is an oriented $l$ component link. In this paper, we define a multi-pointed Heegaard diagram $\mathcal{H} = (\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ for $L$ with the following properties[a]:

- $\Sigma_g$ is a closed oriented surface of genus $g$.
- $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{g+m-1})$ is a collection of disjoint simple closed curves which span a $g$-dimensional lattice of $H_1(\Sigma, \mathbb{Z})$, and the same goes for $\beta = (\beta_1, \ldots, \beta_{g+m-1})$. Thus, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ specify handlebodies $U_\alpha$ and $U_\beta$. We require that $U_\alpha \cup_\Sigma U_\beta = S^3$.
- $\mathbf{z} = (z_1, z_2, \ldots, z_l)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_m)$ are both collections of basepoints in $\Sigma$ where $l \leq m$. We will call $w_{l+1}, w_{l+2}, \ldots w_m$ free basepoints.
- If $\{A_i\}_{i=1}^m$ and $\{B_i\}_{i=1}^m$ are the connected components of $\Sigma \backslash \left( \bigcup_{i=1}^{g+m-1} \alpha_i \right)$ and $\Sigma \backslash \left( \bigcup_{i=1}^{g+m-1} \beta_i \right)$, respectively then $w_i \in A_i \cap B_i$ for any $1 \leq i \leq m$; and there is some permutation $\sigma$ of $\{1, \ldots, l\}$ such that $z_i \in A_i \cap B_{\sigma(i)}$ when $1 \leq i \leq l$.
- The diagram as defined so far specifies the link $L \subset S^3$.
- We require that all of the $\alpha$ and $\beta$ curves intersect transversely and that every non-trivial periodic domain have both positive and negative local multiplicities (see section 3.4 of [23]).

Also recall that for every intersection point $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ there is a Maslov grading $M(\mathbf{x})$ and an Alexander multigrading $A_i(\mathbf{x}) \in \mathbb{H}(L)_i$.

**Definition 3.1** Suppose we have a multi-pointed Heegaard diagram $\mathcal{H} = (\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ for the pair $L$ as above. We define the complex $CF^-(\mathcal{H})$ to be free over $\mathbb{F}$ with generators $[\mathbf{x}, i_1, j_1, \ldots, i_l$

[a] This is identical to the definition given in [23] except we want to allow "spare" basepoints that will arise in the proof of the main theorem.

$, j_l, i_{l+1}, \ldots, i_m]$ where $i_k \in \mathbb{Z}_{\leq 0}$, and $j_k \in \mathbb{Q}$ satisfying $j_k - i_k = A_k(\mathbf{x})$. The differential is, as usual, given by counting holomorphic disks:

$$\partial[\mathbf{x}, i_1, j_1, \ldots, i_l, j_l, i_{l+1}, \ldots, i_m] =$$

$$\sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta} \sum_{\substack{\phi\in\pi_2(\mathbf{x},\mathbf{y})\\ \mu(\phi)=1}} [\mathbf{y}, i_1 - n_{w_1}(\phi), j_1 - n_{z_1}(\phi), \ldots, i_l - n_{w_l}(\phi), j_l - n_{z_l}(\phi), i_{l+1} - n_{w_{l+1}}(\phi), \ldots, i_m - n_{w_m}(\phi)].$$

In the notation of [11], this differential and Heegaard diagram correspond to the maximally colored case.

The complex $CF^-$ is also an $\mathbb{F}[U_1, U_2, \ldots, U_m]$-module. The action of $U_k$ for $1 \leq k \leq l$ is given by:

$$U_k[\mathbf{x}, i_1, j_1, \ldots, i_k, j_k, \ldots i_l, j_l, i_{l+1} \ldots, i_m] = [\mathbf{x}, i_1, j_1, \ldots, i_k - 1, j_k - 1, \ldots i_l, j_l, i_{l+1}, \ldots, i_m];$$

and for $l < k < m$ is given by;

$$U_k[\mathbf{x}, i_1, j_1, \ldots, i_l, j_l, i_{l+1}, \ldots, i_k, \ldots, i_m] = [\mathbf{x}, i_1, j_1, \ldots, i_l, j_l, i_{l+1} \ldots, i_k - 1, \ldots, i_m].$$

We define the Maslov grading of $[\mathbf{x}, i_1, j_1, \ldots, i_k, j_k, \ldots i_l, j_l, i_{l+1} \ldots, i_m]$ by setting it equal to $M(\mathbf{x})$ when all the $i_k$ are 0 and letting the action of each $U_i$ drop the Maslov grading by 2. Note that both as a complex and $\mathbb{F}[U_1, U_2, \ldots, U_m]$-module $CF^-$ is isomorphic to $CF^-$ as defined in [23] via the isomorphism induced by

$$[\mathbf{x}, i_1, j_1, \ldots, i_l, j_l, i_{l+1}, \ldots, i_m] \mapsto U_1^{-i_1} \ldots U_l^{-i_m}\mathbf{x}.$$

And so it follows that $CF^-$ is a chain complex with homology $HF^-(S^3)$.

**Definition 3.2** Suppose that we have a Heegaard diagram $\mathcal{H}$ for $L \subset S^3$ as above. Fix some $\mathbf{s}$ $= (s_1, \ldots, s_l) \in \mathbb{H}(L)$. Now suppose that we restrict $CF^-(\mathcal{H})$ to only those generators $[\mathbf{x}, i_1, j_1, \ldots, i_l$ $, j_l, i_{l+1}, \ldots, i_m]$ which satisfy $A_k(\mathbf{x}) = j_k$ and force the differential to only count holomorphic disks $\phi$ with $n_{z_k}(\phi) = 0$ when $1 \leq k \leq l$. Then this quotient complex of $CF^-(\mathcal{H})$ will be denoted by $CFL^-(\mathcal{H},\mathbf{s})$. Note that $CFL^-$ inherits an $\mathbb{F}[U_1, \ldots U_m]$ module action from $CF^-$.

**Theorem 3.3** *If $CFL^-(\mathcal{H}, \mathbf{s})$ is as above, then its homology is $HFL^-(S^3, L, \mathbf{s})$*

**Proof:** If the diagram $\mathcal{H}$ has no free points, then $CFL^-(\mathcal{H},\mathbf{s})$ is the same as the complex computing $HFL^-$ in [23]. so we only need to show what happens in the case when there are free basepoints in $\mathcal{H}$. Suppose that $\mathcal{H}'$ is another Heegaard diagram that only has $l$-pairs of basepoints (one pair for each link component) and no others. Then we claim that $\mathcal{H}$ can be obtained from $\mathcal{H}'$ via the following moves:

**1** a 3-manifold isotopy

**2** $\alpha$ and $\beta$ curve isotopy

**3** $\alpha$ and $\beta$ handleslide

**4** index one/two stabilization

We may also need the inverses of moves 1-4

**5** free index zero/three stabilization,

but we do not need the inverse of 5.

We follow the argument from proposition 4.13 of [11] which relies on [12] Lemma 2.4 . Basically, we can apply moves $1-4$ to $\mathcal{H}'$ to obtain a Heegaard diagram that differs from a diagram with exactly $l$ pairs of basepoints (one pair for each component) by index zero/three stabilizations only. Then we can apply moves $1-4$ again to obtain the diagram $\mathcal{H}$. Now we know that moves $1-4$ and their inverses give chain homotopy equivalences for the complexes $CFL^-$ by the arguments given in [20] and proposition 3.9 of [23], so we will focus on move 5. Suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ are two Heegaard diagrams for $L$, and $\mathcal{H}_2$ is obtained from $\mathcal{H}_1$ by a free index zero/three stabilization. Then $\mathcal{H}_2$ has an extra free basepoint $w_r$ that $\mathcal{H}_1$ does not have. By the argument of Lemma 6.1 in [23], we see that the complex $CFL^-(\mathcal{H}_2, \mathbf{s})$ is just the mapping cone

$$CFL^-(\mathcal{H}_1,\ \mathbf{s})[U_r] \xrightarrow{U_r - U_k} CFL^-(\mathcal{H}_1,\ \mathbf{s})[U_r],$$

where $k$ is an index corresponding to some $\mathbf{w}$ basepoint in $\mathcal{H}_1$. Now, $k$ may correspond to a free basepoint, or it may correspond to some link component (in which case the action of $U_k$ is trivial); but in either case, the homology of this mapping cone is the same as the homology of $CFL^-(\mathcal{H}_1, \mathbf{s})$. So we see that all of the above 5 Heegaard moves induce quasi-isomorphisms of chain complexes, and this gives the desired result. □

**Definition 3.4** Fix a Heegaard diagram $\mathcal{H}$ for $L$. For a given $\mathbf{s} \in \mathbb{H}(L)$ and $\varepsilon \in E_l$, we define the complex $A^-_{\mathbf{s},\varepsilon}(\mathcal{H})$ to be the quotient complex of $CF^-(\mathcal{H})$ generated by those $[\mathbf{x}, i_1, j_1, \ldots, i_l, j_l, i_{l+1}, \ldots, i_m]$ that satisfy

- $\max\{i_k, j_k - (s_k - 1)\} \leq 0$ if $\varepsilon_k = 0$
- $\max\{i_k, j_k - s_k\} \leq 0$ if $\varepsilon_k = 1$
- $i_k \leq 0$ and $j_k = s_k$ if $\varepsilon_k = 2$.

By $A^-_{\mathbf{s}}(\mathcal{H})$ we mean $A^-_{\mathbf{s},\mathbf{1}}(\mathcal{H})$. We will write $A^-_{\mathbf{s},\varepsilon}$ when the choice of diagram is clear from context. The complex $A^-_{\mathbf{s},\mathbf{1}}(\mathcal{H})$ inherits an $\mathbb{F}[U_1,\ldots U_m]$ action from $CF^-(\mathcal{H})$. When $\mathcal{H}$ is clear from context we will omit $\mathcal{H}$ from the notation.

**Remark 3.5** If we complete $A^-_{\mathbf{s}}(\mathcal{H})$ with respect to the maximal ideal $(U_1,\ldots,U_m)$, there is an isomorphism between the completed version of $A^-_{\mathbf{s}}(\mathcal{H})$ and $\mathfrak{A}^-(\mathcal{H},\mathbf{s})$ as defined in section 4.2 of [11], given by:

$$[\mathbf{x},i_1,j_1,\ldots,i_l,j_l,i_{l+1},\ldots,i_m]\mapsto U_1^{-\max\{i_1,j_1-s_1\}}U_2^{-\max\{i_2,j_2-s_2\}}\ldots U_l^{-\max\{i_l,j_l-s_l\}}U^{-i_{l+1}}\ldots U^{-i_m}\mathbf{x}.$$

We can use the proofs in section 4.3 and 4.4 of [11] to show that the homology of the complex $A^-_{\mathbf{s}}(\mathcal{H})$ does not depend on the choice of a Heegaard diagram. For this reason we will sometimes write $H_*(A^-_{\mathbf{s}}(\mathcal{H}))$ as $H_*(A^-_{\mathbf{s}}(L))$. In this paper we could have just used the complexes $\mathfrak{A}^-_{\mathbf{s}}$ to get the same results about link Floer homology. The choice to use the notation here has been made to make the analogy with the work in [18] and [21] more clear.

**Theorem 3.6** *Suppose that $L\subset S^3$ is an L-space link and $\mathbf{s}\in\mathbb{H}(L)$. Then, as $\mathbb{F}[U_1,U_2,\ldots,U_l]$-modules,*

$$H_*(A^-_{\mathbf{s}})\cong\mathbb{F}[U].$$

*where all of the $U_i$ have the same action as $U$ on the right hand side.*

**Proof:** We can use the proof of Theorem[b] 10.1 in [11] to see that for any $\mathbf{s}\in\mathbb{H}(L)$, $H_*(A^-_{\mathbf{s}})$ is isomorphic (as a module) to $HF^-(Y,\mathfrak{s})$, where $Y$ is some $L$ space obtained by large positive surgery on $L$ and $\mathfrak{s}$ is a Spin$^c$ structure over $Y$. □

**Remark 3.7** The above property characterizes $L$-space links. See also proposition 1.11 of [10].

Suppose that, for a fixed $\mathbf{s}\in\mathbb{H}(L)$, we have $\varepsilon',\varepsilon$ and $\varepsilon''$ in $E_l$ so that they only differ in the $j$th coordinate with $\varepsilon'_j=0,\varepsilon_j=1$ and $\varepsilon''_j=2$. Then, for a given Heegaard diagram $\mathcal{H}$ of $L$, there is a short exact sequence:

$$0\longrightarrow A^-_{\mathbf{s},\varepsilon'}(\mathcal{H})\xrightarrow{i_{\varepsilon'\varepsilon}}A^-_{\mathbf{s},\varepsilon}(\mathcal{H})\xrightarrow{j_{\varepsilon\varepsilon''}}A^-_{\mathbf{s},\varepsilon''}(\mathcal{H})\longrightarrow 0.$$

So, we can define a short exact cube of chain complexes $\boldsymbol{A}^-(\mathcal{H},\mathbf{s})$ by setting $\boldsymbol{A}^-(\mathcal{H},\mathbf{s})_\varepsilon=A^-_{\mathbf{s},\varepsilon}(\mathcal{H})$. Note also that $\overline{\boldsymbol{A}^-(\mathcal{H},\mathbf{s})}$ is just $CFL^-(\mathcal{H},\mathbf{s})$.

**Theorem 3.8** *For any $\mathbf{s}\in\mathbb{H}(L)$, $\boldsymbol{A}^-(\mathcal{H},\mathbf{s})$ is a basic short exact cube when $L$ is an L-space link.*

[b] As was mentioned in Remark 3.5, the only difference between the complex in that paper and this one is that it is defined over $\mathbb{F}[[U_1,U_2,\ldots,U_m]]$ as opposed to $\mathbb{F}[U_1,U_2,\ldots,U_m]$. However the proof of Theorem 10.1 in [11] does not rely on $\mathbb{F}[[U_1,U_2,\ldots,U_m]]$ in any way. See also the proof of Theorem 4.1 in [18].

**Proof:** We want to show properties 4 and 5 in definition 2.1. Note that, by Theorem 3.6, we already know that for all $\varepsilon \in \{0,1\}^n$ we have $H_*(A^-_{\mathbf{s},\varepsilon}) \cong \mathbb{F}[U]$. First, we will examine all maps induced on homology in the cube of inclusions. Suppose that $\varepsilon'$ and $\varepsilon$ are in $\{0,1\}^l$ and differ only in the $j$th coordinate with $\varepsilon'_j = 0$ and $\varepsilon_j = 1$. Also define $\varepsilon''$ to agree in all coordinates with $\varepsilon$ except the $j$th and $\varepsilon''_j = 2$. Now, following the proof of Lemma 3.1 in [21], we define $X$ to be the set of generators $[\mathbf{x}, i_1, j_1, \ldots, i_l, j_l, i_{l+1}, \ldots, l_m]$ of $CF^-$ that satisfy:

**1** $\max\{i_k, j_k - (s_k - 1)\} \leq 0$ if $\varepsilon''_k = 0$

**2** $\max\{i_k, j_k - s_k\} \leq 0$ if $\varepsilon''_k = 1$

**3** $i_k \leq 0$ and $j_k = s_k$ if $\varepsilon''_k = 2$, i.e. when $k = j$.

We define a set $Y$ similarly, except **3** is replaced with;

**3** $i_k = 0$ and $j_k < s_k$ if $\varepsilon''_k = 2$, i.e. when $k = j$.

Note that $X$ naturally generates a sub-complex of a quotient complex of $CF^-$, which we will denote by $C\{X\} = A^-_{\mathbf{s},\varepsilon''}$. Similarly, there are complexes $C\{U_jX\}, C\{Y\}$, $C\{X \cup Y\}$, $C\{U_jX \cup Y\}$ and $C\{X \cup U_jX \cup Y\}$, all of which inherit differentials from $CF^-$. Since $C\{X \cup Y\} = A^-_{\mathbf{s},\varepsilon}/U_j(A^-_{\mathbf{s},\varepsilon})$ its homology is $\widehat{HF}$ of some $L$-space obtained by some large surgery on $L$ (see section 11.2 of [11]). Therefore $H_*(C\{X \cup Y\}) \cong \mathbb{F}$. Similarly $H_*(C\{U_jX \cup Y\}) \cong \mathbb{F}$. Now we have two short exact sequences of complexes:

$$0 \longrightarrow C\{Y\} \xrightarrow{i_1} C\{X \cup Y\} \xrightarrow{j_1} C\{X\} \longrightarrow 0$$

and

$$0 \longrightarrow C\{U_jX\} \xrightarrow{i_2} C\{U_jX \cup Y\} \xrightarrow{j_2} C\{Y\} \longrightarrow 0.$$

We will denote the connecting homomorphims for these two complexes by $\delta_1$ and $\delta_2$, respectively. First note that $\delta_2 \circ \delta_1 = 0$ (this follows from the fact the differential $\partial$ on the quotient complex $C\{X \cup U_jX \cup Y\}$ satisfies $\partial^2 = 0$). Now it follows from the exact same argument as in Lemma 3.1 in [21] that either $H_*(C\{X\}) = H_*(A^-_{\mathbf{s},\varepsilon''})$ is 0 and $H_*(C\{Y\})$ is $\mathbb{F}$, or $H_*(C\{X\})$ is $\mathbb{F}$ and $H_*(C\{Y\})$ is 0 . If $H_*(C\{X\}) = 0$ then the map $i_{\varepsilon'\varepsilon} : A^-_{\mathbf{s},\varepsilon'} \to A^-_{\mathbf{s},\varepsilon}$ clearly induces an isomorphism on homology. If $H_*(A^-_{\mathbf{s},\varepsilon''})$ is $\mathbb{F}$ supported in some degree $k$ then it follows from the first short exact sequence that $H_*(C\{X \cup Y\}) = H_*(A^-_{\mathbf{s},\varepsilon}/U_j(A^-_{\mathbf{s},\varepsilon}))$ is also $\mathbb{F}$ supported in degree $k$. Then, from the second short exact sequence it follows that $H_*(C\{U_jX\}) \cong H_*(C\{U_jX \cup Y\}) = H_*(A^-_{\mathbf{s},\varepsilon'}/U_j(A^-_{\mathbf{s},\varepsilon'}))$ is $\mathbb{F}$ supported in degree $k-2$. So we now have that the top grading in $H_*(A^-_{\mathbf{s},\varepsilon'})$ is two less than the top grading in $H_*(A^-_{\mathbf{s},\varepsilon})$, and we have now completely verified property 5 in the definition of a basic short exact cube.

The only thing that is left to check in property 4 is that for any $\varepsilon \in \{0,1\}^l$, $H_*(\boldsymbol{A}^-(\mathcal{H},\mathbf{s})_\varepsilon) \cong \mathbb{F}[U]$ has even top degree. For any sufficiently large $(s_1, s_2, \ldots, s_l) = \mathbf{s} \in \mathbb{H}(L)$ we have $H_*(A^-_{\mathbf{s}}) \cong HF^-(S^3) = \mathbb{F}_{(0)}[U]$. For any $\mathbf{s}' \leq \mathbf{s}$, we can decrease the $s_j$ by one over finitely many steps to get from $\mathbf{s}$ to $\mathbf{s}'$. By property 5 we know that each of these steps will either preserve the top degree or drop it by 2. The result now follows. □

**Corollary 3.9** *For an L-space link $L \subset S^3$ with Heegaard diagram $\mathcal{H}$, $HC(\boldsymbol{A}^-(\mathcal{H},\, \boldsymbol{s}))$ depends only on $L$ and $\boldsymbol{s}$.*

**Proof:** The top gradings of all the $H_*(A^-_{\mathbf{s},\varepsilon})$ are invariants of $L \subset S^3$ and $\mathbf{s}$. The maps induced by homology in $\boldsymbol{A}^-(L,\, \mathbf{s})^I$ are completely determined by these gradings since we have shown that $\boldsymbol{A}^-(\mathcal{H},\, \mathbf{s})$ is a basic short exact cube. □

Here is another fact that we will use often:

**Lemma 3.10** *Fix some $\mathbf{s} \in \mathbb{H}(L)$ where $L$ in $S^3$ is an arbitrary link (i.e. not necessarily an L-space link). Then, if $HFL^-(L, s{+}\varepsilon)$ is trivial for every $\varepsilon \in \{0,1\}^l, \varepsilon \neq \mathbf{0}$ we get $HFL^-(L,\mathbf{s}) \cong \widehat{HFL}(L,\mathbf{s})$.*

**Proof:** First fix a Heegaard diagram $\mathcal{H}$ for $L \subset S^3$. We define an $l$-dimensional short exact cube $\boldsymbol{C}_{\mathbf{s}}$ as follows: for $\varepsilon \in \{0,1\}^l$ and $\mathbf{s} \in \mathbb{H}(L)$ we define $\boldsymbol{C}_{\mathbf{s},\varepsilon}$ to be a quotient complex of $CF^-(\mathcal{H})$ generated by those $[\mathbf{x}, i_1, j_1, \ldots, i_l, j_l,\, i_{l+1}, \ldots, i_m]$ that satisfy the following:

- $i_k = 0$ and $j_k < s_k$ if $\varepsilon_k = 0$
- $i_k = 0$ and $j_k \leq s_k$ if $\varepsilon_k = 1$
- $i_k = 0$ and $j_k = s_k$ if $\varepsilon_k = 2$.

Then the inclusion and quotient maps of $\boldsymbol{C}_{\mathbf{s}}$ are defined naturally from $CF^-(\mathcal{H})$.

By definition, $H_*(\overline{\boldsymbol{C}}_{\mathbf{s}}) \cong \widehat{HFL}(L,\mathrm{s})$ and for $\varepsilon \in \{0,1\}^l$ we have

$$H_*(\boldsymbol{C}_{\mathbf{s},\varepsilon}) \cong U_1^{1-\varepsilon_1} U_2^{1-\varepsilon_2} \ldots U_l^{1-\varepsilon_l} HFL^-(L, \mathbf{s} + \mathbf{1} - \varepsilon).$$

And so $H_*(\boldsymbol{C}_\varepsilon)$ is only nonzero when $\varepsilon = \mathbf{1}$. So it follows by taking iterated quotients that,

$$HFL^-(L,\mathbf{s}) \cong H_*(\boldsymbol{C}_{\mathbf{1}}) \cong H_*(\overline{\boldsymbol{C}}) \cong \widehat{HFL}(L,\mathrm{s}).$$

□

# Chapter 4

# Proof of the Main Theorems

**Remark 4.1** Suppose that $M = L_{i_1} \sqcup L_{i_2} \sqcup \dots L_{i_k}$ is a sub-link of $L = L_1 \sqcup L_2 \dots \sqcup L_l$ with the inherited orientation. Fix some Heegaard diagram $\mathcal{H}$ for $L$. Now choose any $\mathbf{s} = (s_1, \dots s_l) \in \mathbb{H}(L)$ so that all $s_j$ for $L_j \notin M$ are sufficiently large (for instance larger than $\max\{A_j(\mathbf{x})\}$ for every generator $\mathbf{x}$ in some fixed diagram $\mathcal{H}$ for $L \subset S^3$). Then it is easy to see that for some $\mathbf{r} \in \mathbb{H}(M)$ and any $\varepsilon \in E_l$ the complex $A^-_{\mathbf{s},\varepsilon}(\mathcal{H})$ is the same as $A^-_{\mathbf{r},\varepsilon'}(\mathcal{H}')$ where $\varepsilon' \in E_{l-k}$ is obtained from $\varepsilon$ by deleting $\varepsilon_{i_1}, \dots, \varepsilon_{i_k}$ and reordering and $\mathcal{H}'$ is obtained by deleting $z_{i_1}, \dots, z_{i_k}$ and reordering. The explicit value for $\mathbf{r}$ can be computed by the formula in section 4.5 of [11] (see also section 3.7 of [23]). So $\mathbf{r} = (r_1, \dots, r_k) \in \mathbb{H}(M)$ is given by $r_j = s_{i_j} - \mathrm{lk}(L_{i_j}, L \backslash M)/2$. The next Lemma was observed in [10] Lemma 1.10.

**Lemma 4.2** *Every sub-link of an L-space link is an L-space link.*

**Proof:** Suppose that $M \subset L$ is some sub-link. It suffices to show that, for any $\mathbf{r} \in \mathbb{H}(M)$, we have $H_*(A^-_{\mathbf{r}}(M)) \cong \mathbb{F}[U]$. This is true because $H_*(A^-_{\mathbf{r}}(M)) \cong H_*(A^-_{\mathbf{s}}(L))$ for some $\mathbf{s} \in \mathbb{H}$ as shown above. $\square$

**Lemma 4.3**

$$\sum_{\mathbf{s} \in \mathbb{H}} \chi(HFL^-(L, \mathbf{s})) u^{\mathbf{s}} = P^L_{\emptyset}(u_1, \dots, u_l)$$

**Proof:** It was shown in [23] proposition 9.1 that when $l > 1$

$$\sum_{\mathbf{s} \in \mathbb{H}} \chi(\widehat{HFL}(L, \mathbf{s})) u^{\mathbf{s}} = \pm \left( \prod_{i=1}^{l} u_i^{\frac{1}{2}} - u_i^{-\frac{1}{2}} \right) \Delta_L$$

and we have chosen sign conventions so that

$$\sum_{\mathbf{s} \in \mathbb{H}} \chi(\widehat{HFL}(L, \mathbf{s})) u^{\mathbf{s}} = \left( \prod_{i=1}^{l} u_i^{\frac{1}{2}} - u_i^{-\frac{1}{2}} \right) \Delta_L.$$

and so for $l > 1$ it follows that

$$\begin{aligned}
\sum_{\mathbf{s}\in\mathbb{H}} \chi(HFL^-(L,\mathbf{s}))u^{\mathbf{s}} &= \left(\prod_{(a_1,\ldots,a_l)\in\mathbb{Z}^l_{\leq 0}} u_1^{a_1}\ldots u_l^{a_l}\right)\left(\sum_{\mathbf{s}\in\mathbb{H}}\chi(\widehat{HFL}(L,\mathbf{s}))u^{\mathbf{s}}\right) \\
&= \left(\prod_{(a_1,\ldots,a_l)\in\mathbb{Z}^l_{\leq 0}} u_1^{a_1}\ldots u_l^{a_l}\right)\left(\prod_{i=1}^{l} u_i^{\frac{1}{2}} - u_i^{-\frac{1}{2}}\right)\Delta_L \\
&= \left(\prod_{i=1}^{l}\frac{u_i^{\frac{1}{2}} - u_i^{-\frac{1}{2}}}{1-u_i^{-1}}\right)\Delta_L \\
&= \sqrt{u_1u_2\ldots u_l}\Delta_L \\
&= P^L_{\emptyset}.
\end{aligned}$$

When $l = 1$, it was shown in [18] that:

$$\sum_{\mathbf{s}\in\mathbb{H}}\chi(\widehat{HFL}(L,\mathbf{s}))u^{\mathbf{s}} = \pm\Delta_L(u_1);$$

and so the result follows by the same argument as above. □

**Definition 4.4** Suppose we are given a Heegaard diagram $\mathcal{H}$ for an $L$-space link $L \subset S^3$. Define a directed labeled graph $\mathfrak{T}(\mathcal{H})$ as follows:

- The vertices correspond to the elements of $\mathbb{H}(L)$.
- There is a directed edge from $\mathbf{s} = (s_1, \ldots, s_l)$ to $\mathbf{s}' = (s'_1, \ldots, s'_l)$ if for some $i$ we have $s'_i = s_i + 1$ and $s'_j = s_j$ for every $j \neq i$. We will call this edge $e_{\mathbf{ss}'}$.
- If $\mathbf{s}$ and $\mathbf{s}'$, are as above then define $\varepsilon \in E_l$ so that $\varepsilon_j = 1$ if $j \neq i$ and $\varepsilon_i = 0$. Then the label of edge $e_{\mathbf{ss}'}$ is the same as the label of the edge between $\varepsilon$ and $\mathbf{1}$ in $HC(\boldsymbol{A}^-(L, \mathbf{s}'))$.

Just as in corollary 3.9, the graph $\mathfrak{T}(\mathcal{H})$ is an invariant of $L \subset S^3$. So we will simply say $\mathfrak{T}(L)$. We will denote by ${}^j_s\mathfrak{T}(L)$ the subgraph of $\mathfrak{T}(L)$ that is obtained by restricting to the hyperplane with $j$th coordinate equal to $s$.

**Definition 4.5** Suppose that $L \subset S^3$ is an $L$-space link. Then we recursively define $m(L) \in \mathbb{H}(L)$ as follows. If $L$ has only one component let $m(L)$ be the degree of $\Delta_L$. In general;

$$m(L)_i = \max\left(\{\deg_{u_i}(P^L_{\emptyset})\} \cup \left\{ m(L\backslash L_j)_{i-1} + \frac{\mathrm{lk}(L_i,L_j)}{2}\,\middle|\, j < i\right\} \cup \left\{ m(L\backslash L_j)_i + \frac{\mathrm{lk}(L_i,L_j)}{2}\,\middle|\, j > i\right\}\right)$$

where by $\deg_{u_i}(P^L_{\emptyset})$ we mean the maximal degree of $u_i$ in any monomial of $P^L_{\emptyset}$.

**Proposition 4.6** *For an L-space link $L \subset S^3$ suppose that $s \geq m(L)_j$. Then ${}^j_s\mathfrak{T}(L)$ is completely determined by $\mathfrak{T}(L\backslash L_j)$ and all the edges from ${}^j_s\mathfrak{T}(L)$ to ${}^{\ j}_{s+1}\mathfrak{T}(L)$ must be labeled with* 0.

**Proof:** First note that $\mathfrak{T}(L\backslash L_j)$ only makes sense in light of Lemma 4.2 from which it follows that $L\backslash L_j$ is an $L$-space link. Pick $\mathbf{m} = (m_1, \dots, m_l) \in \mathbb{H}(L)$ so that for any $1 \leq i \leq l$, $m_i > A_i(\mathbf{x})$ for every generator $\mathbf{x}$. Then, we claim that whenever $s_i > m_i$, ${}^{\ i}_{s_i}\mathfrak{T}(L)$ is completely determined by $\mathfrak{T}(L\backslash L_i)$ and all the edges from ${}^{\ i}_{s_i}\mathfrak{T}(L)$ to ${}^{\ \ \ i}_{s_i+1}\mathfrak{T}(L)$ must be labeled with 0. We prove this claim when $i = l$. Since $s_l > m_l$, the inclusion between $A^-_{(s_1,\dots,s_l)}$ and $A^-_{(s_1,\dots,s_l+1)}$ induces an isomorphism on homology. So the edge between $(s_1, \dots, s_l)$ and $(s_1, \dots, s_l + 1)$ is labeled with 0. Following Remark 4.1 we get that the edge between $(s_1, \dots, s_i, \dots, s_l)$ and $(s_1, \dots, s_i + 1, \dots, s_l)$ has the same label as the edge between $\left(s_1 - \frac{\mathrm{lk}(L_1,L_l)}{2}, \dots, s_i - \frac{\mathrm{lk}(L_i,L_l)}{2}, \dots s_{l-1} - \frac{\mathrm{lk}(L_{l-1},L_l)}{2}\right)$ and $\left(s_1 - \frac{\mathrm{lk}(L_1,L_l)}{2}, \dots, s_i - \frac{\mathrm{lk}(L_i,L_l)}{2} + 1, \dots, s_{l-1} - \frac{\mathrm{lk}(L_{l-1},L_l)}{2}\right)$ in $\mathfrak{T}(L\backslash L_l)$ and so this proves the claim.

Now we are ready to prove the proposition.

We will prove this by induction on $l$. If $m_j - 1 \geq m(L)_j$, for some fixed j, the edge between $(s_1, \dots, m_j - 1, \dots, s_l)$ and $(s_1, \dots, m_j, \dots, s_l)$ is labeled zero if $s_i \geq m_i$ for every $i \neq j$ (by induction).

Notice that this determines $\widetilde{HC}(\boldsymbol{A}^-(L, (s_1, \dots, m_j, \dots, s_l)))$. One valid (in the sense of Remark 2.4) labeling of the remaining edges in $HC(\boldsymbol{A}^-(L, (s_1, \dots, m_j, \dots, s_l)))$ is given by setting all the edges between $HC(\boldsymbol{A}^-(L, (s_1, \dots, m_j, \dots, s_l))) \cap {}^{\ \ \ \ j}_{m_j-1}\mathfrak{T}(L)$ and $HC(\boldsymbol{A}^-(L, (s_1, \dots, m_j, \dots, s_l))) \cap {}^{\ j}_{m_j}\mathfrak{T}(L)$ to be zero and letting an edge between $\mathbf{s}_1$ and $\mathbf{s}_2$ in $HC(\boldsymbol{A}^-(L, (s_1, \dots, m_j, \dots, s_l))) \cap {}^{\ \ \ \ j}_{m_j-1}\mathfrak{T}(L)$ have the same labeling as the edge between $\mathbf{s}'_1$ and $\mathbf{s}'_2$ in $HC(\boldsymbol{A}^-(L, (s_1, \dots, m_j, \dots, s_l))) \cap {}^{\ j}_{m_j}\mathfrak{T}(L)$ where $\mathbf{s}'_1$ and $\mathbf{s}'_2$ are the same as $\mathbf{s}_1$ and $\mathbf{s}_2$ after adding one to the $j$th coordinate.

Since $m_j - 1 > \deg_{u_j} P^L_\emptyset$ we must have $\chi(H_*(\overline{\boldsymbol{A}^-(L, (s_1, \dots, m_j, \dots, s_l))})) = 0$ and so the labeling for $HC(\boldsymbol{A}^-(L, (s_1, \dots, m_j, \dots, s_l)))$ described above is the correct one since it yields the correct Euler characteristic (see Remark 2.6 and Lemma 2.5). We can similarly fill in all of ${}^{\ \ \ \ j}_{m_j-1}\mathfrak{T}(L)$ and all the edges between ${}^{\ \ \ \ j}_{m_j-1}\mathfrak{T}(L)$ and ${}^{\ j}_{m_j}\mathfrak{T}(L)$ are labeled 0. Repeating this process by inductively decreasing the $j$th coordinate proves the claim. □

**Lemma 4.7** *For a* 2 *or* 3 *component L-space link,* $\mathfrak{T}(L)$ *completely determines* $HFL^-(L, \mathbf{s})$ *for every* $\mathbf{s} \in \mathbb{H}(L)$.

**Proof:** Note that $\mathfrak{T}(L)$ determines all the hypercube graphs of $\boldsymbol{A}^-(L, \mathbf{s})$ for any $\mathbf{s} \in \mathbb{H}(L)$. Thus, by Lemma 2.7 and Remark 2.4 we get that $\mathfrak{T}(L)$ determines all the $HFL^-(L, \mathbf{s})$ upto an even shift in absolute grading. To fix the grading note that we can pick $\mathbf{s} \in \mathbb{H}(L)$ so that any edge emerging from $\mathbf{s}' \geq \mathbf{s}$ is 0 since for $\mathbf{s}$ sufficiently large $H_*(A^-_{\mathbf{s}}(L)) \cong HF^-(S^3) = \mathbb{F}_{(0)}[U]$. This fixes the grading as required. ☐

**Lemma 4.8** *For an L-space link $L$, the graph $\mathfrak{T}(L)$ is determined by the polynomials $\pm\Delta_M$ and the linking numbers $\operatorname{lk}(L_i, M)$ where $M$ is any sublink of $L$.*

**Proof:** We will prove this by inducting on $l$. First suppose that $l = 1$. Then $\pm\Delta_L$ completely determines $\pm P^L_\emptyset = \sum_{s\in\mathbb{Z}} a_s(u_1)^s$. The only possibilities for $|a_s|$ are either 1 or 0. If $|a_s| = 1$ then this forces the edge between $s-1$ and $s$ to be labeled with 1. If $a_s = 0$ then this forces the edge between $s-1$ and $s$ to be labeled with 0. This proves the case when $l = 1$.
By proposition 4.6, we see that the subgraph of $\mathfrak{T}(L)$ that is induced by all the vertices $\mathbf{s} = (s_1, \ldots, s_l)$ satisfying $\mathbf{s}_i \geq m(L)_i$ for some $i$, is completely determined by the relevant polynomials and linking numbers.

For the rest of $\mathfrak{T}(L)$ note that every edge of $\widetilde{HC}(\boldsymbol{A}^-(L, m(L)))$ is contained inside the part of the graph whose labels we have already determined. By Lemma 2.5, this either completely determines $HC(\boldsymbol{A}^-(L, m(L)))$, or all the edges emerging from $(m(L)_1 - 1, \ldots, m(L)_l - 1)$ are labeled with a 0 or they are all labeled with 1. If $HC(\boldsymbol{A}^-(L, m(L)))$ is not completely determined by $\widetilde{HC}(\boldsymbol{A}^-(L, \mathbf{m}))$, then we can use Lemma 2.6 to see that the absolute values of the coefficients of $\Delta_L$ are enough to determine if all the edges emerging from $(m(L)_1 - 1, \ldots, m(L)_l - 1)$ are labeled with a 0 or 1. Thus, we now have computed $\widetilde{HC}(\boldsymbol{A}^-(L, (m_1, \ldots, m_i - 1, \ldots, m_l))$ for any $i$ and so we can proceed as before to inductively fill out all of $\mathfrak{T}(L)$. This proves the Lemma. ☐

**Proof:** [Proof of Theorem 1.5] This follows immediately from the previous two Lemmas ☐

**Lemma 4.9** *Let $S = \{i_1, \ldots, i_k\} \subsetneq \{1, \ldots, l\}$ and suppose that $\{j_1, \ldots, j_{l-k}\} = \{1, \ldots, l\}\backslash S$ where $j_a < j_b$ when $a < b$. Pick $\mathbf{s} \in \mathbb{H}(L)$ so the $s_{i_p} \geq m(L)_{i_p}$. Then if $a_{s_{j_1}, s_{j_2}, \ldots, s_{j_{l-k}}}$ is the coefficient of $u_{j_1}^{s_{j_1}} \ldots u_{j_{l-k}}^{s_{j_{l-k}}}$ in $P^L_{L_S}$, we have $a_{s_{j_1}, s_{j_2}, \ldots, s_{j_{l-k}}} = \chi(H_*(A^-_{\mathbf{s},\varepsilon}(L)))$, where $\varepsilon \in E_l$ satisfies $\varepsilon_r = 2$ if $r = j_p$ for some $p$ and $\varepsilon_r = 1$ otherwise.*

**Proof:** This follows from Remark 4.1 and Lemma 4.3. ☐

**Proof:** [Proof of Theorem 1.8] We will assume WLOG that $r = 1$. Then let $S = \{i_1, \ldots, i_k\} \subset \{2, \ldots, l\}$ and $\{j_1, \ldots, j_{l-k-1}\} = \{2, \ldots, l\}\backslash S$ with $j_a < j_b$ if $a < b$. $\mathbf{s} = (s_1, \ldots, s_l) \in \mathbb{H}(L)$ is arbitrary. Fix $(m_1, \ldots, m_l) \in \mathbb{H}(L)$ so that $m_i > m(L)_i + 1$. Then we have the following:

$$R_{\substack{\mathbf{s}' \geq \mathbf{s} \\ s_1' = s_1}} \left(P^L_{L_S}\right) = \sum_{\substack{\mathbf{s}' = (s_1', \ldots, s_l') \in \mathbb{H} \\ s_1' = s_1, s_{i_p}' = m_{i_p} \\ m_{j_p} \geq s_{j_p}' \geq s_{j_p}}} \chi\left(H_*\left(A^-_{\mathbf{s}',\rho}\right)\right),$$

where $\rho \in E_l$ is fixed and satisfies $\rho_k = 2$ if $k = j_p$ for some $p$, and $\rho_k = 1$ otherwise. This follows by the previous Lemma. We get that the above quantity is equal to:

$$\sum_{\substack{\mathbf{s}'=(s'_1,\ldots,s'_l)\in\mathbb{H} \\ s'_1=s_1, s'_{i_p}=m_{i_p} \\ m_{j_p}\geq s'_{j_p}\geq s_{j_p}}} \sum_{\substack{\varepsilon\in E_l,\varepsilon_1=2,\varepsilon_{i_p}=1 \\ \varepsilon_{j_p}=1 \text{ or } 0}} (-1)^{\text{number of 0's in } \varepsilon} \chi\left(H_*\left(A^-_{\mathbf{s}',\varepsilon}\right)\right). \tag{4.1}$$

Note that if $\varepsilon \in E_l$ with $\varepsilon_1 = 2$, $\varepsilon_i = 0$ or $1$ if $i \neq 1$ we get:

$$A^-_{\mathbf{s}',\varepsilon} = A^-_{\mathbf{s}'',(2,1,\ldots,1)}.$$

where $\mathbf{s}''$ is given by $\mathbf{s}''_1 = \mathbf{s}'_1$ and $\mathbf{s}''_k = \mathbf{s}'_k + \varepsilon_k - 1$. So all of the terms in (4.1) that correspond to $\mathbf{s}'$ with $s'_i \neq s_i$ or $m_i$ will cancel out. This leaves,

$$\sum_{\substack{\mathbf{s}'=(s'_1,\ldots,s'_l)\in\mathbb{H} \\ s'_1=s_1, s'_{i_p}=m_{i_p} \\ s'_{j_p}=s_{j_p} \text{ or } m_{j_p}}} (-1)^{\text{number of 0's in } \nu(\mathbf{s}')} \chi\left(H_*\left(A^-_{\mathbf{s}',\nu(\mathbf{s}')}(L)\right)\right), \tag{4.2}$$

where here $\nu(\mathbf{s}')_1 = 2, \nu(\mathbf{s}')_{i_p} = 1$ and $\nu(\mathbf{s}')_{j_p} = 1$ if $s'_{j_p} = m_{j_p}$, and $\nu(\mathbf{s}')_{j_p} = 0$ otherwise.
Given $S \subset \{2,\ldots,l\}$, we define $\mathbf{s}(S)$ by setting $\mathbf{s}(S)_1 = s_1$, $\mathbf{s}(S)_k = m_p$ if $p \in S$, and $\mathbf{s}(S)_k = s_p - 1$ otherwise. Then we can rewrite (4.2) as

$$\sum_{S'\subset\{2,\ldots,l\}\setminus S} (-1)^{l-1-|S|-|S'|} \chi\left(H_*\left(A^-_{\mathbf{s}(S\cup S'),(2,1,\ldots,1)}\right)\right). \tag{4.3}$$

Thus, we finally get:

$$\begin{aligned}
\sum_{S\subset\{2,\ldots,l\}} (-1)^{l-1-|S|} R_{\substack{\mathbf{s}'\geq\mathbf{s} \\ s'_1=s_1}}\left(P^L_{L_S}\right) &= \sum_{S\subset\{2,\ldots,l\}} \sum_{S'\subset\{2,\ldots,l\}\setminus S} (-1)^{-|S'|} \chi\left(H_*\left(A^-_{\mathbf{s}(S\cup S'),(2,1,\ldots,1)}\right)\right) \\
&= \sum_{S\subset\{2,\ldots,l\}} \sum_{A\subset S} (-1)^{-|S\setminus A|} \chi\left(H_*\left(A^-_{\mathbf{s}(S),(2,1,\ldots,1)}\right)\right) \\
&= \chi\left(H_*\left(A^-_{\mathbf{s}(\emptyset),(2,1,\ldots,1)}\right)\right). \qquad (4.4)
\end{aligned}$$

Now (4.4) must be either 1 or 0 by Theorem 3.8. □

# Chapter 5

# Application to 2-Bridge Links

We would like to use the recursive formula for the multivariate Alexander polynomial of a 2-bridge link given in [8], so we will use the conventions from that paper. A circle labeled $k$ or $-k$ will represent a braid with $k$ crossings as in Figure 5.1 Suppose we are given a collection of nonzero

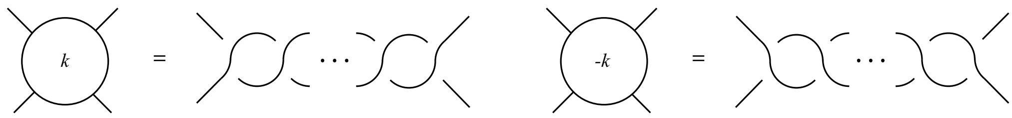



Figure 5.1:

integers $a_1, \ldots, a_n$. Then we can define $\alpha$ and $\beta$ via

$$\frac{\alpha}{\beta} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}} \tag{5.1}$$

where $\alpha > 0$, g.c.d$(\alpha, \beta) = 1$, and $\alpha > |\beta| > 0$. Now, if $\alpha$ is even we can use $(a_1, \ldots, a_n)$ to construct an oriented link $C(a_1, \ldots, a_n)$ as shown in Figure 5.2.

Links of this form are called 2-bridge links, and we have the following classification from [3] and page 144 of [28] (see also chapter 12 in [1]):

**Theorem 5.1** *If $L = C(a_1, \ldots, a_n)$ and $L' = C(b_1, \ldots, b_m)$ are two 2 bridge links where we define $\alpha$ and $\beta$ from $a_1, \ldots, a_n$, as in equation 5.1, and similarly $\alpha'$ and $\beta'$ from $b_1, \ldots, b_m$. Then $L$ and $L'$ are equivalent iff $\alpha' = \alpha$ and $\beta' \equiv \beta^{\pm 1} \mod 2\alpha$. If $\beta' \equiv \beta + \alpha \mod 2\alpha$ or $\beta'\beta \equiv 1 + \alpha \mod 2\alpha$, then $L$ and $L'$ are equivalent after reversing the orientation of one of the components.*

We will denote the 2-bridge link determined by $\alpha$ and $\beta$ as above by $b(\alpha, \beta)$. To use the formulas

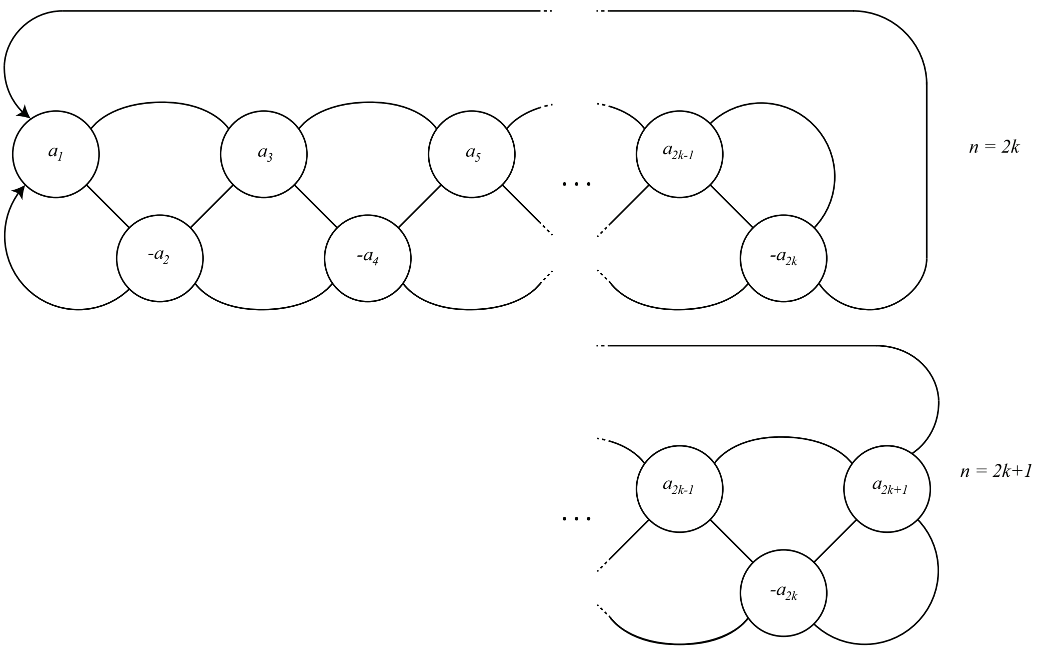



Figure 5.2: Diagram for constructing 2-bridge link given a sequence of non-zero integers.

in [8], we need an expansion of $\frac{\alpha}{\beta}$ of the following form:

$$\frac{\alpha}{\beta} = 2p_1 + \cfrac{1}{2q_1 + \cfrac{1}{2p_2 + \cfrac{1}{2q_2 + \cfrac{1}{\ddots + \cfrac{1}{2p_n}}}}}.$$

We will denote $b(\alpha, \beta) = C(2p_1, 2q_1, \ldots, 2p_{n-1}, 2q_{n-1}, 2p_n)$ by $D(p_1, q_1, p_2, q_2, \ldots, p_n)$ for convenience.

We define two variable polynomials $F_r(u_1, u_2)$ for $r \in \mathbb{Z}$:

$$F_r(u_1, u_2) = \begin{cases} \displaystyle\sum_{i=0}^{r-1} (u_1 u_2)^i & \text{if } r > 0 \\ 0 & \text{if } r = 0 \\ \displaystyle -\sum_{i=r}^{-1} (u_1 u_2)^i & \text{if } r < 0. \end{cases}$$

Now let us define polynomials $\Delta_k \in \mathbb{Z}[u_1^{\pm}, u_2^{\pm}]$ for $0 \leq k \leq n$ recursively as follows:

$$\Delta_0 = 0$$

$$\Delta_1 = F_{p_1}$$

$$\Delta_k = (q_{k-1}(u_1-1)(u_2-1)F_{p_k}+1)\Delta_{k-1} + (u_1u_2)^{p_{k-1}} \frac{F_{p_k}}{F_{p_{k-1}}}(\Delta_{k-1}-\Delta_{k-2}). \quad (5.2)$$

Also set $l_k = \sum_{i=1}^{k} p_i$ and $\tilde{l}_k = \sum_{i=1}^{k} |p_k|$. Then by Theorems 1, 3 and corollary 1 of [8] we have;

**Theorem 5.2** *If $L = D(p_1, q_1, p_2, q_2, \ldots, p_k)$, then:*

$$(u_1u_2)^{\frac{1-l_k}{2}} \Delta_k(u_1, u_2) = \pm \Delta_L(u_1, u_2).$$

*The minimal degree of $u_1$ (or $u_2$) in any monomial of $\Delta_k$ is $\frac{l_k - \tilde{l}_k}{2}$ and the maximal degree of $u_1$ (or $u_2$) in any monomial of $\Delta_k$ is $\frac{l_k+\tilde{l}_k}{2} - 1$.*

Define $q(k) = \prod_{i=1}^{k-1} q_i$ and $F(k) = \prod_{i=1}^{k} F_{p_i}$ where, as usual, the empty product is 1. Also recall that the linking number of $D(p_1, q_1, p_2, q_2, \ldots, p_n)$ is $-l_n$.

Given any $P \in \mathbb{Z}[u_1^{\pm}, u_2^{\pm}]$ where $P = \sum_{r,s\in\mathbb{Z}} a_{r,s} u_1^r u_2^s$, we define $P^{[i]}$ to be the polynomial $\sum_{j\in\mathbb{Z}} a_{j+i,j} u_1^{j+i} u_2^j$. If $P^{[i]} \neq 0$ we say that $P$ is supported on the diagonal $i$. Note that if $Q \in \mathbb{Z}[u_1^{\pm}, u_2^{\pm}]$, then $(P+Q)^{[i]} = P^{[i]} + Q^{[i]}$ and $(PQ)^{[i]} = \sum_{a+b=i} P^{[a]} Q^{[b]}$ . Thus, it follows that if $P^{[0]}$ divides $Q$, then $(Q/P^{[0]})^{[k]} = Q^{[k]}/P^{[0]}$. Using equation (5.2), we get the following identity:

$$\Delta_n^{[k]} = \sum_{i+j=k} (q_{n-1}(u_1-1)(u_2-1)F_{p_n}+1)^{[i]} \Delta_{n-1}^{[j]} + \left( (u_1u_2)^{p_{n-1}} \frac{F_{p_n}}{F_{p_{n-1}}} \right) (\Delta_{n-1} - \Delta_{n-2})^{[k]}. \quad (5.3)$$

This can then be expanded to:

$$\begin{aligned} \Delta_n^{[k]} &= (q_{n-1}(-u_2)F_{p_n})\Delta_{n-1}^{[k+1]} + (q_{n-1}(-u_1)F_{p_n})\Delta_{n-1}^{[k-1]} + (q_{n-1}(u_1u_2+1)F_{p_n}+1)\Delta_{n-1}^{[k]} \\ &+ \left( (u_1u_2)^{p_{n-1}} \frac{F_{p_n}}{F_{p_{n-1}}} \right) (\Delta_{n-1} - \Delta_{n-2})^{[k]}. \end{aligned} \quad (5.4)$$

**Lemma 5.3** *If $t > n-1$ then $\Delta_n$ is not supported on the diagonal $t$. Also:*

$$\Delta_n^{[n-1]} = q(n)(-u_1)^{n-1} F(n).$$

**Proof:** First note that $\Delta_1^{[0]} = \Delta_1 = F_{p_1}$. Now the claim that $\Delta_n^{[t]} = 0$ when $t > n-1$ can be easily seen by induction via equation (5.4). We will prove that $\Delta_n^{[n-1]} = q(n)(-u_1)^{n-1}F(n)$ for $n > 1$ by induction on $n$ using equation (5.4):

$$\begin{aligned}\Delta_n^{[n-1]} &= (q_{n-1}(-u_1)F_{p_n})\left(\prod_{i=1}^{n-2} q_i\right)(-u_1)^{n-2}\left(\prod_{i=1}^{n-1} F_{p_i}\right)\\ &= q(n)(-u_1)^{n-1}F(n).\end{aligned}$$

□

**Lemma 5.4** *For $n \geq 2$:*

$$\Delta_n^{[n-2]} = P_1 + P_2 + P_3$$

*where:*

$$\begin{aligned}P_1 &= (n-1)(u_1u_2+1)q(n)F(n)(-u_1)^{n-2}\\ P_2 &= \sum_{i=2}^{n} \frac{q(n)}{q_{i-1}}\frac{F(n)}{F_{p_i}}(-u_1)^{n-2} \text{ and}\\ P_3 &= \sum_{i=1}^{n-1}(u_1u_2)^{p_i}\frac{q(n)}{q_i}\frac{F(n)}{F_{p_i}}(-u_1)^{n-2}.\end{aligned} \tag{5.5}$$

**Proof:** When $n = 2$, we directly compute that:

$$\Delta_2 = q_1(u_1-1)(u_2-1)F_{p_2}F_{p_1} + F_{p_1} + (u_1u_2)^{p_1}F_{p_2}.$$

For $n > 2$, we can recursively compute $\Delta_n^{[n-2]}$:

$$\begin{aligned}\Delta_n^{[n-2]} &= (q_{n-1}(u_1-1)(u_2-1)F_{p_n}+1)^{[0]}\Delta_{n-1}^{[n-2]} + (q_{n-1}(u_1-1)(u_2-1)F_{p_n}+1)^{[1]}\Delta_{n-1}^{[n-3]}\\ &\quad + (u_1u_2)^{p_{n-1}}\frac{F_{p_n}}{F_{p_{n-1}}}(\Delta_{n-1}^{[n-2]})\\ &= (q_{n-1}(u_1u_2+1)F_{p_n}+1)\frac{q(n)}{q_{n-1}}\frac{F(n)}{F_{p_n}}(-u_1)^{n-2} + q_{n-1}(-u_1)F_{p_n}\Delta_{n-1}^{[n-3]}\\ &\quad + (u_1u_2)^{p_{n-1}}\frac{q(n)}{q_{n-1}}\frac{F(n)}{F_{p_{n-1}}}(-u_1)^{n-2}.\end{aligned}$$

The result now follows by induction. □

**Lemma 5.5** *Let $\Delta_n = \sum_{i,j} a_{ij}u_1^i u_2^j$. Suppose that all the nonzero $a_{ij}$ are $\pm 1$. Suppose also that for fixed $i'$ (or $j'$) the nonzero $a_{i'j}$ (or $a_{j'i}$) alternate in sign. Then we must have $|q_i| = 1$ for every $1 \leq i \leq n-1$. For the $p_i$, one of the following two possibilities holds:*

- *For $i \neq 1$ all $p_i$ are equal. For $i \neq 1$, $p_i = \pm 1$ and $p_i = -q_{i-1}$*
- *For $i \neq n$ all $p_i$ are equal. For $i \neq n$, $p_i = \pm 1$ and $p_i = -q_i$.*

**Proof:** First note that when $n = 1$, the Lemma is vacuously true. So from now on we will assume that $n \geq 2$. If $\Delta_n$ has all coefficients $\pm 1$ or 0, then so does $\Delta_n^{[n-1]} = q(n)F(n)(-u_1)^{n-1}$. For this to happen $|q(n)|$ must be 1 which implies that $q_i = \pm 1$ for every $1 \leq i \leq n-1$. $F(n)$ has coefficients $\pm 1$ if for all but possibly one $i$, we have $p_i = \pm 1$.
Now we focus on $\Delta_n^{[n-2]}$. There are four cases:

**Case 1 (There is some $k \in \{1, 2, \ldots, n\}$ such that $p_k > 1$)** Suppose that $r$ of the $p_i$ are $-1$ (and so except for $p_k$, the rest are 1.) First, we get that:

$$F(n) = (-1)^r \sum_{i=0}^{p_k - 1} (u_1 u_2)^{i-r}.$$

Now, since all the nonzero coefficients of $\Delta_n$ are by assumption $\pm 1$, the same must be true for $\frac{\Delta_n^{[n-2]}}{q(n)(-u_1u_2)^{-r}(-u_1)^{n-2}}$. We will compute the coefficient of $u_1 u_2$ in $\frac{\Delta_n^{[n-2]}}{q(n)(-u_1u_2)^{-r}(-u_1)^{n-2}}$. Now recall that

$$\Delta_n^{[n-2]} = P_1 + P_2 + P_3$$

where $P_1$, $P_2$ and $P_3$ are as defined in equation (5.5). Set $P'_i := \frac{P_i}{q(n)(-u_1u_2)^{-r}(-u_1)^{n-2}}$ for $i = 1, 2$ or 3. Then,

$$P'_1 = (n-1)(u_1u_2 + 1) \sum_{i=0}^{p_k - 1} (u_1 u_2)^i,$$

and so the coefficient of $(u_1 u_2)$ in $P'_1$ is $2(n-1)$. Similarly,

$$P'_2 = q_{k-1} + \sum_{\substack{p_j = 1 \\ 2 \leq j \leq n}} (q_{j-1}) \sum_{i=0}^{p_k - 1} (u_1u_2)^i + \sum_{\substack{p_j = -1 \\ 2 \leq j \leq n}} (-q_{j-1}) \sum_{i=1}^{p_k} (u_1 u_2)^i.$$

So the coefficient of $(u_1 u_2)$ in $P'_2$ is

$$\sum_{\substack{2 \leq j \leq n \\ j \neq k}} p_j q_{j-1},$$

and similarly the coefficient of $(u_1u_2)$ in $P'_3$ is

$$\sum_{\substack{1 \leq j \leq n-1 \\ j \neq k}} p_j q_j.$$

So finally, the coefficient of $u_1u_2$ in $\frac{\Delta_n^{[n-2]}}{q(n)(-u_1u_2)^{-r}(-u_1)^{n-2}}$ is

$$2(n-1) + \sum_{\substack{2 \leq j \leq n \\ j \neq k}} p_j q_{j-1} + \sum_{\substack{1 \leq j \leq n-1 \\ j \neq k}} p_j q_j, \tag{5.6}$$

which must be 1, −1 or 0. Notice first that, if $2 \leq k \leq n-1$ then the sum

$$\sum_{\substack{2 \leq j \leq n \\ j \neq k}} p_j q_{j-1} + \sum_{\substack{1 \leq j \leq n-1 \\ j \neq k}} p_j q_j$$

is bounded above in absolute value by $2(n-2)$, which makes it impossible for equation (5.6) to be equal to 1, −1 or 0. So, we get that $k$ must be 1 or $n$. If $k$ is 1 then equation (5.6) becomes

$$2(n-1) + p_n q_{n-1} + \sum_{j=2}^{n-1} p_j(q_j + q_{j-1}).$$

Notice that the above quantity has smallest possible value 1 and this only occurs if all of the $q_i$ are equal and have opposite sign as all the $p_{i+1}$, which proves the claim in this case. When $k = n$ the argument is similar.

**Case 2 (There is some $k \in \{1, 2, \ldots, n\}$ such that $p_k < -1$)** The argument is the same as in the previous case, except we divide $\Delta_n^{[n-2]}$ by $q(n)(-u_1u_2)^{-r}(-u_1)^{n-2}$ and examine the coefficient of $(u_1u_2)^{-1}$.

**Case 3 (All of the $p_i$ are $\pm 1$ and $n \geq 3$)** We will start by showing that all the $q_i$ are equal. Suppose as in the previous cases that the number of $p_i$ that are −1 is $r$. In this case $\Delta_n^{[n-1]}$ is the monomial

$$(-1)^{n-1+r} q(n) u_1^{n-1-r} u_2^{-r} \neq 0.$$

This has the maximal possible degree for $u_1$ and minimal possible degree for $u_2$ by Theorem 5.2. This immediately forces $\Delta_n^{[n-2]}$ to have at most 2 nonzero coefficients, and $\Delta_n^{[n-3]}$ to have at most 3 nonzero coefficients. So $\Delta_n^{[n-2]}$ is of the form

$$a_{n-2-r,-r} u_1^{n-2-r} u_2^{-r} + a_{n-1-r,1-r} u_1^{n-1-r} u_2^{1-r}$$

Using the symmetry of the Alexander polynomial under the involution $u_i \mapsto u_i^{-1}$, as well as the symmetry given by exchanging $u_1$ and $u_2$ (there is an isotopy of $S^3$ exchanging the two components of a 2 bridge link which is easy to see using the Schubert normal form [28]); we can conclude that $a_{n-2-r,-r} = a_{n-1-r,1-r}$. Suppose that $a_{n-2-r,-r} = a_{n-1-r,1-r} \neq 0$. Then since we have required the signs of $a_{i,j}$ to be alternating for fixed $i$ (and $j$), this forces one of the following possibilities for $\Delta_n^{[n-3]}$

$$\Delta_n^{[n-3]} = \pm(u_1^{n-3-r} u_2^{-r} + u_1^{n-2-r} u_2^{1-r} + u_1^{n-1-r} u_2^{2-r}) \text{ or } \pm(u_1^{n-2-r} u_2^{1-r}) \text{ or } 0.$$

We have ruled out $\pm(u_1^{n-3-r} u_2^{-r} + u_1^{n-1-r} u_2^{2-r})$ due to Theorem 3 (see also definition 2(iv)) in [8].

In all the possibilities for $\Delta_n^{[n-3]}$, we have

$$\Delta_n^{[n-3]}(-1,1) = \pm 1 \text{ or } 0.$$

$F_{p_n}(-1,1)$ is always 1 since we have assumed $p_n = \pm 1$. From this we conclude

$$\Delta_n^{[n-1]}(-1,1) = q(n) \text{ and } \Delta_n^{[n-2]}(-1,1) = 0.$$

Using this in the recursive formula for $\Delta_n^{[n-3]}$ given in equation (5.4), we get

$$\Delta_n^{[n-3]}(-1,1) = -q(n) + q(n-2) + q_{n-1}\Delta_{n-1}^{[n-4]}(-1,1).$$

We manually compute $\Delta_3^{[0]} = 1 - 2q_1q_2$. So this gives the formula

$$\Delta_n^{[n-3]}(-1,1) = \sum_{i=1}^{n-2} \frac{q(n)}{q_i q_{i+1}} - (n-1)q(n).$$

If the above sum is to equal $\pm 1$ (note that it cannot be 0), we must have

$$\sum_{i=1}^{n-2} \frac{1}{q_i q_{i+1}} = n-2,$$

and this can only happen if all the $q_i$ are equal.

Now suppose that $a_{n-2-r,-r} = a_{n-1-r,1-r} = 0$. The constant term of $\frac{\Delta_n^{[n-2]}}{q(n)(-u_1u_2)^{-r}(-u_1)^{n-2}}$ is:

$$(n-1) + \sum_{\substack{2 \le i \le n \\ p_i = 1}} q_{i-1} + \sum_{\substack{1 \le i \le n-1 \\ p_i = -1}} (-q_i), \tag{5.7}$$

which by our assumption must be 0. We can rewrite (5.7) as;

$$(n-1) + \frac{q_{n-1}p_n + q_{n-1}}{2} + \frac{q_1p_1 - q_1}{2} + \sum_{2 \le i \le n-1} \frac{q_{i-1}p_i + q_ip_i + q_{i-1} - q_i}{2}, \tag{5.8}$$

which simplifies to

$$(n-1) + \frac{q_{n-1}p_n + q_1p_1}{2} + \sum_{2 \le i \le n-1} \frac{q_{i-1}p_i + q_ip_i}{2}. \tag{5.9}$$

Note that

$$\frac{q_{n-1}p_n + q_1p_1}{2} + \sum_{2 \le i \le n-1} \frac{q_{i-1}p_i + q_ip_i}{2} \tag{5.10}$$

has a maximum absolute value of $n-1$ which can only happen if all the $q_i$ are equal (and have opposite sign as all the $p_i$).

So we have shown in all cases that all the $q_i$ are equal. This allows us to rewrite equation 5.9 (which is the constant term of $\frac{\Delta_n^{[n-2]}}{q(n)(-u_1u_2)^{-r}(-u_1)^{n-2}}$) as;

$$(n-1)+\sum_{i=2}^{n-1} q_1p_i + q_1\left(\frac{p_1+p_n}{2}\right). \tag{5.11}$$

We must have (5.11) equal to $\pm 1$ or 0. First note that we cannot have $q_1\frac{p_1+p_n}{2}=1$ since $\sum_{i=2}^{n-1} q_1p_i$ is bounded above in absolute value by $n-2$. So we must have that $q_1\frac{p_1+p_n}{2}=-1$ or 0. If $q_1\frac{p_1+p_n}{2}=0$ then $\sum_{i=2}^{n-1} q_1p_i$ must be $-n+2$ which implies that all the $p_i$ for $2\le i\le n-1$ have the opposite sign as $q_1$ and since $q_1\frac{p_1+p_n}{2}=0$ we get that one of $p_1$ and $p_n$ must also have the opposite sign as $q_1$ which proves the claim in this case. If we assume that $q_1\frac{p_1+p_n}{2}=-1$ then we need $\sum_{i=2}^{n-1} q_1p_i \le 3-n$. However $\sum_{i=2}^{n-1} q_1p_i = 3-n$ is impossible since changing the $p_i$ always changes the sum $\sum_{i=2}^{n-1} q_1p_i$ by a multiple of 2. Thus we once again have that $\sum_{i=2}^{n-1} q_1p_i = 2-n$. This along with the fact that $q_1\frac{p_1+p_n}{2}=-1$ implies that all of the $p_i$ have the opposite sign as $q_1$.

**Case 4 ($n=2$ and all the $p_i$ are $\pm 1$)** The only tuples $(p_1,q_1,p_2)$ that do not satisfy the condition given in the Lemma are $(1,1,1)$ and $(-1,-1,-1)$, and we can manually compute $\Delta_2$ in both these cases to check that they do not satisfy that all of the nonzero coefficients are $\pm 1$. In particular for $(1,1,1)$ we have $\Delta_2 = 2-u_1-u_2+2u_1u_2$ and for $(-1,-1,-1)$ we have $\Delta_2 = -\frac{2}{u_1^2u_2^2}+\frac{1}{u_1^2u_2}+\frac{1}{u_1u_2^2}-\frac{2}{u_1u_2}$

□

Now, if an oriented 2-bridge link $L$ is an $L$-space link, it must satisfy the conditions of the Lemma 5.5 by corollary 1.10 and so if $L=D(p_1,q_1,\dots,p_{n-1},q_{n-1},p_n)$, then we have narrowed things down to the following 8 possibilities where $w>0$ is an integer, $q:=2w+1, q':=2w-1$ and $k:=2n-1$.

$$\begin{aligned}
L &= D(-1,1,\dots,-1,1,w) = b(qk-1, q-(qk-1)) && \text{or}\\
&= D(-1,1,\dots,-1,1,-w) = b(q'k+1, q'-(q'k+1)) && \text{or}\\
&= D(1,-1,\dots,1,-1,w) = b(q'k+1, q'k+1-q') && \text{or}\\
&= D(1,-1,\dots,1,-1,-w) = b(qk-1, qk-1-q) && \text{or}\\
&= D(w,-1,1,\dots,-1,1) = b(q'k+1,k) && \text{or}\\
&= D(-w,-1,1,\dots,-1,1) = b(qk-1,-k) && \text{or}\\
&= D(w,1,-1,\dots,1,-1) = b(qk-1,k) && \text{or}\\
&= D(-w,1,-1,\dots,1,-1) = b(q'k+1,-k).
\end{aligned}$$

We can further reduce these 8 possibilities down to 4 by noting $b(qk-1,\pm k) = b(qk-1,\pm(q-(qk-1)))$ which can be seen by rotating the diagram given by 5.2 by 180°, and similarly $b(q'k+1,\pm k) =$

$b(q'k+1, \pm(q'k+1-q'))$. Now we compute the signatures of these four possibilities.

**Lemma 5.6** *When* $q$,$q'$ *and* $k$ *are odd positive integers and* $q \neq 1$ *if* $k = 1$*;*

$$\sigma(b(qk-1, \pm k)) = \pm(q-2) \tag{5.12}$$

$$\sigma(b(q'k+1, \pm k)) = \pm q'. \tag{5.13}$$

**Proof:** First we compute the signature of $b(q'k+1, k)$. Since $\frac{q'k+1}{k} = q' + \frac{1}{k}$, we can use Figure 5.2 to give a diagram $D$ for $b(qk-1, k)$. Now we will use the Gordon-Litherland formula for knot signature(see [4]) on $D$. Since the surface given by a checkerboard coloring of $D$ is orientable, the signature of the link is simply the signature of the Goeritz matrix for $D$ (see the end of the first page in [4]). We denote by $A_n(p)$ the $n \times n$ matrix with $A_{11} = p$, $A_{ii} = 2$ when $2 \leq i \leq n$, $A_{ij} = -1$ when $|j-i| = 1$ and 0 everywhere else. A Goeritz matrix for $D$ is given by $A_q(1+k)$. We claim that if $p > 1$, $A_n(p)$ has signature $n$. This is easy to see inductively; let $B(p) = \begin{pmatrix} 1 & 0 \\ \frac{1}{p} & 1 \end{pmatrix}$, $I_n$ denote the $n \times n$ identity matrix and $B_n(p) = \begin{pmatrix} B(p) & 0 \\ 0 & I_{n-2} \end{pmatrix}$. Then

$$B_n(p)A_n(p)B_n(p)^T = \begin{pmatrix} p & 0 \\ 0 & A_{n-1}(2-1/p) \end{pmatrix}$$

so $\sigma(A_n(p)) = 1 + \sigma(A_n(2-1/p))$ and the claim follows. So the signature of $b(q'k+1, k)$ is $q'$. Since $b(q'k+1, -k)$ is the mirror image of $b(q'k+1, k)$, the signature of $b(q'k+1, -k)$ is $-q'$.
Now we consider $b(qk-1, k)$ where $k > 1$ ($k = 1$ has already been covered above). $\frac{qk-1}{k} = q - \frac{1}{k}$. In this case a Goeritz matrix is $A_q(1-k)$ and

$$B_q(1-k)A_q(1-k)B_q(1-k)^T = \begin{pmatrix} 1-k & 0 \\ 0 & A_{q-1}(2-1/(1-k)) \end{pmatrix}.$$

Now $1-k < 0$ and $2 - 1/(1-k) > 1$, so $\sigma(A_n(1-k)) = -1 + \sigma(A_{q-1}(2-1/(1-k))) = q-2$. Since $b(qk-1, -k)$ is the mirror image of $b(qk-1, k)$, $\sigma(b(qk-1, -k)) = -q+2$ as desired. □

**Proposition 5.7** *If* $L$ *is an* $L$*-space link of the form* $b(qk-1, k) = D(-1, 1, \ldots, -1, 1, w)$ *then* $L = b(2, 1)$

**Proof:** Let us assume that $L = b(qk-1, k)$ is an $L$-space link. Now if $s < n$, it is easy to see by induction that

$$\Delta_s(u_1, u_2) = -\frac{1}{u_1^s u_2^s}\left(\sum_{i=0}^{s-1} u_1^i u_2^{s-1-i}\right).$$

So by equation (5.2) we get

$$\Delta_n(u_1,u_2) = \left((u_1-1)(u_2-1)\left(\sum_{i=0}^{w-1}(u_1u_2)^i\right)+1\right)\left(-\frac{1}{u_1^{n-1}u_2^{n-1}}\left(\sum_{i=0}^{n-2}u_1^iu_2^{n-2-i}\right)\right)+$$
$$\frac{(u_1u_2)^{-1}}{-(u_1u_2)^{-1}}\left(\sum_{i=0}^{w-1}(u_1u_2)^i\right)\left(\left(-\frac{1}{u_1^{n-1}u_2^{n-1}}\left(\sum_{i=0}^{n-2}u_1^iu_2^{n-2-i}\right)\right)-\left(-\frac{1}{u_1^{n-2}u_2^{n-2}}\left(\sum_{i=0}^{n-3}u_1^iu_2^{n-3-i}\right)\right)\right).$$

This simplifies to

$$\Delta_n(u_1,u_2) = \sum_{\substack{0\le i\le w-1\\0\le j\le n-1}} u_1^{i+j+1-n}u_2^{i-j} - \sum_{\substack{0\le i\le w\\0\le j\le n-2}} u_1^{i+j+1-n}u_2^{i-j-1}.$$

Now note that $L = L_1 \sqcup L_2$, where both $L_1$ and $L_2$ are unknots and $\operatorname{lk}(L_1,L_2) = -l_n = -w+n-1$, so we get:

$$P^L_{L_1}(u_2) = (u_2)^{\frac{n-w-1}{2}}\sum_{i=0}^{\infty}(u_2)^{-i} \text{ and } P^L_{L_2}(u_1) = (u_1)^{\frac{n-w-1}{2}}\sum_{i=0}^{\infty}(u_1)^{-i}.$$

Finally, by Theorem 5.2 we also get

$$P^L_\emptyset = \pm(u_1u_2)^{\frac{n-w+1}{2}}\Delta_n(u_1,u_2).$$

Expanding this then gives

$$\pm P^L_\emptyset = (u_1u_2)^{\frac{n-w+1}{2}}\Delta_n(u_1,u_2) = \sum_{\substack{0\le i\le w-1\\0\le j\le n-1}} u_1^{i+j+\frac{3-n-w}{2}}u_2^{i-j+\frac{n-w+1}{2}} - \sum_{\substack{0\le i\le w\\0\le j\le n-2}} u_1^{i+j+\frac{3-n-w}{2}}u_2^{i-j+\frac{n-w-1}{2}}.$$

If $n=1$, we get:

$$\pm P^L_\emptyset = \sum_{0\le i\le w-1} u_1^{i+1\frac{-w}{2}}u_2^{i+1\frac{-w}{2}}.$$

We can then fix the sign for $P^L_\emptyset$ using corollary 1.10 to get

$$P^L_\emptyset = -\sum_{0\le i\le w-1} u_1^{i+1\frac{-w}{2}}u_2^{i+1\frac{-w}{2}}.$$

Then, using the method given in the proof of Theorem 1.5, we can compute $\mathfrak{T}(L)$. In this case $m(L) = (w/2, w/2)$. The edge between $(s_1, w/2-1)$ and $(s_1, w/2)$ is labeled with 0 whenever $s_1 \ge w/2$. Similarly, the edge between $(w/2-1, s_2)$ and $(w/2, s_2)$ is labeled 0 whenever $s_2 \ge w/2$. The coefficient of $u_1^{w/2}u_2^{w/2}$ in $P^L_\emptyset$ is $-1$, which forces both edges from $(w/2-1, w/2-1)$ to be labeled with 1. This along with Lemma 3.10 allows us to compute

$$\widehat{HFL}\left(L, \left(\frac{w}{2}, \frac{w}{2}\right)\right) \cong \mathbb{F}_{(1)}. \tag{5.14}$$

Now, recall that when $L$ is alternating, $\widehat{HFL}(L,\mathbf{s})$ is completely determined by its Euler characteristic and $\sigma(L)$, using Theorem 1.3 in [23]. Specifically, if $\mathbf{s}=(s_1,s_2)$ and $a_{\mathbf{s}}$ is the coefficient of $u^{\mathbf{s}}$ in $(1-u_1^{-1})(1-u_2^{-1})P_\emptyset^L$ then

$$\widehat{HFL}(L,\mathbf{s}) \cong \mathbb{F}^{|a_{\mathbf{s}}|}_{s_1+s_2+\frac{\sigma-1}{2}}.$$

Therefore

$$\widehat{HFL}\left(L,\left(\frac{w}{2},\frac{w}{2}\right)\right) \cong \mathbb{F}_{(2w-1)} \tag{5.15}$$

by Lemma 5.6. Combining equations (5.14) and (5.15) gives $w=1$, which along with $n=1$, gives that $L=b(2,1)$.

If $n\neq 1$, the leading coefficient of $P_\emptyset^L|_{(1,j)}$ and $P_\emptyset^L|_{(1,j+1)}$ have opposite sign iff $j=\frac{w-n+1}{2}$, or in other words there is a sign change in the leading coefficients of $P_\emptyset^L|_{(1,j)}$ at $j=\frac{w-n+1}{2}$. Also note that in $P_{L_2}^L|_{(1,j)}=0$ if $j>\frac{n-w-1}{2}$ and $u_1^j$ otherwise. Combining these facts using corollary 1.10, we must have $w=n-1$. When $w=n-1$ we fix the sign of $P_\emptyset^L$ using corollary 1.10 to get

$$P_\emptyset^L = \sum_{\substack{0\le i\le n-2\\ 0\le j\le n-1}} u_1^{i+j+\frac{3-n-w}{2}} u_2^{i-j+\frac{n-w+1}{2}} - \sum_{\substack{0\le i\le n-1\\ 0\le j\le n-2}} u_1^{i+j+\frac{3-n-w}{2}} u_2^{i-j+\frac{n-w-1}{2}}.$$

We now know enough to compute $\mathfrak{T}(L)$. We will compute the part of $\mathfrak{T}(L)$ inside the region bounded by $s_1+s_2\ge n-2, s_1\ge 0$ and $s_2\ge 0$. This is shown in Figure 5.3. Using this and Lemma 3.10 we compute

$$\widehat{HFL}(L,(1,n-1)) \cong \mathbb{F}_{(1)}. \tag{5.16}$$

Once again, using Theorem 1.3 in [23]: if $\mathbf{s}=(s_1,s_2)$ and $a_{\mathbf{s}}$ is the coefficient of $u^{\mathbf{s}}$ in $(1-u_1^{-1})(1-u_2^{-1})P_\emptyset^L$ then,

$$\widehat{HFL}(L,\mathbf{s}) \cong \mathbb{F}^{|a_{\mathbf{s}}|}_{s_1+s_2+\frac{\sigma-1}{2}}.$$

and therefore

$$\widehat{HFL}(L,(1,n-1)) \cong \mathbb{F}_{(2n-2)}. \tag{5.17}$$

Combining this with equation (5.16) gives a contradiction, since $n$ is an integer. □

**Proposition 5.8** *Suppose $L=b(q'k+1,k)=D(1,-1,\ldots,1,-1,w)$ is an L-space link, then $q'=1$.*

**Proof:** We follow the same proof as the previous proposition. First note that, in this case $\mathrm{lk}(L_1,L_2)=-l_n=-w-n+1$; and so

$$P_{L_1}^L(u_2)=(u_2)^{\frac{-w-n+1}{2}}\sum_{i=0}^{\infty}(u_2)^{-i} \text{ and } P_{L_2}^L(u_1)=(u_1)^{\frac{-w-n+1}{2}}\sum_{i=0}^{\infty}(u_1)^{-i}.$$

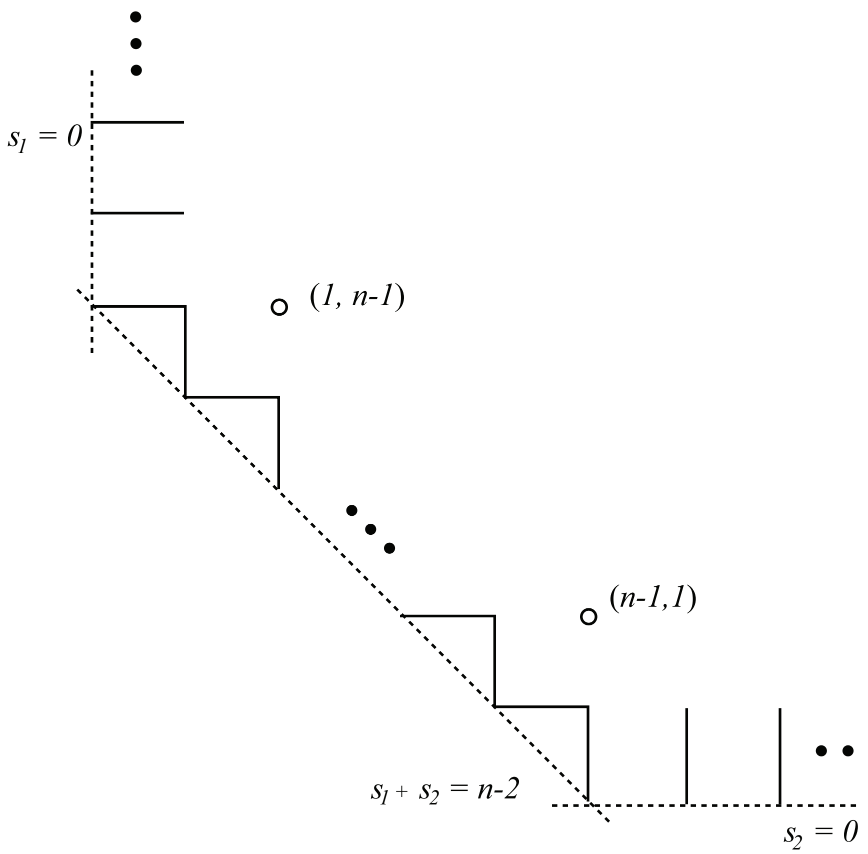



Figure 5.3: Part of $\mathfrak{T}(L)$, for $b(k^2-1,k)$ assuming it is an $L$-space link. Edges labeled with 1 are drawn in black and edges labeled with 0 are not shown.

We can compute

$$\Delta_n(u_1,u_2) = \sum_{\substack{0\le i\le w-1\\0\le j\le n-1}} u_1^{i+j}u_2^{i-j+n-1} - \sum_{\substack{1\le i\le w-1\\0\le j\le n-2}} u_1^{i+j}u_2^{i-j+n-2},$$

which gives

$$P^L_\emptyset = -\sum_{\substack{0\le i\le w-1\\0\le j\le n-1}} u_1^{i+j+\frac{-w-n+3}{2}}u_2^{i-j+\frac{-w+n+1}{2}} + \sum_{\substack{1\le i\le w-1\\0\le j\le n-2}} u_1^{i+j+\frac{-w-n+3}{2}}u_2^{i-j+\frac{-w+n-1}{2}},$$

where the signs are fixed by corollary 1.10. Using this, we compute $\mathfrak{T}(L)$ inside the region bounded by $s_1+s_2 \ge w-2$,$s_1 \ge \frac{w-n+1}{2}$ and $s_2 \ge \frac{w-n+1}{2}$ and it is shown in Figure 5.4.

$$\widehat{HFL}\left(L,\left(\frac{w-n+1}{2},\frac{w+n-1}{2}\right)\right) \cong \mathbb{F}_{(1)} \tag{5.18}$$

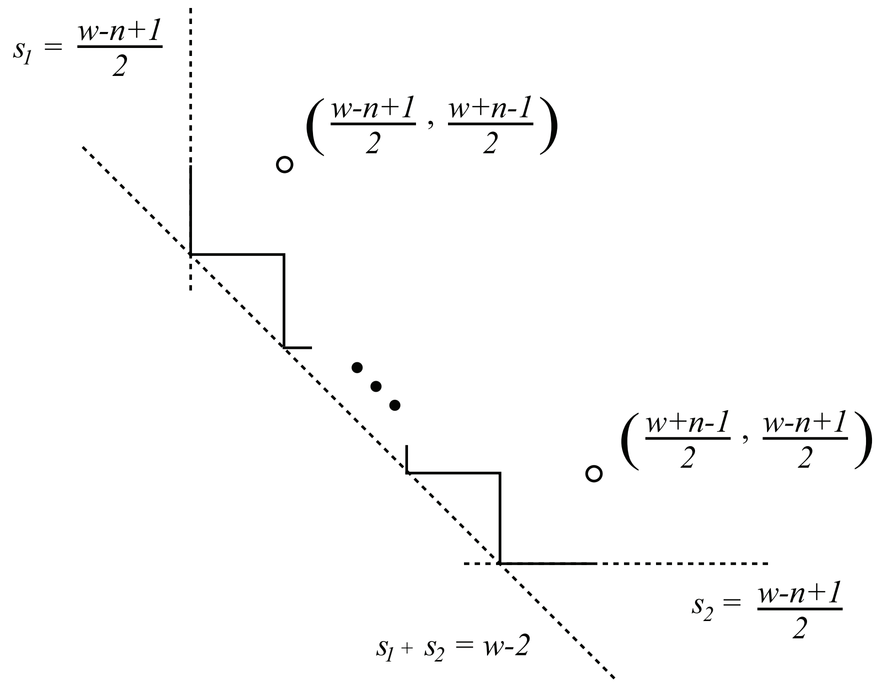



Figure 5.4: Part of $\mathfrak{T}(L)$ for $b(q'k+1,k)$, assuming it is an $L$-space link.

We can do this computation again using the fact that $L$ is alternating and to get

$$\widehat{HFL}\left(L,\left(\frac{w-n+1}{2},\frac{w+n-1}{2}\right)\right)\cong\mathbb{F}_{(2w-1)}. \tag{5.19}$$

combining equation 5.18 and equation 5.19 then gives $w=1$ which implies $q'=1$ as desired $\square$

**Proposition 5.9** *If $L=b(q'k+1,-k)=D(-1,1,\ldots,-1,1,-w)$ is an $L$-space link, then $k=1$.*

**Proof:** Here $\mathrm{lk}(L_1,L_2)=-l_n=w+n-1$, and so

$$P^L_{L_1}(u_2)=(u_2)^{\frac{w+n-1}{2}}\sum_{i=0}^{\infty}(u_2)^{-i} \text{ and } P^L_{L_2}(u_1)=(u_1)^{\frac{w+n-1}{2}}\sum_{i=0}^{\infty}(u_1)^{-i}$$

and

$$P^L_{\emptyset}=-\sum_{\substack{1-w\le i\le -1\\ 0\le j\le n-2}}u_1^{i+j+\frac{w-n+3}{2}}u_2^{i-j+\frac{w+n-1}{2}}+\sum_{\substack{-w\le i\le -1\\ 0\le j\le n-1}}u_1^{i+j+\frac{w-n+3}{2}}u_2^{i-j+\frac{w+n+1}{2}}$$

where we have fixed signs for $P^L_{\emptyset}$, as in the previous two propositions using corollary 1.10. Note that both edges going to $\left(\frac{w+n-1}{2},\frac{w+n-1}{2}\right)$ must be labeled with 1 because they are determined by $P^L_{L_i}$ since $m(L)=\left(\frac{w+n-1}{2},\frac{w+n-1}{2}\right)$. Also notice that when $n>1$, the point $\left(\frac{w+n-1}{2},\frac{w+n-1}{2}\right)$ is outside of the Newton polytope for $P^L_{\emptyset}$. Thus both edges from $\left(\frac{w+n-3}{2},\frac{w+n-3}{2}\right)$ are also labeled with 1. So we get

$$\widehat{HFL}\left(L, \left(\frac{w+n-1}{2}, \frac{w+n-1}{2}\right)\right) \cong \mathbb{F}_{(0)} \oplus \mathbb{F}_{(-1)},$$

which is a contradiction because for an alternating link $L$ we know $\widehat{HFL}(L, \mathbf{s})$ is only supported in one degree. Thus, we must have $n = 1$, which forces $k = 1$ as well. □

**Proof:** [proof of Theorem 1.11] Combining the previous three propositions (also Lemma 5.5) shows that, if $b(\alpha, \beta)$ is an $L$-space link, then it is either $b(qk-1, -k)$ for $q$ and $k$ odd positive integers, or of the form $b(k+1, k)$ where $k$ is odd. Note that reversing the orientation of one of the components of $b(k+1, k)$ gives $b(k+1, -1)$, which proves the Theorem. □

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