On the Construction of Higher étale Regulators

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Abstract

We present three approaches to define the higher étale regulator maps $\Phi_{et}^{r,n} : H_{et}^r(X, \mathbb{Z}(n)) \to H_{\mathcal{D}}^r(X, \mathbb{Z}(n))$ for regular arithmetic schemes. The first two approaches construct the maps on the cohomology level, while the third construction provides a morphism of complexes of sheaves on the étale site, along with a technical twist that one needs to replace the Deligne-Beilinson cohomology by the analytic Deligne cohomology inspired by the work of Kerr, Lewis, and Müller-Stach. A vanishing statement of infinite divisible torsions under $\Phi_{et}^{r,n}$ is established for $r > 2n + 1$.

Contents

Chapter 1

Introduction

Regulator maps have been central tools in the study of special values of Zeta functions. The idea grows out of the classical analytic class number formula that predicts the special value of the Dedekind Zeta function of a number field via the Dirichlet regulator map at $s = 1$. Beilinson [2] generalizes this idea using higher K-theory formulating the celebrated Beilinson conjectures. Bloch [4] later rephased the higher regulator maps in terms of his higher Chow groups. This paper is devoted to the study of the higher étale regulator maps using integral étale motivic cohomology in place of the classical (Zariski) motivic cohomology for regular arithmetic schemes. It is known that the motivic cohomology and the étale motivic cohomology share the same rational structure, which therefore are natural candidates for describing the special values of Zeta functions. Note that the construction of the higher étale regulator maps on the level of complexes permits one to define Arakelov étale motivic cohomology as the hypercohomology of its mapping fibre. This makes the extension of the definition of the Weil-étale motivic complexes describing values of Zeta functions of proper regular arithmetic schemes from $s = 0$ discussed in [9] and [19] to arbitrary $n \in \mathbb{Z}$ possible. Moreover, one can use it to construct a canonical class in the class field theory of finitely generated fields, and this provides a natural candidate for the definition of the Weil group of finitely generated fields. The one-dimensional case has been studied in Burns-Flach [6].

The main results in this paper are Theorem 3.2.3 showing the equivalence of the first two constructions of the higher étale regulator maps, Theorem 3.4.1 describing the vanishing of the higher infinite torsions under the étale regulator maps, and the existence of a morphism of complexes of étale sheaves inducing a higher étale regulator map.

The layout of the paper is as follows. Chapter 2 will be devoted to review various background materials involved in the constructions of higher étale regulators. Three different constructions will then be presented in Chapter 3; among them, we will show that the first two constructions yield the same map, while the third one is structurally different from the first two, as it maps into the analytic Deligne cohomology instead of the Deligne-Beilinson cohomology. We will end our discussion with an example in the case of regular arithmetic toric schemes in the last section.

Chapter 2

Preliminaries

This chapter will be devoted to the background materials to be used in subsequent chapters. The main purpose is to fix the notations we used throughout the paper, and thus without further notification, all notations that appear in this chapter will carry over to the end of the paper.

2.1 Simplicial Homotopy

In this section, we are going to review and fix some notations about simplicial homotopy theory. Most of the results are well known and can be found readily in the literature. However, for the sake of convenience, we will treat them systematically here.

2.1.1 Basic definitions

Let $\mathbf{\Delta}$ be the category of finite ordinal numbers with order preserving maps, consisting of the non-empty finite totally ordered sets

$$[n] = \{0 \leq 1 \leq \cdots \leq n\}$$

as objects for all non-negative integers n, and morphisms being maps

$$\theta : [m] \to [n]$$

such that $\theta(i) \leq \theta(j)$ whenever $i \leq j$.

If we consider the totally ordered sets as small categories in the standard way, $\mathbf{\Delta}$ can be thought of as a full subcategory of the category **Cat** of small categories. In particular, morphisms in $\mathbf{\Delta}$ are generated by the coface morphisms

$$\delta_n^i : [n-1] \to [n] \qquad 0 \leq i \leq n$$
$$j \mapsto \begin{cases} j & , \text{if } j < i, \\ j+1 & , \text{if } j \geq i, \end{cases}$$

and the codegeneracy morphisms

$$\sigma_n^i : [n+1] \to [n] \qquad 0 \leq i \leq n$$
$$j \mapsto \begin{cases} j & , \text{if } j \leq i, \\ j-1 & , \text{if } j > i, \end{cases}$$

subjecting to the following cosimplicial identities:

$$\begin{cases} \delta_{n+1}^j \delta_n^i = \delta_{n+1}^i \delta_n^{j-1} & , \text{for } 0 \leq i < j \leq n+1, \\ \sigma_n^j \delta_{n+1}^i = \delta_n^i \sigma_{n-1}^{j-1} & , \text{for } 0 \leq i < j \leq n, \\ \sigma_n^j \delta_{n+1}^j = 1_{[n]} = \sigma_n^j \delta_{n+1}^{j+1} & , \text{for } 0 \leq j \leq n, \\ \sigma_n^j \delta_{n+1}^i = \delta_n^{i-1} \sigma_{n-1}^j & , \text{for } 0 < j+1 < i \leq n+1, \\ \sigma_{n-1}^j \sigma_n^i = \sigma_{n-1}^i \sigma_n^{j+1} & , \text{for } 0 \leq i \leq j \leq n-1. \end{cases} \tag{2.1.1}$$

Let $\mathcal{C}$ be a small category. A simplicial object in $\mathcal{C}$ is functor $X : \mathbf{\Delta}^{op} \to \mathcal{C}$. Concretely it consists of objects $X_n := X([n])$ for each $n \geq 0$, and face and degeneracy morphisms

$$d_i^n = X\left(\delta_n^i\right) : X_n \to X_{n-1}$$

$$s_i^n = X\left(\sigma_n^i\right) : X_n \to X_{n+1}$$

for each $0 \leq i \leq n$ satisfying the corresponding simplicial identities

$$\begin{cases} d_i^n d_j^{n+1} = d_{j-1}^n d_i^{n+1} & \text{, for } 0 \leq i < j \leq n+1, \\ d_i^{n+1} s_j^n = s_{j-1}^{n-1} d_i^n & \text{, for } 0 \leq i < j \leq n, \\ d_j^{n+1} s_j^n = 1_{X_n} = d_{j+1}^{n+1} s_j^n & \text{, for } 0 \leq j \leq n, \\ d_i^{n+1} s_j^n = s_j^{n-1} d_{i-1}^n & \text{, for } 0 < j+1 < i \leq n+1, \\ s_i^n s_j^{n-1} = s_{j+1}^n s_i^{n-1} & \text{, for } 0 \leq i \leq j \leq n-1. \end{cases} \tag{2.1.2}$$

Given two simplicial objects X, Y in $\mathcal{C}$, a simplicial map $f : X \to Y$ is a natural transformation of the functors, i.e., given by morphisms

$$f_n : X_n \to Y_n, \qquad n \geq 0,$$

such that

$$\begin{cases} d_i^n f_n & = f_{n-1} d_i^n \\ s_i^{n-1} f_{n-1} & = f_n s_i^{n-1} \end{cases} \tag{2.1.3}$$

for all $n > 0, 0 \leq i \leq n$.

Simplicial objects in $\mathcal{C}$ together with simplicial maps between them form a category called the simplicial category of $\mathcal{C}$, denoted by $\mathbf{Simp}(\mathcal{C})$. When $\mathcal{C}$ is taken to be the category **Sets** of sets, it is called the category of simplicial sets, denoted simply by $\mathcal{S}$. For a simplicial set X, we call the elements of X_n n-simplexes of X, while 0-simplexes are commonly called vertices, 1-simplexes are called edges, and so on.

There are distinguished objects in $\mathcal{S}$, namely the standard n-simplex, $\Delta^n := Hom_{\mathbf{\Delta}}(\_, [n])$, which is the contravariant functor represented by $[n]$ in $\mathbf{\Delta}$. By the Yoneda lemma, for any simplicial set $Y : \mathbf{\Delta}^{op} \to \mathbf{Sets}$,

$$Hom_{\mathcal{S}}(\Delta^n, Y) \cong Y([n]) = Y_n,$$

$$f \mapsto f(1_{[n]}),$$

therefore simplicial maps from Δ^n classify n-simplices of simplicial sets. There are some important subobjects of Δ^n of particular interest to us:

(i) the boundary of Δ^n, $\partial\Delta^n$, defined by

$$\partial\Delta^n{}_m = \begin{cases} \Delta^n{}_m & \text{, if } 0 \le m \le n-1, \\ \text{all } m\text{-simplices degenerate} & \text{, if } m \ge n; \end{cases}$$

(ii) the k-th horn of Δ^n, Λ^n_k,for $0 \le k \le n$, defined by

$$(\Lambda^n_k)_m = \begin{cases} \Delta^n{}_m & \text{, if } 0 \le m < n-1, \\ \Delta^n{}_{n-1} - \{d^n_k(1_{[n]})\} & \text{, if } m = n-1, \\ \text{all } m\text{-simplices degenerate} & \text{, if } m \ge n. \end{cases}$$

A simplicial map $p : X \to Y$ in $\mathcal{S}$ is called a Kan fibration, or simply fibration, if for every commutative diagram

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & X \\ \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & Y \end{array}$$

of simplicial maps, there is a morphism $\Delta^n \to X$ such that the following diagram commutes

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & X \\ \downarrow & \overset{\theta}{\nearrow} & \downarrow \\ \Delta^n & \longrightarrow & Y \end{array} .$$

Let $* := \Delta^0$ be the unique simplicial set which has exactly one simplex for each degree with identity face and degeneracy maps. Clearly, $*$ is the terminal object in $\mathcal{S}$, and that every simplicial set X admits a unique simplicial map $X \to *$. A simplicial set X is said to be a Kan complex, or a fibrant simplicial set, if the unique map to $*$ is a Kan fibration.

2.1.2 Simplicial schemes

A simplicial scheme is a simplicial object in the category of schemes. To be more precise, we will consider the category $\mathcal{V}_k^{sm}$ of smooth separated schemes of finite type over a base field k, and its simplicial objects are called smooth simplicial schemes over k.

2.1.3 Skeleta and Coskeleta

On a simplicial category $\mathbf{Simp}(\mathcal{C})$, there are two important concepts called the skeleta and coskeleta of simplicial objects. They form a natural adjoint pair of functors on $\mathbf{Simp}(\mathcal{C})$, and we will need them to define hypercoverings in the next section.

Definition 2.1.1. *Let $\mathcal{C}$ be a category with finite limits and colimits. Define the n-skeleton functor*

$$sk_n : \mathbf{Simp}(\mathcal{C}) \to \mathbf{Simp}(\mathcal{C})$$

and the n-coskeleton functor

$$cosk_n : \mathbf{Simp}(\mathcal{C}) \to \mathbf{Simp}(\mathcal{C})$$

by setting, for each simplicial object $X \in \mathbf{Simp}(\mathcal{C})$,

$$(sk_n X)_m = \varinjlim_{\substack{k \leq n \\ [m] \to [k]}} X_k$$

$$(cosk_n X)_m = \varprojlim_{\substack{k \leq n \\ [k] \to [m]}} X_k$$

endowed with natural structural morphisms.

2.1.4 Coverings and Hypercoverings on sites

We start with recalling the definition of Grothendieck sites, and then we will define the notion of hypercoverings.

Definition 2.1.2. *A Grothendieck site is a (small) category $\mathcal{C}$ together with a Grothendieck topology $\mathcal{J}$ defined on $\mathcal{C}$, consisting of families $\mathcal{J}(U)$ of subfunctors $R \subset Hom_{\mathcal{C}}(\_, U)$ for each object U of $\mathcal{C}$, called covering sieves, satisfying the following axioms:*

(T1) If $s \in \mathcal{J}(U)$ and $f : V \to U$ a morphism in $\mathcal{C}$, then $f^*S \in \mathcal{J}(V)$. Here $f^*S(W) = \{g \in Hom_{\mathcal{C}}(W, V) | fg \in S(W)\}$ for all objects W in $\mathcal{C}$.

(T2) Let $S \in \mathcal{J}(U)$ and $T \subset Hom_{\mathcal{C}}(\_, U)$ any sunfunctor (sieve). Suppose that for each object $V \in \mathcal{C}$, and each $f \in S(V)$, the pullback sieve $f^*T \in \mathcal{J}(V)$. Then $T \in \mathcal{J}(U)$.

(T3) $Hom_{\mathcal{C}}(\_, U) \in \mathcal{J}(U)$.

In most useful cases, a Grothendieck topology can be generated by a Grothendieck pretopology $\mathcal{P}$ consisting of a collection of families of arrows $\mathcal{P}(U)$ mapping to U for each object U of $\mathcal{C}$, called covering families, satisfying the following axioms:

(PT0) For all $U \in \mathcal{C}$, $\{U_\alpha \to U\}_{\alpha \in I} \in \mathcal{P}(U)$, $(V \to U) \in Hom_{\mathcal{C}}(V, U)$, the fibre products $U_\alpha \times_U V$ exist for all $\alpha \in I$.

(PT1) For all $U \in \mathcal{C}$, $\{U_\alpha \to U\}_{\alpha \in I} \in \mathcal{P}(U)$, $(V \to U) \in Hom_{\mathcal{C}}(V, U)$, we have $\{U_\alpha \times_U V \to V\}_{\alpha \in I} \in \mathcal{P}(V)$.

(PT2) If $\{U_\alpha \to U\}_{\alpha \in I} \in \mathcal{P}(U)$, and if $\{U_{\alpha\beta} \to U_\alpha\}_{\beta \in I_\alpha} \in \mathcal{P}(U_\alpha)$ for each $\alpha \in I$, then $\{U_{\alpha\beta} \to U_\alpha \to U\}_{\alpha \in I, \beta \in I_\alpha} \in \mathcal{P}(U)$.

(PT3) If $(f : V \to U) \in Hom_{\mathcal{C}}(V, U)$ is an isomorphism, $\{f\} \in \mathcal{P}(U)$.

Given a pretopology $\mathcal{P}$, one defines a topology $\mathcal{J}$ by setting $\mathcal{J}(U) = \{R \subset Hom_{\mathcal{C}}(\_, U) | \{f_\alpha : U_\alpha \to U\} \in \mathcal{P}(U), f_\alpha \in R(U_\alpha)\}$. For categories with fibre products, every Grothendieck topology on it is generated by some (not necessarily unique) pretopology.

Let $\mathcal{C}$ be a Grothendieck site, and X be an object in $\mathcal{C}$. We call a simplicial object $X_\bullet \in$ **Simp**$(\mathcal{C}/X)$ a hypercovering of X if it satisfies the following:

(i) $X_0 \to X$ is a covering;

(ii) $X_{n+1} \to (cosk_n X_\bullet)_{n+1}$ is a covering for $n \geq 0$.

A typical example is given by the Čech covering. Let $\mathcal{C}$ be a site with finite products. Take a covering $U \to X$ of X. Then the associated Čech covering $\check{C}(U)_\bullet$ is given by $\check{C}(U)_n = U \times_X \cdots \times_X U$ (n copies) with natural projections and diagonals as structure morphisms. In this case $\check{C}(U)_{n+1} = (cosk_n \check{C}(U)_\bullet)_{n+1}$ and thus $\check{C}(U)_\bullet$ is a hypercovering.

The most important property concerning hypercoverings is the Verdier hypercovering theorem:

Proposition 2.1.1 ([1]). *Let $\mathcal{C}$ be a Grothendieck site, and $X \in \mathcal{C}$. Then for any complex $\mathcal{F}^\bullet$ of sheaves of abelian groups on $\mathcal{C}$, we have*

$$\varinjlim_{X_\bullet \in HC(X)} \mathbb{H}^r(X_\bullet, \mathcal{F}^\bullet) \simeq \mathbb{H}^r(X, \mathcal{F}^\bullet).$$

2.2 Deligne Cohomology

2.2.1 Analytic Deligne complex

Let X be a complex analytic manifold. The analytic Deligne cohomology is defined as the hypercohomology

$$H^r_{\mathcal{D},an}(X, \mathbb{Z}(n)) = \mathbb{H}^r(X_{an}, \mathbb{Z}(n)_{\mathcal{D}}),$$

where $\mathbb{Z}(n)_{\mathcal{D}} = \left\{0 \to \mathbb{Z}(n) \to \Omega^0_X \to \cdots \to \Omega^{n-1}_X \to 0\right\}$, $\mathbb{Z}(n) = (2\pi i)^n \mathbb{Z}$.

In case X is a complex algebraic manifold, we can use the algebraic de Rham complex in place of the holomorphic de Rham complex to define the Deligne cohomology via hypercohomology on the Zariski site. However, when X is not proper, the analytic Deligne cohomology groups are far from being finitely generated, which is in general not preferable for arithmetic consideration. To fix this issue, one defines an algebraic version of this Deligne cohomology via good compactifications.

2.2.2 Deligne-Beilinson complex via good compactifications

Our treatment follows closely to [8]. Let X be a complex algebraic manifold. A proper complex algebraic manifold $\overline{X}$ is called a good compactification of X if it contains X as an open submanifold and if the complement $D = \overline{X} - X$ is a simple normal crossing divisor on $\overline{X}$. Such a choice of good compactification enables us to consider the sheaves $\Omega^\bullet_{\overline{X}}(\log D)$ of meromorphic differentials on $\overline{X}$ with logarithmic poles along D. It is well-known that the complex $\Omega^\bullet_{\overline{X}}(\log D)$ has a natural Hodge filtration and that its hypercohomology computes the ordinary cohomology of X with $\mathbb{C}$ coefficients. Let $j : X \hookrightarrow \overline{X}$ be the open embedding. Set

$$\mathbb{Z}(n)_{\mathcal{D},\overline{X}} = Cone(Rj_*\mathbb{Z}(n) \oplus F^n\Omega^\bullet_{\overline{X}}(\log D) \overset{\varepsilon-\iota}{\to} Rj_*\Omega^\bullet_X)[-1],$$

and the Deligne-Beilinson cohomology of X is defined as

$$H^r_{\mathcal{D}}(X,\mathbb{Z}(n)) = \mathbb{H}^r(\overline{X},\mathbb{Z}(n)_{\mathcal{D},\overline{X}}).$$

More generally, for a regular scheme $\mathcal{X}$ defined over $S = \mathrm{Spec}A$ for some subring $A \subset \mathbb{C}$, define

$$H^r_{\mathcal{D}}(\mathcal{X},\mathbb{Z}(n)) = H^r_{\mathcal{D}}(\mathcal{X}_{\mathbb{C}},\mathbb{Z}(n)).$$

It is known that the above definition is independent of the choice of good compactification $\overline{X}$ of X, and it is possible to define it as hypercohomology of complexes of sheaves on either the Zariski or étale site of X. More precisely, we can consider the site Π_τ of pairs $(\overline{X},X)$ of smooth complex varieties, where $\overline{X}$ is complete and contains X as an open subvariety, both endowed with Grothendieck topology τ. Here τ can be either the Zariski or étale topology. If $\mathcal{V}_\tau$ denotes the site of smooth complex varieties with topology τ, then there is a natural projection $\sigma : \Pi_\tau \to \mathcal{V}_\tau$. We can set

$$\mathbb{Z}(n)_{\mathcal{D},\tau} = \varinjlim_{(\overline{X},X)\in\sigma^{-1}(X)} R\sigma_*(F^n\Omega^{\cdot}_{\overline{X}}(\log D), Rj_*Cone(\mathbb{Z}(n) \xrightarrow{\varepsilon} \Omega^{\cdot}_X), -\iota).$$

so that

$$H^r_{\mathcal{D}}(X,\mathbb{Z}(n)) = \mathbb{H}^r(X_\tau,\mathbb{Z}(n)_{\mathcal{D},\tau}).$$

The Deligne-Beilinson cohomology satisfies some nice properties, which we summarize below:

Proposition 2.2.1 ([8]). *Let X be a complex algebraic manifold. Then*

(i) $H^r_{\mathcal{D}}(X,\mathbb{Z}(n)) = 0$ *for* $r \leq 0$ *and* $n \geq 1$*;*

(ii) $H^r_{\mathcal{D}}(X,\mathbb{Z}(0)) = H^r(X,\mathbb{Z})$*;*

(iii) $H^1_{\mathcal{D}}(X,\mathbb{Z}(1)) = \Gamma(X_{Zar},\mathcal{O}^*_X)$*;*

(iv) there is a long exact sequence

$$\cdots \to H^r_{\mathcal{D}}(X, \mathbb{Z}(n)) \to H^r(X, \mathbb{Z}(n)) \to H^r(X, \mathbb{C})/F^n H^r(X, \mathbb{C}) \to H^{r+1}_{\mathcal{D}}(X, \mathbb{Z}(n)) \to \cdots$$

(v) there is a natural homomorphism

$$H^r_{\mathcal{D}}(X, \mathbb{Z}(n)) \to H^r_{\mathcal{D},an}(X, \mathbb{Z}(n)),$$

which is an isomorphism when X is proper, or when $n > \dim X$.

2.2.3 Deligne complex via currents

There is a theory of Deligne homology defined using currents [16] for smooth complex projective varieties, and here we adopt a slightly extended version using Borel-Moore homology. First we recall some facts about Borel-Moore homology. For a (complex) manifold M with locally finite triangulation T we let $C^{BM}_\bullet(M, T, \mathbb{Z}(p))$ be the complex of Borel-Moore chains (subordinate to T). So the group in degree i is the set of (infinite) formal linear combinations $\xi = \sum_\sigma n_\sigma \cdot \sigma$ of oriented i-simplices σ in the triangulation T with coefficients $n_\sigma \in \mathbb{Z}(p)$, and whose support $|\xi| = \bigcup_{n_\sigma \neq 0} |\sigma|$ is closed in M. We let

$$C^{BM}_\bullet(M, \mathbb{Z}(p)) = \varinjlim_T C^{BM}_\bullet(M, T, \mathbb{Z}(p))$$

be the limit over all triangulations T. For an open subset $U \subset M$ there is a restriction map

$$C^{BM}_\bullet(M, \mathbb{Z}(p)) \to C^{BM}_\bullet(U, \mathbb{Z}(p))$$

since U will have a triangulation T' each of whose simplices is contained in a unique simplex of T. So $U \mapsto C^{BM}_\bullet(U, \mathbb{Z}(p))$ is a complex of presheaves that one can moreover show to consist of *soft sheaves.* This implies that its hypercohomology coincides with its cohomology. It is easy to compute the cohomology sheaves since every point of M has a fundamental system of neighborhoods homeomorphic to $\mathbb{R}^d$ for $d = \dim M$ (assumed constant), and the Borel Moore homology of $\mathbb{R}^d$ is $\mathbb{Z}$ in degree d (with canonical generator the fundamental class of $\mathbb{R}^d$) and 0 in all other degrees. So one has a quasi-isomorphism of complexes of sheaves

$$\mathbb{Z}(p)[d] \to C_\bullet^{BM}(-, \mathbb{Z}(p)).$$

Finally there is a quasi-isomorphism

$$C_\bullet^{BM}(M, \mathbb{Z}(p)) \to C_\bullet^{sing,BM}(M, \mathbb{Z}(p))$$

to the complex of locally finite *singular* C^∞-chains with coefficients in $\mathbb{Z}(p)$. All of these facts we just quote from [12][2.4].

Now take M to be a complex manifold X of complex dimension m. ${}'\mathcal{D}^p = {}'\mathcal{D}^p_X$ denotes the sheaf of holomorphic p-currents on X, which is by definition the functional dual of the sheaf ${}^c\Omega^p_X$ of holomorphic p-forms on X with compact support. Every Borel-Moore chain $\xi \in C_i^{BM}(U, \mathbb{Z}(p))$ of dimension i on an open $U \subset X$ gives a current

$$\delta_\xi \in {}'\mathcal{D}^{2m-i}(U) = \bigoplus_{p+q=i} {}'\mathcal{D}^{m-p,m-q}(U)$$

by the formula

$$\omega \mapsto (2\pi i)^p \int_\xi \omega \quad ,$$

which converges since ω has compact support. This gives a morphism of complexes of sheaves

$$\varepsilon : C_{2m-\bullet}^{BM}(U, \mathbb{Z}(p)) \to {}'\mathcal{D}^\bullet$$

for the manifold topology on X. Recall that one also has the Hodge filtration

$$F^p\,{}'\mathcal{D}^\bullet \to {}'\mathcal{D}^\bullet$$

and hence one can define the analytic Deligne complex as the mapping fibre

$$\mathbb{Z}(p-m)_{\mathcal{D}}^{\bullet-2m} = Cone\left(C^{BM}_{2m-\bullet}(-,\mathbb{Z}(p)) \oplus F^{p\prime}\mathcal{D}^{\bullet} \xrightarrow{\varepsilon-\iota} {}^{\prime}\mathcal{D}^{\bullet}\right)[-1](-m).$$

The complex of global sections $\mathbb{Z}(p-m)_{\mathcal{D}}^{\bullet-2m}(X)$ agrees with the complex $\mathcal{C}_{\mathcal{D}}^{\bullet-2m}(X,\mathbb{Z}(p-m))$ defined in [18][5.5]. The groups

$$H_i^{\mathcal{D}}(X,\mathbb{Z}(p)) := H^{-i}(X,\mathbb{Z}(-p)_{\mathcal{D}})$$

deserve to be called the analytic Deligne (Borel-Moore) homology groups of X. Here we can take either cohomology or hypercohomology for the manifold topology, and the two agree since all terms in the Deligne complex are soft sheaves and soft sheaves are acyclic on each open. Note that there is a Poincare duality between the analytic Deligne homology and the analytic Deligne cohomology, namely

$$H^r_{\mathcal{D},an}(X,\mathbb{Z}(n)) \xrightarrow{\cong} H^{\mathcal{D}}_{2m-r}(X,\mathbb{Z}(m-n)).$$

2.3 Higher Chow complex

Higher Chow groups were first introduced in Bloch [3]. They have been shown to be equivalent to the more sophisticated version of the motivic cohomology via Voevodsky's DM construction in the case of equidimensional smooth schemes of finite type over a field k. The advantage of higher Chow groups is that their construction is based on the Bloch's higher cycle complex, from which elements can be represented by explicit algebraic cycles. They are also currently the only definition of motivic cohomology for arithmetic schemes. In this section, we will recall its definition together with some of its important properties. Among them, one of the most important features is that the higher cycle complex satisfies the Zariski descent property, which allows us to reinterpret them via hypercohomologies over the Zariski sites. We will see in the next section that this also gives us a way to define the etale higher Chow groups analogously.

2.3.1 Bloch's higher Chow groups and their functorial properties

To begin with, we have the definitions of the higher Chow complexes and higher Chow groups.

Definition 2.3.1. *Let X be a equidimensional scheme of finite type over a base scheme $S = SpecR$. Here R can be a field or a Dedekind domain. Define the standard algebraic n-simplex over S to be $\Delta^n_S := Spec\,(R[t_0, \ldots, t_n]/(\sum_{i=0}^n t_i - 1))$. The usual boundary and degeneracy maps between the Δ^n_k gives a cosimplicial scheme $\Delta^\cdot_S$. Set $Z^r(X, n)$ to be the free abelian group generated by subvarieties of $X \times \Delta^n_S$ of codimension r that intersect properly with the image of the faces under the boundary maps. Then the cosimplicial structure on $X \times \Delta^\cdot_S$ induces a simplicial structure on $Z^r(X, \cdot)$. From this we obtain a chain complex of abelian groups via the Dold-Kan equivalence. By reindexing it in negative degrees, we defined a cochain complex $\mathbf{Z}^r_X$, called the higher Chow complex of X, namely $\mathbf{Z}^r_X = Z^r(X, -\cdot)$.*

Definition 2.3.2. *Let X be the same as in Definition 2.3.1. The higher Chow group of X, $CH^r(X, n)$, is defined as the $(-n)$-th cohomology of $\mathbf{Z}^r_X$.*

Remark 2.3.1. *Alternatively, one can define the higher cycle complex using cubical structure instead of simplicial structure, and the resulting cochain complex will be quasi-isomorphic to the one in Definition 2.3.1, thus inducing the same cohomology groups. Concretely, set $\square^n_k = \left(\mathbb{P}^1_k \backslash \{1\}\right)^n$ and let $C^r(X, n)$ be the free abelian group generated X and subvarieties of $X \times \square^n_k$ of codimension r that intersect all subfaces properly, and $D^r(X, n)$ be the subgroup of $C^p(X, n)$ generated by those degenerated subvarieties; then we can form $Z^r_\square(X, n) = C^r(X, n)/D^r(X, n)$ together with natural boundary maps induced from the cubical structure of $\square^\cdot$. We then have $CH^r(X, n) = H_n(Z^r(X, \cdot)) = H_n(Z^r_\square(X, \cdot))$. Moreover, when $k = \mathbb{C}$, we can further reduce $Z^r_\square(X, n)$ to a subcomplex*

$$Z^r_{\mathbb{R}}(X, n) = C^r_{\mathbb{R}}(X, n)/(C^r_{\mathbb{R}}(X, n) \cap D^r(X, n))$$

by setting

$$C^r_{\mathbb{R}}(X,n) = \{Z \in C^r(X,n) | Z \text{ intersects properly with } X \times (T_{z_1} \cap \cdots \cap T_{z_j}),$$
$$X \times \{(T_{z_1} \cap \cdots \cap T_{z_j}) \cap \partial^k \square^n\}, 1 \le j \le n, 1 \le k \le n\} \quad ,$$

where $z_j : \square^n \to \mathbb{P}^1$ *is the* j*th coordinate function and* $T_{z_j} = z_j^{-1}(\mathbb{R}^-)$*. The natural inclusion* $Z^r_{\mathbb{R}}(X,\cdot) \to Z^r_{\square}(X,\cdot)$ *is then a quasi-isomorphism.*

Now we will list the main properties of the higher Chow complexes and the higher Chow groups as listed in [4].

Proposition 2.3.2. *Let* X *be the same as in Definition 2.3.1.*

(i) $CH^r(X,n)$ *is covariantly functorial for proper maps, and contravariantly functorial for flat maps.*

(ii) Let $j : Y \hookrightarrow X$ *be a closed immersion of a closed subscheme of pure codimension* e*, and* $i : U = X - Y \hookrightarrow X$ *be the complementary open embedding. Then there is a localization sequence*

$$\cdots \to CH^r(U,n+1) \to CH^{r-e}(Y,n) \to CH^r(X,n) \to CH^r(U,n) \to \cdots$$
$$\to CH^r(U,1) \to CH^{r-e}(Y,0) \to CH^r(X,0) \to CH^r(U,0) \to 0.$$

(iii) $CH^r(X,0) = CH^r(X)$*, the ordinary Chow group of* X*.*

(iv) $CH^r(X,n) = \mathbb{H}^{-n}(X_{Zar}, \mathcal{Z}^r_X)$*, where* $\mathcal{Z}^r_X$ *is the complex of Zariski sheaves given by* $U \mapsto \mathbf{Z}^r_U$*. In particular, there is a standard hypercohomology* E_2*-spectral sequence*

$$E_2^{p,q} = H^p(X_{Zar}, \mathcal{CH}^r(-q)) \Rightarrow CH^r(X, -p-q),$$

where $\mathcal{CH}^r(q)$ *is the Zariski sheaf associated to the presheaf* $U \mapsto CH^r(U,q)$*.*

(v) There are external and internal product structures on $CH^r(X,n)$, namely

$$CH^p(X,n)\otimes CH^q(Y,m)\to CH^{p+q}(X\times Y,n+m),$$

$$CH^p(X,n)\otimes CH^q(X,m)\to CH^{p+q}(X,n+m).$$

For later notational convenience, we will set

$$H^m_{\mathrm{Zar}}(X,\mathbb{Z}(n))=CH^n(X,2n-m)=\mathbb{H}^m(X_{Zar},\mathbb{Z}(n)),$$

where $\mathbb{Z}(n)=\mathcal{Z}^n_X[2n]$ from Proposition 2.3.2. More generally, for any abelain group A, we set $H^m_{\mathrm{Zar}}(X,A(n))=\mathbb{H}^m(X_{Zar},A(n))$, where $A(n)=A\otimes\mathbb{Z}(n)$.

2.3.2 Sheafified higher Chow complex

We will define étale higher Chow groups using the Bloch's higher Chow complexes. They are potential candidates for the etale motivic cohomology that satisfy the Lichtenbaum conjectures. Little has been known of these etale higher Chow groups, except that they share the same rational structure as the higher Chow groups and that over finite coefficients they can be computed using ordinary etale cohomology. We will concentrate on the behaviour of its torsion under the regulator map to be defined in the next section. As we have seen in Proposition 2.3.2, $\mathcal{Z}^r_X$ behaves contravariantly functorial under flat maps, it defines a sheaf on the small etale site X_{et} as well. We can hence define the etale higher Chow groups analogously as hypercohomology groups.

Definition 2.3.3. *Let X be a equidimensional smooth scheme of finite type over a field k. Then the etale higher Chow group, $CH^r_{et}(X,n)$, is defined as $\mathbb{H}^{-n}(X_{et},\mathcal{Z}^r_X)$. For any abelian group A, we set*

$$H^m_{et}(X,A(n))=\mathbb{H}^m(X_{et},A(n)),$$

where $A(n)=A\otimes\mathbb{Z}(n)=A\otimes\mathcal{Z}^n_X[2n]$.

Remark 2.3.3. *Note that the $A(n)$ in both Definitions 2.3.1 and 2.3.3 is not well defined on the big Zariski and étale sites respectively as the higher Chow complexes are not functorial with respect to all maps. However, Kahn [17] has shown that one can replace $A(n)$ by $A(n)'$ well-defined in the derived category of complexes of sheaves over the big Zariski or etale sites, so that their hypercohomologies agree, and that the construction gives quasi-isomorphic complexes to the original higher Chow complexes for smooth quasi-projective varieties.*

There are some known properties:

Proposition 2.3.4 ([20, 11]). *Let X be a smooth quasiprojective variety over k, and $\alpha : X_{et} \to X_{Zar}$ the change of site morphism. Then*

(i) $H^m_{Zar}(X, \mathbb{Q}(n)) \simeq H^m_{et}(X, \mathbb{Q}(n))$ *for all integers m, $n \geq 0$.*

(ii) For any prime $\ell \neq char(k)$, and k being separably closed, the natural map

$$\mathbb{Z}/\ell^r(n) \to \mu_{\ell^r}^{\otimes n}$$

is a quasi-isomorphism. In particular, we have

$$H^m_{et}(X, \mathbb{Z}(n))\{\ell\} \simeq H^{m-1}(X_{et}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$$

for $2n \leq m$ and

$$H^m_{et}(X, \mathbb{Z}(n)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell = 0$$

for $2n \leq m-1$.

(iii) $H^m_{Zar}(X, \mathbb{Q}/\mathbb{Z}(n)) \simeq H^m_{et}(X, \mathbb{Q}/\mathbb{Z}(n))$ *for $m \leq n$.*

(iv) $H^m_{Zar}(X, \mathbb{Z}(n)) \simeq H^m_{et}(X, \mathbb{Z}(n))$ *for $m \leq n+1$.*

2.3.3 Motivic cohomology for smooth simplicial schemes

Now we will extend the definition of motivic cohomology to the case of smooth simplicial schemes. The basic idea is the observation that in Remark 2.3.3 the complex of sheaves $\mathbb{Z}(n)$ can be replaced by $\mathbb{Z}(n)'$, which is well-defined in the derived category $\mathrm{D}^-_\tau(\mathcal{V}^{sm})$ of bounded above complexes of sheaves on $\mathcal{V}^{sm}$ endowed with Grothendieck topology τ, and that

$$H^m_\tau(X, \mathbb{Z}(n)) = Ext^m_{\mathcal{V}^{sm}_\tau}(\mathbb{Z}(X), \mathbb{Z}(n)) = Hom_{\mathcal{V}^{sm}_\tau}(\mathbb{Z}(X), \mathbb{Z}(n)[m]).$$

For a general smooth simplicial scheme $X.$, we can form the representable complex of sheaves $\mathbb{Z}(X.)$ through the Dold-Kan correspondence in the abelian category of complexes of sheaves, and define the motivic cohomology on $X.$ by

$$H^m_\tau(X., \mathbb{Z}(n)) = Hom_{\mathcal{V}^{sm}_\tau}(\mathbb{Z}(X.), \mathbb{Z}(n)[m]).$$

More generally, for an arbitrary commutative ring A, we define

$$H^m_\tau(X., A(n)) = Hom_{\mathcal{V}^{sm}_\tau}(\mathbb{Z}(X.), A \otimes \mathbb{Z}(n)[m]). \tag{2.3.1}$$

The most important property about this motivic cohomology of smooth simplicial schemes is the Quillen spectral sequence that relates them to the motivic cohomology of their components.

Proposition 2.3.5. *Let $X.$ be a smooth simplicial scheme over a field k, and A be a commutative ring. Then there is a E_1-spectral sequence*

$$_\tau E_1^{r,s} = H^s_\tau(X_r, A(n)) \Rightarrow H^{r+s}_\tau(X., A(n)). \tag{2.3.2}$$

Furthermore, if the X_r have uniformly bounded cohomological dimension, then the above spectral sequence converges without taking truncation of simplicial schemes.

Chapter 3

Higher étale regulators

3.1 First construction via spectral sequence

In this section, we will first recast the construction of Bloch's construction of higher cycle maps in a slightly different formulation, which will facilitate our first construction of higher étale regulators. Then we will give the first construction and study some of its properties.

3.1.1 Review of Bloch's construction of higher cycle maps

In [4], Bloch gives a quite general construction of higher cycle maps from the higher Chow groups to any bigraded cohomology theory that satisfies some standard assumptions. His method basically uses the standard technique of tracking through the associated spectral sequence to the canonical cosimplicial scheme structure and passing through the cohomology with supports, which has a natural choice of cycle classes. Here we will give a construction for those bigraded cohomology theories coming from complexes of sheaves $K(n)$ on the big Zariski site $\mathcal{V}_{\mathrm{Zar}}$.

Definition 3.1.1. *Let $K(n)$ be a complex of sheaves on $\mathcal{V}_{Zar}$ for every integer $n \geq 0$. $\{K(n)\}_{n\geq 0}$ is said to be regular if they satisfy*

(i) ($\mathbb{A}^1$-homotopy invairance) $R\Gamma\left(X \times \mathbb{A}^1, K(n)\right) \to R\Gamma\left(X, K(n)\right)$ induced from any inclusion $X \times \{*\} \hookrightarrow X \times \mathbb{A}^1$ is a quasi-isomorphism.

(ii) (Fundamental cycle class) For every $Y \in Z^r(X)$, define the complex of $K(n)$ on X with support

$|Y|$ as

$$R\Gamma_{|Y|}(\_\_, K(n))) := Cone\left(R\Gamma\left(\_\_, K(n)\right) \to R\Gamma\left(\_\_ - |Y|, K(n)\right)\right)[-1]$$

and the cohomology of $K(n)$ on X with support Y as

$$H^m_Y\left(X, K(n)\right) := \mathbb{H}^m\left(X, R\Gamma_{|Y|}\left(\_\_, K(n)\right)\right).$$

Then there is a unique element $cl(Y) \in H^{2r}_Y\left(X, K(r)\right)$ so that cl is contravariant with respect to pullback of morphisms.

(iii) (Weak purity) $H^m_Y\left(X, K(n)\right) = 0$ for every $Y \in Z^r(X)$, $m < 2r$.

Now given a regular complex of sheaves $\{K(n)\}_{n\geq 0}$, our goal is to define higher cycle maps

$$\rho^{r,n} : H^r_{\mathrm{Zar}}(X, \mathbb{Z}(n)) \to \mathbb{H}^r(X, K(n)).$$

As noted in [4], for each X we can replace $K(n)$ by its Godement resolution so that we can assume, without loss of generality, that $\Gamma(X, K(n))$ is acyclic for every $n \geq 0$. Consider the spectral sequence associated to the double complex $\Gamma\left(X \times \Delta^{-p}, K(n)\right)$ constructed from the cosimplicial scheme $X \times \Delta^{\cdot}$, namely

$$E^{p,q}_1 = \mathbb{H}^q\left(X \times \Delta^{-p}, K(n)\right).$$

As $K(n)$ are $\mathbb{A}^1$-homotopy invariance, the differentials $d^{p,q}_1$ are either isomorphism or zero map, so that

$$E^{p,q}_2 = \begin{cases} \mathbb{H}^q(X, K(n)) & \text{, if } p = 0, \\ 0 & \text{, otherwise.} \end{cases}$$

Thus we can resolve the problem of convergence of the spectral sequence by truncating it for $-p \leq N$ for a large even integer N and write

$$E_1^{p,q} \Rightarrow \mathbb{H}^q(X, K(n)).$$

Similarly, define

$$\Gamma_c\left(X \times \Delta^r, K(n)\right) := \varinjlim_{Y \in Z^n(X,r)} \Gamma_{|Y|}\left(X \times \Delta^r, K(n)\right),$$
$$H_c^m\left(X \times \Delta^r, K(n)\right) := \varinjlim_{Y \in Z^n(X,r)} H_Y^m(X \times \Delta^r, K(n)),$$

and consider the corresponding truncated spectral sequence

$${}_cE_1^{p,q} = H_c^q\left(X \times \Delta^{-p}, K(n)\right) \Rightarrow H^{p+q}\left(Tot\left(\Gamma_c\left(X \times \Delta^{-\cdot}, K(n)\right)\right)\right).$$

From the weak purity, we have ${}_cE_1^{p,q} = 0$ for $q < 2n$. We can modify the associated double complex $\Gamma_c\left(X \times \Delta^{-\cdot}, K(n)\right)$ to obtain another double complex $\Gamma'_c\left(X \times \Delta^{-\cdot}, K(n)\right)$ by fixing components of degree greater than $2n$, replacing degree 2n components by

$$coker(d_c^{p,2n-1} : \Gamma_c\left(X \times \Delta^{-p}, K(n)\right)^{2n-1} \to \Gamma_c\left(X \times \Delta^{-p}, K(n)\right)^{2n}),$$

while trvializing all components of degree less than $2n$, so that the canonical quotient morphism $Tot(\Gamma_c\left(X \times \Delta^{-\cdot}, K(n)\right)) \to Tot(\Gamma'_c\left(X \times \Delta^{-\cdot}, K(n)\right))$ is a quasi-isomorphism.

Note that there is a well-defined morphism of complexes

$$Z^n(X, \cdot)[-2n] \to H_c^{2n}\left(X \times \Delta^{2n-\cdot}, K(n)\right) \subset coker(d_c^{2n-\cdot,2n-1}) \subset Tot(\Gamma'_c\left(X \times \Delta^{-\cdot}, K(n)\right))$$

sending Y to its fundamental class $cl(Y)$, and yielding a diagram of morphisms

$$Z^n(X,\cdot)[-2n] \to Tot(\Gamma'_c\left(X \times \Delta^{-\cdot}, K(n)\right)) \xleftarrow{\sim} Tot(\Gamma_c\left(X \times \Delta^{-\cdot}, K(n)\right)) \to Tot(\Gamma(X \times \Delta^{-\cdot}, K(n))) \tag{3.1.1}$$

whose induced morphisms on cohomology give the higher cycle map

$$\rho^{r,n} : H^r_{\mathrm{Zar}}(X, \mathbb{Z}(n)) \to H^r(Tot(\Gamma(X \times \Delta^{-\cdot}, K(n)))) = \mathbb{H}^r(X, K(n)).$$

In particular, if we apply the above construction to the case $K(n) = \mathbb{Z}(n)_{\mathcal{D}}$ we obtain the Bloch-Beilinson regulator maps

$$\Phi^{r,n} : H^r_{\mathrm{Zar}}(X, \mathbb{Z}(n)) \to H^r_{\mathcal{D}}(X, \mathbb{Z}(n)).$$

Note that the sequence of morphisms in 3.1.1 is functorial with respect to X so that we view them as complexes of presheaves on any fixed smooth variety X and consider the induced map on hypercohomologies. In particular, we have

$$\widetilde{\rho}^{r,n} : H^r_{\mathrm{Zar}}(X, \mathbb{Z}(n)) = \mathbb{H}^r(X, \mathbb{Z}(n)_{\mathrm{Zar}}) \to \mathbb{H}^r\left(X, Tot(\Gamma(\_\_ \times \Delta^{-\cdot}, K(n)))\right) \to \mathbb{H}^r(X, K(n)). \tag{3.1.2}$$

From the Zariski descent property of higher Chow groups and the acyclicity of $K(n)$, we have $\widetilde{\rho}^{r,n} = \rho^{r,n}$. However, this does not define the higher cycle maps as induced maps from derived morphisms of complexes of Zariski presheaves, since the above construction depends on the choice of truncation of the cosimplicial scheme $X \times \Delta^{\cdot}$ at some even level N.

3.1.2 Construction of higher étale regulators

We are in the position to define the higher étale regulator maps. As in Section 3.1.1, the morphisms in (3.1.1) are functorial in X, in particular, we can view them as complexes of étale presheaves on $X_{\text{ét}}$ and consider their induced morphisms on étale hypercohomologies. The main obstruction here is that the quasi-isomorphism $Tot(\Gamma_c(X \times \Delta^{-\cdot}, K(n))) \xrightarrow{\sim} Tot(\Gamma'_c(X \times \Delta^{-\cdot}, K(n)))$ may not be quasi-isomorphic as presheaves over $X_{\text{ét}}$ and the acyclicity of $K(n)$ do not carry over to the corresponding étale site. Therefore, in order to resolve this problem, we need to impose an extra condition on $K(n)$, called the étale descent property.

Definition 3.1.2. *Let $K(n)$ be a regular complex of sheaves on $\mathcal{V}_{Zar}$, and $\alpha : \mathcal{V}_{et} \to \mathcal{V}_{Zar}$ the canonical morphism of topoi from the big étale site of smooth varieties over k to the big Zariski site of smooth varieties over k. Then $K(n)$ is said to satisfy étale descent if*

$$\alpha_* : R\Gamma(X_{Zar}, K(n)^q) \to R\Gamma(X_{\acute{e}t}, \alpha^* K(n)^q)$$

is a quasi-isomorphism for each q and smooth variety X.

Given $K(n)$ with étale descent property, it is clear that the same property is also valid for the corresponding complex with support as the quasi-isomorphism is stable under base changes, cone constructions, and filtered direct limits. It follows that we have a quasi-isomorphism of étale presheaves $Tot(\Gamma_c(\_\_ \times \Delta^{-\cdot}, K(n))) \xrightarrow{\sim} Tot(\Gamma'_c(\_\_ \times \Delta^{-\cdot}, K(n)))$ and that the diagram (3.1.1) induces a morphism on hypercohomology

$$H^r_{\text{ét}}(X, \mathbb{Z}(n)) = \mathbb{H}^r(X_{\text{ét}}, \mathbb{Z}(n)_{\text{ét}}) \to \mathbb{H}^r(X_{\text{ét}}, Tot(\Gamma\left(\_\_ \times \Delta^{-\cdot}, K(n)\right))).$$

Note that there is a natural morphism of complexes of étale presheaves

$$Tot(\Gamma\left(\_\_ \times \Delta^{-\cdot}, K(n)\right)) \to \alpha^* K(n),$$

therefore we obtain a morphism

$$H^r_{\text{ét}}(X, \mathbb{Z}(n)) \to \mathbb{H}^r(X_{\text{ét}}, \alpha^* K(n)). \tag{3.1.3}$$

In particular, note that we have an isomorphism of E_2 spectral sequences

$$\begin{array}{ccc} H^p(X_{\mathrm{Zar}}, K(n)^q) & \Longrightarrow & \mathbb{H}^{p+q}(X_{\mathrm{Zar}}, K(n)) \\ \cong\downarrow & & \downarrow \\ H^p(X_{\text{ét}}, \alpha^* K(n)^q) & \Longrightarrow & \mathbb{H}^{p+q}(X_{\text{ét}}, \alpha^* K(n)) \end{array}$$

so that (3.1.3) defines a map

$$\rho_{\text{ét}}^{r,n} : H_{\text{ét}}^r(X, \mathbb{Z}(n)) \to \mathbb{H}^r(X_{\text{ét}}, \alpha_* K(n)) \cong \mathbb{H}^r(X_{\mathrm{Zar}}, K(n)). \tag{3.1.4}$$

As the complexes $\{\mathbb{Z}(n)_{\mathcal{D}}\}_{n\geq 0}$ are regular and satisfy the étale descent property, we have thus obtained the higher étale regulator maps:

Definition 3.1.3. *The higher étale regulator maps*

$$\Phi_{\acute{e}t}^{r,n} : H_{\acute{e}t}^r(X, \mathbb{Z}(n)) \to H_{\mathcal{D}}^r(X, \mathbb{Z}(n))$$

are defined by (3.1.4) taking $K(n) = \mathbb{Z}(n)_{\mathcal{D}}$.

3.2 Second construction via étale hypercovers

In this section, we will present the second construction of the higher étale regulator maps in terms of limit of higher regulator maps of étale hypercovers. We follow mainly the treatment of [20].

3.2.1 Étale descent of motivic cohomology

The main ingredient of this formulation is the following étale descent property of motivic cohomology:

Proposition 3.2.1. *Let X be a smooth variety over k, and let $\pi : X. \to X$ be an étale hypercovering. Then $\mathbb{Z}(X.) \to \mathbb{Z}(X)$ is a quasi-isomorphism in $D^-(Sh(\mathcal{V}_{\acute{e}t}))$. Hence for any complex of étale sheaves*

K, we have a canonical isomorphism

$$\mathbb{H}^r(X_{\acute{e}t}, K) \xrightarrow{\cong} \mathbb{H}^r_{\acute{e}t}(X., K).$$

Proof. First note that it suffices to establish the required quasi-isomorphism on $D^-(Sh(\mathcal{V}_{\text{ét}}/X))$. In this case, $\mathbb{Z}(X)$ is the constant sheaf $\mathbb{Z}$ on X and we would like to show that $\mathbb{Z}(X.)_*$ is a resolution of $\mathbb{Z}$. Note that for every geometric point $p : \text{Sets} \to \mathcal{V}_{et}/X$, $p^*(X.) \to p^*(X)$ is a hypercovering in Sets. As hypercoverings in Sets are contractible, the induced map

$$p^*\mathbb{Z}(X.) = \mathbb{Z}(p^*(X.)) \to \mathbb{Z}(p^*X) = \mathbb{Z}$$

is exact. As $\mathcal{V}_{\text{ét}}/X$ is a site with enough points, this shows that $\mathbb{Z}(X.) \to \mathbb{Z}(X)$ is a quasi-isomorphism in $D^-(Sh(\mathcal{V}_{\text{ét}}))$. □

Using this, we can reinterpret the étale motivic cohomologies as limits of the corresponding motivic cohomologies over the étale hypercoverings, which allows us to represent elements in the étale motivic cohomologies by higher Chow cycles on certain étale coverings.

Theorem 3.2.2. *Let X be a smooth quasiprojective variety over k. Then there is a canonical isomorphism*

$$\varinjlim_{X. \in \acute{E}tCov(X)} \mathbb{H}^r(X., \mathbb{Z}(n)_{Zar}) \cong H^r_{\acute{e}t}(X, \mathbb{Z}(n))$$

Proof. From the distinguished triangle of complexes of sheaves

$$\mathbb{Z}(n) \to \mathbb{Q}(n) \to \mathbb{Q}/\mathbb{Z}(n) \xrightarrow{+1}$$

we have

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H^m_{Zar}(X.,\mathbb{Z}(n)) & \longrightarrow & H^m_{Zar}(X.,\mathbb{Q}(n)) & \longrightarrow & H^m_{Zar}(X.,\mathbb{Q}/\mathbb{Z}(n)) & \longrightarrow & H^{m+1}_{Zar}(X.,\mathbb{Z}(n)) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle \alpha^*} & & \downarrow{\scriptstyle \alpha^*_{\mathbb{Q}}} & & \downarrow{\scriptstyle \alpha^*_{\mathbb{Q}/\mathbb{Z}}} & & \downarrow{\scriptstyle \alpha^*} & \\
\cdots \longrightarrow & H^m_{et}(X.,\mathbb{Z}(n)) & \longrightarrow & H^m_{et}(X.,\mathbb{Q}(n)) & \longrightarrow & H^m_{et}(X.,\mathbb{Q}/\mathbb{Z}(n)) & \longrightarrow & H^{m+1}_{et}(X.,\mathbb{Z}(n)) & \longrightarrow \cdots
\end{array}$$

Note that by Proposition 2.3.4, $\alpha^*_{\mathbb{Q}}$ are isomorphisms, and thus by Five Lemma it suffices to show that $\alpha^*_{\mathbb{Q}/\mathbb{Z}}$ induce isomorphisms after taking limit over all étale hypercoverings $\pi : X. \to X$. Now consider the morphism of the Quillen spectral sequences

$$\begin{array}{ccc}
E_1^{r,s} = \varinjlim_{X.\in \text{ÉtCov(X)}} H^s_{Zar}(X_r,\mathbb{Q}/\mathbb{Z}(n)) & \Longrightarrow & \varinjlim_{X.\in \text{ÉtCov(X)}} H^{r+s}_{Zar}(X.,\mathbb{Q}/\mathbb{Z}(n)) \\
\downarrow & & \downarrow \\
{}_{et}E_1^{r,s} = \varinjlim_{X.\in \text{ÉtCov(X)}} H^s_{et}(X_r,\mathbb{Q}/\mathbb{Z}(n)) & \Longrightarrow & \varinjlim_{X.\in \text{ÉtCov(X)}} H^{r+s}_{et}(X.,\mathbb{Q}/\mathbb{Z}(n))
\end{array}.$$

Note that both $E_1^{r,s}$ and ${}_{et}E_1^{r,s}$ vanish for $s < 0$, so they are first quadrant spectral sequences and thus converge properly. So it suffices to show that $E_1^{r,s} \to {}_{et}E_1^{r,s}$ are isomorphisms. For $s > 0$, both sides vanish. It follows from the fact that both cohomology theories are étale locally trivial. For $s = 0$, they are isomorphic according to Proposition 2.3.4. □

In view of Theorem 3.2.2, we can recast the problem of defining a higher cycle map from the étale motivic cohomology to a $\mathbb{Z}$-bigraded cohomology defined by a regular sequence of complexes of sheaves $\{K(n)\}_{n\geq 0}$ which satisfy the étale descent as extending the original higher cycle map from motivic cohomology to a certain class of regular simplicial schemes, containing those which appear as étale hypercoverings with affine components. The quasiprojective assumption on X is essential here, for it implies that the étale hypercoverings with affine components are cofinal in the poset ÉtCov(X), and that the complexes $\mathbb{Z}(n)'$ defined on $\mathcal{V}_{\text{Zar}}$ give quasi-isomorphic complexes to the original Bloch's higher cycle complexes. In particular, taking $\{K(n)\}_{n\geq 0}$ to be $\{\mathbb{Z}(n)_{\mathcal{D}}\}_{n\geq 0}$, we obtain our second construction of the higher étale regulator map:

Definition 3.2.1. *Let X be a smooth quasiprojective variety over k. We then define the natural* *homomorphisms*

$$\widetilde{\Phi}_{et}^{r,n} = \varinjlim_{X.\in \acute{E}tCov(X)} \Phi^{r,n} : H^r_{et}(X, \mathbb{Z}(n)) \to H^r_{\mathcal{D}}(X, \mathbb{Z}(n)).$$

3.2.2 Equivalence of the constructions

Theorem 3.2.3. *Let X be a smooth quasiprojective variety over k. Then*

$$\Phi_{et}^{r,n}, \widetilde{\Phi}_{et}^{r,n} : H^r_{et}(X, \mathbb{Z}(n)) \to H^r_{\mathcal{D}}(X, \mathbb{Z}(n))$$

are naturally isomorphic.

Proof. Consider the following commutative diagram:

$$\begin{array}{ccc}
\varinjlim_{X.\in \text{ÉtCov(X)}} \mathbb{H}^r(X., \mathbb{Z}(n)'_{Zar}) & \xrightarrow{\simeq} & \varinjlim_{X.\in \text{ÉtCov(X)}} \mathbb{H}^r(X., \mathbb{Z}(n)'_{et}) \\
\downarrow & & \downarrow \\
\varinjlim_{X.\in \text{ÉtCov(X)}} \mathbb{H}^r(X., Tot(\Gamma'_c(__ \times \Delta^{-\cdot}, \mathbb{Z}(n)_{\mathcal{D}}))) & \longrightarrow & \varinjlim_{X.\in \text{ÉtCov(X)}} \mathbb{H}^r(X., Tot(\Gamma'_c((__ \times \Delta^{-\cdot})_{et}, \mathbb{Z}(n)_{\mathcal{D}}))) \\
\uparrow \simeq & & \uparrow \simeq \\
\varinjlim_{X.\in \text{ÉtCov(X)}} \mathbb{H}^r(X., Tot(\Gamma_c(__ \times \Delta^{-\cdot}, \mathbb{Z}(n)_{\mathcal{D}}))) & \longrightarrow & \varinjlim_{X.\in \text{ÉtCov(X)}} \mathbb{H}^r(X., Tot(\Gamma_c((__ \times \Delta^{-\cdot})_{et}, \mathbb{Z}(n)_{\mathcal{D}}))) \\
\downarrow & & \downarrow \\
\varinjlim_{X.\in \text{ÉtCov(X)}} \mathbb{H}^r(X., Tot(\Gamma(__ \times \Delta^{-\cdot}, \mathbb{Z}(n)_{\mathcal{D}}))) & \longrightarrow & \varinjlim_{X.\in \text{ÉtCov(X)}} \mathbb{H}^r(X., Tot(\Gamma((__ \times \Delta^{-\cdot})_{et}, \mathbb{Z}(n)_{\mathcal{D}}))) \\
\downarrow & & \downarrow \\
\varinjlim_{X.\in \text{ÉtCov(X)}} \mathbb{H}^r(X., \mathbb{Z}(n)_{\mathcal{D}}) & \xrightarrow{\simeq} & \varinjlim_{X.\in \text{ÉtCov(X)}} \mathbb{H}^r(X., \alpha^*\mathbb{Z}(n)_{\mathcal{D}})
\end{array}$$

Note that the composition of the left column is just $\widetilde{\Phi}_{et}^{r,n}$, while the composition of the right column gives $\Phi_{et}^{r,n}$ in view of Proposition 3.2.1. The isomorphism on the top and bottom horizontal arrows are given from the étale descent property of $\mathbb{Z}(n)'$ and $\mathbb{Z}(n)_{\mathcal{D}}$ respectively. □

Remark 3.2.4. *Note that the assertion in Theorem 3.2.3 is also valid if we replace the complex*

$\mathbb{Z}(n)_{\mathcal{D}}$ by any regular complex $K(n)$ of sheaves on $\mathcal{V}_{Zar}$ that satisfies étale descent.

3.3 Compatibility with Bloch's higher cycle map

Here we will show that the higher étale regulator maps defined in the last two sections are compatible with Bloch's Beilinson regulator maps in the sense that by pushing forward via the standard change of topology map $\alpha : X_{Zar} \to X_{et}$, we obtain a commutative triangle

$$\begin{array}{ccc} H^r_{Zar}(X,\mathbb{Z}(n)) & \xrightarrow{\Phi^{r,n}} & H^r_{\mathcal{D}}(X,\mathbb{Z}(n)) \\ \downarrow{\scriptstyle \alpha_*} & \nearrow{\scriptstyle \Phi^{r,n}_{et}} & \\ H^r_{et}(X,\mathbb{Z}(n)) & & \end{array} \quad (3.3.1)$$

Proposition 3.3.1. *Let X be a smooth quasiprojective variety over k. Then the digram 3.3.1 commutes.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} \mathbb{H}^r(X_{Zar},\mathbb{Z}(n)) & \xrightarrow{\alpha_*} & \mathbb{H}^r(X_{et},\mathbb{Z}(n)) \\ \downarrow & & \downarrow \\ \mathbb{H}^r(X_{Zar},Tot(\Gamma'_c(\_\_ \times \Delta^{-\cdot},\mathbb{Z}(n)_{\mathcal{D}}))) & \xrightarrow{\alpha_*} & \mathbb{H}^r(X_{et},Tot(\Gamma'_c((\_\_ \times \Delta^{-\cdot})_{et},\mathbb{Z}(n)_{\mathcal{D}}))) \\ \uparrow \simeq & & \simeq \uparrow \\ \mathbb{H}^r(X_{Zar},Tot(\Gamma_c(\_\_ \times \Delta^{-\cdot},\mathbb{Z}(n)_{\mathcal{D}}))) & \xrightarrow{\alpha_*} & \mathbb{H}^r(X_{et},Tot(\Gamma_c((\_\_ \times \Delta^{-\cdot})_{et},\mathbb{Z}(n)_{\mathcal{D}}))) \\ \downarrow & & \downarrow \\ \mathbb{H}^r(X_{Zar},Tot(\Gamma(\_\_ \times \Delta^{-\cdot},\mathbb{Z}(n)_{\mathcal{D}}))) & \xrightarrow{\alpha_*} & \mathbb{H}^r(X_{et},Tot(\Gamma((\_\_ \times \Delta^{-\cdot})_{et},\mathbb{Z}(n)_{\mathcal{D}}))) \\ \downarrow & & \downarrow \\ \mathbb{H}^r(X_{Zar},\mathbb{Z}(n)_{\mathcal{D}}) & \xrightarrow{\alpha_*} & \mathbb{H}^r(X_{et},\alpha_*\mathbb{Z}(n)_{\mathcal{D}}) \end{array}$$

Note that the compositions of arrows on the left and right columns are $\Phi^{r,n}$ and $\Phi^{r,n}_{er}$ respectively, and the botton horizontal arrow is an isomorphism as $\mathbb{Z}_{\mathcal{D}}(n)$ satisfies étale descent. Therefore (3.3.1) commutes. □

3.4 Vanishing of higher infinite torsions under higher étale regulators

Note that the étale motivic cohomology groups $H^r_{et}(X, \mathbb{Z}(n))$ are torsion abelian groups for $r > 2n$. In this section, we will see that their maximal divisible subgroups $H^r_{et}(X, \mathbb{Z}(n))_{div}$ vanisn under the regulator maps $\Phi^{r,n}_{et}$. In view of the conjectural cofinite generation property of these groups, we have

$$H^r_{et}(X, \mathbb{Z}(n)) \simeq (\mathbb{Q}/\mathbb{Z})^{c_{r,n}} \oplus H^r_{et}(X, \mathbb{Z}(n))_{finite}$$

and $H^r_{et}(X, \mathbb{Z}(n))_{div} \simeq (\mathbb{Q}/\mathbb{Z})^{c_{r,n}}$ for some non-negative integers $c_{r,n}$, $r > 2n$.

Theorem 3.4.1. *Let X be a smooth quasi-projective variety over $\mathbb{Q}$. Then $\Phi^{r,n}_{et}(H^r_{et}(X, \mathbb{Z}(n))_{div}) = 0$ when $r > 2n + 1$. In other words, $\Phi^{r,n}_{et}$ factors through its cotorsion quotient $H^r_{et}(X, \mathbb{Z}(n))_{cotor} = H^r_{et}(X, \mathbb{Z}(n))/H^r_{et}(X, \mathbb{Z}(n))_{div}$.*

Proof. Note that as X is defined over $\mathbb{Q}$, the étale regulator map factors through the group of Galois invariants of the étale motivic cohomology of $X_{\overline{\mathbb{Q}}}$,

$$\begin{array}{ccc} H^r_{et}(X, \mathbb{Z}(n)) & \longrightarrow & H^r_{\mathcal{D}}(X, \mathbb{Z}(n)) \\ & \searrow \quad\quad \nearrow & \\ & H^r_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Z}(n))^{G_{\mathbb{Q}}} & \end{array} \tag{3.4.1}$$

Note that $\overline{\mathbb{Q}}$ is a separably closed field of char 0, so it follows from Proposition 2.3.4 that

$$H^r_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Z}(n))^{G_{\mathbb{Q}}} \simeq H^{r-1}_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}/\mathbb{Z}(n))^{G_{\mathbb{Q}}} \simeq \bigoplus_{\ell:\text{prime}} H^{r-1}_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_{\mathbb{Q}}} \quad . \tag{3.4.2}$$

Consider the long exact sequence

$$\cdots \to H^{r-1}_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(n)) \xrightarrow{q} H^{r-1}_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) \xrightarrow{t}$$
$$H^r_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Z}_\ell(n)) \xrightarrow{i} H^r_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(n)) \to \cdots$$

Since the ℓ-adic $G_{\mathbb{Q}}$-representation $H^r_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(n))$ does not have weight 0 components, $H^r_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(n))^{G_{\mathbb{Q}}} = 0$. In particular, $H^r_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Z}_\ell(n))^{G_{\mathbb{Q}}}$ lies in $\ker i = \mathrm{Im}\ t$, which must be torsion as $H^{r-1}_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$ is torsion. Note that the torsion subgroup $H^r(X_{\overline{\mathbb{Q}}}, \mathbb{Z}_\ell(n))_{tor}$ is finite, so the same is true for $H^r_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Z}_\ell(n))^{G_{\mathbb{Q}}}$.

Now observe that

$$\begin{aligned}
&\ker\left(H^{r-1}_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_{\mathbb{Q}}} \xrightarrow{t} H^r_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Z}_\ell(n))^{G_{\mathbb{Q}}}\right)\\
&= H^{r-1}_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_{\mathbb{Q}}} \cap \ker\left(H^{r-1}_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) \xrightarrow{t} H^r_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Z}_\ell(n))\right)\\
&= H^{r-1}_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_{\mathbb{Q}}} \cap \mathrm{Im}\left(H^{r-1}_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(n)) \xrightarrow{q} H^{r-1}_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))\right)\\
&= \mathrm{Im}\left(H^{r-1}_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(n))^{G_{\mathbb{Q}}} \xrightarrow{q} H^{r-1}_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_{\mathbb{Q}}}\right)\\
&= 0.
\end{aligned}$$

Therefore $H^{r-1}_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_{\mathbb{Q}}} \xrightarrow{t} H^r_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Z}_\ell(n))^{G_{\mathbb{Q}}}$ is injective.

It follows that $H^{r-1}_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_{\mathbb{Q}}}$ is a finite torsion group. In particular, (3.4.2) shows that $H^r_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Z}(n))^{G_{\mathbb{Q}}}$ is a direct sum of finite abelian groups which must have a trivial maximal divisible subgroup. Therefore, from (3.4.1) we have $\Phi^{r,n}_{et}(H^r_{et}(X, \mathbb{Z}(n))_{div}) = 0$ for $r > 2n+1$. □

Remark 3.4.2. *Note that when $r = 2n+1$, $\Phi^{2n+1,n}_{et}(H^{2n+1}_{et}(X, \mathbb{Z}(n))_{div}) = 0$ follows from the rational Tate conjecture for all primes ℓ. According to [20] the rational Tate conjecture for prime ℓ is equivalent to $H^{2n+1}_{et}(X, \mathbb{Z}(n))\{\ell\}$ being finite. As $H^{2n+1}_{et}(X, \mathbb{Z}(n))$ is a torsion abelian group,*

$$H^{2n+1}_{et}(X, \mathbb{Z}(n)) \simeq \bigoplus_{\ell:prime} H^{2n+1}_{et}(X, \mathbb{Z}(n))\{\ell\}$$

and thus does not admit nontrivial divisible subgroups.

3.5 Third construction via the Kerr-Lewis-Müller-Stach Map

In this section, we provide a third construction of the étale regulators on the level of étale complexes of sheaves. The idea of the construction is based on the Kerr-Lewis-Müller-Stach map from the higher Chow group to the Deligne homology in [18] taking hypercohomology of morphism of complexes of abelian groups constructed with explicit currents.

Let $z_1, \ldots, z_n$ be the standard affine coordinates on $\square^n$. Define a (Borel-Moore) n-chain T^n on $\square^n$ by

$$T^n = T_{z_1} \cap \cdots \cap T_{z_n} \in C_n^{BM}(\square^n, \mathbb{Z}),$$

a holomorphic n-current

$$\Omega^n = d\log z_1 \wedge \cdots \wedge d\log z_n \in F^n{}'\mathcal{D}^n(\square^n),$$

and a holomorphic $(n-1)$-current

$$R^n = \sum_{j=1}^{n} ((-1)^{n-1} 2\pi i)^{j-1} \log z_j d\log z_{j+1} \wedge \cdots \wedge d\log z_n \cdot \delta_{T_{z_1} \cap \cdots \cap T_{z_{j-1}}} \in {}'\mathcal{D}^{n-1}(\square^n).$$

Then for any higher algebraic cycle $Z = \sum n_j Z_j \in Z^p_{\mathbb{R}}(X, n)$, we define

$$\begin{aligned} T_Z &= \sum n_j \pi_X \{Z_j \cap (X \times T^n)\} \in C^{BM}_{2m+2p-n}(X, \mathbb{Z}) \\ \Omega_Z &= \sum n_j (\pi_X^{Z_j})_* (\pi_{\square^n}^{Z_j})^* \Omega^n \in F^p{}'\mathcal{D}^{2p-n}(X) \\ R_Z &= \sum n_j (\pi_X^{Z_j})_* (\pi_{\square^n}^{Z_j})^* R^n \in {}'\mathcal{D}^{2p-n-1}(X), \end{aligned}$$

where $\pi_X : X \times \square^n \to X, \pi_X^{Z_j} : Z_j \to X$ and $\pi_{\square^n}^{Z_j} : Z_j \to \square^n$ are the natural projections. It is easy to see that T_Z, Ω_Z and R_Z are functorial under pullbacks by étale maps, and hencefore the map of complexes

$$\begin{aligned}\mathcal{R}_X : Z^p_{\mathbb{R}}(X, -\bullet) &\to \mathbb{Z}(p-m)^{2p-2m+\bullet}_{\mathcal{D}}(X)\\ Z &\mapsto \frac{(-2\pi i)^{p-\bullet}}{(2\pi i)^m}\left((2\pi i)^{\bullet} T_Z, \Omega_Z, R_Z\right)\end{aligned}$$

defined in [18][5.5] also makes sense for smooth (but not necessarily proper) complex varieties X and is functorial on the etale site of a fixed such X. We then get a morphism

$$\mathcal{R} : Z^p_{\mathbb{R}}(-, 2p-\bullet) \to \mathbb{Z}(p-m)^{\bullet-2m}_{\mathcal{D}}(-)$$

of complexes of presheaves on $Et(X)$. As noted above, $\mathbb{Z}(p-m)^{\bullet-2m}_{\mathcal{D}}$ consists of soft sheaves for the analytic topology on X and this remains true for each etale $U \to X$. One has the following Lemma

Lemma 3.5.1. *Let X be a separated algebraic variety over $\mathbb{C}$ with associated topological space $X(\mathbb{C})$. Recall that the topos X_{an} of sheaves on $X(\mathbb{C})$ is equivalent to the category of topological spaces $p : Y \to X(\mathbb{C})$ above $X(\mathbb{C})$ so that p is a local homeomorphism. Hence one gets a functor*

$$\epsilon^{-1} : Et(X) \to X_{an}, \qquad (Z \to X) \mapsto (Z(\mathbb{C}) \to X(\mathbb{C}))$$

from étale schemes over X to X_{an}. This functor is continuous for the étale topology on $Et(X)$ and the canonical topology on X_{an}, hence inducing a morphism of topoi

$$\epsilon : X_{an} = Sh(X_{an}, J_{can}) \to X_{et} = Sh(Et(X), J_{et}).$$

If $\mathcal{F}$ is a sheaf on $X(\mathbb{C})$ whose restriction to $U(\mathbb{C})$ is soft for each separated étale $U \to X$, then $\epsilon_*\mathcal{F}$ is an acyclic sheaf on $Et(X)$.

Proof. A soft sheaf $\mathcal{F}$ on a paracompact Hausdorff space is acyclic [22][Thm.3.11] and if $U \to X$ is separated then $U(\mathbb{C})$ is again Hausdorff. By a general formula $R^i\epsilon_*\mathcal{F}$ is the étale sheaf associated to the presheaf $U \mapsto H^i(U(\mathbb{C}), \mathcal{F})$, so $R^i\epsilon_*\mathcal{F} = 0$ for $i > 0$. The Leray spectral sequence for ϵ then implies

$$H^i(X_{et}, \epsilon_*\mathcal{F}) \cong H^i(X_{an}, \mathcal{F}) = H^i(X(\mathbb{C}), \mathcal{F}) = 0$$

for $i > 0$. □

So $\epsilon_*\mathbb{Z}(k-m)_{\mathcal{D}}^{\bullet-2m}$, i.e. $\mathbb{Z}(k-m)_{\mathcal{D}}^{\bullet-2m}$ viewed as a complex of presheaves on $Et(X)$, consists of acyclic sheaves for the étale topology. This implies that its étale hypercohomology coincides with its cohomology (it has "etale descent") and we get our map

$$\begin{aligned} H^i(X_{et}, \mathbb{Z}(p)) :=& \mathbb{H}^i(X_{et}, Z^p_{\mathbb{R}}(-, 2p-\bullet)) \to \mathbb{H}^i(X_{et}, \mathbb{Z}(p-m)_{\mathcal{D}}^{\bullet-2m}) \\ &\cong H^i(\mathbb{Z}(p-m)_{\mathcal{D}}^{\bullet-2m}(X)) \cong H^i_{\mathcal{D},an}(X(\mathbb{C}), \mathbb{Z}(p)). \end{aligned}$$

Concretely, if $i : Z^p_{\mathbb{R}}(-, 2p-\bullet) \to K^\bullet$ is a K-injective resolution of étale sheaves, then by [21][Prop. 1.5] there exists a morphism (unique up to homotopy)

$$r : K^\bullet \to \mathbb{Z}(p-m)_{\mathcal{D}}^{\bullet-2m}$$

such that $r \circ i = \mathcal{R}$.

For a proper, regular arithmetic scheme $\mathcal{X}$ we should get a map of complexes

$$R\Gamma(\mathcal{X}_{et}, \mathbb{Z}(p)) \to R\Gamma((\mathcal{X}_{\mathbb{C}})_{et}, \mathbb{Z}(p)) = K^\bullet(\mathcal{X}_{\mathbb{C}}) \xrightarrow{r(X)} \mathbb{Z}(p-m)_{\mathcal{D}}^{\bullet-2m}(\mathcal{X}_{\mathbb{C}}) = R\Gamma_{\mathcal{D}}(\mathcal{X}_{\mathbb{C}}, \mathbb{Z}(p))$$

whose mapping fibre gives a definition of "Arakelov motivic cohomology" in the sense of Goncharov [14] and Holmstrom-Scholbach [15] but with two refinements: $\mathbb{Z}(p)$-coefficients instead of $\mathbb{R}(p)$-coefficients on the target, and étale instead of Zariski topology on the source.

3.6 Example: Regular arithmetic toric schemes

In this section, we are going to investigate the higher etale regulators for smooth quasi-projective toric varieties. The advantage of this special class of varieties is that they are completely determined by their associated combinatorial structure, called a fan. This gives us a combinatorial way to determine the geometric properties of the varieties, for instance one can construct a good compactification of a smooth toric variety by completing its fan Σ appropriately. Taking advantage of these extra structures, we are able to show that the Deligne-Beilinson cohomologies are finitely generated for suitable degrees. For background and notations about toric varieties, see [7], [10]. Also, due to their pure combinatorial features, we can consider regular arithmetic toric schemes constructed from the same set of data as smooth regular integral models for their corresponding toric varieties; this provides us with an ample source of proper regular arithmetic schemes as testing ground for various conjectures.

Note that from Proposition 2.2.1, the Deligne-Beilinson cohomology $H^r_{\mathcal{D}}(X_\Sigma, \mathbb{Z}(n))$ sits in the long exact sequence

$$\cdots \to H^{r-1}(X_\Sigma, \mathbb{C})/F^n_D H^{r-1}(X_\Sigma, \mathbb{C}) \to H^r_{\mathcal{D}}(X_\Sigma, \mathbb{Z}(n)) \to H^r(X_\Sigma, \mathbb{Z}(n)) \to \cdots .$$

Consider the term $H^{r-1}(X_\Sigma, \mathbb{C})/F^n_D H^{r-1}(X_\Sigma, \mathbb{C}) = \mathbb{H}^{r-1}(\overline{X_\Sigma}, \Omega^{<n}_{X_{\overline{\Sigma}}}(\log D))$. It is well-known that a smooth quasi-projective toric variety X_Σ has a smooth projective toric compactification $X_{\overline{\Sigma}}$ determined by a complete fan $\overline{\Sigma}$ containing Σ as a subfan. Moverover, such compactification is automatically a good compactification of X_Σ and that $\Omega^\bullet_{X_{\overline{\Sigma}}}(\log D))$ are complexes of coherent sheaves on $X_{\overline{\Sigma}}$. By GAGA, it is natural to pass the hypercohomology computation to the associated analytic complex manifold $X_{\overline{\Sigma}}(\mathbb{C})$. Let $\mathcal{A}^n_{X_{\overline{\Sigma}},\mathbb{R}}(\log D)$ be the sheaf of C^∞ n-forms on $X_{\overline{\Sigma}}(\mathbb{C})$ which is invariant under the complex conjugation. Then $\mathcal{A}^\bullet_{X_{\overline{\Sigma}},\mathbb{R}}(\log D)$ forms an acyclic resolution to $\Omega^\bullet_{X_{\overline{\Sigma}},an}(\log D))$ which respects the Hodge filtration. In particular, we have

$$\begin{aligned}\mathbb{H}^{r-1}(\overline{X_\Sigma}, \Omega^{<n}_{X_{\overline{\Sigma}}}(\log D)) &= \mathbb{H}^{r-1}(\overline{X_\Sigma}(\mathbb{C}), \Omega^{<n}_{X_{\overline{\Sigma}}}(\log D)) \\ &= H^{r-1}(\overline{X_\Sigma}(\mathbb{C}), \mathcal{A}^{<n}_{X_{\overline{\Sigma}},\mathbb{R}}(\log D)).\end{aligned}$$

As $\mathcal{A}^{<n}_{X_{\overline{\Sigma}},\mathbb{R}}(\log D)$ is concentrated at degrees less than n, $H^{r-1}(\overline{X_\Sigma}(\mathbb{C}), \mathcal{A}^{<n}_{X_{\overline{\Sigma}},\mathbb{R}}(\log D))$ vanishes whenever $r \geq n+1$. Therefore we have shown

Proposition 3.6.1. *Let X_Σ be a smooth complex quasi-projective toric variety. Then for $r \geq n+1$,*

$$H^r_{\mathcal{D}}(X_\Sigma, \mathbb{Z}(n)) \simeq H^r(X_\Sigma, \mathbb{Z}(n)). \tag{3.6.1}$$

In particular, $H^r_{\mathcal{D}}(X_\Sigma, \mathbb{Z}(n))$ are finitely generated abelian groups for $r \geq n+1$.

As a consequence, we have Theorem 3.4.1 for free in the toric case and extend it further to the range $r \geq n+1$. This is good news, as usually the étale regulator map is quite mysterious in the middle range $n+1 < r \leq 2n$, while from Proposition 2.3.4 $\Phi^{r,n}_{et}$ coincides with $\Phi^{r,n}$ when $r \leq n+1$.

Bibliography

[1] M. Artin, B. Mazur, *Etale Homotopy*, Lecture Notes in Mathematics **100**, Springer-Verlag, Berlin, 1986.

[2] A. A. Beilinson, Higher regulators and values of L-functions, J. Soviet Math. **30** (1985), 2036-2070.

[3] S. Bloch, *Algebraic Cycles and Higher K-theory*, Adv. in Math. **61**, No. 3 (1986), 267-304.

[4] S. Bloch, *Algebraic Cycles and the Beilinson Conjectures*, Comtemp. Math., Vol. **58**, Part I (1986), 65- 79.

[5] J.I. Burgos Gil, E. Feliu, *Higher Arithmetic Chow Groups*, Preprint, available online at, arXiv.org:0907.5169, 2009.

[6] D. Burns, M. Flach, *On Galois structure invariants associated to Tate motives*, Amer. J. Math. **120** (1998), 1343-1397.

[7] D. A. Cox, J. B. Little, H. K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, Vol. **124**, AMS (2011).

[8] H. Esnault, E. Viehweg, *Deligne-Beilinson cohomology, in Beilinson's Conjectures on Special Values of L-functions*, Editors M. Rapaport, P. Schneider and N. Schappacher, Perspect. Math., Vol. **4**, Academic Press, Boston, MA (1988), 43-91.

[9] M. Flach, B. Morin, *On the Weil-étale topos of regular schemes*, Documenta Mathematica **17** (2012), 313-399.

[10] W. Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, Vol. **131**, Princeton University Press (1993).

[11] T. Geisser, M. Levine, *The Bloch-Kato conjecture and a theorem of Suslin-Voevodsky*, J. reine angew. Math. **530** (2001), 55-103.

[12] J. Getz, M. Goresky, *Hilbert Modular Forms with Coefficients in Intersection Homology and Quadratic Base Change*, Progress in Mathematics **298**, Birkhäuser (2012) .

[13] H. Gillet, *Deligne homology and Abel-Jacobi maps*, Bull. AMS **10** (1984), 285-288.

[14] A. Goncharov, *Polylogarithms, regulators, and Arakelov motivic complexes*, J. Amer. Math. Soc.**18** No.1 (2004), 1-60.

[15] A. Holmstrom, J. Scholbach, *Arakelov motivic cohomology I*, Preprint, available online at, arXiv.org:1012.2523v3, 2013.

[16] U. Jannsen, *Deligne homology, Hodge $\mathcal{D}$-conjecture, and motives,* in Beilinson's conjectures on special values of L-functions (Academic Press, Boston, MA 1988), 305-372.

[17] B. Kahn, *The Geisser-Levine method revisted and algebraic cycles over a finite field*, Math. Ann. **324 (2002)**, 581-617.

[18] M. Kerr, J. D. Lewis, S. Müller-Stach, *The Abel-Jacobi map for higher Chow groups*, Compositio **142** (2006), 374-396.

[19] B. Morin, *Zeta functions of regular arithmetic schemes at $s = 0$*, Duke Math. J. **163**, No. 7 (2014), 1263-1336.

[20] A. Rosenschon, V. Srinivas, *Etale Motivic Cohomology and Algebraic Cycles*, in preparation.

[21] N. Spaltenstein, *Resolutions of unbounded complexes*, Compositio **65** (1988), 121-154.

[22] R. O. Wells, *Differential Analysis on Complex Manifolds*, Graduate Texts in Mathematics **65**, Springer, New York, 1980.