

PART I: SYMMETRY BREAKING OF WATER WAVES.  
PART II: ON THE SUPERHARMONIC INSTABILITY  
OF SURFACE WATER WAVES.

Thesis by  
Juan Antonio Zufiria

In Partial Fulfillment of Requirements  
for the Degree of  
Doctor of Philosophy

California Institute of Technology  
Pasadena, California

1987

(Submitted April 16, 1987)



## Acknowledgments

First of all I would like to thank my advisor Dr. Philip G. Saffman for his encouragement, for his many helpful suggestions and for his great patience in guiding my work. I wish also like to thank my undergraduate advisors Dr. J. Jiménez, A. Linan and J.R. Sanmartin for their advice during my several years of research in Spain, as well as for advising me to come to Caltech for my graduate work.

My work at Caltech has been supported by the Ministry of Education of Spain through a Fulbright award, the US-Spanish Joint Committee for Cultural and Educational Cooperation through a scholarship, the Office of Naval Research under contract N00014-79-C-0412, and the National Science Foundation under contract OCE-8415988.

I would also like to thank my parents for all the things that they have done for me throughout these years.

And finally, I wish to thank my wife *Maité* for her love and encouragement since we started this adventure. Without her all this work would not have been possible. This is for her.

## Abstract

### PART I:

*A weakly nonlinear Hamiltonian model for two dimensional irrotational waves on water of finite depth is developed. The truncated model is used to study families of periodic travelling waves of permanent form. It is shown that nonsymmetric periodic waves exist, which appear via spontaneous symmetry breaking bifurcations from symmetric waves.*

*In order to check these results with the full water wave equations, two different methods are used to calculate nonsymmetric gravity waves on deep water. It is found that they exist and the structure of the bifurcation tree is the same as the one found for waves on water of finite depth using the weakly nonlinear Hamiltonian model. One of the methods is based on the quadratic relations between the Stokes coefficients discovered by Longuet-Higgins (1978a). The other method is a new one based on the Hamiltonian structure of the water wave problem.*

*Another weakly nonlinear model is developed from the Hamiltonian formulation of water waves to study the bifurcation structure of gravity-capillary waves on water of finite depth. It is found that, besides a very rich structure of symmetric solutions, nonsymmetric Wilton ripples exist. They appear via spontaneous symmetry breaking bifurcation from symmetric solutions. The bifurcation tree is similar to that for gravity waves. The solitary wave with surface tension is studied with the same model close to a critical depth. It is found that the solution is not unique, and further nonsymmetric solitary waves are possible. The bifurcation tree has the same structure as for the case of periodic waves. The possibility of checking these results*

*in low gravity experiments is discussed.*

## **PART II:**

*Saffman's (1985) theory of the superharmonic stability of two-dimensional irrotational waves on fluid of infinite depth has been generalized to solitary and periodic waves of permanent form on fluid of finite uniform depth. The frame of reference for the calculation of the Hamiltonian for periodic waves of finite depth is found to be the frame in which the mean horizontal velocity is zero.*

*Also, a simple analytical model has been constructed to demonstrate Saffman's (1985) theory. The model shows the change of geometrical and algebraic multiplicity of the eigenvalues and eigenvectors of the stability equation at the critical height. It confirms the existence of Hamiltonian systems with limit points at which there is no change of stability.*

## Table of Contents

Acknowledgments . . . . .	iii
Abstract . . . . .	iv
Table of Contents . . . . .	vi
List of Figures . . . . .	ix
Introduction . . . . .	1
<b>PART I: Symmetry breaking of water waves . . . . .</b>	<b>3</b>
<b>Chapter 1: Nonsymmetric gravity waves</b>	
<b>on water of finite depth . . . . .</b>	<b>4</b>
Section 1.1 Introduction . . . . .	4
Section 1.2 Weakly nonlinear Hamiltonian theory . . . . .	7
Section 1.3 Stokes waves model . . . . .	12
Section 1.4 Computations and results . . . . .	16
References . . . . .	28
<b>Chapter 2: Nonsymmetric gravity waves on water of infinite depth</b>	<b>31</b>
Section 2.1 Introduction . . . . .	31
Section 2.2 Stokes Expansion . . . . .	35
Section 2.2.a General formulation . . . . .	35
Section 2.2.b Symmetric waves . . . . .	40
Section 2.2.c Numerical results . . . . .	42

Section 2.3 Hamiltonian Method . . . . .	50
Section 2.3.a General formulation . . . . .	50
Section 2.3.b Weakly nonlinear form . . . . .	53
Section 2.3.c Numerical algorithm . . . . .	55
Section 2.3.d Numerical results . . . . .	60
Section 2.4 Conclusions . . . . .	64
References . . . . .	68

**Chapter 3: Symmetry breaking in periodic and solitary**

<b>gravity-capillary waves on water of finite depth . . . . .</b>	<b>76</b>
Section 3.1 Introduction . . . . .	76
Section 3.2 Weakly nonlinear model . . . . .	80
Section 3.3 Local analysis . . . . .	87
Section 3.4 Symmetric waves . . . . .	92
Section 3.5 Nonsymmetric waves . . . . .	97
Section 3.6 Solitary wave solutions . . . . .	104
Section 3.7 Conclusions . . . . .	110
References . . . . .	115

**PART II: The superharmonic instability of surface water waves . . . . . 119**

**Chapter 1: The superharmonic instability of finite amplitude**

<b>surface waves on water of finite depth . . . . .</b>	<b>120</b>
Section 1.1 Introduction . . . . .	120
Section 1.2 The Hamiltonian formulation and canonical variables . . . . .	122
Section 1.3 Stability analysis . . . . .	126
References . . . . .	131

**Chapter 2: An example of stability change in a Hamiltonian**

<b>wave system</b>	133
Section 2.1 Introduction	133
Section 3.2 Statement of the problem	134
Section 3.3 Stability	136
References	141



## List of Figures

### PART I:

#### Chapter 1

Figure 1.1 . . . . .	9
Figure 1.2 . . . . .	19
Figure 1.3 . . . . .	21
Figure 1.4 . . . . .	23
Figure 1.5 . . . . .	24
Figure 1.6 . . . . .	25
Figure 1.7 . . . . .	26

#### Chapter 2

Figure 2.1 . . . . .	43
Figure 2.2 . . . . .	45
Figure 2.3 . . . . .	46
Figure 2.4 . . . . .	48
Figure 2.5 . . . . .	49
Figure 2.6 . . . . .	62
Figure 2.7 . . . . .	64
Figure 2.8 . . . . .	65

#### Chapter 3

Figure 3.1 . . . . .	81
Figure 3.2 . . . . .	89

Figure 3.3 . . . . .	93
Figure 3.4 . . . . .	96
Figure 3.5.a . . . . .	98
Figure 3.5.b . . . . .	99
Figure 3.6.a . . . . .	101
Figure 3.6.b . . . . .	102
Figure 3.6.c . . . . .	103
Figure 3.7 . . . . .	106
Figure 3.8 . . . . .	107
Figure 3.9 . . . . .	108
Figure 3.10 . . . . .	110
Figure 3.11 . . . . .	111

PART II:

Chapter 2

Figure 2.1 . . . . .	137
Figure 2.2 . . . . .	139

**List of Tables**

PART I:

Chapter 2

Table 1 . . . . .	71
Table 2 . . . . .	72
Table 3 . . . . .	72

Table 4 . . . . .	73
Table 5 . . . . .	74
Table 6 . . . . .	75

Chapter 3

Table 7 . . . . .	118
-------------------	-----

PART II:

Chapter 2

Table 1 . . . . .	140
Table 2 . . . . .	140

## Introduction

Since early last century one of the problems of fluid mechanics that has attracted a lot of attention in the scientific community is the study of waves propagating on the surface of water. Besides the direct engineering application, this special interest is due to the fact that the equations are highly nonlinear, and possess a rich variety of solutions representing very different physical situations.

Among the great variety of problems related to surface wave propagation on water, a big effort has been devoted to the study of permanent form travelling waves. These waves are basically equilibrium solutions for the water wave problem because the motion is steady in a frame of reference moving with the speed of propagation of the wave.

The present work is devoted to the study of new families of permanent form travelling waves and some stability considerations of these equilibrium solutions in a two dimensional situation. Throughout the entire work the fluid is considered inviscid and the flow irrotational. Under these assumptions a great deal of work has been done since the problem was first considered by Stokes in 1848. However, almost always, the problem has been approached by studying the partial differential equations which follow from the classical water wave formulation.

Here a new tool is introduced to attack the problem. This new tool is the Hamiltonian formulation of the water wave problem, which was introduced by Zakharov

in 1968. Zakharov showed that the classical evolution equations for water waves can be written as evolution equations of a continuous Hamiltonian system.

Discrete Hamiltonian systems of few degrees of freedom have been extensively studied in the field of classical mechanics. Currently, we can say that many aspects of the dynamics of Hamiltonian systems are quite well understood. Thinking about the water wave problem as a Hamiltonian system allows us to apply many of these known results to our problem to answer important questions which have remained open for long time.

The present study has been split into two different parts. The first part deals with the existence of nonsymmetric waves on water. This has been an open question for a very long time. In the first chapter a weakly nonlinear Hamiltonian model for gravity waves on finite depth water is constructed to show that symmetry breaking of water waves is possible. In Chapter 2 the results of Chapter 1 are extended to the full water wave problem in the case of gravity waves on deep water. Finally, Chapter 3 deals with the case of gravity-capillary waves on finite depth water. Again, using a model, the existence of nonsymmetric periodic and solitary waves is shown.

In the second part of the work we deal with the study of superharmonic instabilities of travelling waves. In the first chapter, using the Hamiltonian formulation for waves on water of finite depth, we show how some changes of stability of travelling waves are related to stationary points of the total energy of the waves. This work is an extension to finite depth of results found by Saffman for the case of deep water. In the second chapter, a simple Hamiltonian model is constructed to show examples of the behavior found both in the previous chapter and by Saffman.

**PART I:**

**Symmetry breaking of water waves**

## CHAPTER 1

### Weakly nonlinear nonsymmetric gravity waves on water of finite depth

#### 1.1 Introduction

Frequently in physics the equations that describe a phenomenon are invariant under the action of some symmetry group. In general the solutions have the same symmetry group. In some cases, however, it has been observed that bifurcations can lead to solutions that have a smaller symmetry group. These are examples of spontaneous symmetry breaking. The symmetry group of the equations remains unchanged, but solutions spontaneously break symmetry in the absence of any external perturbation.

A very clear example of this situation is Hopf bifurcation where temporal symmetry is broken. Another physical example of symmetry breaking is the appearance of hexagonal convection cells in Benard convection.

Symmetry breaking plays an important role in several physical disciplines, in-

cluding pattern formation in reaction diffusion problems, convective flows in geophysical phenomena, neurobiology, statistical physics, physical chemistry. For a good analysis of bifurcations in the presence of symmetries using group representation theory the reader is referred to Sattinger(1980,1983).

Since Stokes(1849) discussed the problem of two dimensional progressive symmetric gravity waves, a great deal of work has been done to study all properties of travelling waves of permanent form on water of finite and infinite depth. But to date all the families of waves that have been found correspond to symmetric waves. To define what symmetry means, consider a travelling wave of permanent form that is given by a function  $f(x)$ , which describes the shape. A wave is said to be symmetric if the origin of the wave can be chosen such that  $f(x) = f(-x)$ . This relation defines a symmetry group.

Several attempts have been carried out trying to find nonsymmetric water waves. For example in the case of deep water Chen & Saffman(1980) found that gravity waves of finite amplitude are not unique. They found bifurcations to families of waves in which all crests are not equal. They did in fact find bifurcations into apparently nonsymmetric solutions, but these solutions proved to be just shifted symmetric waves. They searched for symmetry breaking bifurcations in the regular Stokes family and in two families that bifurcate from this one, that they called Class-2 and Class-3 irregular waves. They were not able to find any genuine symmetry breaking bifurcation. Irregular waves of Class-2 on water of finite depth have been computed by Vanden-Broeck(1983).

Recently Longuet-Higgins (1985) carried out calculations using a method based on a Fourier expansion of the shape of the wave in the potential plane. He did not find any bifurcation to nonsymmetric waves from the regular branch. Later



using the same method we looked for symmetry breaking bifurcations in the Class-3 waves. Some were found, but in regions where the convergence of the method was bad, and the bifurcations moved or disappeared as the number of modes used in the expansion was varied.

During the last decade a lot of work has been done trying to understand the local asymptotic expansion near the crest for the Stokes limiting wave of  $120^\circ$  (Norman 1974, Grant 1973, Longuet-Higgins & Fox 1977, 1978, Olfe & Rottman 1980). Again all the work has considered only symmetric expansions. Work in progress shows that even though the crests are always symmetric in first approximation, nonsymmetric solutions are possible when the influence of neighboring crests is considered.

During the last year the Hamiltonian theory of water waves has proved to be very successful in answering some important questions about the stability of travelling waves of permanent form. Saffman(1985), using Zakharov's(1968) Hamiltonian formulation, proved analytically (thereby verifying Tanaka's(1985 a) computations) that an exchange of stability occurs in finite amplitude water waves of permanent form on deep water for a superharmonic disturbance to every wave whose total energy is stationary. In Chapter 6 we extend this result, using Hamiltonian theory again, to periodic waves of finite depth and the solitary wave (again verifying Tanaka's(1985 b) results for this case). Also MacKay & Saffman(1985) determined under which conditions the crossing of eigenvalues for the linearized problem about a wave of permanent form leads to loss of stability.

In the present paper a weakly nonlinear Hamiltonian model is developed for water waves on finite depth by a direct truncation of the complete Hamiltonian for water waves. With this model all the subharmonic bifurcations found by Chen & Saffman(1980) and Saffman(1980) on deep water and Vanden-Broeck (1983) on

finite depth water are reproduced. The model shows also the possibility of more general bifurcations.

Based on some similarities with area preserving maps that were suggested to us by Dr. Robert MacKay, the bifurcation tree for the Stokes family is studied in more detail, and a symmetry breaking bifurcation is found that leads to nonsymmetric waves.

## 1.2 Weakly nonlinear Hamiltonian theory

Consider two dimensional irrotational water waves in a laterally unbounded domain of constant depth. Miles(1977), extending Zakharov's(1968) and Broer's(1974) work, proved that this system has the following Hamiltonian structure

$$\begin{aligned}\frac{\partial \eta}{\partial t} &= \frac{\delta \mathbf{H}}{\delta \psi}, \\ \frac{\partial \psi}{\partial t} &= -\frac{\delta \mathbf{H}}{\delta \eta},\end{aligned}\tag{1.1}$$

where  $\mathbf{H}$  is the total energy of the waves and  $\delta$  represents a functional derivative ( see Goldstein 1980). The canonical variables  $\eta(x, t)$  and  $\psi(x, t)$  are the surface shape and the velocity potential evaluated at the surface.

$$\psi(x, t) = \varphi(x, \eta(x, t), t),\tag{1.2}$$

where  $\varphi(x, y, t)$  is the velocity potential.

$\mathbf{H}$  is given by

$$\mathbf{H} = \int H(\eta, \psi) dx,\tag{1.3}$$

where the integral extends over one period in the case of periodic waves. The energy density  $H$  is given by

$$H = \frac{1}{2} \int_{-h}^{\eta} (\nabla\varphi)^2 dy + \frac{1}{2} g\eta^2. \quad (1.4)$$

$g$  is the acceleration of the gravity and  $h$  the distance from the origin to the bottom.

To make this formulation useful we have to express the energy density in terms of the canonical variables. This transformation leads to very complicated expressions involving integral operators. Considering small amplitude and long waves, a simple expansion for  $H$  can be obtained. In the present case of finite depth, this expansion contains only spatial derivatives of the canonical variables.

In order to obtain this expansion consider the following dimensionless variables,

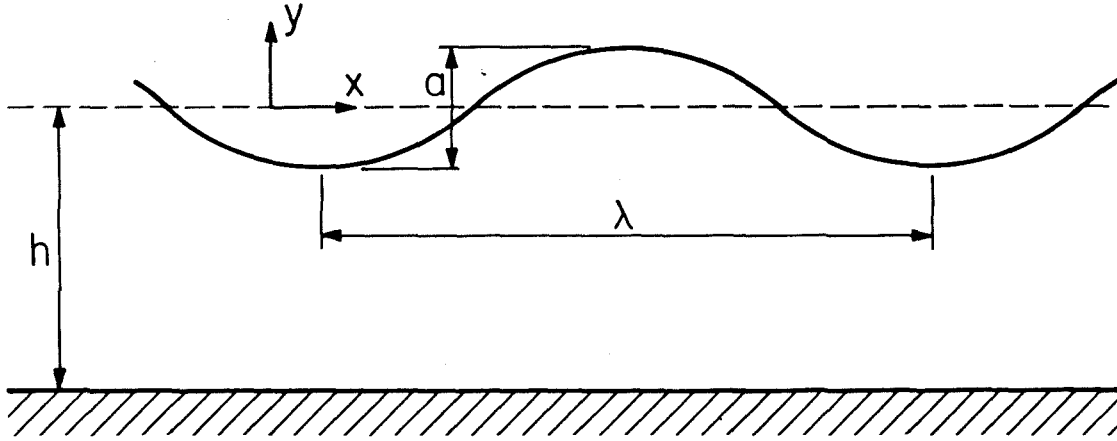
$$\begin{aligned} x' &= \frac{x}{\lambda}, & t' &= t\sqrt{\frac{g}{h}}, & H' &= \frac{\beta^{\frac{1}{2}}}{\alpha^2} \frac{H}{gh^2}, \\ y' &= \frac{y}{h}, & \varphi' &= \frac{h\varphi}{\lambda a\sqrt{gh}}, \\ \eta' &= \frac{\eta}{a}, & \psi' &= \frac{h\psi}{\lambda a\sqrt{gh}}, \end{aligned} \quad (1.5)$$

where  $a$  represents a measure of the amplitude of the waves and  $\lambda$  the typical wave length for the considered waves (see figure 1).  $\alpha$  and  $\beta$  are two parameters defined as follows

$$\alpha = \frac{a}{h}, \quad \beta = \left(\frac{h}{\lambda}\right)^2.$$

$\alpha$  measures the amplitude of the wave and  $\beta$  the dispersion (Whitham 1974).

If the motion is studied in the frame of reference in which the mean horizontal velocity is zero then the dimensionless variables are all of order unity. This change of



**Figure 1.1-** Some notation for periodic travelling waves of permanent form on water of finite depth

variable can be also considered as a canonical transformation (Radder & Dingemans 1985). Dropping the primes the new energy density will be

$$H = \frac{1}{\beta^{\frac{1}{2}}} \left\{ \frac{1}{2} \int_{-1}^{\alpha\eta} (\varphi_x^2 + \frac{1}{\beta} \varphi_y^2) dy + \frac{\eta^2}{2} \right\}. \quad (1.6)$$

If we assume  $\alpha$  and  $\beta$  are small, it is possible to obtain an expansion of  $\varphi_x$  and  $\varphi_y$  in terms of  $\psi, \eta$  and their spatial derivatives.

Taking the Fourier transform, it is clear that

$$\varphi(x, y, t) = \int_{-\infty}^{\infty} \frac{\cosh[k\beta^{\frac{1}{2}}(1+y)]}{\cosh[k\beta^{\frac{1}{2}}]} \hat{\varphi}(k, t) e^{ikx} dk, \quad (1.7)$$

where  $\hat{\varphi}(k, t)$  is the Fourier transform of the value of  $\varphi(x, y, t)$  at  $y = 0$ .

Then  $\psi(x, t)$  is given by

$$\psi(x, t) = \varphi(x, \eta(x, t), t) = \int_{-\infty}^{\infty} \frac{\cosh[k\beta^{\frac{1}{2}}(1 + \alpha\eta)]}{\cosh[k\beta^{\frac{1}{2}}]} \hat{\varphi}(k, t) e^{ikx} dk. \quad (1.8)$$

Expanding the kernel of the integral in powers of  $\alpha$  and inverting the operator it follows

$$\begin{aligned} \hat{\varphi}(k, t) = & \hat{\psi}(k, t) - \alpha \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\eta}(k - k_1) \hat{M}(k_1) \hat{\psi}(k_1, t) dk_1 \\ & + \alpha^2 \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{\eta}(k - k_2) \hat{\eta}(k_2 - k_1) \hat{M}(k_1) \hat{\psi}(k_1, t) dk_1 dk_2 \\ & - \frac{\alpha^2 \beta}{2} \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{\eta}(k - k_1 - k_2) \hat{\eta}(k_2) k_1^2 \hat{\psi}(k_1, t) dk_1 dk_2 \\ & + O(\alpha^3), \end{aligned} \quad (1.9)$$

where  $\hat{\eta}$  and  $\hat{\psi}$  are the Fourier transforms of  $\eta$  and  $\psi$  respectively, and

$$\hat{M}(k) = \beta^{\frac{1}{2}} k \tanh(k\beta^{\frac{1}{2}}) \quad (1.10)$$

Assuming  $\beta \ll 1$  equation (1.7) can be expanded in Taylor series giving

$$\begin{aligned} \varphi(x, y, t) = & \int_{-\infty}^{\infty} \left[ 1 + y \hat{M}(k) + \beta^2 k^2 \frac{y^2}{2} + \beta k^2 \hat{M}(k) \frac{y^3}{6} + \beta^2 k^4 \frac{y^4}{24} \right] \hat{\varphi}(k, t) e^{ikx} dk \\ & + O(\beta^3). \end{aligned} \quad (1.11)$$

From equation (1.9) the previous relation can be rewritten as

$$\begin{aligned} \varphi(x, y, t) = & \psi + (y - \alpha\eta) M \cdot \psi - \frac{\beta}{2} (y^2 - \alpha^2 \eta^2) \psi_{xx} - \alpha (y - \alpha\eta) M \cdot [\eta M \cdot \psi] \\ & - \frac{\beta y^3}{6} M \cdot \psi_{xx} + \frac{\beta^2 y^4}{24} \psi_{xxxx} + O(\beta^3, \alpha^3), \end{aligned} \quad (1.12)$$

where  $M.\psi$  is defined as follows

$$\begin{aligned} M.\psi &= \int_{-\infty}^{\infty} k\beta^{\frac{1}{2}} \tanh(k\beta^{\frac{1}{2}}) \hat{\psi}(k,t) e^{ikx} dk \\ &= -\beta\psi_{xx} - \frac{1}{3}\beta^2\psi_{xxxx} + O(\beta^3). \end{aligned} \quad (1.13)$$

Finally introducing expressions (1.12) and (1.13) into equation (1.6) the following Hamiltonian density is obtained

$$\begin{aligned} \beta^{\frac{1}{2}} H &= \frac{\eta^2}{2} + \left\{ \frac{\psi_x^2}{2} + \frac{\beta}{6}[\psi_{xx}^2 + 2\psi_x\psi_{xxx}] + \right. \\ &\quad \left. \frac{\beta^2}{15}[\psi_{xxx}^2 + \frac{5}{2}(\psi_{xx}\psi_{xxxx} + \psi_x\psi_{xxxxx})] + O(\beta^3) \right\} \\ &\quad + \alpha \left\{ \frac{\eta\psi_x^2}{2} + \frac{\beta}{2}[\eta(\psi_{xx}^2 + 2\psi_x\psi_{xxx}) + 2\psi_x\psi_{xx}\eta_x] + \right. \\ &\quad \left. \frac{\beta^2}{3}[\eta(\psi_{xxx}^2 + \psi_{xx}\psi_{xxxx} + \psi_x\psi_{xxxxx}) + \eta_x(\psi_x\psi_{xxxx} + \psi_{xx}\psi_{xxx})] \right. \\ &\quad \left. + \frac{\beta^2}{2}[(\psi_{xx}\eta)_{xxx}\psi_x + (\psi_{xx}\eta)_{xx}\psi_{xx}] + O(\beta^3) \right\} \\ &\quad + O(\alpha^2). \end{aligned} \quad (1.14)$$

This expansion is accurate up to terms of order  $\alpha\beta^2$ . The evolution equations that can be obtained from this energy density agree with those obtained by Miles(1977) following a similar expansion, and also with the equations obtained by Hunter & Vanden-Broeck(1983) using a direct expansion of the classical water waves equations for  $\alpha$  and  $\beta$  small.

Keeping only first order terms in  $\alpha$  and  $\beta$  in the expansion of  $H$ , the famous Boussinesq approximation is reproduced. Dropping all terms in  $\alpha$  the linear water wave equations are obtained.

In the present analysis the following truncation of energy density will be considered

$$\begin{aligned} \beta^{\frac{1}{2}} H = & \frac{\psi_x^2}{2}(1 + \alpha\eta) + \frac{\eta^2}{2} + \frac{\beta}{6}[\psi_{xx}^2 + 2\psi_x\psi_{xxx}] \\ & \frac{\beta^2}{15}[\psi_{xxx}^2 + \frac{5}{2}(\psi_{xx}\psi_{xxxx} + \psi_x\psi_{xxxxx})]. \end{aligned} \quad (1.15)$$

This energy density contains one more term  $O(\beta^2)$  in the expansion of the dispersion than the Boussinesq approximation in the theory of weakly nonlinear water waves.

### 1.3 Stokes waves model

The dimensionless variables considered in the previous section are good choices for showing the relative magnitude of the different terms that appear in the equations. In order to write the energy density in a form that is independent of the length scale of the waves, it is convenient instead to use the dimensionless variables

$$\begin{aligned} \eta' = \frac{\eta}{h}, \quad \psi' = \frac{\psi}{h\sqrt{gh}}, \\ x' = \frac{x}{h}, \quad H' = \frac{H}{gh^2}. \end{aligned} \quad (1.16)$$

With these new variables expression (1.15) can be written as follows

$$\begin{aligned} H = & \frac{\psi_x^2}{2}(1 + \eta) + \frac{\eta^2}{2} + \frac{1}{6}[\psi_{xx}^2 + 2\psi_x\psi_{xxx}] \\ & \frac{1}{15}[\psi_{xxx}^2 + \frac{5}{2}(\psi_{xx}\psi_{xxxx} + \psi_x\psi_{xxxxx})], \end{aligned} \quad (1.17)$$

where the primes have been dropped.

The evolution equations corresponding to the energy density given by equation (1.17) are

$$\begin{aligned}\eta_t &= \frac{\delta \mathbf{H}}{\delta \psi} = -\psi_{xx} - \frac{1}{3}\psi_{xxxx} - \frac{2}{15}\psi_{xxxxx} - (\eta\psi_x)_x, \\ \psi_t &= -\frac{\delta \mathbf{H}}{\delta \eta} = -\eta - \frac{\psi_x^2}{2}.\end{aligned}\tag{1.18}$$

Travelling waves of permanent form of this system are solutions of the form

$$\begin{aligned}\eta(x, t) &= \eta(x'), \\ \psi(x, t) &= \psi(x'), \quad x' = x - ct,\end{aligned}\tag{1.19}$$

where  $c$  is the phase speed of the train of waves.

Using relations (1.19) and dropping the primes, the evolution equations (1.18) can be rewritten as

$$\begin{aligned}c\eta_x &= \psi_{xx} + \frac{1}{3}\psi_{xxxx} + \frac{2}{15}\psi_{xxxxx} + (\eta\psi_x)_x \\ c\psi_x &= \eta + \frac{\psi_x^2}{2}.\end{aligned}\tag{1.20}$$

The first equation can be integrated once with respect to  $x$  giving

$$(c - \psi_x)\eta = \psi_x + \frac{1}{3}\psi_{xxx} + \frac{2}{15}\psi_{xxxx} + C.\tag{1.21}$$

The vertical position of the origin of the frame of reference for the waves can always be chosen such that  $C = 0$ . Thus in the following we will assume  $C = 0$  without loss of generality.

With the change  $u = \psi_x$  and using the second of equations (1.20), the problem can be reduced to the single equation

$$\frac{1}{3}u_{xx} + \frac{2}{15}u_{xxxx} = u[(c^2 - 1) - \frac{u}{2}(3c - u)].\tag{1.22}$$



This fourth order equation can be rewritten as the following dynamical system of four first order equations

$$\begin{aligned}\frac{du_1}{dx} &= u_2, \\ \frac{du_2}{dx} &= u_3, \\ \frac{du_3}{dx} &= u_4, \\ \frac{du_4}{dx} &= \frac{15}{2}u_1[(c^2 - 1) - \frac{u_1}{2}(3c - u_1)] - \frac{15}{6}u_3,\end{aligned}\tag{1.23}$$

where

$$u_1 = u, \quad u_2 = u_x, \quad u_3 = u_{xx}, \quad u_4 = u_{xxx}.$$

Equation (1.22) has an energy integral that is

$$E = \frac{2}{15}(u_x u_{xxx} - \frac{1}{2}u_{xx}^2) + \frac{1}{3}\frac{u_x^2}{2} + \frac{u^2}{2}[(c^2 - 1) - u(c - \frac{u}{4})].\tag{1.24}$$

Introducing the change of variables

$$\begin{aligned}q_1 &= u, & p_1 &= -(\frac{2}{15}u_{xxx} + \frac{1}{3}u_x), \\ q_2 &= \frac{2}{15}u_{xx}, & p_2 &= u_x,\end{aligned}\tag{1.25}$$

the system (1.23) can be written in the following form

$$Q_t = JD_Q E,\tag{1.26}$$

where  $Q = (p_1, p_2, q_1, q_2)$  and  $D_Q E$  is the gradient of  $E$  with respect to  $Q$ .  $J$  is the following symplectic linear operator(Arnold 1978)

$$J = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \quad (1.27)$$

Therefore the system is Hamiltonian with a parameter  $c$ . The Hamiltonian is the energy

$$E(p_1, p_2, q_1, q_2) = -p_1 p_2 - \frac{1}{2} \left( \frac{15}{2} q_2^2 + \frac{1}{3} p_2^2 \right) + \frac{q_1^2}{2} [(c^2 - 1) - q_1 (c - \frac{q_1}{4})]. \quad (1.28)$$

Notice that we started with a continuous Hamiltonian system evolving in time. Looking for travelling waves of that system we reduced the equations to a fourth order discrete system evolving in  $x$ , and we have found that this new system is also Hamiltonian. Benjamin (1984) has shown that this is a general property for travelling waves of continuous Hamiltonian systems for which Hamiltonian densities can be expressed in terms of finite order derivatives of the canonical variables, as happens in our model.

Our aim is to study periodic orbits of the system (1.23). These periodic orbits correspond to periodic travelling waves of permanent form. The system has three fixed points for all values of the parameter  $c$ . These fixed points are

$$u_1 = \begin{cases} 0 \\ (3c + \sqrt{c^2 + 8})/2 \\ (3c - \sqrt{c^2 + 8})/2 \end{cases} \quad u_2 = u_3 = u_4 = 0. \quad (1.29)$$

Studying the eigenvalues of the linearized systems around the fixed points, it is possible to see that there are several families of periodic orbits around the fixed points. We are interested in the Stokes waves, and these waves correspond to

periodic orbits around the fixed point  $u_1 = u_2 = u_3 = u_4 = 0$  for  $c > 1$ . This family exists in the present model.

It is important to notice that in the case of deep water Stokes waves form a one parameter family of waves. In the finite depth case there is an additional parameter that is the depth of the fluid. As we have normalized the depth to 1, the additional parameter in our formulation is the period  $L$  of the considered periodic wave. We are interested in studying the properties of the Stokes waves as the phase speed  $c$  is changed, therefore we will fix the period and will consider the solutions depending only on one parameter. A change of the phase speed  $c$  corresponds to a change of the wave amplitude.

#### 1.4 Computations and results

To perform the continuation in  $c$  along the family of periodic orbits we have used the program AUTO developed by Doedel & Kernevez(1985). This program can locate Hopf bifurcation points in dynamical systems, and continue the periodic orbits that appear at those bifurcation points in two parameters using a collocation method.

The code AUTO continues periodic orbits of dynamical systems by writing the problem as a boundary value problem for the set of ordinary differential equations that describe the system and imposing periodic boundary conditions on all the variables. The interval of computation is scaled to  $(0, 1)$  by introducing the period  $L$  of the orbit as a new parameter. To solve the boundary value problem numerically the differential equations are approximated by the method of collocation at  $m$  Gauss points with piecewise Lagrange polynomials that belong to the class  $C[0, 1]$ . This

approach is equivalent to an implicit Runge-Kutta method. With this discretization a pseudo-arclength continuation is used for the computations of the solution branches. Even though the code allows nonuniform meshes, in the present computations the mesh for the collocation was chosen to be regular with  $m = 4$ . Up to a maximum of 80 mesh points were used to assure the convergence in the region of interest.

Hamiltonian systems are degenerate dynamical systems, in the sense that they have families of periodic orbits for fixed values of the parameters (Abraham & Marsden 1978). For example our system has infinitely many periodic orbits for a given value of the parameter  $c$ . These periodic orbits are not isolated. They form a continuous family depending on the period  $L$ . In a generic dynamical system the periodic orbits are isolated, and for given values of the parameters the orbit and the period are determined. Because of this degeneracy in computing periodic solutions of Hamiltonian systems singular Jacobians arise. To avoid this condition it is better to perturb the system for numerical purposes. For the computations we consider the following perturbed system

$$\begin{aligned}
 \frac{du_1}{dx} &= u_2 - \epsilon u_1, \\
 \frac{du_2}{dx} &= u_3, \\
 \frac{du_3}{dx} &= u_4, \\
 \frac{du_4}{dx} &= \frac{15}{2}u_1[(c^2 - 1) - \frac{u_1}{2}(3c - u_1)] - \frac{15}{6}u_3.
 \end{aligned}
 \tag{1.30}$$

This is a two parameter  $(c, \epsilon)$  nondegenerate dynamical system. For  $\epsilon = 0$  we reproduce our original system.

We want to continue a travelling wave of fixed period  $L$  in the parameter  $c$ . The additional constraint of fixing the period  $L$  imposes a relation between the two

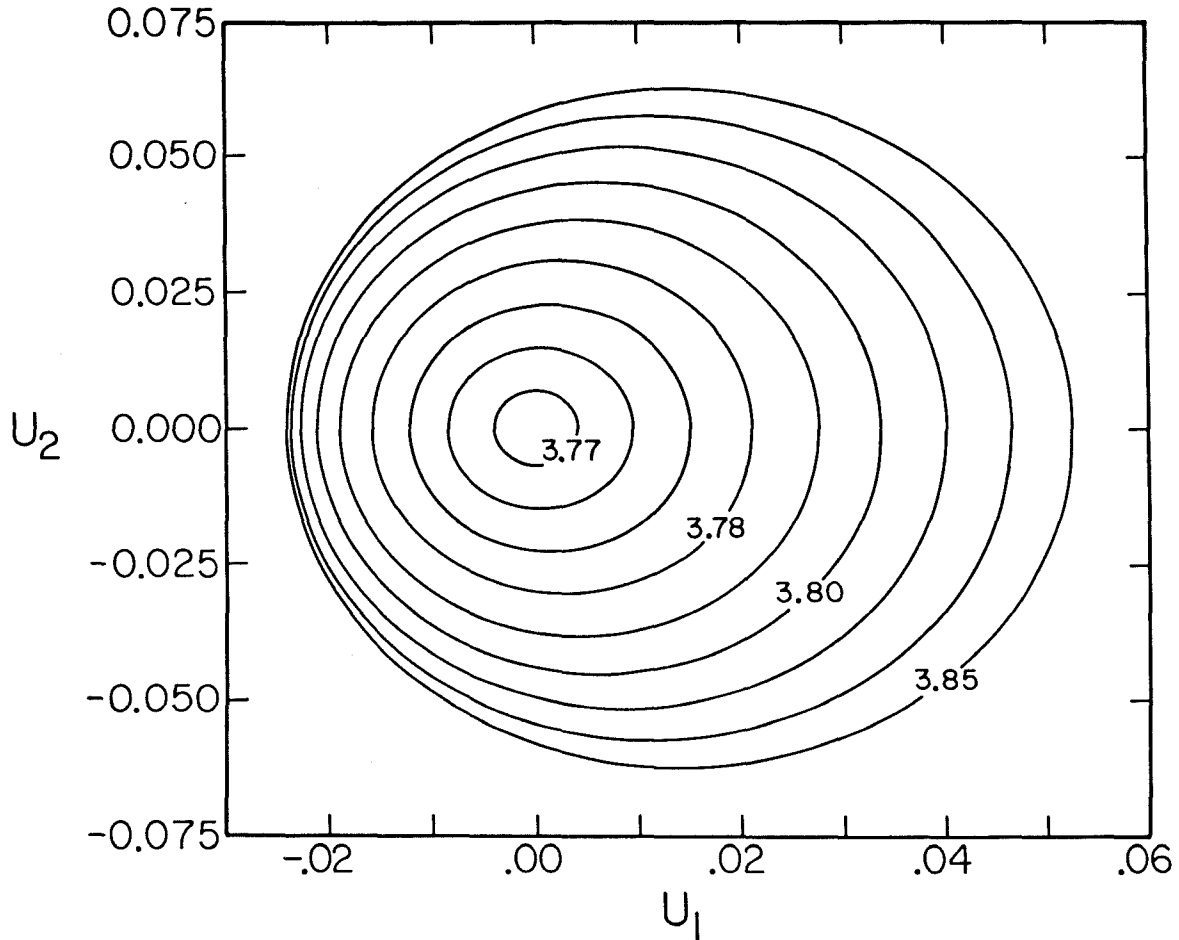
parameters  $c$  and  $\epsilon$  (a codimension one situation). In our case for a given  $c$  and a period  $L$ , the periodic wave and also  $\epsilon$  are determined. Actually we should find that  $\epsilon \equiv 0$  for all solutions because these are in fact solutions of the Hamiltonian system. The value obtained for  $\epsilon$  will serve as a check of the accuracy of the computations.

Looking at the eigenvalues of the linearization of the system (1.30) around the fixed points, it is possible to see that fixed points of our Hamiltonian system (1.23) are Hopf bifurcations in the perturbed system (1.30) when  $\epsilon = 0$ . The bifurcations are in fact vertical Hopf bifurcations (Guckenheimer & Holmes 1983).

To generate a starting orbit for the continuation process, consider the fixed point  $u_1 = u_2 = u_3 = u_4 = 0$  of the perturbed system (1.30). For a fixed value of  $c$  there is a Hopf bifurcation at  $\epsilon = 0$ . We take the branch of periodic orbits that is created at this bifurcation point, and we continue it with  $\epsilon$  as parameter without fixing the period. Notice that the continuation can be done with only one parameter, keeping the other fixed, because the period is not fixed. We find that the branch is vertical, in the sense that  $\epsilon \equiv 0$  in all the branch as we expected. Any point of this branch can be a starting orbit to be continued in  $c$  and  $\epsilon$ . The family of orbits corresponding to this branch are shown in Figure 1.2 .

In all our computations  $\epsilon$  was less than  $10^{-10}$ . In the following we will say that we do the continuation in only one parameter  $c$ , but actually we continue in  $c$  and  $\epsilon$ , and find that  $\epsilon \equiv 0$  for all the solutions.

The program also computes the Floquet multipliers of the orbits. The multipliers give information about the possible bifurcation points and the stability of the orbits. An orbit is said to be *stable* if all the Floquet multipliers lie inside the unit circle, and *unstable* if at least one multiplier lies outside the unit circle. If all Floquet multipliers lie on the unit circle the system is said to be *marginally stable*,



**Figure 1.2-** Periodic orbit for  $c = 1.05$  demonstrating degeneracy of Hamiltonian systems. Numbers denote wavelength

and to determine the stability it is necessary to go to a nonlinear analysis.

It is important to notice that the concept of stability related to the periodic orbits of the dynamical system (1.23) is not the same as the stability of the travelling waves, which is determined by the time dependent analysis of the evolution equations (1.18). The multipliers can be thought of, however, as the magnification of a perturbation in one period.

A periodic orbit always has a multiplier  $+1$  corresponding to sliding a little along the orbit. For Hamiltonian systems  $E$  is conserved so there is always another multiplier  $+1$ . Also we know that the product of the four multipliers has to be  $+1$

because the system is Hamiltonian. Together with the reality of  $E$ , the two other multipliers will be conjugate points  $\sigma, \bar{\sigma}$  on the unit circle or reciprocal points  $\sigma, 1/\sigma$  on the real axis (Green et al. 1981). So our system actually will have two Floquet multipliers not equal to  $+1$ . For proofs of these results the reader is referred to Abraham & Marsden(1978).

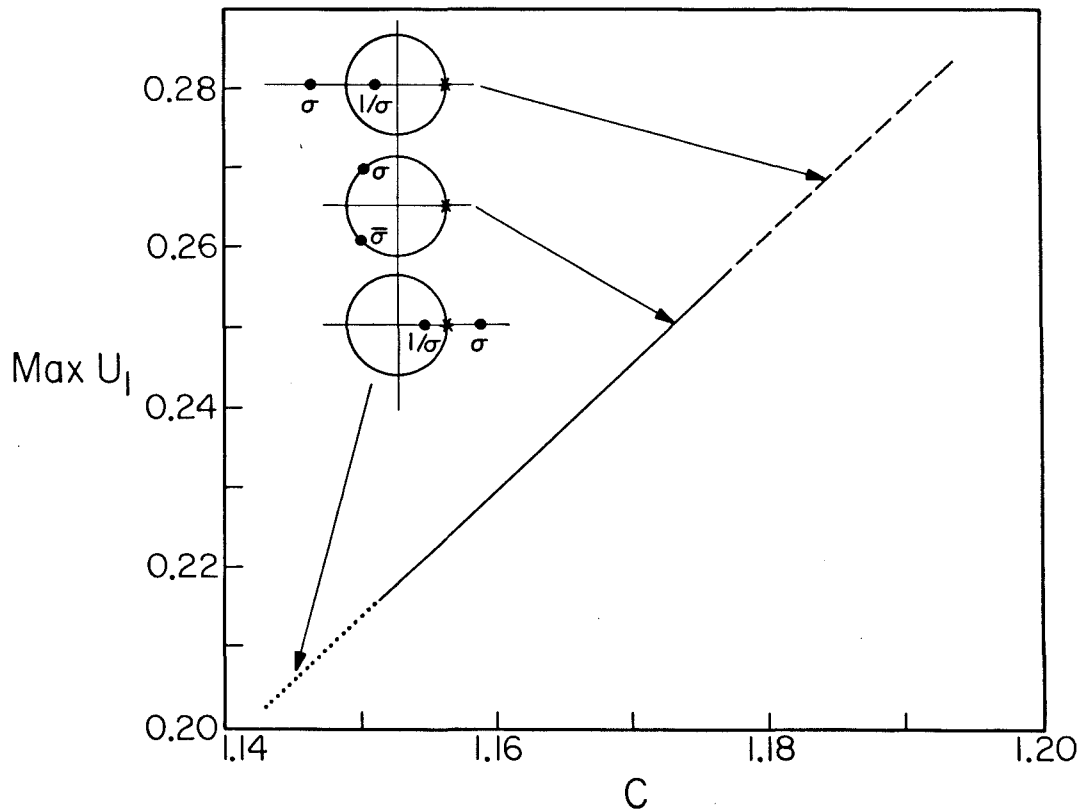
A discrete Hamiltonian system of two degrees of freedom  $(q_1, q_2)$ , via *surface section* (Arnold & Avez 1968) can be reduced to a two dimensional area preserving map. Green et al. (1980), carrying out numerical experiments, found some universal behavior in families of area preserving maps with symmetries. In particular they were able to locate symmetry breaking bifurcations in the bifurcation tree of a universal one parameter map. Because of the analogy with the problem of travelling water waves of permanent form, we expect a similar behavior to occur in the water waves problem. So we study the bifurcation tree of our system with  $c$  as a parameter.

The way in which the starting point was generated fixed the period to be  $L = 3.8$ . With this value of the period we compute the Stokes waves for different values of  $c$ . This branch will be referred to as a period-1 branch (P-1). In this branch for small values of  $c$  the two Floquet multipliers are real and positive. As  $c$  is increased the multipliers go through  $+1$  and lie on the unit circle for a range of values of  $c$ . They then go through  $-1$  and lie again on the real axis, being now negative (see figure 1.3).

When the two multipliers lie on the unit circle, we have possible bifurcation points to new orbits. For example if  $\sigma = e^{2\pi im/n}$ , at that point there is a possible bifurcation to a period- $n$  wave.\* The bifurcations correspond to the subharmonic

---

\* The possibilities of these bifurcations are limited by the conservation of the *Poincare index* (Finn 1974)



**Figure 1.3.-** Maximum of  $u_1(x)$  versus the phase speed for the period  $L$  branch (P-1). Dashed line denotes  $\sigma$  real and negative, dotted line  $\sigma$  real and positive, and solid line  $\sigma$  on the unit circle.

bifurcations that Saffman(1980) found for finite amplitude Stokes waves on deep water. In particular if  $m = 1$  and  $n = 2$  then  $\sigma = -1$  and we have a period doubling bifurcation. The new bifurcated branch corresponds to the Class-2 waves that Chen & Saffman(1980) and Vanden-Broeck (1983) computed.

As our model is qualitatively correct, in the sense that it has the same symmetries and Hamiltonian structure as the original water wave problem, the qualitative agreement with the results given by Saffman (1980), Chen & Saffman(1980) and Vanden-Broeck (1983) is very good. Our results show the same structure in the

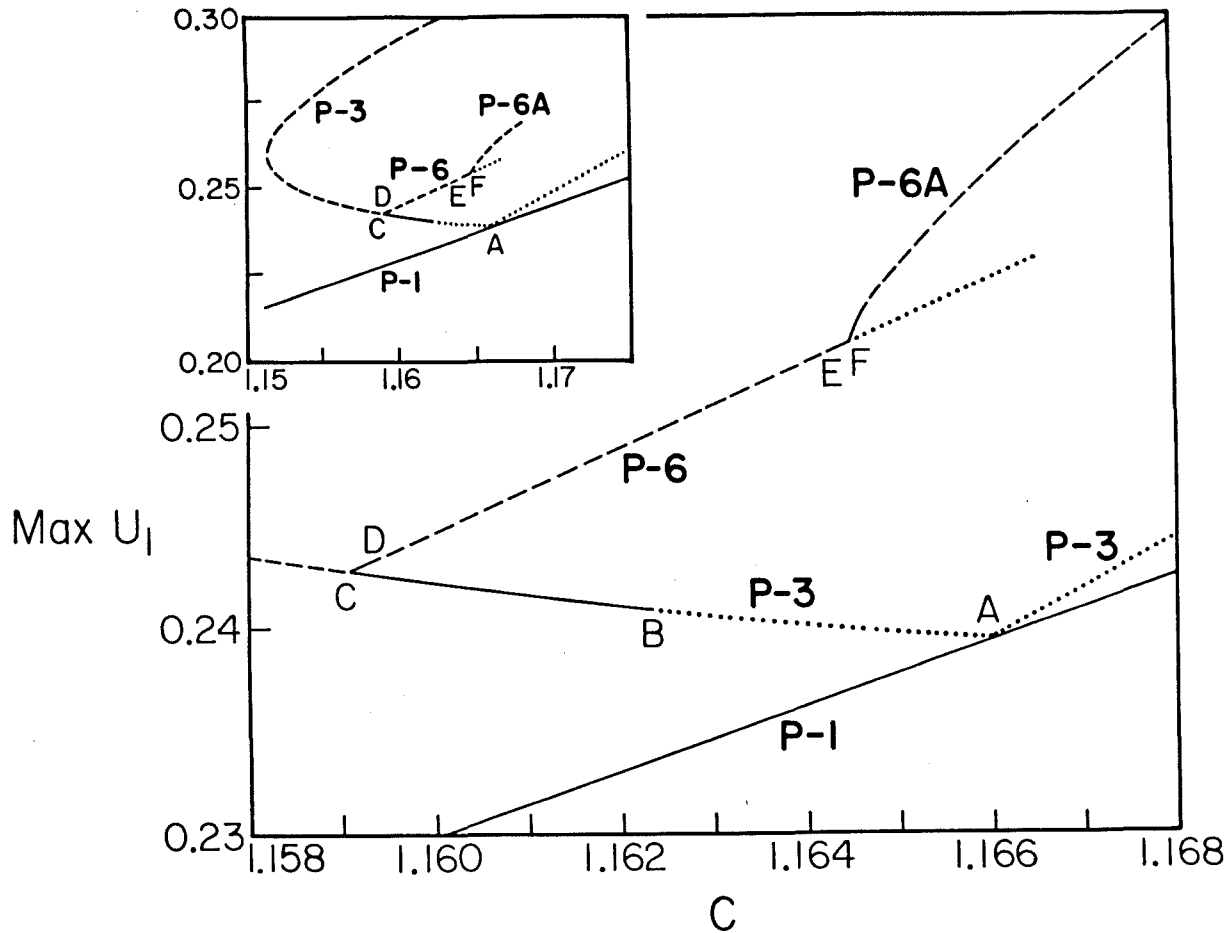


sequence of different crests. However, there are quantitative differences, expected because the model is only an approximation. We find from the results given by Vanden-Broeck (1983) and Cokelet (1977), for the value of the period considered in our computations, the period doubling bifurcation in the full equations occurs at an amplitude  $a/\lambda \approx 0.122$ , whereas the value obtained in our computations is  $a/\lambda \approx 0.095$ .

It is important to say at this point that the Boussinesq approximation (one term less in the expansion of the dispersion) can be reduced to a one degree of freedom Hamiltonian system. In that case there are only two Floquet multipliers and as we stated above they have to stay at  $+1$  for all values of the parameter  $c$ . So the Boussinesq model does not have enough structure to show all this behavior.

Consider now the case  $\sigma = e^{2\pi i/3}$  corresponding to a period tripling bifurcation (point A figure 1.4). We computed the new branch of period  $3L$ . This new branch (P-3) corresponds to the Chen & Saffman Class-3 waves. The waves are still symmetric, but the crests and troughs have unequal levels. A characteristic of this branch is that it exists for smaller and larger values of the phase speed than the value of  $c$  at the period tripling bifurcation point.

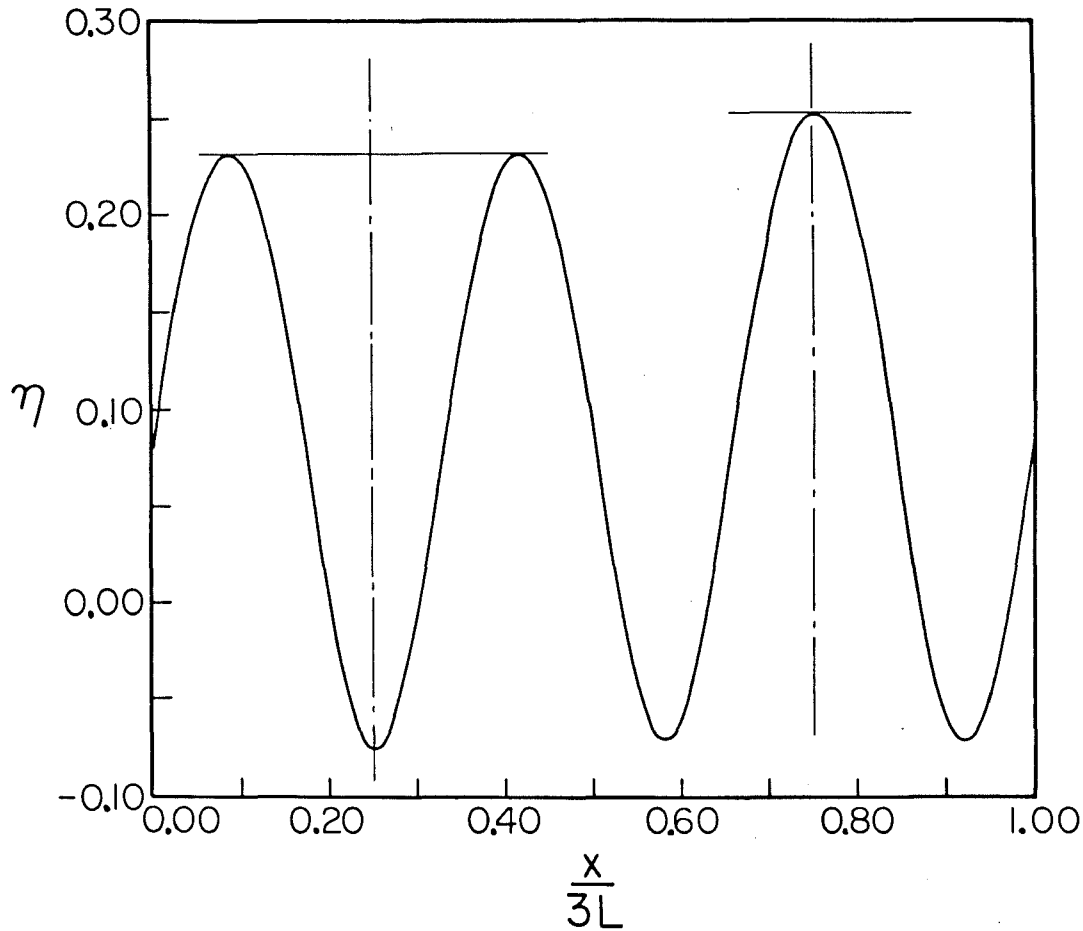
Computing the P-3 branch with the new period  $3L$  we found that at the bifurcation point the Floquet multipliers are at  $+1$ , because for this branch the period tripling bifurcation is a bifurcation point from a wave with the three equal crests to a wave with unequal crests. We found that following the P-3 branch in direction of increasing  $c$  the Floquet multipliers are real positive, and one of them increases ( $\sigma$ ) as the other goes to zero ( $1/\sigma$ ). If the branch is continued in the other direction (decreasing  $c$ ) first the behavior is the same, but  $\sigma$  reaches a maximum and decreases going again through  $+1$  (point B figure 1.4). Then the multipliers move



**Figure 1.4.-** Bifurcation diagram for the family of travelling waves. Insert shows the variation of P-3 branch for larger variation of  $c$ .

along the unit circle, and later leave the unit circle through  $-1$  becoming real and negative. When  $\sigma = -1$ , we have another period doubling bifurcation (point C figure 1.4) This is a bifurcation from a P-3 wave to a P-6 wave. This new bifurcation corresponds to what Chen & Saffman called a secondary bifurcation. Figure 1.5 shows the P-3 wave at point C.

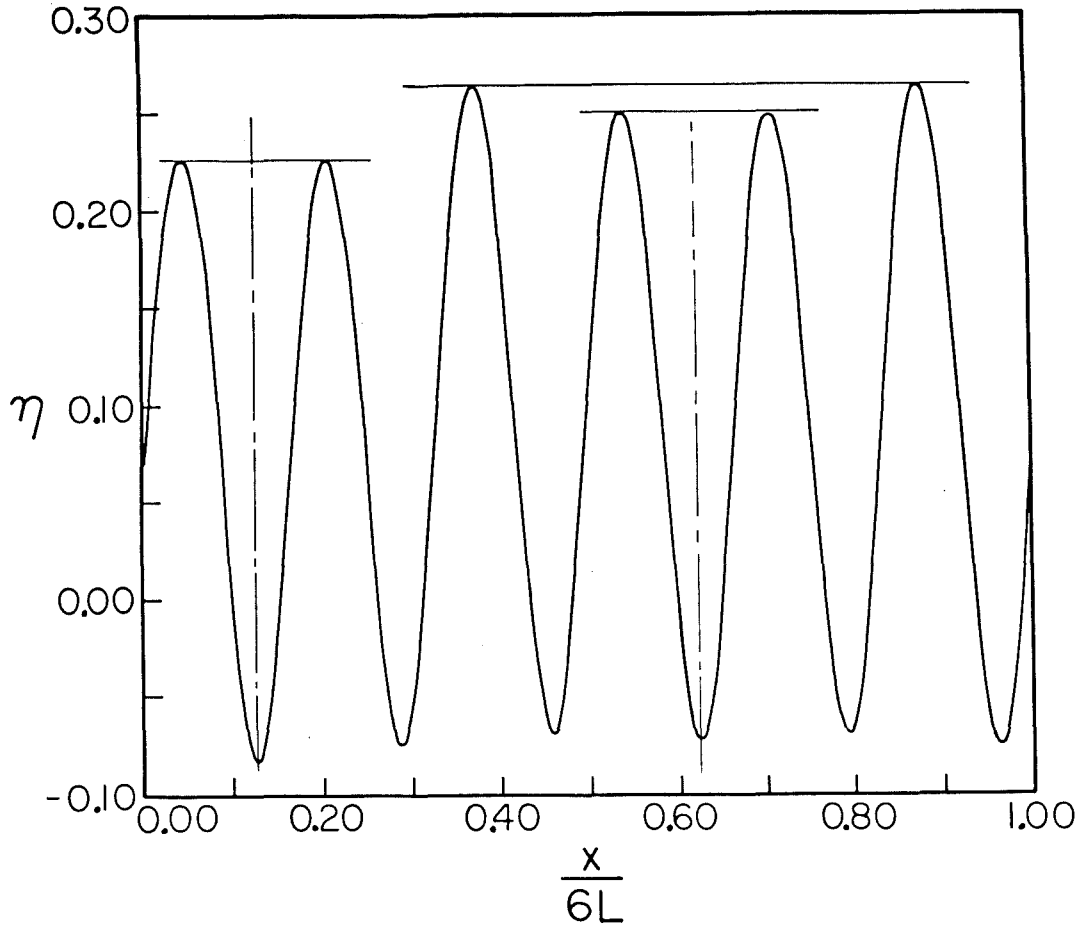
The two branches that are created at the period doubling correspond to the same wave just shifted. Computing with the new period  $6L$  along P-6 we find again that  $\sigma = +1$  at the bifurcation point corresponding to a bifurcation in the P-6 branch. As  $c$  is changed the Floquet multipliers move along the unit circle and go



**Figure 1.5.-** Symmetric travelling wave of period  $3L$  corresponding to point  $C$  in figure 1.4 .

through  $-1$  giving a new period doubling bifurcation to a period  $12L$  wave (point  $D$  figure 1.4). Beyond this period doubling on the  $P-6$  branch, the multipliers are real negative. As  $c$  increases  $\sigma$  increases in absolute value up to a point where  $\sigma$  reaches a maximum and starts decreasing again. The multipliers go again to the unit circle through  $-1$ . So we have a restabilization through an inverse period doubling (point  $E$  figure 1.4).

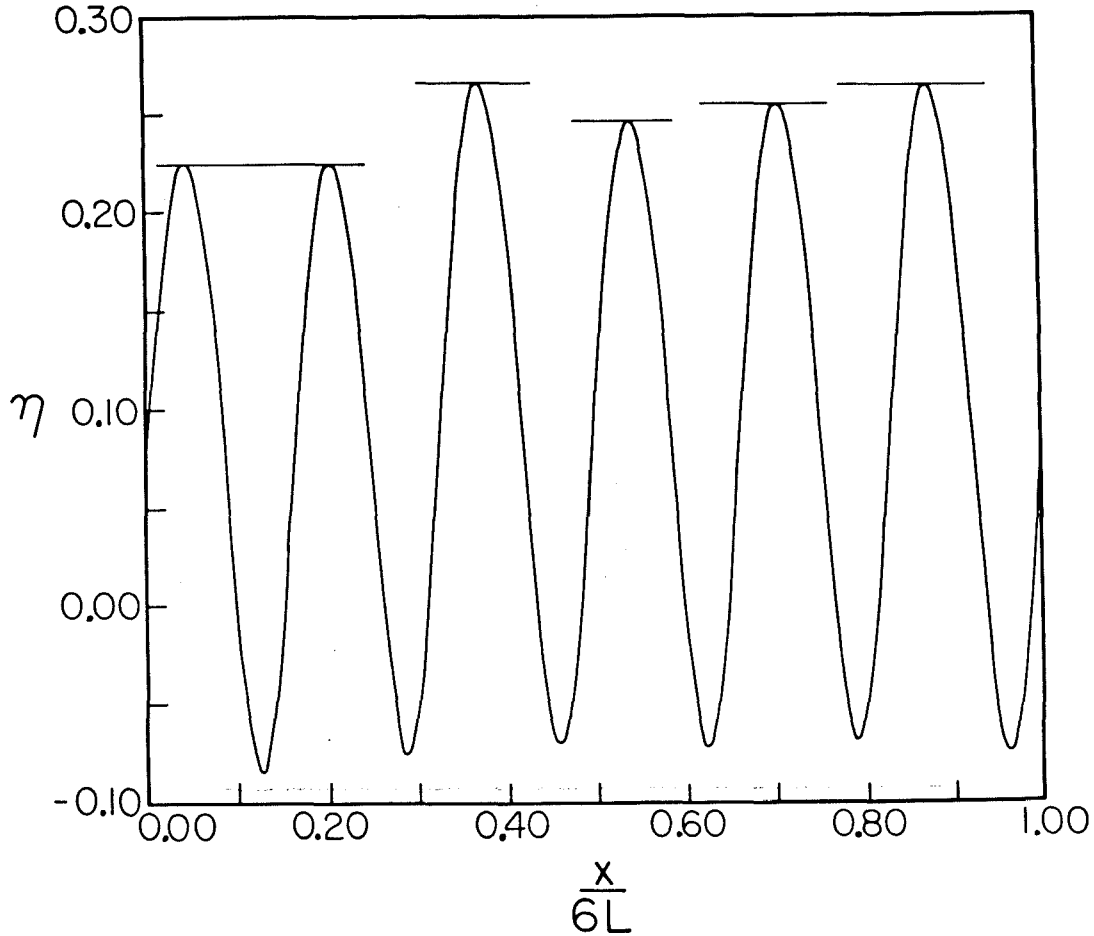
If we keep going on the  $P-6$  branch we find that  $\sigma$  goes again through  $+1$  giving a new bifurcation in which all multipliers are at  $+1$  ( point  $F$  figure 1.4). This kind of bifurcation was studied by Rimmer(1978). The bifurcation can be of two types:



**Figure 1.6.-** Symmetric travelling wave of period  $6L$  corresponding to point  $F$  in figure 1.4 .

a saddle node or a symmetry breaking bifurcation. To determine whether we are in one case or in the other, it is necessary to perform a nonlinear stability analysis of the periodic orbit. A saddle node bifurcation is not actually a bifurcation point in the sense that two families of orbits cross at the bifurcation point, but it indicates an exchange of stability. A saddle node corresponds to a fold on the bifurcation diagram when this is plotted using an appropriated characteristic parameter that the problem has. In general this parameter is the energy of the orbit, and the fold corresponds to an extremum of the energy.

An example of a saddle node bifurcation was found by Tanaka (1985 a) at



**Figure 1.7.-** Nonsymmetric travelling wave of period  $6L$  for  $c = 1.1646$ .

the point of maximum energy for periodic travelling waves of permanent form on deep water ( however Tanaka's concept of stability is different from the one used here). He found that a change of superharmonic stability happens for the wave of maximum energy, and at this point there is no nontrivial bifurcation since the eigenvector, corresponding to the zero eigenvalue, is the eigenvalue corresponding to the horizontal displacement allowed by Galilean invariance. This result was later proved analytically by Saffman (1985) using the Hamiltonian formulation for water waves.

The two previous cases in which we found that all the Floquet multipliers were at

+1 ( branches P-3 (point B figure 1.4) and P-1) are actually saddle node bifurcation points. In the present case (point F figure 1.4) we have found a symmetry breaking bifurcation. The new branch that bifurcates from this point (P-6A) corresponds to period  $6L$  waves, but these are not symmetric any more. Figure 1.6 shows the travelling wave at point F.

In the P-3 branch the sequence of crests was XXY, in the P-6 branch it was XXYZZY. These two sequences are symmetric. In the nonsymmetric branch the sequence is XXYZWY which is not symmetric.

Figure 1.7 shows a wave on the nonsymmetric P-6A branch.

The present study is evidence that the behavior of continuous systems can in some cases be described by a discrete dynamical system of only a few degrees of freedom.

The discovery of nonsymmetric water waves in the truncated model leads us to postulate the existence of nonsymmetric solutions in the full equations for water waves. In particular the existence of these waves on deep water is the subject of the following chapter.

## References

- [1] Abraham, R. and Marsden, J.E.(1978). "Foundations of Mechanics." *2nd ed. Benjamin/Cummings, Massachusetts.*
- [2] Arnold, V.I.(1978). "Mathematical Methods of Classical Mechanics." *Springer-Verlag, New York.*
- [3] Arnold, V.I. and Avez, A.(1968). "Ergodic Problems of Classical Mechanics." *Benjamin, New York.*
- [4] Benjamin, T.B.(1984). "Impulse, Force and Variational Principles." *IMA J. of App. Math.* **32**, 3-68.
- [5] Broer, L.J.F.(1974). "On the Hamiltonian theory of surface waves." *Appl. Sci. Res.* **29**, 430-446.
- [6] Chen, B. and Saffman, P.G.(1980). "Numerical evidence for the existence of new types of gravity waves of permanent form on deep water." *Stud. App. Math.* **62**, 1-21.
- [7] Cokelet, E.D.(1977). "Steep gravity waves in water of arbitrary uniform depth." *Phil. Trans. Roy. Soc. A* **286**, 183-230.
- [8] Doedel, E.J. and Kernevez, J.P.(1985). "Software for continuation problems in ordinary differential equations with applications." (to appear)
- [9] Finn, J.M.(1974) *Doctoral Thesis (U. Maryland)* App. B, p.242.
- [10] Goldstein, H.(1980). "Classical Mechanics." *2nd ed. Addison Wesley, Reading, MA.*

- [11] Grant, M.A.(1973). "The singularity at the crest of a finite amplitude progressive Stokes wave." *J. Fluid Mech.* **59**, 257-262.
- [12] Green, J.M., Mackay, R.S., Vivaldi, F. and Feigenbaum, M.J.(1981). "Universal behavior in families of area-preserving maps." *Physica* **3D**, 468-486.
- [13] Guckenheimer, and Holmes, P.(1983). "Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields." *Springer-Verlag, New York*.
- [14] Hunter, D.H. and Vanden-Broeck, J.M.(1983). "Solitary and periodic gravity-capillary waves of finite amplitude." *J. Fluid Mech.* **134**, 205-219.
- [15] Longuet-Higgins, M.S.(1985). "Bifurcation in gravity waves." *J. Fluid Mech.* **151**, 457-475.
- [16] Longuet-Higgins, M.S. and Fox, M.J.H.(1977). "Theory of the almost-highest wave: the inner solution." *J. Fluid Mech.* **80**, 721-741.
- [17] Longuet-Higgins, M.S. and Fox, M.J.H.(1978). "Theory of the almost-highest wave. Part 2. Matching and analytic extension." *J. Fluid Mech.* **85**, 769-786.
- [18] Mackay, R.S. and Saffman, P.G.(1985). "Stability of water waves." *Proc. Roy. Soc.*(to appear).
- [19] Miles, J.W.(1977). "On Hamilton's principle for surface waves." *J. Fluid Mech.* **83**, 153-158.
- [20] Norman, A.C.(1974). "Expansions for the shape of maximum amplitude Stokes waves." *J. Fluid Mech.* **66**, 261-265.
- [21] Olfe, D.B. and Rottman, J.W.(1980). "Some new highest-wave solutions for deep-water waves of permanent form." *J. Fluid Mech.* **100**, 801-810.



- [22] Radder, A.C. and Dingemans M.W.(1985). "Canonical Equations for almost periodic, weakly nonlinear gravity waves." *Wave Motion* **7**, 473-485.
- [23] Rimmer, R.(1978). "Symmetry and bifurcation of fixed points of area preserving maps." *J. Diff. Equations* **29**, 329-344.
- [24] Saffman, P.G.(1980). "Long wavelength bifurcation of gravity waves on deep water." *J. Fluid Mech.* **101**, 567-581.
- [25] Saffman, P.G.(1985). "The superharmonic instability of finite-amplitude water waves." *J. Fluid Mech.* **159**, 169-174.
- [26] Sattinger, D.H.(1980). "Bifurcation and symmetry breaking in Applied Mathematics." *Bull. Amer. Math. Soc.* **3**, 779-819.
- [27] Sattinger, D.H.(1983). "Branching in the presence of symmetry." *CBMS-NSF Reg. Conf. in Appl. Math. SIAM, Philadelphia.*
- [28] Stokes, G.G.(1849) *Trans. Camb. Phil. Soc.* **8**, 441.
- [29] Tanaka, M.(1985 a). "The stability of steep gravity waves." *J. Fluid Mech.* **156**, 281-289.
- [30] Tanaka, M.(1985 b). "The stability of solitary waves." *Phys. Fluids* (to appear).
- [31] Vanden-Broeck, J.M.(1983). "Some new gravity waves in water of finite depth." *Phys. Fluids* **26**, 2385-2387.
- [32] Whitham, G.B.(1974). "Linear and Nonlinear waves." *Wiley-Interscience, New York.*
- [33] Zakharov, V.E.(1968). "Stability of periodic waves of finite amplitude on the surface of deep fluid." *J. App. Mech. Tech. Phys.* **2**, 190-194.

## CHAPTER 2

### Nonsymmetric gravity waves on water of infinite depth

#### 2.1 Introduction

The problem of two-dimensional irrotational water waves of permanent form was first considered by Stokes (1849). He found, by using successive approximation methods, periodic travelling wave solutions. Stokes results were limited to small amplitude waves. Another important contribution of Stokes in this field is his argument that if a sharp crest is attained by these waves as the amplitude is increased, the sharp crest must have an angle of  $120^\circ$ . During the last decade, with the use of the computers, Stokes's solutions have been extended to finite amplitude waves. It has also been proved that Stokes argument concerning the  $120^\circ$  singular crest is true (Amick *et al.* 1982).

For the case of finite depth, where shallow water approximations have made it possible to introduce simplified models accessible to analytical tools, not only periodic families have been found for finite depth (cnoidal waves), but also solitary

wave solutions have been found (Boussinesq 1871, Rayleigh 1876, Korteweg *et al* 1895).

If  $x$  is taken to be the horizontal coordinate, the equations for travelling waves are invariant under the change  $x \rightarrow -x$ . This relation defines a symmetry group. A large class of solutions of the equations are invariant under the action of the same symmetry group. These solutions are called symmetric. In other words, a wave is said to be symmetric when, if  $f(x)$  represents the shape of the wave, the origin of the  $x$ -axis can be chosen such that  $f(x) = f(-x)$ . Up to date all the travelling wave solutions that are known correspond to symmetric waves.

It is very well known in physics that bifurcations can lead to solutions that are no longer invariant under the action of the symmetry group. Two typical examples are the Hopf bifurcation and Bénard convection. In a Hopf bifurcation the temporal symmetry is broken, and in Bénard convection the spatial symmetry is broken with the appearance of hexagonal convection cells. Bifurcation in the presence of symmetries has been extensively studied by different authors using group representation theory (Sattinger 1980, 1983).

The possibility of finding nonsymmetric solutions for the problem of water waves has been until now an open question. Another open question is, given that these solutions exist, do they appear via a spontaneous symmetry breaking bifurcation from a symmetric family or as isolated branches of solutions?

The rapidly increasing power of numerical methods to study finite amplitude waves has encouraged some authors to search for these kinds of solutions in finite amplitude waves. Chen & Saffman (1980) studied finite amplitude gravity waves on deep water by numerical methods. They found that gravity waves of finite amplitude are not unique. Bifurcations to families of waves in which all crests

are not equal were found. They computed two new families which they called irregular class-2 and class-3 waves. These waves are still symmetric. They did find symmetry breaking bifurcations of the equations but the new solutions proved to be just shifted symmetric waves. In later work, Saffman (1980), using a method based on the quadratic relations between the Stokes coefficients discovered by Longuet-Higgins (1978a), demonstrated that an infinite set of bifurcations to new families exists for the Stokes family, but all the new branches correspond to symmetric waves. Also Longuet-Higgins (1985), using the same method, showed that there are no bifurcations into nonsymmetric waves from the regular Stokes family, over a certain range.

In Chapter 1, using a weakly nonlinear Hamiltonian model based on small amplitude and dispersion approximations, we found that for water of finite depth families of nonsymmetric waves are possible. These waves appear via spontaneous symmetry breaking bifurcations from families of symmetric waves. His results follow from some universal behaviour found by Green *et al.* (1981) on the bifurcation properties of fixed points of two-dimensional area preserving maps in the presence of one symmetry. In particular Green *et al.* (1981) were able to locate some symmetry breaking bifurcations. The Hamiltonian developed in Chapter 1 could be reduced to a 2 degrees of freedom discrete Hamiltonian system, and thus via surface section, can be reduced to an area preserving map (Arnold & Avez, 1968).

The results found in Chapter 1 raise several questions. The first one is whether these results can be generalized to the full equations or are just properties of the approximation. Another question is whether or not these results hold for deep water waves, where assumptions of small amplitude-depth ratio and low dispersion are no longer valid.

In this paper, motivated by these questions, we compute finite amplitude waves on deep water. We show that the same bifurcation scenario occurs. Therefore nonsymmetric permanent form travelling waves on deep water exist, and they appear via symmetry breaking bifurcations. To confirm our results we have done the computations using two different methods, and we have found the same results.

The first method that we consider is the one used by Longuet-Higgins (1985) and Saffman (1980). It consists of an expansion of the shape of the wave in the potential plane, is based on the Longuet-Higgins's (1978a) quadratic relations. This is a very simple method, gives the most accurate results, and can be very easily implemented.

The second method is based in the Hamiltonian structure of the water waves problem. Because the results found on finite depth are related to the Hamiltonian structure of the problem, we thought that it would be interesting to develop a numerical method that kept the Hamiltonian structure of the original formulation. That method would contain all the symmetries of the problem, and the bifurcation scenario would perhaps be better obtained.

The Hamiltonian method of computation is more complicated and time consuming, but it offers some other advantages. The first one is that the superharmonic stability of the wave solutions can be computed directly without the need to write a more complicated code, as happens in other formulations. This is true because the Jacobian of the set of equations is the stability matrix, as we will see below. A second one is that a few number of modes suffice to reproduce the bifurcation behavior of the problem, whereas in the Stokes expansion method it is found (Saffman private communication) that a minimum number of around 50 modes in a class-1 wave are needed to reproduce the bifurcation structure of the problem.

Another motivation to try this numerical approach came from the successful results obtained by several authors during the last few years on the stability of travelling waves using the Hamiltonian formulation for the water waves problem. Some examples are Saffman (1985), MacKay & Saffman (1986).

## 2.2 Stokes Expansion

### 2.2.a. General Formulation

Consider two-dimensional irrotational surface waves of permanent form travelling with speed  $c$  in the  $x$ -direction on water of infinite depth. In a frame of reference moving with the wave, following Stokes (1880) and taking the basic wavelength to be  $2\pi$ , the  $x$  and  $y$  coordinates of the flow can be expressed in the following form (Longuet-Higgins 1985)

$$x + iy = \frac{1}{c}(\Phi + i\Psi) + \frac{i}{2}a_0 + i \sum_{k=1}^{\infty} a_k e^{ik(\Phi+i\Psi)/c}, \quad (2.1)$$

where  $\Phi$  and  $\Psi$  are the velocity potential and the stream function respectively.

For a general wave the coefficients  $a_k$  are complex, except for  $a_0$  that can always be taken to be real. In the case of symmetric waves the origin of the frame of reference can be always chosen such that all the coefficients  $a_k$  are real. This happens when the origin of the reference frame lies on one of the axes of symmetry of the wave train.

If the horizontal level of the frame of reference is chosen such that the mean surface level is  $\bar{y} = -c^2/2g$ , Longuet-Higgins (1978,1985) has shown that the dy-

namical boundary condition of constant pressure at the free surface ( $\Psi = 0$ ) is equivalent to the set of equations

$$\int_{\Phi=0}^{2\pi c} y e^{-ik\Phi} (dx + idy) = \begin{cases} -\pi c^2 & k = 0, \\ 0 & k = 1, 2, \dots, \end{cases} \quad (2.2)$$

where  $g = 1$  has been taken.

Introducing expression (2.1) into equations (2.2) we obtain the following set of quadratic equations for the coefficients

$$\begin{aligned} a_0 + \sum_{k=1}^{\infty} k a_k a_k^* &= -c^2 & m = 0, \\ (1 + m a_0) a_m + \sum_{k=1}^{m-1} k a_k a_{m-k} + \sum_{k=1}^{\infty} (k + m) a_{k+m} a_k^* &= 0 & m > 0, \end{aligned} \quad (2.3)$$

where the star denotes complex conjugate. The first equation is real and the rest are complex. For computational proposes and for the following discussion it is convenient to split the complex equations into real and imaginary parts. Taking  $a_k = p_k + iq_k$ , we have

$$\begin{aligned} p_m(1 + m a_0) + \sum_{k=1}^{m-1} k(p_k p_{m-k} - q_k q_{m-k}) + \sum_{k=1}^{\infty} (k + m)(p_{k+m} p_k + q_k q_{k+m}) &= 0, \\ q_m(1 + m a_0) + \sum_{k=1}^{m-1} k(p_k q_{m-k} + q_k p_{m-k}) - \sum_{k=1}^{\infty} (k + m)(p_{k+m} q_k - p_k q_{k+m}) &= 0. \end{aligned} \quad (2.4)$$

If we truncate the expansion (2.1) at the term  $a_n$  and take  $a_k = 0$  for  $k > n$ , the first equation of (2.3) together with the set of equations (2.4) form a system of  $2n + 1$  nonlinear algebraic equations with  $2n + 1$  unknowns  $(a_0, p_k, q_k)$ ,  $c$  being the only parameter of the problem.

This scheme, simplified for symmetric waves, has been used by several authors. For the family of Stokes waves, which we will call regular waves because all the crests and troughs are equal, the phase speed  $c$  does not behave monotonically through the full range of steepness. This has made several authors introduce other parameters that behave monotonically for range full of steepness. Longuet-Higgins (1985) introduced the parameter  $Q_c$  defined as follows

$$Q_c = 1 - \frac{1}{2}v_{crest}^2, \quad (2.5)$$

where  $v_{crest}$  is the velocity at the crest of the wave. This parameter ranges from 1/2 to 1.  $Q_c = 1/2$  corresponds to the flat surface, and  $Q_c = 1$  corresponds to the 120° Stokes limiting wave.

Chen & Saffman (1980) used another parameter that they called  $b$ . This parameter  $b$  is defined and related to  $Q_c$  through

$$b = 1 - v_{crest}^2/c^2 = 1 - \frac{2(1 - Q_c)}{c^2}. \quad (2.6)$$

We introduce the parameter  $Q$  as

$$Q = 1 + y_0 = 1 + \frac{1}{2}a_0 + \sum_{k=1}^n p_k, \quad (2.7)$$

where  $y_0$  is the height of the wave at the origin of the  $x$  axis. When the origin is chosen on the crest,  $Q=Q_c$ .

Equation (2.7) can be used to complete system (2.4) instead of using the equation for  $c$ , and then the Stokes family can be studied taking  $Q$  as parameter.



As we will see below, some of the bifurcations that we find break the symmetry with respect to the crests, but not with respect to the troughs. Therefore, it is convenient to take the origin of  $x$  on a trough. In our computations we took the origin on a trough and used equation (2.7) to complete the system.

A property of equations (2.3) is that if a set of coefficients  $\{a_k\}$  represents a wave, the set  $\{\hat{a}_k^N\}$  where

$$\begin{aligned}\hat{a}_0^N &= a_0/N, \\ \hat{a}_{Nk}^N &= a_k/N, \\ \hat{a}_k^N &= 0 \quad \textit{otherwise},\end{aligned}\tag{2.8}$$

is also a solution that describes the same wave but of a different scale. The set  $\{a_k\}$  represents a wave of wavelength  $2\pi$ , and the set  $\{\hat{a}_k^N\}$  represents the same wave with wave length  $2\pi/N$ . This waves are called class-N waves (Saffman, 1980) because there are N crests in the window of  $x$  between 0 and  $2\pi$ .

Another property of the system defined by equations (2.3) is that the system is degenerate in the sense that solutions are not isolated. Due to the Galilean invariance, if a set of coefficients  $\{a_k\}$  satisfies equations (2.3) then the set  $\{a_k e^{ik\epsilon}\}$  is also a solution of (2.3) for all  $\epsilon$ . So for a given value of  $Q$  equations (2.3) do not uniquely determine a solution. To remove this degeneracy we thus need to fix the phase. As we will see below, this degeneracy disappears when considering symmetric waves. To avoid this problem for nonsymmetric waves we will have to introduce some modifications into the system of equations.

Because in Chapter 1 we found the symmetry breaking from a class-6 wave, we will start computing with a regular wave of class-6. Therefore the coefficient  $a_6$  will be the dominant coefficient in a solution set  $\{a_k\}$ . To avoid the Galilean invariance

degeneracy, we will force  $a_6$  to be real ( $q_6 = 0$ ) in our computations. This was actually implemented modifying the equation corresponding to  $a_6$  in (2.3) in the following form

$$q_6 + a_6(1 + 6a_0) + \sum_{k=1}^5 ka_k a_{6-k} + \sum_{k=1}^{\infty} (6+k)a_{6+k} a_k^* = 0. \quad (2.9)$$

The modified system has the same solutions as the original system if  $q_6 = 0$ , and the new system is not degenerate anymore. This simple modification has been found to work very well during our computations.

As we are going to study bifurcations, it is important to examine the Jacobian of the system defined by equations (2.4). This Jacobian has the following form

$$J(p, q) = \begin{pmatrix} A_1(p) & B(q) \\ B(q)^T & A_2(p) \end{pmatrix}, \quad (2.10)$$

where  $p = \{p_k\}$  and  $q = \{q_k\}$ . This Jacobian is always singular, and the eigenvector associated to the zero eigenvalue is the shift. As  $A_1$  and  $A_2$  are symmetric  $J$  is symmetric when  $q = 0$ . This symmetry will simplify some of the computations.

Some properties of the waves, such as the kinetic and potential energies, can be expressed in terms of the set of coefficients  $\{a_k\}$ . For example the kinetic energy per one wavelength in the frame of reference in which the fluid is at rest at the bottom ( $y \rightarrow -\infty$ ) is

$$T = \frac{\pi}{2} c^2 \sum_{k=1}^n ka_k a_k^* = -\frac{\pi}{2} c^2 (a_0 + c^2) \quad (2.11)$$

where the first of equations (2.3) has been used.

The potential energy taking the mean horizontal level of the surface as reference level can be written as

$$\begin{aligned}
 V = \pi \left\{ -\frac{(c^2 + a_0)^2}{2} + \frac{1}{2} \sum_{k=1}^n a_k a_k^* \right. \\
 \left. + \frac{1}{8} \sum_{k=1}^n k \left[ 2 \sum_{j=1}^{n-k} (a_k a_j a_{k+j}^* + a_k^* a_j^* a_{k+j}) + \sum_{j=1}^{k-1} (a_k a_j^* a_{k-j}^* + a_k^* a_j a_{k-j}) \right] \right\}. \quad (2.12)
 \end{aligned}$$

A similar expression for  $V$  was obtained by Longuet-Higgins (1984) for the case of symmetric waves.

### 2.2.b. Symmetric waves

All actual calculations that give waves have assumed symmetric waves due to the inability to find nonsymmetric solutions. When symmetric waves are considered, large simplifications can be introduced in the formulation of the problem. The first one is that the shift can be chosen such that  $q_k = 0$  for all  $k$ , and the truncated system of equations defined by (2.4) and (2.7) reduces to

$$\begin{aligned}
 p_m(1 + ma_0) + \sum_{k=1}^{m-1} k p_k p_{m-k} + \sum_{k=1}^n (m+k) p_{m+k} p_k = 0 \quad m > 0, \\
 \frac{a_0}{2} + \sum_{k=1}^n p_k = Q - 1. \quad (2.13)
 \end{aligned}$$

The degeneracy related to the Galilean invariance disappears because the shift is fixed by the condition  $q = 0$ . As we have already pointed out in the general formulation, a wave appears as a continuous set of solutions by varying the shift  $\epsilon$ , but when the wave is symmetric only a discrete and isolated set of those solutions

satisfies the condition  $q = 0$ . For example, for a regular wave, if we take the origin of the  $x$ -axis at a trough or a crest we find  $q = 0$ . Where the origin is taken is determined by the starting point for the calculations.

If  $q = 0$ , the Jacobian defined by relation (2.10) is

$$J(p, 0) = \begin{pmatrix} A_1(p) & 0 \\ 0 & A_2(p) \end{pmatrix} \quad (2.14)$$

because  $B(0) = 0$ . Notice that  $A_1(p)$  is the Jacobian that appears in the symmetric formulation given by (2.13). As Galilean invariance is not allowed in the symmetric formulation  $A_1(p)$  will not be singular in general. As  $J(p, q)$  is always singular  $A_2(p)$  has to be always singular, and the null eigenvector will be the shift. If we now consider the modification introduced by equation (2.9), then  $A_1(p)$  does not change, but  $A_2(p)$  does. Let us call the new matrix  $A'_2(p)$  and the new Jacobian  $J'(p, 0)$ . These matrices will not be singular in general.

A bifurcation from a family of symmetric waves to a different family corresponds to a zero eigenvalue of  $J'$ . This zero eigenvalue can appear in two ways. One way is via a zero eigenvalue of  $A_1$  that corresponds to a new family of waves that are symmetric again with respect to  $x = 0$ . The second way is a zero eigenvalue of the matrix  $A'_2$ . This point corresponds to a bifurcation to a family of waves that are no longer symmetric with respect to  $x = 0$ . This does not mean that the new branch corresponds necessarily to non-symmetric waves. As Chen & Saffman found, these branches can correspond to symmetric waves for which the origin of  $x$  does not lie on one of the axes of symmetry of the wave. But if nonsymmetric waves exist and they appear through symmetry breaking bifurcations from symmetric waves, the symmetry breaking bifurcation has to correspond to a zero eigenvalue of  $A'_2$ .

For a given set of coefficients  $\{a_k\}$ , whether the set represents a symmetric or

nonsymmetric wave can be determined by computing the ratios  $a_k/a_{k+1}$ . If the wave is symmetric we find that  $\arg(a_k/a_{k+1}) = \text{const.}$  for all  $k$ .

### 2.2.c Numerical results

We started by computing the family of regular Stokes waves. Because of the type of structure that we expect in the bifurcation diagram, we start with a regular class-6 wave (6 equal crests in the interval  $(0, 2\pi)$ ). The starting point was generated using Schwartz's (1974) expansion for low amplitude regular waves.

This branch is symmetric and therefore the system defined by (2.13) was used in the computations. The origin of the x-axis was chosen in a trough. A simple arc-length (Keller 1977) continuation method was used with  $Q$  as the parameter along the branches. The algorithm converged very rapidly in all regions, reaching an error less than  $10^{-12}$  for the quadratic norm of the residues of the equations in 3 or 4 iterations for the step used.

Because in our computations we are going to deal with families of waves for which the 6 crests are not equal, it is convenient to introduce a parameter to measure the amplitude of each of the 6 crests. As a generalization of (2.7) we define

$$Q_j = 1 + y_j, \quad i = 1, 2, 3, 4, 5, 6, \quad (2.15)$$

where  $y_j$  is the height of the different crests (figure 2.1). Notice that if  $Q_j = 1$  for some  $j$ , this means that the  $j^{\text{th}}$  crest reaches the limiting  $120^\circ$  Stokes crest. This is because  $y = 0$  corresponds to zero velocity from Bernoulli's equation.

To make the difference among the families of waves that we are going to study

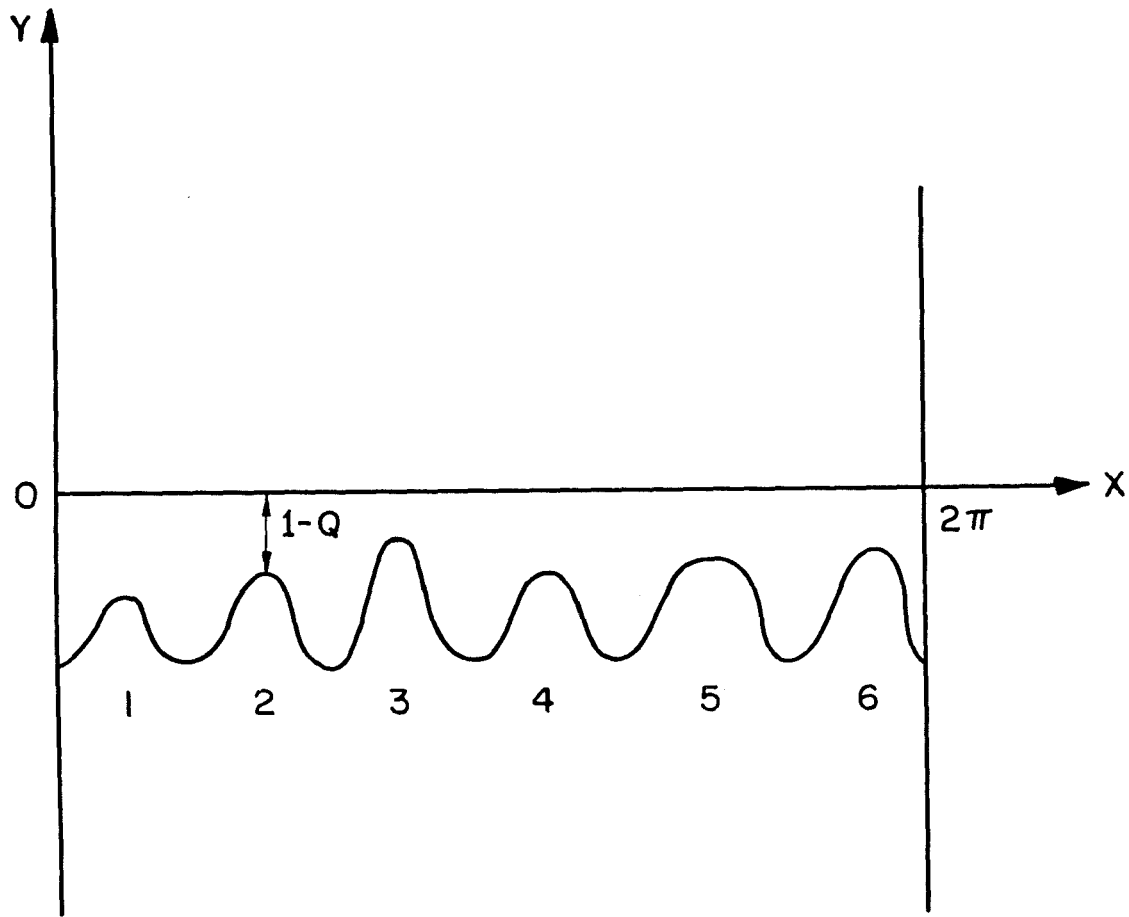


Figure 2.1.- Periodic travelling wave of class-6.

clearer, we introduce the following notation. We associate with each of the 6 crests one of the following letters  $Z, Y, X, W, V, U$  depending on the relative amplitude of the crests. We take  $Z > Y > X > W > V > U$ . A regular wave of class-6 will be denoted by 6 equal letters.

Computations have been performed using  $n=256, 512, 768$  and in some cases up to  $n=1024$ . We have found that for most of the branches in which we are interested  $n=512$  gives good convergence results.

Because the bifurcations that we found are not simple, bifurcation points were detected during the computations by monitoring the determinant and the last 3

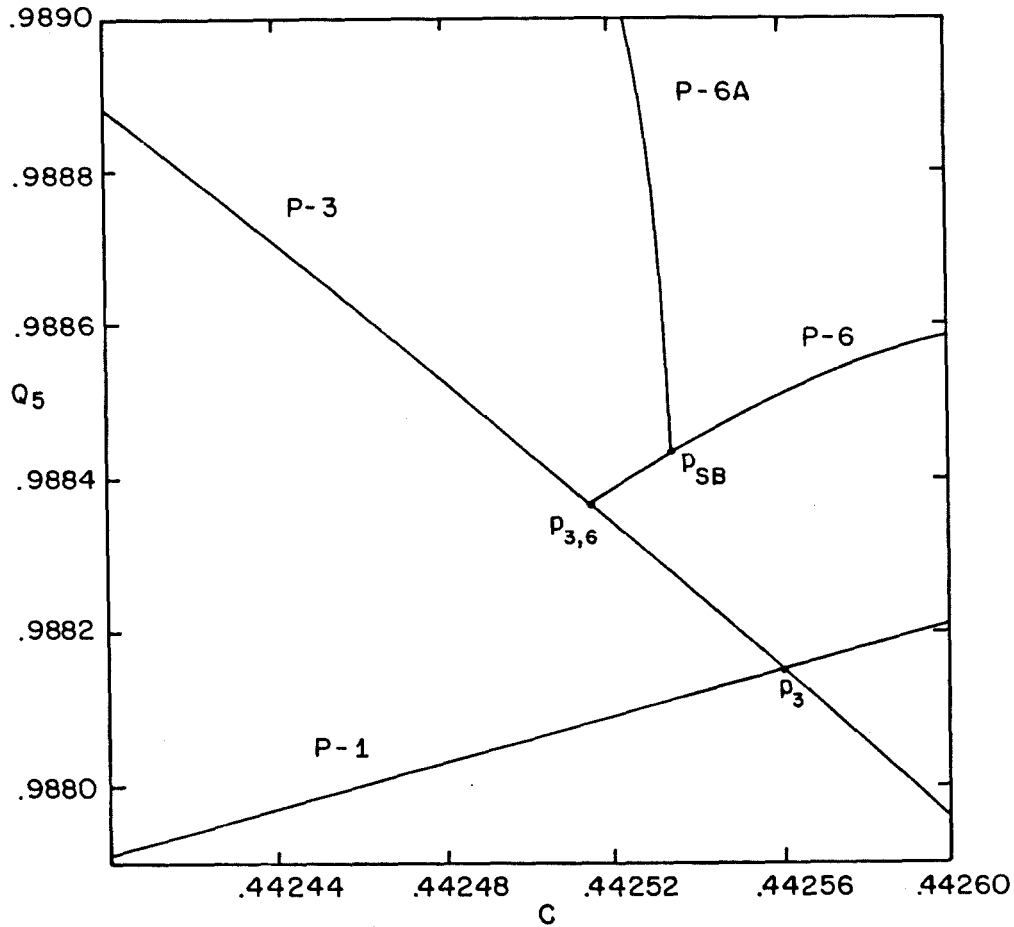
pivots of the Gaussian elimination for the matrices  $A_1$  and  $A'_2$ . In some cases to understand the bifurcations better we also computed the eigenvalues and eigenvectors of  $A_2$ .

To switch branches a code based on Keller's(1977) method was written. But in most of the cases it was found that to switch branches it was enough to perturb the solution at the bifurcation point with the null vector, and impose that the first few Newton iterations of the continuation process be normal to the null vector. This method was found to converge to the new branch in 3 or 4 iterations.

Longuet-Higgins (1985) showed that period doubling and period tripling bifurcations from the regular family appear in very truncated models (  $n=2$  and  $n=3$ ) of equations (2.13). We have found that this is true, but these bifurcations disappear when the number of modes reaches 8, and they do not appear again until the number of modes is around 200 for a class-6 wave.

Starting at low amplitude we computed the regular branch, that we will call the P-1 branch. For this branch the period of the wave is  $L$ , where  $L = 2\pi/6$ . As the amplitude was increased the determinant of  $A_1$  was found to vanish at 3 points, these points corresponding to period six, period three and period two bifurcations. We will refer to these points as  $P_6, P_3$  and  $P_2$  respectively. The point  $P_3$  is shown in figure 2.2 . These bifurcations correspond to the ones that Chen & Saffman(1980) and Saffman (1980) found.  $A'_2$  was singular at the two points  $P_6$  and  $P_3$ . But no non-symmetric families are created at these points. These branches represent just shifted symmetric waves.

The new family of waves that is created at the period tripling bifurcation has the form  $YZYYZY$  (see figure 2.3) and has wavelength  $\pi$ . The branches that appear as nonsymmetric waves with respect to  $x = 0$  have the form  $ZYYZYY$  and

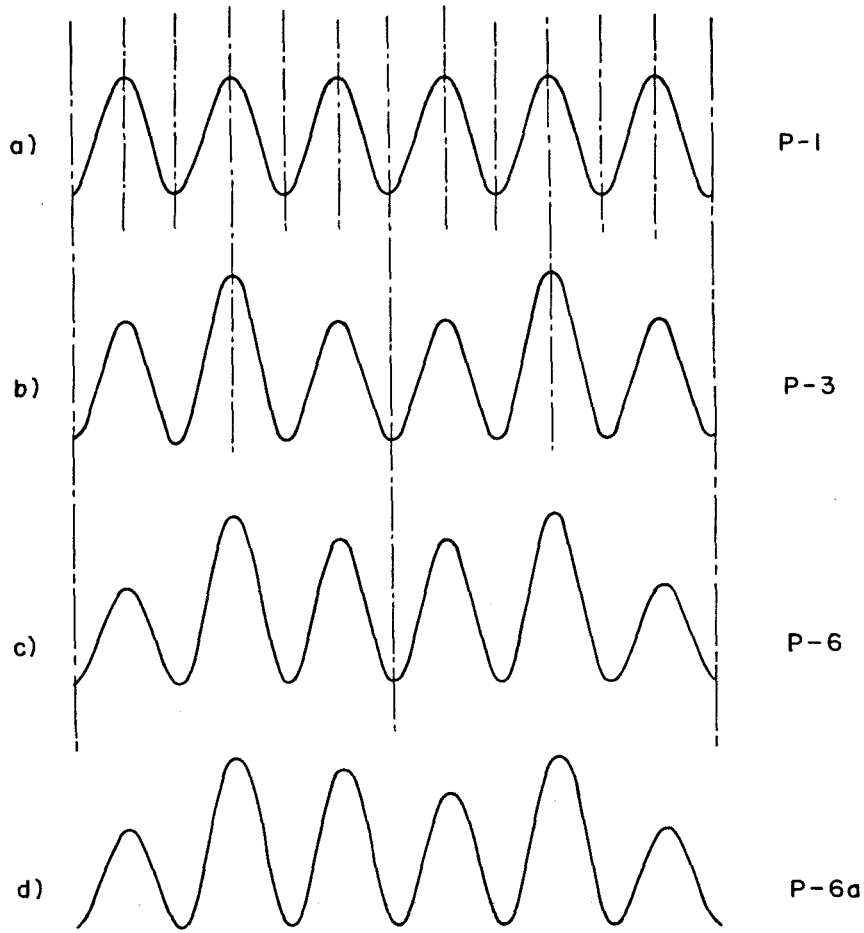


**Figure 2.2.-** Bifurcation diagram for solution branches showing wave height versus wave speed for travelling waves of permanent form on deep water.

$YYZYYZ$ , but are the same waves as  $YZYYZY$ . This  $P-1$  branch was computed using  $n = 256, 512$  and  $768$ . Values of some parameters of the wave at the bifurcation point  $P_3$  are given in table 1. We can see that the agreement with the results of other authors and methods is good. The values obtained for the period six and period doubling bifurcations agree with those given by Saffman (1980) to four figures. In this table,  $ak$  denotes the steepness of the wave which is defined as one half of the waveheight times the wavenumber for the regular class-1 wave.

With  $n = 512$ , computations were carried out up to steeper waves without being





**Figure 2.3.-** Schematic representation showing the relative amplitude of the crests of the travelling waves corresponding to different branches of the bifurcation diagram given in figure 2.2. (a): Wave on branch P-1. (b): Wave on branch P-3. (c): Wave on branch P-6. (d): Wave on branch P-6A. Vertical lines show symmetry axes.

able to find the first maximum in the energy. To obtain convergence in the strongly nonlinear region, it is necessary to increase the number of modes up to around 300 for a class-1 wave (Longuet-Higgins, 1985) that is equivalent to 1800 modes for a class-6 wave.

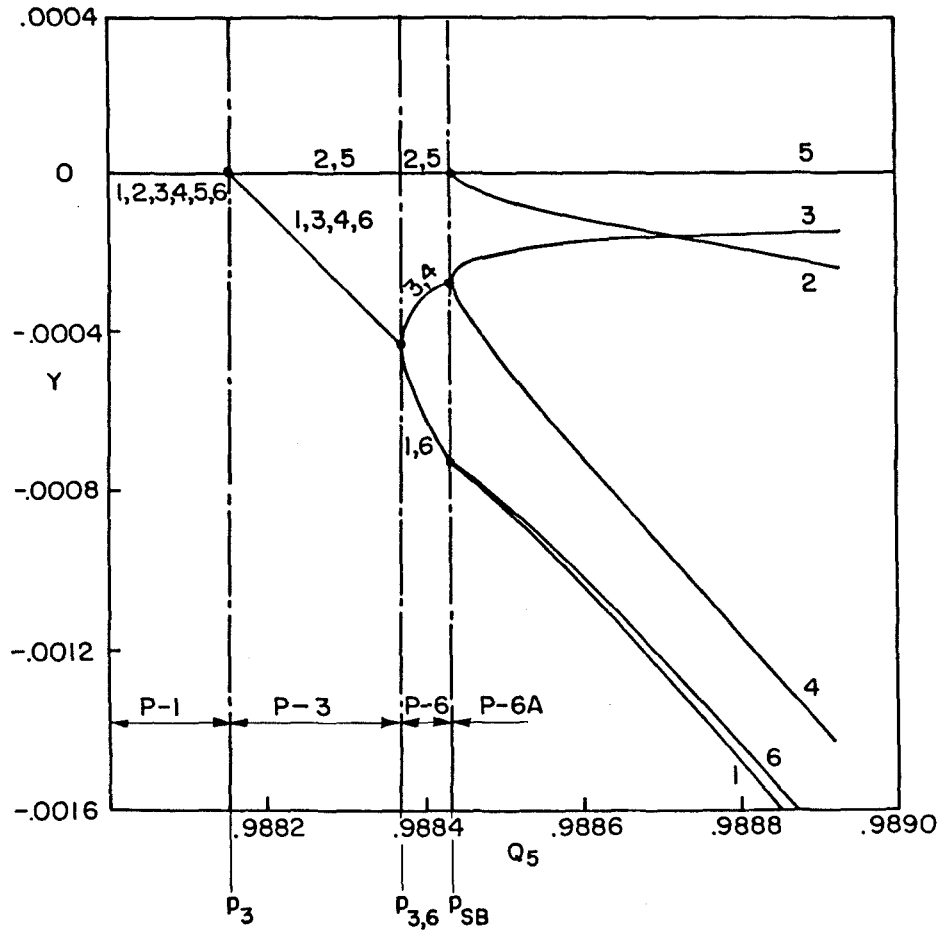
At the period tripling bifurcation point ( $P_3$ ), we switched from the P-1 branch to the P-3 branch (period  $3L$ ). Taking the P-3 branch in the direction of decreasing  $c$  from  $P_3$ , we found a point for which  $A_1$  becomes singular, point  $P_{3,6}$  in figure

2.2 . The bifurcation here corresponds to a period doubling bifurcation. The new branch corresponds to a family of waves of the form  $XZY YZX$ . Notice now the importance of taking the origin of the  $x$ -axis in a trough. If we had taken the origin in a crest, these waves would have appeared as a spurious nonsymmetric branch. This is because the period doubling bifurcation breaks the symmetry with respect to the crests, but not with respect to the troughs. With the origin in a crest, instead of finding  $A_1$  to be singular, we would have found  $A'_2$  to be singular at  $P_{3,6}$ . Thus the new branch could not be computed using system (2.13). The new family of waves is not symmetric with respect to any crest, in contrast to the P-1 and P-3 branches where the waves were symmetric with respect to some crest (see figure 2.3).

On this new branch of period  $6L$ , which we will call the P-6 branch, computations were carried out in both directions from the bifurcation point. We found that the two sides correspond to the same wave, the difference being a shift of  $\pi$  in the origin (see figure 3.2, curves b and c). The matrix  $A_1$  was nonsingular on the whole computed region of the branch, except at the point  $P_{3,6}$ . It was also found that  $A'_2$  vanishes at one point of this branch (point  $P_{sb}$  in figure 2.2). This means that at this point there is bifurcation to a family of waves that are not symmetric with respect to the trough at which we took the origin of  $x$ .

The new branch was examined with a nonsymmetric code based on the system formed by equations (2.4) and (2.7). This time we found that the waves were not just shifted symmetric waves as we found in the other cases in which  $A'_2$  vanished. This new branch corresponds to a family of nonsymmetric waves. At the beginning of the branch the waves have the form described by the sequence  $WZYXZW$  (see figure 2.3), but if we go further along the branch we find that all the crests have different height. The difference among the heights of the crests is very small and

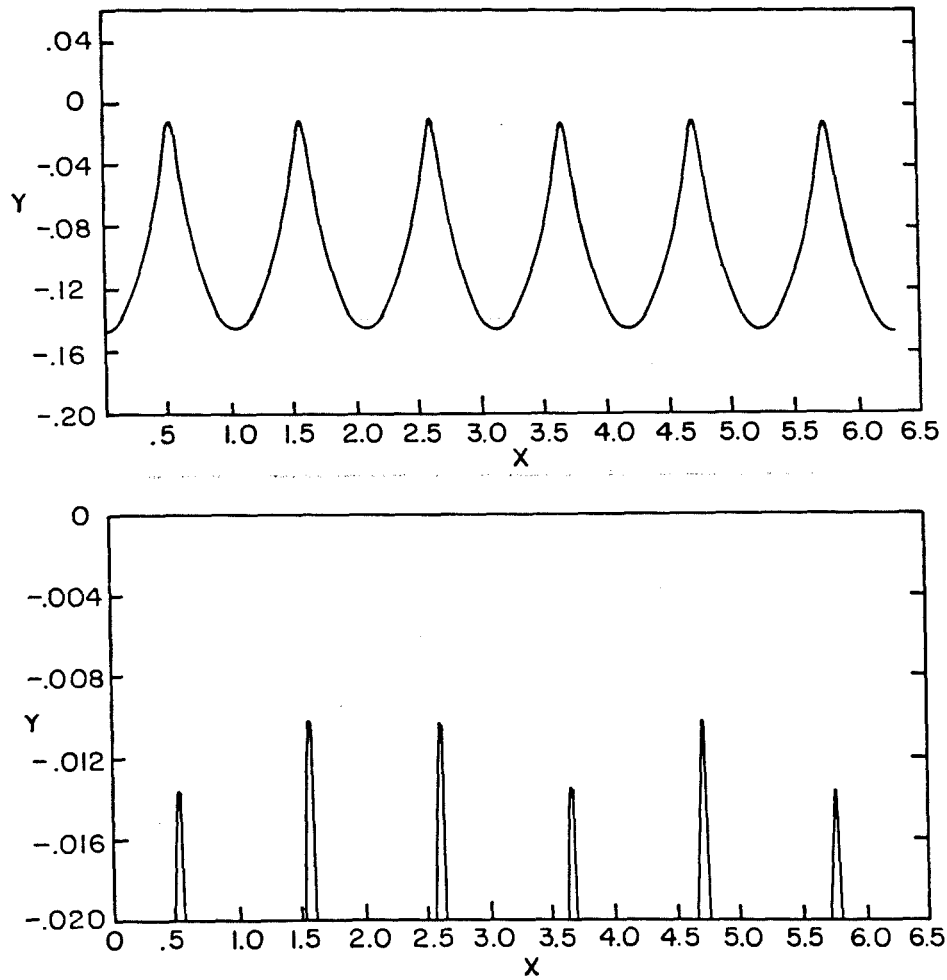
cannot easily be seen by eye. Figure 2.4 shows the relative height of all the crests with respect to the fifth crest for all the branches that we have computed. We actually plot  $(Q_j - Q_5)$  versus  $Q_5$  for all the crests. In figure 2.5 we show the actual shape of a nonsymmetric wave.



**Figure 2.4.-** Relative amplitudes of the crests of the travelling waves along the path P-1, P-3, P-6 an P-6A. Number denote the crests of figure 2.1 .

The P-6 branch was computed using 256, 512, 768 and 1024 to determine with high accuracy the position of the symmetry breaking bifurcation. We found that with 1024 modes very good convergence is achieved. Along this branch the energy increases monotonically, as does the phase speed.

The nonsymmetric branch P-6A was computed using only 256 and 512 modes



**Figure 2.5.-** Nonsymmetric gravity wave for  $c = 0.442497$ . (a): Actual form of the wave. (b): Detail of the relative amplitude of the crests.

in the nonsymmetric code. This is equivalent to use 512 and 1024 modes in the symmetric code from the computational point of view. We can observe that the energy and the phase speed decay very rapidly along the nonsymmetric family.

Properties of the waves along branches P-6 and P-6A are given in tables 4 and 5. Table 6 shows the first 20 coefficients for the nonsymmetric wave given in figure 2.5 .

## 2.3 Hamiltonian Method

### 2.3.a General formulation

Consider two-dimensional irrotational periodic surface waves on water of infinite depth. Zakharov (1968) and Broer (1974) have proved that this system has the following Hamiltonian structure

$$\begin{aligned}\frac{\partial \eta}{\partial t} &= \frac{\delta \mathbf{H}}{\delta \psi}, \\ \frac{\partial \psi}{\partial t} &= -\frac{\delta \mathbf{H}}{\delta \eta},\end{aligned}\tag{2.16}$$

where  $\mathbf{H}$  is the total energy per period of the waves, and  $\delta$  represents a functional derivative (see Goldstein, 1980). The canonical variables  $\eta(x, t)$  and  $\psi(x, t)$  are the surface shape and the velocity potential evaluated at the surface  $\psi(x, t) = \Phi(x, \eta(x, t), t)$ .

The length scale can always be chosen such that the period of the waves is  $2\pi$ . With this choice  $\eta(x, t)$  and  $\psi(x, t)$  are  $2\pi$ -periodic functions, and  $\mathbf{H}$  is given by

$$\mathbf{H} = \int_0^{2\pi} H(\eta, \psi) dx,\tag{2.17}$$

where  $H$  is the energy density. In the case of gravity waves  $H$  is given by

$$H = \frac{1}{2} \int_{-\infty}^{\eta} (\nabla \Phi)^2 dy + \frac{1}{2} g \eta^2,\tag{2.18}$$

where  $g$  is the acceleration of the gravity, which is taken equal to 1 in the following.

As  $\eta(x, t)$  and  $\psi(x, t)$  are periodic functions of  $x$ , they can be expressed in the following form

$$\begin{aligned}\eta(x, t) &= \sum_{-\infty}^{\infty} \eta_k(t) e^{ikx}, \\ \psi(x, t) &= \sum_{-\infty}^{\infty} \psi_k(t) e^{ikx},\end{aligned}\tag{2.19}$$

where  $\eta_k = \eta_{-k}^*$  and  $\psi_k = \psi_{-k}^*$  because  $\eta$  and  $\psi$  are real. The transformation defined by (2.19) can be considered as a canonical transformation from a continuous system with canonical variables  $\eta(x, t)$  and  $\psi(x, t)$  into an infinite dimensional discrete system with the infinite sets  $\{\eta_k\}$  and  $\{\psi_k\}$  as canonical variables (Goldstein 1980).

To write the system in a more compact form, Zakharov (1968) introduced the following transformation

$$\begin{aligned}\eta_k(t) &= \frac{1}{2\sqrt{\pi}} \frac{|k|^{1/2}}{\omega^{1/2}(k)} [a_k(t) + a_{-k}^*(t)], \\ \psi_k(t) &= -\frac{i}{2\sqrt{\pi}} \frac{\omega^{1/2}(k)}{|k|^{1/2}} [a_k(t) - a_{-k}^*(t)],\end{aligned}\tag{2.20}$$

where  $\omega(k)$  is the linear dispersion relation for deep water waves, which is given by

$$\omega(k) = \sqrt{|k|}.\tag{2.21}$$

When the horizontal level of the frame of reference is chosen at the mean horizontal level of the surface,  $a_0 = 0$ .

Using transformations (2.19) and (2.20) the Hamiltonian system (2.16) becomes the single equation

$$\frac{\partial a_k}{\partial t} = -i \frac{\partial \mathbf{H}}{\partial a_k^*},\tag{2.22}$$

where now

$$\mathbf{H} = \int_0^{2\pi} H(\eta, \psi) dx = \mathbf{H}(a, a^*), \quad (2.23)$$

and  $a = \{a_k\}$ . Notice that the functional derivative has become a partial derivative because we now have a discrete system.

As we are interested in the study of permanent form travelling waves, it is convenient to write the system (2.22) in a coordinate system moving with the phase speed  $c$ . Doing this we have

$$\frac{\partial a_k}{\partial t} = -i \frac{\partial \mathbf{H}}{\partial a_k^*} + ick a_k. \quad (2.24)$$

Travelling waves of permanent form are equilibrium solutions (fixed points) of this system. For symmetric wave solutions, the origin of the reference frame can be always chosen such that  $a_k = a_k^*$ . An important property of  $\mathbf{H}$  is that  $\mathbf{H}$  is invariant under the change  $a_k \rightarrow a_k^*$ .

Instead of considering only equation (2.24), it is convenient to consider equation (2.24) together with its complex conjugate and the coefficients  $\{a_k\}$  and  $\{a_k^*\}$  as different independent variables. With these considerations the system is analytic, and the stability analysis becomes clearer.

The stability of these equilibrium solutions of system (2.24) under superharmonic perturbations ( $\propto e^{i\sigma t}$ ) is determined by the eigenvalues of the matrix (Saffman 1985)

$$S(a, a^*) = \begin{pmatrix} M & N \\ -N^* & -M \end{pmatrix}, \quad (2.25)$$

where

$$M = \partial^2 \mathbf{H} / \partial a \partial a^* - ck, \quad N = \partial^2 \mathbf{H} / \partial a^2.$$

When all the eigenvalues are real the system is said to be stable. Stability is lost when some of the eigenvalues leave the real axis. This can occur only by collision of two eigenvalues of different signature (MacKay 1985, MacKay & Saffman 1986).

Some important properties of the matrix  $S$  called “Hamiltonian” (Arnold 1978) are that if  $\sigma$  is an eigenvalue of  $S$ , so is  $-\sigma$ , and so are the complex conjugates  $\sigma^*$  and  $-\sigma^*$ . The matrix  $S$  has always one eigenvalue that is zero due to the Galilean invariance. Because the system is even dimensional  $\sigma = 0$  has to be at least a double eigenvalue. Saffman (1985) proved that at points where the wave energy is an extremum  $\sigma = 0$  is an eigenvalue of algebraic multiplicity four and geometric multiplicity one. This analytical result was checked by our computations as we will see below.

As happened in the Stokes formulation, it is important to notice that if a set of coefficients  $\{a_k\}$  represents an equilibrium solution of (2.24) then the set  $\{a_k^N\}$  where

$$\begin{aligned} a_{kN}^N &= (1/N)^{5/4} a_k, \\ a_k^N &= 0 \quad \textit{otherwise}, \end{aligned} \tag{2.26}$$

is also an equilibrium solution of (2.24), and represents a class-N wave.

### *2.3.b Weakly non-linear form*

Before going into the details of the numerical algorithm, it is worthwhile to work out some linear results that will help us to generate starting points and check the computations.

For linear waves the Hamiltonian  $\mathbf{H}$  is given by



$$\mathbf{H} = \sum_{-\infty}^{\infty} \omega(k) a_k a_k^*. \quad (2.27)$$

Using (2.24) the equations for travelling waves are

$$(\sqrt{|k|} - ck) a_k = 0 \quad -\infty < k < \infty. \quad (2.28)$$

We have an eigenvalue problem. For  $c \neq 1/\sqrt{|k|}$  there is a unique solution  $a_k = 0$  for all  $k$ , that corresponds to the flat surface. For  $c = 1/\sqrt{|k|}$  we have a bifurcation point to a regular class- $k$  wave ( $a_j = 0$  for  $j \neq k$  and  $a_k$  arbitrary).

The eigenvalues for the superharmonic stability of the solutions at the bifurcation points are

$$\sigma_j^k = \pm \left[ \sqrt{|j|} + \frac{j}{\sqrt{|k|}} \right]. \quad (2.29)$$

These values will serve as a first check of the stability code. Each eigenvalue is named by giving the two numbers  $j/k$ . The eigenvector associated to the eigenvalue  $j/k$  is an oscillation with  $|j|$  crests on a class- $k$  wave. Negative values of  $j$  correspond to perturbations travelling in the same sense as the unperturbed wave. Positive values of  $j$  correspond to perturbations travelling in the opposite sense to the unperturbed wave.

As we are interested in the branch corresponding to class-6 waves, in order to generate a starting point for the computations, we calculated the expansion of the energy for small amplitude to second order obtaining

$$\mathbf{H} \simeq \sqrt{6} [a_6 a_6^* + \frac{9}{2\pi} \sqrt{6} a_6^2 a_6^{*2}]. \quad (2.30).$$

With this Hamiltonian, we find that the beginning of the class-6 branch is given by

$$a_6 a_6^* \simeq \frac{\sqrt{6}\pi}{162} [\sqrt{6}c - 1]. \quad (2.31)$$

### 2.3.c Numerical algorithm

We are interested in computing travelling waves of permanent form. In a frame moving with the wave, these waves are solutions of the following steady state equations

$$\left. \begin{aligned} \frac{\partial \mathbf{H}}{\partial a_k^*} - ck a_k &= 0 \\ \frac{\partial \mathbf{H}}{\partial a_k} - ck a_k^* &= 0 \end{aligned} \right\} -\infty < k < \infty, \quad (2.32)$$

where the equations are complex.

We now truncate the system (2.32) by taking  $a_k = 0$  for  $|k| > n$ . Thus we are going to consider only  $2n$  complex coefficients to describe the wave. Then we have to solve an algebraic system of  $4n$  equations in  $4n$  unknowns.

To make this formulation useful, we first need to compute the gradient of the energy  $\mathbf{H}$  in terms of the set of coefficients  $\{a_k\}$ . In other words, we have to evaluate expression (2.18) and its derivatives in terms of the coefficients. The part corresponding to the potential energy is straight forward. The most difficult part is to compute the kinetic energy. Notice that as we need to compute the gradient of the energy we have to compute the energy of a general wave which in general will not correspond to a permanent travelling wave. Hence simple expressions for

the energy similar to the ones shown in equations (2.11) and (2.12), which are valid only for steady waves, cannot be used for the present purpose. Here we describe a way of doing this computation.

Knowing the set of coefficients  $\{a_k\}$ , we can directly compute the sets  $\{\eta_k\}$  and  $\{\psi_k\}$  using (2.20). To evaluate the kinetic energy we need to know the velocity potential  $\Phi$  in the fluid region. Using Fourier analysis and assuming that  $\Phi$  is analytic below the line of mean horizontal height, we can write

$$\Phi(x, y) = \sum_{k=-\infty}^{\infty} \Phi_k e^{|k|y} e^{ikx}, \quad (2.33)$$

where the set  $\{\Phi_k\}$  is the coefficients of the Fourier expansion of the velocity potential on the line  $y = 0$ .

Introducing relation (2.33) into equation (2.18) and performing the integral, we obtain the following expression for the total energy

$$\mathbf{H} = 4\pi \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \frac{kj}{k+j} \Phi_k \Phi_{-j} C_{j-k, k+j} + 2\pi \sum_{k=1}^{\infty} \eta_k \eta_{-k}, \quad (2.34)$$

where the coefficients  $C_{k,m}$  are defined by

$$C_{k,m} = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikx} e^{|m|\eta(x)} dx. \quad (2.35)$$

These coefficients depend only on the set  $\{\eta_k\}$ , and can be computed in the following way. Using an inverse Fast Fourier Transform FFT we can compute  $\eta(x)$  evaluated on equidistant nodes from the set  $\{\eta_k\}$ . For every value of  $m$  we take the exponential, and via a forward FFT we obtain the coefficients  $C_{k,m}$  directly. An alternative way of computing these coefficients consists of expanding the exponential

function in Taylor series, and after doing the algebra among the different modes we can obtain  $C_{k,m}$ . For small amplitude waves the last procedure is faster because we only need to take a few terms in the Taylor expansion of  $\exp(|m|\eta(x))$ . For the range of amplitudes in which we are interested, we found that the expansion has to be carried out up to order 30 to get convergence. Hence for our purposes the algorithm based on the FFT was found to be faster.

To compute the coefficients  $\Phi_k$  we write equations (2.33) on the surface of the wave in the following form

$$\sum_{k=-n}^n \psi_k e^{ikx} = \sum_{j=-\infty}^{\infty} \Phi_j e^{|j|\eta(x)} e^{ijx}. \quad (2.36)$$

Introducing the coefficients  $C_{k,m}$  and equating the same harmonics, equation (2.36) can be rewritten as follows

$$\psi_j = \sum_{k=-\infty}^{\infty} \Phi_k C_{j-k,k} \quad -\infty < j < \infty, \quad (2.37)$$

where  $\psi_j = 0$  for  $j > n$ .

It is important to notice that the series in equations (2.36) and (2.37) will be uniformly convergent if and only if the analytic continuation of  $\Phi(x, y)$  above the surface does not have any singularity below the highest crest of the wave. For the waves that we are going to consider this was found to be true.

Now we have to truncate the set  $\{\Phi_k\}$ , and take  $\Phi_k = 0$  for  $|k| > n_a$ . Thus we have two parameters in the algorithm,  $n$  and  $n_a$ . How the algorithm converges as  $n_a$  is increased serves as a measure of the validity of the statement of the previous paragraph.

To compute the set  $\{\Phi_k\}$  we need to solve the linear system of equations defined by (2.37). This part of the algorithm has a complexity of  $n_a^3$  operations and is the time consuming part of the method.

Until now we have only computed the energy  $\mathbf{H}$ , but for equations (2.32) we actually need to compute the gradient of the energy. This can be done without increasing the complexity of the algorithm as follows.

Taking the derivative of (2.34) with respect to  $a_l^*$  we have

$$\begin{aligned} \frac{\partial \mathbf{H}}{\partial a_l^*} = & 4\pi \sum_{k=1}^{n_a} \sum_{j=1}^{n_a} \frac{kj}{k+j} \left\{ \left[ \frac{\partial \Phi_k}{\partial a_l^*} \Phi_{-j} + \Phi_k \frac{\partial \Phi_{-j}}{\partial a_l^*} \right] C_{j-k, k+j} + \Phi_k \Phi_{-j} \frac{\partial C_{j-k, k+j}}{\partial a_l^*} \right\} \\ & + \frac{\sqrt{\pi}}{2} |l|^{1/4} \eta_l. \end{aligned} \quad (2.38)$$

The derivative of the coefficients  $C_{k,m}$  can be directly obtained by differentiation of equation (2.35) giving

$$\frac{\partial C_{k,m}}{\partial a_l^*} = \frac{\partial C_{k,m}}{\partial \eta_{-l}} \frac{|k|^{1/4}}{2\sqrt{\pi}} = \frac{|k|^{5/4}}{2\sqrt{\pi}} C_{k+l,m}. \quad (2.39)$$

To obtain the derivatives of the coefficients  $\Phi_k$  we take the derivative of equation (2.37) with respect to  $a_l^*$ . We have

$$\frac{\partial \psi_j}{\partial a_l^*} - \sum_{k=-n_a}^{n_a} \phi_k \frac{\partial C_{j-k, k}}{\partial a_l^*} = \sum_{k=-n_a}^{n_a} C_{j-k, k} \frac{\partial \Phi_k}{\partial a_l^*}. \quad (2.40)$$

Therefore to compute the derivatives of  $\Phi_k$  we have to solve  $2n$  linear systems of equations. But notice that we only need to make one back substitution because the matrix of the system is the same as the one that we had before in order to

compute the set  $\{\Phi_k\}$ . Thus to compute the set of derivatives  $\partial\Phi_j/\partial a_l^*$ , we need  $2n$  back substitutions. The total complexity of this part is  $nn_a^2$ .

Notice that using this algorithm, the expansion that we are actually obtaining for the function  $\mathbf{H}(a, a^*)$  is not a Taylor polynomial expansion as happens in the weakly nonlinear expansion given by Zakharov (1968). Due to the properties of the linear system that we are solving, the expansion has the form of the ratio of two polynomials. The structure is similar to the one that is obtained when using Padé approximants to represent a function.

Once we can compute the gradient of the energy in terms of the coefficients  $a_k$ , we can solve equations (2.32) by using arc-length continuation (Keller, 1977) in the parameter  $c$ . As happened in the Stokes expansion method, the system defined by (2.32) is degenerate due to the Galilean invariance. As can be very easily checked, if a set of coefficients  $\{a_k\}$  represents a solution of (2.32) the set  $\{a_k e^{ik\epsilon}\}$  is also solution of (2.32) for all  $\epsilon$ . Therefore, as we have already seen, the Jacobian of equations (2.32) (matrix S) is singular.

To be able to do the continuation with this degeneracy, we modified the continuation algorithm by imposing that the increments in the solution as the parameter is changed be normal to the null vector of the Jacobian. The null vector of the Jacobian is given by the set  $\{ik a_k\}$ . This is equivalent to performing the continuation in a subspace that is the projection of the whole space onto the direction normal to the null vector. Physically this is equivalent to keeping the origin fixed.

Equation (2.32) represents a set of  $4n$  complex equations for the coefficients  $a_k$  and  $a_k^*$ . Using linear combinations among the equations and taking  $a_k = p_k + iq_k$ , the system can be reduced to  $4n$  real equations for the coefficients  $p_k$  and  $q_k$ .

For the continuation process the Jacobian was computed numerically using increments. Even though we used double precision in our calculations, this was the main source of inaccuracy in the computations of the eigenvalues for the stability.

The whole algorithm can be simplified when considering symmetric waves. For symmetrical solutions the coefficients can be taken to be real, with  $a_k = a_k^*$ . Then the original system (2.32) can be reduced to only the first set of  $2n$  real equations. The Jacobian of this real system is just the matrix  $M - N$ , where  $M$  and  $N$  were already defined in equation (2.25).

#### *2.4.d Numerical results*

In order to check our code we first computed the regular branch using a class-1 wave. Taking  $n = 12$ ,  $n_a$  was increased until convergence was obtained for the solution in the range from zero amplitude up to the amplitude at which the energy of the wave reaches the first maximum. We found that  $n_a = 2n = 24$  is sufficient to get convergence in this range.

All the eigenvalues of the matrix  $S$  were computed along this branch. We found that an exchange of stability occurs at the point of maximum energy. This result was first found by Tanaka (1983) who studied the superharmonic stability of regular Stokes waves using a numerical method based on conformal mapping techniques. Later Saffman (1985) proved the result analytically using the Hamiltonian formulation of the water wave problem. He found that this is a property of the Hamiltonian structure of the problem. Because the code keeps the Hamiltonian structure, the coincidence of the critical amplitude for the superharmonic stability and the amplitude of the maximum energy is a property of the structure of the code, and is

independent of the number of modes, even though the location of the actual point is inaccurate unless the number of modes is sufficiently large. On the other hand, other methods like the Stokes expansion used by Longuet-Higgins (1978b) and the methods used by Tanaka (1983,1985) are inaccurate in both respects unless the number of modes is sufficiently large, i.e., when a few number of modes are used in the computations both points do not coincide and their location is not accurate. We also found that the results for the eigenvalues of the the first modes are in very good agreement with computations of Longuet-Higgins (1978b) in the range of steepness within which his results are valid.

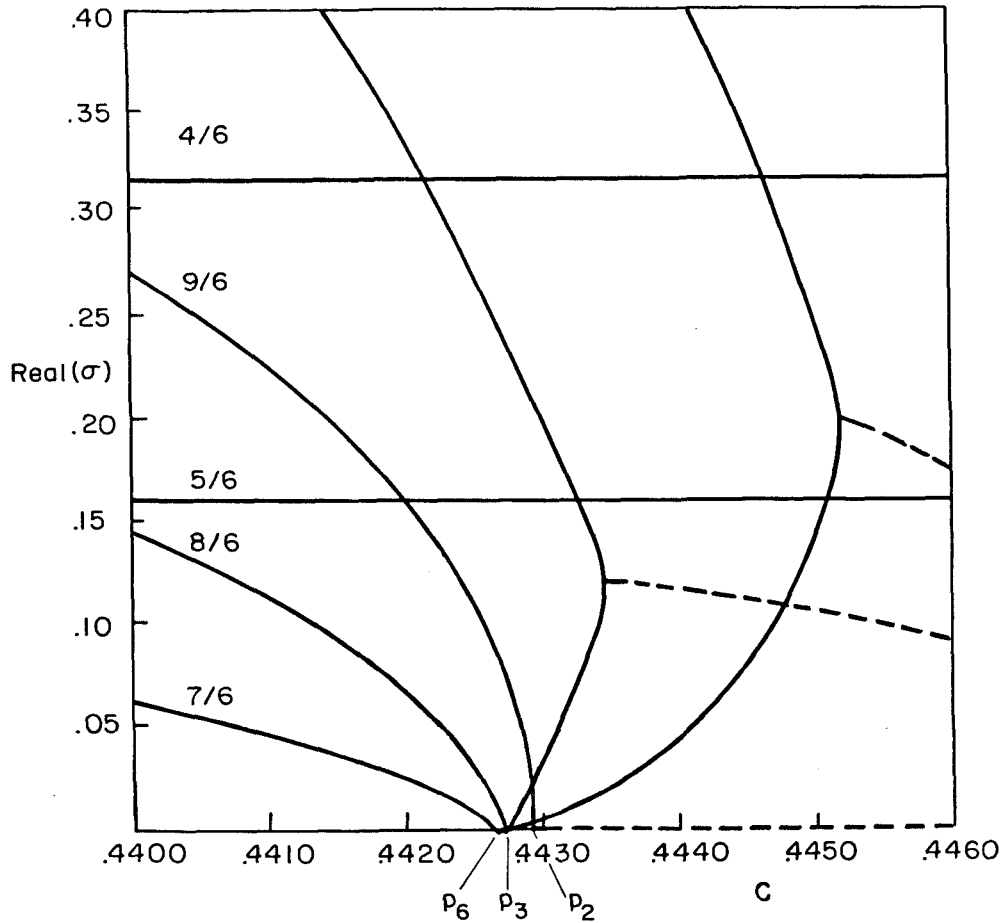
With  $n = 12$  we found good numerical convergence in the region of amplitudes where bifurcations to other families are expected to happen. Actually in this region we found that  $n = n_a$  gives convergence for a given  $n$ , and it is not necessary to go up to  $n_a = 2n$ .

In the region of strong nonlinearity where the extrema of the energy and phase speed appear, with  $n = 12$  and  $n_a = 24$  we were able to reproduce the first maximum of the energy and the phase speed. Numerical values were within a 7% of the real ones for the phase speed and 2.5% for the energy and the amplitude.

Once the code was checked, we tried to reproduce the results about symmetry breaking that we obtained by the Stokes expansion. Using (2.31) we generated a starting point for the regular class-6 family. We increased the amplitude up to the point where bifurcations appear. As the eigenvalues were computed the bifurcations were checked by looking at where eigenvalues become zero. Eigenvalues along this branch are plotted in figure 2.6 .

Because of the complexity of the algorithm computations could be carried out only for up to  $n=72$ . This is equivalent to  $n=12$  in a class-1 wave. To switch





**Figure 2.6.-** Eigenvalues of the superharmonic stability of the class-6 regular family showing the bifurcation points. Dashed lines denotes  $\sigma$  complex.

branches we used the same procedure as in the previous method, and again we found that the method converges to the new branches in 3 or 4 iterations.

Figure 2.6 shows eigenvalues for disturbances to class-6 waves. We can see that there are three bifurcation points. These bifurcations are the same as the ones obtained by the Stokes formulation. Notice that only the period doubling bifurcation leads to a change of stability in the regular family. At the other two points, eigenvalues pass through zero, but they do not give a change of stability. From the behavior of the eigenvalues, it is probable that more bifurcations exist

for higher amplitudes. However in that region we do not get convergence with the number of modes that we are using. These higher-amplitude bifurcations were computed by Vanden-Broek (1983) using conformal mapping techniques for finite depth water waves. Values of the phase speed and total energy for the period tripling bifurcation ( $P_3$ ) are given in table 1.

At the period tripling bifurcation, we switched to the new branch (P-3). This branch was computed in the direction of decreasing phase speed. We found that there is always at least one unstable eigenvalue on this branch. In figure 2.7 we show the two smallest eigenvalues, besides the one due to the shift ( $\sigma = 0$ ). The eigenvalue that leads to the  $P_3$  bifurcation is the one that is always unstable. This was expected because on the  $P - 1$  branch there is no change of stability at the  $P_3$  bifurcation point, as it can be seen in figure 2.6 . We can see that the other eigenvalue also goes through zero giving a new bifurcation. This bifurcation is a period doubling bifurcation. Actually, this is the point  $P_{3,6}$  in figure 2.2 .

At the point  $P_{3,6}$  we switched to the new branch (P-6). The eigenvalues along the new branch are given in figure 2.8 . Again we plot the two smallest eigenvalues finding that another bifurcation appears on this branch. This bifurcation corresponds to the symmetry breaking. In figure 2.8 we can also see that the eigenvalue that gives the  $P_{3,6}$  bifurcation is unstable along the P-6 branch. Notice that, as  $P_{3,6}$  is period doubling bifurcation, the new branch  $P - 6$  starts at the bifurcation point and does not exist for values of the phase speed below the value corresponding to the bifurcation point. Values of the phase speed and energy at the point  $P_{3,6}$  for  $n = 60$  and  $72$  are given in table 3.

Therefore we find the same bifurcation structure using this completely different numerical approach to the problem. This is strong evidence for the validity of our

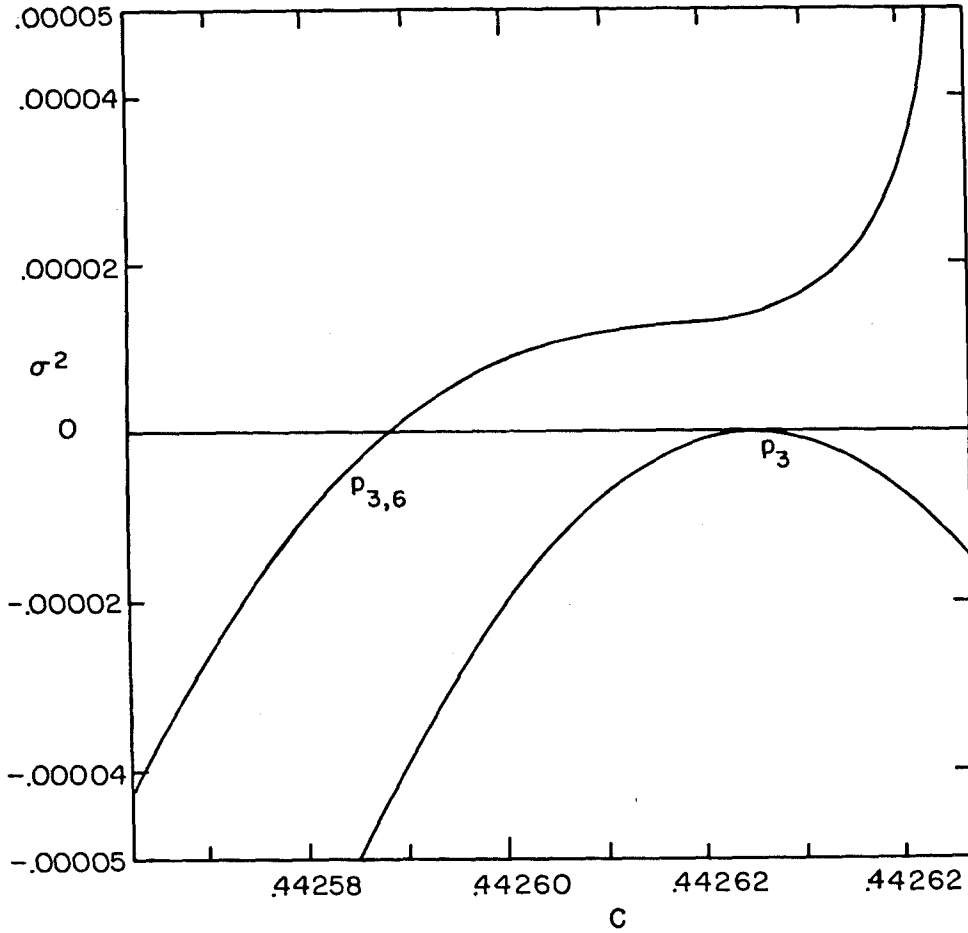
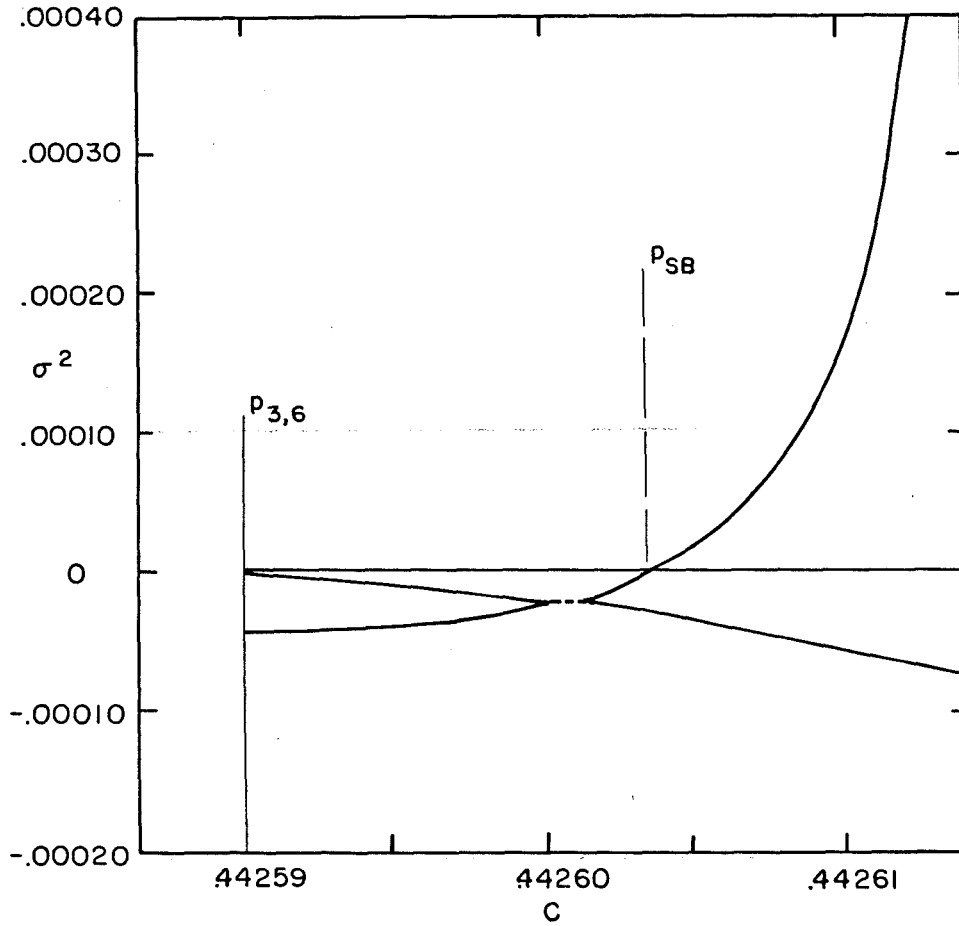


Figure 2.7.- Behavior of the eigenvalues along branch P-3, associated with the bifurcations at  $P_3$  and  $P_{3,6}$ .

results.

## 2.4 Conclusions

We have shown that nonsymmetric gravity waves exist on deep water. They appear from a spontaneous symmetry breaking bifurcation of symmetric waves. To find one of these bifurcations we go from the regular family of waves to a irregular family of class-3, which was found by Chen & Saffman (1980). In this class-3 family there is a period doubling bifurcation to a family of waves that have 6 crests per



**Figure 2.8.-** Behavior of the eigenvalues along branch P-6, associated with the bifurcations at  $P_{3,6}$  and  $P_{sb}$ .

basic wavelength. This class-6 wave can bifurcate to a nonsymmetric family with also 6 crests per period.

Hence we find that results found by in Chapter 1 using a weakly nonlinear Hamiltonian model for finite depth water waves can be completely generalized to the full system and to the deep water case. Chapter 1 results follow from properties of 2 degrees of freedom Hamiltonian systems. This means that in this case results for a finite dimensional Hamiltonian system appear to apply for a continuous system. This raises again the open question of whether or not travelling water waves of permanent form are actually a Hamiltonian system of two degrees of freedom.

In this way Hamiltonian methods have served to answer one more question in the problem of water waves.

In the present paper we have computed only one family of nonsymmetric waves. This does not mean that this family is unique. For area preserving maps many more symmetry breaking bifurcations have been found, but this one is the easiest to compute in the case of water waves. The others bifurcations appear considering waves of higher class. Class-6 is the minimum class to have symmetry breaking.

To confirm all these results we have performed computations using two different methods. The first is based on the Stokes expansion and is shown to be very easy to implement and fast. Therefore many modes can be used in the computations and the method gives the most accurate results. . But we have found that the structure of the equations is poor in the sense that the number of modes needed to reproduce the physical behavior and to find convergence is large.

The new numerical method based on the Hamiltonian formulation of the water waves problem has been found to be much more expensive from the computational point of view because of the complexity of the algorithm to compute the energy. We have been able to run only up to 72 modes. But the bifurcation structure was reproduced using as few as 12 modes.

Also the algorithm gives the stability of the wave solutions without much additional work. The method can be very easily generalized to other cases. For example to consider gravity-capillary waves we only need to add to the total energy the energy due to the surface tension without changing the main part of the code. Also the same code can be used for studying unsteady problems in water waves, and keeping the Hamiltonian structure of the problem. For example, in initial value problems properties that have to be conserved such as energy are directly checked.

We think that more work has to be done to try to speed up the way of computing the kinetic energy, so that the computations can be made more efficient. Also generalization to three dimensional waves is straightforward.

## References

- [1] Amick, L.J., Fraenkel, L.E. and Toland, J.F.(1982). "On the Stokes conjecture for the wave of extreme form." *Acta Mathematica* **148**, 193-214.
- [2] Arnold, V.I.(1978). "Mathematical Methods of Classical Mechanics." *Springer-Verlag, New York*.
- [3] Arnold, V.I. and Avez, A.(1968). "Ergodic Problems of Classical Mechanics." *Benjamin, New York*.
- [4] Broer, L.J.F.(1974). "On the Hamiltonian theory of surface waves." *Appl. Sci. Res.* **29**, 430-446.
- [5] Boussinesq, J.(1871). "Théorie l'intumescence liquide appelée onde solitaire ou de translation se propageant dans an canal rectangulaire." *Comptes Rendus* **72**, 755-759.
- [6] Chen, B. and Saffman, P.G.(1980). "Numerical evidence for the existence of new types of gravity waves of permanent form on deep water." *Stud. App. Math.* **62**, 1-21.
- [7] Goldstein, H.(1980). "Classical Mechanics." *2nd ed. Addison Wesley, Reading, Ma.*
- [8] Green, J.M., Mackay, R.S., Vivaldi, F. and Feigenbaum, M.J.(1981). "Universal behavior in families of area-preserving maps." *Physica 3D*, 468-486.
- [9] Keller, H.B.(1977). Numerical solution of bifurcation and nonlinear eigenvalue problems, in "Applications of Bifurcation Theory." *Academic Press*, pp 359-384.

- [10] Korteweg, P.J. and De Vries, G.(1895).“On the change of form of long waves advancing in a rectangular channel, and on a new type of long stationary waves.” *Phil. Mag. (5)* **39**, 422-443.
- [11] Longuet-Higgins, M.S.(1978a).“Some new relations between Stokes’s waves coefficients in theory of gravity waves.” *J. Inst. Math. Applics.* **22**, 261-273.
- [12] Longuet-Higgins, M.S.(1978b).“The instabilities of gravity waves of finite amplitude in deep water.I. Superharmonics.” *Proc. Roy. Soc. Lond. A* **360**, 471-488.
- [13] Longuet-Higgins, M.S.(1984).“New integral relations for gravity waves of finite amplitude.” *J. Fluid Mech.***149**, 205-215.
- [14] Longuet-Higgins, M.S.(1985).“Bifurcation in gravity waves.” *J. Fluid Mech.* **151**, 457-475.
- [15] MacKay, R.S.(1985).“Stability of equilibria of Hamiltonian systems.” *Proceedings of the RSRE seminar on Dynamical Systems.* (to appear).
- [16] MacKay, R.S. and Saffman, P.G.(1986).“Stability of water waves.” *Proc. Roy. Soc.* (to appear).
- [17] Rayleigh, Lord (1876).“On waves.” *Phil. Mag.(5)* **1**, 257-279.
- [18] Saffman, P.G.(1980).“Long wavelength bifurcation of gravity waves on deep water.” *J. Fluid Mech.* **101**, 567-581.
- [19] Saffman, P.G.(1985).“The superharmonic instability of finite-amplitude water waves.” *J. Fluid Mech.* **159**, 169-174.
- [20] Sattinger, D.H.(1980).“Bifurcation and symmetry breaking in Applied Mathematics.” *Bull. Amer. Math. Soc.* **3**, 779-819.



- [21] Sattinger, D.H.(1983).“Branching in the presence of symmetry.” *CBMS-NSF Reg. Conf. in Appl. Math. SIAM, Philadelphia.*
- [22] Scharwtz, L.W.(1974).“Computer extension and analytic continuation of Stokes’ expansion for gravity waves.” *J. Fluid Mech.* **63**, 553-578.
- [23] Stokes, G.G.(1849).“On the theory of oscillatory waves.” *Trans. Camb. Phil. Soc.* **8**, 441-455.
- [24] Tanaka, M.(1983).“The stability of steep gravity waves.” *J. Phys. Soc. Jpn.* **52**,3047-.
- [25] Tanaka, M.(1985).“The stability of steep gravity waves.II” *J. Fluid Mech.* **156**, 281-289.
- [26] Vanden-Broeck, J.M.(1983).“Some new gravity waves in water of finite depth.” *Phys. Fluids* **26**, 2385-2387.
- [27] Zakharov, V.E.(1968).“Stability of periodic waves of finite amplitude on the surface of deep fluid.” *J. App. Mech. Tech. Phys.* **2**, 190-194.

Table 1.

	$n$	$Q$	$c$	$E = T + V$	$ak$
Saffman (1980)	300*		0.4425	0.01239	0.06744
Chen & Saffman (1980)		0.98815	0.44255	0.01239	0.06745
Longuet-Higgins (1984)	320*				0.067448
Simmen & Saffman (1985)			0.4426		0.06745
Present work					
	256	0.98848	0.44306	0.012518	0.067774
Stokes Expansion	512	0.98815	0.44261	0.012394	0.07446
	768	0.98815	0.44255	0.012393	0.067446
	48		0.44350	0.01266	0.06799
Hamiltonian Method	60		0.44268	0.01243	0.06740
	72		0.44262	0.01240	0.067444

\* Modes on a class-1 wave.

Values of some parameters of the wave at the bifurcation point  $P_3$ .

**Table 2.**

	$n$	$c$	$E = T + V$	$Q_2$
Stokes Expansion	256	.442988	.0125027	.988756
	512	.442515	.0123836	.988367
	768	.442510	.0123825	.988372
	1024	.442510	.0123825	.988373
Hamiltonian Method	60	.442918	.0124842	
	72	.442590	.0124018	

Values obtained for some parameters of the wave at the bifurcation point  $P_{3,6}$ .

**Table 3.**

	$n$	$c$	$E = T + V$	$Q_2$
Stokes Expansion	256	.443014	.0125082	.988823
	512	.442533	.0123874	.988429
	768	.442528	.0123863	.988434
	1024	.442529	.0123866	.988440
Hamiltonian Method	60	.442938	.0124886	
	72	.442606	.0124052	

Values obtained for some parameters of the wave at the bifurcation point  $P_{sb}$ .

Table 4.

$c$	$Q_1$	$Q_2$	$Q_3$	$Q_4$	$Q_5$	$Q_6$	$T$	$V$
0.442515 <sup>1</sup>	0.98794	0.98837	0.98794	0.98794	0.98837	0.98794	0.0064658	0.0059179
0.442525	0.98777	0.98841	0.98810	0.98810	0.98841	0.98777	0.0064670	0.0059189
0.442533 <sup>2</sup>	0.98769	0.98843	0.98816	0.98816	0.98843	0.98769	0.0064682	0.0059197
0.442539	0.98766	0.98845	0.98819	0.98819	0.98845	0.98766	0.0064686	0.0059201
0.442551	0.98758	0.98849	0.98827	0.98827	0.98849	0.98758	0.0064700	0.0059212
0.442571	0.98744	0.98854	0.98838	0.98838	0.98854	0.98744	0.0064724	0.0059229
0.442606	0.98721	0.98859	0.98855	0.98855	0.98859	0.98721	0.0064762	0.0059258
0.442640	0.98695	0.98861	0.98872	0.98872	0.98861	0.98695	0.0064798	0.0059284
0.442691	0.98658	0.98855	0.98897	0.98897	0.98855	0.98658	0.0064849	0.0059320
0.442740	0.98618	0.98839	0.98920	0.98920	0.98839	0.98618	0.0064894	0.0059350
0.442803	0.98562	0.98799	0.98948	0.98948	0.98799	0.98562	0.0064943	0.0059381

1 Point  $P_{3,6}$

2 Point  $P_{sb}$

Values for some parameters of the wave along the  $P - 6$  branch.

Table 5.

$c$	$Q_1$	$Q_2$	$Q_3$	$Q_4$	$Q_5$	$Q_6$	$T$	$V$
0.442533 <sup>1</sup>	0.98769	0.98843	0.98816	0.98816	0.98843	0.98769	0.0064682	0.0059197
0.442316	0.98748	0.98853	0.98850	0.98780	0.98867	0.98750	0.0064673	0.0059189
0.442528	0.98724	0.98865	0.98871	0.98756	0.98887	0.98728	0.0064662	0.0059178
0.442525	0.98715	0.98869	0.98878	0.98748	0.98893	0.98721	0.0064658	0.0059174
0.442517	0.98671	0.98884	0.98910	0.98711	0.98922	0.98683	0.0064630	0.0059144
0.442508	0.98634	0.98891	0.98933	0.98680	0.98943	0.98653	0.0064601	0.0059113
0.442497	0.98595	0.98893	0.98955	0.98650	0.98962	0.98622	0.0064565	0.0059075
0.442483	0.98556	0.98889	0.98976	0.98619	0.98979	0.98592	0.0064524	0.0059031
0.442475	0.98536	0.98885	0.98986	0.98604	0.98987	0.98576	0.0064501	0.0059006
0.442467	0.98515	0.98880	0.98995	0.98589	0.98994	0.98561	0.0064476	0.0058980
0.442450	0.98473	0.98865	0.99012	0.98558	0.99007	0.98530	0.0064423	0.0058923

<sup>1</sup> Point  $P_{sb}$

Values for some parameters of the wave along the  $P - 6A$  branch.

Table 6.

$j$	$p_j(\times 10^{-3})$	$q_j(\times 10^{-3})$
0	-216.795000	0.0
1	-0.092495	0.042022
2	0.190116	0.099751
3	0.003196	0.161178
4	-0.375653	0.194799
5	0.444870	0.204610
6	-49.456600	0.0
7	-0.353888	0.158902
8	0.197979	0.105166
9	0.001469	0.025249
10	0.102077	-0.052641
11	-0.228032	-0.106101
12	16.506100	-0.000028
13	0.280234	-0.124306
14	-0.186820	-0.100413
15	-0.004249	-0.043752
16	-0.423778	0.021337
17	0.154760	0.072821
18	-8.425180	0.000057
19	-0.231066	0.101246
20	0.164546	0.089484

Values of the first coefficients for the nonsymmetric wave shown in figure 5 ( $c=.442497$ ).

## CHAPTER 3

### Symmetry breaking in periodic and solitary gravity-capillary waves on water of finite depth

#### 3.1 Introduction

Since early in this century it was known that the problem of permanent form gravity-capillary waves travelling on water has a very rich structure of solutions. Wilton (1915) discovered that even for very low amplitude waves solutions are not unique, i.e. given the shortest period and height of the wave there may be more than one solution. In particular, he studied the case of waves on deep water, and predicted two different wave profiles of 2.44 cm wavelength. These waves are today called Wilton's ripples. Further, he actually found a set of critical wavelengths where his perturbation solution was not valid.

In the case of gravity waves, it has been shown that for sufficiently low amplitude the solutions are unique. This is because, at low amplitude, resonances are not possible. Two waves are said to be resonant when they travel with the same phase

speed. In general, resonances lead to bifurcations into different families of waves and, therefore, to a lack of uniqueness in the solutions. When capillarity is taken into account, the new parameter allows resonances at low amplitude. These can be observed even with linear theory.

Wilton's results were later extended to higher order by works of Sekerzh-Zenkovich (1956) and Pierson & Fife (1961) using classical perturbation techniques, and by Nayfeh (1970b) using the method of multiple scales. Schooley (1960) photographed short-length wind generated water waves showing that profiles with two dimples predicted by Wilton (1915) exist. He also showed that, under proper conditions, waves of 3,4,5, or more dimples are observed.

Chen & Saffman (1979,1980) reconsidered the problem of gravity-capillary waves on deep water. They performed perturbation expansions for weakly nonlinear waves and computed finite amplitude waves using the full water wave equations. They reinterpreted Wilton's (1915) results as a bifurcation phenomenon and showed that higher order resonances exist and produce bifurcations to new families of solutions with larger wavelengths and more crests per period. Toland & Jones (1985) and Jones & Toland (1986) gave rigorous mathematical account of the existence theory of the weakly nonlinear waves found by Chen & Saffman (1980).

Nayfeh (1970a) considered the problem of periodic gravity-capillary waves on water of finite depth. He obtained third order expansions for the solution using the method of multiple scales. He found that his expansions broke down for two wave numbers if the depth of the fluid was larger than some critical value (his condition on the depth of the fluid is equivalent in our notation to the condition  $\tau < 1/3$ , where  $\tau$  is defined in equation (5)). In the limit of deep water Nayfeh's expansion matches Wilton's. The two points where the expansion breaks down correspond to



the first two critical wavelengths found by Wilton. Recently Jones (1986), using a functional analysis approach based on the Lyapunov-Schmidt reduction procedure, has studied rigorously the existence and multiplicity of small amplitude solutions of the periodic gravity waves-capillary wave problem of finite depth. But, as we will describe below, his work is limited to the study of some particular cases of waves whose shapes are even periodic functions of the horizontal distance (i.e., symmetric waves).

In considering a fluid of finite depth the possibility of solitary wave solutions is added to the problem. Solitary waves in the presence of surface tension were first considered by Korteweg & De Vries (1895). They found that for sufficiently large values of the surface tension a depression solitary wave is possible. Recently, Benjamin (1982) and Vanden-Broeck & Shen (1983) considered the problem again confirming Korteweg & De Vries results. Recently Amick & Kirchgässner (1986) have given a rigorous proof of the existence of the solitary wave of depression when  $\tau > 1/3$ . Also Hunter & Vanden-Broeck (1982) performed numerical computations for the depression solitary wave using the full water wave equations. They studied the effect of the capillary parameter, and found that, as surface tension was decreased below a critical value ( $\tau = 1/3$ ), inflexion points appeared in the solitary wave profile and numerical convergence became very difficult.

Even though there are still many open questions about the problem of Wilton ripples and solitary waves on finite depth, relatively little work has been done. For gravity-capillary water waves on the surface of the earth, resonances and all the interesting behavior appear at depths of approximately 4.8mm. At this scale the waves are strongly affected by viscosity, so that the irrotational model is not realistic and experimental checks are difficult. Benjamin (1982) tried some experiments with water 3 to 4 mm deep, finding that depression solitary waves are realizable, but they

are damped very rapidly.

With the increasing possibility of performing experiments in outer space, where gravity can be reduced several orders of magnitude, and the model becomes quite realistic, such experiments are more feasible and make the problem more attractive.

In the present work we shall consider one of the open theoretical questions. Namely the existence of nonsymmetric gravity-capillary waves. All solutions of permanent form gravity-capillary waves travelling on water known to date describe symmetric waves. A wave is said to be symmetric when, if  $f(x)$  represents the shape of the wave, the origin of the horizontal axis can be chosen such that  $f(x) = f(-x)$ .

In Chapters 1 and 2 we showed that nonsymmetric gravity waves are possible. They appear via a spontaneous symmetry breaking bifurcation from symmetric waves. We first found these waves using a weakly nonlinear model for finite depth water. Later we extended the results to deep water by numerical computations using the full water wave equations. It is our belief that a great part of the success in extending the results from the model to the full equations comes from using a Hamiltonian formulation of the problem, which keeps the symmetries of the water wave problem. We found that most of the physical behavior of the system is related to the Hamiltonian structure and its symmetries.

Motivated by these results, in the present work we develop a weakly nonlinear model for gravity-capillary waves on shallow water. The model is obtained from the Hamiltonian formulation of the water wave problem introduced by Zakharov (1968). With this model we try to increase the understanding of the Wilton's ripples phenomenon by using dynamical systems theory. In particular we show that nonsymmetric waves exist, and they appear via spontaneous symmetry breaking from symmetric waves, as happens for gravity waves.

We start by developing the model, which leads for steady waves to a Hamiltonian system of two degrees of freedom. A local analysis of the solutions around the fixed points of the system is carried out, followed by a numerical global analysis of the model.

In the last part of the work we consider solitary wave solutions. We give numerical evidence that when the depth of the fluid is close to the resonance value ( $\tau = 1/3$ ), the solitary wave is not unique and nonsymmetric solitary waves are possible.

### 3.2 Weakly nonlinear model

Consider two-dimensional irrotational water waves in a laterally unbounded domain of constant depth  $h$ . Taking as variables to describe the state of a wave the shape of the wave  $\eta(x, t)$  and the velocity potential evaluated at the surface of the wave  $\psi(x, t) = \phi(x, \eta(x, t), t)$ , the evolution equations for the system can be written with the following Hamiltonian structure (Zakharov 1968, Broer 1974, Miles 1977)

$$\begin{aligned} \frac{\partial \eta}{\partial t} &= \frac{\delta \mathbf{H}}{\delta \psi}, \\ \frac{\partial \psi}{\partial t} &= -\frac{\delta \mathbf{H}}{\delta \eta}, \end{aligned} \tag{3.1}$$

where  $\mathbf{H}$  is the total energy of the waves and  $\delta$  represents a functional derivative (see Goldstein 1980).

The total energy is defined as

$$\mathbf{H} = \int H(\eta, \psi) dx, \tag{3.2}$$

where the integral extends over one period in the case of periodic waves. In the present study we consider gravity-capillary waves. The energy density  $H$  is

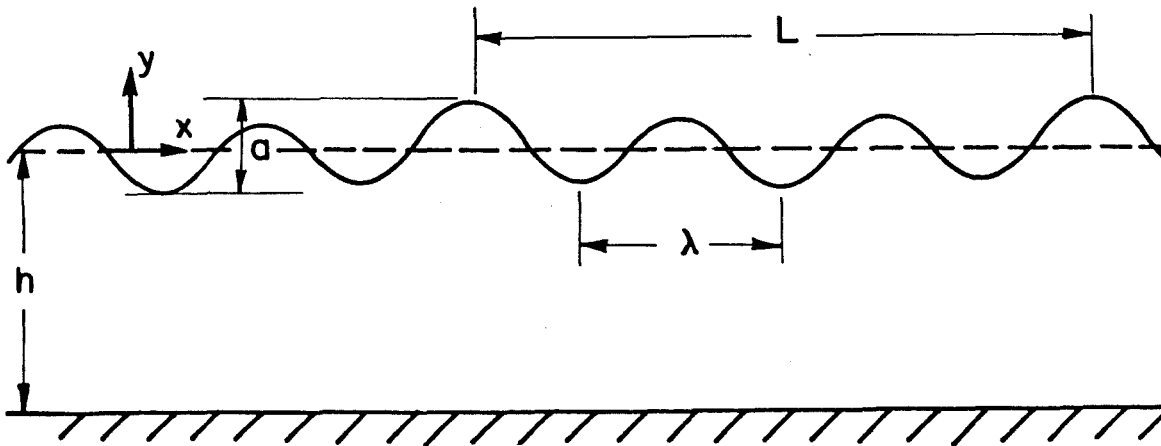
$$H = \frac{1}{2} \int_{-h}^{\eta} (\nabla\phi)^2 dy + \frac{1}{2} g\eta^2 + \frac{T}{\rho} \{\sqrt{1 + \eta_x^2} - 1\}, \quad (3.3)$$

where  $g$  is the acceleration due to gravity,  $\rho$  the density and  $T$  the surface tension.

We consider weakly nonlinear shallow water waves,

$$\alpha = \frac{a}{h} \ll 1, \quad \beta = \left(\frac{h}{\lambda}\right)^2 \ll 1, \quad (3.4)$$

where  $a$  represents a measure of the amplitude and  $\lambda$  of the wavelength (order of magnitude of the distance between crests) of the considered wave (see Figure 3.1). The two dimensionless parameters  $\alpha$  and  $\beta$  measure the amplitude and dispersion of the waves respectively (Whitham 1974).



**Figure 3.1.-** Periodic travelling wave on water of finite depth.

In order to deal with variables of order unity and determine the relative im-

portance of the different terms in the equations, it is convenient to introduce the following dimensionless variables

$$\begin{aligned}\bar{x} &= \frac{x}{\lambda}, & \bar{t} &= t\sqrt{\frac{g}{h}}, & \bar{H} &= \frac{\beta^{1/2}}{\alpha^2} \frac{H}{gh^2}, \\ \bar{y} &= \frac{y}{h}, & \bar{\phi} &= \frac{h\phi}{\lambda a\sqrt{gh}}, & \tau &= \frac{T}{\rho gh^2}, \\ \bar{\eta} &= \frac{\eta}{a}, & \bar{\psi} &= \frac{h\psi}{\lambda a\sqrt{gh}}.\end{aligned}\tag{3.5}$$

For gravity waves, the nonlinearity is of the same order as the dispersion when the waves are such that  $\alpha \approx \beta$ . The same is true for gravity-capillary waves except when we look at the minimum of the dispersion. In the region of minimum dispersion, the nonlinearity and dispersion are of the same order when  $\alpha \approx \beta^2$ . Hunter & Vander-Broeck's(1982) results showed that this region is where all the interesting resonances and bifurcations occur. Therefore in the following we will use  $\epsilon$  instead of  $\alpha$  and  $\beta$ , where  $\epsilon$  is taken as

$$\begin{aligned}\alpha &= \epsilon^2, \\ \beta &= \epsilon.\end{aligned}\tag{3.6}$$

In Chapter 1 we showed that using the variables introduced in equations (3.4) the energy density can be expanded for  $\epsilon \ll 1$  as

$$\begin{aligned}\epsilon^{1/2}\bar{H} &= \frac{\bar{\psi}_{\bar{x}}^2}{2}(1 + \epsilon^2\bar{\eta}) + \frac{\bar{\eta}^2}{2} + \frac{\epsilon}{6}[\bar{\psi}_{\bar{x}\bar{x}}^2 + 2\bar{\psi}_{\bar{x}}\bar{\psi}_{\bar{x}\bar{x}\bar{x}}] + \\ &\quad - \frac{\epsilon^2}{15}[\bar{\psi}_{\bar{x}\bar{x}\bar{x}}^2 + \frac{5}{2}(\bar{\psi}_{\bar{x}\bar{x}}\bar{\psi}_{\bar{x}\bar{x}\bar{x}\bar{x}} + \bar{\psi}_{\bar{x}}\bar{\psi}_{\bar{x}\bar{x}\bar{x}\bar{x}})] + \epsilon\frac{\tau}{2}\bar{\eta}_{\bar{x}}^2 + O(\epsilon^3).\end{aligned}\tag{3.7}$$

Using the energy density defined by equation (3.7) and neglecting terms of order  $\epsilon^3$ , we obtain the following evolution equations for the water wave system

$$\begin{aligned}\bar{\eta}_t &= -\bar{\psi}_{xx} - \frac{1}{3}\epsilon\bar{\psi}_{xxxx} - \frac{2}{15}\epsilon^2\bar{\psi}_{xxxxxx} - \epsilon^2(\bar{\eta}\bar{\psi}_x)_x, \\ \bar{\psi}_t &= -\bar{\eta} + \epsilon\tau\bar{\eta}_{xx} - \frac{1}{2}\epsilon^2\bar{\psi}_x^2.\end{aligned}\tag{3.8}$$

We are looking for permanent form travelling waves of these equations, i.e. solutions of the form

$$\begin{aligned}\bar{\eta} &= \bar{\eta}(\bar{x}, \bar{t}) = \bar{\eta}(\bar{x} - c\bar{t}) = \bar{\eta}(\bar{x}), \\ \bar{\psi} &= \bar{\psi}(\bar{x}, \bar{t}) = \bar{\psi}(\bar{x} - c\bar{t}) = \bar{\psi}(\bar{x}),\end{aligned}\tag{3.9}$$

where  $c$  is the phase speed of the wave train.

Introducing relation (3.9) into equations (3.8), we have

$$\begin{aligned}-c\bar{\eta}_x &= -\bar{\psi}_{xx} - \frac{1}{3}\epsilon\bar{\psi}_{xxxx} - \frac{2}{15}\epsilon^2\bar{\psi}_{xxxxxx} - \epsilon^2(\bar{\eta}\bar{\psi}_x)_x, \\ -c\bar{\psi}_x &= -\bar{\eta} + \epsilon\tau\bar{\eta}_{xx} - \frac{1}{2}\epsilon^2\bar{\psi}_x^2.\end{aligned}\tag{3.10}$$

Notice that the linear dispersion relation associated with equations (3.10) is

$$c^2 = 1 + \left(\tau - \frac{1}{3}\right)k^2 + \left(\frac{2}{15} - \frac{\tau}{3}\right)k^4,\tag{3.11}$$

where  $k = 2\pi h/\lambda$  is the wave number. This dispersion relation agrees up to (but not including) terms of order  $k^6$  with the classical dispersion relation for linear water waves, which is

$$c^2 = \frac{\tanh k}{k}(1 + \tau k^2).\tag{3.12}$$

From (3.11) it follows that system (3.10) has wave trains that travel in both directions with phase speeds  $c \simeq \pm 1$  respectively. To find solutions we will concentrate our study on waves that travel to the right ( $c \simeq 1$ ).

In the present case of gravity-capillary waves on shallow water resonances appear when  $\tau$  is close to  $1/3$  (Nayfeh 1970, Hunter & Vander-Broeck 1982). For  $\tau = 1/3$  the coefficient of the second term of the dispersion relation (11) vanishes, and the term in  $k^4$  becomes important; hence the balance between nonlinearity and dispersion leads to waves such that  $k^4 \approx (a/\lambda)$ , i.e.  $\alpha \approx \beta^2$ .

Considering waves that travel to the right with  $\tau \approx 1/3$ , a travelling wave solution can be expanded in the following way

$$\begin{aligned}\bar{\psi} &= \bar{\psi}_0 + \epsilon \bar{\psi}_1 + \epsilon^2 \bar{\psi}_2 + O(\epsilon^3), \\ \bar{\eta} &= \bar{\eta}_0 + \epsilon \bar{\eta}_1 + \epsilon^2 \bar{\eta}_2 + O(\epsilon^3), \\ \bar{c} &= 1 + \epsilon \bar{c}_1 + \epsilon^2 \bar{c}_2 + O(\epsilon^3), \\ \bar{\tau} &= 1/3 + \epsilon \bar{\tau}_1 + \epsilon^2 \bar{\tau}_2 + O(\epsilon^3).\end{aligned}\tag{3.13}$$

Substituting these relations into equation (3.10), we end up, after some algebra, with

$$\begin{aligned}\bar{c}_1 &= 0, \\ \bar{\psi}_{0\bar{x}} &= \bar{\eta}_0,\end{aligned}\tag{3.14}$$

and the following equation for  $\bar{\eta}_0$

$$2\bar{c}_2 \bar{\eta}_{0\bar{x}} - 3\bar{\eta}_0 \bar{\eta}_{0\bar{x}} + \bar{\tau}_1 \bar{\eta}_{0\bar{x}\bar{x}} - \frac{1}{45} \bar{\eta}_{0\bar{x}\bar{x}\bar{x}\bar{x}} = 0.\tag{3.15}$$

This equation can be integrated once with respect to  $\bar{x}$  giving

$$2\bar{c}_2 \bar{\eta}_0 - \frac{3}{2} \bar{\eta}_0^2 + \bar{\tau}_1 \bar{\eta}_{0\bar{x}\bar{x}} - \frac{1}{45} \bar{\eta}_{0\bar{x}\bar{x}\bar{x}\bar{x}} = C.\tag{3.16}$$

The first three terms of the equation represent the KdV approximation.

The origin of  $y$  can be chosen such that  $C = 0$ . Thus in the following we assume  $C = 0$  without any loss of generality, i.e.  $\eta = 0$  is a solution. Equation (3.16) can be also directly obtained from the classical water wave equations performing the formal expansion for  $\epsilon \ll 1$  (Hunter & Vanden-Broeck 1982).

The dimensionless variables introduced by equation (3.5) are useful to determine which terms of the equations are important in the range of waves of interest and to derive the model equation. However, for computational purposes and understanding the nature of the solutions it is more convenient to scale directly in term of  $h$

$$\begin{aligned}\eta' &= \frac{\eta}{h} = \epsilon^2 \bar{\eta}, \\ x' &= \frac{x}{h} = \epsilon^{1/2} \bar{x}, \\ \tau_1 &= \tau - 1/3 = \epsilon \bar{\tau}_1, \\ c_2 &= c - 1 = \epsilon^2 \bar{c}_2.\end{aligned}\tag{3.17}$$

and obtain the equation

$$2c_2\eta_0 - \frac{3}{2}\eta_0^2 + \tau_1\eta_{0xx} - \frac{1}{45}\eta_{0xxxx} = 0,\tag{3.18}$$

where we have dropped the primes.

In this equation the small parameter  $\epsilon$  has disappeared and each term has its own magnitude (i.e., may be large or small depending on the magnitude of the physical wave).

Since  $\beta \ll 1$ ,  $\partial/\partial x \ll 1$  and the last term in equation (3.18) can be neglected when  $\tau_1$  is of order unity, and we recover the KdV approximation. But when  $\tau_1$  is sufficiently small the fourth term of the equation becomes important.

Equation (3.18) can be rewritten as the following system of four first order



differential equations

$$\begin{aligned}
 \frac{du_1}{dx} &= u_2, \\
 \frac{du_2}{dx} &= u_3, \\
 \frac{du_3}{dx} &= u_4, \\
 \frac{du_4}{dx} &= \frac{45}{2} \left[ (2c_2 - \frac{3}{2}u_1)u_1 + \tau_1 u_3 \right],
 \end{aligned}
 \tag{3.19}$$

where

$$u_1 = \eta_0, \quad u_2 = \eta_{0x}, \quad u_3 = \eta_{0xx}, \quad u_4 = \eta_{0xxx}.
 \tag{3.20}$$

Note that this system is Hamiltonian, with Hamiltonian function

$$H = (c_2 - \frac{1}{2}q_1)q_1^2 + p_1q_2 - \frac{\tau_1}{2}q_2^2 + \frac{45}{2}p_2^2.
 \tag{3.21}$$

and the canonical variables  $(p_1, p_2, q_1, q_2)$  related to the variables  $(u_1, u_2, u_3, u_4)$  by

$$\begin{aligned}
 q_1 &= u_1, & q_2 &= u_2, \\
 p_1 &= \tau_1 u_2 - \frac{u_4}{45}, & p_2 &= \frac{u_3}{45}.
 \end{aligned}
 \tag{3.22}$$

Notice that we started with a continuous Hamiltonian system evolving in time. Looking for travelling waves of that continuous system we finish with a discrete Hamiltonian system that evolves in  $x$ . The same phenomenon was found in Chapter 1 for gravity waves. Benjamin (1984) has shown that this is a general property for travelling waves of continuous Hamiltonian systems for which the Hamiltonian density can be expressed in terms of finite order derivatives, as happens in our model. Hence, the problem of finding travelling waves of permanent form has been

reduced to the study of the properties of a Hamiltonian system of two degrees of freedom with two parameters,  $c_2$  and  $\tau_1$ .

### 3.3 Local analysis

The aim of the present study is to understand the structure of the families of periodic orbits of the dynamical system defined by equations (3.19). These periodic orbits correspond to travelling waves of permanent form in the physical plane.

We start our analysis by examining the fixed points of the dynamical system and the local structure of the phase flow around them. The two-degrees-of-freedom Hamiltonian system defined by equations (3.19) has two fixed points for all values of the parameters  $c_2$  and  $\tau_1$ . These fixed points are

$$\begin{aligned} FP1 : u_1 = u_2 = u_3 = u_4 = 0, \\ FP2 : u_1 = 4c_2/3, u_2 = u_3 = u_4 = 0. \end{aligned} \tag{3.23}$$

Both fixed points represent flat surfaces.  $FP1$  is the origin of the phase space and corresponds to the actual surface of the fluid at rest. The other fixed point represents a flat surface also but with a different location of the horizontal axis. The two fixed points represent states of the system that are conjugate (see Benjamin 1984).

Notice that equations (3.19) have two parameters, but the location of the fixed points depends only on one of the parameters,  $c_2$ .

Linearizing the system in the neighborhood of the fixed points, we find the following eigenvalues for the stability of the fixed points

$$\begin{aligned}
 FP1 : \lambda^2 &= \frac{45}{2} \left[ \tau_1 \pm \sqrt{\tau_1 + \frac{8c_2}{45}} \right], \\
 FP2 : \lambda^2 &= \frac{45}{2} \left[ \tau_1 \pm \sqrt{\tau_1 - \frac{8c_2}{45}} \right].
 \end{aligned}
 \tag{3.24}$$

For each fixed point we have four eigenvalues, which determine the structure of the fixed point. Depending on the values of the parameters, we can have real, pure imaginary or complex eigenvalues. According to the nature of the eigenvalues the  $(\tau_1, c_2)$  plane can be divided in four different regions for each fixed point, see figure 2 (a) and (b). For example, looking at  $FP1$ , we find that for  $c_2 > 0$  two of the eigenvalues are real and two pure imaginary. For  $c_2 < 0$  there are three different possibilities. If  $\tau_1 < -\sqrt{-8c_2/45}$  there are two pairs of pure imaginary eigenvalues ( $\lambda_{1,2} = \pm i\omega_1, \lambda_{3,4} = \pm i\omega_2$ ). For  $\tau_1^2 < -8c_2/45$  the four eigenvalues are complex. And finally, when  $\tau_1 > \sqrt{-8c_2/45}$  all the eigenvalues are real. The situation for  $FP2$  is similar just by changing  $c_2$  to  $-c_2$ .

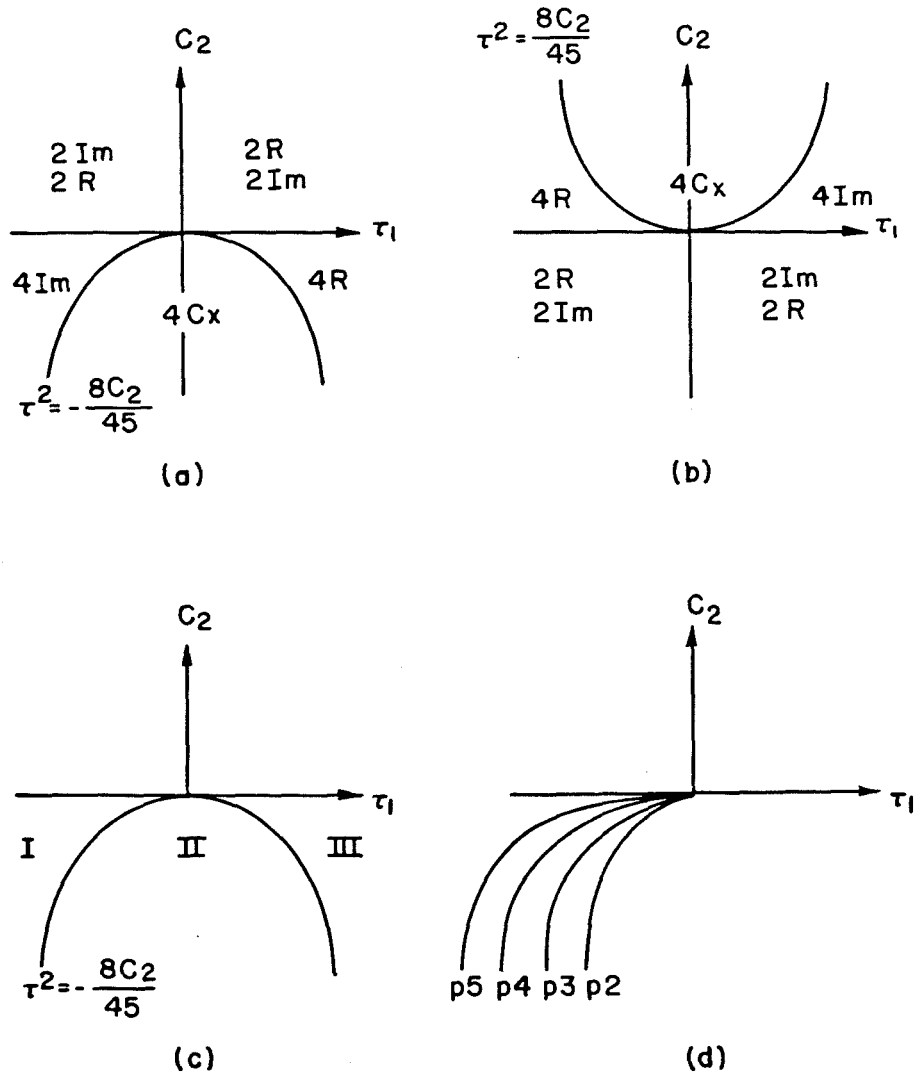
According to the previous paragraph and figures 3.2 (a-b), there are six different regions in the  $(\tau_1, c_2)$  plane. But we find that equation (3.18) is invariant under the transformation

$$\begin{aligned}
 c_2 &\rightarrow -c_2, \\
 u_1 &\rightarrow u_1 - \frac{4c_2}{3}.
 \end{aligned}
 \tag{3.25}$$

This means that the region corresponding to  $c_2 > 0$  is equivalent to the region  $c_2 < 0$  just interchanging the roles of the two fixed points. Due to this symmetry we only need to study three of these regions. Therefore, in the following, we reduce the study to the region  $c_2 < 0$  (see figure 3.2 (c)).

Now consider a local analysis of the three different regions.

- *Region I* ( $c_2 < 0, \tau_1 < -\sqrt{-8c_2/45}$ ) :



**Figure 3.2.-** (a) Eigenvalues for the stability of the fixed point  $FP2$ . (b) Eigenvalues for the stability of the fixed point  $FP2$ . (c) The three different regions of the  $(\tau_1, c_2)$  plane where the study is concentrated. (d) Lines where bifurcations occur at zero amplitude in the  $(\tau_1, c_2)$  plane. P2 denotes period doubling, P3 period tripling and so on.

In this region  $FP1$  is a center. It has two pairs of pure imaginary eigenvalues  $(\lambda_{1,2} = \pm i\omega_1, \lambda_{3,4} = \pm i\omega_2)$ . According to the Liapunov theorem for fixed points of Hamiltonian systems (Abraham & Marsden 1978), for fixed values of the two parameters  $c_2$  and  $\tau_1$  the phase space has two 2-dimensional manifolds containing

*FP1*. Each of these manifolds contains a one-parameter family of periodic orbits whose period approaches  $2\pi/\omega_1$  or  $2\pi/\omega_2$  respectively as the amplitude goes to zero and the orbit approaches the fixed point. This is actually only true when there are no resonance relations between the two frequencies. We say that two frequencies are resonant when there exist two integers  $m$  and  $n$  such that  $m\omega_1 + n\omega_2 = 0$ . In resonant cases linear stability is not enough to determine the local structure of the phase flow around the fixed points.

Note that the Liapunov theorem for Hamiltonian systems is equivalent to the Hopf theorem for generic dynamical systems.

In region *I*, *FP2* has two pure imaginary and two real eigenvalues ( $\lambda_{1,2} = \pm i\omega, \lambda_{3,4} = \pm\alpha$ ). Hence according to the Liapunov theorem there is a single two-dimensional manifold containing *FP2* and a one-parameter family of periodic orbits whose period approach  $2\pi/\omega$  as the amplitude goes to zero.

Due to the fact that there are two families of periodic orbits centered on *FP1*, this region is very rich from the bifurcation point of view, as we will see below.

-Region II ( $c_2 < 0, \tau_1^2 < -8c_2/45$ ):

Inside the parabola (figure 3.2 (c)) *FP1* has four complex eigenvalues ( $\lambda_{1,2} = \pm(\alpha_1 + i\omega_1), \lambda_{3,4} = \pm(\alpha_2 + i\omega_2)$ ). *FP1* has two-dimensional stable and unstable manifolds. On each of the manifolds the flow has a spiral structure. In this case, from the local analysis of *FP1* we can conclude nothing about possible families of periodic orbits. We can only say that locally there are no families of periodic orbits that end at the fixed point.

There exists the possibility of a homoclinic connection for *FP1*. If this homoclinic connection exists, it will correspond to a solitary wave with oscillatory tails.

We will present later numerical evidence that this kind of connection exists.

For  $FP2$  we find the same situation as the previous case. As there is a pair of pure imaginary eigenvalues, we have a one-parameter family of periodic orbits centered on  $FP2$ .

-Region III ( $c_2 < 0, \tau_1 > \sqrt{-8c_2/45}$ ):

$FP1$  has four real eigenvalues ( $\lambda_{1,2} = \pm\alpha_1, \lambda_3 = \pm\alpha_2$ ). No families of periodic orbits coming from  $FP1$  are possible in this region. On the other hand,  $FP2$  has two real and two pure imaginary eigenvalues. Hence, a one-parameter family of periodic orbits comes from  $FP2$ .

Korteweg & De-Vries (1895) showed that solitary waves of depression exist for sufficiently large values of the surface tension. Actually the KdV equation shows that depression solitary waves exist for  $\tau > 1/3$ . The derivation was formalized by Vanden-Broeck & Shen (1983) using a systematic perturbation calculation. In the KdV limit, the phase space is two-dimensional and the two fixed point  $FP1$  and  $FP2$  still exist. For  $\tau > 1/3$  there is a one-parameter family of periodic orbits centered on  $FP2$ .  $FP1$  is a saddle (two real eigenvalues).

We expect our results to match with these previous results when the wave length is increased and  $\tau > 1/3$ . In particular we expect that if the period is increased in the family of periodic orbits that is created at  $FP2$ , the family ends up with an orbit that is homoclinic to  $FP1$ .

With all these local considerations in mind, we can start the global analysis of the system.

### 3.4 Symmetric waves

When considering periodic waves of our system for fixed  $\tau_1$  and a given value of  $c_2$ , the periodic orbits appear as one-parameter families, having the physical period  $L$  (shortest distance over which solution repeats exactly) as parameter. This degeneracy of the system, owing to the Hamiltonian structure of the system, makes it necessary to give three parameters  $(c_2, \tau_1, L)$  to determine a periodic orbit. Note that in the analysis performed below, when  $\tau_1$  or  $c_2$  are changed the phase space is changed. But when  $L$  is changed the phase space remains the same and we just move along one of the one-parameter families of periodic orbits that exist in phase space.

We start the global analysis by studying region I. As discussed above, in phase space there are two one-parameter families of periodic orbits centered on  $FP1$ . These families start at  $FP1$  with periods  $2\pi/\omega_1$  and  $2\pi/\omega_2$  respectively. If we take  $\omega_1 > \omega_2$  then the family corresponding to  $\omega_1$  has a shorter physical plane period than the other family. We call these families, the long-wave family and short-wave family, with respective periods  $L_l = 2\pi/\omega_2$  and  $L_s = 2\pi/\omega_1$ , where  $(L_l > L_s)$ .  $L_s$  and  $L_l$  can be regarded as the periods of the infinitesimal waves for fixed  $\tau_1$  and a given value of  $c_2$ .

The phase speed  $c_2$  determines the location of the fixed points and gives a measure of the amplitude of the waves. The capillarity parameter  $\tau_1$  determines for a given fluid and given gravity the depth of the fluid layer.

In the present analysis we fix the phase speed  $c_2$ , and we study the structure of the phase space varying only the capillarity parameter  $\tau_1$  and the physical period  $L$ . Using these two parameters, the fixed points  $FP1$  and  $FP2$  are represented in

the  $(\tau_1, L)$  plane by two lines  $C_1$  and  $C_2$  as can be seen in figure 3.3 . These lines represent, for a given value of  $c_2$ , the initial period of the families of periodic orbits versus  $\tau_1$ , i.e.  $L = L(\tau_1; c_2)$ . We can see that in region  $I$  for a given value of  $\tau_1$ ,  $C_1$  gives two periods corresponding to the two families that come out of  $FP1$ . We can also see that the periods are ordered such that  $L_l > L_c > L_s$ , where

$$\left(\frac{2\pi}{L_c}\right)^2 = \frac{45}{2} \sqrt{\frac{-8c_2}{45}}. \quad (3.26)$$

For large period  $L$  the two lines  $C_1$  and  $C_2$  match asymptotically with the equivalent lines obtained using the KdV approximation, which is valid for very large wave lengths and away from the resonance region.

To study the possible bifurcations of the families of periodic orbits we need to compute the Floquet multipliers of the orbits. The multipliers are the eigenvalues of the linearized Poincaré map of the flow around a periodic orbit. These eigenvalues can be thought as the magnification of a perturbation in one period (Hartman 1964). The necessary and sufficient condition for a bifurcation is that a multiplier crosses the unit circle.

A periodic orbit always has a multiplier  $+1$  corresponding to sliding a little along the orbit. For Hamiltonian systems, the Hamiltonian is conserved so there is another multiplier at  $+1$ . Also in a Hamiltonian system the flow preserves volume in phase space, thus the product of the all four multipliers is  $+1$ . It follows, from the reality of the Hamiltonian, that the two other multipliers will be conjugate points  $\sigma, \sigma^*$  on the unit circle or reciprocal points  $\sigma, 1/\sigma$  on the real axis (Green *et al.* 1981). When the two multipliers are on the unit circle, we have possible bifurcation to new orbits. For example if  $\sigma = e^{2\pi im/n}$ , at that point there is a possible bifurcation to a period- $n$  wave. When  $arg(\sigma)/2\pi$  is an irrational number we



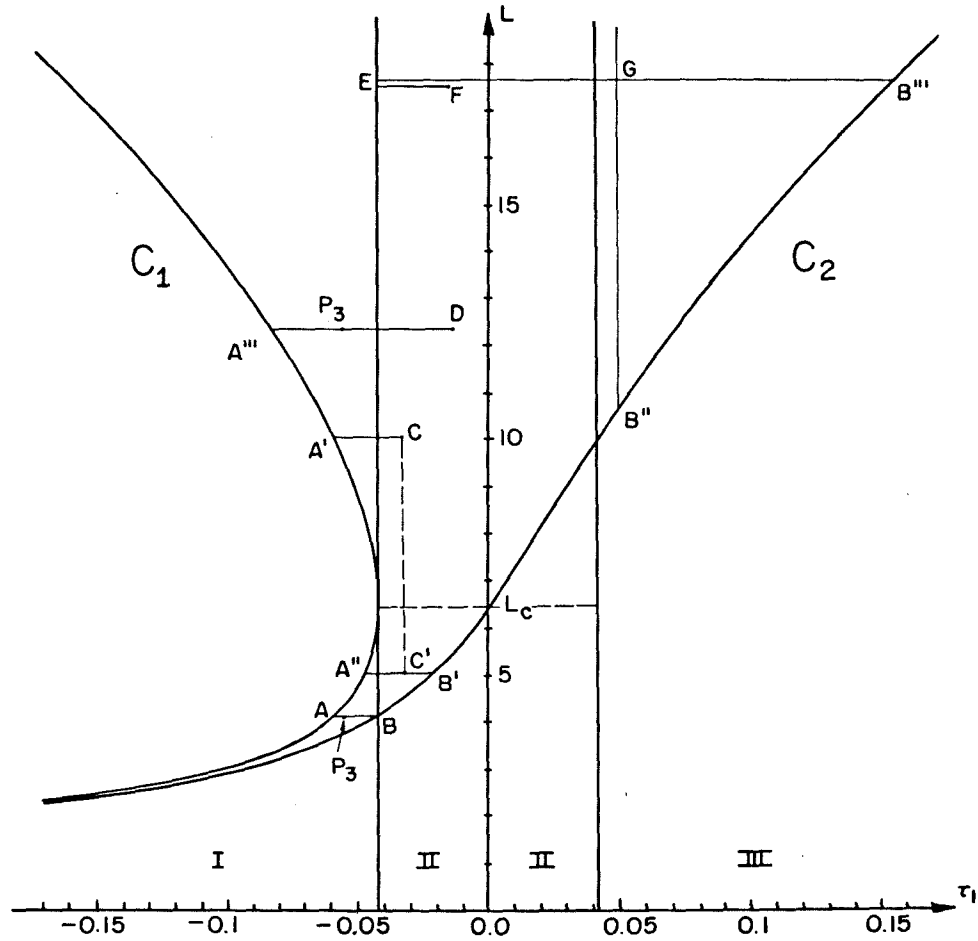


Figure 3.3.-  $(\tau_1, L)$  plane. Lines  $C_1$  and  $C_2$  show the period of the infinitesimal waves at the fixed points versus  $\tau_1$  ( $c_2 = -0.01$ ).

can still talk about bifurcations but the new branch corresponds to a quasiperiodic permanent form travelling wave. Notice that from this point of view bifurcations appear within continuous intervals of the parameter values. When approaches which allow only periodic waves, like the ones used by Chen & Saffman (1980), Toland & Jones (1985), Jones & Toland (1986) and Jones (1986), are considered, only the bifurcations which lead to periodic waves can be detected and they appear in a discrete way. The works by Chen & Saffman (1980) and Jones & Toland (1985) cover the cases where  $\sigma = e^{2\pi in/m}$  for deep water waves. While Jones (1986) covers the bifurcations of the type  $\sigma = e^{2\pi n/(n+1)}$  for finite depth water.

Performing a local analysis of the system around  $FP1$ , we can compute analytically the Floquet multipliers for the beginning of the families of periodic orbits. We find that for the short-wave family the multipliers at  $FP1$  (zero amplitude wave) are

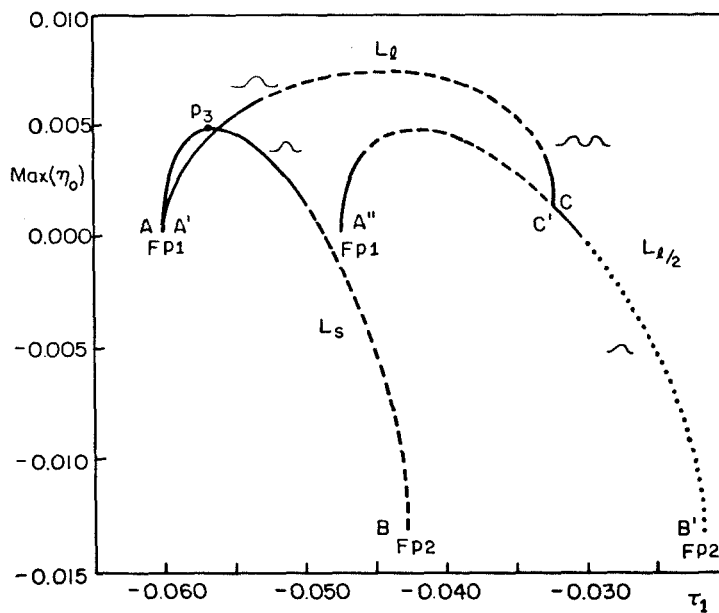
$$\begin{aligned}\sigma_{1,2} &= +1, \\ \sigma_{3,4} &= e^{2\pi i\omega_2/\omega_1}.\end{aligned}\tag{3.27}$$

Hence the multipliers are on the unit circle, and  $FP1$  is a bifurcation point in this region. This means that we can have bifurcation at zero amplitude. We already knew this because at  $FP1$  two different families of periodic orbits cross. When studying gravity waves in Chapter 1 we found that the multipliers go onto the unit circle only for high amplitude waves. Here we have one parameter more, and we can bring the bifurcations to zero amplitude. For example if  $\tau_1$  and  $c_2$  are chosen such that  $\omega_2/\omega_1 = 1/2$  then  $\sigma_{3,4} = -1$ , and we have a period doubling bifurcation at zero amplitude. In general  $\omega_2/\omega_1 = m/n$  corresponds to a bifurcation from a family of period  $L_s$  to a family of period  $nL_s$ . Notice that this relation is equivalent to a resonance relation. In figure 3.2 (d) we show lines where bifurcations occur at zero amplitude.

To understand the global structure of the phase space, we continued numerically the families of periodic orbits by using the program AUTO developed by Doedel & Kernevez (1986). This program performs continuation along families of periodic orbits for dynamical systems by using a collocation method.

In the present analysis, we kept the value of the phase speed fixed at  $c_2 = -0.01$ . We started at the fixed point  $FP1$  by taking  $\tau_1 = -0.06$ . These values correspond to region  $I$ . Starting from a fixed point there are basically two ways for performing the continuation. One is to continue in the period  $L$  keeping the capillarity parameter

$\tau_1$  constant (vertical lines in figure 3.3). The other possibility is to continue in the capillarity parameter keeping the period constant (horizontal lines in figure 3.3). In the present section we consider the latter case.



**Figure 5.-** Amplitude of short-wave and long-wave branches of periodic orbits with constant period versus  $\tau_1$ . Solid line denotes  $\sigma$  on the unit circle, dashed line  $\sigma$  negative and dotted line  $\sigma$  positive. Shape of the wave is sketched next to curve.

We began continuing along the short-wave family coming out from  $FP1$  (point  $A$  figures 3.3 and 3.4). Following the branch with the period  $L$  constant, we found that  $\tau_1$  increases monotonically and the family ends in the fixed point  $FP2$  (point  $B$  figure 3.3 and 3.4). Hence the unique family coming out from  $FP2$  is connected to the short-wave family that starts at  $FP1$ . Along the whole branch the wave is basically sinusoidal with only one crest per period. The wave length of this family is always less than  $L_c$ . This branch is represented in figure 3.3 by the A-B line.

The long-wave family ( $L_l$ ) was found to finish in a period half bifurcation. It starts with a sinusoidal wave form with only one crest per period (point  $A'$  figures 3.3 and 3.4), but as we go along the branch another crest appears. The branch ends

when the two crests are equal which corresponds to a period half bifurcation (point  $C$  figures 3.3 and 3.4). This point corresponds to a period doubling bifurcation on the short-wave family of period  $L_s = L_l/2$  (point  $C'$  figures 3.3 and 3.4). Notice that this is true only if  $L_l/2 < L_c$ . If this relation does not hold, the structure is much more complicated and will not be considered further here. The three branches of periods  $L_s$ ,  $L_l$  and  $L_l/2$  are shown in figure 3.4.

As can be seen in figure 3.4, along these branches the Floquet multipliers of the orbits go onto the unit circle several times. When the multipliers are on the unit circle, bifurcations to families with larger period are possible. In the following chapter we study some of these larger period families.

### 3.5 Nonsymmetric waves

All the gravity-capillary waves that we have studied in the previous section, as well as all the ones studied by other authors (Wilton 1915, Nayfeh 1970, Chen & Saffman 1980, Vanden-Broeck 1982) are symmetric.

In Chapters 1 and 2 we found that for gravity waves on shallow and deep water, the symmetry can be broken via a spontaneous symmetry breaking bifurcation. We found that the sequence of bifurcations that need to be followed to find the symmetry breaking are the same as the ones found by Green *et al.*(1981) for breaking the symmetry of fixed points of area preserving maps with symmetries. We want to determine if nonsymmetric gravity-capillary waves are possible and if the same bifurcation tree is repeated here.

We started again at  $FP1$  and took the short-wave branch. Keeping the period constant we continued in the capillarity parameter  $\tau_1$  as before. At the beginning of

the branch the Floquet multipliers are on the unit circle, and as  $\tau_1$  is increased they leave the unit circle through  $-1$  (see figure 3.4). The point where the multipliers are at  $-1$  corresponds to a period doubling bifurcation to a long-wave family of period  $L_l = 2L_s$ . When the multipliers are at  $120^\circ$  we have a period tripling (point  $P_3$  in figures 3.3, 3.4 and 3.5).

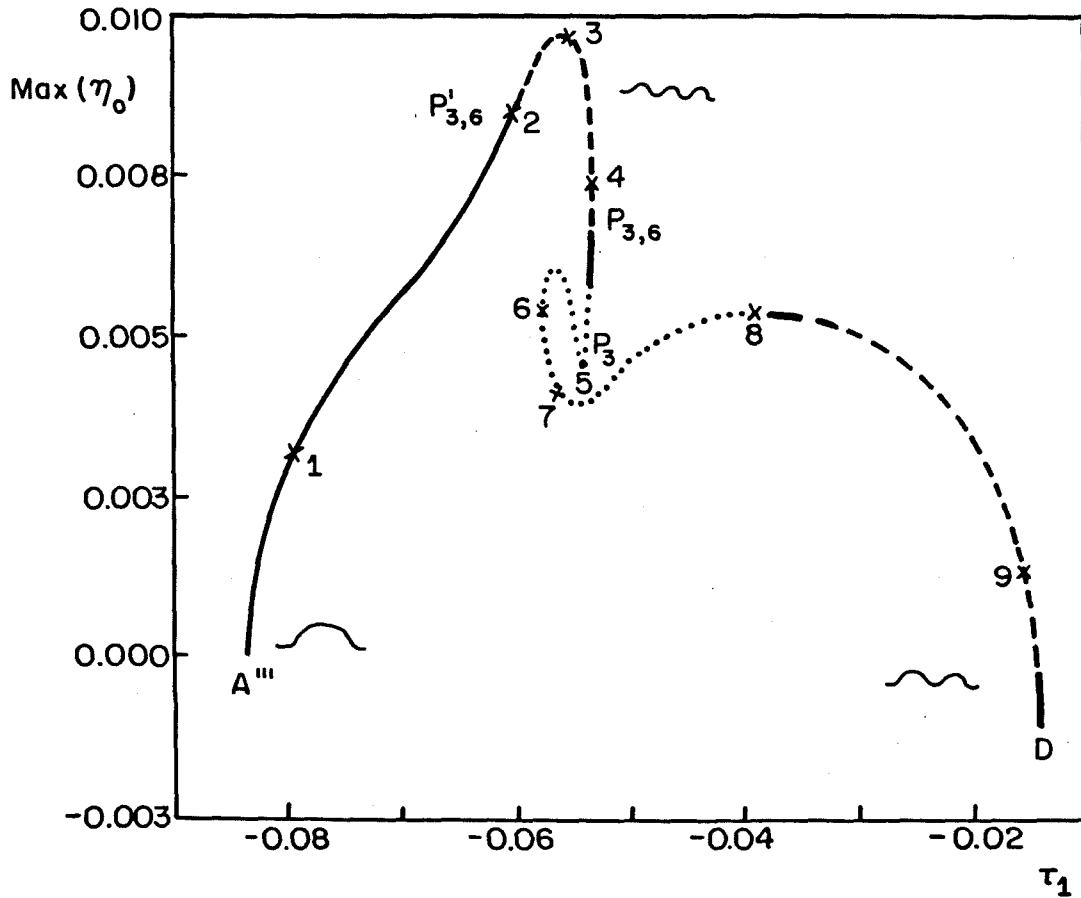
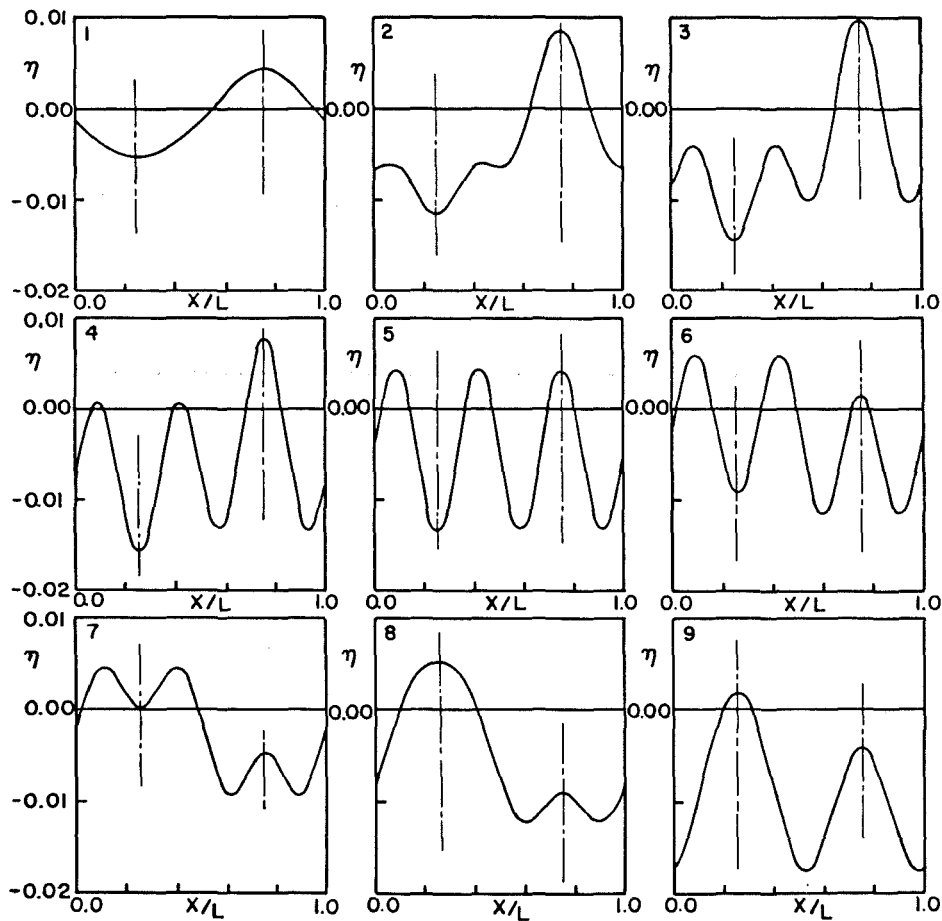


Figure 3.5.a.-  $P - 3$  branch arising from the period tripling  $P_3$  on branch A-B shown in figure 3.4.

We took the new branch of period  $3L_s$ , that we call  $P-3$  branch. Keeping the period constant, we continued in  $\tau_1$  finding that this new branch is just the long-wave branch coming out from  $FP1$  for  $L_l = 3L_s$ . Following the branch in one direction we found that the branch ends in  $FP1$  (point  $A'''$  figures 3.3 and 3.5), and going in the other direction the branch ends in a period half bifurcation (point  $D$



**Figure 3.5.b.-** Shape of the wave at different points along the branch  $P - 3$ . The location of the points is shown in part (a) of the figure. Vertical lines show symmetry axes.

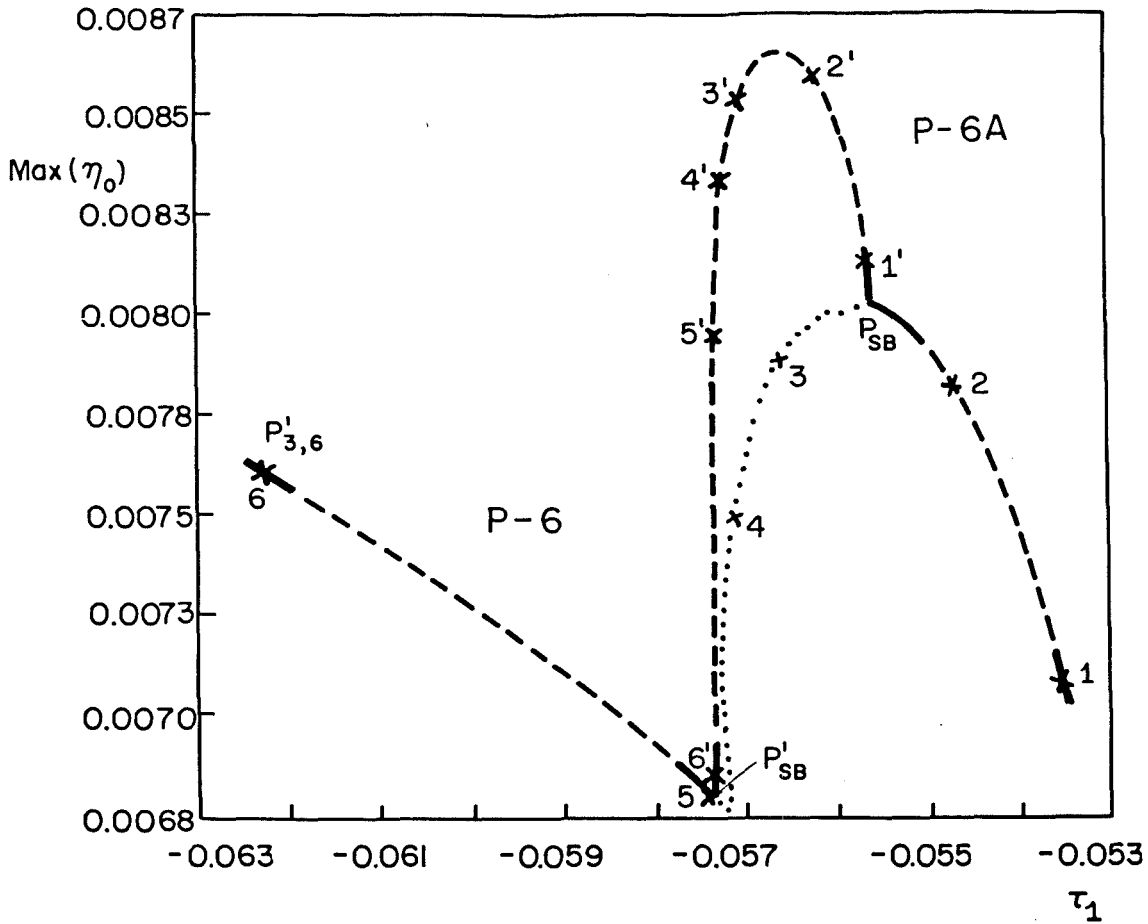
figures 3.3 and 3.5). The whole branch is plotted in figure 3.5 (a). The cusp that appears in the figure is due to the fact that we are plotting the maximum of the amplitude  $\eta$ . Even though the height of each crest changes smoothly with  $\tau_1$ , the maximum does not change smoothly at the points where a crest becomes higher than other. For example looking at the wave 4, 5 and 6 of figure 3.5-(b), we can see that on wave 4 the third crest is the higher. Wave 5 corresponds to the point  $P_3$  where the three crests are equal. And for wave 6 the first two crests are the higher. Hence, the transition through  $P_3$  is not smooth for maximum of  $\eta$ . As we can see in the figure the branch is not single valued in  $\tau_1$ . This means that the solution

is not unique for a given value of the period and the parameters.

At the beginning of the branch, close to  $FP1$ , the wave is sinusoidal and there is only one crest per period. In the region of the nonuniqueness, there are three crests per period. The branch finishes with two equal crests at the half period bifurcation.

On this  $P-3$  branch, at  $P_3$  the four multipliers are at  $+1$  as we expected because along this branch  $P_3$  is a normal bifurcation point. Along the branch there are four regions where the Floquet multipliers lie on the unit circle. Every time that the multipliers leave the unit circle through  $-1$ , we have a period doubling bifurcation into a family of waves of period  $6L_s$ . Following our results of Chapter 1 we took the new branch of period  $6L_s$  ( $P-6$  branch) that starts at the period doubling bifurcation closer to  $P_3$ , which corresponds to the point  $P_{3,6}$  in figure 3.5.

Following figure 3.6-(a), starting at  $P_{3,6}$  and continuing along the new  $P-6$  branch we found that the branch finishes in a half period bifurcation, which corresponds to another period doubling (point  $P'_{3,6}$  in figures 3.5 and 3.6) of the  $P-3$  branch. The whole branch and the behavior of the Floquet multipliers along the branch are shown in figure 3.6. We can see in the figure that the behavior found for gravity waves appears here again. At  $P_{3,6}$  on the  $P-6$  branch the four multipliers are at  $+1$ . As we go along the branch, they move along the unit circle and leave the unit circle through  $-1$ . At this point we have another period doubling bifurcation to a period  $12L_s$  wave. If we keep going along the  $P-6$  branch the multipliers become negative and  $\sigma$  increases in absolute value up to a maximum and starts decreasing again. The multipliers go back onto the unit circle through  $-1$  giving an inverse period doubling. Continuing on the  $P-6$  branch, we find a point where the four Floquet multipliers are at  $+1$ . This kind of bifurcation was studied by Rimmer (1978). In the present case we found that the bifurcation is a symmetry



**Figure 3.6.a.-**  $P - 6$  and  $P - 6A$  branches arising from the period doubling bifurcations of  $P - 3$  branch shown in figure 3.5.a and from symmetry breaking bifurcations of  $P - 6$  respectively. Maximum amplitude of the wave versus  $\tau_1$  along the branches.

breaking bifurcation.

Starting at  $P'_{3,6}$  and following the  $P-6$  branch, we find the same structure again. At  $P'_{3,6}$  the four multipliers are at  $+1$ . As we go along the branch they leave the unit circle through  $-1$ , going back to the unit circle later to give a new symmetry breaking bifurcation at  $P'_{sb}$ . Thus, we find two symmetry breaking bifurcations, point  $P_{sb}$  and  $P'_{sb}$ .

Switching to the nonsymmetric branch, that we call  $P-6A$  branch, we found that



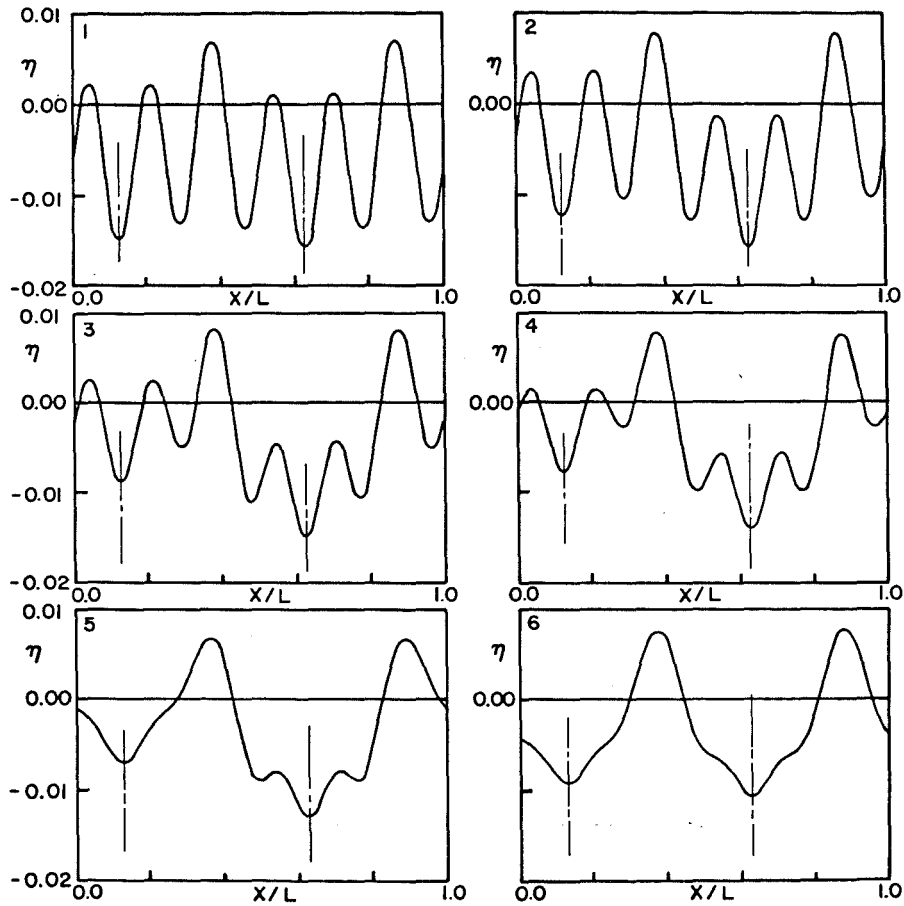
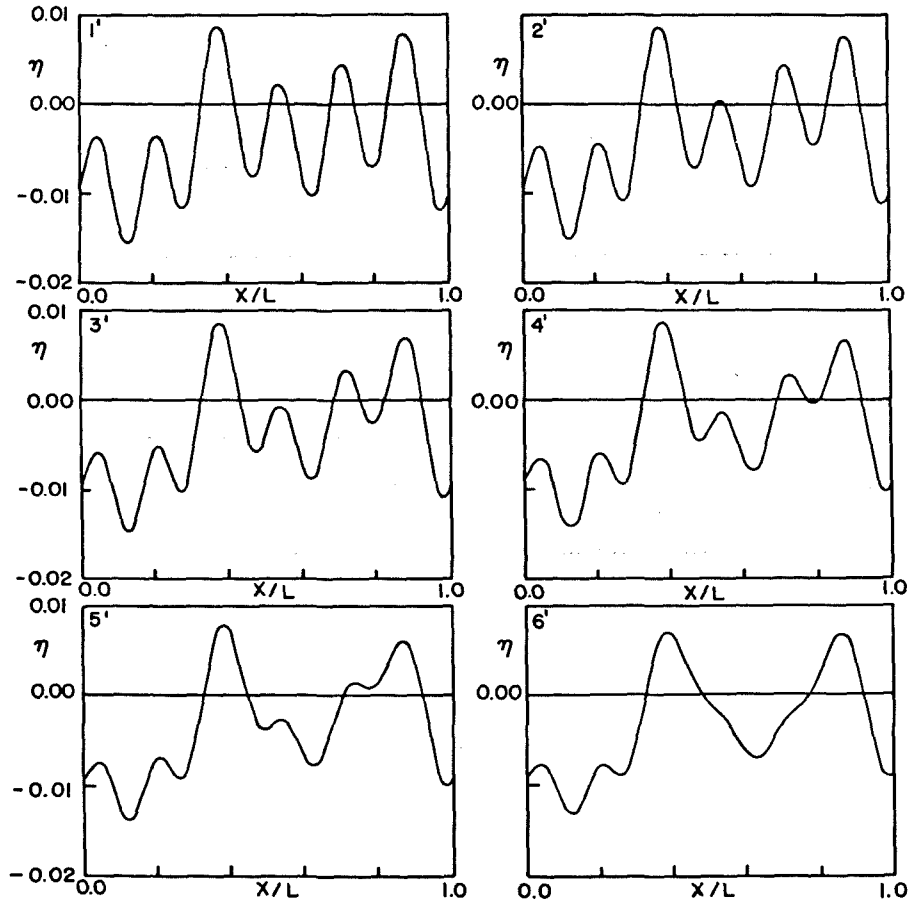


Figure 3.6.b.- Plots of the shape of the wave at different points along the  $P - 6$  branch.

the branch starts at  $P_{sb}$  and finishes at  $P'_{sb}$ . So the nonsymmetric branch connects the two symmetry breaking bifurcations.

In figure 3.6-(c), we show nonsymmetric waves at several points along the branch. We found that, in the present case of gravity-capillary waves, nonsymmetric waves of less than six crests are possible. In Chapter 1 we found that for gravity waves six is the minimum number of crests that is needed to have nonsymmetric waves. When capillarity is introduced, crests can appear and disappear along the branches as the capillarity is changed. In the present case, we can even have nonsymmetric waves with only two crests.



**Figure 3.6.c.-** Plots of the shape of the wave along the non-symmetric  $P - 6A$  branch.

Keeping  $\tau_1$  constant we tried to continue in the period with the idea of determining whether solutions with very large period were possible. We found that the period cannot be increased arbitrarily because the branch ends on a symmetry breaking bifurcation similar to  $P_{3,6}$ . This suggests that nonsymmetric solitary waves are not possible in region  $I$ , and nonsymmetric solutions exist only in a bounded domain on the  $(\tau_1, L)$ -plane.

### 3.6 Solitary wave solutions

The problem of solitary waves travelling on finite depth water with surface tension was first considered by Korteweg & De Vries (1895). They showed that solitary waves exist for sufficiently large values of the surface tension. The waves that they obtained are actually depression waves. Recently, these results have been rederived by Vanden-Broeck & Shen (1983) and Benjamin (1982), using a systematic perturbation expansion. These results left open the question of what happens to the solitary wave as the capillarity is decreased. Hunter & Vanden-Broeck (1982) performed computations for the solitary wave with surface tension using the full water wave equations. They found that as  $\tau$  tends to  $1/3$  from above the solitary wave develops a large number of inflexion points, and the computations become very difficult. They were not able to compute solutions for  $\tau < .21$  with  $c = 0.954$  because too many points were required for convergence.

Another apparent result that follows from the studies described above is that for a given value of the capillarity parameter and phase speed or amplitude, there exists a unique solitary wave.

In the present section we use our model to give evidence that the solitary wave is not unique, and different solutions are possible for the same values of the parameters close to the resonance point, i.e.,  $\tau \approx 1/3$ . We also analyze the influence of the capillarity on the solitary wave and the possibility of nonsymmetric solutions.

Consider the *FP2* point. As already seen in the previous section, for all values of  $\tau_1$ , there is a one-parameter family of periodic orbits coming out from the fixed point. We found that for wave lengths  $L$  such that  $L < L_c$ , as capillarity is changed keeping the period fixed, the family connects the two fixed points. Let us now

analyze what happens when larger wave lengths are considered.

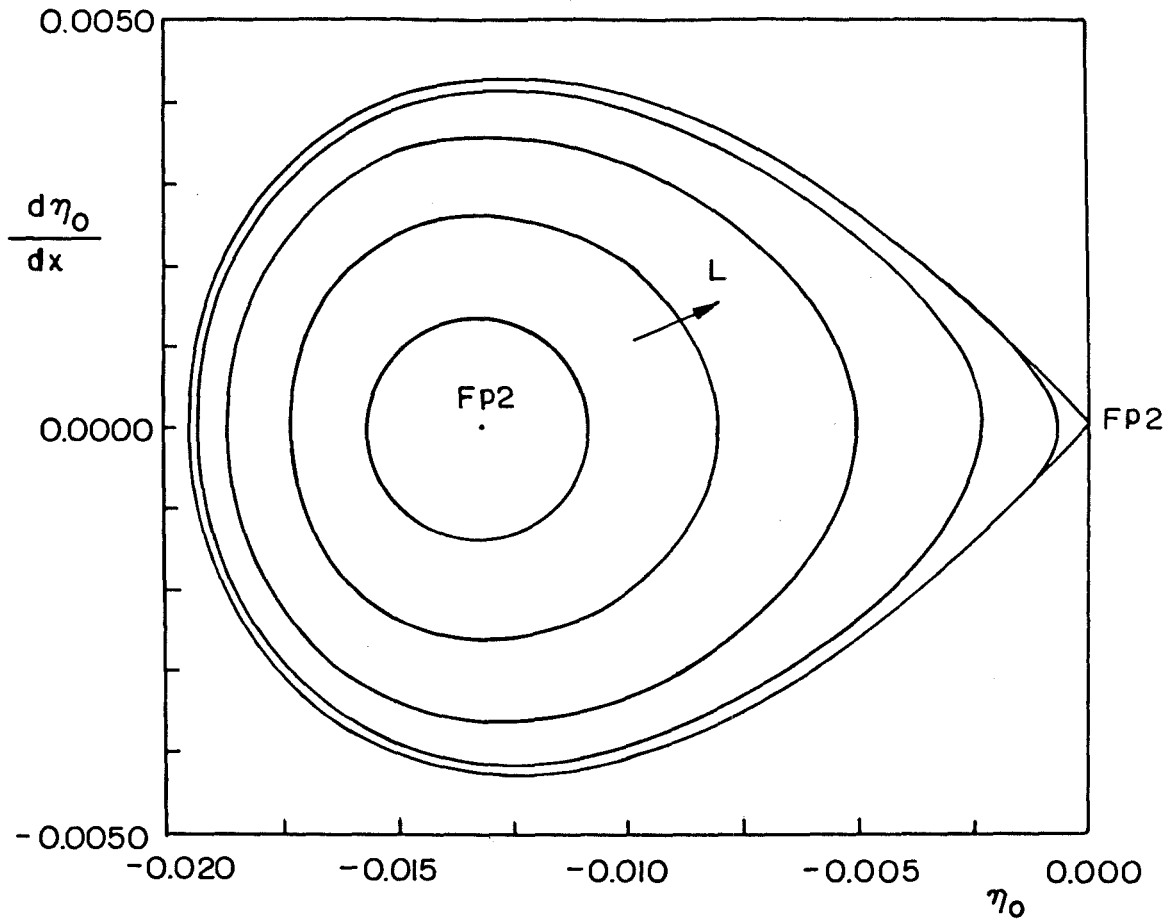
We fixed the value of the capillarity parameter at  $\tau_1 = 0.05$  and the phase speed at  $c_2 = -0.01$ , which corresponds to a point in region *III*. Starting at *FP2* (point *B''* figure 3.3), we continued along the one-parameter family in the period, i.e., following a vertical path in figure 3.3 . We found that the family ends up in an orbit that is homoclinic to the origin *FP1*. This homoclinic orbit corresponds to the depression solitary wave observed by the other authors. In our computations we were able to increase the period of the orbits up to values of order  $10^6$ . We found that orbit already converged to the homoclinic one for periods of order  $10^2$ . Hence, we considered this as good evidence that the orbit being computed corresponds to the homoclinic one.

Several orbits of the family are shown in figure 3.7 . We can see how the homoclinic connection appears as the limiting wave for the family as the period is increased.

In order to find out the effect of the capillarity on this family, we took an orbit of the family, that corresponded to a wave length  $L = 17.7$  (point *G* figures 3.3 and 3.8), and we continued it in the capillarity parameters  $\tau_1$ , keeping the period constant. In figure 3.8 we plot the norm of the solution versus  $\tau_1$ . The norm that has been used is defined as

$$\|u\|^2 = \frac{1}{L} \int_0^L (u_1^2 + u_2^2 + u_3^2 + u_4^2) dx. \quad (3.28)$$

In the direction of increasing capillarity we found, as we already knew, that the family ends in the fixed point *FP2* (point *B'''* figures 3.3 and 3.8). Decreasing  $\tau_1$ , we first found a fold, a minimum in  $\tau_1$  (point *E* figures 3.3 and 3.8), and after the branch finishes in a half period bifurcation (point *F* figures 3.3 and 3.8). Hence for



**Figure 3.7.-** Family of periodic orbits coming from  $FP2$  for  $\tau_1 = 0.05$  and  $c_2 = -0.01$  with the period as parameter.

this wave length, for a given value of  $\tau_1$  and  $c_1$ , the solution is not unique. This suggests that the solitary wave might not be unique (see figure 3.8). Therefore we fixed the value of  $\tau_1$ , and we continued in the period for the two different solutions (points a and b in figure 3.8). We found that both solutions converged to solitary waves, and they were different. Hence depression solitary waves are not unique. In figure 3.9 we show the two solitary waves that we obtained. Computations were again carried out up to periods of order  $10^6$ , finding again convergence for periods of order  $10^2$ .

Notice that the two homoclinic connections lie in region *II*, where the origin

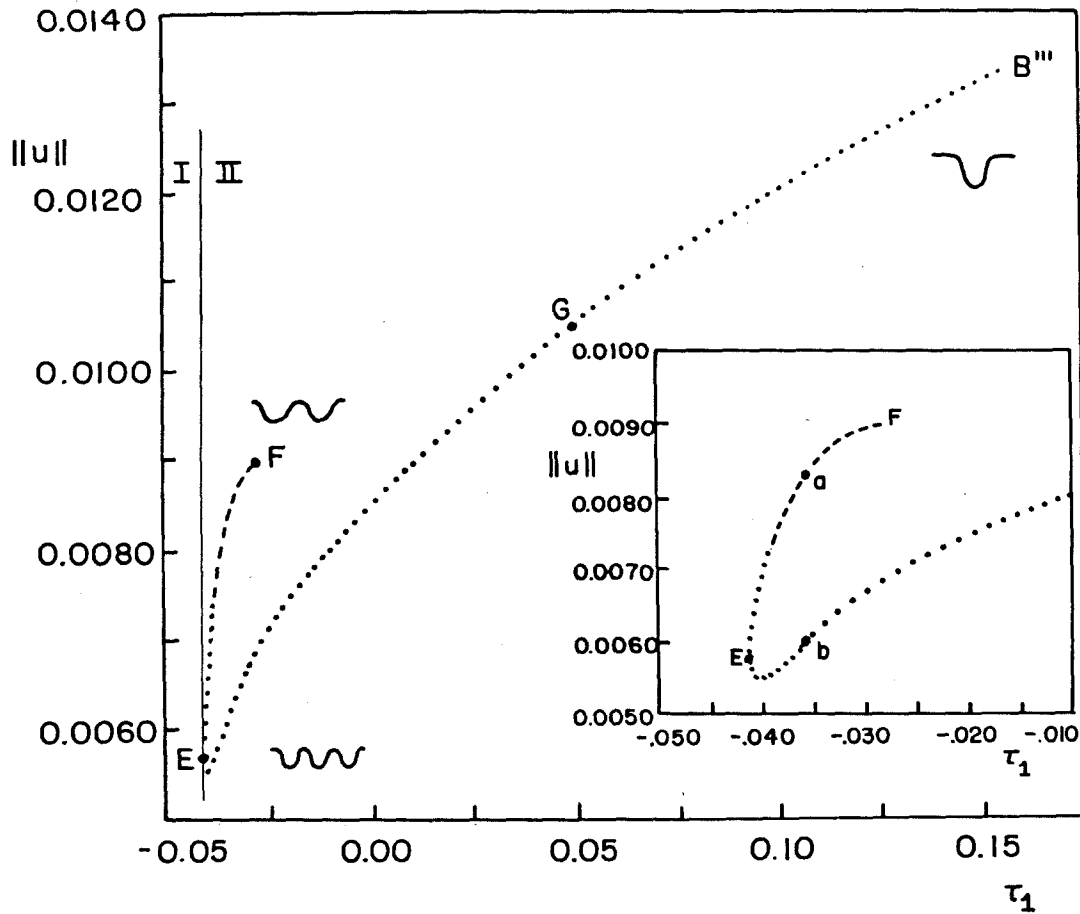


Figure 3.8.- Norm of the solutions versus the parameter  $\tau_1$  for the branch of period  $L = 17.7$  coming from  $FP2$ . Insert shows detail close to de folding point.

$FP1$  has four complex eigenvalues. Therefore we find oscillations at the beginning of the tails of the solitary waves. The number of crests is finite because the real part of the eigenvalues kills the crests and causes the exponentially decaying behavior. The number of crests increases as the capillarity decreases.

Notice also that the minimum of  $\tau_1$  along the branch is located exactly at the separation point between region  $I$  and  $II$ , this being true for whole range of values of  $c_2$  and  $L$ . At this minimum, the real part of the eigenvalues of the origin is zero and the solitary wave has an infinite number of crests. These oscillations explain Hunter

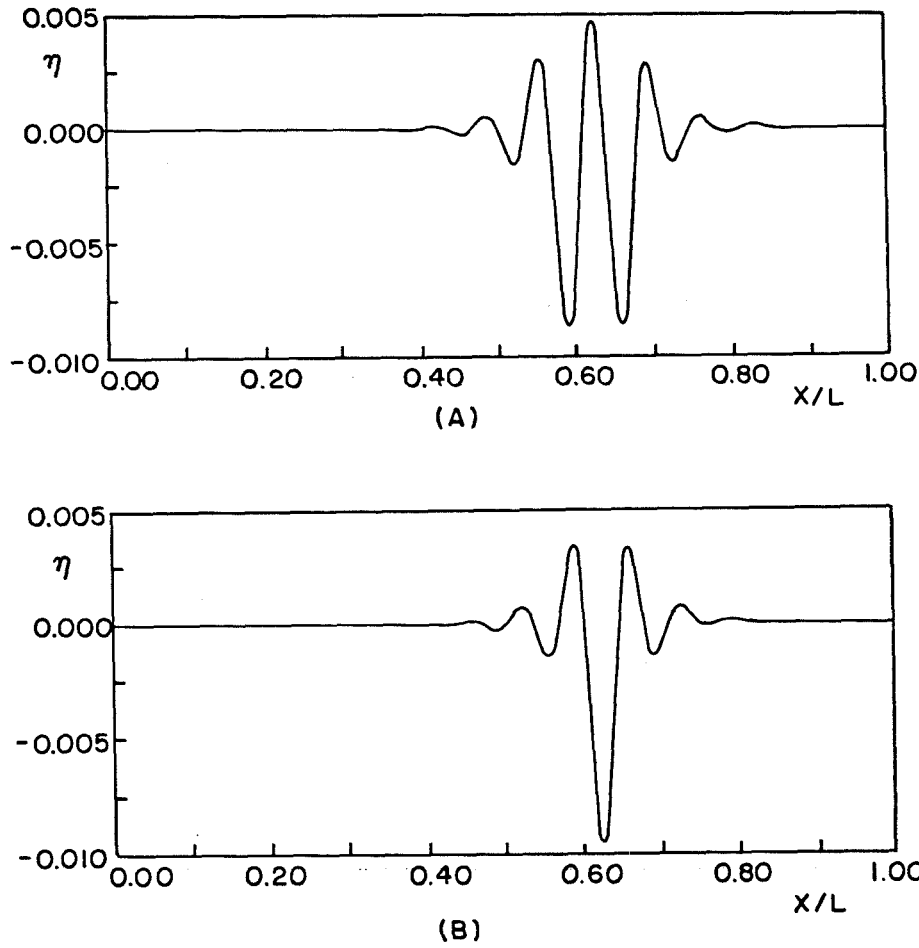


Figure 3.9.- Approximate solitary waves corresponding to points a and b in figure 3.8 ( $L = 100$ ).

& Vanden-Broeck (1982) numerical results, and their problems of convergence as capillarity was decreased below  $\tau = 1/3$ .

Going back to the period  $L = 17.7$  branch and looking at the Floquet multipliers, we found that at  $FP2$  the multipliers are real and positive. As  $\tau_1$  is decreased we find a point where the four multipliers are at  $+1$ , and after they become real and positive again. This is the same behavior as we found on the branch  $P - 3$  of the previous section. Examining the solution at this point, we found that the wave has three equal crests per period. Thus it corresponds to a period tripling on a branch of period  $L/3 = 5.9$ . The period  $L/3$  branch is a short-wave branch. Recall that in

our computations of section 3.5 we found that on a short wave branch there is only one period tripling bifurcation, and this bifurcation gives a wave corresponding to a long-wave branch coming out from  $FP1$ . But those computations were performed using a period of  $L_s = 4.138$ . When the period is larger, on the short wave branch another region where the Floquet multipliers are on the unit circle appears, and therefore there are two period tripling bifurcations. The new one is the one that we have found in the branch coming out from  $FP2$ .

Going along the branch beyond the period tripling point, we found that the multipliers went onto the unit circle. Again we found the same behavior as on the branch  $P-3$ . This suggests the possibility of existence of nonsymmetric solitary waves. On this branch where the multipliers leave the unit circle through  $-1$ , we have a period doubling. We took the new branch that corresponds to a wave of period  $2L = 35.5$ . Computing along this new branch with the period constant, we again found the same behavior as along the branch  $P-6$  and correspondingly a symmetry breaking bifurcation. The bifurcation diagram is shown in figure 3.10 . We took the new branch and computed some points on it. For one of these points, we fixed the capillarity and continued in the period, and we found that the orbit converges to a nonsymmetric orbit homoclinic to the origin. This orbit corresponds to a nonsymmetric solitary wave. In figure 3.11 we show this approximation to a solitary wave for a period of 100.

The nonsymmetric branch was found to finish on another symmetry breaking bifurcation, as it happened in the case studied in the previous section.



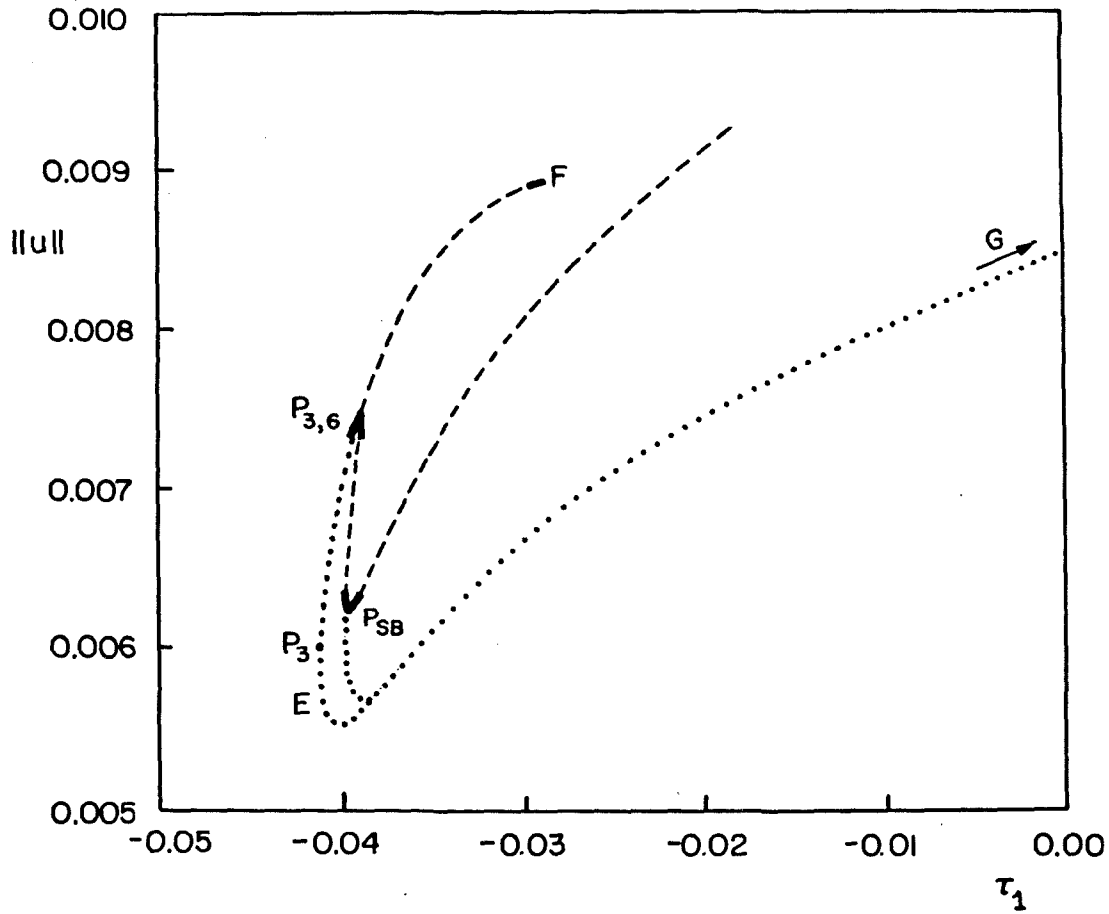
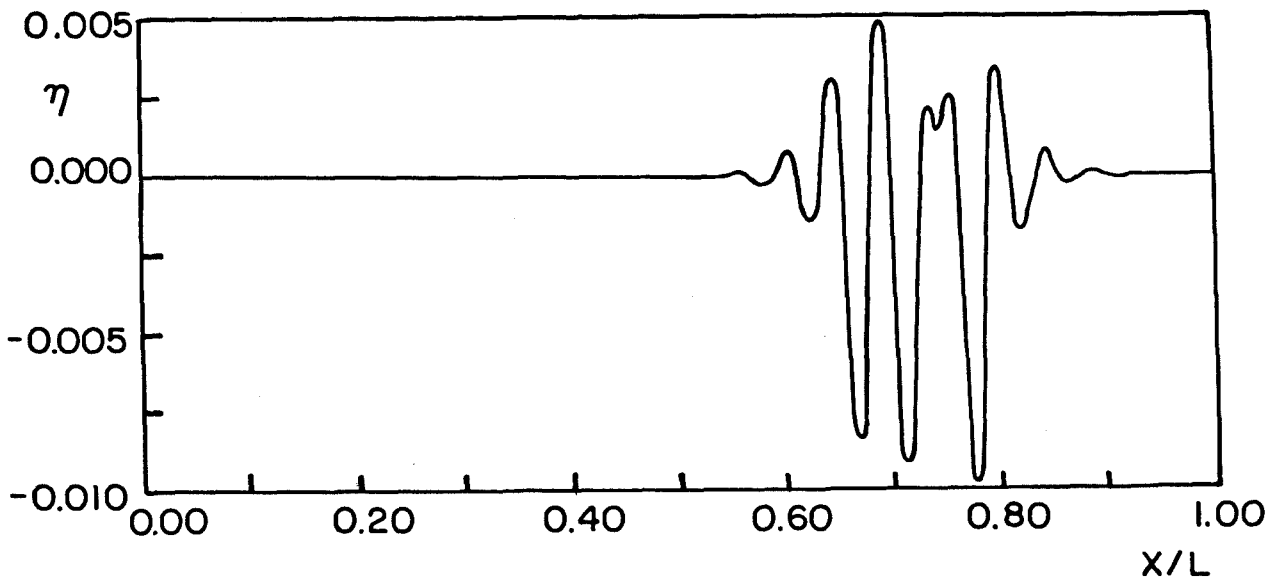


Figure 3.10.- Bifurcation diagram for the branch coming from  $FP2(B''')$  in the neighborhood of the fold.

### 3.7 Conclusions

In the present work, we have developed a weakly nonlinear model to describe low amplitude and small dispersion gravity-capillary waves on water of finite depth. The model has been obtained from the Hamiltonian formulation of the water wave problem. We have found that the analysis of permanent form travelling waves using this model reduces to the study of the properties of a discrete Hamiltonian system



**Figure 3.11.-** Approximate nonsymmetric solitary wave for  $L = 100$ ,  $\tau_1 = -0.039$  and  $c_2 = -0.01$ .

of two degrees of freedom.

Using the dynamical systems theory for Hamiltonian systems, we have studied the bifurcation structure of our model. We have been able to reproduce results about the nonuniqueness of solutions in addition to showing that nonsymmetric Wilton's ripples are possible. They appear, as happened in the case of gravity waves, via a spontaneous symmetry breaking bifurcation. We have found again the same bifurcation tree as the one found in Chapter 1 for finite depth gravity waves, in Chapter 2 for infinite depth gravity waves and Green *et al.* (1981) for fixed point of area-preserving maps in the presence of symmetries. This gives one more example of the universality of this structure in Hamiltonian systems. We have also seen that the symmetric solutions that were known previously are just a subset of the solutions coming from the complex bifurcation structure of Hamiltonian systems.

Numerical evidence is presented showing that solitary waves with sufficiently

large surface tension exist. For  $\tau > 1/3$  the solitary wave is a depression wave. When the surface tension parameter  $\tau$  is decreased below  $1/3$ , inflexion points appear in the profile of the wave. The tails of the solitary wave start having oscillations, the number of crests being finite. It seems from our results that the solitary wave does not exist in region  $I$ , i.e.,  $\tau < 1/3 - \sqrt{-8(c-1)/45}$ . For this value of  $\tau$ , we found a fold in the branch of solutions. This means that the solitary wave is not unique. Actually, we have computed two different solitary waves for the same value of the parameters. Both solitary waves travel with the same speed but have a different crest structure. We have also seen that this is just the beginning of a more complicated structure of solutions. In particular, we have found that a symmetry breaking bifurcation is possible for the solitary wave solution. We have computed solitary waves starting from periodic nonsymmetric solutions, by increasing the period up to very large values ( $L \simeq 10^6$ ). We have found that the solution converges very fast to a nonsymmetric solitary wave.

Once again we have found that the bifurcation properties of water waves are related to the Hamiltonian structure of the problem and its symmetries, and that the Hamiltonian formulation seems to be a powerful tool to analyze this kind of problem.

The stability of all these solutions is an interesting open problem. We leave this question as subject of a future study.

Let us consider the possibility of an experimental check of all these solutions. All these phenomena appear close to the resonance point  $\tau = 1/3$ , which corresponds to a depth

$$h = \left(\frac{3T}{\rho g}\right)^{1/2}. \quad (3.29)$$

The typical phase speed  $c$  of the waves for this depth is given by

$$c = \sqrt{gh} = \left(\frac{3T}{\rho}\right)^{1/4} g^{1/4}. \quad (3.30)$$

From Bernoulli's equation it follows that for waves of amplitude  $a$  and wavelength  $\lambda$ , the characteristic velocity  $v_c$  in the flow is

$$v_c = \frac{a}{h} \sqrt{gh}. \quad (3.31)$$

Given the above definitions, the Reynolds number of the flow can be defined as follows

$$Re = \frac{\text{Inertial forces}}{\text{Viscous forces}} = \frac{v_c^2/\lambda}{\nu v_c/h^2} = \frac{h a}{\nu \lambda} \sqrt{gh}, \quad (3.32)$$

where  $\nu$  is the kinematic viscosity. Using equation (3.29) the Reynolds number  $Re$  can be rewritten as

$$Re = \frac{a}{\lambda \nu} \left(\frac{3T}{\rho}\right)^{3/4} g^{-1/4}. \quad (3.33)$$

Assuming  $\alpha \approx \beta^2$  we find that the characteristic frequency in the flow, which is important in order to generate the waves, can be written as

$$\omega = ck = \frac{2\pi}{h} \beta^{1/2} \sqrt{gh} = 2\pi \left(\frac{a}{\lambda}\right)^{1/5} \left(\frac{3T}{\rho}\right)^{-1/4} g^{3/4}. \quad (3.34)$$

Another parameter which gives a good idea of the validity of the irrotational hypothesis and the possibility of performing experiments is the damping coefficient  $\gamma$  (Landau & Lifshitz, 1978, p.100). This coefficient measures the exponential dissipation of the mechanical energy of the wave with time ( $E \propto e^{-2\gamma t}$ ). For the present case this coefficient is

$$\gamma = 2\nu k^2 = 8\pi^2 \frac{\nu}{h^2} \left(\frac{a}{\lambda}\right)^{2/5} = 8\pi^2 \nu \left(\frac{\rho g}{3T}\right) \left(\frac{a}{\lambda}\right)^{2/5}. \quad (35)$$

In table 7 we show the values of these parameters for four different fluids taking  $a/\lambda = 10^{-2}$  and two values of the acceleration of the gravity. For an experiment using water on the surface of the earth we have that  $h = 4.76mm$  and  $c = 21.6cm/sec$ . The Reynolds number is  $Re = 10$ . Evidently, the Reynolds number is too low to neglect viscosity in the model and assume potential flow. As was experimentally checked by Benjamin (1982), the waves are damped too rapidly for an unequivocal investigation of results of an inviscid theory. We can also see that for liquid helium III the situation is a little bit better, but still not satisfactory.

The possibility of performing experiments in a weightless environment in space has changed completely the problem. In a spacecraft the gravity can be reduced several orders of magnitude, in which case the potential flow hypothesis is better. For example if  $g = 10^{-2}m/sec^2$ , we find that the typical depth is  $h = 14.9cm$  and the phase speed of the waves  $c = 3.86cm/sec$ . For this case the Reynolds number is  $Re = 58$ . As the gravity is decreased, the Reynolds number increases, and the model is better. The typical length of the problem also increases as the gravity decreases. We find that for liquid helium III the Reynolds number is  $Re = 186$  and the typical depth  $h = 2.75cm$ . It seems from the results shown in table 7 that liquid helium is the best candidate for an experiment among the fluids that have been considered. It seems to be possible to find a compromise solution in the scales, that could make an experiment in space possible.

## References

- [1] Abraham, R. and Marsden, J.E. (1978). "Foundations of Mechanics." *2nd ed. Benjamin /Cummings, Massachusetts.*
- [2] Amick C.J. & Kirchgässner K. (1986). "On solitary water-waves in the presence of surface tension." (to appear).
- [3] Benjamin, T.B. (1982). "The solitary wave with surface tension." *Quart. Appl. Math.* **40**, 231-234.
- [4] Benjamin, T.B. (1984). "Impulse, force and variational principles." *IMA J. Appl. Math.* **32**, 3-68.
- [5] Broer, L.J.F. (1974). "On the Hamiltonian theory of surface waves." *Appl. Sci. Res.* **29**, 430-446.
- [6] Chen, B. and Saffman, P.G. (1979). "Steady gravity-capillary waves on deep water.I.- Weakly nonlinear waves." *Studies in Appl. Math.* **60**, 183-310.
- [7] Chen, B. and Saffman, P.G. (1980). "Steady gravity-capillary waves on deep water.II.-Numerical results for finite amplitude." *Studies in Appl. Math.* **62**, 95-111.
- [8] Doedel, E.J. and Kernevez, J.P. (1986). "Software for continuation problems in ordinary differential equations with applications." (to appear)

- [9] Goldstein, H. (1980). "Classical Mechanics." *2nd ed. Addison Wesley, Reading, MA.*
- [10] Green, J.M., MacKay, R.S., Vivaldi, F. and Feigenbaum, M.J. (1981). "Universal behavior in families of area-preserving maps." *Physica 3D*, 468-486.
- [11] Hartman, P. (1964). "Ordinary Differential Equations." *Wiley-Interscience, New York.*
- [12] Hunter, D.H. and Vanden-Broeck, J.M. (1983). "Solitary and periodic gravity-capillary waves of finite amplitude." *J. Fluid Mech.* **134**, 205-219.
- [13] Jones, M.C.W. (1986). "The bifurcation of Wilton ripples and other capillary-gravity waves in a channel of finite depth." (to appear).
- [14] Jones, M.C.W. & Toland, J.F. (1986). "Symmetry and bifurcation of capillary-gravity waves." *Arch. Rat. Mech.* (to appear).
- [15] Keller, W.E. (1969). "Helium-III and Helium-IV." *Plenum Press, New York.*
- [16] Korteweg, D.J. and De Vries, G. (1895). "On the change of form of long waves advancing in a rectangular canal and a new type of long stationary waves." *Phil. Mag. (5)* **39**, 422-443.
- [17] Landau, L.D. and Lifshitz, E.M. (1978). "Fluid Mechanics." *Pergamon Press, London.*
- [18] Miles, J.W. (1977). "On Hamilton's principle for surface waves." *J. Fluid Mech.* **83**, 153-158.
- [19] Nayfeh, A. H. (1970a). "Finite amplitude surface waves in a liquid layer." *J. Fluid Mech.* **40**, 671-684.

- [20] Nayfeh, A. H. (1970b). "Triple and quintuple-dimpled wave profiles in deep water." *Phys. Fluids* **13**, 545-550.
- [21] Pierson, W.J. and Fife, P. (1961). "Some nonlinear properties of long-crested periodic waves with lengths near 2,44 centimetres." *J. Geophys. Res.* **66**, 163.
- [22] Rimmer, R. (1978). "Symmetry and bifurcation of fixed points in area preserving maps." *J. Diff. Equations* **29**, 329-344.
- [23] Sekerzh-Zenkovich, Y.I. (1956). "On the theory of stationary capillary waves of finite amplitude on the surface of heavy fluid." *Doklady Akad. Nauk. SSSR* **109**, 913.
- [24] Schooley, A.H. (1960). "Double, triple and higher-order dimples in the profiles of wind-generated water waves in the in the capillary-gravity transition region." *J. Geophys. Res.* **65**, 4075-4079.
- [25] Toland, J.F. & Jones, M.C.W. (1985). "The bifurcation and secondary bifurcation of gravity-capillary waves." *Proc. Roy. Soc. Lond. A* **399**, 391-417.
- [26] Vanden-Broeck, J.M. and Shen, M.C. (1983). "A note on the solitary and cnoidal waves with surface tension." *J. of Appl. Math. and Phys. (ZAMP)* **34**, 112-117.
- [27] Whitham, G.B. (1974). "Linear and Nonlinear Waves." *Wiley-Interscience, New York*.
- [28] Wilton, J.R. (1915). "On ripples." *Phil. Mag.* **29**, 688.
- [29] Zakharov, V.E. (1968). "Stability of periodic waves of finite amplitude on the surface of deep fluid." *J. App. Mech. Tech. Phys.* **2**, 190-194.



Table 7.

	Water	Mercury	Helium III	Helium IV
$T/\rho(\text{cm}^3/\text{sec}^2)$	74.0	36.8	2.5	1.9
$\nu(\text{cm}^2/\text{sec})$	$10^{-2}$	$10^{-2}$	$2.46 \cdot 10^{-4}$	$2.31 \cdot 10^{-4}$
$g = 982 \text{cm}/\text{sec}^2$				
$h(\text{cm})$	0.475	0.335	0.088	0.077
$c(\text{cm}/\text{sec})$	21.61	18.14	9.29	8.70
$Re$	10	6	33	29
$\omega(\text{rad}/\text{sec})$	113.7	135.4	264.3	282.4
$\gamma(\text{sec}^{-1})$	0.55	1.11	0.40	0.49
$g = 1 \text{cm}/\text{sec}^2$				
$h(\text{cm})$	14.9	10.51	2.76	2.42
$c(\text{cm}/\text{sec})$	3.86	3.24	1.66	1.55
$Re$	58	34	186	163
$\omega(\text{rad}/\text{sec})$	0.648	0.772	1.506	1.610
$\gamma(\text{sec}^{-1})$	$0.563 \cdot 10^{-4}$	$1.134 \cdot 10^{-4}$	$0.405 \cdot 10^{-4}$	$0.495 \cdot 10^{-4}$

Values of some parameters for different fluids and different values of the acceleration of the gravity close to the critical point  $\tau = 1/3$ . The values of  $T$  and  $\nu$  for the helium were taken from Keller (1969).

**PART II :**

**The superharmonic instability  
of surface water waves**

## CHAPTER 1

### The superharmonic instability of finite amplitude surface waves on water of finite depth

#### 1.1 Introduction

During the last seven years, great progress has been achieved in understanding the stability of irrotational water waves of permanent form to infinitesimal disturbances. The problem was first studied by Longuet-Higgins (1978 a,b), who considered the stability of two-dimensional periodic gravity waves to subharmonic and superharmonic two-dimensional disturbances. In the superharmonic case he suggested that a change of stability occurs as the height is increased when the phase speed  $c$  becomes a maximum.

Since  $c$  maximum is a limit point solution of the family of permanent waves, Longuet-Higgins's suggestion agreed with the fact that Chen & Saffman (1980) were not able to find superharmonic bifurcation for gravity waves, as implied by Garabedian's (1965) uniqueness proof.

Using a different numerical method, Tanaka (1983) again computed the superharmonic stability and found that there is an exchange of stability for the wave whose energy is a maximum. As this result appeared to conflict with the previous results and the theoretical understanding, Tanaka(1985 a) and Longuet-Higgins (private communication) repeated the calculations. The new results confirmed that an exchange of stability to superharmonic disturbances occurs when the energy is stationary. Tanaka also computed the eigenvector associated with the critical eigenvalue. He found that this eigenvector is identical to the eigenvector corresponding to horizontal displacement, allowed by Galilean invariance, so that there is no conflict with the Garabedian and Chen & Saffman conclusion of no superharmonic bifurcation.

Tanaka's and Longuet-Higgins' results were obtained by large scale computations. Because of the slow convergence for steep waves, the calculations used harmonic expansions of up to 600 modes for the shape of the wave. Recently, Saffman (1985) proved analytically that an exchange of stability occurs for superharmonic disturbances for every wave whose total energy is stationary. Longuet-Higgins & Fox (1978) showed that there are infinitely many such waves as the wave of greatest height is approached. The argument used the Hamiltonian formulation for water waves (Zakharov 1968). Another demonstration of the usefulness of the Hamiltonian approach is the recent work by Mackay & Saffman (1985) determining under which conditions the crossing of the eigenvalues for the linearized problem about the permanent form wave leads to loss of stability.

Saffman's theory was done only for periodic two-dimensional deep water waves. Recently ,using an integrodifferential method, Tanaka (1985 b) has computed the stability of finite amplitude, two-dimensional solitary waves of permanent form on water of uniform depth to two-dimensional infinitesimal disturbances. He has found

that an exchange of stability occurs again at a stationary point of the energy.

These results lead to two open questions about Saffman's approach. The first one is whether the proof can be generalized to include solitary and periodic waves on finite depth. The second one is if the proof can be generalized to the finite depth case, then in which frame of reference should the energy be calculated. Notice that for the cases of deep water periodic waves and solitary waves, there is a special frame of reference that is the frame in which the fluid is at rest at infinity. This frame does not exist in the case of periodic waves on finite depth.

In the present paper we generalize Saffman's argument to include these two cases. We also find that the frame of reference, in which the motion has to be studied, is the frame in which the mean horizontal velocity is zero.

## 1.2 The Hamiltonian formulation and canonical variables

Consider two-dimensional irrotational water waves in an unbounded domain of constant depth. Let  $\varphi(x, y, t)$  and  $\eta(x, t)$  be the velocity potential and the surface shape respectively. The most general representation for a traveling wave of permanent form is (Whitham 1974)

$$\begin{aligned}\varphi(x, y, t) &= \beta x - \gamma t + \Phi(\theta, y), \\ \eta(x, t) &= N(\theta), \quad \theta = x - ct,\end{aligned}\tag{1.1}$$

where  $\beta$  is the mean horizontal velocity of the waves in the frame of reference in which the motion is studied, and  $c$  is the phase speed in that frame.  $\Phi$  and  $N$  are periodic functions of  $\theta$ , and  $\gamma$  is related to the mean height of the wave  $\bar{\eta}$ . Actually there are three constants that are related to each other. These constants are  $\gamma$ ,  $\bar{\eta}$  and the constant  $K$  of the Bernoulli's equation

$$\varphi_t + \frac{1}{2}(\nabla\varphi)^2 + g\eta = K. \quad (1.2)$$

Two of these constants can be chosen arbitrarily, and the third one is determined by the other two. We will take  $K = \gamma = 0$ . With this choice we have

$$\begin{aligned} \varphi(x, y, t) &= \beta x + \Phi(\theta, y), \\ \eta(x, t) &= N(\theta). \end{aligned} \quad (1.3)$$

Miles (1977), extending Zakharov's (1968) work, proved that irrotational waves on water of finite depth have the following Hamiltonian structure

$$\begin{aligned} \frac{\partial\eta}{\partial t} &= \frac{\delta\mathbf{H}}{\delta\psi}, \\ \frac{\partial\psi}{\partial t} &= -\frac{\delta\mathbf{H}}{\delta\eta}, \end{aligned} \quad (1.4)$$

where  $\delta$  represents a functional derivative ( see Goldstein 1950).  $\psi$  is the velocity potential at the surface  $\psi(x, t) = \varphi(x, \eta(x, t), t)$ .  $\mathbf{H}$  is the total energy of the waves in the considered frame, given by

$$\mathbf{H} = \int H(\eta, \psi) dx, \quad (1.5)$$

where the integral is over a period in the case of periodic waves and all the real axis in the case of solitary waves. The energy density is given by

$$H(\eta, \psi) = \int_{-d}^{\eta} \frac{1}{2}(\nabla\varphi)^2 dy + \frac{1}{2}g\eta^2, \quad (1.6)$$

where  $d$  is the distance from the origin to the bottom.

Consider now the following transformation

$$\begin{aligned}\psi(x, t) &= \beta x + f(\theta, t), \\ \eta(x, t) &= g(\theta, t).\end{aligned}\tag{1.7}$$

This transformation can be considered as a canonical transformation from the variables  $\eta$  and  $\psi$  to the variables  $g$  and  $f$ . This change is equivalent to writing the equations in a coordinate system moving with speed  $c$ . With this transformation equations (1.4) become

$$\begin{aligned}\frac{\partial g}{\partial t} &= \frac{\delta \mathbf{H}}{\delta f} + c \frac{\partial g}{\partial \theta}, \\ \frac{\partial f}{\partial t} &= -\frac{\delta \mathbf{H}}{\delta g} + c \frac{\partial f}{\partial \theta}.\end{aligned}\tag{1.8}$$

Permanent travelling waves are the equilibrium solutions of this system.

For stability considerations it is convenient to introduce the canonical transformation to the complex coefficients  $a(k, t)$  and  $a^*(k, t)$  proposed by Zakharov (1968).  $a(k, t)$  and  $a^*(k, t)$  are linear combinations of the Fourier transforms of  $g(\theta, t)$  and  $f(\theta, t)$ . The transformation is given by

$$\begin{aligned}g(\theta, t) &= \frac{1}{2\pi\sqrt{2}} \int_{-\infty}^{\infty} \frac{|k|^{\frac{1}{2}}}{\omega^{\frac{1}{2}}(k)} [a(k, t)e^{ik\theta} + a^*(k, t)e^{-ik\theta}] dk, \\ f(\theta, t) &= -\frac{i}{2\pi\sqrt{2}} \int_{-\infty}^{\infty} \frac{\omega^{\frac{1}{2}}(k)}{|k|^{\frac{1}{2}}} [a(k, t)e^{ik\theta} - a^*(k, t)e^{-ik\theta}] dk,\end{aligned}\tag{1.9}$$

where  $\omega(k)$  is the linear dispersion relation given by

$$\omega(k) = \sqrt{gk \tanh(kd)}.\tag{1.10}$$

With this transformation equations (1.8) become the single equation

$$\frac{\partial a(k, t)}{\partial t} = -i \frac{\delta \mathbf{H}}{\delta a^*} + ick a(k, t). \quad (1.11)$$

Now the energy is

$$\mathbf{H} = \int_{-\infty}^{\infty} H(\beta, a, a^*) dk. \quad (1.12)$$

In the case of periodic waves there is only a discrete set of modes, and  $a(k, t)$  has the following form

$$a(k, t) = \sum_{n=-\infty}^{\infty} a_n(t) \delta(k - nk_0), \quad (1.13)$$

where  $\frac{2\pi}{k_0}$  is the wavelength.

It is important to notice that now

$$\frac{\delta \mathbf{H}}{\delta a} = \frac{\partial H}{\partial a}. \quad (1.14)$$

As we will see below, this represents a very important simplification for studying stability.

In this formulation periodic travelling waves are the solutions of the following equation

$$-i \frac{\partial H}{\partial A^*(k)} + ick A(k) = 0. \quad (1.15)$$

For symmetric wave solutions the origin can always be chosen such that  $A(k) = A^*(k)$ . An important property of  $H$  is that  $H$  is invariant under the change  $a \rightarrow a^*$ .



It is useful to regard  $c$  as the independent variable and the properties of steady waves as function of  $c$ .

### 1.3 Stability analysis

Consider a time dependent perturbation of the steady wave of the form

$$\begin{aligned} a(k, t) &= A(k) + \epsilon b^+(k) e^{i\sigma t}, \\ a^*(k, t) &= A^*(k) + \epsilon b^-(k) e^{i\sigma t}. \end{aligned} \tag{1.16}$$

$\sigma$  and  $(b^+, b^-)$  will be the eigenvalue and eigenfunction respectively of the following eigenvalue problem

$$(L + \sigma I) \begin{pmatrix} b^+(k) \\ b^-(k) \end{pmatrix} \equiv \begin{pmatrix} M + \sigma & N \\ -N^* & -M + \sigma \end{pmatrix} \begin{pmatrix} b^+(k) \\ b^-(k) \end{pmatrix} = 0, \tag{1.17}$$

where  $M = \frac{\partial^2 H}{\partial A \partial A^*} - ck$  and  $N^* = \frac{\partial^2 H}{\partial A^2}$

Because of Galilean invariance  $\sigma = 0$  is always an eigenvalue of this problem. The corresponding eigenfunction is the shift, that in this case is

$$\vec{\alpha} = \begin{pmatrix} ikA(k) \\ -ikA^*(k) \end{pmatrix}. \tag{1.18}$$

Following Saffman's (1985) arguments, in order to have an exchange of stability the eigenvalue  $\sigma = 0$  has to have algebraic multiplicity at least four. Taking the derivative of equation (1.15) with respect to  $c$  we obtain

$$L \begin{pmatrix} \frac{dA}{dc} \\ \frac{dA^*}{dc} \end{pmatrix} = \vec{\alpha} - \begin{pmatrix} \frac{\partial}{\partial \beta} \left( \frac{\partial H}{\partial A^*} \right) \\ \frac{\partial}{\partial \beta} \left( \frac{\partial H}{\partial A} \right) \end{pmatrix} \frac{d\beta}{dc}. \tag{1.19}$$

Let us focus our attention on the last term of this equation.  $\beta$  can not be determined by the problem. The mean horizontal velocity is determined when we specify the frame of reference to which we refer the motion. As we change  $c$ , if the waves are referred to a frame of reference in which  $\beta$  changes, then  $d\beta/dc$  will not be zero. On the other hand, if all the waves are referred to to a frame in which  $\beta$  is the same for all  $c$ , then the additional term will disappear. A particular case is the frame in which  $\beta = 0$ ; this is the frame that we shall consider. Notice that the frame in which the motion is steady is not valid because the mean horizontal velocity relative to this frame changes as  $c$  changes. If  $d\beta/dc = 0$  then

$$\vec{\gamma} = \begin{pmatrix} \frac{dA}{dc} \\ \frac{dA^*}{dc} \end{pmatrix} \quad (1.20)$$

is a generalized eigenfunction of  $\sigma = 0$ , and  $\sigma = 0$  has therefore algebraic multiplicity of at least two.

To continue the proof, consider the following equation

$$L\vec{v} = \vec{\gamma}. \quad (1.21)$$

If this equation has a solution, then  $\sigma = 0$  is an eigenvalue with algebraic multiplicity of at least three. The necessary and sufficient condition for (1.21) to have a solution is for  $\vec{\gamma}$  to be orthogonal to the eigenfunction of the adjoint problem corresponding to  $\sigma = 0$  (Fredholm's alternative).

To define the adjoint operator we have to introduce an inner product. We will consider the following inner product

$$\langle \vec{p}, \vec{q} \rangle = \int_{-\infty}^{\infty} (p_1^* q_1 + p_2^* q_2) dk, \quad (1.22)$$

where  $\vec{p} = (p_1(k), p_2(k))$  and  $\vec{q} = (q_1(k), q_2(k))$ . Once we have the inner product the adjoint operator  $L^+$  is defined by

$$\langle \vec{p}, L\vec{q} \rangle = \langle L^+\vec{p}, \vec{q} \rangle. \quad (1.23)$$

With this inner product the adjoint operator  $L^+$  is

$$L^+ = \begin{pmatrix} M^* & -N \\ N^* & -M^* \end{pmatrix} = \begin{pmatrix} M & -N \\ N^* & -M \end{pmatrix}. \quad (1.24)$$

We can directly see that

$$\vec{\alpha}^+ = \begin{pmatrix} ikA(k) \\ ikA^*(k) \end{pmatrix} \quad (1.25)$$

is the eigenfunction of  $\sigma = 0$  for the adjoint operator. This result agrees with the general property of discrete Hamiltonian system that if  $(p, p^*)$  is a right eigenvector of the linearized flow in the neighborhood of an equilibrium point, then  $(p, -p^*)$  is a left eigenvector of the linearized flow (Mackay 1985). So the system (1.21) will have a solution if and only if

$$\langle \vec{\alpha}^+, \vec{\gamma} \rangle = -i \int_{-\infty}^{\infty} (A^*(k) \frac{dA}{dc} + A(k) \frac{dA^*}{dc}) k dk = 0. \quad (1.26)$$

It is easy to show that this condition is equivalent to the condition of having a maximum in the energy when the motion is studied in the frame of reference in which  $\beta \equiv 0$ , for taking the derivative with respect to  $c$  of equation (1.12) we have

$$\frac{dH}{dc} = \int_{-\infty}^{\infty} \left( \frac{\partial H}{\partial A} \frac{dA}{dc} + \frac{\partial H}{\partial A^*} \frac{dA^*}{dc} + \frac{\partial H}{\partial \beta} \frac{d\beta}{dc} \right) dk. \quad (1.27)$$

As  $A(k)$  is an equilibrium solution of (1.21) it follows that

$$\frac{d\mathbf{H}}{dc} = \int_{-\infty}^{\infty} (ckA^*(k) \frac{dA}{dc} + ckA(k) \frac{dA^*}{dc} + \frac{\partial H}{\partial \beta} \frac{d\beta}{dc}) dk. \quad (1.28)$$

If the frame of reference is such that  $\beta \equiv 0$ , stationarity of energy is equivalent to (1.26). In this case, the eigenvalue  $\sigma = 0$  will have algebraic multiplicity at least three.

To complete the proof we have to look at the following equation

$$L\vec{\chi} = \vec{\nu}. \quad (1.29)$$

If this equation has a solution,  $\sigma = 0$  will have algebraic multiplicity of at least four, and the proof will be finished.

From the symmetries of equation (1.21) it is always possible to choose the solution such that  $\vec{\nu} = (\nu_1, -\nu_1^*)$ . The Fredholm alternative in this case becomes

$$\int_{-\infty}^{\infty} (A^* \nu_1 - A \nu_1^*) k dk = 0. \quad (1.30)$$

If the problem is restricted to symmetric waves  $A(k) = A^*(k)$ , and equation (1.21) becomes

$$\begin{pmatrix} M & N \\ -N & -M \end{pmatrix} \begin{pmatrix} \nu_1 \\ -\nu_1^* \end{pmatrix} = \begin{pmatrix} \frac{dA}{dc} \\ \frac{dA}{dc} \end{pmatrix}. \quad (1.31)$$

From this equation it follows that  $\nu_1 = \nu_1^*$ , and condition (1.27) is satisfied. So, for symmetric waves, at a point in which the energy is stationary the eigenvalue

$\sigma = 0$  has algebraic multiplicity at least four. That point corresponds to an exchange of stability.

## References

- [1] Chen, B. and Saffman, P.G.(1980). "Numerical evidence for the existence of new types of gravity waves of permanent form on deep water." *Stud. App. Math.* **62**, 1-21.
- [2] Garabedian, P.R.(1965). "Surface waves of finite depth." *J. D'Analyse Math.* **14**, 161-169.
- [3] Goldstein, H.(1980). "Classical Mechanics." 2nd ed. Addison Wesley, Reading, MA.
- [4] Longuet-Higgins, M.S.(1978 a). "The instabilities of gravity waves of finite amplitude in deep water.I-Superharmonics." *Proc. Roy. Soc. A* **360**, 471-488.
- [5] Longuet-Higgins, M.S.(1978 b). "The instabilities of gravity waves of finite amplitude in deep water.II-Subharmonics." *Proc. Roy. Soc. A* **360**, 489-505.
- [6] Longuet-Higgins, M.S. and Fox, M.J.M.(1978). "Theory of the almost-highest wave.Part 2. Matching and analytic extension." *J. Fluid. Mech.* **85**, 769-786.
- [7] Mackay, R.S.(1985). "Stability of equilibria of Hamiltonian systems." *Proc. RSRE Seminar on Dynamical Systems*.(to appear).
- [8] Mackay, R.S. and Saffman P.G.(1985). "Stability of water waves." *Proc. Roy. Soc.* (submitted).
- [9] Miles, J.W.(1977). "On Hamilton's principle for surface waves." *J. Fluid Mech.* **83**, 153-158.
- [10] Saffman, P.G.(1985). "The superharmonic instability of finite amplitude water waves." *J. Fluid Mech.*(to appear).

- [11] Tanaka, M.(1983).“The stability of steep gravity waves.” *J. Phys. Soc. Japan* **52**, 3047-3055.
- [12] Tanaka, M.(1985 a).“The stability of steep gravity waves.” *J. Fluid Mech.* **156**, 281-289.
- [13] Tanaka, M.(1985 b).“The stability of solitary waves.” *Phys. Fluids*.(to appear).
- [14] Whitham, G.B.(1974).“Linear and Nonlinear waves.” *Wiley-Interscience, New York*.
- [15] Zakharov, V.E.(1968).“Stability of periodic waves of finite amplitude on the surface of deep fluid.” *J. App. Mech. Tech. Phys.* **2**, 190-194.

## CHAPTER 2

### An example of stability exchange in a Hamiltonian wave system

#### 2.1 Introduction

In a study of the stability of superharmonic disturbances on finite amplitude two dimensional deep water gravity waves of permanent form, calculations by Longuet-Higgins (1978) suggested that a change of stability occurs as the wave height is increased when the wave speed  $c$  becomes a maximum. This result agreed with the general belief that changes of stability occur at limit points. In the water wave case, an extremum value of  $c$  signifies a limit point because it implies the existence of two neighboring solutions moving with the same speed, whose difference is a null stationary eigenvector in the frame moving with the wave.

Tanaka's (1983) calculations showed, however, that this is not true. He found that a change of superharmonic stability happens first for the wave of maximum energy, which is a less steep wave with smaller wave speed. Tanaka's results, however, were questioned because of the conflict with the belief about stability changes at limit points, and also because they appear to imply superharmonic bifurcation into



a new family of waves with crests and troughs of the same height. These according to Garabedian's (1965) uniqueness proof and the Chen and Saffman's (1980) computations do not exist. With improved accuracy, Tanaka (1985) repeated the calculations and confirmed his results. He also explained why there is no non-trivial bifurcation at the wave of maximum energy since the eigenvector, corresponding to the zero eigenvalue, is linearly dependent on the eigenvector corresponding to the horizontal displacement (shift) allowed by Galilean invariance. It was also shown by Longuet-Higgins (1984) that the existence of a zero eigenvalue implied that either the wave speed is an extremum, or the associated eigenvector is the shift. All these results (except Garabedian's) depend on extensive computations for steep waves.

Using Zakharov's (1968) Hamiltonian formulation of water waves, Saffman (1985) proved analytically without any computations that the change of stability occurs at the point of maximum energy and is not associated with a non-trivial bifurcation. He also showed that a change of stability need not occur at the limit points. Saffman's arguments are general and can be applied to any Hamiltonian system which possesses the necessary symmetries.

In view of the subtle misconceptions which hindered the acceptance of Tanaka's results, it was felt desirable to construct a simple Hamiltonian model maintaining all the symmetries, which demonstrates explicitly the behavior of the eigenvalues and eigenvectors at points of maximum energy and speed.

## 2.2 Statement of the problem

In Zakharov's Hamiltonian formulation of water waves, the canonical variables are complex quantities  $a_n$  and  $a_n^*$  ( $-\infty < n < \infty$ ), which are linear combinations of the Fourier components of the wave shape  $\eta(x,t)$  and the velocity potential

$\varphi(x, \eta(x, t), t)$  evaluated at the surface. The total energy of the wave is  $E(a_n, a_n^*)$ .

The evolution equations in a coordinate system moving with speed  $c$  can be written as (Saffman 1985)

$$\frac{\partial \vec{a}}{\partial t} = -i \frac{\partial E}{\partial \vec{a}^*} + ic I_n \vec{a} \quad (2.1)$$

$$\frac{\partial \vec{a}^*}{\partial t} = i \frac{\partial E}{\partial \vec{a}} - ic I_n \vec{a}^*$$

where  $I_n$  is a diagonal matrix with  $n$  on the  $n$ -th row and  $n$ -th column. The equilibrium solutions for steady waves are given by

$$\frac{\partial E}{\partial \vec{A}^*} - c I_n \vec{A} = 0 \quad (2.2)$$

$$\frac{\partial E}{\partial \vec{A}} - c I_n \vec{A}^* = 0$$

where  $\vec{A}, \vec{A}^*$  are the equilibrium values. As the energy is real, the phases can be chosen so that  $\vec{A}$  and  $\vec{A}^*$  are real, ( $\vec{A} = \vec{A}^*$ ).

This Hamiltonian system has the following properties:

$E(\vec{a}, \vec{a}^*)$  is invariant under the change  $a_n \rightarrow a_n^*$ . (Reality)

$E(\vec{a}, \vec{a}^*)$  is invariant under the change  $a_n \rightarrow a_{-n}$ . (Time reversal)

If  $\vec{A} = \{A_n\}$  is a solution, then  $\vec{A}^\xi = \{A_n e^{in\xi}\}$  is also a solution. (Galilean invariance)

We have constructed a model that satisfies these properties. We take  $|n| \leq 2$ , and write  $\vec{a} = (a_2, a_1, a_{-1}, a_{-2})$ . The Hamiltonian has the form

$$H(\vec{a}, \vec{a}^*) = H_1(a_2, a_1, a_1^*, a_2^*) + H_1(a_{-2}, a_{-1}, a_{-1}^*, a_{-2}^*)$$

where

$$\begin{aligned}
 H_1(a_2, a_1, a_1^*, a_2^*) = & \frac{1}{277} \left[ \frac{1}{3} a_1 a_1^* (-509 + \frac{3557}{4} a_1 a_1^* - \frac{257}{3} (a_1 a_1^*)^2) \right. \\
 & - 922 a_2 a_2^* + (a_1^2 a_2^* + a_1^{*2} a_2) \\
 & \left. (269 - \frac{353}{4} a_1 a_1^* + \frac{21}{4} (a_1 a_1^*)^2) \right]
 \end{aligned} \tag{2.3}$$

The coefficients are chosen so that the values of the solutions at the critical points are small integers.

Equations (2.2) give the equilibrium solutions as a function of the wave speed  $c$ . This is appropriate for theoretical discussion, but awkward for algebraic manipulation when there is a limit point in  $c$ . In this case, it is convenient to take one of the canonical variables as the independent parameter. For the present example we use  $a_1$ .

Only solutions with  $A_{-1} = A_{-2} = A_{-1}^* = A_{-2}^* = 0$  have been considered. These correspond to propagating waves. This Hamiltonian (2.3) possesses equilibrium solutions with  $A_1 = A_2 = 1, c=2$ , for which the energy is a maximum, and with  $A_1=2, A_2=0, c=2$  which is a limit point. See figure 2.1 .

### 2.3 Stability

In order to study the stability of the equilibrium waves, we consider solutions of the form

$$\begin{aligned}
 \vec{a} &= \vec{A} + \epsilon \vec{b}^+ e^{i\sigma t} \\
 \vec{a}^* &= \vec{A} + \epsilon \vec{b}^- e^{i\sigma t}
 \end{aligned} \tag{2.4}$$

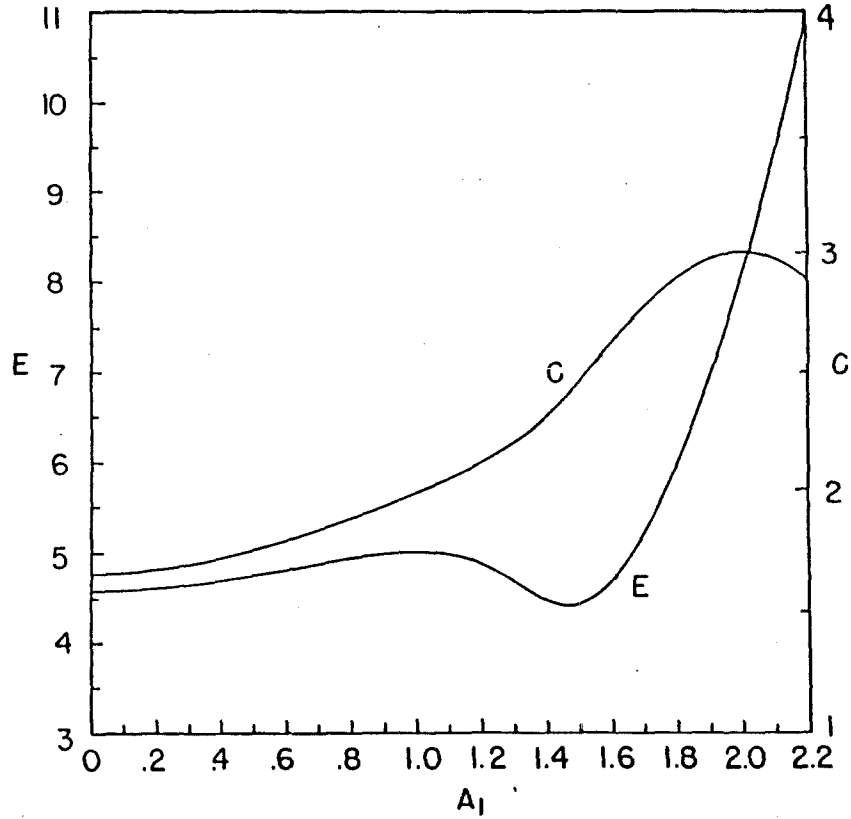


Figure 2.1.- Variation of the energy  $E$  and speed  $c$  with amplitude of fundamental mode for non-linear propagating waves by the Hamiltonian of equation (2.3).

Substituting these expressions into equations (2.1) we obtain the following eigenvalue problem for  $\sigma, \vec{b}^+, \vec{b}^-$

$$J(\sigma) \begin{pmatrix} \vec{b}^+ \\ \vec{b}^- \end{pmatrix} \equiv \begin{pmatrix} \sigma + M & N \\ -N & \sigma - M \end{pmatrix} \begin{pmatrix} \vec{b}^+ \\ \vec{b}^- \end{pmatrix} = 0. \quad (2.5)$$

where  $M = \frac{\partial^2 E}{\partial A \partial A^*} - c I_n$  and  $N^* = \frac{\partial^2 E}{\partial A \partial A}$ .

Because of the symmetry and Galilean invariance,  $\sigma=0$  is always an eigenvalue with algebraic multiplicity at least two and eigenvector  $\vec{\alpha} = (I_n \vec{A}, -I_n \vec{A})$ . This is called the shift, and corresponds to the trivial bifurcation due to Galilean invariance.

a) *Wave of Maximum Energy*

Saffman (1985) proved that for this wave  $\sigma=0$  is an eigenvalue with algebraic

multiplicity four, and geometric multiplicity one. The corresponding eigenvector is the shift, which at this point is

$$\vec{\alpha} = \left(1, \frac{1}{2}, 0, 0, -1, \frac{-1}{2}, 0, 0\right) \quad (2.6)$$

For a general wave  $\sigma=0$  is a double eigenvalue, with only one eigenvector. For the wave of maximum energy, another two eigenvalues becomes zero, giving a change of stability. As the geometric multiplicity does not change, there is no nontrivial bifurcation and there is no new family of waves.

We have solved exactly the eigenvalue problem for our Hamiltonian, and the results are given in table 1. We can see that the results confirm Saffman's proof.

*b) Wave of Maximum Speed.*

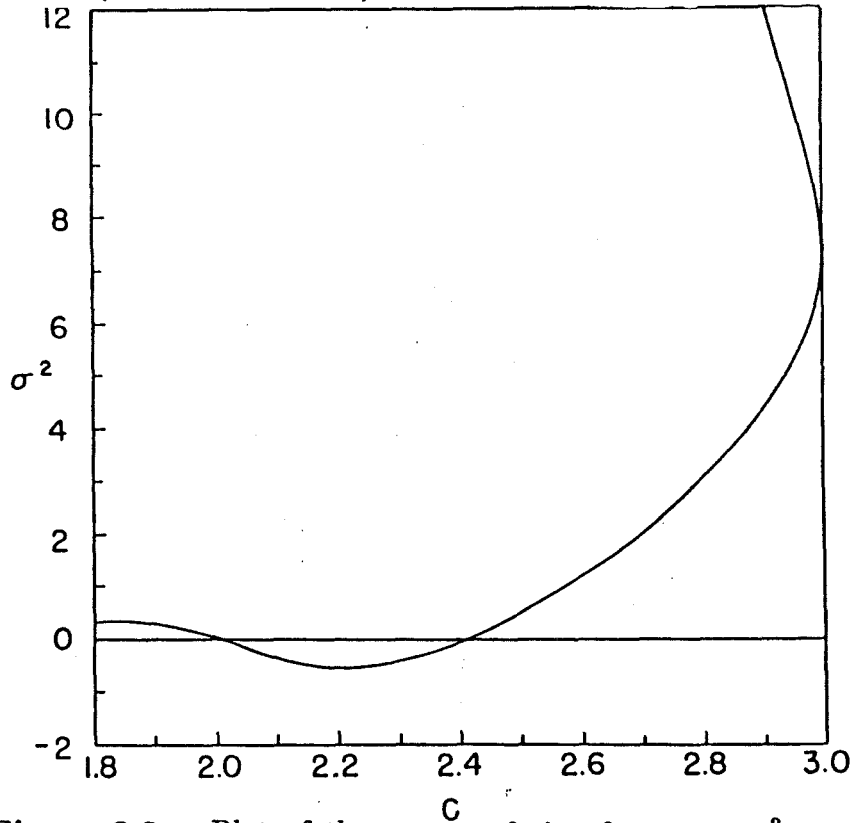
The results of eigenvalue calculation for this wave are shown in table 2. In this case,  $\sigma =0$  is a double eigenvalue, but it has two independent eigenvectors. This is the unique wave for which the algebraic and geometric multiplicities of the zero eigenvalue are equal. We can determine the reason for this result from examination of the equilibrium equation. If we take the derivative of equations (2.2) with respect to  $c$ , we have

$$\begin{pmatrix} M & N \\ -N & -M \end{pmatrix} \begin{pmatrix} \frac{d\vec{A}}{dc} \\ \frac{d\vec{A}}{dc} \end{pmatrix} = \begin{pmatrix} I_n \vec{A} \\ -I_n \vec{A} \end{pmatrix}. \quad (2.7)$$

From this relation it follows that away from the limit point  $(\frac{d\vec{A}}{dc}, \frac{d\vec{A}}{dc})$  is a generalized eigenvector of  $J(0)$ . At the limit point this vector becomes an eigenvector of  $J(0)$ . If we take, for example,  $\alpha_1$  as parameter we can rewrite equation (2.7) as

$$\begin{pmatrix} M & N \\ -N & -M \end{pmatrix} \begin{pmatrix} \frac{d\vec{A}}{da_1} \\ \frac{d\vec{A}}{da_1} \end{pmatrix} = \frac{dc}{da_1} \begin{pmatrix} I_n \vec{A} \\ -I_n \vec{A} \end{pmatrix}. \quad (2.8)$$

As we approach the limit point  $\frac{dc}{da_1} \rightarrow 0$ , so  $(\frac{d\vec{A}}{dc}, \frac{d\vec{A}}{dc})$  is a new eigenvector for  $\sigma=0$ . To check all this behavior, we have expanded the solution in powers of  $(A_1 - 2)$  around this point. Keeping the two first terms of the expansion we have solved equation (2.8). In the limit  $a_1 \rightarrow 2$ , it is found that a solution is  $(1, -1, 0, 0, 1, -1, 0, 0)$  which corresponds to the new eigenvector for  $\sigma=0$  seen in table 2. For this wave the shift eigenvector is  $(0, 1, 0, 0, 0, -1, 0, 0)$ .



**Figure 2.2.-** Plot of the square of eigenfrequency  $\sigma^2$  versus wavenumber  $c$  for the fundamental mode. Note exchange of stability at extrema of the energy.

Figure 2.2 shows a plot of  $\sigma^2$  versus  $c$  for the fundamental oscillation, i.e. disturbances for which the  $n = \pm 1$  components are dominant and which exchange stability at extrema of the energy

TABLE 1

$\sigma$	$\frac{-2030}{277}$	$\frac{2030}{277}$	$\frac{-1735}{544}$	$\frac{1735}{544}$	0
order of root	1	1	1	1	4
$\vec{a}_2$	0	0	0	0	1
$\vec{a}_1$	0	0	0	0	$\frac{1}{2}$
$\vec{a}_{-1}$	0	0	0	1	0
$\vec{a}_{-2}$	0	1	0	0	0
$\vec{a}_2^*$	0	0	0	0	-1
$\vec{a}_1^*$	0	0	0	0	$-\frac{1}{2}$
$\vec{a}_{-1}^*$	0	0	1	0	0
$\vec{a}_{-2}^*$	1	0	0	0	0

Eigenvalues and eigenvectors for the maximum energy wave.

TABLE 2

$\sigma$	$\frac{-2584}{277}$	$\frac{2584}{277}$	$\frac{-1292}{277}$	$\frac{1292}{277}$	$\frac{-740}{277}$	$\frac{740}{277}$	0
order of root	1	1	1	1	1	1	2
$a_2$	0	0	0	0	1	0	0 1
$a_1$	0	0	0	0	$\frac{1}{2}$	1	1 -1
$a_{-1}$	0	0	0	1	0	0	0 0
$a_{-2}$	0	1	0	0	0	0	0 0
$a_2^*$	0	0	0	0	0	-2	0 1
$a_1^*$	0	0	0	0	$-\frac{1}{2}$	-1	-1 1
$a_{-1}^*$	0	0	1	0	0	0	0 0
$a_{-2}^*$	1	0	0	0	0	0	0 0

Eigenvalues and eigenvectors for the maximum speed wave.

### References

- [1] Chen, B. and Saffman, P.G.(1980). "Numerical evidence for the existence of new type of gravity waves of permanent form in deep water." *Stud. App. Math.* **62**, 1-21.
- [2] Garabedian, P.R.(1965). "Surface waves of finite depth." *J. D'Analyse Math.* **14**, 161-169.
- [3] Longuet-Higgins, M.S.(1978). "The instabilities of gravity waves of finite amplitude in deep water.I-Superharmonics." *Proc. Roy. Soc. A* **360**, 471-488.
- [4] Longuet-Higgins, M.S.(1984). "On the stability of steep gravity waves." *Proc. Roy. Soc. A* **396**, 269-280.
- [5] Saffman, P.G.(1985). "The superharmonic instability of finite amplitude water waves." *J. Fluid Mech.*(to appear)
- [6] Tanaka, M.(1983). "The stability of steep water waves." *J. Phys. Cos. Japan* **52**, 3047-3055.
- [7] Tanaka, M.(1985). "The stability of steep gravity waves.II." *J. Fluid Mech.*(to appear)
- [8] Zakharov, V.E.(1968). "Stability of periodic waves of finite amplitude on the surface of deep fluid." *J. App. Mech. Tech. Phys.* **2**, 190-194.