SMOOTH SETS FOR BOREL EQUIVALENCE RELATIONS AND THE COVERING PROPERTY FOR σ-IDEALS OF COMPACT SETS

Thesis by

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Abstract

This thesis is divided into three chapters. In the first chapter we study the smooth sets with respect to a Borel equivalence realtion E on a Polish space X. The collection of smooth sets forms a σ-ideal. We think of smooth sets as analogs of countable sets and we show that an analog of the perfect set theorem for $\mathbf{\Sigma}^1_1$ sets holds in the context of smooth sets. We also show that the collection of $\mathbf{\Sigma}^1_1$ smooth sets is Π^1_1 on the codes. The analogs of thin sets are called sparse sets. We prove that there is a largest Π^1_1 sparse set and we give a characterization of it. We show that in L there is a Π^1_1 sparse set which is not smooth. These results are analogs of the results known for the ideal of countable sets, but it remains open to determine if large cardinal axioms imply that $\mathbf{\Pi}^1_1$ sparse sets are smooth. Some more specific results are proved for the case of a countable Borel equivalence relation. We also study $I(E)$, the σ-ideal of closed E-smooth sets. Among other things we prove that E is smooth iff $I(E)$ is Borel.

In chapter 2 we study σ-ideals of compact sets. We are interested in the relationship between some descriptive set theoretic properties like thinness, strong calibration and the covering property. We also study products of σ-ideals from the same point of view. In chapter 3 we show that if a σ-ideal I has the covering property (which is an abstract version of the perfect set theorem for $\mathbf{\Sigma}^1_1$ sets), then there is a largest Π^1_1 set in I^{int} (i.e., every closed subset of it is in I). For σ-ideals on 2^ω we present a characterization of this set in a similar way as for C_1, the largest thin Π^1_1 set. As a corollary we get that if there are only countable many reals in L, then the covering property holds for Σ^1_2 sets.

Notation

Throughout we will be working with methods of effective descriptive set theory and in the context of ZFC. Our notation is standard as in Moschovakis' book [15]. Any descriptive set theoretic notions or notation not defined in this thesis can be found in [15].

We will review briefly some basic facts. A Polish space is a complete separable metric space. When its metric is effective it is called recursively presented (see [15] 3.B). The Borel sets are those sets in the least σ-algebra containing the open sets. We will use the notation $\mathbf{\Sigma}^0_\eta$, $\mathbf{\Pi}^0_\eta$ for the Borel hierarchy as in [15] 1.B. For instance F_σ sets are denoted also by $\mathbf{\Sigma}^0_2$ and G_δ sets are denote by $\mathbf{\Pi}^0_2$. The analytic sets, denoted by $\mathbf{\Sigma}^1_1$, are the continuous images of Borel sets. Coanalytic sets, denoted by $\mathbf{\Pi}^1_1$, are the complements of $\mathbf{\Sigma}^1_1$ sets. $\mathbf{\Sigma}^1_1$ sets are very well behaved: they are universally measurable and have the property of Baire. The projective sets, denoted by $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$, are defined by induction on n by taking continuous images and complements. But we will not go beyond $\mathbf{\Sigma}^1_2$, i.e., continuous images of $\mathbf{\Pi}^1_1$ sets. $\mathbf{\Delta}^1_1$ is the collection of sets which are both $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$. Suslin's theorem says that the Borel sets are exactly the $\mathbf{\Delta}^1_1$ sets. Throughout we will use the standard lightface-boldface notation of effective descriptive set theory, for instance Σ^1_1, $\Sigma^1_1(x)$ and $\mathbf{\Sigma}^1_1$ (see [15]).

For each compact Polish space X, $\mathcal{K}(X)$ denotes the collection of closed subsets of X. The Hausdorff topology on $\mathcal{K}(X)$ is generated by the sets

$$\{K \in \mathcal{K}(X) : K \cap V \neq \emptyset\}\,, \{K \in \mathcal{K}(X) : K \subseteq V\},$$

where V is an open set in X. So the basic open sets are of the form

$$\{K \in \mathcal{K}(X) : K \subseteq V_0 \;\&\; K \cap V_1 \neq \emptyset \;\&\ldots\&\; K \cap V_n \neq \emptyset\},$$

where V_0,..., V_n are open sets in X. This is a compact, metrizable space with the following metric

$$\delta(K,L) = \begin{cases} \text{Sup}\{\ \max\ \{\text{dist}(\text{x},\text{K}),\ \text{dist}(y,L)\}\text{: } x \in L, y \in K\} & \text{if } K,L \neq \emptyset \\ \text{diam }(X) & \text{otherwise} \end{cases}.$$

The basic facts about this topology that we are going to use can be found in chapter IV,§2 of [13].

Contents

Chapter 1

Smooth and Sparse sets for Borel equivalence relations

A Borel equivalence relation E on a Borel set B (in a Polish space X) is said to be *smooth* if it admits a countable Borel separating family, i.e., a collection (A_n) of E-invariant Borel subsets of B such that for all $x, y \in B$

$$xEy \text{ iff } (\forall n)(x \in A_n \leftrightarrow y \in A_n).$$

Given an arbitrary Borel equivalence relation E on X, a set $A \subseteq X$ is called *E-smooth* if there is a Borel set $B \supseteq A$ such that the restriction of E to B is a smooth equivalence relation. The collection of E-smooth sets forms a $\sigma-$ideal. Thus we consider smoothness a notion of smallness. Smooth sets are a generalization of the notion of wandering sets in ergodic theory.

In this chapter we will study the descriptive set theoretic and definability properties of the collection of smooth sets. We will use the notion of countability as a paradigm of a notion of smallness. We will see that most of the definability properties of the collection of countable sets can be translated to the context of smooth sets.

There are, however, two open questions regarding this analogy with countable sets. First, it is a well known fact that large cardinals axioms imply that the Perfect Set Theorem holds also for Π^1_1 sets (see [15]). We have the analog of the Perfect Set Theorem for smoothness (theorem 1.1.6), but we do not know how to extended it to Π^1_1 sets. In the case of the countable sets this fact can be deduced from a (seemingly) stronger form of the Perfect Set Theorem called the covering property (we will study this property in chapters 2 and 3). This is the second open question: does the covering property hold in the context of smoothness ?

We will be working with methods of effective descriptive set theory. The standard reference is Moschovakis' book [15].

1.1 Smoothness as a notion of smallness

The basic result about smooth equivalence relations is the Glimm-Effros type Dichotomy Theorem proved by Harrington, Kechris and Louveau in [6], which characterizes the smooth Borel equivalence relations and thus the Borel smooth sets. This theorem can be extended to $\mathbf{\Sigma}^1_1$ sets as we show in §1.1 (theorem 1.1.4). This result can be considered as an analog of the Perfect Set Theorem in the context of smoothness. Also we have what could be thought as an analog of the hyperarithmetic reals (see §1.2).

Theorem 1.1.4 will also provide the basic representation of $\mathbf{\Sigma}^1_1$ smooth sets as the common null sets for the family of E-ergodic non-atomic measures. This in particular says that smoothness for $\mathbf{\Sigma}^1_1$ sets is a notion concentrated on closed sets, i.e., a $\mathbf{\Sigma}^1_1$ set A is smooth iff every closed subset of A is smooth. We called the sets with this property *sparse* sets and they are the analog of thin sets (i.e., sets without perfect subset). We will see some of their properties in §1.3 and §1.4. Also we will see that for Π_1^1 sets smoothness and sparseness are not equivalent in general in ZFC, a similar result as in the case of thin sets. In §1.5 we will look at the particular case of a countable equivalence relation (i.e., one all of whose equivalence class are countable).

1.1.1 Smooth Σ_1^1 sets

First we will define the basic concepts and state some basic facts. Let X be a Polish space (i.e., a complete separable metric space). E will always denote an equivalence relation on X. $[x]_E$ or sometimes E_x will denote the E-equivalence class of x. $[A]_E$ is the saturation of A, i.e., $[A]_E = \{y \in X : \exists x \in A(xEy)\}$. A set A 1is called E-invariant (or just invariant, if there is no confusion about E), if $A = [A]_E$.

One fact that we will use, without explicit mention, is that given a Δ_1^1 equivalence relation E, (i.e., E as a subset of $X \times X$ is a Δ_1^1 set) and $A \subseteq B$, with B a Π_1^1 invariant set and A a Σ_1^1 set, then there is a Δ_1^1 invariant set C with $A \subseteq C \subseteq B$. In other words, the separation theorem holds in an invariant form for Δ_1^1 equivalence relations (actually it holds for Σ_1^1 equivalence relations). A proof of this can be found in [6] (lemma 5.1).

The main notion that we will be dealing with is the notion of a Γ-separated equivalence relation. First, we introduce the following notation: Script capital letters will denote a countable family of subsets of X, i.e., $\mathcal{A} = (A_n)$, with $A_n \subseteq X$ for $n \in \mathbf{N}$. For each of these collections we define the following equivalence relation:

$$x \, E_{\mathcal{A}} \, y \text{ iff } (\forall n)(x \in A_n \longleftrightarrow y \in A_n).$$

We have the following

Definition 1.1.1 *Let* Γ *be a pointclass*

(i) E *is* Γ-separated *iff there is a countable collection* $\mathcal{A} = (A_n)$ *with each* $A_n \in \Gamma$, *such that:* $\forall x \forall y (xEy \longleftrightarrow x\, E_{\mathcal{A}}\, y)$, *i.e.,* $E = E_{\mathcal{A}}$.

(ii) A subset A *of* X *is* Γ-separated, *iff there is a collection* $\mathcal{A} = (A_n)$ *of* E*-invariant sets, with each* $A_n \in \Gamma$, *and* $\forall x \in A, \forall y \in A(xEy \longleftrightarrow x\, E_{\mathcal{A}}\, y)$. *In this case we say that* $\mathcal{A}$ separates A.

(iii) A *is called* strongly Γ-separated *if* $\forall x \in A\, \forall y (xEy \longleftrightarrow x\, E_{\mathcal{A}}\, y)$*; and we say that* $\mathcal{A}$ strongly separates A.

Remarks: (1) Notice that in (i), each A_n has to be E-invariant (because if $x \in A_n$ and yEx, then $x\, E_{\mathcal{A}}\, y$. Hence $y \in A_n$).

(2) Denote by $[x]_{\mathcal{A}}$ the $E_{\mathcal{A}}$-equivalence class of x. Then $\mathcal{A}$ separates A iff for all $x \in A$, $[x]_E \cap A = [x]_{\mathcal{A}} \cap A$; and $\mathcal{A}$ strongly separates A iff for all $x \in A$, $[x]_E = [x]_{\mathcal{A}}$.

(3) If $\mathcal{A} = (A_n)$ and each A_n is invariant then $E \subseteq E_{\mathcal{A}}$, thus only one direction in (ii) is not trivial.

A finite, positive Borel measure μ on X is called *E-ergodic* if for every μ-measurable invariant set A, $\mu(A) = 0$ or $\mu(X - A) = 0$. It is called *E-non atomic*, or just *non atomic*, if for every $x \in X$ $\mu([x]_E) = 0$. A basic fact about E-ergodic non-atomic measure is that if μ is such a measure, then there is no μ-measurable separating family for E. In fact: if $\mathcal{A} = (A_n)$ is a collection of E-invariant, μ-measurable sets, put $A_n^* = A_n$ if $\mu(A_n) > 0$, otherwise put $A_n^* = X - A_n$. As each A_n^* is invariant and μ is E-ergodic, then $\mu(\bigcap_n A_n^*) > 0$. As μ is non-atomic, then $\mu([x]_E) = 0$. But $\bigcap A_n^* = [x]_{\mathcal{A}}$ for some x, hence $[x]_E \neq [x]_{\mathcal{A}}$. So (A_n) does not separate E.

A typical example of an equivalence relation with a non atomic ergodic measure is E_0, which is defined on 2^ω by

$$xE_0y \text{ iff } (\exists m)(\forall n > m)(x(n) = y(n)).$$

The usual product measure on 2^ω is non atomic and E_0-ergodic (the so called 0-1 law).

One way of defining ergodic measures is through embeddings. Let E and E' be two equivalence relations on X and Y respectively. An *embedding* from E into E' is a 1-1 map $f : X \to Y$ such that for all $x, y \in X$, $xEy \longleftrightarrow f(x)E'f(y)$. For Borel equivalence relations we define $E \sqsubseteq E'$ if there is a Borel embedding of E into E'. Notice, if there is an E-ergodic, non atomic measure μ on X and $E \sqsubseteq E'$ then there is an E'-ergodic non atomic measure ν in Y. Namely, if $f : X \to Y$ is the embedding from E into E', define ν by $\nu(A) = \mu(f^{-1}(A))$. Clearly ν is E'-ergodic and non atomic.

The fundamental result about these notions is the following theorem of Harrington, Kechris and Louveau (see [6]). We will refer to it as the HKL theorem.

Theorem 1.1.2 *(Harrington,Kechris,Louveau [6]) Let X be a recursively presented perfect Polish space, E a Δ^1_1 equivalence relation on X. Then exactly one of the following holds:*

(1) *E has a Δ^1_1 separating family $\mathcal{A} = (A_n)$, such that the relation "$x \in A_n$", is Δ^1_1.*

(2) *$E_0 \sqsubseteq E$ (via a continuous embedding).*

□

In this section we are going to deal with the following question: Let $A \subseteq X$ be a Σ^1_1 subset of X. Does the HKL theorem hold for the restriction of E to A? The answer of this question will lead to the notion of smooth set. First we need

Definition 1.1.3 *Let Γ be a pointclass*

(i) Let $A \subseteq X$ and define $E\lceil A$ to be the restriction of E to A, i.e., $E\lceil A = E \cap (A \times A)$. $E\lceil A$ is an equivalence relation on A. And, naturally, we say $E\lceil A$ is Γ-separated if there is a countable collection $\mathcal{A} = (A_n)$ of Γ-subsets of A such that for all $x, y \in A\,(xE_{\mathcal{A}}\,y \longleftrightarrow xEy)$.

(ii) A measure μ on X is called $E\lceil A -$ ergodic if $\mu(X - A) = 0$ and for every $B \subseteq A$ which is $E\lceil A$-invariant and μ-measurable, we have $\mu(B) = 0$ or $\mu(X - B) = 0$. Notice that $\mu(X - B) = 0$ iff $\mu(A - B) = 0$.

If $A \in \Gamma$ (for Γ a pointclass closed under intersections) is invariant, then it is clear that A is Γ-separated iff $E\lceil A$ is Γ-separated. The next theorem, among other things, says that for a Borel equivalence relation all the natural variations for a notion of countable separation for $\mathbf{\Sigma}^1_1$ sets are equivalent.

Theorem 1.1.4 *Let X be a recursively presented Polish space, E a Δ^1_1 equivalence relation on X, and A a Σ^1_1 subset of X. The following are equivalent:*

(1) *There is a Δ^1_1 invariant set B such that $A \subseteq B$ and B is (strongly) Δ^1_1-separated. Moreover, the separating family for B is uniformly Δ^1_1, i.e., the relation "$x \in A_n$" is Δ^1_1 .*

(2) *A is strongly Δ^1_1-separated.*

(3) *$[A]_E$ is Σ^1_1-separated.*

(4) *A is Σ^1_1-separated.*

(5) *$E\lceil A$ is Σ^1_1-separated.*

(6) *A is universally measurable separated.*

(7) *$E\lceil A$ is universally measurable separated.*

(8) *For every E-ergodic non atomic measure μ, $\mu(A) = 0$.*

(9) *For every $E\lceil A$-ergodic, non atomic measure μ, $\mu(A) = 0$.*

(10) $E_0 \not\sqsubseteq E\lceil A$.

Similarly, the same equivalence holds by relativization for a $\mathbf{\Sigma^1_1}$ set A and a $\mathbf{\Delta^1_1}$ equivalence relation.

Proof: (1) $\Rightarrow$ (2) The family that strongly separates B in (1) also separates A.

(2) $\Rightarrow$ (3) Let $\mathcal{A} = (A_n)$ be a family of Δ^1_1 invariant sets which strongly separates A. Then $\mathcal{A}$ also separates $[A]_E$. In fact: let $x, y \in [A]_E$; say xEx' and yEy' with $x', y' \in A$. If $x\, E_{\mathcal{A}}\, y$, then we easily get that $x'\, E_{\mathcal{A}}\, y'$, and hence xEy.

(3) $\Rightarrow$ (4) Obvious, as $A \subseteq [A]_E$.

(4) $\Rightarrow$ (5) If $\mathcal{A} = (A_n)$ is a collection of Σ^1_1 invariant sets which separates A, put $B_n = A \cap A_n$ and $\mathcal{B} = (B_n)$. $\mathcal{B}$ separates $E\lceil A$, as it can be easily shown.

(5) $\Rightarrow$ (6) It is enough to show (5) $\Rightarrow$ (4). Let $\mathcal{A} = (A_n)$ be a collection of Σ^1_1 subsets of A which separates $E\lceil A$. Let $B_n = [A_n]_E$ and $\mathcal{B} = (B_n)$. Then $\mathcal{B}$ separates A, in fact: just observe that $B_n \cap A = A_n$ (if $x \in A_n, x \in A$ and yEx, then $x\, E_{\mathcal{A}}\, y$. Hence $y \in A_n$). Thus for $x, y \in A$, $x\, E_{\mathcal{B}}\, y$ iff xEy.

(6) $\Rightarrow$ (7) By a similar argument as in (4) $\Rightarrow$ (5).

(6) $\Rightarrow$ (8) Let μ be a E-ergodic non atomic measure on X, and $\mathcal{A} = (A_n)$ be a universally measurable separating family for A. Then either $\mu(A_n) = 0$ or $\mu(X - A_n) = 0$. Put $B_n = A_n$, if $\mu(X - A_n) = 0$ and $B_n = X - A_n$ otherwise. Let $B = \cap B_n$. As $\mathcal{A}$ separates A, we get $A \cap B = \emptyset$ or $A \cap B = [x]_E$, for some $x \in X$. Hence as μ is non atomic, $\mu(A \cap B) = 0$; but $\mu(X - B) = 0$ thus $\mu(A) = 0$.

(7) $\Rightarrow$ (9) By a similar argument as in (6) $\Rightarrow$ (8). Now working on A and observing that if (A_n) separates $E\lceil A$, then each A_n is $E\lceil A$-invariant.

(8) $\Rightarrow$ (10) We will show the contrapositive. Suppose $E_0 \sqsubseteq E\lceil A$ via a continuous embedding (even a Borel embedding works) $f : 2^\omega \to X$. Define a measure on X by $\mu(B) = \lambda(f^{-1}(B))$, where λ is the usual product measure on 2^ω and $B \subseteq 2^\omega$. Then $\mu(A) = 1$ and it is easy to check that μ is an E-ergodic, non atomic measure.

(9) $\Rightarrow$ (10) By a similar argument as in (8) $\Rightarrow$ (10). And observing that the measure defined through the embedding is $E\lceil A$-ergodic, non atomic and concentrated in A.

For (10) $\Rightarrow$ (1) we will use the following lemma, which is coming from the proof of the HKL theorem. This proof uses the Gandy-Harrington topology (also called the Σ^1_1-topology). The basis for this topology is the collection of Σ^1_1 sets. This is a Baire topology (i.e., it satisfies the Baire category theorem). The basic facts about it can be found in [6].

Lemma A: *Let τ be the Gandy-Harrington topology on X and $\overline{E}$ the $\tau \times \tau$-clousure of E. Let A be a Σ^1_1 subset of X. If $\{x : E_x \neq (\overline{E})_x\} \cap A \neq \emptyset$ then $E_0 \sqsubseteq E\lceil A$, via a continuous embedding.*

Proof: In the proof of the HKL theorem was shown that if $\{x : E_x \neq (\overline{E})_x\} \cap A \neq \emptyset$, then E is meager in $(A \times A) \cap \overline{E}$ (see lemma 5.3). Hence the construction of the embedding from E_0 into $E \lceil A$ can be carried out in A.

($\Box$ lemma A)

We need also the following

Lemma B: *Let $D = \{x : E_x = (\overline{E})_x\}$, D is a Π^1_1 strongly Δ^1_1-separated invariant set. Actually, the separating family for D is* $\{A \subseteq X : A$ is a Δ^1_1 invariant set $\}$.

Proof: First, $\overline{E}$ is a Σ^1_1 equivalence relation (see lemma 5.2 in the proof of the HKL theorem). And we have: $x \in D$ iff $(\forall y)(x\overline{E}y \rightarrow xEy)$. Thus D is Π^1_1 . Also, as $E \subseteq \overline{E}$, then D is E-invariant (actually $\overline{E}$-invariant). On the other hand, we know $\overline{E} =\sim \bigcup\{A \times \sim A : A$ is Δ^1_1 invariant set $\}$. So, if $\mathcal{A} = \{A : A$ is a Δ^1_1 invariant set $\}$, then $\overline{E} = E_{\mathcal{A}}$. And we get: $\forall x \in D(E_x = (\overline{E})_x = (E_{\mathcal{A}})_x)$. Thus $\forall x \in D \forall y(xE_{\mathcal{A}}y \longleftrightarrow xEy)$, i.e., D is strongly separated by $\mathcal{A}$.

($\Box$ lemma B)

Now we finish the proof of (10) $\Rightarrow$ (1). Suppose (10) holds. Then by Lemma A $A \subseteq D$. By separation there is a Δ^1_1 invariant set B with $A \subseteq B \subseteq D$. Hence, by lemma B B is strongly Δ^1_1 separated by $\{A \subseteq X : A$ is Δ^1_1 invariant set $\}$. Now, $\mathcal{A}$ is clearly a Π^1_1 collection, so by a separation argument (see [6]) we can easily show that there is a Δ^1_1 subsequence of $\mathcal{A}$ which also separates B, so (1) holds.

$\Box$

In view of theorem 1.1.4, we introduce the following

Definition 1.1.5 *Let E be a Borel equivalence relation on X. A $\mathbf{\Sigma}^1_1$ subset $A \subseteq X$ is called* E-smooth *(or smooth with respect to E) if any of the equivalent conditions of theorem 1.1.4 holds.*

It is clear that a $\mathbf{\Sigma}^1_1$ subset of a $\mathbf{\Sigma}^1_1$ smooth set is also smooth and countable unions of smooth sets are smooth. So, we regard smooth sets as small sets. And, we have what can be thought as an analog of the Perfect Set Theorem for $\mathbf{\Sigma}^1_1$ sets in the context of smooth sets. It summarizes the most important part of theorem 1.1.4.

Theorem 1.1.6 (Analog of The Perfect Set Theorem for $\mathbf{\Sigma}^1_1$ sets) *Let E be a Δ^1_1 equivalence relation on a recursively presented Polish space X. Let $A \subseteq X$ be a Σ^1_1 set. Then either A is smooth or $E_0 \sqsubseteq E \lceil A$ (via a continuous embedding). Similarly the same result holds by relativization for a $\mathbf{\Sigma}^1_1$ set A and a $\mathbf{\Delta}^1_1$ equivalence relation E.* □

Another feature of the ideal of countable sets is that it is Π^1_1 on the codes of $\mathbf{\Sigma}^1_1$ sets. A similar definability result holds for $\mathbf{\Sigma}^1_1$ smooth sets. This is also a consequence of theorem 1.1.4 (i).

Theorem 1.1.7 *Let E be a Δ^1_1 equivalence relation on a recursively presented Polish space X. Then the collection of $\mathbf{\Sigma}^1_1$ smooth sets is Π^1_1 on the codes of $\mathbf{\Sigma}^1_1$ sets.*

Proof : Given a $\Sigma^1_1(\alpha)$ smooth set A, by theorem 1.1.4 there is a $\Delta^1_1(\alpha)$ separating family for A consisting of $\Delta^1_1(\alpha)$ invariant sets. Let $\mathcal{U}$ be a Σ^1_1 universal set, then

$$\mathcal{U}_\alpha \text{ is smooth } \text{ iff } \exists \mathcal{A} \in \Delta^1_1(\alpha)[\forall x, y \in \mathcal{U}_\alpha(xEy \longleftrightarrow x\, E_{\mathcal{A}}\, y)] \qquad (*)$$

To see that (*) is indeed a Π^1_1 relation, we need to code sequences of $\Delta^1_1(\alpha)$ invariant sets. For that end consider the following relations: Let $C \subseteq \omega^\omega \times \omega \times X$, $W \subseteq \omega^\omega \times \omega$ such that (C, W) parametrizes the $\mathbf{\Delta}^1_1$ subsets of X i.e.,

(1) C and W are Π^1_1.

(2) For every $\alpha \in \omega^\omega$, if $A \subseteq X$ is $\Delta^1_1(\alpha)$, then there is n such that $W(\alpha, n)$ and $A = \{x : C(\alpha, n, x)\}$.

(3) There is a Σ^1_1 relation D such that if $W(\alpha, n)$ holds, then $C(\alpha, n, x) \Leftrightarrow D(\alpha, n, x)$, i.e., $C_{\alpha,n}$ is $\Delta^1_1(\alpha)$.

Define

$$SF(\gamma, \alpha) \iff (\forall n)[W(\alpha, \gamma(n)) \;\;\&\;\; C_{\alpha,\gamma(n)} \text{ is invariant}].$$

Since we have that

$$C_{\alpha,\gamma(n)} \text{ is invariant} \iff \forall x, y\,(\, x \in C_{\alpha,\gamma(n)} \,\&\, yEx \Rightarrow y \in C_{\alpha,\gamma(n)}).$$

Then from (3) we get that SF is Π^1_1. Define

$$ER(x, y, \gamma, \alpha) \iff (\forall n)[C(\alpha, \gamma(n), x) \leftrightarrow C(\alpha, \gamma(n), y)].$$

Notice, if $SF(\gamma, \alpha)$ holds, then the equivalence relation given by $ER(x, y, \gamma, \alpha)$ is $\Delta^1_1(\alpha, \gamma)$. Thus we finally get:

$$\mathcal{U}_\alpha \text{ is smooth iff } \exists\gamma \in \Delta^1_1(\alpha)[\,(SF(\gamma, \alpha)) \,\&\, \forall x, y \in \mathcal{U}_\alpha\,(xEy \leftrightarrow ER(x, y, \gamma, \alpha))]$$

which is a Π^1_1 relation.

□

Remark: The proof of (10) ⇒ (1) in 1.1.4 was based on the proof of the HKL theorem (1.1.2). However one can prove that (10) ⇒ (4) directly granting the following slightly stronger version of the HKL theorem (which follows very easily from the proof): If E is a Δ^1_1 equivalence relation and P is a Π^0_1 set which is not Δ^1_1 separated, then $E_0 \sqsubseteq E\lceil P$ via a continuous embedding. The same holds by relativization for a $\boldsymbol{\Delta}^1_1$ equivalence relation and a $\boldsymbol{\Pi}^0_1$ set. First, let us observe that we actually have proved (in 1.1.4) that (4) and (5) are equivalent. Also it is easy to see directly that (4) $\Rightarrow$ (3). In fact: Suppose $\mathcal{A} = (A_n)$ separates A, where each A_n is a Σ^1_1 invariant set. Let $x, y \in [A]_E$, then for some $\overline{x}, \overline{y} \in A$ we have $xE\overline{x}$ and $yE\overline{y}$. Now, if $x\, E_{\mathcal{A}}\, y$ then $\overline{x}\, E_{\mathcal{A}}\, \overline{y}$. But as $\mathcal{A}$ separates A, then $\overline{x}E\overline{y}$. Thus xEy, i.e., $\mathcal{A}$ also separates $[A]_E$. So we can assume that A is an invariant set.

Let E be a Δ^1_1 equivalence relation and A a Σ^1_1 invariant set. Suppose A is not Σ^1_1-separated, we will show that $E_0 \sqsubseteq E\lceil A$. Let R be a Π^0_1 subset of $X \times \omega^\omega$ such that $A = proj(R)$. Define an equivalence relation $\tilde{E}$ on $X \times \omega^\omega$ as follows:

$$(x,\alpha)\tilde{E}(y,\beta) \text{ iff } xEy \text{ and } (x,\alpha),\, (y,\beta) \in R.$$

$\tilde{E}$ is clearly Δ^1_1 and R is $\tilde{E}$-invariant.

Claim 1: *If R is Σ^1_1-separated (with respect to $\tilde{E}$), then A is Σ^1_1-separated (with respect to E).*

Proof: Let $\mathcal{A} = (A_n)$ be a Σ^1_1-separating family for R. Since R is $\tilde{E}$-invariant we can assume that each $A_n \subseteq R$. So, we have

$$\forall (x,\alpha),(y,\beta) \in R[(x,\alpha)\, E_{\mathcal{A}}\, (y,\beta) \leftrightarrow (x,\alpha)\tilde{E}(y,\beta)].$$

Put $B_n = proj(A_n)$ and $\mathcal{B} = (B_n)$. As A is E-invariant we easily get that each B_n is E-invariant. We claim that $\mathcal{B}$ separates A. In fact we only need to observe that

$$(\forall(x,\alpha),(y,\beta) \in R)[(x,\alpha)\, E_{\mathcal{A}}\,(y,\beta) \leftrightarrow x\, E_{\mathcal{B}}\, y]. \qquad (*)$$

From this we get that

$$(\forall x, y \in A)[x\, E_{\mathcal{B}}\, y \leftrightarrow xEy].$$

To see that (*) holds, let (x,α),$(y,\beta) \in R$ be such that $(x,\alpha)\, E_{\mathcal{A}}\,(y,\beta)$. Suppose $x \in B_n$ and let γ be such that $(x,\gamma) \in A_n$. Then $(x,\alpha)\tilde{E}(x,\gamma)$, thus $(x,\alpha) \in A_n$. Therefore $(y,\beta) \in A_n$, so $y \in B_n$.

Conversely, let (x,α),$(y,\beta) \in R$ be such that $x\, E_{\mathcal{B}}\, y$ and suppose $(x,\alpha) \in A_n$. Thus $x \in B_n$ and hence $y \in B_n$. Let γ be such that $(y,\gamma) \in A_n$. As $(y,\beta)\tilde{E}(y,\gamma)$ then we get that $(y,\beta)\, E_{\mathcal{A}}\,(y,\gamma)$. Hence $(y,\beta) \in A_n$.

($\square$ Claim 1)

Claim 2: *Let $P \subseteq X$, $Q \subseteq R$ with $P = proj(Q)$, and $\mathcal{A} = (A_n)$ with each A_n E-invariant. Put $B_n = R \cap (A_n \times \omega^\omega)$ and $\mathcal{B} = (B_n)$. If $\mathcal{A}$ separates P then $\mathcal{B}$ separates Q.*

Proof: First, observe that each B_n is $\tilde{E}$-invariant. It is easy to check that if (x,α),$(y,\beta) \in Q$ and $(x,\alpha)\, E_{\mathcal{B}}\,(y,\beta)$, then $x\, E_{\mathcal{A}}\, y$.

($\square$ Claim 2)

To finish the proof, assume A is not Σ^1_1-separated. Then by claim 1 R is not Δ^1_1-separated. Hence by the version of the HKL theorem mentioned at the begining we have that $E_0 \sqsubseteq \tilde{E}\lceil R$, via a continuous embedding. Say $f : 2^\omega \to R$. Put $Q = f[2^\omega]$ and $P = proj(Q)$. Q is not $\mathbf{\Sigma}^1_1$-separated. Hence, by claim 2, P is not $\mathbf{\Sigma}^1_1$-separated. Since P is compact, by the HKL theorem $E_0 \sqsubseteq E\lceil P$, via a continuous embedding.

1.1.2 A possible analog of the Hyperarithmetic reals

The effective perfect set theorem for Σ^1_1 sets says that a Σ^1_1 countable set contains only Δ^1_1 reals. Looking at the proof of theorem 1.1.4, we observe that the set D defined on 1.1.4 seems to play in the context of smooth sets, the same role as the set of Δ^1_1 reals does in the context of countable sets. The next proposition makes more precise this remark and summarizes what we know about D.

Proposition 1.1.8 *Let be E a Δ^1_1 equivalence relation on X and $\overline{E}$ be the $\tau \times \tau$-closure of E, where τ is the GH-topology on X. Put*

$$D = \{x : E_x = (\overline{E})_x\}$$

then: (i) D is a Π^1_1 set.

(ii) For every Σ^1_1 set A, A is smooth iff $A \subseteq D$.

(iii) D is the largest strongly Δ^1_1-separated set.

Proof: (i) We already saw in 1.1.4 lemma B that D is Π^1_1 and strongly Δ^1_1-separated by $\{A : A \text{ is } \Delta^1_1 \text{ invariant set }\}$.

(ii) This follows from lemmas A and B in 1.1.4.

(iii) It remains to show that every strongly Δ^1_1 separated set is a subset of D. Let $\mathcal{A} = \{A : A \text{ is } \Delta^1_1 \text{ invariant set }\}$ and B a strongly Δ^1_1-separated set, say by a family $\mathcal{B}$ of Δ^1_1 invariant sets. Let $D_{\mathcal{B}} = \{x : [x]_E = [x]_{\mathcal{B}}\}$, i.e., $x \in D_{\mathcal{B}}$ iff for all $y(x\, E_{\mathcal{B}}\, y \longleftrightarrow xEy)$. Analogously we define $D_{\mathcal{A}}$. We saw in 1.1.4 lemma B that $D = D_{\mathcal{A}}$. By definition of strong separation $B \subseteq D_{\mathcal{B}}$. But as $\mathcal{B} \subseteq \mathcal{A}$, then $E_{\mathcal{A}} \subseteq E_{\mathcal{B}}$ and thus $D_{\mathcal{B}} \subseteq D_{\mathcal{A}}$. Therefore $B \subseteq D_{\mathcal{A}}$.

□

Let us recall here that the collection of hyperarithmetic reals, denoted by $\Delta^1_1(X)$, is a true Π^1_1 set and is equal to $\bigcup\{A : A$ is a countable Δ^1_1 set $\}$. Continuing the analogy with countable sets (see also proposition 2.1.22 in chapter 2) we have the following natural questions:

(i) Is $D = \bigcup\{A : A$ is Δ^1_1 smooth set$\}$? Equivalently, is D the union of Σ^1_1 sets?

(ii) Is D a true Π^1_1 set ?

We will show in §5 that for a countable Δ^1_1 equivalence relation the answer for (i) is yes. And as a consequence of a theorem of Kechris, this is also true for a Δ^1_1 equivalence relation generated by the action of a locally compact group of Δ^1_1 automorphisms of X. Regarding question (ii), we know that for E_0 D is a true Π^1_1 set. The proof of this is as follows: Let us observe that every Δ^1_1 point $x \in 2^\omega$ belongs to D; this is because $\{x\}$ is a Δ^1_1 smooth set. Also, D has measure zero with respect to the standard product measure on 2^ω (because this measure is E_0-ergodic). Then by a basis theorem it cannot be Δ^1_1 : otherwise its complement would contain a Δ^1_1 point. So for this case the analogy between D and the hyperarithmetic reals is quite clear.

In the next section we will see a relation of D with the notion of smoothness for Π^1_1 sets.

1.1.3 Sparse $\boldsymbol{\Pi}^1_1$ sets

In the context of countable sets there is another notion of smallness that turns out to be quite useful. A set A is called thin if every closed subset of A is countable, i.e., A does not have a perfect subset. The perfect set theorem for $\boldsymbol{\Sigma}^1_1$ sets asserts that being countable and thin is equivalent for $\boldsymbol{\Sigma}^1_1$ sets.

In view of theorem 1.1.4 to say that every closed subset of a Σ^1_1 set A is smooth is equivalent to say that $E_0 \not\sqsubseteq E\lceil A$. We introduce the following

Definition 1.1.9 *A set $A \subseteq X$ is* E-sparse *(or sparse with respect to E) if $E_0 \not\sqsubseteq E\lceil A$.*

Notice that thin sets are clearly E-sparse. Looking at theorem 1.1.4, we observe that for an arbitrary set A, (10) is implied by all the other statements (because any of the conditions (1)-(9) can be translated to 2^ω through the embedding, but we know E_0 is not smooth). And if A is a universally measurable set then (10) and (8) are equivalent, because we are dealing with Borel measures (hence regular). Thus a universally measurable set A is E-sparse iff for every E-ergodic non atomic measure μ, $\mu(A) = 0$. Equivalentely, a universally measurable set A is sparse iff every closed subset of A is smooth.

In 1.1.5 we have introduced the notion of smoothness for Σ^1_1 sets and in terms of the notion of sparseness, theorem 1.1.4 says that a Σ^1_1 set is smooth if and only if it is sparse. Thus to continue with the analogy with the countable sets, we introduce the following

Definition 1.1.10 *A set $A \subseteq X$ is called* E-smooth *(or smooth with respect to E) if there is a Borel smooth set B such that $A \subseteq B$.*

Thus the analog of countable is smooth. Since it is consistent that there are Π^1_1 thin sets which are not countable, one should not expect that the equivalences in 1.1.4 will hold (in ZFC) for Π^1_1 sets. This is showed in the next proposition.

Proposition 1.1.11 *Let* $E = \equiv_{\mathrm{T}}$ *(Turing equivalence) and* $\mathrm{C}_1 = \{\alpha : \alpha \in \mathrm{L}_{\omega_1^\alpha}\}$. *Then :*

(i) C_1 *is a* Π^1_1 *E-invariant sparse set.*

(ii) (in L*)* C_1 *is not contained in a Borel E-invariant smooth set, i.e.,* C_1 *is not E-smooth.*

Proof : (i) Since C_1 is thin (actually, the largest Π^1_1 thin set, see [8]), then C_1 is E-sparse. And it is clearly closed under $\equiv_{\mathrm{T}}$.

(ii) Let B be an E-invariant Borel set with $\mathrm{C}_1 \subseteq B$. As C_1 is unbounded in $\equiv_{\mathrm{T}}$ (see 5A.11 in [15]), then by a result of Martin B contains a cone of Turing degrees, i.e., there is $\beta \in \omega^\omega$ such that $\hat{\beta} = \{\alpha \in \omega^\omega : \beta \leq_{\mathrm{T}} \alpha\} \subseteq B$. But this implies that B is not smooth, because Turing cones are not smooth for $\equiv_{\mathrm{T}}$. To see this, let $\beta \in \omega^\omega$, and consider the equivalence relation $\tilde{E}$ on 2^ω defined by $\alpha \tilde{E} \gamma$ iff $< \alpha, \beta > \equiv_{\mathrm{T}} < \gamma, \beta >$. Clearly $\equiv_{\mathrm{T}} \subseteq \tilde{E}$. As $\equiv_{\mathrm{T}}$ is not smooth and every equivalence class with respect to $\tilde{E}$ is countable, then $\tilde{E}$ can not be smooth (an alternative argument is as follows: Since Martin's measure is concentrated on the cones and it is $\equiv_{\mathrm{T}}$-ergodic and non-atomic, then every cone is not smooth).

□

This proposition shows that in L the notions of smoothness and sparseness for Π^1_1 sets are not equivalent. Which naturally rises the following

Question : Is there a Perfect set theorem for Π^1_1 sets, in the context of smoothness ? That is to say: If A is a Π^1_1 sparse set, is there a Borel smooth set B contaning A ? This is of course assuming (in view of 1.1.11) some large cardinal axiom. We will come back to this question at the end of §1.2.

It seems natural to investigate which of the equivalences of theorem 1.1.4 remain valid for Π^1_1 sets. In the next section we will study the relation between smoothness and strong separation, where we will show the following

Theorem: *Let E be a countable Borel equivalence relation on X. Let A be an arbitrary subset of X. The following are equivalent:*

(1) There is a Borel invariant E-smooth set containing A, i.e., A is smooth.

(2) A is strongly Borel separated.

□

We do not know if this theorem holds for an arbitrary Borel equivalence relation. However, if the answer for the question left after proposition 1.1.8 is positive, then it does hold. In fact: if $D = \bigcup\{A : A$ is a Δ^1_1-invariant set $\}$, then D is clearly Borel. We know by theorem 1.1.8 that D is strongly Δ^1_1-separated and it contains every strongly Δ^1_1-separated set. Thus, in this case being smooth and strongly Δ^1_1-separated would be equivalent. As we have observed before this holds for an equivalence relation generated by the action of a locally compact group.

Now we are going to look at the relation between the notions of separation and strong separation.

Proposition 1.1.12 *Let E be a Δ^1_1 equivalence relation on X, Γ a pointclass such that $\Delta^1_1 \subseteq \Gamma$ and Γ is closed under intersections. $\hat{\Gamma}$ will denote the dual pointclass. Then for $A \subseteq X$*

(i) If A is an E-invariant set in $\boldsymbol{\Gamma}$ and $\boldsymbol{\Gamma}$-separated, then A is $\boldsymbol{\Gamma}$-strong separated.

(ii) If A is Γ-separated (resp. strong separated), then A is $\hat{\Gamma}$-separated (resp. strong separated).

Proof: (i) $\mathcal{A} = (A_n)$ be a collection of invariant sets in $\mathbf{\Gamma}$ which separates A. Put $B_{n+1} = A \cap B_n$ and $B_0 = A - \bigcup A_n$. Since $\mathcal{A}$ separates A, then either $B_0 = \emptyset$ or $B_0 = [x]_E$, for some $x \in A$. In either case B_0 is in $\mathbf{\Delta}^1_1$, hence (by hypothesis) in $\mathbf{\Gamma}$. Put $\mathcal{B} = (B_n)$, we claim that $\mathcal{B}$ strongly separates A. In fact: First let us observe that for $x, y \in A$, if $x\, E_{\mathcal{B}}\, y$ then $x\, E_{\mathcal{A}}\, y$. So, it suffices to show that if $x \in A$ and $x\, E_{\mathcal{B}}\, y$ then $y \in A$. Let $x \in A$ and $y \in X$ such that $x\, E_{\mathcal{B}}\, y$, then there is n such that $x \in B_n$. Hence $y \in B_n$ and so $y \in A$.

(ii) If $\mathcal{A} = (A_n)$ separates A, then obviously so does $(\sim A_n)$.

□

Corollary 1.1.13 *If A is a Π^1_1 set $\mathbf{\Sigma}^1_1$-separated, then A is $\mathbf{\Sigma}^1_1$-strong separated.*

□

Remark: (i) above seems to be only a boldface fact. Except for the case $\Gamma = \Delta^1_1$, where we can get strong Δ^1_1 separation by requiring that the separating family is uniformly Δ^1_1, i.e., the relation "$x \in A_n$" has to be Δ^1_1; then the same proof applies.

Next we want to see what happens with $E\lceil A$ when A is Π^1_1. We have the following

Proposition 1.1.14 *Let E be a Δ^1_1 equivalence relation on X and A a Π^1_1 subset of A. Then* *(i) if A is Σ^1_1-separated, then $E\lceil A$ is Π^1_1-separated.*

(ii) if $E\lceil A$ is Σ^1_1-separated, then A is $\mathbf{\Sigma^1_1}$ (and hence A is a Borel set).

(iii) if $E\lceil A$ is Π^1_1-separated, then A is Σ^1_2-separated.

Proof:(i) Let $\mathcal{A} = (A_n)$ be a collection of Σ^1_1 invariant sets which separates A, then $(\sim A_n)$ also separates A. Put $B_n = A \cap (\sim A_n)$, it is straightforward to check that $\mathcal{B} = (B_n)$ separates $E\lceil A$.

(ii) if $\mathcal{A} = (A_n)$ is a sequence of Σ^1_1 subsets of A separating $E\lceil A$, then $A - \bigcup A_n$ is either empty or equal to $[x]_E$ for some $x \in A$. Hence A is $\mathbf{\Sigma^1_1}$.

(iii) Suppose $\mathcal{A} = (A_n)$ separates $E\lceil A$ and each A_n is a Π^1_1 subset of A. Let $B_n = [A_n]_E$. Each B_n is Σ^1_2 and it is easy to check that (B_n) separates A.

□

Remarks : (i) We will see in the next section that in general Π^1_1 sparse sets are not Δ^1_1-strong separated, but we do not know whether or not it is provable in ZFC that they are Σ^1_1-separated.

(ii) If P is a Π^1_1 invariant sparse set, let $P = \bigcup\{P_\alpha : \alpha < \omega_1\}$, with P_α a Borel invariant set. Each P_α is sparse and hence smooth. So, let $\{A^\alpha_n : n \in \mathbf{N}\}$ be a Borel separating family for each P_α, $\alpha < \omega_1$. It is easy to check that

$$\forall x \in P, \forall y \in P[(\forall n)(\forall \alpha < \omega_1)(x \in A^\alpha_n \leftrightarrow y \in A^\alpha_n) \longleftrightarrow xEy].$$

Thus P is separated by a collection of $\aleph_1$ Borel invariant sets (we can get strong separation if the (A^α_n) strong separates P_α). We do not know if this is the best that can be proved in ZFC. This is a (seemingly) weaker form of a perfect set theorem for Π^1_1 sparse sets. Following the analogy with the notion of thin sets, this is the analog of: Every $\mathbf{\Pi}^1_1$ thin set has cardinality at most $\aleph_1$.

1.1.4 The largest Π^1_1 sparse set

Sparse sets are, as we said, the analog of thin sets, and it is well known that $C_1 = \{\alpha \in \omega^\omega : \alpha \in \mathrm{L}_{\omega_1^\alpha}\}$ is the largest Π^1_1 thin set (see [8]). A similar result holds for sparse sets, i.e., there is a largest Π^1_1 sparse set which can be described in a similar way as C_1, as we will show next.

First we need to recall some standard facts about codes for Borel sets which we will use in the sequel. One way of coding Borel sets is with elements of the Baire space: Put

$$B_0 = \{\alpha \in \omega^\omega : \alpha(0) = 0\}$$

and for each $\eta < \omega_1$, put

$$B_\eta = \{\alpha \in \omega^\omega : (\forall n)((\alpha^*)_n \in \bigcup_{\xi<\eta} B_\xi) \ \&\ \alpha(0) = 1\}$$

where $\alpha^*(t) = \alpha(t+1)$. For each $\eta < \omega_1$ define a function π_η by:

$$\pi_\eta(\alpha) = \begin{cases} N(X, \alpha(1)) & \text{if } \eta = 0 \\ \bigcup_n [X - \pi_{\xi(n)}((\alpha^*)_n)] & \text{otherwise.} \end{cases}$$

Where $N(X, j)$ is the j-th open basic nbhd of X and $\xi(n) =$ least ξ such that $(\alpha^*)_n \in B_\xi$, if $\alpha \in B_\eta$.

A set A is in $\mathbf{\Sigma}^0_\xi$ iff there is $\alpha \in B_\xi$ such that $\pi_\xi(\alpha) = A$. Such α is called a *Borel code* for A. Let $BC := \bigcup\{B_\xi : \xi < \omega_1\}$. BC is the collection of Borel codes.

A second way of coding Borel sets is by ξ-codes. For each countable limit ordinal $\xi > \omega$ a ξ-code is a well founded tree T on some countable ordinal together with an assignment $\varphi : T \to \omega$ such that:

(1) For all $u \in T$, $\varphi(u)$ takes one of the values $< 0, s >$, 1 or 2; where $s \in \omega^{<\omega}$ and $\varphi(\emptyset) = 2$.

(2) $\varphi(u) =< 0, s >$, for some s iff u is a terminal node.

(3) If $\varphi(u) = 1$, then there is exactly one immediate extension $u\frown\eta \in T$ and $\varphi(u\frown\eta) \neq 1$.

(4) If $\varphi(u) = 2$, then there is at least one immediate extension $u\frown\eta$ and for all such $u\frown\eta$, $\varphi(u\frown\eta) \neq 2$.

(5) If for each $u \in T$ we define $\| u \|_T$ by induction as follows:

$\| u \|_T = 0$ if u is terminal.

$\| u \|_T = \| u\frown\eta \|_T$, if $\varphi(u) = 1$.

$\| u \|_T = sup\{\| u\frown\eta \|_T + 1 : u\frown\eta \in T\}$, if $\varphi(u) = 2$.

Then $\| \emptyset \|_T \leq \xi$.

If T and φ are as in (1)-(4), then T is called a labeled tree (φ is the label). A ξ-code is then a wellfounded labeled tree with rank $\leq \xi$.

Analogously as before to each ξ-code $< T, \varphi >$ we associate a Borel set B_T as follows: First we define by induction B_T^u for $u \in T$ and then we let $B_T = B_T^{\emptyset}$

$$B_T^u = N_s, \text{ if } \varphi(u) =< 0, s >$$

$$B_T^u =\sim B_T^{u\frown\eta}, \text{ if } \varphi(u) = 1.$$

$$B_T^u = \bigcup_\eta B_T^{u\frown\eta}, \text{ if } \varphi(u) = 2.$$

Also we have that a set $B \subseteq \omega^\omega$ is $\mathbf{\Sigma}^0_\xi$ iff it is of the form B_T for some ξ-code T. One can also go from one type of code to the other in an effective way, provided that we have a code for the countables ordinal involved.

Recall there is a Π^1_1 recursive function $M : WO \to \omega^\omega$ such that if $\mathrm{w} \in WO$ then $M(\mathrm{w})$ codes $\mathrm{L}_{|\mathrm{w}|}$. We will identify $M(\mathrm{w})$ with the structure it codes. We will need the next proposition in order to translate one type of coding into the other.

Proposition 1.1.15 *There is a Π^1_1 recursive function F such that for all $\mathrm{w} \in WO$ with $|\mathrm{w}|$ limit, if $M(\mathrm{w}) \models$ "$m = < m_T, m_\varphi, <_m >$ is a labeled tree on some ordinal", then $F(m, \mathrm{w})$ codes a labeled tree $< T, \varphi >$ on ω isomorphic to the tree coded by $< m_T, m_\varphi >$.*

Proof: Let us define $\bar{T}$, $\bar{\varphi}$ and $<_{\bar{T}}$ as follows:

$$n \in \bar{T} \text{ iff } M(\mathrm{w}) \models \text{“}n \in m_T\text{”}$$

$$\bar{\varphi}(n) = a \text{ iff } M(\mathrm{w}) \models \text{“}m_\varphi(n) = a\text{”}$$

$$n <_{\bar{T}} k \text{ iff } M(\mathrm{w}) \models \text{“}n <_m k\text{”}$$

Then $< \bar{T}, \bar{\varphi}, <_{\bar{T}} >$ is a countable labeled tree with height $\leq \omega$. Hence it is isomorphic to a ordinary (i.e., a subset of $\omega^{<\omega}$) labeled tree on ω which is easily seen to be $\Delta^1_1(\mathrm{w})$, since its definition can be expressed using the satisfaction relation on $M(\mathrm{w})$.

□

Suppose P is a Π^1_1 relation on ω^ω and consider the following relation

$$R(\alpha) \text{ iff } (\exists \xi < \omega_1^\alpha)(\exists \text{ a } \xi\text{-code } T \text{ in } L_{\omega_1^\alpha})[\alpha \in B_T \,\&\, P(\alpha)].$$

We claim that R is also Π^1_1. In fact, this type of quantifier is equivalent to saying: there is $\mathrm{w} \in WO$ with $\mathrm{w} \in \Delta^1_1(\alpha)$ and m such that in $M(\mathrm{w})$ "m is a labeled tree" and $F(m, \mathrm{w})$ is a code of a wellfounded labeled tree T on ω

(hence it is a ξ-code for some $\xi < \omega_1^\alpha$) such that $\alpha \in B_T$, where F is the function defined on the previous proposition.

The existence of a largest Π_1^1 set with some "thinness" property can be guaranteed in a quite general context (see [8]and also chapter 3). The next theorem characterizes this set for ideals on ω^ω represented as the null sets of a collection of Borel positive measures.

Theorem 1.1.16 *Let J be an ideal of subsets of ω^ω such that for some collection of Borel positive measures M on ω^ω we have $J = Null(M)$, i.e., $A \in J$ iff for all $\mu \in M$, $\mu(A) = 0$. Assume also that J is Π_1^1 on the codes of Σ_1^1 sets. Then there is a largest Π_1^1 set in J , which is characterized by*

$$C = \{\alpha \in \omega^\omega : \exists T \in \mathrm{L}_{\omega_1^\alpha} (T \textit{ is a } \xi\textit{-code for some } \xi < \omega_1^\alpha)(\alpha \in B_T \ \&\ B_T \in J)\}.$$

Proof: Since the relation $B_T \in J$ is Π_1^1 , then from the observation above we get that C is Π_1^1. Next we show that C is in J. This is like proving that J is Π_1^1 additive (see [8]). Suppose not, let $\mu \in M$ be a Borel measure such that $\mu(C) > 0$. Define the following prewellordering on C:

$$\alpha \leq \beta \text{ iff } \alpha,\ \beta \in C \text{ and } \omega_1^\alpha \leq \omega_1^\beta.$$

Since the relation " $\omega_1^\alpha \leq \omega_1^\beta$" is Σ_1^1, then $\leq$ is μ-measurable.

For each $\alpha \in C$, we have that $\{\beta \in \omega^\omega : \beta \leq \alpha\} \subseteq \bigcup\{B : B$ is a Borel set with a ξ-code in $L_{\omega_1^\alpha}$ for some $\xi < \omega_1^\alpha$, and $B \in J\}$. But since $L_{\omega_1^\alpha}$ is countable and every set in J has μ measure zero, then $\mu\{\beta : \beta \leq \alpha\} = 0$. Hence by Fubini's theorem we get that for almost all α , $\mu\{\beta : \alpha \leq \beta\} = 0$, which contradicts that $\mu(C) > 0$.

Finally we show that every Π^1_1 set in J is a subset of C. Let A be a Π^1_1 set in J and let T be a recursive tree on $\omega \times \omega$ such that : $\alpha \in A$ iff $T(\alpha)$ is wellfounded. Fix $\alpha \in A$ and let $\xi = \mid T(\alpha) \mid$, then $\xi < \omega_1^\alpha$. Consider $B = \{\beta \in \omega^\omega : \mid T(\beta) \mid \leq \xi\}$. B is a Borel subset of A and hence $B \in J$. By standard descriptive set theory we can find a ξ-code for B in $L_{\omega_1^\alpha}$ (see 8G.5 in Moschovakis's book). This finishes the proof of the theorem.

□

Corollary 1.1.17 *Let E be a Δ^1_1 equivalence relation on ω^ω. Then the largest Π^1_1 sparse set is*

$$C = \{\alpha \in \omega^\omega : \exists T \in \mathrm{L}_{\omega_1^\alpha} (T \text{ is a } \xi\text{-code for some } \xi < \omega_1^\alpha \text{ such that}$$

$$\alpha \in B_T \;\&\; B_T \text{ is smooth })\}$$

□

Remark: It is clear that every thin set is sparse, hence $C_1 \subseteq C$. We have defined in §1.2 a Π^1_1 set D, which is the largest strongly Δ^1_1-separated set. Clearly D is sparse and thus $D \subseteq C$. However, we will see in the next section that in L for $E = \equiv_T$ (Turing equivalence) C_1 is not strongly $\boldsymbol{\Delta}^1_1$-separated. Therefore, in this case $C_1 \not\subseteq D$ and hence $D \neq C$.

1.1.5 The case of a countable Borel equivalence relation

In this section we will look at the particular case of a countable Borel equivalence relation, i.e., one for which every equivalence class is countable. Typical examples are equivalence relations generated by a Borel automorphism (i.e., hyperfinite equivalence relations), and more generally by the action of a countable group of Borel automorphisms. In fact, a theorem of Feldman-Moore (see [5]) says that for every countable Borel equivalence relation E on a Polish X there is a countable group G of Borel automorphisms of X such that $E = E_G$, where

$$xE_Gy \text{ iff } g(x) = y, \text{ for some } g \in G.$$

It is a classical fact that for every Borel subset B of X there is a Polish topology τ, extending the given topology of X, for which B is τ-clopen. Moreover, τ admits a basis consisting of Borel sets with respect to the original topology of X. Thus the Borel structure of X is not changed. As a corollary we get that for every countable Borel equivalence relation E there is a Polish topology τ and a countable group G of τ-homeomorphisms of X such that $E = E_G$, τ extends the original topology of X and the Borel structure of X remains the same. We will use this fact to study the smooth sets with respect to a countable Borel equivalence relation. We will prove that, for a countable Borel equivalence relation, an arbitrary set is strongly Borel separated iff it is contained in a smooth Borel set.

First, we will show an effective version of the result from topology mentioned above. We will follow the exposition given in [17]. The definition of these topologies is by induction on the complexity of the Borel set. The first step is

Proposition 1.1.18 *Let τ_0 be the topology on X, and F a τ_0-closed subset of X. Put*

$$\tau = \tau_0 \cup \{V \cap F : V \in \tau_0\};$$

τ is the least Polish topology extending τ_0 for which F is τ-clopen.

Proof: This follows from the proof of Lemma 4 in [17]. □

Next, we handle countable unions of Polish topologies.

Proposition 1.1.19 *Let τ_n be a sequence of Polish topologies on X with $\tau_0 \subseteq \tau_n$ for all $n \geq 1$. Let τ_∞ be the topology generated by the collection $\{\bigcap_{i=1}^k G_i : G_i \in \tau_i, k \in \mathbf{N}\}$. τ_∞ is the least Polish topology which extends every τ_n, for $n \geq 0$.*

Proof: See lemma 3 in [17].

□

We have

Theorem 1.1.20 *For every Borel set $B \subseteq X$ there is a Polish topology τ_B extending τ_0 such that B is τ_B-clopen. Moreover, there is a total recursive function $h : \omega^\omega \times \omega^\omega \to \omega^\omega$ such that if γ is a Borel code of B, then $\{h(\gamma, n) : n \in \mathbf{N}\}$ is a collection of Borel codes for a basis of τ_B, where B is the Borel set coded by γ. In particular (X, τ_0) and (X, τ_B) have the same Borel structure.*

Proof: We will sketch the proof to make clear that such function h exists.

For each Borel code $\gamma \in BC$ we will define a topology τ_γ by induction on the definition of γ.

Case (i) $\gamma(0) = 0$. In this case $\pi_0(\gamma) = N(X, \gamma(1))$ (see the definition of the Borel codes in §1.4), thus by proposition 1.1.18

$$\tau_\gamma = \tau_0 \cup \{\pi_0(\gamma) \bigcap N(X, m) : m \in \mathbf{N}\}$$

is a basis for the topology for $\pi_0(\gamma)$.

Case (ii). Suppose we have defined the topology τ_γ for every $\gamma \in B_\xi$ with $\xi < \eta$, as in the statement of the theorem.

Let $\gamma \in B_\eta$. Then $\pi_\eta(\gamma) = \bigcup_n X - \pi_{\xi(n)}((\gamma^*)_n)$. Let τ_∞ be the topology generated by $\{\bigcap_{n=1}^k V_n : V_n \in \tau_{(\gamma^*)_n},\ k \in \mathbf{N}\}$. By proposition 1.1.19 $\pi_\eta(\gamma)$ is τ_∞-open. To make it closed we apply again proposition 1.1.18 to $X - \pi_\eta(\gamma)$ and we get the desired Polish topology for $\pi_\eta(\gamma)$.

In order to show that we can effectively find Borel codes for the basis of these topologies recall there are total recursive functions f_1 , $f_2 : \omega^\omega \to \omega^\omega$ such that if $\gamma \in BC$, then $f_1(\gamma)$ is a Borel code for $X - \pi(\gamma)$; and if $\gamma_1, ..., \gamma_k$ are Borel codes, then $f_2(< \gamma_1, ..., \gamma_k >)$ is a Borel code for $\pi(\gamma_1) \bigcup .... \bigcup \pi(\gamma_k)$.

Using these two functions and by a standard application of the Kleene recursion theorem one can show there is a total recursive function $h : \omega^\omega \times \to \omega^\omega$ as in the statement of the theorem.

□

Corollary 1.1.21 *The collection of Δ^1_1 sets forms a basis for a Polish topology τ such that every Δ^1_1 set is τ-clopen.*

Proof: From the previous theorem we get that for every Δ^1_1 set A there is a Polish topology τ which admits a basis consisting of Δ^1_1 sets and such that A is τ-clopen. Now apply 1.1.18 and 1.1.19 (as in the proof of the previous theorem) and observe that the basis given there consists of Δ^1_1 sets.

□

Remark : Let us observe that the basis given by the previous theorem may contain the empty set. In fact, there is not an effective way of enumerating a basis for these topologies without including the empty set. This is because one needs to determine whether or not two Borel sets have non empty intersection.

As we mentioned at the begining of this section, a corollary of this result is the following

Theorem 1.1.22 *Let g be a Δ^1_1 automorphism of X. There is a Polish topology τ on X extending the given topology on X such that g is a τ-homeomorphism. Moreover, τ admits a basis consisting of Δ^1_1 sets effectively enumerated.*

Proof: Let $\{V_n : n \in \mathbf{N}\}$ be an enumeration of a basis for the given topology on X. We want to close this collection under g, g^{-1}. This can be done by virtue of 1.1.20 as follows: For each n let σ_n,ρ_n be the Polish topologies given by 1.1.20 for the Borel set $g^{-1}(V_n)$ and $g(V_n)$ respectively. Let τ_1 be the Polish topology given by 1.1.19 for the collection $\{\sigma_n, \rho_n : n \in \mathbf{N}\}$. Repeat this process now starting with τ_1. After countable many iterations we will get the desired topology.

We will show next that this can be done effectively. We will do it only for g, since it is analogous for g^{-1}. As g is Δ^1_1 then g is effectively Borel (see 7B.9 in [15]). Hence there is a total recursive function $h_1 : \omega^\omega \to \omega^\omega$ such that whenever γ is a Borel code for a set A then $h_1(\gamma)$ is a Borel code for $g^{-1}(A)$.

Fix a recursive enumeration of Borel codes for a basis of the given topology on X, say $\{\gamma_n : n \in \mathbf{N}\}$. Define by simultaneous recursion functions h_2, τ ,σ as follows:

$$h_2(0, n) = h_1(\gamma_n).$$

$$\tau(0, n, i) = \text{Borel code of the } i^{th} \text{ open set in the basis}$$

of the topology given by 1.1.20 for the Borel
set coded by $h_2(0,n)$.

$\sigma(0,i) =$ Borel code of the i^{th} open set in the basis
for the topology given by 1.1.19 for
$\{\tau(0,n) : n \in \mathbf{N}\}$, where $\tau(0,n)$ is the
topology generated by $\{\tau(0,n,i) : i \in \mathbf{N}\}$.

In general we define

$$h_2(m+1,n) = h_1(\sigma(m,n)).$$

$\tau(m+1,n,i)=$ Borel code of the i^{th} open set in the basis
of the topology given by 1.1.20 for the Borel
set coded by $h_2(m,n)$.

$\sigma(m+1,n,i) =$ Borel code of the i^{th} open set in the basis
for the topology given by 1.1.19 for
$\{\tau(m,n) : n \in \mathbf{N}\}$, where $\tau(m,n)$ is the
topology generated by $\{\tau(m,n,i) : i \in \mathbf{N}\}$.

It is clear that except for the initial value $\{h_1(\gamma_n) : n \in \mathbf{N}\}$ all these functions are recursive, as can be easily shown as in the proof of 1.1.20.

Finally, since the topologies $\sigma_m = \{\sigma(m,i) : i \in \mathbf{N}\}$ are increasing, then as in 1.1.19 we conclude that $\bigcup_m \sigma_m$ is a basis for a Polish topology extending each σ_m. By construction g is a τ-homeomorphism.

□

The Feldman-Moore result quoted above has an effective proof. That is to say: If E is a Δ^1_1 countable equivalence relation, then there is a countable group G of Δ^1_1-automorphisms of X such that $E = E_G$. Moreover, there is a Δ^1_1 relation $R(x, y, n)$ on $X \times X \times \omega$ such that for all n, R_n is a graph of some $g \in G$. And vice versa, for all $g \in G$ there is n such that $\text{graph}(g) = R_n$. By an abuse of the language we will say that the relation $R(g, x, y) \Leftrightarrow g(x) = y$ is Δ^1_1.

Notice that in this case if $Q(x)$ is a Δ^1_1 relation, then $(\exists g \in G)Q(g(x))$, $(\forall g \in G)Q(g(x))$ are also Δ^1_1. In other words $(\exists y \in [x]_E)Q(y)$ and $(\forall y \in [x]_E)Q(y)$ are Δ^1_1.

We have also an analog of 1.1.22 for a countable group of Δ^1_1 automorphisms. Let $R(x, y, n)$ be a Δ^1_1 enumeration of G as above, then there is a Polish topology τ extending that on X such that every $g \in G$ is a τ-homeomorphism and τ admits a basis of Δ^1_1 sets effectively enumerated. The proof of this is as in 1.1.22; we only need to observe that G is, in this case, uniformly effectively Borel. That is to say: if g_n is the n^{th} element of G, i.e., $g_n(x) = y$ iff $R(x, y, n)$, then there is a recursive map $v : \omega \times \omega \to \omega^\omega$ such that for all s and n, $v(s, n)$ is a Borel code of $g_n^{-1}(N(X, s))$. Such a function v can be defined using the Souslin-Kleene theorem as follows: Put

$$P(x, s, n) \text{ iff } \exists y \in \Delta^1_1(x)[R(x, y, n) \& y \in N(X, s)]$$
$$\text{iff } (\exists y)[R(x, y, n) \& y \in N(X, s)].$$

By the theorem of restricted quantification (see 4D.3 in [15]) P is Δ^1_1, so let ε_1,ε_2 be recursive elements of ω^ω such that

$$P(x,n,s) \text{ iff } U(x,n,s,\varepsilon_1) \text{ iff } \sim U(x,n,s,\varepsilon_2)$$

where U is a good Σ^1_1 universal set. By the Souslin-Kleene theorem and the s-m-n theorem we can easily get a recursive function v which computes a Borel code for $P_{n,s}$ from n,s, ε_1 and ε_2.

We are also interested in computing effectively a Polish topology for which $[B]_E$ is clopen, where B is a Borel set and E is a countable equivalence relation. This is done in the following

Proposition 1.1.23 *Let E be a Δ^1_1 countable equivalence relation on X, $B \subseteq X$ a Δ^1_1 set and G a countable group of Δ^1_1 automorphisms of X such that $E = E_G$ with "$g(x) = y$ " a Δ^1_1 relation (as it was explained above). There is a Polish topology τ extending that on X such that every $g \in G$ is a τ-homeomorphism and $[B]_E$ is τ-clopen. Moreover, τ admits a basis of Δ^1_1 sets effectively enumerated.*

Proof: As we have remarked above, G is uniformly effectively Borel, hence we can find effectively a Borel code for $[B]_E$ and a Polish topology τ_1 for which $[B]_E$ is τ_1-clopen. By the remark above we also can find effectively a Polish topology τ_2 for which every $g \in G$ is a τ_2-homeomorphism. By 1.1.19 there is a Polish topology τ extending τ_i, i=1,2. This is the desired topology.

□

Definition 1.1.24 *Let E be a countable Borel equivalence relation and B a Borel subset of X. We call the topology given by 1.1.23 the* canonical

Polish topology for $[B]_E$. *Even in the case that B is E-invariant we say the topology for $[B]_E$, to distinguish it from the topology given by 1.1.20 which we call the* canonical Polish topology for B.

Now we start applying this result to study the smooth sets. The following definitions will play a crucial role in the sequel.

Definition 1.1.25 *Let τ be a Polish topology on X. Put*

$$P(\tau) = \{x \in X : [x]_E \text{ has an isolated point with respect to } \tau\ \}$$

i.e., $x \in P(\tau)$ *iff* $(\exists V \in \tau)(\mid V \cap [x]_E \mid = 1)$.

Remark: In the case of E generated by a single homeomorphism of (X, τ), $X - P(\tau)$ is a generalization of the notion of recurrent points (see [17]).

Definition 1.1.26 *For each countable collection $\mathcal{A} = (A_n)$ of E-invariant sets put*

$$D_{\mathcal{A}} = \{x \in X : [x]_E = [x]_{\mathcal{A}}\}$$

i.e., $x \in D_{\mathcal{A}}$ *iff* $(\forall y)(xEy \longleftrightarrow x\, E_{\mathcal{A}}\, y)$ *where*

$$x\, E_{\mathcal{A}}\, y \text{ iff } (\forall n)(x \in A_n \longleftrightarrow y \in A_n).$$

Notice that a set B is strongly separated by $\mathcal{A}$ iff $B \subseteq D_{\mathcal{A}}$. The following lemma will be very important in the sequel.

Lemma 1.1.27 *Let E be a countable equivalence relation on X, τ a Polish topology on X with basis $\{W_n : n \in \mathbf{N}\}$ such that the E-saturation of every τ-open set is τ-open. Put $B_n = [W_n]_E$ and $\mathcal{B} = (B_n)$. Then $P(\tau) = D_{\mathcal{B}}$.*

Proof: First we prove that if $y \notin D_{\mathcal{B}}$, then $y \notin P(\tau)$. It suffices to show that if $x \notin D_{\mathcal{B}}$ and $x \in W_n$, then $| W_n \bigcap [x]_E |> 1$. This is because if $y \notin D_{\mathcal{B}}$ and $W_n \bigcap [y]_E \neq \emptyset$, say $x \in W_n \bigcap [y]_E$, then as $D_{\mathcal{B}}$ is invariant $x \notin D_{\mathcal{B}}$, and so $| W_n \bigcap [y]_E |=| W_n \bigcap [x]_E |> 1$.

So, suppose $x \notin D_{\mathcal{B}}$ and let y be such that $x\, E_{\mathcal{B}}\, y$ but $x \not\!E y$. Let n be such that $x \in W_n$. So, in particular $W_n \neq \{x\}$. Otherwise $x \in D_{\mathcal{B}}$ (let us observe that (X, τ) can have isolated points). As $y \in [W_n]_E$, there is $w \in W_n$ with yEw. Clearly $x \not\!E w$ and $x\, E_{\mathcal{B}}\, w$. Put $V = [W_n]_E - \{x\}$; V is τ-open and $V \cap W_n \neq \emptyset$. Thus there is m such that $w \in W_m \subseteq V \cap W_n$, but as $x\, E_{\mathcal{B}}\, w$ then $x \in [W_m]_E$. Therefore for some $z \in W_m$ zEx. Clearly $x \neq z$, hence $| W_n \bigcap [x]_E |> 1$, i.e., $x \notin P(\tau)$.

Second, we show that if $x \in D_{\mathcal{B}}$ then $x \in P(\tau)$. Let $x \in D_{\mathcal{B}}$. Then $[x]_E = [x]_{\mathcal{B}}$ and hence $[x]_E = \{y : (\forall n)(x \in B_n \leftrightarrow y \in B_n)\}$. As each B_n is τ-open, $[x]_E$ is a τ-G_δ set. Since $[x]_E$ is countable, by the Baire category theorem we conclude that $[x]_E$ has a τ-isolated point, i.e., $x \in P(\tau)$.

□

Remark: $P(\tau) \subseteq D_{\mathcal{B}}$ is always true, without assuming that E is countable.

Our first aplication is the following

Theorem 1.1.28 *Let τ be a Polish topology on X with a basis consisting of Borel sets with respect to the original topology on X. Let G be a countable group of τ-homeomorphisms of X and $E = E_G$. Then a τ-G_δ E-invariant set H is E-smooth iff $H \subseteq P(\tau)$.*

Proof: Let B be as in lemma 1.1.27, then $P(\tau) \subseteq D_{\mathcal{B}}$. As each element of the basis of τ is Borel, we get that $P(\tau)$ is strongly Borel separated.

On the other hand, suppose H is E-smooth, by a result of Effros [4] we get that for every $x \in H$, $[x]_E$ is τ-locally closed in H. But as H is τ-G_δ and $[x]_E$ is countable, then $[x]_E$ has a τ-isolated point, i.e., $x \in P(\tau)$.

□

As a corollary we get the following characterization of Borel smooth sets.

Corollary 1.1.29 *Let E be a $\mathbf{\Delta}^1_1$ countable equivalence relation on X and B a $\mathbf{\Delta}^1_1$ subset of X. Let τ_B be the canonical Polish topology for $[B]_E$ (given by 1.1.23). Then B is smooth iff $B \subseteq P(\tau_B)$.*

Proof: Since $[B]_E$ is τ_B-clopen, by the previous theorem $[B]_E$ is smooth iff $[B]_E \subseteq P(\tau_B)$. And by 1.1.4 B is smooth iff $[B]_E$ is smooth. Finally observe that $P(\tau)$ is an invariant set, thus $B \subseteq P(\tau_B)$ iff $[B]_E \subseteq P(\tau_B)$.

□

Remark: (i) This corollary can be seen as a Borel analog of 1.1.8. That is to say for Borel smooth sets $P(\tau)$ plays the same role as D does for Σ^1_1 smooth sets. We will show below that in this case we have that $D = P(\tau)$ for some topology.

(ii) On the other hand this is a generalization of a result of Weiss (see [17]) which says that the equivalence relation induced by an aperiodic homeomorphism is not smooth iff there is a recurrent point.

Our next theorem answers a question raised in §2.

Theorem 1.1.30 *Let E be a countable Δ^1_1 equivalence relation on a recursively presented Polish space X. Let D be the set defined on 1.1.8 and ρ be the Polish topology generated by the Δ^1_1 sets (see 1.1.21). Then*

(i) $D = P(\rho)$

(ii) $D = \bigcup\{A : A$ *is a* Δ^1_1 *smooth set*$\}$

Proof: Let us show first that (i) implies (ii). Let $x \in D$. We want to show that there is a Δ^1_1 smooth set A with $x \in A$. Since $[x]_E$ has a ρ-isolated point, let B be a Δ^1_1 set such that $\mid B \cap [x]_E \mid = 1$. Put $A = \{y :\mid B \cap [y]_E \mid = 1\}$. It is easy to check that A is Δ^1_1: just recall that $\exists z \in [y]_E$ and $\forall z \in [y]_E$ are number quantifiers. Clearly $A \subseteq P(\rho) = D$, so A is smooth and $x \in A$.

Let $\mathcal{A} = (A_n)$ be the collection of Δ^1_1 invariant sets. By 1.1.8 we know that $D = D_{\mathcal{A}}$. For every Δ^1_1 set A, $[A]_E$ is Δ^1_1. Hence from 1.1.27 we get that $D = P(\rho)$.

□

As we have observed before, the previous theorem implies that strong Borel separation and smoothness are equivalent. This can also be proved directly as we show next. But first let us notice that the previous theorem can be extended to Borel equivalence relations generated by the action of a locally compact group of Borel automorphisms of X by using a theorem of Kechris that says that these equivalence relations admit a Borel quasitransversal (that is to say a Borel set B such that for every $x \in X$, $B \cap [x]_E$ is countable).

Theorem 1.1.31 *Let* E *be a* Δ^1_1 *countable equivalence relation on* X *and* C *be an arbitrary subset of* X*. The following are equivalent*

(i) There is a Δ^1_1 *invariant smooth set* B *with* $C \subseteq B$*, i.e.,* C *is* E*-smooth.*

(ii) C *is strongly* Δ^1_1 *separated by a collection of* Δ^1_1 *sets which is uniformly* Δ^1_1.

Hence, by relativization, we get that a subset of X *is smooth iff it is strongly Borel separated.*

Proof: (i)$\Rightarrow$ (ii) is a consequence of 1.1.4, as Δ^1_1 **smooth** set are clearly Δ^1_1 strongly separated.

(ii)$\Rightarrow$ (i). Let $\mathcal{A} = (A_n)$ be a collection of Δ^1_1 invariant sets such that C is strongly separated by $\mathcal{A}$. As we have observed before this is equivalent to say that $C \subseteq D_{\mathcal{A}}$. The idea of the proof is to find another collection $\mathcal{B}$ of Δ^1_1 sets such that $D_{\mathcal{B}}$ is a Δ^1_1 set and $D_{\mathcal{A}} \subseteq D_{\mathcal{B}}$. With this idea in mind we introduce the following partial order: Given two collections of invariant subsets of X, say $\mathcal{A} = (A_n)$ and $\mathcal{B} = (B_n)$, we say that $\mathcal{A} \leq \mathcal{B}$ if for all $n \in \mathbf{N}$, there is a sequence $(n_i)_i$ such that $A_n = \bigcup_i B_{n_i}$, i.e., $\mathcal{B}$ "refines" $\mathcal{A}$.

First we have the following

Claim: (i) if $\mathcal{A} \leq \mathcal{B}$, then $E_{\mathcal{B}} \subseteq E_{\mathcal{A}}$.

(ii) if $E_{\mathcal{B}} \subseteq E_{\mathcal{A}}$, then $D_{\mathcal{A}} \subseteq D_{\mathcal{B}}$.

Proof: (i) Let x,y be such that $x \, E_{\mathcal{B}} \, y$, fix $n \in \mathbf{N}$ and $(n_i)_i$ such that $A_n = \bigcup_i B_{n_i}$. Then we have the following equivalences

$$x \in A_n \text{ iff } (\exists i)(x \in B_{n_i}) \text{ iff } (\exists i)(y \in B_{n_i}) \text{ iff } y \in A_n$$

(ii) As $E_{\mathcal{B}} \subseteq E_{\mathcal{A}}$ then $[x]_{\mathcal{B}} \subseteq [x]_{\mathcal{A}}$. Hence $D_{\mathcal{A}} \subseteq D_{\mathcal{B}}$.

($\Box$ claim)

By the results at the begining of this section there is a Polish topology τ and a countable group G of τ-homeomorphisms of X such that $E = E_G$, every A_n is a τ-clopen set and τ admits a basis of Δ^1_1 sets, say $\{W_n : n \in \mathbf{N}\}$. Put $B_n = [W_n]_E$ and $\mathcal{B} = (B_n)$. Since each A_n is τ-open and invariant, then $\mathcal{A} \leq \mathcal{B}$. Hence by the claim $D_{\mathcal{A}} \subseteq D_{\mathcal{B}}$ and by hypothesis $C \subseteq D_{\mathcal{A}}$. From lemma 1.1.27 we get that $D_{\mathcal{B}} = P(\tau)$ and we easily see that $P(\tau)$ is Δ^1_1, in fact

$$x \in P(\tau) \text{ iff } (\exists n)(\exists g \in G)[g(x) \in W_n \& (\forall h \in G)(h(x) \in W_n \rightarrow h(x) = g(x))].$$

Observe now that as $\mathcal{A}$ is effectively Δ^1_1 (i.e., each A_n is Δ^1_1 and the relation "$x \in A_n$" is Δ^1_1) the basis for τ is effectively enumerated and hence $P(\tau)$ is Δ^1_1 (recall here the observation we made about the quantifier $\exists g \in G$ after theorem 1.1.22).

□

Remark: (i) For a $\mathbf{\Sigma}^1_1$ set C the conclusion of the previous theorem follows from theorem 1.1.4.

(ii) We do not know if this theorem holds for any Borel equivalence relation. Recall the remark we made after 1.1.14, i.e., if $D = \{x : E_x = \overline{E}_x\}$ is equal to $\bigcup\{A : A \text{ is a } \Delta^1_1 \text{ smooth set }\}$, then every strongly Δ^1_1-separated set is contained in a smooth Borel set, namely D.

Here is the corollary we have mentioned after 1.1.17

Corollary 1.1.32 *Let $E = \equiv_{\mathrm{T}}$. Then in* L, *C_1 is not strongly Borel separated.*

Proof: It follows from 1.1.31 and 1.1.11, which says that (in L) C_1 is not contained in a Borel smooth set.

□

For a Borel equivalence relation induced by the action of a Polish group, a Borel set A is smooth iff it has a Borel transversal. That is to say, there is a Borel subset T of X such that for all $x \in T$, $[x] \cap T = \{x\}$ and $A \subseteq [T]_E$ (see [2]). We will show below an effective version of this fact for a countable Borel equivalence relation E.

By the Feldman-Moore theorem and the result at the begining of this section there is a Polish topology τ on X and a countable group G of τ-homeomorphisms of X such that $E = E_G$. Also τ admits a basis $\{W_n : n \in \mathbf{N}\}$ consisting of Borel sets. Define the following relation on $X \times \omega$:

$R(x, m)$ iff m is the least n such that $| W_n \cap [x]_E |= 1$ (if it exists). (*)

Now, we have

$$R(x,m) \text{ iff } (\exists g \in G)[(g(x) \in W_m)\&(\forall f \in G)(f(x) \in W_m \rightarrow f(x) = g(x))]$$

$$\&(\forall n < m)[(\exists g \in G)(g(x) \in W_m) \rightarrow (\exists h \in G)(h(x) \in W_m \& h(x) \neq g(x))]$$

Thus R is Borel and clearly $P(\tau) = \exists^\omega R$. Put

$$x \in T \text{ iff } (\exists m)R(x,m) \ \& \ x \in W_m \qquad (**)$$

It is easy to check that T is a transversal for $P(\tau)$. From this we get the following

Theorem 1.1.33 *Let E be a Δ^1_1 countable equivalence relation on X. There is a total recursive function F such that whenever γ is a Borel code for a Borel smooth set B, then $F(\gamma)$ is a Borel code for a transversal for B.*

Proof: If B is a Borel smooth set, then by corollary 1.1.29 $B \subseteq P(\tau_B)$. Let T_B be the transversal given by (**). Since τ_B has an effectively enumerated basis (1.1.23) then we can effectively get a Borel code for T_B from its definition (**). Finally $T_B \cap [B]_E$ is a transversal for B.

$\square$

Remark: The preceding result can be actually generalized to the more general context of a Borel equivalence relation with K_σ equivalence classes. This is a corollary of the results on [6]. The argument is as follows:

Let B be a smooth invariant Borel set and let $F : X \to 2^\omega$ be a Borel map such that: $(\forall x, y \in B)(F(x) = F(y) \leftrightarrow xEy)$.

We will define a transversal for B. We can assume without loss of generality that B is Δ^1_1 and that F is Δ^1_1-recursive. Fix $x \in B$ and let $y = F(x)$. Then $[x]_E = F^{-1}(y)$ and hence $[x]_E$ is $\Delta^1_1(y)$. Since $[x]_E$ is K_σ, there is a $z \in [x]_E$ with $z \in \Delta^1_1(y)$ (see 4F.15 in [15]). We want to chose such a z in a Δ^1_1 canonical way.

First, let us observe that the argument above shows that $F[B]$ is Δ^1_1. In fact:

$$y \in F[B] \quad \text{iff} \quad (\exists x)(x \in B \ \& \ F(x) = y)$$
$$y \in F[B] \quad \text{iff} \quad (\exists x \in \Delta^1_1(y))(x \in B \ \& \ F(x) = y).$$

Hence by the theorem of restricted quantification (see 4D.3 in [15]) we get that $F[B]$ is both Σ^1_1 and Π^1_1. Now, consider the following relation:

$$P(y, x) \quad \text{iff} \quad x \in B \ \& \ F(x) = y.$$

For every $y \in F[B]$, $P_y = [x]_E$, hence it is a K_σ set. Therefore there is a Δ^1_1 uniformizing function G (see 4F.16 in [15]), i.e., for every $y \in F[B]$, $P(y, G(y))$. Put $H(x) = G(F(x))$. H is clearly E-invariant on B, i.e., for every $x, x' \in B$, if xEx' then $H(x) = H(x')$. Hence the set $\{x \in B : H(x) = x\}$ is a Δ^1_1 transversal for B.

We finish this section by looking at the particular case of closed smooth sets.

Proposition 1.1.34 *Let E be a Borel equivalence relation on X generated by a countable group of homeomorphisms of X. Assume also that for every $x \in X$, $[x]_E$ is dense in X. Let F be a closed smooth set. Then* *(i)* $[F]_E = P(\tau_F)$, *where* τ_F *is the canonical Polish topology for* $[F]_E$ *given by 1.1.23.*

(ii) F *has a* $\mathbf{\Sigma}^0_4$ *transversal.*

Proof: Both results are based in the following fact about the canonical Polish topology for F_σ sets.

Lemma: Let $F = \bigcup_n F_n$ be an F_σ set. Let τ_0 be the given topology on X, put $H = X - F$ and let τ be the topology generated by

$$\tau_0 \cup \{\bigcap_{i=1}^{n} F_{k_i} \cap W : W \in \tau_0 \text{ and } n, k_i \in \mathbf{N}\} \cup \{H \cap V : V \in \tau\}.$$

Then τ is the canonical topology for F.

Proof: For each $n \in \mathbf{N}$, let τ_n be the topology generated by $\tau_0 \cup \{V \cap F_n : V \in \tau_0\}$. By proposition 1.1.18 F_n is τ_n-clopen. Let τ_∞ be the topology generated by $\{\bigcap_{i=1}^n V_i : V_i \in \tau_i\}$. By proposition 1.1.19 F is τ_∞-open, and every $\tau_n \subseteq \tau_\infty$. It is easy to check that τ_∞ is generated by $\tau_0 \cup \{\bigcap_{i=1}^n F_{k_i} \cap W : W \in \tau_0; n, k_i \in \mathbf{N}\}$. Let τ be the topology generated by $\tau_\infty \cup \{H \cap V : V \in \tau_\infty\}$. By 1.1.18 F is τ-clopen, and it is easy to check that τ is actually generated by $\tau_\infty \cup \{H \cap V : V \in \tau_0\}$.

(Lemma □)

Now we start the proof of the proposition. Let G be a group of homeomorphisms of X which generate E. Then $[F]_E = \bigcup_{g \in G} g[F]$. Let τ by the topology generated by

$$\tau_0 \cup \{\bigcap_{i=1}^{n} g_i[F] \cap V : V \in \tau_0, g_i \in G\} \cup \{H \cap V : V \in \tau_0\},$$

where $H = X - [F]_E$. By the lemma $[F]_E$ is τ-clopen, and it is easy to see that also every $g \in G$ is a τ-homeomorphism (just observe that H is

E-invariant). Hence we actually have that τ is the canonical topology for $[F]_E$. By corollary 1.1.29 $[F]_E \subseteq P(\tau)$. Conversely if $x \notin [F]_E$ and $V \in \tau$ we will show that either $V \cap [x]_E = \emptyset$ or it is infinite. There are three cases:

(a) if $V \in \tau_0$,then by hypothesis $[x]_E \cap V$ is dense in V.

(b) if $V = H \cap W$ with $W \in \tau_0$, then as H is invariant the same argument shows that $V \cap [x]_E$ is infinite.

(c) Finally if $V = \bigcap_{i=1}^{n} g_i[F] \cap W$, for some $W \in \tau_0$, then $V \cap H = \emptyset$. Therefore $x \notin P(\tau)$.

This proves (i).

(ii) From the lemma we get that τ admits a basis consisting of G_δ sets. From (*) and (**) above (just before 1.1.33) we easily get a $\mathbf{\Sigma}_4^0$ transversal for F.

□

Corollary 1.1.35 *Every closed smooth set with respect to* E_0 *has a* $\mathbf{\Sigma}_4^0$ *transversal.*

Proof: We only need to show that $E_0 = E_G$ for some group of homeomorphisms of 2^ω. For each finite sequence $n_1, ..., n_k \in \mathbf{N}$ define a function from 2^ω into 2^ω by

$$f_{n_1,\dots,n_k}(\alpha)(m) = \begin{cases} 1-\alpha(m) & \text{if } m = n_i \text{ for some } i \\ \alpha(m) & \text{otherwise} \end{cases}$$

These functions clearly work.

□

1.2 The σ-ideal of closed smooth sets

As we have already pointed out, theorem 1.1.4 implies that the notion of smoothness for $\mathbf{\Sigma}^1_1$ is concentrated on closed sets, i.e., a $\mathbf{\Sigma}^1_1$ set A is smooth iff every closed subset of A is smooth. In this part we will deal with the collection of closed smooth sets. To be more precise, let E be a Borel equivalence relation on a compact Polish space X. The collection of closed subsets of X, which is denoted by $\mathcal{K}(X)$, equipped with the Hausdorff topology is a Polish space. Let

$$I(E) = \{K \in \mathcal{K}(X) : K \text{ is smooth with respect to } E\}.$$

It is clear that $I(E)$ is a $\sigma-$ideal. We are interested in studying the complexity of $I(E)$ as well as some of its structural properties like calibration, the covering property and Borel basis. One of the results of this section is that E is smooth iff $I(E)$ is Borel. We will also look at the particular case of $I(E_0)$.

1.2.1 A definability result

A Π^1_1 $\sigma-$ideal I satisfies the so called dichotomy theorem (see [14]), namely either I is a true Π^1_1 set or a G_δ set. In the next theorem we compute the complexity of $I(E)$.

Theorem 1.2.1 *Let E be a non smooth Δ^1_1 equivalence relation on a compact Polish space X. Then $I(E)$ is a strongly calibrated, locally non Borel, Π^1_1 $\sigma-$ideal.*

Proof: It is clear that $I(E)$ is a $\sigma-$ideal and since the smooth sets are the common null sets of all E-ergodic, non atomic measures on X, by a standard capacitability argument we get that $I(E)$ is strongly calibrated. A similar argument as in the proof of 1.1.7 (i.e., the collection of $\mathbf{\Sigma}^1_1$ smooth sets is Π^1_1 on the codes of $\mathbf{\Sigma}^1_1$ sets) shows that $I(E)$ is Π^1_1.

To show that $I(E)$ is locally non Borel we need the following two lemmas.

Lemma A: *Let $f : 2^\omega \to X$ be a continuous embedding from E_0 into E. For every closed set $K \subseteq 2^\omega$*

$$K \in I(E_0) \textit{ iff } f[K] \in I(E).$$

Proof: Let $K \notin I(E_0)$ and put $E_1 = E_0 \lceil K$. By 1.1.4, $E_0 \sqsubseteq E_1$ via a continuous embedding. But clearly $E_1 \sqsubseteq E \lceil f[K]$ and $\sqsubseteq$ is transitive, hence $E_0 \sqsubseteq E \lceil f[K]$, i.e., $f[K] \notin I(E)$.

Conversely, suppose $K \in I(E_0)$ and let $\mathcal{A} = (A_n)$ be a separating family of Σ^1_1 sets for $E_0 \lceil K$. Put $B_n = f[A_n]$ and $\mathcal{B} = (B_n)$. We claim that $\mathcal{B}$ is a separating family for $E \lceil f[K]$. In fact: as f is 1-1 one easily gets that $(\forall x, y \in K)(f(x)\, E_{\mathcal{B}}\, f(y) \leftrightarrow x\, E_{\mathcal{A}}\, y)$. Hence $(\forall z, w \in f[K])(z\, E_{\mathcal{B}}\, w \leftrightarrow zEw)$. Therefore from 1.1.4 we get that $f[K]$ is E-smooth.

(□ lemma A)

Lemma B: $I(E_0)$ *is not Borel.*

We show first that this implies $I(E)$ is locally not Borel. Let $K \in \mathcal{K}(X)$ then we have that

$$I(E) \cap \mathcal{K}(K) = \{F \in \mathcal{K}(K) : F \text{ is } E\text{-smooth }\} = I(E \lceil K).$$

From lemma A we get that $I(E_0)$ is not Borel iff $I(E \lceil K)$ is not Borel. Now the conclusion follows from lemma B.

By the dichotomy theorem for $\sigma-$ideals (see [14]), it suffices to show that $I(E_0)$ is not G_δ. We will actually show that $I(E_0)$ has no non trivial $\mathbf{\Sigma}^1_1$

subideals. We recall here that every $\mathbf{\Sigma}^1_1$ σ-ideal is actually G_δ (see [14]).

Lemma 1.2.2 *For every $x \in 2^\omega$ there is a continuous map $f : 2^\omega \to \mathcal{K}(2^\omega)$ such that*

(i) if γ is eventually zero, then $f(\gamma)$ is a finite subset of $[x]_{E_0}$.

(ii) if γ is not eventually zero, then $f(\gamma)$ is a non-smooth closed set (with respect to E_0).

In other words, there is a continuous reduction of $\{\alpha \in 2^\omega : \alpha$ is eventually zero $\}$ into the collection of finite subsets of $[x]_{E_0}$ and $\sim I(E_0)$. In particular $I(E_0)$ is not G_δ.

Proof: Consider the following function

$$f(\gamma) = \{\alpha \in 2^\omega : (\forall n)(\gamma(n) = 0 \to \alpha(n) = x(n))\}.$$

Clearly if γ is eventually zero, then (i) holds. On the other hand if γ has infinite many 1's, then $f(\alpha)$ is a perfect set. Let $g : 2^\omega \to 2^\omega$ be the canonical bijection of 2^ω onto $f(\gamma)$. It is not difficult to see that g is actually an embedding from E_0 into $E_0 \lceil f(\gamma)$, i.e., for all $\alpha, \beta \in 2^\omega$

$$\alpha E_0 \beta \quad \text{iff} \quad g(\alpha) E_0 g(\beta).$$

Just observe that if T is the tree of $f(\gamma)$ and some sequence in T of length n splits, then every sequence in T of length n splits.

Finally, let us check that f is continuous. For each $s \in 2^{<\omega}$ put

$$A_s = \{\alpha \in 2^\omega : (\forall n < lh(s))(s(n) = 0 \Rightarrow \alpha(n) = x(n))\},$$

each A_s is closed and if $t \prec s$, then $A_s \subseteq A_t$. We have that $f(\gamma) = \bigcap_n A_{\gamma\lceil n}$ and also that for every $s \in 2^{<\omega}$

$$f(\gamma) \bigcap N_s \neq \emptyset \quad \text{iff} \quad \forall n < lh(s)(s(n) = 0 \Rightarrow \gamma(n) = x(n))$$

which easily implies that f is continuous.

($\square$ lemma 1.2.2)

To finish the proof of the theorem we just need to recall that by the Baire category theorem there are no countable dense G_δ sets. Hence lemma 1.2.2 says that $I(E_0)$ is not G_δ. $\square$ Theorem 1.2.1

As a corollary of lemma 1.2.2 we get the following

Corollary 1.2.3 *Let E be a non smooth Borel equivalence relation on X, then*

*(i) If $J \subseteq I(E_0)$ is a dense $\sigma-$*ideal, *then J is not $\mathbf{\Sigma}^1_1$.*

*(ii) If $J \subseteq I(E)$ is a $\sigma-$*ideal *such that for every $x \in X$ $\{x\} \in J$, then J is not $\mathbf{\Sigma}^1_1$.*

Proof: (ii) follows from (i), because if $f : 2^\omega \to X$ is an embedding witnessing that E is not smooth and $J \subseteq I(E)$ is a $\sigma-$ideal containing all singletons, then $J^* = f^{-1}[J]$ is a dense $\sigma-$ideal and it is contained in $I(E_0)$.

(i) Let J be as in the hypothesis of (i). As we said before it suffices to show that J is not G_δ. Suppose toward a contradiction that $J \subseteq I(E_0)$ is a G_δ dense $\sigma-$ideal. Let $H = \{x \in 2^\omega : \{x\} \in J\}$, H is a G_δ dense set. Let G be a countable collection of homeomorphisms of 2^ω generating E_0. Put $H^* = \bigcap_{g \in G} g[H]$, H^* is an invariant dense G_δ subset of H. Let $x \in H^*$. For every y such that yE_0x, we have $\{y\} \in J$. From lemma 1.2.2 we get that J is not a G_δ set, a contradiction.

$\square$

Remarks: (1) (i) above implies that there are no dense G_δ smooth sets with respect to E_0, because if H is such a set then $\mathcal{K}(H)$ would be a dense G_δ subideal of $I(E_0)$. Actually we will see in the next section that every Baire measurable E_0-smooth set is of the first category.

(2) (ii) above is best possible in the sense that there is a non smooth Borel equivalence relation E and a dense G_δ set H which is smooth with respect to E, hence as before we get $\mathcal{K}(H)$ is a dense Borel subideal of $I(E)$. Such an equivalence relation will be constructed in the next section.

(3) Kechris (see [12]) has proved that the $\sigma-$ideal of closed sets of extended uniqueness also satisfies this hereditary property but even in a stronger form, i.e., for every perfect set M of restricted multiplicity the $\sigma-$ideal $U_0 \bigcap \mathcal{K}(M)$ has no dense $\mathbf{\Sigma}^1_1$ subideals. We do not know if this holds for $I(E_0)$.

Since for E smooth $I(E)$ is trivial, we get the following nice characterization of a smooth Borel equivalence relation.

Corollary 1.2.4 *Let E be a Borel equivalence relation on X. Then E is smooth iff $I(E)$ is Borel.*

□

1.2.2 Relation between smoothness and category

In any topological space there is a natural notion of smallness : to be a set of first category. In this section we are interested in the relation between smoothness and category. We will show that in general we do not have that smooth sets are of first category, but it is true for some equivalence relations generated by the action of a collection of homeomorphisms. We will start with this case.

Let G be a collection of homeomorphisms of X. We say that G satisfies (*) if the following condition holds:

$$(\forall O \subseteq X \text{ open})(\exists g \in G)[(g[O] = O \ \& \ (\exists x \in O)(g(x) \neq x))]. \qquad (*)$$

For instance E_0 is generated by the following collection of homeomorphisms of 2^ω: For each $s, t \in 2^n$, $n \in \mathbf{N}$ let $f_{s,t} : 2^\omega \to 2^\omega$ defined by

$$f_{s,t}(\alpha) = \begin{cases} t^\frown\gamma & \text{if } \alpha = s^\frown\gamma \\ s^\frown\gamma & \text{if } \alpha = t^\frown\gamma \\ \alpha & \text{otherwise.} \end{cases}$$

This collection $\{f_{s,t} : s, t \in 2^n, n \in \mathbf{N}\}$ clearly generates E_0 and it satisfies (*).

Lemma 1.2.5 *Let E be an equivalence relation on X generated by a collection G of homeomorphisms of X which satisfies (*). Then for every open set $O \subseteq X$ and every dense G_δ subset H of O there are $x, y \in H$ with xEy and $x \neq y$, i.e., H is not a transversal.*

Proof: Let $g \in G$ such that $g[O] = O$ as in (*) and let $H_1 = g^{-1}[H]$. Then H_1 is a dense G_δ subset of O and so is $H_2 = H_1 \cap H$. By hypothesis $g \neq id$ on O, hence there is $z \in H_2$ with $g(z) \neq z$, i.e., H is not a transversal.

□

We immediately get the following

Corollary 1.2.6 *Let E be an equivalence relation generated by a collection G of homeomorphisms of X which satisfies (*). Then*

(i) Every transversal (with respect to E) with the property of Baire is of first category.

(ii) If in addition G is countable and E is Borel, then every smooth set is of first category.

Proof: (i) Let T be a transversal with the property of Baire. Thus there is an open set O such that $T \triangle O \subseteq F$, with F a set of first category. So, let $H \subseteq T$ be a G_δ set such that $\overline{H} = \overline{O}$. By the previous lemma this can only happen if O is empty i.e., T is of first category.

(ii) Let A be a smooth set and T a Borel transversal for A i.e., $A \subseteq \bigcup_{g \in G} g[T]$ (such T exists because E is a countable Borel equivalence relation, see 1.1.33). Then by (i) each $g[T]$ is of first category.

□

Since E_0 satisfies these conditions we immediately get

Corollary 1.2.7 *Every smooth set with respect to E_0 is of first category.*

□

One property that the majority of 'complicated' $\sigma-$ideals do not have is the c.c.c property or in other words they are not thin. Recall that an ideal I of closed sets is called thin if any disjoint collection of closed sets not in I is at most countable. From the corollary above we get

Corollary 1.2.8 *Let E be a non smooth Borel equivalence relation on X then $I(E)$ is not thin.*

Proof: First, it suffices to show it for $I(E_0)$. Because if $f : 2^\omega \to X$ is an embedding witnessing that E is not smooth, then we have seen in the proof of theorem 1.2.1 (lemma A) that for every K, $K \in I(E_0)$ iff $f[K] \in I(E)$. Hence if $I(E_0)$ is not thin, then $I(E)$ is not thin either.

Now for $I(E_0)$ it follows from a result in chapter 2 (see remark after 2.1.7) which says that if every Borel set in I^{int} is meager then I is not thin (where I is a $\sigma-$ideal and $B \in I^{int}$ if $\mathcal{K}(B) \subseteq I$).

□

Now, we will show there is a non-smooth Borel equivalence relation on 2^ω for which there is a dense G_δ smooth set (and hence of the second category).

Example 1.2.9 (A Σ_2^0 countable equivalence relation with a smooth dense G_δ)

Let $\{\beta_n : n \in \mathbf{N}\}$ be a countable dense subset of 2^ω. Put $\tilde{\beta_n}(m) = \beta_n(2m)$ and $F_n = \{< \tilde{\beta}_n, \gamma >: \gamma \in 2^\omega\}$, where $< \beta_n, \gamma > (2n) = \beta(n)$ and $< \beta_n, \gamma > (2n+1) = \gamma(n)$.

Claim: For every $n \in \mathbf{N}$, $\beta_n \in F_n$ and F_n is a locally non-smooth (for E_0) nowhere dense set.

Granting this claim we finish the argument. Let $F = \bigcup_n F_n$. As each F_n is nowhere dense and $\beta_n \in F_n$, F is dense and of the first category. Define E as follows

$$xEy \text{ iff } x = y \text{ or } (x, y \in F \ \& \ xE_0y).$$

Then E is clearly a Σ_2^0 equivalence relation and $E\lceil F = E_0\lceil F$. Hence E is not smooth. Put $H = 2^\omega - F$. H is a dense G_δ and a transversal for E.

So it remains to show the claim. It is clear that each F_n is meager and that $\beta_n \in F_n$. To see that each F_n is locally non-smooth, let $s \in 2^{<\omega}$ be such that $N_s \cap F_n \neq \emptyset$, say $< \tilde{\beta}_n, \gamma > \in N_s \cap F_n$. Let $t = \gamma\lceil lh(s)$. The map $\delta \mapsto < \tilde{\beta}_n, \delta >$ from N_t into $N_s \cap F_n$ is a continuous embedding and is easy to check that it preserves E_0.

Let us observe that every closed smooth set for E is nowhere dense. In fact: if $V \subseteq 2^\omega$ is open, put $V = \bigcup_n \overline{V}_n$ with V_n an open set. Since F is dense there are n,m such that $F_n \cap V_m \neq \emptyset$. As F_n is locally non-smooth (for E_0), so is $\overline{F_n \cap V_m}$, which easily implies that $\overline{F_n \cap V_m} \notin I(E)$. In particular, this implies that H cannot be covered by countably many sets in $I(E)$.

($\square$ example 1.2.9)

One of the consequences of 1.1.6 is that a $\mathbf{\Sigma}_1^1$ set A is smooth for E iff every closed subset of A is smooth, i.e., $A \in I(E)^{\text{int}}$. In the abstract setting of a $\sigma-$ideal I consisting of closed meager sets the question of whether or not a given $\mathbf{\Sigma}_1^1$ set in I^{int} is of first category is solved by proving that I has the covering property (see chapter 2 for the corresponding definitions). The example above shows that for some Borel equivalence relation E, $I(E)$ does not have the covering property and it is straightfoward to check that if $I(E_0)$ does not have the covering property, then for every non smooth Borel equivalence relation E, $I(E)$ does not have the covering property (just translate the counterexample with the embedding). However, since every smooth set with respect to E_0 is of first category, it is possible that $I(E_0)$ has the covering property. We will look at this question in the next section.

1.2.3 Some properties of $I(E_0)$

As we have said in the previous section it is quite natural to ask whether or not $I(E_0)$ has the covering property. The only criterion known to show this is the following

Theorem: (Debs-Saint Raymond [3]) *Let I be a Π_1^1 $\sigma-$*ideal *of compact sets. Suppose I is calibrated, locally non-Borel and has a Borel basis. Then I has* *the covering property.*

A proof of it can also be found in [13]. In view of theorem 1.2.1 we are left with the question of whether or not $I(E_0)$ has a Borel basis. In general, the question about the existence of Borel basis for a given ideal is a hard question, and for this particular case we do not know the answer yet.

From now on we will be working only with E_0. A possible candidate for a basis for $I(E_0)$ is the collection of closed transversals. Let $B = \{F \in \mathcal{K}(2^\omega) : F \text{ is a transversal }\}$, then we have

$$F \in B \quad \text{iff} \quad (\forall x, y)(x, y \in F \ \&\ xE_0y \rightarrow x = y).$$

Since the relation

$$R(x, y, F) \quad \text{iff} \quad (x, y \in F \ \&\ xE_0y \rightarrow x = y)$$

is Π^0_2, then we have that B is also Π^0_2. Denote by $I_t = (B)_\sigma$ the $\sigma-$ideal generated by B, observe that B is a dense set in $\mathcal{K}(X)$. By a result in [14] I_t is a Π^1_1 $\sigma-$ideal. Since it is a dense subset of $I(E_0)$ by 1.2.3 it is not Borel. The next propositions show a bit more about I_t, in particular we will see that $I_t \neq I(E_0)$. But before let us observe that as B is a dense G_δ set, $I(E_0)$ is not meager and therefore, by a result in [14], $I(E_0)$ does not have a Σ^0_2 basis.

Proposition 1.2.10 *I_t is a locally non-Borel Π^1_1 $\sigma-$ideal.*

Proof: We have already seen that I_t is a Π^1_1 $\sigma-$ideal. The proposition will easily follow from the following

Lemma : *Let F be a closed set which is locally not in B. There is a continuous function $f : 2^\omega \rightarrow \mathcal{K}(F)$ such that* *(i) if γ is eventually zero, then $f(\gamma)$ is finite.*

(ii) if γ is not eventually zero, then $f(\gamma)$ is locally not a transversal.

From this lemma we get that $I_t \cap \mathcal{K}(F)$ is not G_δ and hence by the dichotomy theorem it is not Borel i.e., I_t is locally non-Borel.

Proof: Let F be a closed set locally not in B. For every $s \in 2^{<\omega}$ such that $N_s \cap F \neq \emptyset$ there are α_s, $\beta_s \in N_s \cap F$ such that $\alpha_s E_0 \beta_s$ and $\{\alpha_s \neq \beta_s\}$. Fix such a collection $\{\alpha_s, \beta_s\}$.

We will define a sequence F_s, $s \in 2^{<\omega}$ such that

(i) F_s is a finite subset of F.

(ii) if $s \prec t$, then $F_s \subseteq F_t$.

(iii) if $s \prec t$, then dist$(F_s, F_t) \leq 2^{-lh(s)}$.

(iv) for every $s \in 2^{<\omega}$ $F_{s\frown(0)} = F_s$.

(v) if $m = 2^{lh(s)+1}$ and $\gamma \in F_s$, put $t = \gamma\lceil m$; then α_t,$\beta_t \in F_{s\frown(1)}$.

Suppose we have defined such sequence F_s, then put

$$f(\gamma) = \overline{\bigcup_n F_{\gamma\lceil n}}.$$

It is not difficult to see that (iii) implies that f is continuous (see lemma 2.1.24 in chapter 2).

If γ is eventually zero, then it is clear that $f(\gamma)$ is finite. On the other hand, let us assume that γ has infinite many 1's. We will show that $f(\gamma)$ is locally not a transversal. Let $u \in 2^{<\omega}$ be such that $N_u \cap f(\gamma) \neq \emptyset$. It suffices to show that there is $t \succ u$ such that α_t,$\beta_t \in f(\gamma)$. Let n be such that $N_u \cap F_{\gamma\lceil n} \neq \emptyset$ and let $\delta \in N_u \cap F_{\gamma\lceil n}$. Let $m > n$ such that $\gamma(m) = 1$. Put $s = \gamma\lceil m$ and $t = \delta\lceil 2^{m+1}$. Then by (v) α_t,$\beta_t \in F_{s\frown(1)}$.

So, it remains to show that such sequence F_s exists. Fix $\alpha_\emptyset \in F$ and put

$F_\emptyset = \{\alpha_\emptyset\}$. Suppose we have defined F_s for every $s \in 2^n$ satisfying (i), (ii), (iv) and (v). Let $m = 2^{lh(s)+1}$, then put

$$F_{s\frown(1)} = F_s \cup \{\alpha_t, \beta_t : (\exists\gamma \in F_s)(t = \gamma\lceil m)\}. \qquad (*)$$

The only condition that remains to be checked is (iii). But (*) implies that $\text{dist}(F_s, F_{s\frown(1)}) \leq 2^{-lh(s)-1}$ which easily implies that if $s \prec t$, then $\text{dist}(F_s, F_t)$ $\leq 2 \cdot 2^{-lh(s)-1}$.

□

Let us observe that I_t is not calibrated iff $I_t \neq I(E_0)$. In fact, one direction is trivial since $I(E_0)$ is calibrated. Now, suppose that I_t is calibrated. Then by the Debs-Saint Raymond theorem quoted above we get that I_t has the covering property. Let $F \in I(E_0)$, then there is a Borel transversal T such that $F \subseteq [T]_{E_0}$. It is clear that $T \in I_t^{int}$, hence there is a countable sequence $(K_n)_n$ of closed transversals such that $T \subseteq \bigcup_n K_n$. This clearly implies that $[T]_{E_0}$ can also be covered by countably many closed transversals (just take the images of the K_n's under a group that generates E_0), hence $F \in I_t$. In fact the same argument shows that if $J \subseteq I(E_0)$ is a $\sigma-$ideal with the covering property containing all closed transversals and such that for every $F \in J$, $[F]_{E_0} \in J^{ext}$, then $J = I(E_0)$.

Now we will show that $I_t \neq I(E_0)$. For every $x \in 2^\omega$ we will define a tree $T = T_x$ such that $[T]$ is smooth but not in I_t. We will use the following notation: for every $s \in 2^{<\omega}$, x_s denotes the real obtained from x by substituting $x\lceil n$ by s, where $n = lh(s)$.

We will define by induction a set of sequences T_n. For $n = 0$ let $T_0 = \{\emptyset\}$ and let $T_1 = \{x\lceil 1, < 1 - x(0) >\}$. There is k_1 and sequences $u_s^1 \in 2^{k_1}$ for $s \in T_1$ such that: if $s \neq t$ then $u_s^1 \neq u_t^1$, and $x\lceil[1, k_1] \neq u_s^1$ for every s in T_1.

Put

$$T_2 = \{x_s\lceil k_1 + 1 : s \in T_1\} \cup \{s^\frown u^1_s : s \in T_1\}.$$

Notice that every $t \in T_2$ has length equal to $(k_1 + 1)$. The reason to add $x_s\lceil k_1 + 1$ is in order to get at the end a closed set which is locally not a transversal. And by asking that the u^1_s's are different we make sure there are no more equivalent elements. We define T_3 and the pattern to define T_n should be clear. There is an integer k_2 and sequences $u^2_s \in 2^{k_2}$ for each $s \in T_2$ such that: if $s \neq t$, then $u^2_s \neq u^2_t$; and also $u^2_t \neq x\lceil[k_1 + 1, k_2 + k_1 + 1]$, for every $s \in T_2$. Put

$$T_3 = \{x_s\lceil k_2 + k_1 + 1 : s \in T_2\} \cup \{s^\frown u^2_s : s \in T_2\}.$$

Put T = smallest tree contaning $\bigcup_n T_n$.

We claim that $[T]$ is smooth and not in I_t. In fact, notice first that $[T] - [x]_{E_0}$ is a transversal. Because if $\alpha \in [T] - [x]_{E_0}$, then in infinite many pieces α is equal to some u_s, and they were chosen to form a transversal. Since $[T] = ([T] - [x]_{E_0}) \cup ([T] \cap [x]_{E_0})$, clearly $[T]$ is smooth. On the other hand, by construction, for every $s \in T$, $\mid [T] \cap N_s \cap [x]_{E_0} \mid\geq 2$, hence $[T]$ is locally not a transversal.

Let us observe that we have actually shown that I_t is not calibrated since $[T]$ is a counterexample to the definition of calibration.

Since every $\mathbf{\Sigma}^1_1$ E_0-smooth set is of first category, then every $\mathbf{\Sigma}^1_1$ set in I_t^{int} is also of the first category. Hence, by proposition 2.1.7 in chapter 2, I_t is not thin. We will collect these facts in the following

Proposition 1.2.11 *I_t is neither thin nor calibrated. Therefore $I_t \neq I(E_0)$.*

□

One can give a simple description of I_t-perfect sets as follows: Let T be a tree on 2 . For every $s \in 2^{<\omega}$ let $T_s = \{t \in 2^{<\omega} : s\frown t \in T\}$. Then $[T]$ is I_t-perfect iff for every $s \in T$ there are $s_1, s_2 \in 2^{<\omega}$ such that $lh(s_1) = lh(s_2)$, $s_1 \neq s_2$ and $[T_{s\frown s_1}] \cap [T_{s\frown s_2}] \neq \emptyset$. This collection of trees might define an interesting notion of forcing.

Remark: The existence of a Borel basis for $I(E_0)$ would have a very interesting consequence. Recall that we have left the question of whether every Π^1_1 sparse set is smooth. Clearly it suffices to answer this question only for the largest Π^1_1 sparse set C defined in §1.4. Now, if $I(E_0)$ has a Borel basis then it has the covering property. We will show in chapter 3 that in this case we have that $\alpha \in C$ iff there is $T \in L_{\omega_1^\alpha}$ such T is a tree on 2, $\alpha \in [T]$ and $[T]$ is smooth for E_0. This is, roughly speaking, because every Borel subset of C is smooth and hence it can be covered by countable many closed smooth sets. So, under the hypothesis that there are only countable many reals in L, we get that C is covered by countable many smooth sets, hence it is smooth.

Chapter 2

On σ-ideals of compact sets

In this chapter we will present some results related to $\sigma-$ideals of compact sets. Such $\sigma-$ideals occur very naturally in Analysis as notions of smallness. We are interested in their descriptive set theoretic properties. This approach was initiated by Kechris, Louveau and Woodin on [14], where the basic theory was developed. We are especially interested in the so called covering property, which can be thought as an abstract version of the Perfect Set Theorem for Σ^1_1 sets. We will look at it in §1, where we show that some definability and structural properties like strong calibration, thinness and control can be deduced from the covering property. Most of the $\sigma-$ideals we know do not have the covering property. However, there are two very important ideals that do have it: The ideal of countable closed subsets of a perfect Polish space and the ideal of closed sets of extended uniqueness in the unit circle (see [13]). A main open question is to characterize the $\sigma-$ideals with the covering property. In §2 we present some result about product of ideals from the same point of view.

We will follow the notation of [14]. The letter I will always denote a $\sigma-$ideal of closed sets on a compact Polish space X. The collection of compact subsets of X is denoted by $\mathcal{K}(X)$. With the Hausdorff metric it is a compact Polish space.

2.1 The covering property and related notions

With each ideal I of closed subsets of X, there are two classes of (arbitrary) subsets of X associated with I. Define I^{int} as follows: a subset A of X is in I^{int} if every closed subset of A belongs to I, i.e., $\mathcal{K}(A) \subseteq I$. In this case we say that A belongs to I from the interior. And define I^{ext} by: $A \in I^{ext}$ if there is a countable collection $\{F_n\}$ of closed sets in I such that $A \subseteq \bigcup_n F_n$. In this case we say that A belongs to I from the exterior.

Definition 2.1.1 *We say that I has the* covering property, *if for every $\mathbf{\Sigma}^1_1$ set $A \in I^{int}$, there is a countable collection $\{F_n\}$ of closed sets in I such that $A \subseteq \bigcup_n F_n$.*

Since every set in I^{ext} is trivially in I^{int}, then I has the covering property if for a $\mathbf{\Sigma}^1_1$ set A , $A \in I^{int}$ iff $A \in I^{ext}$.

The classical Perfect Set Theorem for $\mathbf{\Sigma}^1_1$ sets says that if A is a $\mathbf{\Sigma}^1_1$ subset of X and every closed subset of A is countable, then A is countable. In other words, the $\sigma-$ideal of closed countable subsets of X has the covering property. So, we can regard this property as an abstraction of the content of the Perfect Set Theorem. Since in ZFC this theorem cannot be extended to Π^1_1 sets, we do not expect to have (in ZFC) a covering property for Π^1_1 sets (we will look at this problem in chapter 3).

Let us observe that for a $\sigma-$ideal I consisting of meager sets the covering property implies that $\mathbf{\Sigma}_1^1$ sets in I^{int} are of first category, i.e., they are also small in the sense of category.

The following notion is closely related to the covering property.

Definition 2.1.2 *An ideal I is* calibrated *if for every closed set F the following holds: If for some collection $\{F_n\}$ of closed sets in I, $F - \bigcup_n F_n \in I^{int}$, then $F \in I$.*

A typical calibrated ideal is the collection of closed null sets with respect to some Borel measure. On the other hand, the ideal of closed meager sets is not calibrated.

Let B be a hereditary subset of $\mathcal{K}(X)$, i.e., downward closed under inclusion. B_σ denotes the smallest $\sigma-$ideal (of closed sets) containing B, i.e., $K \in B_\sigma$ if there is a sequence $\{K_n\}$ of elements of B such that $K = \bigcup_n K_n$. We say that I has a *Borel basis* if there is a Borel hereditary set $B \subseteq I$ such that $I = B_\sigma$. I is called *locally non-Borel* if for every closed set $F \notin I$, $I \cap \mathcal{K}(F)$ is not Borel.

The only criterion known to show that an ideal has the covering property is the following theorem, which was originally used to show that the ideal of set of uniqueness does not have a Borel basis (see [13] for a proof of both results).

Theorem 2.1.3 *(Debs-Saint Raymond [3]). Let I be a calibrated, locally non-Borel, Π_1^1 $\sigma-$*ideal. *If I has a Borel basis, then I has the covering property.*

□

Kechris [11] has asked the question of characterizing the $\sigma-$ideals which have the covering property. It clearly implies calibration, but it is not known if the other hypotheses of the previous theorem are necessary. Let us recall here that a Π_1^1 $\sigma-$ideal I satisfies the so called dichotomy theorem: It is either a true Π_1^1 set or a G_δ set (see [14]). Hence, the first step in reversing the Debs-Saint Raymond theorem would be to show that there are no G_δ (hence Borel) σ-ideals with the covering property. This has been the main motivation for the results presented in this section.

The usual way to show that the covering property fails for a $\sigma-$ideal I consisting of meager sets is by finding a dense G_δ set G with $G \in I^{int}$. In fact, let us suppose such a G can be covered by a collection $\{F_n\}$ of sets in I. Then by the Baire category theorem there is an n and an open set V such that $V \cap G \neq \emptyset$ and $V \cap G \subseteq F_n$. As G is dense, we get $V \subseteq \overline{V \cap G} \subseteq F_n$, which contradics that F_n is meager. In other words, the covering property fails for a G_δ set. This is the case, for instance, when I consists of the null sets with respect to a Borel measure.

We will see later on that it is convenient to restrict attention to $\mathbf{\Pi}_2^0$ sets. So, we say that a $\sigma-$ideal I has the *covering property for* $\mathbf{\Pi}_2^0$ sets, if for every $\mathbf{\Pi}_2^0$ set $G \in I^{int}$, there is a countable collection $\{K_n\}$ of sets in I such that $G \subseteq \bigcup_n K_n$. We also need the following notion: A set M is said to be *locally not in I* (or *I-perfect*), if for every open set V with $V \cap M \neq \emptyset$, we have that $\overline{V \cap M} \notin I$. Given a closed set $F \notin I$, there is $F' \subseteq F$ such that F' is locally not in I. In fact, let $O = \bigcup\{V \subseteq X : V \text{ is open and } F \cap V \in I^{ext}\}$. Put $F' = F - O$. It is easy to check that F' is locally not in I. F' is the I-perfect kernel of F.

We have the following useful characterization of this notion

Proposition 2.1.4 *Let I be a $\sigma-$ideal of compact sets. The following are equivalent:*

(i) I has the covering property for $\mathbf{\Pi}^0_2$ sets.

(ii) For each $\mathbf{\Pi}^0_2$ set G such that $\overline{G}$ is locally not in I, we have $G \notin I^{int}$.

Proof: (i)$\Rightarrow$ (ii). Let G be a G_δ set such that $M = \overline{G}$ is locally not in I. Suppose, towards a contradiction that $G \in I^{int}$. By (i) there is a sequence $\{F_n\}$ of sets in I such that $G \subseteq \bigcup_n F_n$. By the Baire category theorem there is an n and an open set V such that $\emptyset \neq G \cap V \subseteq F_n$. Hence $\overline{V \cap M} = \overline{V \cap G} \subseteq F_n$. So, $\overline{V \cap M} \in I$, which contradicts that M is locally not in I.

(ii)$\Rightarrow$ (i). Let G be a $\mathbf{\Pi}^0_2$ set in I^{int}. Assume towards a contradiction that $G \notin I^{ext}$. Let $O = \bigcup\{V \subseteq X : V$ is an open set and $V \cap G \in I^{ext}\}$. Let $G' = G - O$. As $G \notin I^{ext}$, then $G' \neq \emptyset$. It is clear that for all V open, if $V \cap G' \neq \emptyset$ then $V \cap G' \notin I^{ext}$. Clearly G' is a $\mathbf{\Pi}^0_2$ set in I^{int} and for every open set V, if $V \cap G' \neq \emptyset$ then $\overline{V \cap G'} \notin I$. Therefore $M = \overline{G}$ is locally not in I, which contradicts (ii).

□

The following result is a partial answer to the question of whether a G_δ $\sigma-$ideal can have the covering property. First we need the following

Lemma 2.1.5 *Let $D \subseteq \mathcal{K}(X)$ be an open hereditay set such that if $F \in D$ and $x \in X$, then $F \cup \{x\} \in D$. Then there is an open dense set U such that $\mathcal{K}(U) \subseteq D$.*

Proof: Let $\{x_n\}$ be a countable dense subset of X. We will define a sequence $\{O_n\}$ of open sets such that $x_n \in O_n$ and $\bigcup_{j=1}^N \overline{O_n} \in D$, for each N.

First, observe that if $F \in D$, then there is an open set O such that $F \subseteq O$ and $\mathcal{K}(O) \subseteq D$. To see this, note that since D is open, there is an open nghd W in $\mathcal{K}(X)$, such that $F \in W$ and $W \subseteq D$. Say $W = \{K \in \mathcal{K}(X) : K \subseteq V_0 \& K \cap V_i \neq \emptyset, 1 \leq i \leq n\}$, where each V_i is an open subset of X. We claim that $\mathcal{K}(V_0) \subseteq D$: if $K \subseteq V_0$, let $y_i \in V_i$ for $1 \leq i \leq n$; then $K \cup \{y_i : 1 \leq i \leq n\} \in W$, hence $K \cup \{y_i : 1 \leq i \leq n\} \in D$. But as D is hereditary, then $K \in D$.

We define $\{O_n\}$ by induction on n. For $n = 0$: as $\{x_0\} \in D$, there is an open set O such that $x_0 \in O$ and $\mathcal{K}(O) \subseteq D$. Let O_0 be an open set such that $x_0 \in O_0$ and $\overline{O_0} \subseteq O$.

Suppose we have defined O_n for $0 \leq n \leq N$ such that $x_n \in O_n$ and $\bigcup_{j=0}^{N} \overline{O_j} \in D$. Then by hypothesis $\bigcup_{j=0}^{N} \overline{O_j} \cup \{x_{N+1}\} \in D$. By the observation above, there is an open set V such that $\bigcup_{j=0}^{N} \overline{O_j} \cup \{x_{N+1}\} \subseteq V$ and $\mathcal{K}(V) \subseteq D$. Let O_{N+1} be an open set such that $x_{N+1} \in O_{N+1}$ and $\overline{O_{N+1}} \subseteq V$. Clearly $\bigcup_{j=0}^{N+1} \overline{O_j} \in D$.

Finally, put $U = \bigcup_{j=0}^{\infty} O_j$. U is clearly an open dense set. Now, if $F \subseteq U$, by compactness, there is N such that $F \subseteq \bigcup_{j=0}^{N} O_j \subseteq \bigcup_{j=0}^{N} \overline{O_j}$. Since D is hereditary $F \in D$, i.e., $\mathcal{K}(U) \subseteq D$.

□

Theorem 2.1.6 *Let I be a $\mathbf{\Pi}_2^0$ hereditary collection of compact sets. Assume there are open sets D_n in $\mathcal{K}(X)$ such that $I = \bigcap_n D_n$ and for all $F \in D$ and all $x \in X$ we have $F \cup \{x\} \in D_n$. Then there is a dense G_δ set G such that $\mathcal{K}(G) \subseteq D$, i.e., $G \in I^{int}$. In particular, if I is a $\mathbf{\Pi}_2^0$ ideal of closed meager sets as above, then I does not have the covering property for $\mathbf{\Pi}_2^0$ sets.*

Proof: First, we can assume that each D_n is hereditary. In fact, consider the following sets:

$$J_n = \{K \in \mathcal{K}(X) : (\forall F)(F \subseteq K \rightarrow F \in D_n)\}.$$

Recall that the relation $R(F, K)$ iff $F \subseteq K$ is closed in $\mathcal{K}(X) \times \mathcal{K}(X)$. Thus J_n is open and it is clearly a hereditary subset of D_n. Notice also that if $F \in J_n$ and $x \in X$, then $F \cup \{x\} \in J_n$. Now, as I is hereditary if $F \in I$, then $F \in J_n$ for all n, i.e., $I = \bigcap_n J_n$.

To prove the theorem, we have by the previous lemma that there are open dense sets O_n such that $\mathcal{K}(O_n) \subseteq D_n$. Put $G = \bigcap_n O_n$. G is a dense G_δ in I^{int}.

Finally, we have already seen that the Baire category theorem implies that such G can not be covered by countable many meager closed sets.

□

Remark: We do not know of any $\mathbf{\Pi}^0_2$ ideal which does not satisfy the hypothesis of the previous theorem, even in the following weaker form: there is a dense countable set D such that the condition about $\{x\} \cup F$ holds only for $x \in D$.

The next type of ideals that we are going to consider are the thin ideals. This notion was introduced in [14] and it corresponds dually to the countable chain condition. We say that I is *thin* if every collection of pairwise disjoint closed sets not in I is at most countable. The typical example of thin ideal is the collection of null sets for some Borel measure. The next theorem relates thinness with the covering property.

Theorem 2.1.7 *Let I be a σ-ideal of closed sets which satisfies one of the following non triviality conditions:*

(i) $I \neq \mathcal{K}(X)$ and for every $x \in X$, $\{x\} \in I$.

(ii) Every $K \in I$ is a meager set.

If I is thin, then I does not have the covering property for $\mathbf{\Pi}^0_2$ sets. Actually, if (ii) holds, then there is a dense G_δ set in I^{int}.

Proof: Assume first that (i) holds. Let $O = \bigcup\{V \subseteq X : V$ is open and $V \in I^{ext}\}$. Put $K = X - O$, K is locally not in I (if $V \bigcap K \neq \emptyset$, then $\overline{V \bigcap K} \notin I$, otherwise $V \subseteq O$). As $I \neq \mathcal{K}(X)$ and every singleton is in I, then K is a perfect set. Let G be a dense G_δ subset of K with empty interior with respect to the relative topology of K. Let $\{K_n\}$ be a maximal collection of pairwise disjoint closed subsets of G with each $K_n \notin I$. Each K_n is meager in K. Put $F = \bigcup_n K_n$ and $H = G - F$. Then H is a dense (in K) G_δ subset of K. Clearly $H \in I^{int}$, hence by 2.1.4 I does not have the covering property for $\mathbf{\Pi}^0_2$ sets.

Now if (ii) holds, then X is locally not in I, hence the same proof applies. Finally, let's observe that in this case we get a dense G_δ set in I^{int}.

□

Remark: (i) Besides $I \neq \mathcal{K}(X)$, some other non-triviality condition has to be imposed on I in order to get the conclusion of 2.1.7, as the following example shows: let $F \subseteq X$ be a countable closed set and $V = X - F$. Put $I = \mathcal{K}(V)$. I is thin, because $K \notin I$ iff $K \cap F \neq \emptyset$. Thus there are only countable many disjoint sets not in I. However, I trivially satisfies the covering property (because $V \in I^{ext}$ and if $H \in I^{int}$ then $H \subseteq V$).

(ii) We will use 2.1.7 usually as follows. Suppose that every Borel set in I^{int} is of the first category ($\mathbf{\Pi}_2^0$ sets suffice). Then I is not thin. Just notice that in this case every set in I is meager.

The following notion was introduced in [14]. A set $A \subseteq X$ is called *I-thin* if there is no uncountable family of pairwise disjoint closed subsets of A which are not in I. In other words, A is I-thin if the restriction of I to $\mathcal{K}(A)$ is a thin ideal. Given an ideal I define another ideal J_I as follows:

$$K \in J_I \text{ iff } K \text{ is } I\text{-thin.}$$

It was proved in [14] that if I is a Π_1^1 calibrated $\sigma-$ideal then so is J_I. It was asked there to find out for a given I whether $J_I = I$. In relation with this question we have the following

Corollary 2.1.8 *Let I be a $\sigma-$*ideal *of closed subsets of X containing all singletons. If I has the covering property for $\mathbf{\Pi}_2^0$ sets, then $I = J_I$.*

Proof: It is clear that $I \subseteq J_I$. Now, let F be a closed set not in I. We want to show that $F \notin J_I$. We can assume without loss of generality that F is locally not in I. Hence as I contains all singletons, F is perfect. Put $\tilde{I} = \mathcal{K}(F) \cap I$. $\tilde{I}$ is non trivial in the sense of 2.1.7 (i) and it has the covering property for $\mathbf{\Pi}_2^0$ sets: if $H \subseteq F$ is a $\mathbf{\Pi}_2^0$ set in $\tilde{I}^{int}$ then $H \in I^{int}$. Hence, by the covering property for I, $H \in I^{ext}$. This clearly implies that $H \in \tilde{I}^{ext}$. Therefore, by 2.1.7 $\tilde{I}$ is not thin, i.e., $F \notin J_I$.

□

Corollary 2.1.9 *(Kaufman) Let U_0 denote the $\sigma-$*ideal *of closed set of extended uniqueness in the unit circle. Then $U_0 = J_{U_0}$.*

Proof: Debs and Saint Raymond [3] have shown that U_0 has the covering property. □

Theorem 2.1.7 says that a non trivial Π^1_1 thin σ−ideal I does not have the covering property. In [14] it was asked whether for an I that was also calibrated we have that I has to be $\mathbf{\Pi}^0_2$. The next theorem is a partial answer to this question.

Theorem 2.1.10 *If I is a calibrated, thin, Π^1_1 σ-ideal of closed sets* with a Borel basis, *then I is $\mathbf{\Pi}^0_2$.*

Proof: Let $\{F_n\}$ be a maximal pairwise disjoint countable collection of closed sets such that for each n, $F_n \notin I$ and $I \cap \mathcal{K}(F_n)$ is $\mathbf{\Pi}^0_2$. Put $F = \bigcup_n F_n$ and $H = X - F$. We claim that $H \in I^{int}$. Granting this claim we have:

$$K \in I \text{ iff } (\forall n)(K \cap F_n \in I). \tag{*}$$

The direction ($\Rightarrow$) is trivial. On the other hand, let $K \subseteq X$ be a closed set. Then $K = (K \cap H) \cup \bigcup_n (K \cap F_n)$. Suppose that each $K \cap F_n \in I$. As I is calibrated and $K \cap H \in I^{int}$, then $K \in I$.

Now, the map $K \mapsto K \cap F_n$ is Borel, so (*) says that I is Borel. Therefore by the Dichotomy theorem (see [14]) I is $\mathbf{\Pi}^0_2$.

It remains to show that H is in I^{int}. Suppose not towards a contradiction. Let $M \subseteq H$ be a closed set locally not in I. Since $\{F_n\}$ is maximal then for every $x \in M$, $\{x\} \in I$. Hence M is a perfect set. Consider the σ−ideal $I_0 = \mathcal{K}(M) \cap I$. I_0 is clearly a Π^1_1, calibrated, thin (non-trivial as in 2.1.7) σ−ideal with a Borel basis. As $\{F_n\}$ is maximal, for every $F \subseteq M$ with $F \notin I_0$ we have that $\mathcal{K}(F) \cap I_0 = \mathcal{K}(F) \cap I$ is not $\mathbf{\Pi}^0_2$. Hence I_0 is locally non Borel and thus all the hypotheses of the Debs-Saint Raymond theorem

(2.1.3) are satisfied. Therefore I_0 has the covering property, but also it is non trivial and thin which contradicts 2.1.7.

□

This raises the following question: Does every calibrated, thin Π^1_1 σ−ideal have a Borel basis ?

Theorem 2.1.7 also connects the covering property with the notion of controlled ideal. Let's recall this notion. Let $G \subseteq 2^\omega \times X$ be a $\mathbf{\Pi}^0_2$ universal set for $\mathbf{\Pi}^0_2$ subsets of X. A code for a $\mathbf{\Pi}^0_2$ set H is an $\alpha \in 2^\omega$ such that $H = G_\alpha$. A collection A of $\mathbf{\Pi}^0_2$ subsets of X is *compatible* with I if the least σ−ideal J of $\mathbf{\Pi}^0_2$ sets containing I and A extends I, i.e., it satisfies $J \cap \mathcal{K}(X) = I$. An ideal I is said to be *controlled* if there is a $A \subseteq \mathbf{\Pi}^0_2(X)$ such that $\emptyset \in A$, A is compatible with I and A is $\mathbf{\Sigma}^1_1$ in the codes of $\mathbf{\Pi}^0_2$ sets (i.e., $\{\alpha \in 2^\omega : G_\alpha \in A\}$ is $\mathbf{\Sigma}^1_1$). Such set A is called a *control set* for I.

Observe that for a calibrated σ−ideal I, A is compatible with I iff $A \subseteq I^{int} \cap \mathbf{\Pi}^0_2(X)$. The following theorem was proved in [14].

Theorem (Kechris, Louveau, Woodin see [14]): *Let I be a controlled Π^1_1 σ-ideal of closed subsets of X. Then I is $\mathbf{\Pi}^0_2$ and thin.* □

From this and 2.1.7 we immediately get the following

Corollary 2.1.11 *Let I be a Π^1_1 σ−ideal non trivial in the sense of 2.1.7. If I has the covering property for $\mathbf{\Pi}^0_2$ sets, then I is not controlled.*

□

We do not know yet if there are $\mathbf{\Pi}^0_2$ σ−ideals with the covering property. However, the corollary above implies that every non trivial Π^1_1 σ−ideal of closed sets with the covering property has to be true Π^1_1 on the codes of $\mathbf{\Pi}^0_2$

sets. This will follow from the following lemma:

Lemma 2.1.12 *Let G be a $\mathbf{\Pi}^0_2$ universal sets for $\mathbf{\Pi}^0_2$ subsets of X and I a* Π^1_1 $\sigma-$ideal *of closed subsets of X. Then*

(i) $\{\alpha \in 2^\omega : G_\alpha \in I^{int}\}$ *is* Π^1_1.

(ii) $\{\alpha \in 2^\omega : G_\alpha$ *is closed* $\}$ *is* Π^1_1.

Proof: (i) First, we have that

$$G_\alpha \in I^{int} \text{ iff } (\forall F \in \mathcal{K}(X))(F \subseteq G_\alpha. \Rightarrow F \in I)$$

Now, the relation $R(F,\alpha) \Leftrightarrow F \subseteq G_\alpha$ is $\mathbf{\Pi}^0_2$, because

$$F \subseteq G_\alpha \text{ iff } (\forall x)(x \notin F \text{ or } (\alpha, x) \in G).$$

And recall that the projection of a F_σ subset of a compact space is F_σ. Hence, " $G_\alpha \in I$" is Π^1_1.

(ii) Fix a countable open basis for the topology of X, say $\{V_n : n \in \mathbf{N}\}$. Then

$$G_\alpha \text{ is closed iff } (\forall x)[(\forall n)(x \in V_n \Rightarrow V_n \cap G_\alpha \neq \emptyset) \Rightarrow x \in G_\alpha]. \qquad (*)$$

Now, the following relation is clearly $\mathbf{\Sigma}^1_1$.

$$R(n,\alpha,x) \text{ iff } (x \in V_n \Rightarrow V_n \cap G_\alpha \neq \emptyset)$$
$$\text{iff } x \notin V_n \text{ or } (\exists y)(y \in V_n \;\&\; (\alpha, y) \in G).$$

Hence (*) is Π^1_1.

□

Proposition 2.1.13 *Let I be a Π^1_1 σ-ideal of closed subsets of X, which is non trivial in the sense of 2.1.7. If I has the covering property, then* $\{\alpha \in 2^\omega : G_\alpha$ *is closed and* $G_\alpha \in I\}$ *is a true* Π^1_1 *set.*

Proof: Let $A = \{G_\alpha : G_\alpha \text{ is closed and } G_\alpha \in I\}$, then $\emptyset \in A$ and $A \subseteq \mathbf{\Pi}^0_2(X) \cap I^{int}$. As I is not controlled (by 2.1.11), then A is not $\mathbf{\Sigma}^1_1$ on the codes of $\mathbf{\Pi}^0_2$ sets. Hence from 2.1.12 we get the conclusion of the proposition.

□

Remark: It would be interesting to determine if for every calibrated controlled Π^1_1 $\sigma-$ideal I, $\{\alpha \in 2^\omega : G_\alpha \in I^{int}\}$ is a Borel set. For instance, for $I = Null(\mu)$ where μ is a measure, this set is $\mathbf{\Pi}^0_3$. This is because the relation $M(\alpha, r)$ iff $\mu(F_\alpha) > r$ is $\mathbf{\Sigma}^0_2$, where F is a $\mathbf{\Sigma}^0_2$ universal set (see [7]).

There is a stronger notion of calibration which also follows from the covering property.

Definition 2.1.14 *An ideal I is* strongly calibrated *if for every closed set $F \subseteq X$ with $F \notin I$ and every $\mathbf{\Pi}^0_2$ set $H \subseteq X \times 2^\omega$ such that $proj(H) = F$, there is a closed set $K \subseteq H$ such that $proj(K) \notin I$.*

This notion was introduced in [14]. It resembles the conclusion of Choquet's capacitability theorem and in fact this theorem implies that the $\sigma-$ideal of closed measure zero sets for a collection of Borel measures is strongly calibrated: Let $\mathcal{M}$ be a collection of Borel measures on X and let $I = Null(\mathcal{M})$. Let $Q \subseteq X \times 2^\omega$ be a $\mathbf{\Pi}^0_2$ set such that $proj(Q) = F \notin I$, and say $\mu(F) > 0$ for some $\mu \in \mathcal{M}$. Define a capacity γ on $X \times 2^\omega$ as follows:

$$\gamma(A) = \mu^*(proj(A)), \text{ for } A \subseteq X \times 2^\omega.$$

As Q is $\mathbf{\Pi}^0_2$ and $\gamma(Q) > 0$, by Choquet's capacitability theorem there is a compact set $K \subseteq Q$ such that $\gamma(K) > 0$. Hence $proj(K) \notin I$. These type of ideals have the property that the collection of $\mathbf{\Sigma}^1_1$ sets in I^{int} is Π^1_1 on the codes of $\mathbf{\Sigma}^1_1$ sets (assuming that I is Π^1_1). The usual argument to show this uses the capacitability theorem. We show next that strongly calibrated σ-ideals also have this property.

Proposition 2.1.15 *Let I be a Π^1_1 strongly calibrated σ-*ideal *of closed subsets of X. Then the collection of $\mathbf{\Sigma}^1_1$ sets in I^{int} is Π^1_1 in the codes of $\mathbf{\Sigma}^1_1$ sets.*

Proof: Let $\mathcal{U} \subseteq 2^\omega \times X$ be a $\mathbf{\Sigma}^1_1$ universal set for $\mathbf{\Sigma}^1_1$ subsets of X. Let $Q \subseteq (2^\omega \times X) \times 2^\omega$ be a $\mathbf{\Pi}^0_2$ set such that $\mathcal{U} = proj(Q)$. Consider the following relation

$$R(F, \alpha) \text{ iff } F \subseteq \mathcal{U}_\alpha \ \& \ F \notin I.$$

Then we have

$$\mathcal{U}_\alpha \notin I^{int} \text{ iff } (\exists F) R(F, \alpha).$$

Hence it suffices to show that R is $\mathbf{\Sigma}^1_1$. We claim that

$$R(F, \alpha) \text{ iff } (\exists K \in \mathcal{K}(X))(K \subseteq Q^\alpha \ \& \ proj(K) \notin I). \qquad (*)$$

The direction $\Leftarrow$ clearly holds. For the other, suppose that $R(F, \alpha)$ holds and put $H = Q^\alpha \cap (2^\omega \times F)$. Then $proj(H) = F$. As H is $\mathbf{\Pi}^0_2$, by strong calibration there is a closed $K \subseteq H$ such that $proj(K) \notin I$, this set K clearly works.

To see that (*) is $\mathbf{\Sigma}^1_1$ recall that the function $K \mapsto proj(K)$ is continuous and it is easy to check that $K \subseteq Q^\alpha$ is a $\mathbf{\Pi}^0_2$ relation of K and α.

□

Strong calibration implies calibration (see [14]). Also, one can take projections of $\mathbf{\Sigma}^1_1$ subsets of any compact Polish space in the definition of strong calibration as the following proposition shows. This sometimes makes this notion easier to use.

Proposition 2.1.16 *Strong calibration is equivalent to any of the following statements.*

(i) If $F \subseteq X$ is a closed set not in I and $Q \subseteq X \times 2^\omega$ is a $\mathbf{\Sigma}^1_1$ set such that $proj(Q) = F$, then there is a closed set $K \subseteq Q$ such that $proj(K) \notin I$.

(ii) Let Y be a compact Polish space. If $F \subseteq X$ is a closed set not in I and $Q \subseteq X \times Y$ is a $\mathbf{\Sigma}^1_1$ set such that $proj(Q) = F$, then there is a closed set $K \subseteq Q$ such that $proj(K) \notin I$.

Proof: (ii) follows from (i) because for any compact Polish space Y there is a continuous surjection $f : 2^\omega \to Y$.

To prove (i), let $Q \subseteq X \times 2^\omega$ be a $\mathbf{\Sigma}^1_1$ set as in the hypothesis of (i). Let $P \subseteq X \times 2^\omega \times 2^\omega$ be a $\mathbf{\Pi}^0_2$ set such that $proj(P) = Q$. Let $f : 2^\omega \to 2^\omega \times 2^\omega$ be an homeomorphism, say $f = (f_0, f_1)$. Define $P^* \subseteq X \times 2^\omega$ by

$$(x, \alpha) \in P^* \text{ iff } (x, f_0(\alpha), f_1(\alpha)) \in P.$$

Then P^* is $\mathbf{\Pi}^0_2$ and clearly $proj(P^*) = F$. So by strong calibration, there is a closed $K^* \subseteq P^*$ such that $proj(K^*) \notin I$. Define $K \subseteq X \times 2^\omega$ by $(x, \alpha) \in K$ iff $(\exists \beta)((x, f^{-1}(\alpha, \beta))) \in K^*$. It is easy to check that K is a closed subset of Q and $proj(K) = proj(K^*)$.

□

As we said before we have the following

Theorem 2.1.17 *Let I be a $\sigma-$*ideal *of closed subsets of X. If I has the covering property for $\mathbf{\Pi}^0_2$ sets, then I is strongly calibrated.*

Proof: Let F be a closed set locally not in I and $Q \subseteq X \times 2^\omega$ be a $\mathbf{\Pi}^0_2$ set such that $F = proj(Q)$. By the von Neumann selection theorem (see 4E.9 in [15]) there is a Baire measurable function f such that for all $x \in F$,$(x, f(x)) \in Q$. By the analog of the Lusin's theorem for category (see [16]), there is a G_δ set $G \subseteq F$ dense in F, such that f is continuous on G. Since I has the covering property for $\mathbf{\Pi}^0_2$ sets, then by 2.1.4, $G \notin I^{int}$. Thus, there is a closed set $K \subseteq F$ with $K \notin I$. Let K^*=graph of f restricted to K. As f is continuous on K, then K^* is a closed set and clearly $proj(K^*) = K$. This finishes the proof.

□

Corollary 2.1.18 *Let I be a Π^1_1 locally non Borel σ-ideal with a Borel basis. Then I is calibrated iff I is strongly calibrated.*

Proof: It was proved in [14] that strong calibration implies calibration. On the other hand, by the Debs-Saint Raymond theorem (2.1.3) every $\sigma-$ideal as in the hypothesis of the corollary has the covering property. Hence, by previous theorem it is strongly calibrated.

□

From the proof of 2.1.17 one gets the following: Let's say that an ideal I has the *continuity property* if for every Baire measurable function f with $dom(f) = F \notin I$ (F a closed set), there is a closed set $K \subseteq F$,$K \notin I$ and f continuous on K.

Corollary 2.1.19 *(of the proof of 2.1.17) Let I be a $\sigma-$*ideal *of closed subsets of X.*

(i) If I has the covering property for $\mathbf{\Pi}_2^0$ sets, then I has the continuity property.

(ii) If I has the continuity property, then I is strongly calibrated.

□

Remark: Observe that if I is strongly calibrated, then I has the continuity property for Borel functions: Just apply the strong calibration to the graph of f.

Strong calibration is not equivalent to the covering property for $\mathbf{\Pi}_2^0$ sets, because as we have already mentioned $Null(\mu)$ is strongly calibrated but it does not have the covering property.

Calibration is equivalent to saying that $I^{int} \cap \mathbf{\Pi}_2^0(X)$ is a $\sigma-$ideal (see Proposition 1 §3 in [14]). The next proposition shows that for strong calibration we get a similar result for $\mathbf{\Sigma}_1^1$ sets.

Proposition 2.1.20 *Let I be a strongly calibrated $\sigma-$ideal. Then*

(i) If F is a closed set such that $F = P \cup \bigcup_n F_n$, for some $\mathbf{\Sigma}_1^1$ set P in I^{int} and each F_n in I, then $F \in I$. In particular I is calibrated.

(ii) $\{P \subseteq X : P$ is a $\mathbf{\Sigma}_1^1$ set in $I^{int}\}$ is a $\sigma-$ideal.

(iii) Define a collection $J \subseteq \mathcal{K}(X \times 2^\omega)$ as follows:

$$K \in J \textit{ iff } proj(K) \in I$$

Then J is a calibrated $\sigma-$ideal.

Proof: (i) Let $F = P \cup \bigcup_n F_n$ be a closed set not in I with P a $\mathbf{\Sigma}_1^1$ set and each F_n in I. We will show that $P \notin I^{int}$. Let $G \subseteq X \times 2^\omega$ be a $\mathbf{\Pi}_2^0$ set such that $proj(G) = P$. Put

$$Q = (G \times \{0\}) \cup \bigcup_n (F_n \times 2^\omega \times \{1\})$$

$Q \subseteq X \times (2^\omega \times (\omega + 1))$ and $proj(Q) = F$. By strong calibration there is $K \subseteq Q$ closed such that $proj(K) \notin I$. Now, we have

$$K = K \cap (G \times \{0\}) \cup \bigcup_n K \cap (F_n \times 2^\omega \times \{1\}).$$

Hence

$$proj(K) = proj(K \cap (G \times \{0\})) \cup \bigcup_n proj(K \cap (F_n \times 2^\omega \times \{1\})).$$

Since $K \cap (G \times \{0\})$ is closed in $X \times (2^\omega \times (\omega + 1))$ and $proj(K \cap (F_n \times 2^\omega \times \{1\})) \subseteq F_n \in I$, then $proj(K \cap (G \times \{0\})) \notin I$. Thus $proj(G) = P \notin I^{int}$.

We show (iii) first. It is clear that J is a $\sigma-$ideal. Let $K = G \cup \bigcup_n K_n$, where $K \subseteq X \times 2^\omega$ is closed, G is a $\mathbf{\Pi}^0_2$ set in J^{int} and each K_n is in J. Now, $proj(K) = proj(G) \cup \bigcup_n proj(K_n)$. As $proj(K_n)$ is a closed set in I, it suffices to show that $proj(G) \in I^{int}$ and then apply (i). Let $F \subseteq proj(G)$ and suppose toward a contradiction that $F \notin I$. By strong calibration there is $K \subseteq (F \times 2^\omega) \cap G$ closed such that $proj(K) \notin I$. This contradicts that G in J^{int}.

(ii) It is easy to check (as in (iii)) that strong calibration implies that

$$\{P \subseteq X : P \in \mathbf{\Sigma}^1_1(X) \cap I^{int}\} = \{proj(G) : G \in \mathbf{\Pi}^0_2(X \times 2^\omega) \cap J^{int}\}.$$

Since J is calibrated the collection of $\mathbf{\Pi}^0_2$ sets in J^{int} is a $\sigma-$ideal (see Proposition 1§3 in [14]), from which the claim follows.

□

The next proposition relates the covering property of I and J.

Proposition 2.1.21 *Let I be a $\sigma-$*ideal *and J be the $\sigma-$*ideal *defined in 2.1.20 (iii). Then the following are equivalent:*

(i) J has the covering property.

(ii) J has the covering property for $\mathbf{\Pi}^0_2$ sets.

(iii) I has the covering property.

Proof: Clearly (i) $\Rightarrow$ (ii).

(ii)$\Rightarrow$ (iii). Let P be a $\mathbf{\Sigma}^1_1$ set in I^{int} and $G \subseteq X \times 2^\omega$ be a $\mathbf{\Pi}^0_2$ set such that $proj(G) = P$. Clearly $G \in J^{int}$. Hence there are closed sets $K_n \in J$ such that $G \subseteq \bigcup_n K_n$. Each $proj(K_n) \in I$ and $proj(G) \subseteq \bigcup_n proj(K_n)$.

(iii)$\Rightarrow$ (i). Let $G \subseteq X \times 2^\omega$ be a $\mathbf{\Sigma}^1_1$ set with $G \in J^{int}$. By 2.1.17 I is strongly calibrated, hence (as in the proof of (ii) in 2.1.20) $proj(G) \in I^{int}$. So, there are closed sets F_n in I such that $proj(G) \subseteq \bigcup_n F_n$. Thus $G \subseteq \bigcup_n F_n \times 2^\omega$ and clearly for all $F_n \times 2^\omega \in J$.

□

If I has the covering property then for every $\mathbf{\Sigma}^1_1$ set $A \in I^{int}$ there is a Borel (actually an F_σ) set $B \in I^{int}$ with $A \subseteq B$. This is also a consequence of strong calibration.

Proposition 2.1.22 *Let I be a strongly calibrated Π^1_1 $\sigma-$*ideal. *Let A be a Σ^1_1 set in I^{int}. Then there is a Δ^1_1 set $B \in I^{int}$ such that $A \subseteq B$. Therefore if we let*

$$H(I) = \bigcup\{B \subseteq X : B \text{ is } \Delta^1_1 \text{ and } B \in I^{int}\},$$

we have

(i) $H(I)$ is a Π^1_1 set in I^{int}.

(ii) For every Σ^1_1 set A, $A \in I^{int}$ iff $A \subseteq H(I)$.

Proof: This follows from the reflection principle but we give a direct proof anyway. Let A be a Σ^1_1 set in I^{int} and put $P = X - A$. Let φ be a Π^1_1 norm on P and consider

$$M = \{ x \in X : \{y : \neg(y <^*_\varphi x)\} \in I^{int}\}.$$

As in the proof of proposition 2.1.15 we have that M is Π^1_1. We claim that $A \subseteq M$. In fact, if $x \in A$ then by definition of $<^*_\varphi$ we have that

$$\{y : \neg(y <^*_\varphi x)\} = A.$$

By separation, let $B \subseteq M$ be a Δ^1_1 set with $A \subseteq B \subseteq M$. If $A = B$ we are done. Else let ξ be the least ordinal in $\{\varphi(x) : x \in B\}$ and let $x \in B$ with $\varphi(x) = \xi$. Then

$$B \subseteq \{y : \neg(y <^*_\varphi x)\}.$$

Hence $B \in I^{int}$.

From the proposition 2.1.20 we know that the collection of $\mathbf{\Sigma}^1_1$ sets in I^{int} form a $\sigma-$ideal, so $H(I) \in I^{int}$. As in the proof of 2.1.15 we can show that $H(I)$ is Π^1_1. This proves (i). And (ii) follows from the first claim.

□

The set $H(I)$ can be thought as an abstract version of the hyperarithmetic reals. By Theorem 2.1.17 the covering property for G_δ sets implies strong calibration, thus we immediately get

Theorem 2.1.23 *Let I be a Π^1_1 $\sigma-$ideal. If I has the covering property for Borel sets, then it has the covering property.* □

The covering property for $\mathbf{\Pi}^0_2$ sets can be deduced from a strong form of local-non-Borelness, as we will see next. We will show two versions of this result. The first works for ideals with a $\mathbf{\Sigma}_2^0$ basis. The second proof is due to A. Louveau and it is for meager ideals. We include both since it is not completely clear what extra information can be obtained from the construction given in the first proof.

Fisrt we need the following topological lemma.

Lemma 2.1.24 *Let $\{F_n\}$ be an increasing sequence of closed sets such that for all n and all $m > n$, $dist(F_n, F_m) \leq 1/2^n$. Then $\overline{\bigcup F_n} = lim_n F_n$.*

Proof: Let $K = \overline{\bigcup F_n}$. As $F_n \subseteq K$, then for all n, $dist(K, F_n) = sup\{d(y, F_n) : y \in K\}$. So it suffices to show that for all $y \in K$ and all n, $d(y, F_n) \leq 1/2^n$. Let $y \in K$ and fix n. Fix also a sequence $\{y_m\}$ such that $y_m \in F_m$ and $y = lim_m y_m$. For every k there is $m_k \geq n$ such that $d(y, y_{m_k}) \leq 1/k$. Now, as $dist(F_n, F_{m_k}) \leq 1/2^n$, then in particular $dist(y_{m_k}, F_n) \leq 1/2^n$. Thus there is $z_k \in F_n$ such that $d(y_{m_k}, z_k) \leq 1/2^n$. So we have that

$$d(y, z_k) \leq d(y, y_{m_k}) + d(y_{m_k}, z_k) \leq 1/k + 1/2^n.$$

By compactness, there is a subsequence $\{z'_k\}$ of $\{z_k\}$ and $z \in F_n$ such that $z'_k \to z$. Hence $d(y, z) \leq 1/2^n$. Thus $d(y, F_n) \leq 1/2^n$.

□

Theorem 2.1.25 *Let I be a $\mathbf{\Pi}_1^1$ dense $\sigma-$*ideal *of closed meager sets with a $\mathbf{\Sigma}_2^0$ basis. Then there is a continuous function $f : 2^\omega \to \mathcal{K}(X)$ such that*

(i) If α is eventually zero, then $f(\alpha)$ is a finite set.

(ii) If α is not eventually zero, then $f(\alpha) \notin I$.

Actually, for every given dense set D we can find f so that if α is eventually zero, then $f(\alpha) \subseteq D$.

In particular, if $J \subseteq I$ is a dense $\sigma-$ideal then J is not Borel. Moreover, the same holds locally, i.e., if F is a closed set locally not in I, and I' is the restriction of I to $\mathcal{K}(F)$, then every dense (in $\mathcal{K}(F)$) subideal of I' is not Borel.

Proof: Let $B = \bigcup_m L_m$ be a basis for I, with each L_m a closed set. Since $Her(L_m) = \{K : \exists F \in L_m \text{ such that } K \subseteq F\}$ is also a closed subset of I, we can assume without loss of generality that each L_m is hereditary. Also assume that $L_m \subseteq L_{m+1}$.

We claim that each L_m is meager: Suppose, towards a contradiction, that $W \subseteq L_m$ is an open set. As L_m is hereditary there is an open set $V \subseteq X$ such that $\mathcal{K}(V) \subseteq L_m$, which contradicts that every set in I is meager.

Fix a dense set $D \subseteq X$. We will define a sequence F_s for $s \in 2^{<\omega}$ such that

(1) F_s is a finite subset of D.

(2) If $s \prec t$, then $F_s \subseteq F_t$ and $dist(F_s, F_t) \leq 1/2^{lh(s)}$.

(3) For all $x \in F_s$ there is $K_x^s \notin L_{lh(s)}$ such that $K_x^s \subseteq F_{s\frown(1)}$ and $diam(K_x^s) \leq 1/2^{lh(s)+2}$.

(4) $F_{s\frown(0)} = F_s$.

Assuming this sequence has been defined we finish the proof. Put

$$f(\alpha) = \overline{\bigcup_n F_{\alpha\lceil n}}.$$

By the previous lemma we have that

$$f(\alpha) = lim_n F_{\alpha\lceil n}.$$

This clearly implies that f is continuous: In fact, we easily get that if $\alpha\lceil n = \beta\lceil n$, then $dist(F_{\alpha\lceil m}, F_{\beta\lceil m}) \leq 2/2^n$ for all $m > n$.

By (4), it is clear that if α is enventually zero, then $f(\alpha)$ is a finite subset of D. Now, suppose that α has infinite many 1's. We will show that $f(\alpha)$ is locally not in I. Put $F = f(\alpha)$. Let V be an open subset of X with $F \cap V \neq \emptyset$. Then there is n such that $F_{\alpha\lceil n} \cap V \neq \emptyset$. Let $x \in F_{\alpha\lceil n} \cap V$, thus $x \in F_{\alpha\lceil m} \cap V$, for all $m \geq n$. As $diam(K_x^{\alpha\lceil m}) \to 0$, then there is N such that for all $m \geq N$,

$$K_x^{\alpha\lceil m} \subseteq V \cap F_{\alpha\lceil m} \subseteq V \cap F.$$

Therefore for all $m \geq N$ $\overline{V \cap F} \notin L_m$, which implies that $\overline{V \cap F} \notin I$.

We define the sequence F_s by induction on the length of $s \in 2^{<\omega}$. Fix $x_0 \in D$ and let $F_\emptyset = \{x_0\}$. Suppose we have defined F_s for all $s \in 2^n$ and (1)-(4) are satisfied. Put $F_{s\widehat{\ }(0)} = F_s$. To define $F_{s\widehat{\ }(1)}$ consider the following: For every $x \in F_s$ let V_x^s be an open ball such that $x \in V_x^s$ and $diam(V_x^s) \leq 1/2^{lh(s)+2}$. As $L_{lh(s)}$ is meager, then there is $T_x^s \subseteq V_x^s$ such that $T_x^s \notin L_{lh(s)}$. As D is dense there is $K_x^s \subseteq D$ finite such that $K_x^s \subseteq V_x^s$. Now, one of those K_x^s's is not in $L_{lh(s)}$: Otherwise, as $L_{lh(s)}$ is closed, then T_x^s would be in $L_{lh(s)}$.

So put

$$F_{s\widehat{\ }(1)} = F_s \cup \{K_x^s : x \in F_s\}.$$

Notice, for every $y \in F_{s\widehat{\ }(1)}$ there is $x \in F_s$ such that $y \in K_x^s \cup F_s$ and $d(x,y) \leq 1/2^{lh(s)+1}$. Hence $dist(y, F_s) \leq 1/2^{lh(s)+1}$.

Thus $F_{s\widehat{\ }(1)}$ satisfies (1)-(4). This finishes the construction of f.

To finish the proof of the theorem, let $J \subseteq I$ be a dense $\sigma-$ideal. We will show that J is not Borel. By the dichotomy theorem it suffices to show that J is not $\mathbf{\Pi}_2^0$. Let $D = \{x \in X : \{x\} \in J\}$. As J is dense, so is D. We just have proved that there is a continuous reduction of the eventually zero sequences into the collection of finite subsets of D and the complement of I. In particular it says that we cannot separate with a G_δ set the collection of finite subsets of D from the complement of J. Hence J is not $\mathbf{\Pi}^0_2$.

Finally, let F be a closed set locally not in I and I' be the restriction of I to $\mathcal{K}(F)$. I' clearly has a Σ^0_2 basis and since F is locally not in I, then every set in I' is meager in F. Hence the same argument applies.

□.

As we have said before A. Louveau has given a more general argument:

Let I be a $\mathbf{\Pi}^1_1$ dense $\sigma-$ideal of closed meager sets which is meager (as a subset of $\mathcal{K}(X)$). For every dense set $D \subseteq X$ there is a continuous function $f : 2^\omega \to \mathcal{K}(X)$ as in the statement of the previous theorem and such if α is eventually zero, then $f(\alpha)$ is a finite subset of D. In particular, if $J \subseteq I$ is a dense $\sigma-$ideal then J is not Borel.

Let D be a countable dense subset of X such that for all $x \in D$ $\{x\} \in I$. Let $G \subseteq \mathcal{K}(X)$ be a G_δ dense set such that $I \cap G = \emptyset$. Put $A = \{F \in \mathcal{K}(X) : F \text{ is a finite subset of } D\}$. A is a dense F_σ set. By the Baire category theorem no F_σ set L separates G from A (i.e., $G \subseteq L$ and $L \cap A = \emptyset$). Hence by the Hurewicz-type theorem (see [14] theorem 4§1) there is a continuous function $f : 2^\omega \to \mathcal{K}(X)$ such that

(i) If α is eventually zero, then $f(\alpha) \in A$.

(ii) If α is not eventually zero, then $f(\alpha) \in G$.

This function clearly works.

Let us observe that if I has a $\mathbf{\Sigma}^0_2$ basis, then the collection of I-perfect sets is a $\mathbf{\Pi}^0_2$ dense set. Hence I is meager.

Remark: Suppose I is a $\sigma-$ideal which does not have non-trivial dense Borel subideal and suppose also that this holds locally i.e., if M is locally not in I, then $I \cap \mathcal{K}(M)$ does not have non-trivial dense (in $\mathcal{K}(M)$) Borel subideal. In particular, if $G \subseteq X$ is G_δ dense set, then $\mathcal{K}(G) \not\subseteq I$ i.e., $G \notin I^{int}$ and the same happens locally. That is to say, I has the covering property for $\mathbf{\Pi}^0_2$ sets. By the theorem 2.1.25 this is the case of a $\sigma-$ideal I with a $\mathbf{\Sigma}^0_2$ basis, in fact in [11] it was shown that such I has the covering property.

2.2 Products of σ-ideals

In this section we are going to present some results on products of $\sigma-$ideals from the definability point of view and also in relation with the covering property. At the end we will make a remark in relation with the Fubini theorem in this abstract setting of $\sigma-$ideals of compact sets.

Definition 2.2.1 *Let X and Y be compact Polish spaces. Let I and J be* $\sigma-$ideals *on X and Y respectively. Define the product of I and J as follows: Let $K \subseteq X \times Y$ be a closed set, denote by K_x the x-section of K, i.e.,* $K_x = \{y \in Y : (x, y) \in K\}$

$$K \in I \times J \text{ iff } \{x \in X : K_x \notin J\} \in I^{int}.$$

If J is $\mathbf{\Pi}^0_2$, then for every closed subset K of $X \times Y$ $\{x : K_x \notin J\}$ is $\mathbf{\Sigma}^0_2$. So $\{x : K_x \notin J\} = \bigcup_n F_n$ for some closed sets F_n. Then $K \in I \times J$ iff for all n, $F_n \in I$. We will see below that if I is also $\mathbf{\Pi}^0_2$, then $I \times J$ is a $\mathbf{\Pi}^0_2$ $\sigma-$ideal.

On the other hand if J is Π^1_1, then $\{x : K_x \notin J\}$ is $\mathbf{\Sigma}^1_1$. So, in order to get that $I \times J$ is a $\sigma-$ideal we need that the collection of $\mathbf{\Sigma}^1_1$ sets in I^{int} forms a $\sigma-$ideal. This happens, for instance, when I is strongly calibrated (by 2.1.20). We will show that under this hypothesis we also get that $I \times J$ is a Π^1_1 calibrated $\sigma-$ideal.

Proposition 2.2.2 *Let I and J be $\mathbf{\Pi}^0_2$ $\sigma-$*ideal*s of closed subsets of X and Y respectively. Then $I \times J$ is a $\mathbf{\Pi}^0_2$ $\sigma-$*ideal *of closed subset of $\mathcal{K}(X) \times \mathcal{K}(Y)$.*

Proof: Consider the following relation on $X \times \mathcal{K}(X)$

$$P_J(x, K) \iff K_x \in J.$$

Claim: P_J is $\mathbf{\Pi}^0_2$.

Proof: We have that

$$P_J(x, K) \iff (\forall L \in \mathcal{K}(Y))[L \subseteq K_x \Rightarrow L \in J].$$

Now, consider the relation: $R(x, K, L) \Leftrightarrow L \subseteq K_x$. Then

$$R(x, K, L) \iff (\forall V \text{ open in } Y)\ [K_x \subseteq V \Rightarrow L \subseteq \overline{V}].$$

For every open set V let $R_V(L) \Leftrightarrow L \subseteq \overline{V}$ and $R'_V(x, K) \Leftrightarrow K_x \subseteq V$. Clearly R_V is closed in $\mathcal{K}(Y)$ and

$$R'_V(x, K) \iff (\forall y \in Y)[(x, y) \in K \Leftrightarrow y \in V].$$

Thus $\sim R'_V$ is the projection of a compact set . Hence R'_V is open. Therefore R is closed and thus P_J is $\mathbf{\Pi}^0_2$. (□ Claim)

Put

$$\sim P_J(x, K) = \bigcup_n F_n(x, K)$$

with each F_n closed in $X \times \mathcal{K}(X \times Y)$. Put $P_J(K) = \{x : P_J(x, K)\}$, thus

$$\sim P_J(K) = \bigcup_n F_n(K).$$

Then

$$\begin{aligned} K \in I \times J \quad & \text{iff} \quad \{x : K_x \notin J\} \in I^{int} \\ & \text{iff} \quad [\textstyle\bigcup_n F_n(K)] \in I^{int} \\ & \text{iff} \quad (\forall n)[F_n(K) \in I]. \end{aligned}$$

As before we have that $\{K \in \mathcal{K}(X \times Y) : F_n(K) \in I\}$ is $\mathbf{\Pi}^0_2$. Therefore $I \times J$ is $\mathbf{\Pi}^0_2$.

It is clear that $I \times J$ is hereditary. Let $K = \bigcup K_n$ be a closed set with each $K_n \in \mathcal{K}(X \times Y)$. As before we get that

$$\{x : K_x \notin J\} = \bigcup_m \{x : (K_m)_x \notin J\} = \bigcup_{n,m} F_n(K_m).$$

Thus

$$\begin{aligned} K \in I \times J \quad & \text{iff} \quad (\forall n)(\forall m) F_n(K_m) \in I \\ & \text{iff} \quad (\forall m) K_m \in I \times J. \end{aligned}$$

Hence $I \times J$ is a $\sigma-$ideal.

□

As we said before in the case that I and J are Π^1_1 we need an extra hypothesis to get a similar result as in 2.2.2.

Proposition 2.2.3 *Suppose I is a strongly calibrated Π^1_1 $\sigma-$ideal on X and J a Π^1_1 calibrated $\sigma-$ideal on Y. Then $I \times J$ is a calibrated Π^1_1 $\sigma-$ideal on $X \times Y$.*

Proof: For every $K \in \mathcal{K}(X \times Y)$ $\{x : K_x \notin J\}$ is a $\mathbf{\Sigma}^1_1$ set. By 2.1.20 we know that the collection of $\mathbf{\Sigma}^1_1$ sets in I^{int} is a $\sigma-$ideal. From this we easily get that $I \times J$ is a $\sigma-$ideal.

To show that $I \times J$ is Π^1_1 consider the following relation: Let $Q \subseteq \mathcal{K}(Y) \times 2^\omega$ be a $\mathbf{\Pi}^0_2$ set such that

$$F \notin J \text{ iff } \exists \alpha Q(F, \alpha).$$

Then given $K \in \mathcal{K}(X \times Y)$ and $x \in X$ we have

$$K_x \notin J \text{ iff } \exists \alpha \exists F(F = K_x \ \& \ Q(F, \alpha)).$$

So consider the following relation on $X \times \mathcal{K}(Y) \times 2^\omega \times \mathcal{K}(X \times Y)$

$$R(x, F, \alpha, K) \Leftrightarrow F = K_x \ \& \ Q(F, \alpha).$$

It is easy to check that R is $\mathbf{\Pi}^0_2$. We get

$$\{x : K_x \notin J\} = \{x : \exists \alpha \exists F(R(x, F, \alpha, K))\}.$$

Since I is strongly calibrated we get

$$\{x : K_x \notin J\} \notin I^{int} \text{ iff } \exists P \in \mathcal{K}(X \times \mathcal{K}(Y) \times 2^\omega)[proj(P) \notin I \ \& \ P \subseteq R_K]$$

where

$$R_K = \{(x, F, \alpha) \in X \times \mathcal{K}(Y) \times 2^\omega : R(x, F, \alpha, K)\}.$$

And we have

$$P \subseteq R_K \text{ iff } \forall x \in X \forall F \in \mathcal{K}(Y) \forall \alpha \in 2^\omega((x, F, \alpha) \in P \Rightarrow R(x, F, \alpha, K))$$

which clearly is a $\mathbf{\Pi}^0_2$ relation on P and K. Hence $\{x : K_x \notin J\} \notin I^{int}$ is a $\mathbf{\Sigma}^1_1$ relation on K, i.e., $I \times J$ is Π^1_1.

To finish we will show that $I \times J$ is calibrated. We will need the following

Claim: Let $G \subseteq X \times Y$ be a $\mathbf{\Pi}^0_2$ set. Then $G \in (I \times J)^{int}$ iff $\{x : G_x \notin J^{int}\} \in I^{int}$.

Proof: First suppose $\{x : G_x \notin J^{int}\} \in I^{int}$. Let $K \subseteq G$ be a closed set. Then

$$\{x : K_x \notin J\} \subseteq \{x : G_x \notin J^{int}\}$$

hence $K \in I \times J$, i.e., $G \in (I \times J)^{int}$.

Conversely, suppose $\{x : G_x \notin J^{int}\} \notin I^{int}$ and let $H \subseteq \{x : G_x \notin J^{int}\}$ with $H \notin I$. Consider the following relation on $X \times \mathcal{K}(Y)$

$$R(x, F) \Leftrightarrow F \subseteq G_x \ \& \ F \notin J \ \& \ x \in H.$$

R is $\mathbf{\Sigma}^1_1$ and $proj(R) = H$. As I is strongly calibrated there is a closed $Q \subseteq R$ such that $proj(Q) \notin I$. Define $P \subseteq X \times Y$ as follows

$$P(x, y) \Leftrightarrow \exists F \in \mathcal{K}(Y)(y \in F \& (x, y) \in Q).$$

As $Q \subseteq R$ then P is a (closed) subset of G and $proj(Q) = \{x : P_x \notin J\} \notin I$. Hence $P \notin I \times J$, i.e., $G \notin (I \times J)^{int}$.

(Claim $\square$)

Let $K = G \cup \bigcup_n H_n$ be a closed set , where $G \in (I \times J)^{int}$ is $\mathbf{\Pi}^0_2$ and each H_n is in $I \times J$. We want to show that $K \in I \times J$. For all x we have

$$K_x = G_x \cup \bigcup_n (H_n)_x.$$

Since J is calibrated one easily gets that

$$K_x \notin J \text{ iff } G_x \notin J^{int} \text{ or } (\exists n)[(H_n)_x \notin J.$$

That is to say

$$\{x : K_x \notin J\} = \{x : G_x \notin J^{int}\} \cup \bigcup_n \{x : (H_n)_x \notin J\}.$$

By the claim $\{x : G_x \notin J^{int}\} \in I^{int}$ and since every $H_n \in I \times J$ then $\{x : (H_n)_x \notin J\} \in I^{int}$. As I is strongly calibrated, the collection of $\mathbf{\Sigma}^1_1$ sets in I^{int} is a σ−ideal. So we get $\{x : K_x \notin J\} \in I^{int}$, i.e., $K \in I \times J$.

□

In relation with the covering property we have the following

Proposition 2.2.4 *Let I and J be σ−ideals of meager closed sets on X and Y respectively. If $I \times J$ has the covering property for $\mathbf{\Pi}^0_2$ sets, then I and J has the covering property for $\mathbf{\Pi}^0_2$ sets.*

Proof: Suppose I does not have the covering property for $\mathbf{\Pi}^0_2$ sets. By 2.1.4 there is a locally non in I closed set M and a $\mathbf{\Pi}^0_2$ set G with $\overline{G} = M$ and $G \in I^{int}$. Put $H = G \times Y$. Clearly H is a $\mathbf{\Pi}^0_2$ set and $G \in (I \times J)^{int}$ (if $K \subseteq H$, then $\{x : K_x \notin J\} = G$). Also $\overline{H} = M \times Y$. So, it remains to show that $\overline{H}$ is locally not in $I \times J$. Let $V \subseteq X$,$W \subseteq Y$ be open sets. Then $(V \times W) \cap H = (V \cap G) \times W$. Thus

$$\{x : [\overline{(V \times W) \cap H}]_x \notin J\} = \{x : [(\overline{V} \cap M) \times \overline{W}]_x \notin J\} = \overline{V} \cap M \notin I$$

(since for every open set W, $\overline{W} \notin J$).

Analogously, if J does not have the covering property, then a similar argument shows that $I \times J$ does not have the covering property.

□

Given two ideals I and J on X there is a natural question regarding the definition of $I \times J$: Let $K \subseteq X \times X$ be a closed set, does the following hold:

$$\{x : K_x \notin J\} \in I^{int} \text{ iff } \{y : K_y \notin I\} \in J^{int}. \qquad (*)$$

In other words is $I \times J = J \times I$?

In particular if $I = J$ we say that I has the *Fubini property* if (*) holds for every closed $K \subseteq X \times X$. For instance, if $I = Null(\mu)$ for a measure μ on X then Fubini theorem says that I has the Fubini property. Also, if I is the ideal of meager sets, the Kuratowski- Ulam theorem (see [16]) implies that I has the Fubini property. In relation with this property we have the following

Proposition 2.2.5 *Let I be a* Π^1_1 $\sigma-$ideal *of closed subsets of 2^ω. If I is not thin, then I does not have the Fubini property. In particular, if I has the Fubini property and is non trivial in the sense of 2.1.7, then I does not have the covering property for $\mathbf{\Pi}^0_2$ sets.*

Proof: By theorem 2§3 on [14], as I is not thin, there is a continuous function $f : 2^\omega \to \mathcal{K}(2^\omega)$ such that

(i) For all $\alpha \in 2^\omega$ $f(\alpha) \notin I$.

(ii) For all $\alpha, \beta \in 2^\omega$, if $\alpha \neq \beta$ then $f(\alpha) \cap f(\beta) = \emptyset$.

Consider the following subset of $2^\omega \times 2^\omega$

$$K(\alpha, \beta) \text{ iff } \alpha \in f(\beta)$$

then

$$K(\alpha, \beta) \text{ iff } (\exists F)(\alpha \in F \& f(\beta) = F).$$

As f is continuous K is closed. We have that

$$\{\beta : K^\beta \notin I\} = 2^\omega \text{ and } \{\alpha : K_\alpha \notin I\} = \emptyset.$$

Hence I does not have the Fubini property. The last part of the proposition follows directly from 2.1.7.

$\square$

Remark: For an arbitrary compact Polish space X we can analogously get that there is a Borel set $B \subseteq X \times X$ such that $\{\beta : B^\beta \notin I^{int}\} = 2^\omega$ and $\{\alpha : B_\alpha \notin I\} = \emptyset$ (but actually every section B^β and B_α is closed). The reason is that in this case the thickness witness $f : X \to \mathcal{K}(X)$ is a Borel function.

Chapter 3

The covering property for Σ_2^1 sets

In this chapter we are going to present some results related to the covering property for Σ_2^1 sets. Throughout X will be a compact, perfect recursively presented Polish space. As we have already mentioned, given a Π_1^1 $\sigma-$ideal I of closed subsets of X, it is not provable in ZFC that every Π_1^1 set in I^{int} can be covered by countably many sets in I. We will prove that (as in the case of the ideal of countable sets) if there are only countable many reals in L, then every Π_1^1 $\sigma-$ideal of closed meager subsets of 2^ω with the covering property also has this property for Σ_2^1 sets.

The proof is based in a generalization of well known facts about the ideal of countable sets. In particular we will show that for every Π_1^1 $\sigma-$ideal of meager sets with the covering property there is a largest Π_1^1 set in I^{int}, which for ideals on 2^ω it has a similar characterization as the one for the largest Π_1^1 set without perfect subset. In §1 we present this generalization and in §2 we get as a corollary the result mentioned above. Also, we get a generalization of the well known result of Solovay that if there are only countable many reals in L, then $\omega^\omega \cap L$ is the largest countable Σ^1_2 set.

The only criterion known to show that a $\sigma-$ideal has the covering property is a theorem due to Debs and Saint Raymond. This theorem can be naturally extended to κ-Suslin sets. We present this result in §3.

3.1 The largest Π^1_1 set in I^{int}

In this section we will prove the following theorem:

Theorem 3.1.1 *Let I be a Π^1_1 $\sigma-$ideal of meager subsets of 2^ω with the covering property. Then there is a largest Π^1_1 set $C_1(I)$ in I^{int} which is characterized by*

$$x \in C_1(I) \text{ iff } \exists T \in L_{\omega_1^x} (\ T \text{ is a tree on } 2 \ \& \ x \in [T] \ \& \ [T] \in I).$$

This is a generalization of C_1, the largest Π^1_1 set without perfect subset which is characterized by $\alpha \in C_1$ iff $\alpha \in L_{\omega_1^\alpha}$ (see [8] and [9] for similar results on $\sigma-$ideals on ω^ω defined by games).

Before we give the proof of 3.1.1 we will present some results related to the general case of $\sigma-$ideals on an arbitrary recursively presented perfect Polish space X.

There is a theorem due to Kechris (see [8] 1A-2) that gives sufficient conditions for the existence of such a largest Π^1_1 set for $\sigma-$ideals of subsets of X. One of these conditions is the so called Π^1_1-additivity. We will show next that for every $\sigma-$ideal I of meager subsets of X, if I has the covering property, then I^{int} is Π^1_1-additive. The proof is based on a representation of I as the common meager closed sets for a collection of Polish topologies on X.

Definition 3.1.2 *For every topology τ on X, let Meager(τ) be the collection of τ-closed τ-meager sets. We say that a topology τ on X is* compatible with I *if τ extends the original topology on X, every τ-open set is Borel and $I \subseteq$ Meager(τ).*

Observe that in this case the Borel structure of X and (X, τ) are the same. In particular every C-measurable subset $B \subseteq X$ has the property of Baire with respect to τ (C is the least σ-algebra containing the open sets and closed under the Suslin operation).

Lemma 3.1.3 *Let I be a $\sigma-$*ideal *of meager closed subsets of a compact Polish space X. Then we have*

$$I = \bigcap\{Meager(\tau) \cap \mathcal{K}(X) \colon \tau \text{ is a Polish topology on } X \text{ compatible with } I\}.$$

Proof: One direction is obvious. Let $K \notin I$. We want to find a Polish topology τ on X compatible with I and such that K is not τ-meager. Without loss of generality we assume that K is locally not in I. Let τ_0 be the given topology on X and consider the topology τ generated by

$$\tau_0 \cup \{V \cap K : V \in \tau_0\}.$$

It is a standard fact that τ is the least Polish topology for which K is τ-clopen. It remains only to show that $I \subseteq Meager(\tau)$. But this is clear, because as K is locally not in I, for every $V \in \tau_0$ if $V \cap K \neq \emptyset$,then $\overline{V \cap K} \notin I$. Hence for every $F \in I$, $V \cap K \not\subseteq F$.

□

Also we get a characterization of I^{int}.

Theorem 3.1.4 *Let I be a $\sigma-$*ideal *of meager subsets of X with the covering property and let B be a subset of X with the property of Baire with respect to every Polish topology compatible with I. The following are equivalent:*

(i) $B \in I^{int}$.

(ii) B is τ-meager for every topology on X compatible with I.

Proof: (i) $\Rightarrow$(ii). Suppose that B is not τ-meager for some topology τ compatible with I. As B has the property of Baire for τ, then there is a τ-open set V such that B is τ-comeager in V. So, let G be a τ-G_δ set τ-dense in V and $G \subseteq B$. As τ consists of Borel sets then G is also Borel. We claim that $G \notin I^{int}$. Otherwise, as I has the covering property, there are closed sets $\{F_n\}$ in I such that $G \subseteq \bigcup_n F_n$. Each F_n is τ-closed, hence by the Baire category theorem there is a τ-open set W and an n such that $\emptyset \neq W \cap G \subseteq F_n$. But as G is τ-dense in V we get that F_n is not τ-meager, which contradicts that τ is compatible with I.

(ii) $\Rightarrow$ (i). It follows directly from the previous lemma.

□

Let us recall the definition of Π^1_1-additivity (see [8]): A hereditary collection J of subsets of X is called Π^1_1-*additive* if for every sequence $\{A_\xi\}_{\xi<\theta}$ of sets in J such that the associated prewellordering

$$x \preceq y \text{ iff } x, y \in \bigcup_{\xi<\theta} A_\xi \ \ \& \ \text{ least } \xi\,(x \in A_\xi) \leq \ \text{ least } \xi\,(y \in A_\xi)$$

is $\mathbf{\Pi^1_1}$, we have that $\bigcup_{\xi<\theta} A_\xi \in J$. As we said before, we have the following

Corollary 3.1.5 *Let I be a $\sigma-$*ideal *of closed meager subsets of X with the covering property. Then I^{int} is Π^1_1-additive.*

Proof: The proof is the same as in the case of the $\sigma-$ideal of closed meager sets (see [8]). Towards a contradiction, assume θ is the least ordinal such that there is a sequence $\{A_\xi\}_{\xi<\theta}$ of sets in I^{int} such that the associated prewellordering $\preceq$ is $\mathbf{\Pi}^1_1$, but $\bigcup_{\xi<\theta} A_\xi \notin I^{int}$.

First we observe that θ is a limit ordinal: Otherwise let $\theta = \eta + 1$ and pick $x \in A_\eta - \bigcup_{\xi<\eta} A_\xi$. The associated prewellordering of $\{A_\xi : \xi < \eta\}$, $\preceq_\eta$, is also $\mathbf{\Pi}^1_1$, because $z \preceq_\eta w$ iff $z \preceq w$ & $w \prec x$. By the minimality of θ we have that $\bigcup_{\xi<\eta} A_\xi \in I^{int}$. Also we have that $A_\eta = \{z : y \preceq z\}$, where y is any point in $A_\eta - \bigcup_{\xi<\eta} A_\xi$. Thus A_η and $\bigcup_{\xi<\eta} A_\xi$ are in the σ-algebra generated by the $\mathbf{\Pi}^1_1$ sets and therefore they have the property of Baire for every Polish topology compatible with I. Therefore by the previous theorem they are τ meager for any of such topologies. Thus $\bigcup_{\xi<\eta+1} A_\xi \in I^{int}$, which is a contradiction.

Let $K \subseteq \bigcup_{\xi<\theta} A_\xi$ with $K \notin I$ and fix a Polish topology τ compatible with I such that K is not τ meager. The restriction of $\preceq$ to $K \times K$ is $\mathbf{\Pi}^1_1$ and hence it has the property of Baire with respect to τ. We can assume that we are working in (K, τ). For every $x \in K$ we have

$$S_x = \{y \in K : y \preceq x\} \subseteq \bigcup_{\xi<\eta} A_\xi$$

for some $\eta < \theta$ (as θ is limit). Hence by the minimality of θ we have that $S_x \in I^{int}$. From the previous theorem we get S_x is τ-meager. By the Kuratowski-Ulam theorem (see for instance [16]) we know that for τ-comeager many y's, $S^y = \{x \in K : y \preceq x\}$ is τ-meager. So as $K = S_y \cup S^y$, then K is τ-meager, which is a contradiction.

□

And then we get the following

Corollary 3.1.6 *Let I be a Π^1_1 $\sigma-$*ideal *of closed meager subsets of X with the covering property. There exists a largest Π^1_1 set in I^{int}.*

Proof: In order to apply theorem 1A-2 in [8] we need only to show that the collection of Σ^1_1 sets in I^{int} is Π^1_1 on the codes. This is a consequence of the fact that I is strongly calibrated, as we have shown this in chapter 2 (Proposition 2.1.15 and 2.1.17).

□

Remark: If we trace back how much the covering property is needed to prove these theorems we see that it would be sufficient with the covering property for G_δ sets. This is because the topologies used in the proof of 3.1.3 admit a basis consisting of G_δ sets in the original topology of X.

From now on we fix a Π^1_1 $\sigma-$ideal I of closed meager subsets of 2^ω with the covering property. There is a derivative operator on closed sets similar to the Cantor-Bendixson derivative which will provide us with canonical closed sets to cover a given Σ^1_1 set in I^{ext}.

Definition 3.1.7 *Let S be a tree on $2 \times \omega$; define a derivative as follows*

$$(s,u) \in S^{(1)} \textit{ iff } \overline{p[S_{(s,u)}]} \notin I.$$

By transfinite recursion we define S^η for every ordinal η.

Notice that S^η is also a tree on $2\times\omega$ and $S^{\eta+1} \subseteq S^\eta$. There is a countable ordinal θ such that $S^{\theta+1} = S^\theta$. We denote this fixed point by S^∞.

Lemma 3.1.8 $S^\infty = \emptyset$ *iff* $p[S] \in I^{ext}$.

Proof: Suppose that $S^\infty = \emptyset$. Let θ be a countable ordinal such that $S^\theta = \emptyset$. Since $([S^\eta])$ is a decreasing sequence of sets, we have

$$p[S] \subseteq \bigcup\{\overline{p[S^\alpha_{(s,u)}]} : \overline{p[S^\alpha_{(s,u)}]} \in I \;\&\; \alpha < \theta \;\&\; (s,u) \in S\}.$$

This clearly shows that $p[S] \in I^{ext}$.

On the other hand suppose that $p[S] \in I^{ext}$. Say $p[S] \subseteq \bigcup K_n$ with $K_n \in I$. Let $L = [S^\infty]$. We have that $L \subseteq \bigcup(K_n \times \omega^\omega)$. Towards a contradiction suppose that $L \neq \emptyset$. By the Baire category theorem there is an n, $(s,u) \in S^\infty$ such that $\emptyset \neq L \cap (N_s \times N_u) \subseteq K_n \times \omega^\omega$. Hence $\overline{p[S^\infty_{(s,u)}]} \in I$, which contradicts that $(s,u) \in S^\infty$.

□

Before proving the necessary lemmas to prove theorem 3.1.1 let us give an idea of how the proof goes. Fix a Π^1_1 set $A \in I^{int}$. Let T be a recursive tree on $2 \times \omega$ such that

$$x \in A \text{ iff } T(x) \text{ is wellfounded.}$$

Let $x \in A$ and let $\xi = \mid T(x) \mid$. There is a canonical way of defining a tree S_ξ on $2 \times \xi$ such that

$$\mid T(x) \mid \leq \xi \text{ iff } S_\xi(x) \text{ is not wellfounded.}$$

Put $S = S_\xi$. As $p[S]$ is a $\boldsymbol{\Sigma}^1_1$ subset of A and $A \in I^{int}$, then $p[S] \in I^{ext}$. We can easily translate the definition of the derivative to the space $2 \times \xi$. Hence by 3.1.8 $S^\infty = \emptyset$. Thus the closed sets $\overline{p[S^\alpha_{(s,u)}]}$, as in the proof of 3.1.8, cover $p[S]$. The key of the proof is the fact that for each of these closed sets we can find a tree $T^\alpha_{(s,u)}$ in the least admisible set containing ξ such that

$$\overline{p[S^\alpha_{(s,u)}]} \subseteq [T^\alpha_{(s,u)}] \in I.$$

Since clearly $\xi < \omega_1^x$, this tree belongs to $L_{\omega_1^x}$, and we are done.

We will define the trees S_ξ uniformly on the codes of ξ using the following

Lemma 3.1.9 *(Shoenfield see [15]) Let T be a recursive tree on $2 \times \omega$. Let $A \subseteq 2^\omega$ be defined by*

$$x \in A \text{ iff } T(x) \text{ is wellfounded.}$$

Define also for each countable ordinal ξ

$$x \in A_\xi \text{ iff } \mid T(x) \mid \leq \xi.$$

There is a recursive relation $S \subseteq \omega^\omega \times 2^{<\omega} \times \omega^{<\omega}$ such that

(i) if $\mathrm{w} \in WO$ and $\mid \mathrm{w} \mid = \xi$, then $S(\mathrm{w}) = \{(t, s) : S(\mathrm{w}, t, u)\}$ is a tree on $2 \times \omega$ such that

$$x \in A_\xi \text{ iff } S(\mathrm{w})(x) \text{ is not wellfounded.}$$

(ii) There is a tree S_ξ on $2 \times \xi$ (as we mentioned before) such that $p[S_\xi] = A_\xi$ and this tree belongs to the least admisible set containing ξ. Moreover, given a sequence $u \in \omega^{<\omega}$, we can think that u codes a sequence of ordinals h by using the wellorder of ω given by w and such that

$$(t, u) \in S(\mathrm{w}) \text{ iff } (t, h) \in S_\xi.$$

Thus if w, $\mathrm{z} \in WO$ and $\mid \mathrm{w} \mid = \mid \mathrm{z} \mid = \xi$, then $S(\mathrm{w})$ and $S(\mathrm{z})$ code essentially the same tree S_ξ.

□

In the following lemma we compute the complexity of the derivative defined above.

Lemma 3.1.10 *Let I be a Π^1_1 $\sigma-$*ideal *of closed subsets of 2^ω with the covering property. Let T and S as in lemma 3.1.9.*

(i) There is a Σ^1_1 relation P on $\omega \times \omega \times \omega^\omega$ such that for v,w$\in WO$ *we have*

$$P(t,u,\mathrm{v},\mathrm{w}) \textit{ iff } (t,u) \in [S(\mathrm{w})]^{|\mathrm{v}|}.$$

Here $[S(\mathrm{w})]^{|\mathrm{v}|}$ is defined as in 3.1.7.

(ii) Let A and A_ξ be defined as in 3.1.9 and suppose that $A \in I^{int}$. For every $\xi < \omega_1$ and every $\mathrm{w} \in WO$ with $| \mathrm{w} | = \xi$, the closure ordinal of $S(\mathrm{w})$ is $< \xi^+$ (the least admissible ordinal bigger than ξ).

Proof: First we claim there is a Σ^1_1 relation D on $\omega \times \omega \times \omega^\omega$ such that

$$D(t,u,J) \text{ iff } J \text{ is a tree on } 2 \times \omega \ \& \ (t,u) \in J^{(1)}.$$

To see this, consider the following relation

$$B(x,J) \text{ iff } J \text{ is a tree on } 2 \times \omega \ \ \& \ x \in \overline{proj[J]}.$$

B is clearly Σ^1_1 and $D(t,u,J)$ iff $B(J_{(t,u)}) \notin I^{int}$. We have shown in chapter 2 (proposition 2.1.15) that the collection of Σ^1_1 sets in I^{int} is Π^1_1 on the codes of $\mathbf{\Sigma}^1_1$ sets; this easily implies that D is Σ^1_1.

We will use the recursion theorem to define P. Let $\mathcal{U}$ be a Σ^1_1 universal set on $\omega \times \omega \times \omega^\omega \times \omega^\omega \times \omega^\omega$. Consider the following relation

$$\begin{aligned} Q(t,u,\mathrm{v},\mathrm{w},\rho) \text{iff } & \mathrm{v} \notin WO \text{ or } (\mathrm{v} \in LO \ \& \ \mathrm{v} \equiv \emptyset \ \& \ S(t,u,\mathrm{w})) \\ & \text{or } (\exists \mathrm{z})(\mathrm{v},\mathrm{z} \in LO \ \& \ \mathrm{v} \equiv \mathrm{z}+1 \ \& \ D(t,u,\{(l,k) : \mathcal{U}(l,k,\mathrm{z},\mathrm{w},\rho)\})) \\ & \text{or } (\forall n)\mathcal{U}(t,u,\mathrm{v}\lceil n,\mathrm{w},\rho) \end{aligned}$$

where $\mathrm{v} \equiv \emptyset$ means that v codes the empty order; $\mathrm{v} \equiv \mathrm{z}+1$ means that the linear order coded by v has a last element and z is the linear order obtained by deleting this last element and $\mathrm{v}\lceil n$ is the linear order obtained by restricting v to $\{m : m <_{\mathrm{v}} n\}$.

Notice that $D(t,u,A)$ holds iff $\exists B(B \subseteq A \ \& \ D(t,u,B))$ (i.e., it is a monotone operator), hence Q is Σ^1_1. By the recursion theorem there is a recursive ρ^* such that

$$Q(t,u,\mathrm{v},\mathrm{w},\rho^*) \longleftrightarrow \mathcal{U}(t,u,\mathrm{v},\mathrm{w},\rho^*).$$

As usual, put

$$P(t,u,\mathrm{v},\mathrm{w}) \longleftrightarrow \mathcal{U}(t,u,\mathrm{v},\mathrm{w},\rho^*).$$

By induction on the length of $\mathrm{v} \in WO$ one can easily show that if $\mathrm{w} \in WO$, then

$$P(t,u,\mathrm{v},\mathrm{w}) \longleftrightarrow (t,u) \in [S(\mathrm{w})]^{|\mathrm{v}|}.$$

(ii) Let $\mathrm{w} \in WO$ with $|\mathrm{w}| = \xi$ and let $S = S(\mathrm{w})$. $A_\xi = p[S]$ is a Σ^1_1 set in I^{int}. As I has the covering property, then by lemma 3.1.8 $S^\infty = \emptyset$. Since the derivative operartor is Σ^1_1 it is an standard fact that in this case the closure ordinal of S is recursive in S, hence recursive in w.

From 3.1.9 we also get the following: Let $\mathrm{z} \in WO$ with $|\mathrm{w}| = |\mathrm{z}| = \xi$ and let $u,v \in \omega^{<\omega}$. If u,v code the same sequence of ordinals with respect to the wellorder of ω given by w and z respectively, then

$$(t,u) \in S(\mathrm{w})^{(1)} \text{ iff } (t,v) \in S(\mathrm{z})^{(1)}.$$

In particular the clousure ordinal of $S(\mathrm{w})$ and of $S(\mathrm{z})$ are the same. Let then z be a generic (with respect to the partial order that collapses ξ to ω)

ordinal code for ξ. It is an standard fact that $\omega_1^z = \xi^+$. This finishes the proof of (ii).

□

A key fact in the proof is that the trees $S(\mathrm{w})$ in the previous lemma have an invariant definition in the following sense.

Definition 3.1.11 *Let $\sim$ be an equivalence relation on ω^ω and Γ be a pointclass. We say that a set A is $\sim$-invariantly-$\Gamma(\alpha)$ if there is a Γ relation R on $X \times \omega^\omega$ such that for every $\beta \sim \alpha$ we have*

$$x \in A \text{ iff } R(x, \beta).$$

In particular A is called $\sim$-invariantly-$\Delta^1_1(\alpha)$, if A is both $\sim$-invariantly-$\Sigma^1_1(\alpha)$ and $\sim$-invariantly-$\Pi^1_1(\alpha)$.

Consider the following equivalence relation on ω^ω : Let LO be the collection of codes of linear orders of ω . We say that two codes α and β in LO are isomorphic if the linear orders coded by them are isomorphic. Define $\equiv$ by

$$\alpha \equiv \beta \text{ iff } \alpha, \beta \in \mathrm{LO} \ \& \ \alpha \text{ and } \beta \text{ are isomorphic.}$$

It is an standard fact that $\equiv$ is a Σ^1_1 relation (see [15]). The following two lemmas make clear why it is interesting to look at the notion of $\equiv$-invariantly definable sets.

Lemma 3.1.12 *Let ξ be a countable ordinal and w an ordinal code for ξ. Let $T \subseteq \omega$ be a $\equiv$-invariantly-$\Delta^1_1(\mathrm{w})$ set. Then T belongs to the least admissible set containing ξ.*

Proof: Let M denote the least admissible set containing ξ. We will show that T is $\underset{\sim}{\Delta}_1$ definable over M. Let $R \subseteq \omega \times \omega^\omega$ be a Π^1_1 set such that for all ordinal codes w with $|\mathrm{w}| = \xi$, we have

$$s \in T \text{ iff } R(s, \mathrm{w}).$$

Let ψ be a Σ_1 formula (in ZF) such that if N is an admissible set and w$\in N$, then

$$R(s, \mathrm{w}) \text{ iff } N \models \psi(s, \mathrm{w}) \tag{*}$$

Consider the notion of forcing $\mathbf{P}$ that collapses ξ to ω. If G is $\mathbf{P}$-generic, let w_G be the corresponding ordinal code, i.e.,

$$\mathrm{w}_G(n, m) = 0 \text{ iff } \exists p \in G(p(n) < p(m)).$$

Consider the following name

$$\tau = \{\langle \sigma, p \rangle : \sigma = \langle (n, m)\check{}, 0 \rangle \text{ and for some ordinals } \alpha < \beta,$$
$$\langle n, \alpha \rangle, \langle m, \beta \rangle \in p\}.$$

Then for every $\mathbf{P}$-generic G, $i_G(\tau) = \mathrm{w}_G$. Since for every admissible set N, $N[G]$ is also admissible, then from (*) we get

$$R(s, \mathrm{w}_G) \text{ iff } M[G] \models \psi(s, \mathrm{w}_G). \tag{**}$$

As (**) holds for every G $\mathbf{P}$-generic, then

$$s \in T \text{ iff } \Vdash \psi(\check{s}, \tau).$$

Since ψ is Σ_1, the relation $B(s, \tau)$ iff $\Vdash \psi(\check{s}, \tau)$ is Σ_1 over M. Hence T is $\underset{\sim}{\Sigma}_1$ over M. Similarly we have that $s \notin T$ is $\underset{\sim}{\Sigma}_1$ over M. This finishes the proof.

□

There is another basic fact about Σ^1_1 equivalence relations and Π^1_1 sets that we are going to use.

Definition 3.1.13 *(Solovay [10]) Let $\sim$ be an equivalence relation on ω^ω and $P \subseteq \omega^\omega$ be a $\sim$-invariant set, i.e., if $x \in P$ and $y \sim x$ then $y \in P$. A norm $\varphi : P \to$ ordinals is called* $\sim$-invariant *if*

$$x \sim y \ \& \ x \in P \Rightarrow \varphi(x) = \varphi(y).$$

Let Γ be a pointclass. We say that Γ is invariantly normed *if for every equivalence relation $\sim$ in $\check{\Gamma}$ and every $\sim$-invariant set P in Γ, P admits a $\sim$-invariant norm.*

It was proved by Solovay (see [10]) that Π^1_1 is invariantly normed.

Let K be a closed subset of 2^ω, recall that the tree of K, T_K is defined as follows:

$$s \in T_K \text{ iff } K \cap N_s \neq \emptyset.$$

Conversely, given any $T \subseteq 2^{<\omega}$ we define a closed set $[T]$ by

$$x \in [T] \text{ iff } (\forall n)(\exists s \in T \text{ such that } x\lceil n \prec s).$$

Notice that for every closed K, $[T_K] = K$.

The following result will be crucial for the proof of 3.1.1.

Lemma 3.1.14 *(see Barua-Srivatsa [1])Let $\sim$ be a Σ^1_1 equivalence relation on ω^ω and $A \subseteq 2^\omega$ be a $\sim$-invariantly $\Sigma^1_1(\alpha)$ set. If $\overline{A} \in I$, then there is a $\sim$-invariantly-$\Delta^1_1(\alpha)$ tree T such that $A \subseteq [T]$ and $[T] \in I$.*

Proof: Suppose not, towards a contradiction. So, in particular the tree of $\overline{A}$, $T_A = \{s : N_s \cap A \neq \emptyset\}$ is not $\sim$-invariantly $\Delta^1_1(\alpha)$. Let R be a Σ^1_1 relation such that for all $\beta \sim \alpha$

$$x \in A \text{ iff } R(x, \beta).$$

Then for every $\beta \sim \alpha$

$$s \in T_A \text{ iff } (\exists y)(y \in N_s \& R(y, \beta)).$$

Hence T_A is $\sim$-invariantly-$\Sigma^1_1(\alpha)$. We will show that it is also $\sim$-invariantly-$\Pi^1_1(\alpha)$. Put $T = T_A$. Let $Q \subseteq \omega \times \omega^\omega$ be a Σ^1_1 set such that for all $\beta \sim \alpha$ we have that

$$s \in T \text{ iff } Q(s, \beta). \qquad (*)$$

Consider the following equivalence relation on $\omega \times \omega^\omega$:

$$(s, \alpha) \approx (t, \beta) \text{ iff } s, t \text{ codes binary sequences, } s = t \text{ and } \alpha \sim \beta$$

then $\approx$ is Σ^1_1. We want to put an $\approx$-invariant norm on $\sim Q$. For that end we need to make Q $\approx$-invariant. So let P be the $\approx$-saturation of Q, i.e.,

$$P(s, \gamma) \text{ iff } (\exists \beta)(\gamma \sim \beta \& Q(s, \beta)).$$

Observe that for every $\beta \sim \alpha$, (*) above still holds for P. Let φ be a $\approx$-invariant norm on $\sim P$. We claim that for every $\beta \sim \alpha$ we have

$$s \in T \text{ iff } (\forall S)\{[S \subseteq 2^{<\omega} \& (\forall t)(t \in S \Rightarrow \neg((t, \beta) <^*_\varphi (s, \beta))] \Rightarrow [S] \in I\}.$$

Assuming this claim we clearly have that T is $\sim$-invariantly $\Pi^1_1(\alpha)$. To prove the claim let $s \in T$ and let $S \subseteq 2^{<\omega}$ be such that for all $t \in S$, $\neg((t, \beta) <^*_\varphi (s, \beta))$. As $\beta \sim \alpha$, then $Q(s, \beta)$ holds and hence $P(s, \beta)$

also holds. Thus, by definition of $<^*_\varphi$, we get that $S \subseteq \{t : P(t,\beta)\} = T$. Thus $[S] \subseteq [T] \in I$.

On the other hand, let $s \notin T$. Put

$$S = \{t : \neg((t,\alpha) <^*_\varphi (s,\alpha))\}.$$

We claim that S is $\sim$-invariantly $\Delta^1_1(\alpha)$. In fact, for every $\beta \sim \alpha$ we clearly have that

$$t \in S \text{ iff } \neg((t,\beta) <^*_\varphi (s,\beta)).$$

But since $\sim P(s,\alpha)$, we also have that for all $\beta \sim \alpha$

$$t \in S \text{ iff } (s,\beta) \leq^*_\varphi (t,\beta).$$

Finally, by definiton of $\leq^*_\varphi$ we have that $T \subseteq S$. Hence by hypothesis $[S] \notin I$. This finishes the proof.

□

Now we are ready to give the

Proof of theorem 3.1.1: First we want to show that $C_1(I)$ is a Π^1_1 set in I^{int}

$$x \in C_1(I) \text{ iff } \exists T \in L_{\omega_1^x}(T \text{ is a tree } \& \ x \in [T] \ \& \ [T] \in I).$$

It is clearly Π^1_1, since

$$T \in L_{\omega_1^x} \text{ iff } \exists \gamma, \beta \in \Delta^1_1(x)[\gamma \in WO \ \& \ \beta \in L_{|w|} \ \& \ \beta = T].$$

Now we show that $C_1(I) \in I^{int}$. Put $C = C_1(I)$. By 3.1.4 it suffices to show that C is τ-meager for every topology τ compatible with I. Fix such a topology τ. Define the following prewellordering on C

$$x \leq y \text{ iff } x, y \in C \text{ and } \omega_1^x \leq \omega_1^y.$$

Since this prewellordering is in the σ-algebra generated by the $\mathbf{\Sigma}_1^1$ sets, it has the property of Baire with respect to τ. Now for every $y \in C$

$$\{x \in C : x \leq y\} \subseteq \bigcup\{[T] : T \in L_{\omega_1^y} \ \&\ [T] \in I\}.$$

As every $L_{\omega_1^y}$ is countable, $\{x \in C : x \leq y\}$ is τ-meager. Thus by the Kuratowski-Ulam theorem we have that except for a τ-meager set of x's $\{y \in C : x \leq y\}$ is τ-meager. Thus C is τ-meager.

Finally, we need only to show that every Π_1^1 set A in I^{int} is a subset of $C_1(I)$. Fix such an A and let T be a recursive tree on $2 \times \omega$ such that

$$x \in A \text{ iff } T(x) \text{ is wellfounded.}$$

Fix $x \in A$ and let $\mid T(x) \mid = \xi$. Notice that $\xi^+ < \omega_1^x$. Let S as in 3.1.9, then for every ordinal code w with $|\mathrm{w}| = \xi$ we have that

$$A_\xi = p[S(\mathrm{w})].$$

As $A_\xi \in I^{int}$ and I has the covering property, from lemma 3.1.8 we get that $S(\mathrm{w})^\infty = S(\mathrm{w})^\theta = \emptyset$. Hence as in the proof of 3.1.8

$$A_\xi \subseteq \bigcup \{\overline{p[S(\mathrm{w})^\alpha_{(s,u)}]} : \overline{p[S(\mathrm{w})^\alpha_{(s,u)}]} \in I \ \&\ \alpha < \theta \ \&\ (s,u) \in S(\mathrm{w})\}.$$

We want to show that the sets $[S(\mathrm{w})^\alpha_{(s,u)}]$ have an invariant definition in order to apply 3.1.14. Let P as in 3.1.10. Consider the following relations

$$(\mathrm{z}_1, \ldots, \mathrm{z}_m) \equiv_\mathrm{w} r \text{ iff } (r \in \omega^{<\omega}) \ \&\ (\forall i \leq m)(\mathrm{z}_i \in LO \ \ \&\ \ \mathrm{w} \in LO \ \ \&\ \ \mathrm{w}\lceil r(i) \equiv \mathrm{z}_i)$$

where $\mathrm{w}\lceil r(i)$ is the initial segment of the linear order coded by w determined by $r(i)$, i.e.,

$$\mathrm{w}\lceil r(i) = \{(l,k) : \mathrm{w}(l,k) = \mathrm{w}(l, r(i)) = \mathrm{w}(k, r(i)) = 0\}.$$

Put

$R(s,u,t,\mathrm{z},\mathrm{w},\mathrm{v})$ iff $t \in 2^{<\omega}$ & $lh(t) = n$ & $t \prec s$ &

$(\exists r \in \omega^{<\omega})((\mathrm{z}_1,\ldots,\mathrm{z}_n) \equiv_{\mathrm{w}} r$ & $r \prec u$ & $P(s,u,\mathrm{w},\mathrm{v}))$.

Now consider the following equivalence relation on $\omega^\omega \times \omega^\omega \times \omega^\omega$

$(\mathrm{z},\mathrm{w},\mathrm{v}) \sim (\mathrm{z}',\mathrm{w}',\mathrm{v}')$ iff

$\mathrm{z}_0(0) = \mathrm{z}'_0(0)$ & $(\forall 0 < i \le \mathrm{z}_0(0))(\mathrm{z}_i, \mathrm{z}'_i \in LO$ & $\mathrm{z}_i \equiv \mathrm{z}'_i$ & $\mathrm{w}_i \equiv \mathrm{w}'_i$ & $\mathrm{v}_i \equiv \mathrm{v}'_i)$.

Let $(t,r) \in S(\mathrm{w})$ such that

$$x \in \overline{p[S(\mathrm{w})^{|\mathrm{v}|}_{(t,r)}]}$$

and put

$$B = p[S(\mathrm{w})^{|\mathrm{v}|}_{(t,r)}].$$

Now if z codes a sequence of ordinals such that $(\mathrm{z}_1,\ldots,\mathrm{z}_m) \equiv_{\mathrm{w}} r$, then

$$x \in B \text{ iff } (\exists\alpha)(\forall n)R(x\lceil n, \alpha\lceil n, t, \mathrm{z}, \mathrm{w}, \mathrm{v}).$$

Hence B is $\sim$-invariantly-Σ^1_1 with respect to the variables $(\mathrm{z},\mathrm{w},\mathrm{v})$. Also $\overline{B} \in I$, thus by lemma 3.1.14 we have that there is a $\sim$-invariantly-Δ^1_1 tree T on 2 such that $B \subseteq [T]$ and $[T] \in I$.

By a similar argument as in the proof of lemma 3.1.12 we know that T belongs to the least admissible set containing all the ordinals coded by w,z,v (we need only to use the product of the notion of forcing defined in 3.1.12, one for each of the m ordinals coded in $(\mathrm{z},\mathrm{w},\mathrm{v})$, where $m = lh(r) + 2$).

But from lemma 3.1.10(ii) we know these ordinals are less than $\xi^+ < \omega_1^x$. Therefore $T \in L_{\omega_1^x}$. This finishes the proof of the theorem.

□

Remark: This proof clearly works for ideals on $(2^\omega)^m$.

3.2 On the strength of the covering property for Σ^1_2 sets

It is well known that the perfect set theorem for $\mathbf{\Pi}^1_1$ sets is equiconsistent with the existence of an inaccesible cardinal (Solovay). In fact, $\omega_1^L < \omega_1$ iff the perfect set theorem holds for Π^1_1 sets. In this section we will show that under the assumption that there are only countable many reals in L, any Π^1_1 $\sigma-$ideal of closed meager subsets of 2^ω with the covering property has also the covering property for Σ^1_2 sets. Also, we will see that for some $\sigma-$ideals the covering property for Π^1_1 sets fails in L and thus it is independent of ZFC.

Theorem 3.2.1 *Let I be a Π^1_1 $\sigma-$ideal of meager closed subsets of 2^ω with the covering property. If $\omega_1^L < \omega_1$, then I has the covering property for Π^1_1 sets. And by relativization, given $x \in \omega^\omega$, if $\omega_1^{L(x)} < \omega_1$, then the covering property holds for $\Pi^1_1(x)$ sets.*

Also the same result holds for σ-ideals of closed meager subsets of $(2^\omega)^m$.

Proof: It clearly suffices to show that the largest Π^1_1 set $C_1(I)$ in I^{int} belongs to I^{ext}. But if $\omega_1^L < \omega_1$, then there are only countable many binary trees in L. Hence from theorem 3.1.1 we easily get that $C_1(I) \in I^{ext}$.

□

The next result is a generalization of the result of Solovay that says that if there are only countable reals in L, then $\omega^\omega \cap L$ is the largest countable Σ^1_2 set. A similar result holds for some $\sigma-$ideals defined by games (see [9]).

Theorem 3.2.2 *Under the hypothesis of 3.2.1 the largest* Σ^1_2 *in* I^{ext} *and in* I^{int} *is*

$$C_2(I) = \{x \in 2^\omega : \exists T \in L(\ T \text{ is a tree on } 2\ \&\ x \in [T]\ \ \&\ \ [T] \in I)\}.$$

In particular, the covering property holds for Σ^1_2 *sets. And by relativization, given* $x \in \omega^\omega$, *if* $\omega_1^{L(x)} < \omega_1$, *then the covering property holds for* $\Sigma^1_2(x)$ *sets.*

Proof: If there are only countable many reals in L, then there are only countable many binary trees in L. Thus $C_2(I)$ is clearly a Σ^1_2 set in I^{ext}.

Let A be a Σ^1_2 set in I^{int} and let $B \subseteq X \times 2^\omega$ be a Π^1_1 set such that $x \in A$ iff $\exists\alpha(x,\alpha) \in B$. Let J be the $\sigma-$ideal of closed subsets of $2^\omega \times 2^\omega$ defined in chapter 2 proposition 1.20 , i.e.,

$$K \in J \text{ iff } proj(K) \in I. \qquad (*)$$

By proposition 2.1.21 J has the covering property and clearly J is a Π^1_1 $\sigma-$ideal of meager sets. Hence by the previous theorem J has the covering property for Π^1_1 sets. As $A \in I^{int}$, then $B \in J^{int}$ (if $K \subseteq B$, then $proj(K) \subseteq A$). Let $C_1(J)$ be the largest Π^1_1 set in J^{int}, i.e.,

$$C_1(J) = \{(x,\alpha) : \exists S \in L_{\omega_1^{(x,\alpha)}}(S \text{ is a tree on } 2 \times 2$$

$$\&\,(x,\alpha) \in [S]\,\&\,proj([S]) \in I)\}.$$

It is clear that $A \subseteq proj(C_1(J))$. Now, let K be a closed subset of $2^\omega \times 2^\omega$ and let S be the tree of K. Put $T = \{t : \exists s(t,s) \in S\}$. It is easy to check that T is a tree and $[T] = proj([S])$. Clearly if $S \in L$, then so does T. Hence

$$A \subseteq proj(C_1(J)) \subseteq \{x \in 2^\omega : \exists T \in L(\ x \in [T]\ \ \&\ \ [T] \in I)\}.$$

□

The next proposition will be used in the proof that for some ideals the covering property for Π^1_1 set fails in L. These results are due to Dougherty and Kechris.

Let us denote by $\leq_T$ the relation of Turing reducibility, i.e., $x \leq_T y$ iff x is recursive in y.

Proposition 3.2.3 *(Dougherty, Kechris) Let μ be the product probability measure on 2^ω and let I be the $\sigma-$ideal of closed μ-measure zero subsets of 2^ω. Then for every $x \in 2^\omega$, $\{y : x \leq_T y\} \notin I^{ext}$.*

Proof: Let $\{K_n\}$ be a countable collection of sets in I. We will define $y \notin \bigcup_n K_n$ such that $x \leq_T y$.

By the n-th block we mean the interval $[2^n, 2^{n+1})$. Call $z \in 2^\omega$ *good* if for infinite many n's, z is constant in the n-th block. If z is good let $\tilde{z}$ be defined as follows : Let $n_0 < n_1 < ...$ be an enumeration of the blocks on which z is constant; put $\tilde{z}(i) = j$ if z is constantly equal to j in the n_i-th block.

We will define by induction a good $y \notin \bigcup_n K_n$ such that $\tilde{y} = x$. Clearly $x \leq_T y$ and we will be done. For every n and every sequence $s \in 2^{2^n}$ and $k > n$ let

$$F^s_k = \{z \in 2^\omega : z \text{ is not constant in the j-th block for } n \leq j \leq k \ \ \& \ s \prec z\}.$$

There are exactly $2^{2^n} - 2$ non constant sequences of length 2^n. Therefore, if $z \in F^s_n$, then z can take $2^{2^j} - 2$ possible values in the j-th block. From this, one easily gets that

$$\mu(F^s_k) = (2^{2^n} - 2)(2^{2^{n+1}} - 2) \cdots (2^{2^k} - 2)/2^{2^{k+1}}.$$

Hence

$$\mu(F_k^s) = \frac{1}{2^{2^n}} \prod_{j=n}^{k}(1 - \frac{2}{2^{2^j}}). \qquad (*)$$

If $k \to \infty$ the infinite product (*) is equiconvergent with

$$\sum_{j=n}^{\infty} \frac{1}{2^{2^j}}.$$

Hence, for every $s \in 2^n$ we have

$$\mu(\bigcap_{k=n}^{\infty} F_k^s) > 0.$$

Let $F^s = \bigcap_{k=n}^{\infty} F_k^s$. Now we start defining y. As $\mu(F^{\emptyset}) > 0$, there is $z \in F^{\emptyset} - K_0$. Choose n_0 large enough such that if $z\lceil 2^{n_0} \prec w$, then $w \notin K_0$. Define $t_0 \in 2^{n_0+1}$ by $t_0\lceil 2^{n_0} = z\lceil 2^{n_0}$ and $t(i) = x(0)$ for every $i \in [2^{n_0}, 2^{n_0+1})$. Put $y\lceil 2^{n_0+1} = t_0$. Notice that t_0 is not constant in any j-block for $j < n_0$. Clearly we can repeat this for K_1 and F_{t_0}. So let $z \in F^{t_0} - K_1$ and $n_1 > n_0 + 1$ large enough such that if $z\lceil 2^{n_1} \prec w$, then $w \notin K_1$. Define as before $t_1 \in 2^{n_1+1}$ by $t_1\lceil 2^{n_1} = z\lceil 2^{n_1}$ and $t_1(i) = x(1)$ for every $i \in [2^{n_1}, 2^{n_1+1})$. Put $y\lceil 2^{n_1+1} = t_1$. The induction step should be now clear. So we get $y \notin \bigcup_n K_n$ and $\tilde{y} = x$. This finishes the proof.

□

For the $\sigma-$ideal of countable closed subsets of 2^ω the largest Π_1^1 set without perfect subset is characterized by

$$C_1 = \{\alpha \in 2^\omega : \alpha \in L_{\omega_1^\alpha}\}.$$

The next theorem shows that (in L) C_1 cannot be covered by countable closed of (Lebesgue) measure zero. Let us observe however that as C_1 has no perfect subsets, it clearly has measure zero and also belongs to I^{int} for every ideal containing all singletons.

Theorem 3.2.4 *(Dougherty, Kechris) Let μ and I as in 3.2.3. In L, $C_1 \notin I^{ext}$. Therefore, if J is a $\sigma-$*ideal *on 2^ω such that J contains all singletons and $J \subseteq I$, then (in L) J does not have the covering property for Π^1_1 sets.*

Proof: Let $\{K_n\}$ be a countable collection of closed sets of μ-measure zero. We will show that there is $y \in C_1$ and $y \notin \bigcup_n K_n$.

Let $\{T_n\}$ be the corresponding trees and let $\alpha < \omega_1^L$ be an ordinal such that each $T_n \in L_\alpha$. We can assume without loss of generality that α is an index (i.e., there is $x \in \omega^\omega$ such that $x \in L_{\alpha+1} - L_\alpha$). Let x be a complete set of index α (that is: $x \in L_{\alpha+1} - L_\alpha$ and any $y \in \omega^\omega \cap L_{\alpha+1}$ is arithmetical in x), in particular $\alpha < \omega_1^x$.

Let y be as in the proof of the previous proposition. It is easy to check that y can be found in $L_{\alpha+\omega}$. As $\omega_1^x \leq \omega_1^y$ (because $x \leq_T y$), $\alpha + \omega \leq \omega_1^y$. Hence $y \in L_{\omega_1^y}$, so $y \in C_1$. By construction $y \notin \bigcup_n K_n$.

□

These theorems can be easily transfered to compact intervals of the real line as follows: Say we are working on $[0,1]$ and consider the function $f : 2^\omega \to [0,1]$ defined by

$$f(\varepsilon) = \sum_{i=0}^{\infty} \varepsilon(i) 2^{-(i+1)};$$

f is continuous and surjective. Now, given a $\sigma-$ideal I of closed meager subsets of $[0,1]$ define an ideal J of closed subsets of 2^ω, as follows:

$$K \in J \text{ iff } f[K] \in I.$$

Observe that J consists of meager sets (because for every nbhd N_s on 2^ω we have that $f[N_s]$ contains an interval).

Lemma 3.2.5 *If I has the covering property, then so does J.*

Proof: First we show that if A is a $\mathbf{\Sigma}^1_1$ set, then $A \in J^{int}$ iff $f[A] \in I^{int}$.The direction $\Leftarrow$ is obvious by the definition of J.

Let A be a $\mathbf{\Sigma}^1_1$ set such that $f[A] \notin I^{int}$, say $K \subseteq f[A]$ is a closed set and $K \notin I$. Define R as follows:

$$R(x, \alpha) \text{ iff } \alpha \in A \,\&\, x \in K \,\&\, f(\alpha) = x.$$

Then $x \in K$ iff $\exists \alpha R(x, \alpha)$. Hence, as I is strongly calibrated, there is a closed set $F \subseteq R$ such that

$$K_0 = \{x : \exists \alpha (x, \alpha) \in F\} \notin I$$

Notice that $K_0 \subseteq K$. Put $L = \{\alpha : \exists x (x, \alpha) \in F\}$. Then $f[L] = K_0$ and $L \subseteq A$, so $A \notin J^{int}$.

The covering property for J now follows: If $A \in J^{int}$ is a $\mathbf{\Sigma}^1_1$ set, then $f[A] \in I^{int}$. Hence $f[A] \in I^{ext}$, which clearly implies that $A \in J^{ext}$.

□

Theorem 3.2.6 *Let I be a Π^1_1 $\sigma-$ideal of closed meager subsets of [0,1] with the covering property. Let f be the function defined above. The largest Π^1_1 set in I^{int} is*

$$C_1(I) = \{x \in [0,1] : \exists T \in L_{\omega_1^x} (T \textit{ is a tree on } 2 \;\&\; x \in f[T] \;\&\; f[T] \in I)\}$$

and the largest $\Sigma^1_2 \in I^{ext}$ is characterized by

$$C_2(I) = \{x \in [0,1] : \exists T \in L (T \textit{ is a tree on } 2 \;\&\; x \in f[T] \;\&\; f[T] \in I)\}.$$

In particular, if $\omega_1^L < \omega_1$, then I has the covering property for Σ^1_2 sets. And by relativization, given $x \in \omega^\omega$, if $\omega_1^{L(x)} < \omega_1$, then the covering property holds for $\Sigma^1_2(x)$ sets.

Proof: First, as in the proof of theorem 3.1.1 we have that $C_1(I)$ is a Π^1_1 set in I^{int}. To see that it is the largest, consider the $\sigma-$ideal J defined on 2^ω as in 3.2.5. J has the covering property. Let $C_1(J)$ be the largest Π^1_1 set in J^{int} given by theorem 3.1.1. i.e.,

$$C_1(J) = \{\alpha \in 2^\omega : \exists T \in L_{\omega_1^\alpha} (T \text{ is a tree on } 2 \ \& \ \alpha \in [T] \ \& \ [T] \in J)\}.$$

Let A be a Π^1_1 set in I^{int}. Put $B = f^{-1}(A)$, B is a Π^1_1 set in J^{int}. So $B \subseteq C_1(J)$, hence it suffices to show that $f(C_1(J)) \subseteq C_1(I)$. Let $\alpha \in C_1(J)$ and let $T \in L_{\omega_1^\alpha}$ such that $\alpha \in [T]$ and $[T] \in J$. As f is Δ^1_1, then $\omega_1^\alpha = \omega_1^{f(\alpha)}$. So $T \in L_{\omega_1^{f(\alpha)}}$. Thus $f(\alpha) \in f[T]$ and also $f[T] \in I$.

The proof for $C_2(I)$ is similar.

□

Theorem 3.2.4 can also be transfered to [0,1] as follows: Let us observe that for every basic nghd N_s in 2^ω we have that $\mu(N_s) = \lambda(f[N_s])$, where μ is the standard product measure on 2^ω and λ is the Lebesgue measure on $[0,1]$. One easily checks that if $f[C_1]$ can be covered by countably many closed sets of Lebesgue measure zero, then C_1 can also be covered by countably many closed of μ-measure zero. It is also clear that this set does not contain a perfect subset. We collect these facts in the following

Theorem 3.2.7 *Let I be a $\sigma-$*ideal *of closed subsets of [0,1] such that every set in I has Lebesgue measure zero. In L, I does not have the covering property for Π^1_1 sets.*

□

Remark: As we have already mentioned the $\sigma-$ideal of closed set of extended uniqueness has the covering property (see [3]). Hence, from 3.2.6 and 3.2.7 we get that the covering property for Π^1_1 sets of extended uniqueness is not provable in ZFC, but can be proved from the hypothesis that there are only countably many reals in L. Also we get a characterization of the largest Π^1_1 set of extended uniqueness as in 3.2.6.

3.3 The covering property for κ-Suslin sets

The only criterion known to show that a Π^1_1 $\sigma-$ideal has the covering property is a theorem due to Debs and Saint Raymond (see [3]) which says that every Π^1_1 locally non Borel, calibrated $\sigma-$ideal with a Borel basis has the covering property. The proof can be easily extended to κ-Suslin sets as we are going to show in this section.

Given an infinite cardinal κ, put in κ^ω the product topology. A subset $A \subseteq X$ is called κ-Suslin if there is a closed $F \subseteq X \times \kappa^\omega$ such that $A = proj(F)$, i.e.,

$$x \in A \text{ iff } \exists f \in \kappa^\omega[(x, f) \in F].$$

We will write in this case $A = p[F]$.

Theorem 3.3.1 *Let I be a Π^1_1, locally non Borel, calibrated $\sigma-$*ideal *of closed meager subsets of X with a Borel basis. If A is a κ-Suslin set in I^{int}, then A can be covered by less that κ^+ many closed sets in I.*

Proof: We define a derivative on closed subsets of $X \times \kappa^\omega$ as follows: Let $F \subseteq X \times \kappa^\omega$ be a closed set and let V_s be an enumeration of an open basis for X

$$(x, f) \in F^{(1)} \text{ iff } (\forall s)(\forall n)(x \in V_s \Rightarrow \overline{p[(V_s \times N_{f\lceil n}) \cap F]} \notin I).$$

By transfinite recursion we define $F^{(\alpha)}$ for all ordinals α. Observe that $F^{(1)}$ is also a closed set. Hence there is an ordinal $\theta < \kappa^+$ such that $F^{(\theta)} = F^{(\theta+1)}$. We denote by $F^{(\infty)}$ this fixed point.

Let A be a κ-Suslin set and let $F \subseteq X \times \kappa^\omega$ be a closed set such that $A = p[F]$.

Claim 1: *If $F^\infty = \emptyset$, then $p[F]$ can be covered by less than κ^+ many closed sets in I.*

Proof: Let $\theta < \kappa^+$ be such that $F^{(\theta)} = \emptyset$. For each $(x, f) \in F$ there is $\alpha < \theta$ such that $(x, f) \in F^{(\alpha)} - F^{(\alpha+1)}$, thus there is n and s such that $x \in V_s$ and $\overline{p[(V_s \times N_{f\lceil n}) \cap F^{(\alpha)}]} \in I$. Then we have

$$p[F] \subseteq \bigcup\{\overline{p[V_s \times N_u \cap F^{(\alpha)}]} : s \in \omega \,\&\, u \in \kappa^{<\omega} \& \alpha < \theta$$

$$\& \overline{p[(V_s \times N_u) \cap F^{(\alpha)}]} \in I\}.$$

This clearly proves the claim.

□

Claim 2: *If $F^\infty \neq \emptyset$, then $p[F] \notin I^{int}$.*

Proof: We will show that if $F \subseteq X \times \kappa^\omega \neq \emptyset$ is closed and $F^{(1)} = F$, then $p[F] \notin I^{int}$.

Let $B \subseteq I$ be a Borel basis for I. We will construct for each $t \in \omega^{<\omega}$ an element $u_t \in \kappa^{<\omega}$, an open set V_t and $K_t \in \mathcal{K}(X)$ such that

(i) $K_t \subseteq L_t = \overline{p[(V_t \times N_{u_t}) \cap F]}$ and $K_t \in I - B$.

(ii) $diam(V_{t^\frown(n)}) \leq 2^{-lh(t)}$, for all $n \in \omega$.

(iii) $V_{t^\frown(m)} \cap K_t = \emptyset$ for all $m \in \omega$.

(iv) $\overline{V_{t^\frown(n)}} \cap \overline{V_{t^\frown(m)}} = \emptyset$, for all $n \neq m$.

(v) $K_t \cup \bigcup_n L_{t^\frown(n)} = \overline{\bigcup_n L_{t^\frown(n)}}$.

(vi) $\overline{V_{t^\frown(n)}} \subseteq V_t$, $u_{t^\frown(n)}$ strictly extends u_t and $Lim_n diam(V_{t^\frown(n)}) = 0$.

For $t = \emptyset$, put $s_t = u_t = \emptyset$. Thus $L_\emptyset \notin I$. Since I is locally non Borel, there is $K_\emptyset \subseteq L_\emptyset$ such that $K_\emptyset \in I - B$.

Assume we have defined K_t, V_t and u_t for all $t \in \omega^{<\omega}$ with $lh(t) = k$. Notice that L_t is locally not in I, hence K_t is nowhere dense in L_t. It is not difficult to find (see [13] page 202) a countable discrete set $D_t \subseteq p[(V_t \times N_{u_t}) \cap F]$ such that

$$D_t \cap K_t = \emptyset \text{ and } K_t \cup D_t = \overline{D_t}.$$

Let $\{x_n\}$ be an enumeration of D_t. For each n find an open set $V_{t^\frown(n)}$, $u_{t^\frown(n)} \in \kappa^{<\omega}$ properly extending u_t so that

$$x_n \in p[(V_{t^\frown(n)} \times N_{u_{t^\frown(n)}}) \cap F]$$

and also

$$L_{t^\frown(n)} = \overline{p[(V_{t^\frown(n)} \times N_{u_{t^\frown(n)}}) \cap F]}$$

satisfies (ii), (iii), (iv), (v) and (vi) (for (v) observe that $diam(L_{t^\frown(n)}) \to 0$, when $n \to \infty$).

Now we want to define $K_{t^\frown(n)}$ for each n. Since $L_t \notin I$, as before we can find $K_{t^\frown(n)} \subseteq L_{t^\frown(n)} \in I - B$. Clearly all conditions (i)-(vi) are satisfied.

Subclaim: *Let $K = \overline{\bigcup_t K_t}$. Then $K \notin I$.*

Proof: We will show that if V is an open set and $V \cap K \neq \emptyset$ then $\overline{V \cap K} \notin B$, which says that K is locally not in I. Let V be an open set such that

$V \cap K \neq \emptyset$. For some $t \in \omega^{<\omega}$, $V \cap K_t \neq \emptyset$. Since $diam(L_{t^\frown(n)}) \to 0$, when $n \to \infty$, then from (v) we get that for some n, $L_{t^\frown(n)} \subseteq V$. Thus $K_{t^\frown(n)} \subseteq V$ and in consequence $K_{t^\frown(n)} \subseteq \overline{V \cap K}$. Therefore from (i) we get that $K_{t^\frown(n)} \notin B$.

(□ Subclaim.)

As I is calibrated there is a closed set $M \subseteq K - \bigcup_t K_t$ with $M \notin I$. We will show that $M \subseteq p[F]$ and we will be done.

Put

$$F_n = \bigcup\{K_t : lh(t) < n\} \cup \bigcup\{L_t : lh(t) = n\}.$$

We claim that each F_n is closed: we show it for $n = 2$, the other cases are similar. Let $\{y_i\}$ be a sequence in F_2 and suppose that $y_i \to y$. Assume $y \notin \bigcup\{K_t : lh(t) < 2\}$, we will show that $y \in L_t$ for some t with $lh(t) = 2$. By (v) we can assume that $y_i \in L_{t_i}$ with $lh(t_i) = 2$ (or replace $\{y_i\}$ by other sequence satisfying this condition and with the same limit). From (ii) and since every D_t is a discrete set, it is easy to show that there is n such that $y_i \in L_{<n,m_i>}$ for infinite many i's. From (v) and since $y \notin K_{<n>}$, we get that $y \in L_{<n,m>}$ for some m.

From (v) we get that $K \subseteq F_n$ for every n. Therefore $M \subseteq F_n$ for every n and thus $M \subseteq \bigcap_n F_n$. Hence

$$M \subseteq \bigcap_n \bigcup_{lh(t)=n} L_t.$$

From this and (vi) it is easy to see that $M \subseteq p[F]$.

(□ Claim 2)

This finishes the proof of the theorem. □

And we immediately get this result for the $\sigma-$ideal U_0 of closed set of extended uniqueness.

Corollary 3.3.2 *If A is a universally measurable κ-Suslin set in $\mathcal{U}_0$ (i.e., a set of extended uniqueness), then A can be covered by less that κ^+ many closed sets of extended uniqueness.*

□

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