TOPICS IN QUANTUM GRAVITY

Thesis by
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In Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy

California Institute of Technology
Pasadena, California

1990
(Submitted May 24, 1990)

ACKNOWLEDGEMENTS

Many people were responsible for making graduate school a wonderful phase in my life and it is a pleasure to thank them here. To begin with, I would like to thank my teachers who communicated not just the body of knowledge but the spirit of research and instilled in me a deep love for the subject. I would especially like to thank my advisor, Prof. John Preskill, who has guided me through these formative years with a sure but gentle touch. Besides being a patient teacher, he has always been generous with his ideas and insightful comments. Some of my happiest moments have been spent in discussing physics with him, be it in the corridors or over lunch or sometimes even the men's room! I would also like to thank Prof. Mark Wise for some wonderful collaborations. I hope in the years to come I will continue to share his zeal and enthusiasm for research. I wish I could have thanked Prof. Richard Feynman for the opportunity of working with him. The high standards he set by his insistence on understanding physics in one's own way are a constant source of grief to me!

If my teachers inspired me, it was my friends and colleagues who had to bear the brunt of my daily questions and confusion. They made physics lively and exciting. It is a pleasure to thank Ted Allen, Glenn Boyd, Sumit Das, Michael Douglas, Olivier Espinosa, John Feiler, Jim Hughes, Arun Gupta, Desmond Johnston, Elias Kiritsis, Eric Raiten, Keke Li, David Montano, Alexis Polychronakos, Soo Jong Rey, Alex Ridgway, Gill Rivlis, Ryan Rohm, Martin Savage, Patricia Schwarz, Roxanne Springer, Brian Warr, Osamu Yasuda and Tim Yen. Some of the work presented in this thesis was done in collaboration with Mike Douglas. I learnt a lot in conversations with him, not the least being an interesting new attitude towards driving! Arun Gupta, who shared an office with me for four years and Olivier Espinosa, whose books I have always regarded as my own had more than their fair share of trouble and deserve to be thanked separately. It is also a great pleasure to thank Donna Driscoll, Valerie Mandeville, Helen Tuck and Judith Barnett for all the help and sound advise. I cannot resist from thanking Helen Tuck especially, for the comfortable chairs!

Vidyadhar Mudkavi, Raghu Madhavan, Ranajit Sahu and Murali Sharma were forced to share quarters with me at some or the other stage. Their stoicism in the face of much provocation can only be commended. I hope this thesis does not mark an end to the generous impulses of Anamitra Makur, Vidyadhar and Sandhya Mudkavi, Olivier and Nina Espinosa, Shrinivas and Hiromi Kulkarni and Sumati Shankar who often took pity and invited me for dinner.

I wish my grandfather, Mama, could have been here today to see this thesis. He helped put life in the correct perspective for me and I hope I carry within me some of his gentleness. I cannot possibly express what I owe to my parents. Let me just say that I would not have even got to Caltech without their unswerving faith, in the face of all odds, that I would make it through school. I have also learnt a lot, over the years, from my sister, Nandini. The wonderful, though, usually heated arguments I have had with her, are no doubt in part, responsible for my interest in physics. I would like to thank her and also her husband, Mohit Randeria, for the weekend calls which sustained me through some of my worst moments. Finally, I would like to thank my wife Shubha, for showing me that there was after all, much more to life than physics.

ABSTRACT

We study some aspects of conformal field theory, wormhole physics and two-dimensional random surfaces. Inspite of being rather different, these topics serve as examples of the issues that are involved, both at high and low energy scales, in formulating a quantum theory of gravity. In conformal field theory we show that fusion and braiding properties can be used to determine the operator product coefficients of the non-diagonal Wess-Zumino-Witten models. In wormhole physics we show how Coleman's proposed probability distribution would result in wormholes determining the value of θ_{QCD}. We attempt such a calculation and find the most probable value of θ_{QCD} to be π. This hints at a potential conflict with nature. In random surfaces we explore the behaviour of conformal field theories coupled to gravity and calculate some partition functions and correlation functions. Our results throw some light on the transition that is believed to occur when the central charge of the matter theory gets larger than one.

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1. Introduction

Formulating a quantum theory of gravity remains one of the outstanding challenges in physics today. Despite the success of the classical theory, we know it is incomplete. It does not provide an adequate description of the interactions of gravity with matter which we know is quantized. Neither does it provide an understanding of singularities which can occur in the course of stellar evolution or the evolution of the universe. Over the years, considerable effort has been put into formulating the quantum theory, but it still remains poorly understood. The conceptual issues that confront us involve a better understanding, not only of physics at energies much higher than the Planck scale, but surprisingly at much lower energies too. Consequently, and this point is worth emphasising, such a theory can have profound consequences at both high and low energies.

From the point of view of physics at high energies we seek a quantum theory which is renormalisable and therefore has predictive power. As is well known, in recent years string theory has emerged as a promising candidate for such a theory. It incorporates not only gravity but all the other known interactions too. A study of this theory has revealed that in perturbation theory it is well behaved; in fact its ultra- violet behaviour is finite and it is free from anomalies. Another interesting consequence of this study has been the discovery of a large number of classical ground states, i.e., vacua. Conformal field theory (C.f.t.) provides a convenient framework for studying these vacua and for doing perturbation theory, i.e., evaluating S-matrix elements. A string moving in space time sweeps out a two dimensional world sheet and the behaviour of the string (at least in perturbation theory) can be understood in terms of a two dimensional field theory living on the world sheet. It is a distinguishing feature of this two dimensional theory that it is scale invariant. In two dimensions scale invariance is a powerful constraint and very often implies a very large (in fact infinite) set of symmetries. Theories possessing these symmetries are called conformal field theories. Besides being relevant for string theory, they are also useful in studying two dimensional critical phenomena.

In chapter two we study some of the properties of C.f.t.'s. As is well known each C.f.t. has two sets of infinite symmetry algebras called chiral algebras. States can be characterised as primaries or descendents depending on the action of these algebra's. Correlation functions in these theories are of course single valued functions of the positions of the various fields involved, but it helps to think of them as being built up of blocks which are not single valued. These blocks, often obtained as solutions to differential equations, transform into one another as the fields are moved around one another in the correlation functions. We explore in this chapter how the fusion and braiding properties which arise as a consequence of this can be used to compute the operator product expansions in C.f.t.'s. More specifically we deal with theories with an $SU(2)$ symmetry and calculate the operator product coefficients for the D and E invariants. Since a C.f.t is completely characterised by its chiral algebra, the primary fields with respect to this algebra and the operator product expansion coefficients of the primary fields, this completes the task of categorising these theories.

In chapter three we turn to some conceptual issues confronting us in an understanding of the low energy theory. Specifically we focus on wormholes and explore some of their consequences. Some time ago Coleman made an interesting proposal that wormholes would make the constants of nature quantum mechanically indeterminate. He then proposed a probability distribution for them and showed that this distribution was very, very strongly peaked at the cosmological constant being zero. This was a dramatic prediction for low energy physics coming from the quantum theory of gravity. Moreover, it agreed with experiment. We show in this chapter that yet another consequence of his proposal would be to determine the value of θ_{QCD}. We attempt such a calculation in chiral perturbation theory and find that θ_{QCD} is overwhelmingly likely to be close to π . Since experimentally it is commonly believed that θ_{QCD} is close to zero, this hints at a potential conflict between experiment and Coleman's proposal.

Finally in chapter three we turn to a study of random surfaces. As a theory of two dimensional gravity, these provide us with a simple model in which to explore the problem of defining a suitable measure by regulating in a generally covariant manner. Such problems plague higher dimensional gravity too. Quite apart from this, random surfaces are also of interest from the point of view of fundamental string theory and the strong interactions . In recent months this subject has gained much attention. We will not be able to do justice to all the progress made but will touch on a few issues within the context of the continuum approach. All the progress made so far has been restricted to the case of conformally invariant matter with a central charge, not exceeding 1, coupled to gravity. It is commonly believed that a phase transition occurs as the central extension is increased any further. Working in the conformal gauge we calculate partition functions and some correlation functions in these models, for genus 0 and 1 surfaces. When the central charge of the matter theory equals 1, some interesting things happen. The partition function on the sphere vanishes, and the partition function on the torus indicates the appearance of a divergence associated with the tachyon in string theory.

2. The Operator Product Expansion In Wess-Zumino-Witten Models

2.1 Introduction

Conformal field theories (C.f.t.'s)[1] are scale invariant theories in which the trace of the stress energy tensor vanishes. Together with the conservation of the stress energy tensor, i.e.,

$$\partial^{\mu} T_{\mu\nu} = 0 \ , \tag{1}$$

this implies that

$$\partial_{\bar{z}} T_{zz} = 0 \tag{2}$$

and similarly

$$\partial_{z} T_{\bar{z}\bar{z}} = 0 \ . \tag{3}$$

In fact, as we shall see below, the whole theory decomposes into a left moving sector dependent on z and a right moving sector dependent on $\bar{z}$, and this separation between the right and left moving parts is largely responsible for making these theories solvable. Since T_{zz} is an analytic function of z, it can be Laurent expanded in powers of z to give :

$$T_{zz} = \sum \frac{L_n}{z^{n+2}} \ . \tag{4}$$

The L_n's are generators of conformal transformations and satisfy the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(m^3 - m)\delta_{m+n=0} \tag{5}$$

where the constant c is called the central charge and characterises the conformal field theory. States in the theory can then be primary or descendents depending on the action of this algebra. A primary state is one for which

$$L_n|\phi\rangle = 0 \tag{6}$$

for all $n > 0$. Descendents are then obtained by acting on primary states with the L_{-n}'s . Primary fields are also eigenvectors of L_0, i.e.,

$$L_0|\phi\rangle = \Delta|\phi\rangle. \tag{7}$$

We have so far talked only about the left moving sector, but (3) shows that there is a similar Virasoro algebra that can be defined in the right moving sector, too, and states will transform as primary fields or as descendents under this algebra.

One invaluable tool in the study of C.f.t.'s is the operator product expansion (o.p.e.) which governs the behaviour of two operators as they come together. Thus the o.p.e. of $T(z)$ and a primary field ϕ_Δ is

$$T(z)\ \phi_\Delta(w) = \frac{\phi_\Delta(w)}{(z-w)^2} + \frac{\partial\phi_\Delta}{(z-w)} + \dots \tag{8}$$

where the dots indicate nonsingular terms containing descendents of ϕ_Δ and can be worked out once the singular pieces and the Virasoro algebra (5) are known. Similarly the o.p.e. of the stress tensor with itself gives:

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2\ T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \dots \tag{9}$$

The o.p.e. of two primary fields will, in general, contain several primary field, and their descendents :

$$\phi_a(z,\bar{z})\phi_b(w,\bar{w}) = \sum C^c_{ab}(z-w)^{\Delta_c-\Delta_a-\Delta_b}(\bar{z}-\bar{w})^{\bar{\Delta}_c-\bar{\Delta}_a-\bar{\Delta}_b}\phi_c(w,\bar{w}) + \dots \tag{10}$$

The coefficients $C^c_{a,b}$ are called operator product coefficients and characterise the leading terms in the o.p.e. Once again successive terms in the expansion can be derived from these leading terms. Armed with the o.p.e. we can, in principle at any rate, evaluate correlation functions involving primary fields and then also descendents by doing the expansion successively. Thus once the primary fields in the theory and their operator product coefficients are known, the theory can be solved completely. In practice though, the above mentioned method of calculating correlations is almost impossible to carry out and one has to resort to other techniques like looking for differential equations connected with the existence of null states in the theory.

We have so far only considered the Virasoro algebra, but it if often convenient to consider theories which have enhanced symmetries. One such example is of theories that have a current algebra corresponding to some Lie algebra G[2,3] , i.e., dimension one left moving (and similarly also right moving) primary fields $J^a(z)$ which have an o.p.e.

$$J(z)J(w) = \frac{k/2\delta^{ab}}{(z-w)^2} + \frac{i\ f_{abc}\ J_c}{(z-w)} + \ldots \tag{11}$$

where 'k' is the extension of the Kac-Moody algebra and f_{abc}'s are the structure constants of the Lie algebra G. The stress energy tensor in these theories is given by the Sugawara construction in terms of the currents J^a as :

$$T(z) = \frac{1}{(c_v + k)} : J^a(z)J^a(z) : \tag{12}$$

where c_v is the Casimir of the adjoint representation and the dots on the right hand side indicate normal ordering (removal of singular pieces). This means that the central charge of the theory can be expressed in terms of 'k' the extension of the Kac-Moody algebra . The relation is

$$c = \frac{k\ dim_G}{(c_v + k)} \tag{13}$$

where dim_G is the dimension of the group G.

As before we can mode expand $J^a(z)$ in powers of z to get

$$J^a(z) = \sum \frac{J^a_n}{z^{(n+1)}} \tag{14}$$

and then characterise states as primary or descendent by their behaviour under the current algebra. A primary state would satisfy the relations

$$J^a_n|\phi_i\rangle = 0 \tag{15}$$

for all $n > 0$ and

$$J^a_0|\phi_i\rangle = t^a_{ij}|\phi_j\rangle \tag{16}$$

where t^a is a representation of the Lie algebra G. The fields in the theory that are primary with respect to the current algebra fall into representations of G and their dimension in terms of this representation is given as

$$\Delta_a = \frac{c_a}{(c_v + k)} \tag{17}$$

where c_a is the Casimir of the representation to which the primary field belongs. Once again correlations would be determined by the operator product coefficients of fields primary with respect to the current algebra.

One important class of C.f.t.'s are unitary conformal field theories. The Hilbert space in these theories does not contain any negative norm states. Just as in the representation theory of ordinary Lie algebras, this puts a severe limitation on the set of allowed C.f.t.'s. For the case of theories with Kac-Moody symmetry, it in fact implies that the central extension 'k' must be an integer (when the roots of the algebra are normalized so that the largest root has a length of 2).

In the rest of this chapter we will be considering unitary C.f.t.'s which have a $SU(2)$ symmetry. Such theories are known to be important not only in their own right, but also as building blocks for other kinds of theories like the N=2 minimal superconformal field theories[4] and coset conformal field theories[5–8] which in turn are useful in constructing realistic ground states for string theory. These theories are well known to fall into the ADE series[9]. The field content of these theories can be read off from their partition functions which are as follows:

$k \geq 1$ $\quad \sum_{l=0}^{k} \chi_l^* \chi_l$ $\quad (A_{k+1})$

$k = 4j$ $\quad \sum_{l=0}^{j-1} |\chi_{2l} + \chi_{k-2l}|^2 + 2\ \chi_{k/2}^* \chi_{k/2}$ $\quad (D_{2j+2})$

$k = 4j-2$ $\quad \sum_{l=0}^{k/2} |\chi_{2l}|^2 + \sum_{l=0}^{2j-1} \chi_{2l+1} \chi_{k-2l-1}^*$ $\quad (D_{2j+1})$

$k = 10$ $\quad |\chi_0 + \chi_6|^2 + |\chi_3 + \chi_7|^2 + |\chi_4 + \chi_{10}|^2$ $\quad (E_6)$

$k = 16$ $\quad |\chi_0 + \chi_{16}|^2 + |\chi_4 + \chi_{12}|^2 + |\chi_6 + \chi_{10}|^2$
$\quad |\chi_8|^2 + (\chi_2 + \chi_{14})\chi_8^* + \chi_8(\chi_2 + \chi_{14})^*$ $\quad (E_7)$

$k = 28$ $\quad |\chi_0 + \chi_{10} + \chi_{18} + \chi_{28}|^2 + |\chi_6 + \chi_{12} + \chi_{16} + \chi_{22}|^2$ $\quad (E_8)$

Given the partition function and chiral algebra, these theories are completely characterized once the operator product coefficients of primary fields are known. These coefficients were first calculated for the A series by Fateev and Zamolodchikov[10]. Since the E_6 theory is known to be the diagonal invariant associated with $SO(5)$ at level 1 and the E_8 theory is similarly the diagonal invariant associated with G_2 at level 1 calculating their o.p.e is straightforward. Here we show how properties of the fusion matrices can be used to calculate these coefficients for the D and E_7 invariants, thus completing the task of characterising these theories[11]. The techniques we use are very general and in fact can be used to calculate the o.p.e. in any conformal field theory.

2.2 Outline of calculations

Operator product coefficients in conformal field theory are determined by the duality properties of four-point functions[12,13]. It suffices to consider four-point functions of primary fields; a primary field ϕ_a is characterized by its left and right

SU(2) quantum numbers (j_L , m_L , and j_R , m_R respectively), and its operator products as

$$\phi_a\phi_b \to \sum_c C^c_{ab}\phi_c. \tag{18}$$

Duality says that every four-point function must be the same when expanded in terms of either s-channel or t-channel intermediate states. Thus:

$$\begin{aligned}\langle\phi_a(\infty,\infty)\,\phi_b(1,1)\,\phi_c(z,\bar z)\,\phi_d(0,0)\rangle &= \sum C^a_{bi}C^i_{cd}I_{iL}(z)\bar I_{iR}(\bar z) &(19)\\ &= \sum C^a_{jd}C^j_{bc}\tilde I_{jL}(1-z)\bar{\tilde I}_{jR}(1-\bar z) &(20)\end{aligned}$$

Here I and $\bar I$'s are conformal blocks in the s-channel whereas $\tilde I$ and $\bar{\tilde I}$'s are conformal blocks in the t-channel. They are determined by solving a linear differential equation as shown in [2] .

Furthermore, we know that each block in the s-channel can be expressed as a linear combination of blocks in the t-channel. Thus:

$$I_{iL}(z) = \sum F(a_L, b_L, c_L, d_L)(j_L, i_L)\tilde I_{jL}(1-z)\,, \tag{21}$$

where $F(a_L, b_L, c_L, d_L)$ are the so-called fusion matrices. The condition of duality can then be rephrased to say:

$$\sum C^a_{bi}\,C^i_{cd}F(a_L, b_L, c_L, d_L)(j_L, i_L)F(a_R, b_R, c_R, d_R)(j_R, i_R) = C^a_{jd}C^j_{bc}\,. \tag{22}$$

Where in the above equation the field j on the right-hand side has SU(2) spins j_L and j_R respectively. If there is no primary field in the theory with these SU(2) spins, then the right-hand side vanishes.

More schematically equation(22) reads:

$$FCF^T = C\,, \tag{23}$$

where C above stands for a matrix formed by products of the operator product coefficients.

Once the fusion matrices are known we can solve for the operator product coefficients. These fusion matrices are usually found by constructing the conformal blocks explicitly and calculating their transformation laws. It is much easier, though, to find a simple fusion matrix, one involving only spin 1/2 fields for example, in this manner, and generate the others by using various identities these fusion matrices are known to satisfy. We describe below how this technique can be used to find the operator product coefficients of the E_7 model. The identity which will prove most crucial and which we use throughout all our work is called the pentagon identity[14] (Fig. 1) and says that

$$F(a,j_1,d,e)(j_2,i_2)F(a,b,c,i_2)(j_1,i_1) =$$

$$\sum_{j_3} F(j_2,b,c,d)(j_1,j_3)F(a,b,j_3e)(j_2,i_1)F(i_1,c,d,e)(j_3,i_2)\,. \qquad (24)$$

It is known however that the D and E_7 models are simply related to the A models by extensions in the chiral algebra or automorphisms in the fusion rules[15], and one may wonder whether their operator product coefficients are not simply related to those of the A models. Since these o.p.e.s of the A models are known, this would be an easy way to calculate them. We show below how by thinking along these lines we can calculate the operator product coefficients of the D models. In the actual calculation we use both the extended algebra in terms of which the models are diagonal and the original algebra in terms of which the models are not diagonal but have blocks determined by the same differential equations as for the A-models. The F matrix for the extended algebra acts on a subspace of that for the original algebra and is in principle determined by Ward identities with respect to new operators in the chiral algebra. Unfortunately these techniques cannot quite be driven through for the E_7 model. This model is related to the D_{10} model by an automorphism of the fusion rules which must be reflected in symmetries of the fusion matrices when a canonical basis for the conformal blocks is chosen. We found no *a priori* way to determine this basis, however, and have had to do the E_7 invariant by a brute force technique.

2.3 Symmetry properties of operator product coefficients

Before we go any further, let us state some of our conventions (these have been adopted from the paper by Fateev and Zamalodchikov[16]). A primary field is denoted by $\phi_{\substack{J_L \\ m_L}\substack{J_R \\ m_R}}$ (although sometimes we may suppress the m indices) and is normalized by taking the two-point functions as:

$$\langle \phi_{\substack{J_L \\ -m_L}\substack{J_R \\ -m_R}}(\infty) \quad \phi_{\substack{J_L \\ m_L}\substack{J_R \\ m_R}}(0)\rangle = (-1)^{J_L - m_L + J_R - m_R} . \tag{25}$$

The operator product coefficient is defined by:

$$\begin{aligned}\langle \phi_{\substack{J_{3L} \\ m_{3L}}\substack{J_{3R} \\ m_{3R}}}(\infty)\, \phi_{\substack{J_{1L} \\ m_{1L}}\substack{J_{1R} \\ m_{1R}}}(1)\, \phi_{\substack{J_{2L} \\ m_{2L}}\substack{J_{2R} \\ m_{2R}}}(0)\rangle &= C((j_{1L}, j_{1R}),\, (j_{2L}, j_{2R}),\, (j_{3L}, j_{3R})) \\ &\quad \sqrt{C_{FZ}(j_{1L}, j_{2L}, j_{3L}) C_{FZ}(j_{1R}, j_{2R}, j_{3R})} \\ &\quad \times C.G.(j_{1L}, m_{1L}, j_{2L}, m_{2L}, j_{3L}, m_{3L}) \\ &\quad \times C.G.(j_{1R}, m_{1R}, j_{2R}, m_{2R}, j_{3R}, m_{3R})\end{aligned} \tag{26}$$

where $C.G.$ denotes the Clebsch–Gordon coefficient and is taken to be

$$\begin{aligned}&C.G.(j_1, m_1, j_2, m_2, j_3, m_3) = \\ &\qquad \left[\frac{(j_1 + j_2 + j_3 + 1)!(j_1 + j_3 - j_2)!(j_3 + j_2 - j_1)!(j_1 + j_2 - j_3)!}{2j_1!2j_2!2j_3!}\right]^{1/2} \\ &\qquad \times \mathrm{Wig}\begin{bmatrix} j_3 & j_1 & j_2 \\ -m_3 & m_1 & m_2 \end{bmatrix} .\end{aligned} \tag{27}$$

Here 'Wig' stands for the usual Wigner $3 - j$ symbol[17] and $C_{FZ}(j_{1L}, j_{2L}, j_{3L})$ for the operator product coefficients of the left-right symmetric A models calculated by Fateev and Zamalodchikov. With these conventions our operator product coefficients have the property that

$$\begin{aligned}C((j_{1L}, j_{1R}),\, (j_{2L}, j_{2R}),\, (j_{3L}, j_{3R})) &= (-1)^{\text{Parity}(1)+\text{Parity}(2)+\text{Parity}(3)} \\ &\quad \times C((j_{1L}, j_{1R}),\, (j_{3L}, j_{3R}),\, (j_{2L}, j_{2R}))\end{aligned} \tag{28}$$

where

$$\text{Parity}(1) = \Delta(j_{1L}) - \bar{\Delta}(j_{1R}) + j_{1L} + j_{1R} . \tag{29}$$

Fusion matrices involving a spin 1/2 field have been worked out by Tsuchiya and Kanie[18] and are also rather simple in this basis of conformal blocks. They are given by:

$$F(j_4,1/2,j_2,j_1)(j_2+1/2,j_4-1/2)= \left[\frac{S(1-2j_1+2j_2+2j_4)}{S(1+2j_2)}\right]^{1/2} \times\left[\frac{S(3+2j_1+2j_2+2j_4)}{S(1+2j_4)}\right]^{1/2} \tag{30}$$

$$F(j_4,1/2,j_2,j_1)(j_2+1/2,j_4+1/2)= \left[\frac{S(1+2j_1-2j_2+2j-4)}{S(1+2j_2)}\right]^{1/2} \times\left[\frac{S(1+2j_1+2j_2-2j_4)}{S(1+2j_4)}\right]^{1/2} \tag{31}$$

$$F(j_4,1/2,j_2,j_1)(j_2-1/2,j_4+1/2)= F(j_4,1/2,j_2,j_1)(j_2+1/2,j_4-1/2) \tag{32}$$

$$F(j_4,1/2,j_2,j_1)(j_2-1/2,j_4-1/2)= -F(j_4,1/2,j_2,j_1)(j_2+1/2,j_4+1/2) \tag{33}$$

where

$$S(j)=\sin\left(\frac{\pi j}{2(k+2)}\right). \tag{34}$$

The operator product coefficients also show a symmetry under the exchange of primary fields $\phi_{j_R}\rightarrow\phi_{k/2-j_R}$, where the exchange acts on any two of the three fields involved. This symmetry is related to the 'diagram automorphism' in the theory of affine Lie algebras which relates these representations of affine $SU(2)$. We now explain this property as a special case of the pentagon identities. Consider a theory like the $D_{2\rho+2}$ invariants which have primary fields $\phi_{0,k/2}$ and $\phi_{k/2,0}$ in the spectrum. Then, by considering Fig. 2 we conclude that

$$\begin{aligned}
C((j_{1L},j_{1R}),\,(j_{2L},j_{2R}),\,(j_{3L},j_{3R})) =& \\
&\times F(j_{3L},k/2,k/2,j_{3L})(0,k/2-j_{3L}) \\
&\times F(j_{1L},j_{2L},k/2,k/2-j_{3L})(k/2-j_{2L},j_{3L}) \\
&\times C((j_{1L},j_{1R}),\,(k/2-j_{2L},j_{2R}),\,(k/2-j_{3L},j_{3R})) \\
&\times C((k/2-j_{2L},j_{2R}),\,(j_{2L},j_{2R}),\,(k/2,0)) \\
&\times C((k/2-j_{3L},j_{3R}),\,(k/2,0),\,(j_{3L},j_{3R}))\,.
\end{aligned} \tag{35}$$

This allows us to relate $C((j_{1L},j_{1R}),\,(j_{2L},j_{2R}),\,(j_{3L},j_{3R}))$ to $C((j_{1L},j_{1R}),\,(k/2-j_{2L},j_{2R})\,,(k/2-j_{3L},j_{3R}))$.

Of course we could do a similar transformation on the right isospin. If the theory does not have an extended algebra with fields $(0,k/2)$ and instead has only the field $(k/2,k/2)$, then a similar identity holds; only now we reflect the left- and right-isospins around $k/2$ for the two fields simultaneously. This gives us

$$\begin{aligned}
C((j_{1L},j_{1R}),\,(j_{2L},j_{2R})\,(j_{3L},j_{3R})) =& \\
&F(j_{3R},k/2,k/2,j_{3R})(0,k/2-j_{3R}) \\
&\times F(j_{3L},k/2,k/2,j_{3L})(0,k/2-j_{3L}) \\
&\times F(j_{1R},j_{2R},k/2,k/2-j_{3R})(k/2-j_{2R},j_{3R}) \\
&\times F(j_{1L},j_{2L},k/2,k/2-j_{3L})(k/2-j_{2L},j_{3L}) \\
&\times C((j_{1L},j_{1R}),(k/2-j_{2L},k/2-j_{2R}),(k/2-j_{3L},k/2-j_{3R})) \\
&\times C((k/2-j_{2L},k/2-j_{2R}),(j_{2L},j_{2R}),(k/2,k/2)) \\
&\times C((k/2-j_{3L},k/2-j_{3R}),(k/2,k/2),(j_{3L},j_{3R}))\,.
\end{aligned} \tag{36}$$

These relations are not as complicated as they seem; with our choice of Clebsch–Gordon coefficients, a fusion matrix where one of the four fields has an isospin $k/2$ has a single non-zero coefficient, which is either ± 1. $C((j_L,j_R),(0,k/2),(j_L,k/2-j_R))$

is simple too; either ± 1 or $\pm i$. Thus we get for the $D_{2\rho+2}$ invariants that

$$C((j_{1L},j_{1R}),(j_{2L},j_{2R}),(j_{3L},j_{3R})) = \alpha\, C((j_{1L},j_{1R}),(j_{2L},\, k/2-j_{2R}),(j_{3L},k/2-j_{3R})) \tag{37}$$

where

$$\alpha = \pm 1 \text{ or} \pm i \tag{38}$$

and similarly for the other cases. By considering a particular case of the pentagon identity, i.e., equation(24) for $a=j_1$, $b=k/2-j_2$, $c=k/2$, $d=j_3$, $e=j_4$, we can also conclude that

$$F(j_1,j_2,j_3,j_4)(j_{\rm new},j_{\rm old}) = \pm F(j_1,k/2-j_2,k/2-j_3,j_4)(j_{\rm new},k/2-j_{\rm old}) \tag{39}$$

and similarly that

$$F(j_1,j_2,j_3,j_4)(j_{\rm new},j_{\rm old}) = \pm F(k/2-j_1,k/2-j_2,j_3,j_4)(k/2-j_{\rm new},j_{\rm old}). \tag{40}$$

This completes our digression on diagram automorphism.

Finally there is a simple rule which allows us to decide which operator product coefficients are real and which, imaginary. If we take three self conjugate primary fields ϕ_a, ϕ_b and ϕ_c, then $C(\phi_a,\, \phi_b,\, \phi_c)$ is real if the sum of the three conformal spins is even and imaginary otherwise. A simple argument for this is as follows. Let us denote the state $\phi_b(1/2,1/2)\phi_c(0,0)\,|0\rangle$ by $|B>$, then the hermitian conjugate state in conformal field theory is a state like $<0|\phi_c(\infty,\infty)\phi_b(2,2) = \langle B|$. Similarly, we can take the state $|\phi_a\rangle = \phi_a(0,0)\,|0\rangle$ and its hermitian conjugate state $\langle\phi_a|$. Now $\langle 0|\phi_c(\infty,\infty)\phi_b(2,2)\phi_a(0,0)|0\rangle$ $\langle 0|\phi_a(\infty,\infty)\phi_b(1/2,1/2)\phi_c(0,0)|0\rangle$ then corresponds to $\langle B|A\rangle\,\langle A|B\rangle$ and so must be positive in a unitary theory. However

$$\langle 0|\,\phi_c(\infty,\infty)\phi_b(2,2)\phi_a(0,0)\,|0\rangle = (-1)^{\sum \text{conformal spins}} \times (\frac{1}{4})^{\Delta_b+\bar{\Delta}_b} \times \langle 0|\,\phi_a(\infty,\infty)\phi_b(1/2,1/2)\phi_c(0,0)\,|0\rangle \tag{41}$$

and so we have

$$(-1)^{\sum \text{ conformal spins}} \left(\langle 0 | \, \phi_a(\infty,\infty) \phi_b(1/2,1/2) \phi_c(0,0) \, |0\rangle \right)^2 \; \geq \; 0 \; . \qquad (42)$$

This means the three point function and thus the operator product coefficients will be real or imaginary depending on the sum of the three conformal spins. A simple example is the three-point function of currents; if we define the two-point function of currents by the convention above, the three-point function is proportional to structure constants f^{abc} which are purely imaginary, while the conventional 3-j symbols are purely real.

2.4 Detailed Calculations

1.$D_{2\rho+1}$ models:

The D invariants are of two kinds: the $D_{2\rho+1}$ and $D_{2\rho+2}$ invariants. Operator product coefficients for the $D_{2\rho+1}$ invariant were calculated by Di Francesco[19] who used the results of Rehren and Schroer[20] to claim that since there was an automorphism in the fusion rules under which the half integer isospin 'j' primary fields were replaced by their images with respect to $k/2$ i.e. $k/2-j$ and integer spins were left unchanged; we would expect the fusion matrices to also share this symmetry property. This can be understood as a consequence of equation(39) and equation(40) which allow us to conclude that

$$F\,(j_1,j_2,j_3,j_4)(j_{\text{new}},j_{\text{old}}) \;=\; \pm F(\sigma(j_1),\sigma(j_2),\sigma(j_3),\sigma(j_4))(\sigma(j_{\text{new}},\sigma(j_{\text{old}})) \qquad (43)$$

where $\alpha(j)$ is the image of j under the automorphism. This means that the o.p.e.'s for the $D_{2\rho+2}$ invariant are ± 1 or $\pm i$.

2. $D_{2\rho+2}$ invariants:

The $D_{2\rho+2}$ invariants have $k = 4\rho$ and break up into two kinds: ρ_{even} and ρ_{odd}. They are characterized by an extended chiral algebra and have fusion rules which were derived by Moore and Seiberg[15]

$k/4 = \rho_{\rm even}$

$$\begin{aligned} \ell_1 \times \ell_2 &= \sum j + (\rho^+ + \rho^-) \\ \ell \times \rho^+ &= \sum j + x_l \rho^+ + y_l \rho^- \\ \rho^+ \times \rho^+ &= \sum x_l \ell + \rho^+ \\ \rho^+ \times \rho^- &= \sum y_l \ell \end{aligned} \tag{44}$$

$k/4 = \rho_{\rm odd}$

$$\begin{aligned} \ell_1 \times \ell_2 &= \sum j + (\rho^+ + \rho^-) \\ \ell \times \rho^+ &= \sum j + x_l \rho^+ + y_l \rho^- \\ \rho \times \rho^+ &= \sum y_l \ell + \rho^- \\ \rho \times \rho^- &= \sum x_l \ell \end{aligned} \tag{45}$$

where x_l is one if l is even and zero if l is odd and $y_l = 1 - x_l$. Here ρ^+ and ρ^- refer to the two fields with isospin $k/4$ that exist in these theories.

To illustrate how the operator product coefficients are calculated, we take the case when $k/4 =$ even, the other case being very similar. First consider four fields with isospin (i, i) where $i \neq k/4$. We denote the relevant fusion matrix by F', and the related fusion matrix with the same external fields but intermediate states in the A model as F. The simplest case is where the intermediate fields all have isospin $j < k/4$, in which case $F' = F$ and $C((i,i),(i,i),(j,j)) = 1$.

Suppose we have the intermediate fields ρ^+ and ρ^- as well with isospin $k/4$. Their conformal blocks are determined by $SU(2)$ current algebra; from the point of view of $SU(2)$ these fields are exactly the same, so their conformal blocks are the same. Therefore, a possible solution to the duality relation is to take the A invariant solution, with the coefficient $C^2((i,i),(i,i),(k/4,k/4))$ replaced by $C^2((i,i),(i,i),(\rho^+,\rho^+)) + C^2((i,i),(i,i),(\rho^-,\rho^-))$. In fact, this is the only possible solution, as we shall argue below. Since the D model has a symmetry $\rho^+ \leftrightarrow \rho^-$, this determines $C^2((i,i),(i,i),(\rho^+,\rho^+)) = C^2((i,i),(i,i),(\rho^-,\rho^-)) = 1/2$.

Finally we consider intermediate fields with isospin $j > k/4$. In terms of the A model conformal blocks, there are off-diagonal terms in the matrix C^2. If we work instead in terms of the D model conformal blocks, these are cases in which there are several conformal blocks for the same intermediate field (in the language of Verlinde and Vafa $N_{iij} > 1$[21−23]). In principle this complicates matters, but since the equation we are trying to solve is just a generalization of the equation for the A model (with extra possible non-zero o.p.e.'s), we see that the matrix C^2 from the A model is still a solution. Since there is a unique solution to this duality relation, this is the only possible solution.

To prove the uniqueness we adapt an argument due to Brustein et al.[24]. Equation(22) for the case of four spin (i,i) fields says that:

$$\sum_{\mathrm{old_R}} \sum_{\mathrm{old_L}} F(\mathrm{new_L}, \mathrm{old_L})\, F(\mathrm{new_R}, \mathrm{old_R})\, C^2((i,i),(i,i),(\mathrm{old_L},\mathrm{old_R})) = C^2((i,i),(i,i),(\mathrm{new_L},\mathrm{new_R})) \tag{46}$$

where the right - hand side is 0 if there is no field in the theory with isospin $(\mathrm{new_L}, \mathrm{new_R})$. One of the intermediate fields which occurs in the t- channel is the identity, so let us take $\mathrm{new_L} = 0$ and in this subspace consider the equations. Since $C^2((i,i),(i,i),(0,0)) = 1$,

$$\sum_{\mathrm{old_R}} \left(\sum_{\mathrm{old_L}} C^2((i,i),(i,i),(\mathrm{old_L},\mathrm{old_R}))\, F(0,\mathrm{old_L}) \right) F(\mathrm{new_R}, \mathrm{old_R}) = \delta_{\mathrm{new_R},0} \quad . \tag{47}$$

Solving this gives

$$\sum_{\mathrm{old_L}} F(0,\mathrm{old_L}) C^2((i,i),(i,i),(\mathrm{old_L},\mathrm{old_R})) = F^{-1}(\mathrm{old_R},0). \tag{48}$$

If we think in terms of the original SU(2) algebra, this tells us immediately that $C^2((i,i),(i,i),(\rho^+,\rho^+)) + C^2((i,i),(i,i),(\rho^-,\rho^-))$ is equal to the coefficient $C^2((i,i),(i,i),(k/4,k/4))$ in the A model as claimed above.

Furthermore, although we used fusion matrices F of the SU(2) algebra in the duality relations, it is clear that a similar equation will be valid for any basis in the space of chiral vertex operators. In particular, if we can choose a basis which diagonalises the operator product coefficients and denote the fusion matrix with respect to this basis as $\tilde{F}$, then the o.p.e. are uniquely determined by equation(48) and are given to be

$$C^2((i,i),(i,i),(\mathrm{old_L},\mathrm{old_R})) = \frac{\tilde{F}^{-1}(\mathrm{old_L},0)}{\tilde{F}(0,\mathrm{old_L})}. \tag{49}$$

Consider now the case when in terms of the extended chiral algebra $N_{i,i,j} > 1$; it is easy to see that in our model then $N_{i,i,j} = 2$. In this case there are two channels, say α and β, for the same intermediate conformal field and things get complicated because $C^2((i,i),(i,i),(\alpha,\beta)) \neq 0$ in general. However, since the operator product coefficient can be thought of as a bilinear form (symmetric matrix) C in the space of chiral vertex operators, we can always diagonalise it. This is true even if we restrict ourselves to the subspace spanned by the channels α and β. Therefore, let us consider the new basis α' and β' which diagonalises C in this subspace. We know from equation(49) that $C^2((i,i),(i,i),(\alpha',\alpha'))$ and $C^2((i,i),(i,i),(\beta',\beta'))$ are uniquely determined, so if the basis α' and β' is unique, the o.p.e. in terms of the original basis α and β is also uniquely determined. But if α' and β' are not unique, then the eigenvalues of C must be degenerate; since we are working in a two dimensional subspace this means C must be a multiple of the identity. But then it doesn't really matter what basis we choose C will be the same multiple of the identity and the solution is unique once again. This completes our proof of uniqueness.

We conclude that

$$C^2((i,i),(i,i),(l_L,l_R)) = \delta_{l_L,l_R} \quad \text{if } i < k/4 \quad \text{and } l_L \neq \rho^{\pm} \tag{50}$$

and

$$\begin{aligned} C^2((i,i),(i,i),(\rho^+,\rho^+)) &= C^2((i,i),(i,i),(\rho^-,\rho^-)) \\ &= 1/2\,. \end{aligned} \tag{51}$$

Once these o.p.e.'s are known, we can consider four- point functions like $\langle(i,i),(i,i),(j,j),(j,j)\rangle$ and show that for $i,j < k/4$

$$C^2(i,i),(i,i),(l_L,l_R)) = \delta_{l_L,l_R} \quad \text{if} \quad l_L \neq k/4 \\ = 1/2\ \delta_{l_L,l_R} \quad \text{if}\, l_L = \rho^{\pm} \quad . \tag{52}$$

Operator product coefficients with two or three ρ fields can be obtained from four-point functions with all external ρ fields. Here we will need the coefficients of F'; they are sums of coefficients of F which correspond to different intermediate fields related by the extended algebra. These are just fields related by diagram automorphism, and so to find F' we need to fix the sign in equation(39) and equation(40) when $j_1 = j_2 = j_3 = j_4 = \rho$. For the $D_{2\rho+2}$ invariants a comparison of the relevant Clebsch's tells us that

$$F(j_4,k/2,j_2,j_1)(k/2-j_2,k/2-j_1) = (-1)^{j_4+j_2+j_3}\,. \tag{53}$$

so that

$$F(\rho,\rho,\rho,\rho)(\text{new},\text{old}) = (-1)^{\text{new}} F(\rho,\rho,\rho,\rho)(\text{new},k/2-\text{old}) \tag{54}$$

similarly for 'new'. Thus the sums which appear in F' are easy to find. For example, if we consider a four- point function in which all the four fields are (ρ^+,ρ^+), i.e.,

$$\langle\,(\rho^+,\rho^+),\,(\rho^+,\rho^+),\,(\rho^+,\rho^+),\,(\rho^+,\rho^+)\rangle\ , \tag{55}$$

we see that only even isospin can occur as intermediate states. The fusion matrix F' is given as

$$F' = \begin{pmatrix} 2F(0,0) & 2F(0,2) & \ldots & 2F(0,2m) & \ldots & F(0,k/4) \\ 2F(2,0) & 2F(2,2) & \ldots & 2F(2,2m) & \ldots & F(2,k/4) \\ \vdots & & & & & \\ 2F(k/4,0) & 2F(k/4,2) & \ldots & & \ldots & F(k/4,k/4) \end{pmatrix} \tag{56}$$

and the duality relation is

$$F'_{ij}\, C^2((\rho^+,\rho^+),(\rho^+,\rho^+),(j,j))\, F'_{kj} \;=\; C^2((\rho^+,\rho^+),(\rho^+,\rho^+),(k,k)) \quad \delta_{i,k} \quad . \tag{57}$$

It has a unique solution as shown above :

$$C^2((\rho^+,\rho^+),(\rho^+,\rho^+),(i,i)) = \frac{F'^{-1}_{i,0}}{F'_{0,i}} \quad . \tag{58}$$

For this case, obviously $F'^2 = 1$ and we get

$$\begin{aligned} C^2((\rho^+,\rho^+),(\rho^+,\rho^+),(i,i)) &= 1 \quad \text{if} \quad i \neq \rho^+ \\ &= 2 \quad \text{if} \quad i = \rho^+ \end{aligned} \quad . \tag{59}$$

Similarly we can now consider a four point-function $\langle(\rho^+,\rho^+),(\rho^-,\rho^-),(\rho^-,\rho^-),(\rho^+,\rho^+)\rangle$ and again, by thinking in terms of the extended chiral algebra, we get that

$$C^2((\rho^+,\rho^+),(\rho^-,\rho^-),(i,i)) = 1 \quad \text{for all odd } i\,. \tag{60}$$

All the other operator product coefficients can be determined by diagram automorphism.

The case with $k/4$ odd is very similar. The only differences which arise are because the field $(0,k/2)$ has odd spin and because the fields ρ^+ and ρ^- are not self conjugate but conjugate to each other instead. Our results are summarised towards the end of the paper.

3. E_7 invariant:

We started with fusion matrices involving one spin 1/2 field which were described earlier in equation(30) to equation(33) and used the pentagon identity to generate fusion matrices with higher spins. Once these fusion matrices are known up to high enough spin (actually spin 2 is good enough) the o.p.e.s can be found. Calculations were done on the computer using SMP[25].

The results we obtain are shown in table I. The o.p.e.s for this model are simply related to those in the D_{10} and E_7 models. The connection between the E_7 and D_{10} models is worth understanding better. Both share the same extended chiral algebra and they can be related by an automorphism in the fusion rules in which one of the isospin 4 fields is exchanged with the isospin 1 field. This automorphism in the fusion rules can be expected to translate into a linear automorphism on the chiral vertex operators, but *a priori* there is no way to determine the exact matrix elements of this automorphism, which are needed to calculate o.p.e.s. We were able to determine the automorphism explicitly for the $A \to D_{odd}$ case (it is exactly the diagram automorphism above) because it can be represented as the action of a primary (but unphysical) field in theory, namely $\phi_{0,k/2}$. The automorphism which relates D_{10} and $E7$ is clearly not of this form (since it preserves the chiral vertex operator 0+8, and the only operators which do this are in the block 0+8, which preserve everything), and we see from the exact solution that its matrix elements are not all 1. For example, in the D_{10} model $C^2((3,3),(3,3),(2,2)) = 1$ and $C^2((3,3),(3,3),(2,6)) = 0$, but in the E_7 model $C^2((3,3),(3,3),(2,2)) = 1/4$ and $C^2((3,3),(3,3),(2,6)) = 3/4$.

2.5 Results

Our results for the $D_{2\rho+2}$ and the E_7 models are as follows.

(1) $D_{2\rho+2}$ invariants:

(A) $k/4 = \rho$ even:

(*a*)

$$C^2((i,i),(j,j),(l,l)) = 1 \quad \text{if} \quad i,j < k/4 \quad \text{and} \quad l \neq \rho^{\pm}$$

(*b*)

$$C^2((i,i),(j,j),(l_L,l_R)) = 0 \quad \text{if}\ i,j < k/4 \quad \text{and} \quad l_L \neq l_R$$

(*c*)

$$C^2((i,i),(j,j),(l,l)) = 1/2 \quad \text{if} \quad l = \rho^{\pm}$$

(d)

$$C^2((\rho^\pm,\rho^\pm),(\rho^\pm,\rho^\pm),(i,i)) = 1$$

if $i < k/4$ and this is allowed by the fusion rules.

(e)

$$\begin{aligned} C^2((\rho^+,\rho^+),(\rho^+,\rho^+),(\rho^+,\rho^+)) &= C^2((\rho^-,\rho^-),(\rho^-,\rho^-),(\rho^-,\rho^-)) \\ &= 2 \end{aligned}$$

(f)

$$\begin{aligned} C^2((k/2-j_{1L},j_{1R}),(k/2-j_{2L},j_{2R}),(j_{3L},j_{3R})) &= (-1)^{j_{1L}+j_{2L}} \\ &\times C^2((j_{1L},j_{1R}),(j_{2L},j_{2R}),(j_{3L},j_{3R})) \end{aligned}$$

and similarly for diagram automorphisms on the right isospins.

(B) $k/4 = \rho_{\text{odd}}$:

The rules are almost the same except for rule (f) which now reads

(f′)

$$\begin{aligned} C^2((j_{1L},j_{1R}),(j_{2L},j_{2R}),(j_{3L},j_3)) = \\ &(-1)^{j_{1L}+j_{2L}} \\ &\times C^2((\frac{k}{2}-j_{1L},j_{1R}),(\frac{k}{2}-j_{2L},j_{2R}),(j_{3L},j_{3R})) \end{aligned}$$

only if j_{1L} and $j_{2L} \neq \rho^\pm$ and

$$\begin{aligned} C^2((j_{1L},j_{1R}),(j_{2L},j_{2R}),(j_{3L},j_3)) &= (-1)^{j_{2L}} \\ &\times C^2((\frac{k}{2}-j_{1L},j_{1R}),(\frac{k}{2}-j_{2L},j_{2R}),(j_{3L},j_{3R})) \end{aligned}$$

if $j_{1L} = \rho^\pm$ and $j_{2L} \neq \rho^\pm$

and similarly for the right isospins too.

These operator product coefficients we get are different from those quoted in Di Francesco's paper[19]. Take as an example the case when $\rho = 4$ (corresponding to the D_{10} invariant). Then we have $C^2((3,3),(3,3),(3,3)) = 1$ whereas Di Francesco quotes it to be $= 1/2$.

(2) E_7 invariant:

Our results are shown in Table I. As equation (26) shows we use conventions where the Clebsch Gordon coefficients include the square roots of the operator product coefficients of the left- right symmetric A invariants. All the remaining operator product coefficients can be obtained by using two properties:

(a) Diagram automorphism discussed earlier which says:

$$C^2((j_{1L}, j_{1R}), (j_{2L}, j_{2R}), (j_{3L}, j_{3R})) =$$

$$(-1)^{j_{1L}+j_{2L}} C^2((k/2 - j_{1L}, j_{1R}), (k/2 - j_{2L}, j_{2R}), (j_{3L}, j_{3R})) \qquad (61)$$

and similarly on the right isospin indices and

(b) Symmetry under left- right exchange in which $\phi_{j_L, j_R} \rightarrow \phi_{j_R, j_L}$, i.e.,

$$C^2((j_{1L}, j_{1R}), (j_{2L}, j_{2R}), (j_{3L}, j_{3R})) = C^2((j_{1R}, j_{1L}), (j_{2R}, j_{2L}), (j_{3R}, j_{3L})) \qquad (62)$$

2.6 Conclusions

This chapter is in praise of fusion matrices and the various identities they satisfy, in particular the pentagon identity. This identity allows us to calculate operator product coefficients in the E_7 model by generating higher isospin fusion matrices from those involving fields with lower isospins. It also allows us to understand diagram automorphism very simply and thus the operator product coefficients in the $D_{2\rho+1}$ models; furthermore, when coupled to the notion of extended chiral algebras (e.c.a.) and fusion matrices with respect to this e.c.a., it allows us to understand the structure constants of the $D_{2\rho+2}$ models. In all these cases we find that the operator product coefficients are simply related to those in the simplest left-right symmetric A models.

Our work for the D invariants closely parallels that of Di Francesco[19] although our results for the $D_{2\rho+2}$ models are different from his. After we completed our calculation for the E_7 model, we received a paper by Kato and Kitazawa[26] where the same calculation is done. They use the conventional method of writing out the conformal blocks and seeing how they transform explicitly. Our answers essentially agree (up to factors of i which are rather trivial to determine). This agreement between two rather complex calculations is reassuring.

The techniques we have used involving fusion matrices are in fact very general and can be used to calculate the operator product coefficients in other C.f.t.'s too. These coefficients determine, as emphasised in the beginning of the chapter, all correlation functions in the theory. Thus, from the string theory point of view, they determine the scattering amplitudes (at least perturbatively) and so the Yukawa and gauge couplings of the effective field theory to which the string theory gets mapped at low energies. From the point of view of C.f.t., fusion matrices and braiding matrices (which we have not talked about here but which are related to fusion matrices) store information about the monodromy properties of conformal blocks in theory and carry a lot of information. They have been used to some very powerful results about rational C.f.t.'s. These include the famous theorem by Verlinde that the S-matrix diagonalises the fusion rules, the fact that all non diagonalinvariants must be associated with either an extended chiral algebra and/or automorphisms of the fusion rules and finally a very strong form of naturality which says that in a rational C.f.t. the operator product coefficient vanishes if and only if the corresponding fusion rules vanish in the maximally extended chiral algebra, i.e., if and only if there is a symmetry that makes them vanish.

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Table I

$C^2((1,4),(1,4),(1,4)) =$	$-\sqrt{2}$	$C^2((1,4),(1,4),(2,2)) =$	1
$C^2((1,4),(2,2),(3,3)) =$	$\frac{-1}{\sqrt{2}}$	$C^2((1,4),(2,2),(2,2)) =$	$\frac{-1}{\sqrt{2}}$
$C^2((1,4),(4,4),(3,3)) =$	$\frac{-1}{\sqrt{2}}$	$C^2((1,4),(4,1),(3,3)) =$	$\frac{1}{2}$
$C^2((1,4),(3,3),(3,3)) =$	$\frac{-1}{\sqrt{2}}$	$C^2((1,4),(2,2),(3,3)) =$	$\frac{-1}{\sqrt{2}}$
$C^2((1,4),(4,1),(4,4)) =$	1	$C^2((1,4),(4,4),(4,4)) =$	0
$C^2((6,6),(3,3),(3,3)) =$	$\frac{1}{4}$	$C^2((4,4),(2,2),(2,2)) =$	$\frac{1}{2}$
$C^2((2,2),(2,2),(3,3)) =$	1	$C^2((2,2),(2,2),(2,2)) =$	1
$C^2((4,4),(2,2),(3,3)) =$	$\frac{1}{2}$	$C^2((4,4),(4,4),(2,2)) =$	1
$C^2((2,2),(3,3),(5,5)) =$	$\frac{1}{4}$	$C^2((4,4),(2,2),(2,2)) =$	$\frac{1}{2}$
$C^2((3,3),(3,3),(3,3) =$	1	$C^2((3,3),(3,3),(5,5)) =$	1
$C^2((3,3),(3,3),(3,5)) =$	0	$C^2((4,4),(3,3),(3,3)) =$	$\frac{1}{2}$
$C^2((4,4),(4,4),(4,4)) =$	2	$C^2((4,4),(4,4),(3,3)) =$	0
$C^2((2,2),(3,3),(3,3)) =$	$\frac{1}{4}$	$C^2((2,2),(3,3),(3,5)) =$	$-\frac{3}{4}$

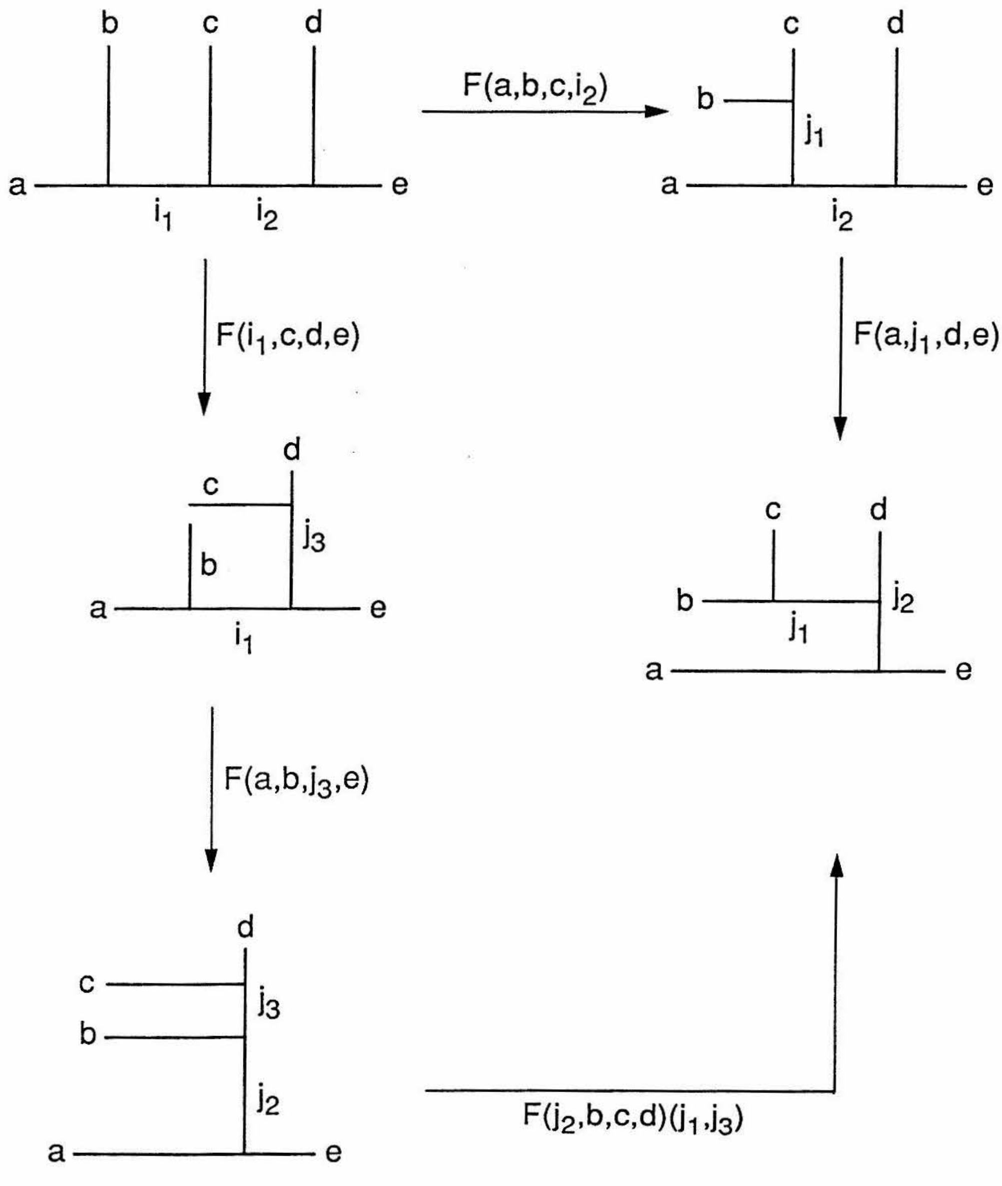

Fig. 1. The pentagon identity

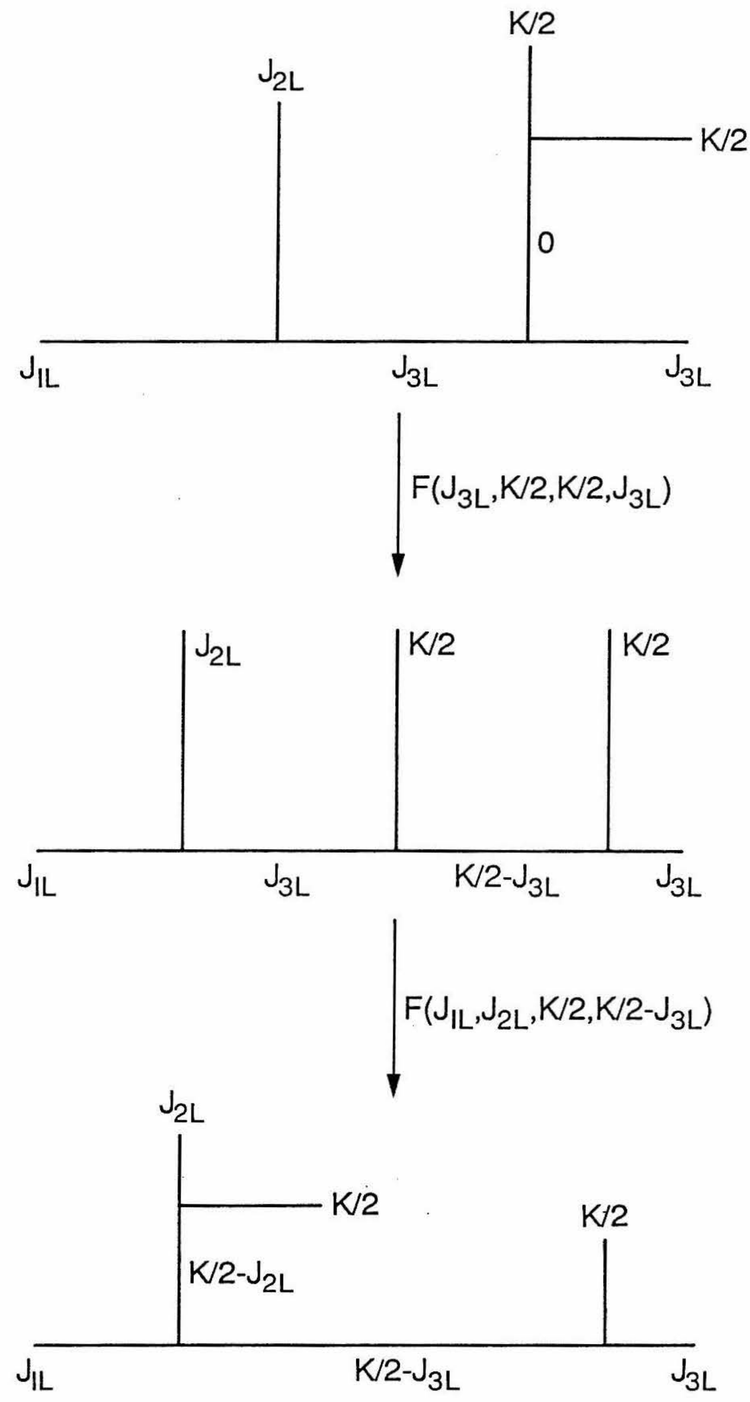

Fig. 2. The diagram shows how eq (35) is derived

3. Wormholes and θ_{QCD}

3.1 Introduction

In this chapter we turn to a study of the low energy effective theory of gravity. We might have expected that our lack of knowledge of physics at high energies would not have impeded our ability to study the behaviour of gravity at low energies if we parametrised our ignorance in terms of a few renormalisable couplings. This chapter will, however, serve to emphasise that even the low energy effective theory of gravity is ill defined and has a number of open questions. The issue we will be primarily concerned with is that of fluctuations in the topology of space time which can connect two otherwise distantly separated points. We will often refer to such fluctuations as wormholes. It should be mentioned at the very outset that there is no compelling reason known today for allowing such fluctuations to occur . Hopefully, the consistency of the theory will eventually dictate whether or not they should be allowed. For the moment, we will assume that they are present and study their consequences.

Over the past two years some interesting results have come out of the study of wormholes. For example, Coleman [2] showed how they would not lead to a loss of locality or a loss of quantum coherence, as might have been expected [1,3], but would instead alter the fundamental constants of nature and, in fact, inflict them with a fundamental quantum indeterminacy. This raised the possibility of low energy physics completely decoupling from high energy physics; even the fundamental constants of nature for low energy physics would not be determined by high energy physics; in fact, they would be quantum mechanically indeterminate. Coleman then went on to propose a way of avoiding this unfortunate situation [4]. He proposed a probability distribution for the occurrence of a universe, characterised by a set of coupling constants, and showed that it was very strongly peaked at the cosmological constant being zero. This prediction was in accord with nature. The idea was further pursued by Grinstein and Wise [5] and Preskill [6] who showed that the proposed probability distribution would cause G^{-1} to be maximised too and that this criterion was sufficient to fix all the constants of nature. In practice, though, the dependence of G on these constants was very sensitive to short distance physics and so could not be calculated in terms of low energy physics alone.

We summarise the developments mentioned above in the next section. In the subsequent section, we show that there is one fundamental constant of nature $\theta_{\rm QCD}$ which escapes this fate. The dependence of G on θ can be determined in terms of low energy physics alone since this dependence is nonperturbative in the QCD coupling and so exponentially suppressed at high energies due to asymptotic freedom.

2.2 The vanishing of the cosmological constant

Coleman made two assumptions about wormholes: (a) that they are characterised by a scale M_w and (b) that their action is independent of the distance between the two points they connect, where the distance is measured in terms of the background manifold. These imply that at scales much below the wormhole scale M_w, the effect of wormholes can be integrated out and they induce in the low energy effective action (at least in the dilute wormhole approximation) a term :

$$\delta\ S = \int O_i(x)\ \ \Delta_{ij}\ \ O_j(y)\ \sqrt{g(x)}\ \sqrt{g(y)}\ d^4x\ d^4y \tag{63}$$

where Δ_{ij} is independent of x or y. The path integral then contains a factor

$$= \int Dg\ e^{-\delta S}. \tag{64}$$

We can rewrite this by introducing Lagrange multipliers as

$$= \int d\alpha\ exp\ [\frac{1}{2}\sum_{i,j}\ \alpha_i\ \Delta_{ij}^{-1}\ \alpha_j]\quad exp\ [\sum_i \alpha_i \textstyle\int d^4x\ \sqrt{g(x)}O_i(x)]. \tag{65}$$

Rewritten in this way we see that paradoxically, the extreme nonlocality of wormhole effects is, in a sense, responsible for the fact that they do not cause a loss of locality or quantum coherence but instead alter the values of the fundamental constants of nature by shifting them by α dependent factors. In any particular universe we could measure the constants to as much precision as we desire and so any one particular universe is characterised by a set of α 's. The expression above can then be interpreted as a sum over all possible universes with a probability distribution weighting them. Wormholes, it seems, would make it impossible to predict the fundamental constants of nature. This is a very important conclusion. It raises the possibility that low energy physics is completely independent of high energy physics. Our universe is what it is as a matter of chance; even knowing the theory of everything at high energies would not let us predict the low energy behaviour of our universe.

Having suggested such an unfortunate possibility, Coleman went on to propose a way of avoiding it. So far our discussion has been fairly general, but from now on we will work within the context of the Euclidean path integral approach to quantum gravity. In this approach the expectation value of an operator O is given by

$$\langle O \rangle = N \int_{c1}^{c2} Dg \; D\phi \; e^{-S} O(x) \tag{66}$$

where the path integral is a sum over all histories that take us from an 'initial' configuration c1 to 'final' configuration c2. c1 and c2 depend on our choice of the wavefunction of the ground state. One common choice is that of Hartle and Hawking which consists of having no boundaries so that the integral above is over closed manifolds that have no boundaries. We should mention that this approach to quantum gravity, like any other, is far from well understood. For example, the Euclidean action for gravity (unlike that of a scalar field or a gauge field) is unbounded from below thereby making the path integral ill defined even in the semiclassical approximation. There are also the problems associated with gauge fixing and regulating it in a generally covariant manner. In spite of all this we will persist in using it for it does provide us with a convenient framework for studying topology change. To return to our main train of thought, Coleman's insight was that the actual distribution of α 's was governed by a probability distribution different from what one naively expected by looking at equation (65), i.e.,

$$P(\alpha) = exp - \frac{1}{2}[\alpha_i \, \Delta_{ij}^{-1} \, \alpha_j]. \tag{67}$$

If we assume Hartle Hawking boundary conditions (our conclusions will not be sensitively dependent on this choice) the sum over histories in eq(66) is over all closed manifolds. The expectation value of the operator O can then be written as the sum over all connected manifolds which contain the point x times a sum over all closed manifolds. The sum over all closed manifolds exponentiates so we have

$$\langle O \rangle = N \int d\alpha \;\; P(\alpha) \;\; exp[\textstyle\int Dg' \;\; e^{-S[g',\alpha]}] \displaystyle\int Dg \;\; e^{-S[g,\alpha]} O(x). \tag{68}$$

Thus we see that the probability distribution in α space is given by not just $P(\alpha)$ but rather the product of $P(\alpha)$ and $Q(\alpha)$ where

$$Q(\alpha) = exp[\textstyle\int Dg \;\; e^{-S[g,\alpha]}] \;\; . \tag{69}$$

To proceed further Coleman assumed that the sum over all closed connected manifolds is dominated by large smooth geometries . We can then integrate out all the massive degrees of freedom in the theory and get a theory of gravity valid in the far infra-red regime (it probably contains a few other degrees of freedom like the photon which we will neglect here). In this theory we can then attempt to evaluate the path integral and thus $Q(\alpha)$ by a semiclassical approximation. Unfortunately, as mentioned above, this is a potentially treacherous step since the path integral even in Euclidean gravity, is not bounded below. The action can always be lowered by making a suitable conformal transformation on the metric. We will have to assume that there is some correct way of dealing with these fluctuations in the conformal factor and that when this is done the "stationary" points we consider here will continue to dominate the path integral. Given all these caveats, though, we can derive a rather spectacular result. For now if

$$S_E(g) = \frac{1}{16\pi G} \int d^4x\sqrt{g}[2\lambda - R] \ , \tag{70}$$

we get that the classical equations of motion are given by

$$R_{\mu\nu} = \lambda \ g_{\mu\nu} \ . \tag{71}$$

The action then at the stationary point becomes

$$S = \frac{1}{16\pi G} \int d^4x\sqrt{g}[\lambda - 2\lambda] \ , \tag{72}$$

i.e.,

$$S = -\frac{1}{16\pi G} \lambda \int d^4x\sqrt{g} \ . \tag{73}$$

If λ is positive we know that the solution to the classical equations which maximises the volume is the four sphere with volume V given by

$$V = 24\pi^2/\lambda^2 \ . \tag{74}$$

This gives that

$$Q(\alpha) = exp[\ exp(3\pi/G\lambda)\ ] \ . \tag{75}$$

We see that this is a function peaked very strongly at $\lambda = 0$. Compared to this, $P(\alpha)$ is a smoothly varying function, so we see that the cosmological constant is overwhelmingly likely to be zero. Actually we have computed $Q(\alpha)$ only for $\lambda > 0$ when $\lambda < 0$, solutions to the classical equations which minimise the volume are required and these are not known. We are assuming here that in this regime $Q(\alpha)$ takes on values obtained by analytically continuing from the regime $\lambda > 0$.

Despite all the assumptions made, this is a stunning result. The quantum indeterminacy of the constants of nature has proved to be a boon. Coleman's proposal has shown that at least in the case of the cosmological constant, this indeterminacy is really not there. Wormholes cause the probability distribution $Q(\alpha)$ to be such a sharply peaked function that they in fact single out a value of the cosmological constant as being overwhelmingly probable, and moreover this value is in accord with observations and solves a long standing problem.

What about the other constants of nature? Does Coleman's suggested probability distribution determine the other constants of nature too? Such a line of thinking was pursued by Grinstein and Wise and Preskill. They found that $Q(\alpha)$ would maximise G^{-1} too. At first one might wonder if this statement has any physical content since G^{-1} is a dimensionful quantity. Presumably before the wormholes are integrated out giving rise to α dependent parameters, we have a theory of gravity (like say string theory) with some characteristic length scale which determines the characteristic scale of wormholes M_W . What we then mean is that G^{-1} in units of M_W or the characteristic length scale of the fundamental theory is maximised. G here refers to the value of Newton's constant in the low energy theory of gravity referred to above obtained by integrating out all massive degrees of freedom. As equation (75) shows, this claim does look sensible at first, but it might be argued that since $\lambda = 0$ is very strongly favoured it does not matter what value G takes. To settle this issue one can consider surfaces in the space of α 's corresponding to λ being constant and study the behaviour of Q as this constant value tends to zero. For any nonzero value we see that $G^{-1}(\alpha)$ is indeed maximised. It is then natural to assume that this tendency of G^{-1} to be maximised is also retained as λ tends to 0. An explicit argument can also be given by regulating the theory with a maximum volume infra-red regulator but we do do so here.

Now the constraint that G^{-1} is maximised is much more restrictive than the constraint $\lambda = 0$ since it singles out a particular point in the space of α 's rather than a surface of codimension one. Is it possible, though, that there are some α 's which are still left undetermined? The α 's, as we saw, feed into the bare couplings which in turn, when we integrate out the massive degrees of freedom, feed into G_{ren}^{-1}. Since everything couples to gravity we expect G_{ren}^{-1} to depend on all the couplings and so on all the α 's. We see then that $Q(\alpha)$ is a very sharply peaked function of the α's. The quantum indeterminacy we feared wormholes might induce has gone away. There is an overwhelmingly likely set of values the α 's must take.

To determine these values we could, in principle, proceed as follows. First we could integrate out the wormholes and determine the α dependence of the bare couplings and then integrate out degrees of freedom to find out how these bare couplings feed into a infra- red theory of gravity thus determining G_{ren}. By minimising this function we could then find the most probable values of $\alpha' s$. In practice, this is, to say the least, a rather ambitious programme. For one thing, we do not know (beyond the dilute wormhole approximation) how the bare couplings depend on α 's. We could try to avoid this difficulty by assuming as a first approximation that all the bare couplings (with the exception of G^{-1}) are allowed to vary over their whole range as the α 's are varied and simply ask what were the most probable values of the bare couplings. Unfortunately the dependence of G_{ren}^{-1} on the bare couplings is very sensitive to short wavelength physics and therefore not possible to compute, at least at present, with an incomplete theory of gravity. In the next section we will discuss an exception to this rule and show that the functional dependence of G on θ_{QCD}, is insensitive to Planck scale physics and can be determined in terms of low energy physics alone.

But before this we would like to mention, although only briefly, the large wormhole problem which has arisen on account of Coleman's proposal. We had assumed at the very outset that wormholes have a characteristic size R_w. It is natural to expect that wormholes with a size much larger than the Planck length would be suppressed since their action would be much too large. We would then expect the corresponding Δ_L in equation (67) to be very small and this would make the α_L very strongly peaked at 0. Large α_L dependent shifts in the couplings will then be highly improbable. This is good; we do not want large wormholes to play a very significant role for this would contradict the observed locality of low energy physics.

But Coleman's proposal was that the distribution of α's is governed not by just $P(\alpha)$ but by the product of $P(\alpha)$ and $Q(\alpha)$. Since $Q(\alpha)$ is a very singular function in the vicinity of $\lambda = 0$, this would mean large wormholes, despite their suppression by $P(\alpha)$ would try to maximise G^{-1} on the surface $\lambda = 0$. Since their contribution to G^{-1} is maximised when these wormholes are dense, this would lead to the above mentioned problem with non locality at low energies. Despite several attempts this remains one of the biggest loopholes in Coleman's proposal. Unfortunately, we will not have much more to say on this important issue in the subsequent discussion.

3.3 Wormholes and $\theta_{\rm QCD}$

We now turn to showing how the dependence of the θ-parameter of quantum chromodynamics [9] *can* be computed in terms of low-energy physics alone [21]. The point is that θ dependence arises only through nonperturbative strong interaction effects, and these are presumably exponentially small at the Planck scale, because of asymptotic freedom. We have calculated the θ-dependence of G in an approximation that is valid if the masses of the light up and down quarks are sufficiently small. We find, assuming the validity of this approximation, that the minimum of G occurs for θ very near π. (The minimum would be at exactly $\theta = \pi$ were it not for small CP-violating effects due to the weak interactions.)

We therefore expect that $\theta \simeq \pi$ is overwhelmingly favored by the probability distribution equation (1). Since the phenomenological evidence suggests that instead $\theta \simeq 0$, our calculation indicates a potentially serious conflict between current ideas about wormholes and observed low-energy physics. We will comment further below about how this conflict might be resolved.

Because the θ parameter is CP-odd, and the strong interactions conserve CP to remarkable accuracy, it has long been recognized that θ must be extremely close to either 0 or π. ($\theta = \pi$ is a CP-conserving value because θ is a periodic variable defined modulo 2π.) The experimental limit on the electric dipole moment of the neutron indicates that θ deviates from 0 or π by an amount at most of order 10^{-9} [10]. Given that large CP-violating phases do infect the weak interactions, this inclination of the strong interaction to conserve CP poses a serious puzzle. The most satisfying explanation for the CP conservation by the strong interaction is that originally suggested by Peccei and Quinn [11]. They proposed that θ is actually a dynamical variable, and therefore assumes that value that minimizes the energy density of the vacuum. A powerful nonperturbative argument shows that the vacuum energy density of QCD, as a function of θ, is minimized at $\theta = 0$ [12]. Thus, the Peccei-Quinn mechanism naturally explains why θ is very close to the CP-conserving value. (The minimum of the energy density is perturbed slightly away from $\theta = 0$ by CP-violating effects due to the weak interaction. The amount of the perturbation depends on the detailed nature of CP violation; in the Kobayashi-Maskawa model, one can estimate that the minimum occurs for $\theta \sim 10^{-14}$, which is well within the experimental limit.) Another interesting consequence of this mechanism is that there exists a very light, very weakly interacting particle, the axion, associated with the oscillations of θ about the minimum [13,14]. It has even been proposed that these axions comprise the dark matter of the universe [15].

Within the context of wormhole physics, the Peccei-Quinn explanation for $\theta \simeq 0$ is problematic. Their mechanism relies on the existence of an approximate global symmetry, the Peccei-Quinn symmetry, that is intrinsically broken only by a color anomaly. But wormholes have no respect for global symmetries (whether exact or approximate). Rather, wormhole effects are expected to generate α-dependent couplings of all types consistent with the *local* symmetries of fundamental physics [1,2]. It will not do, then, to invoke a Peccei-Quinn symmetry by fiat; the symmetry itself requires an explanation.

(A similar remark applies to another explanation that is sometimes proposed for the small value of the electric dipole moment of the neutron — that the mass of the up quark is zero, or very close to zero. This is no explanation unless one understands *why* the up quark is massless. Indeed, this proposal is closely related to the Peccei-Quinn mechanism, for if the up quark is massless, then there is an approximate global symmetry that is intrinsically broken only by a color anomaly.)

In spite of the above comments, wormholes and the Peccei-Quinn mechanism might be reconcilable. Two possibilities come to mind. Perhaps an approximate Peccei-Quinn mechanism arises in low-energy physics as an accidental consequence of local symmetries, which are not disturbed by wormhole effects. (This would be like the approximate conservation of baryon number in the standard model that is an automatic consequence of local SU(3) $\times$ SU(2) $\times$ U(1) invariance.) It is not so easy to make this idea workable, however. The problem is that it does not suffice for the accidental Peccei-Quinn symmetry to apply to the operators in the effective action that are of renormalizable type (dimension four or less). If the Peccei-Quinn mechanism is to ensure that θ is very small, then nonperturbative strong interaction effects must swamp all other effects that break the Peccei-Quinn symmetry; this constraint typically requires that the symmetry be satisfied by operators of quite high dimension [16]. There is another possible way to rescue the Peccei-Quinn mechanism, in spite of the tendency of wormhole effects to break global symmetries. Although the Peccei-Quinn symmetry is badly broken for generic values of α, it may become a good approximate symmetry for that particular "standard" value of α that minimizes G.

At any rate, there appears to be ample motivation to consider whether, within the context of wormhole physics, the CP conservation of the strong interactions can be explained without appealing to the Peccei-Quinn mechanism, and without requiring the existence of a light axion. Indeed, the crucial feature of the Peccei-Quinn mechanism is that it makes θ an adjustable quantity, a dynamical variable that seeks the minimum of the energy density at $\theta = 0$. And wormhole effects also make θ an adjustable quantity, not a dynamical variable, but an α-dependent coupling constant that seeks the sharp peak in the probability distribution equation (1). Furthermore, as Nielsen and Ninomiya [17] recently stressed, G is CP even while θ is CP odd; therefore, strong interaction effects generate a dependence of G on θ that is an even function of θ. This function is stationary at both $\theta = 0$ and $\theta = \pi$, and so it is reasonable to expect that its minimum occurs either at $\theta = 0$ or at $\theta = \pi$. Since the peak in the probability distribution occurs where G is minimized, the CP conservation by the strong interactions is naturally explained. (As for the Peccei-Quinn mechanism, CP-violating effects due to the weak interventions perturb the minimum, but only slightly.)

The dependence of θ on α arises as follows: The Yukawa couplings of the quarks to the Higgs doublet are modified by wormhole effects, and hence are α-dependent in both modulus and phase. When the electroweak gauge symmetry is spontaneously broken, this α-dependence enters the quark mass matrix. Some of the phases in the mass matrix are unobservable, because they can be removed by a redefinition of the phases of the quark fields. But there remain, as observable parameters, the values of the quark masses and the Kobayashi-Maskawa angles and phases that infect the charged weak current. Finally, there is one phase that can be removed from the quark mass matrix only by a field redefinition that has a color anomaly. This phase is θ. It is irrelevant in all orders of perturbation theory, but nonperturbative strong interactions do depend on θ.

We will assume in the ensuing discussion that it is possible to adjust the α-parameters so that θ changes, while all other couplings in the effective Lagrangian remain fixed. It is easy to construct toy models that behave this way, and we expect that this behavior is reasonably generic. When the α-dependence of Newton's constant G is considered, one finds that perturbative renormalization effects induce large contributions to G that depend on the quark masses and the KM angles. These contributions are of order M_{Pl}^2, where M_{Pl} is the Planck mass scale, and are sensitive to the details of Planck-scale physics. The criterion that $G(\alpha)$ is at its minimum on the surface $\Lambda(\alpha) = 0$, then, determines these quantities, but only in a manner that cannot be computed based on a knowledge of low-energy physics alone [6]. But since the dependence of G on θ arises only from nonperturbative strong interaction effects, θ is calculable based on low-energy physics alone, at least in principle.

Before we proceed with our calculation of $G(\theta)$, one more point needs emphasis. We asserted above that θ can be determined by finding the minimum of $G(\theta)$, but the actual criterion that determines the constants of Nature is that $G(\alpha)$ is minimized on the surface where $\Lambda(\alpha) = 0$. We must explain why it is an excellent approximation to disregard the requirement that $\Lambda(\alpha) = 0$. The crucial point is that the dependence of Λ and G on θ is characterized by the strong interaction scale, rather than the Planck scale. If we perturb θ by a small amount $\delta\theta$, Λ and G change according to

$$\begin{aligned} \delta\Lambda &= a(\theta)\delta\theta \quad , \\ \delta\left(\frac{1}{16\pi G}\right) &= b(\theta)\delta\theta \quad ; \end{aligned} \tag{2}$$

we will calculate $a(\theta)$ and $b(\theta)$ below, in chiral perturbation theory. But when a generic α parameter is perturbed by $\delta\alpha$, we have instead, schematically,

$$\begin{aligned} \delta\Lambda &\sim M_{Pl}^4\delta\alpha \quad , \\ \delta\left(\frac{1}{16\pi G}\right) &\sim M_{Pl}^2\delta\alpha \quad . \end{aligned} \tag{3}$$

Thus, if we perturb θ and adjust α slightly to remain on the $\Lambda = 0$ surface, the change of G is given by

$$\delta\left(\frac{1}{16\pi G}\right)\bigg|_{\delta\Lambda=0} \sim \left(b(\theta) - \frac{a(\theta)}{M_{Pl}^2}\right)\delta\theta \quad . \tag{4}$$

Because a and b are very small in Planck units, the second term in equation (4) is negligible. We may just as well minimize $G(\theta)$ without regard for the $\Lambda = 0$ constraint.

Now we are finally prepared to describe the calculation of $G(\theta)$, in chiral perturbation theory. The main idea that underlies the calculation is quite simple. If the pion mass were very small, as would be true if the up and down quarks were sufficiently light, then the strong interaction contribution to G would be dominated by a one-pion-loop diagram that has a calculable logarithmic sensitivity to m_π^2. Then, when the pion is light enough, the dependence of G on θ can be calculated from the dependence of m_π^2 on θ. One finds that $G(\theta)$ is minimized when $m_\pi^2(\theta)$ is minimized. And it is easy to see, again in the limit where the pion is sufficiently light, that $m_\pi^2(\theta)$ is minimized at $\theta = \pi$.

To perform the calculation, we make use of a chiral Lagrangian that describes the self-interactions at low momenta of the pseudo-Goldstone bosons π^+, π^-, π^0. This chiral Lagrangian respects a nonlinearly realized $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ chiral symmetry. It can be expressed in terms of a field $\Sigma(x)$ that is a 2×2 unitary matrix with determinant one, and that transforms under chiral symmetry as

$$\Sigma \to V_L \Sigma(x) V_R^\dagger \quad , \tag{5}$$

where $V_L \in \mathrm{SU}(2)_L$ and $V_R \in \mathrm{SU}(2)_R$. In terms of the pion fields, Σ can be expressed as

$$\Sigma = \exp\left(2i\Pi/f\right), \ \Pi = \begin{pmatrix} \pi^0/\sqrt{2} & \pi^+ \\ \pi^- & -\pi^0/\sqrt{2} \end{pmatrix} \quad , \tag{6}$$

where f is the pion decay constant. The chiral Lagrangian can be expanded in powers of the derivatives acting on the Σ field; terms with more derivatives are suppressed at low energy by additional powers of the pion momentum.

The effects of the explicit breaking of chiral symmetry by quark masses can also be systematically incorporated in the chiral Lagrangian. If m is the 2×2 mass matrix of the light quarks, then QCD respects a formal symmetry in which equation (5) is accompanied by

$$m \to V_R m V_L^\dagger \quad . \tag{7}$$

By demanding invariance under this formal symmetry, we find that the leading mass-dependent terms in the chiral Lagrangian, in a curved spacetime background, are

$$\mathcal{L}_{\mathrm{mass}} \quad = \quad v \ \mathrm{tr}\ (m\Sigma + \Sigma^\dagger m^\dagger)(1 + cR) + \dots \quad . \tag{8}$$

Here v is a quantity with the dimensions of $(\mathrm{mass})^3$, and c is a quantity with the dimensions of $(\mathrm{mass})^{-2}$; both are determined by nonperturbative strong interaction effects. R is the curvature scalar of the background spacetime. In equation (8), we have neglected terms that contain derivatives of the Σ field, more powers of the light quark mass matrix m, or more powers of the curvature R.

The θ parameter enters the chiral Lagrangian in the light quark mass matrix m, through the relation

$$\theta = \arg(\det m) \quad . \tag{9}$$

We find the precise form of m by performing a chiral rotation of m that ensures that $\mathcal{L}_{\text{mass}}$ contains no "tadpole" terms linear in the pion fields. The result is

$$m = \begin{bmatrix} m_u e^{i\phi} & 0 \\ 0 & m_d e^{i(\theta-\phi)} \end{bmatrix} \quad , \tag{10}$$

where

$$\sin\phi = \frac{m_d \sin\theta}{(m_u^2 + m_d^2 + 2m_u m_d \cos\theta)^{1/2}} \quad , \quad \sin(\theta - \phi) = \frac{m_u}{m_d} \sin\phi \quad . \tag{11}$$

By expanding Σ in powers of the pion field, we find, in tree approximation in the chiral Lagrangian and to lowest order in light quark masses,

$$\begin{aligned} m_\pi^2(\theta) &= \frac{4v}{f^2}\left(m_u^2 + m_d^2 + 2m_u m_d \cos\theta\right)^{1/2} \quad , \\ \Lambda(\theta) &= \Lambda_0 - (1/2) f^2 m_\pi^2(\theta) \quad , \\ \frac{1}{16\pi G} &= \frac{1}{16\pi G_0} - (1/2) c f^2 m_\pi^2(\theta) \end{aligned} \tag{12}$$

Here Λ_0 and G_0 are constants independent of θ.

Arguments based on QCD inequalities show that v is nonnegative [12]. Hence, the vacuum energy is evidently minimized at $\theta = 0$, as is required for the Peccei-Quinn mechanism to work. But the expression for G in equation (12) could be minimized at either $\theta = 0$ or $\theta = \pi$, depending on the sign of c. Though the sign of c is determined in principle by the nonperturbative strong interactions, we do not know how to compute it reliably. Nonetheless, what we have found is consistent with the expectation of Nielsen and Ninomiya, that the minimum of G occurs at a CP-conserving value of θ.

In fact, it is possible to go further, because the tree approximation contribution to the θ-dependence of $(16\pi G)^{-1}$ in equation (12) is not actually the leading contribution when m_π^2 is very small. There is a contribution from one pion loop that is enchanced by a logarithm of the pion mass. This logarithmically enhanced contribution is [18, 5, 6]

$$\delta\left(\frac{1}{16\pi G}\right) = -\frac{1}{64\pi^2} m_\pi^2(\theta) \ell n \left[M_{\rm CSB}^2 / m_\pi^2(\theta)\right] \quad , \tag{13}$$

where $M_{\rm CSB}$ is the "chiral symmetry breaking scale" of QCD; a naive estimate of it is $M_{\rm CSB} \sim 4\pi f \sim 1$ GeV. (Equation (13) is the one-loop contribution to $(16\pi G)^{-1}$ that arises from the minimal coupling of the pion to gravity. There is also a one-loop contribution that involves the nonminimal coupling of the pion to R in equation (8), but this contribution is of order $(m_\pi^4 \ell n m_\pi^2)$ and hence higher order in chiral perturbation theory.)

If the pion mass is sufficiently small, then the one-loop contribution to $(16\pi G)^{-1}$ in equation (13) dominates the tree contribution in equation (12). The calculated sign of the one-loop contribution shows that $m_\pi^2(\theta)$ seeks the smallest possible value in order to minimize $G(\theta)$. In view of the expression for $m_\pi^2(\theta)$ in equation (12), this means that $\theta = \pi$ is the preferred value. We have shown, then, that at least in a world in which the light quark masses are sufficiently small, the criterion that $G(\alpha)$ is at its minimal value on the surface in α-space where $\Lambda(\alpha) = 0$ requires θ to be very close to π. (As in the Peccei-Quinn model, weak interactions perturb θ slightly away from value chosen by QCD, by an amount of order 10^{-14} in the KM model of CP violation.)

The approximation of neglecting the contribution to $(16\pi G)^{-1}$ in equation (12) compared to the contribution in equation (13) is justified provided that

$$\frac{1}{32\pi^2 c f^2} \ell n [M_{\rm CSB}^2 / m_\pi^2] >> 1; \tag{14}$$

it is not clear whether it is justified for realistic values of the light quark masses. To get some insight about whether the conclusion that G is minimized at $\theta = \pi$

survives beyond the approximation of very light quark masses, we have considered the opposite limit of infinite quark masses, or pure Yang-Mills theory. In pure Yang-Mills theory, we have computed $G(\theta)$ in the dilute instanton gas approximation. Unlike chiral perturbation theory, which can be justified when the quark masses are small enough, the dilute instanton gas approximation cannot really be justified. Nonetheless, it is known to give the right answer for the vacuum energy; namely, that the minimum occurs at $\theta = 0$, in agreement with the QCD inequality argument.

To calculate $G(\theta)$ we compute the connected two-point function of the energy momentum tensor and extract its leading behavior at low momentum. The calculation turns out to involve a subtlety concerning the trace anomaly in the presence of instantons; we will not report on the details here. The result is that the minimum of $G(\theta)$ occurs at $\theta = \pi$. Thus, the dilute instanton gas calculation lends support to the view that $G(\theta)$ is minimized at $\theta = \pi$ generically, irrespective of the value of quark masses. Perhaps it will eventually be possible to resolve this issue by doing numerical calculations in lattice QCD.

Finally, let us consider whether our conclusion that $\theta \simeq \pi$ is in conflict with experiment. There is suggestive evidence that θ is actually close to zero in Nature [10]. But one should recall that this evidence is based on chiral perturbation theory calculations of the pseudoscalar meson masses that treat the *strange* quark mass as a small parameter, a somewhat dubious procedure [19]. If the corrections to leading order perturbation theory in the strange quark mass turn out to be surprisingly large, then it may be that θ is really close to π in Nature after all, as wormhole considerations indicate. Again, this issue may ultimately be resolved by lattice QCD calculations.

To summarize, we have argued that, at least in an approximation in which the masses of the up and down quarks are taken to be very small, wormhole fluctuations in the topology of spacetime drive the θ parameter of QCD to $\theta \simeq \pi$. Since $\theta \simeq 0$ appears to be satisfied in Nature, this prediction poses a possible conflict between wormhole physics and experiment. We have noted several ways in which this conflict might be resolved. Perhaps a Peccei-Quinn symmetry can survive in spite of wormhole effects, allowing θ to relax dynamically to the value $\theta = 0$. Perhaps chiral perturbation theory is misleading, and wormholes actually prefer $\theta \simeq 0$ for realistic values of the light quark masses. And finally, it is at least conceivable that $\theta \simeq \pi$ really is satisfied in Nature, in accord with our prediction.

After completion of this work we found that Choi and Holman have also concluded (using different methods) that wormholes favor $\theta = \pi$ [20].

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4. Random Surfaces

4.1 Introduction

The previous chapter emphasised how poorly the quantum theory of gravity is understood. We dealt there with the consequences of wormholes. Here we deal with another ambiguity associated with defining an appropriate measure in such theories. These ambiguities arise on account of problems in regulating them in a generally covariant fashion. We will not work with the four dimensional theory but instead focus on the much simpler example of gravity in two dimensions. Such a theory, also provides a convenient framework to study some of the ambiguities associated with fluctuations in the conformal factor which we referred to in the previous chapter. But quite apart from it's interest as a theory of gravity, this theory is also interesting from the point of view of strings in non–critical dimensions and the statistical mechanics of systems on random surfaces. Inspired by the seminal work of Polyakov, the subject has seen some progress recently. Besides the work done in the light cone gauge by Polyakov [1] and subsequently by Knizhnik, Polyakov and Zamolodchikov [2], there has been progress in the conformal gauge by David, Distler and Kawai (DDK) [3,4] and on the lattice using random matrix methods [5,6]. The anomalous dimensions and thus scaling exponents derived in these three ways agree. Given the status of our understanding, though, it would help if other quantities can be calculated and compared in these three approaches. There also has been some work in conformal gauge on random surfaces with the topology of the disc using operator methods [7].

In this chapter we will restrict ourselves to the conformal gauge and calculate both in the semiclassical limit [8] and exactly, following DDK, partition functions and some correlation functions [20]. The fixed area partition function on surfaces with the topology of the torus is straightforward to compute and we find that it diverges for $d > 1$; the divergence occurs for large imaginary values of the modular parameter and is analogous to the divergence due to the tachyon in string theory. On surfaces with the topology of the sphere in the DDK approach, we will calculate the partition function only for some "magic" values* of the central extension d. We find that it vanishes for $d = 1$. This result is analogous to the vanishing of the zero, one and two point functions on the sphere in string theory. Similarly, we will calculate some correlation functions exactly, for any matter theory; however, these must involve a specific number of operators of fixed dimensions. We also calculate correlation functions of the operator R^2 in the semiclassical limit. We will find that semiclassically the operator R^2 does not acquire an anomalous dimension.

The partition function for conformally invariant matter coupled to gravity in two dimensions is (for manifolds of fixed topology) [9,10]

$$Z = \frac{1}{\mathrm{V_{CKV}}} \int (d\tau)[D_g\phi][D_gX]Je^{-[S_M(X,g)+S_{\mathrm{grav}}]} \quad . \tag{1.1}$$

Here ϕ stands for the Liouville field, $(d\tau)$ denotes the integration over modular parameters (this is absent for manifolds with the topology of the two–sphere), $(\mathrm{V_{CKV}})$ is the volume generated by the conformal killing vectors, and $J/(\mathrm{V_{CKV}})$ is a determinant that arises from gauge fixing. $S_{\mathrm{grav}}[g]$, the Einstein–Hilbert action in two dimensions, is given by

$$S_{\mathrm{grav}}[g] = \int d^2\xi\sqrt{g}\left(\frac{\ell n\lambda}{8\pi}R + \tilde{\mu}\right) \tag{1.2}$$

where λ and $\tilde{\mu}$ are the "string coupling constant" and the cosmological constant, respectively. In two dimensions

$$\frac{1}{8\pi}\int d^2\xi\sqrt{g}R = (1-h) \tag{1.3}$$

where h is the genus of the surface. So

$$S_{\mathrm{grav}}[g] = \ell n\lambda\,(1-h) + \int d^2\xi\sqrt{g}\,\tilde{\mu} \quad . \tag{1.4}$$

⋆ These values of the central extension also seem to play an important role in the work of Ref. [7].

The conformal invariance of the matter action implies that if

$$g = e^{\phi}\hat{g} \quad , \tag{1.5}$$

where $\hat{g}$ is an arbitrary reference metric, then

$$S_M[X,g] \;=\; S_M[X,\hat{g}] \quad . \tag{1.6}$$

However, since the path–integral in equation (1.1) must be regulated in a generally covariant fashion, the measure $D_g X$ depends on ϕ. Since, as far as X is concerned, ϕ can be treated as a background field, it is straightforward to explicitly display the dependence of the measure on ϕ

$$\int [D_g X]\; e^{-S_M[X,g]} \;=\; \int\; [D_{\hat{g}} X]\; e^{-S_M[X,\hat{g}]}\; e^{+\frac{d}{48\pi}S_L[\phi,\hat{g}]} \tag{1.7}$$

where d is the central charge of the matter theory and $S_L[\phi,\hat{g}]$ is the Liouville action

$$S_L[\phi,\hat{g}] \;=\; \int\; d^2\xi\sqrt{\hat{g}}\left[\frac{1}{2}\hat{g}^{ab}\partial_a\phi\partial_b\phi + \hat{R}\phi + \bar{\mu}e^{\phi}\right] \quad . \tag{1.8}$$

Similarly, the dependence of the determinant J on the conformal factor ϕ can be determined

$$Z \;=\; \frac{1}{(\mathrm{V_{CKV}})}\int\; (d\tau)[D_g\phi][D_{\hat{g}}X]\hat{J}\; e^{-S_M[X,\hat{g}]}\; e^{+\left(\frac{d-26}{48\pi}\right)\; S_L[\phi,\hat{g}]}\; e^{-S_{\mathrm{grav}}[g]} \quad . \tag{1.9}$$

In the measure for the Liouville field ϕ, the metric cannot be treated as a background field. This makes it more difficult to deduce the Jacobian involved in going from $[D_g\phi]$ to $[D_{\hat{g}}\phi]$.

DDK assume that it can be expressed as a sum over local terms involving the ϕ field and that these terms are of the same form as those originally appearing in the

Liouville action. After appropriate rescalings this ansatz gives

$$Z = \frac{\lambda^{(h-1)}}{(\mathrm{V_{CKV}})} \int (d\tau)[D_{\hat{g}}\phi][D_{\hat{g}}X]\hat{J}\, e^{\frac{1}{8\pi}\int d^2\xi\sqrt{\hat{g}}[\phi\,\hat{\Box}\,\phi+Q\hat{R}\phi+\mu e^{\alpha\phi}]}\, e^{-S_M[X,\hat{g}]} \quad . \tag{1.10}$$

Treating μ perturbatively, they show that the choice of the background metric $\hat{g}$ is irrelevant if

$$Q = \sqrt{\frac{25-d}{3}} \tag{1.11a}$$

$$\alpha = \frac{-1}{2\sqrt{3}}\left[\sqrt{25-d} \pm \sqrt{1-d}\right] \quad . \tag{1.11b}$$

Comparison with results of the semiclassical approximation reveals that the minus sign should be chosen in equation (1.11b).

Similarly, if an insertion of the average over the two–dimensional space of a primary scalar field, O_M, of the matter theory is desired, then

$$\int d^2\xi\sqrt{g(\xi)}\; O_M(\xi) \tag{1.12a}$$

is replaced by

$$\int d^2\xi\sqrt{\hat{g}}\; O_M(\xi)e^{\beta\phi} \tag{1.12b}$$

Dependence on the choice of background metric drops out of correlations functions if the operator $O_M(\xi)e^{\beta\phi}$ is a (1,1) operator. This condition determines (again perturbatively in μ) that

$$\beta = \frac{-1}{2\sqrt{3}}\left[\sqrt{25-d} \pm \sqrt{1-d+24\Delta}\right] \tag{1.13}$$

where Δ is the scaling dimension of O_M. Comparison with semiclassical results determines that the minus sign is appropriate in equation (1.13).

The partition function in equation (1.10) doesn't exist for genus zero and one surfaces since the action (for a positive cosmological constant) is minimized as $\phi \rightarrow$

$-\infty$; implying that the partition function is dominated by surfaces with very small area. A quantity which is well–defined (and one we will study) is the partition function for surfaces with fixed area A. With zero cosmological constant it is

$$Z[A] = \frac{\lambda^{(h-1)}}{(\mathrm{V_{CKV}})} \int [D_{\hat{g}}\phi][D_{\hat{g}}X]\hat{J}\; e^{\frac{1}{8\pi}\int d^2\xi\sqrt{\hat{g}}[\phi\hat{\Box}\phi+Q\hat{R}\phi]}\; e^{-S_M[X,\hat{g}]}$$
$$\delta\left(\int d^2\xi\sqrt{\hat{g}}\; e^{\alpha\phi} - A\right) \quad . \tag{1.14}$$

Typically, the delta function couples together the various modes of ϕ, making this theory non–trivial.

4.2 Partition function on manifolds with the topology of the torus

The genus one partition function is calculated following the methods developed by Polchinski [11], which were used to do the analogous calculation in string theory. The partition function factorizes into a matter contribution, a contribution due to the "ghosts" (i.e., $\hat{J}$), and the contribution from the Liouville field. The latter is

$$Z_L[A,\tau] \;=\; \int [D_{\hat{g}}\phi]\; e^{\frac{1}{8\pi}\int d^2\xi\sqrt{\hat{g}}(\phi\hat{\Box}\phi+Q\hat{R}\phi)}\;\delta\left(\int d^2\xi\sqrt{\hat{g}}\; e^{\alpha\phi} - A\right) \quad . \tag{2.1}$$

It is convenient to choose $\hat{g}$ to be a constant curvature metric ($\hat{g}_{11} = 1$, $\hat{g}_{12} = \tau_1$, $\hat{g}_{22} = \tau_1^2 + \tau_2^2$) in which case $\hat{R} = 0$. In this case the constant mode of ϕ doesn't appear in the action and so the delta function can be used to do this integration without introducing interactions in the theory. Expanding the Liouville field in eigenfunctions of the Laplacian $\hat{\Box}$:

$$\phi = \frac{C_{0,0}}{\sqrt{\tau_2}} \;+\; \sum_{n_1,n_2} C_{n_1,n_2}\;\phi_{n_1,n_2} \tag{2.2}$$

where

$$\int d^2\xi\sqrt{\hat{g}}\phi^*_{n_1,n_2}\;\phi_{n_1',n_2'} \;=\; \delta_{n_1,n_1'}\delta_{n_2,n_2'} \tag{2.3}$$

the measure is

$$[D_{\hat{g}}\phi] \; = \; \prod_{n_1,n_2} \left(\frac{dC_{n_1,n_2}}{\sqrt{2\pi}}\right) \quad . \tag{2.4}$$

The path integral (2.1) then yields

$$Z_L[A,\tau] = \frac{1}{\sqrt{2\pi}} \cdot \frac{\sqrt{\tau_2}}{\alpha A} \left[\det{}'\left(-\frac{\hat{\Box}}{4\pi}\right)\right]^{-1/2} \tag{2.5}$$

$$= \left(\frac{1}{\sqrt{8\pi}} \cdot \frac{\sqrt{\tau_2}}{\alpha A}\right) \left[\tau_2^2 \; e^{-\pi\tau_2/3} \left|f(e^{2\pi i\tau})\right|^4\right]^{-1/2} \tag{2.6}$$

with

$$f(e^{2\pi i\tau}) \; = \; \prod_n (1 - e^{2\pi i n\tau}) \tag{2.7}$$

and $\tau = \tau_1 + i\tau_2$ the modular parameter of the torus. Thus the τ–dependence of the Liouville field's contribution to the partition function is the same as that of a free scalar field. The contribution from the ghosts is the same as in string theory

$$\hat{J} \; = \; \frac{1}{2\tau_2^2} \left[\tau_2 \; e^{-\pi\tau_2/3} |f|^4\right] \tag{2.8}$$

while the matter contribution $Z_M(\tau)$ depends on the particular conformal field theory chosen.

Putting these results together gives

$$Z(A) \; = \; \left(\frac{1}{2\sqrt{8\pi}} \frac{1}{\alpha A}\right) \int \frac{d^2\tau}{\tau_2^2} \left(\tau_2 e^{-\pi\tau_2/3} |f|^4\right)^{1/2} \; Z_M(\tau) \tag{2.9}$$

where the τ integral is over the usual fundamental region.

The genus one partition function is independent of the "string coupling" λ. However in the formalism of DDK rescaling of the fields has been performed and so it is not clear that the absolute normalization of Z can be determined.

The τ dependence of $Z(A)$ arising from integrating over the Liouville field and the ghosts is independent of the matter sector and modular invariant on its own. For large τ_2 we expect that (for unitary matter) [12]

$$Z_M(\tau) \to e^{(\pi\tau_2\ d/6)} \tag{2.10}$$

which implies that the exponential dependence of the complete integrand of equation (2.9) on τ_2 is

$$e^{-\pi\tau_2\ (1-d)/6} \quad . \tag{2.11}$$

This shows that for $d > 1$ the surfaces tend to degenerate to those with infinite values of τ_2 (e.g., infinitely long thin tubes, etc).* There are of course other indications of a change in the character of the theory for $d > 1$, which manifest themselves even for manifolds with the topology of the sphere. For example, anomalous dimensions and the string susceptibility turn complex in this region.†

The divergence encountered here is analogous to the one encountered in bosonic string theory [14] due to the tachyon. There is no spacetime interpretation when $d \leq 1$, nonetheless this may be an indication of the instability of the vacuum.

4.3 Partition function on manifolds with the topology of the sphere

Choosing the background metric $\hat{g}$ to correspond to the sphere of unit radius the Liouville sector corresponds to a free field theory with imaginary background charge. It is the fixed area constraint, which couples the various modes together, that makes the theory nontrivial.

⋆ This agrees with simple physical pictures of the nature of the transition at $d = 1$ [13] which suggest that for $d > 1$ the surfaces become branched polymers.

† Unlike these the divergence arising from (2.11) occurs even in the semiclassical approximation where fluctuations in the metric are neglected in the measure for ϕ.

The genus zero partition function factorizes into a contribution from the ghosts, the matter and the Liouville field. The latter gives

$$Z_L[A] = \int [D_{\hat{g}}\phi] \exp\left[\frac{1}{8\pi}\int d^2\xi\sqrt{\hat{g}}(\phi\hat{\Box}\phi + Q\hat{R}\phi)\right] \cdot \delta\left(\int d^2\xi\sqrt{\hat{g}}\; e^{\alpha\phi} - A\right) . \tag{3.1}$$

Proceeding as in the case of the torus, we expand the Liouville field in eigenfunctions of the Laplacian $\hat{\Box}$ and use the delta function to do the integral over the constant mode. This gives

$$Z_L[A] = \frac{\sqrt{2}}{\alpha A}\int [D_{\hat{g}}\phi'] \exp\left(\frac{1}{8\pi}\int d^2\xi\sqrt{\hat{g}}(\phi'\hat{\Box}\phi')\right)\left[\frac{\int d^2\xi\sqrt{\hat{g}}\; e^{\alpha\phi'}}{A}\right]^{-Q/\alpha} \tag{3.2}$$

where ϕ' denotes the nonzero modes of ϕ. When the central extension d is such that $-Q/\alpha$ is a non–negative integer n, there is a dramatic simplification. The path integral then corresponds to a free field theory with n–vertex operator insertions. In these cases the path integral is straightforward to perform. Mapping the unit sphere into the complex plane by stereographic projection, we find that

$$Z_L[A] = \frac{\left(e^{-(\frac{n^2\alpha^2}{2})}\cdot\sqrt{2}\right)}{\alpha} \cdot \frac{1}{A^{n+1}} \cdot (\epsilon^2)^{\frac{-n\alpha^2}{2}} \cdot \left[\det{}'\left(-\frac{\hat{\Box}}{4\pi}\right)\right]^{-1/2} \cdot \int d^2\, z_1 \ldots \int d^2\, z_n \prod_{i<j} \frac{1}{|z_i - z_j|^{2\alpha^2}} \tag{3.3}$$

where ϵ^2 is an invariant short distance cutoff‡ on the unit sphere. There are ϵ^2 dependent pieces which arise from the determinant in equation (3.3) and from the ghost and matter sectors too. Together these can be absorbed in the bare string coupling to define the renormalized partition function in terms of a renormalized string coupling (which is dimensionful).

‡ The two–point correlation for the Liouville field is

$$\langle\phi'(z_1)\phi'(z_2)\rangle = -\left\{\ell n\left(|z_1 - z_2|^2 + \epsilon^2 e^{-\sigma\,(z_+)}\right) + \frac{\sigma(z_1)}{2} + \frac{\sigma(z_2)}{2} + 1\right\}$$

where $e^{-\sigma(z)} = (1 + |z|^2)^2$ and $z_+ = (z_1 + z_2)/2$.

In close analogy with string theory scattering amplitudes on the sphere, the quantity $Z_L(A)$ possess a $SL(2,C)$ symmetry which enables us to cancel the factor of $1/(\mathrm{V_{CKV}})$ in $Z(A)$ (see equation (1.14)) against three of the integrals in equation (3.3). Naively, when $n < 3$, the partition function on the sphere vanishes.[§] The relationship between the central extension and n is

$$d \; = \; -\frac{6n^2 - 25n + 25}{n-1} \quad . \tag{3.4}$$

For $d = 1$ the number of integrals over the complex plane is only two and the partition function vanishes. This may be related to the logarithmic contribution to scaling seen in the random matrix approach to the $d = 1$ model [6].

The methods introduced here can also be applied to the partition function for manifolds with the topology of the projective plane (i.e., P^2). In this case the number of vertex operator insertions is $n/2$ so n must be an even (non–negative) integer. Also the volume of conformal killing vectors is finite for P^2, so the $d = 1$ partition function doesn't vanish in this case. This is analogous to bosonic string theory where the contribution to the vacuum energy from S^2 vanishes but the contribution from P^2 doesn't [16].

The quantities Q and α were chosen so that the partition function $Z(A)$ is independent of the background metric $\hat{g}$. However, the arguments of DDK were based on "free field" theory while the fixed area constraint couples the modes of the Liouville field together. We close this section by explicitly showing that when $-Q/\alpha = n$, a non–negative integer, $Z[A]$ (for surfaces with the topology of the sphere) is independent of the choice of reference metric.

We start with a fiducial metric $\bar{\hat{g}} = e^{\sigma}\hat{g}$, where $\hat{g}$ is the (constant curvature) metric on the unit sphere. Then $Z_L[A]$ is given by

$$Z_L[A] = \int [D_{\bar{\hat{g}}}\phi] \exp\left[\frac{1}{8\pi}\int d^2\xi\sqrt{\bar{\hat{g}}}\left[\phi\bar{\hat{\Box}}\phi + Q\bar{\hat{R}}\phi\right]\right]\cdot\delta\left(\int d^2\xi\sqrt{\bar{\hat{g}}}\; e^{\alpha\phi} - A\right) \quad . \tag{3.5}$$

§ Such "naive" arguments sometimes fail in string theory. See ref. [15].

Since σ can be treated as a background field

$$[D_{\bar{g}}\phi] \;=\; [D_{\hat{g}}\phi]\; e^{\frac{1}{48\pi}S_L[\sigma]} \quad . \tag{3.6}$$

Expanding ϕ in eigenfunctions of $\hat{\Box}$, as before, gives

$$Z_L[A] = \exp\left(\frac{1}{48\pi}S_L[\sigma]\right)\frac{\sqrt{2}}{\alpha A}\int [D_{\hat{g}}\phi']\; e^{\frac{1}{8\pi}\int d^2\xi\sqrt{\hat{g}}[\phi'\hat{\Box}\phi' - Q\hat{\Box}\sigma\phi']} \cdot \left[\frac{\int d^2\xi\sqrt{\hat{g}}\; e^{\alpha\phi'}\; e^{\sigma}}{A}\right]^n , \tag{3.7}$$

where ϕ' denotes the nonzero mode part of ϕ. Shifting the integration variable ϕ' to $\phi' - Q\sigma'/2$ (the measure is invariant under this) gives

$$Z_L[A] = \exp\left(\frac{1}{48\pi}S_L[\sigma]\right)\exp\left[\frac{1}{48\pi}3Q^2\int d^2\xi\sqrt{\hat{g}}\sigma\hat{\Box}\sigma\right] \cdot \frac{\sqrt{2}}{\alpha A}\int [D_{\hat{g}}\phi'] e^{\frac{1}{8\pi}\int d^2\xi\sqrt{\hat{g}}\phi'\hat{\Box}\phi'}\left[\frac{1}{A}\int d^2\xi\sqrt{\hat{g}}e^{\sigma}\; e^{\alpha(\phi'+Q\frac{\sigma'}{2})}\; e^{\alpha\phi'}\right]^n \tag{3.8}$$

Note that σ' denotes the nonzero mode part of σ. Performing the functional integral, regulating the short distance singularities with a cutoff that is invariant with respect to the full metric $\bar{g}$, gives

$$Z_L[A] = \left\{\exp\frac{1}{48\pi}(1+3Q^2)S_L[\sigma]\right\}\frac{1}{\alpha A}\left\{\prod_{i=1}^{n}\int d^2\xi_i\sqrt{\hat{g}(\xi_i)}\; e^{\sigma(\xi_i)}e^{\left(\frac{\alpha^2}{2}+\frac{\alpha Q}{2}\right)\;\sigma(\xi_i)}\right\}\times(\text{terms independent of }\sigma) \quad . \tag{3.9}$$

In the derivation of equation (3.9) we redistributed some of the dependence on the zero mode of σ between the two terms in brace brackets. Since $\frac{1}{2}\alpha(\alpha+Q) = -1$ the σ–dependence drops out of the second term in brace brackets. Furthermore, in $Z[A]$, the matter and ghost contributions dependence on σ cancels that of $Z_L[A]$ since

$$1 + 3Q^2 + (d - 26) \;=\; 0 \quad . \tag{3.10}$$

4.4 Correlation functions

The method used to derive, for certain "magic" values of the central extension d, an explicit expression for the partition function on the sphere can also be used to calculate some correlation functions. Consider the N–point correlation of an operator in the matter sector $O_M(\xi)$ with dimensions (Δ, Δ). In the formalism of DDK this is replaced by the operator $O(\xi) = e^{\beta\phi(\xi)}O_M(\xi)$, where the dressing factor $e^{\beta\phi}$ is determined by demanding that $O(\xi)$ be a $(1,1)$ operator. Gauge invariant quantities are formed by integrating this operator over the surface. Since we are assuming that the matter is conformally coupled, correlation functions factorize into the product of a correlation function in the Liouville sector with one in the matter sector. The Liouville sector correlation function is given by

$$\int [D_{\hat{g}}\phi] e^{\frac{1}{8\pi}\int d^2\xi\sqrt{\hat{g}}[\phi\hat{\Box}\phi+Q\hat{R}\phi]}\, \delta\left(\int d^2\xi\sqrt{\hat{g}}\; e^{\alpha\phi} - A\right)\cdot e^{\beta\phi(\xi_1)}\ldots e^{\beta\phi(\xi_N)}\;. \tag{4.1}$$

Performing the integral over the constant mode of ϕ gives that the above is equal to

$$\frac{\sqrt{2}}{\alpha A}\int [D_{\hat{g}}\phi']\; e^{\frac{1}{8\pi}\int d^2\xi\sqrt{\hat{g}}[\phi'\hat{\Box}\phi']}\; e^{\beta\phi'(\xi_1)}\ldots e^{\beta\phi'(\xi_N)}\left[\frac{\int d^2\xi\sqrt{\hat{g}}e^{\alpha\phi'}}{A}\right]^{-\frac{Q}{\alpha}-\frac{N\beta}{\alpha}} \tag{4.2}$$

where ϕ' is the non–constant part of the Liouville field. As before, if

$$-\left(\frac{Q}{\alpha}+\frac{N\beta}{\alpha}\right) = n \quad , \tag{4.3}$$

where n is a non–negative integer, the above path integral is straightforward to do. The restriction in equation (4.3) means that only correlation functions involving specific numbers of operators can be done with this method. Once the Liouville correlation function is known, it can be combined with the matter correlation, deduced from conformal field theory [17], and then integrations over the surface can be performed to get a gauge invariant quantity.

In this process, though, two or more operators can come arbitrarily close together giving rise to divergences which have to be regulated.* This situation is familiar to us from string theory. In that case, though, since we are calculating S–matrix elements, we can make sense of these expressions by analytically continuing the external particle momenta, a privilege we do not have here. However, even in the case of string theory we can think of computing these amplitudes by using a worldsheet regulator [18]. We have, for example, computed the four–point tachyon amplitude with an invariant short distance cutoff; the usual answer obtained by analytic continuation then corresponds to subtracting all the cutoff dependent pieces and keeping the finite piece which is left over. This is actually only true for power law divergences; logarithmic divergences which cannot be subtracted without introducing another scale in the theory have to be handled differently. When they occur, the divergences cannot be subtracted; the amplitude diverges and this divergence is interpreted as a pole in the S–matrix. Work on understanding this issue in the context of random surfaces is in progress. However, we feel that, at least in cases where there are no logarithmic divergences, a similar procedure of regulating using a naive short distance cutoff and keeping the finite parts will work. It is straightforward to show that such a procedure gives answers independent of the background metric.

There are several examples where the constraints on the dimensions of the fields are met and no logarithmic divergences appear. Since the operators have anomalous dimensions, though, physical quantities will consist of ratios of Green's functions from which the cutoff dependence drops out.

As an example, take the $d = 25/28$ model in the minimal series, and consider the fields (3,5) and (3,6), where we are following the notation of ref. (17) to label primary fields as (p, q), so that $1 \leq p \leq 7$ and $1 \leq q \leq 8$ in this case.

For the four–point function of the (3,5) field we have then $-Q/\alpha - (4\beta_1)/\alpha = 1$. Similarly, for the four–point function of the (3,6) field we have $-Q/\alpha - (4\beta_2)/\alpha = 3$. Finally, for the four–point function consisting of two (3,6) and two (3,5) fields, we

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⋆ In the case of the partition function on the sphere these divergences don't occur.

have $-Q/\alpha - (2\beta_1)/\alpha - (2\beta_2)/\alpha = 2$. A systematic study of the operator product expansion shows there are no logarithmic divergences in all these cases. Thus we could compute with our naive regulator the physically meaningful ratio:

$$\frac{\langle O_1 O_2 O_1 O_2 \rangle^2}{\langle O_1 O_1 O_1 O_1 \rangle \langle O_2 O_2 O_2 O_2 \rangle} \tag{4.4}$$

where the expectation values also denote integrals over the surface. In practice, of course, this will involve doing several difficult integrals over the surface.

It is clear that the above example has many analogues in other $d < 1$ minimal models.

So far we have outlined a method that can be used to calculate some correlation functions. However, there are correlation functions which at the present time cannot be computed using the formalism of DDK. For large negative values of the central extension d, the semiclassical approximation is valid. Using the semiclassical approximation, correlations that are not accessible with the method of DDK can be computed in conformal gauge. In the remainder of this section we compute semiclassically, all the connected correlations of the operator R^2 (integrated over the surface). The generating functional for these is

$$Z(\alpha) \;=\; \frac{1}{(\mathrm{V_{CKV}})} \int [D_g\phi]\, e^{-(\frac{26-d}{48\pi})S_L} \delta\left(\int d^2\xi\sqrt{g} - A\right)\, e^{-\alpha\int d^2\xi\sqrt{g}R^2} \tag{4.5}$$

and the connected correlations are determined by differentiation with respect to α

$$\left\langle \left(\int d^2\xi\sqrt{g}R^2\right)^n \right\rangle_c \;=\; \frac{1}{n!}\left(\frac{-d}{d\alpha}\right)^n \ell n Z(\alpha)\Big|_{\alpha=0} \quad . \tag{4.6}$$

In equation(4.5) we have suppressed dependence on the string coupling and put the cosmological constant to zero. The Liouville mode is given by

$$g_{ab}(\xi) \;=\; e^{\phi(\xi)}\hat{g}_{ab}(\xi) \tag{4.7}$$

and the reference metric, $\hat{g}_{ab}(\xi)$, is taken to be the metric on the sphere of unit radius. In the semiclassical approximation, $d \to -\infty$, the path integral, for $Z(\alpha)$, is dominated by small fluctuations about the sphere of area A. Then the measure $[D_g\phi]$ is defined by expanding $\phi(\xi)$ in orthonormal modes on the sphere of area A

$$\phi(\xi) = \sum_{\ell,m} C_{\ell,m}\, Y_{\ell,m}\,(\xi)\sqrt{\frac{4\pi}{A}} \tag{4.8}$$

$$[D_g\phi] = \prod_{\ell,m} \frac{dC_{\ell,m}}{\sqrt{2\pi}} \quad . \tag{4.9}$$

The generating function $Z(\alpha)$ is evaluated by using the delta function to do the integral over $C_{0,0}$. In the semiclassical approximation the resulting "effective action" is expanded to quadratic order in the coefficients $C_{\ell,m}, \ell \geq 1$, yielding for large negative d,

$$Z(\alpha) = \frac{1}{(\mathrm{V_{CKV}})}\, e^{\frac{(8\pi)^2\alpha}{A}} \int \prod_{\substack{\ell,m\\ \ell\geq 1}} \frac{dC_{\ell m}}{\sqrt{2\pi}} \exp\Bigg\{ -\left(\frac{-d}{12A}\right)\sum_{\ell,m}\frac{1}{2}|C_{\ell m}|^2$$
$$\left(\Big[\ell(\ell+1)-2\Big] + \frac{384\pi^2\alpha}{(-d)A}\Big[\ell(\ell+1)-2\Big]^2\right)\Bigg\} \quad . \tag{4.10}$$

Since the connected correlations of $\int d^2\xi\sqrt{g(\xi)}R^2(\xi)$ are determined by differentiating $\ell n\ Z(\alpha)$ with respect to α, α–independent constants of proportionality in equation (4.10) have been dropped. Note that the argument of the exponential (in brace brackets) is independent of the coefficients of the $\ell = 1$ modes, C_{1m}. This is because of the $SL(2, C)$ symmetry generated by the conformal killing vectors on the sphere. In the semiclassical approximation the integral over the $\ell = 1$ modes is proportional to the volume generated by the conformal killing vectors ($\mathrm{V_{CKV}}$).

The integral over the coefficients $C_{\ell,m}$ with $\ell \geq 2$, produces a determinant which is defined using zeta function regularization. Hence

$$\ell n Z(\alpha) \;=\; \left(\frac{(8\pi)^2\alpha}{A}\right) \;-\; \frac{1}{2}\,\zeta'(0) + \frac{1}{2}\,\ell n\left[\left(\frac{-d}{12A}\right)\right]\zeta(0) + \ldots \tag{4.11}$$

where

$$\zeta(s) = \sum_{\ell=2}^{\infty} (2\ell+1)\left\{\left[\ell(\ell+1)-2\right] + \frac{384\pi^2\alpha}{(-d)A}\left[\ell(\ell+1)-2\right]^2\right\}^{-s} \tag{4.12}$$

and a prime denotes differentiation with respect to s. In equation (4.11) the ellipses denote factors independent of α. Now we expand in α:

$$\zeta(s) = \sum_{k=1}^{\infty}\left(\frac{-384\pi^2\alpha}{(-d)A}\right)^k \frac{s(s+1)\ldots[s+k-1]}{k!}\sum_{\ell=2}^{\infty}(2\ell+1)\left[\ell(\ell+1)-2\right]^{-s+k}+\ldots \tag{4.13}$$

where the ellipses denote the term independent of α. The sum over ℓ can be put in a convenient form for our purposes using the methods of Weisberger [19]

$$\sum_{\ell=2}^{\infty}(2\ell+1)[\ell(\ell+1)-2]^{-s+k} = \sum_{n=0}^{\infty}\frac{(-1)^n 3^n \Gamma(s-k+n-1)}{n!\Gamma(s-k)}(2s-2k+n-2)\zeta_R(2s-2k+n-1)\,. \tag{4.14}$$

Combining equations (4.13) and (4.14) gives that $\zeta(0)$ is independent of α and that

$$\zeta'(0) = \sum_{k=1}^{\infty}\Gamma(k)\left[\frac{-384\pi^2\alpha}{(-d)A}\right]^k \sum_{m=0}^{[(k+1)/2]}\Gamma(2m+1)\frac{3^{2m}}{\Gamma(k+2-2m)}B_{2k+2-2m} \tag{4.15}$$

where B_m denotes the m'th Bernoulli number and $[(k+1)/2]$ denotes the largest natural number less than or equal to $(k+1)/2$. Because $\zeta(0)$ is independent of α, semiclassically, the operator $\int d^2\xi\sqrt{g(\xi)}R^2(\xi)$ has no anomalous scaling while equation (4.15) implies that its connected k–point correlations are given by

$$\left\langle\left(\int d^2\xi\sqrt{g(\xi)}R^2(\xi)\right)^k\right\rangle_c = \left(\frac{(8\pi)^2}{A}\right)\delta_{k,1} - \left[\frac{384\pi^2}{(-d)A}\right]^k\Gamma(k)\sum_{m=0}^{[(k+1)/2]}\frac{3^{2m}}{\Gamma(2m+1)}\frac{B_{2k+2-2m}}{2\Gamma(k+2-2m)}\,. \tag{4.16}$$

So, for example, using $B_2 = 1/6$ and $B_4 = -1/30$ gives

$$\left\langle \int d^2\xi \sqrt{g(\xi)} R^2(\xi) \right\rangle = \frac{(8\pi)^2}{A} - \frac{192\pi^2}{(-d)A} \left[\frac{11}{15}\right] \quad . \tag{4.17}$$

For large negative d fluctuations decrease the average value of $\int d^2\xi \sqrt{g(\xi)} R^2(\xi)$. Similarly, correlations of higher powers of the curvature can be calculated and we find no anomalous scaling in the semiclassical approximation for these operators.

4.5 Concluding remarks

In this chapter we have shown that for certain values of the central extension the formalism of DDK can be used to exactly calculate the partition function on manifolds with the topology of the sphere, the projective plane, and the torus. (We find that for $d = 1$ the partition function vanishes on the sphere.) Also some correlation functions on these manifolds can (in principle) be computed exactly; however, these must involve specific numbers of operators of fixed dimensions. In these cases we see that the prescription of DDK gives results independent of the choice of background metric, even though the fixed area constraint makes the theory nontrivial. It would be interesting to pursue the issue of independence of the choice of background metric in other cases.

When (integrated) correlations are calculated, divergences occur since operators can come arbitrarily close to each other on the surface. Work on understanding this issue is in progress. However, a preliminary investigation reveals that power law divergences can be subtracted unambiguously. Logarithmic divergences, on the other hand, cannot be subtracted without introducing another scale in the theory. The analogous logarithmic divergences in string theory are true infinities corresponding to poles in the S–matrix. It is possible that in the case of random surface theory these logarithmic divergences also cause some (integrated) correlations to be infinite.

When the genus one partition function was examined we found a divergence for $d > 1$, which signaled the domination of the partition function by manifolds with very large values of the imaginary part of the modular parameter (e.g., long thin tubes, etc.). This divergence is analogous to the divergence in bosonic string theory associated with the tachyon. This analogy may give a clue to the origin of the transition at $d = 1$ in random surface theory.

At the present time the extension of the path integral conformal gauge approach of DDK to open surfaces has not been made. One issue that arises in that case is the choice of boundary condition appropriate for the Liouville field. A preliminary examination (using the semiclassical approximation) of this problem for manifolds with the topology of the disk suggests that Neumann boundary conditions give results that are consistent with the general coordinate invariance of the underlying theory (while other simple choices of boundary conditions do not).

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5. Concluding Remarks

The preceding chapters have served to illustrate that much more needs to be understood about quantum gravity. We are confronted with important issues at length scales both larger and smaller than the Planck scale. Consequently, a proper understanding of quantum gravity could lead to a better understanding of physics, not only at high energies, but also at much lower energies. Specifically, we dealt with three fairly different topics, conformal field theory (C.f.t.), wormholes and two dimensional random surfaces. As pointed out earlier conformal field theory and random surfaces are subjects of interest from other points of view too. All three have been areas of much research in the past few years. We do not presume to be experts in all these fields, but before concluding will try to briefly describe some of the obvious questions that have remained unanswered.

In C.f.t. there has been much progress in understanding the structure of rational C.f.t.'s. It is commonly believed that all such rational C.f.t.'s can be obtained using the coset construction from Wess Zumino Witten models. Part of the reason for believing in this is simply that no other rational C.f.t.'s have been found now for a long time, but it is also true that many of the properties of these theories can be understood from a higher dimensional point of view, and this suggests that the most general rational C.f.t. is indeed associated with a coset model. It has proved much more difficult to understand non-rational C.f.t.'s. This is unfortunate since from the point of view of string theory non rational C.f.t.'s are as good examples of classical vacua. Recently some progress has been made in understanding non-critical two dimensional theories by perturbing C.f.t's with operators that still keep them integrable. It has been suggested that the space of all two dimensional theories would provide a convenient setting for formulating some issues in string theory and this would be yet another step in understanding it.

We turn now for a brief look at wormholes. Within the context of Coleman's, proposal the large wormhole problem has still not met with a satisfactory solution. In the more general context, the understanding of the Euclidean path integral continues to be precarious. It is conceivable that some of the progress being made in understanding the conformal factor in two dimensions would shed some light on this. Finally, it remains to be seen if a consistence criterion can determine whether topology fluctuations are allowed at all.

The subject of random surfaces has seen some rapid progress in recent months. Random matrix techniques have been used to gain non-perturbative information about gravity in the $d \leq 1$ phase. Furthermore, there is now good reason to believe that the $d < 1$ phase is topological. But the problem of understanding the phase transition as d gets larger than 1 still remains as daunting as ever. In fact, recent progress made showing that the $d < 1$ phase is topological has made this even more worthwhile to understand for as an example of a transition from a topological to a non topological phase it might have an echo in string theory.

Clearly the issues alluded to above are only a few of the many that need to be understood before a proper understanding of quantum gravity emerges. It is conceivable that the battle will not be won for many years to come and when it is, some of our most cherished beliefs including those of quantum mechanics will have to be given up. Then again it is conceivable that we are on the threshold of victory, that our belief in string theory is justified and that it will lead to a complete understanding of quantum gravity. Either way it will be a privilege to participate in this struggle !