I. ON DUST DEVILS

II. LINEARIZED THEORY OF CONICAL TURBOMACHINES

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SUMMARY

Dust devils are small-scale atmospheric motions of instability arising from the development of large thermal stratifications in the lowest layers above the surface of the earth. A theory is proposed in Part I to describe the conditions of formation of these motions and depends on the new result that shear provides a powerful stabilizing influence even in non-viscous fluid motions in which denser fluid is situated above less dense. Those features of the flow which can be predicted by the theory and compared with observations are found to be in reasonably good agreement, and it is therefore indicated that the theory, which is based on a highly simplified model of flow, furnishes at least a qualitatively correct correlation of the basic ideas involved in the stable flow of very slightly viscous fluids containing density inversions. Applications to technically interesting flows of this type, in large-scale atmospheric motions as well as in high speed aerodynamic boundary layers, are indicated but not analyzed in detail.

In Part II the perfect fluid flow is determined for a turbomachine of conical shape and prescribed blade loading. On the basis of the assumption that the stream surfaces are conical in shape, a linear, elliptic partial differential equation of the second order is obtained. The associated boundary value problem is of the Sturm-Liouville type and is solved completely. An asymptotic representation of the solution is determined which is convenient for computational purposes.

I. ON DUST DEVILS

Recent developments in aerodynamics and meteorology have stimulated interest in fluid flows of variable density with denser fluid above. In aerodynamics the advent of high speed flight has introduced flows in which boundary layer fluid is less dense than its overlying free stream (see, for example, Ref. 1) and similar configurations in wakes and jets. Research in connection with control of the North American potato blight recently revealed that on clear days lapse rates in the lowest five feet above agricultural land generally exceed the adiabatic value (which represents the threshhold of static stability) by a factor commonly of the order of one hundred, and consequently produce a similar stratification (Ref. 2); observations of the same sort made at the bottom of the atmosphere have been well known in meteorological circles for many years and have continued to defy explanation. Further, there are indications that the energy of tornadoes is derived from the flow of masses of cold air over warm and less dense air masses, though an insufficient understanding of such flows has impeded general acceptance of their importance as a factor in meteorological dynamics.

Attempting to account for the remarkable convection patterns observed by Benard in liquids heated from below, Lord Rayleigh (Ref. 3) in 1916 initiated a series of investigations of the convection flows resulting from density defects in lower layers of liquids, and discovered stability criteria which depend on the molecular viscosity of the liquid. A large amount of hydrodynamic research, notably including that of Low (Ref. 4), Jeffreys (Ref. 5), Southwell (Ref. 6), Brunt (Ref. 7), and Sutton (Ref. 8), has been devoted to refinement of Rayleigh's analysis.

Brunt (Ref. 9) analogously suggested that the existence and persistence of superadiabatic lapse rates in the atmosphere is attributable to an effective turbulent viscosity.

In his investigation of the influence of heat stratification on the development of turbulence in aerodynamic boundary layer flow, Schlichting (Ref. 10) examined the hydrodynamic stability of such flows to the extent of determining the dependence of the threshhold of stability on the Reynolds number of the flow and the density difference. By means of an elaborate numerical calculation it was thus indicated that a flow containing a density inversion could be stable under disturbances of sufficiently short wave length and low Reynolds number. The measurements made by Prandtl and Reichardt (Ref. 11) in the Gottingen hot-cold wind tunnel, the results of which Schlichting compared with his theory, do not include flows with density inversions, however. By confining his interest to the indifference curves in the stability diagrams, Schlichting made no distinction between the regions of very weak instability and the regions in which the stability characteristics could determine the course of an entire flow field at later instants of time.

Numerous atmospheric phenomena suggest that viscosity may be largely unessential to the stability of flow in fluids with greater density above. This conjecture is confirmed in the following analysis, in which it is shown that a simple transverse shearing motion furnishes a strong stabilizing influence even in the laminar motion of non-viscous fluids. The specific problem discussed is the conditions of formation of those small-scale, vertical, atmospheric, whirling motions which are commonly observed on hot days in the deserts of the southwest United States (see, for example, Ref. 12) and in India.* The spectacular, unpredictable, and destructive nature of these motions has earned for them the name "dust devil" which is also their accepted meteorological designation. Dust devils are at times strongly developed for a height of a hundred feet or more above the earth's surface, and are persistently and characteristically violent disturbances in an otherwise abnormally calm atmosphere containing only very gentle breezes. Dust devils in the earliest stages of development are preferred for analysis rather than other technically more important variable density flows of the same fundamental character, partly on account of the apparent simplicity of observational conditions in which they develop. Dust devils have been carefully observed, and experimental evidence has long disproved the fable that the earth's rotation is an essential feature of the motion, as in the case of the much larger scale and geometrically similar major (geostrophic) wind systems of the earth. From the simplicity of the atmospheric conditions which generate dust devils it is not to be inferred that the motion itself is in all other respects also a simple one; in some senses quite the opposite is true. Dust devils are unsteady and unstable motions resulting from a disturbance in a heterogeneous, turbulent, compressible, heat-conducting fluid in an initially rotational motion. It is of further interest to note that in the present development of fluid dynamics a flow which is described by any one of the

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* Rudyard Kipling has immortalized the dust devil in India in his "Ballad of the East and West".

adjectives of the previous sentence is beset with great analytical difficulties; every one of the seven plays an essential role in the development of dust devils, as will be demonstrated. In view of the extremely small number of flows which are well understood and involve combinations of the features mentioned, any qualitative information which can be obtained regarding flows involving all of them interacting simultaneously is of some interest entirely apart from its application.

It is well known that of all possible hydrodynamical motions of real fluids, those which have been satisfactorily explained in terms of fundamental dynamical principles form a very small fraction of the whole. Essential features of dust devils are that they are unsteady, rotational, and three-dimensional motions in a heterogeneous medium. These facts alone present insuperable technical difficulties if one attempts a complete analysis of the hydrodynamics of the motion - - other important features of the flow would complicate the analysis even further. If any progress is to be made toward understanding the physical mechanisms of the motion, therefore, one must be content with examining separately different aspects of the problem, and considering drastically idealized models of the real motions and processes. Following this procedure sufficient deductions have been made to constitute a more or less complete picture of the circumstances under which dust devils can develop, and of the motion in its earliest stages. From the fact that dust devils are unstable motions involving the release of relatively large amounts of energy, it appears that there must be a storage of energy preceding the development of one of these motions.

Lapse rates in the lowest layers of the atmosphere have repeatedly been observed which are so large that stratifications exist with density significantly greater above than below. In the sense that under some circumstances a negative amount of work would be required to be supplied externally in order to effect an overturning, there is then a static instability associated with such a stratification; but the mere fact that these stratifications are not immediately destroyed, but rather do persist and intensify, indicates that they must in fact possess some degree of stability. A logical first step would then appear to be to determine the stabilizing influence in such configurations. If stability criteria can also be found, these may delineate the parameters which are decisive in creating the unstable motions. It will be shown that shearing action furnishes the stabilizing effect and a criterion is found thereby which suggests several important features of the mechanism of dust devils.

For this purpose a highly idealized model of the undisturbed flow is considered. It involves several arbitrary simplifying assumptions based on the real flow, but does retain its most essential features. The reasons for making these assumptions will be plain from the analysis which follows, and their à postiori justification will be indicated.

The method of small perturbations is applied to the basic flow which is steady and two-dimensional motion of two incompressible and homogeneous fluids, one in a layer of finite thickness above a plane boundary which represents the earth's surface, the second a semi-infinite sea of denser homogeneous liquid above the first. The interface is the plane $z = 0$, above which there is a uniform motion with speed U parallel to the positive direction of the x-axis, below which the shear velocity decreases linearly from the value U at $z = 0$ to nothing at the plane boundary at $z = -h$.

The flow considered is indicated in Fig. 1, in which the shear

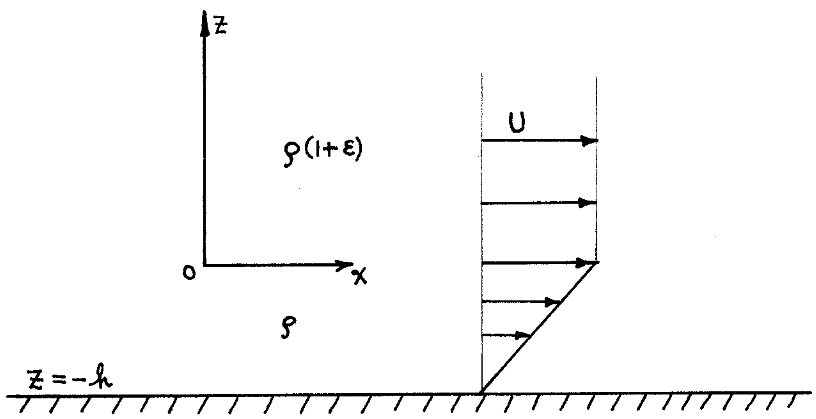

Fig. 1

velocity profile is indicated, as well as the density discontinuity at the plane $z = 0$. Greater density in the upper fluid corresponds to a positive increment $\varepsilon > 0$.

It is convenient for later purposes first to write the hydrodynamic equations in a form slightly more general than is required for the flow just described; continuous density variations and non-vanishing shear velocity profile curvatures will not be excluded from consideration for the present.

Taking the origin of coordinates at h units of height above the surface, the z-axis directed upward, and the x-axis in the direction of the local wind, the undisturbed motion is governed by the equations of momentum which reduce to

$$\frac{1}{\rho_0}\frac{dp_0}{dz} = -g$$

in the usual notation where the subscript zero denotes values in the undisturbed state, and g is the gravity force per unit mass. The equation of continuity is identically satisfied, as is also the equation which expresses the assumed incompressibility.

The stability of the flow pattern described is investigated by examining the consequences of a small perturbation of the flow which is assumed to be such that the resulting total flow obeys the dynamical equations. Denoting small disturbance quantities by lower case letters with no subscript or superscript, the equations of momentum become

$$\rho_0\left(\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + w\frac{\partial U}{\partial z}\right) + \frac{\partial p}{\partial x} = 0 \qquad (1)$$

$$\rho_0\left(\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x}\right) + \frac{\partial p}{\partial z} + \rho g = 0 \qquad (2)$$

when products of small quantities are neglected in the usual manner. The equations of continuity and incompressibility likewise become

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \qquad (3)$$

$$\frac{\partial \rho}{\partial t} + U\frac{\partial \rho}{\partial x} + w\frac{\partial \rho_0}{\partial z} = 0. \qquad (4)$$

The last four equations contain four unknowns, viz., the velocity components u and w, and the perturbation density ρ and pressure p. The coefficients of all unknowns are independent of time t and coordinate x, so that dependence on these variables can be written in the form $e^{i(\omega t + \lambda x)}$. In this notation ω is the (possibly complex) frequency and λ the wave number corresponding to a single harmonic component; it is well known that disturbances of a more general nature can be resolved into components in this manner. Thus the principle of superposition of solutions of linear differential equations permits the consideration of more general motions by compounding different harmonic components. Eliminating the velocity component u and the pressure and density from the Equations (2), (3), and (4), and using the assumed dependence on the x and t variables, a single equation is obtained for the remaining velocity component:

$$\frac{d^2 W}{d z^2} + \frac{1}{\rho_0}\frac{d\rho_0}{dz}\frac{dW}{dz} - \left\{\lambda^2 + \frac{1}{\rho_0}\frac{d\rho_0}{dz}\left[\frac{\lambda U'(z)}{\omega+\lambda U} + \frac{g\lambda^2}{(\omega+\lambda U)^2}\right] + \frac{\lambda U''(z)}{\omega+\lambda U}\right\} W = 0 . \qquad (5)$$

This equation, which will be fundamental in the analysis which follows, reduces to the equation considered by Love in 1891, when U = 0 and the notation is slightly modified (Ref. 13).

Each fluid element is required to satisfy the dynamical and conservation principles implied in Equation (5). In addition it is assumed that the vertical component of velocity w is continuous at all points including, in particular, those at the interface between one fluid and another, and at solid boundaries. The same requirement is imposed on the pressure perturbation p, and these requirements together suffice to determine the flow. The condition on the pressure can be expressed in terms of w by requiring the continuity of the quantity

$$\rho_0 \left[(\omega+\lambda U)^2 \frac{dw}{dz} - (\lambda g + \{\omega+\lambda U\} U')\lambda w\right]. \tag{6}$$

A boundary value problem in the mathematical sense is determined by Equation (5) and the two auxiliary conditions just mentioned.

It is clear that in certain cases the requirement (6) of pressure continuity reduces to requiring continuity of the derivative $\frac{dw}{dz}$.

Returning to consideration of the particular flow model already described, it is seen that Equation (5) is greatly simplified for each region of flow, and becomes

$$\frac{d^2w}{dz^2} - \lambda^2 w = 0 \tag{5a}$$

since in both the upper fluid and the lower the density has been assumed constant and the wind shear has been chosen to contain zero curvature. Finiteness of disturbances at great distances from the origin requires the solution of Equation (5) in the upper region to be simply

$$w = c_1 e^{-\lambda z}$$

while the condition of zero normal velocity at the solid boundary requires

$$w = c_2 \sinh \lambda(z+h)$$

in the lower fluid. Continuity of w at the interface requires, within the present approximation, that

$$C_2 = \frac{C_1}{\sinh \lambda h} \quad .$$

The pressure continuity condition then reduces to an equation for the determination of the frequency which becomes

$$\omega + \lambda U = \frac{\frac{U}{h} \pm \sqrt{\frac{U^2}{h^2} - 4\lambda g \varepsilon (1+\varepsilon+\coth \lambda h)}}{2(1+\varepsilon+\coth \lambda h)} \tag{7}$$

and furnishes stability criteria. The frequency is a real quantity when the expression under the radical sign is greater than zero, and the disturbance is then said to be stable with respect to small disturbances of wave length $L = \frac{2\pi}{\lambda}$. If the quantity under the radical sign is less than zero, however, the frequency has a non-vanishing imaginary part which indicates a motion which is increasing in amplitude exponentially with time and therefore represents an instability with respect to small perturbations.

Two features of the stability criteria thus obtained from Equation (7) must be realized, and these are limitations of the criteria. It must be understood, first, that no information is provided concerning the response to a large disturbance; stratifications which appear to be stable from the criteria Equation (7) may behave quite differently under larger perturbations than can be considered by the present analysis. Conversely, a prediction of instability based on Equation (7) offers no assurance that the motion remains unstable when perturbations reach finite values. In short one may say that instability criteria deduced from Equation (7) are necessary but not sufficient conditions for the development of finite motions of instability from small perturbations, and that predictions of positive stability which are based on Equation (7) are necessary but not sufficient in a corresponding sense.

The stability criterion obtained from Equation (7) is thus that a disturbance is stable if the condition

$$\frac{U^2}{h^2} - 4\lambda g \varepsilon (1 + \varepsilon + \coth \lambda h) \geqq 0 \tag{8}$$

is satisfied. Several qualitative features of the criterion are apparent.

Since it is sufficient to consider only positive values of the wave number λ and since, moreover, the gravity constant g and depth h are intrinsically positive quantities, then it is clear that the condition (8) is satisfied whenever lighter fluid is superimposed on denser, i.e., whenever $\varepsilon < 0$ (in this connection it may be emphasized that $\varepsilon > -1$ is a necessary consequence of the fact that mass is always a positive quantity). The presence of wind shear is irrelevant in guaranteeing stability. In the absence of wind shear when denser fluid is above ($\varepsilon > 0$), however, disturbances corresponding to all wave lengths are unstable. This is a familiar result and displays the characteristic that instability is greatest for disturbances of small wave lengths (see for example, Lamb, Ref. 14). The presence of wind shear modifies the instability, weakening it for all wave lengths, and presenting the possibility of repressing the instability of disturbances of wave lengths greater than a critical value. This occurs whenever the shear is sufficiently great so that the condition (8) is satisfied. This demonstrates that shearing motion stabilizes a statically unstable configuration in which more dense fluid overlies less dense. Before considering more closely the qualitative behavior of the criterion (8) for the limit of very small wave lengths, and before attempting numerical calculations using parameter values corresponding to conditions in which dust devils can form it is of interest to reinforce one's concepts of the fundamental basis of the last result with the following observation. The question of the stability of a disturbance appears from (8) to be determined by the relative magnitudes of two forms of energy: when the kinetic energy of shearing motion exceeds a sort of potential energy of stratification, the disturbance is a stable one. The quantity U^2 is a measure of the shear motion kinetic energy per unit mass, and $4\lambda g h^2 \varepsilon(1+\varepsilon+\coth\lambda h)$ has the form and dimension of a gravitational potential energy which is positive when denser fluid is above. It will be seen later that in practically significant numerical examples the quantity ε satisfies the condition $0<\varepsilon \ll 1$, from which it appears that a very delicate balance of energies determines stability of motions.

Two limiting cases of the criterion (8) require special attention; these are the behaviours for very small wave numbers (large wave lengths), and for very large wave numbers (small wave length components). For small wave numbers (more precisely, for $\lambda h \ll 1$), on account of the singularity of the hyperbolic cotangent at the origin, if one observes that

$$\lim_{\lambda h \to 0} \lambda h \coth \lambda h = 1 + \frac{\lambda^2 h^2}{3} + \cdots$$

then it follows that flows are stable for $0 < \varepsilon \ll 1$ provided the shear is sufficiently great:

$$\frac{U^2}{h^2} \geqq \frac{4g\varepsilon}{h} \quad .$$

In other words, it is verified that the condition that sufficient shear stabilizes the motion continues to apply to long wave components, i.e., when $\frac{2\pi}{\lambda} = L \gg 2\pi h$. It is doubtful if this limit is of great importance in dust devil formation, since instabilities caused by destruction of shear and action of long wave components represents a very weak instability, whereas disturbances of smaller wave length and correspondingly greater time factor of instability determine the growth of the complete motion.

It remains to consider the opposite limit; when anomalous behavior of the criterion (8) for small wave lengths has been clarified it will be seen to be possible to describe a more or less complete mechanism of the formation of a dust devil. In this connection it is evident from (8) for any values of the shear parameter and for any $\varepsilon > 0$, that for sufficiently small wave lengths (i.e., for sufficiently large values of the wave number λ , or for $\lambda h \gg 1$), an instability is always indicated. In view of the ubiquity of very short wave length disturbances in nature, this fact seems to require further investigation. In this connection it will be demonstrated next that on account of friction the criterion is invalid for extremely large wave number components of a disturbance and that, in fact, these produce an effect opposite to that indicated in (8) by providing a measure of positive stability. It will be seen that the creation of stratifications of denser fluid above can be explained in terms of these components, thereby completing the suggested mechanism for dust devil formation.

The action of very small scale eddies differs essentially from that of larger eddies on account of the fact that temperature and velocity gradients in the former are of greater importance than in the latter. The smaller dimensions increase gradients and therefore also friction and heat conduction effects. In the absence of definite knowledge about the form of turbulent eddies of small dimension, it is impossible to specify a critical dimension below which the effect may not be neglected. The argument which follows, however, appears to be plausible for "sufficiently" small eddies and it may be supposed that for somewhat larger eddies the effects found are of lesser strength and so on for larger and larger eddies, so that a continuous state of affairs exists for all eddy sizes. The real justification of the assumptions which follow, however, is the agreement of the conclusion with observations which will be cited.

It is assumed that the kinetic energy acquired by a warm eddy as a result of the action of buoyant forces is entirely transformed to internal energy as a result of the action of friction and heat conduction. The buoyant force on unit mass of fluid element is $\frac{\delta\rho}{\rho} g$ where $\delta\rho$ is the excess of the density of the surroundings over that of the element. For small eddies the density variation can be taken to be proportional to the temperature variation, since the accompanying pressure variations, while sufficient to create motions, are entirely negligible in their thermodynamic effects. This is Boussinesq's celebrated "hydrostatic assumption", (Ref. 15). Then

$$\frac{\delta\rho}{\rho} \propto \frac{\delta T}{T} = \frac{dT}{dz}\ell$$

where ℓ is a characteristic mixing length for the motion. The work done by the buoyant force is then proportional to $\ell^2 \frac{dT}{dz}$ and this term is then also proportional to the kinetic energy $\frac{w^2}{2}$ acquired by unit mass of the eddy. The turbulent heat conduction coefficient is $w\ell$ and the heat flux through unit horizontal surface is $w\ell \frac{dT}{dz}$. When equilibrium of heat flow is established this quantity is constant; i.e.,

$$w\ell \frac{dT}{dz} \propto \sqrt{\ell^2 \frac{dT}{dz}} \cdot \ell \frac{dT}{dz} = \ell^2 \left(\frac{dT}{dz}\right)^{3/2} = \text{const.} \tag{9}$$

If the mixing length is given by the von Kármán law $\ell = k(z + z_o)$ in terms of the universal constant k and characteristic roughness length z_o, then integration of (9) leads at once to

$$T \propto (z + z_o)^{-1/3}. \tag{10}$$

When the ground surface is hot, as in cloudless desert calm at high sun, with conservative estimates of the roughness parameter z_o, the temperature law (10) results in extreme gradients in the lowest layers, greatly exceeding the adiabatic lapse rate and thereby creating a stratification containing less dense air at low levels beneath layers of greater density air.

The above argument was first pointed out to the author by Professor W. D. Hayes, in private communication. Professor Hayes also stated that

Dr. Prandtl made a similar deduction in about 1946. Measurements made by Johnson and Heywood in England have subsequently confirmed the qualitative result (10) to a high degree of accuracy (Ref. 16).

The result (10) is of importance in the present problem in demonstrating that very small eddies provide the mechanism for producing a density inversion and, therefore, a stratification which is statically unstable. This transformation of energy from a thermal form to gravitational potential energy, in transferring heat upward and increasing the supply of energy which drives the unstable motion, can, however, be regarded as a stabilizing influence on the dynamical configuration. Dynamic instability, which represents a further conversion producing kinetic energy at the expense of the density inversion, then proceeds at a rate determined by the strength of longer wave components for which the criterion (8) is expected to apply. While it would be difficult to estimate precisely an upper limit to the size of the small eddies which are thus viewed as stabilizing in their influence and for which (8), derived by neglecting frictional effects, is not a valid dynamical criterion, it seems plausible that some such limit does exist. Schlichting's findings, cited above, may be taken as a confirmation of this conjecture.

Refinements of the argument leading to (10) and of this result itself can be made but are not needed for the present problem. It is of interest to calculate numerical values from (10) for conditions in which dust devils are known to develop. Taking one inch as a conservative value of the roughness parameter (see, for example, Ref. 9, p. 248), and the temperature one inch from the surface to be 125° F, then the lapse rate at one foot above the surface exceeds the adiabatic value by a factor of about 17,000. This may be compared with the rate of 2,000 measured by Best (Ref. 17) above grassland in England. While the drastic assumptions used in deriving the result (10) above may exaggerate the lapse rates which actually exist, the extreme intensity of the sun's rays in the desert may represent a greater energy source than is found elsewhere and hence produce lapse rates sufficiently much greater than those observed by Best so that dust devils are formed in one place but not the other.

Returning to consideration of the criterion (8), the results deduced above concerning the effects of eddies of small scale permit the mechanism of formation of motions of the dust devil type to be described in the following manner.

When the desert sun shines on flat land containing sparse foliage which exists only in the lowest few feet above the surface of the earth, that surface is heated very intensely. The layers of air immediately above the surface are heated by conduction, with irregularities of temperature from one point to another which depend on surface conditions. The Reynolds number of the atmosphere being sufficiently great even in the presence of the gentlest breezes, motion on the smallest scale is turbulent, so that small and relatively warm and expanded elements of air are readily lifted by buoyant forces and carry heat upward. Short wave length components of ubiquitous small disturbances thus tend to create steep lapse rates which are strongly superadiabatic in the lowest layers and therefore statically unstable. In the absence of a sufficiently strong transverse shearing motion, some components of intermediate length are also dynamically unstable, according to (8) and overturning results through motions of instability which are of greater intensity for relatively shorter wave lengths. This prevents the development of density inversions or destroys them more or less as quickly as they are created. If a gentle breeze is blowing, friction at the surface produces a convex shearing profile (and therefore a rotational main flow). The presence of the shearing flow represses the instability of disturbances on that flow in such a manner that some disturbance components become stable. This means that density inversions may be produced, corresponding to the law (10) or possibly milder forms. Greater shears produce the possibility for development of stronger density inversions, the existence of which represents a storage of gravitational potential energy. When such a stratified flow is disturbed, its response differs according to the strength of the different wave lengths which are present. Relatively long wave lengths (λ small) are relatively ineffective in violating the condition (8) by diminshing the term under the radical sign. Very small wave components were shown to result in an intensification of the density inversion, when friction and heat conduction effects were considered. In contrast with instabilities of greater wave lengths which will be seen to destroy the density inversion, these small wave components actually develop it. For intermediately long wave components of a disturbance, the criterion (8) indicates that if the shear is sufficiently great, they produce no instability. For some smaller values of the shear, components of length greater than a critical value are stable, while smaller wave lengths are unstable. The instability is more intense (the negative of the imaginary part of ω , which is the time scale of the instability, being greater) for the shorter wave length components. When the shear is locally reduced by an obstacle or gust, say, if it is reduced sufficiently so that some wave lengths which are present create an instability, then vertical motions occur and this further reduces the shear. Then longer wave lengths, if present, also become unstable causing even greater reduction of shear. If it should ensue that the shear is entirely destroyed in this manner, then more wave length components of an initial disturbance become unstable, and ultimately all greater than some small value are unstable, each with a time factor proportional to the square root of the wave number. For a given amplitude, the relatively shorter unstable wave components are more influential in determining the rate of growth of the motion, since they correspond to greater time factors.

It has been shown that when no wind is blowing, or when it is too weak, instabilities are sufficiently great to prevent the development of strong density inversions. For somewhat greater values of the wind shear, density inversions may be stable and this leads to the possibility of subsequent sustained unstable motions. For still stronger winds, it appears plausible that vertical mixing is increased by the turbulent shear itself, and prevents the initial development of density inversions and subsequent instability. Hence it appears that a gentle breeze may be necessary for the development of a dust devil, but both more calm and also more strong winds are unsuitable for its development. This accords with observations (Ref. 12).

In the intermediate case when sustained unstable motions become possible, it seems credible that the motions which develop should have the form actually observed in dust devils. The destruction of shear being a local effect, the resulting motion should be of limited horizontal extent, rather than of significantly greater extent in one horizontal direction than in another. Unstable motion proceeds at the expense of a store of energy, which in the present case is gravitational potential energy depletion by ascent of less dense air in a chimney-like motion and its replacement by subsiding air surrounding the ascending column. The rotation about a vertical axis appears as a consequence of the principle of conservation of moment of momentum, and the whirling motion becomes more intense as air is sucked in from larger and larger distances from the axis of motion. The direction of the whirling motion depends on the form of the initial flow obstruction or other disturbance which creates the motion; there is no preferred direction, as in the case of the geostrophic winds. The inherent limitations of the linearized dynamical theory which has been employed in formulating the mechanism suggested prevents the detailed analysis of later phases of the motion just described. The early stages have been described qualitatively in detail by means of the results (8) and (10), but the mechanism described so far lacks that convincing aspect which is only obtained by verifying quantitatively the correctness of the description which has been given. This can and will be considered next, before an attempt is made to evaluate the assumptions which have been made.

A verification of the qualitative correctness of the mechanism described above for the genesis of dust devil motions can be obtained by examining the criterion (8) in the following manner. For a given ground temperature and an assumed superadiabatic lapse rate in a layer above the earth's surface, as well as an assumed wave length, that value of the wind shear can be calculated which corresponds to neutral stability, by considering (8) with the equality sign. The parameter values chosen are necessarily arbitrary, and on this account it seems well to indicate in detail the reason for the particular choices which are made.

In an atmosphere in which there is a constant temperature lapse rate n times greater than the adiabatic value $\Gamma = \frac{\gamma-1}{\gamma}\frac{g}{R} = 0.98\ ^{\circ}\mathrm{C}/100$ m., the density $\rho(z)$ at height z meters above the surface is related to the value at the surface, ρ_0 , and the surface temperature T_0, by the formula

$$\rho(z) = \rho_0 \left(\frac{T_0 - n\Gamma z}{T_0}\right)^{\frac{\gamma}{n(\gamma-1)} - 1} \tag{11}$$

where, as usual, γ is the ratio of specific heats, equal to 1.4 for air. One deduces (11) from the equations of state and of hydrostatic equilibrium. The density increment can be calculated from (11), or, since this quantity is very small compared with unity in cases of interest to the present problem, it can be approximated very well by retaining only the first two terms of the binomial expansion, as

$$\frac{\rho(z)-\rho_0}{\rho_0} = \left(1 - \frac{\gamma}{n(\gamma-1)}\right)\frac{n\Gamma z}{T_0} \tag{12}$$

from which it is apparent that density inversions, corresponding to $\rho(z) > \rho_0$ result when the lapse rate $n\Gamma$ exceeds the so-called auto-convective value $\frac{\gamma}{\gamma-1}\Gamma = 3.5\Gamma$. If the lapse rate exceeds the adiabatic value by a factor of n = 10 in the lowest ten feet with a ground temperature $T_0 = 125^\circ$ F $= 325^\circ$ K, then the density increment at ten feet above the surface is $\left.\frac{\rho(10)-\rho_0}{\rho_0}\right|_{n=10} = 0.0006$. The same value obtains if a layer 19 feet thick is considered in which the lapse rate factor n = 7, or for other combinations which can be easily calculated from (11) or its approximation (12). In the model considered in deducing (8) the lower layer was assumed homogeneous, which suggests that for the atmospheric conditions indicated just above, a reasonable mean value is obtained by taking one half the value calculated above, $\varepsilon = 0.0003$, say. If the thickness of the lower layer is taken to be ten feet, the required wind shear depends only on the wave number. Remembering that the smallest wave numbers present in appreciable strength in an initial disturbance correspond to most rapid unstable growths, it seems reasonable that these will be the most important ones. If, then, following the suggestion of von Kármán's similarity hypothesis, we take the wave length to be four tenths of the thickness of the lower layer, the wave number $\lambda = 0.64$/ft. and the required wind speed is 3.5 ft./sec. This represents a gentle breeze and seems to be at least of the same order of magnitude as the observed gentle breezes which have been reported when dust devils have been observed. Recognizing that the wind speed value just obtained may have an appreciable quantitative error, it is of interest, nevertheless, to continue calculations using this value for want of a better one. If the wind were this strong then, or stronger, a density inversion corresponding to a lapse rate ten times greater than the adiabatic value would be stable with respect to small disturbance components of wave number smaller than $\lambda = 0.64$/ft., i.e., for relatively longer wave lengths. If a flow obstruction diminished the cross wind, an instability would appear for the same wave length. The strength of the instability, given by the time factor from (8), clearly depends on the extent of the wind diminution. In the limiting case in which the wind is entirely destroyed by the supposed flow obstruction and subsequent unstable motion, the time factor is then

$$|\omega| = \sqrt{-\varepsilon g \lambda (1+\varepsilon+\coth \lambda h)} = \left|\frac{U}{h}\right| \doteq 0.36/\text{sec.}$$

This indicates that the amplitude of a disturbance would increase by a factor of $e \doteq 2.7$ in approximately 2.8 seconds. This value, representing an upper limit as it does, also seems to be in reasonable agreement with the observed slowness of dust devil growth in earliest stages. A very much smaller value, corresponding to much larger time, would be less acceptable, since other flow irregularities might then be expected to predominate.

Similar calculations could be made for either steeper or less steep temperature gradients, for different thicknesses of the rarefied layer, and for various supposed wave lengths, but for two reasons such calculations would seem to be unjustified. In the first place it would be difficult to state which values might be more representative of actual conditions which prevail in the atmosphere when dust devil motions begin -- such values are, furthermore, of little importance. In the second place, the special nature of the assumptions made in deducing the criterion (8) may have an important influence on quantitative conclusions obtained from it. It seems at any rate to be satisfactorily established by the numerical calculations above that the resulting mechanism which has been suggested for the formation of dust devils is not in error by entire orders of magnitude -- further information is not desired of numerical calculations at present. The general character of the criterion (8) can be displayed, however, by plotting it in dimensionless form for one value of the dimensionless density parameter ε and for the case of neutral stability in which the quantity under the radical in (8) vanishes. This is done in Fig. 2 in which the ratio of kinetic to potential energies, $\frac{U^2}{gh}$ (which is the square of a Froude Number), is plotted against the dimensionless wave length parameter $\frac{L}{h}$. The solid line then separates a stable region above the curve from an unstable region below. The curve cannot be continued to the origin of the abscissa for reasons discussed previously Further, on account of the smallness of the parameter $\varepsilon \ll 1$ in atmospheric density inversions, it is apparent that the curves corresponding to different values of this parameter are essentially parallel to each other, and it suffices to plot only one curve.

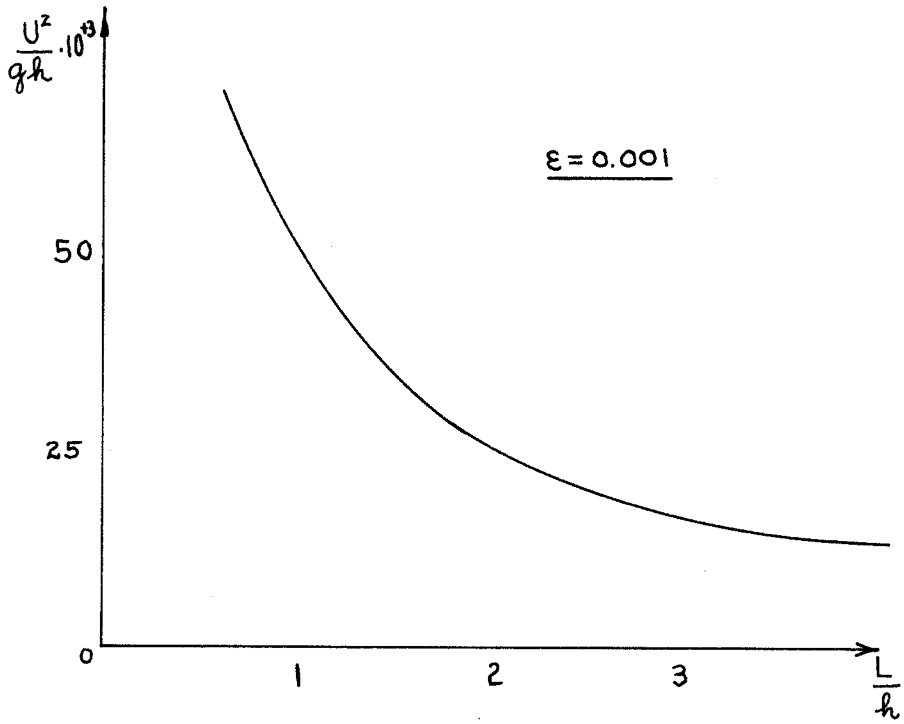

Fig. 2

The significance of several of the assumptions made in the flow model which has been analyzed can now be discussed and evaluated.

The fluids involved have been assumed incompressible, whereas the compressibility of a fluid element is essential for the thermal expansion which is basic for the formation of a density inversion. The neglect of compressibility in dynamical equations is justified for the present problem, however. Just as in high speed aerodynamic flows in which compressibility is negligible when the Mach number is small compared with unity, likewise the neglect of compressibility in the present problem depends on the smallness of a corresponding dimensionless ratio. The phase velocity $\frac{\omega}{\lambda}$ is the quantity which requires comparison with the local sound speed c, which is the measure of compressibility. The phase velocity is appropriate as the comparison quantity since it is the only ratio of the same dimension as sound speed which can be formed from the intrinsic flow parameters which occur in the analysis. Compressibility is negligible when

$$\left|\frac{\omega}{\lambda}\right| \ll c$$

where the sound speed $c = 1200$ ft./sec. at the assumed surface temperature of 125° F. For the assumed wave number $\lambda = 0.64$/ft., then, compressibility can be neglected for frequencies

$$|\omega| \ll \lambda c \doteqdot 770/\text{sec};$$

the frequency which was found in the calculated example $|\omega| = 0.36$/sec., certainly satisfies this condition. Alternatively, the assumption is justified for frequencies of the order of $\frac{1}{3}$/sec. whenever the wave number

$$\lambda \gg \frac{|\omega|}{c}$$

or, equivalently, whenever the wave length

$$L = \frac{2\pi c}{|\omega|} = 4.3 \text{ miles};$$

as it appears very unlikely that such large wave lengths are of any importance in the formation of dust devils, the assumption of incompressibility appears to be completely justified in the dynamical analysis. It is apparent, also, that in the study of longer atmospheric waves one may not neglect compressibility, and indeed classical formulations of the analysis are frequently in terms of the divergence of the velocity, sometimes called the condensation, which is a measure of the compressibility of a gas. A formulation of the present problem in terms of the condensation, would therefore appear to be inappropriate.

The novelty of the result that a density inversion is not unstable with respect to disturbances of all wave lengths, which is concluded from (8), is clearly a consequence of the assumed shear pattern. Undisturbed flow with less simple wind shears are well known in hydrodynamics and are in many cases unstable. In the absence of a shearing wind, $U \equiv 0$ and density inversions ($\varepsilon > 0$) are unstable with respect to disturbances of all wave lengths, the shortest wave lengths displaying the strongest instabilities, in the usual manner. Less dense fluid above more dense ($\varepsilon < 0$) is of course stable, but even this situation is radically modified by shear. It is shown by Lamb, for example (Ref. 14, p. 373), that if the two fluids are moving each with a uniform but different speed from that of the other, then disturbances of sufficiently small wave length are unstable, and this is true regardless of whether the upper fluid is more dense than the lower or less. In either case there is a sheet of vorticity at the interface. In reality, the discontinuity, if it could ever be originated, would be immediately abolished by viscosity, and the vortex-sheet replaced by a film of vorticity. From this point of view, the conclusion drawn from (8) can be expressed as follows: If a fluid is moving with uniform velocity with respect to and over a less dense layer of fluid, then any small disturbance precipitates an initially unstable motion. As transverse momentum is exchanged and vorticity diffuses from the discontinuity surface, the shear profile created by the early instability becomes a smoother one. If this continues, a convex pattern forms, which may resemble the polygonal one which has been considered in deducing (8).

In this case the unstable motion continues until a stable flow regime is established, and this may involve the continued storage of statically unstable gravitational potential energy: the dynamic instability is held in abeyance by the shear pattern. Destruction of this shear pattern then would unharness a further unstable motion (of the dust devil type, perhaps). The mechanism just described is of interest as a description of the later and greatly altered stages of a classical hydrodynamic instability, with possibly important applications in meteorology and elsewhere, but not in the genesis of dust devils since the assumed initial wind discontinuity is of course not realized in the atmosphere.

The simple polygonal velocity profile which was chosen for analysis in the model considered was selected for the resulting simplicity of the differential equation. If a curved profile were considered, the Equation (5) would no longer have constant coefficients and simple exponential solutions. While the more complex equation can readily be solved, yielding hypergeometric functions for solutions, it also would entail a secular equation which would furnish the frequency as a transcendental function of the wave length, rather than the extremely simple algebraic form (8) which is obtained by neglecting profile curvature and continuous density variations. It appears reasonable from the form of (5) that shear profile curvature would involve drastic changes in the stability properties only if sufficiently great negative curvatures were present so that the coefficient of w experienced a change in sign. As wind profiles are characteristically not sharply curved in the lowest layers, this possibility can be excluded from consideration in the present problem. Alternatively, we can accept Schlichting's results as an indication that profile curvature does not nullify the conclusions drawn.

By reason of the form of Equation (5) it is difficult to examine rigorously the consequences of the assumption of uniform density in each of the two layers of fluid considered and a density discontinuity at the interface. This results from the fact that when continuous density variations are admitted in addition to a variable shear, the coefficients in Equation (5) are no longer constants and solutions are not simple exponential functions. Solutions, instead, are then hypergeometric functions the parameters of which depend on the frequency. The resulting frequency equation obtained from the pressure condition (6) is then a transcendental function of the frequency, the properties of which are not nearly so transparent as those of the quadratic equation which led to the solutions (7). Consequently, although solutions and a frequency equation can be obtained in the case of continuously varying density, comparison of the result with Equation (7) would be more difficult in practice than circumstances justify, and the calculation has not been carried out. The effect of the assumption of a density discontinuity can be estimated in another manner, however, which is only slightly less satisfactory. By specializing the result, Equation (7) to the case with no shear, $U \equiv 0$, the familiar result is obtained that

$$\omega^2 = \frac{-\varepsilon g \lambda}{1+\varepsilon+\coth\lambda h} \tag{12}$$

which represents an instability for density inversions $\varepsilon > 0$, and of strength which is greatest for small wave lengths (large wave numbers λ). The instability represented by (12) is a classical one, and is discussed by Lamb (Ref. 14, p. 372). The result (8) then shows how shear stabilizes this flow. It seems reasonable therefore to compare (12) with the instability obtained by neglecting shear in Equation (5), but by considering exponential density variations. The resulting equation then has constant coefficients and exponential solutions which depend on the frequency ω . The frequency equation resulting from the pressure condition (6) is then a transcendental equation for the frequency, but one which an involved calculation shows to possess the simple solution

$$\omega^2 = \frac{-\varepsilon' g}{h\left(1+\frac{\varepsilon'^2}{4h^2\lambda^2}\right)} \tag{13}$$

in which ε' denotes the dimensionless density surplus at z = 0 over that at z = -h and the other symbols retain their former meanings. For purposes of comparison with (12) it seems reasonable to assign, as an appropriate mean value for ε' the value $\frac{\varepsilon}{2}$, since, $|\varepsilon'| \ll 1$ in all cases of practical interest. Then (13) becomes

$$\omega^2 = \frac{-\varepsilon g}{2h\left(1+\frac{\varepsilon'^2}{4h^2\lambda^2}\right)} \tag{14}$$

and it is therefore clear that the instability of a continuously varying density stratification, for fixed values of ε is of the same general form as in the case (12) and even of almost precisely the same strength when the depth h in one case has the same value as the reciprocal of the wave number λ in the other. In other words, the instability is of the same form in both cases, whenever the depth of the lower layer is of the same order of magnitude as the wave length $L = \frac{2\pi}{\lambda}$. In the case of continuously varying density, the strength of the instability depends more sensitively on the depth of the lower layer than in the case of a density discontinuity in which the wave number is the more important quantity. If one supposes, as seems reasonable, that the stabilizing influence of shear is not greatly different in the two cases, then it appears that the result (8) for the density discontinuity is not greatly influenced by this assumption, at least for wave lengths of the same order of magnitude as the depth of the lower layer. The inadequacy of (8) for much longer and for much shorter wave lengths has already been discussed.

In the preceding analysis consideration has been confined to a two-dimensional basic flow and to disturbances in the same two-dimensions. It is not difficult to extend the argument and to show, thereby, that the possibility of a stable shearing motion in a density inversion is preserved. The minor modification resulting from the admission of variations in the third dimension is also put in evidence in a simple manner.

By choosing the x-axis again in the direction of the local wind, the z-axis vertical upward, and the y-axis perpendicular to both, the undisturbed motion remains two-dimensional. Adding to the previous notation the symbol v for the component of disturbance velocity in the y direction, a third equation of momentum is obtained in addition to Equations (1) and (2), and an additional term is introduced into the continuity Equation (3). The coefficients in the resulting system of equations are independent of y, and it is therefore appropriate to assume a dependence of all disturbance quantities on y in the form $e^{i\mu y}$, in the usual manner. Formulating the problem in terms of the vertical component of disturbance velocity, the equation is replaced by

$$\frac{d^2W}{dz^2} + \frac{\rho_0'}{\rho_0}\frac{dW}{dz} - \left[\lambda^2 + \mu^2 + \frac{\rho_0'}{\rho_0}\left\{\frac{g(\lambda^2+\mu^2)}{(\omega+\lambda U)^2} + \frac{\lambda U'}{\omega+\lambda U}\right\} + \frac{\lambda U''}{\omega+\lambda U}\right]W = 0 \tag{15}$$

and the condition of continuity of pressure disturbance requires the continuity of

$$\frac{\rho_0}{\lambda^2+\mu^2}\left[(\omega+\lambda U)^2\frac{dW}{dz} - \left\{g(\lambda^2+\mu^2) + \lambda(\omega+\lambda U)U'\right\}W\right]. \tag{16}$$

It is clear that the two-dimensional case is subsumed under Equations (15) and (16) by letting the wave number μ approach zero, or equivalently, by requiring infinitely large wave lengths in the y direction. In the case of zero shear, $U \equiv 0$, it is also apparent that the three-dimensional formulation is obtained from that in two-dimensions by replacing λ by $\sqrt{\lambda^2+\mu^2}$, so that the only asymmetry of the motion arises from differences between μ and λ .

Assuming the same density variations and shear patterns which were previously considered, it follows that a disturbance is not unstable if the condition

$$\frac{U^2}{h^2} - 4\varepsilon\lambda g\left(1+\frac{\mu^2}{\lambda^2}\right)^{3/2}\left(1+\varepsilon+\coth\lambda h\sqrt{1+\frac{\mu^2}{\lambda^2}}\right) \geqq 0 \tag{17}$$

is satisfied. The condition (17) thus replaces the condition (8), and of course reduces to it when μ vanishes. The difference between the two criteria can be expressed in the following manner: the shear which is required to stabilize a density inversion subject to three-dimensional disturbances is increased, as compared to the two-dimensional case, according to the smallness of the wave lengths in the y direction. More importantly, however, it is apparent that the two-dimensional analysis is not essentially modified in its principal conclusions by the consideration of two dimensions only.

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II. LINEARIZED THEORY OF CONICAL TURBOMACHINES

The flow of a perfect fluid through an axial turbomachine with prescribed blade loading has been analyzed by means of a linearized theory by Marble (Ref. 1). In order to estimate the influence of radial flow components in turbomachines, a similar analysis can be carried out. Taking the walls of the turbomachine to consist of portions of two coaxial cones with a common vertex, the flow problem can be linearized by assuming that the stream surfaces consist of conical surfaces.

To discuss the flow in detail, it is convenient to write the partial differential equation of the stream function ψ in spherical polar coordinates R, θ, ϕ with corresponding velocity components U, v, W and vorticity components Ξ, η, Z. Letting $\mu \equiv \cos\phi$, the equation becomes

$$\frac{\partial^2\psi}{\partial R^2}+\frac{1-\mu^2}{R^2}\frac{\partial^2\psi}{\partial\mu^2}=R\sqrt{1-\mu^2}\left\{(\omega R\sqrt{1-\mu^2}-v)\frac{\partial}{\partial\psi}(vR\sqrt{1-\mu^2})+\frac{F_\mu}{U_m}-\frac{\partial H_o}{\partial\psi}-\frac{\partial}{\partial\psi}(\omega R v_o\sqrt{1-\mu^2})\right\} \tag{1}$$

Here the stream function has the properties $U=\frac{1}{R\sin\phi}\frac{\partial\psi}{R\partial\phi}$, $W=\frac{-1}{R\sin\phi}\frac{\partial\psi}{\partial R}$, and the symbols F_μ, H_o, and ω represent, respectively, the force component exerted by the blade row in the ϕ (or μ) direction, the total head per unit mass of fluid, and blade angular velocity. The right side of this equation is linearized by considering $U_o(R)$ as the basic flow corresponding to a source or sink at the origin R = 0, only slightly perturbed by the action of the blade row. Then on the right side of Equation (1) choose

$$d\Psi \doteq U_o(R)\ R \sin\phi\ d\phi = -U_o(R)\ R\ d\mu$$

and consequently the linearized equation becomes

$$\frac{\partial^2\psi}{\partial R^2} + \frac{1-\mu^2}{R^2}\frac{\partial^2\psi}{\partial\mu^2} = \sqrt{1-\mu^2}\left\{\left(\frac{V-\omega R\sqrt{1-\mu^2}}{U_o(R)}\right)\frac{\partial}{\partial\mu}\left(vR\sqrt{1-\mu^2}\right) + \frac{R F_\mu}{U_o(R)} + \frac{\partial H_o}{\partial\mu} + \frac{\partial}{\partial\mu}\left(\omega R V_o\sqrt{1-\mu^2}\right)\right\}. \tag{2}$$

If the radial velocity $U_o(R)$ has a value of $U_o(R)$ at a reference radius R_o, it follows that at any other radius,

$$\Psi_o = U_o(R_o)\ R_o^2\ (\mu_1-\mu). \tag{3}$$

The boundary conditions to be satisfied by the stream function Ψ are those of vanishing tangential derivative at the inner and outer cone angles, say μ_1 and μ_2, vanishing disturbances in the μ direction both for upstream and downstream, and certain conditions at the blade depending upon whether the blade shape, blade loading, or angular momentum distribution is prescribed. These conditions may be given analytically: At the inner and outer boundaries,

$$\begin{aligned} \Psi(R, \mu_1) &= 0 \\ \Psi(R, \mu_2) &= U_o(R_o)R_o^2\ (\mu_1 - \mu_2). \end{aligned} \tag{4}$$

As $R \to 0$ and $R \to \infty$ the flow becomes conical, and therefore

$$\lim_{R\to 0}\left(\frac{\partial\psi}{\partial R}\bigg/\frac{\partial\psi}{R\partial\phi}\right) = \lim_{R\to\infty}\left(\frac{\partial\psi}{\partial R}\bigg/\frac{\partial\psi}{R\partial\phi}\right) = 0. \tag{5}$$

The values of H_o and v_o are assumed to be known at some station upstream of the blade row, and the angular velocity of the blade is given. Concerning the conditions prescribed at the blade row, only the case where the angular momentum is prescribed will be treated. The extension to the other cases may be effected in a manner analogous to that used for the axial turbomachine.

The mathematical problem to be considered is therefore the partial differential equation

$$\frac{\partial^2\psi}{\partial R^2} + \frac{1-\mu^2}{R^2}\frac{\partial^2\psi}{\partial\mu^2} = f(R,\mu) + \left[\frac{\partial H_o}{\partial\mu} + \frac{\partial}{\partial\mu}\left(\omega R V_o\sqrt{1-\mu^2}\right)\right]\sqrt{1-\mu^2}$$

where

$$f(R,\mu) = \sqrt{1-\mu^2}\left\{\frac{V-\omega R\sqrt{1-\mu^2}}{U_o(R)}\frac{\partial}{\partial\mu}\left(VR\sqrt{1-\mu^2}\right) + \frac{RF_\mu}{U_o(R)}\right\}$$
$$\psi(R,\mu_1) = 0 \qquad \psi(R,\mu_2) = U_o(R_o)R_o^2(\mu_1-\mu_2) \tag{6}$$

The functions $v(R,\mu)$, $H_o(\mu)$, $v_o(\mu)$ are given. It is convenient to choose the stream function to be the sum of two partial stream functions

$$\psi = \psi^{(1)} + \psi^{(2)} \tag{7}$$

where $\psi^{(1)}$ corresponds to the flow which would exist for the same initial and boundary conditions, but with the blade removed. The function $\psi^{(2)}$ is then the perturbation stream function corresponding to the effect of the blade row on the U and W velocity components.

Clearly $\frac{\partial\psi^{(1)}}{\partial R} = 0$ so that $\psi^{(1)}$ is found by a simple quadrature, for according to the relations (6)

$$\frac{\sqrt{1-\mu^2}}{R^2}\frac{\partial^2\psi^{(1)}}{\partial\mu^2} = \frac{\partial H_o}{\partial\mu} + \frac{\partial}{\partial\mu}\left(\omega R V_o\sqrt{1-\mu^2}\right) \tag{8}$$

which, with the conditions

$$\psi^{(1)}(R,\mu_1) = 0$$

$$\psi^{(1)}(R,\mu_2) = U_0(R_0)R_0^2(\mu_1-\mu_2)$$

determines $\psi^{(1)}(R,\mu)$ completely. Then the stream function $\psi^{(2)}$ satisfies the homogeneous problem

$$\frac{\partial^2\psi^{(2)}}{\partial R^2} + \frac{1-\mu^2}{R^2}\frac{\partial^2\psi^{(2)}}{\partial\mu^2} = f(R,\mu) \qquad (9)$$

$$\psi^{(2)}(R,\mu_1) = \psi^{(2)}(R,\mu_2) = \psi^{(2)}(0,\mu) = \psi^{(2)}(\infty,\mu) = 0.$$

Solution of the corresponding homogeneous equation may be written in the form $R^{n_i+1}H_{n_i}(\mu), R^{-n_i}H_{n_i}(\mu)$ where $H_{n_i}(\mu)$ are linear combinations of associated Legendre Functions (Ref. 2) of order 1, degree n_i, of both first and second kinds:

$$H_{n_i}(\mu) = P_{n_i}^{(1)}(\mu)\,Q_{n_i}^{(1)}(\mu_1) - P_{n_i}^{(1)}(\mu_1)\,Q_{n_i}^{(1)}(\mu). \qquad (10)$$

This clearly vanishes identically when $\mu = \mu_1$. The characteristic functions of the problem are thus determined by finding those values of the degree n_i such that

$$H_{n_i}(\mu_2) \equiv P_{n_i}^{(1)}(\mu_2)\,Q_{n_i}^{(1)}(\mu_1) - P_{n_i}^{(1)}(\mu_1)\,Q_{n_i}^{(1)}(\mu_2) = 0. \qquad (11)$$

The resulting infinite set of values n_i are the characteristic numbers which range between $-\infty$ and $+\infty$. However, it is possible to restrict the necessary values of n_i by noting (Ref. 2) that

$$P_{-n_i-1}^{(1)}(\mu) = P_{n_i}^{(1)}(\mu)$$

$$Q_{-n_i-1}^{(1)}(\mu) = \frac{\sin(n_i+1)\pi}{\sin(n_i-1)\pi}\,Q_{n_i}^{(1)}(\mu) + \frac{\pi\cos n_i\pi}{\sin(n_i-1)\pi}\,P_{n_i}^{(1)}(\mu)$$

and therefore the value of $H_{-n_i-1}(\mu)$ is simply

$$H_{-n_i-1}(\mu) = \frac{\sin(n_i+1)\pi}{\sin(n_i-1)\pi} H_{n_i}(\mu).$$

Consequently it is not necessary to consider values of n_i for which $n_i < -1$. The corresponding set of characteristic functions $H_{n_i}(\mu)$ is complete and possesses orthogonality properties common to functions satisfying a Sturm-Liouville problem.

A solution to the inhomogeneous partial differential equation (9) is obtained from solutions $R^{n_i+1} H_{n_i}(\mu)$ and $R^{-n_i} H_{n_i}(\mu)$ of the homogeneous equation. If $I(\alpha,\varepsilon)$ is an impulse function with properties

$$I(\alpha,\varepsilon) = \begin{cases} 1\ ; & \alpha-\varepsilon < R < \alpha+\varepsilon \\ 0\ ; & 0 \le R \le \alpha-\varepsilon\ ;\ R > \alpha+\varepsilon \end{cases}$$

the contribution to the solution of the function $f(R,\mu)$ in the range $\alpha-\varepsilon < R < \alpha+\varepsilon$ may be found to be

$$\begin{aligned} &\int_{\mu_1}^{\mu_2} I(\alpha,\varepsilon)\, f(\alpha,\mu) \sum_1^{\infty} \frac{H_{n_i}(\beta)\, H_{n_i}(\mu)}{\nu_{n_i}^2\,(2n_i+1)} \sqrt{\frac{1-\mu^2}{1-\beta^2}}\, \frac{R^{n_i+1}}{\alpha^{n_i}}\, d\beta \qquad R < \alpha \\ &\int_{\mu_1}^{\mu_2} I(\alpha,\varepsilon)\, f(\alpha,\mu) \sum_1^{\infty} \frac{H_{n_i}(\beta)\, H_{n_i}(\mu)}{\nu_{n_i}^2\,(2n_i+1)} \sqrt{\frac{1-\mu^2}{1-\beta^2}}\, \frac{R^{-n_i}}{\alpha^{-n_i-1}}\, d\beta \qquad R > \alpha \end{aligned} \tag{12}$$

where the choice of the solutions is determined by the boundary conditions at $R = 0$ and $R = \infty$. The numbers $\nu_{n_i}^2$ are norms of the $H_{n_i}(\mu)$ functions

$$\nu_{n_i}^2 = \int_{\mu_2}^{\mu_1} H_{n_i}^2(\beta)\, d\beta \tag{13}$$

The complete solution to the problem is simply the sum of the solutions of type (12) for each element of the range where $f(R,\mu) \neq 0$. Consequently if we define the function $L(R,\mu;\ \alpha,\beta)$ to be

$$L(R,\mu;\alpha,\beta)=\begin{cases}\sum_{1}^{\infty}\frac{H_{n_i}(\beta)H_{n_i}(\mu)}{\nu_{n_i}^2(2n_i+1)}\sqrt{\frac{1-\mu^2}{1-\beta^2}}\,\frac{R^{n_i+1}}{\alpha^{n_i}} & R<\alpha \\ \sum_{1}^{\infty}\frac{H_{n_i}(\beta)H_{n_i}(\mu)}{\nu_{n_i}^2(2n_i+1)}\sqrt{\frac{1-\mu^2}{1-\beta^2}}\,\frac{R^{-n_i}}{\alpha^{-n_i-1}} & R>\alpha\end{cases} \tag{14}$$

The complete solution is

$$\psi^{(2)}(R,\mu)=\int_0^{\infty}\int_{\mu_1}^{\mu_2} f(\alpha,\beta)\,L(R,\mu;\alpha,\beta)\,d\alpha\,d\beta \tag{15}$$

so long as the function $f(\alpha,\beta)$ is integrable.

Although this procedure is formally quite simple for any distribution of angular momentum (or for any other manner of prescribing information at the blade row) the details of the calculations involving the Legendre functions are somewhat cumbersome. The difficulties lie principally in the lack of extensive tabulations. Therefore it is appropriate, and usually sufficiently accurate, to use an asymptotic expression for the functions $H_{n_i}(\mu)$. This representation is

$$H_{n_i}(\mu) \doteq \frac{n_i \sin(n_i+\frac{1}{2})(\phi-\phi_1)}{\sqrt{\sin\phi \quad \sin\phi_1}} \tag{16}$$

where ϕ_1 is the semi-vertex angle of the root cone. Consequently the values of n_i are determined so that $H_{n_i}(\mu_2)$ vanishes, or if ϕ_2 is the semi-vertex angle of the tip cone

$$\sin(n_i+\tfrac{1}{2})(\phi_2-\phi_1)=0$$

or clearly

$$n_i = \frac{i\pi}{\phi_2-\phi_1}-\frac{1}{2} \quad . \tag{17}$$

Therefore , except for impractical included angles $(\phi_2 - \phi_1)$, only positive values of n_i will enter into the problem. The norms of the functions are easily calculated

$$\nu_{n_i}^2 = \int_{\phi_1}^{\phi_2} \frac{n_i^2 \sin^2(n_i+\frac{1}{2})(\phi-\phi_1)}{(\sin\phi \sin\phi_1)^2} \, d(\cos\phi) = \frac{-n_i^2}{(2n_i+1)\sin\phi_1} . \tag{18}$$

Then in the asymptotic representation, the Green's function becomes

$$L(R,\mu;\alpha,\beta) = \begin{cases} -\sum_{1}^{\infty} \frac{\sin(n_i+\frac{1}{2})(\phi-\phi_1)\sin(n_i+\frac{1}{2})(\bar{\phi}-\bar{\phi}_1)}{\sqrt{\sin\phi \sin\bar{\phi}}} \frac{R^{n_i+1}}{\alpha^{n_i}} \frac{\sin\phi}{\sin\bar{\phi}} & R<\alpha \\ -\sum_{1}^{\infty} \frac{\sin(n_i+\frac{1}{2})(\phi-\phi_1)\sin(n_i+\frac{1}{2})(\bar{\phi}-\bar{\phi}_1)}{\sqrt{\sin\phi \sin\bar{\phi}}} \frac{R^{-n_i}}{\alpha^{-n_i-1}} \frac{\sin\phi}{\sin\bar{\phi}} & R>\alpha \end{cases} \tag{19}$$

where $\bar{\phi}$ is the variable of integration corresponding to ϕ , that is, $\cos\bar{\phi} = \beta$. By means of these results, the perturbation stream function $\psi^{(2)}(R,\mu)$ may be evaluated directly from equation (15). The integrations offer no essential difficulty.

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