Chapter 2

BACKGROUND INFORMATION

Periodic structure is found in many materials and systems. It is the regular arrangement of atoms, particles, or unit cells which results in both dispersion and band gaps. Due to the possibility of enhanced control over wave propagation, these remarkable properties have fed the growth of fields such as photonics[11,38,39] and phononics[40,41]. In addition, it is the periodic structure that allows scientists in photonics, phononics, and semiconductors to take inspiration from each other's previous work.

As a mathematical construct, periodicity continues ad infinitum. However, real world systems are not infinite, and the spatial periodicity of a lattice ends at some point. In atomic systems, the idea of periodicity can only be maintained if the grain size of a crystal is significantly large enough. In macroscopic systems, there is a similar problem. How big is big enough? Can we consider a system of 2, 3, or 4, or even a hundred repeating units as periodic? This finite size has a significant effect on the frequency band structure that originates from periodicity.

Granular crystals are an example of a nonlinear periodic mechanical structure, which is inherently finite and may have defects. In the following sections I present a review on media with broken periodicities, focusing on the literature for granular crystals and proceeding from nearly linear to strongly nonlinear.

2.1 Linear

Wave dispersion, in which the wave velocity depends on frequency[10], is a result of the discreteness of a periodic system. This is in direct contrast with most continuous media waves which are typically non-dispersive, meaning the wave speed is constant and the frequency and wavelength are determined by the well-known relationship, $c = \lambda f$. In discrete media long wavelengths do not 'sense' the discrete nature of the media. However, when the wavelength grows shorter and begins to approach the size of the lattice spacing, the discreteness becomes extremely important. This dispersion causes wave packets with finite bandwidth to spread out spatially.

In addition, periodic media does not support wave propagation at all frequency ranges. Bands of frequencies that are exponentially attenuated and reflected are called band gaps. In 1987, the idea of a band gap inspired the field of photonics, where the original interest lay in using periodic dielectric materials for localizing and trapping light[11,38]. In 1993 Kushwaha et al. presented the concept of acoustic bands in periodic composites[42] and in 2000 Lui et al. demonstrated this in a sonic crystal as a method to reflect propagating sound[17]. In addition, phononic crystals are being used to enhance control over high frequency sound and heat[43]. In periodic mechanical systems, such as the granular crystal, frequency band gaps have been introduced as a passive filtering mechanism[30,31].

In a monoatomic or homogenous 1D system, in which all particles are identical, there is a single acoustic band. Figure 2.1a shows a schematic of the acoustic band, indicated in red, for a homogenous granular crystal. The acoustic band rises up to a frequency that depends on the coupling stiffness and masses in the lattice. Above this cutoff frequency waves are reflected and do not propagate. The phase velocity, defined as the velocity of each frequency component, $v_p = \omega(k)/k$, decreases for increasing frequency, a consequence of dispersion. The speed of sound of the discrete media is defined by the long-wavelength approximation.

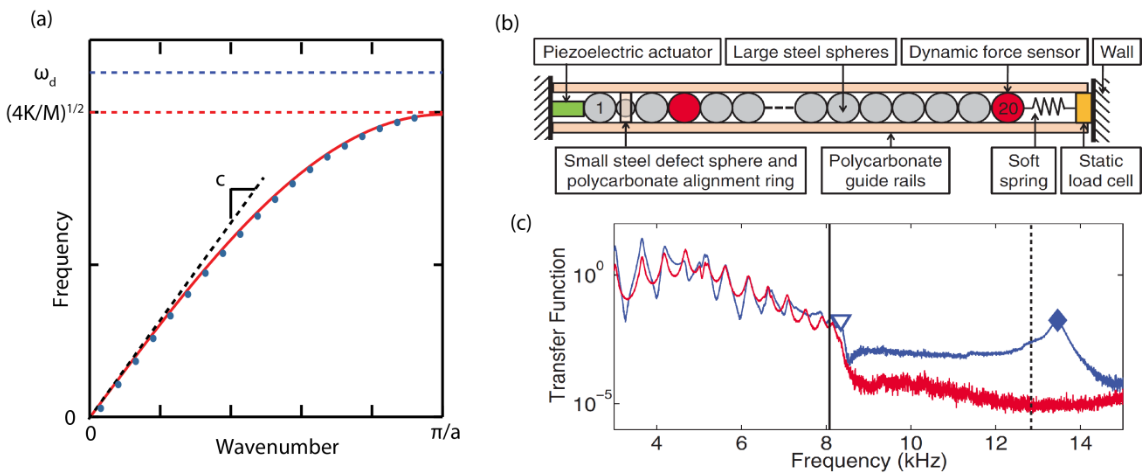

Figure 2.1: a) Schematic for the acoustic band (red) of a monoatomic granular crystal. Effects of broken periodicity, finite size, and a defect mode are shown in blue. The long wavelength speed of sound is shown as the tangent as the wavenumber approaches zero (black). b) Setup by Man et al.[27] to measure (c) the linear acoustic band and localized defect mode in a finite, 20 particle long, chain. Reprinted pending permission from American Physical Society[27], copyright (2012).

Broken periodic symmetry affects the frequency bands primarily in two ways: through finite size and defects. Finite size limits the number of degrees of freedom, so what previously was an acoustic band with an infinite number of modes becomes a collection of a finite number of modes with frequencies up to the acoustic band edge. This is schematically shown in Figure 2.1a. Because the system is no longer infinitely periodic, all waves are not completely reflected. Instead the system filtering rolls off, with the imaginary component of the wavenumber also affected. As the lattice gets smaller the filtering effects are reduced, and the discreetness play a larger role. While Man et al. explored defect modes in a linear finite granular crystal, they also presented the finite size effects in the linear monoatomic chain. The transfer function in Figure 2.1c shows the individual resonant peaks, and the localized defect mode.

Defects have an enormous effect on material properties: electrical conductivity in semiconductors[12], thermal conductivity[13], and mechanical strength[14,15] are just a few examples. These effects may be due to a strain the defect imposes on the lattice, impurity scattering, or even defect-defect interactions. In otherwise periodic materials, defects destroy the lattice's translational symmetry and alter the fundamental modes of the system, which carry propagating wave information. Localized modes may result from a defect in an otherwise periodic media[44-47]. As briefly mentioned above, periodicity leads to frequency band gaps that do not support propagating waves. Therefore, if a mode exists in this forbidden frequency range, it will be spatially confined, i.e., any energy at this frequency cannot travel in either direction and will be locally trapped. In granular crystals, when a mass defect is smaller than the other particles in the lattice it may result in a defect mode, where the mode frequency and spatial profile depend only on the mass ratio between the defect and the rest of the lattice[27]. Man et al. provides an approximation to finding the frequency of the localized mode using a reduced three particle model. In our analysis for local to extended transitions using resonant defects[37], we provide an analytic approach to finding the frequency and spatial profiles for weakly localized waves, in which the three particle approximation breaks down.

Acoustic and mechanical metamaterials have been proposed for protecting structures against low frequency waves, by engineering sub-wavelength structures as local resonances. In these materials the locally resonant structure can be approximated as a frequency dependent effective mass[48]. At certain frequencies the effective mass appears negative and this causes a low frequency band gap to open. This was designed in granular crystals as a proof of concept and has also been shown in other structures with local resonances as a means to control wave speed[19,49-51].

2.2 Weakly nonlinear

Nonlinearity affects a broad range of disciplines and describes the response of real systems, for example, the amplitude dependence of a simple pendulum[52], the reaction-diffusion in biological systems[53], the butterfly effect in weather systems[54], the synchronization between fireflies[55], and the contact law for spheres compressed against each other[22]. Frequently in applications nonlinearity is either essential to how a device works or how it breaks down. Here we look at nonlinear effects and then at how they manifest themselves in nonlinear lattices.

2.2.1 Introduction to nonlinear phenomena

Almost all systems have some sort of frequency dependence. In linear system theory, this is embodied in the transfer function of a system, and in materials science this frequency dependence presents itself in variety of ways: two examples are the band gaps in periodic materials or the plasma frequency of a metal. But until now we have disregarded any notion of a dependence on amplitude. This is where nonlinearity becomes essential.

Nonlinearity describes how a system, e.g., a material's force response, a child's swinging, or a stress wave, depends on amplitude[8]. This is in direct contrast with a linear response, in which the force-displacement, or stress-strain relationship, is amplitude independent. However, when the incremental change in spring constant, stiffness, or modulus are no longer constant but depend on the excitation amplitude, the system is characterized as nonlinear.

The idea of nonlinearity should be somewhat familiar. In mechanics we typically think of this as originating from one of two sources, either a constitutive nonlinear response or a geometric nonlinearity. Rubbers, foams, and carbon nanotube foams[56] present examples of materials with constitutive nonlinear responses. In these examples the material microstructure results in the nonlinearity, and the material does not follow Hooke's Law under compression or tension. For example, a foam's densification causes an overall increase in the stiffness of the material. For contrast, a geometric nonlinearity develops through a change in the geometry. An example of this occurs for a beam in axial compression. As the compression is increased, there is a point where the boundaries can no longer be ignored and the beam would rather bend than undergo further compressive strain[55]. This causes a change in the incremental stiffness. Granular crystals present a geometric nonlinearity when two spheres come into contact[22]. Initially the contact is a point. As the beads are compressed the contact area grows and the stiffness correspondingly increases[22]. Due to either geometric or constitutive effects, nearly all mechanical responses becomes nonlinear at high enough amplitudes. Therefore nonlinear systems and their dynamics are an essential part of our physical world.

But nonlinear systems are also quite difficult to study. Many of the mathematical tools used to study linear systems are no longer available: linear superposition of states does not apply, tools from Fourier analysis which rely on linear transnationally invariant systems are no longer accessible, and in some cases multiple solutions to the same problem exist. Nonlinear systems are relatively difficult to fully characterize and even more difficult to use in an engineering application. However, nonlinearity presents dynamics that are not accessible in linear systems that could allow for new applications and designs.

One particular advantage of nonlinear systems over their linear counterparts is the ability to transfer energy between modes at different frequencies[52,57] . In optical systems this led to sum frequency generation[57] and is used today in mechanical systems for nondestructive evaluation techniques and failure prediction[58,59]. The nonlinearity communication between modes allows them to exchange energy[52], and nonlinear wave mixing has been used for granular crystals for acoustic logic elements[60]. We show how energy transfer between modes can lead to the breakdown of finite size periodic mechanical filters.

Weakly nonlinear dynamics are characterized by a smooth transition from linear dynamics in which a perturbation scheme presents an appropriate method to solving the problem. In dynamical nonlinear systems, such as the classic nonlinear Duffing oscillator or a granular crystal, this can be seen as a slow transition away from the linear dynamics. The system is initially linearized at some point, but as the amplitude of oscillations grow, the potential is not strictly harmonic (i.e., not linear) and higher order terms become important. As oscillation amplitudes grow resonance responses become asymmetric and the amplitude can change rapidly due to small system changes.

2.2.2 Weakly nonlinear lattices

Up until know I have presented nonlinearity and periodicity as two separate phenomena. However in a granular chain, the lattice we study, these two phenomena meet. Before introducing a granular chain, it is important to present a brief overview of the different nonlinear lattices that have been studied, influential problems and results, as well as the approaches to solving these problems, and follow up with a discussion of how our research in granular crystals fits in among these other mechanical lattices.

Initially, interest in nonlinear lattices primarily grew from two distinct events: the first observation of a soliton in 1834 by John Scott Russell, and a surprising computational experiment by Fermi, Pasta, and Ulam more than a hundred years later[61,62]. A soliton is a type nonlinear wave that travels at a constant velocity, maintaining a constant spatial profile[61]. John Scott Russell first observed this type of wave in a shallow water channel and followed the pulse for miles before he finally lost it. This inspired some of the initial work on nonlinear waves[63]. The constant spatial profile of the wave is achieved through a system that has both nonlinearity and dispersion[64]. Because nonlinear lattices contain both these properties they can also support this class of unique waves. The nonlinearity has a tendency for higher amplitude waves to travel faster, causing a steepening of the wave, while dispersion, due to the discreet nature of the lattice, tends to spread out a wave packet. When these two phenomena balance a soliton results. In granular crystals, solitary waves have been extensively studied and were first proposed by Nesterenko using a long wavelength approximation for the discrete granular chain[35].

The computational result by E. Fermi, J. Pasta, and S. Ulam in 1955 examines the energy transferred between the modes of a simple mass spring lattice, with a weakly nonlinear nearest neighbor coupling. They initialized the system with a long-wavelength oscillation, and expected the system to effectively thermalize by distributing its energy over all modes, a consequence of ergodicity. Instead, they found energy transferred to the other modes of the system, but periodically returned back to the initial mode. This was later explained in the context of solitons breaking up and coming back together by Zabusky and Kruskal in 1965[65]. Together, the discovery of solitons and this computational result inspired significant interest in the dynamics of 1-D discrete lattice systems. Important for our purposes, the granular lattice can be approximated by dynamics in a the weakly nonlinear FPU type lattice[66].

However, even in perfectly periodic systems many of the dynamics depend on the underlying lattice. One way nonlinear lattices can differ is due to different coupling potentials, e.g., the 6-12 Lennard Jones Potential, the Morse Potential, or the potential from a magnetic dipole interaction. In the weakly nonlinear limit this results in Taylor expansions with different coefficients. An additional significant differentiating factor is how the nonlinearity enters. In a lattice of pendulums coupled through torsional springs, the nonlinearity appears as a geometric nonlinearity at each pendulum site. As the pendulum swings to larger amplitudes the linear approximation breaks down. The nonlinearity is not in the coupling, but in an onsite potential introduced at each lattice point. This type of lattice can be characterized as a Klein-Gordon lattice and can be written as,

$$\ddot{u}_i = k(u_{i-1} + u_{i+1} - 2u_i) + V'(u_i). \qquad (2.1)$$

Other lattices, such as the Fermi-Pasta-Ulam(FPU) lattices or the Toda lattice, may have no onsite potential but instead the lattice is coupled through nonlinear springs. The granular lattice that we study fits into this second category. These can be written as,

$$\ddot{u}_i = V'(u_{i+1} - u_i) - V'(u_i - u_{i-1}), \qquad (2.2)$$

Where u_i represents the displacement of the particle at lattice site i, k is a coupling parameter, and V is either the onsite potential for the Klein Gordon lattice or the coupling potential for the FPU lattices.

In driven weakly nonlinear crystals, there has been extensive study of physics previously predicted in FPU lattices, including phenomena related to broken lattice periodicity[67]. Marin et al. discuss the effects of finite size on discrete breather instabilities, a type of localized modulation instability in diatomic chains[28,29]. They discuss how shorter lattices increased the magnitude of the instability. Boechler et al. demonstrated discrete breathers in a diatomic granular chain[68], and explored stability in the different dynamical regimes[66]. In diatomic granular chains Hoogeboom et al looked at the hysteresis loop that occurs when driving the system past a bifurcation[69]. Theocharis et al. looked at the stability of nonlinear localized impurity modes that resulted from mass defects[70]. Finally, Boechler et al. used a granular crystal with a local defect mode and broken mirror symmetry to engineer an acoustic rectifier[32]. This design was based on using local resonances to transfer energy to lower frequency propagating modes (Figure 2.2). These dynamics are for the granular crystal in the weakly nonlinear regime with a broken periodicity due to either defects or finite size.

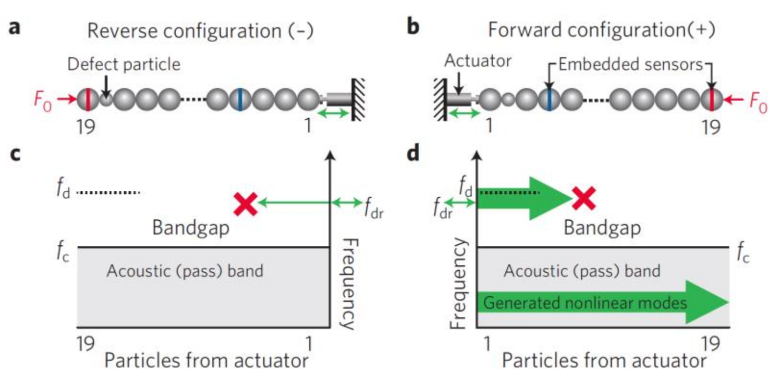

Figure 2.2: Acoustic rectification by Boechler et al.[32] in the weakly nonlinear regime. The nonlinear resonance of a localized defect mode is only excited in one direction due to the broken mirror symmetry. Energy is transferred through nonlinearity to lower frequency propagating modes when the defect mode is excited. Reprinted by permission from Macmillan Publishers Ltd: Nature Materials[32] , copyright (2011).

2.3 Strongly nonlinear

In weakly nonlinear systems, a common approach is to initially linearize the system and then use a perturbation analysis to study the effect of increasing amplitude when nonlinear terms cannot be ignored. In contrast, strongly nonlinear systems cannot be studied through a perturbation scheme. The dynamics are drastically different from the linear case and must be approached differently. In the granular crystal this can easily be seen when the system has no initial pre-compression. The Hertz law is piecewise continuous at this point and is essentially nonlinear. The granular crystal in the absence of pre-compression has no speed of sound, i.e., a sonic vacuum[33]. Nesterenko shows how this leads to the inability for acoustic waves to propagate, but instead energy propagates as highly localized pulses[71,72].

One key advantage of granular crystals is that they present both weak and strong nonlinear dynamic responses[35]. The granular crystal can be tuned to have an effectively linear response (at high pre-compressions), weakly nonlinear response (at intermediate pre-compressions), or essentially strongly nonlinear response (in the absence of pre-compression). This tunability, i.e., the variability of their dynamic response, has attracted significant interest to the granular crystal system, especially in its ability to propagate solitary waves[35,73] and be used as a shock absorbing protection[74]. This has been extended to two dimensional granular crystals as a mechanism for controlling the directionality of wave fronts[75].

In granular crystals with broken periodicity the work can be separated into either driven dynamics or solitary wave propagation interacting with defect or boundaries. Jayaprakash et al. examined the time periodic nonlinear normal modes in finite size, essentially nonlinear granular chains[76]. The exploration of solitary waves interacting with broken periodicity began with Nesterenko studying the breakup of a solitary wave at the boundary between two different essentially nonlinear media[72]. Daraio et al. experimentally investigated this in an array of granular chains adjacent to an array of stainless steel spheres[74,77]. Job et al. have studied how solitary waves in essentially nonlinear granular chains interact with boundaries[78] and localized mass defects[79]. Especially interesting in this case of the mass defect is that because the interactions are strongly nonlinear, the mass defect can oscillate at frequencies higher than the incident wave spectrum.

2.4 Energy harvesting

The nonlinear phenomena that we just explained present a host of advantages over linear systems. They have the ability to transfer energy between frequencies, to show amplitude dependent behavior, and to go through sharp bifurcation transitions. In many energy harvesting these differences could mean new paradigms for harvesting systems with significantly greater efficiency.

Energy harvesting is the practice of converting ambient energy, i.e., energy sources that are nonconventional, small, and or broadband, into a more usable form. In vibrational energy harvesting, ambient vibrations are typically converted into electrical energy by using coupled piezoelectric devices or some other electromechanical coupling[80,81]. While linear systems have been a good first step, recent nonlinear designs suggest leveraging nonlinearity could lead to greater efficiencies.

The traditional approach is to enhance the total energy harvested by matching the linear resonant frequency of the piezoelectric mass system with the resonance of the mechanical structure. This then causes an increase in the strain on the piezoelectric, a voltage across the terminals, and the possibility for energy to be transferred. In this approach the dynamics are completely linear. This allows for easy implementation, but does not necessarily address some of the inherent differences of energy harvesting, for which a linear system is not ideally suited. Roundy talks about the general approach of defining efficiency in electromechanical harvesting systems[81], while Anton and Sodano review the application and implementation of piezoelectric energy harvesting[82].

A main issue in energy harvesting is that sources are not ideal energy sources; they are both not harmonic and not infinite in size. Linear systems are inherently limited by a quality factor bandwidth relationship. When the damping of the system is high the harvesting occurs better over a broader frequency range, but also now has more internal damping. When the damping is low, the energy harvester harvests only from a narrow range of frequencies[83].

Nonlinear systems have the possibility of overcoming this limitation, since in nonlinear systems energy can be transferred between frequencies. This has inspired a broad range of interest in the field of nonlinear energy harvesting. For highly broadband signals, Cottone, Vocca, and Gammaitoni presented an seminal approached based on a bistable duffing oscillator in which noise is converted to higher amplitude motion as the system jumps between two stable minima[84,85]. Here the authors show that this bistable system has the possibility to effectively convert energy from a noise signal to a resonance through stochastic resonance. Although the authors include electromechanical coupling in the equation, the effect is negligible for the parameters chosen. The nonlinear mechanics of similar systems have been subsequently studied[86-89]. A different approach is based on driving a Duffing oscillator into its high amplitude state[90]. This approach relies on tuning the device to follow the resonant frequency of excitation and requires that the electrical coupling be small enough to stay in the high amplitude state.

In general in the nonlinear energy harvesting community, the energy sink that is introduced by electrically coupling the electrical system to the mechanical is swept to the side or ignored. However, this is inherently one of the most important pieces to the problem; damping affects amplitude, and in nonlinear systems, the amplitude affects the state of the device operation and solution. In addition, by ignoring electromechanical coupling, or choosing poor weak back coupling, the energy harvesting device is doing inherently bad. In this case the energy in the mechanical system is not being effectively transferred to the electrical load.

We present two nonlinear mechanical systems to address some of the limitations and problems in vibrational energy harvesting. Both ideas are based on pushing a system into an instability in order to create transient dynamics. When the dynamics of the system are transient and the amplitude of the solution grows, there is a mismatch between the neoconservative forces. This allows us to use electrical dissipation to move the system from an unstable state to a stable one and harvest this additional energy. This is presented in chapter 10.