Chapter 10

CONCLUSIONS & RESEARCH SUMMARY

The granular crystal is a great model system to study dynamics that result from the combination of nonlinearity and broken periodicity. The similarity of the modeling and dynamical equation makes these results easily relatable to other materials systems. In the linear approximation, the granular crystal is a coupled mass spring model, the same as that used to explain the heat capacity and phonon branches in perfect crystals[95,140]. In the nonlinear case, the interaction law can be approximated through a Taylor expansion, which helps extend the physics to other systems with nonlinearity.

In this thesis, we demonstrate new phenomena that result from the combination of nonlinearity and finite size. We show that the local defect mode, introduced from a resonant defect, has a spatial profile that can be tuned using an external compression. This could be used for applications in designing for tunable wave speed propagation. We demonstrate how the stiffness of a material can be tuned at any displacement or strain point. In a granular crystal this tuning pushes the incremental stiffness in the negative direction, however in lattices with other potentials, this could be used to engineer positive infinite stiffness. We show how nonlinearity in finite granular chains allows energy to propagate instead of the system acting as a mechanical filter. This can be used as a way to selectively transfer high frequency energy through lower frequency phonon modes. And finally we demonstrate two approaches to nonlinear energy harvesting, in which the nonlinearity is used to overcome the fundamental limitations in linear systems. We demonstrate how nonlinearity can and should be used when designing materials or systems for targeted properties.

While the granular crystal is a specific example of a nonlinear lattice with broken periodicity, these results apply to a broader range of fields. Our results on tuning a resonant defect mode extend to other condensed matter systems where defect modes are important. For example, in optical systems, the arrangement of defects supporting localized modes has been used to achieve slow group velocities in coupled optical resonant wave guides[50]. One question is, how can a local resonant defect be implemented and controlled in similar optical system to achieve tunable ultraslow group velocity?

Our results of extraordinary tunable stiffness in lattices (chapter 6) shows how the incremental stiffness can be tuned to arbitrary values. This effect would be quite interesting in a nonlinear optical system. Specifically, is it possible to tune the effective dielectric constant of a material to arbitrary values? In our analysis on energy harvesting, we show a defect mode driven at a high frequency to control the effective response at lower frequencies. This is especially interesting in enhanced coupling, where frequencies are not necessarily matched. Could this mechanism enable targeted energy transfer between a high frequency optical mode and lower frequency phonon mode?

The last chapter discussed two future applications of this work in the field of energy harvesting. While the ideas have been inspired by our studies of the physics in nonlinear lattices with broken periodicity, they are presented in simplified nonlinear systems, i.e., a coupled string-cantilever or single parametric oscillator. The models are simplified with the hope that they can be applied to other dynamical systems. The next step for these energy harvesting examples is figuring out how to implement the physics into realistic materials and structural systems. Since many energy sources are stochastic, it is also important to study the dynamics of these physical systems under stochastic excitations.

These are just some of the questions that could be explored in future research projects that follow this research thesis. More generally this research shows that it is important to look at the physics that may be missed when making linear approximations. Oftentimes the nonlinearity provides rich dynamics that have potential in applications.

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