THEORY OF VISCous AND THERMAL ATTENUATION
OF SOUND BY SMALL SPHERES

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ABSTRACT

The problem is to calculate the attenuation of plane sound waves passing through a viscous, heat-conducting fluid containing small spherical inhomogeneities. The attenuation is calculated by evaluating the rate of increase of entropy caused by two irreversible processes: (1) the mechanical work done by the viscous stresses in the presence of velocity gradients, and (2) the flow of heat down the thermal gradients. The method is first applied to a homogeneous fluid with no spheres and shown to give the classical Stokes-Kirchhoff expressions. The method is then used to calculate the additional viscous and thermal attenuation when small spheres are present. The viscous attenuation agrees with Epstein's result obtained in 1941 for a non-heat-conducting fluid. The thermal attenuation is found to be similar in form to the viscous attenuation and, for gases, of comparable magnitude. The general results are applied to the case of water drops in air and air bubbles in water.

For water drops in air the viscous and thermal attenuations are comparable; the thermal losses occur almost entirely in the air, the thermal dissipation in the water being negligible. The theoretical values are compared with Knudsen's experimental data for fogs and found to agree in order of magnitude and dependence on frequency. For air bubbles in water the viscous losses are negligible and the calculated attenuation is almost completely due to thermal losses occurring in the air inside the bubbles, the thermal dissipation in the water being relatively small. (These results apply only to non-resonant bubbles whose radius changes but slightly during the acoustic cycle.)
SYMBOLS*

A = viscous wave potential

\( c \) = normal sound speed

\( c, c_2, c_1 \) = phase velocity of thermal and shear waves

\( c_v, c_p \) = specific heats

f = frequency

\( j_n, h_n \) = spherical Bessel and Hankel functions

\( k, k_2, K \) = wave numbers of acoustic, thermal and viscous waves

\( l, l_2, l_1 \) = damping lengths of acoustic, thermal and viscous waves

n = concentration of scatterers

p = pressure

\( P_{mn} \) = total stress tensor

q = heat current

r = radial coordinate

R = radius of scatterers

s = entropy/unit mass

\( S_{irr} \) = irreversible entropy density

t = time

T = temperature

u = internal energy/unit mass

\( \alpha \) = attenuation coefficient

\( \alpha_v \) = coefficient of volume expansion

\( \gamma \) = ratio of specific heats \((c_p/c_v)\)

\( \eta \) = coefficient of shear viscosity

\( \theta \) = polar angle

\( \bar{\kappa} \) = \( \sigma/\rho_0 c_v \) = thermometric conductivity

\( \lambda \) = \( \sigma/\rho_0 c_v \) = \( \gamma \bar{\kappa} \)

\( \lambda_1, \lambda_2, \lambda_1 \) = wavelength of acoustic, thermal, and viscous waves

\( \mu \) = coefficient of dilatational viscosity

\( \nu = \eta/\rho = \) kinematic viscosity

\( \rho \) = density

\( \sigma \) = thermal conductivity

\( \Gamma_{mn} \) = viscous stress tensor

\( \Phi_1, \Phi_2 \) = acoustic and thermal wave potentials

\( \varphi \) = azimuthal angle

\( \Phi \) = viscous dissipation function

\( \Psi \) = thermal dissipation function

\( \omega = 2\pi f \) = circular frequency

*See Table I for values of the physical constants.*
\( v \) = specific volume \( (\ )_0 \) = unperturbed quantities
\( \nu \) = particle velocity \( (\ )^* \) = complex conjugate
\( (\ )_{AV} \) = time average
\( \Re(\ ) \) = real part

In Sections IV and V unprimed quantities \((\rho, \phi, \ldots)\) refer to the medium outside the small spheres; primed quantities \((\rho', \phi', \ldots)\) refer to the medium inside the spheres; primes on the Bessel and Hankel functions \((j', h')\) indicate differentiation with respect to the argument.
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I. INTRODUCTION

When a plane sound wave passes through a homogeneous fluid it is attenuated by two types of irreversible processes which increase the internal energy of the fluid at the expense of the mechanical acoustic energy of the wave. The first of these is the irreversible mechanical work done by the viscous forces. The second is the irreversible flow of heat down the thermal gradients.

If the fluid contains spherical inhomogeneities the primary sound wave is scattered and the secondary waves are then attenuated by viscosity and thermal conduction. In addition, the secondary waves carry off acoustic energy in all directions, so that the primary wave suffers an additional loss by scattering as well as by enhanced viscous and thermal dissipation.

These dissipative processes lead to an expression for the average energy flux $E$ (the acoustic intensity) of the wave having the form

$$E = E_0 e^{-\alpha x}$$  \hfill (0.1)

where $\alpha$ is the attenuation constant (cm$^{-1}$) and is the sum of the attenuations due to viscosity, heat conduction, and scattering:

$$\alpha = \alpha_\eta + \alpha_\sigma + \alpha_s.$$  \hfill (0.2)

For a plane sound wave in a homogeneous medium Stokes (Ref. 1) first obtained the viscous attenuation

$$\alpha_\eta = \frac{4}{3} \frac{\eta \omega^3}{\rho c^3} = \frac{4}{3} \frac{\nu \omega}{c^2} k_1.$$  \hfill (0.3)
Kirchhoff (Ref. 2) was the first to point out in 1866 that in gases the effects of thermal conductivity are of the same order of magnitude as those of viscosity and should be taken into account in accurate work. Kirchhoff's result for the thermal attenuation was

\[ \alpha = \frac{(\gamma-1)\sigma_0 \omega^2}{\rho_0 c_p c^4} = (\gamma-1) \frac{\omega}{c} \frac{\omega}{c} \cdot k_1. \] (0.4)

Here \( \nu \) is the kinematic viscosity, \( \bar{\kappa} \) the thermometric conductivity, \( c \) the velocity of sound, \( \omega \) the circular frequency and \( k_1 = \omega/c \) the wave number. Thus, for a plane sound wave the attenuation due to viscosity and heat conduction is

\[ \alpha = \left[ \frac{4}{3} \frac{\nu \omega}{c^2} + (\gamma-1) \frac{\bar{\kappa} \omega}{c^4} \right] k_1. \] (0.5)

The scattering of sound from small spheres was first calculated by Lord Rayleigh (Ref. 3) and for fixed rigid spheres his result (including zero- and first-order harmonics) is

\[ \alpha_s = \frac{7}{12} V_p k_1 a_1^3 = \frac{7}{12} V_p R^3 \left( \frac{2\pi}{\lambda} \right)^4, \] (0.6)

where the dimensionless quantity \( V_p \) is the total volume of the small spheres per unit volume of the medium, \( R \) the radius of the spheres and \( \lambda \) the wavelength of the sound. In the second form \( k_1 = 2\pi/\lambda \) is the wave number and \( a_1 = k_1 R \) is very small since \( R \ll \lambda \) for small spheres.

In 1910 Sewell (Ref. 4) calculated the additional viscous dissipation suffered by a plane wave passing through a medium containing small rigid fixed spheres. Sewell's result was
This result, while satisfactory at high frequencies, gives a finite attenuation at the low frequency limit $\omega = 0$. The reason for this erroneous result lies in the assumption that the spheres are fixed; this is not the case in aerosols; as Lamb (Ref. 5) puts it, "Owing to its great inertia in comparison with that of an equal volume of air, a globule of water in suspension, if not too small, may remain practically at rest as the air waves beat upon it. If, however, the radius ($R$) be diminished, the inertia diminishes as $R^3$ while the surface on which viscosity acts diminishes as $R^2$, and it is to be expected that a stage will be reached when the globule will simply drift to and fro with the vibrating air, and so cause little or no loss in energy". In an addendum Sewell tried to correct his results for the oscillatory motion of the particles, but the treatment was not completely satisfactory.

In 1941 Epstein (Ref. 6) published an important theoretical paper dealing with the viscous attenuation of sound by spherical particles suspended in liquids or gases. Epstein employed a more general method than Sewell which automatically included the oscillations of the particles in the acoustic field. This step was partly necessitated by the experimental work of Hartman and Fock (Ref. 15) on aqueous suspensions, since in their work the density ratio of the medium to the particles was near unity, so that Sewell's theory of fixed particles was not at all valid. Epstein considered three cases in which the spherical obstacles were (1) rigid,
(2) viscous fluids, (3) elastic solids, and showed that at sufficiently high frequencies so that the oscillations of the drops could be neglected, Sewell's equation was a close first approximation. In the case of fluid spheres in air Epstein's result was

\[ \alpha = \frac{9}{4} V_p k, \frac{1+y}{y^2} \left[ \frac{16y^4}{16y^4 + 72y^3 + 81y^2(1+2y+2y^2)} \right]. \]  

(0.8)

In view of the relative importance of the thermal effects it is of theoretical interest to extend Epstein's treatment to include the attenuation due to heat conduction. Recent experimental work by Knudsen (Ref. 7) on the attenuation of sound in fogs provides an additional reason for the inclusion of thermal effects. Finally, it may be pointed out that war research on sound transmission in the sea showed that bubbles may contribute strongly to the absorption/sound. As shown by Carstensen and Foldy (Ref. 8) this effect is strongest near the resonant frequency of the bubble; while resonant absorption is not included in the present paper, it is still of considerable interest to examine the viscous and thermal absorption of bubbles for frequencies outside the resonant region.

There are seen to be both theoretical and experimental reasons for investigating the present problem and we now turn to the development of the basic theory underlying the solution.

The essence of the method employed was first given by Prof. P. S. Epstein in his lectures on "Mechanics of Continuous Media". An attempt to carry it through was made in the thesis of W. Garvin (California Institute of Technology 1948). However, Garvin used certain short cuts and simplifications whose validity is not entirely above suspicion. Therefore, it
seemed desirable to reopen the problem and to carry it through in a completely systematic and rigorous manner. The essential point in the new approach lies in the use of the entropy associated with the irreversible processes.
II. THE BASIC THEORY

In this chapter we shall discuss the fundamental conservation equations for a compressible, viscous, heat-conducting fluid. From these equations an entropy equation is derived in which both reversible and irreversible processes are represented. The rate of increase of irreversible entropy depends on velocity and thermal gradients and leads to expressions for the rate of increase of the internal energy at the expense of the kinetic and potential energy of the fluid motion. It is thus possible to develop expressions for the viscous dissipation, corresponding to irreversible mechanical work done by viscous forces, and for the thermal dissipation, corresponding to the irreversible flow of heat down thermal gradients.

Although the mass, momentum, and energy equations to be derived are found in the literature (c.f. Busemann, Ref. 14), the entropy equation is not usually given, and it seems useful to derive all four basic equations in a systematic way. The development will also present a number of relations of use in later sections.

Assumptions

We shall make the following simplifying assumptions in developing the theory:

(1) The viscosity $\eta$, thermal conductivity $\sigma$, specific heats, etc., will be assumed constant, and in particular they will be considered independent of the temperature.

(2) Because of recent work by Liebermann (Ref. 13) involving the role of the compressional (or "second") coefficient of viscosity in absorption of sound, we shall not make the usual assump-
tion that this coefficient vanishes, but instead shall retain it throughout our analysis.

(3) We shall assume the medium to be in a "pseudo"-steady state so that the gradual changes in the unperturbed temperature, pressure, etc., brought about by the increasing internal energy (due to absorption) can be neglected over the time of interest for which the average dissipation is calculated.

(4) We shall use the usual form of the Newtonian stress tensor, modified to include the second coefficient of viscosity (Ref. 14). This assumption is an approximation, since it excludes thermal stresses which a careful statistical-kinetic theory would include in the stress tensor. For gases, however, the error is very small (~ 1%), so that in view of the additional analytical complexity which would be introduced by the thermal stress terms, it seems entirely reasonable to omit them.

We now choose a Cartesian inertial system \( x_\alpha (\alpha = 1, 2, 3) \) with respect to which the fluid particle velocity is \( \mathbf{v} \) with components \( v_\alpha \). The position of any fluid element at any time \( t \) will be given by its Eulerian coordinates \( x(t) \). To derive the basic equations in the simplest way we consider a volume \( V' \) of the fluid whose surface \( S' \) has at every point the velocity of the fluid at that point. Thus \( V' \) moves with the fluid and since no fluid crosses its surface, \( V' \) contains a constant mass of fluid.

The total mass within $V'$ is then

$$M = \int \rho \, dV'$$

and the conservation law then assumes the form

$$\frac{dM}{dt} = \frac{d}{dt} \int \rho \, dV' = 0$$

or

$$\int \frac{\partial \rho}{\partial t} \, dV' + \int \rho \mathbf{u} \cdot dS' = 0$$

The first term represents the rate of mass increases arising from the time dependence of $\rho$. The second term is the rate of mass increase due to the changing volume of integration. Using the divergence theorem we obtain

$$\int \left[ \frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{u} \right] dV' = 0$$

and since $V'$ is an arbitrary volume the integrand must vanish and we have the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{u} = 0$$

We define the stream derivative

$$\frac{D}{Dt}(\cdot) = \frac{\partial}{\partial t}(\cdot) + \mathbf{u} \cdot \nabla (\cdot)$$
and obtain a useful alternate form of the continuity equation:

Mass

$$\frac{dP}{dt} + \rho \nabla \cdot \mathbf{v} = 0.$$  \hfill (1.7)

We note a useful relation following directly from (1.5): If $f$ is any scalar quantity

$$\rho \frac{df}{dt} = \frac{\partial \rho f}{\partial t} + \nabla \cdot (\rho f \mathbf{v}).$$  \hfill (1.8)

2. Conservation of Momentum (Force Equation)

The vector momentum of the fluid in the volume $V'$ is

$$\mathbf{M} = \int \rho \mathbf{v} \, dV'$$  \hfill (2.1)

or, using Cartesian tensor notation,

$$\mathbf{M}_e = \int \rho v_e \, dV'$$  \hfill (2.2)

and thus the rate of change of momentum is

$$\frac{dM_e}{dt} = \frac{d}{dt} \int \rho v_e \, dV'$$  \hfill (2.3)

$$= \int \frac{\partial}{\partial t} (\rho v_e) \, dV' + \int \rho v_e v_\alpha \, dS'_\alpha$$  \hfill (2.4)

$$= \int \left[ \frac{\partial}{\partial t} \rho v_e + \nabla_\alpha (\rho v_e v_\alpha) \right] \, dV'.$$  \hfill (2.5)

In the last form (and hereafter) we use the Einstein convention and sum over doubled Greek subscripts only. We now use (1.8) with $f = v_e$ and obtain
\[
\frac{dM_e}{dt} = \frac{d}{dt} \int \rho v_e \, dV' = \int \rho \frac{Dv_e}{Dt} \, dV'. \tag{2.6}
\]

The last equality is a particular example of a useful general theorem proved in the same manner as the derivation just made:

**Theorem** If \( f \) is any scalar function and \( V' \) is a volume moving with the fluid and containing a constant mass of fluid then

\[
\frac{d}{dt} \int \rho f \, dV' = \int \rho \frac{Df}{Dt} \, dV'. \tag{2.7}
\]

We must now formulate the force acting on the fluid in \( V' \). We assume that there are no body forces so that only the stress forces are acting and the total force components are

\[
F_e = \int dS^\alpha \rho_{\alpha\varepsilon} = \int \nabla_\alpha \rho_{\alpha\varepsilon} \, dV', \tag{2.8}
\]

where \( \rho_{\alpha\varepsilon} \) is the symmetric stress tensor. Then the force equation states that

\[
F_e = \frac{dM_e}{dt} \tag{2.9}
\]

or using (2.8) and (2.6)

\[
\int \rho \frac{Dv_e}{Dt} \, dV' = \int \nabla_\alpha \rho_{\alpha\varepsilon} \, dV', \tag{2.10}
\]

and since \( V' \) is arbitrary we obtain
\[
\rho \frac{D v_\epsilon}{D t} = \nabla_\alpha p_{\alpha \epsilon}
\]

which is the equation of motion.

Now the total stress tensor \( p_{\alpha \epsilon} \) for an isotropic homogeneous fluid is given by

\[
p_{\alpha \epsilon} = - \rho \delta_{\alpha \epsilon} + T_{\alpha \epsilon}
\]

with

\[
\delta_{\alpha \epsilon} = \begin{cases} 
0 & (\alpha \neq \epsilon) \\
1 & (\alpha = \epsilon), 
\end{cases}
\]

the viscous stress tensor \( T_{\alpha \epsilon} \) by

\[
T_{\alpha \epsilon} = -\frac{2}{3} (\eta - \mu) (\nabla \cdot v) \delta_{\alpha \epsilon} + 2\eta \epsilon_{\alpha \epsilon},
\]

and the velocity strain tensor \( \epsilon_{\alpha \epsilon} \) (in Cartesian coordinates) by

\[
\epsilon_{\alpha \epsilon} = \frac{1}{2} (\nabla_\alpha v_\epsilon + \nabla_\epsilon v_\alpha)
\]

with \( \eta \) the shear coefficient of viscosity and \( \mu \) the compressional ("second") coefficient of viscosity. The components of the stress tensor are thus given by

\[
p_{mn} = \eta (\nabla_m v_n + \nabla_n v_m) \quad (m \neq n)
\]

\[
p_{nn} = -\rho - \frac{2}{3} (\eta - \mu) (\nabla \cdot v) + 2\eta \nabla_n v_n
\]

where, in accordance with our summation convention, (2.16) is not summed
on n.

Equation (2.11) with (2.12) now gives the usual force equation:

\[ \rho \frac{Dv_e}{Dt} = -\nabla \cdot p + \nabla \cdot \tau_{de} \]  

(2.17)

Substituting \( \tau_{de} \) from (2.13) and \( \epsilon_{de} \) from (2.14) we obtain

\[ \nabla \cdot \tau_{de} = -\frac{2}{3} (\eta - \mu) \nabla \cdot (v \cdot v) + \eta \nabla \cdot (\nabla \cdot v) + \eta \nabla \cdot (v \cdot v) \]  

(2.18)

\[ \nabla \cdot \tau_{de} = -\frac{2}{3} (\eta - \mu) \nabla \cdot (v \cdot v) + \eta \nabla^2 v + \eta \nabla \cdot (v \cdot v) \]  

(2.19)

and the equation of motion, written in vector form is

\[ \rho \frac{Dv}{Dt} = -\nabla \cdot p - \frac{2}{3} (\eta - \mu) \nabla \cdot (v \cdot v) + \eta \nabla^2 v + \eta \nabla \cdot (v \cdot v) \]  

(2.20)

Now using the vector relation

\[ \nabla \times \nabla \times v = \nabla \left( v \cdot v \right) - \nabla^2 v \]

we obtain the force equation in its final form:

\[ \text{Momentum} \quad \rho \frac{Dv}{Dt} = -\nabla \cdot p + \frac{4N}{3} \eta \nabla \cdot (v \cdot v) - \eta \nabla \times \nabla \times v \]  

(2.21)

where the constant \( N \) in the second term is defined by

\[ N = 1 + \mu / 2 \eta \]  

(2.22)

so that when the second coefficient of viscosity vanishes we have \( N = 1 \) and the force equation reduces to the standard form usually given in the literature.
3. Conservation of Energy (First Law of Thermodynamics)

To formulate the energy equation we assume that heat is added to the volume $V'$, the heat current $q$ being given by

$$q = -\sigma \nabla T.$$  \hfill (3.1)

Then the total rate of heat addition is

$$Q = \int q \cdot dS' = \int \sigma \nabla T \cdot dS' = \int \sigma \nabla^2 T \, dV'.$$  \hfill (3.2)

Now the total energy of the fluid in $V'$ is the sum of the total internal energy $U$ and total kinetic energy $K$:

$$U + K = \int \rho u \, dV' + \int \frac{1}{2} \rho v^2 \, dV',$$  \hfill (3.4)

and using the theorem (2.7) we obtain the rate of increase of the total energy of the fluid in $V'$:

$$\frac{dU}{dt} + \frac{dK}{dt} = \int \rho \frac{Du}{dt} \, dV' + \int \rho \frac{Dv^2/2}{dt} \, dV'.$$  \hfill (3.5)

Finally, the rate at which the fluid inside $V'$ does work on the outside medium is

$$W = -\int dS' \rho \alpha \epsilon u \, \dot{v}_\epsilon = -\int \nabla (\rho \alpha \epsilon \dot{v}_\epsilon) \, dV'.$$  \hfill (3.6)

Then the conservation of energy requires that the rate at which energy (heat) is added to $V'$ be equal to the rate of energy increase of the fluid in $V'$ plus the rate at which this fluid does work:

$$Q = \frac{dU}{dt} + \frac{dK}{dt} + W.$$  \hfill (3.7)

or, using the above equations,

$$\sigma \nabla^2 T = \rho \frac{Du}{dt} + \rho \frac{Dv^2/2}{dt} - \nabla (\rho \alpha \epsilon \dot{v}_\epsilon).$$  \hfill (3.8)
Now multiplying the force equation (2.11) by $v_\epsilon$ and summing we get

$$\rho \frac{Dv^2}{Dt} = v_\epsilon \nabla_\alpha \mathbf{p}_\alpha \epsilon$$

and this together with (3.8) gives

$$\sigma \nabla^2 T = \rho \frac{Du}{Dt} - \mathbf{p}_\alpha \epsilon \nabla_\alpha v_\epsilon.$$  

Finally, using the definition of $\mathbf{p}_\alpha \epsilon$ we get the energy equation as usually given in the literature:

$$\sigma \nabla^2 T = \rho \frac{Du}{Dt} + \rho \nabla \cdot \mathbf{v} - \Phi$$

where the viscous dissipation function $\Phi$ is defined by

$$\Phi = \tau_\alpha \epsilon \nabla_\alpha v_\epsilon$$

and will be discussed later. A more useful form of the energy equation for our purposes is obtained by defining the specific volume

$$Y = \frac{1}{\rho}$$

so that, using the continuity equation (1.7), we have

$$\rho \frac{Dv}{Dt} = \frac{1}{Y} \frac{DY}{Dt} = - \frac{1}{\rho} \frac{DP}{Dt} = \nabla \cdot \mathbf{v}.$$  

From (3.11) we then obtain the desired form of the energy equation:

**Energy**

$$\sigma \nabla^2 T = - \nabla \cdot \mathbf{q} = \rho \frac{Du}{Dt} + \rho \frac{Dv}{Dt} - \Phi.$$  

4. Equations of State and Internal Energy

We have now derived three basic conservation equations for mass momentum, and energy: two scalar equations and one vector equation, or a total of five scalar equations. Seven variables appear in these equations: $\rho, v_1, v_2, v_3, p, u, T$. We therefore require two more equations to complete our set of basic equations. These are furnished by the thermodynamic equation of state and the specification of the internal energy ($u$) which we assume to be given
in the form

\[ p = \rho(p, T) \]  \hspace{1cm} (4.1)

and

\[ u = u(p, T) \]  \hspace{1cm} (4.2)

In the case of a perfect gas these become

\[ p = \rho \frac{RT}{\gamma-1} \]  \hspace{1cm} (4.3)

\[ u = c_v T \]  \hspace{1cm} (4.4)

where \( R = c_p - c_v \) is the gas constant/gm, \( c_p, c_v \) the specific heats/gm and \( \gamma = c_p/c_v \).

It may be pointed out here that the entropy equation (derived in the next section) does not provide another independent equation since it involves the new variable \( s \) (entropy/gm). If we include the entropy equation we then have eight variables and eight scalar equations which we summarize schematically as follows:

- \( \rho \) — Continuity Equation (Scalar)
- \( \mathbf{u} \) — Force Equation (Vector)
- \( T \) — Energy Equation (Scalar)
- \( p \) — State Equation (Scalar)
- \( u \) — Internal Energy Function (Scalar)
- \( s \) — Entropy Equation (Scalar).

5. Entropy Equation (Second Law of Thermodynamics)

Following Eckart (11) and Tolman (12) we express the second law of thermodynamics as an equality, using the concept of irreversible entropy:

\[ \frac{d}{dt} \int \rho s \, dV' = -\int \frac{1}{T} \mathbf{q} \cdot d\mathbf{S}' + \int \frac{D\mathbf{s}_{irr}}{Dt} \, dV'. \]  \hspace{1cm} (5.1)
The left term is the rate of increase of the total entropy within $V'$, $s$ being the entropy per unit mass of fluid. The right side is the sum of (a) the rate of entropy increase due to the entropy current ($\frac{1}{T} \mathbf{u}$) flowing in through the surface, and (b) the rate of increase of entropy throughout the volume due to irreversible processes, $\frac{D s_{irr}}{Dt}$ being the rate of production of "irreversible entropy" per unit volume of the fluid. In accordance with the second law as usually stated we require this quantity to be positive.

Using the theorem (2.7) on the left side and converting the surface integral to a volume integral we have

$$\int \rho \frac{D s}{Dt} dV' = \int -\nabla \cdot \left( \frac{1}{T} \mathbf{q} \right) dV' + \int \frac{D s_{irr}}{Dt} dV',$$

and since the volume $V'$ is arbitrary we obtain the entropy equation:

**Entropy**

$$\rho \frac{D s}{Dt} = -\nabla \cdot \left( \frac{1}{T} \mathbf{q} \right) + \frac{D s_{irr}}{Dt}.$$  \hspace{1cm} (5.3)

Now we substitute for the rate of change of the entropy density of the fluid, as a consequence of its known dependence on the internal energy and volume of the fluid, the expression

$$\frac{D s}{Dt} = \frac{1}{T} \frac{Du}{Dt} + \frac{p}{T} \frac{D V}{Dt}.$$  \hspace{1cm} (5.4)

But from the energy equation (3.15) the right side may be replaced and we obtain

$$\rho \frac{D s}{Dt} = \frac{1}{T} \left[ \rho \frac{Du}{Dt} + \rho \frac{p}{T} \frac{D V}{Dt} \right] = \frac{1}{T} \left[ -\nabla \cdot \mathbf{q} + \Phi \right].$$  \hspace{1cm} (5.5)

Comparing this with (5.3) we then have an expression involving $\frac{D s_{irr}}{Dt}$:

$$\frac{D s_{irr}}{Dt} = \nabla \cdot \left( \frac{1}{T} \mathbf{q} \right) - \frac{1}{T} \nabla \cdot \mathbf{q} + \frac{1}{T} \Phi$$  \hspace{1cm} (5.6)

$$= -\frac{1}{T^2} \mathbf{q} \cdot \nabla T + \frac{1}{T} \Phi,$$  \hspace{1cm} (5.7)
and using the definition of the heat current $q$,

$$q = -\sigma \nabla T$$  \hspace{1cm} (5.8)

we obtain the desired expression for the rate of production of irreversible entropy per unit volume:

$$\frac{DS_{irr}}{Dt} = \sigma \left( \frac{\nabla T}{T} \right)^2 + \frac{1}{T} \Phi . \tag{5.9}$$

The first term on the right is the rate of irreversible entropy production associated with the thermal conduction due to the presence of temperature gradients. Since $\sigma$ is positive, this term is clearly positive, so that it is in accordance with the requirements of the second law as usually stated, i.e., that the entropy associated with irreversible processes must always increase.

The second term on the right is the rate of irreversible entropy production associated with the work done by the viscous forces due to the presence of velocity gradients. Moreover it is easily shown from the definitions of $\Phi$ and $e_{\alpha\varepsilon}$ that

$$\Phi = \tau_{\alpha\varepsilon} e_{\alpha\varepsilon} \tag{5.10}$$

$$= 2\eta \left[ e_{\alpha\varepsilon} e_{\varepsilon\alpha} - \frac{2}{3} (\eta - \mu) (\nabla \cdot \mathbf{v}) \delta_{\alpha\varepsilon} e_{\alpha\varepsilon} \right] \tag{5.11}$$

$$= 2\eta \left[ e_{\alpha\varepsilon}^2 + e_{\varepsilon\alpha}^2 + e_{33}^2 \right] + 4\eta \left[ e_{23}^2 + e_{31}^2 + e_{12}^2 \right] - \frac{2}{3} (\eta - \mu) (\nabla \cdot \mathbf{v})^2 \tag{5.12}$$

or

$$\Phi = 2\eta \left[ (e_{\alpha\varepsilon} + a)^2 + (e_{\varepsilon\alpha} + a)^2 + (e_{33} + a)^2 + 2(e_{23}^2 + e_{31}^2 + e_{12}^2) \right] \tag{5.13}$$

with

$$a = \frac{1}{3} \left[ \left( \frac{\mu}{\eta} \right)^{1/2} - 1 \right] \nabla \cdot \mathbf{v} \tag{5.14}$$

$$\nabla \cdot \mathbf{v} = e_{\alpha\alpha} = e_{11} + e_{22} + e_{33} \tag{5.15}$$

so that, since $\eta$ and $T$ are positive we see from (5.13) that the second term
of (5.9) is also always positive.

It is to be noted that we have not proved the second law, but have only shown that if we assume the above definition of the stress tensor (2.12) and assume that $\sigma, \eta, T$ are positive, then the rate of production of irreversible entropy $D_{s_{irr}}/Dt$ is indeed positive. To prove the second law we would have to justify the above assumptions regarding $p_\infty, \sigma, \eta, T$, by a careful statistical-kinetic treatment, including also any effects of thermal stresses.

6. The Viscous and Thermal Dissipation

From the results of the preceding section we have seen that the presence of velocity and thermal gradients in a viscous heat conducting fluid leads to a rate of production of irreversible entropy per unit volume given by $D_{s_{irr}}/dt$ in (5.9). From this we obtain the total rate of dissipation of energy per unit volume $dE/dt$ due to irreversible processes:

$$\frac{dE}{dt} = T \frac{D_{s_{irr}}}{dt} = \Psi + \Phi$$

(6.1)

where

$$\Psi = \frac{\sigma}{T} (\nabla T)^2$$

(6.2)

and $\Phi$ is given by (3.12).

We shall call $\Psi$ the thermal dissipation function and $\Phi$ the viscous dissipation function. $\Psi$ and $\Phi$ represent respectively the rate of increase of the internal energy per unit volume of the fluid due to (a) irreversible heat flow in the presence of the thermal gradients and (b) irreversible mechanical work done by the viscous forces in the presence of the velocity gradients. This increase in the internal energy is at the expense of the ordered macroscopic energy of the heat currents and the mechanical flow energy of the fluid; this fact justifies the use of the term "dissipation".
If we now consider a volume \( V \) fixed in space we obtain the rate of dissipation of the acoustic energy of a sound wave sweeping over the volume:

\[
\frac{dE_\eta}{dt} = \int \Phi \, dV \quad (6.3)
\]

\[
\frac{dE_\sigma}{dt} = \int \Psi \, dV \quad (6.4)
\]

where \( dE_\eta /dt \) is the total viscous dissipation and \( dE_\sigma /dt \) is the total thermal dissipation, both in units of energy/sec. (We have dropped the prime in the volume element \( dV \) since the volume of integration \( V \) is fixed in space rather than moving with the fluid).

It is particularly to be noted that since \( \Psi \) and \( \Phi \) are quadratic, we need only solve the first-order equations of the acoustic field in order to obtain \( \Psi \) and \( \Phi \) to second order.

This completes our derivation of the general basic equation for a viscous heat-conducting fluid. We now turn to the acoustic case and apply these results.
III. THE SOUND FIELD

In this chapter we shall consider the general acoustic equations, obtained as the first-order approximations to the basic equations of Chapter II. From these equations we shall derive the wave equations for the acoustic, thermal, and shear waves and discuss their solutions. Finally we shall list some useful auxiliary relations relating the first-order pressure and temperature with the velocity potentials.
7. The Linearized Equations

The general first-order acoustic equations are obtained by linearizing the basic conservation equations for mass, momentum and energy. We expand our variables to first order:

\[
\begin{align*}
\rho &= \rho_o + \rho_1 \\
\mathbf{v} &= \mathbf{v}_o + \mathbf{v}_1 \\
\mathbf{p} &= \mathbf{p}_o + \mathbf{p}_1 \\
T &= T_o + T_1 \\
u &= u_o + u_1
\end{align*}
\] (7.1)

where the \(o\)-subscript refers to the unperturbed quantities in the absence of the sound field (\(\mathbf{v} = 0\)), and the \(1\)-subscript denotes the first-order quantities. We assume the fluid to be at rest in the unperturbed state so that \(\mathbf{v}_o = 0\). Substituting these quantities in the basic equations (1.7), (2.21), and (3.15) and noting that to first order \(\mathbf{D} = 0\) and \(\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t}\) (there are no transport phenomena since the term \(\mathbf{v} \cdot \nabla \mathbf{v}\)) is second-order), we obtain

Continuity Equation:

\[
\frac{\partial \rho_1}{\partial t} + \rho_o \mathbf{v} \cdot \nabla \mathbf{v}_1 = 0
\] (7.2)

Force Equation:

\[
\rho_o \frac{\partial \mathbf{v}_1}{\partial t} = -\nabla p_1 + \frac{4N}{3} \eta \nabla (\mathbf{v} \cdot \mathbf{v}) - \eta \nabla \times \nabla \times \mathbf{v}_1
\] (7.3)

Energy Equation:

\[
\sigma \nabla^2 T_1 = \rho_o \frac{\partial u_1}{\partial t} + \rho_o \rho_o \frac{\partial v_1}{\partial t}
\] (7.4)

State Equation:

\[
\mathbf{p}_1 = \mathbf{p}_1 (\mathbf{v}_1, T_1)
\] (7.5)
Internal Energy Equation:

\[ u_i = u_i(v_i, T_i) \quad (7.6) \]

We now eliminate \( p_1, u_1, p_1 \) from the equations, to obtain two equations in the two fundamental variables \( v_i \) and \( T_i \).

To eliminate \( u_1 \) we differentiate \( (7.6) \) (using \( \dot{u} \) to denote \( \partial u / \partial t \)) to obtain

\[ \dot{u}_i = \left( \frac{\partial u}{\partial T_i} \right)_v \dot{T}_i + \left( \frac{\partial u}{\partial v_i} \right)_T \dot{v}_i \quad (7.7) \]

and substitute this into the energy equation:

\[ \sigma \nabla^2 T_i = \rho_0 \left( \frac{\partial u}{\partial T_i} \right)_v \dot{T}_i + \rho_0 \left[ \frac{\partial u}{\partial v_i} + \left( \frac{\partial u}{\partial T_i} \right)_T \right] \dot{v}_i \quad (7.8) \]

To eliminate \( p_1 \) from the force equation we differentiate \( (7.5) \) and use the continuity equation for \( p_1 \):

\[ \dot{p}_i = \left( \frac{\partial p}{\partial v_i} \right)_T \dot{v}_i + \left( \frac{\partial p}{\partial T_i} \right)_v \dot{T}_i \quad (7.9) \]

so

\[ \dot{p}_i = -\rho_0 \left( \frac{\partial p}{\partial v_i} \right)_T \nabla v_i + \left( \frac{\partial p}{\partial T_i} \right)_v \dot{T}_i \quad (7.10) \]

Now we differentiate the force equation with respect to \( t \) and substitute \( (7.10) \) to obtain

\[ \rho_0 \ddot{v}_i = \rho_0 \left( \frac{\partial p}{\partial v_i} \right)_T \nabla (v_i) - \left( \frac{\partial p}{\partial T_i} \right)_v \dot{T}_i + \frac{4N}{b} \nabla \nabla (v_i) - \nabla \times \nabla \times \nabla v_i. \quad (7.11) \]

Equations \((7.8)\) and \((7.11)\) are the two desired equations in \( v_i \) and \( T_i \).

We now note the definitions

\[ c_v = \left( \frac{\partial u}{\partial T_i} \right)_v \quad (7.12) \]

\[ \gamma_v = \frac{1}{v_0} \left( \frac{\partial v}{\partial T_i} \right)_v, \quad -\frac{1}{\rho_0} \left( \frac{\partial p}{\partial T_i} \right)_v \quad (7.13) \]

\[ c^2 = \left( \frac{\partial p}{\partial p} \right)_s = \gamma \left( \frac{\partial p}{\partial T} \right)_T \quad (7.14) \]
for the specific heat at constant volume \( c_v \), the volume coefficient of expansion at constant pressure \( \alpha_v \) and the adiabatic velocity of sound \( c \). We then use the general thermodynamic relations

\[
\left( \frac{\partial p}{\partial T} \right)_V = \frac{1}{\gamma} \left( \frac{\partial p}{\partial T} \right)_P = -\left( \frac{\partial p}{\partial T} \right)_P \left( \frac{\partial p}{\partial T} \right)_P
\]  

(7.15)

\[
\gamma \left( \frac{\partial p}{\partial T} \right)_V = \rho_0 + \left( \frac{\partial u}{\partial \rho} \right)_T
\]  

(7.16)

and the above definitions to obtain

\[
\left( \frac{\partial p}{\partial T} \right)_P = \frac{1}{\gamma} c^2 \rho_0 \alpha_v
\]  

(7.17)

\[
\gamma \left( \frac{\partial u}{\partial \rho} \right)_T = T_0 \frac{1}{\gamma} c^2 \rho_0 \alpha_v.
\]  

(7.18)

Substituting these relations in equations (7.8) and (7.11) we obtain the working forms of the velocity and energy equations:

\[
\ddot{v}_i = \frac{c^2}{\rho} \nabla (\rho \dot{v}_i) - \frac{c^2 \alpha_v}{\rho} \rho \dot{T}_i + \frac{4N}{3} \rho \nabla (\rho \dot{v}_i) - \nabla \times \nabla \times \dot{v}_i
\]  

(7.19)

\[
\kappa \nabla^2 T_i = \dot{T}_i + \frac{T_0 c^2 \alpha_v}{c_p} \nabla \cdot \dot{v}_i
\]  

(7.20)

where \( \kappa = \sigma / \rho_0 c_v \), and \( \nu = n / \rho_0 \) is the kinematic viscosity, and we have used \( \rho_0 \dot{v}_i = \nabla \cdot \dot{v}_i \) from (3.14).

We now introduce the periodic time dependence of all first order quantities:

\[
\rho_i = \rho e^{-i \omega t}
\]

\[
\dot{v}_i = \dot{v} e^{-i \omega t}
\]

\[
\rho_0 \dot{v}_i = \nabla \cdot \dot{v}_i
\]

\[
T_i = T e^{-i \omega t}
\]

\[
\rho_0 \dot{v}_i = \nabla \cdot \dot{v}_i
\]  

(7.21)
Here $p, \mathbf{v}, p, T, u$ are pure space functions independent of time*.

Noting that $\frac{\partial}{\partial t} = -i\omega$ we obtain from (7.19) and (7.20) two space equations for $\mathbf{v}$ and $T$:

$$-i\omega \mathbf{v} = -\frac{c^2\omega}{\gamma} \mathbf{v} T + \left( -\frac{c^2}{i\omega \gamma} + \frac{4N}{3} \right) \nabla(\mathbf{v} \cdot \mathbf{v}) - \nabla \times \nabla \times \mathbf{v}$$  \hspace{1cm} (7.22)

$$\nabla^2 T = -i\omega T + \frac{T_0 c^2 \omega}{C_p} \mathbf{v} \cdot \mathbf{v}$$  \hspace{1cm} (7.23)

Our problem is now to solve these two space equations for $\mathbf{v}$ and $T$. It may be noted that the first-order quantities are all real; for convenience, however we have assumed them to be general complex functions in (7.21), so that after (7.22) and (7.23) have been solved for $\mathbf{v}$ and $T$ we must use for the first-order quantities $\mathbf{v}_1$ and $T_1$ the real part of the quantities in (7.21).

8. Derivation and Solution of Wave Equations for the Acoustic, Thermal and Viscous Waves

In order to solve the two space equations (7.22) and (7.23) we assume for $\mathbf{v}$ the general vector field

$$\mathbf{v} = -\nabla \phi + \nabla \times \mathbf{A}$$  \hspace{1cm} (8.1)

where $\phi$ is a scalar potential and $\mathbf{A}$ a vector potential, and where we take $\nabla \cdot \mathbf{A} = 0$ without loss of generality. The force equation (7.22) becomes

$$i\omega \nabla \phi - i\omega \nabla \times \mathbf{A} = -\frac{c^2\omega}{\gamma} \mathbf{v} T + \left[ \frac{c^2}{i\omega \gamma} - \frac{4N}{3} \right] \nabla(\nabla \phi) + \nabla \times (\nabla^2 \mathbf{A})$$  \hspace{1cm} (8.2)

or, equating the vector and scalar parts separately we get

---

*The use of these symbols for the amplitudes of the first-order quantities will cause no confusion with Chapter II since throughout the remainder of the work we shall use only the first-order quantities.
The energy equation (7.23) becomes

\[ \text{Re} \nabla^2 T = -i\omega T - T_0 \frac{c^2 \omega v}{c_p} \nabla^2 \phi. \]  

(8.4)

From (8.3) we obtain \( T \) as a function of the scalar velocity potential:

\[ T = -\frac{i\omega \phi}{c^2 \omega v} + \left( \frac{1}{i\omega \omega v} - \frac{4N\omega^3}{3c^2 c^2 v} \right) \nabla^2 \phi \]

(8.5)

and by substituting this into (8.4) we eliminate \( T \) and obtain the biquadratic or double wave equation in \( \phi \):

\[ \frac{c^4}{\omega^4} \left[ \frac{4N\omega^4}{3c^4} + i \frac{2\omega^2}{c^2} \right] \nabla^4 \phi - \frac{c^2}{\omega^2} \left[ \frac{1}{i\left( \frac{4N\omega^4}{3c^4} + \frac{2\omega^2}{c^2} \right)} \right] \nabla^2 \phi - \phi = 0 \]

(8.6)

We now make the substitutions

\[ e = \frac{4N}{3} \frac{\omega v}{c^2} \ll 1, \quad f = \frac{2\omega}{c^2} \ll 1 \]

(8.7)

where for most fluids the magnitudes of the dimensionless quantities \( e \) and \( f \) are generally very small for values of \( \omega < 10^6 \). (See Table I for the values of the various constants for water and air.) We then have to solve the biquadratic equation

\[ \frac{c^4}{\omega^4} \left[ ef + if/v \right] \nabla^4 \phi - \frac{c^2}{\omega^2} \left[ 1 - i(e+f) \right] \nabla^2 \phi - \phi = 0 \]

(8.8)

**Solution of Biquadratic Equation**

A solution of (8.8) may be obtained in the form

\[ \phi = \phi_1 + \phi_2 \]

(8.9)

where \( \phi_1 \) and \( \phi_2 \) are wave functions satisfying the equations...
\[ \nabla^2 \phi_1 + k_1^2 \phi_1 = 0, \quad \nabla^2 \phi_2 + k_2^2 \phi_2 = 0 \quad (8.10) \]

and where \( k_1^2 \) and \( k_2^2 \) are the two roots of an equation obtained from (8.8) as follows: We substitute

\[ \nabla^2 \phi + k^2 \phi = 0 \quad (8.11) \]

in (8.8) and get

\[ \left\{ \frac{c^4}{\omega^4} k^4 [e f + i f / \gamma] + \frac{c^2}{\gamma^2} k^2 \left[ 1 - i (e + f) \right] - 1 \right\} \phi = 0. \quad (8.12) \]

Since \( \phi \) is not identically zero the curly bracket must vanish, so that, putting \( x = (c k / \omega)^2 \), we must solve the quadratic in \( x \) (or \( k^2 \)):

\[ x^2 [e f + i f / \gamma] + x [1 - i (e + f)] - 1 = 0. \quad (8.13) \]

Since \( e \) and \( f \) are very small we see immediately that one root is approximately \( x_1 = 1 \). We therefore put

\[ x = 1 + q \quad (8.14) \]

where \( q \ll 1 \) and solve for \( q \), retaining only first-order terms in \( e, f, g \). The result is

\[ q = i \left[ e f - f / \gamma \right] \quad (8.15) \]

so that one root is approximately

\[ x_1 = 1 + i \left[ e f - f / \gamma \right]. \quad (8.16) \]

To obtain the second root we note that if we divide (8.13) by the coefficient of \( x^2 \), the constant term gives the product of the two roots:

\[ x_1 x_2 = (e f + i f / \gamma)^{-1}. \quad (8.17) \]
Using (8.16) and again keeping only first-order terms we obtain
\[ x_2 = i \sqrt{\frac{c}{c^2}}. \] (8.18)

Replacing \( x_1 \) by \( (ck_1/\omega)^2 \) and \( x_2 \) by \( (ck_2/\omega)^2 \) we finally obtain
\[ k_1^2 = \frac{\omega^2}{c^2} \left[ 1 + i \left( \frac{e-f-f/\xi}{f} \right) \right] \] (8.19)
\[ k_2^2 = \frac{\omega^2}{c^2} \left[ i \sqrt{\frac{c}{c^2}} \right] \] (8.20)

or, replacing \( e \) and \( f \) we have
\[ k_1^2 = \frac{\omega^2}{c^2} \left[ 1 + i \left( \frac{4N}{\beta} \frac{v\omega}{c^2} + \frac{2\rho\omega}{c^2} (1-1/\xi) \right) \right] \] (8.21)
\[ k_2^2 = i \sqrt{\frac{c}{c^2}}. \] (8.22)

These are the desired values of the two roots of the quadratic in \( k^2 \) inside the curly bracket of (8.12).

**Summary.** We now collect our results: We have found that the velocity field is given by two scalar potentials \( \phi_1 \) and \( \phi_2 \) and the vector potential \( A \), all of which satisfy the wave equation:

**Velocity** \[ \mathbf{v} = -\nabla \phi_1 - \nabla \phi_2 + \nabla \times A \] (8.23)

**Acoustic Wave** \[ \nabla^2 \phi_1 + k_1^2 \phi_1 = 0 \] (8.24)

**Thermal Wave** \[ \nabla^2 \phi_2 + k_2^2 \phi_2 = 0 \] (8.25)

**Viscous Wave** \[ \nabla^2 A + K^2 A = 0 \] (8.26)
where from (8.2), (8.21) and 8.22) the wave numbers are given by

$$k_1^2 = \frac{\omega^2}{c_1^2} \left\{ 1 + i \left[ \frac{4N}{3} \frac{\nu \omega}{c_1^2} + (\nu-i) \frac{\tilde{\varepsilon} \omega}{c_1^2} \right] \right\}$$  (8.27)

$$k_2^2 = i \omega / \tilde{\varepsilon}$$  (8.28)

$$K^2 = i \omega / \nu$$  (8.29)

and where in place of \( \varepsilon \) we have for convenience used the thermometric conductivity \( \tilde{\varepsilon} \) defined by

$$\tilde{\varepsilon} = \sigma / \rho_0 c_p = \varepsilon / \gamma .$$  (8.30)

(Since the imaginary part of the wave numbers gives rise to the damping effect for the waves, it was necessary to retain the small terms in \( k_1^2 \) to exhibit this effect.) Taking the square roots we obtain

$$k_1 = \frac{\omega}{c_1} \left\{ 1 + i \frac{1}{2} \left[ \frac{4N}{3} \frac{\nu \omega}{c_1^2} + (\nu-i) \frac{\tilde{\varepsilon} \omega}{c_1^2} \right] \right\}$$  (8.31)

$$k_2 = (1+i)(\omega / 2 \tilde{\varepsilon})^{1/2} = (1+i)(\omega \rho_0 c_p / 2 \sigma)^{1/2}$$  (8.32)

$$K = (1+i)(\omega / 2 \nu)^{1/2} = (1+i)(\omega \rho_0 / 2 \eta)^{1/2} ,$$  (8.33)

where in each case we use the root corresponding to damping of the wave (i.e., positive imaginary part).

The corresponding phase velocities and wavelengths for the acoustic, thermal and viscous waves are*

* The subscript \( \perp \) denotes the "traverse" character of the viscous or shear wave.
The velocity and wave length of the thermal and viscous waves are seen to be frequency-dependent. Values for these quantities at various frequencies for water and air are listed in Table I at the end of the text. The imaginary part of the wave numbers \( k_2 \) and \( K \) give rise to the "damping lengths" discussed in Sec. 10.

We have called \( \phi_1 \) the acoustic wave because it is propagated with the normal sound speed \( c \); \( \phi_2 \) is called the thermal wave since its character is determined by the thermal conductivity; finally \( \phi_3 \) is the viscous or shear wave since its properties depend on the viscosity.

To examine the physical nature of the three waves and their effects in dissipating acoustic energy we shall treat the one dimensional plane wave in the following chapter. Before doing this however, we consider briefly the manner in which the first-order temperature and pressure depend on the velocity potentials, and derive a useful expression for the viscous and thermal dissipation in terms of the first-order acoustic quantities.

9. Expressions for First-Order Temperature and Pressure and for the Viscous and Thermal Dissipation

First-Order Temperature

Substituting our expression for \( \nu \) from (8.23) into the temperature equation (8.5) we obtain

\[
\frac{\alpha}{\alpha} = \alpha \phi_1 + \alpha \phi_2
\]
where the constants \( \alpha_1 \) and \( \alpha_2 \) are given by

\[
\alpha_1 = -i \omega (3 - i \gamma)/c^2 \quad = -i \omega T_o \alpha_\nu /c_p
\]  
(9.2)

\[
\alpha_2 = - (\alpha_\nu \bar{\alpha}_\nu)^{-1}
\]  
(9.3)

Thus the space part of the first-order temperature \( T_1 = T e^{-i \omega t} \) is a linear function of the two scalar potentials \( \phi_1 \) and \( \phi_2 \). The same result is reached if we substitute (9.1) in the energy equation (7.23) and equate the coefficients of \( \phi_1 \) and \( \phi_2 \) to obtain \( \alpha_1 \) and \( \alpha_2 \). The reductions above are facilitated by the useful relation

\[
T_o c^2 \alpha_\nu^2 = c_p (3 - i \gamma)
\]  
(9.4)

which is easily proved from the definitions of \( \alpha_\nu \) and \( c^2 \) in (7.13) and (7.14), together with the familiar thermodynamic relation for the specific heats:

\[
c_p - c_\nu = T_o \left( \frac{\partial p}{\partial T} \right)_T \left( \frac{\partial f}{\partial T} \right)_T
\]  
(9.5)

**First-Order Pressure**

To derive the pressure dependence on \( \phi_1 \) and \( \phi_2 \) we write \( \nu = \nu e^{-i \omega t} \) in the linear force equation (7.3) to obtain

\[
-i \omega \rho_o \nu = -\nabla p + \frac{4N}{3} \eta \nabla (\nu \cdot \nu) - \eta \nabla \times \nabla \times \nu
\]  
(9.6)

We then substitute \( \nu \) from (8.23) and separate the vector (curl) and scalar (divergence) parts, the latter giving

\[
i \omega \rho_o \nabla (\phi_1 + \phi_2) = -\nabla p - \frac{4N}{3} \eta \nabla (\nabla^2 \phi_1 + \nabla^2 \phi_2)
\]  
(9.7)
or
\[ i\omega p_0 (\phi_1 + \phi_2) = -p - \frac{4N}{3} \eta (\nabla^2 \phi_1 + \nabla^2 \phi_2), \]  

(9.8)
and using the wave equations (8.24) and (8.25) we obtain
\[ p = -i\omega p_0 (\gamma_1 \phi_1 + \gamma_2 \phi_2), \]  

(9.9)
where the constants \( \gamma_1 \) and \( \gamma_2 \) are given by
\[ \gamma_1 = 1 + i \frac{4N}{3} \frac{\nu \omega}{c^2}, \]  

(9.10)
\[ \gamma_2 = 1 - \frac{4N}{3} \frac{\nu}{c^2}. \]  

(9.11)

We drop the small imaginary term in \( \gamma_1 \) but retain the second term in \( \gamma_2 \) since in general it is of order 1 for fluids. Thus the space part (p) of the first-order pressure \( p_1 = p e^{-i\omega t} \), like the temperature, is a linear function of the two scalar potentials \( \phi_1 \) and \( \phi_2 \).

The Viscous Dissipation

We have seen in Sec. 6 that the total viscous dissipation is given by the volume integral
\[ \frac{dE_n}{dt} = \int \Phi dV. \]  

(9.12)

It is in general a tedious process to evaluate this integral and a great simplification is introduced by converting part of the integral to a surface integral. This we now proceed to do.

The dissipation function \( \Phi \) is defined by (3.12):
So

\[ \Phi = \tau_{e\alpha} \nabla_{\alpha} v_e \]  

(9.13)

\[ = p_{\alpha e} \nabla_{\alpha} v_e + \rho \delta_{\alpha e} \nabla_{\alpha} v_e \]  

(9.14)

\[ = \nabla_{\alpha} (p_{\alpha e} v_e) - v_e \delta_{\alpha} p_{\alpha e} + \rho \nabla \cdot \mathbf{v} . \]  

(9.15)

where we have used the definition of the total stress tensor \( p_{\alpha e} \) from (2.12) and the relation (3.9) for the kinetic energy. From (9.16) we then obtain the desired general form:

\[ \Phi = \nabla_{\alpha} (p_{\alpha e} v_e) - \rho \frac{Dv^{2/2}}{Dt} + \rho \nabla \cdot \mathbf{v} , \]  

(9.16)

where the first volume integral has been converted to a surface integral over the surface \( S \) bounding the fixed volume \( V \).

For the acoustic case we expand the stress to first order terms

\[ p_{\alpha e} = -p_0 \delta_{\alpha e} + \rho_{h} \epsilon_{\alpha e} \]  

(9.18)

where

\[ \rho_{h} \epsilon_{\alpha e} = -\rho_0 \delta_{\alpha e} + \tau_{\alpha e} \]  

(9.19)

and use the previous expansions in \( \rho, \mathbf{v} \) and \( p \) from (7.1)

\[ \rho = \rho_0 + \rho_i \]  

(9.20)

\[ \mathbf{v} = \mathbf{v}_i \]  

(9.21)

\[ p = p_0 + p_i \]  

(9.22)
These give for $\Xi$ in (9.16)

$$\Xi = \nabla \alpha (p_{1 \alpha e} u_{1 e}) - \rho_0 \frac{\partial u_{1 e}^2}{\partial t} + p_1 \nabla \cdot \mathbf{u},$$

(9.21)

so that $\Xi$ is determined by products of the first-order acoustic quantities each of which is periodic in time. Since we are interested in the average rate of dissipation we take the time average of $\Xi$, using the relations in Appendix I, and obtain

$$\Xi_{AV} = \frac{1}{2} \Re \left[ \nabla \alpha (p_{1 \alpha e}^* u_{1 e}) - \rho_0 u_{1 e}^* \cdot \mathbf{u}_1 + p_{1 e}^* \nabla \cdot \mathbf{u}_1 \right].$$

(9.22)

The second term in (9.22) may be dropped since it is pure imaginary $(\rho_0 i \omega \mathbf{u}_1^* \cdot \mathbf{u}_1)$ and its time average is zero (see Appendix I). Here $\Re$ denotes the real part of the expression following and the star * denotes complex conjugate. As before we write the first-order quantities in the form

$$p_i = p e^{-i \omega t}, \quad u_i = \mathbf{u} e^{-i \omega t}, \quad p_{\alpha e} = p_{\alpha e} e^{-i \omega t}$$

(9.23)

where $p$, $\mathbf{u}$, $p_{\alpha e}$ (the amplitudes of the corresponding first-order quantities) are complex space functions only and do not depend on the time. Thus

$$\Xi_{AV} = \frac{1}{2} \Re \left[ \nabla \alpha (p_{\alpha e}^* u_{\alpha e}) + p_{\alpha e}^* \nabla \cdot \mathbf{u} \right].$$

(9.24)

so that, integrating this we have

$$\frac{dE_n}{dt} = \int \Xi_{AV} dV = \frac{1}{2} \Re \int p_{\alpha e}^* u_{\alpha e} dS_{\alpha} + \frac{1}{2} \Re \int p_{\alpha e}^* \nabla \cdot \mathbf{u} dV,$$

(9.25)
and this is our general computational form for the viscous dissipation in an acoustic field.

**Thermal Dissipation**

The thermal dissipation is given by

\[
\frac{dE_\sigma}{dt} = \int \Psi \, dV. \tag{9.26}
\]

This can also be transformed to a surface integral and a simpler volume integral for computation. We have

\[
\Psi = \frac{\sigma}{T^o} (\nabla T \cdot \nabla T) \tag{9.27}
\]

\[
= \frac{\sigma}{T^o} \left[ \nabla \cdot (\nabla T \cdot T - T \nabla^2 T) \right]. \tag{9.28}
\]

Now \( \nabla^2 T \) is given by the energy equation (7.20):

\[
\nabla^2 T = \frac{1}{\kappa} \left( \dot{T} + T^o \frac{c^2_0 \alpha v}{c_p} \nabla \cdot \nabla \right). \tag{9.29}
\]

We therefore have for \( \Psi \)

\[
\Psi = \frac{\sigma}{T^o} \nabla \cdot (\nabla T \cdot T) - \frac{\sigma}{T^o \kappa} (\dot{T} + T^o \frac{c^2_0 \alpha v}{c_p} \nabla \cdot \nabla), \tag{9.30}
\]

and \( \Psi_{\text{AV}} \) is

\[
\Psi_{\text{AV}} = \frac{\sigma}{2 T^o} R \left[ \nabla \cdot (\nabla^* T \cdot T) \right] - \frac{p_0 c^2_0 \alpha v}{2 \gamma} R \left[ \nabla^* \nabla \cdot \nabla \right], \tag{9.31}
\]

where the \( T^* \) being pure imaginary has been dropped, since its real part is zero, and we have used the definition \( \alpha = \frac{\sigma}{\rho_o c_v} \) to obtain the new...
form for the coefficient of the last term. Integrating this we have for the thermal dissipation

\[ \frac{dE}{dt} = \int \nabla^2 \Phi \, dV = \frac{\sigma}{2 \tau_0} \int (T^* \nabla T) \cdot dS - \frac{\rho c^2}{2 \nu} \int T^* \nabla \cdot \mathbf{v} \, dV, \]  

(9.32)

and this is the desired general computational form. The surface integral is extended over the closed surface S bounding the volume of integration V.

10. Plane Wave Solutions

We now apply the foregoing theory to the case of a one-dimensional plane wave moving in the positive x-direction. The appropriate solution of the scalar wave equations is

\[ \Phi = \Phi_1 + \Phi_2 = A e^{ik_1 x} + B e^{ik_2 x} \]  

(10.1)

where A and B are constants. Since \( k_1 \) and \( k_2 \) are complex the amplitudes will diminish exponentially. Using (8.31) and (8.32) for \( k_1 \) and \( k_2 \),

\[ \Phi_1 = A \exp \left[ i \frac{\omega}{c} x - \frac{k_1 \omega}{2c} \left( \frac{4N}{3} \frac{\nu \omega}{c^2} + \frac{\kappa c^2}{c^2} (\chi - 1) \right) \right] \]  

(10.2)

\[ \Phi_2 = B \exp \left[ i x (\omega/2c)^{1/2} - x (\omega/2c)^{1/2} \left( \frac{4N}{3} \frac{\nu \omega}{c^2} + \frac{\kappa c^2}{c^2} (\chi - 1) \right) \right] \]  

(10.3)

so that if \( l \) is the absorption length, i.e., the distance in which the wave amplitude declines to 1/e of its initial value, we have

\[ l_1 = \frac{2c}{\omega} \left[ \frac{4N}{3} \frac{\nu \omega}{c^2} + (\chi - 1) \frac{\kappa c^2}{c^2} \right]^{-1} \]  

(10.4)

\[ l_2 = \left( \frac{2 \kappa c}{\omega} \right)^{1/2} \]  

(10.5)
Since \( l_1 \propto \omega^{-2} \), \( l_2 \propto \omega^{-1/2} \), the absorption lengths both decrease with frequency, and the absorption increases with frequency, the increase being faster for \( \phi_1 \) than for \( \phi_2 \). The absorption lengths for air and water are tabulated in Table I at the end of the thesis. It is apparent from Table I that \( l_1 \) and \( l_2 \) are of entirely different orders of magnitude, \( \phi_1 \) being damped only slightly over distances of the order of centimeters, while \( \phi_2 \) is very strongly damped.

For air \( \nu \) and \( \bar{\kappa} \) are about equal so that the thermal absorption is comparable to the viscous absorption for the acoustic wave \( \phi_1 \), a result previously obtained by Kirchhoff.

While we have no viscous shear wave in a one-dimensional sound field, we see from the wave equation that the wave number \( K \) \((8.33)\) gives an absorption length for this transverse \((l)\) wave of
\[
\ell_l = \left( \frac{2 \nu}{\omega} \right)^{1/2}
\]
\((10.6)\)
in the same way as the thermal wave number \( k_2 \) \((8.32)\). As is seen from Table I, the viscous wave is also rapidly damped, the absorption of the viscous and thermal waves being comparable in magnitude.

It therefore appears that the main source of attenuation of the incident plane wave on small spherical obstacles is due to the conversion of acoustic waves into thermal and viscous waves which are rapidly absorbed within small distances from the spheres.

To obtain the attenuation \( \alpha \) we first proceed in the usual way to derive the acoustic intensity in the form \( E = E_0 e^{-\alpha x} \). We consider only \( \phi_1 \) since \( \phi_2 \) is rapidly damped in any case, and note that if we neglect small transport and conduction terms, the energy flux is given by the average rate
at which work is done by the stresses on unit area normal to the wave:

\[ E = \left( p, v \right)_{\text{Av}} = \frac{1}{2} R \left( p^* v \right) \]  \hspace{1cm} (10.7)

and using (9.9) for the pressure and (10.2) for the potential we have

\[ E = \frac{1}{2} R \left[ (i \omega \rho_o \Phi, \left( \frac{\partial \Phi}{\partial x} \right) \right] = \frac{1}{2} \omega \rho_o k, A A^* e^{i(k, -k^*)} \]  \hspace{1cm} (10.8)

or

\[ E = E_0 e^{-\alpha x} \]  \hspace{1cm} (10.9)

where

\[ \alpha = \frac{2}{k,} = \frac{\omega}{c} \left[ \frac{4 N}{3} \frac{v \omega}{c^2} + (v - i) \frac{\omega}{c^2} \right] \]  \hspace{1cm} (10.10)

is obtained from the real part of $ik, \chi$ in (10.2). This is then the attenuation desired and it is seen to consist of the viscous attenuation obtained by Stokes (but modified by N to include the effects of the compressional viscosity), together with the Kirchhoff thermal attenuation. For future use we note that over distances small compared to $l_1$ the wave is attenuated negligibly and the intensity is

\[ E = E_0 = \frac{1}{2} k, \omega \rho_o A A^* \]  \hspace{1cm} (10.11)

for a plane wave whose velocity potential is $\phi_1 = A e^{i k, x}$.

A second method of obtaining the attenuation is to evaluate the average viscous and thermal dissipation integrals (6.3) and (6.4):

\[ \frac{d\tilde{E}^n}{dt} = \int \tilde{\Xi} dV \]  \hspace{1cm} (10.12)

\[ \frac{d\tilde{E}^r}{dt} = \int \tilde{\Psi} dV \]  \hspace{1cm} (10.13)
For the one-dimensional case $\Omega$ is found from (5.12) to be

$$\Omega = \frac{4N}{3} \eta \left( \frac{2\nu}{\xi x} \right)^2$$  \hspace{1cm} (10.14)

or

$$\Omega_{AV} = \frac{1}{2} R \left[ \frac{4N}{3} \eta \left( \frac{2\nu}{\xi x} \right) \left( \frac{2\nu^*}{\xi x} \right) \right]$$  \hspace{1cm} (10.15)

$$\approx \frac{2N}{3} \eta \frac{\omega^4}{c^4} AA^*.$$  \hspace{1cm} (10.16)

From (10.11) this gives

$$\frac{dE}{dt} = \frac{2N}{3} \eta \frac{\omega^4}{c^4} AA^* L$$  \hspace{1cm} (10.17)

for a volume of integration of unit cross section normal to the wave and length $L$, where $L \ll L_1$, so that the attenuation of $\phi_1$ is negligible over the volume.

For the thermal dissipation we have from 6.2)

$$\chi_{AV} = \frac{1}{2} R \left[ \frac{\tau}{\tau_0} \chi T^* \cdot \chi T \right]$$  \hspace{1cm} (10.18)

Using our temperature relation (9.1)

$$T = \alpha, \phi_1$$  \hspace{1cm} (10.19)

this becomes

$$\chi_{AV} = \frac{1}{2} \frac{\sigma}{\tau_0} (\alpha, \phi^*_1)(k,k^*_1) AA^*$$  \hspace{1cm} (10.20)
From (10.13) we then have the thermal dissipation:

\[ \frac{dE_r}{dt} = \frac{\sigma}{2T_0} \left[ \frac{\omega^2 (\gamma - 1) T_0}{c^2 c_p} \right] \frac{\omega^3}{c^3} A A^* L \]  

(10.21)

where (9.2) has been used for \((\alpha - \alpha^*)\) and again \(S\) is a unit area normal to the direction of the wave and \(L\) is the length of the volume of integration. To obtain the attenuation \(\alpha\) per unit length we divide the total dissipation per unit length by the incident energy flux \(E_0\) from (10.11):

\[ \alpha = \frac{1}{E_0} \left( \frac{1}{L} \frac{dE_n}{dt} + \frac{1}{L} \frac{dE_r}{dt} \right) \]  

(10.22)

or

\[ \alpha = \frac{\omega}{c} \left[ \frac{4N}{3} \frac{\nu \omega}{c^2} + (\gamma - 1) \frac{\omega c}{c^2} \right] \]  

(10.23)

Since this agrees with our previous result, we have thus verified that the two methods of obtaining the attenuation are consistent.
IV. ATTENUATION OF PLANE SOUND WAVES BY SMALL SPHERES

We now attack our main problem of calculating the attenuation of sound waves by small uniform spherical inhomogeneities embedded in an infinite fluid medium. We shall start by considering a single sphere of radius $R$ and finding the velocity potentials. Knowing these we can then calculate the additional viscous and thermal dissipation caused by the presence of the sphere. Finally, from the dissipation we shall compute the attenuation for a number of such spheres.

11. Assumptions

The problem will be solved under the following assumptions, the first three being the assumptions made in the development of the general theory of Part II:

(1) The viscosity, thermal conductivity specific heats, etc., will be assumed to be constants, in particular, they do not depend on the temperature.

(2) The coefficient of compressional viscosity will be retained and will not be assumed to vanish.

(3) The medium will be assumed to be in a pseudo-steady state so that the imperturbed temperature, pressure, etc., are constant.

(4) We shall assume that over the region occupied by the small spheres the attenuation is so small that the intensity of the incident wave may be considered constant (See Sec. 15).

(5) The spheres, of radius $R$, will be assumed very small compared to the acoustic wave length $\lambda$. 
6. We shall assume the deformation of the spheres by the stress variations to be negligible so that in applying the boundary conditions at the surface we shall put the radial coordinate \( r = R \).

7. We shall not include the attenuation caused by scattering. As shown by Epstein (Ref. 6) this effect is very small compared to the viscous and thermal attenuation because of assumption (4). (Compare \( \alpha_s \) with \( \alpha_\eta \) and \( \alpha_\sigma \) in the Introduction.) If the scattering term is required it can easily be calculated.

8. The spheres will be assumed to be sparsely and randomly distributed so that the interaction effects between spheres (secondary scattering, collisions, aggregation, etc.) can be neglected.

12. Velocity Potentials

Consider a plane sound wave propagating in an infinite fluid medium and impinging on a fluid sphere. We shall denote quantities relating to the outer medium by unprimed letters and those pertaining to the interior by primed letters. Outside the sphere we have the incident wave represented by a potential \( \phi_1 \) and the scattered wave, represented by the acoustic and thermal scalar potentials \( \phi_1 \) and \( \phi_2 \) and the shear vector potential \( \Delta \). Inside the sphere we have two scalar potentials \( \phi'_1 \) and \( \phi'_2 \) and a vector potential \( \Delta' \). We thus have seven potentials.

We choose a spherical coordinate system \((r, \theta, \varphi)\) with origin at the center of the sphere and polar axis in the direction of propagation of the incident wave. Because of the axial symmetry two components of the vector potential vanish \((A_r = A_\theta = 0)\) and only the azimuthal component \( A_\varphi \) is different from zero. Again, because of the axial symmetry, there is no dependence on \( \varphi \). Under these conditions the appropriate solutions of the wave
equations
\[
\nabla^2 (\phi_i + \phi) + k_i (\phi_i + \phi) = 0 \\
\nabla^2 \phi_i + k_i^2 \phi_i = 0 \\
\nabla^2 A_i + k_i^2 A_i = 0
\]
\[
\nabla^2 \phi_i' + k_i^2 \phi_i' = 0 \\
\nabla^2 \phi_i + k_i^2 \phi_i = 0 \\
\nabla^2 A_i + k_i^2 A_i = 0
\]
\[(12.1)\]

can be written in the form
\[
\phi_i = e^{ik_ir \cos \theta} = \sum_{n=0}^{\infty} i^n (2n+1) P_n (\cos \theta) j_n (k_i r)
\]
\[
\phi_i = \sum_{n=0}^{\infty} i^n (2n+1) P_n (\cos \theta) B_n h_n (k_i r)
\]
\[
\phi_i = \sum_{n=0}^{\infty} i^n (2n+1) P_n (\cos \theta) C_n j_n (k_i r)
\]
\[
A_p = A = \sum_{n=0}^{\infty} i^n (2n+1) P_n j_n (k_i r)
\]
\[
\phi_i' = \sum_{n=0}^{\infty} i^n (2n+1) P_n (\cos \theta) B_n' j_n (k_i r)
\]
\[
\phi_i' = \sum_{n=0}^{\infty} i^n (2n+1) P_n (\cos \theta) C_n' j_n (k_i r)
\]
\[
A_p' = A' = \sum_{n=0}^{\infty} i^n (2n+1) P_n (\cos \theta) B_n' j_n (k_i r)
\]
\[(12.2)\]

Here the form for \( \phi_1 \) is the standard expansion for a plane wave in spherical coordinates; the remaining expressions are convenient forms of the wave functions in spherical coordinates, with the six constants \( B, C, D, B', C', D' \) to be determined from the boundary conditions. The radial functions \( j_n \) and \( h_n \) are the spherical Bessel Hankel functions discussed in Appendix II. The choice of \( h_n \) (Hankel function of the first kind) for the outside scattered potentials insures that the \( \phi_2 \) and \( A \) waves decline exponentially at large distances. Inside the sphere we use the Bessel functions which are the solutions regular at \( r = 0 \). The functions \( P_n (\cos \theta) \) and \( P_n^{-1} (\cos \theta) \) are Legendre functions and are discussed in Appendix III.
13. Velocity, Temperature, Pressure, and Stresses

The velocity in the outside medium is given by (8.23):

\[ \mathbf{v} = -\nabla \phi + \nabla \times \mathbf{A} \]  \[ (13.1) \]

where \( \phi = \phi_1 + \phi_2 \). Writing \( \mathbf{A} = A_y \), we have the velocity components:

\[ v_r = -\frac{3\phi}{2\pi} + \frac{1}{\pi \sin \theta} \frac{\partial}{\partial \theta} (A \sin \theta) \]

\[ v_\theta = -\frac{1}{\pi} \frac{\partial \phi}{\partial \theta} - \frac{1}{\pi} \frac{\partial}{\partial \theta} (A \lambda) \]

\[ v_\phi = 0 \]

with corresponding expressions for \( v_r', v_\theta ', v_\phi ' \).

From Sec. 9 we obtain the expressions for the temperature and pressure:

\[ T = \omega_1 (\phi_1 + \phi_2) + \omega_2 \phi_2 \]

\[ T' = \omega_1' \phi_1 + \omega_2' \phi_2 \]  \[ (13.3) \]

and

\[ p = -i \omega \rho_0 \left[ \gamma_1 (\phi_1 + \phi_2) + \gamma_2 \phi_2 \right] \]

\[ p' = -i \omega \rho_0' \left[ \gamma_1' \phi_1 + \gamma_2' \phi_2 \right] \]  \[ (13.4) \]

We now obtain the radial stress components from (2.15), (2.16), and (13.2) using standard tensor methods (Ref. 9):

\[ \tau_{rr} = \frac{\eta}{r} \left\{ -\frac{3}{\eta} \frac{\partial}{\partial \lambda} \left( \frac{1}{\lambda} \frac{\partial \phi}{\partial \lambda} - \frac{\phi}{\lambda^2} \right) - \left( \frac{3}{\lambda^2} \frac{\partial A}{\partial \lambda} - \frac{2A}{\lambda^2} \right) + \frac{1}{\lambda^2} \frac{\partial}{\partial \theta} \left[ \frac{1}{\sin \theta} \left( \frac{A}{\lambda^2} + \frac{\lambda}{\eta} \frac{\partial}{\partial \lambda} (A \sin \theta) \right) \right] \right\} \]

\[ (13.5) \]

\[ \tau_{\lambda\lambda} = \frac{\eta}{r} \left[ \beta_1 (\phi_1 + \phi_2) + \beta_2 \phi_2 \right] + \frac{2}{r} \left[ -\frac{3}{\lambda^2} \frac{\partial \phi}{\partial \lambda} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left[ \sin \theta \left( \frac{A}{\lambda^2} + \frac{\lambda}{\eta} \frac{\partial}{\partial \lambda} (A \sin \theta) \right) \right] \right] \]

\[ (13.6) \]

\[ \tau_{\rho\phi} = 0 \]  \[ (13.7) \]
where
\[ \beta_1 = 1 + 2i N \frac{\nu_0}{c^2} \quad \beta_2 = 1 - 2 \frac{N}{c^2}. \quad (13.8) \]

14. Boundary Conditions

The boundary conditions at the surface of the sphere are obtained from the physical requirements that the velocity, temperature, radial heat current, and normal (radial) stress components all be continuous across the surface \( r = R \). We thus have the six boundary conditions

\[
\begin{align*}
(1) \quad & \nu_\lambda = \nu_\lambda' \\
(2) \quad & \nu_\theta = \nu_\theta' \\
(3) \quad & T = T' \\
(4) \quad & \sigma \nabla_r T = \sigma' \nabla_r T' \\
(5) \quad & p_{r\theta} = p_{r\theta}' \\
(6) \quad & p_{rr} = p_{rr}' 
\end{align*}
\quad (14.1)
\]

and these are sufficient to solve for the six coefficients for the potentials. For simplicity we introduce the abbreviations

\[
\begin{align*}
& a_1 = k_1 R \\
& a_2 = k_2 R \\
& b = KR \\
& a_1' = k_1' R \\
& a_2' = k_2' R \\
& b' = K'R
\end{align*}
\quad (14.2)
\]

Using the standard relations for the Legendre functions (Appendix III):

\[
\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta P_n^1) = n(n+1) P_n^1, \quad \frac{\partial P_n}{\partial \theta} = -P_n^1
\quad (14.3)
\]

and indicating differentiation of the Bessel and Hankel functions with respect to their arguments by primes, we obtain from (14.1) and the previous section the six equations:
\[(v): \quad B_n a_i h_n(a) - B'_n a_i' j'_n(a) + C_{n,a} h'_n(a) - C'_{n,a} j'_n(a) = -a_i j'_n(a) \quad (14.4)\]

\[(v_9): \quad B_n h_n(a) - B'_n j_n(a) + C_{n,h} h_n(a) - C'_{n,j} j_n(a) - D_n [b h_n(b) + h_n(b)] + D'_n [b'_j_n(b) + j_n(b)] = -j_n(a) \quad (14.5)\]

\[(T): \quad B_n \alpha \cdot h_n(a) - B'_n \alpha ' \cdot j_n(a) + C_{n,\alpha} h_n(a) - C'_{n,\alpha'} j_n(a) = -\omega_1 j_n(a) \quad (14.6)\]

\[(q_r): \quad B_n \not\omega w_i h_n(a) - B'_n \not\omega w_i' j'_n(a) + C_{n,\not\omega w} h'_n(a) - C'_{n,\not\omega w'} j'_n(a) = -\not\omega w_i j'_n(a) \quad (14.7)\]

\[(p_{\not\omega}): \quad B_n \not\omega = [a_n h_n(a) - h_n(a)] - B'_n [a_i j_i(a) - j_i(a)] + C_n [a_i h_i(a) - h_i(a)] - C'_n [a_i j_i'(a) - j_i'(a)]
- D_n \not\omega \gamma [b h_n(b) + (n^2 + n - 2) j_n(b)] + D'_n \not\omega \gamma [b'_j_n(b) + (n^2 + n - 2) j_n(b)] = \epsilon [a_i j_i'(a) - j_i'(a)] \quad (14.8)\]

\[(p_{\not\omega}): \quad B_n \not\omega = b^2 \rho_i h_n(a) - 2 a^2_j h''_n(a) - B'_n [b^2 \rho_i j_n(a) - 2 a^2_j j''_n(a)] + C_n [b^2 \rho_j h_n(a) - 2 a^2_j h''_n(a)] - C'_n [b^2 \rho_j j_n(a) - 2 a^2_j j''_n(a)]
+ D_n \not\omega \gamma [b h_n(b) - h_n(b)] - D'_n \not\omega \gamma [b'_j_n(b) - j_n(b)] = \epsilon [b^2 \rho_i j_n(a) - 2 a^2_j j''_n(a)] \quad (14.9)\]

where \[\epsilon = \eta / \eta' \quad \chi = \sigma / \sigma' \quad (14.10)\]

For \(n=0\) there is no vector potential \((D'_o = D''_o = 0)\) and no \(Q\)-dependence, so that the \(v_9\) and \(p_{\not\omega}\) equations do not appear. In this case there are four equations in \(B'_0, B''_0, C'_0, C_0\). For \(n \geq 1\) all six equations are needed for the six unknown coefficients \(B_n, B'_n, C_n, C'_n, D_n, D'_n\).

15. Approximations

The problem has thus far been completely general; we have made no approximations and have not used our condition that \(R\) is small compared to \(\lambda_1\).
We now introduce assumptions (4) and (5) and make various approximations.

Assumption (4) states that the acoustic waves are not appreciably damped over the region occupied by the scattering spheres. This means that we neglect the imaginary part of $k_1$:

$$k_1 = \frac{\omega}{c} \left[ 1 + i \frac{1}{2} \left( \frac{4N \nu \omega}{c^2} + (y-1) \frac{\rho \omega}{c^2} \right) \right] \approx \frac{\omega}{c}$$

which amounts to neglecting the extremely small quantities

$$\frac{4N \nu \omega}{c^2} << 1 \quad (y-1) \frac{\rho \omega}{c^2} << 1 \quad (15.1)$$

compared to unity as was done in the derivation of $k_1$ and $k_2$ (Sec. 8).

Physically we are making the very reasonable assumption that the extent of the region occupied by the spheres is very small compared to the damping wavelength $\lambda_1$ of the acoustic waves. The same assumption is even more valid inside the spheres. Thus from this point on we assume that the acoustic wave numbers $k_1$ and $k_1'$ are real:

$$k_1 = \frac{\omega}{c} \quad k_1' = \frac{\omega}{c} \quad (15.2)$$

Our assumption (5) that $R << \lambda_1$ then leads to

$$a_1 = k_1 R << 1 \quad a'_1 = k'_1 R << 1 \quad (15.3)$$

We shall make no assumptions regarding the magnitudes of the quantities $a_2, a_2', b, b'$ in obtaining the general expressions for the attenuation.

Using our values of $k_2$ and $K$ from Sec. 8 we have

$$a_2 = k_1 R = (1+i) (\omega R^2/2 \nu) \frac{1}{2} \quad b = KR = (1+i) (\omega R^2/2 \nu) \frac{1}{2}$$

$$a'_2 = k'_1 R = (1+i) (\omega R^2/2 \nu) \frac{1}{2} \quad b' = K'R = (1+i) (\omega R^2/2 \nu) \frac{1}{2} \quad (15.4)$$
Finally using the approximations (15.1) we have the following relations for the constants appearing in the temperature, stress, and pressure expressions of Sec. 13:

\[
\alpha_1 = -i \omega (x-1)/\alpha_\nu c^2 \\
\beta_1 = 1 \\
\gamma_1 = 1 \\
\alpha_2 = -(\alpha_\nu \delta \varepsilon)^{-1} \\
\beta_2 = 1 - 2N \nu /\delta \varepsilon \\
\gamma_2 = 1 - 4N \nu /3 \delta \varepsilon
\]  

(15.5)

\[|\frac{a_1}{\alpha_1}|^2 \sim \frac{|k_1|}{k_1} \sim |\frac{\alpha_1}{\alpha_2}| \sim \frac{4N \nu \omega}{3 \delta \varepsilon} \sim \frac{\omega \delta \varepsilon}{c^2}(x) << 1
\]  

(15.6)

with similar relations for the primed quantities.

16. Values of Potential Coefficients

It is shown in Appendices IV and V that with the above approximations the potential coefficients for \( n=0 \) and \( n=1 \) are as follows:

For \( n=0 \)

\[
B_0 = i \frac{a_2^3}{3} [\delta (\frac{a_1}{a_2})^2 - 1] + (\gamma \frac{\alpha_1}{\alpha_2} - 1) i a_1 C_0 a_2 h_0(a_3)
\]

(16.1)

\[
B_0' = \delta
\]

(16.2)

\[
C_0 = \frac{\alpha_1}{\alpha_2} \frac{(\delta \frac{a_1}{a_2} - 1)}{h_0(a_2)(1 - \gamma Z)}
\]

(16.3)

\[
C_0' = \gamma \left( \frac{\alpha_1}{\alpha_2} \right) \frac{a_2 h_0(a_3)}{a_2^2 j_0(a_4)} C_0
\]

(16.4)

where in \( C_0 \)

\[
Z = \frac{j_0(a_4)}{a_2^2 j_0(a_4)} \frac{a_2 h_0(a_3)}{h_0(a_2)} = \frac{1 - i a_2}{1 - a_2^2 \cot a_2}
\]

(16.5)
The last form is easily verified from Appendix II and will be used later.

For \( n = 1 \)

\[
B_1 = \frac{i}{3} \left( 1 - \delta \right) G = \frac{i}{3} \left( 1 - \delta \right) \frac{N_{00} + N_{01} + N_{10} + N_1}{D_{00} + D_{01} + D_{10} + D_1}
\]

\[
B_1' = \frac{a_1}{a_1'} G' = \frac{a_1}{a_1'} 3 \delta G
\]

\[
C_1 = \frac{a_1}{3} a_1' \left[ \left( 2 + \chi - \frac{3}{2} \alpha \right) G' - 2 \left( 1 - \gamma \right) (1 - \delta) G \right] j_1(a_1) + \frac{a_1'}{a_1} \left[ \frac{1}{2} a_1' G' - 1 - (1 - \delta) G \right] a_2 j_0(a_2)
\]

\[
C_1' = -\frac{a_1}{3} a_1' \left[ \left( 1 + 2 \chi \right) \frac{1}{2} a_1' G' - 3 \chi \right] h_0(a_1) + \chi \left[ \left( 1 + (1 - \delta) G - \frac{1}{2} a_1' G' \right) a_2 h_0(a_2) - \chi j_1(a_2) a_2 h_0(a_2) \right]
\]

For \( G \) the numerator and denominator terms (N's and D's) are given by

\[
N_{00} = \left[ 1 - \epsilon \right] b j_0(b') \cdot b h_0(b)
\]

\[
N_{01} = \left[ \left( 1 - \epsilon \right) - \frac{1}{2} \epsilon b^2 \right] b j_0(b') \cdot h(b)
\]

\[
N_{10} = \left[ \left( 1 - \epsilon \right) + \frac{1}{2} b^2 \right] j_1(b') \cdot b h_0(b)
\]

\[
N_{11} = \left[ \left( 1 - \epsilon \right) + \frac{3}{2} b^2 \right] j_1(b') \cdot h(b)
\]

\[
D_{00} = \left[ \left( 1 - \epsilon \right) (2 + \delta) \right] b j_0(b') \cdot b h_0(b)
\]

\[
D_{01} = \left[ \left( 1 - \epsilon \right) - \frac{1}{2} \epsilon b^2 \right] b j_0(b') \cdot h(b)
\]

\[
D_{10} = \left[ \left( 1 - \epsilon \right) (2 + \delta) + \frac{1}{2} b^2 \right] j_1(b') \cdot b h_0(b)
\]

\[
D_{11} = \left[ \left( 1 - \epsilon \right) (2 + \delta) + \frac{3}{2} b^2 \right] j_1(b') \cdot h(b)
\]

and the various constants satisfy the relations

\[
\epsilon = \eta / \eta' \quad \chi = \sigma / \sigma' \quad \delta = \rho / \rho' \quad \delta b^2 = \epsilon b^2
\]

For later use we note that a simpler expression for \( B_1 \) may be derived using the last relation (16.11) and the relations

\[
gh(b) = 3 h(b) - bh_0(b)
\]

and

\[
b j_1(b') = 3 j_1(b) - b' j_0(b')
\]

from

\[
(2.3) \text{ in Appendix II.}
\]

Using the above values for the N's and D's we find.
It is interesting to note that while the amplitude $B_0$ of the zero-order scattered acoustic wave depends on the thermal conductivity $\sigma$ (in the $a_2$ term), the first order amplitude $B_1$ is independent of $\sigma$ and agrees with Epstein's result ($0.8$) in which the conductivity was ignored ($\sigma = 0$).

It is useful for later calculations to list the general order of magnitude of the coefficients

$$
B_0 \sim a_1^3, \quad B_1 \sim a_1^3, \quad C_0 \sim a_1^3, \quad C_1 \sim a_1^3,
$$

Interpreting these results physically we see that for an incident acoustic wave of unit amplitude, the scattered acoustic wave outside has an amplitude of the order of $a_1^3 << 1$, and the inside acoustic wave has an amplitude proportional to the density ratio $\delta$ of the outside to the inside medium. The scattered thermal waves inside and outside are of comparable magnitude but those of first order ($C_1$ and $C_1'$) are smaller than the zero-order amplitudes ($C_0$ and $C_0'$) by the factor $a_1 << 1$. Thus the principal thermal waves are of order zero and are consequently purely radial waves. This result
is of great importance in evaluating the thermal dissipation. It can also be shown that the shear waves of first order (there are none of zero order) are of magnitude \( D_1 \sim a_1, D_1' \sim a_1' \), so that the scattered shear waves outside and inside the obstacle are of order \( a_1 \).

Because of the smallness of \( a_1 \) and \( a_1' \) the coefficients \( B_n \) and \( C_n \) decrease so rapidly with \( n \) that to order \( a_1^3 \) we need retain only \( B_0, B_1, C_0, C_1 \) given above. In evaluating the attenuation inside the sphere the contributions of the terms involving \( B_o \), \( c', B_1 \), \( C_1 \) appear in volume integrals over the sphere, and since the volume is \( \propto R^3 \) we again obtain terms of the order \( a_1^3 \). Thus for results of order \( a_1^3 \) we require only the \( n=0 \) and \( n=1 \) terms in the potential expansions. This rapid convergence of the series (12.2) is a characteristic feature of diffraction by small obstacles.

We have thus found the velocity potentials from which the acoustic field quantities are completely determined to our order of approximation. We now turn to the calculation of the viscous and thermal dissipation.

17. Calculation of Viscous Dissipation

To calculate the dissipation caused by viscous forces we use the computational form (9.25):

\[
\frac{dE_n}{dt} = \frac{1}{2} \mathbf{R} \int p^* \xi \nu \mathbf{dS}_\xi + \frac{1}{2} \mathbf{R} \int \mathbf{p}^* \nabla \xi \mathbf{dV}.
\]  

(17.1)

For our fixed volume of integration we choose a large sphere of radius \( \bar{R} \) concentric with the small scattering sphere. More precisely we wish \( \bar{R} \) to be (1) much larger than the thermal and viscous damping lengths \( l_2 \) and \( l_1 \), (2) much smaller than the acoustic damping length \( l_2' \), and (3) much larger than the acoustic wavelength \( \lambda_1 \). The requirement (1) insures that,
due to the rapid exponential damping of the thermal and viscous waves with
distance, the potentials $\phi_2$ and $A$ may be neglected at the surface of the
large sphere of integration. This is easily seen by writing the Hankel
functions for the two waves at large distances. Using Appendix II and the
values of $k_2$ and $K$ in terms of $l_2$ and $l_1$ (Sec. 10) we find

$$k_2 \bar{R} = (1+i) \bar{R}/l_2 \quad K \bar{R} = (1+i) \bar{R}/l_1$$

so that for $k_2 \bar{R} \ll 1$, $K \bar{R} \ll 1$,

$$|h_n(k_2 \bar{R})| \rightarrow \frac{1}{|k_2 \bar{R}|} e^{-\bar{R}/l_2}$$

$$|h_n(K \bar{R})| \rightarrow \frac{1}{|K \bar{R}|} e^{-\bar{R}/l_1}$$

and $\phi_2$ and $A$ are seen to be negligible.

The second requirement that $\bar{R} \ll l_1$ insures that the acoustic wave is
not damped appreciably over the sphere of integration, in accordance with
our assumption that $k_1$ is real (Sec. 15). The requirement (3) that $k_1 \bar{R} \gg 1$
allows us to use the asymptotic forms of $j_n(k_1 \bar{R})$ and $h_n(k_1 \bar{R})$ for the acous-
tic wave in evaluating the surface integral. We thus have the chain of in-
EQUALITIES between the radius $\bar{R}$, the acoustic wavelength $\lambda_1$, and the three
damping lengths $l_1$, $l_2$, $l_1$:

$$l_1, \lambda_1 \ll \bar{R} \ll l$$

Examination of Table I shows that this inequality can always be satisfied.

For definiteness we may take
where we have omitted absolute magnitude signs and have used the relation
\[ \ell_1 \sim \frac{k_2}{k_1} \gg 1 \]
obtained from the definition of \( \ell_1 \) (Sec. 10) and the approximate equalities (15.6). We then find the approximate magnitude relation

\[ (k, \bar{R})(k, R) \sim k_1 R = a_2 \]

which we shall use later in determining the order of magnitude of expressions involving \( k_1 \bar{R} \).

We now write the viscous dissipation (17.1) as the sum of one surface and two volume integrals: (see note on page 60)

\[ \frac{dE_\nu}{dt} = I_s + I_v + I'_v \]

where

\[ I_s = \frac{1}{2} R \int \mathbf{p}^* \mathbf{v}_e dS_\nu \quad \lambda = \bar{R} \]

\[ I_v = \frac{1}{2} R \int \mathbf{p}^* \mathbf{v} \cdot dV \quad \bar{R} \leq \lambda \leq \bar{R} \]  

\[ I'_v = \frac{1}{2} R \int \mathbf{p}^* \mathbf{v} \cdot dV \quad 0 \leq \lambda \leq \bar{R} \]  

(17.7)

\( I_v \) is the volume integral outside the small sphere and \( I'_v \) is the volume integral inside the small sphere. These preliminaries being understood we now proceed to the evaluation of the integrals.

**Evaluation of Surface Integral**

The surface integral \( I_s \) on our sphere of radius \( \bar{R} \) assumes the form

\[ I_s = \frac{1}{2} R \int (\mathbf{p}^* \mathbf{v}_r + \mathbf{p}^* \mathbf{v}_r) dS_r \]  

(17.8)
From the previous discussion we see that only $\phi_1$ and $\phi_2$ need be considered in calculating the stresses and velocities. The general stress and velocity equations of Sec. 13 then give ($\beta_1 = 1$):

$$
\begin{align*}

\nu_r &= -\frac{3}{2\pi} (\phi_1 + \phi_2) \\
\nu_\theta &= -\frac{1}{2\pi} \frac{3}{\partial \theta} (\phi_1 + \phi_2) \\

p_{rr} &= \eta K^2 (\phi_1 + \phi_2) - 2\eta \frac{3}{2\pi} (\phi_1 + \phi_2) \\

p_{r\theta} &= -2\eta \frac{3}{\partial \theta} \left[ \frac{1}{r} \frac{3}{\partial \phi_2} (\phi_1 + \phi_2) - \frac{1}{r^2} (\phi_1 + \phi_2) \right].
\end{align*}
$$

(17.9)

Using the asymptotic forms of $j_n(k_1 r)$ and $h_n(k_1 r)$ for large arguments it is easily found that $\nu_\theta$ and $p_{r\theta}$ are negligible compared to $\nu_r$ and $p_{rr}$ and that the second term in $p_{rr}$ is negligible compared to the first. Our surface integral then becomes (with $\eta K^2 = i\omega \rho_0$):

$$
I_s = \frac{1}{2} \mathcal{R} \int \left[ i\omega \rho_0 (\phi_1 + \phi_2)^* \right] \left[ -\frac{3}{\partial \phi_2} (\phi_1 + \phi_2) \right] dS_n.
$$

(17.10)

This integral is evaluated in Appendix VI; the result is

$$
I_s = -2\pi \rho_0 c \sum_{n=0}^{\infty} (2n+1) \mathcal{R} \left[ B_n + B_n B_n^* \right]
$$

(17.11)

or, retaining only the terms for $n=0, n=1$,

$$
I_s = -2\pi \rho_0 c \mathcal{R} \left[ B_0 + 3B_1 + B_0 B_0^* + 3B_1 B_1^* \right].
$$

(17.12)

Using the magnitudes of the coefficients from Sec. 16 the last two terms are seen to be of order $a^6_1$ while the first two are of order $a^3_1$; hence to order $a^3_1$ the surface integral is

$$
I_s = -2\pi \rho_0 c \mathcal{R} \left[ B_0 + 3B_1 \right].
$$

(17.13)
These results are in agreement* with Epstein's evaluation of the same integral. (Ref. 6, Eq. 59).

It should be pointed out that in evaluating the various dissipation integrals we do not include any contributions arising from products $\phi_1 \phi_1^*$ which are due solely to the incident wave. The reason for this is that these terms represent the dissipation in the absence of the scattering sphere, whereas we are interested not in the total dissipation (given by the integral including the $\phi_1 \phi_1^*$ terms), but only in the additional dissipation due to the presence of the spheres, so that the $\phi_1 \phi_1^*$ terms must be excluded.

In our integrals this question does not arise because we have assumed $k_1 = \omega/c$ to be real, so that there is no attenuation, and the $\phi_1 \phi_1^*$ terms contribute nothing. In $I_s$, for example, the term drops out because we take the real part of the product which turns out to be imaginary (See Eq. (6.7) in Appendix VI).

**Evaluation of Volume Integrals**

The two volume integrals $I_v$ and $I'_v$ are evaluated in Appendix VII, where it is shown that the dominant part of each integral is of the same order of magnitude as the surface integral $I_s$:

$$I_v = -2\pi \rho_0 c \mathcal{R}[ia, C_0 a_1 h, (a_1)]$$

$$I'_v = -2\pi \rho_0 c \mathcal{R}[-\frac{\lambda}{\omega^2} ia, C_0 a_1 h, (a_1)]$$

where the second integral has been expressed in terms of $C_0$ instead of $B'_0 C'_0$ for reasons which will be clear in the next paragraph. From the

* A factor 3 should be supplied in the $B_1 B_1^*$ term of Epstein's Eq. (59).
derivations in Appendix VII it is clear that the dominant part of the outside integral $I_v$ represents the interference effects between the incident acoustic wave $\phi_1$ and the scattered thermal wave $\phi_2$, while the inside integral represents the interference effects between the scattered acoustic wave $\phi_1'$ and the scattered thermal wave $\phi_2'$. 

Final Result for Viscous Dissipation

Combining $I_s$, $I_v$, and $I_v'$ we have

$$\frac{dE_n}{dt} = -2\pi \rho_0 c \mathcal{R} \left[ 3B_1 + B_0 + (1 - \frac{x^2}{a^2})i a_C a_k h(a_2) \right] \tag{17.16}$$

But now we reach a surprising result: from the value of $B_0$ in Sec. 16 we see that the $c_0$ terms in (17.16) arising from the volume integral of $p^\ast \nabla \cdot \mathbf{v}$ exactly cancel the second term of $B_0$, so that

$$\frac{dE_n}{dt} = -2\pi \rho_0 c \mathcal{R} \left[ 3B_1 + i \frac{a^3}{3} \left( \delta(a_i/a_0)^2 - 1 \right) \right] \tag{17.17}$$

a result which agrees exactly with Epstein's derivation of the viscous dissipation ignoring thermal conductivity. Since the second term is pure imaginary our final result for the viscous dissipation is

$$\frac{dE_n}{dt} = -2\pi \rho_0 c \mathcal{R} [3B_1] \tag{17.18}$$

in agreement with Epstein. Thus we see that only the first order ($n=1$) scattered waves contribute to the dominant part of the viscous dissipation; the terms for $n=0$ do not contribute significantly.

This result can be checked by a direct calculation of the volume integral of the viscous dissipation function $\Phi$. Using the expression (5.12) for $\Phi$ we easily find for the case $n=0$ (in which the velocity depends only
on $r$ and is independent of $\varphi$ and $\Theta$ the result
\begin{equation}
\Phi = \frac{2}{3} \mu (\nabla \cdot \mathbf{u})^2 + \frac{4}{3} \eta \left( \frac{\partial \mathbf{u}}{\partial t} - \frac{\mathbf{u}}{\alpha} \right)^2 .
\end{equation}

The time average of $\Phi$ in the acoustic case is then
\begin{equation}
\Phi_{AV} = \frac{1}{2} \mathcal{R} \left[ \frac{2}{3} \mu (\nabla \cdot \mathbf{u}) (\nabla \cdot \mathbf{u}^*) + \frac{4}{3} \eta \left( \frac{\partial \mathbf{u}}{\partial t} - \frac{\mathbf{u}}{\alpha} \right) \left( \frac{\partial \mathbf{u}^*}{\partial t} - \frac{\mathbf{u}^*}{\alpha} \right) \right]
\end{equation}
and using the expressions for the potentials of Sec. 12 for $n=0$ it can be shown that
\begin{equation}
\frac{dE_n}{dt} = \frac{1}{2} \mathcal{R} \int_{n=0}^{n=R} \Phi_{AV}(n=0) dV
\end{equation}
is of higher order in $\alpha_1$ and $\alpha_1^*$ than the first order result (17.18) and may therefore be neglected.

It was pointed out in Sec. 16 that $B_1$ does not depend on $\sigma$, the thermal conductivity. Thus even in a heat conducting medium the viscous dissipation (17.18) is completely independent of the thermal effects. This result would not be true if the $B_0$ and $C_0$ terms of $dE/\eta$, had not cancelled, since these terms depend on $\sigma$. It follows that the only way in which the effects of heat flow enter into the dissipation, either directly or indirectly, is through the thermal dissipation term $dE/\sigma$, to which we now turn.

18. Calculation of Thermal Dissipation

The thermal dissipation within a volume $V$ bounded by the closed surface $S$ is given by the general computational form (9.32).
\begin{equation}
\frac{dE_\sigma}{dt} = \frac{\sigma}{2T_0} \mathcal{R} \int (T^* \nabla T) \cdot dS - \frac{\rho_0 c_0^2 \alpha}{2T} \mathcal{R} \int T^* \nabla \mathbf{u} \cdot dV .
\end{equation}

Just as in the case of the viscous dissipation the volume integral splits into two integrals $J_V$ and $J^V$ taken over the volumes outside and inside the
small sphere. Moreover putting $\sigma$ inside the integral sign the integrand of the surface integral can be written $T \sigma \nabla T = -T \nabla q$, so that since both the temperature and the heat current $q$ are continuous on the boundary of the small sphere, the integrand is also, and the surface integrals at this boundary cancel, leaving only the surface integral over the large sphere $r = \bar{R}$. The thermal dissipation for our problem is therefore

$$\frac{dE}{dt} = J_s + J_v + J'_v$$

where

$$J_s = \frac{\sigma}{2T_0} R \int (T^* \frac{\partial T}{\partial n}) dS_n$$

$$J_v = -\frac{\rho c^2 \alpha_v}{2} R \int T^* \nabla \cdot \nabla' dV$$

$$J'_v = -\frac{\rho' c' \alpha' \alpha_v}{2} R \int T' \nabla \cdot \nabla' dV$$

(18.2)

**Evaluation of Surface Integral**

As before, the thermal potential $\phi_2$ is negligible on the surface $r = \bar{R}$ so that only $\phi_1 + \phi$ need be considered. From (13.3) the temperature is then

$$T = \alpha_1 (\phi + \phi_1)$$

(18.3)

and

$$J_s = \frac{\sigma}{2T_0} \alpha_1 \alpha_1^* R \int (\phi_1 + \phi) \frac{\partial}{\partial n} (\phi_1 + \phi) dS_n$$

(18.4)

Comparing this with the corresponding integral $I_s$ for the viscous dissipation (17.10) we see that

$$J_s/I_s \sim \frac{\sigma \alpha \alpha^*}{T_0 \omega \rho_0} \sim (r-1) \frac{\bar{R} \omega}{c^2} << 1$$

(18.5)
So that $J_s$ is negligible compared to the terms of order $I_s$ which we are retaining in our treatment. $J_s$ contributes negligibly to the thermal dissipation and will therefore be dropped.

**Evaluation of Volume Integrals**

The two volume integrals $J_v$ and $J'_v$ are evaluated in Appendix VIII, where it is shown that they are of the same order as $I_s$: The volume integrals $J_v$ and $J'_v$ give a thermal dissipation which is of the same order as the viscous dissipation. The results are

\[
J_v = \frac{1}{2} \frac{\rho_0 \omega^2}{\sigma} R \int (\phi_1 + \phi_2) \phi_2 \, dV
\]  
\[= \frac{1}{2} \frac{\rho_0 \omega^2}{\sigma} R \left[ -\frac{4\pi R}{k_2^2} C_0 a_2 h_1(a_2) \right] \tag{18.6}
\]

\[
J'_v = \frac{1}{2} \frac{p'_0 \omega^2}{\sigma'} R \int \phi'_1 \phi'_2 \, dV
\]  
\[= \frac{1}{2} \frac{p'_0 \omega^2}{\sigma'} R \left[ \frac{4\pi R}{k'_2^2} B'_0 C'_0 a'_2 j'_1(a'_2) \right] \tag{18.7}
\]

Substituting $C_0$, $C'_0$, and $B'_0$ from Sec. 16 we find, after reducing the coefficients,

\[
J_v = 4\pi R \left( \frac{1}{2} p_0 k_1 \right) \alpha E(x-i)(1-\delta \frac{\alpha'_1}{\alpha'_2}) R \left[ \frac{a_2 h_1(a_2)/h_0(a_2)}{1 - \gamma' Z} \right] \tag{18.10}
\]

\[
J'_v = -\left( \delta \frac{\alpha'_1}{\alpha'_2} \right) J_v \tag{18.11}
\]

$Z$ being given by (16.5).
Final Results for Thermal Dissipation

Combining $J_v$ and $J'_v$ we have

$$\frac{dE}{dt} = 4\pi R \left( \frac{1}{2} \rho_0 k_0^2 \right) \delta (s-i) \left( 1 - \delta \frac{x_1}{x_2} \right)^2 R \left[ \frac{a_2 h_1(a_2)}{h_0(a_2)} \right]$$

This is our desired result for the thermal dissipation of a plane sound wave by a small sphere. The bracket can be simplified by using the general relations $a_2 h_1(a_2)/h_1(a_2) = 1 - ia_2$ and $a_2 j_1(a_2)/j_1(a_2) = 1 - a_2 \cot a_2$; this gives

$$\frac{dE}{dt} = 4\pi R \left( \frac{1}{2} \rho_0 k_0^2 \right) \delta (s-i) \left( 1 - \delta \frac{x_1}{x_2} \right)^2 R \left[ \frac{1 - ia_2}{1 - a_2 \cot a_2} \right]$$

as an alternate form for $\frac{dE}{dt}$. We shall not attempt further reduction for these general forms, since the algebra, while straightforward, leads to very complicated expressions.

This concludes our calculation of the thermal and viscous dissipation.

We now turn to the evaluation of the attenuation coefficient.

19. The Attenuation Coefficient

The attenuation coefficient $\alpha$ for a plane sound wave traversing a medium containing $n$ small scattering spheres per unit volume, the energy loss per unit time due to each sphere being $\frac{dE}{dt}$, is defined as the ratio of the total energy $n\frac{dE}{dt}$ lost per unit time to the energy flux $E_0$ of the plane wave:

$$\alpha = \frac{n}{E_0} \frac{dE}{dt}$$

Since $\frac{dE}{dt}$ is the sum of the viscous and thermal dissipations
\[
\frac{dE}{dt} = \frac{dE_n}{dt} + \frac{dE_\sigma}{dt} \tag{19.2}
\]

we have from (19.1)

\[
\alpha = \alpha_n + \alpha_\sigma \tag{19.3}
\]

where

\[
\alpha_n = \frac{n}{E_0} \frac{dE_n}{dt} \quad \alpha_\sigma = \frac{n}{E_0} \frac{dE_\sigma}{dt} \quad . \tag{19.4}
\]

From (10.11) we have \( E_0 = \frac{1}{2} k_1 \rho_0 \omega \), and using \( \frac{dE_n}{dt} \) and \( \frac{dE_\sigma}{dt} \) from the previous sections we obtain the general expressions for the viscous and thermal attenuations:

\[
\alpha_n = \frac{4\pi n}{k_1^3} \mathcal{R} \left[ -3B_i \right] \tag{19.5}
\]

\[
\alpha_n = \frac{4\pi n}{k_1^3} \mathcal{R} \left[ -i \frac{b h_2(b) \left[ \delta h_2(b) - \frac{1}{2} b_1^2 \right] - \left( \frac{1}{2} \delta b^2 \right) b j_2(b) h_1(b) }{[-(\alpha + \delta) b h_2(b) + 9 \delta h_1(b)] \left[ \delta h_2(b) - \frac{1}{2} b_1^2 \right] + \left( \frac{1}{2} \delta b^2 \right) (2\alpha + \delta) b j_2(b) h_1(b) } \right] \tag{19.6}
\]

and

\[
\alpha_\sigma = \frac{4\pi n R}{c} \frac{\mathcal{R}(z-i)}{(1-\delta \frac{\alpha_1'}{\alpha_1})^2} \mathcal{R} \left[ \frac{a_2 h_1(a_2)}{h_0(a_2)} \right] \frac{1-i a_2}{1-a_2' \cot a_2'} \tag{19.7}
\]

where

\[
Z = \frac{j_0(a_2')}{a_2' j_1(a_2')} \frac{a_2 h_1(a_2)}{h_0(a_2)} = \frac{1-i a_2}{1-a_2' \cot a_2'} . \tag{19.8}
\]
The explicit general expressions (19.6) and (19.7) for $\alpha_\eta$ and $\alpha_\sigma$ conclude our main objective; together they give the additional attenuation of a plane sound wave due to small fluid spheres in a fluid medium. We shall apply these general results in the following chapter to calculate the attenuation by water drops in air and by air bubbles in water.

(Note: Both factors in the integrand $p^* n \eta e$ of the surface integral in (17.1) are continuous at the surface of the small sphere; the two surface integrals (bounding the inside and outside media) therefore cancel at this boundary, leaving only the outside surface integral in (17.7) at the boundary of the large sphere.)
V. APPLICATIONS TO WATER DROPS IN AIR AND AIR BUBBLES IN WATER

In this chapter we shall apply our general results to two limiting cases of great practical importance. For water drops in air the density, viscosity, and thermal conductivity of the drops are large compared to the outside medium; for air bubbles in water the reverse is true. In each case the general expressions for the viscous and thermal attenuation simplify considerably.

20. Attenuation by Water Drops in Air

In this section we shall reduce our theoretical results for comparison with Knudsen's data on the attenuation of sound in fogs (Ref. 7). Since the average drop radius is about $10^{-3}$ m we see that our assumption $a_1 = \frac{\omega R}{c} < 1$ will be reasonably valid up to $\omega = 10^6$ sec$^{-1}$ (for which $a_1 \sim 0.03$).

Since water is denser, more viscous, and a better thermal conductor than air we see that the outside-to-inside density ratio $\delta$, viscosity ratio $\epsilon$, and conductivity ratio $\chi$ are all small. The actual values appropriate to Knudsen's data are

\[ \delta = 0.00117, \quad \epsilon = 0.0167, \quad \chi = 0.040 \]  \hspace{1cm} (20.1)

It is shown in Appendix IX that for small $\delta$, $\epsilon$, $\chi$ the general expression (19.6) for $\alpha_\eta$ reduces to

\[
\alpha_\eta = \frac{4\pi n}{k_i} a_1^3 R \left[ -i \frac{b h_i(b)}{-2 b h_o(b) + 9 \delta h_i(b)} \right] \]  \hspace{1cm} (20.2)

\[
= \frac{4\pi n R}{c} \frac{3}{2} v(i+y) \left[ \frac{16y^4}{16y^4 + 72\delta y^3 + 81\delta^2(1+2y+2y^2)} \right] \]  \hspace{1cm} (20.3)
\[ \alpha_\eta = \frac{9}{2} V_p \frac{\nu}{c R^2} (1+y) \left[ \frac{16y^4}{16y^4 + 12\delta y^3 + 81\delta^2(1+2y+2y^2)} \right] . \quad (20.4) \]

The second form is obtained in Appendix IX by substituting the explicit expressions for the h's and defining the dimensionless real variable \( y \) by

\[ y^2 = \frac{b^2}{2} = \omega R^2/2 \nu \quad b = (1+i) y . \quad (20.5) \]

In the third form the dimensionless quantity \( V_p \) is the total volume of the water drops per unit volume of air:

\[ V_p = \frac{4}{3} \pi n R^3 \quad (20.6) \]

In both (20.3) and (20.4) the coefficient before the bracket is Sewell's attenuation discussed in the introduction (0.7):

\[ \alpha_\text{sew} = \frac{4\pi n R}{c} \frac{3}{2} \nu (1+y) = \frac{9}{2} V_p \frac{\nu}{c R^2} \left[ 1 + (\omega R^2/2 \nu)^{1/2} \right] . \quad (20.7) \]

The expression (20.4) for \( \alpha_\eta \) agrees with Knudsen's reduction (Ref. 7) of Epstein's result (Ref. 6), and since \( y \to \infty \) as \( \omega \to \infty \) the bracket approaches unity and \( \alpha_\eta \) is seen to reduce to Sewell's result at high frequencies. At low frequencies \( y \to 0 \) and the bracket approaches zero as \( y^4 \), so that \( \alpha_\eta \) vanishes, as it should for zero frequency, in contrast to Sewell's result which is finite at \( \alpha = 0 \). The physical reasons for this were discussed in the introduction and we thus see that the correct expression (20.4), originally obtained by Epstein is also valid in our problem which includes thermal conduction.

Turning to the thermal attenuation and imposing the condition that \( \delta, \epsilon, \chi \) be small we have from (19.7):
\[ \alpha_t = \frac{4 \pi n R}{c} \bar{\kappa}(y-1) R \left[ \frac{a_1 h(a_1) / h_0(a_1)}{1 - \chi Z} \right] \]  
\hspace{1cm} (20.8)

since \( \alpha_t / \alpha_i \approx 0.014 \) so that \( \delta \alpha_t / \alpha_t \ll 1 \). In passing we note that, using (18.11), the last inequality shows that \( J' \ll J \), i.e., that the thermal dissipation inside the drops is negligible compared to that outside.

It is shown in Appendix IX that for small \( \chi \) the real part of the bracket reduces to

\[ (1 + z) \left[ 1 + \frac{3}{2} \frac{\delta c_p / c_p}{z^2} + \frac{9}{4} \left( \frac{\delta c_p / c_p}{z^2} \right)^2 \right]^{-1} \]  
\hspace{1cm} (20.9)

where the dimensionless variable \( z \) is defined by

\[ z^2 = a^2 / 2 \omega = \omega R^2 / 2 \bar{\omega} \]  
\[ a_2 = (1 + i) \bar{\omega} \]

and we finally have for the thermal attenuation the results

\[ \alpha_0 = \frac{4 \pi n R}{c} \bar{\kappa}(y-1)(1 + z) \left[ \frac{4 z^4}{4 z^4 + 12 (\delta c_p / c_p) z^3 + 9 (\delta c_p / c_p)^2} \right] \]  
\hspace{1cm} (20.10)

\[ = 3 V_p \frac{\bar{\kappa}(y-1)}{c R^2} (1 + z) \left[ \frac{4 z^4}{4 z^4 + 12 (\delta c_p / c_p) z^3 + 9 (\delta c_p / c_p)^2} \right] \]  
\hspace{1cm} (20.11)

which correspond to the expressions for \( \alpha_\eta \) in (20.3) and (20.4).

We note that at low frequencies \( \alpha_0 \propto z^4 \to 0 \) as it should, there being no reason for any thermal losses at zero frequency. Thus the rigorous present treatment avoids the anomaly of a constant thermal attenuation at \( \omega = 0 \), corresponding to Sewell's values of \( \alpha_\eta \). The limiting values of the attenuations at low frequencies are
and the ratio of the thermal to the viscous attenuation is

\[
\frac{\alpha_{\sigma}}{\alpha_{\eta}} \rightarrow \frac{3}{2} \frac{\eta}{\delta e} \left( \frac{c_p}{c_p} \right)^2 \sim 7.2
\]  

(20.13)

At high frequencies \( \alpha_{\sigma} \) approaches the value

\[
\frac{\alpha_{\sigma}}{\omega \rightarrow \infty} \rightarrow \frac{4\pi n R}{c} \frac{3}{2} \delta e (\gamma - 1) \left( \frac{c_p}{c_p} \right)^2 \left( \frac{1}{1+z} \right) = \frac{3}{2} \sqrt{\frac{\delta e}{\nu}} \left( \gamma - 1 \right) \left( \frac{\delta e}{\nu} \right)^{1/2} \sim 0.3
\]  

(20.14)

corresponding to Sewell's result (20.7) for \( \alpha_{\eta} \). Finally we note that the ratio of \( \alpha_{\sigma} \) to \( \alpha_{\eta} \) at high frequencies is asymptotic to the value

\[
\frac{\alpha_{\sigma}}{\alpha_{\eta}} \rightarrow \frac{\delta e}{\nu} \left( \frac{3}{2} \right) \gamma \sim 0.3
\]  

(20.15)

From (20.10) and (20.12) we see that at low frequencies the thermal attenuation of water drops in air dominates the viscous attenuation by a factor of seven, while at high frequencies the viscous attenuation dominates by a factor of three.

A convenient way to show the dependence of the attenuation on frequency \( \omega \) and radius \( R \) of the drop is to plot the quantities

\[
\sum_{\gamma} = \frac{\alpha_{\eta} c}{4\pi n R} = \frac{3}{2} \nu (1+y) \left[ \frac{16y^4}{16y^4 + 72\delta y^3 + 81\delta^2(1+2y+2y^2)} \right]
\]  

(20.16)
and
\[
\Sigma = \frac{\alpha c}{4\pi n R} = \frac{\delta e(y-i)(1+z)}{4 \pi^4} \left[ \frac{4 \pi^4}{4 \pi^4 + 12(\delta c_p/c_p)^2 z^3 + 9(\delta c_p/c_p)^2} \right]
\]

(20.17)

against \(\omega R^2\) (which is a factor of \(y^2\) and \(z^2\)) and thus obtain general curves for all \(\omega\) and \(R\). In addition we define

\[
\Sigma = \Sigma_\eta + \Sigma_\sigma = \frac{\alpha c}{4\pi n R}
\]

(20.18)

corresponding to the total attenuation and

\[
\Sigma_{\text{Sewell}} = \frac{\alpha_{\text{Sewell}} c}{4\pi n R} = \frac{3}{2} \omega (1+y)
\]

(20.19)

corresponding to Sewell's value.

These quantities are plotted against \(\omega R^2\) in Fig. 1. It is seen that the \(\Sigma_\eta\) and \(\Sigma_\sigma\) curves are quite similar throughout the entire range. At high frequencies \(\Sigma_\eta \sim 3 \Sigma_\sigma\) as remarked above, and both are proportional to \((\omega R^2)^{1/2}\). At low frequencies \(\Sigma_\sigma \sim 7 \Sigma_\eta\). The critical frequency separating the high and low frequency regions is determined by the point at which \(\Sigma_\sigma = \Sigma_\eta\). This is seen to be \(\omega R^2 \sim 6 \times 10^{-4}\), so that

\[
\omega_{\text{crit}} = 6 \times 10^{-4}/R^2
\]

(20.20)

where \(R\) is the drop radius in cm and \(\omega = 2\pi f\) is the circular frequency in sec\(^{-1}\).

The overall curve \(\Sigma\) has the same asymptotic dependence at high and low frequencies as \(\Sigma_\eta\) and \(\Sigma_\sigma\) have. The Sewell curve \(\Sigma_{\text{Sewell}}\) is seen
to level off to a constant value: \( \sum_{\text{const}} \alpha_{\text{Se}} = 0.23 \) as \( \omega \to 0 \), corresponding to the constant value of \( \alpha_{\text{Se}} \) at low frequencies already discussed. Finally, it should perhaps be emphasized that the expressions for \( \alpha_\eta \) and \( \alpha_\sigma \) refer to a fog consisting of water drops all having the same radius \( R \).

21. Comparison with Knudsen's Data

The absorption of sound in fogs has been observed by many workers but in most cases no analysis of drop size was made and this precludes any close comparison with the theory. Recently, however, Knudsen (Ref. 7) carried out absorption measurements in an artificially produced fog, using the reverberation chamber technique, and in this work the distribution in drop-size was determined. The experimental results and the calculated attenuations (see below) are given in Table 21.1, together with the corresponding physical data.
Table 21.1 - Comparison of Observed and Calculated Attenuation

<table>
<thead>
<tr>
<th>f (sec^-1)</th>
<th>Observed (db/cm)</th>
<th>Calculated (db/cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>300</td>
<td>((1.4 \pm 0.2) \times 10^{-5})</td>
<td>(1.32 \times 10^{-5})</td>
</tr>
<tr>
<td>1000</td>
<td>((2.0 \pm 0.3) \times 10^{-5})</td>
<td>(1.74 \times 10^{-5})</td>
</tr>
<tr>
<td>2000</td>
<td>((2.7 \pm 0.4) \times 10^{-5})</td>
<td>(1.89 \times 10^{-5})</td>
</tr>
<tr>
<td>4000</td>
<td>((2.9 \pm 0.4) \times 10^{-5})</td>
<td>(2.07 \times 10^{-5})</td>
</tr>
<tr>
<td>6000</td>
<td>((3.5 \pm 0.4) \times 10^{-5})</td>
<td>(2.19 \times 10^{-5})</td>
</tr>
<tr>
<td>8000</td>
<td>((3.8 \pm 0.4) \times 10^{-5})</td>
<td>(2.27 \times 10^{-5})</td>
</tr>
</tbody>
</table>

\(T = 22^\circ C = 295^\circ A\) \hspace{1cm} \(p_o = 755 \text{ mm Hg}\) \hspace{1cm} \(V_p = (2.00 \pm 0.10) \times 10^{-6}\)

**Water Drops:**
- \(\rho_o = 1 \text{ gm/cm}^3\)
- \(c_p = 1 \text{ cal/oA}\)
- \(\bar{\kappa} \vec{t} = 1.43 \times 10^{-3} \text{ cm}^2/\text{sec}\)
- \(\vec{\nu} = 0.011 \text{ cm}^2/\text{sec}\)
- \(c' = 1.45 \times 10^5 \text{ cm/sec}\)

**Air:**
- \(\rho_o = 1.17 \times 10^{-3} \text{ gm/cm}^3\)
- \(\vec{\eta} = 1.32 \times 10^{-4} \text{ gram/cm sec}\)
- \(\vec{\nu} = \vec{\eta}/\rho = 0.156 \text{ cm}^2/\text{sec}\)
- \(\vec{\kappa} = \sigma/\rho c_p = 0.206 \text{ cm}^2/\text{sec}\)
- \(c_p = 0.24 \text{ cal/oA}\)
- \(c = 3.44 \times 10^4 \text{ cm/sec}\)

The drops were grouped into five size classes and the mean radius and fractional volume \((\Delta V/V_p)\) for each class calculated. The results are summarized in Table 21.2.
Table 21.2 - Drop Size Data

<table>
<thead>
<tr>
<th>No. drops</th>
<th>Fractional Number</th>
<th>Fract.Vol. ( \Delta V/V_p )</th>
<th>Mean Radius (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) 102</td>
<td>0.121</td>
<td>0.005</td>
<td>3.75 ( \times 10^{-4} )</td>
</tr>
<tr>
<td>(2) 375</td>
<td>0.445</td>
<td>0.034</td>
<td>6.25 ( \times 10^{-4} )</td>
</tr>
<tr>
<td>(3) 230</td>
<td>0.272</td>
<td>0.211</td>
<td>10.0 ( \times 10^{-4} )</td>
</tr>
<tr>
<td>(4) 93</td>
<td>0.110</td>
<td>0.298</td>
<td>15.0 ( \times 10^{-4} )</td>
</tr>
<tr>
<td>(5) 43</td>
<td>0.052</td>
<td>0.402</td>
<td>21.5 ( \times 10^{-4} )</td>
</tr>
<tr>
<td>843</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The attenuation \( \tilde{\alpha} \) in decibels per centimeter (db/cm) is given by

\[
\tilde{\alpha} = 4.34 \alpha = 4.34(\alpha_{\eta} + \alpha_{\sigma})
\]

(21.1)

where \( \alpha_{\eta} (\text{cm}^{-1}) \) and \( \alpha_{\sigma} (\text{cm}^{-1}) \) are given by (20.4) and (20.11). At each frequency the attenuation \( \tilde{\alpha} \) was calculated for each class, and the \( \tilde{\alpha} \)'s for each class were then added after weighing them by the fractional volume factor. The calculated and observed attenuations are plotted against frequency in Fig. 2. The agreement is seen to be fair, the calculated and measured values having the same order of magnitude and the same frequency dependence, but with the calculated values lying below the observed values. The reason for this is not clear. It is known that the results would be strongly affected by changes in the drop-size distribution but whether more accurate size distribution data would remove the discrepancy is doubtful. A careful analysis of the experimental methods would be required to assess
other factors affecting the attenuation, and this would be out of place in the present work.

22. Attenuation by Air Bubbles in Water

Turning to the attenuation by air bubbles in water we have the opposite case in which the outside-to-inside density, viscosity and conductivity ratios are large:

\[ \delta = 855, \quad \varepsilon = 60, \quad \lambda = 25 \]  

(22.1)

It is shown in Appendix X that for large \( \delta, \varepsilon, \lambda \) the general expression (19.6) for \( \alpha_\eta \) reduces to

\[ \alpha_\eta = \frac{4\pi n}{k_i^2} a^3 R \left[ -i \frac{b h_0(b) + \frac{1}{2} b^2 h_1(b)}{b h_0(b) - 9 h_1(b) + \frac{1}{2} b^2 h_1(b)} \right] \]  

(22.2)

or

\[ \alpha_\eta = \frac{4\pi n R}{c} 2 \nu \left[ \frac{2 y^7 + 12 y^6 + 6 y^5}{2 y^6 + 6 y^5 + 9 y^4 + 36 y^3 + 162 y^2 + 162 y + 81} \right] \]  

(22.3)

For the thermal attenuation we note that now \( \delta \alpha_1/\alpha_1 \sim 6 \times 10^4 \gg 1 \) so that \( J_v \ll J'_v \); i.e., the thermal dissipation in the water is negligible compared to that in the air inside the bubbles. The attenuation is given by (19.7):

\[ \alpha_\sigma = \frac{4\pi n R}{c} \bar{\sigma} (\gamma - 1) (\delta \omega_i/\omega_i)^2 R \left[ \frac{a_h(a_1)/h_0(a_2)}{1 - \gamma \frac{j_0(a_1)}{a''_j(a_2)} \frac{a_h(a_1)}{a''_j(a_2)}} \right] \]  

(22.4)

Since \( \gamma \gg 1 \) we see that for both low and high frequencies \( (\alpha_2, a_2 \rightarrow 0 \) or \( \infty \) the second term in the denominator dominates the first. Neglecting the first term we can then cancel the common factor \( a_2 h_1(a_2)/h_0(a_2) \), and after rearranging the constants with the aid of \( (\alpha_1/\alpha_1)^2 = (\delta \nu \cdot c_c p)/(\delta \nu \cdot c' c_p) \),
(22.4) reduces to
\[ \alpha' = \frac{4\pi nR}{c} \tilde{e}'(z') \left( \rho_o c^2 / \rho'_o c^2 \right) R \left[ -a'_2 \frac{j(\alpha'_2)}{j_0(\alpha'_2)} \right] \quad (22.5) \]
or
\[ \alpha' = \frac{4\pi nR}{c} \tilde{e}'(z') \left( \rho_o c^2 / \rho'_o c^2 \right) R \left[ a'_2 \cot \alpha'_2 - 1 \right] \quad (22.6) \]

Again introducing the real dimensionless \( z' \)
\[ z'^2 = \frac{a'_2^2}{2i} = \omega R^2 / 2 \tilde{e}' \quad a'_2 = (1+i) z' \quad (22.7) \]

we find, after expanding the cotangent and taking the real part,
\[ \alpha' = \frac{4\pi nR}{c} \tilde{e}'(z') \left( \rho_o c^2 / \rho'_o c^2 \right) \left[ z' \frac{\sinh 2z'}{\cosh 2z'} - \cos 2z' - 1 \right] \quad (22.8) \]

This is the final desired form for the thermal attenuation by air bubbles in water. Numerical comparison of \( \alpha'_\eta \) and \( \alpha'_\sigma \) shows that for all values of \( y \) and \( z' \) (or of \( \omega R^2 \)),
\[ \alpha'_\eta / \alpha'_\sigma \sim 10^{-5} \ll 1 \]
this result being primarily due to the large factor
\[ \rho_o c^2 / \rho'_o c^2 = 1.5 \times 10^4 \quad (22.9) \]

Thus the thermal dissipation inside the air bubbles completely dominates the viscous dissipation. This is an important and interesting result. Rewriting (22.8) we have for the attenuation
\[ \alpha = \alpha' = \frac{4\pi nR}{c} A X(z') \quad (22.10) \]
and for the quantity
\[ \Sigma = \Sigma' = \frac{\alpha' c}{4\pi nR} = A X(z') \quad (22.11) \]
Where
\[ A = \frac{\rho c^2}{\rho_0 c^2} = 1.27 \times 10^4 \text{ cm}^2/\text{sec} \]
\[ X(z') = R \left[ -a_z j_z(a_z')/j_z(a_z') \right] = R \left[ a_z' \cot a_z' - 1 \right] \]
\[ = z' \frac{\sinh 2z' + \sin 2z'}{\cosh 2z' - \cos 2z'} - 1 \]  

A general plot of + against is shown in Fig. 3. The function \( X(z') \) lies close to its asymptotic values
\[ X \rightarrow \frac{4}{45} z' \quad \text{as } z' \rightarrow 0 \]
\[ X \rightarrow z' - 1 \quad \text{as } z' \rightarrow \infty \]  

except in a small region near \( z' = 2 \).

At the high and low frequency limits \( \sum \) has the corresponding asymptotic forms (light lines in Fig. 3):
\[ \sum \rightarrow A \frac{4}{45} z' \quad \text{as } z' \rightarrow 0 \]
\[ \sum \rightarrow A z' \quad \text{as } z' \rightarrow \infty \]  

Since we do not have experimental data on both absorption by bubbles and bubble size distribution we shall not attempt to discuss the correspondence between theory and experiment.

We must note in conclusion that the results obtained here are valid only for bubbles whose radius varies only slightly during the passage of the primary sound wave. Our value of the attenuation is therefore not valid in the neighborhood of the resonance peak where large oscillations occur (Ref. 9).
PART II

Part II contains the various mathematical appendices referred to in the text; as well as the References, Table I (Physical Constants), and the Figures.
APPENDIX I

Time Average of Products of First-Order Quantities

We wish to list several useful expressions involving the time averages of products of two first-order quantities $A_1$ and $B_1$ of the form (see Sec. 7):

\[ A_1 = A e^{-i\omega t} \quad B_1 = B e^{-i\omega t} \]  \hspace{1cm} (1.1)

where $A$ and $B$ are complex space functions independent of time and having absolute values $A$ and $B$:

\[ A = |A| e^{i\alpha} \quad B = |B| e^{i\beta} \]  \hspace{1cm} (1.2)

so

\[ A_1 = |A| e^{i(\alpha - \omega t)} \quad B_1 = |B| e^{i(\beta - \omega t)} . \]

Now as pointed out in Sec. 7, the first-order quantities are all real; for convenience however we represent them as complex quantities. Consequently, only the real parts $\Re(A_1)$, $\Re(B_1)$ are to be used in evaluating the products. Thus the time average of $A_1 B_1$ is

\[ (A_1 B_1)_\text{Av} = [\Re(A_1) \Re(B_1)]_{\text{Av}} \]  \hspace{1cm} (1.3)

so that from (1.2)

\[ (A_1 B_1)_\text{Av} = |A| |B| [\cos(\alpha - \omega t) \cos(\beta - \omega t)] \]  \hspace{1cm} (1.4)

\[ = \frac{1}{2} |A| |B| [\cos(\alpha + \beta - 2\omega t) + \cos(\alpha - \beta)] \]  \hspace{1cm} (1.5)

\[ = \frac{1}{2} |A| |B| \cos(\alpha - \beta) , \]  \hspace{1cm} (1.6)

since the time average of the first quantity in the bracket of (1.5) is zero. But now, using (1.2)
\[ \frac{1}{2} (AB^* + A^*B) = \frac{1}{2} |A| |B| \left[ e^{i(\alpha - \beta)} + e^{-i(\alpha - \beta)} \right] \tag{1.7} \]

\[ = |A| |B| \cos(\alpha - \beta) \tag{1.8} \]

so that (1.6) becomes

\[ (A, B)_{AV} = \frac{1}{4} (AB^* + BA^*) \tag{1.9} \]

We now use

\[ \frac{1}{2} (AB^* + A^*B) = R(AB^*) = R(AB^*) \tag{1.10} \]

to obtain our final result

\[ (A, B)_{AV} = \frac{1}{2} R(AB^*) = \frac{1}{2} R(AB^*) \tag{1.11} \]

which may also be written

\[ (A, B)_{AV} = \frac{1}{2} R(AB^*) = \frac{1}{2} R(AB^*) \tag{1.12} \]

Two special cases of interest are

\[ (A, A)_{AV} = \frac{1}{2} R(AA^*) = \frac{1}{2} |A|^2 = \frac{1}{2} |A|^2, \tag{1.13} \]

\[ (\hat{A}, A)_{AV} = \left[ (-i\omega A) A \right]_{AV} = \frac{\omega}{2} R[(iA)A^*] = 0 \tag{1.14} \]

(1.14) follows from the fact that the quantity in the last bracket is pure imaginary, so that its real part vanishes.
APPENDIX II

Spherical Bessel and Hankel Functions

The radial part \( R(kr) \) of the wave function satisfies the spherical Bessel Equation

\[
\frac{1}{\rho^2} \frac{d}{d\rho} \left( \rho^2 \frac{dR}{d\rho} \right) + \left( k^2 - \frac{n(n+1)}{\rho^2} \right) R = 0
\]

or

\[
\frac{1}{\rho^2} \frac{d}{d\rho} \left( \rho^2 \frac{dR}{d\rho} \right) + \left( 1 - \frac{n(n+1)}{\rho^2} \right) R = 0
\]

or

\[
\rho^2 R'' + 2\rho R' + \left[ \rho^2 - n(n+1) \right] R = 0
\]

where \( \rho = kr, \) \( k \) being the complex wave number, and \( R' = dR/d\rho \) etc. The solutions of this equation are the spherical Bessel functions \( j_n(\rho) \) (regular at \( \rho=0 \)), the spherical Neumann functions \( y_n(\rho) \) (singular at \( \rho=0 \)), and the first and second spherical Hankel functions \( h_n(\rho) \) and \( \tilde{h}_n(\rho) \), defined by

\[
\begin{align*}
  j_n(\rho) &= (\pi/2\rho)^{1/2} J_{n+1/2}(\rho) \\
  y_n(\rho) &= (\pi/2\rho)^{1/2} Y_{n+1/2}(\rho) \\
  h_n(\rho) &= j_n(\rho) + iy_n(\rho) \\
  \tilde{h}_n(\rho) &= j_n(\rho) - iy_n(\rho)
\end{align*}
\]

where \( J_{n+1/2} \) and \( Y_{n+1/2} \) are the ordinary cylindrical Bessel functions of half-odd-integer order. Letting \( R_n(\rho) \) denote any solution of (2.1), the following relations are satisfied by \( j_n, y_n, h_n, \) and \( \tilde{h}_n \) for \( n \geq 0 \):
\[
\rho R_{n+1}(\rho) = (2n+1) R_n(\rho) - \rho R_{n-1}(\rho)
\]
(2.3)

\[
\rho R'_n(\rho) = \rho R_{n-1}(\rho) - (n+1) R_n(\rho)
\]
(2.4)

\[
\rho R''_n(\rho) = n R_n(\rho) - \rho R_{n+1}(\rho)
\]
(2.5)

\[
\rho R''_n(\rho) = 2 \rho R_{n+1}(\rho) + n(n-1) R_n(\rho) - \rho^2 R_n(\rho)
\]
(2.6)

\[
\rho R''_n(\rho) = (n+1)(n+2) R_n(\rho) - \rho^2 R_n(\rho) - 2 \rho R_{n-1}(\rho)
\]
(2.7)

For \( n=0 \), terms of the form \( R_{n-1}(\rho) \) can be evaluated by using the general relations

\[
y_n(\rho) = (-1)^{n+1} j_{-n-1}(\rho) \quad y_n(\rho) = (-1)^{n+1} j_{n-1}(\rho)
\]
(2.8)

to obtain \( y \) and \( j \) for negative orders.

Explicit expressions for the first three \( j \)'s, \( y \)'s and \( h \)'s are

\[
j_0(\rho) = \frac{\sin \rho}{\rho} \quad j_1(\rho) = \frac{\sin \rho}{\rho^2} - \frac{\cos \rho}{\rho}
\]
(2.9)

\[
j_2(\rho) = \left( \frac{3}{\rho^3} - \frac{1}{\rho} \right) \sin \rho - \frac{3}{\rho^2} \cos \rho
\]
(2.9)

\[
y_0(\rho) = -\frac{\cos \rho}{\rho} \quad y_1(\rho) = -\frac{\cos \rho}{\rho^2} - \frac{\sin \rho}{\rho}
\]
(2.10)

\[
y_2(\rho) = -\left( \frac{3}{\rho^3} - \frac{1}{\rho} \right) \cos \rho - \frac{3}{\rho^2} \sin \rho
\]
(2.11)

\[
h_0(\rho) = -i \frac{e^{i\rho}}{\rho} \quad h_1(\rho) = -e^{i\rho} \left( \frac{i}{\rho^2} + \frac{1}{\rho} \right)
\]
(2.11)

\[
h_2(\rho) = -e^{i\rho} \left( \frac{3i}{\rho^3} + \frac{3}{\rho^2} - \frac{i}{\rho} \right)
\]
(2.11)

For small arguments the following expansions are valid for \( n \geq 0 \):

\[
j_n(\rho) \xrightarrow{\rho \to 0} \frac{\rho^n}{1 \cdot 3 \cdot 5 \ldots (2n+1)} \left[ 1 - \frac{\rho^2}{2(2n+3)} + \frac{\rho^4}{2 \cdot 4(2n+3)(2n+5)} - \ldots \right]
\]
(2.12)
The general series expansion for $h_n(p)$ is

$$h_n(p) = i^{-n+1} \frac{e^{ip}}{p} \left[ 1 + \frac{(n+1)!}{(n-1)!} \left( \frac{i}{2p} \right) + \frac{(n+2)!}{(n-2)!} \left( \frac{i}{2p} \right)^2 + \cdots + \frac{(2n)!}{n!} \left( \frac{i}{2p} \right)^n \right]$$

For large values of $p$ the asymptotic expansions are

$$j_n(p) \xrightarrow{p \to \infty} \frac{i}{p} \cos \left[ p - (n+\frac{1}{2}) \right]$$

$$y_n(p) \xrightarrow{p \to \infty} \frac{1}{p} \sin \left[ p - (n+\frac{1}{2}) \right]$$

$$h_n(p) \xrightarrow{p \to \infty} \frac{1}{p} e^{i \left[ p - (n+\frac{1}{2}) \right]}$$

We also note that since $j$ and $y$ are real functions of $p$, the complex conjugates of $j$, $y$, and $h$, are given by

$$j_n^*(p) = j_n(\rho^*) \quad j_n^*(\rho) = j_n(\rho^*)$$

$$y_n^*(\rho) = y_n(\rho^*)$$

$$h_n^*(\rho) = j_n(\rho^*) + i y_n(\rho^*) = j_n(\rho^*) - i y_n(\rho^*) = h_n(\rho^*)$$

so that in the case of $h_n(p)$, the complex conjugate $h_n^*(k\lambda) = h(k^*\lambda)$ is also a solution of the differential equation, with the complex conjugate wave number, a fact we shall use later.
Finally we list several useful integrals. All are easily derived from the differential equations for the two spherical Bessel functions in the integrand, or they may be obtained from the corresponding integrals for the cylindrical functions, using the definitions (2.2). (See Ref. 16). Let $R_n(k\lambda)$ and $\bar{R}_n(l\lambda)$ be any two spherical Bessel functions, $k$ and $l$ being general complex wave numbers and $r$ the real radial coordinate. Then

$$\int^\infty R_n^2(k\lambda) \lambda^2 d\lambda = \frac{\lambda^3}{2} \left[ R_n^2(k\lambda) - R_{n-1}(k\lambda) R_{n+1}(k\lambda) \right], \quad (2.20)$$

$$\int^\infty R_n(k\lambda) \bar{R}_n(k\lambda) \lambda^2 d\lambda = \frac{\lambda^3}{4} \left[ 2R_n(k\lambda) \bar{R}_n(k\lambda) - R_{n-1}(k\lambda) \bar{R}_{n+1}(k\lambda) - R_{n+1}(k\lambda) \bar{R}_{n-1}(k\lambda) \right], \quad (2.21)$$

$$\int^\infty R_n(k\lambda) \bar{R}_n(l\lambda) \lambda^2 d\lambda = \frac{\lambda^2}{k^2 - l^2} \left[ l \bar{R}_{n-1}(l\lambda) R_n(k\lambda) - k R_{n-1}(k\lambda) \bar{R}_n(l\lambda) \right] \quad (2.22)$$

$$= \frac{\lambda^2}{k^2 - l^2} \left[ k R_{n+1}(k\lambda) \bar{R}_n(l\lambda) - l \bar{R}_{n+1}(l\lambda) R_n(k\lambda) \right] \quad (2.23)$$

A useful special case of (2.23) is the following integral, obtained by use of (2.19):

$$\int^\infty h_n^*(k\lambda) h_n(k\lambda) \lambda^2 d\lambda = \frac{\lambda^2}{k^2 - k^*^2} \left[ k h_{n-1}^*(k\lambda) h_n^*(k\lambda) - k^* h_{n+1}^*(k\lambda) h_n(k\lambda) \right], \quad (2.24)$$

The above integrals are all valid for $n \geq 0$ where again, for the case $n=0$, the relations (2.8) are to be used to evaluate $j_{-1}$ or $y_{-1}$ in the $R_{n-1}$ terms.
Legendre Functions

The Legendre polynomial $P_n(\cos \Theta)$ satisfies the $\Theta$-part of the wave equation:

$$\frac{1}{\sin \Theta} \frac{d}{d\Theta} \left( \frac{d}{d\Theta} P_n(\cos \Theta) \right) + n(n+1)P_n = 0 \tag{3.1}$$

and the associated Legendre function $P_n^m(\cos \Theta) = P_n^m(\mu)$ may be defined by

$$P_n^m(\mu) = (1-\mu^2)^{m/2} \frac{d^m P_n(\mu)}{d\mu^m} \tag{3.2}$$

where $\mu = \cos \Theta$. For $n=1$ we have from (3.2)

$$P_1^1(\cos \Theta) = - \frac{dP_1}{d\Theta} \tag{3.3}$$

so that (3.1) gives the relation

$$\frac{1}{\sin \Theta} \frac{d}{d\Theta} \left( \frac{d}{d\Theta} P_1^1(\cos \Theta) \right) = n(n+1)P_1 \tag{3.4}$$

which is used in the boundary equations involving the vector potential $A$ in Sec. 14.

The orthogonality property of the Legendre functions requires that the integral of the product $P_n^m P_r^m \sin \Theta d\Theta$ vanish unless $n=r$, when it has the value

$$\int_0^\pi [P_n^m(\cos \Theta)]^2 \sin \Theta d\Theta = \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!} \tag{3.5}$$
APPENDIX IV

Calculation of the Potential Coefficients for \( n=0 \)

We wish to determine the potential coefficients \( B_0, B'_0, C_0, C'_0 \) from the six general boundary-condition equations in Sec. 14 for \( n=0 \), under the assumptions that \( a_1 \ll 1 \), and that the order of magnitude relations of Sec. 15 hold:

\[
\left( \frac{a_1}{a_2} \right)^2 \sim \left( \frac{k_1}{k_2} \right)^2 \sim \frac{\alpha_1}{\alpha_2} \ll 1, \quad \beta_1 = 1.
\]

and where we impose no restriction on the magnitudes of \( a_2 \) and \( b \). Similar relations hold for the corresponding primed quantities.

We shall indicate the general procedure for the \( B_0 \) coefficient and then indicate only the main steps for the remaining quantities.

For \( n=0 \) the general equations in Sec. 14 reduce to the following four equations (the \( \psi_0' \) and \( p_{r0} \) equations do not appear for \( n=0 \) and \( D_0 = D_0' = 0 \)):

\[
B_0 a_1 h_0'(a_1) - B'_0 a_1 j_0'(a_1) + C_0 a_1 h_0'(a_1) - C'_0 a_1 j_0'(a_1) = -a_1 j_0(a_1) \quad (4.1)
\]

\[
B_0 \alpha_1 h_0(a_1) - B'_0 \alpha_1 j_0(a_1) + C_0 \alpha_1 h_0(a_1) - C'_0 \alpha_1 j_0(a_1) = -\alpha_1 j_0(a_1) \quad (4.2)
\]

\[
B_0 \chi \alpha_1 a_1 h_0'(a_1) - B'_0 \chi \alpha_1 a_1 j_0'(a_1) + C_0 \chi \alpha_1 a_1 h_0'(a_1) - C'_0 \chi \alpha_1 a_1 j_0'(a_1) = -\chi \alpha_1 a_1 j_0(a_1) \quad (4.3)
\]

\[
B_0 \epsilon \left[ \beta_1^2 h_0(a_1) - 2a_1^2 h_0''(a_1) \right] + C_0 \epsilon \left[ \beta_1^2 h_0(a_1) - 2a_1^2 h_0''(a_1) \right] - B'_0 \left[ \beta_1^2 j_0'(a_1) - 2a_1^2 j_0''(a_1) \right] - C'_0 \left[ \beta_1^2 j_0'(a_1) - 2a_1^2 j_0''(a_1) \right] = -\epsilon \left[ \beta_1^2 j_0(a_1) - 2a_1^2 j_0''(a_1) \right] \quad (4.4)
\]
These equations are exact. We now write the array of coefficients of the unknowns $B_0$, $B'_0$, $C_0$, $C'_0$ and the constants on the right symbolically as

$$
\begin{pmatrix}
B_0 & B'_0 & C_0 & C'_0 \\
A_1 & A_2 & A_3 & A_4 \\
C_1 & C_2 & C_3 & C_4 \\
D_1 & D_2 & D_3 & D_4 \\
F_1 & F_2 & F_3 & F_4
\end{pmatrix} = 
\begin{pmatrix}
A_7 \\
C_7 \\
D_7 \\
F_7
\end{pmatrix}
$$

(4.5)

where, using the relations in Appendix II and imposing the conditions that $a_1 \ll 1$, $a'_1 \ll 1$, $\beta_1 = \beta'_1 = 1$, we find for the values of the above constants (putting $J_0(a_1) = 1$):

- $A_1 = -h_0(a_1)$
- $A_2 = a_1 j_1(a_1)$
- $A_3 = -a_2 h_1(a_2)$
- $A_4 = a_2 j_1(a_2)$
- $A_7 = a_1 j_1(a_1)$
- $C_1 = \alpha_1 h_0(a_1)$
- $C_2 = -\alpha'_1$
- $C_3 = \alpha_2 h_0(a_2)$
- $C_4 = -\alpha'_2 J_0(a_2)$
- $C_7 = -\alpha'_1$
- $D_1 = -\chi_0 h_0(a_1)$
- $D_2 = \alpha'_1 a_1 j_1(a_1)$
- $D_3 = -\chi_2 a_2 h_0(a_2)$
- $D_4 = \alpha'_2 a_2 j_1(a_2)$
- $D_7 = -\chi_0 a_1 j_1(a_1)$
- $F_1 = \epsilon (b^2 - 4) h_0(a_1)$
- $F_2 = -b^2$
- $F_3 = \epsilon \beta_2 b^2 h_0(a_2) + 2 \epsilon a_2^2 h_0(a_2) - 4 \epsilon a_2 h_0(a_2)$
- $F_4 = -\beta'_2 b^2 J_0(a_2) - 2 a_2^2 J_0(a_2) + 4 a_2 j_1(a_2)$

(4.6)
Then the unknown $B_0$ is given by

$$B_0 = \Delta(B_0)/\Delta_0$$

(4.7)

where $\Delta_0$ is the determinant of the coefficients on the left of (4.5) and $\Delta(B_0)$ is the same determinant with the first column replaced by the column on the right of (4.5).

We now evaluate $\Delta_0$. Since the first two columns contain small quantities while the second two are of arbitrary magnitude, we shall write only the first two columns explicitly in order to see which elements are negligible:

$$\Delta_0 = \alpha_2^2 h_0(a_1).$$

(4.8)

where we have divided out $h_0(a_1)$ from the first column and $\alpha_2$ from the second and third rows. Now consider the magnitudes of the elements in the first two columns: In the first column the top and bottom elements are of order unity and the other three are of order $a_1^2$ since $\varepsilon(=\eta/\eta')$, $\chi(=\sigma/\sigma')$ and $b^2$ are unrestricted and $\alpha_1/\alpha_2$ is of order $a_1^2 << 1$. Similarly, in the second column the first two elements are of order $a_1^2 << 1$, the third is of order $a_1^3 << 1$, while the fourth ($-b_1^2$) being unrestricted may be considered of order unity.
It follows that if the determinant is expanded by second-order determinants in the first two columns, only the \((-1)(-b^2)\) product is of order unity. Since this product represents the \(A_1 F_2\) term we see that \(\Delta_o\) can be expanded as

\[
\Delta_o = A_1 F_2 = \begin{vmatrix} C_3 & C_4 \\ D_3 & D_4 \end{vmatrix}
\] (4.9)

After some algebra we then find

\[
\Delta_o = h_o(a) b^2 \begin{vmatrix} C_3 & C_4 \\ D_3 & D_4 \end{vmatrix} = h_o(a) b^2 (\alpha_2 \alpha'_2) h_o(a_1) a'_2 j_2(a_1) .
\] (4.10)

Following the same procedure for \(\Delta(B_0)\) we find

\[
\Delta(B_0) = \begin{vmatrix} A_7 & A_2 & A_3 & A_4 \\ C_7 & C_2 & C_3 & C_4 \\ D_7 & D_2 & D_3 & D_4 \\ F_7 & F_2 & F_3 & F_4 \end{vmatrix} = \alpha_2^2 \begin{vmatrix} a_j(a) & a'_j(a) & A_3 & A_4 \\ -\alpha_1/\alpha_2 & -\alpha'_1/\alpha_2 & C_3/\alpha_2 & C_4/\alpha_2 \\ \gamma \alpha_2 a_j(a) & \gamma' \alpha'_2 a'_j(a) & D_3/\alpha_2 & D_4/\alpha_2 \\ -\epsilon b^2 & -b^2 & F_3 & F_4 \end{vmatrix}
\] (4.11)

If we now indicate elements of order unity by 1 and order \(a_1^2\) by \(x << 1\) we have

\[
\Delta(B_0) = \alpha_2^2 \begin{vmatrix} x & x & A_3 & A_4 \\ x & x & C_3/\alpha_2 & C_4/\alpha_2 \\ x^2 & x^2 & D_3/\alpha_2 & D_4/\alpha_2 \\ 1 & 1 & F_3 & F_4 \end{vmatrix}
\] (4.12)

We see that in the first two columns only two determinants are of order \(x\):

\[
\begin{vmatrix} A_7 & A_2 \\ F_7 & F_2 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} C_7 & C_2 \\ F_7 & F_2 \end{vmatrix}
\]
the rest being of order $x^2$ or $x^3$. Expanding $\Delta(B_0)$ in this manner we therefore have

$$\Delta(B_0) = \begin{vmatrix} A_7 & A_2 \\ F_7 & F_2 \end{vmatrix} \begin{vmatrix} C_3 & C_4 \\ D_3 & D_4 \end{vmatrix} + \begin{vmatrix} F_7 & F_2 \\ C_7 & C_2 \end{vmatrix} \begin{vmatrix} A_3 & A_4 \\ D_3 & D_4 \end{vmatrix}$$

(4.13)

In evaluating the determinants we use the relation $\delta b^2 = eb^2$ and find

$$\begin{vmatrix} A_7 & A_2 \\ F_7 & F_2 \end{vmatrix} = b^2 \left[ \delta a_{ij} (a_i) - a_{ij} (a_i) \right]$$

$$\begin{vmatrix} C_3 & C_4 \\ D_3 & D_4 \end{vmatrix} = \alpha_2 \alpha'_1 \left[ a_{ij} (a_i) h_0 (a_i) - \chi_j (a_i) \cdot a_{ij} h_1 (a_i) \right]$$

(4.14)

$$\begin{vmatrix} F_7 & F_2 \\ C_7 & C_2 \end{vmatrix} = b^2 \left( \delta \alpha'_1 - \alpha'_1 \right)$$

$$\begin{vmatrix} A_3 & A_4 \\ D_3 & D_4 \end{vmatrix} = \left( \chi \alpha'_2 - \alpha'_2 \right) a_{ij} (a_i) \cdot a_{ij} h_1 (a_i)$$

Finally, with $\Delta(B_0)$ from (4.13) and $\Delta_0$ from (4.10) we find $B_0$ from (4.7)

$$B_0 = i \frac{a_3}{3} \left[ \delta \left( \frac{a'_1}{a_i} \right)^2 - 1 \right] + \left( \chi \frac{a_3}{a_i} - 1 \right) i a_1 \left[ \frac{a_3 (\delta \frac{a'_1}{a_i} - 1)}{1 - \chi Z} \frac{a_{i1} h_1 (a_i)}{h_0 (a_i)} \right]$$

or

$$B_0 = i \frac{a_3}{3} \left[ \delta \left( \frac{a'_1}{a_i} \right)^2 - 1 \right] + \left( \chi \frac{a_3}{a_i} - 1 \right) i a_1 C_0 a_{ij} h_1 (a_i)$$

(4.15)
where
\[ Z = \frac{j_0(a_1)}{a_1 j_1(a_1)} \cdot \frac{a_2 h_1(a_2)}{h_0(a_2)} \] (4.16)
and \( C_0 \) is given below. In the reduction we have again used Appendix II to evaluate \( j_1(a_1), h_0(a_1), \) etc.

It may be noted that for zero thermal conductivity \( \sigma = 0, \bar{\omega} = 0, \) so that \( \alpha_2 = -(\alpha_1 \bar{\omega})^{-1} \rightarrow \infty \) and the second term in (4.14) vanishes; the remaining first term then agrees with Epstein's corresponding coefficient, given by his Eq. (47) in Ref. 6 for the case \( \sigma = 0. \) Unlike the first term which is pure imaginary and contributes nothing to the viscous dissipation (see (17.12) in Sec. 17), the second term has a real part, and this later plays an important role in our results for the viscous dissipation (Sec. 17).

Turning now to \( B'_0 = \Delta(B'_0)/\Delta_0, \) we proceed the same way as before to evaluate \( \Delta(B'_0); \) we find

\[
\Delta(B'_0) = \begin{vmatrix}
A_1 & A_7 & A_3 & A_4 \\
C_1 & C_7 & C_3 & C_4 \\
D_1 & D_7 & D_3 & D_4 \\
F_1 & F_7 & F_3 & F_4
\end{vmatrix} \sim \begin{vmatrix}
1 & x & A_3 & A_4 \\
x & x & C_3 & C_4 \\
x^2 & D_3 & D_4 \\
1 & 1 & F_3 & F_4
\end{vmatrix}.
\] (4.17)

Clearly \( A_1 F_7 \) is the dominant product in the first two columns; hence we have

\[
\Delta(B'_0) = (A_1 F_7) \cdot \begin{vmatrix}
C_3 & C_4 \\
D_3 & D_4
\end{vmatrix} = b^2 h_0(a_1) \begin{vmatrix}
C_3 & C_4 \\
D_3 & D_4
\end{vmatrix}.
\] (4.13)
and using $\epsilon b^2 = \delta b' r^2$ and $\Delta_o$ we obtain

$$B'_o = \frac{\Delta(B'_o)}{\Delta_o} = \delta . \quad (4.19)$$

For $C_o = \Delta(C_o)/\Delta_o$ we have

$$\Delta(C_o) = \begin{vmatrix} A_1 & A_2 & A_7 & A_4 \\ C_1 & C_2 & C_7 & C_4 \\ D_1 & D_2 & D_7 & D_4 \\ F_1 & F_2 & F_7 & F_4 \end{vmatrix} \sim \begin{vmatrix} 1 & x & A_4 \\ x & x & C_4 \\ x & x^2 & D_4 \\ 1 & 1 & F_4 \end{vmatrix} \quad (4.20)$$

and we see that the dominant term in the expansion will be of order $x$ and is

$$\Delta(C_o) = -A_1 D_4 \begin{vmatrix} C_2 & C_7 \\ F_2 & F_7 \end{vmatrix} = b'^2 \alpha'_2 (\delta \alpha'_i - \alpha_i) h_o(a_i) a'_j(a'_i), \quad (4.21)$$

so that

$$C_o = \frac{\alpha_i}{\alpha'_2} \left( \frac{\delta \alpha'_i}{\alpha'_i} - 1 \right) \frac{1}{h_o(a_i)} \cdot \frac{1}{1-x^2 Z}, \quad (4.22)$$

where $Z$ is given above in (4.16).

Similarly, for $C'_o = \Delta(C'_o)/\Delta_o$ we have

$$\Delta(C'_o) = \begin{vmatrix} A_1 & A_2 & A_3 & A_7 \\ C_1 & C_2 & C_3 & C_7 \\ D_1 & D_2 & D_3 & D_7 \\ F_1 & F_2 & F_3 & F_7 \end{vmatrix} \sim \begin{vmatrix} 1 & x & A_3 \\ x & x & C_3 \\ x & x^2 & D_3 \\ 1 & 1 & F_3 \end{vmatrix} \quad (4.23)$$
or

\[ \Delta(C'_0) = A_1 D_3 \begin{vmatrix} C_2 & C_7 \\ F_2 & F_7 \end{vmatrix} = \chi' b' \alpha_2 \left( \delta \alpha'_1 - \alpha'_2 \right) h_0(a_2) \cdot a_2 h_1(a_2), \]  \hspace{1cm} (4.24)

and finally

\[ C'_0 = \chi' \frac{\alpha_2}{\alpha'_2} \frac{a_2 h_1(a_2)}{a'_2 h_j(a'_2)} C_0. \]  \hspace{1cm} (4.25)

This completes the calculation of the coefficients for \( n=0 \).
APPENDIX V

Calculation of the Potential Coefficients for \( n=1 \)

Setting \( n=1 \) in the general boundary condition equations of Sec. 14 we may write, in the same way as for \( n=0 \), the symbolic equation

\[
\begin{bmatrix}
A_1 & A_2 & A_3 & A_4 & A_5 & A_6 & A_7 \\
B_1 & B_2 & B_3 & B_4 & B_5 & B_6 & B_7 \\
C_1 & C_2 & C_3 & C_4 & 0 & 0 & C_7 \\
D_1 & D_2 & D_3 & D_4 & 0 & 0 & D_7 \\
E_1 & E_2 & E_3 & E_4 & E_5 & E_6 & E_7 \\
F_1 & F_2 & F_3 & F_4 & F_5 & F_6 & F_7
\end{bmatrix}
\]

(5.1)

for the six unknown coefficients \( B_1, B_1, C_1, C_1, D_1, D_1 \). Putting \( a_1 \ll 1 \), \( a_1' \ll 1 \) etc., as before, the constants \( A_1 \ldots F_7 \) are given by

\[
\begin{align*}
A_1 &= -2h(a) \\
A_2 &= -j(a) \quad A_3 = a_2h(a_z) - 2h(a_z) \quad A_4 = -a_2'j(a_z) + 2j(a_z) \\
A_5 &= -2h(b) \\
A_6 &= 2j(b) \\
A_7 &= -j(a) \\
B_1 &= h(a) \\
B_2 &= -j(a) \\
B_7 &= -j(a) \\
C_1 &= \alpha h(a) \\
C_2 &= -\alpha' j(a) \\
C_3 &= \alpha' h(a_z) \\
C_4 &= -\alpha' j(a_z) \\
C_7 &= -\alpha j(a) \\
\end{align*}
\]

(5.2)
We first calculate the determinant $\Delta_1$ of the coefficients, i.e., the determinant on the left of (5.1). Factoring out $h_1(a_1)$ from the first column, $j_1(a'_1)$ from the second, and writing $x$ for quantities of order $a_1^2$ in the first two columns we have as in Appendix IV

\[
\begin{vmatrix}
-2 & -1 & A_3 & A_4 & A_5 & A_6 \\
1 & -1 & B_3 & B_4 & B_5 & B_6 \\
x & -x & C_3 & C_4 & 0 & 0 \\
x & -x & D_3 & D_4 & 0 & 0 \\
-3\epsilon & x & E_3 & E_4 & E_5 & E_6 \\
\epsilon(b^2-12) & -b^2 & F_3 & F_4 & F_5 & F_6 \\
\end{vmatrix}
\]

(5.3)

Neglecting elements of order $x$ or smaller we see that in the second and third rows only $C_3 C_4$ and $D_3 D_4$ survive; expanding by these rows we have
After tedious but straightforward algebra the fourth-order determinant is found to be

\[
\Delta_1 = h_1(a_1) j_1(a_1) \begin{vmatrix} C_3 & C_4 \\ D_3 & D_4 \end{vmatrix} \frac{1}{-2 -1 A_5 A_6} \begin{vmatrix} 1 & -1 B_5 B_6 \\ -3\epsilon & 0 E_5 E_6 \end{vmatrix} \begin{vmatrix} (b^2-12)\epsilon & b^2 F_5 F_6 \end{vmatrix}
\]

(5.4)

Using the relation \( \epsilon b^2 = \delta b_1^2 \) we can factor \( b_1^2 \) out of each \( \Delta \) and rewrite the fourth-order determinant as

\[
b_1^2 \left( D_{00} + D_{01} + D_{10} + D_{11} \right)
\]

(5.7)

where

\[
D_{00} = [(i-\epsilon)(2+\delta)] b_j(\delta)^2 b_4(b)
\]

\[
D_{10} = \left[ 3(\epsilon-\frac{1}{2}\delta) \right] (2+\delta) j_1(\delta) b_4(b)
\]

\[
D_{01} = 9\delta(\epsilon-1) \left[ \frac{1}{2}(e-b^2) + 9(-\epsilon) \right]
\]

\[
D_{11} = 27\delta(\epsilon-1) \left[ \frac{1}{2}(e-b^2) + 9\delta \right]
\]

(5.8)

Thus we finally have for the determinant of the coefficients

\[
A_1 = b_1^2 h_1(a_1) j_1(a_1) \begin{vmatrix} C_3 \ D_4 \\ D_3 \ D_4 \end{vmatrix} \left( (D_{00} + D_{01} + D_{10} + D_{11}) \right).
\]

(5.9)

We now calculate \( B_1 \). It is given by
\[ B_1 = \Delta(B_1) / \Delta_1 \]  
(5.10)

where

\[
\Delta(B_1) = \begin{vmatrix}
A_7 & A_2 & A_3 & A_4 & A_5 & A_6 \\
B_7 & B_2 & B_3 & B_4 & B_5 & B_6 \\
C_7 & C_2 & C_3 & C_4 & 0 & 0 \\
D_7 & D_2 & D_3 & D_4 & 0 & 0 \\
E_7 & E_2 & E_3 & E_4 & E_5 & E_6 \\
F_7 & F_2 & F_3 & F_4 & F_5 & F_6 \\
\end{vmatrix}
\]

\[
\Delta_1 = \begin{vmatrix}
1 & 1 & A_3 & A_4 & A_5 & A_6 \\
1 & 1 & B_3 & B_4 & B_5 & B_6 \\
x & x & C_3 & C_4 & 0 & 0 \\
x & x & D_3 & D_4 & 0 & 0 \\
-x & -x & E_3 & E_4 & E_5 & E_6 \\
e^{b^2} + b_1^2 & F_3 & F_4 & F_5 & F_6 \\
\end{vmatrix}
\]  
(5.11)

The second order determinants of order unity in the first two columns arise from the AF and BF products; expanding \( \Delta(B_1) \) we get

\[
\Delta(B_1) = \begin{vmatrix}
A_7 & A_2 \\
F_7 & F_2 \\
\end{vmatrix} \cdot \begin{vmatrix}
C_3 & C_4 \\
D_3 & D_4 \\
\end{vmatrix} \left( \begin{vmatrix}
B_5 & B_6 \\
E_5 & E_6 \\
\end{vmatrix} - \begin{vmatrix}
A_5 & A_6 \\
E_5 & E_6 \\
\end{vmatrix} \right) 
\]  
(5.12)

where we have used the fact that \( B_7 = A_7 \), \( B_2 = A_2 \) to eliminate \( B_7 \) and \( B_2 \).

After carrying out the algebra and using \( \epsilon b^2 = \delta b_1^2 \) we find for the quantity in the parenthesis the value

\[ N_{00} + N_{01} + N_{10} + N_{11} \]  
(5.13)

where the \( N \)'s are given by

\[
N_{00} = \left[ 1 - \epsilon \right] b_1^2 j_1(b) \cdot h_1(b) \\
N_{01} = \left[ 3(\epsilon - \frac{1}{2} b^2 \delta - \frac{1}{2} \delta^2) \right] j_1(b) \cdot h_1(b) \\
N_{10} = \left[ 3(\epsilon - \frac{1}{2} b^2 \delta - \frac{1}{2} \delta^2) \right] j_1(b) \cdot h_1(b) \\
N_{11} = \left[ 3(\epsilon - \frac{1}{2} b^2 \delta - \frac{1}{2} \delta^2) \right] j_1(b) \cdot h_1(b) \\
\]  
(5.14)

Since

\[
\begin{vmatrix}
A_7 & A_2 \\
F_7 & F_2 \\
\end{vmatrix} = b_1^2 (1 - \delta) j_1(a_1) j_1(a_1) \],
\]

we finally find
\[ \Delta(B_1) = b'i^2 j_1(a_1')j_1(a_1')(1 - \delta) \left| \begin{array}{cc} C_3 & C_4 \\ D_3 & D_4 \end{array} \right| (N_{00} + N_{01} + N_{10} + N_{11}) \] (5.15)

and with \( \Delta_1 \) we obtain \( B_1 \) from (5.10)

\[ B_1 = i \frac{a_1}{3} (i - \delta) \left[ \frac{N_{00} + N_{01} + N_{10} + N_{11}}{D_{00} + D_{01} + D_{10} + D_{11}} \right] = i \frac{a_1}{3} (1 - \delta) G \] (5.16)

after canceling \( j_1(a_1') \) in \( \Delta(B_1) \) and \( \Delta_1 \) and using \( h_1(a_1')/j(a_1') = i \frac{1}{3} a_1^2 \) from Appendix II. \( G \) is a useful abbreviation for the quantity in the bracket.

The coefficient \( B_1' \) is found in an exactly similar manner. We replace column two in the determinant on the left of (5.1) by column seven and obtain exactly the same order-of-magnitude scheme as for \( \Delta_1 \) (5.3) except that \( j_1(a_1') \) is replaced by \( j_1(a_1) \) and \( b'i^2 \) by \( \epsilon b^2 \). It follows that \( \Delta(B_1') \) can be written in the form (5.4) with the same replacements. We then expand the fourth-order determinant; after the algebra has died down there finally emerges a form similar to (5.13) above in which each \( N \) is \( 3 \epsilon b^2 = 3 \delta b'i^2 \) times the corresponding \( N \) of \( B_1 \) in (5.14). This gives for \( \Delta(B_1') \) the result

\[ \Delta(B_1') = 3 \delta b'i^2 h_1(a_1)j_1(a_1) \left| \begin{array}{cc} C_3 & C_4 \\ D_3 & D_4 \end{array} \right| (N_{00} + N_{01} + N_{10} + N_{11}) \] (5.17)

so that using \( \Delta_1 \) we finally get \( B_1' \):

\[ B_1' = \frac{a_1}{a_i} 3 \delta \left[ \frac{N_{00} + N_{01} + N_{10} + N_{11}}{D_{00} + D_{01} + D_{10} + D_{11}} \right] = \frac{a_1}{a_i} 3 \delta G = \frac{a_1}{a_i} G' \] (5.18)
after canceling \( h_1'(a_1) \) in \( \Delta(B_1') \) and \( \Delta_1 \) and writing \( j_1'(a_1)/j_1(a_1) = a_1/a_1' \).

Having found \( B_1 \) and \( B_1' \) we can now obtain \( C_1 \) and \( C_1' \) from (14.6) and (14.7) of Sec. 14 expressing the continuity of temperature and normal heat flow at the surface of the scattering sphere, since these equations do not involve \( D_1 \) and \( D_1' \). We write these equations for \( n=1 \) in the form

\[
C_1 \alpha_2 h_1(a_2) - C_1' \alpha_2' j_1'(a_2') = P_1 \tag{5.19}
\]

\[
C_1 \chi \alpha_2 a_2 h_1(a_2) - C_1' \alpha_2' j_1'(a_2') = Q_1 \tag{5.20}
\]

where

\[
P_1 = B_1' \alpha_1' j_1'(a_1') - \alpha_1 j_1(a_1) - B_1 \alpha_1 h_1(a_1) \tag{5.21}
\]

\[
Q_1 = B_1' \alpha_1' j_1'(a_1') - \chi \alpha_1 j_1(a_1) + 2 B_1 \chi \alpha_1 h_1(a_1) \tag{5.22}
\]

Here the approximate relations \( a_1' j_1'(a_1) = j_1(a_1) \) and \( a_1 h_1(a_1) = -2h_1(a_1) \) for small arguments have been used in the \( Q_1 \) expression. Substituting the values of \( B_1 \) and \( B_1' \) we find, again using the small argument relations of Appendix II:

\[
P_1 = \frac{1}{3} a_1 \alpha_1 \left[ \frac{\alpha_1'}{\alpha_1} G' - 1 - (1-\delta)G \right] \tag{5.23}
\]

\[
Q_1 = \frac{1}{3} a_1 \alpha_1 \left[ \frac{\alpha_1'}{\alpha_1} G' - \chi - 2\chi(1-\delta)G \right] \tag{5.22}
\]

Now we use (2.4) from Appendix II to obtain \( h_1' \) and \( j_1' \) in (5.20) above and find
\[ \Delta = -\alpha_2 \alpha_2' a_2 j(a_2') h_1(a_2) + \alpha_2 \alpha_2' \gamma j(a_2') a_2 h_1(a_2) \]
\[ = -\alpha_2 \alpha_2' \left[ 2 (\gamma - 1) j(a_2') h_1(a_2) + a_2' j_0(a_2') h_1(a_2) - \gamma a_2 h_1(a_2) j(a_2') \right] \]

\[ \Delta(c_1) = -P_1 \alpha_2' a_2 j(a_2') + Q_1 \alpha_2' j(a_2') \]
\[ = -\frac{1}{3} \alpha_2 \alpha_2' a_1 \left\{ j(a_2') \left[ (2 + \gamma) - 3 \frac{\alpha_1'}{a_1} G' + 2 (1 - \gamma) (1 - \delta) G \right] \right. \]
\[ + a_2' j_0(a_2') \left[ \frac{\alpha_1'}{a_1} G' - 1 - (1 - \delta) G \right] \left\} \right. \]

\[ \Delta(c_1') = Q_1 \alpha_2 h_1(a_2) - P_1 \gamma \alpha_2 a_2 h_1(a_2) \]
\[ = \frac{1}{3} \alpha_2 \alpha_2' a_1 \left\{ h_1(a_2) \left[ (1 + 2 \gamma) \frac{\alpha_1'}{a_1} G' - 3 \gamma \right] + a_2 h_1(a_2) \left[ 1 + (1 - \delta) G - \frac{\alpha_1 G'}{a_1} \right] \right\} \]

Combining these equations we find

\[ C_1 = \Delta(c_1) / \Delta \]
\[ C_1' = \Delta(c_1') / \Delta, \]

the expressions given in (16.8) and (16.9) of Sec. 16.

Since we do not need \( D_1 \) or \( D_1' \) we do not calculate them but merely note that by the same method as above it is easily found that both are of order \( a_1 \). This completes our calculation of the potential coefficients for \( n=1 \). As will be seen in Secs. 17 and 18, only \( B_1 \) is required explicitly; however the order of magnitudes of \( B_1, C_1, C_1' \) are required and the full expressions are given since they were worked out and may be of interest in connection with other problems.
APPENDIX VI

Calculation of Viscous Dissipation Surface Integral

We wish to calculate the surface integral $I_s$ of Sec. 17:

$$I_s = \frac{1}{2} \mathcal{R} \int i \omega \rho_0 \phi \frac{\partial \phi}{\partial n} dS$$

(6.1)

over a sphere of radius $\bar{R}$ for which $k_1 \bar{R} >> 1$. Here we have written

$$\phi = \phi_1 + \phi_1$$

$$\phi = \sum_{n=0}^{\infty} i^n (2n+1) P_n(\cos \theta) \left[ j_n(\rho_1) + B_n h_n(\rho_1) \right]$$

(6.2)

with $\rho_1 = k_1 r >> 1$. For the derivative of $\phi$ we have

$$\frac{\partial \phi}{\partial n} = k_1 \frac{\partial \phi}{\partial \rho_1} = \sum_{n=0}^{\infty} i^n (2n+1) P_n k_1 \left[ j_n'(\rho_1) + B_n h_n'(\rho_1) \right]$$

(6.3)

where the prime signifies differentiation with respect to $\rho_1$. From Appendix II, with $\delta_n = (n + 1) \frac{\pi}{2}$, we have the asymptotic forms for large arguments:

$$j_n(\rho_1) = \frac{1}{\rho_1} \cos(\rho_1 - \delta_n) = \frac{1}{\rho_1} \cos \sigma$$

$$j_n'(\rho_1) = -\frac{1}{\rho_1} \sin(\rho_1 - \delta_n) = -\frac{1}{\rho_1} \sin \sigma$$

$$h_n(\rho_1) = \frac{1}{\rho_1} e^{i(\rho_1 - \delta_n)} = \frac{1}{\rho_1} e^{i\sigma}$$

$$h_n'(\rho_1) = \frac{i}{\rho_1} e^{i(\rho_1 - \delta_n)} = \frac{i}{\rho_1} e^{i\sigma}$$

(6.4)

where $\sigma$ is real and we have taken the "phase derivative" only, since the derivative of $1/\rho_1$ is negligible compared to $1/\rho_1$. We now use the
orthogonality property of the Legendre functions of different orders (Appendix III) in the surface integral to conclude that only products of the same order will survive in the integration.

Thus the integrand evaluated at the surface is

\[
(i \omega p \phi_{\frac{\partial}{\partial n}}) = \sum_{n=0}^{\infty} i \omega p (2n+1) (p_n)^2 k_1 \left[ j_n(p) + B_n h_n(p) \right] \left[ j_n(p) + B_n^* h_n(p) \right] \quad (6.5)
\]

\[
= \sum_{n=0}^{\infty} i \omega p (2n+1) (p_n)^2 \frac{k_1}{p_1^2} \left[ -\frac{1}{2} \sin 2\sigma + i B_n e^{i\sigma} - B_n^* e^{-i\sigma} + i B_n B_n^* \right] \quad (6.6)
\]

where (6.4) has been used. Putting the \( i \) inside the bracket we get

\[
\frac{1}{2} R \left( i \omega p \phi_{\frac{\partial}{\partial n}} \right) = \sum_{n=0}^{\infty} \omega p (2n+1) (p_n)^2 \frac{k_1}{p_1^2} \frac{1}{2} R \left[ -\frac{1}{2} \sin 2\sigma - B_n e^{i\sigma} \right. - i B_n^* e^{-i\sigma} \left. - B_n B_n^* \right] \quad (6.7)
\]

But for any complex number \( A \) we have \( R(A) = R(A^*) \); applying this to the third term, \( R(-i B_n^* e^{-i\sigma} \sin \sigma) = R(i B_n e^{i\sigma} \sin \sigma) \) and (6.7) reduces to

\[
\frac{1}{2} R \left( i \omega p \phi_{\frac{\partial}{\partial n}} \right) = \sum_{n=0}^{\infty} \omega p (2n+1) (p_n)^2 \frac{k_1}{p_1^2} \frac{1}{2} R \left[ -B_n - B_n B_n^* \right] \quad , \quad (6.8)
\]

where the first term in the bracket in (6.7) has been dropped since it is pure imaginary. Putting \( \rho_1 = k_1 \bar{R} \), integrating over the surface and using the Legendre product integral we obtain
\[ I_s = \sum_{n=0}^{\infty} (2n+1)^2 \left\{ \left( \frac{\pi}{P_0} \right)^2 \frac{2}{2n+1} \sin \theta d\theta \right\} \frac{k_i}{(k_i R)^2} \frac{\omega \rho_0}{2} \frac{1}{2} R \left[ -B_n - B_n^* \right] \]

\[ = \sum_{n=0}^{\infty} (2n+1)^2 \left\{ \frac{4\pi}{2n+1} \right\} \frac{\omega \rho_0}{k_i} \frac{1}{2} R \left[ -B_n - B_n^* \right], \]

or, using \( k_1 = \frac{\omega}{c} \)

\[ I_s = -2\pi \rho_0 c \sum_{n=0}^{\infty} (2n+1) R \left[ B_n + B_n^* \right] , \quad (6.9) \]

which is the desired result.
APPENDIX VII

Calculation of Viscous Dissipation Volume Integrals

Outside Integral

The outside volume integral of Sec. 17 to be evaluated is

\[ I_v = \frac{1}{2} \mathcal{R} \int_{\Omega} \rho^* \mathbf{v} \cdot \mathbf{v} \, d\Omega \quad \text{for} \quad R \leq r \leq \infty \]  

(7.1)

where, from Secs. 8, 9, 15 we have

\[ \begin{align*}
\mathbf{v} \cdot \mathbf{v} &= -\nabla^2 \phi = k_1^2 (\phi_i + \phi) + k_2^2 \phi_2 \\
\rho^* &= i \omega \rho_o \left[ \phi_i (\phi_i + \phi) + y_i \phi_2 \right]
\end{align*} \]  

(7.2)

(\(a_1 \ll 1, \quad \gamma_i \approx 1, \quad \gamma_2 = 1 - \frac{4N \nu}{3 \varrho}, \quad k_1^2 = \frac{\alpha^2}{c^2}, \quad k_2^2 = i \frac{\omega}{\varrho}, \quad \frac{k_2^2}{k_1^2} \ll 1\))

(7.3)

Thus

\[ \begin{align*}
\frac{1}{2} \mathcal{R} (\rho^* \mathbf{v} \cdot \mathbf{v}) &= \frac{1}{2} \omega \rho_o \mathcal{R} \left[ i k_1^2 (\phi_i + \phi)^*(\phi_i + \phi) + i k_2^2 \phi_2^*(\phi_i + \phi) \\
&\quad + i \gamma_i k_1^2 \phi_2^*(\phi_i + \phi) + i \gamma_i k_2^2 \phi_2 \phi_2^* \right]
\end{align*} \]  

(7.4)

\[ = \frac{1}{2} \omega \rho_o \mathcal{R} \left[ i k_2^2 \phi_2^*(\phi_i + \phi) + i \gamma_i k_2 \phi_2 \phi_2^* \right] \]  

(7.5)

since in (7.4) the first term is pure imaginary and the third term is negligible compared to the second.

We now integrate the first term in (7.5) using the potentials of Sec. 12, and recalling the orthogonality of the wave functions:
\[ \int dV \phi_2(\phi_1 + \phi_\ast) = \int dV \sum_{n=0}^{\infty} \left[ (-1)^n (2n+1) P_n C_n h_n(k_2 \lambda) \right] \left[ C_n \int R \left( j_n^*(k_1 \lambda) h_\lambda(k_1 \lambda) \lambda^2 d\lambda \right) + B_n C_n \int R h_n^*(k_1 \lambda) h_\lambda(k_2 \lambda) \lambda^2 d\lambda \right] \]  

\[ \text{(7.6)} \]

The first integral in the bracket has the value (Appendix II):

\[ \int R j_n^*(k_1 \lambda) h_\lambda(k_2 \lambda) \lambda^2 d\lambda = \left\{ \frac{\lambda^2}{k_1 - k_2} \left[ k_1 j_{n+1}(k_1 \lambda) h_\lambda(k_2 \lambda) - k_2 h_{n+1}(k_2 \lambda) j_n(k_1 \lambda) \right] \right\} \]

\[ \text{(7.7)} \]

\[ = \frac{R}{k_2} \left[ a_1 j_{n+1}(a_1) h_\lambda(a_2) - a_2 h_{n+1}(a_2) j_n(a_1) \right] \]

\[ \text{(7.8)} \]

where we have used the facts that \( k_1 \) is real so that \( j_n^*(k_1 r) = j_n(k_1 r) \), that \( k_1^2 << k_2^2 \), and that the Hankel functions are negligibly small at the upper limit \( R \), as discussed in Sec. 17. Moreover since \( a_1 j_{n+1}(a_1)/j_n(a_1) \sim a_2 << 1 \), the second term dominates the first in (7.8) so that the first integral in (7.6) is

\[ C_n \int R j_n^*(k_1 \lambda) h_\lambda(k_2 \lambda) \lambda^2 d\lambda = -C_n \frac{R}{k_2} j_n(a_1) a_2 h_{n+1}(a_2) \]

\[ \text{(7.9)} \]

Evaluating the second integral of (7.6) in the same way we find
where now both terms in the bracket are of the same order so that

\[
B_n^* C_n \int_R \bar{R} \frac{k^2}{k_z^2} \left[ a_1 h_{n+1}^*(a_1) h_n(a_2) - a_2 h_{n+1}^*(a_2) h_n(a_1) \right] \text{ d}x. \tag{7.12}
\]

Thus the integral of the first term in (7.5) is

\[
\int dV \phi_1(\phi_1 + \phi_1)^* = \sum_{n=0}^{\infty} \frac{4 \pi (2n+1)}{C_n C_n^*} \int_R h_n^*(k_z \xi) h_n(k_z \xi) \text{ d}x \tag{7.13}
\]

Turning to the integral of the second term of (7.5) we have in the same way, using the expression in Sec. 12 for \( \phi_2^* \):

\[
\int dV \phi_2 \phi_2 = \sum_{n=0}^{\infty} \frac{4 \pi (2n+1)}{C_n C_n^*} \int_R h_n^*(k_z \xi) h_n(k_z \xi) \text{ d}x \tag{7.14}
\]

\[
= \sum_{n=0}^{\infty} \frac{4 \pi (2n+1)}{C_n C_n^*} \left\{ \frac{k_z^2}{k_z^2-k_2^2} \left[ k_z h_{n+1}(k_z \xi) h_n^*(k_z \xi) - k_2 h_{n+1}(k_2 \xi) h_n(k_2 \xi) \right] \right\} \text{ d}x \tag{7.15}
\]

\[
= \sum_{n=0}^{\infty} \frac{4 \pi (2n+1)}{C_n C_n^*} \int_R \frac{k_z^2}{-2k_2} \left[ a_2 h_{n+1}(a_2) h_n^*(a_2) - a_2 h_{n+1}(a_2) h_n(a_2) \right] \text{ d}x. \tag{7.16}
\]
In (7.14) we have used (2.24) of Appendix II to evaluate the integral; in (7.15) $\frac{k^2}{2} - \frac{k_x^2}{2} = 2k^2$ since $k^2$ is pure imaginary, and the minus sign appears because the bracket is evaluated only at the lower limit as above in (7.8).

Collecting our results we finally have from (7.5), (7.13), and (7.16):

$$I_v = \frac{1}{2} \omega \rho_0 R \left\{ i k_z^2 \int dV \phi_2 (\phi_1 + \phi_1^*) + i k_z^2 \int dV \phi_2 \phi_2^* \right\}$$  \hspace{1cm} (7.17)

$$= 2\pi \omega \rho_0 R \left\{ \sum_{n=0}^{\infty} (2n+1) \left\{ -i C_n j_n (a_1) a_z h_{n+1} (a_1) + \right. \right.$$  

$$+ i B_n^* C_n \left[ a_z h_{n+1}^*(a_1) h_n (a_1) - a_z h_{n+1} (a_1) h_n^*(a_1) \right]$$  

$$\left. - \frac{1}{2} i \left[ a_z h_{n+1}^*(a_1) h_n (a_1) - a_z h_{n+1} (a_1) h_n^*(a_1) \right] \right\}$$  \hspace{1cm} (7.18)

We now wish to evaluate this for $n=0$, $n=1$. Using the approximations of Appendix II for the small argument $a_1$ in the $j$'s and $h$'s and the order-of-magnitude results for the coefficients from Sec. 15 ($C_0 \sim a_1^2, C_1 \sim B_0 \sim B_1 \sim a_1^3$) we easily find that for $n=0$ the $C_0$ term in (7.18) is of order $a_1^2$, while the $B_0^* C_0$ and $C_0 C_0^*$ terms are of order $a_1^4$. Moreover, for $n=1$, the $C_1$ and $B_1^* C_1$ terms are of order $a_1^4$ and the $C_1 C_1^*$ term of order $a_1^6$. Thus only the $C_n$ term for $n=0$ is significant, and we finally obtain for the outside volume integral (with $j_0 (a_1) = 1$):

$$I_v = -2\pi \rho_0 c R \left[ i a_1 C_0 a_z h_n (a_1) \right]$$  \hspace{1cm} (7.19)

Since the surviving term represents the product $\phi_2^* \phi_1$ in (7.17), the significant contribution to the volume integral of $p^* \psi \cdot \psi$ arises from the
interference effects between the incident acoustic wave \( \phi_i \) and the scattered thermal wave \( \phi_2 \). As we shall now see, a similar result holds for the inside integral, taken over the interior of the scattering sphere.

**Inside Integral**

The inside volume integral of Sec. 17 is

\[
I' = \frac{1}{2} \Re \left( \int \phi^* \cdot \phi' \, dV \right) \quad 0 \leq \lambda \leq R. \tag{7.20}
\]

In the same way as before we find

\[
\frac{1}{2} \Re \left( \phi^* \cdot \phi' \right) = \frac{1}{2} \omega \rho' R \left[ i(\phi_i + \phi_2) (k_i \phi_i + k_2 \phi_2) \right] \tag{7.21}
\]

\[
= \frac{1}{2} \omega \rho' R \left[ i k_i^2 \phi_i \phi_i^* + i k_2^2 \phi_2 \phi_2^* + i k_i \phi_i \phi_2^* + i k_2 \phi_i \phi_2^* \right] \tag{7.22}
\]

\[
= \frac{1}{2} \omega \rho' R \left[ i k_i^2 \phi_i \phi_i^* + i k_2^2 \phi_2 \phi_2^* \right], \tag{7.23}
\]

where again the first term in the bracket of (7.22) is pure imaginary and the third term negligible compared to the second. Integrating the first term of (7.23) we get

\[
\int dV \phi_i^* \phi_i^* = \int dV \sum_{n=0}^{\infty} \left[ i^n (zn+i) P_n C_n j_n(k \lambda) \right] \left[ i^n (zn+i) P_n C_n^* j_n^*(k' \lambda) \right] \tag{7.24}
\]

\[
= \sum_{n=0}^{\infty} 4 \pi (zn+i) B_n^* C_n j_n^*(k \lambda) j_n^*(k' \lambda) \lambda^2 d\lambda \tag{7.25}
\]

\[
= \sum_{n=0}^{\infty} 4 \pi (zn+i) B_n C_n' \left\{ \frac{\lambda}{k_i - k_2} \left[ k_{i,n} j_n(k, \lambda) j_n(k', \lambda) - k_{l,n} j_n(k, \lambda) j_n(k', \lambda) \right] \right\} \tag{7.26}
\]
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\[ I = \sum_{n=0}^{\infty} 4\pi (2n+1) B_n C_n^* \frac{R}{k_z^2} \left[ a_n j_{n+1}(a_1) j_n(a_1) - d_{n+1} j_{n+1}(a_2) j_n(a_2) \right] \quad (7.27) \]

\[ = \sum_{n=0}^{\infty} 4\pi (2n+1) B_n C_n^* \frac{R}{k_z^2} \left[ a_n j_{n+1}(a_2) j_n(a_1) \right] \quad (7.28) \]

since in (7.27) the first term is of order \( a_1^2 \) smaller than the second.

Integrating the second term of (7.23) we have

\[ \int dV \phi_1^* \phi_1^* = \int dV \sum_{n=0}^{\infty} (2n+1)^2 (P_n^*)^2 C_n^* C_n j_n(k_z a_1) j_n^{*}(k_z a_2) \quad (7.29) \]

\[ = \sum_{n=0}^{\infty} 4\pi (2n+1) C_n^* C_n \int_0^R j_n(k_z a_1) j_n^{*}(k_z a_2) k^2 dn \quad (7.30) \]

\[ = \sum_{n=0}^{\infty} 4\pi (2n+1) C_n^* C_n \left\{ \frac{1}{2} \left[ 2 j_n(k_z a_1) j_n^{*}(k_z a_2) - j_{n+1}(k_z a_1) j_{n+1}^{*}(k_z a_2) \right] \right\} \quad (7.31) \]

\[ = \sum_{n=0}^{\infty} 4\pi (2n+1) C_n^* C_n \frac{1}{2} R^3 \left[ 2 j_n(a_2) j_n^{*}(a_2) - j_{n+1}(a_2) j_{n+1}^{*}(a_2) \right] \quad (7.32) \]

Collecting our results (7.28) and (7.32) and using (7.23) in (7.20) we find for the inside integral the general result:

\[ I_{V} = \frac{1}{2} \omega p_0 R \left[ i k_z^2 \int dV \phi_1^* \phi_1^* + i k_z^2 \int dV \phi_2^* \phi_2^* \right] \quad (7.33) \]

\[ = (2\pi \omega p_0 R) \left\{ \sum_{n=0}^{\infty} (2n+1) \left\{ i B_n^* C_n a_2 j_n(a_1) j_n^{*}(a_1) + \frac{1}{2} i k_z^2 C_n^* C_n a_2^2 \left[ 2 j_n(a_2) j_n^{*}(a_2) - j_{n+1}(a_2) j_{n+1}^{*}(a_2) - j_{n+1}(a_2) j_{n+1}^{*}(a_2) \right] \right\} \right\} \quad (7.34) \]

To evaluate this for \( n=0 \) and \( n=1 \) we use the order of magnitude relations from Sec. 15 \((B_1^2 B_1 \sim 1, C_0 \sim a_1^2, C_1 \sim a_1^3)\) and \( j_n(a_1) \sim a_1^n \). For \( n=0 \)
We see that the $B'_{0} C'_{0}$ term is of order $a_{1}^{-2}$ and the $C_{0} C'_{0}$ term of order $a_{1}^{-1}$; for $n=1$ the terms are of order $a_{1}^{-1/2}$ and $a_{1}^{-1/2}$. Thus we keep only the $B_{n} C_{n}$ term for $n=0$ and get ($j_{n}(a_{1}) = 1$)

$$I'_{v} = (2\pi \omega \rho'_{0} R) \mathcal{R}[i B_{0} C'_{0} a_{2} j_{1}(a_{2})];$$

(7.35)

but $B'_{0} = \delta = \rho_{0}/\rho'_{0}$ so, after multiplying numerator and denominator by $k_{1}$ we get

$$I'_{v} = -2\pi \rho_{0} c \mathcal{R}[-i a_{1} C_{0} a_{2} j_{1}(a_{2})].$$

(7.36)

We now note from Appendix IV that $C'_{0}$ is given by

$$C'_{0} = \frac{\alpha_{1}}{\alpha_{2}} \frac{a_{2} h_{j}(a_{2})}{a_{2} j_{1}(a_{2})} C_{0};$$

so that we finally obtain for the inside volume integral of the viscous dissipation the result

$$I'_{v} = -2\pi \rho_{0} c \mathcal{R}[-\frac{\alpha_{1}}{\alpha_{2}} i a_{1} C_{0} a_{2} h_{j}(a_{2})],$$

(7.37)

which is in a more convenient form than (7.35).

As pointed out before, the dominant contribution to this integral arises from the $\phi'_{2} \phi'_{1}$ term in (7.33), which represents the interference between the scattered acoustic and thermal waves in the interior of the small scattering sphere.
APPENDIX VIII

Calculation of Thermal Dissipation Volume Integrals

From Sec. 18 we have the two volume integrals $J_v$ and $J'_v$ to evaluate:

$$J_v = - \frac{p_c^{\prime} c^2 \alpha_v}{2} \mathcal{R} \int_{\mathcal{R}} T^* \mathbf{\cdot} \mathbf{v} \, dV$$  \hspace{1cm} (8.1)$$

$$J'_v = - \frac{p_0 c^2 \alpha_v'}{2} \mathcal{R} \int_{0}^{\mathcal{R}} T'\mathbf{^*} \mathbf{\cdot} \mathbf{v}' \, dV .$$  \hspace{1cm} (8.2)$$

These integrals are easily reduced to the corresponding integrals of $I_v$ and $I'_v$ in Appendix VII (with integrand $p^* \mathbf{v} \cdot \mathbf{v}$ instead of $T^* \mathbf{v} \cdot \mathbf{v}$).

Using the general relations of Secs. 8 and 9

$$T^* \mathbf{\cdot} \mathbf{v} = - T^* \mathbf{\cdot} \nabla^2 (\phi_1 + \phi_2) = T^* (\kappa^1_1 \phi_1 + \kappa^1_2 \phi_2)$$  \hspace{1cm} (8.3)$$

$$= (\alpha_1^1 \phi_1 + \alpha_1^2 \phi_2) (\kappa^1_1 \phi_1 + \kappa^1_2 \phi_2)$$  \hspace{1cm} (8.4)$$

$$= \alpha^*_1 k^1_1 \phi_1 \phi_1^* + \alpha^*_1 k^1_2 \phi_2 \phi_1^* + \alpha^*_2 k^2_1 \phi_1 \phi^*_2 + \alpha^*_2 k^2_2 \phi_2 \phi^*_2 .$$  \hspace{1cm} (8.5)$$

We have used $\phi_1$ for the acoustic wave ($\phi_1 + \phi_1'$ outside, $\phi_1'$ inside).

Now $k^2_1$, $\alpha_2$ are real and $k^2_2$, $\alpha_1$ are imaginary. Hence the first and last terms are pure imaginary with zero real parts and may be dropped.

We also have the relation

$$\alpha^*_1 k^2_2 = (\gamma -1) \alpha_2^2 k^2_1 .$$  \hspace{1cm} (8.6)$$
as is easily verified. Thus we have

\[
\mathcal{R} \left[ T^{*} \phi \cdot \phi' \right] = \mathcal{R} \left[ \alpha_{2} k_{i}^{2} (y_{-i}) \phi_{i} \phi' + \alpha_{2} k_{i} \phi_{i} \phi' \right]
\]

\[
= \alpha_{2} k_{i} \mathcal{R} \left[ (y_{-i}) \phi_{i} \phi' + \phi_{i} \phi' \right]
\]

\[
= \alpha_{2} k_{i} \gamma \mathcal{R} \left[ \phi_{i} \phi' \right].
\] (8.7)

The corresponding expression with primes holds for \( \mathcal{R} \left[ T'^{*} \phi \cdot \phi' \right] \), and the integrals thus become

\[
J_{v} = - \frac{\rho_{0} c^{2} \omega^{2}}{2 \gamma} \mathcal{R} \left[ (\alpha_{2} k_{i}^{2} y_{-i}) \mathcal{R} \int \mathcal{R} (\phi_{i} + \phi'_{i}) \phi_{i} \, d\mathbf{v} \right]
\] (8.8)

\[
J_{v}' = - \frac{\rho_{0} c^{2} \omega^{2}}{2 \gamma} \mathcal{R} \left[ (\alpha_{2}' k_{i}^{2} y_{-i}') \mathcal{R} \int \mathcal{R} \phi_{i}' \phi_{i}' \, d\mathbf{v} \right].
\] (8.9)

The coefficient in \( J_{v} \) is easily reduced to \( \left( -\frac{1}{2} \rho_{0} \omega^{2}/\gamma \right) \) and the integral is given by (7.13) of Appendix VII. Similar remarks hold for \( J_{v}' \); the integral is given by (7.27) of Appendix VII. Thus the exact expressions are

\[
J_{v} = \frac{1}{2} \frac{\rho_{0} \omega^{2}}{\gamma} \mathcal{R} \sum_{n=0}^{\infty} 4\pi (2n+1) \left\{ - C_{n} \frac{R}{k_{i}^{2}} j_{n}(a_{i}) a_{2} h_{n+1}(a_{i}) + \right.
\]

\[
+ B_{n} C_{n} \frac{R}{k_{i}^{2}} [a_{i} h_{n+1}^{*}(a_{i}) h_{n}(a_{i}) - a_{2} h_{n+1}(a_{i}) h_{n}^{*}(a_{i})] \right\}
\] (8.10)

and

\[
J_{v}' = \frac{1}{2} \frac{\rho_{0}' \omega^{2}}{\gamma} \mathcal{R} \sum_{n=0}^{\infty} 4\pi (2n+1) B_{n} C_{n}^{'} \frac{R}{k_{i}^{2}} \left[ a_{i} j_{n}(a_{i}) j_{n}(a_{i}) - a_{2} j_{n+1}(a_{i}) j_{n}(a_{i}) \right]
\] (8.11)

As in Appendix VII, we evaluate the terms for \( n=0, 1 \) and using the order
of magnitude of the coefficients and the asymptotic forms for the \( j \)'s and \( h \)'s, it is easily shown that in \( J_v \), only the first term, for \( n=0 \), is important, while for \( J'_v \), the second term in the bracket dominates:

\[
J_v = \frac{1}{2} \frac{\rho \omega^2}{2} \Re \left[ - \frac{4\pi R}{k_2^2} C_0 a_2 h_1(a_2) \right] \\
J'_v = \frac{1}{2} \frac{\rho' \omega^2}{2\epsilon'} \Re \left[ \frac{4\pi R}{k_2^2} B'_0 C'_0 a'_2 j_1(a'_2) \right].
\]

(8.12)  
(8.13)

These are the desired expressions for \( J_v \) and \( J'_v \) in Sec. 18.
APPENDIX IX

Calculation of Attenuation by Water Drops in Air

a. Reduction of $\alpha_\eta$

From (19.6) the general expression for $\alpha_\eta$ is

$$
\alpha_\eta = \frac{4\pi n a_i^3}{k^2} \mathcal{R} \left[ -i \frac{b h_a(b) [(1 - \delta) b_j(b) - (\frac{1}{2} b^3) j_i(b)] - (\frac{1}{2} b^3) b_j^3(b) h_i(b)}{[-(2 + \delta) b h_a(b) + 9 \delta h_i(b)] [(1 - \delta) b_j(b) - (\frac{1}{2} b^3) j_i(b)] + (\frac{1}{2} b^3) (2 + \delta) b_j^3(b) h_i(b)} \right]
$$

(9.1)

For $\delta, \epsilon \ll 1$ this may be reduced as follows. We first note that for small arguments $h_1(b) \propto 1/b^2 = \frac{2}{\omega R^2} \rightarrow \infty$ as $\omega \rightarrow 0$, so that even though $h_1$ is multiplied by $\delta$ in the last terms of the numerator and denominator, it cannot be dropped without further examination. Using the small argument approximations however, it is found that the last numerator term is of order $\delta b'^3$ smaller than the first and is therefore completely negligible for all values of $b'$. A similar result is found for the second term in the denominator so that it may also be neglected. Canceling a common factor we then have

$$
\alpha_\eta = \frac{4\pi n a_i^3}{k^2} \mathcal{R} \left[ -i \frac{b h_a(b)}{-2 b h_a(b) + 9 \delta h_i(b)} \right]
$$

(9.2)

where we have dropped $\delta$ in the $(1 - \delta)$ and $(2 + \delta)$ terms. The attenuation is seen to be independent of $b'$ and hence also of $\eta'$, the viscosity of the medium inside the scattering sphere.
We now substitute for the Hankel functions their explicit expressions from Appendix II and make the change of variable

\[ b^2 = 2i y^2 \quad b = (1+i)y \quad y = (\omega R^2 / 2\nu)^{1/2} \quad (9.3) \]

where \( y \) is real, to obtain:

\[ \alpha_n = \frac{4\pi n}{k_i} a^3 R \left[ -i \left( \frac{(2y^2 + 3) + i 3(1+y)}{(4y^2 + 9\delta y) + i 9\delta (1+y)} \right) \right] \quad (9.4) \]

\[ = \frac{4\pi n}{k_i} a^3 \left( \frac{3}{4} \frac{1+y}{y^2} \right) \left[ \frac{16y^4}{16y^4 + 72\delta y^3 + 81\delta^2 (1+2y+2y^2)} \right] \quad (9.5) \]

or

\[ \alpha_n = \frac{4\pi n R}{c} \frac{3}{2} y (1+y) \left[ \frac{16y^4}{16y^4 + 72\delta y^3 + 81\delta^2 (1+2y+2y^2)} \right] \quad (9.6) \]

Equation (9.6) is the desired form for the viscous attenuation of water drops in air.

b. Reduction of \( \alpha_\sigma \)

To evaluate \( \alpha_\sigma \) given in (20.8) we must calculate the real part of the expression:

\[ \Re \left[ \frac{a_2 h_1(a_2) / h_0(a_2)}{1 - \chi Z} \right] = \Re \left[ \frac{1 - ia_2}{1 - \chi Z} \right] \quad (9.7) \]
where from (19.8) we have $Z$:

$$Z = \frac{j_0(a'_2)}{a'_{1j}(a'_2)} \frac{a_2 h_n(a_2)}{h_0(a_2)} = \frac{1-ia_2}{1-a_2 \cot a'_2} \quad (9.8)$$

Now since $\chi$ is small we see that $\chi Z$ will be small except at low frequencies where $a'_2 \to 0$, $a''_2 \to 0$; in this case $Z$ has the limiting form

$$Z \xrightarrow{\omega \to 0} \frac{1-ia_2}{\frac{a'_2^2}{3} + \frac{a'_2^4}{45}} \quad (9.9)$$

where the numerator is exact and in the denominator we have retained the first two terms of the expansion for $1 - a'_1 \cot a'_2$ for small $a'_2$. (Since $a'_2^2$ is pure imaginary it is necessary to retain the first two terms in order to have a real part in the expansion). Now define the dimensionless real quantities $z$ and $z'$ by

$$a'_2 = 2iz^2 \quad a_2 = (1+i)z \quad z = (\omega R^2/2 \bar{x})^{1/2} \quad (9.10)$$

$$a''_2 = 2iz'^2 \quad a'_2 = (1+i)z' \quad z' = (\omega R^2/2 \bar{x'})^{1/2} \quad (9.11)$$

Substituting these into $Z$ and keeping only the largest real and imaginary parts in both numerator and denominator we find

$$Z \xrightarrow{\omega \to 0} \frac{(1+z) - i\bar{z}}{i\frac{3}{2}z^2 - \frac{a'}{45}z'^4} \xrightarrow{z \to 0} \frac{-\frac{3}{2}z + i}{z'^2}, \quad (9.12)$$

so that (9.7) becomes
but since \( \chi \) is small, the second term in the numerator is negligible compared to the first for all \( z, z' \); similarly the third denominator term is negligible compared to the fourth. Dropping these terms and putting

\[
\chi / z^2 = (\delta c_p / c_p) / z^2
\]

into (9.14) we finally have

\[
R \left[ \frac{1 - i a_z}{1 - \chi Z} \right] \xrightarrow{\omega \to 0} (1 + z) \left[ 1 + 3(\delta c_p / c_p) \frac{1}{z} + \frac{9}{4}(\delta c_p / c_p)^2 \frac{1}{z^2} \right]^{-1}.
\] (9.16)

But now we see that this expression is also valid at higher frequencies, for although our low-frequency approximations \( (z, z' \to 0) \) in \( Z \) are of course no longer valid, the whole term \( \chi Z \) is small compared to unity anyway, so that the exact form of \( Z \) is unimportant. Thus

\[
R \left[ \frac{a_z h_z(a_z) / h_0(a_z)}{1 - \chi Z} \right] = (1 + z) \left[ 1 + 3(\delta c_p / c_p) \frac{1}{z} + \frac{9}{4}(\delta c_p / c_p)^2 \frac{1}{z^2} \right]^{-1}
\] (9.17)

is the final desired result (20.9) of Sec. 20.
APPENDIX X

Calculation of Attenuation by Air Bubbles in Water

a. Reduction of $\alpha_\eta$

From (19.6) we have the general expression for $\alpha_\eta$:

$$\alpha_\eta = \frac{4\pi n}{k_i} a_\eta (1 - \varepsilon) R \left[ -i \frac{b h(b) \left[ (1 - \varepsilon) b j_2(b) - \frac{1}{2} b^2 j_1(b) \right] - \left( \frac{1}{2} \varepsilon b^2 \right) b j_0(b) h_1(b)}{\left[ -(1 + \varepsilon) b h(b) + 9 b h_1(b) \right] \left[ (1 - \varepsilon) b j_2(b) - \frac{1}{2} b^2 j_1(b) \right] + \left( \frac{1}{2} \varepsilon b^2 \right) (1 + \varepsilon) b j_0(b) h_1(b)} \right]$$ \hspace{1cm} (10.1)

For $\delta \gg 1, \varepsilon \gg 1$ the bracket reduces to

$$-i \frac{b h(b) (1 - \varepsilon) b j_2(b) - \left( \frac{1}{2} \varepsilon b^2 \right) b j_0(b) h_1(b)}{\delta \left[ b h(b) + 9 b h_1(b) \right] \left[ -\varepsilon b j_2(b) \right] + \left( \frac{1}{2} \varepsilon b^2 \right) \delta b j_0(b) h_1(b)}$$ \hspace{1cm} (10.2)

where we have used $\delta b^2 = \varepsilon b^2$. Canceling common factors and replacing $(1 - \delta)$ by $-\delta$ in (10.1) we obtain

$$\alpha_\eta = \frac{4\pi n}{k_i} a_\eta R \left[ -i \frac{b h(b) + \frac{1}{2} b^2 h_1(b)}{b h(b) - 9 b h_1(b) + \frac{1}{2} b^2 h_1(b)} \right]$$ \hspace{1cm} (10.3)

or

$$\alpha_\eta = \frac{4\pi n}{k_i} a_\eta R \left[ -i \frac{ib^2 - 6i + b^3 - 6b}{18b - 9b^3 + 18i - 3ib^2} \right]$$ \hspace{1cm} (10.4)

We now change to the variable $y$ defined by (9.3) in Appendix IX and obtain

$$\alpha_\eta = \frac{4\pi n}{k_i} a_\eta R \left[ -i \frac{(y^3 + y^2 + 3y) + i(y^3 - 3y - 3)}{(y^3 + 3y^2 + 9y) + i(-y^3 + 9y + 9)} \right].$$ \hspace{1cm} (10.5)
After taking the real part we finally have

\[ \alpha_n = \frac{4\pi n}{k_i} a_i \frac{1}{y^2} \left[ \frac{2y^7 + 12y^6 + 6y^5}{2y^6 + 6y^5 + 9y^4 + 36y^3 + 162y^2 + 162y + 81} \right] \]  \hspace{1cm} (10.6)

or

\[ \alpha_n = \frac{4\pi n R}{c} 2\nu \left[ \frac{2y^7 + 12y^6 + 6y^5}{2y^6 + 6y^5 + 9y^4 + 36y^3 + 162y^2 + 162y + 81} \right] \]  \hspace{1cm} (10.7)

the desired result (22.3) of Sec. 22 for the viscous attenuation by air bubbles in water.
REFERENCES

2. Kirchhoff, S., "Uber den einfluss der Warmeleitung in einem Gase auf die Schallbewegung" (or, Ges. avh. i. 540).
Table I - Physical Constants

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<td>( l_2 ) cm</td>
<td>( 7.6 \times 10^{-3} ) ( 2.4 \times 10^{-3} ) ( 7.6 \times 10^{-4} ) ( 2.4 \times 10^{-4} )</td>
<td>( 6.7 \times 10^{-4} ) ( 2.1 \times 10^{-4} ) ( 6.7 \times 10^{-5} ) ( 2.1 \times 10^{-5} )</td>
</tr>
<tr>
<td>( c_2 ) cm/sec</td>
<td>( 48.5 ) ( 150 ) ( 48.5 ) ( 1500 )</td>
<td>( 4.2 ) ( 13 ) ( 42 ) ( 133 )</td>
</tr>
</tbody>
</table>
Fig. 1. Theoretical attenuation function for water drops in air (Sec. 20).
Fig. 2 Comparison of theoretical attenuation with Knudsen's experimental values for water drops in air (Sec. 21).
Fig. 3. Theoretical attenuation function for air bubbles in water (Sec. 22).