THEORY OF VISCOUS AND THERMAL ATTENTUATION OF SOUND BY SMALL SPHERES

Thesis by

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ABSTRACT

The problem is to calculate the attenuation of plane sound waves passing through a viscous, heat-conducting fluid containing small spherical inhomogeneities. The attenuation is calculated by evaluating the rate of increase of entropy caused by two irreversible processes: (1) the mechanical work done by the viscous stresses in the presence of velocity gradients, and (2) the flow of heat down the thermal gradients. The method is first applied to a homogeneous fluid with no spheres and shown to give the classical Stokes-Kirchhoff expressions. The method is then used to calculate the additional viscous and thermal attenuation when small spheres are present. The viscous attenuation agrees with Epstein's result obtained in 1941 for a non-heat-conducting fluid. The thermal attenuation is found to be similar in form to the viscous attenuation and, for gases, of comparable magnitude. The general results are applied to the case of water drops in air and air bubbles in water.

For water drops in air the viscous and thermal attenuations are comparable; the thermal losses occur almost entirely in the air, the thermal dissipation in the water being negligible. The theoretical values are compared with Knudsen's experimental data for fogs and found to agree in order of magnitude and dependence on frequency. For air bubbles in water the viscous losses are negligible and the calculated attenuation is almost completely due to thermal losses occurring in the air inside the bubbles, the thermal dissipation in the water being relatively small. (These results apply only to non-resonant bubbles whose radius changes but slightly during the acoustic cycle.)

SYMBOLS*

$\underline{A}$	= viscous wave potential
c	= normal sound speed
$c_2, c_\perp$	= phase velocity of thermal and shear waves
c_v, c_p	= specific heats
f	= frequency
j_n, h_n	= spherical Bessel and Hankel functions
$k_1 k_2 K$	= wave numbers of acoustic, thermal and viscous waves
$\ell_1 \ell_2 \ell_\perp$	= damping lengths of acoustic, thermal and viscous waves
n	= concentration of scatterers
p	= pressure
P_{mn}	= total stress tensor
$\underline{q}$	= heat current
r	= radial coordinate
R	= radius of scatterers
s	= entropy/unit mass
s_{irr}	= irreversible entropy density
t	= time
T	= temperature
u	= internal energy/unit mass

α	= attenuation coefficient
α_v	= coefficient of volume expansion
γ	= ratio of specific heats (c_p/c_v)
η	= coefficient of shear viscosity
θ	= polar angle
$\bar{\varkappa}$	$= \sigma/\rho_o c_p$ = thermometric conductivity
$\varkappa$	$= \sigma/\rho_o c_v = \gamma \bar{\varkappa}$
$\lambda_1 \lambda_2 \lambda_\perp$	= wavelength of acoustic, thermal, and viscous waves
μ	= coefficient of dilatational viscosity
$\nu = \eta/\rho$	= kinematic viscosity
ρ	= density
σ	= thermal conductivity
τ_{mn}	= viscous stress tensor
$\phi_1 \phi_2$	= acoustic and thermal wave potentials
φ	= azimuthal angle
Φ	= viscous dissipation function
Ψ	= thermal dissipation function
ω	$= 2\pi f$ = circular frequency

*See Table I for values of the physical constants.

$\mathscr{v}$	= specific volume	$(\ )_o$	= unperturbed quantities
$\underline{v}$	= particle velocity	$(\ )^*$	= complex conjugate
		$(\ )_{Av}$	= time average
		$\Re(\ )$	= real part

In Sections IV and V unprimed quantities ($\rho, \phi, \ldots$) refer to the medium outside the small spheres; primed quantities ($\rho', \phi', \ldots$) refer to the medium inside the spheres; primes on the Bessel and Hankel functions (j', h') indicate differentiation with respect to the argument.

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I. INTRODUCTION

When a plane sound wave passes through a homogeneous fluid it is attenuated by two types of irreversible processes which increase the internal energy of the fluid at the expense of the mechanical acoustic energy of the wave. The first of these is the irreversible mechanical work done by the viscous forces. The second is the irreversible flow of heat down the thermal gradients.

If the fluid contains spherical inhomogeneities the primary sound wave is scattered and the secondary waves are then attenuated by viscosity and thermal conduction. In addition, the secondary waves carry off acoustic energy in all directions, so that the primary wave suffers an additional loss by scattering as well as by enhanced viscous and thermal dissipation.

These dissipative processes lead to an expression for the average energy flux E (the acoustic intensity) of the wave having the form

$$E = E_o \, e^{-\alpha x} \tag{0.1}$$

where α is the attenuation constant (cm^{-1}) and is the sum of the attenuations due to viscosity, heat conduction, and scattering:

$$\alpha = \alpha_\eta + \alpha_\sigma + \alpha_s \, . \tag{0.2}$$

For a plane sound wave in a homogeneous medium Stokes (Ref. 1) first obtained the viscous attenuation

Stokes $$\alpha_\eta = \frac{4}{3}\frac{\eta\omega^2}{\rho_o c^3} = \frac{4}{3}\frac{\nu\omega}{c^2} k_1 \, . \tag{0.3}$$

Kirchhoff (Ref. 2) was the first to point out in 1866 that in gases the effects of thermal conductivity are of the same order of magnitude as those of viscosity and should be taken into account in accurate work. Kirchhoff's result for the thermal attenuation was

<u>Kirchhoff</u> $$\alpha_{\sigma} = \frac{(\gamma-1)\sigma\,\omega^2}{\rho_0 c_p c^3} = (\gamma-1)\frac{\bar{\varkappa}\,\omega}{c^2}k_1 . \qquad (0.4)$$

Here ν is the kinematic viscosity, $\bar{\varkappa}$ the thermometric conductivity, c the velocity of sound, ω the circular frequency and $k_1 = \omega/c$ the wave number. Thus, for a plane sound wave the attenuation due to viscosity and heat conduction is

$$\alpha = \left[\frac{4}{3}\frac{\nu\omega}{c^2} + (\gamma-1)\frac{\bar{\varkappa}\omega}{c^2}\right]k_1 . \qquad (0.5)$$

The scattering of sound from small spheres was first calculated by Lord Rayleigh (Ref. 3) and for fixed rigid spheres his result (including zero- and first-order harmonics) is

<u>Rayleigh</u> $$\alpha_s = \frac{7}{12}V_p k_1 a_1^3 = \frac{7}{12}V_p R^3\left(\frac{2\pi}{\lambda}\right)^4 , \qquad (0.6)$$

where the dimensionless quantity V_p is the total volume of the small spheres per unit volume of the medium, R the radius of the spheres and λ the wavelength of the sound. In the second form $k_1 = 2\pi/\lambda$ is the wave number and $a_1 = k_1 R$ is very small since $R \ll \lambda$ for small spheres.

In 1910 Sewell (Ref. 4) calculated the additional viscous dissipation suffered by a plane wave passing through a medium containing small rigid fixed spheres. Sewell's result was

<u>Sewell</u> $$\alpha_{Sewell} = \frac{9}{2} V_p \frac{\eta}{\rho_0 c R^2}\left[1 + \left(\frac{\omega R^2 \rho_0}{2\eta}\right)^{1/2}\right] = \frac{9}{4} V_p k_1 \frac{1+y}{y^2} \qquad (0.7)$$

$$y^2 = \omega R^2/2\nu \qquad \nu = \eta/\rho_0 .$$

This result, while satisfactory at high frequencies, gives a finite attenuation at the low frequency limit $\omega = 0$. The reason for this erroneous result lies in the assumption that the spheres are <u>fixed</u>; this is not the case in aerosols; as Lamb (Ref. 5) puts it, "Owing to its great inertia in comparison with that of an equal volume of air, a globule of water in suspension, if not too small, may remain practically at rest as the air waves beat upon it. If, however, the radius (R) be diminished, the inertia diminishes as R^3 while the surface on which viscosity acts diminishes as R^2, and it is to be expected that a stage will be reached when the globule will simply drift to and fro with the vibrating air, and so cause little or no loss in energy". In an addendum Sewell tried to correct his results for the oscillatory motion of the particles, but the treatment was not completely satisfactory.

In 1941 Epstein (Ref. 6) published an important theoretical paper dealing with the viscous attenuation of sound by spherical particles suspended in liquids or gases. Epstein employed a more general method than Sewell which automatically included the oscillations of the particles in the acoustic field. This step was partly necessitated by the experimental work of Hartman and Fock (Ref. 15) on aqueous suspensious, since in their work the density ratio of the medium to the particles was near unity, so that Sewell's theory of fixed particles was not at all valid. Epstein considered three cases in which the spherical obstacles were (1) rigid,

(2) viscous fluids, (3) elastic solids, and showed that at sufficiently high frequencies so that the oscillations of the drops could be neglected, Sewell's equation was a close first approximation. In the case of fluid spheres in air Epstein's result was

<u>Epstein</u> $$\alpha_\eta = \frac{9}{4} V_p k_1 \frac{1+y}{y^2} \left[\frac{16y^4}{16y^4 + 72\delta y^3 + 81\delta^2(1+2y+2y^2)} \right] . \qquad (0.8)$$

In view of the relative importance of the thermal effects it is of theoretical interest to extend Epstein's treatment to include the attenuation due to heat conduction. Recent experimental work by Knudsen (Ref. 7) on the attenuation of sound in fogs provides an additional reason for the inclusion of thermal effects. Finally, it may be pointed out that war research on sound transmission in the sea showed that bubbles may contribute strongly to the absorption of sound. As shown by Carstensen and Foldy (Ref. 8) this effect is strongest near the resonant frequency of the bubble; while resonant absorption is <u>not</u> included in the present paper, it is still of considerable interest to examine the viscous and thermal absorption of bubbles for frequencies <u>outside</u> the resonant region.

There are seen to be both theoretical and experimental reasons for investigating the present problem and we now turn to the development of the basic theory underlying the solution.

The essence of the method employed was first given by Prof. P. S. Epstein in his lectures on "Mechanics of Continuous Media". An attempt to carry it through was made in the thesis of W. Garvin (California Institute of Technology 1948). However, Garvin used certain short cuts and simplifications whose validity is not entirely above suspicion. Therefore, it seemed desirable to reopen the problem and to carry it through in a completely systematic and rigorous manner. The essential point in the new approach lies in the use of the entropy associated with the irreversible processes.

II. THE BASIC THEORY

In this chapter we shall discuss the fundamental conservation equations for a compressible, viscous, heat-conducting fluid. From these equations an entropy equation is derived in which both reversible and irreversible processes are represented. The rate of increase of irreversible entropy depends on velocity and thermal gradients and leads to expressions for the rate of increase of the internal energy at the expense of the kinetic and potential energy of the fluid motion. It is thus possible to develop expressions for the viscous dissipation, corresponding to irreversible mechanical work done by viscous forces, and for the thermal dissipation, corresponding to the irreversible flow of heat down thermal gradients.

Although the mass, momentum, and energy equations to be derived are found in the literature (c.f. Busemann, Ref. 14), the entropy equation is not usually given, and it seems useful to derive all four basic equations in a systematic way. The development will also present a number of relations of use in later sections.

Assumptions

We shall make the following simplifying assumptions in developing the theory:

(1) The viscosity η, thermal conductivity σ, specific heats, etc., will be assumed constant, and in particular they will be considered independent of the temperature.

(2) Because of recent work by Liebermann (Ref. 13) involving the role of the compressional (or "second") coefficient of viscosity in absorption of sound, we shall not make the usual assump-

tion that this coefficient vanishes, but instead shall retain it throughout our analysis.

(3) We shall assume the medium to be in a "pseudo"-steady state so that the gradual changes in the unperturbed temperature, pressure, etc., brought about by the increasing internal energy (due to absorption) can be neglected over the time of interest for which the average dissipation is calculated.

(4) We shall use the usual form of the Newtonian stress tensor, modified to include the second coefficient of viscosity (Ref. 14). This assumption is an approximation, since it excludes thermal stresses which a careful statistical-kinetic theory would include in the stress tensor. For gases, however, the error is very small ($\sim 1\%$), so that in view of the additional analytical complexity which would be introduced by the thermal stress terms, it seems entirely reasonable to omit them.

We now choose a Cartesian inertial system x_α ($\alpha = 1, 2, 3$) with respect to which the fluid particle velocity is $\underline{v}$ with components v_α. The position of any fluid element at any time t will be given by its Eulerian coordinates x (t). To derive the basic equations in the simplest way we consider a volume V' of the fluid whose surface S' has at every point the velocity of the fluid at that point. Thus V' moves with the fluid and since no fluid crosses its surface, V' contains a constant mass of fluid.

1. Conservation of Mass (Continuity Equation).

The total mass within V' is then

$$M = \int \rho \, dV' \tag{1.1}$$

and the conservation law then assumes the form

$$\frac{dM}{dt} = \frac{d}{dt} \int \rho \, dV' = 0 \tag{1.2}$$

or

$$\int \frac{\partial \rho}{\partial t} \, dV' + \int \rho \underline{v} \cdot d\underline{S}' = 0 \tag{1.3}$$

The first term represents the rate of mass increases arising from the time dependence of ρ. The second term is the rate of mass increase due to the changing volume of integration. Using the divergence theorem we obtain

$$\int \left[\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot \rho \underline{v} \right] dV' = 0 \tag{1.4}$$

and since V' is an arbitrary volume the integrand must vanish and we have the continuity equation

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot \rho \underline{v} = 0 \, . \tag{1.5}$$

We define the stream derivative

$$\frac{D}{Dt}(\;) = \frac{\partial}{\partial t}(\;) + \underline{v} \cdot \underline{\nabla}(\;) \tag{1.6}$$

and obtain a useful alternate form of the continuity equation:

Mass $$\frac{D\rho}{Dt} + \rho \underline{\nabla} \cdot \underline{v} = 0 . \tag{1.7}$$

We note a useful relation following directly from (1.5): If f is any scalar quantity

$$\rho \frac{Df}{Dt} = \frac{\partial \rho f}{\partial t} + \underline{\nabla} \cdot \rho f \underline{v} . \tag{1.8}$$

2. Conservation of Momentum (Force Equation)

The vector momentum of the fluid in the volume V' is

$$\underline{M} = \int \rho \underline{v} \, dV' \tag{2.1}$$

or, using Cartesian tensor notation,

$$M_\epsilon = \int \rho v_\epsilon \, dV' \tag{2.2}$$

and thus the rate of change of momentum is

$$\frac{dM_\epsilon}{dt} = \frac{d}{dt} \int \rho v_\epsilon \, dV' \tag{2.3}$$

$$= \int \frac{\partial}{\partial t}(\rho v_\epsilon) dV' + \int \rho v_\epsilon v_\alpha \, dS'_\alpha \tag{2.4}$$

$$= \int \left[\frac{\partial}{\partial t} \rho v_\epsilon + \nabla_\alpha (\rho v_\epsilon v_\alpha) \right] dV' . \tag{2.5}$$

In the last form (and hereafter) we use the Einstein convention and sum over doubled Greek subscripts only. We now use (1.8) with f = v_ϵ and obtain

$$\frac{dM_\epsilon}{dt} = \frac{d}{dt}\int \rho v_\epsilon \, dV' = \int \rho \frac{Dv_\epsilon}{Dt}\, dV' . \tag{2.6}$$

The last equality is a particular example of a useful general theorem proved in the same manner as the derivation just made:

<u>Theorem</u> If f is any scalar function and V' is a volume moving with the fluid and containing a constant mass of fluid then

$$\frac{d}{dt}\int \rho f \, dV' = \int \rho \frac{Df}{Dt}\, dV' . \tag{2.7}$$

We must now formulate the force acting on the fluid in V'. We assume that there are no body forces so that only the stress forces are acting and the total force components are

$$F_\epsilon = \int dS'_\alpha \, p_{\alpha\epsilon} = \int \nabla_\alpha p_{\alpha\epsilon} \, dV' , \tag{2.8}$$

where $p_{\alpha\epsilon}$ is the symmetric stress tensor. Then the force equation states that

$$F_\epsilon = \frac{dM_\epsilon}{dt} \tag{2.9}$$

or using (2.8) and (2.6)

$$\int \rho \frac{Dv_\epsilon}{Dt}\, dV' = \int \nabla_\alpha p_{\alpha\epsilon}\, dV' , \tag{2.10}$$

and since V' is arbitrary we obtain

$$\rho \frac{Dv_\epsilon}{Dt} = \nabla_\alpha p_{\alpha\epsilon} \tag{2.11}$$

which is the equation of motion.

Now the total stress tensor $p_{\alpha\epsilon}$ for an isotropic homogeneous fluid is given by

$$p_{\alpha\epsilon} = -p\,\delta_{\alpha\epsilon} + T_{\alpha\epsilon} \tag{2.12}$$

with
$$\begin{aligned} \delta_{\alpha\epsilon} &= 0 \quad (\alpha \neq \epsilon) \\ &= 1 \quad (\alpha = \epsilon), \end{aligned}$$

the viscous stress tensor $T_{\alpha\epsilon}$ by

$$T_{\alpha\epsilon} = -\tfrac{2}{3}(\eta-\mu)(\underline{\nabla}\cdot\underline{v})\delta_{\alpha\epsilon} + 2\eta\, e_{\alpha\epsilon}\ , \tag{2.13}$$

and the velocity strain tensor $e_{\alpha\epsilon}$ (in Cartesian coordinates) by

$$e_{\alpha\epsilon} = \tfrac{1}{2}\left(\nabla_\alpha v_\epsilon + \nabla_\epsilon v_\alpha\right) \tag{2.14}$$

with η the shear coefficient of viscosity and μ the compressional ("second") coefficient of viscosity. The components of the stress tensor are thus given by

$$p_{mn} = \eta(\nabla_m v_n + \nabla_n v_m) \qquad (m \neq n) \tag{2.15}$$

$$p_{nn} = -p - \tfrac{2}{3}(\eta-\mu)(\underline{\nabla}\cdot\underline{v}) + 2\eta\nabla_n v_n \tag{2.16}$$

where, in accordance with our summation convention, (2.16) is <u>not</u> summed on n.

Equation (2.11) with (2.12) now gives the usual force equation:

$$\rho \frac{D v_\epsilon}{Dt} = -\nabla_\epsilon p + \nabla_\alpha \tau_{\alpha\epsilon} . \tag{2.17}$$

Substituting $\tau_{\alpha\epsilon}$ from (2.13) and $e_{\alpha\epsilon}$ from (2.14) we obtain

$$\nabla_\alpha \tau_{\alpha\epsilon} = -\tfrac{2}{3}(\eta-\mu)\nabla_\epsilon(\underline{\nabla}\cdot\underline{v}) + \eta\nabla_\alpha(\nabla_\alpha v_\epsilon + \nabla_\epsilon v_\alpha) \tag{2.18}$$

$$= -\tfrac{2}{3}(\eta-\mu)\nabla_\epsilon(\underline{\nabla}\cdot\underline{v}) + \eta\nabla^2 v_\epsilon + \eta\nabla_\epsilon(\underline{\nabla}\cdot\underline{v}) \tag{2.19}$$

and the equation of motion, written in vector form is

$$\rho\frac{D\underline{v}}{Dt} = -\underline{\nabla}p - \tfrac{2}{3}(\eta-\mu)\underline{\nabla}(\underline{\nabla}\cdot\underline{v}) + \eta\nabla^2\underline{v} + \eta\underline{\nabla}(\underline{\nabla}\cdot\underline{v}) . \tag{2.20}$$

Now using the vector relation

$$\underline{\nabla}\times\underline{\nabla}\times\underline{v} = \underline{\nabla}(\underline{\nabla}\cdot\underline{v}) - \nabla^2\underline{v}$$

we obtain the <u>force equation</u> in its final form:

<u>Momentum</u>
$$\rho\frac{D\underline{v}}{Dt} = -\underline{\nabla}p + \frac{4N}{3}\eta\underline{\nabla}(\underline{\nabla}\cdot\underline{v}) - \eta\,\underline{\nabla}\times\underline{\nabla}\times\underline{v} \tag{2.21}$$

where the constant N in the second term is defined by

$$N = 1 + \mu/2\eta \tag{2.22}$$

so that when the second coefficient of viscosity vanishes we have N=1 and the force equation reduces to the standard form usually given in the literature.

3. Conservation of Energy (First Law of Thermodynamics)

To formulate the energy equation we assume that heat is added to the volume V', the heat current $\underline{q}$ being given by

$$\underline{q} = -\sigma \underline{\nabla} T . \tag{3.1}$$

Then the total rate of heat addition is

$$Q = \int -\underline{q} \cdot d\underline{S}' = \int \sigma \underline{\nabla} T \cdot d\underline{S}' \tag{3.2}$$

$$= \int \sigma \nabla^2 T \, dV' . \tag{3.3}$$

Now the total energy of the fluid in V' is the sum of the total internal energy U and total kinetic energy K:

$$U + K = \int \rho u \, dV' + \int \tfrac{1}{2} \rho v^2 dV' , \tag{3.4}$$

and using the theorem (2.7) we obtain the rate of increase of the total energy of the fluid in V':

$$\frac{dU}{dt} + \frac{dK}{dt} = \int \rho \frac{Du}{Dt} dV' + \int \rho \frac{D v^2/2}{Dt} dV' . \tag{3.5}$$

Finally, the rate at which the fluid inside V' does work on the outside medium is

$$W = -\int dS'_\alpha \, p_{\alpha\epsilon} v_\epsilon = -\int \nabla_\alpha (p_{\alpha\epsilon} v_\epsilon) \, dV' . \tag{3.6}$$

Then the conservation of energy requires that the rate at which energy (heat) is added to V' be equal to the rate of energy increase of the fluid in V' plus the rate at which this fluid does work:

$$Q = \frac{dU}{dt} + \frac{dK}{dt} + W \tag{3.7}$$

or, using the above equations,

$$\sigma \nabla^2 T = \rho \frac{Du}{Dt} + \rho \frac{D v^2/2}{Dt} - \nabla_\alpha (p_{\alpha\epsilon} v_\epsilon) . \tag{3.8}$$

Now multiplying the force equation (2.11) by v_ϵ and summing we get

$$\rho \frac{D v^2/2}{Dt} = v_\epsilon \nabla_\alpha p_{\alpha\epsilon} \tag{3.9}$$

and this together with (3.8) gives

$$\sigma \nabla^2 T = \rho \frac{Du}{Dt} - p_{\alpha\epsilon} \nabla_\alpha v_\epsilon . \tag{3.10}$$

Finally, using the definition of $p_{\alpha\epsilon}$ we get the energy equation as usually given in the literature:

$$\sigma \nabla^2 T = \rho \frac{Du}{Dt} + p \underline{\nabla} \cdot \underline{v} - \Phi \tag{3.11}$$

where the viscous dissipation function Φ is defined by

$$\Phi = \tau_{\alpha\epsilon} \nabla_\alpha v_\epsilon \tag{3.12}$$

and will be discussed later. A more useful form of the energy equation for our purposes is obtained by defining the specific volume

$$\vartheta = \frac{1}{\rho} \tag{3.13}$$

so that, using the continuity equation (1.7), we have

$$\rho \frac{D\vartheta}{Dt} = \frac{1}{\vartheta} \frac{D\vartheta}{Dt} = -\frac{1}{\rho} \frac{D\rho}{Dt} = \underline{\nabla} \cdot \underline{v} . \tag{3.14}$$

From (3.11) we then obtain the desired form of the <u>energy equation</u>:

<u>Energy</u>
$$\sigma \nabla^2 T = -\underline{\nabla} \cdot \underline{q} = \rho \frac{Du}{Dt} + \rho p \frac{D\vartheta}{Dt} - \Phi . \tag{3.15}$$

4. Equations of State and Internal Energy

We have now derived three basic conservation equations for mass momentum, and energy: two scalar equations and one vector equation, or a total of five scalar equations. Seven variables appear in these equations: ρ, v_1, v_2, v_3, p, u, T. We therefore require two more equations to complete our set of basic equations. These are furnished by the thermodynamic equation of state and the specification of the internal energy (u) which we assume to be given in the form

$$p = p(\rho, T) \tag{4.1}$$

and

$$u = u(\rho, T), \tag{4.2}$$

In the case of a perfect gas these become

$$p = \rho R T = \rho c_v (\gamma - 1) T \tag{4.3}$$

$$u = c_v T, \tag{4.4}$$

where $R = c_p - c_v$ is the gas constant/gm, c_p, c_v the specific heats/gm and $\gamma = c_p/c_v$.

It may be pointed out here that the entropy equation (derived in the next section) does not provide another independent equation since it involves the new variable s (entropy/gm). If we include the entropy equation we then have eight variables and eight scalar equations which we summarize schematically as follows:

ρ ——— Continuity Equation (Scalar)

$\underline{v}$ ——— Force Equation (Vector)

T ——— Energy Equation (Scalar)

p ——— State Equation (Scalar)

u ——— Internal Energy Function (Scalar)

s ——— Entropy Equation (Scalar).

5. Entropy Equation (Second Law of Thermodynamics)

Following Eckart (11) and Tolman (12) we express the second law of thermodynamics as an equality, using the concept of irreversible entropy:

$$\frac{d}{dt}\int \rho s \, dV' = -\int \frac{1}{T}\underline{q} \cdot d\underline{S}' + \int \frac{D s_{irr}}{Dt} dV'. \tag{5.1}$$

The left term is the rate of increase of the total entropy within V' , s being the entropy per unit mass of fluid. The right side is the sum of (a) the rate of entropy increase due to the entropy current ($\frac{1}{T}\underline{q}$) flowing in through the surface, and (b) the rate of increase of entropy throughout the volume due to irreversible processes, $\frac{Ds_{irr}}{Dt}$ being the rate of production of "irreversible entropy" per unit volume of the fluid. In accordance with the second law as usually stated we require this quantity to be positive.

Using the theorem (2.7) on the left side and converting the surface integral to a volume integral we have

$$\int \rho \frac{Ds}{Dt} dV' = \int -\underline{\nabla}\cdot\left(\frac{1}{T}\underline{q}\right) dV' + \int \frac{Ds_{irr}}{Dt} dV', \tag{5.2}$$

and since the volume V' is arbitrary we obtain the <u>entropy equation:</u>

<u>Entropy</u>

$$\rho \frac{Ds}{Dt} = -\underline{\nabla}\cdot\left(\frac{1}{T}\underline{q}\right) + \frac{Ds_{irr}}{Dt} . \tag{5.3}$$

Now we substitute for the rate of change of the entropy density of the fluid, as a consequence of its known dependence on the internal energy and volume of the fluid, the expression

$$\frac{Ds}{Dt} = \frac{1}{T}\frac{Du}{Dt} + \frac{p}{T}\frac{D\vartheta}{Dt} . \tag{5.4}$$

But from the energy equation (3.15) the right side may be replaced and we obtain

$$\rho\frac{Ds}{Dt} = \frac{1}{T}\left[\rho\frac{Du}{Dt} + \rho p \frac{D\vartheta}{Dt}\right] = \frac{1}{T}\left[-\underline{\nabla}\cdot\underline{q} + \Phi\right] . \tag{5.5}$$

Comparing this with (5.3) we then have an expression involving Ds_{irr}/Dt :

$$\frac{Ds_{irr}}{Dt} = \underline{\nabla}\cdot\left(\frac{1}{T}\underline{q}\right) - \frac{1}{T}\underline{\nabla}\cdot\underline{q} + \frac{1}{T}\Phi \tag{5.6}$$

$$= -\frac{1}{T^2}\underline{q}\cdot\underline{\nabla}T + \frac{1}{T}\Phi , \tag{5.7}$$

and using the definition of the heat current $\underline{q}$,

$$\underline{q} = -\sigma \underline{\nabla} T \tag{5.8}$$

we obtain the desired expression for the rate of production of irreversible entropy per unit volume:

$$\frac{D s_{irr}}{D t} = \sigma \left(\frac{\nabla T}{T} \right)^2 + \frac{1}{T} \Phi . \tag{5.9}$$

The first term on the right is the rate of irreversible entropy production associated with the thermal conduction due to the presence of temperature gradients. Since σ is positive, this term is clearly positive, so that it is in accordance with the requirements of the second law as usually stated, i.e., that the entropy associated with irreversible processes must always increase.

The second term on the right is the rate of irreversible entropy production associated with the work done by the viscous forces due to the presence of velocity gradients. Moreover it is easily shown from the definitions of Φ and $e_{\alpha\epsilon}$ that

$$\Phi = \tau_{\alpha\epsilon} e_{\alpha\epsilon} \tag{5.10}$$

$$= 2\eta \left[e_{\alpha\epsilon} e_{\alpha\epsilon} - \tfrac{2}{3} (\eta - \mu)(\underline{\nabla} \cdot \underline{v}) \delta_{\alpha\epsilon} e_{\alpha\epsilon} \right] \tag{5.11}$$

$$= 2\eta \left[e_{11}^2 + e_{22}^2 + e_{33}^2 \right] + 4\eta \left[e_{23}^2 + e_{31}^2 + e_{12}^2 \right] - \tfrac{2}{3} (\eta - \mu)(\underline{\nabla} \cdot \underline{v})^2 \tag{5.12}$$

or

$$\Phi = 2\eta \left[(e_{11} + a)^2 + (e_{22} + a)^2 + (e_{33} + a)^2 + 2 \left(e_{23}^2 + e_{31}^2 + e_{12}^2 \right) \right] \tag{5.13}$$

with

$$a = \tfrac{1}{3} \left[(\mu / \eta)^{1/2} - 1 \right] \underline{\nabla} \cdot \underline{v} \tag{5.14}$$

$$\underline{\nabla} \cdot \underline{v} = e_{\alpha\alpha} = e_{11} + e_{22} + e_{33} , \tag{5.15}$$

so that, since η and T are positive we see from (5.13) that the second term of (5.9) is also always positive.

It is to be noted that we have not proved the second law, but have only shown that if we assume the above definition of the stress tensor (2.12) and assume that σ, η, T are positive, then the rate of production of irreversible entropy Ds_{irr}/Dt is indeed positive. To prove the second law we would have to justify the above assumptions regarding $p_{\alpha\epsilon}$, σ, η, T, by a careful statistical-kinetic treatment, including also any effects of thermal stresses.

6. The Viscous and Thermal Dissipation

From the results of the preceding section we have seen that the presence of velocity and thermal gradients in a viscous heat conducting fluid leads to a rate of production of irreversible entropy per unit volume given by Ds_{irr}/Dt in (5.9). From this we obtain the total rate of dissipation of energy per unit volume dE/dt due to irreversible processes:

$$\frac{dE}{dt} = T\frac{Ds_{irr}}{Dt} = \Psi + \Phi \tag{6.1}$$

where

$$\Psi = \frac{\sigma}{T}(\nabla T)^2 \tag{6.2}$$

and Φ is given by (3.12).

We shall call Ψ the thermal dissipation function and Φ the viscous dissipation function. Ψ and Φ represent respectively the rate of increase of the internal energy per unit volume of the fluid due to (a) irreversible heat flow in the presence of the thermal gradients and (b) irreversible mechanical work done by the viscous forces in the presence of the velocity gradients. This increase in the internal energy is at the expense of the ordered macroscopic energy of the heat currents and the mechanical flow energy of the fluid; this fact justifies the use of the term "dissipation".

If we now consider a volume V <u>fixed in space</u> we obtain the rate of dissipation of the acoustic energy of a sound wave sweeping over the volume:

$$\frac{dE_\eta}{dt} = \int \Phi \, dV \tag{6.3}$$

$$\frac{dE_\sigma}{dt} = \int \Psi \, dV \tag{6.4}$$

where dE_η/dt is the total viscous dissipation and dE_σ/dt is the total thermal dissipation, both in units of energy/sec. (We have dropped the prime in the volume element dV since the volume of integration V is fixed in space rather than moving with the fluid).

It is particularly to be noted that since Ψ and Φ are quadratic, <u>we need only solve the first-order equations of the acoustic field in order to obtain Ψ and Φ to second order.</u>

This completes our derivation of the general basic equation for a viscous heat-conducting fluid. We now turn to the acoustic case and apply these results.

III. THE SOUND FIELD

In this chapter we shall consider the general acoustic equations, obtained as the first-order approximations to the basic equations of Chapter II. From these equations we shall derive the wave equations for the acoustic, thermal, and shear waves and discuss their solutions. Finally we shall list some useful auxiliary relations relating the first-order pressure and temperature with the velocity potentials.

7. The Linearized Equations

The general first-order acoustic equations are obtained by linearizing the basic conservation equations for mass, momentum and energy. We expand our variables to first order:

$$
\begin{aligned}
\rho &= \rho_0 + \rho_1 \\
\underline{v} &= \underline{v}_0 + \underline{v}_1 \qquad (\underline{v}_0 = 0) \\
p &= p_0 + p_1 \\
T &= T_0 + T_1 \\
u &= u_0 + u_1
\end{aligned}
\tag{7.1}
$$

where the o-subscript refers to the unperturbed quantities in the absence of the sound field ($\underline{v} = 0$), and the 1-subscript denotes the first-order quantities. We assume the fluid to be at rest in the unperturbed state so that $\underline{v}_0 = 0$. Substituting these quantities in the basic equations (1.7), (2.21), and (3.15) and noting that to first order $\Phi = 0$ and $D/Dt = \partial/\partial t$ (there are no transport phenomena since the term $\underline{v} \cdot \underline{\nabla}\,(\ )$ is second-order), we obtain

Continuity Equation:

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \underline{\nabla} \cdot \underline{v}_1 = 0 \tag{7.2}$$

Force Equation:

$$\rho_0 \frac{\partial \underline{v}_1}{\partial t} = -\underline{\nabla} p_1 + \frac{4N}{3} \eta \underline{\nabla}(\underline{\nabla} \cdot \underline{v}_1) - \eta \underline{\nabla} \times \underline{\nabla} \times \underline{v}_1 \tag{7.3}$$

Energy Equation:

$$\sigma \nabla^2 T_1 = \rho_0 \frac{\partial u_1}{\partial t} + \rho_0 p_0 \frac{\partial \mathcal{V}_1}{\partial t} \tag{7.4}$$

State Equation:

$$p_1 = p_1(\mathcal{V}_1, T_1) \tag{7.5}$$

Internal Energy Equation:

$$u_1 = u_1(\mathcal{V}_1, T_1) \tag{7.6}$$

We now eliminate ρ_1, u_1, p_1 from the equations, to obtain two equations in the two fundamental variables $\underline{v}_1$ and T_1.

To eliminate u_1 we differentiate (7.6) (using $\dot{u}_1$ to denote $\partial u_1/\partial t$) to obtain

$$\dot{u}_1 = \left(\frac{\partial u}{\partial T}\right)_{\mathcal{V}} \dot{T}_1 + \left(\frac{\partial u}{\partial \mathcal{V}}\right)_T \dot{\mathcal{V}}_1 \tag{7.7}$$

and substitute this into the energy equation:

$$\sigma \nabla^2 T_1 = \rho_0 \left(\frac{\partial u}{\partial T}\right)_{\mathcal{V}} \dot{T}_1 + \rho_0 \left[p_0 + \left(\frac{\partial u}{\partial \mathcal{V}}\right)_T \right] \dot{\mathcal{V}}_1 \tag{7.8}$$

To eliminate p_1 from the force equation we differentiate (7.5) and use the continuity equation for $\dot{\rho}_1$:

$$\dot{p}_1 = \left(\frac{\partial p}{\partial \rho}\right)_T \dot{\rho}_1 + \left(\frac{\partial p}{\partial T}\right)_\rho \dot{T}_1 \tag{7.9}$$

so

$$\dot{p}_1 = -\rho_0 \left(\frac{\partial p}{\partial \rho}\right)_T \underline{\nabla}\cdot\underline{v}_1 + \left(\frac{\partial p}{\partial T}\right)_\rho \dot{T}_1 \tag{7.10}$$

Now we differentiate the force equation with respect to t and substitute (7.10) to obtain

$$\rho_0 \ddot{\underline{v}}_1 = \rho_0 \left(\frac{\partial p}{\partial \rho}\right)_T \underline{\nabla}(\underline{\nabla}\cdot\underline{v}_1) - \left(\frac{\partial p}{\partial T}\right)_\rho \underline{\nabla}\dot{T}_1 + \frac{4N}{3}\eta \underline{\nabla}(\underline{\nabla}\cdot\underline{v}_1) - \eta \underline{\nabla}\times\underline{\nabla}\times\underline{v}_1 \ . \tag{7.11}$$

Equations (7.8) and (7.11) are the two desired equations in $\underline{v}_1$ and T_1.

We now note the definitions

$$c_v = \left(\frac{\partial u}{\partial T}\right)_{\mathcal{V}} \tag{7.12}$$

$$\alpha_v = \frac{1}{\mathcal{V}_0}\left(\frac{\partial \mathcal{V}}{\partial T}\right)_p = -\frac{1}{\rho_0}\left(\frac{\partial \rho}{\partial T}\right)_p \tag{7.13}$$

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \gamma \left(\frac{\partial p}{\partial \rho}\right)_T \tag{7.14}$$

for the specific heat at constant volume (c_v), the volume coefficient of expansion at constant pressure (α_v) and the adiabatic velocity of sound (c). We then use the general thermodynamic relations

$$\left(\frac{\partial p}{\partial T}\right)_{\vartheta} = \left(\frac{\partial p}{\partial T}\right)_{\rho} = -\left(\frac{\partial p}{\partial \rho}\right)_{T}\left(\frac{\partial \rho}{\partial T}\right)_{p} \tag{7.15}$$

$$T_0\left(\frac{\partial p}{\partial T}\right)_{\vartheta} = p_0 + \left(\frac{\partial u}{\partial \vartheta}\right)_{T} \tag{7.16}$$

and the above definitions to obtain

$$\left(\frac{\partial p}{\partial T}\right)_{\rho} = \frac{1}{\gamma} c^2 \rho_0 \alpha_v \tag{7.17}$$

$$p_0 + \left(\frac{\partial u}{\partial \vartheta}\right)_{T} = T_0 \frac{1}{\gamma} c^2 \rho_0 \alpha_v \,. \tag{7.18}$$

Substituting these relations in equations (7.8) and (7.11) we obtain the working forms of the velocity and energy equations:

$$\ddot{\underline{v}}_1 = \frac{c^2}{\gamma}\underline{\nabla}(\underline{\nabla}\cdot\underline{v}_1) - \frac{c^2\alpha_v}{\gamma}\underline{\nabla}\dot{T}_1 + \frac{4N}{3}\nu\,\underline{\nabla}(\underline{\nabla}\cdot\dot{\underline{v}}_1) - \nu\,\underline{\nabla}\times\underline{\nabla}\times\dot{\underline{v}}_1 \tag{7.19}$$

$$\varkappa \nabla^2 T_1 = \dot{T}_1 + \frac{T_0 c^2 \alpha_v}{c_p}\underline{\nabla}\cdot\underline{v}_1 \tag{7.20}$$

where $\varkappa = \sigma/\rho_0 c_v$, and $\nu = \eta/\rho_0$ is the kinematic viscosity, and we have used $\rho_0\dot{\vartheta}_1 = \underline{\nabla}\cdot\underline{v}_1$ from (3.14).

We now introduce the periodic time dependence of all first order quantities:

$$\begin{aligned} \rho_1 &= \rho\, e^{-i\omega t} \\ \underline{v}_1 &= \underline{v}\, e^{-i\omega t} \\ p_1 &= p\, e^{-i\omega t} \\ T_1 &= T\, e^{-i\omega t} \\ u_1 &= u\, e^{-i\omega t} \end{aligned} \tag{7.21}$$

Here ρ, $\underline{v}$, p, T, u are pure space functions independent of time*. Noting that $\partial/\partial t = -i\omega$ we obtain from (7.19) and (7.20) two space equations for $\underline{v}$ and T:

$$-i\omega\underline{v} = -\frac{c^2\alpha_v}{\gamma}\underline{\nabla}T + \left(-\frac{c^2}{i\omega\gamma} + \frac{4N}{3}\nu\right)\underline{\nabla}(\underline{\nabla}\cdot\underline{v}) - \nu\underline{\nabla}\times\underline{\nabla}\times\underline{v} \tag{7.22}$$

$$\kappa\nabla^2 T = -i\omega T + \frac{T_0 c^2\alpha_v}{c_p}\underline{\nabla}\cdot\underline{v} \tag{7.23}$$

Our problem is now to solve these two space equations for $\underline{v}$ and T. It may be noted that the first-order quantities are all real; for convenience, however we have assumed them to be general complex functions in (7.21), so that after (7.22) and (7.23) have been solved for $\underline{v}$ and T we must use for the first-order quantities v_1 and T_1 the real part of the quantities in (7.21).

8. Derivation and Solution of Wave Equations for the Acoustic, Thermal and Viscous Waves

In order to solve the two space equations (7.22) and (7.23) we assume for $\underline{v}$ the general vector field

$$\underline{v} = -\underline{\nabla}\phi + \underline{\nabla}\times\underline{A} \tag{8.1}$$

where ϕ is a scalar potential and $\underline{A}$ a vector potential, and where we take $\underline{\nabla}\cdot\underline{A} = 0$ without loss of generality. The force equation (7.22) becomes

$$i\omega\underline{\nabla}\phi - i\omega\underline{\nabla}\times\underline{A} = -\frac{c^2\alpha_v}{\gamma}\underline{\nabla}T + \left[\frac{c^2}{i\omega\gamma} - \frac{4N}{3}\nu\right]\underline{\nabla}(\nabla^2\phi) + \nu\underline{\nabla}\times(\nabla^2\underline{A}) \tag{8.2}$$

or, equating the vector and scalar parts separately we get

*The use of these symbols for the amplitudes of the first-order quantities will cause no confusion with Chapter II since throughout the remainder of the work we shall use only the first-order quantities.

$$-i\omega \underline{A} = \nu \nabla^2 \underline{A}$$

$$i\omega\phi = -\frac{c^2\alpha_v}{\gamma} T + \left[\frac{c^2}{i\omega\gamma} - \frac{4N}{3}\nu\right] \nabla^2\phi \; . \tag{8.3}$$

The energy equation (7.23) becomes

$$\varkappa \nabla^2 T = -i\omega T - T_0 \frac{c^2\alpha_v}{C_p} \nabla^2\phi \; . \tag{8.4}$$

From (8.3) we obtain T as a function of the scalar velocity potential:

$$T = -\frac{i\omega\gamma}{c^2\alpha_v}\phi + \left(\frac{1}{i\omega\alpha_v} - \frac{4N\nu\gamma}{3c^2\alpha_v}\right)\nabla^2\phi \tag{8.5}$$

and by substituting this into (8.4) we eliminate T and obtain the <u>biquadratic</u> or <u>double wave equation in ϕ :</u>

$$\frac{c^4}{\omega^4}\left[\frac{4N\nu\varkappa\omega^2}{3c^4} + i\frac{\varkappa\omega^2}{\gamma c^2}\right]\nabla^4\phi - \frac{c^2}{\omega^2}\left[1 - i\left(\frac{4N\nu\omega}{3c^2} + \frac{\varkappa\omega}{c^2}\right)\right]\nabla^2\phi - \phi = 0 \tag{8.6}$$

We now make the substitutions

$$e = \frac{4N}{3}\frac{\nu\omega}{c^2} << 1 \qquad , \qquad f = \frac{\varkappa\omega}{c^2} << 1 \tag{8.7}$$

where for most fluids the magnitudes of the dimensionless quantities e and f are generally very small for values of $\omega < 10^6$. (See Table I for the values of the various constants for water and air.) We then have to solve the biquadratic equation

$$\frac{c^4}{\omega^4}\left[ef + if/\gamma\right]\nabla^4\phi - \frac{c^2}{\omega^2}\left[1 - i(e+f)\right]\nabla^2\phi - \phi = 0 \tag{8.8}$$

<u>Solution of Biquadratic Equation</u>

A solution of (8.8) may be obtained in the form

$$\phi = \phi_1 + \phi_2 \tag{8.9}$$

where ϕ_1 and ϕ_2 are wave functions satisfying the equations

$$\nabla^2\phi_1 + k_1^2\phi_1 = 0\,, \qquad \nabla^2\phi_2 + k_2^2\phi_2 = 0 \tag{8.10}$$

and where k_1^2 and k_2^2 are the two roots of an equation obtained from (8.8) as follows: We substitute

$$\nabla^2\phi + k^2\phi = 0 \tag{8.11}$$

in (8.8) and get

$$\left\{ \frac{c^4}{\omega^4} k^4[ef + if/\gamma] + \frac{c^2}{\omega^2}k^2[1 - i(e+f)] - 1 \right\} \phi = 0\,. \tag{8.12}$$

Since ϕ is not identically zero the curly bracket must vanish, so that, putting $\varkappa = (ck/\omega)^2$, we must solve the quadratic in $\varkappa$ (or k^2):

$$\varkappa^2[ef + if/\gamma] + \varkappa[1 - i(e+f)] - 1 = 0\,. \tag{8.13}$$

Since e and f are very small we see immediately that one root is approximately $\varkappa_1 = 1$. We therefore put

$$\varkappa = 1 + g \tag{8.14}$$

where $g \ll 1$ and solve for g, retaining only first-order terms in e, f, g. The result is

$$g = i[e + f - f/\gamma] \tag{8.15}$$

so that one root is approximately

$$\varkappa_1 = 1 + i[e + f - f/\gamma]\,. \tag{8.16}$$

To obtain the second root we note that if we divide (8.13) by the coefficient of $\varkappa^2$, the constant term gives the product of the two roots:

$$\varkappa_1\varkappa_2 = (ef + if/\gamma)^{-1}\,. \tag{8.17}$$

Using (8.16) and again keeping only first-order terms we obtain

$$\varkappa_2 = i\gamma/f \quad . \tag{8.18}$$

Replacing $\varkappa_1$ by $(ck_1/\omega)^2$ and $\varkappa_2$ by $(ck_2/\omega)^2$ we finally obtain

$$k_1^2 = \frac{\omega^2}{c^2}\left[1 + i(e + f - f/\gamma)\right] \tag{8.19}$$

$$k_2^2 = \frac{\omega^2}{c^2}\left[i\gamma/f\right] \tag{8.20}$$

or, replacing e and f we have

$$k_1^2 = \frac{\omega^2}{c^2}\left\{1 + i\left[\frac{4N}{3}\frac{\nu\omega}{c^2} + \frac{\varkappa\omega}{c^2}(1 - 1/\gamma)\right]\right\} \tag{8.21}$$

$$k_2^2 = i\gamma\omega/\varkappa \quad . \tag{8.22}$$

These are the desired values of the two roots of the quadratic in k^2 inside the curly bracket of (8.12).

Summary. We now collect our results: We have found that the velocity field is given by two scalar potentials Φ_1 and Φ_2 and the vector potential $\underline{A}$, all of which satisfy the wave equation:

Velocity $$\underline{v} = -\underline{\nabla}\Phi_1 - \underline{\nabla}\Phi_2 + \underline{\nabla}\times\underline{A} \tag{8.23}$$

Acoustic Wave $$\nabla^2\Phi_1 + k_1^2\Phi_1 = 0 \tag{8.24}$$

Thermal Wave $$\nabla^2\Phi_2 + k_2^2\Phi_2 = 0 \tag{8.25}$$

Viscous Wave $$\nabla^2\underline{A} + K^2\underline{A} = 0 \tag{8.26}$$

where from (8.2), (8.21) and 8.22) the wave numbers are given by

$$k_1^2 = \frac{\omega^2}{c^2}\left\{1 + i\left[\frac{4N}{3}\frac{\nu\omega}{c^2} + (\gamma-1)\frac{\bar{\varkappa}\omega}{c^2}\right]\right\} \tag{8.27}$$

$$k_2^2 = i\omega/\bar{\varkappa} \tag{8.28}$$

$$K^2 = i\omega/\nu \tag{8.29}$$

and where in place of $\varkappa$ we have for convenience used the thermometric conductivity $\bar{\varkappa}$ defined by

$$\bar{\varkappa} = \sigma/\rho_0 c_p = \varkappa/\gamma \ . \tag{8.30}$$

(Since the imaginary part of the wave numbers gives rise to the damping effect for the waves, it was necessary to retain the small terms in k_1^2 to exhibit this effect.) Taking the square roots we obtain

$$k_1 = \frac{\omega}{c}\left\{1 + i\,\frac{1}{2}\left[\frac{4N}{3}\frac{\nu\omega}{c^2} + (\gamma-1)\frac{\bar{\varkappa}\omega}{c^2}\right]\right\} \tag{8.31}$$

$$k_2 = (1+i)(\omega/2\bar{\varkappa})^{1/2} = (1+i)(\omega\rho_0 c_p/2\sigma)^{1/2} \tag{8.32}$$

$$K = (1+i)(\omega/2\nu)^{1/2} = (1+i)(\omega\rho_0/2\eta)^{1/2}, \tag{8.33}$$

where in each case we use the root corresponding to damping of the wave (i.e., positive imaginary part).

The corresponding phase velocities and wavelengths for the acoustic, thermal and viscous waves are*

---

* The subscript $\perp$ denotes the "traverse" character of the viscous or shear wave.

$$
\begin{aligned}
c &= \omega/k_1 & \lambda_1 &= 2\pi/k_1 \\
c_2 &= \omega/\mathcal{R}(k_2) = (2\omega\varkappa)^{1/2} & \lambda_2 &= 2\pi/\mathcal{R}(k_2) = 2\pi(2\varkappa/\omega)^{1/2} \\
c_\perp &= \omega/\mathcal{R}(K) = (2\omega\nu)^{1/2} & \lambda_\perp &= 2\pi/\mathcal{R}(K) = 2\pi(2\nu/\omega)^{1/2}.
\end{aligned}
\tag{8.34}
$$

The velocity and wave length of the thermal and viscous waves are seen to be frequency-dependent. Values for these quantities at various frequencies for water and air are listed in Table I at the end of the text. The imaginary part of the wave numbers k_2 and K give rise to the "damping lengths" discussed in Sec. 10.

We have called ϕ_1 the acoustic wave because it is propagated with the normal sound speed c; ϕ_2 is called the thermal wave since its character is determined by the thermal conductivity; finally $\underline{A}$ is the viscous or shear wave since its properties depend on the viscosity.

To examine the physical nature of the three waves and their effects in dissipating acoustic energy we shall treat the one dimensional plane wave in the following chapter. Before doing this however, we consider briefly the manner in which the first-order temperature and pressure depend on the velocity potentials, and derive a useful expression for the viscous and thermal dissipation in terms of the first-order acoustic quantities.

9. Expressions for First-Order Temperature and Pressure and for the Viscous and Thermal Dissipation

First-Order Temperature

Substituting our expression for $\underline{v}$ from (8.23) into the temperature equation (8.5) we obtain

$$T = \alpha_1\phi_1 + \alpha_2\phi_2 \tag{9.1}$$

where the constants α_1 and α_2 are given by

$$\alpha_1 = -i\omega(\gamma-1)/c^2\alpha_v = -i\omega T_0 \alpha_v / c_p \qquad (9.2)$$

$$\alpha_2 = -(\alpha_v \bar{\varkappa})^{-1} . \qquad (9.3)$$

Thus the space part of the first-order temperature $T_1 = T e^{-i\omega t}$ is a linear function of the two scalar potentials ϕ_1 and ϕ_2. The same result is reached if we substitute (9.1) in the energy equation (7.23) and equate the coefficients of ϕ_1 and ϕ_2 to obtain α_1 and α_2. The reductions above are facilitated by the useful relation

$$T_0 c^2 \alpha_v^2 = c_p(\gamma-1) \qquad (9.4)$$

which is easily proved from the definitions of α_v and c^2 in (7.13) and (7.14), together with the familiar thermodynamic relation for the specific heats:

$$c_p - c_v = T_0 \left(\frac{\partial p}{\partial T}\right)_{\vartheta} \left(\frac{\partial \vartheta}{\partial T}\right)_p . \qquad (9.5)$$

First-Order Pressure

To derive the pressure dependence on ϕ_1 and ϕ_2 we write $\underline{v}_1 = \underline{v}\, e^{-i\omega t}$ in the linear force equation (7.3) to obtain

$$-i\omega\rho_0 \underline{v} = -\underline{\nabla} p + \frac{4N}{3}\eta \underline{\nabla}(\underline{\nabla}\cdot\underline{v}) - \eta \underline{\nabla}\times\underline{\nabla}\times\underline{v} . \qquad (9.6)$$

We then substitute $\underline{v}$ from (8.23) and separate the vector (curl) and scalar (divergence) parts, the latter giving

$$i\omega\rho_0 \underline{\nabla}(\phi_1 + \phi_2) = -\underline{\nabla} p - \frac{4N}{3}\eta \underline{\nabla}(\nabla^2\phi_1 + \nabla^2\phi_2) \qquad (9.7)$$

or

$$i\omega\rho_0(\phi_1+\phi_2) = -p - \frac{4N}{3}\eta(\nabla^2\phi_1+\nabla^2\phi_2) , \tag{9.8}$$

and using the wave equations (8.24) and (8.25) we obtain

$$p = -i\omega\rho_0(\gamma_1\phi_1+\gamma_2\phi_2) \tag{9.9}$$

where the constants γ_1 and γ_2 are given by

$$\gamma_1 = 1 + i\,\frac{4N}{3}\,\frac{\nu\omega}{c^2} \tag{9.10}$$

$$\gamma_2 = 1 - \frac{4N}{3}\,\frac{\nu}{\chi} \quad . \tag{9.11}$$

We drop the small imaginary term in γ_1 but retain the second term in γ_2 since in general it is of order 1 for fluids. Thus the space part (p) of the first-order pressure $p_1 = p\,e^{-i\omega t}$, like the temperature, is a linear function of the two scalar potentials ϕ_1 and ϕ_2.

The Viscous Dissipation

We have seen in Sec. 6 that the total viscous dissipation is given by the volume integral

$$\frac{dE_\eta}{dt} = \int \Phi\, dV \ . \tag{9.12}$$

It is in general a tedious process to evaluate this integral and a great simplification in introduced by converting part of the integral to a surface integral. This we now proceed to do.

The dissipation function Φ is defined by (3.12):

$$\Phi = \tau_{\alpha\epsilon} \nabla_\alpha v_\epsilon \tag{9.13}$$

$$= p_{\alpha\epsilon} \nabla_\alpha v_\epsilon + p\,\delta_{\alpha\epsilon} \nabla_\alpha v_\epsilon \tag{9.14}$$

$$= \nabla_\alpha (p_{\alpha\epsilon} v_\epsilon) - v_\epsilon \nabla_\alpha p_{\alpha\epsilon} + p\,\underline{\nabla}\cdot\underline{v} \ . \tag{9.15}$$

So
$$\Phi = \nabla_\alpha (p_{\alpha\epsilon} v_\epsilon) - \rho \frac{D v^2/2}{Dt} + p\,\underline{\nabla}\cdot\underline{v} \ , \tag{9.16}$$

where we have used the definition of the total stress tensor $p_{\alpha\epsilon}$ from (2.12) and the relation (3.9) for the kinetic energy. From (9.16) we then obtain the desired general form:

$$\frac{dE_n}{dt} = \int p_{\alpha\epsilon} v_\epsilon \, dS_\alpha + \int p\,\underline{\nabla}\cdot\underline{v}\, dV - \int \rho \frac{D v^2/2}{Dt}\, dV \ , \tag{9.17}$$

where the first volume integral has been converted to a surface integral over the surface S bounding the fixed volume V.

For the acoustic case we expand the stress to first order terms

$$p_{\alpha\epsilon} = - p_0 \delta_{\alpha\epsilon} + p_{1\alpha\epsilon} \tag{9.18}$$

where

$$p_{1\alpha\epsilon} = - p_1 \delta_{\alpha\epsilon} + \tau_{1\alpha\epsilon} \tag{9.19}$$

and use the previous expansions in ρ, $\underline{v}$ and p from (7.1)

$$\rho = \rho_0 + \rho_1 \qquad \underline{v} = \underline{v}_1 \qquad p = p_0 + p_1 \quad . \tag{9.20}$$

These give for Φ in (9.16)

$$\Phi = \nabla_\alpha \left(p_{1\alpha\epsilon} v_{1\epsilon} \right) - \rho_0 \frac{\partial v_1^2/2}{\partial t} + p_1 \nabla \cdot v_1 \tag{9.21}$$

so that Φ is determined by products of the first-order acoustic quantities each of which is periodic in time. Since we are interested in the average rate of dissipation we take the time average of Φ, using the relations in Appendix I, and obtain

$$\Phi_{Av} = \frac{1}{2} \mathcal{R} \left[\nabla_\alpha \left(p_{1\alpha\epsilon}^* v_{1\epsilon} \right) - \rho_0 \dot{v}_1^* \cdot v_1 + p_1^* \nabla \cdot v_1 \right] . \tag{9.22}$$

The second term in (9.22) may be dropped since it is pure imaginary ($\rho_0 i\omega v_1^* \cdot v_1$) and its time average is zero (see Appendix I). Here $\mathcal{R}$ denotes the real part of the expression following and the star * denotes complex conjugate. As before we write the first-order quantities in the form

$$p_1 = p e^{-i\omega t} \qquad v_1 = v e^{-i\omega t} \qquad p_{1\alpha\epsilon} = p_{\alpha\epsilon} e^{-i\omega t} \tag{9.23}$$

where p, v, $p_{\alpha\epsilon}$ (the amplitudes of the corresponding first-order quantities) are complex space functions only and do not depend on the time. Thus

$$\Phi_{Av} = \frac{1}{2} \mathcal{R} \left[\nabla_\alpha \left(p_{\alpha\epsilon}^* v_\epsilon \right) + p^* \nabla \cdot v \right] . \tag{9.24}$$

so that, integrating this we have

$$\frac{dE_n}{dt} = \int \Phi_{Av} dV = \frac{1}{2} \mathcal{R} \int p_{\alpha\epsilon}^* v_\epsilon \, dS_\alpha + \frac{1}{2} \mathcal{R} \int p^* \nabla \cdot v \, dV , \tag{9.25}$$

and this is our general computational form for the viscous dissipation in an acoustic field.

Thermal Dissipation

The thermal dissipation is given by

$$\frac{dE_\sigma}{dt} = \int \Psi \, dV . \tag{9.26}$$

This can also be transformed to a surface integral and a simpler volume integral for computation. We have

$$\Psi = \frac{\sigma}{T_0} (\underline{\nabla} T \cdot \underline{\nabla} T) \tag{9.27}$$

$$= \frac{\sigma}{T_0} \left[\underline{\nabla} \cdot (T \underline{\nabla} T) - T \nabla^2 T \right] . \tag{9.28}$$

Now $\nabla^2 T$ is given by the energy equation (7.20):

$$\nabla^2 T = \frac{1}{\varkappa} \left(\dot{T} + T_0 \frac{c^2 \alpha_v}{c_p} \underline{\nabla} \cdot \underline{v} \right). \tag{9.29}$$

We therefore have for Ψ

$$\Psi = \frac{\sigma}{T_0} \underline{\nabla} \cdot (T \underline{\nabla} T) - \frac{\sigma}{T_0 \varkappa} \left(T \dot{T} + T_0 \frac{c^2 \alpha_v}{c_p} T \, \underline{\nabla} \cdot \underline{v} \right), \tag{9.30}$$

and Ψ_{Av} is

$$\Psi_{Av} = \frac{\sigma}{2T_0} \mathcal{R} \left[\underline{\nabla} \cdot (T^* \underline{\nabla} T) \right] - \frac{\rho_0 c^2 \alpha_v}{2 \gamma} \mathcal{R} \left[T^* \underline{\nabla} \cdot \underline{v} \right], \tag{9.31}$$

where the $T\mathring{T}$ being pure imaginary has been dropped, since its real part is zero, and we have used the definition $\varkappa = \sigma / \rho_0 c_v$ to obtain the new form for the coefficient of the last term. Integrating this we have for the thermal dissipation

$$\frac{dE_\sigma}{dt} = \int \Psi_{Av}\, dV = \frac{\sigma}{2T_0} \mathcal{R} \int (T^* \underline{\nabla} T) \cdot d\underline{S} - \frac{\rho_0 c^2 \alpha_v}{2\gamma} \mathcal{R} \int T^* \underline{\nabla} \cdot \underline{v}\, dV, \qquad (9.32)$$

and this is the desired general computational form. The surface integral is extended over the closed surface S bounding the volume of integration V.

10. Plane Wave Solutions

We now apply the foregoing theory to the case of a one-dimensional plane wave moving in the positive x-direction. The appropriate solution of the scalar wave equations is

$$\phi = \phi_1 + \phi_2 = A e^{ik_1 x} + B e^{ik_2 x} \qquad (10.1)$$

where A and B are constants. Since k_1 and k_2 are complex the amplitudes will diminish exponentially. Using (8.31) and (8.32) for k_1 and k_2,

$$\phi_1 = A \exp\left[i \frac{\omega}{c} x - \frac{x\omega}{2c}\left(\frac{4N}{3}\frac{\nu\omega}{c^2} + \varkappa \frac{\omega}{c^2}(\gamma - 1)\right)\right] \qquad (10.2)$$

$$\phi_2 = B \exp\left[i x (\omega/2\varkappa)^{1/2} - x(\omega/2\varkappa)^{1/2}\right], \qquad (10.3)$$

so that if ℓ is the absorption length, i.e., the distance in which the wave amplitude declines to 1/e of its initial value, we have

$$\ell_1 = \frac{2c}{\omega}\left[\frac{4N}{3}\frac{\nu\omega}{c^2} + (\gamma-1)\frac{\varkappa\omega}{c^2}\right]^{-1} \qquad (10.4)$$

$$\ell_2 = (2\varkappa/\omega)^{1/2}. \qquad (10.5)$$

Since $\ell_1 \propto \omega^{-2}$, $\ell_2 \propto \omega^{-1/2}$, the absorption lengths both decrease with frequency, and the absorption increases with frequency, the increase being faster for ϕ_1 than for ϕ_2. The absorption lengths for air and water are tabulated in Table I at the end of the thesis. It is apparent from Table I that ℓ_1 and ℓ_2 are of entirely different orders of magnitude, ϕ_1 being damped only slightly over distances of the order of centimeters, while ϕ_2 is very strongly damped.

For air ν and $\mathfrak{K}$ are about equal so that the thermal absorption is comparable to the viscous absorption for the acoustic wave ϕ_1, a result previously obtained by Kirchhoff.

While we have no viscous shear wave in a one-dimensional sound field, we see from the wave equation that the wave number K (8.33) gives an absorption length for this transverse ($\perp$) wave of

$$\ell_{\perp} = (2\nu/\omega)^{1/2} \tag{10.6}$$

in the same way as the thermal wave number k_2 (8.32). As is seen from Table I, the viscous wave is also rapidly damped, the absorption of the viscous and thermal waves being comparable in magnitude.

It therefore appears that the main source of attenuation of the incident plane wave on small spherical obstacles is due to the conversion of acoustic waves into thermal and viscous waves which are rapidly absorbed within small distances from the spheres.

To obtain the attenuation α we first proceed in the usual way to derive the acoustic intensity in the form $E = E_o e^{-\alpha x}$. We consider only ϕ_1 since ϕ_2 is rapidly damped in any case, and note that if we neglect small transport and conduction terms, the energy flux is given by the average rate at which work is done by the stresses on unit area normal to the wave:

$$E = (p_1 v_1)_{Av} = \frac{1}{2}\mathcal{R}(p^* v) \tag{10.7}$$

and using (9.9) for the pressure and (10.2) for the potential we have

$$E = \frac{1}{2}\mathcal{R}\left[(i\omega\rho_0\phi_1^*)\left(-\frac{\partial\phi_1}{\partial x}\right)\right] = \frac{1}{2}\omega\rho_0 k_1 AA^* e^{i(k_1-k_1^*)x} \tag{10.8}$$

or

$$E = E_0 e^{-\alpha x} \tag{10.9}$$

where

$$\alpha = \frac{2}{\ell_1} = \frac{\omega}{c}\left[\frac{4N}{3}\frac{\nu\omega}{c^2} + (\gamma-1)\frac{\chi\omega}{c^2}\right], \tag{10.10}$$

is obtained from the real part of ik_1x in (10.2). This is then the attenuation desired and it is seen to consist of the viscous attenuation obtained by Stokes (but modified by N to include the effects of the compressional viscosity), together with the Kirchhoff thermal attenuation. For future use we note that over distances small compared to ℓ_1 the wave is attenuated negligibly and the intensity is

$$E = E_0 = \frac{1}{2}k_1\omega\rho_0 AA^* \tag{10.11}$$

for a plane wave whose velocity potential is $\phi_1 = A\, e^{ik_1x}$.

A second method of obtaining the attenuation is to evaluate the average viscous and thermal dissipation integrals (6.3) and (6.4):

$$\frac{dE_\eta}{dt} = \int \Phi \, dV \tag{10.12}$$

$$\frac{dE_\sigma}{dt} = \int \Psi \, dV . \tag{10.13}$$

For the one-dimensional case Φ is found from (5.12) to be

$$\Phi = \frac{4N}{3}\eta\left(\frac{\partial v}{\partial x}\right)^2 \tag{10.14}$$

or

$$\Phi_{Av} = \frac{1}{2}\mathcal{R}\left[\frac{4N}{3}\eta\left(\frac{\partial v}{\partial x}\right)\left(\frac{\partial v^*}{\partial x}\right)\right] \tag{10.15}$$

$$\approx \frac{2N}{3}\eta\frac{\omega^4}{c^4}AA^* \; . \tag{10.16}$$

From (10.11) this gives

$$\frac{dE_n}{dt} = \frac{2N}{3}\eta\frac{\omega^4}{c^4}AA^*L \tag{10.17}$$

for a volume of integration of unit cross section normal to the wave and length L, where $L \ll \ell_1$, so that the attenuation of ϕ_1 is negligible over the volume.

For the thermal dissipation we have from 6.2)

$$\Psi_{Av} = \frac{1}{2}\mathcal{R}\left[\frac{\sigma}{T_0}\nabla T^* \cdot \nabla T\right] \tag{10.18}$$

Using our temperature relation (9.1)

$$T = \alpha_1\phi_1 \tag{10.19}$$

this becomes

$$\Psi_{Av} = \frac{1}{2}\frac{\sigma}{T_0}(\alpha_1\alpha_1^*)(k_1k_1^*)AA^* \tag{10.20}$$

From (10.13) we then have the thermal dissipation:

$$\frac{dE_\sigma}{dt} = \frac{\sigma}{2T_0}\left[\frac{\omega^2(\gamma-1)T_0}{c^2 c_p}\right]\frac{\omega^2}{c^2} AA^* L \tag{10.21}$$

where (9.2) has been used for $(\alpha_1\alpha_1^*)$ and again S is a unit area normal to the direction of the wave and L is the length of the volume of integration. To obtain the attenuation α per unit length we divide the total dissipation per unit length by the incident energy flux E_0 from (10.11):

$$\alpha = \frac{1}{E_0}\left(\frac{1}{L}\frac{dE_\eta}{dt} + \frac{1}{L}\frac{dE_\sigma}{dt}\right) \tag{10.22}$$

or

$$\alpha = \frac{\omega}{c}\left[\frac{4N}{3}\frac{\nu\omega}{c^2} + (\gamma-1)\frac{\bar{\varkappa}\omega}{c^2}\right] \tag{10.23}$$

Since this agrees with our previous result, we have thus verified that the two methods of obtaining the attenuation are consistent.

IV. ATTENUATION OF PLANE SOUND WAVES BY SMALL SPHERES

We now attack our main problem of calculating the attenuation of sound waves by small uniform spherical inhomogeneities embedded in an infinite fluid medium. We shall start by considering a single sphere of radius R and finding the velocity potentials. Knowing these we can then calculate the additional viscous and thermal dissipation caused by the presence of the sphere. Finally, from the dissipation we shall compute the attenuation for a number of such spheres.

11. Assumptions

The problem will be solved under the following assumptions, the first three being the assumptions made in the development of the general theory of Part II:

(1) The viscosity, thermal conductivity specific heats, etc., will be assumed to be constants, in particular, they do not depend on the temperature.

(2) The coefficient of compressional viscosity will be retained and will not be assumed to vanish.

(3) The medium will be assumed to be in a pseudo-steady state so that the imperturbed temperature, pressure, etc., are constant.

(4) We shall assume that over the region occupied by the small spheres the attenuation is so small that the intensity of the incident wave may be considered constant (See Sec. 15).

(5) The spheres, of radius R, will be assumed very small compared to the acoustic wave length λ_1.

(6) We shall assume the deformation of the spheres by the stress variations to be negligible so that in applying the boundary conditions at the surface we shall put the radial coordinate $r = R$.

(7) We shall not include the attenuation caused by scattering. As shown by Epstein (Ref. 6) this effect is very small compared to the viscous and thermal attenuation because of assumption (4). (Compare α_s with α_η and α_σ in the Introduction.) If the scattering term is required it can easily be calculated.

(8) The spheres will be assumed to be sparsely and randomly distributed so that the interaction effects between spheres (secondary scattering, collisions, aggregation, etc.) can be neglected.

12. Velocity Potentials

Consider a plane sound wave propagating in an infinite fluid medium and impinging on a fluid sphere. We shall denote quantities relating to the outer medium by unprimed letters and those pertaining to the interior by primed letters. Outside the sphere we have the incident wave represented by a potential ϕ_i and the scattered wave, represented by the acoustic and thermal scalar potentials ϕ_1 and ϕ_2 and the shear vector potential $\underline{A}$. Inside the sphere we have two scalar potentials ϕ_1' and ϕ_2' and a vector potential $\underline{A}'$. We thus have seven potentials.

We choose a spherical coordinate system (r, θ, φ) with origin at the center of the sphere and polar axis in the direction of propagation of the incident wave. Because of the axial symmetry two components of the vector potential vanish ($A_r = A_\theta = 0$) and only the azimuthal component A_φ is different from zero. Again, because of the axial symmetry, there is no dependence on φ . Under these conditions the appropriate solutions of the wave equations

$$\nabla^2(\phi_i+\phi_1)+k_1^2(\phi_i+\phi_1)=0 \qquad \nabla^2\phi_2+k_2^2\phi_2=0 \qquad \nabla^2\underline{A}+K^2\underline{A}=0$$

$$\nabla^2\phi_1'+k_1'^2\phi_1'=0 \qquad \nabla^2\phi_2'+k_2'^2\phi_2'=0 \qquad \nabla^2\underline{A}'+K'^2\underline{A}'=0 \tag{12.1}$$

can be written in the form

$$\begin{aligned}
\phi_i = e^{ik_1 r\cos\theta} &= \sum_{n=0}^{\infty} i^n(2n+1)\,P_n(\cos\theta)\,j_n(k_1 r)\\
\phi_1 &= \sum_{n=0}^{\infty} i^n(2n+1)\,P_n(\cos\theta)\,B_n h_n(k_1 r)\\
\phi_2 &= \sum_{n=0}^{\infty} i^n(2n+1)\,P_n(\cos\theta)\,C_n h_n(k_2 r)\\
A_\varphi = A &= \sum_{n=1}^{\infty} i^n(2n+1)\,P_n^1(\cos\theta)\,D_n h_n(K r)\\
\phi_1' &= \sum_{n=0}^{\infty} i^n(2n+1)\,P_n(\cos\theta)\,B_n' j_n(k_1' r)\\
\phi_2' &= \sum_{n=0}^{\infty} i^n(2n+1)\,P_n(\cos\theta)\,C_n' j_n(k_2' r)\\
A_\varphi' = A' &= \sum_{n=1}^{\infty} i^n(2n+1)\,P_n^1(\cos\theta)\,D_n' j_n(K' r)
\end{aligned} \tag{12.2}$$

Here the form for ϕ_i is the standard expansion for a plane wave in spherical coordinates; the remaining expressions are convenient forms of the wave functions in spherical coordinates, with the six constants $B_n C_n D_n B'_n C'_n D'_n$ to be determined from the boundary conditions. The radial functions j_n and h_n are the spherical Bessel Hankel functions discussed in Appendix II. The choice of h_n (Hankel function of the first kind) for the outside scattered potentials insures that the ϕ_2 and A waves decline exponentially at large distances. Inside the sphere we use the Bessel functions which are the solutions regular at r = 0. The functions $P_n(\cos\theta)$ and $P_n^1(\cos\theta)$ are Legendre functions and are discussed in Appendix III.

13. Velocity, Temperature, Pressure, and Stresses

The velocity in the outside medium is given by (8.23):

$$\underline{v} = -\underline{\nabla}\phi + \underline{\nabla}\times\underline{A} \tag{13.1}$$

where $\phi = \phi_i + \phi_1 + \phi_2$. Writing $A = A_\varphi$ we have the velocity components:

$$\begin{aligned} v_r &= -\frac{\partial\phi}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A\sin\theta) \\ v_\theta &= -\frac{1}{r}\frac{\partial\phi}{\partial\theta} - \frac{1}{r}\frac{\partial}{\partial r}(Ar) \end{aligned} \tag{13.2}$$

$$v_\varphi = 0$$

with corresponding expressions for $v_r', v_\theta', v_\varphi'$.

From Sec. 9 we obtain the expressions for the temperature and pressure:

$$T = \alpha_1(\phi_i + \phi_1) + \alpha_2\phi_2 \qquad T' = \alpha_1'\phi_1' + \alpha_2'\phi_2' \tag{13.3}$$

and

$$p = -i\omega\rho_0[\gamma_1(\phi_i+\phi_1) + \gamma_2\phi_2] \qquad p' = -i\omega\rho_0'[\gamma_1'\phi_1' + \gamma_2'\phi_2'] . \tag{13.4}$$

We now obtain the radial stress components from (2.15), (2.16), and (13.2) using standard tensor methods (Ref. 9):

$$p_{r\theta} = \eta\left\{-2\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial\phi}{\partial r} - \frac{\phi}{r^2}\right) - \left(\frac{\partial^2 A}{\partial r^2} - \frac{2A}{r^2}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(A\sin\theta)\right]\right\} \tag{13.5}$$

$$p_{rr} = \eta K^2\left[\beta_1(\phi_i+\phi_1) + \beta_2\phi_2\right] + 2\eta\left\{-\frac{\partial^2\phi}{\partial r^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\left(-\frac{A}{r^2} + \frac{1}{r}\frac{\partial A}{\partial r}\right)\right]\right\} \tag{13.6}$$

$$p_{r\varphi} = 0 \tag{13.7}$$

where

$$\beta_1 = 1 + 2iN\frac{\nu\omega}{c^2} \qquad\qquad \beta_2 = 1 - 2\frac{N\nu}{\varkappa} \,. \tag{13.8}$$

14. Boundary Conditions

The boundary conditions at the surface of the sphere are obtained from the physical requirements that the velocity, temperature, radial heat current, and normal (radial) stress components all be continuous across the surface r = R. We thus have the six boundary conditions

$$\begin{array}{llll} (1) & v_r = v_r' & (4) & \sigma\nabla_r T = \sigma'\nabla_r T' \\ (2) & v_\theta = v_\theta' & (5) & p_{r\theta} = p_{r\theta}' \\ (3) & T = T' & (6) & p_{rr} = p_{rr}' \end{array} \tag{14.1}$$

and these are sufficient to solve for the six coefficients forthe potentials. For simplicity we introduce the abbreviations

$$\begin{array}{lll} a_1 = k_1 R & a_2 = k_2 R & b = KR \\ a_1' = k_1' R & a_2' = k_2' R & b' = K'R \end{array} \tag{14.2}$$

Using the standard relations for the Legendre functions (Appendix III):

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, P_n^1\right) = n(n+1)P_n \,, \qquad \frac{\partial P_n}{\partial\theta} = -P_n^1 \tag{14.3}$$

and indicating differentiation of the Bessel and Hankel functions with respect to their arguments by primes, we obtain from (14.1) and the previous section the six equations:

$$(v_r):\quad \begin{aligned} & B_n a_1 h_n'(a_1) - B_n' a_1' j_n'(a_1') + C_n a_2 h_n'(a_2) - C_n' a_2' j_n'(a_2') \\ & \quad - D_n n(n+1) h_n(b) + D_n' n(n+1) j_n(b') = -a_1 j_n'(a_1) \end{aligned} \tag{14.4}$$

$$(v_\theta):\quad \begin{aligned} & B_n h_n(a_1) - B_n' j_n(a_1') + C_n h_n(a_2) - C_n' j_n(a_2') \\ & \quad - D_n[b h_n'(b) + h_n(b)] + D_n'[b' j_n'(b') + j_n(b')] = -j_n(a_1) \end{aligned} \tag{14.5}$$

$$(T):\quad B_n \alpha_1 h_n(a_1) - B_n' \alpha_1' j_n(a_1') + C_n \alpha_2 h_n(a_2) - C_n' \alpha_2' j_n(a_2') = -\alpha_1 j_n(a_1) \tag{14.6}$$

$$(q_r):\quad B_n \chi \alpha_1 a_1 h_n'(a_1) - B_n' \alpha_1' a_1' j_n'(a_1') + C_n \chi \alpha_2 a_2 h_n'(a_2) - C_n' \alpha_2' a_2 j_n'(a_2) = -\chi \alpha_1 a_1 j_n'(a_1) \tag{14.7}$$

$$(p_{r\theta}):\quad \begin{aligned} & B_n \epsilon [a_1 h_n'(a_1) - h_n(a_1)] - B_n' [a_1' j_n'(a_1') - j_n(a_1')] \\ & + C_n \epsilon [a_2 h_n'(a_2) - h_n(a_2)] - C_n' [a_2' j_n'(a_2') - j_n(a_2')] \\ & - D_n \tfrac{\epsilon}{2} [b^2 h_n''(b) + (n^2+n-2) h_n(b) + D_n' \tfrac{1}{2} [b'^2 j_n''(b') + (n^2+n-2) j_n(b')] \\ & \qquad = \epsilon [a_1 j_n'(a_1) - j_n(a_1)] \end{aligned} \tag{14.8}$$

$$(p_{rr}):\quad \begin{aligned} & B_n \epsilon [b^2 \beta_1 h_n(a_1) - 2a_1^2 h_n''(a_1)] - B_n' [b'^2 \beta_1' j_n(a_1') - 2a_1'^2 j_n''(a_1')] \\ & + C_n \epsilon [b^2 \beta_2 h_n(a_2) - 2a_2^2 h_n''(a_2)] - C_n' [b'^2 \beta_2' j_n(a_2') - 2a_2'^2 j_n''(a_2')] \\ & + D_n \epsilon 2n(n+1) [b h_n'(b) - h_n(b)] - D_n' 2n(n+1) [b' j_n'(b') - j_n(b')] = -\epsilon [b^2 \beta_1 j_n(a_1) - 2a_1^2 j_n''(a_1)] \end{aligned} \tag{14.9}$$

where

$$\epsilon = \eta/\eta' \qquad \chi = \sigma/\sigma' \; . \tag{14.10}$$

For n=0 there is no vector potential ($D_o = D_o' = 0$) and no Θ-dependence, so that the v_Θ and $p_{r\Theta}$ equations do not appear. In this case there are four equations in B_o, B'_o, C_o, C'_o. For $n \geq 1$ all six equations are needed for the six unknown coefficients B_n, B'_n, C_n, C'_n, D_n, D'_n.

15. Approximations

The problem has thus far been completely general; we have made no approximations and have not used our condition that R is small compared to λ_1.

We now introduce assumptions (4) and (5) and make various approximations.

Assumption (4) states that the acoustic waves are not appreciably damped over the region occupied by the scattering spheres. This means that we neglect the imaginary part of k_1:

$$k_1 = \frac{\omega}{c}\left\{1 + i\,\frac{1}{2}\left[\frac{4N}{3}\frac{\nu\omega}{c^2} + (\gamma-1)\frac{\varkappa\omega}{c^2}\right]\right\} \approx \frac{\omega}{c}$$

which amounts to neglecting the extremely small quantities

$$\frac{4N}{3}\frac{\nu\omega}{c^2} << 1 \qquad\qquad (\gamma-1)\frac{\varkappa\omega}{c^2} << 1 \tag{15.1}$$

compared to unity as was done in the derivation of k_1 and k_2 (Sec. 8). Physically we are making the very reasonable assumption that the extent of the region occupied by the spheres is very small compared to the damping wavelength ℓ_1 of the acoustic waves. The same assumption is even more valid inside the spheres. Thus from this point on we assume that the acoustic wave numbers k_1 and k_1' are real:

$$k_1 = \frac{\omega}{c} \qquad\qquad k_1' = \frac{\omega}{c'} \; . \tag{15.2}$$

Our assumption (5) that $R << \lambda_1$ then leads to

$$a_1 = k_1 R << 1 \qquad\qquad a_1' = k_1' R << 1 \; . \tag{15.3}$$

We shall make no assumptions regarding the magnitudes of the quantities a_2 a_2', b, b' in obtaining the general expressions for the attenuation. Using our values of k_2 and K from Sec. 8 we have

$$\begin{aligned} a_2 &= k_2 R = (1+i)(\omega R^2/2\varkappa)^{1/2} & \qquad b &= KR = (1+i)(\omega R^2/2\nu)^{1/2} \\ a_2' &= k_2' R = (1+i)(\omega R^2/2\varkappa')^{1/2} & \qquad b' &= K'R = (1+i)(\omega R^2/2\nu')^{1/2} \; . \end{aligned} \tag{15.4}$$

Finally using the approximations (15.1) we have the following relations for the constants appearing in the temperature, stress, and pressure expressions of Sec. 13:

$$\begin{aligned} \alpha_1 &= -i\omega(\gamma-1)/\alpha_v c^2 & \alpha_2 &= -(\alpha_v \bar{\varkappa})^{-1} \\ \beta_1 &= 1 & \beta_2 &= 1 - 2N\nu/\bar{\varkappa} \\ \gamma_1 &= 1 & \gamma_2 &= 1 - 4N\nu/3\bar{\varkappa} \end{aligned} \tag{15.5}$$

$$\left|\frac{a_1}{a_2}\right|^2 \sim \left|\frac{a_1}{b}\right|^2 \sim \left|\frac{k_1}{k_2}\right|^2 \sim \left|\frac{\alpha_1}{\alpha_2}\right| \sim \frac{4N\nu\omega}{3c^2} \sim \frac{\omega\bar{\varkappa}}{c^2}(\gamma-1) << 1 \tag{15.6}$$

with similar relations for the primed quantities.

16. Values of Potential Coefficients

It is shown in Appendices IV and V that with the above approximations the potential coefficients for n=0 and n=1 are as follows:

For n=0

$$B_0 = i\frac{a_1^3}{3}\left[\delta\left(\frac{a_1'}{a_1}\right)^2 - 1\right] + \left(\chi\frac{\alpha_2}{\alpha_2'} - 1\right) i a_1 C_0 a_2 h_1(a_2) \tag{16.1}$$

$$B_0' = \delta \tag{16.2}$$

$$C_0 = \frac{\alpha_1}{\alpha_2} \frac{\left(\delta\frac{\alpha_1'}{\alpha_1} - 1\right)}{h_0(a_2)(1-\chi Z)} \tag{16.3}$$

$$C_0' = \chi\left(\frac{\alpha_2}{\alpha_2'}\right)\frac{a_2 h_1(a_2)}{a_2' j_1(a_2')} C_0 \tag{16.4}$$

where in C_0

$$Z = \frac{j_0(a_2')}{a_2' j_1(a_2')} \cdot \frac{a_2 h_1(a_2)}{h_0(a_2)} = \frac{1 - ia_2}{1 - a_2' \cot a_2'} \; . \tag{16.5}$$

The last form is easily verified from Appendix II and will be used later.

<u>For n=1</u>

$$B_1 = i\frac{a_1^3}{3}(1-\delta)G = i\frac{a_1^3}{3}(1-\delta)\frac{N_{00}+N_{01}+N_{10}+N_{11}}{D_{00}+D_{01}+D_{10}+D_{11}} \tag{16.6}$$

$$B_1' = \frac{a_1}{\alpha_1'}G' = \frac{a_1}{\alpha_1'}3\delta G \tag{16.7}$$

$$C_1 = \frac{a_1}{3}\frac{\alpha_1}{\alpha_2}\frac{\left[2+\chi-3\frac{\alpha_1'}{\alpha_1}G'+2(1-\chi)(1-\delta)G\right]j_1(a_2')+\left[\frac{\alpha_1'}{\alpha_1}G'-1-(1-\delta)G\right]a_2'j_0(a_2')}{2(\chi-1)j_1(a_2')h_1(a_2)+a_2'j_0(a_2')h_1(a_2)-\chi j_1(a_2')a_2h_0(a_2)} \tag{16.8}$$

$$C_1' = -\frac{a_1}{3}\frac{\alpha_1}{\alpha_2'}\frac{\left[(1+2\chi)\frac{\alpha_1'}{\alpha_1}G'-3\chi\right]h_1(a_2)+\chi\left[1+(1-\delta)G-\frac{\alpha_1'}{\alpha_1}G'\right]a_2h_0(a_2)}{2(\chi-1)j_1(a_2')h_1(a_2)+a_2'j_0(a_2')h_1(a_2)-\chi j_1(a_2')a_2h_0(a_2)} \tag{16.9}$$

For G the numerator and denominator terms (N's and D's) are given by

$$\begin{aligned}
N_{00} &= [1-\epsilon]\,b'j_0(b')\cdot bh_0(b) & D_{00} &= [(1-\epsilon)(2+\delta)]\,b'j_0(b')\cdot bh_0(b)\\
N_{01} &= [3(\epsilon-1)-\tfrac{1}{2}\epsilon b^2]\,b'j_0(b')\cdot h_1(b) & D_{01} &= [9\delta(\epsilon-1)-\tfrac{1}{2}\epsilon b^2(\delta+2]\,b'j_0(b')\cdot h_1(b)\\
N_{10} &= [3(\epsilon-1)+\tfrac{1}{2}b'^2]\;j_1(b')\cdot bh_0(b) & D_{10} &= [3(\epsilon-1)(\delta+2)+\tfrac{1}{2}b'^2(\delta+2)]\,j_1(b')\cdot bh_0(b)\\
N_{11} &= [9(1-\epsilon)+\tfrac{3}{2}b'^2(\delta-1)]\,j_1(b')\cdot h_1(b) & D_{11} &= [27\delta(1-\epsilon)+\tfrac{3}{2}\epsilon b^2(\delta-1)]\,j_1(b')\cdot h_1(b).
\end{aligned} \tag{16.10}$$

and the various constants satisfy the relations

$$\epsilon = \eta/\eta' \qquad \chi = \sigma/\sigma' \qquad \delta = \rho_0/\rho_0' \qquad \delta b'^2 = \epsilon b^2 . \tag{16.11}$$

For later use we note that a simpler expression for B_1 may be derived using the last relation (16.11) and the relations

$bh_2(b) = 3h_1(b) - bh_0(b)$ and $b'j_2(b') = 3j_1(b') - b'j_0(b')$ from (2.3) in Appendix II. Using the above values for the N's and D's we find

$$
\begin{aligned}
N_{00}+N_{01}+N_{10}+N_{11} &= bh_2(b)\left[(1-\epsilon)b'j_2(b') - \tfrac{1}{2}b'^2 j_1(b')\right] - \left(\tfrac{1}{2}\delta b'^2\right)b'j_2(b')h_1(b) \\
D_{00}+D_{10} &= -(2+\delta)\,bh_0(b)\left[(1-\epsilon)b'j_2(b') - \tfrac{1}{2}b'^2 j_1(b')\right] \\
D_{01}+D_{11} &= 9\delta h_1(b)\left[(1-\epsilon)b'j_2(b') - \tfrac{1}{2}b'^2 j_1(b')\right] + \left(\tfrac{1}{2}\delta b'^2\right)(2+\delta)b'j_2(b')h_1(b)
\end{aligned}
\tag{16.12}
$$

from which

$$
B_1 = i\frac{a_1^3}{3}(1-\delta)\left[\frac{bh_2(b)\left[(1-\epsilon)b'j_2(b') - \frac{1}{2}b'^2 j_1(b')\right] - \left(\frac{1}{2}\delta b'^2\right)b'j_2(b')\cdot h_1(b)}{\left[-(2+\delta)bh_0(b)+9\delta h_1(b)\right]\left[(1-\epsilon)b'j_2(b') - \frac{1}{2}b'^2 j_1(b')\right] + (2+\delta)\left(\frac{1}{2}\delta b'^2\right)b'j_2(b')h_1(b)}\right] \tag{16.13}
$$

It is interesting to note that while the amplitude B_0 of the zero-order scattered acoustic wave depends on the thermal conductivity σ (in the a_2 term), the first order amplitude B_1 is independent of σ and agrees with Epstein's result (0.8) in which the conductivity was ignored ($\sigma = 0$).

It is useful for later calculations to list the general order of magnitude of the coefficients

$$
\begin{array}{llll}
B_0 \sim a_1^3 & B_1 \sim a_1^3 & C_0 \sim a_1^2 & C_1 \sim a_1^3 \\
B_0' \sim \delta & B_1' \sim \delta & C_0' \sim a_1'^2 & C_1' \sim a_1'^3
\end{array}
\tag{16.14}
$$

Interpreting these results physically we see that for an incident acoustic wave of unit amplitude, the scattered acoustic wave outside has an amplitude of the order of $a_1^3 \ll 1$, and the inside acoustic wave has an amplitude proportional to the density ratio δ of the outside to the inside medium. The scattered thermal waves inside and outside are of comparable magnitude but those of first order (C_1 and C_1') are smaller than the zero-order amplitudes (C_0 and C_0') by the factor $a_1 \ll 1$. <u>Thus the principal thermal waves are of order zero and are consequently purely radial waves.</u> This result is of great importance in evaluating the thermal dissipation. It can also be shown that the shear waves of first order (there are none of zero order) are of magnitude $D_1 \sim a_1$, $D_1' \sim a_1'$, so that the scattered shear waves outside and inside the obstacle are of order a_1.

Because of the smallness of a_1 and a_1' the coefficients B_n and C_n decrease so rapidly with n that to order a_1^3 we need retain only B_o B_1 C_o C_1 given above. In evaluating the attenuation inside the sphere the contributions of the terms involving B_o' C_o' B_1' C_1' appear in volume integrals over the sphere, and since the volume is $\propto R^3$ we again obtain terms of the order $a_1^3 \sim a_1'^3$. Thus for results of order a_1^3 we require only the n=0 and n=1 terms in the potential expansions. This rapid convergence of the series (12.2) is a characteristic feature of diffraction by small obstacles.

We have thus found the velocity potentials from which the acoustic field quantities are completely determined to our order of approximation. We now turn to the calculation of the viscous and thermal dissipation.

17. Calculation of Viscous Dissipation

To calculate the dissipation caused by viscous forces we use the computational form (9.25):

$$\frac{dE_\eta}{dt} = \frac{1}{2} R \int p^*_{\alpha\epsilon} v_\epsilon \, dS_\alpha + \frac{1}{2} R \int p^* \underline{\nabla} \cdot \underline{v} \, dV. \tag{17.1}$$

For our fixed volume of integration we choose a large sphere of radius $\bar{R}$ concentric with the small scattering sphere. More precisely we wish $\bar{R}$ to be (1) much larger than the thermal and viscous damping lengths ℓ_2 and $\ell_\perp$, (2) much smaller than the acoustic damping length ℓ_1, and (3) much larger than the acoustic wavelength λ_1. The requirement (1) insures that, due to the rapid exponential damping of the thermal and viscous waves with distance, the potentials ϕ_2 and A may be neglected at the surface of the large sphere of integration. This is easily seen by writing the Hankel functions for the two waves at large distances. Using Appendix II and the values of k_2 and K in terms of ℓ_2 and $\ell_\perp$ (Sec. 10) we find

$$k_2\bar{R} = (1+i)\bar{R}/\ell_2 \qquad\qquad KR = (1+i)\bar{R}/\ell_\perp \tag{17.2}$$

so that for $k_2\bar{R} \ll 1$, $K\bar{R} \ll 1$,

$$\begin{aligned} |h_n(k_2\bar{R})| &\longrightarrow \frac{1}{|k_2\bar{R}|}\, e^{-\bar{R}/\ell_2} \\ |h_n(K\bar{R})| &\longrightarrow \frac{1}{|K\bar{R}|}\, e^{-\bar{R}/\ell_\perp} \end{aligned} \tag{17.3}$$

and ϕ_2 and $\underline{A}$ are seen to be negligible.

The second requirement that $\bar{R} \ll \ell_1$ insures that the acoustic wave is <u>not</u> damped appreciably over the sphere of integration, in accordance with our assumption that k_1 is real (Sec. 15). The requirement (3) that $k_1\bar{R} \gg 1$ allows us to use the asymptotic forms of $j_n(k_1\bar{R})$ and $h_n(k_1\bar{R})$ for the acoustic wave in evaluating the surface integral. We thus have the chain of inequalities between the radius $\bar{R}$, the acoustic wavelength λ_1, and the three damping lengths ℓ_1, ℓ_2, $\ell_\perp$:

$$\ell_2, \ell_\perp \ll \lambda_1 \ll \bar{R} \ll \ell_1 \tag{17.4}$$

Examination of Table I shows that this inequality can always be satisfied. For definiteness we may take

$$\bar{R} = (\ell_1 \lambda_1)^{1/2} \sim \frac{1}{k_1}(\ell_1 k_1)^{1/2} \sim \frac{1}{k_1}\frac{k_2}{k_1} \quad , \tag{17.5}$$

where we have omitted absolute magnitude signs and have used the relation $\ell_1 k_1 \sim k_2^2/k_1^2 \gg 1$ obtained from the definition of ℓ_1 (Sec. 10) and the approximate equalities (15.6). We then find the approximate magnitude relation

$$(k_1\bar{R})(k_1 R) \sim k_2 R = a_2 \tag{17.6}$$

which we shall use later in determining the order of magnitude of expressions involving $k_1\bar{R}$.

We now write the viscous dissipation (17.1) as the sum of one surface and two volume integrals: (see note on page 60)

$$\frac{dE_r}{dt} = I_s + I_v + I_v'$$

where

$$\begin{aligned} I_s &= \tfrac{1}{2}\mathcal{R}\int p^*_{\alpha\epsilon} v_\epsilon \, dS_\alpha & r &= \bar{R} \\ I_v &= \tfrac{1}{2}\mathcal{R}\int p^* \underline{\nabla}\cdot\underline{v}\, dV & R \leq r &\leq \bar{R} \\ I_v' &= \tfrac{1}{2}\mathcal{R}\int p'^* \underline{\nabla}\cdot\underline{v}'\, dV & 0 \leq r &\leq \bar{R} \end{aligned} \tag{17.7}$$

I_v is the volume integral outside the small sphere and I_v' is the volume integral inside the small sphere. These preliminaries being understood we now proceed to the evaluation of the integrals.

Evaluation of Surface Integral

The surface integral I_s on our sphere of radius $\bar{R}$ assumes the form

$$I_s = \tfrac{1}{2}\mathcal{R}\int (p^*_{rr} v_r + p^*_{r\theta} v_\theta)\, dS_r \quad . \tag{17.8}$$

From the previous discussion we see that only ϕ_i and ϕ_1 need be considered in calculating the stresses and velocities. The general stress and velocity equations of Sec. 13 then give ($\beta_1 = 1$):

$$v_r = -\frac{\partial}{\partial r}(\phi_i + \phi_1) \qquad v_\theta = -\frac{1}{r}\frac{\partial}{\partial \theta}(\phi_i + \phi_1)$$

$$p_{rr} = \eta K^2(\phi_i + \phi_1) - 2\eta \frac{\partial^2}{\partial r^2}(\phi_i + \phi_1) \tag{17.9}$$

$$p_{r\theta} = -2\eta \frac{\partial}{\partial \theta}\left[\frac{1}{r}\frac{\partial}{\partial r}(\phi_i + \phi_1) - \frac{1}{r^2}(\phi_i + \phi_1)\right] .$$

Using the asymptotic forms of $j_n(k_1 r)$ and $h_n(k_1 r)$ for large arguments it is easily found that v_θ and $p_{r\theta}$ are negligible compared to v_r and p_{rr} and that the second term in p_{rr} is negligible compared to the first. Our surface integral then becomes (with $\eta K^2 = i\omega\rho_0$):

$$I_s = \frac{1}{2}\mathcal{R}\int \left[i\omega\rho_0(\phi_i + \phi_1)\right]^*\left[-\frac{\partial}{\partial r}(\phi_i + \phi_1)\right] dS_r . \tag{17.10}$$

This integral is evaluated in Appendix VI; the result is

$$I_s = -2\pi\rho_0 c \sum_{n=0}^{\infty}(2n+1)\mathcal{R}\left[B_n + B_n B_n^*\right] \tag{17.11}$$

or, retaining only the terms for n=0, n=1,

$$I_s = -2\pi\rho_0 c\, \mathcal{R}\left[B_0 + 3B_1 + B_0 B_0^* + 3B_1 B_1^*\right] . \tag{17.12}$$

Using the magnitudes of the coefficients from Sec. 16 the last two terms are seen to be of order a_1^6 while the first two are of order a_1^3; hence to order a_1^3 the surface integral is

$$I_s = -2\pi\rho_0 c\, \mathcal{R}\left[B_0 + 3B_1\right] . \tag{17.13}$$

These results are in agreement* with Epstein's evaluation of the same integral. (Ref. 6, Eq. 59).

It should be pointed out that in evaluating the various dissipation integrals we do <u>not</u> include any contributions arising from products $\phi_i \phi_i^*$ which are due solely to the incident wave. The reason for this is that these terms represent the dissipation in the absence of the scattering sphere, whereas we are interested not in the total dissipation (given by the integral including the $\phi_i \phi_i^*$ terms), but only in the additional dissipation due to the presence of the spheres, so that the $\phi_i \phi_i^*$ terms must be excluded.

In our integrals this question does not arise because we have assumed $k_1 = \omega/c$ to be real, so that there is no attenuation, and the $\phi_i \phi_i^*$ terms contribute nothing. In I_s, for example, the term drops out because we take the real part of the product which turns out to be imaginary (See Eq. (6.7) in Appendix VI).

<u>Evaluation of Volume Integrals</u>

The two volume integrals I_v and I_v' are evaluated in Appendix VII, where it is shown that the dominant part of each integral is of the same order of magnitude as the surface integral I_s:

$$I_v = -2\pi\rho_o c \, \mathcal{R}[i a_1 C_o a_2 h_1(a_2)] \tag{17.14}$$

$$I_v' = -2\pi\rho_o c \, \mathcal{R}\left[-\gamma \frac{\alpha_2}{\alpha_2'} i a_1 C_o a_2 h_1(a_2)\right], \tag{17.15}$$

where the second integral has been expressed in terms of C_o instead of $B_o'^* C_o'$ for reasons which will be clear in the next paragraph. From the

---

* A factor 3 should be supplied in the $B_1 B_1^*$ term of Epstein's Eq. (59).

derivations in Appendix VII it is clear that the dominant part of the outside integral I_v represents the interference effects between the incident acoustic wave ϕ_i and the scattered thermal wave ϕ_2, while the inside integral represents the interference effects between the scattered acoustic wave ϕ_i' and the scattered thermal wave ϕ_2'.

Final Result for Viscous Dissipation

Combining I_s, I_v, and I_v' we have

$$\frac{dE_\eta}{dt} = -2\pi\rho_0 c\,\mathcal{R}\left[3B_1 + B_0 + \left(1-\gamma\frac{\alpha_2}{\alpha_2'}\right) i a_1 C_0 a_2 h_1(a_2)\right] \tag{17.16}$$

But now we reach a surprising result: from the value of B_o in Sec. 16 we see that the C_o terms in (17.16) arising from the volume integral of $p^*\underline{\nabla}\cdot\underline{v}$ exactly cancel the second term of B_o, so that

$$\frac{dE_\eta}{dt} = -2\pi\rho_0 c\,\mathcal{R}\left\{3B_1 + i\,\frac{a_1^3}{3}\left[\delta(a_1'/a_1)^2-1\right]\right\}, \tag{17.17}$$

a result which agrees exactly with Epstein's derivation of the viscous dissipation ignoring thermal conductivity. Since the second term is pure imaginary our final result for the viscous dissipation is

$$\frac{dE_\eta}{dt} = -2\pi\rho_0 c\,\mathcal{R}[3B_1], \tag{17.18}$$

in agreement with Epstein. <u>Thus we see that only the first order (n=1) scattered waves contribute to the dominant part of the viscous dissipation; the terms for n=0 do not contribute significantly.</u>

This result can be checked by a direct calculation of the volume integral of the viscous dissipation function Φ. Using the expression (5.12) for Φ we easily find for the case n=0 (in which the velocity depends only on r and is independent of φ and Θ) the result

$$\Phi = \frac{2}{3}\mu(\underline{\nabla}\cdot\underline{v})^2 + \frac{4}{3}\eta\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)^2 \quad . \tag{17.19}$$

The time average of Φ in the acoustic case is then

$$\Phi_{Av} = \frac{1}{2}\mathcal{R}\left[\frac{2}{3}\mu(\underline{\nabla}\cdot\underline{v})(\underline{\nabla}\cdot\underline{v}^*) + \frac{4}{3}\eta\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)\left(\frac{\partial v^*}{\partial r} - \frac{v^*}{r}\right)\right] \tag{17.20}$$

and using the expressions for the potentials of Sec. 12 for n=0 it can be shown that

$$\frac{dE_\eta}{dt} = \frac{1}{2}\mathcal{R}\int_{r=0}^{r=\bar{R}} \Phi_{Av}(n=0)\,dV$$

is of higher order in a_1 and a_1' than the first order result (17.18) and may therefore be neglected.

It was pointed out in Sec. 16 that B_1 does not depend on σ, the thermal conductivity. Thus even in a heat conducting medium the viscous dissipation (17.18) is completely independent of the thermal effects. This result would not be true if the B_o and C_o terms of dE_η/dt, had not cancelled, since these terms depend on σ. It follows that the only way in which the effects of heat flow enter into the dissipation, either directly or indirectly is through the thermal dissipation term dE_σ/dt, to which we now turn.

18. Calculation of Thermal Dissipation

The thermal dissipation within a volume V bounded by the closed surface S is given by the general computational form (9.32).

$$\frac{dE_\sigma}{dt} = \frac{\sigma}{2T_o}\mathcal{R}\int(T^*\underline{\nabla}T)\cdot d\underline{S} - \frac{\rho_o c^2 \alpha_v}{2\gamma}\mathcal{R}\int T^*\underline{\nabla}\cdot\underline{v}\,dV\,. \tag{18.1}$$

Just as in the case of the viscous dissipation the volume integral splits into two integrals J_v and J_v' taken over the volumes outside and inside the small sphere. Moreover putting σ inside the integral sign the integrand of the surface integral can be written $T^*\sigma \underline{\nabla} T = -T^* \underline{q}$, so that since both the temperature and the heat current $\underline{q}$ are continuous on the boundary of the small sphere, the integrand is also, and the surface integrals at this boundary cancel, leaving only the surface integral over the large sphere $r = \bar{R}$. The thermal dissipation for our problem is therefore

$$\frac{dE_\sigma}{dt} = J_s + J_v + J_v'$$

where

$$\begin{aligned} J_s &= \frac{\sigma}{2T_0} \mathcal{R} \int \left(T^* \frac{\partial T}{\partial r} \right) dS_r && r = \bar{R} \\ J_v &= -\frac{\rho_0 c^2 \alpha_v}{2\gamma} \mathcal{R} \int T^* \underline{\nabla} \cdot \underline{v} \, dV && R \leq r \leq \bar{R} \\ J_v' &= -\frac{\rho_0' c'^2 \alpha_v'}{2\gamma'} \mathcal{R} \int T'^* \underline{\nabla} \cdot \underline{v}' \, dV && 0 \leq r \leq R \end{aligned} \tag{18.2}$$

Evaluation of Surface Integral

As before, the thermal potential ϕ_2 is negligible on the surface $r = \bar{R}$ so that only $\phi_i + \phi_1$ need be considered. From (13.3) the temperature is then

$$T = \alpha_1 (\phi_i + \phi_1) \tag{18.3}$$

and

$$J_s = \frac{\sigma}{2T_0} \alpha_1 \alpha_1^* \mathcal{R} \int (\phi_i + \phi_1)^* \frac{\partial}{\partial r} (\phi_i + \phi_1) \, dS_r \ . \tag{18.4}$$

Comparing this with the corresponding integral I_s for the viscous dissipation (17.10) we see that

$$J_s / I_s \sim \frac{\sigma \alpha_1 \alpha_1^*}{T_0 \omega \rho_0} = (\gamma - 1) \frac{\varkappa \omega}{c^2} << 1 , \tag{18.5}$$

So that J_s is negligible compared to the terms of order I_s which we are retaining in our treatment. J_s contributes negligibly to the thermal dissipation and will therefore be dropped.

Evaluation of Volume Integrals

The two volume integrals J_v and J'_v are evaluated in Appendix VIII, where it is shown that they are of the same order as I_s: The volume integrals J_v and J'_v give a thermal dissipation which is of the same order as the viscous dissipation. The results are

$$J_v = \frac{1}{2} \frac{\rho_0 \omega^2}{\varkappa} \mathcal{R} \int (\phi_i + \phi_1)^* \phi_2 \, dV \tag{18.6}$$

$$= \frac{1}{2} \frac{\rho_0 \omega^2}{\varkappa} \mathcal{R} \left[- \frac{4\pi R}{k_2^2} C_0 a_2 h_1(a_2) \right] \tag{18.7}$$

$$J'_v = \frac{1}{2} \frac{\rho'_0 \omega^2}{\varkappa'} \mathcal{R} \int \phi_1'^* \phi'_2 \, dV \tag{18.8}$$

$$= \frac{1}{2} \frac{\rho'_0 \omega^2}{\varkappa'} \mathcal{R} \left[\frac{4\pi R}{k_2'^2} B_0'^* C'_0 a'_2 j_1(a'_2) \right] . \tag{18.9}$$

Substituting C_o, C'_o, and B'^*_o from Sec. 16 we find, after reducing the coefficients,

$$J_v = 4\pi R \left(\tfrac{1}{2} \rho_0 k_1^2\right) \varkappa (\gamma - 1)\left(1 - \delta \frac{\alpha'_1}{\alpha_1}\right) \mathcal{R} \left[\frac{a_2 h_1(a_2) / h_0(a_2)}{1 - \gamma Z} \right] \tag{18.10}$$

$$J'_v = -\left(\delta \frac{\alpha'_1}{\alpha_1}\right) J_v \quad , \tag{18.11}$$

Z being given by (16.5).

Final Results for Thermal Dissipation

Combining J_v and J_v' we have

$$\frac{dE_\sigma}{dt} = 4\pi R\left(\tfrac{1}{2}\rho_0 k_1^2\right)\bar{\mathscr{H}}(\gamma-1)\left(1-\delta\frac{\alpha_1'}{\alpha_1}\right)^2 \mathcal{R}\left[\frac{a_2 h_1(a_2)/h_0(a_2)}{1-\chi\dfrac{j_0(a_2')}{a_2' j_1(a_2')}\cdot\dfrac{a_2 h_1(a_2)}{h_0(a_2)}}\right] \tag{18.12}$$

This is our desired result for the thermal dissipation of a plane sound wave by a small sphere. The bracket can be simplified by using the general relations $a_2 h_1(a_2)/h_1(a_2) = 1 - ia_2$ and $a_2' j_1(a_2')/j_0(a_2') = 1 - a_2' \cot a_2'$; this gives

$$\frac{dE_\sigma}{dt} = 4\pi R\left(\tfrac{1}{2}\rho_0 k_1^2\right)\bar{\mathscr{H}}(\gamma-1)\left(1-\delta\frac{\alpha_1'}{\alpha_1}\right)^2 \mathcal{R}\left[\frac{1-ia_2}{1-\chi\dfrac{1-ia_2}{1-a_2'\cot a_2'}}\right] \tag{18.12}$$

as an alternate form for dE_σ/dt. We shall not attempt further reduction for these general forms, since the algebra, while straightforward, leads to very complicated expressions.

This concludes our calculation of the thermal and viscous dissipation. We now turn to the evaluation of the attenuation coefficient.

19. The Attenuation Coefficient

The attenuation coefficient α for a plane sound wave traversing a medium containing n small scattering spheres per unit volume, the energy loss per unit time due to each sphere being dE/dt, is defined as the ratio of the total energy $n\,dE/dt$ lost per unit time to the energy flux E_0 of the plane wave:

$$\alpha = \frac{n}{E_0}\frac{dE}{dt} \quad . \tag{19.1}$$

Since dE/dt is the sum of the viscous and thermal dissipations

$$\frac{dE}{dt} = \frac{dE_\eta}{dt} + \frac{dE_\sigma}{dt} \tag{19.2}$$

we have from (19.1)

$$\alpha = \alpha_\eta + \alpha_\sigma \tag{19.3}$$

where

$$\alpha_\eta = \frac{n}{E_o}\frac{dE_\eta}{dt} \qquad\qquad \alpha_\sigma = \frac{n}{E_o}\frac{dE_\sigma}{dt} \quad . \tag{19.4}$$

From (10.11) we have $E_o = \frac{1}{2} k_1 \rho_o \omega$, and using dE_η/dt and dE_σ/dt from the previous sections we obtain the general expressions for the viscous and thermal attenuations:

$$\alpha_\eta = \frac{4\pi n}{k_1^2} \mathcal{R}\left[-3B_1\right] \tag{19.5}$$

$$\alpha_\eta = \frac{4\pi n}{k_1^2} a_1^3 (1-\delta)\, \mathcal{R}\left[-i\,\frac{b h_2(b)\left[(1-\epsilon)b' j_2(b') - \frac{1}{2}b'^2 j_1(b')\right] - \left(\frac{1}{2}\delta b'^2\right) b' j_2(b')\, h_1(b)}{\left[-(2+\delta) b h_o(b) + 9\delta h_1(b)\right]\left[(1-\epsilon) b' j_2(b') - \frac{1}{2} b'^2 j_1(b')\right] + \left(\frac{1}{2}\delta b'^2\right)(2+\delta) b' j_2(b')\, h_1(b)}\right] \tag{19.6}$$

and

$$\alpha_\sigma = \frac{4\pi n R}{c}\, \varkappa (\gamma - 1)\left(1 - \delta\frac{\alpha_1'}{\alpha_1}\right)^2 \mathcal{R}\left[\frac{a_2 h_1(a_2)/h_o(a_2)}{1 - \chi Z}\right] \tag{19.7}$$

where

$$Z = \frac{j_o(a_2')}{a_2' j_1(a_2')}\cdot\frac{a_2 h_1(a_2)}{h_o(a_2)} = \frac{1 - i a_2}{1 - a_2' \cot a_2'} \quad . \tag{19.8}$$

The explicit general expressions (19.6) and (19.7) for α_η and α_σ conclude our main objective; together they give the additional attenuation of a plane sound wave due to small fluid spheres in a fluid medium. We shall apply these general results in the following chapter to calculate the attenuation by water drops in air and by air bubbles in water.

(Note: Both factors in the integrand $p^{*}_{\alpha\epsilon} v_\epsilon$ of the surface integral in (17.1) are continuous at the surface of the small sphere; the two surface integrals (bounding the inside and outside media) therefore cancel at this boundary, leaving only the outside surface integral in (17.7) at the boundary of the large sphere.)

V. APPLICATIONS TO WATER DROPS IN AIR AND AIR BUBBLES IN WATER

In this chapter we shall apply our general results to two limiting cases of great practical importance. For water drops in air the density, viscosity, and thermal conductivity of the drops are large compared to the outside medium; for air bubbles in water the reverse is true. In each case the general expressions for the viscous and thermal attenuation simplify considerably.

20. Attenuation by Water Drops in Air

In this section we shall reduce our theoretical results for comparison with Knudsen's data on the attenuation of sound in fogs (Ref. 7). Since the average drop radius is about 10^{-3} cm we see that our assumption $a_1 = \frac{\omega R}{c} << 1$ will be reasonably valid up to $\omega = 10^6 \ \mathrm{sec}^{-1}$ (for which $a_1 \sim 0.03$).

Since water is denser, more viscous, and a better thermal conductor than air we see that the outside-to-inside density ratio δ, viscosity ratio ϵ, and conductivity ratio χ are all small. The actual values appropriate to Knudsen's data are

$$\delta = 0.00117, \qquad \epsilon = 0.0167 \qquad \chi = 0.040 \tag{20.1}$$

It is shown in Appendix IX that for small δ, ϵ, χ the general expression (19.6) for α_η reduces to

$$\alpha_\eta = \frac{4\pi n}{k_1^2} a_1^3 \mathcal{R}\left[-i\,\frac{b h_2(b)}{-2bh_0(b) + 9\delta h_1(b)}\right] \tag{20.2}$$

$$= \frac{4\pi n R}{c}\,\frac{3}{2}\,\nu(1+y)\left[\frac{16y^4}{16y^4 + 72\delta y^3 + 81\delta^2(1+2y+2y^2)}\right] \tag{20.3}$$

$$\alpha_\eta = \frac{9}{2} V_p \frac{\nu}{cR^2}(1+y)\left[\frac{16y^4}{16y^4 + 72\delta y^3 + 81\delta^2(1+2y+2y^2)}\right] . \tag{20.4}$$

The second form is obtained in Appendix IX by substituting the explicit expressions for the h's and defining the dimensionless real variable y by

$$y^2 = b^2/2i \quad = \omega R^2/2\nu \qquad\qquad b = (1+i)y \quad . \tag{20.5}$$

In the third form the dimensionless quantity V_p is the total volume of the water drops per unit volume of air:

$$V_p = \frac{4}{3}\pi n R^3 \tag{20.6}$$

In both (20.3) and (20.4) the coefficient before the bracket is Sewell's attenuation discussed in the introduction (0.7):

$$\alpha_{Sew} = \frac{4\pi n R}{c}\,\frac{3}{2}\nu(1+y) \quad = \frac{9}{2} V_p \frac{\nu}{cR^2}\left[1 + (\omega R^2/2\nu)^{1/2}\right] . \tag{20.7}$$

The expression (20.4) for α_η agrees with Knudsen's reduction (Ref. 7) of Epstein's result (Ref. 6), and since $y \longrightarrow \infty$ as $\omega \longrightarrow \infty$ the bracket approaches unity and α_η is seen to reduce to Sewell's result at high frequencies. At low frequencies $y \longrightarrow 0$ and the bracket approaches zero as y^4, so that α_η vanishes, as it should for zero frequency, in contrast to Sewell's result which is finite at $\omega = 0$. The physical reasons for this were discussed in the introduction and we thus see that the correct expression (20.4), originally obtained by Epstein is also valid in our problem which includes thermal conduction.

Turning to the thermal attenuation and imposing the condition that δ, ϵ, χ be small we have from (19.7):

$$\alpha_\sigma = \frac{4\pi n R}{c}\,\bar{\varkappa}(\gamma-1)\,\mathcal{R}\left[\frac{a_2 h_1(a_2)/h_0(a_2)}{1-\chi Z}\right] \tag{20.8}$$

since $\alpha_1'/\alpha_1 \simeq 0.014$ so that $\delta\alpha_1'/\alpha_1 \ll 1$. In passing we note that, using (18.11), the last inequality shows that $J_v' \ll J_v$, i.e., that the thermal dissipation inside the drops is negligible compared to that outside.

It is shown in Appendix IX that for small χ the real part of the bracket reduces to

$$(1+z)\left[1+3(\delta c_p/c_p')\frac{1}{z}+\frac{9}{4}(\delta c_p/c_p')^2\frac{1}{z^4}\right]^{-1} \tag{20.9}$$

where the dimensionless variable z is defined by

$$z^2 = a_2^2/2i = \omega R^2/2\bar{\varkappa} \qquad\qquad a_2 = (1+i)z$$

and we finally have for the thermal attenuation the results

$$\alpha_\sigma = \frac{4\pi n R}{c}\,\bar{\varkappa}(\gamma-1)(1+z)\left[\frac{4z^4}{4z^4+12(\delta c_p/c_p')z^3+9(\delta c_p/c_p')^2}\right] \tag{20.10}$$

$$= 3V_p\,\frac{\bar{\varkappa}(\gamma-1)}{cR^2}(1+z)\left[\frac{4z^4}{4z^4+12(\delta c_p/c_p')z^3+9(\delta c_p/c_p')^2}\right] \tag{20.11}$$

which correspond to the expressions for α_η in (20.3) and (20.4).

We note that at low frequencies $\alpha_\sigma \propto z^4 \longrightarrow 0$ as it should, there being no reason for any thermal losses at zero frequency. Thus the rigorous present treatment avoids the anomaly of a constant thermal attenuation at $\omega = 0$, corresponding to Sewell's values of α_η. The limiting values of the attenuations at low frequencies are

$$\alpha_\eta \xrightarrow[\omega\to 0]{} \frac{4\pi n R}{c}\,\frac{3\nu}{2}\,\frac{16y^4}{81\delta^2} = \frac{4\pi n R}{c}\,\frac{6}{81}\,\frac{(\omega R^2)^2}{\delta^2\nu}$$

$$\alpha_\sigma \xrightarrow[\omega\to 0]{} \frac{4\pi n R}{c}\,\bar{\varkappa}(\gamma-1)\frac{4z^4}{9\delta^2}\left(\frac{c_p'}{c_p}\right)^2 = \frac{4\pi n R}{c}\,\frac{(\gamma-1)}{9}\,\frac{(\omega R^2)^2}{\delta^2\bar{\varkappa}}\left(\frac{c_p'}{c_p}\right)^2 \tag{20.12}$$

and the ratio of the thermal to the viscous attenuation is

$$\frac{\alpha_\sigma}{\alpha_\eta} \xrightarrow[\omega\to 0]{} (\gamma-1)\frac{3}{2}\,\frac{\nu}{\bar{\varkappa}}\left(\frac{c_p'}{c_p}\right)^2 \sim 7.2 \tag{20.13}$$

At high frequencies α_σ approaches the value

$$\alpha_\sigma \xrightarrow[\omega\to\infty]{} \frac{4\pi n R}{c}\,\bar{\varkappa}(\gamma-1)(1+z) = 3V_p\,\frac{\bar{\varkappa}(\gamma-1)}{c^2 R}\left[1+(\omega R^2/2\bar{\varkappa})^{1/2}\right], \tag{20.14}$$

corresponding to Sewell's result (20.7) for α_η. Finally we note that the ratio of α_σ to α_η at high frequencies is asymptotic to the value

$$\frac{\alpha_\sigma}{\alpha_\eta} \xrightarrow[\omega\to\infty]{} (\gamma-1)\frac{\bar{\varkappa}\,z}{\frac{3}{2}\nu y} = \frac{2}{3}(\gamma-1)(\bar{\varkappa}/\nu)^{1/2} \sim 0.3 \ . \tag{20.15}$$

From (20.10) and (20.12) we see that at low frequencies the thermal attenuation of water drops in air dominates the viscous attenuation by a factor of seven, while at high frequencies the viscous attenuation dominates by a factor of three.

A convenient way to show the dependence of the attenuation on frequency ω and radius R of the drop is to plot the quantities

$$\Sigma_\eta = \frac{\alpha_\eta c}{4\pi n R} = \frac{3}{2}\nu(1+y)\left[\frac{16y^4}{16y^4+72\delta y^3+81\delta^2(1+2y+2y^2)}\right] \tag{20.16}$$

and

$$\sum_{\sigma} = \frac{\alpha_{\sigma} c}{4\pi n R} = \varkappa(\gamma-1)(1+z)\left[\frac{4z^4}{4z^4 + 12(\delta c_p/c_p')z^3 + 9(\delta c_p/c_p')^2}\right] \tag{20.17}$$

against ωR^2 (which is a factor of y^2 and z^2) and thus obtain general curves for all ω and R. In addition we define

$$\sum = \sum_{\eta} + \sum_{\sigma} = \frac{\alpha c}{4\pi n R} \tag{20.18}$$

corresponding to the total attenuation and

$$\sum_{\text{Sewell}} = \frac{\alpha_{\text{Sewell}}\, c}{4\pi n R} = \frac{3}{2}\nu(1+y) \tag{20.19}$$

corresponding to Sewell's value.

These quantities are plotted against ωR^2 in Fig. 1. It is seen that the Σ_η and Σ_σ curves are quite similar throughout the entire range. At high frequencies $\Sigma_\eta \sim 3\,\Sigma_\sigma$ as remarked above, and both are proportional to $(\omega R^2)^{1/2}$. At low frequencies $\Sigma_\sigma \sim 7\,\Sigma_\eta$. The critical frequency separating the high and low frequency regions is determined by the point at which $\Sigma_\sigma = \Sigma_\eta$. This is seen to be $\omega R^2 \sim 6 \times 10^{-4}$, so that

$$\omega_{\text{crit}} = 6 \times 10^{-4}/R^2 \tag{20.20}$$

where R is the drop radius in cm and $\omega = 2\pi f$ is the circular frequency in $\sec^{-1}$.

The overall curve Σ has the same asymptotic dependence at high and low frequencies as Σ_η and Σ_σ have. The Sewell curve Σ_{Sew} is seen to level off to a constant value: $\Sigma_{Sew} = 0.23$ as $\omega \longrightarrow 0$, corresponding to the constant value of α_{Sew} at low frequencies already discussed. Finally, it should perhaps be emphasized that the expressions for α_η and α_σ refer to a fog consisting of water drops all having the <u>same</u> radius R.

21. Comparison with Knudsen's Data

The absorption of sound in fogs has been observed by many workers but in most cases no analysis of drop size was made and this precludes any close comparison with the theory. Recently, however, Knudsen (Ref. 7) carried out absorption measurements in an artifically produced fog, using the reverberation chamber technique, and in this work the distribution in drop-size was determined. The experimental results and the calculated attenuations (see below) are given in Table 21.1, together with the corresponding physical data.

Table 21.1 - Comparison of Observed and Calculated Attenuation

f sec^{-1}	(db/cm) observed	(db/cm) calculated
300	$(1.4 \pm 0.2) \times 10^{-5}$	1.32×10^{-5}
1000	$(2.0 \pm 0.3) \times 10^{-5}$	1.74×10^{-5}
2000	$(2.7 \pm 0.4) \times 10^{-5}$	1.89×10^{-5}
4000	$(2.9 \pm 0.4) \times 10^{-5}$	2.07×10^{-5}
6000	$(3.5 \pm 0.4) \times 10^{-5}$	2.19×10^{-5}
8000	$(3.8 \pm 0.4) \times 10^{-5}$	2.27×10^{-5}

$T = 22^{\circ}C = 295^{\circ}A$ $p_o = 755$ mm Hg $V_p = (2.00 \pm 0.10) \times 10^{-6}$

Water Drops:

$\rho'_o = 1$ gm/cm^3

$c'_p = 1$ cal/oA

$\bar{\varkappa}' = 1.43 \times 10^{-3}$ cm^2/sec

$\nu' = 0.011$ cm^2/sec

$c' = 1.45 \times 10^5$ cm/sec.

Air

$\rho_o = 1.17 \times 10^{-3}$ gm/cm^3

$\eta = 1.82 \times 10^{-4}$ gram/cm sec

$\nu = \eta/\rho = 0.156$ cm^2/sec

$\bar{\varkappa} = \sigma/\rho c_p = 0.206$ cm^2/sec

$c_p = 0.24$ cal/oA.

$c = 3.44 \times 10^4$ cm/sec

The drops were grouped into five size classes and the mean radius and fractional volume $(\Delta V_p/V_p)$ for each class calculated. The results are summarized in Table 21.2.

Table 21.2 - Drop Size Data

	No. drops	Fractional Number	Fract.Vol. $\Delta V_p/V_p$	Mean Radius (cm)
(1)	102	0.121	0.005	3.75×10^{-4}
(2)	375	0.445	0.084	6.25×10^{-4}
(3)	230	0.272	0.211	10.0×10^{-4}
(4)	93	0.110	0.298	15.0×10^{-4}
(5)	43	0.052	0.402	21.5×10^{-4}
	843			

The attenuation $\bar{\alpha}$ in decibels per centimeter (db/cm) is given by

$$\bar{\alpha} = 4.34\,\alpha = 4.34(\alpha_\eta + \alpha_\sigma) \tag{21.1}$$

where $\alpha_\eta(\mathrm{cm}^{-1})$ and $\alpha_\sigma(\mathrm{cm}^{-1})$ are given by (20.4) and (20.11). At each frequency the attenuation $\bar{\alpha}$ was calculated for each class, and the $\bar{\alpha}$'s for each class were then added after weighing them by the fractional volume factor. The calculated and observed attenuations are plotted against frequency in Fig. 2. The agreement is seen to be fair, the calculated and measured values having the same order of magnitude and the same frequency dependence, but with the calculated values lying below the observed values. The reason for this is not clear. It is known that the results would be strongly affected by changes in the drop-size distribution but whether more accurate size distribution data would remove the discrepancy is doubtful. A careful analysis of the experimental methods would be required to assess other factors affecting the attenuation, and this would be out of place in the present work.

22. Attenuation by Air Bubbles in Water

Turning to the attenuation by air bubbles in water we have the opposite case in which the outside-to-inside density, viscosity and conductivity ratios are large:

$$\delta = 855\,, \qquad \epsilon = 60\,, \qquad \chi = 25\,. \tag{22.1}$$

It is shown in Appendix X that for large δ, ϵ, χ the general expression (19.6) for α_η reduces to

$$\alpha_\eta = \frac{4\pi n}{k_1^2} a_1^3 \mathcal{R}\left[-i\,\frac{b h_2(b) + \frac{1}{2} b^2 h_1(b)}{b h_0(b) - 9 h_1(b) + \frac{1}{2} b^2 h_1(b)}\right] \tag{22.2}$$

or

$$\alpha_\eta = \frac{4\pi n R}{c}\, 2\nu \left[\frac{2y^7 + 12y^6 + 6y^5}{2y^6 + 6y^5 + 9y^4 + 36y^3 + 162y^2 + 162y + 81}\right] \tag{22.3}$$

For the thermal attenuation we note that now $\delta\alpha_1'/\alpha_1 \sim 6 \times 10^4 \gg 1$ so that $J_v \ll J_v'$; i.e., <u>the thermal dissipation in the water is negligible compared to that in the air inside the bubbles</u>. The attenuation is given by (19.7):

$$\alpha_\sigma = \frac{4\pi n R}{c}\, \bar{\varkappa}(\gamma-1)(\delta\alpha_1'/\alpha_1)^2\, \mathcal{R}\left[\frac{a_2 h_1(a_2)/h_0(a_2)}{1 - \chi\,\dfrac{j_0(a_2')}{a_2' j_1(a_2')}\cdot\dfrac{a_2 h_1(a_2)}{h_0(a_2)}}\right] \tag{22.4}$$

Since $\chi \gg 1$ we see that for both low and high frequencies (a_2', $a_2 \longrightarrow 0$ or ∞) the second term in the denominator dominates the first. Neglecting the first term we can then cancel the common factor $a_2 h_1'(a_2)/h_0(a_2)$, and after rearranging the constants with the aid of $(\alpha_1'/\alpha_1)^2 = (\gamma'-1)c^2 c_p/(\gamma-1)c'^2 c_p'$, (22.4) reduces to

$$\alpha_\sigma = \frac{4\pi n R}{c}\,\bar{\varkappa}'(\gamma'-1)\left(\rho_0 c^2/\rho_0' c'^2\right)\,\mathcal{R}\left[-a_2' j_1(a_2')/j_0(a_2')\right] \tag{22.5}$$

or

$$\alpha_\sigma = \frac{4\pi n R}{c}\,\bar{\varkappa}'(\gamma'-1)\left(\rho_0 c^2/\rho_0' c'^2\right)\,\mathcal{R}\left[a_2' \cot a_2' - 1\right]. \tag{22.6}$$

Again introducing the real dimensionless z'

$$z'^2 = a_2'^2/2i = \omega R^2/2\bar{\varkappa}' \qquad\qquad a_2' = (1+i)z' \tag{22.7}$$

we find, after expanding the cotangent and taking the real part,

$$\alpha_\sigma = \frac{4\pi n R}{c}\,\bar{\varkappa}'(\gamma'-1)\left(\rho_0 c^2/\rho_0' c'^2\right)\left[z'\,\frac{\sinh 2z' + \sin 2z'}{\cosh 2z' - \cos 2z'} - 1\right]. \tag{22.8}$$

This is the final desired form for the thermal attenuation by air bubbles in water. Numerical comparison of α_η and α_σ shows that for all values of y and z' (or of ωR^2),

$$\alpha_\eta/\alpha_\sigma \sim 10^{-5} \ll 1\ ,$$

this result being primarily due to the large factor

$$\rho_0 c^2/\rho_0' c'^2 = 1.5 \times 10^4\ . \tag{22.9}$$

<u>Thus the thermal dissipation inside the air bubbles completely dominates the viscous dissipation.</u> This is an important and interesting result. Rewriting (22.8) we have for the attenuation

$$\alpha \doteq \alpha_\sigma = \frac{4\pi n R}{c}\,A\,X(z') \tag{22.10}$$

and for the quantity

$$\Sigma \doteq \Sigma_\sigma = \frac{\alpha_\sigma c}{4\pi n R} = A\,X(z') \tag{22.11}$$

where

$$A = \varkappa'(\gamma'-1)(\rho_0 c^2/\rho_0' c'^2) = 1.27 \times 10^4 \text{ cm}^2/\text{sec}$$

$$X(z') = \mathcal{R}\left[-a_2' j_1(a_2')/j_0(a_2')\right] = \mathcal{R}\left[a_2' \cot a_2' - 1\right] \tag{22.12}$$

$$= z' \frac{\sinh 2z' + \sin 2z'}{\cosh 2z' - \cos 2z'} - 1 \quad .$$

A general plot of Σ_σ against ωR^2 is shown in Fig. 3. The function $X(z')$ lies close to its asymptotic values

$$X \xrightarrow[z' \to 0]{} \frac{4}{45} z'^4, \qquad X \xrightarrow[z' \to \infty]{} z' - 1. \tag{22.13}$$

except in a small region near $z' = 2$.

At the high and low frequency limits Σ_σ has the corresponding asymptotic forms (light lines in Fig. 3):

$$\Sigma_\sigma \xrightarrow[\omega \to 0]{} A \frac{4}{45} z'^4 = A \frac{4}{45} \left(\frac{\omega R^2}{2\varkappa'}\right) \propto (\omega R^2)^2$$

$$\Sigma_\sigma \xrightarrow[\omega \to \infty]{} A z' = A \left(\frac{\omega R^2}{2\varkappa'}\right)^{1/2} \propto (\omega R^2)^{1/2} . \tag{22.14}$$

Since we do not have experimental data on both absorption by bubbles <u>and</u> bubble size distribution we shall not attempt to discuss the correspondence between theory and experiment.

We must note in conclusion that the results obtained here are valid only for bubbles whose radius varies only slightly during the passage of the primary sound wave. Our value of the attenuation is therefore not valid in the neighborhood of the resonance peak where large oscillations occur (Ref. 9).

PART II

Part II contains the various mathematical appendices referred to in the text; as well as the References, Table I (Physical Constants), and the Figures.

APPENDIX I

Time Average of Products of First-Order Quantities

We wish to list several useful expressions involving the time averages of products of two first-order quantities A_1 and B_1 of the form (see Sec. 7):

$$A_1 = A e^{-i\omega t} \qquad\qquad B_1 = B e^{-i\omega t} \tag{1.1}$$

where A and B are complex space functions independent of time and having absolute values A and B :

$$A = |A| e^{i\alpha} \qquad\qquad B = |B| e^{i\beta} \tag{1.2}$$

so

$$A_1 = |A| e^{i(\alpha-\omega t)} \qquad\qquad B_1 = |B| e^{i(\beta-\omega t)} \quad .$$

Now as pointed out in Sec. 7, the first-order quantities are all real; for convenience however we represent them as complex quantities. Consequently, only the real parts $\mathcal{R}(A_1)$, $\mathcal{R}(B_1)$ are to be used in evaluating the products. Thus the time average of A_1 B_1 is

$$(A_1 B_1)_{Av} = [\mathcal{R}(A_1)\mathcal{R}(B_1)]_{Av} \tag{1.3}$$

so that from (1.2)

$$(A_1 B_1)_{Av} = |A|\cdot|B| \left[\cos(\alpha-\omega t)\cos(\beta-\omega t)\right] \tag{1.4}$$

$$= \frac{1}{2}|A|\cdot|B| \left[\cos(\alpha+\beta-2\omega t) + \cos(\alpha-\beta)\right] \tag{1.5}$$

$$= \frac{1}{2}|A|\cdot|B| \cos(\alpha-\beta) , \tag{1.6}$$

since the time average of the first quantity in the bracket of (1.5) is zero. But now, using (1.2)

$$\tfrac{1}{2}(AB^* + A^*B) = \tfrac{1}{2}|A|\cdot|B|\left[e^{i(\alpha-\beta)} + e^{-i(\alpha-\beta)}\right] \tag{1.7}$$

$$= |A|\cdot|B|\cos(\alpha-\beta) \tag{1.8}$$

so that (1.6) becomes

$$(A_1 B_1)_{Av} = \tfrac{1}{4}(AB^* + BA^*) \quad . \tag{1.9}$$

We now use

$$\tfrac{1}{2}(AB^* + A^*B) = \mathcal{R}(A^*B) = \mathcal{R}(AB^*) \tag{1.10}$$

to obtain our final result

$$(A_1 B_1)_{Av} = \tfrac{1}{2}\mathcal{R}(A_1^* B_1) = \tfrac{1}{2}\mathcal{R}(A_1 B_1^*) \tag{1.11}$$

which may also be written

$$(A_1 B_1)_{Av} = \tfrac{1}{2}\mathcal{R}(A^*B) = \tfrac{1}{2}\mathcal{R}(AB^*) \quad . \tag{1.12}$$

Two special cases of interest are

$$(A_1 A_1)_{Av} = \tfrac{1}{2}\mathcal{R}(AA^*) = \tfrac{1}{2}|A|^2 = \tfrac{1}{2}|A_1|^2 , \tag{1.13}$$

$$(\dot{A}_1 A_1)_{Av} = \left[(-i\omega A_1)A_1\right]_{Av} = \tfrac{\omega}{2}\mathcal{R}\left[(iA)A^*\right] = 0 \quad ; \tag{1.14}$$

(1.14) follows from the fact that the quantity in the last bracket is pure imaginary, so that its real part vanishes.

APPENDIX II

Spherical Bessel and Hankel Functions

The radial part R(kr) of the wave function satisfies the spherical Bessel Equation

$$
\left.
\begin{aligned}
&\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left(k^2 - \frac{n(n+1)}{r^2}\right)R = 0 \\
\text{or}\quad &\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right) + \left(1 - \frac{n(n+1)}{\rho^2}\right)R = 0 \\
\text{or}\quad &\rho^2 R'' + 2\rho R' + [\rho^2 - n(n+1)]R = 0
\end{aligned}
\right\}
\qquad (2.1)
$$

where $\rho = kr$, k being the complex wave number, and $R' = dR/d\rho$ etc. The solutions of this equation are the spherical Bessel functions $j_n(\rho)$ (regular at $\rho=o$), the spherical Neumann functions $y_n(\rho)$ (singular at $\rho=o$), and the first and second spherical Hankel functions $h_n(\rho)$ and $\hbar_n(\rho)$, defined by

$$
\begin{aligned}
j_n(\rho) &= (\pi/2\rho)^{1/2}\, J_{n+1/2}(\rho) \\
y_n(\rho) &= (\pi/2\rho)^{1/2}\, Y_{n+1/2}(\rho) \\
h_n(\rho) &= j_n(\rho) + i y_n(\rho) \\
\hbar_n(\rho) &= j_n(\rho) - i y_n(\rho)
\end{aligned}
\qquad (2.2)
$$

where $J_{n+1/2}$ and $Y_{n+1/2}$ are the ordinary cylindrical Bessel functions of half-odd-integer order. Letting $R_n(\rho)$ denote any solution of (2.1), the following relations are satisfied by j_n, y_n, h_n, and $\hbar_n$ for $n \geq 0$:

$$\rho R_{n+1}(\rho) = (2n+1) R_n(\rho) - \rho R_{n-1}(\rho) \tag{2.3}$$

$$\rho R_n'(\rho) = \rho R_{n-1}(\rho) - (n+1) R_n(\rho) \tag{2.4}$$

$$\rho R_n'(\rho) = n R_n(\rho) - \rho R_{n+1}(\rho) \tag{2.5}$$

$$\rho^2 R_n''(\rho) = 2\rho R_{n+1}(\rho) + n(n-1) R_n(\rho) - \rho^2 R_n(\rho) \tag{2.6}$$

$$\rho^2 R_n''(\rho) = (n+1)(n+2) R_n(\rho) - \rho^2 R_n(\rho) - 2\rho R_{n-1}(\rho) \tag{2.7}$$

For n=0, terms of the form $R_{n-1}(\rho)$ can be evaluated by using the general relations

$$y_n(\rho) = (-1)^{n+1} j_{-n-1}(\rho) \qquad y_{-n}(\rho) = (-1)^{n-1} j_{n-1}(\rho) \tag{2.8}$$

to obtain y and j for negative orders.

Explicit expressions for the first three j's, y's and h's are

$$j_0(\rho) = \frac{\sin\rho}{\rho} \qquad j_1(\rho) = \frac{\sin\rho}{\rho^2} - \frac{\cos\rho}{\rho} \tag{2.9}$$
$$j_2(\rho) = \left(\frac{3}{\rho^3} - \frac{1}{\rho}\right)\sin\rho - \frac{3}{\rho^2}\cos\rho$$

$$y_0(\rho) = -\frac{\cos\rho}{\rho} \qquad y_1(\rho) = -\frac{\cos\rho}{\rho^2} - \frac{\sin\rho}{\rho} \tag{2.10}$$
$$y_2(\rho) = -\left(\frac{3}{\rho^3} - \frac{1}{\rho}\right)\cos\rho - \frac{3}{\rho^2}\sin\rho$$

$$h_0(\rho) = -i\,\frac{e^{i\rho}}{\rho} \qquad h_1(\rho) = -e^{i\rho}\left(\frac{i}{\rho^2} + \frac{1}{\rho}\right) \tag{2.11}$$
$$h_2(\rho) = -e^{i\rho}\left(\frac{3i}{\rho^3} + \frac{3}{\rho^2} - \frac{i}{\rho}\right) \quad .$$

For small arguments the following expansions are valid for $n \geq 0$:

$$j_n(\rho) \xrightarrow[\rho\to 0]{} \frac{\rho^n}{1\cdot 3\cdot 5\cdots(2n+1)}\left[1 - \frac{\rho^2}{2(2n+3)} + \frac{\rho^4}{2\cdot 4(2n+3)(2n+5)} - \cdots\right] \tag{2.12}$$

$$y_n(\rho) \xrightarrow[\rho \to 0]{} -\frac{1\cdot 1\cdot 3\cdots(2n-1)}{\rho^{n+1}}\left[1 + \frac{\rho^2}{2(2n-1)} + \frac{\rho^4}{2\cdot 4(2n-1)(2n-3)} + \cdots\right] . \tag{2.13}$$

The general series expansion for $h_n(\rho)$ is

$$h_n(\rho) = i^{-n-1}\frac{e^{i\rho}}{\rho}\left[1 + \frac{(n+1)!}{(n-1)!\,1!}\left(\frac{i}{2\rho}\right) + \frac{(n+2)!}{(n-2)!\,2!}\left(\frac{i}{2\rho}\right)^2 + \cdots + \frac{(2n)!}{n!}\left(\frac{i}{2\rho}\right)^n\right] \tag{2.14}$$

$$= i^{-n-1}\frac{e^{i\rho}}{\rho}\sum_{r=0}^{n}\frac{(n+r)!}{(n-r)!\,r!}\left(\frac{i}{2\rho}\right)^r .$$

For large values of ρ the asymptotic expansions are

$$j_n(\rho) \xrightarrow[\rho \to \infty]{} \frac{1}{\rho}\cos\left[\rho - (n+1)\frac{\pi}{2}\right] \tag{2.15}$$

$$y_n(\rho) \xrightarrow[\rho \to \infty]{} \frac{1}{\rho}\sin\left[\rho - (n+1)\frac{\pi}{2}\right] \tag{2.16}$$

$$h_n(\rho) \xrightarrow[\rho \to \infty]{} \frac{1}{\rho}e^{i[\rho - (n+1)\frac{\pi}{2}]} . \tag{2.17}$$

We also note that since j and y are real functions of ρ, the complex conjugates of j, y, and h, are given by

$$j_n^*(\rho) = j_n(\rho^*) \qquad\qquad y_n^*(\rho) = y_n(\rho^*)$$
$$h_n^*(\rho) = j_n^*(\rho) + i^* y_n^*(\rho) = j_n(\rho^*) - i y_n(\rho^*) = \hbar_n(\rho^*) \tag{2.19}$$

so that in the case of $h_n(\rho)$, the complex conjugate $h_n^*(kr) = \hbar(k^*r)$ is also a solution of the differential equation, with the complex conjugate wave number, a fact we shall use later.

Finally we list several useful integrals. All are easily derived from the differential equations for the two spherical Bessel functions in the integrand, or they may be obtained from the corresponding integrals for the cylindrical functions, using the definitions (2.2). (See Ref. 16). Let $R_n(kr)$ and $\bar{R}_n(\ell r)$ be any two spherical Bessel functions, k and ℓ being general complex wave numbers and r the real radial coordinate. Then

$$\int^r R_n^2(kr)r^2dr = \frac{r^3}{2}\left[R_n^2(kr) - R_{n-1}(kr)R_{n+1}(kr)\right], \tag{2.20}$$

$$\int^r R_n(kr)\bar{R}_n(kr)r^2dr$$
$$= \frac{1}{4}r^3\left[2R_n(kr)\bar{R}_n(kr) - R_{n-1}(kr)\bar{R}_{n+1}(kr) - R_{n+1}(kr)\bar{R}_{n-1}(kr)\right] \tag{2.21}$$

$$\int^r R_n(kr)\bar{R}_n(\ell r)r^2dr = \frac{r^2}{k^2-\ell^2}\left[\ell\bar{R}_{n-1}(\ell r)R_n(kr) - kR_{n-1}(kr)\bar{R}_n(\ell r)\right]_{k\neq\ell} \tag{2.22}$$

$$= \frac{r^2}{k^2-\ell^2}\left[kR_{n+1}(kr)\bar{R}_n(\ell r) - \ell\bar{R}_{n+1}(\ell r)R_n(kr)\right]_{k\neq\ell} \tag{2.23}$$

A useful special case of (2.23) is the following integral, obtained by use of (2.19):

$$\int^r h_n(kr)h_n^*(kr)r^2dr = \frac{r^2}{k^2-k^{*2}}\left[k h_{n+1}(kr)h_n^*(kr) - k^*h_{n+1}^*(kr)h_n(kr)\right]. \tag{2.24}$$

The above integrals are all valid for $n \geq 0$ where again, for the case n=0, the relations (2.8) are to be used to evaluate j_{-1} or y_{-1} in the R_{n-1} terms.

APPENDIX III

Legendre Functions

The Legendre polynomial $P_n(\cos \Theta)$ satisfies the Θ-part of the wave equation:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP_n}{d\theta}\right) + n(n+1)P_n = 0 \tag{3.1}$$

and the associated Lengendre function $P_n^{\ m}(\cos \Theta) = P_n^{\ m}(\mu)$ may be defined by

$$P_n^m(\mu) = (1-\mu^2)^{\frac{m}{2}} \frac{d^m P_n(\mu)}{d\mu^m} \tag{3.2}$$

where $\mu = \cos \Theta$. For n=1 we have from (3.2)

$$P_n^1(\cos\theta) = -\frac{dP_n}{d\theta} \tag{3.3}$$

so that (3.1) gives the relation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\, P_n^1\right) = n(n+1)P_n \tag{3.4}$$

which is used in the boundary equations involving the vector potential A in Sec. 14.

The orthogonality property of the Legendre functions requires that the integral of the product $P_n^{\ m} P_r^{\ m} \sin \Theta \, d\, \Theta$ vanish unless n=r, when it has the value

$$\int_0^\pi \left[P_n^m(\cos\theta)\right]^2 \sin\theta\, d\theta = \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!} \quad . \tag{3.5}$$

APPENDIX IV

Calculation of the Potential Coefficients for n=0

We wish to determine the potential coefficients B_o B'_o C_o C'_o from the six general boundary-condition equations in Sec. 14 for n=0, under the assumptions that $a_1 \ll 1$, and that the order of magnitude relations of Sec. 15 hold:

$$\left(\frac{a_1}{a_2}\right)^2 \sim \left(\frac{a_1}{b}\right)^2 \sim \left(\frac{k_1}{k_2}\right)^2 \sim \frac{\alpha_1}{\alpha_2} << 1 , \qquad \beta_1 = 1 .$$

and where we impose no restriction on the magnitudes of a_2 and b. Similar relations hold for the corresponding primed quantities.

We shall indicate the general procedure for the B_o coefficient and then indicate only the main steps for the remaining quantities.

For n=0 the general equations in Sec. 14 reduce to the following four equations (the v_θ and $p_{r\theta}$ equations do not appear for n=0, and $D_o = D'_o = 0$):

$$B_o a_1 h_0'(a_1) - B_o' a_1' j_1'(a_1') + C_o a_2 h_0'(a_2) - C_o' a_2' j_0'(a_2') = - a_1 j_0'(a_1) \tag{4.1}$$

$$B_o \alpha_1 h_0(a_1) - B_o' \alpha_1' j_0(a_1') + C_o \alpha_2 h_0(a_2) - C_o' a_2' j_0(a_2') = - \alpha_1 j_0(a_1) \tag{4.2}$$

$$B_o \gamma \alpha_1 a_1 h_0'(a_1) - B_o' \alpha_1' a_1' j_0'(a_1') + C_o \gamma \alpha_2 a_2 h_0'(a_2) - C_o' \alpha_2' a_2' j_0'(a_2') = -\gamma \alpha_1 a_1 j_0'(a_1) \tag{4.3}$$

$$\begin{aligned} &B_o \epsilon\left[b^2\beta_1 h_0(a_1) - 2a_1^2 h_0''(a_1)\right] + C_o \epsilon\left[b^2\beta_2 h_0(a_2) - 2a_2^2 h_0''(a_2)\right] \\ &- B_o'\left[b'^2\beta_1' j_0(a_1') - 2a_1'^2 j_0''(a_1')\right] - C_o'\left[b'^2\beta_2' j_0(a_2') - 2a_2'^2 j_0''(a_2')\right] \\ &= -\epsilon\left[b^2\beta_1 j_0(a_1) - 2a_1^2 j_0''(a_1)\right] . \end{aligned} \tag{4.4}$$

These equations are exact. We now write the array of coefficients of the unknowns B_o B_o' C_o C_o' and the constants on the right symbolically as

$$\begin{array}{cccc} (B_o) & (B_o') & (C_o) & (C_o') \\ \end{array}$$

$$\begin{vmatrix} A_1 & A_2 & A_3 & A_4 \\ C_1 & C_2 & C_3 & C_4 \\ D_1 & D_2 & D_3 & D_4 \\ F_1 & F_2 & F_3 & F_4 \end{vmatrix} = \begin{vmatrix} A_7 \\ C_7 \\ D_7 \\ F_7 \end{vmatrix} \tag{4.5}$$

where, using the relations in Appendix II and imposing the conditions that $a_1 \ll 1$, $a_1' \ll 1$, $\beta_1 = \beta_1' = 1$, we find for the values of the above constants (putting $j_o(a_1) = 1$):

$$A_1 = -h_o(a_1) \qquad A_2 = a_1' j_1(a_1') \qquad A_3 = -a_2 h_1(a_2) \qquad A_4 = a_2' j_1(a_2')$$

$$A_7 = a_1 j_1(a_1)$$

$$C_1 = \alpha_1 h_o(a_1) \qquad C_2 = -\alpha_1' \qquad C_3 = \alpha_2 h_o(a_2) \qquad C_4 = -\alpha_2' j_o(a_2')$$

$$C_7 = -\alpha_1$$

$$D_1 = -\chi\alpha_1 h_o(a_1) \qquad D_2 = \alpha_1' a_1' j_1(a_1') \qquad D_3 = -\chi\alpha_2 a_2 h_1(a_2) \qquad D_4 = \alpha_2' a_2' j_1(a_2')$$

$$D_7 = \chi\alpha_1 a_1 j_1(a_1) \tag{4.6}$$

$$F_1 = \epsilon(b^2-4)h_o(a_1) \qquad F_2 = -b'^2 \qquad F_3 = \epsilon\beta_2 b^2 h_o(a_2) + 2\epsilon a_2^2 h_o(a_2) - 4\epsilon a_2 h_1(a_2)$$

$$F_7 = -\epsilon b^2 \qquad F_4 = -\beta_2' b'^2 j_o(a_2') - 2a_2'^2 j_o(a_2') + 4a_2' j_1(a_2')$$

Then the unknown B_o is given by

$$B_o = \Delta(B_o)/\Delta_o \tag{4.7}$$

where Δ_o is the determinant of the coefficients on the left of (4.5) and $\Delta(B_o)$ is the same determinant with the first column replaced by the column on the right of (4.5).

We now evaluate Δ_o. Since the first two columns contain small quantities while the second two are of arbitrary magnitude, we shall write only the first two columns explicitely in order to see which elements are negligible:

$$\Delta_o = \alpha_2^2 h_o(a_1) \cdot \begin{vmatrix} -1 & a_1' j_1(a_1') & A_3 & A_4 \\ \alpha_1/\alpha_2 & -\alpha_1'/\alpha_2 & \dfrac{C_3}{\alpha_2} & \dfrac{C_4}{\alpha_2} \\ \chi\alpha_1/\alpha_2 & \dfrac{\alpha_1'}{\alpha_2} a_1' j_1(a_1') & \dfrac{D_3}{\alpha_2} & \dfrac{D_4}{\alpha_2} \\ \epsilon(b^2-4) & -b'^2 & F_3 & F_4 \end{vmatrix} \tag{4.8}$$

where we have divided out $h_o(a_1)$ from the first column and α_2 from the second and third rows. Now consider the magnitudes of the elements in the first two columns: In the first column the top and bottom elements are of order unity and the other three are of order a_1^2 since $\epsilon\,(=\eta/\eta')$, $\chi\,(=\sigma/\sigma')$ and b^2 are unrestricted and α_1/α_2 is of order $a_1^2 \ll 1$. Similarly, in the second column the first two elements are of order $a_1'^2 \ll 1$, the third is of order $a_1'^3 \ll 1$, while the fourth $(-b'^2)$ being unrestricted may be considered of order unity.

It follows that if the deterimant is expanded by second-order determinants in the first two columns, only the $(-1)(-b'^2)$ product is of order unity. Since this product represents the A_1F_2 term we see that Δ_o can be expanded as

$$\Delta_o = A_1F_2 \begin{vmatrix} C_3 & C_4 \\ D_3 & D_4 \end{vmatrix} \qquad (4.9)$$

After some algebra we then find

$$\Delta_o = h_o(a_1)b'^2 \begin{vmatrix} C_3 & C_4 \\ D_3 & D_4 \end{vmatrix} = h_o(a_1)b'^2(\alpha_2\alpha_2')h_o(a_2)\, a_2' j_1(a_2') \quad . \qquad (4.10)$$

Following the same procedure for $\Delta(B_o)$ we find

$$\Delta(B_o) = \begin{vmatrix} A_7 & A_2 & A_3 & A_4 \\ C_7 & C_2 & C_3 & C_4 \\ D_7 & D_2 & D_3 & D_4 \\ F_7 & F_2 & F_3 & F_4 \end{vmatrix} = \alpha_2^2 \begin{vmatrix} a_1 j_1(a_1) & a_1' j_1(a_1') & A_3 & A_4 \\ -\alpha_1/\alpha_2 & -\alpha_1'/\alpha_2 & C_3/\alpha_2 & C_4/\alpha_2 \\ \gamma\frac{\alpha_1}{\alpha_2} a_1 j_1(a_1) & \frac{\alpha_1'}{\alpha_2} a_1' j_1(a_1') & D_3/\alpha_2 & D_4/\alpha_2 \\ -\epsilon b^2 & -b'^2 & F_3 & F_4 \end{vmatrix} \qquad (4.11)$$

If we now indicate elements of order unity by 1 and order a_1^2 by $x << 1$ we have

$$\Delta(B_o) = \alpha_2^2 \begin{vmatrix} x & x & A_3 & A_4 \\ x & x & C_3/\alpha_2 & C_4/\alpha_2 \\ x^2 & x^2 & D_3/\alpha_2 & D_4/\alpha_2 \\ 1 & 1 & F_3 & F_4 \end{vmatrix} \quad . \qquad (4.12)$$

We see that in the first two columns only two determinants are of order x:

$$\begin{vmatrix} A_7 & A_2 \\ F_7 & F_2 \end{vmatrix} \text{ and } \begin{vmatrix} C_7 & C_2 \\ F_7 & F_2 \end{vmatrix} \quad ,$$

the rest being of order x^2 or x^3. Expanding $\Delta(B_o)$ in this manner we therefore have

$$\Delta(B_o) = \begin{vmatrix} A_7 & A_2 \\ F_7 & F_2 \end{vmatrix} \cdot \begin{vmatrix} C_3 & C_4 \\ D_3 & D_4 \end{vmatrix} + \begin{vmatrix} F_7 & F_2 \\ C_7 & C_2 \end{vmatrix} \cdot \begin{vmatrix} A_3 & A_4 \\ D_3 & D_4 \end{vmatrix} \tag{4.13}$$

In evaluating the determinants we use the relation $\delta b'^2 = \epsilon b^2$ and find

$$\begin{vmatrix} A_7 & A_2 \\ F_7 & F_2 \end{vmatrix} = b'^2 \left[\delta\, a_1' j_1(a_1') - a_1 j_1(a_1) \right]$$

$$\begin{vmatrix} C_3 & C_4 \\ D_3 & D_4 \end{vmatrix} = \alpha_2 \alpha_2' \left[a_2' j_1(a_2') h_0(a_2) - \chi\, j_0(a_2') \cdot a_2 h_1(a_2) \right] \tag{4.14}$$

$$\begin{vmatrix} F_7 & F_2 \\ C_7 & C_2 \end{vmatrix} = b'^2 \left(\delta \alpha_1' - \alpha_1 \right)$$

$$\begin{vmatrix} A_3 & A_4 \\ D_3 & D_4 \end{vmatrix} = \left(\chi \alpha_2 - \alpha_2' \right) a_2' j_1(a_2') \cdot a_2 h_1(a_2)$$

Finally, with $\Delta(B_o)$ from (4.13) and Δ_o from (4.10) we find B_o from (4.7)

$$B_o = i \frac{a_1^3}{3} \left[\delta \left(\frac{a_1'}{a_1} \right)^2 - 1 \right] + \left(\chi \frac{\alpha_2}{\alpha_2'} - 1 \right) i a_1 \left[\frac{\alpha_1}{\alpha_2} \left(\delta \frac{\alpha_1'}{\alpha_1} - 1 \right) \frac{a_2 h_1(a_2)/h_0(a_2)}{1 - \chi Z} \right]$$

or

$$B_o = i \frac{a_1^3}{3} \left[\delta \left(\frac{a_1'}{a_1} \right)^2 - 1 \right] + \left(\chi \frac{\alpha_2}{\alpha_2'} - 1 \right) i a_1 C_o\, a_2 h_1(a_2) \tag{4.15}$$

where
$$Z = \frac{j_0(a_2')}{a_2' j_1(a_2')} \cdot \frac{a_2 h_1(a_2)}{h_0(a_2)} \tag{4.16}$$

and C_o is given below. In the reduction we have again used Appendix II to evaluate $j_1(a_1)$, $h_o(a_1)$, etc.

It may be noted that for zero thermal conductivity $\sigma = 0$, $\bar{\varkappa} = 0$, so that $\alpha_2 = -(\alpha_v \bar{\varkappa})^{-1} \longrightarrow \infty$ and the second term in (4.14) vanishes; the remaining first term then agrees with Epstein's corresponding coefficient, given by his Eq. (47) in Ref. 6 for the case $\sigma = 0$. Unlike the first term which is pure imaginary and contributes nothing to the viscous dissipation (see (17.12) in Sec. 17), the second term has a real part, and this later plays an important role in our results for the viscous dissipation (Sec. 17).

Turning now to $B_o' = \Delta(B_o')/\Delta_o$, we proceed the same way as before to evaluate $\Delta(B_o')$; we find

$$\Delta(B_o') = \begin{vmatrix} A_1 & A_7 & A_3 & A_4 \\ C_1 & C_7 & C_3 & C_4 \\ D_1 & D_7 & D_3 & D_4 \\ F_1 & F_7 & F_3 & F_4 \end{vmatrix} \sim \begin{vmatrix} 1 & x & A_3 & A_4 \\ x & x & C_3 & C_4 \\ x & x^2 & D_3 & D_4 \\ 1 & 1 & F_3 & F_4 \end{vmatrix} . \tag{4.17}$$

Clearly A_1F_7 is the dominant product in the first two columns; hence we have

$$\Delta(B_o') = (A_1F_7) \cdot \begin{vmatrix} C_3 & C_4 \\ D_3 & D_4 \end{vmatrix} = b^2 h_o(a_1) \begin{vmatrix} C_3 & C_4 \\ D_3 & D_4 \end{vmatrix} \tag{4.18}$$

and using $\epsilon b^2 = \delta b'^2$ and Δ_o we obtain

$$B'_o = \Delta(B'_o)/\Delta_o = \delta . \tag{4.19}$$

For $C_o = \Delta(C_o)/\Delta_o$ we have

$$\Delta(C_o) = \begin{vmatrix} A_1 & A_2 & A_7 & A_4 \\ C_1 & C_2 & C_7 & C_4 \\ D_1 & D_2 & D_7 & D_4 \\ F_1 & F_2 & F_7 & F_4 \end{vmatrix} \sim \begin{vmatrix} 1 & x & x & A_4 \\ x & x & x & C_4 \\ x & x^2 & x^2 & D_4 \\ 1 & 1 & 1 & F_4 \end{vmatrix} \tag{4.20}$$

and we see that the dominant term in the expansion will be of order x and is

$$\Delta(C_o) = - A_1 D_4 \begin{vmatrix} C_2 & C_7 \\ F_2 & F_7 \end{vmatrix} = b'^2 \alpha_2' (\delta\alpha_1' - \alpha_1) h_0(a_1) a_2' j_1(a_2'), \tag{4.21}$$

so that

$$C_o = \frac{\alpha_1}{\alpha_2}\left(\delta \frac{\alpha_1'}{\alpha_1} - 1\right) \frac{1}{h_0(a_2)} \cdot \frac{1}{1 - \varkappa Z} , \tag{4.22}$$

where Z is given above in (4.16).

Similarly, for $C'_o = \Delta(C'_o)/\Delta_o$ we have

$$\Delta(C'_o) = \begin{vmatrix} A_1 & A_2 & A_3 & A_7 \\ C_1 & C_2 & C_3 & C_7 \\ D_1 & D_2 & D_3 & D_7 \\ F_1 & F_2 & F_3 & F_7 \end{vmatrix} \sim \begin{vmatrix} 1 & x & A_3 & x \\ x & x & C_3 & x \\ x & x^2 & D_3 & x^2 \\ 1 & 1 & F_3 & 1 \end{vmatrix} \tag{4.23}$$

or

$$\Delta(C_0') = A_1 D_3 \begin{vmatrix} C_2 & C_7 \\ F_2 & F_7 \end{vmatrix} = \chi\, b'^2 \alpha_2 (\delta\alpha_1' - \alpha_1) h_0(a_1) \cdot a_2 h_1(a_2) , \tag{4.24}$$

and finally

$$C_0' = \chi \frac{\alpha_2}{\alpha_2'} \frac{a_2 h_1(a_2)}{a_2' j_1(a_2')} C_0 . \tag{4.25}$$

This completes the calculation of the coefficients for n=0.

APPENDIX V

Calculation of the Potential Coefficients for n=1

Setting n=1 in the general boundary condition equations of Sec. 14 we may write, in the same way as for n=0, the symbolic equation

$$\begin{array}{cccccc} (B_1) & (B_1') & (C_1) & (C_1') & (D_1) & (D_1') \end{array}$$

$$\begin{vmatrix} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \\ B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \\ C_1 & C_2 & C_3 & C_4 & 0 & 0 \\ D_1 & D_2 & D_3 & D_4 & 0 & 0 \\ E_1 & E_2 & E_3 & E_4 & E_5 & E_6 \\ F_1 & F_2 & F_3 & F_4 & F_5 & F_6 \end{vmatrix} = \begin{vmatrix} A_7 \\ B_7 \\ C_7 \\ D_7 \\ E_7 \\ F_7 \end{vmatrix} \qquad (5.1)$$

for the six unknown coefficients B_1 B_1' C_1 C_1' D_1 D_1'. Putting $a_1 \ll 1$, $a_1' \ll 1$ etc., as before, the constants $A_1 \ldots F_7$ are given by

$$\left.\begin{array}{lll}
A_1 = -2h_1(a_1) & A_3 = a_2 h_0(a_2) - 2h_1(a_2) & A_5 = -2h_1(b) \\
A_2 = -j_1(a_1') & A_4 = -a_2' j_0(a_2') + 2j_1(a_2') & A_6 = 2j_1(b') \\
A_7 = -j_1(a_1) & & \\
B_1 = h_1(a_1) & B_3 = h_1(a_2) & B_5 = h_1(b) - b h_0(b) \\
B_2 = -j_1(a_1') & B_4 = -j_1(a_2') & B_6 = -j_1(b') + b' j_0(b') \\
B_7 = -j_1(a_1) & & \\
C_1 = \alpha_1 h_1(a_1) & C_3 = \alpha_2 h_1(a_2) & \\
C_2 = -\alpha_1' j_1(a_1') & C_4 = -\alpha_2' j_1(a_2') & \\
C_7 = -\alpha_1 j_1(a_1) & &
\end{array}\right\} \qquad (5.2)$$

$$\left.\begin{aligned}
&D_1 = -2\gamma\alpha_1 h_1(a_1) && D_3 = \gamma\alpha_2 a_2 h_0(a_2) - 2\gamma\alpha_2 h_1(a_2) \\
&D_2 = -\alpha_1' j_1(a_1') && D_4 = -\alpha_2' a_2' j_0(a_2') + 2\alpha_2' j_1(a_2') \\
&D_7 = -\gamma\alpha_1 j_1(a_1) && \\
&E_1 = -3\epsilon h_1(a_1) && E_3 = \epsilon a_2 h_0(a_2) - 3\epsilon h_1(a_2) \qquad E_5 = -\tfrac{1}{2}\epsilon(6-b^2)h_1(b) + \epsilon b h_0(b) \\
&E_2 = \tfrac{1}{5}a_1'^2 j_1(a_1') && E_4 = -a_2' j_0(a_2') + 3 j_1(a_2') \qquad E_6 = \tfrac{1}{2}(6-b'^2)j_1(b') - b' j_0(b') \\
&E_7 = \tfrac{1}{5}\epsilon a_1^2 j_1(a_1) && \\
&F_1 = \epsilon(b^2-12)h_1(a_1) && F_3 = \epsilon b^2\beta_2 h_1(a_2) + 2\epsilon a_2^2 h_1(a_2) - 4\epsilon a_2 h_2(a_2) \qquad F_5 = -4\epsilon b h_2(b) \\
&F_2 = -b'^2 j_1(a_1') && F_4 = -\beta_2 b'^2 j_1(a_2') - 2a_2'^2 j_1(a_2') + 4a_2' j_2(a_2') \qquad F_6 = 4b' j_2(b') \\
&F_7 = -\epsilon b^2 j_1(a_1) &&
\end{aligned}\right\} \quad (5.2)$$

We first calculate the determinent Δ_1 of the coefficients, i.e., the determinent on the left of (5.1). Factoring out $h_1(a_1)$ from the first column, $j_1(a_1')$ from the second, and writing x for quantities of order a_1^2 in the first two columns we have as in Appendix IV

$$\Delta_1 = h_1(a_1)j_1(a_1') \begin{vmatrix} -2 & -1 & A_3 & A_4 & A_5 & A_6 \\ 1 & -1 & B_3 & B_4 & B_5 & B_6 \\ x & -x & C_3 & C_4 & 0 & 0 \\ -x & -x & D_3 & D_4 & 0 & 0 \\ -3\epsilon & x & E_3 & E_4 & E_5 & E_6 \\ \epsilon(b^2-12) & -b'^2 & F_3 & F_4 & F_5 & F_6 \end{vmatrix} \qquad (5.3)$$

Neglecting elements of order x or smaller we see that in the second and third rows only C_3C_4 and D_3D_4 survive; expanding by these rows we have

$$\Delta_1 = h_1(a_1) j_1(a_1') \begin{vmatrix} C_3 & C_4 \\ D_3 & D_4 \end{vmatrix} \cdot \begin{vmatrix} -2 & -1 & A_5 & A_6 \\ 1 & -1 & B_5 & B_6 \\ -3\epsilon & 0 & E_5 & E_6 \\ (b^2-12)\epsilon & b'^2 & F_5 & F_6 \end{vmatrix} \tag{5.4}$$

After tedious but straightforward algebra the fourth-order determinant is found to be

$$\Delta_{00} b' j_0(b') \cdot b h_0(b) + \Delta_{01} b' j_0(b') \cdot h_1(b) + \Delta_{10} j_1(b') \cdot b h_0(b) + \Delta_{11} j_1(b') \cdot h_1(b) \tag{5.5}$$

with

$$\begin{aligned} \Delta_{00} &= [1-\epsilon][\epsilon b^2 + 2b'^2] & \Delta_{01} &= -\epsilon b^2[\tfrac{1}{2}\epsilon b^2 + 9(1-\epsilon) + b'^2] \\ \Delta_{10} &= [b'^2 - 6(1-\epsilon)][b'^2 + \tfrac{1}{2}\epsilon b^2] & \Delta_{11} &= 3\epsilon b^2[\tfrac{1}{2}(\epsilon b^2 - b'^2) + 9(1-\epsilon)] \end{aligned} \tag{5.6}$$

Using the relation $\epsilon b^2 = \delta b'^2$ we can factor b'^2 out of each Δ and rewrite the fourth-order determinent as

$$b'^2 \; (D_{00} + D_{01} + D_{10} + D_{11} \tag{5.7}$$

where

$$\begin{aligned} D_{00} &= [(1-\epsilon)(2+\delta)]\, b' j_0(b') \cdot b h_0(b) & D_{01} &= [9\delta(\epsilon-1) - \tfrac{1}{2}\epsilon b^2(2+\delta)]\, b' j_0(b') \cdot h_1(b) \\ D_{10} &= [3(\epsilon-1) + \tfrac{1}{2}b'^2](2+\delta)\, j_1(b') \cdot b h_0(b) & D_{11} &= [27\delta(1-\epsilon) + \tfrac{3}{2}\delta b'^2(\delta-1)]\, j_1(b') \cdot h_1(b)\,. \end{aligned} \tag{5.8}$$

Thus we finally have for the determinent of the coefficients

$$A_1 = b'^2 h_1(a_1) \cdot j_1(a_1') \begin{vmatrix} C_3 & D_4 \\ D_3 & D_4 \end{vmatrix} (D_{00} + D_{01} + D_{10} + D_{11}) \;. \tag{5.9}$$

We now calculate B_1. It is given by

$$B_1 = \Delta(B_1)/\Delta_1 \tag{5.10}$$

where

$$\Delta(B_1) = \begin{vmatrix} A_7 & A_2 & A_3 & A_4 & A_5 & A_6 \\ B_7 & B_2 & B_3 & B_4 & B_5 & B_6 \\ C_7 & C_2 & C_3 & C_4 & 0 & 0 \\ D_7 & D_2 & D_3 & D_4 & 0 & 0 \\ E_7 & E_2 & E_3 & E_4 & E_5 & E_6 \\ F_7 & F_2 & F_3 & F_4 & F_5 & F_6 \end{vmatrix} = j_1(a_1)\, j_1(a_1') \begin{vmatrix} 1 & 1 & A_3 & A_4 & A_5 & A_6 \\ 1 & 1 & B_3 & B_4 & B_5 & B_6 \\ x & x & C_3 & C_4 & 0 & 0 \\ x & x & D_3 & D_4 & 0 & 0 \\ -x & -x & E_3 & D_4 & E_5 & E_6 \\ \epsilon b^2 & +b'^2 & F_3 & F_4 & F_5 & F_6 \end{vmatrix} \tag{5.11}$$

The second order determinants of order unity in the first two columns arise from the AF and BF products; expanding $\Delta(B_1)$ we get

$$\Delta(B_1) = \begin{vmatrix} A_7 & A_2 \\ F_7 & F_2 \end{vmatrix} \cdot \begin{vmatrix} C_3 & C_4 \\ D_3 & D_4 \end{vmatrix} \left(\begin{vmatrix} B_5 & B_6 \\ E_5 & E_6 \end{vmatrix} - \begin{vmatrix} A_5 & A_6 \\ E_5 & E_6 \end{vmatrix} \right) \tag{5.12}$$

where we have used the fact that $B_7 = A_7$, $B_2 = A_2$ to eliminate B_7 and B_2. After carrying out the algebra and using $\epsilon b^2 = \delta b'^2$ we find for the quantity in the parenthesis the value

$$N_{00} + N_{01} + N_{10} + N_{11} \tag{5.13}$$

where the N's are given by

$$\begin{aligned} N_{00} &= [1-\epsilon]\, b' j_0(b') \cdot b h_0(b) & N_{10} &= [3(\epsilon-1)+\tfrac{1}{2}b'^2]\, j_1(b') \cdot b h_0(b) \\ N_{01} &= [3(\epsilon-1)-\tfrac{1}{2}\epsilon b^2]\, b' j_0(b') \cdot h_1(b) & N_{11} &= [9(1-\epsilon)+\tfrac{3}{2}b'^2(\delta-1)]\, j_1(b') \cdot h_1(b) \end{aligned} \tag{5.14}$$

Since $\begin{vmatrix} A_7 & A_2 \\ F_7 & F_2 \end{vmatrix} = b'^2(1 - \delta) j_1(a_1) j_1(a_1')$, we finally find

$$\Delta(B_1) = b'^2 j_1(a_1) j_1(a_1')(1-\delta) \begin{vmatrix} C_3 & C_4 \\ D_3 & D_4 \end{vmatrix} (N_{00} + N_{01} + N_{10} + N_{11}) \qquad (5.15)$$

and with Δ_1 we obtain B_1 from (5.10)

$$B_1 = i\frac{a_1^3}{3}(1-\delta)\left[\frac{N_{00}+N_{01}+N_{10}+N_{11}}{D_{00}+D_{01}+D_{10}+D_{11}}\right] = i\frac{a_1^3}{3}(1-\delta)\,G \qquad (5.16)$$

after canceling $j_1(a_1')$ in $\Delta(B_1)$ and Δ_1 and using $h_1(a_1)/j(a_1) = i\frac{1}{3}a_1^3$ from Appendix II. G is a useful abbreviation for the quantity in the bracket.

The coefficient B_1' is found in an exactly similar manner. We replace column two in the determinant on the left of (5.1) by column seven and obtain exactly the same order-of-magnitude scheme as for Δ_1 (5.3) except that $j_1(a_1')$ is replaced by $j_1(a_1)$ and b'^2 by ϵb^2. It follows that $\Delta(B_1')$ can be written in the form (5.4) with the same replacements. We then expand the fourth-order determinant; after the algebra has died down there finally emerges a form similar to (5.13) above in which each N is $3\epsilon b^2 = 3\delta b'^2$ times the corresponding N of B_1 in (5.14). This gives for $\Delta(B_1')$ the result

$$\Delta(B_1') = 3\delta b'^2 h_1(a_1) j_1(a_1) \begin{vmatrix} C_3 & C_4 \\ D_3 & D_4 \end{vmatrix} (N_{00} + N_{01} + N_{10} + N_{11}), \qquad (5.17)$$

so that using Δ_1 we finally get B_1':

$$B_1' = \frac{a_1}{a_1'}\,3\delta\left[\frac{N_{00}+N_{01}+N_{10}+N_{11}}{D_{00}+D_{01}+D_{10}+D_{11}}\right] = \frac{a_1}{a_1'}\,3\delta G = \frac{a_1}{a_1'}\,G' \qquad (5.18)$$

after canceling $h_1(a_1)$ in $\Delta(B_1')$ and Δ_1 and writing $j_1(a_1)/j_1(a_1') = a_1/a_1'$.

Having found B_1 and B_1' we can now obtain C_1 and C_1' from (14.6) and (14.7) of Sec. 14 expressing the continuity of temperature and normal heat flow at the surface of the scattering sphere, since these equations do not involve D_1 and D_1'. We write these equations for n=1 in the form

$$C_1 \alpha_2 h_1(a_2) \quad - C_1' \alpha_2' j_1(a_2') \quad = P_1 \tag{5.19}$$

$$C_1 \chi \alpha_2 a_2 h_1'(a_2) - C_1' \alpha_2' j_1(a_2') \quad = Q_1 \tag{5.20}$$

where

$$P_1 = B_1' \alpha_1' j_1(a_1') - \alpha_1 j_1(a_1) - B_1 \alpha_1 h_1(a_1) \tag{5.21}$$

$$Q_1 = B_1' \alpha_1' j_1(a_1') - \chi \alpha_1 j_1(a_1) + 2 B_1 \chi \alpha_1 h_1(a_1) \quad . \tag{5.22}$$

Here the approximate relations $a_1' j_1'(a_1) = j_1(a_1)$ and $a_1 h_1'(a_1) = -2h_1(a_1)$ for small arguments have been used in the Q_1 expression. Substituting the values of B_1 and B_1' we find, again using the small argument relations of Appendix II:

$$\begin{aligned} P_1 &= \frac{1}{3} a_1 \alpha_1 \left[\frac{\alpha_1'}{\alpha_1} G' - 1 - (1-\delta) G \right] \\ Q_1 &= \frac{1}{3} a_1 \alpha_1 \left[\frac{\alpha_1'}{\alpha_1} G' - \chi - 2\chi (1-\delta) G \right] \quad . \end{aligned} \tag{5.23}$$

Now we use (2.4) from Appendix II to obtain h_1' and j_1' in (5.20) above and find

$$\left.\begin{aligned}
\Delta &= -\alpha_2\alpha_2' a_2' j_1'(a_2') h_1(a_2) + \alpha_2\alpha_2' \chi\, j_1(a_2')\cdot a_2 h_1'(a_2) \\
&= -\alpha_2\alpha_2'\Big[2(\chi-1) j_1(a_2')h_1(a_2) + a_2' j_0(a_2')h_1(a_2) - \chi a_2 h_0(a_2)\cdot j_1(a_2')\Big] \\[2ex]
\Delta(C_1) &= -P_1\alpha_2' a_2' j_1'(a_2') + Q_1\alpha_2' j_1(a_2') \\
&= -\tfrac{1}{3}\alpha_1\alpha_2' a_1\Big\{ j_1(a_2')\Big[(2+\chi) - 3\tfrac{\alpha_1'}{\alpha_1}G' + 2(1-\chi)(1-\delta)G\Big] \\
&\qquad\qquad + a_2' j_0(a_2')\Big[\tfrac{\alpha_1'}{\alpha_1}G' - 1 - (1-\delta)G\Big]\Big\} \\[2ex]
\Delta(C_1') &= Q_1\alpha_2 h_1(a_2) - P_1\chi\alpha_2 a_2 h_1'(a_2) \\
&= \tfrac{1}{3}\alpha_1\alpha_2 a_1\Big\{h_1(a_2)\Big[(1+2\chi)\tfrac{\alpha_1'}{\alpha_1}G' - 3\chi\Big] + a_2 h_0(a_2)\Big[1+(1-\delta)G - \tfrac{\alpha_1'}{\alpha_1}G'\Big]\Big\}
\end{aligned}\right\} \quad (5.24)$$

Combining these equations we find

$$C_1 = \Delta(C_1)/\Delta \qquad\qquad C_1' = \Delta(C_1')/\Delta \ ,$$

the expressions given in (16.8) and (16.9) of Sec. 16.

Since we do not need D_1 or D_1' we do not calculate them but merely note that by the same method as above it is easily found that both are of order a_1. This completes our calculation of the potential coefficients for n=1. As will be seen in Secs. 17 and 18, only B_1 is required explicitely; however the order of magnitudes of B_1', C_1, C_1' are required and the full expressions are given since they were worked out and may be of interest in connection with other problems.

APPENDIX VI

Calculation of Viscous Dissipation Surface Integral

We wish to calculate the surface integral I_s of Sec. 17:

$$I_s = \frac{1}{2}\mathrm{R}\int i\omega\rho_0\phi^*\frac{\partial\phi}{\partial r}\,dS \tag{6.1}$$

over a sphere of radius $\bar{R}$ for which $k_1\bar{R} \gg 1$. Here we have written $\phi = \phi_i + \phi_1$:

$$\phi = \sum_{n=0}^{\infty} i^n(2n+1)P_n(\cos\theta)\left[j_n(\rho_1) + B_n h_n(\rho_1)\right] \tag{6.2}$$

with $\rho_1 = k_1 r \gg 1$. For the derivative of ϕ we have

$$\frac{\partial\phi}{\partial r} = k_1\frac{\partial\phi}{\partial\rho_1} = \sum_{n=0}^{\infty} i^n(2n+1)P_n k_1\left[j_n'(\rho_1) + B_n h_n'(\rho_1)\right] \tag{6.3}$$

where the prime signifies differentiation with respect to ρ_1. From Appendix II, with $\delta_n = (n+1)\frac{\pi}{2}$, we have the asymptotic forms for large arguments:

$$\begin{aligned}
j_n(\rho_1) &= \frac{1}{\rho_1}\cos(\rho_1-\delta_n) &&= \frac{1}{\rho_1}\cos\sigma \\
j_n'(\rho_1) &= -\frac{1}{\rho_1}\sin(\rho_1-\delta_n) &&= -\frac{1}{\rho_1}\sin\sigma \\
h_n(\rho_1) &= \frac{1}{\rho_1}e^{i(\rho_1-\delta_n)} &&= \frac{1}{\rho_1}e^{i\sigma} \\
h_n'(\rho_1) &= \frac{i}{\rho_1}e^{i(\rho_1-\delta_n)} &&= \frac{i}{\rho_1}e^{i\sigma}
\end{aligned} \tag{6.4}$$

where σ is real and we have taken the "phase derivative" only, since the derivative of $1/\rho_1$ is negligible compared to $1/\rho_1$. We now use the orthogonality property of the Legendre functions of different orders (Appendix III) in the surface integral to conclude that only products of the same order will survive in the integration.

Thus the integrand evaluated at the surface is

$$\left(i\omega\rho_0\phi^*\frac{\partial\phi}{\partial r}\right) = \sum_{n=0}^{\infty} i\omega\rho_0(2n+1)^2(P_n)^2 k_1\left[j_n(\rho_1)+B_n h_n(\rho_1)\right]^*\left[j_n'(\rho_1)+B_n h_n'(\rho_1)\right] \tag{6.5}$$

$$= \sum_{n=0}^{\infty} i\omega\rho_0(2n+1)^2(P_n)^2\frac{k_1}{\rho_1^2}\left[-\tfrac{1}{2}\sin 2\sigma + iB_n e^{i\sigma}\cos\sigma - B_n^* e^{-i\sigma}\sin\sigma + iB_nB_n^*\right] \tag{6.6}$$

where (6.4) has been used. Putting the i inside the bracket we get

$$\frac{1}{2}\mathcal{R}\left(i\omega\rho_0\phi^*\frac{\partial\phi}{\partial r}\right) = \sum_{n=0}^{\infty} \omega\rho_0(2n+1)^2(P_n)^2\frac{k_1}{\rho_1^2}\,\frac{1}{2}\mathcal{R}\Big[-\frac{i}{2}\sin 2\sigma - B_n e^{i\sigma}\cos\sigma - iB_n^* e^{i\sigma}\sin\sigma - B_nB_n^*\Big] \tag{6.7}$$

But for any complex number A we have R(A) = R(A*); applying this to the third term, $R(-i\, B_n^*\, e^{-i\sigma}\sin\sigma) = R(iB_n\, e^{i\sigma}\sin\sigma)$ and (6.7) reduces to

$$\frac{1}{2}\mathcal{R}\left(i\omega\rho_0\phi^*\frac{\partial\phi}{\partial r}\right) = \sum_{n=0}^{\infty} \omega\rho_0(2n+1)^2(P_n)^2\frac{k_1}{\rho_1^2}\,\frac{1}{2}\mathcal{R}\left[-B_n - B_nB_n^*\right] , \tag{6.8}$$

where the first term in the bracket in (6.7) has been dropped since it is pure imaginary. Putting $\rho_1 = k_1\bar{R}$, integrating over the surface and using the Legendre product integral we obtain

$$I_s = \sum_{n=0}^{\infty}(2n+1)^2\left\{\int_0^{\pi}(P_n)^2 2\pi(\bar{R})^2\sin\theta\, d\theta\right\}\frac{k_1}{(k_1\bar{R})^2}\,\omega\rho_0\frac{1}{2}\mathcal{R}\left[-B_n-B_nB_n^*\right]$$

$$= \sum_{n=0}^{\infty}(2n+1)^2\left\{\frac{4\pi}{2n+1}\right\}\frac{\omega\rho_0}{k_1}\,\frac{1}{2}\mathcal{R}\left[-B_n-B_nB_n^*\right],$$

or, using $k_1 = \omega/c$

$$I_s = -2\pi\rho_0 c\sum_{n=0}^{\infty}(2n+1)\mathcal{R}\left[B_n+B_nB_n^*\right], \tag{6.9}$$

which is the desired result.

APPENDIX VII

Calculation of Viscous Dissipation Volume Integrals

Outside Integral

The outside volume integral of Sec. 17 to be evaluated is

$$I_v = \frac{1}{2}\mathcal{R}\int p^* \underline{\nabla}\cdot\underline{v}\, dV \qquad R \leq r \leq \overline{R} \tag{7.1}$$

where, from Secs. 8, 9, 15 we have

$$\left.\begin{aligned} \underline{\nabla}\cdot\underline{v} &= -\nabla^2\phi = k_1^2(\phi_i+\phi_1) + k_2^2\phi_2 \\ p^* &= i\omega\rho_0\left[\gamma_1(\phi_i+\phi_1)+\gamma_2\phi_2\right] \end{aligned}\right\} \tag{7.2}$$

$$a_1^2 << 1, \quad \gamma_1 \approx 1, \quad \gamma_2 = 1 - \frac{4N}{3}\frac{\nu}{\mathcal{K}}, \quad k_1^2 = \frac{\omega^2}{c^2}, \quad k_2^2 = i\frac{\omega}{\mathcal{K}}, \quad \frac{k_1^2}{k_2^2} << 1 \tag{7.3}$$

Thus

$$\begin{aligned} \frac{1}{2}\mathcal{R}\left(p^*\underline{\nabla}\cdot\underline{v}\right) = \frac{1}{2}\omega\rho_0\mathcal{R}\Big[& ik_1^2(\phi_i+\phi_1)^*(\phi_i+\phi_1) + ik_2^2\phi_2(\phi_i+\phi_1)^* \\ & + i\gamma_2 k_1^2\phi_2^*(\phi_i+\phi_1) + i\gamma_2 k_2^2\phi_2\phi_2^* \Big] \end{aligned} \tag{7.4}$$

$$= \frac{1}{2}\omega\rho_0\mathcal{R}\left[ik_2^2\phi_2(\phi_i+\phi_1)^* + i\gamma_2 k_2^2\phi_2\phi_2^*\right] \tag{7.5}$$

since in (7.4) the first term is pure imaginary and the third term is negligible compared to the second.

We now integrate the first term in (7.5) using the potentials of Sec. 12, and recalling the orthogonality of the wave functions:

$$\int dV\, \phi_2(\phi_i+\phi_1)^* = \int dV \sum_{n=0}^{\infty}\left[i^{n}(2n+1)P_n C_n h_n(k_2 r)\right]\left[i^{-n}(2n+1)P_n\left(j_n^*(k_1 r)+B_n^* h_n^*(k_1 r)\right)\right]$$

$$= \sum_{n=0}^{\infty} 4\pi(2n+1)\left[C_n \int_R^{\bar{R}} j_n^*(k_1 r)h_n(k_2 r)r^2 dr + B_n^* C_n \int_R^{\bar{R}} h_n^*(k_1 r)h_n(k_2 r)\, r^2 dr\right] \quad . \tag{7.6}$$

The first integral in the bracket has the value (Appendix II):

$$\int_R^{\bar{R}} j_n^*(k_1 r)h_n(k_2 r)r^2 dr = \left\{\frac{r^2}{k_1^2-k_2^2}\left[k_1 j_{n+1}(k_1 r)h_n(k_2 r) - k_2 h_{n+1}(k_2 r)\, j_n(k_1 r)\right]\right\}_R^{\bar{R}} \tag{7.7}$$

$$= \frac{R}{k_2^2}\left[a_1 j_{n+1}(a_1)h_n(a_2) - a_2 h_{n+1}(a_2)\, j_n(a_1)\right] \quad , \tag{7.8}$$

where we have used the facts that k_1 is real so that $j_n^*(k_1 r) = j_n(k_1 r)$, that $k_1^2 << k_2^2$, and that the Hankel functions are negligibly small at the upper limit $\bar{R}$, as discussed in Sec. 17. Moreover since $a_1 j_{n+1}(a_1)/j_n(a_1) \sim a_1^2 << 1$, the second term dominates the first in (7.8) so that the first integral in (7.6) is

$$C_n \int_R^{\bar{R}} j_n^*(k_1 r)h_n(k_2 r)r^2 dr = -\, C_n \frac{R}{k_2^2}\, j_n(a_1)\, a_2 h_{n+1}(a_2) \quad . \tag{7.9}$$

Evaluating the second integral of (7.6) in the same way we find

$$\int_R^{\bar{R}} h_n^*(k_1r)h_n(k_2r)r^2dr = \left\{\frac{r^2}{k_1^2-k_2^2}\left[k_1h_{n+1}^*(k_1r)h_n(k_2r) - k_2h_{n+1}(k_2r)h_n^*(k_1r)\right]\right\}_R^{\bar{R}} \tag{7.10}$$

$$= \frac{R}{k_2^2}\left[a_1h_{n+1}^*(a_1)h_n(a_2) - a_2h_{n+1}(a_2)h_n^*(a_1)\right] \tag{7.11}$$

where now both terms in the bracket are of the same order so that

$$B_n^*C_n\int_R^{\bar{R}} h_n^*(k_1r)h_n(k_2r)r^2dr = B_n^*C_n\frac{R}{k_2^2}\left[a_1h_{n+1}^*(a_1)h_n(a_2) - a_2h_{n+1}(a_2)h_n^*(a_1)\right] . \tag{7.12}$$

Thus the integral of the first term in (7.5) is

$$\int dV\,\phi_2(\phi_i+\phi_1)^* = \sum_{n=0}^{\infty} 4\pi(2n+1)\left\{-C_n\frac{R}{k_2^2}j_n(a_1)a_2h_{n+1}(a_2)\right.$$

$$\left. + B_n^*C_n\frac{R}{k_2^2}\left[a_1h_{n+1}^*(a_1)h_n(a_2) - a_2h_{n+1}(a_2)h_n^*(a_1)\right]\right. . \tag{7.13}$$

Turning to the integral of the second term of (7.5) we have in the same way, using the expression in Sec. 12 for ϕ_2:

$$\int dV\,\phi_2\phi_2^* = \sum_{n=0}^{\infty} 4\pi(2n+1)C_nC_n^*\int_R^{\bar{R}} h_n^*(k_2r)h_n(k_2r)r^2dr \tag{7.14}$$

$$= \sum_{n=0}^{\infty} 4\pi(2n+1)C_nC_n^*\left\{\frac{r^2}{k_2^2-k_2^{*2}}\left[k_2h_{n+1}(k_2r)h_n^*(k_2r) - k_2^*h_{n+1}(k_2r)h_n(k_2r)\right]\right\}_R^{\bar{R}} \tag{7.15}$$

$$= \sum_{n=0}^{\infty} 4\pi(2n+1)C_nC_n^*\frac{R}{-2k_2^2}\left[a_2h_{n+1}(a_2)h_n^*(a_2) - a_2^*h_{n+1}^*(a_2)h_n(a_2)\right] . \tag{7.16}$$

In (7.14) we have used (2.24) of Appendix II to evaluate the integral; in (7.15) $k_2^2 - k_2^{*2} = 2k_2^2$ since k_2^2 is pure imaginary, and the minus sign appears because the bracket is evaluated only at the lower limit as above in (7.8).

Collecting our results we finally have from (7.5), (7.13), and (7.16):

$$I_v = \frac{1}{2}\omega\rho_0 \mathcal{R}\left\{ i k_2^2 \int dV \phi_2 (\phi_i + \phi_1)^* + i\gamma_2 k_2^2 \int dV \phi_2 \phi_2^* \right\} \tag{7.17}$$

$$\begin{aligned} = 2\pi\omega\rho_0 R\, \mathcal{R}\Big[\sum_{n=0}^{\infty} (2n+1) \Big\{ & -i C_n j_n(a_1) a_2 h_{n+1}(a_2) + \\ & + i B_n^* C_n \left[a_1 h_{n+1}^*(a_1) h_n(a_2) - a_2 h_{n+1}(a_2) h_n^*(a_1) \right] \\ & - \tfrac{1}{2} i \gamma_2 C_n C_n^* \left[a_2 h_{n+1}(a_2) h_n^*(a_2) - a_2^* h_{n+1}(a_2) h_n(a_2) \right] \Big\} \Big] \end{aligned} \tag{7.18}$$

We now wish to evaluate this for n=0, n=1. Using the approximations of Appendix II for the small argument a_1 in the j's and h's and the order-of-magnitude results for the coefficients from Sec. 15 ($C_0 \sim a_1^2, C_1 \sim B_0 \sim B_1 \sim a_1^3$) we easily find that for n=0 the C_0 term in (7.18) is of order a_1^2, while the $B_0^* C_0$ and $C_0 C_0^*$ terms are of order a_1^4. Moreover, for n=1, the C_1 and $B_1^* C_1$ terms are of order a_1^4 and the $C_1 C_1^*$ term of order a_1^6. Thus only the C_n term for n=0 is significant, and we finally obtain for the outside volume integral (with $j_0(a_1) = 1$):

$$I_v = -2\pi\rho_0 c\, \mathcal{R}\left[i a_1 C_0 a_2 h_1(a_2) \right] \quad . \tag{7.19}$$

Since the surviving term represents the product $\phi_2^* \phi_1$ in (7.17), the significant contribution to the volume integral of $p^* \underline{\nabla} \cdot \underline{v}$ arises from the interference effects between the incident acoustic wave ϕ_i and the scattered thermal wave ϕ_2. As we shall now see, a similar result holds for the inside integral, taken over the interior of the scattering sphere.

Inside Integral

The inside volume integral of Sec. 17 is

$$I'_V = \frac{1}{2}\mathcal{R}\int p'^{*}\,\underline{\nabla}\cdot\underline{v}'\,dV \qquad 0 \leq r \leq R \ . \tag{7.20}$$

In the same way as before we find

$$\frac{1}{2}\mathcal{R}\left(p'^{*}\,\underline{\nabla}\cdot\underline{v}'\right) = \frac{1}{2}\omega\rho'_0\,\mathcal{R}\left[i(\phi'_1+\gamma_2\phi'_2)^{*}(k_1'^2\phi'_1+k_2'^2\phi'_2)\right] \tag{7.21}$$

$$= \frac{1}{2}\omega\rho'_0\,\mathcal{R}\left[ik_1'^2\phi_1'^{*}\phi'_1 + ik_2'^2\phi'_2\phi_1'^{*} + i\gamma_2'^{*}k_1'^2\phi'_1\phi_2'^{*} + i\gamma_2'^{*}k_2'^2\phi'_2\phi_2'^{*}\right] \tag{7.22}$$

$$= \frac{1}{2}\omega\rho'_0\,\mathcal{R}\left[ik_2'^2\phi'_2\phi_1'^{*} + i\gamma_2'^{*}k_2'^2\phi'_2\phi_2'^{*}\right] \ , \tag{7.23}$$

where again the first term in the bracket of (7.22) is pure imaginary and the third term negligible compared to the second. Integrating the first term of (7.23) we get

$$\int dV\,\phi'_2\phi_1'^{*} = \int dV \sum_{n=0}^{\infty}\left[i^{n}(2n+1)P_nC'_nj_n(k'_2r)\right]\left[i^{-n}(2n+1)P_nB_n'^{*}j_n^{*}(k'_1r)\right] \tag{7.24}$$

$$= \sum_{n=0}^{\infty} 4\pi(2n+1)B_n'^{*}C'_n\int_0^R j_n^{*}(k'_1r)j_n(k'_2r)\,r^2\,dr \tag{7.25}$$

$$= \sum_{n=0}^{\infty} 4\pi(2n+1)B_n'^{*}C'_n\left\{\frac{r^2}{k_1'^2-k_2'^2}\left[k'_1j_{n+1}(k'_1r)j_n(k'_2r) - k'_2j_{n+1}(k'_2r)j_n(k'_1r)\right]\right\}_0^R \tag{7.26}$$

$$= \sum_{n=0}^{\infty} 4\pi(2n+1) B_n'^{*} C_n' \frac{R}{-k_2'^2} \left[a_1' j_{n+1}(a_1') j_n(a_2') - a_2' j_{n+1}(a_2') j_n(a_1') \right] \tag{7.27}$$

$$= \sum_{n=0}^{\infty} 4\pi(2n+1) B_n'^{*} C_n' \frac{R}{k_2'^2} a_2' j_{n+1}(a_2') j_n(a_1') \ , \tag{7.28}$$

since in (7.27) the first term is of order $a_1'^2$ smaller than the second. Integrating the second term of (7.23) we have

$$\int dV \phi_2' \phi_2'^{*} = \int dV \sum_{n=0}^{\infty} (2n+1)^2 (P_n)^2 C_n' C_n'^{*} j_n(k_2' r) j_n^{*}(k_2' r) \tag{7.29}$$

$$= \sum_{n=0}^{\infty} 4\pi(2n+1) C_n' C_n'^{*} \int_0^R j_n(k_2' r) j_n^{*}(k_2' r) r^2 dr \tag{7.30}$$

$$= \sum_{n=0}^{\infty} 4\pi(2n+1) C_n' C_n'^{*} \left\{ \frac{1}{4} r^3 \left[2 j_n(k_2' r) j_n^{*}(k_2' r) - j_{n-1}(k_2' r) j_{n+1}^{*}(k_2' r) - j_{n+1}(k_2' r) j_{n-1}^{*}(k_2' r) \right] \right\}_0^R \tag{7.31}$$

$$= \sum_{n=0}^{\infty} 4\pi(2n+1) C_n' C_n'^{*} \frac{1}{4} R^3 \left[2 j_n(a_2') j_n^{*}(a_2') - j_{n-1}(a_2') j_{n+1}^{*}(a_2') - j_{n+1}(a_2') j_{n-1}^{*}(a_2') \right] . \tag{7.32}$$

Collecting our results (7.28) and (7.32) and using (7.23) in (7.20) we find for the inside integral the general result:

$$I_V' = \frac{1}{2} \omega \rho_0' \mathcal{R} \left[i k_2'^2 \int dV \phi_2' \phi_1' + i k_2'^2 \gamma_2' \int dV \phi_2' \phi_2'^{*} \right] \tag{7.33}$$

$$= (2\pi\omega\rho_0' R) \mathcal{R} \left\{ \sum_{n=0}^{\infty} (2n+1) \left\{ i B_n'^{*} C_n' a_2' j_{n+1}(a_2') j_n(a_1') + \frac{1}{4} i \gamma_2' C_n' C_n'^{*} a_2'^2 \left[2 j_n(a_2') j_n^{*}(a_2') - j_{n-1}(a_2') j_{n+1}^{*}(a_2') - j_{n+1}(a_2') j_{n-1}^{*}(a_2') \right] \right\} \right\} \tag{7.34}$$

To evaluate this for n=0 and n=1 we use the order of magnitude relations from Sec. 15 ($B_0' \sim B_1' \sim 1$, $C_0' \sim a_1'^2$, $C_1' \sim a_1'^3$) and $j_n(a_1') \sim a_1'^n$. For n=0 we see that the $B_o'^* C_o'$ term is of order $a_1'^2$ and the $C_o C_o'^*$ term of order $a_1'^4$; for n=1 the terms are of order $a_1'^4$ and $a_1'^6$. Thus we keep only the $B_n'^* C_n'$ term for n=0 and get $(j_n(a_1') = 1)$

$$I_v' = (2\pi\omega\rho_o' R)\,\mathcal{R}\left[iB_o'^* C_o' a_2' j_1(a_2')\right] ; \tag{7.35}$$

but $B_o' = \delta = \rho_o/\rho_o'$ so, after multiplying numerator and denominator by k_1 we get

$$I_v' = -2\pi\rho_o c\,\mathcal{R}\left[-ia_1 C_o' a_2' j_1(a_2')\right] . \tag{7.36}$$

We now note from Appendix IV that C_o' is given by

$$C_o' = \cancel{\chi}\frac{\alpha_2}{\alpha_2'}\,\frac{a_2 h_1(a_2)}{a_2' j_1(a_2')}\,C_o$$

so that we finally obtain for the inside volume integral of the viscous dissipation the result

$$I_v' = -2\pi\rho_o c\,\mathcal{R}\left[-\cancel{\chi}\frac{\alpha_2}{\alpha_2'}\,ia_1 C_o a_2 h_1(a_2)\right] , \tag{7.37}$$

which is in a more convenient form than (7.35).

As pointed out before, the dominant contribution to this integral arises from the $\phi_2'\ \phi_1'^*$ term in (7.33), which represents the interference between the scattered acoustic and thermal waves in the interior of the small scattering sphere.

APPENDIX VIII

Calculation of Thermal Dissipation Volume Integrals

From Sec. 18 we have the two volume integrals J_v and J'_v to evaluate:

$$J_v = -\frac{\rho_0 c^2 \alpha_v}{2\gamma} \mathcal{R} \int_R^{\bar{R}} T^* \underline{\nabla} \cdot \underline{v} \, dV \tag{8.1}$$

$$J'_v = -\frac{\rho'_0 c'^2 \alpha'_v}{2\gamma'} \mathcal{R} \int_0^R T'^* \underline{\nabla} \cdot \underline{v}' \, dV . \tag{8.2}$$

These integrals are easily reduced to the corresponding integrals of I_v and I'_v in Appendix VII (with integrand $p^* \underline{\nabla} \cdot \underline{v}$ instead of $T^* \underline{\nabla} \cdot \underline{v}$). Using the general relations of Secs. 8 and 9

$$T^* \underline{\nabla} \cdot \underline{v} = -T^* \nabla^2 (\phi_1 + \phi_2) = T^* (k_1^2 \phi_1 + k_2^2 \phi_2) \tag{8.3}$$

$$= (\alpha_1 \phi_1 + \alpha_2 \phi_2)^* (k_1^2 \phi_1 + k_2^2 \phi_2) \tag{8.4}$$

$$= \alpha_1^* k_1^2 \phi_1 \phi_1^* + \alpha_1^* k_2^2 \phi_1^* \phi_2 + \alpha_2^* k_1^2 \phi_1 \phi_2^* + \alpha_2^* k_2^2 \phi_2 \phi_2^* . \tag{8.5}$$

We have used ϕ_1 for the acoustic wave ($\phi_i + \phi_1$ outside, ϕ'_1 inside). Now k_1^2 , α_2 are real and k_2^2 , α_1 are imaginary. Hence the first and last terms are pure imaginary with zero real parts and may be dropped. We also have the relation

$$\alpha_1^* k_2^2 = (\gamma - 1) \alpha_2 k_1^2 \tag{8.6}$$

as is easily verified. Thus we have

$$\begin{aligned}\mathcal{R}[T^{*}\underline{\nabla}\cdot\underline{v}] &= \mathcal{R}\left[\alpha_2 k_1^2(\gamma-1)\phi_1^{*}\phi_2 + \alpha_2 k_1^2 \phi_1\phi_2^{*}\right]\\ &= \alpha_2 k_1^2 \mathcal{R}\left[(\gamma-1)\phi_1^{*}\phi_2 + \phi_1^{*}\phi_2\right]\\ &= \alpha_2 k_1^2 \gamma\, \mathcal{R}[\phi_1^{*}\phi_2]\,.\end{aligned} \tag{8.7}$$

The corresponding expression with primes holds for $\mathcal{R}[T'^{*}\underline{\nabla}\cdot\underline{v}']$, and the integrals thus become

$$J_v = -\frac{\rho_0 c^2 \alpha_v}{2\gamma}(\alpha_2 k_1^2\gamma)\,\mathcal{R}\int_R^{\bar{R}}(\phi_i+\phi_1)^{*}\phi_2\, dV \tag{8.8}$$

$$J_v' = -\frac{\rho_0' c'^2 \alpha_v'}{2\gamma'}(\alpha_2' k_1'^2\gamma')\,\mathcal{R}\int_0^R \phi_1'^{*}\phi_2'\, dV\ . \tag{8.9}$$

The coefficient in J_v is easily reduced to $(\frac{1}{2}\rho_0\omega^2/\bar{\varkappa})$ and the integral is given by (7.13) of Appendix VII. Similar remarks hold for J_v'; the integral is given by (7.27) of Appendix VII. Thus the exact expressions are

$$\begin{aligned}J_v = \frac{1}{2}\frac{\rho_0\omega^2}{\bar{\varkappa}}\,\mathcal{R}\sum_{n=0}^{\infty}4\pi(2n+1)\Big\{&-C_n\frac{R}{k_2^2}\,j_n(a_1)a_2h_{n+1}(a_2)\ + \\ &+ B_n^{*}C_n\frac{R}{k_2^2}\left[a_1h_{n+1}^{*}(a_1)h_n(a_2)-a_2h_{n+1}(a_2)h_n^{*}(a_1)\right]\Big\}\end{aligned} \tag{8.10}$$

and

$$J_v' = \frac{1}{2}\frac{\rho_0'\omega^2}{\bar{\varkappa}'}\,\mathcal{R}\sum_{n=0}^{\infty}4\pi(2n+1)B_n'^{*}C_n'\frac{R}{-k_2'^2}\left[a_1'j_{n+1}(a_1')j_n(a_2')-a_2'j_{n+1}(a_2')j_n(a_1')\right] \tag{8.11}$$

As in Appendix VII, we evaluate the terms for n=0, 1 and using the order of magnitude of the coefficients and the asymptotic forms for the j's and h's, it is easily shown that in J_v only the first term, for n=0, is important, while for J'_v, the second term in the bracket dominates:

$$J_v = \frac{1}{2} \frac{\rho_0 \omega^2}{\mathscr{H}} \mathscr{R}\left[- \frac{4\pi R}{k_2^2} C_0 a_2 h_1(a_2) \right] \tag{8.12}$$

$$J'_v = \frac{1}{2} \frac{\rho'_0 \omega^2}{\mathscr{H}'} \mathscr{R}\left[\frac{4\pi R}{k_2'^2} \overset{*}{B'_0} C'_0 a'_2 j_1(a'_2) \right] , \tag{8.13}$$

These are the desired expressions for J_v and J'_v in Sec. 18.

APPENDIX IX

Calculation of Attenuation by Water Drops in Air

a. Reduction of α_η

From (19.6) the general expression for α_η is

$$\alpha_\eta = \frac{4\pi n}{k_1^2} a_1^3 (1-\delta) \mathcal{R}\left[-i \frac{b h_2(b)\left[(1-\epsilon)\, b' j_2(b') - (\frac{1}{2} b'^2)\, j_1(b')\right] - (\frac{1}{2}\delta b'^2)\, b' j_2(b')\, h_1(b)}{\left[-(2+\delta) b h_0(b) + 9\delta h_1(b)\right]\left[(1-\epsilon)\, b' j_2(b') - (\frac{1}{2} b'^2)\, j_1(b')\right] + (\frac{1}{2}\delta b'^2)(2+\delta)\, b' j_2(b')\, h_1(b)} \right] \tag{9.1}$$

For δ , $\epsilon << 1$ this may be reduced as follows. We first note that for small arguments $h_1(b) \propto 1/b^2 = \dfrac{2\nu}{\omega R^2} \longrightarrow \infty$ as $\omega \longrightarrow 0$, so that even though h_1 is multiplied by δ in the last terms of the numerator and denominator, it cannot be dropped without further examination. Using the small argument approximations however, it is found that the last numerator term is of order $\delta b'^3$ smaller than the first and is therefore completely negligible for all values of b'. A similar result is found for the second term in the denominator so that it may also be neglected. Canceling a common factor we then have

$$\alpha_\eta = \frac{4\pi n}{k_1^2} a_1^3 \mathcal{R}\left[-i \frac{b h_2(b)}{-2 b h_0(b) + 9\delta h_1(b)} \right] \tag{9.2}$$

where we have dropped δ in the $(1 - \delta)$ and $(2 + \delta)$ terms. The attenuation is seen to be independent of b' and hence also of η', the viscosity of the medium inside the scattering sphere.

We now substitute for the Hankel functions their explicit expressions from Appendix II and make the change of variable

$$b^2 = 2iy^2 \qquad b = (1+i)y \qquad y = (\omega R^2/2\nu)^{1/2} \tag{9.3}$$

where y is real, to obtain:

$$\alpha_\eta = \frac{4\pi n}{k_1^2} a_1^3 \mathcal{R}\left[-i\,\frac{(2y^2+3) + i\,3(1+y)}{(4y^2+9\delta y) + i\,9\delta(1+y)}\right] \tag{9.4}$$

$$= \frac{4\pi n}{k_1^2} a_1^3 \left(\frac{3}{4}\,\frac{1+y}{y^2}\right)\left[\frac{16y^4}{16y^4 + 72\delta y^3 + 81\delta^2(1+2y+2y^2)}\right] \tag{9.5}$$

or

$$\alpha_\eta = \frac{4\pi n R}{c}\,\frac{3}{2}\,\nu(1+y)\left[\frac{16y^4}{16y^4 + 72\delta y^3 + 81\delta^2(1+2y+2y^2)}\right] \tag{9.6}$$

Equation (9.6) is the desired form for the viscous attenuation of water drops in air.

b. Reduction of α_σ

To evaluate α_σ given in (20.8) we must calculate the real part of the expression:

$$\mathcal{R}\left[\frac{a_2 h_1(a_2)/h_0(a_2)}{1-\chi Z}\right] = \mathcal{R}\left[\frac{1-ia_2}{1-\chi Z}\right] \tag{9.7}$$

where from (19.8) we have Z:

$$Z = \frac{j_0(a_2')}{a_2' j_1(a_2')} \frac{a_2 h_1(a_2)}{h_0(a_2)} = \frac{1 - i a_2}{1 - a_2' \cot a_2'} \quad . \tag{9.8}$$

Now since χ is small we see that χZ will be small except at low frequencies where $a_2 \longrightarrow 0$, $a_2' \longrightarrow 0$; in this case Z has the limiting form

$$Z \xrightarrow[\omega \to 0]{} \frac{1 - i a_2}{\frac{a_2'^2}{3} + \frac{a_2'^4}{45}} \quad , \tag{9.9}$$

where the numerator is exact and in the denominator we have retained the first two terms of the expansion for $1 - a_2' \cot a_2'$ for small a_2'. (Since $a_2'^2$ is pure imaginary it is necessary to retain the first <u>two</u> terms in order to have a real part in the expansion). Now define the dimensionless real quantities z and z' by

$$a_2^2 = 2i z^2 \qquad a_2 = (1 + i) z \qquad z = (\omega R^2 / 2 \bar{\varkappa})^{1/2} \tag{9.10}$$

$$a_2'^2 = 2i z'^2 \qquad a_2' = (1 + i) z' \qquad z' = (\omega R^2 / 2 \bar{\varkappa}')^{1/2} \quad . \tag{9.11}$$

Substituting these into Z and keeping only the largest real and imaginary parts in both numerator and denominator we find

$$Z \xrightarrow[\omega \to 0]{} \frac{(1+z) - i z}{i \frac{2}{3} z'^2 - \frac{4}{45} z'^4} \longrightarrow \frac{-3}{2} \frac{z + i}{z'^2} \quad , \tag{9.12}$$

so that (9.7) becomes

$$\mathcal{R}\left[\frac{1-ia_2}{1-\chi Z}\right] \xrightarrow[\omega\to 0]{} \mathcal{R}\left[\frac{(1+z)-iz}{1+\chi\frac{3}{2}\frac{z+i}{z'^2}}\right] \tag{9.13}$$

$$= (1+z)\left[\frac{1+\frac{3}{2}\chi\frac{z}{1+z}\frac{1}{z'^2}}{1+3\chi\frac{z}{z'^2}+\frac{9}{4}\chi^2\frac{z^2}{z'^4}+\frac{9}{4}\chi^2\frac{1}{z'^4}}\right], \tag{9.14}$$

but since χ is small, the second term in the numerator is negligible compared to the first for all z, z'; similarly the third denominator term is negligible compared to the fourth. Dropping these terms and putting

$$\chi/z'^2 = (\delta c_p/c_p')/z^2 \tag{9.15}$$

into (9.14) we finally have

$$\mathcal{R}\left[\frac{1-ia_2}{1-\chi Z}\right] \xrightarrow[\omega\to 0]{} (1+z)\left[1+3(\delta c_p/c_p')\frac{1}{z}+\frac{9}{4}(\delta c_p/c_p')^2\frac{1}{z^4}\right]^{-1}. \tag{9.16}$$

<u>But now we see that this expression is also valid at higher frequencies</u>, for although our low-frequency approximations ($z, z' \longrightarrow 0$) in Z are of course no longer valid, the whole term χZ is small compared to unity anyway, so that the exact form of Z is unimportant. Thus

$$\mathcal{R}\left[\frac{a_2 h_1(a_2)/h_0(a_2)}{1-\chi Z}\right] = (1+z)\left[1+3(\delta c_p/c_p')\frac{1}{z}+\frac{9}{4}(\delta c_p/c_p')^2\frac{1}{z^4}\right]^{-1} \tag{9.17}$$

is the final desired result (20.9) of Sec. 20.

APPENDIX X

Calculation of Attenuation by Air Bubbles in Water

a. Reduction of α_η

From (19.6) we have the general expression for α_η:

$$\alpha_\eta = \frac{4\pi n}{k_1^2} a_1^3 (1-\delta) \mathcal{R} \left[-i \frac{b h_2(b) \left[(1-\epsilon) b' j_2(b') - \frac{1}{2} b'^2 j_1(b')\right] - \left(\frac{1}{2}\delta b'^2\right) b' j_2(b') h_1(b)}{\left[-(2+\delta) b h_0(b) + 9\delta h_1(b)\right]\left[(1-\epsilon) b' j_2(b') - \frac{1}{2} b'^2 j_1(b')\right] + \left(\frac{1}{2}\delta b'^2\right)(2+\delta) b' j_2(b') h_1(b)} \right] \tag{10.1}$$

For $\delta >> 1$, $\epsilon >> 1$ the bracket reduces to

$$-i \frac{b h_2(b)(-\epsilon) b' j_2(b') - \left(\frac{1}{2}\epsilon b^2\right) b' j_2(b') h_1(b)}{\delta\left[-b h_0(b) + 9 h_1(b)\right]\left[-\epsilon b' j_2(b')\right] + \left(\frac{1}{2}\epsilon b^2\right)\delta b' j_2(b') h_1(b)} \tag{10.2}$$

where we have used $\delta b'^2 = \epsilon b^2$. Canceling common factors and replacing $(1 - \delta)$ by $-\delta$ in (10.1) we obtain

$$\alpha_\eta = \frac{4\pi n}{k_1^2} a_1^3 \mathcal{R}\left[-i \frac{b h_2(b) + \frac{1}{2} b^2 h_1(b)}{b h_0(b) - 9 h_1(b) + \frac{1}{2} b^2 h_1(b)} \right] \tag{10.3}$$

or

$$\alpha_\eta = \frac{4\pi n}{k_1^2} a_1^3 \mathcal{R}\left[-i \frac{i b^2 - 6i + b^3 - 6b}{18b - b^3 + 18i - 3ib^2} \right] \tag{10.4}$$

We now change to the variable y defined by (9.3) in Appendix IX and obtain

$$\alpha_\eta = \frac{4\pi n}{k_1^2} a_1^3 \mathcal{R}\left[-i \frac{(y^3 + y^2 + 3y) + i(y^3 + 3y - 3)}{(y^3 + 3y^2 + 9y) + i(-y^3 + 9y + 9)} \right] . \tag{10.5}$$

After taking the real part we finally have

$$\alpha_\eta = \frac{4\pi n}{k_1^2} a_1^3 \frac{1}{y^2} \left[\frac{2y^7 + 12y^6 + 6y^5}{2y^6 + 6y^5 + 9y^4 + 36y^3 + 162y^2 + 162y + 81} \right] \qquad (10.6)$$

or

$$\alpha_\eta = \frac{4\pi n R}{c} 2v \left[\frac{2y^7 + 12y^6 + 6y^5}{2y^6 + 6y^5 + 9y^4 + 36y^3 + 162y^2 + 162y + 81} \right] \qquad (10.7)$$

the desired result (22.3) of Sec. 22 for the viscous attenuation by air bubbles in water.

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Table I - Physical Constants

		Air				Water			
Acoustic Wave	f sec^{-1}	10^3	10^4	10^5	10^6	10^3	10^4	10^5	10^6
	ω sec^{-1}	6.28×10^3	6.28×10^4	6.28×10^5	6.28×10^6	6.28×10^3	6.28×10^4	6.28×10^5	6.28×10^6
	c cm/sec	3.30×10^4				1.45×10^5			
	ρ_o g/cm^3	1.29×10^{-3}				1.00			
	$\rho_o c^2$ $g/cm\ sec^2$	1.41×10^6				2.10×10^{10}			
	$k = \frac{\omega}{c}$ cm^{-1}	0.19	1.90	19.0	190	0.043	0.43	4.3	43.
	λ_1 cm	33.0	3.30	0.33	0.033	145	14.5	1.45	0.145
	ℓ_1 cm	7×10^6	7×10^4	7×10^2	7	10^{10}	10^8	10^6	10^4
Viscous Wave	η g/cm sec	1.82×10^{-4}				1.10×10^{-2}			
	ν cm^2/sec	0.141				0.011			
	$\omega\nu/c^2$	8.1×10^{-7}	8.1×10^{-6}	8.1×10^{-5}	8.1×10^{-4}	3.3×10^{-9}	3.3×10^{-8}	3.3×10^{-7}	3.3×10^{-6}
	$\|K\|$ cm^{-1}	2.1×10^2	6.6×10^2	2.1×10^3	6.6×10^3	7.5×10^2	2.4×10^3	7.5×10^3	2.4×10^4
	$\lambda_\perp$ cm	4.2×10^{-2}	1.3×10^{-2}	4.2×10^{-3}	1.3×10^{-3}	1.2×10^{-2}	3.7×10^{-3}	1.2×10^{-3}	3.7×10^{-4}
	$\ell_\perp$ cm	6.7×10^{-3}	2.1×10^{-3}	6.7×10^{-4}	2.1×10^{-4}	1.9×10^{-3}	5.9×10^{-4}	1.9×10^{-4}	5.9×10^{-5}
	c cm/sec	42	133	420	1330	11.7	37	117	370
Thermal Wave	σ cal/cm sec°C	5.8×10^{-5}				1.43×10^{-3}			
	γ	1.4				1.00336 (15°C) (Ref. 10)			
	c_p cal/g°C	0.24				1.0			
	α_v $(^oC)^{-1}$	3.66×10^{-3}				1.4×10^{-4} (15°C) (Ref. 10)			
	$\bar{\kappa}$ cm^2/sec	0.187				1.43×10^{-3}			
	$\omega\bar{\kappa}/c^2$	1.1×10^{-6}	1.1×10^{-5}	1.1×10^{-4}	1.1×10^{-3}	4.3×10^{-10}	4.3×10^{-9}	4.3×10^{-8}	4.3×10^{-7}
	k_2 cm^{-1}	1.8×10^2	5.8×10^2	1.8×10^3	5.8×10^3	2.1×10^3	6.6×10^3	2.1×10^4	6.6×10^4
	λ_2 cm	4.8×10^{-2}	1.5×10^{-2}	4.8×10^{-3}	1.5×10^{-3}	4.2×10^{-3}	1.3×10^{-3}	4.2×10^{-4}	1.3×10^{-4}
	ℓ_2 cm	7.6×10^{-3}	2.4×10^{-3}	7.6×10^{-4}	2.4×10^{-4}	6.7×10^{-4}	2.1×10^{-4}	6.7×10^{-5}	2.1×10^{-5}
	c_2 cm/sec	48.5	150	48.5	1500	4.2	13	42	133

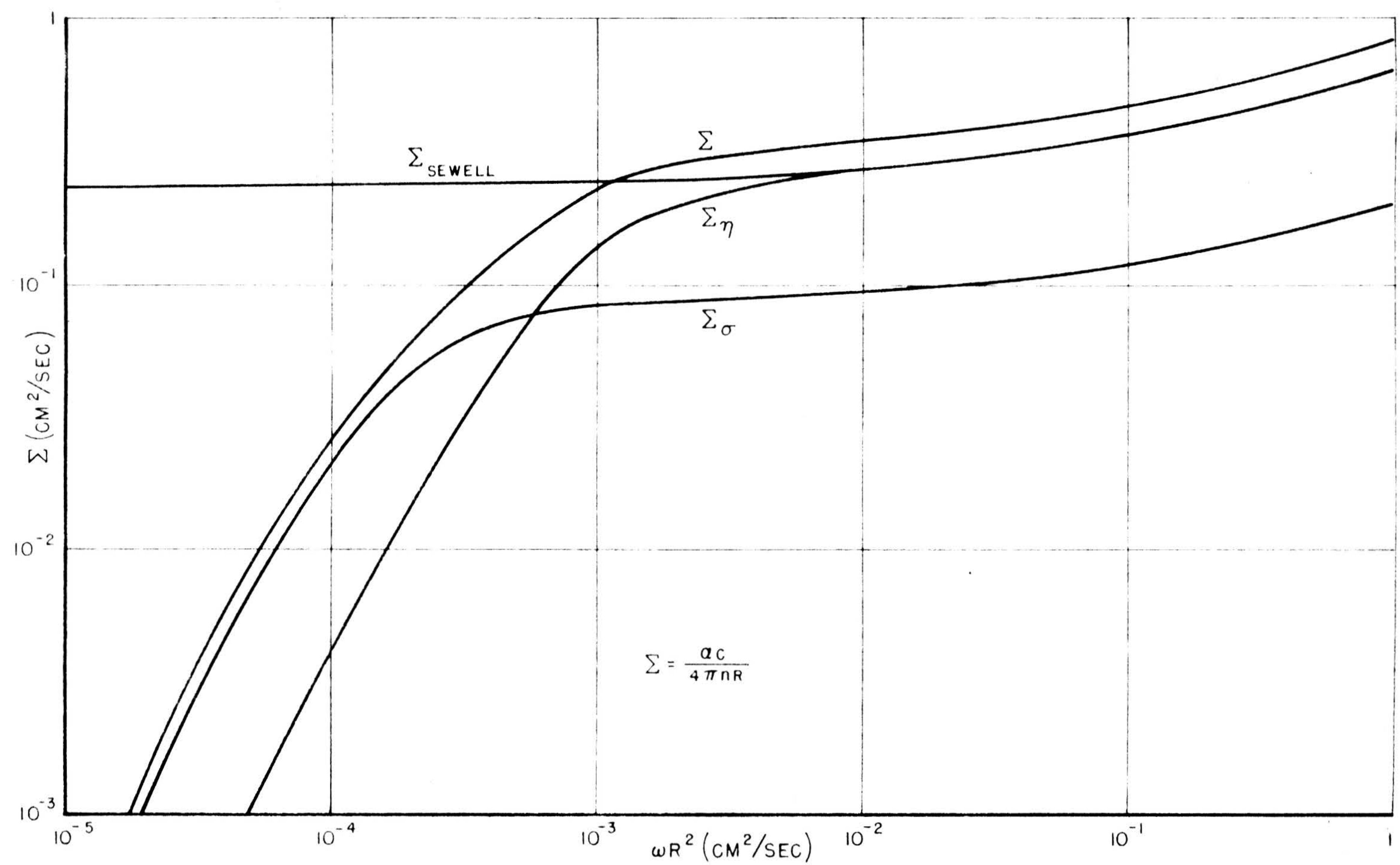

Fig. 1. Theoretical attenuation function for water drops in air (Sec. 20).

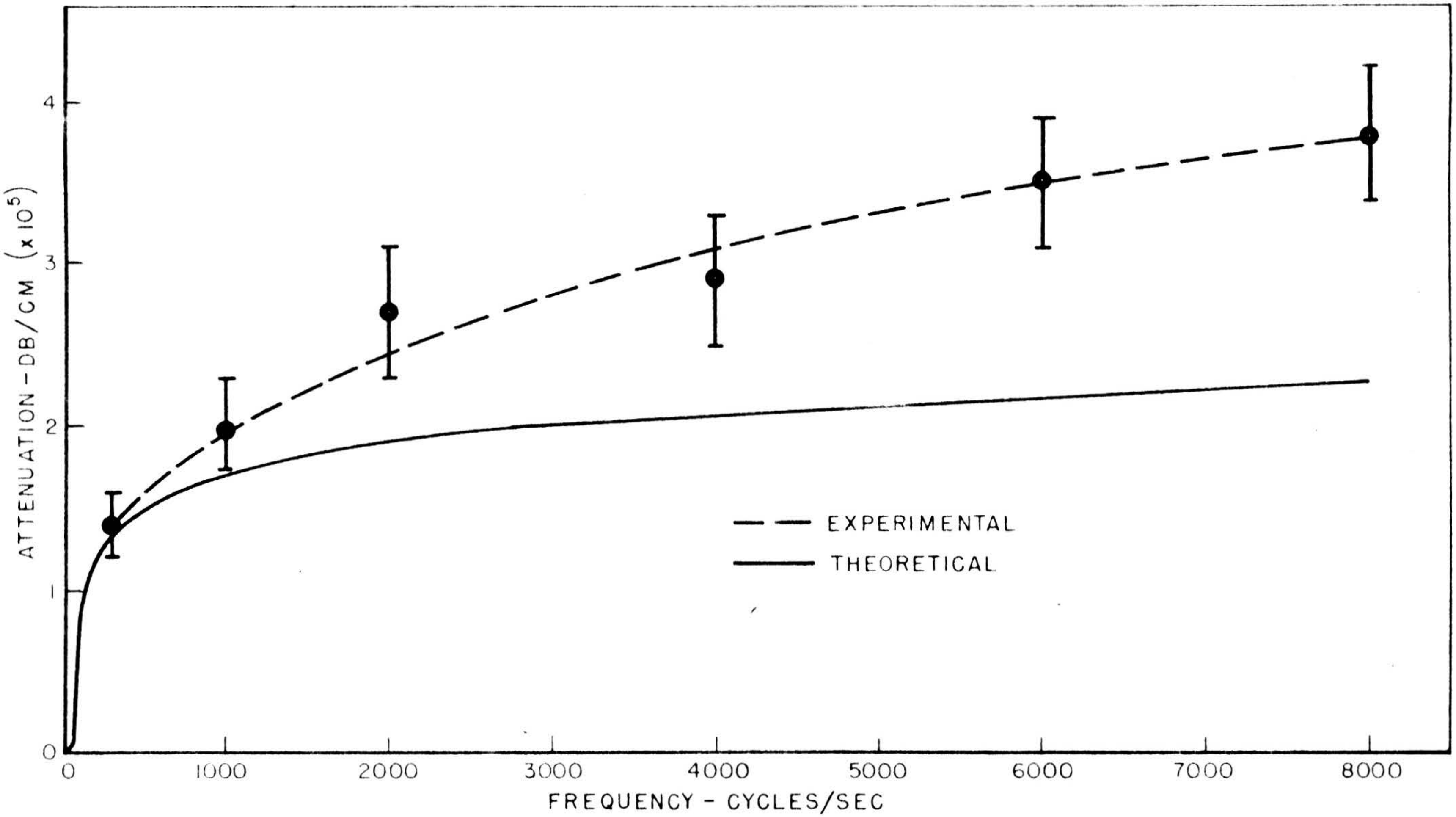

Fig. 2 Comparison of theoretical attenuation with Knudsen's experimental values for water drops in air (Sec. 21).

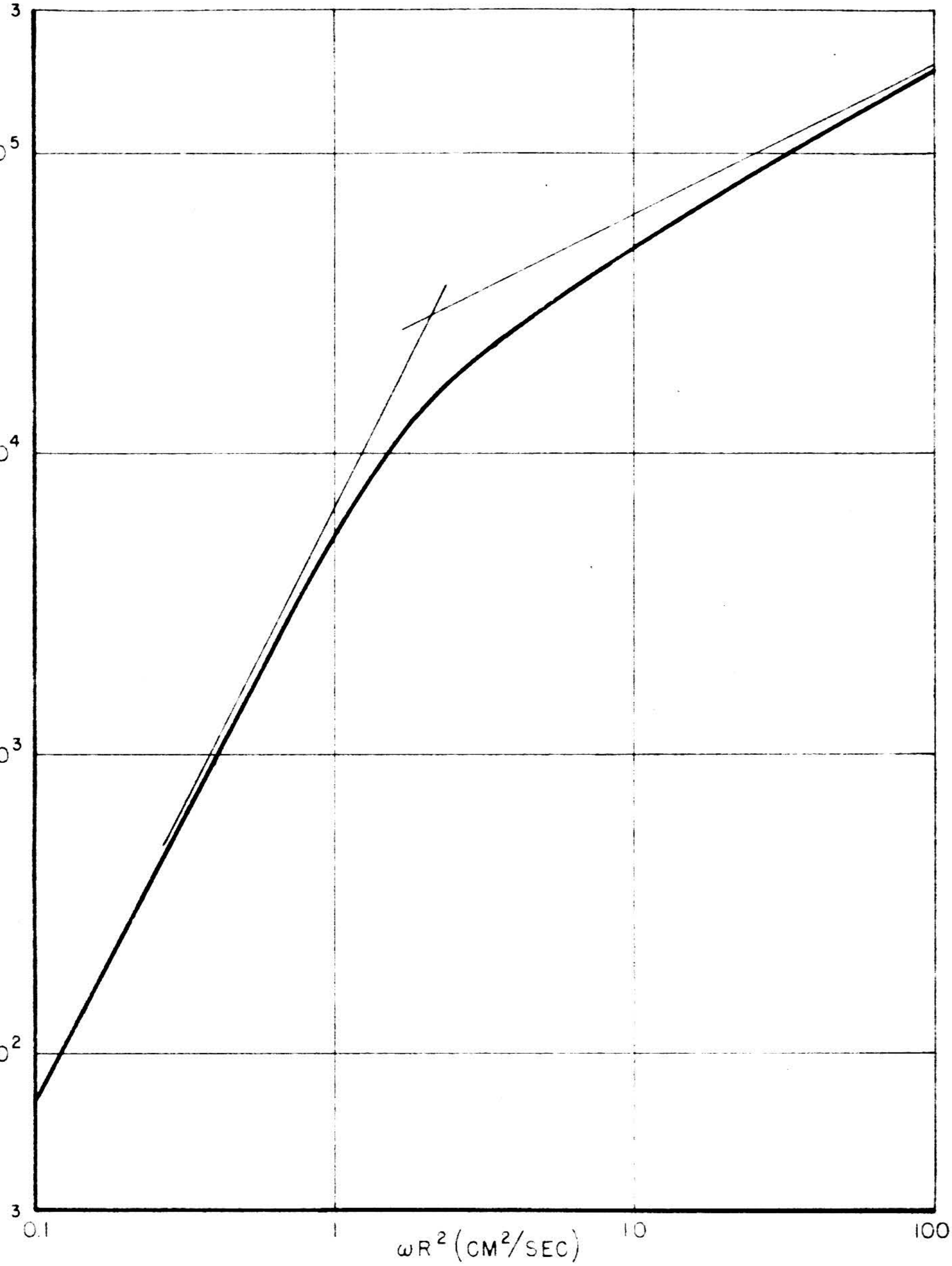

Fig. 3 Theoretical attenuation function for air bubbles in water (Sec. 22).