Macroscopically Dissipative Systems with Underlying Lossless Dynamics: Properties and Limits of Measurement

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Abstract

While some of the deepest results in nature are those that give explicit bounds between important physical quantities, some of the most intriguing and celebrated of such bounds come from fields where there is still a great deal of disagreement and confusion regarding even the most fundamental aspects of the theories. For example, in quantum mechanics, there is still no complete consensus as to whether the limitations associated with Heisenberg's Uncertainty Principle derive from an inherent randomness in physics, or rather from limitations in the measurement process itself, resulting from phenomena like back action. Likewise, the second law of thermodynamics makes a statement regarding the increase in entropy of closed systems, yet the theory itself has neither a universally-accepted definition of equilibrium, nor an adequate explanation of how a system with underlying microscopically Hamiltonian dynamics (reversible) settles into a fixed distribution.

Motivated by these physical theories, and perhaps their inconsistencies, in this thesis we use dynamical systems theory to investigate how the very simplest of systems, even with no physical constraints, are characterized by bounds that give limits to the ability to make measurements on them. Using an existing interpretation, we start by examining how dissipative systems can be viewed as high-dimensional lossless systems, and how taking this view necessarily implies the existence of a noise process that results from the uncertainty in the initial system state. This fluctuation-dissipation result plays a central role in a measurement model that we examine, in particular describing how noise is inevitably injected into a system during a measurement, noise that can be viewed as originating either from the randomness of the many degrees of freedom of the measurement device, or of the environment. This noise constitutes one component of measurement back action, and ultimately imposes limits on measurement uncertainty. Depending on the assumptions we make about active devices, and their limitations, this back action can be offset to varying degrees via control. It turns out that using active devices to reduce measurement back action leads to estimation problems that have non-zero uncertainty lower bounds, the most interesting of which arise when the observed system is lossless. One such lower bound, a main contribution of this work, can be viewed as a classical version of a Heisenberg uncertainty relation between the system's position and momentum. We finally also revisit the murky question of how macroscopic dissipation appears from lossless dynamics, and propose alternative approaches for framing the question using existing systematic

methods of model reduction.

Contents

Chapter 1

Introduction

Despite the now well-developed state of the theories of both quantum mechanics and statistical physics, with confirmation having come from countless physical experiments over the last century, there is surprisingly still a great deal of disagreement and confusion over some of the most fundamental aspects of the theories. In statistical mechanics, how to reconcile a microscopic world, described by time-reversible lossless dynamics, with a macroscopic one in which there is energy dissipation and in which processes seem to have a preferred direction in time, is still an open problem. Statistical mechanics is hugely successful in that it can make statements about the behavior of inconceivably large ensembles of particles through simple relations between just a handful of macroscopic variables. That there is such a 'reduction' in the description of very large systems is truly remarkable. However, much of this theory is predicated on the concept of a 'thermodynamic equilibrium,' and a consistent microscopic explanation of how such an equilibrium is reached, or even of how to properly define it, has been out of reach. Microscopically, if one assumes an ensemble of particles undergoing Hamiltonian dynamics, with a probability density on every particle's position and momentum, this probability distribution never reaches a steady state distribution by the mere fact that the dynamics are time-reversible. So, this naturally opens the question of how to properly define thermodynamic steady state from a microscopic perspective. One approach might be to define thermodynamic equilibrium as one in which the density of the system is close to a stationary distribution most of the time as in [1], and using arguments from ergodic theory, but it turns out that the recurrence times grow exponentially with the number of particles, so time averaging arguments would only hold on scales much longer than those over which physical observations are actually made. Other proposed explanations for increasing entropy include the argument that physical constants cannot be known to arbitrary precision, and entropy increase in time comes from this 'fuzzy' knowledge of the Hamiltonian [2].

The research outlined in this thesis is not so ambitious as to try to explain these sorts of deep gaps in the theory, but to perhaps get a better understanding of simpler problems from a purely dynamical systems perspective and examine similar questions like, under what conditions dissipative systems

can be approximated by lossless systems, and under what conditions very large 'random' lossless systems can be approximated by low order dissipative ones? The answer to the first question, in the case of linear systems, is answered by Sandberg, Delvenne and Doyle [3, 4]. The second question is still an active area of our research, and is examined in Chapter 3 of this thesis.

Examining the connection between high-dimensional lossless systems and lower-dimensional dissipative ones from a purely dynamical systems point of view gives rise to a chain of results that, at their end, make statements about the ability to be able to 'measure' one dynamical system via an interconnection with another. Although at first glance these results, presented in Chapter 2, might seem to represent too much of an abstraction, or to be based on assumptions in a measurement model that are too far removed from physical reality, it is precisely the fact that the assumptions made for these results are so minimalistic that makes them interesting. What we find is that even after stripping away all the constraints of physical laws, and looking at the measurement problem on its own as a dynamical systems problem, the interaction between 'measurement device' and 'observed system' is such that there is a back action effect during measurement which yields non zero lower bounds on the ability to make arbitrarily precise measurements. This sort of interaction, where the system under observation is influenced during measurement, is one that is very often given as an explanation for one interpretation of the Heisenberg Uncertainty Principle in quantum mechanics.

In many axiomatic formulations of quantum mechanics, the measurement process is addressed as an independent axiom, considered to be associated with an instantaneous "collapse" of the wave function to an eigenstate of the operator corresponding to whatever physical quantity is being measured [5]. Measurement of observables with operators that happen not to commute with the operator associated with the measurement in turn is not compatible with this process. An inherent problem with treating wavefunction collapse as an independent axiom is that doing so represents a sort of contradiction of another necessary axiom in any formulation of quantum mechanics, namely that the time evolution of closed systems is represented as an application of a unitary operator (without dissipation of energy). Of course, instantaneous wavefunction collapse is not a unitary operation, and the inclusion of an external observer means that the system is no longer closed. In more general theories of quantum measurement, such as those presented in [6, 7], uncertainty arises as a result of a phenomenon known as "decoherence," which is a byproduct of the entanglement between the *observed system, its environment, and the measurement device (observer).* While in this thesis we do not use quantum mechanics, we do adopt this decomposition of measurement into the three components above, albeit conceptually.

The theme of this thesis is one that has been repeated many times in the dynamical systems research literature, namely to take a physical idea and define mathematical objects that can be viewed as more general versions of the motivating physical concept. For instance, the physical notion of energy can be thought of as motivating Hamiltonian dynamical systems, or more generally,

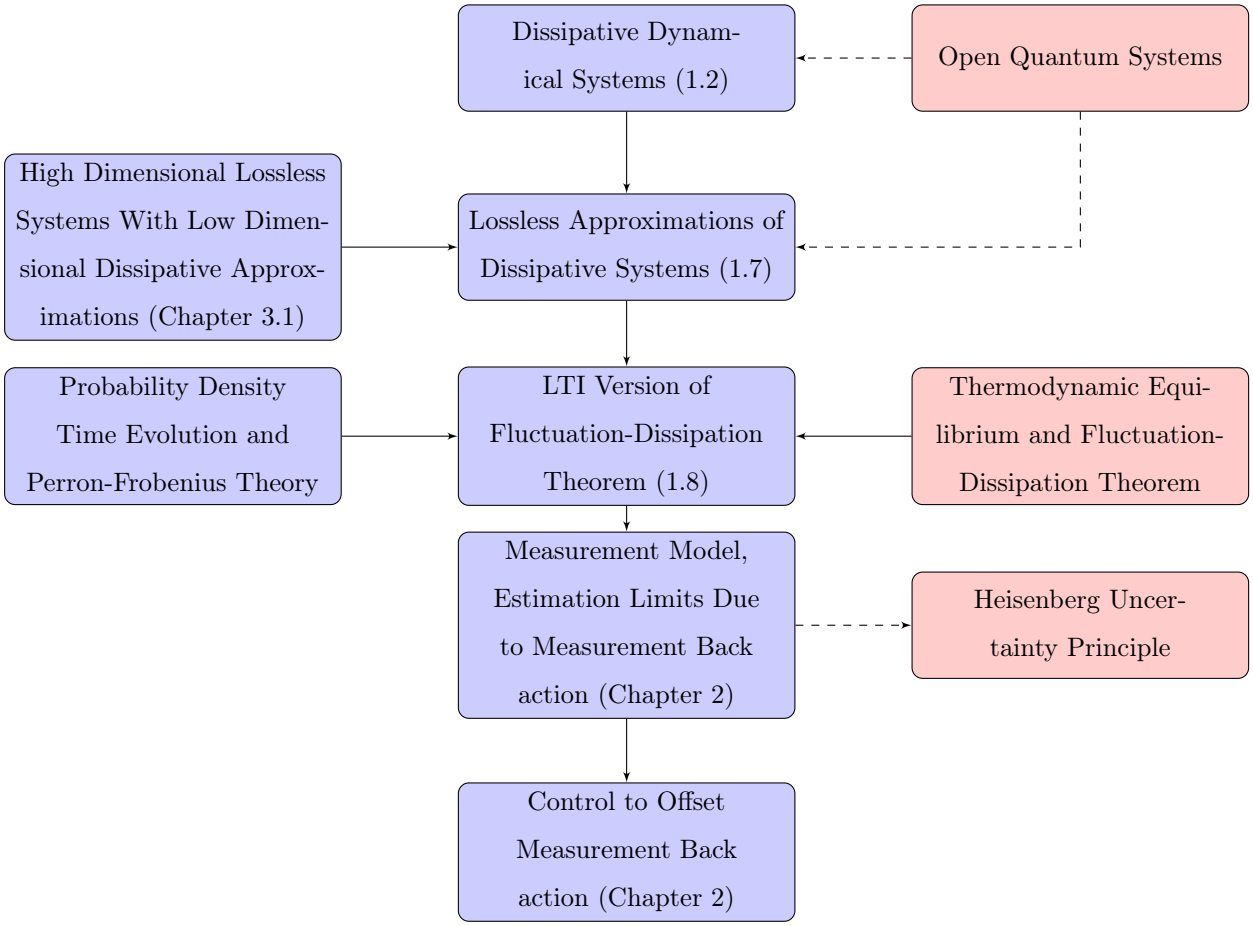
lossless dynamical systems. In [8, 9], an entire rich general theory based on energy conservation is built up from merely two definitions, which gives a precise meaning to the characterization of a system as 'dissipative.' We give an overview of this general theory later in this introduction, as it is a foundation for the rest of our work, but now briefly outline the logical interconnection between all the topics covered in this thesis.

1.1 Outline of Topics

Because the original motivation for our research was to connect ideas from a broad range of fields, in this introduction, we will very briefly touch on elements from statistical mechanics, quantum mechanics, systems theory, estimation and stochastic control that are relevant to the problem formulation in Chapter 2.

Starting with tools from dissipative dynamical systems theory presented later in this introduction, we introduce results from [3] concerning how large lossless systems can arbitrarily well approximate any dissipative system when viewed on any finite time horizon, before any recurrences. From this, we show the implications of assuming that any dissipative system has underlying microscopically lossless dynamics, the most important of which is the celebrated Fluctuation-Dissipation Theorem (FDT) known in statistical mechanics. A short summary of the derivation and implications of the FDT is given later in this introduction. The FDT imposes a noise source that is unavoidably associated with any dissipative element in a measurement model based on [4], which we analyze in Chapter 2 using basic estimation tools. We find that for this abstract measurement model, the observer inevitably perturbs the system being measured in such a way that there is a trade-off between the information extracted about the state of the observed system and the amount that the system is affected by the measurement. We analyze various methods of minimizing the effects of the measurement on the system of interest. In Chapter 3, we look at the approximations between microscopically lossless and macroscopically dissipative systems from a viewpoint that is in a sense is different from the result in [10].

The figure below summarizes the logical interconnection between the concepts treated in the rest of this thesis. Subjects in red boxes represent topics pertaining to the physical world, while those in blue are general mathematical abstractions. Solid lines connecting topics indicate that there is a direct mathematical connection between the subjects, made clear in this work, while dotted lines are used to show connections that are more motivating in nature.



1.2 Dissipative Dynamical Systems

The notions of dissipation and losslessness have been treated extensively in dynamical systems research, and varying definitions of these ideas have been given that heuristically coincide with the behavior of damped systems in nature. One approach, which is well-summarized in [11], is to look at closed systems and characterize whether a system is dissipative according to the flows of the system $\Phi^t(x)$. With this view, systems can be considered dissipative if there exists a bounded set Ω such that $\Phi^t(\Omega) \subseteq \Omega$, and for which any trajectory eventually enters Ω. Another, which is particularly interesting from the viewpoint of analyzing the time evolution of probability densities over the states of the system, is to consider what happens to phase space volume as the system evolves, in particular, looking at whether the Jacobian of the vector field is negative, positive or zero to describe whether a system is dissipative, non-dissipative or lossless. We will briefly comment on this matter in section 1.5 of this introduction.

The view of dissipation that we will take in most of this thesis is that outlined in the seminal work

[8], [9]. Readers will see that this manner of defining dissipative systems has a great deal in common with Lyapunov stability theory, and perhaps philosophically speaking, the theory that results could be considered more important, as it is used to make statements about the input-output properties of systems, rather than just addressing their behavior in isolation. Any physical experiment compares inputs to outputs, so not only is the theory introduced in this section necessary for the mathematical derivations of our work, it is also motivational in that it provides generalizations of physical concepts like energy and power input, in the same way that we hope to examine generalizations of the measurement process. We start with a series of definitions (stated for completeness) for treating input-output systems with dynamics that can be expressed in the state space form

$$\dot{x}(t) = f(x,u,t) \qquad x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m \tag{1.2.1}$$

$$y(t) = h(x,u,t) \qquad y(t) \in \mathbb{R}^p \tag{1.2.2}$$

where $u(t)$ and $y(t)$ are the input and output functions, respectively. We assume a general locally-integrable *work rate function* associated with the system, which is a function of both the input and output functions,

$$w(t) = w(u(t), y(t), t) \in \mathbb{R}. \tag{1.2.3}$$

With respect to this work rate function a dissipative system can be generally defined as follows,

Definition 1.2.1. A system is said to be ***dissipative*** with respect to $w(t)$ if there exists some function $S(x,t) \geq 0$ such that

$$S(x_0,t_0) + \int_{t_0}^{t_1} w(t)dt \geq S(x_1,t_1) \tag{1.2.4}$$

for all t_0, t_1, x_0, x_1 and $u(t)$, which takes the system from $x_0 \to x_1$.

Similarly, a general lossless system can be defined as follows,

Definition 1.2.2. A system is said to be ***lossless*** with respect to $w(t)$ if there exists some function $S(x,t) \geq 0$ such that

$$S(x_0,t_0) + \int_{t_0}^{t_1} w(t)dt = S(x_1,t_1) \tag{1.2.5}$$

for all t_0, t_1, x_0, x_1 and $u(t)$, which takes the system from $x_0 \to x_1$.

Note the sign convention is such that $w(t) > 0$ represents work being done *on* the system. An important point to keep in mind is that, in general, storage functions are *not* unique for a given

system and work rate function.

Two additional important definitions that define the maximum amount of 'work' that can be *extracted* from a system, or the minimum amount of work needed to be done on a system to change its state, are the *available storage* and the *required supply* functions,which are defined as follows,

Definition 1.2.3. The **available storage function** and **required supply function** $S_{avail}, S_{req} : \mathbb{R}^n \to \mathbb{R}$ are defined by the relations

$$S_{avail}(x) = \sup_{u(t) \text{ s.t. } x(0)=x \to x(t_f)=0} \int_0^{t_f} -w(t)dt \tag{1.2.6}$$

$$S_{req}(x) = \inf_{u(t) \text{ s.t. } x(0)=0 \to x(t_f)=x} \int_0^{t_f} w(t)dt. \tag{1.2.7}$$

An immediate result is that,

Theorem 1.2.4. *A system is dissipative iff* $S_{avail}(x) < \infty$.

This statement follows from the observation that $S_{avail}(x)$ is a possible storage function itself, satisfying the inequality in Eq.(1.2.4). That it satisfies Eq.(1.2.4) is seen by considering the process of taking the state x_0 to some state x_1 with total work done $\int_{t_0}^{t_1} w(t)dt$ and then extracting the available storage from state x_1, which by definition applied to the combined process, extracts less work than $S_{avail}(x_0)$. A similar argument can be used to prove that a bounded S_{req} implies dissipativity, which can be argued in the same manner to also be a storage function.

Rather remarkably, these simple and physically-intuitive definitions are enough of a starting point to allow for an entire theory of deep but easily described results, particularly in the case of linear time-invariant systems (LTI). Applied to linear systems, research in dissipative systems has given rise or been closely related to extremely important results in control theory; in particular, the theory of linear matrix inequalities (LMIs) and the Kalman-Yakubovich-Popov Lemma. Readers who are interested in these connections are encouraged to read [12] and [13]. Since a majority of the analysis in this thesis will be done on LTI systems in order to take advantage of all the tools of systems theory, we will rather pedagogically state that by linear time-invariant dynamical systems, we are to mean systems that can be described by a single *transfer function* $g(t)$ which entirely describes the mapping from all inputs $u(t) \to$ outputs $y(t)$ via a convolution integral, with the input-output relationship having the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \quad x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m \\ y(t) &= Cx(t) + Du(t) \qquad\qquad\quad y(t) \in \mathbb{R}^p \end{aligned} \tag{1.2.8}$$

where A, B, C, D are time-invariant matrices of appropriate dimension. For the rest of this work,

we will also assume that the work rate function is always quadratic,

$$w(t) = \begin{pmatrix} u^T & y^T \end{pmatrix} \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} u \\ y \end{pmatrix}. \tag{1.2.9}$$

Under these conditions, and assuming the system is controllable, the optimization problem associated with finding the optimal $u(t)$ in Definition 1.2.3 is one in which *the objective function is quadratic and the dynamics constraint is linear, so we would expect the optimal values to be quadratic functions.* We could solve this optimization explicitly, but since the actual solution is of no use to us, we omit it but mention that the methodology to solving it is almost identical to the optimization done in Section (3.1). This observation, along with Theorem 1.2.4, implies that the system is dissipative if and only if there exists a storage function of the form $S(x) = x^T \Sigma x$ where $\Sigma = \Sigma^T$. A simple criterion for checking whether a system is dissipative immediately follows.

Theorem 1.2.5. *A dynamical system of the form 1.2.8, for which the pair (A, B) is reachable, with an associated work rate function given by 1.2.9, is dissipative iff the LMI in Σ given by*

$$\begin{pmatrix} 0 & B^T\Sigma \\ \Sigma B & A^T\Sigma + \Sigma A \end{pmatrix} \leq \begin{pmatrix} I & D^T \\ 0 & C^T \end{pmatrix} \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} I & 0 \\ D & C \end{pmatrix} \tag{1.2.10}$$

is feasible.

Proof. If the system is dissipative, Eq.(1.2.4) implies that in particular $\dot{S}(x) \leq w(t)$, so that

$$\dot{S}(x) = (Ax + Bu)^T \Sigma x + x^T \Sigma (Ax + Bu) = \begin{pmatrix} u^T & x^T \end{pmatrix} \begin{pmatrix} 0 & B^T\Sigma \\ \Sigma B & A^T\Sigma + \Sigma A \end{pmatrix} \begin{pmatrix} u \\ x \end{pmatrix} \tag{1.2.11}$$

$$\leq \begin{pmatrix} u^T & y^T \end{pmatrix} \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} u \\ y \end{pmatrix} \tag{1.2.12}$$

for all (x, u, y). The right-hand side can be rewritten by eliminating $y = Ax + Du$,

$$\begin{pmatrix} u^T & y^T \end{pmatrix} \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} u \\ y \end{pmatrix} = \begin{pmatrix} u^T & x^T \end{pmatrix} \begin{pmatrix} I & D^T \\ 0 & C^T \end{pmatrix} \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} I & 0 \\ D & C \end{pmatrix} \begin{pmatrix} u \\ x \end{pmatrix}. \tag{1.2.13}$$

Combining this with 1.2.11 gives the desired result. The converse is immediate, as the feasibility of the given LMI implies $\dot{S}(t) \leq 0$, which implies the dissipation inequality 1.2.4. □

Example 1.2.6. Among quadratic work rate functions, an illustrative form to look at is the product $w(t) = u^T(t)y(t)$, where $u(t)y(t) \in \mathbb{R}^m$. This quantity might, for instance, represent the power going into a circuit network where $u(t)$ is the vector of input currents and $y(t)$ are the output voltages.

For such a $w(t)$, the LMI from theorem 1.2.5 simplifies to ($Q = R = 0$ and $S = I/2$)

$$\begin{pmatrix} -D - D^T & B^T\Sigma - C \\ \Sigma B - C^T & A^T\Sigma + \Sigma A \end{pmatrix} \leq 0, \tag{1.2.14}$$

which is a well known LMI in the controls systems literature known as the positive-real lemma. If we assume $D + D^T > 0$, this inequality can additionally be transformed by taking the Schur complement of the block $-D - D^T$ to give

$$A^T\Sigma + \Sigma A + (\Sigma B - C^T)(D + D^T)^{-1}(B^T\Sigma - C) \leq 0. \tag{1.2.15}$$

The above matrix inequalities can be directly related to frequency domain conditions for a system to be dissipative via the Kalman-Yakubovich-Popov Lemma, which the interested reader can find the details of in [13]. Since going into this would be too far of an aside for an introduction, we will state that the LMI conditions stated in Theorem(1.2.5) can be shown to be equivalent to the condition

$$\begin{pmatrix} I & G^T(-j\omega) \end{pmatrix} \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} I \\ G(j\omega) \end{pmatrix} \geq 0. \tag{1.2.16}$$

For the specific work rate function given by $w(t) = u^T(t)y(t)$, this becomes the well-known condition from the theory of electrical networks that states that a system is necessarily dissipative (passive) if and only if

$$G(i\omega) + G^T(-i\omega) \geq 0 \quad \text{dissipative system.} \tag{1.2.17}$$

Rather expectedly, *lossless* systems can be treated in an almost identical manner, where the only changes in the theory involve substituting the set of aforementioned LMIs and frequency domain inequalities, to equalities. For lossless systems with above work rate function, the frequency domain condition for a system to be lossless is

$$G(i\omega) + G^T(-i\omega) = 0 \quad \text{lossless system.} \tag{1.2.18}$$

We will make use of these properties throughout the rest of this thesis, specifically in our examples. Before concluding this section, we highlight one additional result regarding systems that do not have a direct term ($D = 0$ in 1.2.8). For such systems, the LMI given in Eq.(1.2.5) is equivalent to the conditions

$$A^T\Sigma + \Sigma A \leq 0 \tag{1.2.19}$$

$$\Sigma = \Sigma^T > 0 \tag{1.2.20}$$

$$\Sigma B = C^T. \tag{1.2.21}$$

Factoring $\Sigma = QQ^T$, where Q is invertible, and applying a state transformation $\tilde{x} = Qx$ yields a system with system matrices $(\tilde{A} = QAQ^{-1}, \tilde{B} = QB, \tilde{C} = CQ^{-1})$ and storage function $\tilde{\Sigma} = I$. Conditions 1.2.19, 1.2.21 then imply that a dissipative LTI system without a direct term can necessarily be written in the form

$$\begin{aligned} \dot{x}(t) &= (J-K)x(t) + Bu(t) \\ y(t) &= B^T x(t) \end{aligned} \tag{1.2.22}$$

where J, D are skew-symmetric $J = -J^T, D = -D^T$, $K > 0$, (J, B) is a controllable pair, and where the storage function is given by $\tilde{\Sigma}(x) = x^T x$.

Similarly, under the same conditions, a lossless system without a direct term can be represented in the form

$$\begin{aligned} \dot{x}(t) &= Jx(t) + Bu(t) \\ y(t) &= B^T x(t) \end{aligned} \tag{1.2.23}$$

and the energy function is given by $\tilde{\Sigma}(x) = x^T x$.

While these conditions and definitions are sufficient to pose and solve the types of problems we examine in this thesis, an in-depth continued introduction to dissipative systems can be found both in the earlier mentioned references, as well as [14].

1.3 Lossless Approximations of Dissipative Systems on Finite Time Horizons

With a formal characterization of dissipation and losslessness in hand, we can begin to formulate and (hopefully) begin to answer how macroscopic dissipation appears. What we will find is that as a byproduct of assuming systems to be microscopically lossless, that uncertainty about the state of these lossless systems has implications on the ability to extract information about the states of dynamical systems via interconnections with other dynamical systems. Before stating the result in [3], we present an additional result that gives a time domain condition on the input-output

relationship of dissipative systems. In some literature, this condition is actually the starting point as a *definition* of a system being passive, and it is equivalent to the definition given earlier in this introduction. For a proof, see [8].

Theorem 1.3.1. *If the LTI dynamical system is reachable, the system is dissipative with respect to the work rate function $w(t) = u^T(t)y(t)$ if and only if for all $u(t)$, $t_f \geq 0$*

$$\int_0^{t_f} u^T(t)y(t)dt \geq 0. \tag{1.3.1}$$

An immediate observation deduced from 1.3 is that 'resistors,' by which we mean systems with input-output relationships $y(t) = ku(t)$, are dissipative if and only if $k > 0$. One way in which macroscopic 'resistors' can appear from lossless dynamics, is given in the following theorem, which is also a justification for the measurement model introduced in Chapter 2.

Theorem 1.3.2. *If the LTI system described by the input-output relation $y(t) = \int_0^t g(t-s)u(s)ds$ is **dissipative**, for a given fixed time interval $[0, \tau]$ there exist **lossless** approximations of it, $y_N(t) = \int_0^t g_N(t-s)u(s)ds$, of state dimension N, such that*

$$\int_0^{\tau} ||g(s) - g_N(s)||^2 ds \leq \epsilon(N, \tau) \tag{1.3.2}$$

such that ϵ can be made arbitrarily small for sufficiently large N.

Simply stated, the proof of this result in [4] makes use of the fact that any impulse response $g(t)$ can be approximated arbitrarily well over a fixed time interval by a Fourier series expansion in sinusoids with base frequency $\frac{\pi}{T}$. While in Chapter 3 we give an alternative view on how systems approximating ideal resistors can arise from lossless elements, to demonstrate the type of construction used in the proof of Theorem (1.3.2), we present the following example.

Example 1.3.3. Lossless Approximation of a Resistor

The approximation of the resistor transfer function can be done by approximating an impulse train by the truncated series

$$g_N(t) = \left(\frac{k}{2\tau} + \sum_{l=1}^{N} \frac{k}{\tau} \cos(l\omega_0 t) \right)^+ \qquad \omega_0 = \frac{\pi}{\tau} \tag{1.3.3}$$

with realization

$$\dot{x}(t) = J_N x(t) + B_N u(t) \qquad J_N = -J_N^T \tag{1.3.4}$$

$$y(t) = B_N^T x(t) \tag{1.3.5}$$

where

$$J_N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \Omega_N \\ 0 & -\Omega_N & 0 \end{pmatrix} \tag{1.3.6}$$

$$\Omega_N = \text{diag}(\omega_0, 2\omega_0, \ldots, N\omega_0) \tag{1.3.7}$$

$$B_N^T = \sqrt{\frac{k}{\tau}} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \ldots & 0 & 1 & \ldots & 1 \end{pmatrix}. \tag{1.3.8}$$

This example represents a very extreme situation where a high-dimensional lossless system approximates a memoryless one of zero state dimension, and as such might give us insight into the microscopically lossless $\rightarrow$ macroscopically dissipative transition. What we can extract from this example, at least heuristically, is that the lossless system should be able to be directly excited across a wider and wider range of frequencies.

1.4 Separation of System and Environment in Quantum Mechanics

Motivated by the previous section, we outline how quantum and statistical physics theories treat the appearance of dissipation in a system embedded as part of a larger lossless system (with its environment). For the quantum mechanics perspective, the density matrix formalism is introduced and it is explained how the dynamics of open systems are obtained by 'tracing out' states of the environment. A thorough introduction to this approach can be found in [5] and [15]. As a completely classical analogue of this, we also very briefly discuss the time evolution of probability density functions from given trajectory dynamics, and how closed systems with Hamiltonian dynamics do not converge to a stationary distribution, a point that is somewhat of a thorn in equilibrium statistical mechanics. Finally, we contrast these two 'distributions' views of dissipation from the 'trajectories' view presented in the previous section. While treating the topics in this section is somewhat removed from the main results in Chapter 2 and 3, it does serve a purpose of giving another view of how dissipation is accounted for when viewing subsystems of larger lossless systems.

Quantum mechanics postulates that a quantum system in a *pure state* is described by a *normalized state vector* in a Hilbert space denoted $|\psi\rangle$. In addition, it postulates that the time evolution of this vector $|\psi(t)\rangle$ is simply given by a unitary transformation $U(t, t_0)$ on that state, the transformation being an element of a semigroup with infinitesimal generator $\frac{1}{i\hbar}H$, where H is a Hermitian operator. Two obvious observations from this is that the norm, denoted in this notation as $(\langle\psi(t)|\psi(t)\rangle)^{1/2}$, is preserved, and that $H|\psi(t)\rangle = U(t, t_0)H|\psi(t_0)\rangle$. If we associate $\langle\psi(t)|H|\psi(t)\rangle)$

as the total energy of the system and $\langle\psi(t)|\psi(t)\rangle)^{1/2}$ as a total probability, we can just see this postulate as specifying conservation of energy and probability.

The problem of measurement of quantum mechanics remains unsolved according to most standard interpretations, which take as a postulate the following statement: that any physical observable quantity of the system has a corresponding operator on the Hilbert space of quantum states, and that the result of any measurement yields an outcome that is necessarily an eigenvalue of the operator, and takes the system from $|\ \psi\rangle$ to an eigenstate of that operator. Part of the motivation for our later work is taking **a classical view of the measurement process** and seeing how the resulting limitations parallel those that arise from the quantum mechanical formulation. This view is explained in more detail later.

If we look at the *density operator* formed as the outer product

$$\rho_i := |\psi_i\rangle\langle\psi_i| \tag{1.4.1}$$

that obviously has all the information content of $|\psi_i\rangle$, we can consider more general quantum states that are *statistical mixtures* of *pure states* formed from the weighted sum of outer products of pure states

$$\rho = \sum_{i=1}^{N} p_i \rho_i \qquad \text{where} \quad \sum_{i=1}^{N} p_i = 1. \tag{1.4.2}$$

Remark 1.4.1. At this point, it is important to avoid any confusion and note that the statistical nature of this weighted sum is different from that of the probabilistic nature of the measurement outcome on given $|\psi\rangle$.

The time evolution of the density operator $\rho(t)$ is derived immediately from the unitary time evolution postulate on the individual states $|\psi_i\rangle$, which states that

$$\rho_i(t) = U(t,t_0)|\psi_i(t_0)\rangle\langle\psi_i(t_0)|U^{-1}(t,t_0). \tag{1.4.3}$$

Consequently,

$$\begin{aligned}\dot{\rho}_i(t) &= \dot{U}(t,t_0)|\psi_i(t_0)\rangle\langle\psi_i(t_0)|U^{-1}(t,t_0) + U(t,t_0)|\psi_i(t_0)\rangle\langle\psi_i(t_0)\dot{U}^{-1}(t,t_0) \\ &= \frac{1}{i\hbar}HU(t,t_0)|\psi_i(t_0)\rangle\langle\psi_i(t_0)|U^{-1}(t,t_0) - \frac{1}{i\hbar}U(t,t_0)|\psi_i(t_0) \\ &= \frac{1}{i\hbar}[H,\rho(t)]\end{aligned} \tag{1.4.4}$$

where the commutator $[H,\rho(t)] = H\rho(t) - \rho(t)H$. Since for any function f, $\frac{d}{dt}Tr[f(H)\rho(t)] = 0$, we see the 'losslessness' in the dynamics in the density operator picture simply translates to the trace

conservations $Tr[H\rho(t)] = \text{constant}$ and $Tr[\rho(t)] = 1$.

While 1.4.4 represents the dynamics of a closed system, the dynamics of open systems are necessarily treated with an additional postulate. Specifically, quantum mechanics assumes that a composite system made up of two subsystems A and B is an element of the Hilbert space that is the tensor product of the Hilbert space in which the states of A and B individually lie, $H_{AB} = H_A \otimes H_B$, the feature that is responsible for quantum entanglement. Linear operators $\mathcal{O}_{AB}$ on H_{AB} are identically formed as a product of the operators $\mathcal{O}_{AB} = \mathcal{O}_A \otimes \mathcal{O}_B$, and the trace operator is then defined as the sum, as it normally would be,

$$tr[\mathcal{O}_{AB}] = \sum_{m,n} \langle m_A, n_B | \mathcal{O}_{AB} | m_A, n_B \rangle \tag{1.4.5}$$

where $|m_A\rangle, |n_B\rangle$ form an orthonormal basis. For the case of the density operator of the composite system ρ_{AB}, the density operator associated with either subsystem $\rho_{A,B}$ can then be obtained as a 'partial trace" on ρ_{AB} where all the basis states not in that subsystem are integrated out

$$\rho_A = Tr_B[\rho_{AB}] = \sum_n \langle \langle m_A, n_B | \rho_{AB} | m_A, n_B \rangle. \tag{1.4.6}$$

The question now is, what are the dynamics of $\rho_A(t)$ individually? We can be virtually certain that the dynamics will usually no longer be unitary with a trace energy conservation law, but how will the 'dissipative' terms show up in the dynamics? The question has the completely classical analogue of the question of what do the dynamics of the probability distribution of a system look like when projected onto a subspace that corresponds to the states of the subsystem of interest? In the previous section, we looked at dissipation from a trajectories viewpoint, but we would like to know how the two pictures relate. In the special case in which the subsystem of interest interacts with a much larger composite system that has already reached an equilibrium state, that is, has evolved to reach a stationary probability distribution, the **fluctuation dissipation theorem** will tell us exactly that.

For the purpose of completeness, we will look at the form of the dynamics of ρ_A, but will be unsatisfied in the general difficulty in actually integrating the expression. If we take

$$H_{AB} = H_A \otimes I_B + I_A \otimes B + H_{ABcross} \tag{1.4.7}$$

where $I_{A,B}$ is the identity operator on the respective subsystem, and $H_{ABcross}$ is the portion of the Hamiltonian corresponding to the subsystem interactions, then

$$\begin{aligned}\dot{\rho}_A(t) &= Tr_B[\dot{\rho}_{AB}(t)] \\ &= \frac{1}{i\hbar}[H_{AB}, \rho_{AB}(t)] \\ &= \frac{1}{i\hbar}Tr_B[[(H_A, \rho_A(t)] \otimes \rho_B(t)] + \frac{1}{i\hbar}Tr_B[\rho_A(t) \otimes [I_B, \rho_B(t)]] + \frac{1}{i\hbar}Tr_B[[H_{ABcross}, \rho_{AB}(t)]] \\ &= \underbrace{\frac{1}{i\hbar}[H_A, \rho_A(t)]}_{\text{unitary dynamics}} + \underbrace{\frac{1}{i\hbar}Tr_B[[H_{ABcross}, \rho_{AB}(t)]]}_{\text{dissipative term}}.\end{aligned} \tag{1.4.8}$$

Solving the above *master equation* is generally done with major simplifying assumptions, as those taken in [6] and [7], the most important being the Markov property that $\rho_A(t+dt)$, given $\rho_A(t)$, is independent of past values of $\rho_A(s)$, in which case the master equation can be shown to reduce to a 'dissipative' *Lindblad* master equation. This same simplification is also used in deriving the fluctuation-dissipation theorem that is the focus of the next section, and typically involves assuming that system B is much 'larger' than A.

1.5 Time Evolution of Probability Densities

As a classical analogue of the density matrix dynamics, at this point we introduce the *Perron-Frobenius* operator P associated with a given mapping $y = h(x)$ of some random variable 'x.' which maps $P_h : \rho_x \to \rho_y$, where ρ_x, ρ_y are the density measures for the random variables x and y [16]. For example, in the simple special case in which h is one-to-one and continuously differentiable, $P_h\rho_x(y) = \frac{1}{|J_h|}\rho_x(h^{-1}(y))$ where $|J_h|$ is the determinant of the Jacobian of h.

If the variable x_0 is evolved according to the *deterministic dynamics*

$$\dot{x} = f(x), \tag{1.5.1}$$

then it is a standard result that the dynamics of the density $\rho_x(t) = P\rho_{x_0}$ evolve according to the advection PDE

$$\frac{\partial\rho(x,t)}{dt} = -\sum_{i=1}^{N}\frac{\partial(\rho(x)f_i(x))}{\partial x_i}. \tag{1.5.2}$$

[17]. Suppose now that we are looking specifically at the time evolution of a **closed** Hamiltonian system and considering how the probability density on the entire system evolves. The deterministic dynamics are given by

$$\dot{x} = \begin{pmatrix}\dot{q}\\ \dot{p}\end{pmatrix} = \begin{pmatrix}\frac{\partial H}{\partial p}\\ -\frac{\partial H}{\partial q}\end{pmatrix}. \tag{1.5.3}$$

Using 1.5.2,

$$\frac{\partial \rho(q,p,t)}{dt} = -\left[\sum_{i=1}^{N} \frac{\partial \rho}{\partial q_i}\frac{\partial H}{\partial p_i} - \rho\frac{\partial^2 H}{\partial p_i \partial q_i} - \frac{\partial \rho}{\partial p_i}\frac{\partial H}{\partial q_i} + \rho\frac{\partial^2 H}{\partial q_i \partial p_i}\right] \tag{1.5.4}$$

$$= -\left[\sum_{i=1}^{N} \frac{\partial \rho}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial \rho}{\partial p_i}\frac{\partial H}{\partial q_i}\right] \tag{1.5.5}$$

$$:= -\sum_{i=1}^{N} \{\rho, H\}_i \tag{1.5.6}$$

which is the *classical Liouville equation*, where the object in Eq.(1.5.6) is the well-known Poisson bracket for canonical coordinates, the classical analogue for the commutator in Eq.(1.4.4). In order, then, for a density ρ_* to be a *stationary distribution* (a fixed point of the Perron-Frobenius operator for these dynamics), it must necessarily satisfy

$$\{\rho_*, H\} = 0, \tag{1.5.7}$$

or in other words, ρ_* must be a function of the Hamiltonian itself.

At this point, perhaps the most troubling issue in statistical mechanics becomes abundantly clear. Since the microscopic dynamics are Hamiltonian, they are time-reversible. That is, the dynamics are time reversed by the mapping $(q,p) \to (q,-p)$. So, while Eq.(1.5.7) characterizes all the stationary distributions, 1.5.5 does not explain how a system evolves from an arbitrary non-stationary distribution to one that is stationary. Stationary distributions are not attractors. The fundamental assumption of equilibrium thermodynamics itself is that the system distribution is stationary (in particular, the Boltzman distribution). Since this is not related to our work, we will say that readers interested in this paradox are encouraged to read [18], [19], [20]. A systems theory attempt to explain this discrepancy can be found in [1].

One feature that we can take from Eq. (1.5.2) is the suggestion, at least, that dissipative/lossless systems can be characterized by how phase-space is transformed under the given dynamics. What allowed the simplification from 1.5.4 to 1.5.5 is the fact that areas of phase space regions remain the same for all time. We can see this more generally as the case where the term in Eq.(1.5.2) given by

$$\rho(x)\sum_{i=1}^{N} \frac{\partial f_i(x)}{\partial x_i} = 0. \tag{1.5.8}$$

i.e., the divergence of the vector field of the dynamical system is zero, or $det(J_{h:x(t)\to x(t+\tau)}) = 1$, where $J_{h:x(t)\to x(t+\tau)}$ is the Jacobian of the system dynamics mapping. While it would be nice to broadly identify systems as lossless/dissipative based on the Jacobian of the system dynamics, unfortunately this is not the case (see [11]). Nevertheless, for LTI systems, this viewpoint applies trivially for lossless systems, as the divergence condition in Eq.(1.5.8) implies $Tr(A) = 0$, or for

dissipative dynamical systems, that they are lossless, and can be written in the form given in Eq.(1.2.23).

1.6 Fluctuation-Dissipation Theorem

The basic tools and results presented so far in this introduction are sufficient to rather generally explain a fundamental result in physics that relates the linearized response of a dissipative system to statistical properties of the variables of that system. The outline of a mathematical derivation of this statement, the *fluctuation-dissipation theorem*, is included here both because it is easily accessible and because a simplified version of it, for linear time-invariant systems, is a fundamental component of our analysis in Chapter 2.

The root of the theorem is directly connected to a subject covered in the previous sections, that is, how to analyze open systems. Applying a method similar to the separation of system and environment as done in the quantum master equation, we want to be able to better understand, classically, how dissipation arises and manifests itself in the density evolution of subsystems that have been projected out from the dynamics of a larger closed system.

The *fluctuation dissipation theorem* partially addresses this by considering a Hamiltonian system with Hamiltonian H_0, which is *assumed at the outset* to be in an equilibrium distribution (fixed point of the Liouville equation) that could be of the particular form of the Boltzman distribution

$$\rho_0 = \frac{e^{-k_B T H_0}}{Z}, \tag{1.6.1}$$

where Z is a normalization factor [20]. Justification for this particular (of the infinitely many) fixed point of the Liouville equation comes from 'canonical ensemble' arguments (in which the entropy of a composite system composed of the system of interest and a much larger reservoir, is expanded to first order).

If at time t_0 the system is perturbed by an external force $F_{ext}(t)$, and G is its general conjugate quantity, then

$$H_0 \to H_0 - GF(t) := H_0 + H_{ext}(t) \tag{1.6.2}$$

where $H_{ext}(t) \ll H_0$. Expanding $\rho_{sys}(t)$ to first order around ρ_0

$$\rho_{sys}(t) = \rho_0 + \rho_\Delta(t) \tag{1.6.3}$$

and using the Liouville equation, we find that

$$\dot{\rho}_\Delta(t) = -[H_0, \rho_\Delta(t)] - [H_{ext}(t), \rho_0] - [H_0, \rho_0] - [H_{ext}(t), \rho_\Delta(t)] \tag{1.6.4}$$

$$= -[H_0, \rho_\Delta(t)] - [H_{ext}(t), \rho_0] \tag{1.6.5}$$

$$:= C_0\rho_\Delta(t) + C_{ext}(t)\rho_0 \tag{1.6.6}$$

where $C_k : \rho \to [H_k, \rho]$ is skew-Hermitian, where the third term in 1.6.4 is identically zero (by assumption on ρ_0), and where the fourth term is of second order and neglected. After integrating the above equation, we get

$$\rho_\Delta(t) = \int_{t_0}^{t} e^{C_0(t-s)} C_{ext}(s)\rho_0 ds. \tag{1.6.7}$$

$$\tag{1.6.8}$$

If we wanted to calculate the expected value of some arbitrary function of the internal variables $V(q,p)$, the mean value of V due to the application of the perturbing force would in this approximation be

$$\mathbb{E}\, V_\Delta(t) = Tr[V\rho_\Delta(t)] = \int_{t_0}^{t} ds \int_{\{q,p\}} V(q,p) e^{C_0(t-s)} C_{ext}(s)\rho_0 dqdp \tag{1.6.9}$$

$$= \int_{t_0}^{t} ds \int_{\{q,p\}} \rho_0 C^*_{ext} e^{C_0(s-t)} V(q,p) dqdp \tag{1.6.10}$$

since the trace operation is invariant under the transpose operation, and C_0 is skew-Hermitian. Repeating our assumption that the applied force is small enough to not affect the conjugate quantity's value, we can say $G(t) = G(t_0)$ and

$$\mathbb{E}\, V_\Delta(t) = \int_{t_0}^{t} ds \int_{\{q,p\}} \rho_0 C^*_{ext}(s) V(s-t) dqdp \tag{1.6.11}$$

$$= \int_{t_0}^{t} F(s) ds \int_{\{q,p\}} \rho_0 [G(t_0), V(s-t)] dqdp \tag{1.6.12}$$

$$= \int_{t_0}^{t} F(s)\phi_{VG}(t-s) ds \tag{1.6.13}$$

where $[G(t_0), V(s-t)]$ is the commutator as used before, $F(s)$ was removed from the integration over the system variables since it is an external force, and the following function definitions were made

$$V(t) \equiv e^{C_0 t} V \tag{1.6.14}$$

$$\phi_{VG}(t) \equiv \mathbb{E}\,[G(t_0), V(t)]. \tag{1.6.15}$$

This is a form of the *fluctuation-dissipation theorem* that shows the direct relationship between two quantities

1. The time evolution of the *the mean* difference in the value of any function of the internal variables of the composite system from its steady mean value had the system not been perturbed.

2. The correlation function $\phi_{VG}(t)$, which is the correlation between the variables conjugate to the force at the time when the external force was applied, G_0, and the function $V(t)$.

Specifically, the linear response of the mean of $V(q, p)$ with respect to the applied external force has impulse response $\phi_{VG}(t)$.

This particular presentation and derivation of the fluctuation-dissipation theorem is due to Kubo [21], and is actually a more general statement about a phenomenon first observed in the 19th century by Robert Brown, who noticed the random motion of pollen particles floating in water [22]. Various mathematical descriptions given separately by Einstein, Smoluchowski and Langevin derived the squared mean of the displacements of these particles in time, and specifically gave a relationship between the resistive properties (drag due to the viscosity of the liquid) of the motion, the temperature of the liquid and the magnitude of the fluctuations. Similarly, in 1927, John Johnson at Bell Labs observed white noise voltage across an open resistor, with magnitude proportional to the temperature and resistance, an ingenious explanation of which was given by Harry Nyquist using arguments involving the modes of a transmission line terminated by resistors on each end [23]. Other early explanations can be found, including those due to Callen [24]. More modern, general versions of the fluctuation-dissipation theorem than those presented here apply to systems that are not assumed to be in thermodynamic equilibrium [25], but we end our discussion of the subject here, as our use of fluctuation-dissipation relation in the next chapter will be limited to the special case of LTI systems in thermodynamic equilibrium.

Chapter 2

Measurement Across a Medium Satisfying a Fluctuation-Dissipation Relation

2.1 Introduction

That the very process of extracting information about a system must necessarily influence the system's behavior is an idea generally not considered in controls-systems problems formulations. Such *measurement back action* effects might be considered negligible or perhaps too specific to the physical aspects of a given problem to address using any general theory. Nevertheless, motivated by some of the theory mentioned earlier in the introduction, in this chapter we look at various ways to abstractly think about the measurement process. Our approach is to initially strip away any physical constraints and think about measurement from a purely dynamical systems perspective using systems tools for simplicity. Surely, if we happen to encounter any measurement limits using even these simplest of models, adding any more realistic constraints can only make for measurement bounds that are worse.

In the current chapter, we will look at both *passive* and *active* approaches to measurement. By *active*, we mean that we cast the measurement in such a way that we actively try to preserve the observed state in some fashion. To start, we analyze a passive measurement model based on [3], which takes elements from statistical mechanics, in particular implicitly adding a heat bath as a component in the measurement. While there have been numerous prior investigations looking at the estimation process in a statistical mechanics setting, among them [26], [27], [10], the measurement model in [3] is most closely tied in with our interest in Chapter 3, as it stems from examining the transition from lossless to dissipative systems.

2.2 Fluctuation-Dissipation Theorem for LTI Systems

From Theorem (1.3.2), we know that any dissipative system can be arbitrarily well-approximated over an arbitrarily long time horizon by a sufficiently high-dimensional lossless system in the norm $L_2(0,T)$, and in Section 1.3, we saw how this approximation can be explicitly constructed. The fact that these approximations have a very specific, perhaps contrived, construction may not be entirely satisfying to some. This is a matter that we address in Chapter 3. However, in taking the view that all dissipative systems are truly microscopically lossless, either using Theorem (1.2.3) or the results of Chapter 3, the arguments used in the linear-response theory based proof of the fluctuation dissipation theorem in Section 1.6 can be directly appled for LTI systems, almost trivially.

Beginning with a lossless system that is an approximation to a dissipative system, the input-output relation is given from Eq.(1.2.23) as

$$y(t) = B^T e^{Jt} x_0 + \int_0^t B^T e^{J(t-s)} B u(s) ds \quad x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}, y(t) \in \mathbb{R}, \tag{2.2.1}$$

with the energy function of the form,

$$\Sigma(x) = \frac{x^T x}{2}. \tag{2.2.2}$$

Taking the initial state x_0 of the system to be a random variable with initial variance X_0 the autocovariance of $y(t)$ can be written as

$$R_y(t,s) = E[y(t) - E(y(t))][y(s) - E(y(s))]^T = B^T e^{Jt} X_0 e^{-Js} B. \tag{2.2.3}$$

The only assumption that we will make to be able to exactly follow the arguments in Section 1.6 is that the lossless system has somehow reached an equilibrium distribution that maximizes the entropy for the given total expected energy. This density is the Boltzman distribution, and is a fixed point of the Liouville equation. We might question how the system reached an equilibrium in the first place, a concern that was mentioned in the introduction, but we can assume that it did so through interactions with another large heat bath. Nevertheless, in this case, the entropy-maximizing probability is the Boltzman distribution given in Eq.(1.6.1), and is a Gaussian with the energy evenly distributed across the degrees of freedom,

$$X_0 = \frac{E[\Sigma(x_0)]}{n} I_n. \tag{2.2.4}$$

In this case the autocovariance in Eq. (2.2.3) becomes

$$R_y(t,s) = 2\frac{E(\Sigma(x_0))}{n} B^T e^{J(t-s)} B. \tag{2.2.5}$$

Reiterating, for the purposes of any experiment based on observing inputs and outputs, linear time-invariant systems that are dissipative with respect to the scalar work-rate function $w.r.(t) = y^T(t)u(t)$ can be viewed from the standpoint of actually being lossless LTI systems of very high order, with long recurrence times and in thermal equilibrium. Such systems satisfy a fluctuation theorem, namely that their input-output relation takes the form

$$y(t) = \int_0^t g(t-s)u(s)ds + n(t) \text{ where } \mathbb{E}\, n(t) = 0 \tag{2.2.6}$$

$$\text{and} \quad \mathbb{E}\, n(t)n(s) = \begin{cases} 2kTg(t-s) & t-s \geq 0 \\ 2kTg^T(s-t) & t-s < 0 \end{cases} \tag{2.2.7}$$

where $y(t), u(t), n(t) \in \mathbb{R}^p$, $g(t) \in \mathbb{R}^{p\times p}$.

For the case where the lossless system approximates a resistor, $y(t) = k_m u(t)$, we have

$$R_y(t,s) = 2\frac{\mathbb{E}(H(x_0))}{n}\delta(t-s). \tag{2.2.8}$$

That is, the uncertainty in the initial condition of the large lossless system underlying the resistor manifests itself in the input-output relationship of the resistor as an additional white noise term $w(t)$ with autocovariance $R_y(t,s)$ being added to the output. The resulting input-output relationship 2.2.6 can then be written as the *Langevin equation*,

$$y(t) = k_m u(t) + \sqrt{k_m \frac{\mathbb{E}(H(x_0))}{n}} w(t), \tag{2.2.9}$$

which is a relation that is exactly as specified by more general versions of the fluctuation dissipation theorem, as in Section 1.6.

2.3 A Measurement Model

The fluctuation-dissipation theorem tells us that a dissipative system is necessarily accompanied with noise derived from the unknown initial state of the underlying high dimensional lossless system, with the form of the time correlation of the noise being directly related to the linear response of its mean to the external perturbation. The view of measurement taken in this chapter, and for which we derive our results, is taken from [4], in which the following assumptions are made:

1. A system's output cannot be directly measured, but rather information about it can be obtained by interaction through an intermediate system.

2. This intermediate medium, if it is dissipative, satisfies a fluctuation-dissipation relation.

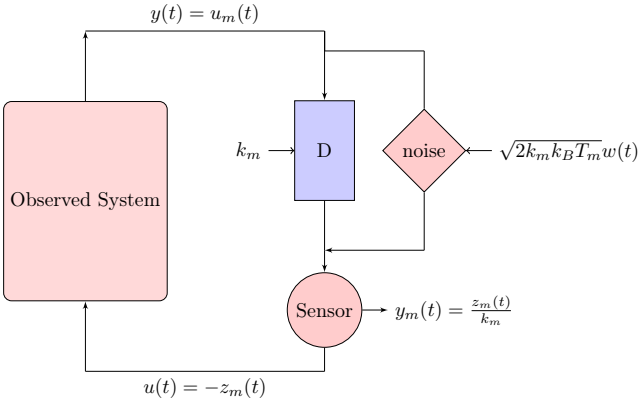


Figure 2.1: Passive Measurement Structure

3. The output of the system being measured is *scalar*, and the intermediate system is a simple generalized resistor $y(t) = ku(t)$ with $k > 0$.

We intentionally did not combine assumptions 1 and 3 for the purpose of a commentary that we make at the end of this chapter. The measurement structure is illustrated in Figure 2.1, in which the intent is to measure an output potential by observing its conjugate flow across a generalized resistor of the type given in Section 1.3.3 . As such, the components of the measurement are the observed system, denoted $\mathcal{S}$, with identical input/output ports and of the form

$$\dot{x}(t) = Fx(t) + Bu(t) \qquad x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R} \tag{2.3.1}$$

$$y(t) = B^T x(t) \qquad y(t) \in \mathbb{R} \tag{2.3.2}$$

interconnected with a measurement device we denote by $\mathcal{M}$, consisting of a resistor satisfying the fluctuation-dissipation theorem, along with an ideal flow meter. As a notational change, we will make the defining substitution $\mathbb{E}[H(x_0)] = \frac{n}{2} k_B T$ in the Langevin equation 2.2.9 where we introduce T_m, which we will call "temperature," along with the constant k_b. The rationale for making this change is to connect the average energy per degree of freedom of the underlying high state dimension resistor with the term $\frac{k_B T}{2}$, which is the well-known average energy per degree of freedom of physical systems with quadtratic Hamiltonians in thermodynamic equilibrium. With this notation, the dynamics of the measurement device are taken to be

$$z_m(t) = k_m u_m(t) + \sqrt{2k_m k_B T_m} w(t) \tag{2.3.3}$$

where the idealized sensor sees the signal $y_m(t) = \frac{z_m(t)}{k_m}$.

Interconnecting these two components such that $u_m(t) = y(t)$ and $u(t) = -z_m(t)$, the dynamics *during measurement* of the interconnected system become

$$\dot{x}(t) = (F - k_m BB^T)x(t) - B\sqrt{2k_m k_B T_m}w(t) \tag{2.3.4}$$

$$y_m(t) = B^T x(t) + \sqrt{\frac{2k_B T_m}{k_m}}w(t). \tag{2.3.5}$$

This is the linear measurement model that we will use throughout the rest of this chapter. What is immediate from this form is that during measurement, the dynamics have been changed *deterministically* such that the system matrix is now $F - k_m BB^T$. We will refer to this change as the *deterministic back action*. We will call the unavoidable input noise term into the system as the *stochastic back action*.

2.4 Estimation and Passive Measurement

As a first step, let's look at the estimation problem associated with the given measurement model. As the system is linear, and the noise process Gaussian noise, we know that a Kalman filter estimator will give us the absolute minimum mean square estimation error [28]. Because of the correlation (perfect) between the process and observation noise, we can rewrite Eq. (2.3.4) by adding to it the term

$$-Bk_m y_m(t) + k_m BB^T x(t) + k_m\sqrt{\frac{2k_B T_m}{k_m}}Bw(t) = 0$$

so that we have

$$\begin{aligned}\dot{x}(t) &= Fx(t) - k_m By_m(t)\\ y_m(t) &= B^T x(t) + \sqrt{\frac{2k_B T_m}{k_m}}w(t).\end{aligned} \tag{2.4.1}$$

The estimation problem associated with this measurement model is thus one that is equivalent to one with no process noise. As such, we can directly write the error-covariance and estimator filter using the standard Kalman filter equations (with no process noise),

$$\dot{P}(t) = FP(t) + P(t)F^T - \frac{k_m}{2k_B T_m}P(t)BB^T P(t) \tag{2.4.2}$$

$$K(t) = \frac{k_m}{2k_B T_m}(P(t)B - 2k_B T_m B) \tag{2.4.3}$$

where $P(t)$ is the estimation error covariance $E||x(t) - \hat{x}(t)||^2$, and $K(t)$ is the multiplicative term in the filter evolution equation

$$\dot{\hat{x}}(t) = \left(F - k_m BB^T\right)\hat{x}(t) + K(t)[y(t) - B^T\hat{x}(t)]. \tag{2.4.4}$$

If we define $M(t) := P^{-1}(t)$ and note that $\dot{P}^{-1}(t) = -P^{-1}(t)\dot{P}(t)P^{-1}(t)$, we find that the dynamics of $M(t)$ are described by the Lyapunov differential equation

$$\dot{M}(t) = \frac{k_m}{2k_B T_m} BB^T - M(t)F - F^T M(t) \tag{2.4.5}$$

with solution

$$M(t) = \frac{k_m}{2k_B T_m} \int_0^t e^{-F^T s} BB^T e^{-Fs} ds + e^{-Ft} M(0) e^{-Ft}. \tag{2.4.6}$$

This form of $P^{-1}(t)$ will ultimately allow us to gain some insight into the passive measurement problem for some simple examples, as well as to immediately notice that

- If F is Hurwitz, and (F, B^T) is observable, then all eigenvalues of $M(t) \to \infty$ as $t \to \infty$, and hence $P(t) \to 0$.
- If the system is lossless $F = J = -J^T$, then $M(t)$ can be viewed as the variance of the state of a lossless system (J, B, B^T) injected with white noise, and also grows indefinitely, albeit much more slowly with eigenvalues growing as order t. So, in this case, $P(t) \to 0$ as well.
- If F is unstable, the non-zero steady state estimation error covariance is just the solution of the Lyapunov Equation

  $$\frac{k_m}{2k_B T_m} BB^T - P_\infty^{-1} F - F^T P_\infty^{-1} = 0 \tag{2.4.7}$$

  given by

  $$P_\infty^{-1} = \frac{k_m}{2k_B T_m} \int_0^\infty e^{-F^T s} BB^T e^{-Fs} ds. \tag{2.4.8}$$

Remark 2.4.1 *In the case where F unstable, the integral on the right side of Eq. (2.4.8) is the* controllability gramian *of the pair $(-F^T, B)$ [29], and specifically the cross-Gramian for the system $(-F^T, B, B^T)$. Among results concerning this integral, it happens that it can be used as the basis for an approach to finding reduced order models of systems, and it would be interesting to investigate if the object P_∞, that is, the steady state estimation error for an unstable system with observation noise, is an interpretation of this integral that might be used in model reduction methods for unstable systems on finite time horizons. A similar reduction technique, balanced truncation applied to*

unstable systems, has been atempted in [30] and [31]. We will mention this topic again in Chapter 3.

While for lossless or dissipative systems that are observable we know that in this model we can eventually estimate the state of the observed system arbitrarily well, the critical point is that this state is one that has been disturbed by the measurement process itself. We will make the differentiation between the measured state and unmeasured state by the labeling x_{meas} and x_{unmeas}, and using Eq.(2.3.4), find that

$$x_{meas}(t) - x_{unmeas}(t) = e^{Ft}x_0 - \underbrace{e^{(F-k_mBB^T)t}x_0}_{\beta_D(t)} + \underbrace{\sqrt{2k_mk_BT_m}\int_0^t e^{(F-k_mBB^T)(t-s)}Bw(s)ds}_{\beta_S(t)} \quad (2.4.9)$$

where we have defined $\beta_D(t)$ and $\beta_S(t)$ as the deterministic and stochastic back action, respectively, mentioned earlier. The stochastic back action effectively 'heats up' the observed system by increasing the variance of the state during measurement by the amount

$$\mathbb{E}[\beta_S\beta_S^T](t) = 2k_mk_BT_m\int_0^t e^{(F-k_mBB^T)s}BB^Te^{(F^T-k_mBB^T)s}ds. \quad (2.4.10)$$

Example 2.4.1. *Measurement of an LC Circuit*

In this example, we consider measuring the voltage and current in an LC circuit with transfer function given by $j\omega L||\frac{1}{j\omega C}$ where $L = C = \frac{1}{\omega}$. In this case,

$$B = \sqrt{\omega}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}. \quad (2.4.11)$$

Using our expression for $P^{-1}(t)$,

$$M(t) = \frac{1}{2k_BRT_m}\int_0^t e^{Fs}BB^Te^{F^Ts}ds + e^{F^Tt}M(0)e^{Ft}, \quad (2.4.12)$$

and writing out

$$\begin{aligned} e^{Ft} &= \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix} \\ e^{Fs}BB^Te^{F^Ts} &= \omega\cdot\begin{pmatrix} \cos^2\omega t & -\cos\omega t\sin\omega t \\ -\cos\omega t\sin\omega t & \sin^2\omega t \end{pmatrix} \\ \int_0^t e^{Fs}BB^Te^{F^Ts}ds &= \frac{1}{2}\cdot\begin{pmatrix} \omega t+\cos\omega t\sin\omega t & -\sin^2\omega t \\ -\sin^2\omega t & \omega t-\cos\omega t\sin\omega t \end{pmatrix}. \end{aligned}$$

Now, if we assume that we initially have $M(0) = 0$, we can write

$$P(t) = 4k_B R T_m \frac{1}{\omega^2 t^2 - \sin^2 \omega t} \begin{pmatrix} \omega t - \cos \omega t \sin \omega t & +\sin^2 \omega t \\ +\sin^2 \omega t & \omega t + \cos \omega t \sin \omega t \end{pmatrix}. \tag{2.4.13}$$

We see that $P(t)$ decays to 0 as $t \to \infty$, algebraically in t. What this simple example illustrates nicely, and which we can deduce more generally from 2.4.10 and 2.4.6, is that the initial very fast decrease of $P(t)$ in some sense makes it advantageous to take very short measurements before the effects of back action, which vary exponentially, become more significant.

2.4.1 Smoothed Estimate of x_0

In the previous section, we saw an explicit tradeoff between the decrease in the state estimation error $P(t)$ over the measurement time interval, and the increase of the measurement back action, $\beta_S(t)$ and $\beta_D(t)$ in that same time period. Perhaps it is not important that the state of the measurement system gets destroyed in the measurement process, and the only objective is to measure x_0. With this objective in mind, in this section, we examine how the smoothed estimate of x_0 evolves over time. Although formulas for smoothing estimates are standard and directly applicable, since we are interested in an estimate of the state at a fixed point in time, we can easily address the estimation problem on x_0 by forming the system

$$\begin{bmatrix} \dot{x(t)} \\ \dot{\xi}(t) \end{bmatrix} = \begin{bmatrix} F - k_m B B^T & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} [u(t) - \sqrt{2k_m k_B T} w(t))] \tag{2.4.14}$$

$$y(t) = \begin{bmatrix} B^T & 0 \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} + \sqrt{\frac{2k_B T_m}{k_m}} w(t) \tag{2.4.15}$$

where $x(0) = \xi(0)$, and looking at the error covariance of the MMSE of $\begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}$ by solving the homogeneous Riccati equation 2.4.2 given by

$$\begin{aligned} \dot{P}(t) = \begin{bmatrix} \dot{P}_{11} & \dot{P}_{12} \\ \dot{P}_{12}^T & \dot{P}_{22} \end{bmatrix} &= \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \begin{bmatrix} F^T & 0 \\ 0 & 0 \end{bmatrix} - \\ &\frac{k_m}{2k_B T_m} \begin{bmatrix} P_{11} B B^T P_{11} & P_{11} B B^T P_{12} \\ P_{12}^T B B^T P_{11} & P_{12}^T B B^T P_{12} \end{bmatrix}, \\ P(0) = \begin{bmatrix} P_{11}(0) & P_{12}(0) \\ P_{12}^T(0) & P_{22}(0) \end{bmatrix} &= \begin{bmatrix} P(0) & P(0) \\ P(0) & P(0) \end{bmatrix}, \end{aligned} \tag{2.4.16}$$

which we can solve in sequence $P_{11} \to P_{12} \to P_{22}$,

$$\begin{aligned}
\dot{P}_{11}(t) =& FP_{11} + P_{11}F^T - \frac{k_m}{2k_BT_m}P_{11}BB^TP_{11} \\
\dot{P}_{12}(t) =& (F - \frac{k_m}{2k_BT_m}P_{11}BB^T)P_{12} \\
\dot{P}_{22}(t) =& -\frac{k_m}{2k_BT_m}P_{12}^TBB^TP_{12}.
\end{aligned} \tag{2.4.17}$$

While we have already analyzed the the matrix differential equation for $P_{11}(t)$ in the previous section, and concluded that for a *lossless* measured system the state uncertainty goes to zero in time, it is not clear from the coupled matrix ODEs of Eq.(2.4.17) exactly how $P_{22}(t)$, the smoothed estimate of the measured system before onset of the mesurement process, evolves. Without resorting to analyzing this ODE, though, we can deduce the limiting form of the smoothed estimate of x_0 from simple arguments.

Lemma 2.4.1 *For measurement of a lossless system, the error in the smoothed estimate of x_0 goes to zero as $t \to \infty$. In particular, $P_2(t) \to 0$ in Eq. (2.4.17).*

To facillitate showing this, let's first ask a similar question, which itself can be viewed as another type of inquiry in this measurement problem formulation. What levels of uncertainty could we achieve if we attempted to estimate the state *to which the system would have evolved* had the measurement not been made? The answer to this is can be seen from Eq. (2.4.1), which if we integrate, gives us the relationship

$$x_{meas}(t) = e^{Ft}x(0) - k_m\int_0^t e^{F(t-s)}By(s)ds. \tag{2.4.18}$$

That is, the difference between the measured state $x_{meas}(t)$ and state of the unmeasured system $x_{unmeas}(t) = e^{Ft}x(0)$ is simply a deterministic function of the observations $y(t)$. As such, the minimum mean squared error of the estimate $x_{unmeas}(t)$ must be the same as for $x_{meas}(t)$. This point can also be seen by modifying the composite equations in Eq.(2.4.1), changing the composite system state matrix from

$$\begin{bmatrix} F - k_mBB^T & 0 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} F - k_mBB^T & 0 \\ 0 & F \end{bmatrix}, \tag{2.4.19}$$

in which case the coupled covariance equations would become,

$$\begin{aligned}\dot{P}_{11}(t) =& FP_{11} + P_{11}F^T - \frac{k_m}{2k_BT_m}P_{11}BB^TP_{11}\\ \dot{P}_{12}(t) =& FP_{12} + P_{12}F^T - \frac{k_m}{2k_BT_m}P_{11}BB^TP_{12}\\ \dot{P}_{22}(t) =& FP_{22} + P_{22}F^T - \frac{k_m}{2k_BT_m}P_{12}^TBB^TP_{12},\end{aligned} \tag{2.4.20}$$

and from which we see the trivial result that $P_{11}(t) = P_{12}(t) = P_{22}(t)$. That is, all of our analysis for the estimation error of the measured system is the same as that we would encounter if we were interested in the time evolution of the unmeasured system. In the special case where the system is *lossless*, this means that *we can measure the state to which the system would have evolved arbitrarily well.* What does this tell us about our first question regarding the behavior of $P_{22}(t)$ in Eq.(2.4.17), i.e., the error in the smoothed estimate of x_0? For a lossless sytem, we have just shown that the estimation error, $\tilde{x}_{unmeas}(t)$, defined as

$$\tilde{x}_{unmeas}(t) = x_{unmeas}(t) - \hat{x}_{unmeas}(t), \tag{2.4.21}$$

satisfies

$$\mathbb{E}\left(\tilde{x}_{unmeas}(t)\tilde{x}_{unmeas}(t)^T\right) \to 0 \quad \text{as} \quad t \to \infty. \tag{2.4.22}$$

So, taking $\hat{x}_0(t) = e^{-Jt}\hat{x}_{unmeas}(t)$ as *an* estimator of the system initial condition, we find that for this estimator

$$\tilde{x}_0(t) = x_0 - \hat{x}_0(t) = e^{-Jt}\left(e^{Jt}x_0 - \hat{x}_{unmeas}(t)\right) = e^{-Jt}\tilde{x}_{unmeas}(t). \tag{2.4.23}$$

In particular,

$$\mathbb{E}\left(\tilde{x}_0(t)\tilde{x}_0(t)^T\right) = e^{-Jt}\,\mathbb{E}\left(\tilde{x}_{unmeas}(t)\tilde{x}_{unmeas}(t)^T\right)e^{Jt}, \tag{2.4.24}$$

which from Eq. (2.4.22) goes to zero as $t \to \infty$. Thus, we also find that we can determine the initial state of a lossless sytem arbitrarily well with sufficiently long measurements. Looking back at our simple LC example Ex.(2.4), we can integrate 2.4.17 and see the exact decay of $P_{22}(t)$ that we would expect from Eq. (2.4.22) and Eq.(2.4.24) in Figure (2.2).

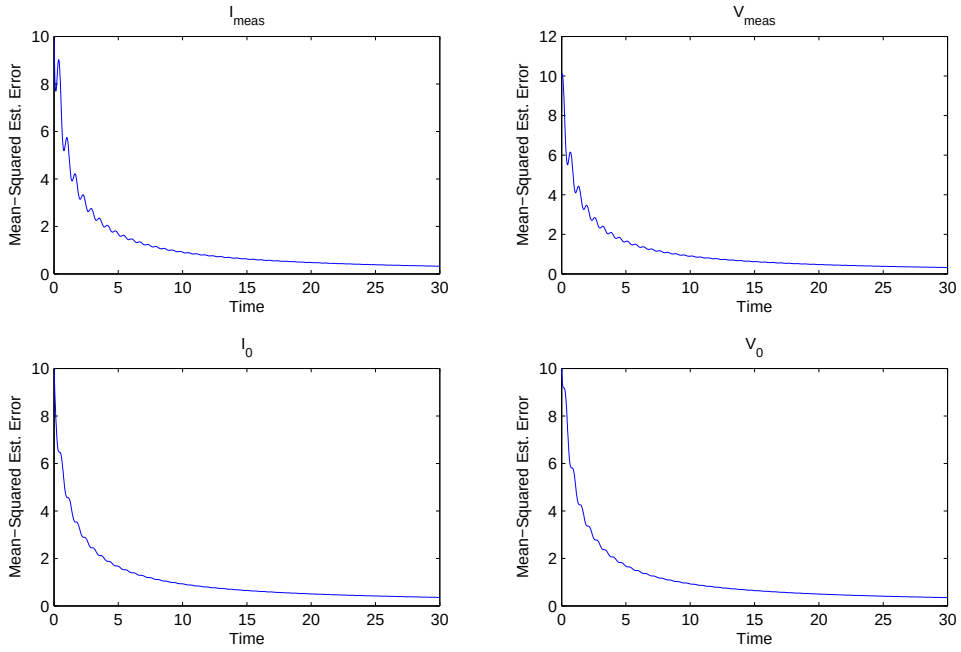


Figure 2.2: Minimum mean-squared estimation error for measured state and initial condition of LC circuit with $L = C = 5$,$B = (1, 1)^T$, $R = 1$, $2kT = 1$, $P_0 = (10, 2; 2, 10)$.

2.5 Active Measurement

In the last section, we completely characterized the estimation problem for the given measurement model, and for lossless systems found that we can measure a state arbitrarily well if we are willing to destroy it and wait long enough. In this section, we examine possible ways to measure a state while preserving it in some way. We might want to ask the question of what would be considered a 'minimally disturbing' measurement? Defining this is open to a great deal of freedom, but intuitively, we would want to keep the state of the system 'close' to its unperturbed value either throughout the measurement or after the measurement process is complete. Given the form of the back action in the measurement model in the last section, and its decomposition into deterministic and stochastic components, if we allow ourselves the ability to apply an external signal to offset a portion of this back action, then we have a measurement problem that involves both estimation and control. This is an issue that traditionally is not addressed in controls, where the measurement process is usually taken for granted. In the following section, we look at multiple approaches to reducing the effects of measurement back action, with their own idealized assumptions, and emphasize the limitations of each.

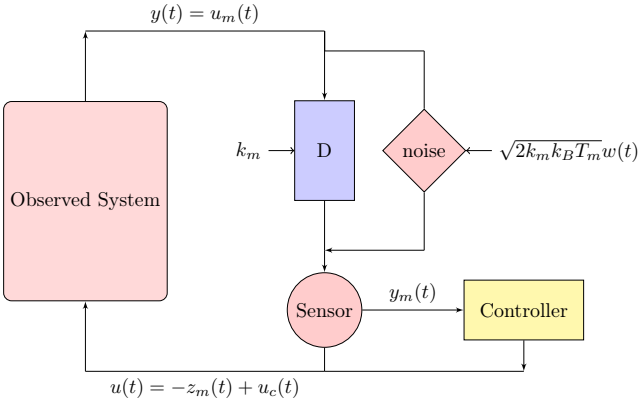


Figure 2.3: Active Measurement Structure

2.5.1 Impulsive Control

In some active measurement schemes, we will refer to an actual controller placed in the original measurement model, as shown in Figure 2.2. From 2.4.18, we know that we can measure the given system, and at the end of the measurement apply an impulsive control

$$u(t) = \left(k_m \int_0^{t_f} e^{F(t_f - s)} B y_m(s) ds \right) \delta(t - t_f) \tag{2.5.1}$$

to 'place' the system back to where it would have been had the measurement not taken place. This sort of trivial solution to our 'minimally disturbing' measurement problem illustrates the point that in order to get bounds of any meaning, we have to impose constraints on our controller that correspond, in some way, to reality, restricting the types of abstract dynamical systems that we can employ as 'controllers.' For instance, we might want to see what limits to measurement we end up with if we restrict our control effort to have finite power. There are also more subtle issues to consider, such as the implementation of the Kalman filter in the previous section. Since the dynamics of the filter are those of the measured system during measurement, the filter must itself consist of a resistor which would, through the fluctuation-dissipation theorem, be accompanied by another independent noise source. Maybe this filter could be run at a lower temperature, or maybe it could be implemented 'digitally' with underlying non-linear dynamical systems. All sorts of issues can be imagined that complicate this picture, so we will try to avoid them in the next section by employing very simple active dynamical systems as components in an active measurement device.

2.5.2 Deterministic Back Action Cancellation

In this formulation, we will assume that an active device can be inserted in parallel with the measurement device such that the deterministic portion of the back action is eliminated, and examine

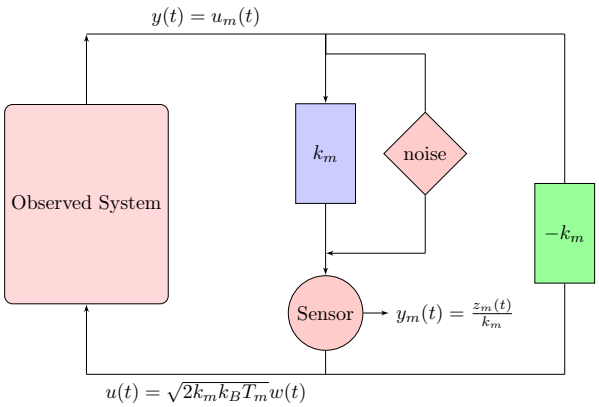


Figure 2.4: Deterministic Back Action Cancellation

the limitations imposed by the stochastic back action. This 'negative resistor' is inserted as shown in Figure 2.3. While it would be easy to initially dismiss the admission of such a device in our measurement problem as allowing us too much freedom, in fact, simple non-linear dynamical systems can be made that arbitrarily well approximate such devices. We again emphasize the point that active devices must necessarily be nonlinear. An easy construction found in [3] is given by

$$\dot{x}(t) = \frac{k}{\sqrt{2E}} u(t)^2, \quad x(0) = \sqrt{2E} \tag{2.5.2}$$

$$y(t) = \frac{k}{\sqrt{2E}} x(t) u(t), \tag{2.5.3}$$

where k can be negative and where E represents the initial energy of the system, which will be assumed large. The input-output relationship for this system is given by

$$y(t) = kx(t) + \frac{k^2}{2E} u(t) \int_0^t u(s)^2 ds, \tag{2.5.4}$$

which can arbitrarily well approximate a negative resistor in the limiting case where the available energy in the initial condition of the system goes infinity. Generalizing this construction to include noise on the initial system state can also be handled easily, and is explicitly given in, [3], but we don't need to include it in the subsequent analysis, as we will shortly find estimation error lower bounds even for the case where this 'negative resistor' is assumed noiseless.

With the inclusion of the system in Eq.(2.5.4), in the large E limit, into the measurement structure shown in Figure 2.3, the state and output equations become

$$\dot{x}(t) = Fx(t) - B\sqrt{2k_m k_B T_m}w(t) \tag{2.5.5}$$

$$y_m(t) = B^T x(t) + \sqrt{\frac{2k_B T_m}{k_m}}w(t). \tag{2.5.6}$$

The resulting Kalman filter equations become

$$\dot{P}(t) = FP(t) + P(t)F + 2k_m k_B T_m BB^T - \frac{2k_B T_m}{k_m}K(t)K^T(t) \tag{2.5.7}$$

$$K(t) = \frac{k_m}{2k_B T_m}(P(t)B - 2k_B T_m B) \quad , \tag{2.5.8}$$

which, almost identically to before,

$$\dot{P}(t) = FP(t) + P(t)F + 2k_m k_B T_m BB^T - \frac{k_m}{2k_B T_m}P(t)BB^T P(t) \tag{2.5.9}$$

$$-2k_b k_m T_m BB^T + k_m(P(t)BB^T + BB^T P(t)) \tag{2.5.10}$$

so that

$$\dot{P}(t) = (F + k_m BB^T)P(t) + P(t)(F^T + k_m BB^T) \tag{2.5.11}$$

$$- \frac{k_m}{2k_B T_m}P(t)BB^T P(t). \tag{2.5.12}$$

Again, the equation for $P^{-1}(t)$ is a Lyapunov differential equation

$$\dot{M}(t) = \frac{k_m}{2k_B T_m}BB^T - M(t)(F + k_m BB^T) - (F^T + k_m BB^T)M(t). \tag{2.5.13}$$

of which we get the solution

$$M(t) = \frac{k_m}{2k_B T_m}\int_0^t e^{-(F^T + k_m BB^T)s}BB^T e^{-(F + k_m BB^T)s}ds + e^{-(F^T + k_m BB^T)t}M(0)e^{-(F + k_m BB^T)t}. \tag{2.5.14}$$

A noteworthy feature of this approach to measurement is that it does not require computation of the active control signal, so analysis of this type of measurement does not burden us with the analysis of limits that would arise from computation . As such, since the stochastic back action is not accounted for, the measured system is simply 'heated up' during measurement. The consequences of this fact are manifested in our ability to estimate the state of a system measured in this manner, and are explicitly stated in the following theorem.

Theorem 2.5.1 (**Uncertainty Lower Bound for Lossless Systems Measured with Cancellation of Deterministic Back Action**). *For any measurement of an* 2*-dimensional lossless*

system concurrent with active cancellation of the deterministic component of back action,

$$Tr[P^{-1}(t)] \leq \frac{1}{2k_B T_m}$$

for all t provided that the ***initial uncertainty*** *satisfies*

$$P(0) \geq 4k_B T_m I.$$

A specific consequence of this statement is that:

Corollary 2.5.1 *If (q, p) are the state variables of the system transformed so that the energy function is given by $\Sigma(p, q) = p^2 + q^2$, then both*

$$\mathbb{E}[\Delta q^2] \geq 4k_B T_m \quad \textit{and} \quad \mathbb{E}[\Delta p^2] \geq 4k_B T_m, \tag{2.5.15}$$

for all t, where Δq^2 and Δp^2 are the mean squared errors of the estimates of q, p.

Proof. From Eq.(2.5.13) and the results in Section 1.2, we know that $F = J$, and since (J, B) is controllable that the inverse of the state estimation error covariance in the limit as $t \to \infty$ is the unique solution of

$$\frac{k_m}{2k_B T_m} BB^T - M_\infty(J + k_m BB^T) - (J^T + k_m BB^T)M_\infty = 0, \tag{2.5.16}$$

which can be easily verified to be

$$M_\infty = \frac{I}{4k_B T_m}. \tag{2.5.17}$$

Taking the trace of Eq.(2.5.13), we get

$$\begin{aligned} Tr[\dot{M}(t)] &= \frac{k_m}{2k_B T_m}\|B\|^2 - Tr[M(t)J + J^T M(t)] - k_m Tr[M(t)BB^T + BB^T M(t)] \\ &= \frac{k_m}{2k_B T_m}\|B\|^2 - k_m Tr[M(t)BB^T + BB^T M(t)], \end{aligned} \tag{2.5.18}$$

where the last equality comes from the fact that the trace of the product of a symmetric and skew-symmetric matrix is zero. Also, since $M(t)$ is positive semidefinite, we have that

$$Tr[M(t)BB^T] \leq \lambda_{max}[M(t)]\|B\|^2 \leq Tr[M(t)]\|B\|^2. \tag{2.5.19}$$

Thus, we get

$$Tr[\dot{M}(t)] \geq k_m\|B\|^2 \left(\frac{1}{2k_B T_m} - Tr[M(t)]\right). \tag{2.5.20}$$

By assumption,

$$0 < M(0) \leq \frac{I}{4k_B T_m} \implies Tr[M(0)] \leq \frac{1}{2k_B T_m}.$$

From Eq.(2.5.17) and the fact that the trace is a continuous function, we also have that

$$\lim_{t\to\infty} Tr[M(t)] = Tr[M_\infty] = \frac{1}{2k_B T_m}.$$

Given Eq.(2.5.20), we deduce that for all t,

$$Tr[\dot{M}(t)] \geq 0 \quad \text{and} \quad Tr[M(t)] \leq Tr[M_\infty] = \frac{1}{2k_B T_m}.$$

That is, $Tr[M(t)]$ increases monotonically to its limit, which is the desired result. □

Corollary 2.5.1 can be immediately seen by observing that

$$\lambda_{min}\left(P(t)\right) \geq \frac{2}{Tr[M(t)]}$$

.

and since $P(t) \geq 0$, that $P_{11}(t), P_{22}(t) \geq \lambda_{min}\left(P(t)\right)$.

Example 2.5.2. *Measurement of an LC Circuit with Cancellation of Deterministic Back Action* To demonstrate the tightness of this bound, we can go back to the *LC* circuit example in the previous section, although this time analyze it numerically in the case where

$$\mathbb{E}\begin{pmatrix} I_L^2(0) & I_L(0)V_C(0) \\ I_L(0)V_C(0) & V_C^2(0) \end{pmatrix} = 2k_b T \begin{pmatrix} 10 & 7.9 \\ 7.9 & 10 \end{pmatrix}. \tag{2.5.21}$$

The time evolution of the error covariances are shown in Figure 2.4.

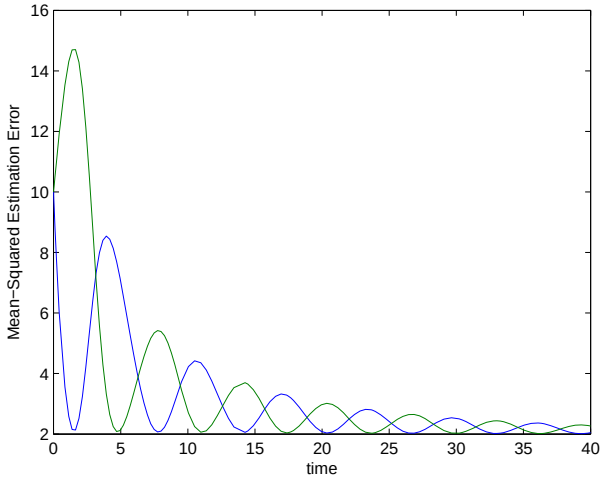


Figure 2.5: Mean squared estimation error for V_c and I_L in Example 2.5.2

We can also extract additional information about this sort of measurement by explicitly solving for the error covariances using the formulas given in the previous section. For instance, if we were to take $B = [0,1]^T$ and assume that the initial error covariance is infinite, we would find that the minimum-mean squared error estimate of the state of the LC circuit would be given by

$$P^{-1} = M(t) = \begin{pmatrix} m_{11}(t) & m_{12}(t) \\ m_{21}(t) & m_{22}(t) \end{pmatrix}, \tag{2.5.22}$$

where

$$\begin{aligned} m_{11}(t) =& \frac{k_m}{2k_BT_m} \cdot \left(\frac{1}{2k_m} \cdot (1 - e^{-k_m t}) - \frac{1}{4\omega_d^2} \cdot e^{-k_m t} \left(k_m \sin^2 \omega_d t - 2\omega_d \cdot \cos \omega_d t \sin \omega_d t \right) \right) \\ m_{12}(t) =& \frac{k_m}{2k_BT_m} \cdot \left(\frac{-\omega}{2\omega_d^2} \cdot e^{-k_m t} \sin^2 \omega_d t \right) \\ m_{22}(t) =& \frac{k_m}{2k_BT_m} \cdot \left(\frac{1}{2k_m} \cdot \left(1 - e^{-k_m t}\right) - \frac{1}{4\omega_d^2} \cdot e^{-k_m t} \left(k_m \sin^2 \omega_d t + 2\omega_d \cdot \cos \omega_d t \sin \omega_d t \right) \right), \end{aligned}$$

expressions that allow us to directly read off the entries of $P(t)$ by noting the simplification,

$$\det M(t) = \left(\frac{k_m}{2k_BT_m} \right)^2 \cdot \left(\left(\frac{1}{2k_m} \right)^2 \left(1 - e^{-k_m t}\right)^2 - \frac{e^{-k_m t}}{4\omega_d^2} \sin^2 \omega_d t \right). \tag{2.5.23}$$

We have to be mindful of that fact that with this sort of measurement, the system is heated up, and in particular

$$x_{meas}(t) = x_{unmeas}(t) + \beta_S(t)$$

where the stochastic back action $\beta_S(t)$ is a normal random variable with covariance given by

$$\begin{aligned} \mathbb{E}[\beta_S \beta_S^T](t) &= 2k_m k_B T_m \int_0^t e^{Fs} B B^T e^{F^T s} ds \\ &= k_m k_b T_m \cdot \begin{pmatrix} t + \frac{1}{\omega} \cos \omega t \sin \omega t & -\frac{1}{\omega} \sin^2 \omega t \\ -\frac{1}{\omega} \sin^2 \omega t & t - \frac{1}{\omega} \cos \omega t \sin \omega t \end{pmatrix}. \end{aligned} \tag{2.5.24}$$

If we want to compare decay of the error covariance matrix to the stochastic back action with increasingly large k_m, we note that for $k_m \gg \omega$,

$$\omega_d = \frac{1}{2}\sqrt{k_m^2 - 4\omega^2} \sim \frac{k_m}{2} - \frac{\omega^2}{k_m} \tag{2.5.25}$$

and that in this limiting condition, that the right side of Eq. (2.5.23) is approximately

$$\begin{aligned}
&= \left(\frac{k_m}{2k_BT_m}\right)^2 \cdot \left(\left(\frac{1}{2k_m}\right)^2 \left(1-e^{-k_m t}\right)^2 + \frac{e^{-k_m t}}{4\omega_d^2}\sinh^2\omega_d t\right) \\
&\sim \left(\frac{k_m}{2k_BT_m}\right)^2 \cdot \left(\frac{1}{2k_m}\right)^2 \cdot \left(\left(1-e^{-k_m t}\right)^2 + \left(e^{-2k_m t} + e^{-\frac{\omega^2}{k_m}t} - 2e^{-k_m t}\right)\right) \\
&= \left(\frac{1}{4k_BT_m}\right)^2 \cdot \left(\left(1-e^{-k_m t}\right)^2 + \left(e^{-2k_m t} + e^{-\frac{\omega^2}{k_m}t} - 2e^{-k_m t}\right)\right).
\end{aligned}$$

We can directly use Eq. 2.5.23 to get the small t expansion of the determinant as

$$\begin{aligned}
\det M(t) &\sim \left(\frac{k_m}{2k_BT_m}\right)^2 \cdot \left(\left(\frac{1}{2k_m}\right)^2 \left(k_m^2t^2 - k_m^3t^3 + o_1(t^4)\right) - \frac{1}{4\omega_d^2}\cdot\left(\omega_d^2t^2 - k_m\omega_d^2t^3 + o_2(t^5)\right)\right) \\
&= \left(\frac{k_m}{2k_BT_m}\right)^2 \cdot \left(\left(\frac{1}{2k_m}\right)^2 \left(o_1(t^4)\right) - \frac{1}{4\omega_d^2}\cdot\left(o_2(t^4)\right)\right),
\end{aligned}$$

as all the terms up to order t^3 vanish. The resulting order t^4 terms end up being

$$\begin{aligned}
o_1(t^4) &= \frac{7}{12}k_m^4t^4 \\
o_2(t^4) &= \left(\frac{k_m^2\omega_d^2}{2} - \frac{\omega_d^4}{3}\right)t^4,
\end{aligned}$$

which gives us, making the substitution $4\omega^2 = 4\omega_d^2 + k_m^2$,

$$\det M(t) \sim \left(\frac{k_m}{2k_BT_m}\right)^2 \cdot \frac{\omega^2}{12}\cdot t^4.$$

Using this, the short measurement time expansions of the individual entries of $P(t)$ can be read off taking the lowest-order expansions of the entries given for $M(t)$ earlier, to give

$$\begin{aligned}
p_{11}(t) &\sim \frac{2k_BT_m}{k_m}\cdot\frac{4}{t} \\
p_{12}(t) &\sim \frac{2k_BT_m}{k_m}\cdot\left(\frac{6}{\omega t^2}\right) \\
p_{22}(t) &\sim \frac{2k_BT_m}{k_m}\cdot\left(\frac{12}{\omega^2t^3}\right).
\end{aligned}$$

From these expressions, we can directly deduce two non-obvious features specific to measurement of a harmonic oscillator. Firstly, for short t, the magnitude of the resistance used only appears as a multiplicative constant in the error covariance. Secondly, for each individual state, we get a relationship between the back action and error covariance such that

$$\mathbb{E}[\beta_S\beta_S^T]_{ii}(t)\cdot P_{ii}(t) = \left(\frac{2k_BT_m}{k_m}\right)^2\cdot c_i, \tag{2.5.26}$$

where $c_1 = 4$ and $c_2 = 12$. That is, for short time, this product relationship between back action

and estimation error is independent of the resistance of the measurement medium, and does not change with respect to time.

2.5.3 LQG Control to Meausure and Preserve Pre-Measurement State

The advantage to analyzing active measurement with only deterministic back action is two-fold. As already mentioned, when we cancel the deterministic back action, the Kalman filter itself does not have a resistor in it, so we are not introducing additional noise sources in the realization of the estimator. Additionally, the implementation of the negative resistor can be accomplished using a simple dynamical system with a large stored internal energy. Of course, the non-optimality of this technique is that input to the measured system from the active advice does not make use of information about the time history of the noise process.

In this section, we consider a measurement which, unlike the deterministic-back action cancellation approach of the last section, utilizes information about the noise process injected into the system, and attempts to offset it. To accomplish this end, we must take a leap in both the freedom to perform computation, and the ability to generate arbitrary control signals without the introduction of additional noise. However, what makes this measurement approach worth examining is that it is not so extreme an idealization as the impulsive control technique of Section 2.5, but rather, depending on the cost we choose to place on control effort, is characterized by having finite power and bandwidth requirements.

We will assume that we have an infinite-resistance device that can generate arbitrary inputs according to a control law, and consider a measurement method where the objective is to maintain the state of a measured *lossless* system as close to its original value over the course of the measurement. By close, we will mean in a mean-squared sense, so we pose this problem as a standard LQG control tracking problem. For the rest of this section, we will examine the minimization of

$$J = \mathbb{E} \int_0^{t_f} \left(x_m(t) - x_0\right)^T Q \left(x_m(t) - x_0\right) + ||u(t)||^2 dt, \tag{2.5.27}$$

where $x_m(t)$ and x_0 denote the state of the measured system and pre-measurement state, respectively. The main question we want to answer is, in the case where the measured system is *lossless*, how does the cost we put on control effort prevent the LQG controller from squeezing the state of the measured system to the pre-measurement state?

The minimization in 2.5.27 would proceed just as in Section 2.41, by introducing the state

$$v(t) = \begin{bmatrix} x_m(t) \\ x_0 \end{bmatrix}. \tag{2.5.28}$$

Defining quantities as in Section 2.41, we write

$$\tilde{F} = \begin{bmatrix} J - k_m BB^T & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \tilde{Q} = \begin{bmatrix} Q & -Q \\ -Q & Q \end{bmatrix} \tag{2.5.29}$$

$$P(t) = \begin{bmatrix} P_{11}(t) & P_{12}(t) \\ P_{12}^T(t) & P_{22}(t) \end{bmatrix}. \tag{2.5.30}$$

From Eq. (2.4.1), we know how to solve for $P(t)$. The state feedback control law for the LQG minimization follows (see [28] or [32]) by partitioning the matrix

$$S(t) = \begin{bmatrix} S_{11}(t) & S_{12}(t) \\ S_{12}^T(t) & S_{22}(t) \end{bmatrix} \tag{2.5.31}$$

and solving the Riccati differential equation

$$-\dot{S}(t) = \tilde{F}^T S + S\tilde{F} - S\tilde{B}\tilde{B}^T S + \tilde{Q}, \tag{2.5.32}$$

which yields the coupled matrix ODEs

$$\begin{aligned} -\dot{S}_{11}(t) =& (J - k_m BB^T)^T S_{11} + S_{11}(J - k_m BB^T) - S_{11} BB^T S_{11} + Q \\ -\dot{S}_{12}(t) =& (J - k_m BB^T)^T S_{12} - S_{11} BB^T S_{12} - Q \\ -\dot{S}_{22}(t) =& - S_{12}^T BB^T S_{12} + Q. \end{aligned} \tag{2.5.33}$$

The LQG control law then is $u(t) = -\tilde{B}^T S(t)\hat{v}(t)$. So, how does the LQG-controlled state $x(t)$ evolve in time? How well does the LQG controller keep $x(t)$ near x_0 when there is non-zero cost on control, in the $t \to \infty$ limit?

We argue that since $P(t) \to 0$, and specifically, we eventually know x_0 arbitrarily well, that in the limit as $t \to \infty$ that the analysis for this question is the same as asking how well the controller can squeeze the state $x(t)$ to zero. This is the avenue we take. So, how does the LQG-controlled state $x(t)$ evolve in time? Writing the estimate error as $\tilde{x} = x - \hat{x}$, we have that

$$\mathbb{E}\left(x(t)x^T(t)\right) = P_{11}(t) + \mathbb{E}\left(\hat{x}(t)\hat{x}^T(t)\right), \tag{2.5.34}$$

since the state estimate and estimate error are independent. We also have that

$$\tilde{B}u(t) = \begin{bmatrix} -BB^T & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} S_{11}(t) & S_{12}(t) \\ S_{12}^T(t) & S_{22}(t) \end{bmatrix} \hat{v}(t) = \begin{bmatrix} -BB^T S_{11}(t) & -BB^T S_{12}(t) \\ 0 & 0 \end{bmatrix} \hat{v}(t) \quad (2.5.35)$$

so that in closed loop,

$$\dot{\hat{x}}(t) = \left(J - k_m BB^T - BB^T S_{11}(t)\right) \hat{x}(t) + K(t)e(t) - BB^T S_{12}\hat{x}_0(t) \quad (2.5.36)$$

where $e(t)$ is the innovations process of the estimation problem, and $K(t)$ is as given in Eq.(2.4.2). We reiterate, that since our concern in this analysis is the ability of the infinite-horizon controller to account for the process and observation noise, we will assume that $x_0 = 0$. Also, since we know that the error covariance of the smoothed estimate of the initial state of a lossless system goes to zero for large t, that for steady state analysis of the infinite-horizon problem, the limiting behavior of the controller can be deduced by analyzing the limiting form of the equation,

$$\dot{\hat{x}}(t) = \left(J - k_m BB^T - BB^T S_{11}(t)\right) \hat{x}(t) + K(t)e(t) \quad (2.5.37)$$

rather than Eq.(2.5.36). From this, we can immediately derive (since $e(t)$ is a white noise process)

$$\begin{aligned} \dot{\hat{X}}(t) :=& \frac{d\,\mathbb{E}(\hat{x}(t)\hat{x}^T(t))}{dt} \\ =& \left(J - G(t)\right) \hat{X} + \hat{X} \left(J - G(t)\right)^T + \frac{2k_b T_m}{k_m} K(t)K^T(t), \end{aligned} \quad (2.5.38)$$

where we have defined

$$G(t) := k_m BB^T + BB^T S_{11}(t). \quad (2.5.39)$$

Again, since this analysis is for the limit as $t \to \infty$, we use the infinite horizon LQG solution with

$$\tilde{B}u(t) = -B^T S_{11}^{\infty} x(t), \quad (2.5.40)$$

where S_{11}^{∞} is the limit of the backwards integrated equation 2.5.33. Since $P(t) \to 0$ for large t, $K(t) \to -k_m B$ (from 2.4.2), we can use Eq. (2.5.34) along with Eq. (2.5.38) to show that (see for example [33], [34])

$$\lim_{t\to\infty} \mathbb{E}(x(t)x^T(t)) = X_{\infty}, \quad (2.5.41)$$

where X_∞ satisfies

$$0 = (F - G_\infty)X_\infty + X_\infty(F - G_\infty)^T + 2k_bT_mk_mBB^T. \tag{2.5.42}$$

This is the same Lyapunov equation that we would obtain for the mean squared value of the controlled state at large time in the case where we had ***complete state feedback with process noise*** of magnitude $\sqrt{2k_bT_mk_m}$. We illustrate the performance of the LQG controller for an LC circuit, but again emphasize that the process of computing the steady state value of $\mathbb{E}\|x(t)\|^2$ under control simply involves solving the ARE in S_{11},

$$0 = (F - k_mBB^T)^TS_{11}^\infty + S_{11}^\infty(F - k_mBB^T) - S_{11}^\infty BB^TS_{11}^\infty + Q \tag{2.5.43}$$

and the Lyapunov equation 2.5.42 where

$$G_\infty = k_mBB^T + BB^TS_{11}^\infty. \tag{2.5.44}$$

Example 2.5.3 (LC Circuit Revisited)**.** We numerically look at the tradeoff between state weighting and the ability of the LQG controller to squeeze the steady state in the case where the measured system is our LC circuit. Again, the motivation in this is to see how imposing cost on control affects our ability to minimally disturb the system being measured while extracting information. Note that in this example the state is being squeezed to zero, but the steady state result would be the same as in the case where we were attempting to squeeze the state to x_0, i.e., the pre-measurement state. In this example, we have chosen parameters such that $2k_BT = 1, Q = q \cdot I, P = 10 \cdot I, k_m = 0.1$ and $2k_BT = 1$.

The diagonal entries of $P_{11}(t)$ and the LQG-controlled $\mathbb{E}[\hat{i}_L(t)]^2(t)$ and $\mathbb{E}[\hat{v}_C(t)]^2$ are shown in Figures 2.5 and 2.6, respectively. From Eq. (2.5.34), the sum of the individual values from the two figures is how the mean squared value of the state evolves in time under the LQG control. Although $P_{11}(t)$ goes to zero, the mean squared estimate of the state never reaches zero as the controller fights to offset the effects of the process noise, so the tradeoff between control cost (in this formulation, penalty on state) and steady state mean squared value is shown in Figures 2.7 and 2.8 for various levels of process noise amplitude. Notice that in this example, with the form of B, that the state x_1 is directly subject to the process noise, so the steady state mean squared is larger than that of the controlled x_2. This holds no matter what the relative state weighting we might apply in Q.

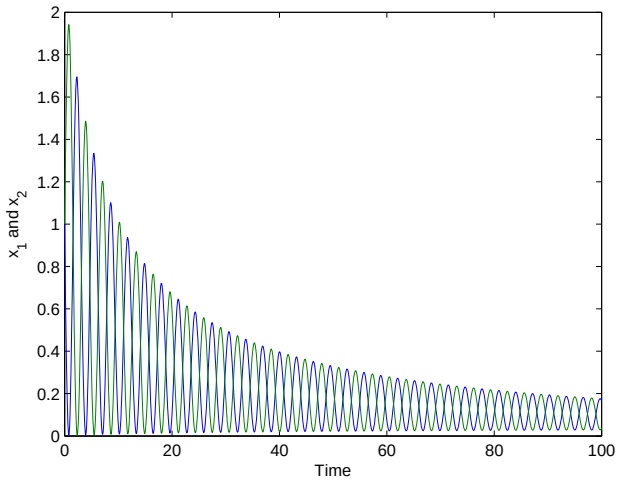


Figure 2.6: Mean squared estimation error for $i_L(t)$ (blue) and $v_C(t)$ (green)

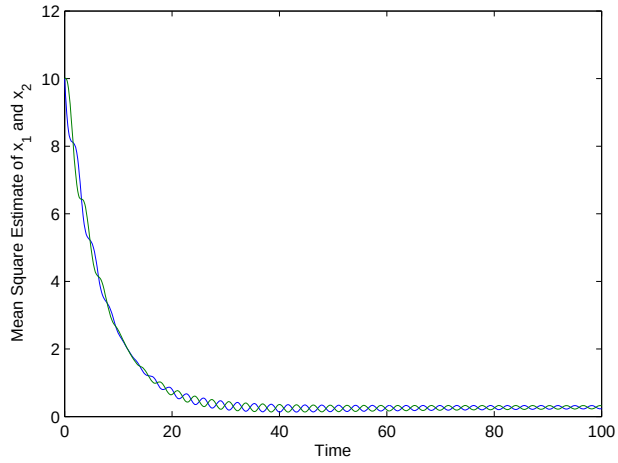


Figure 2.7: Mean squared estimate of controlled $i_L(t)$ (blue) and $v_C(t)$ (green) with $\mathbb{E}[i_L(0)]^2 = \mathbb{E}[v_C(0)]^2 = 10$

.

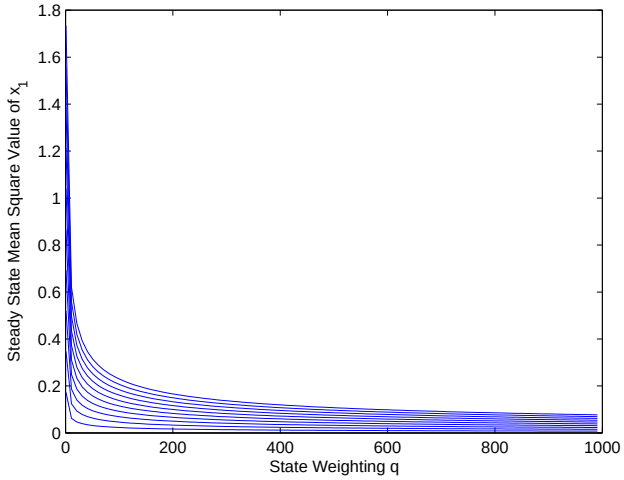


Figure 2.8: Steady state mean squared value of LQG-controlled $i_L(t)$ versus weighting placed on $i_L(t)$ in LQG cost functional. Plots cover range of $2k_bT_mk_m$ from values 1 to 10.

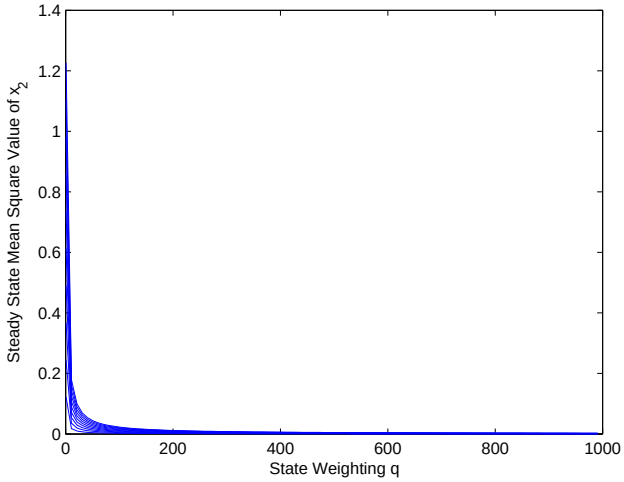


Figure 2.9: Steady state mean squared value of LQG-controlled $v_C(t)$ versus weighting placed on $v_C(t)$ in LQG cost functional. Plots cover range of $2k_bT_mk_m$ from values 1 to 10.

2.5.4 LQG Control to Meausure and Preserve Pre-Measurement State

In this section, we address another measurement problem similar to the one posed in the previous section, in which we attempt to use LQG control to maintain the lossless system's state close to the state it would have evolved to without the measurement taking place. In attempting to avoid being repetitive, we refer back to Eq. (2.4.18), and note that for a given control input $u(t)$,

$$x_{meas}(t) = e^{Ft}x(0) - k_m \int_0^t e^{\left(J - k_m BB^T\right)(t-s)} By(s)ds := x_{unmeas}(t) + \eta(t), \tag{2.5.45}$$

where $\eta(t)$ satisfies the ODE

$$\dot{\eta}(t) = (J - k_m BB^T)\eta(t) - k_m By(t) = (J - k_m BB^T)\eta(t) + B\left(B^T x(t) + \sqrt{\frac{2k_B T_m}{k_m}} w(t)\right). \tag{2.5.46}$$

For the objective of keeping $x_{unmeas}(t)$ close to $x_{meas}(t)$, we form the cost functional

$$\begin{aligned} J =& \mathbb{E}\int_0^{t_f} \left(x_{meas}(t) - x_{unmeas}(t)\right)^T Q \left(x_{meas}(t) - x_{unmeas}(t)\right) + ||u(t)||^2 dt, \\ =& \mathbb{E}\int_0^{t_f} \eta(t)^T Q\eta(t) + ||u(t)||^2 dt \end{aligned}$$

and recall by the separation principle that the optimal control is obtained by separating the estimation and deterministic control problems. We can solve this proceeding in a similar fashion to the last, where we introduce the composite state,

$$v(t) = \begin{bmatrix} x_{meas}(t) \\ \eta(t) \end{bmatrix}. \tag{2.5.47}$$

Now, technically speaking, $\eta(t)$ is always known *exactly*, since it is a deterministic function of both $y(t)$ and $u(t)$, but we will nevertheless still write out an equation for the estimator of $\eta(t)$, using Eq. (2.5.46) as

$$\begin{aligned} \dot{\hat{\eta}}(t) &= \dot{\eta}(t) \\ &= (J - k_m BB^T)\hat{\eta}(t) - k_m By(t) \\ &= (J - k_m BB^T)\hat{\eta}(t) + k_m BB^T \hat{x}_{meas}(t) - k_m Be(t) \end{aligned} \tag{2.5.48}$$

where again we use the innovations process $e(t) = y(t) - B^T \hat{x}_{meas}(t)$. Put together, we get

$$\begin{aligned} \dot{\hat{v}}(t) &= \begin{bmatrix} \dot{\hat{x}}_{meas}(t) \\ \dot{\hat{\eta}}(t) \end{bmatrix} \\ &= \begin{bmatrix} J - k_m BB^T & 0 \\ k_m BB^T & J - k_m BB^T \end{bmatrix} \begin{bmatrix} \hat{x}_{meas}(t) \\ \hat{\eta}(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} K(t) \\ -k_m B \end{bmatrix} e(t). \end{aligned} \tag{2.5.49}$$

where $K(t)$ is given just as in Eq.(2.5.7). Looking now at the controls aspect of this problem, we consider the controller associated with minimizing,

$$J_{det} = \int_0^{t_f} v(t)^T Rv(t) + ||u(t)||^2 dt,$$

where now Q is given by

$$R = \begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix}, \tag{2.5.50}$$

and where the dynamics of $v(t)$ are given by

$$\begin{aligned} \dot{v}(t) &= \begin{bmatrix} \dot{x}_{meas}(t) \\ \dot{\eta}(t) \end{bmatrix} \\ &= \begin{bmatrix} J - k_m BB^T & 0 \\ k_m BB^T & J - k_m BB^T \end{bmatrix} \begin{bmatrix} \hat{x}_{meas}(t) \\ \hat{\eta}(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t). \end{aligned} \tag{2.5.51}$$

Recast in this form, the solution to the original problem of minimizing the cost functional in Eq. (2.5.27) is a rather straightforward application of the equations used in the last section. Defining

$$\tilde{F} = \begin{bmatrix} J - k_m BB^T & 0 \\ k_m BB^T & J - k_m BB^T \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix} \tag{2.5.52}$$

and solving

$$-\dot{S}(t) = \tilde{F}^T S + S\tilde{F} - S\tilde{B}\tilde{B}^T S + R \tag{2.5.53}$$

and partitioning $S(t)$ as in Eq. (2.5.31), we get the optimal deterministic control law, again, as

$$u(t) = \begin{bmatrix} -B^T S_{11}(t) & -B^T S_{12}(t) \end{bmatrix} v(t), \tag{2.5.54}$$

where the individual sub-matrices of $S(t)$ satisfy

$$\begin{aligned} -\dot{S}_{11}(t) =& (J - k_m BB^T)^T S_{11} + S_{11}(J - k_m BB^T) - S_{11} BB^T S_{11} + k_m BB^T S_{12}^T + k_m S_{12} BB^T \\ -\dot{S}_{12}(t) =& (J - k_m BB^T)^T S_{12} + S_{12}(J - k_m BB^T) + k_m BB^T S_{22} - S_{11} BB^T S_{12} \\ -\dot{S}_{22}(t) =& (J - k_m BB^T)^T S_{22} + S_{22}(J - k_m BB^T) - S_{12}^T BB^T S_{12} + Q. \end{aligned} \tag{2.5.55}$$

While these equations taken in their entirety are not amenable to analytic solution, and are not particularly insightful regarding the controller's behavior, they do in fact represent the complete solution of the measurement problem in the case where the intention is to preserve the lossless system's unperturbed trajectory. We could do the sort of analysis on these equations that we did in the previous section to determine the controller's ability to squeeze the individual conjugate variables of the lossless system, but we omit it since the arguments would be nearly identical, and the end result is the same. As we would expect given the sort of ideal control strategy given by Eq.(2.5.1)

and the observations in Section 2.5.1, with sufficiently low cost on control effort, the LQG controller can squeeze states arbitrarily well.

2.6 Concluding Remarks

Summarizing what we have shown, in the studied measurement model, the correlation between the process and observation noise allows for eventual perfect measurement of the state as long as (F, B) is controllable and $F \leq k_m I$. This state is of course one of a system that has been disturbed by the measurement process itself, so this property of the estimation problem may not be of benefit, depending on the intention of the experiment. However, in the case where F is skew-Hermitian, the smoothed estimate of the initial state also goes to zero, so an observer could measure a state to arbitrary precision if she were to accept destroying the state and waiting an arbitrarily long time. To address measurement scenarios where it would be desirable to preserve the state while extracting information about it, we must envision adding active elements to the measurement model to account for the effects of measurement back action. Whereas the passive measurement model and associated estimation problem were formulated with very basic assumptions, with pretext coming from physical theory, an analysis of active measurement is much more susceptible to the allowance of unrealistic idealized devices. In most cases, the mere addition of an active device is accompanied by additional noise sources [35], annulling the special properties of the estimation problem and the perfect measurement scenarios associated with it. In addition, proper bookkeeping of uncertainty must also account for the error in computation that comes with calculating any sort of optimal control signal, as is in the case of the LQG measurement problem that we examined, and we do not know how to quantify this in any general sense. In light of this, by far the most realistic approach to actively measuring back action that we have looked at is the cancelling of the deterministic component of the measurement back action using a non-linear system, as this does not require any computation beyond the estimation filter. For this type of measurement, in the case where the system being measured is lossless, there is a non-zero temperature dependent lower bound on estimation errors of the conjugate variables. Stepping into the realm of active measurement with highly idealized active devices, we looked at measurement with LQG control to offset the measurement back action, without reference to how control signals would be generated or what noise they might have. Rather unsurprisingly, in the limit of no cost on control, this controller can eventually hold the state to its pre-measured value or to the state to which it would have evolved to if the measurement had not been done.

Chapter 3

Physical Constraints and Low-Order Approximations of Random High-Dimensional Lossless Systems

In the previous chapter, we derived results on measurements with components including dissipative elements satisfying a fluctuation-dissipation relation, with our justification for doing so coming from Theorem 1.3.2. While that theorem states that in the topology induced by the $L_{2[0,T]}$ norm that the lossless systems are dense in the set of dissipative systems, the construction of these lossless approximations have properties that are very different from what how we would expect dissipative structures like macroscopic resistors to look like in nature. The lossless approximations constructed in [3] are such that the system frequencies are perfectly-spaced integer multiples of each other in a Fourier series expansion of the dissipative impulse response, and with the recurrence manifesting itself as a perfect copy of the original output at a period T later. By contrast, most macroscopic models [36], [37] of real resistors involve elements like the random motion of thermally excited electrons in a lattice, which heuristically seems very far removed from the exactly 'designed' abstract resistor presented in Example 1.3.3. That circuits can, and often are, intentionally designed so that they nearly appear as resistors is well-known. For instance, early supercomputers were designed for signal balance, with even individual wire lengths being chosen so as to minimize switching noise, to the point where power supplies for these devices usually did not need be regulated at all.

In this chapter, we will focus on the question of: what are the properties of large lossless systems such that they admit low dimensional dissipative approximations on finite time horizons? We will first introduce a well-known approach for finding reduced-order models for which there are error bounds for the approximation. Next, we will look at the most extreme case of model order reduction; that is whether pure resistors (zero order systems) approximate any class of random lossless LTI systems. The answer in the linear case seems to be in the negative. Before doing so, we will first introduce concepts in model reduction techniques to be able to pose problems that might answer

these questions.

3.1 Finite Time Horizon Model Reduction by Balanced Truncation

We will briefly outline one method for obtaining reduced-order models of linear systems. Our intention for introducing this procedure is to be able to pose a problem concerning the reduction of random lossless systems in the next section. Given an LTI system of the form 1.2.8, in the case where A is stable, and the pairs (A, B) and (A, C) are controllable and observable respectively, the objects

$$G_c^\infty = \int_0^\infty e^{At} B B^T e^{A^T t} dt \tag{3.1.1}$$

$$G_o^\infty = \int_0^\infty e^{A^T t} C^T C e^{At} dt, \tag{3.1.2}$$

denoted the controllability and observability gramians respectively, are well-defined and satisfy the algebraic Lyapunov equations

$$A G_c^\infty + G_c^\infty A^T + B B^T = 0 \tag{3.1.3}$$

$$A^T G_o^\infty + G_o^\infty + C^T C = 0. \tag{3.1.4}$$

This is seen immediately by differentiating the *'finite time'* gramian

$$G_o(t) = \int_0^t e^{A^T s} C^T C e^{As} ds \tag{3.1.5}$$

(in the case of G_0^∞) under the integral sign, taking $t \to \infty$ and observing that A is stable. Recall we encountered these objects in Section 2.4, and in the case of our estimation problem found that for A *unstable* that the Gramian of $(-A, C)$ represented the inverse of the steady state minimum estimation error when the system was observed with additive white noise. Likewise, the controllability and observability gramians themselves have simple interpretations. For an arbitrary initial state x_0, the quadratic form

$$x_0^T G_o(t) x_0 = G_o(t) = \int_0^t x_0^T e^{A^T s} C^T C e^{st} x_0 ds = \int_0^t \|C e^{As} x_0\|^2 ds = \int_0^t \|y(s)\|^2 ds \tag{3.1.6}$$

so that the eigenvalues of $G_0(t)$ can be interpreted as the 2-norm of the system output integrated over time $[0, t]$ when the system initial condition is the corresponding eigenvector of G_0. The dual

of this scenario can be seen by asking the question: given a state $x(0) = x_0$, and given a finite time period to work with $[-t, 0]$, among all inputs $u(t)$ restricted to be nonzero on $[-t, 0]$ what is the minimum value of

$$\int_{-t}^{0} \|u(s)\|^2 ds \tag{3.1.7}$$

such that $u(t)$ takes the system from state $x(-t) = 0$ to $x(0) = x_0$? To answer this, consider the two operators $K : x \to f(t)$ and $L : f(t) \to x$, where $f(t)$ are functions in $L_2[-t, 0]$ given by

$$K(x) = \begin{cases} B^T e^{-A^T s} x & -t \leq s \leq 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.1.8}$$

and

$$L : f(s) \to \int_{-t}^{0} e^{-As} B f(s) ds. \tag{3.1.9}$$

These operators are adjoints of each other, since for any given pair $x, f(s)$

$$x^T \left(Lf(s)\right) = \int_{-t}^{0} x^T e^{-As} B f(s) ds = \int_{-t}^{0} f^T(s) B^T e^{-A^T s} x ds = \int_{-t}^{0} f^T K(x) ds \tag{3.1.10}$$

Noticing

$$L \circ K = \int_{-t}^{0} e^{-As} B B^T e^{-A^T s} ds = G_c(t), \tag{3.1.11}$$

we make the assertion that the minimizing $u(s)$ is given by

$$u_{min}(s) = K G_c^{-1}(t) x_0. \tag{3.1.12}$$

Clearly, this input $u(t)$ takes the state of the system from $x(-t) = 0$ to $x(0) = x_0$, since

$$x(t) = \int_{-t}^{0} e^{-As} B B^T e^{-A^T s} G_c^{-1} x_0 ds = \int_{-t}^{0} e^{-As} B B^T e^{-A^T s} ds G_c^{-1} x_0 = x_0. \tag{3.1.13}$$

To show that this is the $L_2[-t, 0]$ minimizing $u(t)$, consider the operator $P = K \circ G_c^{-1} \circ L$, and notice using Eq. (3.1.11) that $P^2 = P$ and that $P u_{min}(t) = u_{min}(t)$. Thus, the operators P and $I - P$ can be used to decompose a given function on $[-t, 0]$ into components that are orthogonal with respect to the inner product

$$\int_{-t}^{0} v^T(s) w(s) ds. \tag{3.1.14}$$

So, take any other function $u(t)$ satisfying $Lu(t) = x_0$, and decompose it as $u(t) = Pu(t)+(I-P)u(t)$. Then,

$$\begin{aligned}\int_{-t}^{0}\|u(s)\|^2 ds \geq \int_{-t}^{0}\|Pu(s)\|^2 ds = \int_{-t}^{0}\|KG_c^{-1}(t)x_0\|^2 = \int_{-t}^{0}\|u_{min}(s)\|^2 ds \\ = \int_{-t}^{0} u_{min}^T(s)KG_c^{-1}(t)x_0 ds = x_0^T G_c^{-1} L u_{min} = x_0^T G_c^{-1} x_0,\end{aligned} \tag{3.1.15}$$

where the second to the last equality follows from Eq. (3.1.10). Thus, an interpretation of the controllability gramian is that through quadratic forms of its inverse, it tells us the minimum effort required, expressed as the $L_2[-t, 0]$ energy of the input function, that it takes to 'create' a state at time $t = 0$ starting with $x(-t) = 0$. This sort of minimization is the same procedure one would use to prove that linear systems have quadratic available storage functions, as asserted in Section 1.2.

It turns out that the states of the system can be transformed such that in this new representation the gramians satisfy $G_c(t) = G_o(t) = D$ where D is a diagonal matrix that has as entries the eigenvalues of $G_c(t) \cdot G_o(t)$. For an explicit formula of the transformation, see [38]. In this form, there is a 'balance' in the weighting placed on control effort and output energy. One approach to obtaining a reduced-order model involves, in essence, 'throwing away' those states that take a great deal of energy to control, and those for which their effect on the output energy of the system is low. The singular values of the aforementioned gramians give a measure of exactly those two quantities for individual states, so it is not unreasonable to presume that the input-output properties of a reduced-order system obtained in this manner will approximate those of the original [39]. Indeed, this is the case, and we will very briefly outline this 'balanced truncation' method.

Assuming $G(s)$ is *stable* and that we have a minimal state space realization (i.e., there are no unobservable or uncontrollable states) with state dimension n, and assuming the states have been transformed such that $G_c^\infty = G_o^\infty = D$, a reduced-order system $G_r(s)$ obtained by eliminating the $n - m$ states corresponding to the $n - m$ smallest singular values of D, will necessarily satisfy

$$2 \cdot d_{m+1} \leq \|G - G_r\|_\infty \leq 2 \cdot \sum_{i=m+1}^{n} d_i \tag{3.1.16}$$

where we are labeling the diagonal elements of D such that $d_1 \geq d_2 \geq \ldots d_n$.

Unfortunately, the bounds above only hold for reduction on an infinite time-horizon, and is based on the infinite-time Gramians, which precludes a direct application of this technique to unstable systems. However, there are avenues to circumvent this issue. In [40], it is shown that by starting with an unstable system, by shifting the state matrix $A \to A - aI$, where 'a' is a constant such that $A_a = A - aI$ is stable, performing balanced truncation on this stable system, and shifting the resulting A_r from reduction back $A_r \to A_r + aI$, that the resulting model approximates the original

such that

$$\|G - G_{r,a}\|_{L_2[0,t],ind} \leq 2e^{at} \sum_{i=m+1}^{n} d_i \tag{3.1.17}$$

where again, the d_i are obtained from the infinite-time-horizon balanced gramians of the shifted system, and where $\| \quad \|_{L_2[0,t],ind}$ is the $L_2[0,t]$ induced norm.

For the purposes of computation, this result is useful; however, for generalizing how large lossless systems admit low-order dissipative approximations, the numerous steps in the construction of the reduced model makes its use in a theoretical investigation difficult. Another approach for reducing unstable systems is to directly perform the balanced truncation technique using the *finite time* gramians $G_o(t), G_c(t)$ [41], [31]. Yet another, for discrete time systems, is outlined in [42]. In practice, if the input and output dimensions are much smaller than the state dimension, this approach generally has two salient features that have yet to be proven. First, the resulting lower-order model usually very well approximates the original over the time horizon $(0,t)$, after which the approximation abruptly becomes very poor. Second, in practice the spectrum of the singular values d_i usually have a sharp cutoff after the first few largest values. We illustrate these features in the following example of a single-input single output lossless system with $C = B^T$.

Example 3.1.1 (Circularly Coupled Oscillators)**.** We will consider a chain of N identical masses and springs such that the chain closes in on itself and for which the input and output are the input velocity and output position of just one of the masses. That is, the dynamics are simply

$$k\ddot{x}_n = m(x_{n+1} + x_{n-1} - 2x_n), \tag{3.1.18}$$

for which it is simple to show that the impulse response is given by the expression

$$\frac{1}{N}\sum_{l=0}^{N-1} \cos(t\sqrt{2\frac{k}{m}(1-\cos(\frac{2l\pi}{N}))}) \to J_0(2t\sqrt{k/m}) \quad \text{as} \quad n \to \infty. \tag{3.1.19}$$

Rather remarkably, performing balanced truncation model reduction on this system, regardless of how large a number of oscillators are introduced, yields excellent approximations using only two-dimensional dissipative systems. The singular values of the gramians in this case drop off precipitously.

What this suggests is that to gain some understanding of the lossless $\to$ dissipative transition, it might be beneficial to attempt to generalize the behavior of the eigenvalue spectrum of integrals of the form,

$$\int_0^t e^{Js} B B^T e^{-Js} ds, \tag{3.1.20}$$

where $J = -J^T$ and B is a vector, to see if there is a structure on J that leads to fast eigenvalue decay of the above integral. Unfortunately, we have not been able to make any such characterization.

3.2 Resistors from Random Lossless Systems

In this section, we take a glimpse at a rather extreme notion of model reduction, that is, whether perhaps 'resistors' derive as approximations to some class of large lossless random systems. We will first focus on how the frequencies of A impact the ability to make such an approximation. As a first step, let's consider a SISO lossless system with random frequencies and an impulse response of the form

$$g_N^+(t) = \left(\sum_{l=1}^{N} \cos \omega_l t \right)^+, \tag{3.2.1}$$

where ω_l are *i.i.d.* random variables with probability density $\phi(\omega)$, and where as before, $()^+$ refers to the causal portion of the argument. The characteristic function of the random variable $\cos(\omega_l t)$ defined as [43]

$$F_l(x) = \mathop{\mathbb{E}}_{\omega_l}[e^{ix\cos(\omega_l t)}]$$

can be calculated immediately using the identity

$$e^{ix\cos(\theta)} = \sum_{n=-\infty}^{\infty} i^n J_n(x) e^{in\theta},$$

where $J_n(x)$ are Bessel functions of the first kind, and the convergence is uniform in x. Using this identity, and taking expectation with respect to $\theta = \omega t$, we get

$$\begin{aligned} \mathop{\mathbb{E}}_{\theta}[e^{ix\cos(\theta)}] = \sum_{n=-\infty}^{\infty} i^n J_n(x) \mathop{\mathbb{E}}_{\theta}[e^{in\theta}] \implies \\ F_l(x) = \sum_{n=-\infty}^{\infty} i^n J_n(x) \mathop{\mathbb{E}}_{\omega}[e^{in\omega t}] = \sum_{n=-\infty}^{\infty} i^n J_n(x) \Phi_l(nt), \end{aligned}$$

where we have defined $\Phi_l(v) = \mathbb{E}_{\omega_l}[e^{vy\omega_l}]$. Since the ω_l are assumed to have the same density $\phi(\omega)$, $\Phi_l(v) = \Phi(v) = \mathcal{F}(\phi(\omega))$, where $\mathcal{F}$ is the Fourier transform operator. So we have the characteristic function for each term in the sum 3.2.1, and since the characteristic function for a sum of independent random variables is the product of the characteristic functions of those random variables, we get

$$\mathop{\mathbb{E}}_{\{\omega_l\}}[e^{ixg_N(t)}] = \left(\sum_{n=-\infty}^{\infty} i^n J_n(x) \Phi_l(nt) \right)^N := L(x). \tag{3.2.2}$$

From this expression, we can easily calculate all the moments of $g(t)$ by the relation

$$\mathbb{E}[g_N^k(t)] = (-i)^k \frac{d^k L}{dx^k}\bigg|_{x=0} \tag{3.2.3}$$

using the identity

$$J'(x) = \frac{(J_{n-1}(x) - J_{n+1}(x))}{2} \quad n \neq 0, \quad J_0'(x) = -J_1(x) \tag{3.2.4}$$

and the fact that $J_0(0) = 1$, $J_n(0) = 0$ for $n \neq 0$. We can immediately calculate the mean since $\phi(\omega)$ is a real pdf, and so

$$\mathbb{E}[g_N(t)] = N\left(\Phi(0)\right)^{N-1} \frac{\Phi_N(t) + \Phi_N(-t)}{2} = N\Phi_N(t) \tag{3.2.5}$$

Now, if we are at least heuristically motivated by the approximations referenced throughout this thesis, where we saw that lossless approximations of dissipative systems required systems with ever increasing frequencies, we might try to examine the case where the random frequencies are distributed uniformly over a support that increases linearly with N. That is, taking

$$\phi_N(\omega) = \frac{1}{2N}\chi_{[-N,N]}(\omega) \xrightarrow{\mathcal{F}} \Phi_N(v) = \frac{\sin Nv}{Nv} \tag{3.2.6}$$

where $\chi_{[-N,N]}$ is the characteristic function over the interval $[-N, N]$, and using Eq. (3.2.5), we get

$$\mathbb{E}[g_N(t)] = \left(\frac{\sin Nt}{t}\right), \tag{3.2.7}$$

so that for a smooth L^1 integrable $u(t)$,

$$\mathop{\mathbb{E}}_{\{\omega_l\}} y(t) = \int_{-\infty}^{t} \mathbb{E}[g_N^+(t-\tau)]u(\tau)d\tau = \int_{-\infty}^{t} \left(\frac{\sin N(t-\tau)}{(t-\tau)}\right)^+ u(\tau)d\tau. \tag{3.2.8}$$

In the limit as $N \to \infty$, it is well-known that this integral converges in L^1 to $\pi u(t)/2$. So while this random system appears as a resistor in the mean as $N \to \infty$, what can we say about the second-order moment? We can calculate $\mathbb{E}[g_N(t)]^2$ from the moment-generating function already given.

$$\mathbb{E}[g_N^2(t)] = -\left.\frac{d^k L}{dx^k}\right|_{x=0} = \left(N(N-1)\Phi_N^2(t) + \frac{N}{2}\left(1 + \Phi_N(2t)\right)\right) \tag{3.2.9}$$

So, the variance of $g_N(t)$ is

$$\mathbb{E}[g_N^2(t)] - \left(\mathbb{E}[g_N(t)]\right)^2 = \frac{N}{2}(\Phi_N(2t) - 2\Phi^2(t) + 1), \tag{3.2.10}$$

which cannot go to zero as $N \to \infty$.

3.2.1 Random Lossless Systems Approximating Band-Limited Resistors

What we have gathered from the previous example is that it is too much of a leap to take the simultaneous limits of a large number of oscillators, and a corresponding increase in the support of the density of the *i.i.d* frequencies in the random cosine system. As a simple illustrative example of what we would more generally like to show for random lossless sytems, let's look instead at the same problem, but this time fixing the distribution on the system frequencies as

$$\phi(\omega_l) = \frac{1}{2M}\chi_{[-M,M]}(\omega_l), \tag{3.2.11}$$

where again $\chi_{[-M,M]}$ is the characteristic funcition over the interval $[-M, M]$, and taking

$$g_N^+(t) = \frac{M}{N}\left(\sum_{l=1}^{N}\cos\omega_l t\right)^+. \tag{3.2.12}$$

By the L^2 Weak Law of Large Numbers [43] we have that

$$\begin{aligned}\frac{1}{N}\left(\sum_{l=1}^{N}\cos\omega_l t\right) \xrightarrow{\mathcal{L}^2} \mathbb{E}\left(\cos\left(\omega_l t\right)\right) &= \int_{-\infty}^{\infty}\phi(\omega_l)e^{j\omega_l t}d\omega_l \\ &= \mathcal{F}.\mathcal{T}.\left(\frac{1}{2M}\chi_{[-M,M]}(\omega_l)\right) \\ &= \frac{\sin Mt}{Mt},\end{aligned} \tag{3.2.13}$$

for any finite t. That is, $g_N^+(t)$ approximates the causal portion of an ideal low-pass filter with cutoff frequency M given by

$$g_N^+(t) = \left(\frac{\sin Mt}{t}\right)^+. \tag{3.2.14}$$

In fact, we can immediately determine just how $g_N(t)$ converges in L^2 as a function of t:

$$\begin{aligned}\mathbb{E}\left(\frac{M}{N}\sum_{l=1}^{N}\cos(\omega_l t) - \mathbb{E}\,g_N(t)\right)^2 &= \left(\frac{M}{N}\sum_{l=1}^{N}\left(\cos(\omega_l t) - \mathbb{E}\cos(\omega_l t)\right)\right)^2 \\ &= \frac{M^2}{N^2}\cdot N\cdot\mathbb{E}\left(\cos(\omega_l t) - \mathbb{E}\cos(\omega_l t)\right)^2 \\ &= \frac{M^2}{N}\left(\mathbb{E}\cos^2(\omega_l t) - \left(\mathbb{E}\cos(\omega_l t)\right)^2\right) \\ &= \frac{M^2}{N}\left(\frac{1}{2} - \frac{\mathbb{E}\cos(2\omega_l t)}{2} - \frac{\sin^2(Mt)}{M^2t^2}\right) \\ &= \frac{M^2}{N}\left(\frac{1}{2} + \frac{\sin(2Mt)}{8Mt} - \frac{\sin^2(Mt)}{M^2t^2}\right),\end{aligned} \tag{3.2.15}$$

where the second inequality follows from the independence of the ω_l.

As a simple illustration, the impulse response of a random cosine system with cutoff frequency $M = 10$ rad/s and $N = 2 \times 10^4$ is shown in Figure (3.1). The response of a random cosine system, with $M = 10$ rad/s and $N = 2500$, to the band-limited function $sinc(t)$ is shown overlaid with the system input in Figure (3.2).

What this simple example shows is that for linear systems with state space representation of the form (A, B, B^T), it should be possible to characterize the distributions on random A, B such that the input-ouput relationship of the system is that of a band-limited resistor for sufficiently high state dimension. For instance, replacing the coefficients in the sum in 3.2.12 by identical independent random variables b_l, giving the form

$$g_N^+(t) = \frac{M}{N}\left(\sum_{l=1}^{N} b_l \cos \omega_l t\right)^+, \tag{3.2.16}$$

does not change the result as long as the distribution on b_l is such that $\mathbb{E}\, b_l = 1$. We have at least a hint of an answer to the question we asked in the beginning of the chapter: namely, what statistical properties of random linear systems are necessary to admit low-dimensional dissipative approximations? This is an open question that we hope would be a topic of future research. What we are also left with from this simple construction is additional justification for the measurement model in Chapter 2, an explanation that goes beyond the rationale of [3] and the theorem on lossless approximations of dissipative systems stated in Theorem (1.3.2).

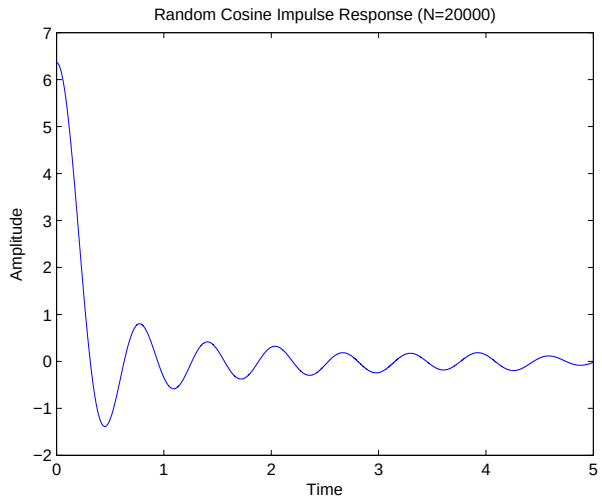


Figure 3.1: Impulse response of random cosine system with $M = 10$ rad/s and $N = 2e4$

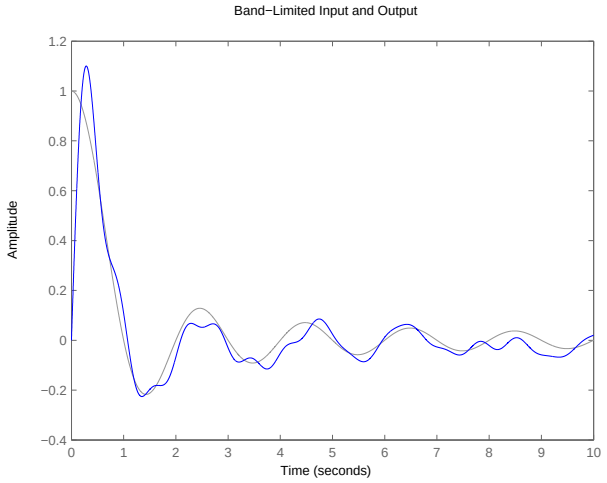


Figure 3.2: Overlaid input and output with $u(t) = sinc(t)$ and $M = 10$, $N = 2500$

3.2.2 Extension to Arbitrary Dissipative Systems and Connections with Model Reduction

The types of random lossless systems treated in the previous sections, and the form of the underlying distributions of the parameters of these systems, demonstrate one straightforward manner in which an arbitrary dissipative system can appear from lossless dynamics; the distributions on the frequencies of the random cosine system can simply be chosen as the Fourier transform of the impulse response of the dissipative realization. What we have attempted to investigate, but still do not have a grasp on, is how the techniques of model reduction from Section 3.1 might tie into these types of realizations. More specifically, if we *a priori* choose the distribution of the frequencies of a lossless system such that the impulse response approximates a low-order dissipative system, how do the singular values of the finite-horizon Gramians, as in Eq. (3.1.20), behave with a fixed time interval and increasing N? Is the distribution of singular values such that there is a clear cutoff that corresponds to the dimension of the dissipative system? These are open questions that we would propose for future investigation.

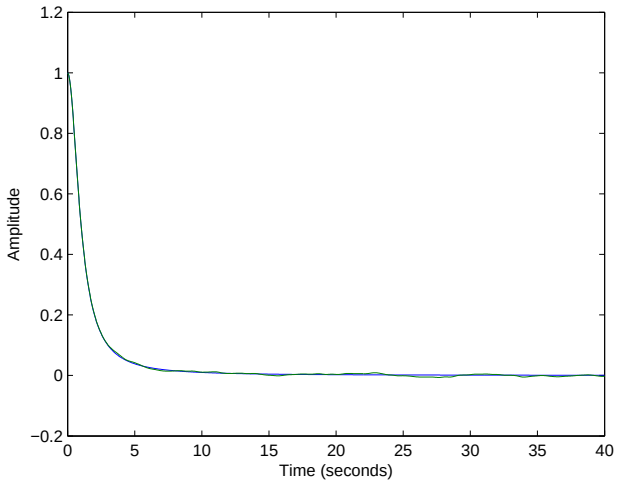


Figure 3.3: Impulse response of random cosine system with exponential density on frequencies, plotted against $h(t) = \frac{1}{1+t^2}$

Chapter 4

Conclusion and Future Research

What we have found is that from a purely mathematical view, characterizing dissipation in open systems and the limits of even idealized measurements, at least for the simple case of linear systems, can be well-understood. However, an important limitation of our research is that we still need to better understand how the dynamics of those mathematical models that have the input-output relationships of common physical devices, like the generalized resistors, can be matched to the underlying physical processes that are responsible for the macroscopic responses of these real devices. It is important to note that the generalized resistor example given in [3] is not realized in the form of a repeating LC ladder; rather, its realization would take a more complicated interconnection structure of inductances and capacitances. Solid-state physics models of electronic devices abound, each with varying assumptions about their structure and operating regimes, and it would be interesting to be able to see whether some of these models themselves can be approximated by lossless systems with energy sources. This is not an avenue we investigated deeply enough, but is an important topic for us, as it would give the results presented in this thesis more practical, physical relevance.

An obvious extension of our work would be to consider nonlinear systems, but of course by doing so, we would lose most of our more elementary tools from estimation theory, tools that allowed us to draw the conclusions that we ultimately did on measurement with back action. The special structure of the measurement model we used, which allowed for a type of 'cancellation' of the process and observations noises in the linear case, would no longer be present in the non-linear case, and generally we would not be able to estimate the measured system arbitrarily well, as we showed was possible under certain conditions where the observation was taken over a long enough time period. The measurement problem we analyzed in this thesis represents the most lax possible, affording the most freedom in measurement precision. Despite this, we found that adding even the slightest constraint on control to offset measurement back action ruins the ability to estimate perfectly. Back action ultimately does lead to measurement uncertainty, uncertainty that in any practical sense cannot be reduced beyond a certain amount, a bound which we were able to explicitly calculate for certain cases.

We would have liked to explain how the types of model reduction methods presented in Chapter 4 almost invariably yield very low-order accurate approximations to the dynamics of subsystems of much larger systems on finite time horizons. While heuristically we would expect this to be the case, showing this systematically has proven to be difficult, and doing so for the balanced truncation approach to reduction would entail proving a result on the decay of the singular values of the finite time gramians over a class of 'typical' dynamical systems, however that concept is to be properly defined. Perhaps an approach using random matrix theory, where a probability measure is defined on some class of system matrices would be a first step, but so far this has been untractable for us. Conceptually, this reduction problem is essentially the same as 'tracing out' the states in the quantum Liouville equation presented in Section (1.4), but the similarity does not end there. While Eq. (1.4.8) neatly gives an expression for the dynamics of an open subsystem, actually calculating the operators in that equation that are associated with dissipation is generally not possible. Developing any theory towards the end of some more general formulas for some class of systems would not only be interesting from a dynamical systems theory view, but could be extremely useful in some physics problems.

The areas in which we were able to make progress include providing a complete characterization of the solutions of the estimation problem arising from the measurement model in [3], and providing an alternative explanation for the transition from microscopically lossless to macroscopically dissipative systems, using a class of random linear systems. With regards to the latter contribution, it could be argued that the random systems interpretation has properties that are more desirable for making connections to the features of dissipative physical systems than does the fixed-time horizon deterministic Fourier series expansion outlined in Section 1.3.2. This interpretation is more robust, in that small changes in frequencies to any individual oscillator do not 'ruin' any particular realization, and the realizations are not dependent on the time horizon of interest, with no fixed recurrence times. In addition, this alternative view lends additional justification to the assumptions made in forming the measurement model in Chapter 2.

Concerning the estimation problem arising this measurement model, we were able to show generally that a system can be measured arbitrarily well, even in the presence of a single environmental thermal noise source. Even though the inclusion of the environment as a component of the measurement process requires that the measured system is affected, we can eventually perfectly estimate the state of a lossless system, at any given time, even before measurement, albeit with delay. One possible alternative way to interpret the measurement dynamics in the case where the deterministic back action in our measurement model is offset is to think of measurement as not necessarily including a dissipative element at all, but simply assuming that the system is heated up during any measurement, as it must be coupled to the environment in some way for any experiment to be done on it. As our results showed, taking this interpretation would of course imply non-zero lower bounds

on measurement error, but these bounds would not include a product tradeoff in uncertainty on conjugate variables, as in the case of measurement of quantum systems.

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