STUDIES IN NUCLEAR REACTOR DYNAMICS

I. THE ACCURACY OF POINT KINETICS

II. THE EFFECT OF DELAYED NEUTRONS ON THE SPECTRUM OF THE GROUP-DIFFUSION OPERATOR

Thesis by

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ABSTRACT

This thesis is a theoretical work on the space-time dynamic behavior of a nuclear reactor without feedback. Diffusion theory with G-energy groups is used.

In the first part the accuracy of the point kinetics (lumped-parameter description) model is examined. The fundamental approximation of this model is the splitting of the neutron density into a product of a known function of space and an unknown function of time; then the properties of the system can be averaged in space through the use of appropriate weighting functions; as a result a set of ordinary differential equations is obtained for the description of time behavior. It is clear that changes of the shape of the neutron-density distribution due to space-dependent perturbations are neglected. This results to an error in the eigenvalues and it is to this error that bounds are derived. This is done by using the method of weighted residuals to reduce the original eigenvalue problem to that of a real asymmetric matrix. Then Gershgorin-type theorems are used to find discs in the complex plane in which the eigenvalues are contained. The radii of the discs depend on the perturbation in a simple manner.

In the second part the effect of delayed neutrons on the eigenvalues of the group-diffusion operator is examined. The delayed neutrons cause a shifting of the prompt-neutron eigenvalues and the appearance of the delayed eigenvalues. Using a simple perturbation method this shifting is calculated and the delayed eigenvalues are predicted with good accuracy.

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I. INTRODUCTION

A. Problems of Nuclear Reactor Dynamics

The prediction of the time-response of a nuclear reacting system under given circumstances is one of the fundamental problems in reactor physics. It is intimately related to the question of safety which in recent years has been of great importance in connection with the development of large fast reactors.

Unfortunately the problem is extremely complex. To solve the transport equation along with the ordinary or partial differential equations which describe the energy transfer in the system is a rather impossible task. Faced with insurmountable difficulties reactor physicists have developed many approximate methods of dealing with the problem. The next question, of course, is when do these approximations give satisfactory results and when do they fail to describe the situation at hand. Definite answers to these questions are very difficult to obtain; numerical investigations, intuition and experience help to find the optimum model in each case.

In recent years the need to investigate the space-time problem has been recognized. In the new large reactors the transients depend very much on the spatial distribution of the perturbations. For example, the accidental withdrawal of a control rod from a large reactor causes a power rise and a distortion of the flux shape. To estimate the feedback effects it is essential that this tilting be taken into account. Also, in a Liquid Metal Cooled Fast Breeder Reactor (LMFBR) the effects of

loss of the sodium coolant depend on where the sodium was lost. If the loss occurred in an inner region positive feedback reactivity coefficient might result. With the disastrous effects that an uncontrolled transient in an LMFBR may have, it is clear how important the understanding of the space-time behavior of the reactor is. Approximate methods are employed again and the need for analytical estimates of their dependability becomes even more pronounced.

B. Review of Theoretical Work

The quantity of interest in a nuclear reactor is the neutron angular density. It describes the distribution of neutrons in space, velocity space and time. Conservation of neutrons is expressed by the transport equation (Boltzmann equation) as follows[2]

$$\frac{\partial n}{\partial t} + v\underline{\Omega}\cdot\nabla n + \Sigma v n = \int\int \Sigma' f v' n' d\underline{\Omega}\, dE' + Q$$

where $n(\underline{r}, \underline{\Omega}, E, t)$ is the number of neutrons at $\underline{r}$ moving with direction $\underline{\Omega}$ and having energy E at time t per unit volume per unit solid angle per unit energy.

v = speed of neutrons

$\Sigma(\underline{r}, E)$ = total macroscopic cross section for all interactions

$Q(\underline{r}, \underline{\Omega}, E, t)$ = source density of neutrons

Total probability density of neutron transfer from $\underline{\Omega}'$, E' to $\underline{\Omega}$, E =

$$\Sigma(\underline{r}, E')\, f(\underline{r}; \underline{\Omega}', E' \rightarrow \underline{\Omega}, E)$$

For convenience we have defined

$$\Sigma' \equiv \Sigma(\underline{r}, E')$$

$$n' \equiv n(\underline{r}, \underline{\Omega}', E', t)$$

The above equation must be modified to account for the delayed neutrons. These are emitted during the decay of fission products and they are distinguished from the prompt neutrons which are emitted immediately after fission. If ν neutrons result from a fission on the average, a fraction $\beta_i \nu$ are emitted by the fission products of the i^{th} kind (i^{th} precursor) with normalized spectrum $\chi_i(E)$ and $(1-\beta)\nu$, $\beta = \sum_{i=1}^{6} \beta_i$, are prompt neutrons with spectrum $\chi_p(E)$.

Then the appropriate equations are

$$\frac{\partial n}{\partial t} + v\underline{\Omega} \cdot \nabla n + \Sigma v n = \iint \sum_{x \neq f} \Sigma'_x f_x n' v' d\underline{\Omega}' dE' +$$

$$+ \frac{\chi_p}{4\pi}(1-\beta)\nu \iint \Sigma'_f v' n' d\underline{\Omega}' dE' + \sum_i \lambda_i C_i(\underline{r},t) \frac{\chi_i}{4\pi} + Q$$

$$\frac{\partial C_i}{\partial t} + \lambda_i C_i = \beta_i \nu \iint \Sigma'_f v' n' d\underline{\Omega}' dE' \qquad i = 1,2,\ldots,6$$

where

$\Sigma_x(\underline{r},E,t)$ = cross section for the x process (absorption, etc.)

$\Sigma_f(\underline{r},E,t)$ = fission cross section

$C_i(\underline{r},t)$ = density of precursors

λ_i = decay constant of precursors

If the boundary of the reactor is such that once a neutron has escaped it cannot come back, the boundary condition is

$$n(\underline{R},\hat{\Omega},E,t) = 0 \quad \text{if} \quad \hat{k} \cdot \hat{\Omega} < 0$$

where $\hat{k}$ is the unit vector normal to the boundary surface $\underline{R}$ pointing outwards. Initial conditions on n and C_i must also be given.

These equations are extremely difficult to work with and useful approximations have been made to render them to further analysis. In reactor dynamics we are mainly interested in the space-time problem in large systems, so we eliminate the dependence on $\underline{\Omega}$ by expanding in spherical harmonics and retaining the first two terms; the energy dependence is treated with the multigroup method, i. e., the energy range is divided into G groups and in each group we define the neutron density and the appropriate averages of the above parameters[2]. This leads to a matrix formulation of the problem which with the notation of Reference 1, is

$$\left(\nabla \cdot \underset{\sim}{D}\nabla - \underset{\sim}{R}_a - \underset{\sim}{R}_s + \underset{\sim}{S} + (1-\beta)\underline{\chi}_p \underline{F}^T\right)\underline{N} + \sum_{i=1}^{6} \lambda_i \underline{\chi}_i C_i + \underline{Q} = \frac{\partial}{\partial t}\underline{N} \qquad (1.1)$$

$$\beta_i \underline{F}^T \underline{N} - \lambda_i C_i = \frac{\partial C_i}{\partial t} \quad , \quad i = 1,\ldots,6 \qquad (1.2)$$

where

$\underline{N}(\underline{r},t)$ = G-dimensional column vector of neutron densities

v_g = average speed of neutrons in group g

$C_i(\underline{r},t)$ = concentration of the i^{th} precursor

β_i = fraction of the i^{th} precursor

λ_i = decay constant of the i^{th} precursor

$\underset{\sim}{D}(\underline{r},t)$ = G×G diagonal matrix of group diffusion coefficients $D^g v_g$

$\underset{\sim}{R}_a(\underline{r},t)$ = G×G diagonal matrix of macroscopic absorption cross sections $\Sigma_a^g v_g$

$\underset{\sim}{R}_s(\underline{r},t)$ = G×G diagonal matrix of scattering cross sections $\Sigma_s^g v_g$

$\underset{\sim}{S}(\underline{r},t)$ = G×G matrix of removal cross sections $\Sigma_s^{g' \to g} v_{g'}$

$\underline{\chi}_p, \underline{\chi}_i$ = column vectors of prompt fission and precursor decay spectra

$\underline{F}(\underline{r},t)$ = G×1 column vector with elements $\nu^g \Sigma_f^g v_g$, where ν^g is the average number of neutrons per fission induced by a neutron of the energy group g and Σ_f^g the macroscopic fission cross section of group g

$\underline{Q}(\underline{r},t)$ = column vector of external sources

The above equations are supplemented with boundary conditions which here are taken to be

$$\underline{N}(\underline{R},t) = \underline{0}$$

where $\underline{R}$ describes the boundary of the system. At interfaces the usual requirements of continuity of density and current apply, i. e. $\underline{N}$ and $\underset{\sim}{D}\nabla\underline{N}$ are continuous throughout the reactor.

The problem in its general form is nonlinear due to the temperature dependence of the parameters of the reactor and, of course, more equations are needed to express the energy balance. For many situations this dependence can be neglected and, in any case, the study of the linear equations (cross sections independent of temperature) is important to our understanding of the behavior of the system. In this thesis it is the linear problem that interests us.

Equations (1. 1) and (1. 2) suggest that we look into the possibility of exponential solutions of the form $e^{\omega_n t} \underline{N}_n(\underline{r})$ and $e^{\omega_n t} C_{i,n}(\underline{r})$ for the

source free problem. If ω_0 is the fundamental eigenvalue, the asymptotic behavior of the system will be $e^{\omega_0 t}\underline{N}_0(\underline{r})$. Furthermore such solutions will give a convenient set of functions $\{\underline{N}_n(\underline{r})\}$ for series solutions of the form

$$\underline{N}(\underline{r},t) = \sum_j a_j(t)\underline{N}_j(\underline{r})$$

when $\underline{\underline{Q}} \neq 0$.

Thus we replace $\partial\underline{N}/\partial t$ by $\omega_n \underline{N}_n$ and $\partial C_i/\partial t$ by $\omega_n C_{i,n}$. The equations in matrix form are

$$\begin{bmatrix} \left[(1-\beta)\underline{\underline{M}}_p - \underline{\underline{L}}\right] & \lambda_1\underline{\chi}_1 & \cdots & \lambda_6\underline{\chi}_6 \\ \beta_1\underline{F}^T & -\lambda_1 & 0\cdots & 0 \\ & 0 & & \vdots \\ & \vdots & & 0 \\ \beta_6\underline{F}^T & 0 & \cdots\ 0 & -\lambda_6 \end{bmatrix} \begin{bmatrix} \underline{N}_n \\ C_{1,n} \\ \\ C_{6,n} \end{bmatrix} = \omega_n \begin{bmatrix} \underline{N}_n \\ C_{1,n} \\ \\ C_{6,n} \end{bmatrix} \tag{1.3}$$

(Period- or ω-eigenvalue problem)

where we have defined the operators

$$\underline{\underline{M}}_p \equiv \underline{\chi}_p\underline{F}^T$$

$$\underline{\underline{L}} \equiv -\nabla\cdot\underline{\underline{D}}\nabla + \underline{\underline{R}}_a + \underline{\underline{R}}_s - \underline{\underline{S}}$$

Equation (1.3) is written in compact form as

$$\underline{\underline{Q}}\underline{\psi}_n = \omega_n\underline{\psi}_n \tag{1.4}$$

with obvious notation.

A major difficulty in dealing with the operator $\underline{\underline{Q}}$ is the fact that it is non-self-adjoint. This is clear from the matrix form (1.3). We point out that also $\underline{\underline{L}}$ is asymmetric, because $\underline{\underline{S}}$ is asymmetric and this

is a consequence of the irreversibility of the slowing down of neutrons (they only lose energy through collisions).

By taking the complex conjugate transpose of the operators in (1.3) we form the adjoint problem

$$\underset{\sim}{Q}^{*}\underline{\psi}_m^{*} = \omega_m^{*}\underline{\psi}_m^{*} \tag{1.5}$$

The boundary condition for $\underline{N}_m^{*}$ is again $\underline{N}_m^{*}(\underline{R},t) = 0$.

Very little is known about the eigenfunctions and eigenvalues of (1.4) and (1.5). To be able to proceed the following assumptions are made[2,10].

1. The eigenvalues are discrete and simple.
2. The eigenfunctions of $\underset{\sim}{Q}$ are linearly independent; the same is true for $\underset{\sim}{Q}^{*}$.
3. For every eigenfunction $\underline{\psi}_n$ there exists a $\underline{\psi}_n^{*}$ such that

$$\langle \underline{\psi}_n^{*} | \underline{\psi}_n \rangle \neq 0$$

where the inner product in a G-dimensional space is defined by

$$\langle \underline{v} | \underline{w} \rangle = \int_{\substack{\text{volume} \\ \text{of reactor}}} (\bar{v}_1 w_1 + \ldots + \bar{v}_G w_G)\, d\underline{r}$$

and the bar denotes complex conjugate.

4. The eigenfunctions are complete.

Using these assumptions we can prove that

1. $$\langle \underline{\psi}_m^{*} | \underline{\psi}_n \rangle \sim \delta_{mn} \quad \text{(biorthogonality)} \tag{1.6}$$

where δ_{mn} is Kronecker's delta.

2. The eigenvalues of $\underset{\sim}{Q}^*$ are the complex conjugates of those of $\underset{\sim}{Q}$

3. The values $-\lambda_1, \ldots, -\lambda_6$ are not eigenvalues of $\underset{\sim}{Q}$.

Proof. Pre-multiplying (1.4) by $\underline{\psi}_m^*$ and (1.5) by $\underline{\psi}_n$, integrating over the volume of the reactor and subtracting the results we arrive at

$$\langle \underline{\psi}_m^* | \underset{\sim}{Q} | \underline{\psi}_n \rangle - \langle \underline{\psi}_m^* | \underset{\sim}{Q} | \underline{\psi}_n \rangle = \left(\omega_n - \overline{\omega}_n^*\right) \langle \underline{\psi}_m^* | \underline{\psi}_n \rangle = 0$$

For $\omega_n \neq \overline{\omega}_m^*$ (1.6) follows and under assumption (3) for m=n the inner product is nonzero, hence $\omega_n = \overline{\omega}_m^*$. To prove that $-\lambda_i$ $i=1,\ldots,6$ are not eigenvalues we assume that one of them, say $-\lambda_i$, is. Then the corresponding equation of (1.3) (that is, the (G+i) equation) gives

$$\underline{F}^T \underline{N}_n = 0$$

But the dimension of $\underline{F}^T$ is G, i.e., the number of groups used and this is at our disposal. We can change the number of groups, but still $\underline{N}_n$ (which also has G components) must be orthogonal to $\underline{F}$. This just implies that $\underline{N}_n \equiv \underline{0}$. From the remaining equations of (1.3) it is readily seen that also $C_{i,n} \equiv 0$, $i=1\ldots6$, that is $\underline{\psi}_n \equiv \underline{0}$. Therefore $-\lambda_i$ cannot be an eigenvalue.

The problems (1.4) and (1.5) can be written in another form involving only the neutron densities. This is due to the fact that there is no differential operator in the last six lines of (1.3), so we can solve for $C_{i,n}$, i.e.

$$C_{i,n} = \frac{\beta_i}{\omega_n + \lambda_i} \underline{F}^T \underline{N}_n$$

Similarly

$$C^*_{i,m} = \frac{\lambda_i}{\bar{\omega}_m + \lambda_i} \underline{\chi}_i^T \underline{N}^*_m$$

Using these we first modify the biorthogonality relation (1.6)

$$\langle \underline{N}^*_m | \underline{N}_n \rangle + \sum_{i=1}^{6} \frac{\lambda_i \langle \underline{N}^*_m | \underline{\underline{M}}_i | \underline{N}_n \rangle}{(\lambda_i + \bar{\omega}_m)(\lambda_i + \omega_n)} = \delta_{mn} \tag{1.7}$$

where $\underline{\underline{M}}_i \equiv \beta_i \underline{\chi}_i \underline{F}^T$ (see Appendices A and B for details).

The important thing to observe is that the sets $\{\underline{N}^*_m\}$ and $\{\underline{N}_n\}$ are not biorthogonal.

We now consider the first line of (1.3)

$$[(1-\beta)\underline{\underline{M}}_p - \underline{\underline{L}}]\underline{N}_n + \sum_{i=1}^{6} \lambda_i \underline{\chi}_i C_{i,n} = \omega_n N_n$$

Using the expression for $C_{i,n}$ this can be written as

$$[(1-\beta)\underline{\underline{M}}_p - \underline{\underline{L}}] + \sum_{i=1}^{6} \frac{\lambda_i \underline{\underline{M}}_i}{\lambda_i + \omega_n} \underline{N}_n = \omega_n \underline{N}_n \tag{1.8}$$

This is again a non-self-adjoint eigenvalue problem for $\underline{N}_n$ only. The additional complication is the presence of the dyadics $\underline{\underline{M}}_i$. Nevertheless it can be used to yield information for the modes.

1. If the reactor is a bare homogeneous system the eigenfunctions can be written as (one dimension is assumed for simplicity)

$$\underline{N}_n(x) = \underline{w}_n u_n(B_n x)$$

where $\underline{w}_n$ is a constant vector and B_n is the buckling defining the spatial mode. The function $u_n(B_n x)$ is given by

$$\nabla^2 u_n + B_n^2 u_n = 0$$

$$u_n = 0 \text{ on the boundary}$$

But then in (1.8) the terms $-\nabla^2$ of $\underset{\sim}{L}$ can be replaced by B_n^2, thus yielding an ordinary matrix eigenvalue problem. The eigenvalues are given by

$$\det\left\{\left[(1-\beta)\underset{\sim}{M}_p - \underset{\sim}{L}\left(B_n^2\right)\right] + \sum_{i=1}^{6} \frac{\lambda_i \beta_i}{\lambda_i + \omega} \underline{\chi}_i \underline{F}^T - \omega I\right\} = 0$$

$I \equiv G \times G$ identity matrix. The above determinant gives $G+6$ eigenvalues. The intervals $I_6 \equiv (-\lambda_6, -\lambda_5)$, $I_5 \equiv (-\lambda_5, -\lambda_4)$, ... $I_1 \equiv (-\lambda_1, \infty)$ contain exactly one eigenvalue each and the other G are in the interval $I_7 \equiv (-\infty, -\lambda_6)$. Each eigenvalue yields its own eigenvector $\underline{w}_{n,j}$. We have used two indices to indicate the buckling by n and the eigenvectors with the same buckling by j. For fixed n the index j takes on the values $1, \ldots, G+6$. The above distribution of eigenvalues holds for every buckling, therefore each of these intervals contains an infinite number of eigenvalues. As $n \to \infty$ and $B_n \to \infty$ the eigenvalues in each interval I_j, $j=1,\ldots,6$, tend to $-\lambda_j$, while those in $(-\infty, -\lambda_6)$ tend to $-\infty$. In the general case of a multiregion reactor there is not a single buckling for the whole system. However the above results can be carried over, if we talk in terms of spatial modes (each defined by its zeros in the domain of the reactor). Thus for each spatial mode there will be $G+6$ eigenvalues distributed in the same way as before. The spectrum of (1.8) consists of seven infinite sets of eigenvalues each in one of the intervals I_j, $j=1,\ldots,7$ with limit points $-\infty$ and $-\lambda_j$, $j=1,\ldots,6$.

2. For a given spatial mode the six eigenvalues closer to $-\lambda_j$, $j=1,\ldots,6$ are called delayed eigenvalues. They are of the order of $|\lambda_j|$, i.e. less than $2\sim3$ $\sec^{-1}$. The corresponding eigenfunctions exhibit the very important property of being approximately the same. If we use two indices n and j, where n indicates the spatial mode and $j=1,\ldots,6$, then we claim that

$$\underline{N}_{n,j} \cong \underline{N}_{n,k} \quad , \quad j,k=1,\ldots,6$$

To see this we observe that the diagonal elements of $\underset{\sim}{L}$ due to R_a+R_s are the total cross sections for each group $\Sigma_g v_g$, which even for thermal neutrons are at least of the order 10^3 $\sec^{-1}$. But ω_n is close to $-\lambda_i$, therefore $1/\lambda_i+\omega_n$ is very large and sensitive to changes in ω_n. In view of these facts the term $\omega_n \underline{N}_n$ in the right-hand side of (1.8) can be omitted compared to $(R_a+R_s)\underline{N}_n$. Furthermore, the differences in the spectra $\underline{\chi}_i$ are very small and we can assume for the moment that all $\underline{\chi}_i$ are equal. With these assumptions we rewrite Equation (1.8) as

$$\left\{\left[(1-\beta)\underset{\sim}{M}_p-\underset{\sim}{L}\right]+\left(\underline{\chi}_i \underline{F}^T\right)\sum_{i=1}^{6}\frac{\lambda_i\beta_i}{\lambda_i+\omega_n}\right\}\underline{N}_n = 0$$

But now the whole term

$$\sum_{i=1}^{6}\frac{\lambda_i\beta_i}{\lambda_i+\omega_n}$$

is an eigenvalue of this equation, say k, with eigenfunction $\underline{N}_n$. The ω_n will be found by

$$\sum_{i=1}^{6}\frac{\lambda_i\beta_i}{\lambda_i+\omega_n} = k$$

which yields six ω's all having the same $\underline{N}_n$. By including the differences in $\underline{\chi}_i$ and the term $\omega_n \underline{N}_n$ in the right-hand side the degeneracy is removed and to every eigenvalue there corresponds one eigenfunction $\underline{N}_{n,j}$, j=1,...,6. Still the splitting is expected (no rigorous proof exists) to be small, so that the six eigenfunctions are almost the same, i. e. the dependence on j is very weak.

3. The rest of the spectrum consists of eigenvalues determined primarily by $(1-\beta)\underset{\sim}{M}_p - \underset{\sim}{L}$ with magnitude at least 10^3 sec^{-1}. Then the summation term acts merely as a perturbation shifting these ω's.

4. The fundamental assumption is made now that for every group of similar six eigenfunctions corresponding to the delayed ω's there exists a very similar eigenfunction with large ω. Referring to the the original problem (1. 4) we can state that some (not all) of the eigenfunctions appear to come in clusters of seven all members of a cluster having approximately the same neutron-density vector $\underline{N}_n$ while they differ in the remaining six components $C_{i,n}$, i=1,...,6. Numerical investigations have verified this assumption[34].

5. We assume further that in the above mentioned clusters of seven the neutron-density vectors are exactly the same[10]. These define the inhour modes which satisfy useful orthogonality properties and are examined in more detail in Appendix B. To point out their usefulness we state here that they are biorthogonal and, in lieu of (1. 7), they satisfy

$$\langle \underline{N}^*_{m,j} | \underline{N}_{n,j} \rangle \sim \delta_{mn} \quad \text{and} \quad \langle \underline{N}^*_{m,j} | \underset{\sim}{M}_i | \underline{N}_{n,j} \rangle \sim \delta_{mn}$$

The inhour modes form a subspace; they are complete in space, but

not in energy, hence they can be used for series solutions of problems where energy transients are not important (e.g. control-rod movement). They are crucial and instrumental to the work presented in this thesis.

The big difference in magnitude of the large and delayed eigenvalues can be attributed to the two time scales in the multiplication of the neutron population[5,6]. As we have mentioned some of the neutrons are emitted immediately after fission (prompt neutrons). These have a short generation time (roughly, the time it takes a neutron to produce another fission neutron) of the order of 10^{-5} sec. As a consequence they excite modes with eigenvalues very large in absolute values (larger than 10^{3} sec^{-1} for thermal reactors and 10^{7} sec^{-1} for fast ones). On the other hand the neutrons which are emitted as a result of the decay of the precursors appear in the reactor after a time which is of the order of the decay constant (several seconds). These excite modes with eigenvalues close to $|\lambda_i|$ (less than 2 - 3 sec^{-1}).

To further clarify these results some of the eigenvalues of a large water-cooled reactor are presented in Figure I.1[34].

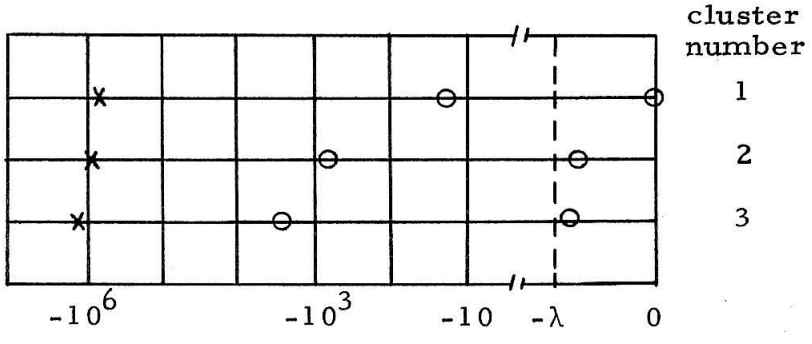

Figure I.1. Eigenvalues and Clusters of a Large Reactor

In this example two-group theory is used (G=2) and one group of precursors with decay constant $\lambda = 0.08\ \text{sec}^{-1}$. There are three eigenfunctions with the same spatial distribution (in the sense of zeros in the domain of the reactor). Of these one has eigenvalue close to $-\lambda$ while the other two have eigenvalues much larger. In each group of three the two with the smallest $|\omega|$ form the cluster of eigenfunctions with very similar neutron density distributions. The eigenfunctions corresponding to the eigenvalues denoted by a cross in the figure are excluded from the set of inhour modes.

The different behavior of prompt and delayed neutrons serves as a motivation for the definition of two new eigenvalue problems. Thus ignoring the delayed neutrons, the prompt neutron eigenvalues and eigenfunctions are given by[1,2]

$$\left[(1-\beta)\underset{\sim}{M}_p - \underset{\sim}{L}\right] \underline{N}_n^{(p)} = \omega_n^{(p)} \underline{N}_n^{(p)} \tag{1.9}$$

and the adjoint equation

$$\left[(1-\beta)\underset{\sim}{M}_p^T - \underset{\sim}{L}^T\right] \underline{N}_n^{*(p)} = \overline{\omega}_n^{(p)} \underline{N}_n^{*(p)}$$

Habetler and Martino[3] have proved completeness of these eigenfunctions for the homogeneous reactor and for heterogeneous systems but only in one dimensional geometries in the latter case. Again the conjecture of completeness in the general geometry is employed. The biorthogonality property is now

$$\langle \underline{N}_n^{*(p)} | \underline{N}_m^{(p)} \rangle = 0 \qquad n \neq m \tag{1.10}$$

The use of these functions for series solutions of the form

$$\underline{N}(\underline{r},t) = \sum_j a_j(t)\underline{N}_j^{(p)}(\underline{r})$$

in Equations (1.1) and (1.2) does not lead to finality[4] (i.e. the resulting equations for the coefficients are not decoupled). The $\omega_n^{(p)}$ approximate the large eigenvalues of $\underset{\sim}{Q}$ very well, and $\underline{N}_n^{(p)}$ is very similar to $\underline{N}_n$ with large $|\omega|$ of $\underset{\sim}{Q}$. Numerical investigations have verified this assertion[7,8]. Mika[9] has studied the effect of delayed neutrons on the spectrum of the one-speed transport operator, which corresponds to one energy group in the present formalism. We take up the general case in Chapter III.

The second eigenvalue problem corresponds to the period modes which have small eigenvalues (delayed eigenvalues). It is called the k-problem and it is

$$\underset{\sim}{L}\underline{N}_n = \frac{1}{k_n}\underset{\sim}{M}_p\underline{N}_n \tag{1.9}$$

with adjoint

$$\underset{\sim}{L}^T\underline{N}_n^* = \frac{1}{k_n}\underset{\sim}{M}_p^T N_n^*$$

The biorthogonality property is

$$\langle \underline{N}_n^* | \underset{\sim}{M}_p | \underline{N}_m \rangle = 0 \qquad n \neq m \tag{1.10}$$

Again when used in series solutions of (1.1) and (1.2) these modes do not decouple the equations. The interesting property they have is that the modes are very similar to the modes with small $|\omega|$ of $\underset{\sim}{Q}$.

We now turn our attention to the solution of the general problem given in (1.1) and (1.2). The previously mentioned eigenvalue problems

give sets of linearly independent functions which are used in the development of approximate methods of solution of (1.1) and (1.2). These methods are special cases of the very general weighted-residual method[11,12] as demonstrated in Reference 13.

In the nodal approach the reactor is divided into several regions and each one interacts with its neighbors through leakage[14].

In the modal[11,15,16] approach the neutron flux is expanded in known functions of the space variables with coefficients of time. Using weighting functions, equations for the coefficients are derived. Various choices of the trial functions have been proposed in the literature. Also variations of the method are available as the "space-time" synthesis[17] in which one space variable is transferred from the trial functions to the unknown coefficients.

The crudest and yet the most widely used approximation is the point kinetics model[1,2,18,19]. An outline of its derivation is given in Appendix A. The idea behind the model is to assume that the neutron density is the product of a known function of space, the "shape function", and an unknown function of time, the "amplitude function". Then the space-dependent parameters of (1.1) and (1.2) are averaged over the reactor using the known shape function and ordinary differential equations result for the amplitude function. Thus the reactor is treated as a point in space ("lumped parameter" description). A major result of this thesis is the development of analytic methods of testing the accuracy of the model.

The fundamental assumption in point kinetics is the separability of neutron density into a product of a function of position and a function

of time. Strictly speaking this is true only if the reactor is on an asymptotic period. In all other cases the shape function changes with time. Still if we can assume that the change of the shape function is slower than the change of the amplitude function the model can be improved by calculating the shape function at selected time intervals. Such improvements are the adiabatic model[20] and the quasistatic approximation[21].

There exist many methods for solving point reactor equations in the computer. A review of these methods is given in Reference 22. Along with computational techniques for solving the space-dependent equations they have served to test the accuracy of point kinetics.

Yasinsky and Henry[23] have compared exact and approximate solutions of the linear two-group diffusion equations. Space-dependent perturbations were considered and the point kinetics solutions were found to be in considerable error, especially for large cores. For prompt critical excursions point kinetics failed to give satisfactory results even for small cores.

Jackson and Kastenberg[24] examined the accuracy of point kinetics in fast reactors. With uniform feedback they found that the peak power is underestimated while for prompt critical transients use of the shape at prompt critical as shape function gives improved results. However it is very difficult for point kinetics to account for the spatial effects induced by space-dependent feedback.

Although numerical investigations prove that point kinetics is not adequate for the description of space-time effects in nuclear

reactors there is no analytical work establishing quantitatively under what conditions the model fails. Nor is there any quantitative estimate of the error introduced by the inability to describe exactly the changes in the shape of the neutron density in time. Only qualitatively one can say that for large "loosely-coupled" cores spatial effects are important and a more sophisticated analysis than point kinetics is required.

As mentioned before, the lack of analytical estimates of the errors is mainly due to the great difficulties one is confronted with when dealing with unbounded non-self-adjoint operators. For self-adjoint cases such errors can be bounded usually with the use of variational principles where the minimum (or maximum) of certain functionals is sought[25,26]. For non-self-adjoint operators no such extremum property exists. As an example, Ronen[27] considered the case of one-speed diffusion theory with no delayed neutrons, i.e.

$$-\nabla D(\underline{r})\nabla\varphi(\underline{r}) + \Sigma_a(\underline{r})\varphi(\underline{r}) - \nu\Sigma_f(\underline{r})\varphi(\underline{r}) = \lambda\varphi(\underline{r}) \qquad (1.11)$$

If the absorption cross section is increased by $\delta\Sigma_a(\underline{r})$ in a volume ΔV, then the change of the fundamental eigenvalue is bounded by

$$\lambda_0 - \lambda_0' \leq \left(\int_{\Delta V} | \delta\Sigma_a(\underline{r})\varphi_0'(\underline{r}) |^2 \, d\underline{r} \right)^{1/2}$$

where λ_0' and $\varphi_0'(\underline{r})$ are the fundamental eigenvalue and eigenfunction of the unperturbed system.

Such a bound is possible to obtain because the problem (1.11) is self-adjoint. The derivation of bounds in the general case concerns us in Chapter II.

C. Motivation for this Thesis

As stated in the previous section, the point kinetics model is the most widely used approximation in dealing with the temporal behavior of a reactor. Numerical experiments showing its inefficiency were reported and the lack of analytical criteria was pointed out.

In Chapter II we derive bounds to the error in reactivity which results from the inability of point kinetics to account for spatial effects. The linear case is examined and step and ramp insertions of reactivity are considered. The bound is easy to find and is useful in the sense that it vanishes when point kinetics gives exact results and it follows quite closely the true error. Numerical examples are provided to test the prediction of the error.

In Chapter III we use a simple perturbation method to derive corrections to the prompt eigenvalues of a reactor so that the periods of the system can be predicted with good accuracy. Furthermore, the accuracy of the inhour-modes approximation can be tested through simple analytical criteria. Numerical examples demonstrate the applicability of these results.

II. ERROR BOUNDS

The ω-modes of a reactor with nuclear characteristics independent of time satisfy the inhour equation (Appendix B, Eqn. B.6). For the fundamental mode we have

$$\left(\frac{\rho}{\Lambda}\right)_{Ex} = \omega_{Ex} + \sum_{i=1}^{6} \left(\frac{\beta_i}{\Lambda}\right)_{Ex} \left(\frac{\omega_{Ex}}{\omega_{Ex}+\lambda_i}\right)$$

or

$$\left(\frac{\rho}{\Lambda}\right)_{Ex} = f_{Ex}(\omega_{Ex}) \tag{2.1}$$

The subscript denotes that the exact period would result, if the parameters of the equation were exact, that is, calculated with the use of the exact fundamental eigenfunction and its adjoint (see Appendix A for details). However in conventional point kinetics the eigenfunctions of another critical reference reactor are used to evaluate the parameters $(\rho/\Lambda)_{pk}$ and $(\beta_i/\Lambda)_{pk}$ and the resulting periods are given by

$$\left(\frac{\rho}{\Lambda}\right)_{pk} = f_{pk}(\omega_{pk}) \tag{2.2}$$

The question then arises by how much ω_{pk} deviates from ω_{Ex}. To derive an estimate of the error we subtract (2.2) from (2.1) and we get

$$\left(\frac{\rho}{\Lambda}\right)_{Ex} - \left(\frac{\rho}{\Lambda}\right)_{pk} = f_{Ex}(\omega_{Ex}) - f_{pk}(\omega_{pk})$$

$$= (\omega_{Ex} - \omega_{pk}) + \sum \left(\frac{\beta_i}{\Lambda}\right)_{Ex} \left(\frac{\omega_{Ex}}{\omega_{Ex}+\lambda_i}\right) - \sum \left(\frac{\beta_i}{\Lambda}\right)_{pk} \left(\frac{\omega_{pk}}{\omega_{pk}+\lambda_i}\right)$$

As we will explain later the main source of error is the reactivity term $(\rho/\Lambda)_{pk}$ while $(\beta_i/\Lambda)_{pk}$ is insensitive to the spatial shape used in its calculation. In a wide range of cases we can safely assume that

$$f_{Ex}(\omega) \cong f_{pk}(\omega) = f(\omega)$$

If we also have a bound for the error in (ρ/Λ), i. e.

$$\left|\left(\frac{\rho}{\Lambda}\right)_{Ex} - \left(\frac{\rho}{\Lambda}\right)_{pk}\right| \leq b$$

The error in the period will be given by

$$\left|f(\omega_{Ex}) - f(\omega_{pk})\right| \leq b$$

This equation can give the error $|\omega_{Ex}-\omega_{pk}|$ either graphically by plotting the function $f(\omega)$ or by expansion in Taylor series, i. e.

$$f(\omega_{Ex}) \cong f(\omega_{pk}) + (\omega_{Ex}-\omega_{pk}) \left.\frac{df(\omega)}{d\omega}\right|_{\omega=\omega_{pk}}$$

hence

$$\left|(\omega_{Ex}-\omega_{pk}) \left.\frac{df(\omega)}{d\omega}\right|_{\omega=\omega_{pk}}\right| \leq b$$

It is such a bound b that we seek in this chapter.

A. The Method of Weighted Residuals

Eliminating the precursor densities from the ω-eigenvalue problem we arrive at the equation (Eqn. B. 5)

$$\underset{\sim}{\mathcal{A}}\,\underline{N}_n = \left(\omega_n + \sum_{i=1}^{6} \frac{\omega_n}{\omega_n+\lambda_i}\,\underset{\sim}{M}_i\right)\underline{N}_n \qquad (2.3)$$

where the operators are defined in Appendix A. Equation (2. 3)

represents an eigenvalue problem more complicated than the usual cases due to the summation term present in the right-hand side. Nevertheless, the method of weighted residuals can be used to obtain approximate values for ω_n. The most important eigenvalue is the fundamental, which ultimately gives the asymptotic behavior of the system. We will concentrate our attention on approximate estimates of this ω_0.

The method of weighted residuals consists of expanding the eigenfunction $\underline{N}_n$ in a set of trial functions $\underline{\eta}_n$, i. e.

$$\underline{N}_n = \sum_{j=0}^{M} c_j \underline{\eta}_j$$

Substitution in (2. 3) yields the residual

$$\underline{R}(\underline{r}) = \sum_{j=0}^{M} c_j \left(\underset{\sim}{\mathscr{A}} \underline{\eta}_j - \omega \underline{\eta}_j - \sum_{i=1}^{6} \frac{\omega}{\omega+\lambda_i} \underset{\sim}{M}_i \underline{\eta}_j \right) \tag{2.4}$$

We now use a set of weighting functions $\underline{w}_j(\underline{r})$ to distribute the residual over the whole reactor. So, multiplication from the left of (2. 4) with $\underline{w}_j(\underline{r})$ and integration over the reactor volume gives the following equation by setting the integral of $\underline{R}(\underline{r})$ equal to zero

$$\underset{\sim}{A}\,\underline{c} = \left(\underset{\sim}{B}\omega + \sum_{i=1}^{6} \frac{\omega}{\omega+\lambda_i} \underset{\sim}{G}^i \right) \underline{c} \tag{2.5}$$

where

$$A_{mn} \equiv \langle \underline{w}_m | \underset{\sim}{\mathscr{A}} | \underline{\eta}_n \rangle$$

$$B_{mn} \equiv \langle \underline{w}_m | \underline{\eta}_n \rangle \tag{2.6}$$

$$G^i_{mn} \equiv \langle \underline{w}_m | \underset{\sim}{M}_i | \underline{\eta}_n \rangle$$

$$\underline{c} \equiv (c_0, \ldots, c_M)^T$$

It is clear now that the eigenvalues of (2.5) can be obtained from

$$\det \left[\underset{\sim}{A} - \underset{\sim}{B}\omega - \sum_{i=1}^{6} \frac{\omega}{\omega+\lambda_i} \underset{\sim}{G}^i \right] = 0 \tag{2.7}$$

where det ≡ determinant.

Finally the trial and weighting functions must be chosen. As trial functions we will use the inhour modes of a reference critical reactor (if the reactor at hand is perturbed from a critical state then the unperturbed modes are used). With the subscript zero denoting the reference system the trial functions satisfy

$$\underset{\sim}{\mathcal{A}}_0 \underline{\eta}_j \equiv \left(\underset{\sim}{H}_0 + \sum_{i=1}^{6} \underset{\sim}{M}_i^0 \right) \underline{\eta}_j = \left(\omega_j^0 + \sum_{i=1}^{6} \frac{\omega_j^0}{\omega_j^0 + \lambda_i} \underset{\sim}{M}_{0,i} \right) \underline{\eta}_j$$

with $\omega_0^0 = 0$ (critical system). The weighting functions are the adjoint of $\underline{\eta}_j$, i.e. $\underline{w}_j \equiv \underline{\eta}_j^*$, where

$$\underset{\sim}{\mathcal{A}}_0^T \underline{\eta}_j^* = \left(\overline{\omega}_j^0 + \sum_{i=1}^{6} \frac{\overline{\omega}_j^0}{\overline{\omega}_j^0 + \lambda_i} M_{0,i}^{0T} \right) \underline{\eta}_j^*$$

This choice of the inhour modes as trial functions and their adjoints permits us to use the orthogonality properties (B.7), (B.8) and (B.9) which here become

$$\langle \underline{\eta}_n^* | \underline{\eta}_m \rangle \sim \delta_{nm} \tag{2.8}$$

$$\langle \underline{\eta}_n^* | \underset{\sim}{M}_i^0 | \underline{\eta}_m \rangle \sim \delta_{nm}$$

$$\langle \underline{\eta}_n^* | \underset{\sim}{H}_0 | \underline{\eta}_m \rangle \sim \delta_{nm}$$

Then the first rows of the matrices A, B and G^i, which are of interest, are

$$A_{0n} = \langle \underline{\eta}_0^* | \delta \underset{\sim}{\mathcal{H}} | \underline{\eta}_n \rangle$$

$$B_{0n} = \langle \underline{\eta}_0^* | \underline{\eta}_n \rangle$$

$$G^i_{0n} = \langle \underline{\eta}_0^* | \underset{\sim}{M}_i^0 + \delta \underset{\sim}{M}_i | \underline{\eta}_n \rangle$$
$$n = 0, 1, \ldots, M$$

where the prefix δ denotes difference of the properties of the actual reactor from those of the reference reactor.

One remark is in order here. Due to the non-self-adjoint character of the neutron diffusion operator it is impossible to locate the approximate eigenvalues given by (2.7) relative to the true ones. When the same method is applied to self-adjoint systems, for instance, the Sturm-Liouville equation, the resulting matrices are symmetric and the approximate eigenvalues are upper bounds to the actual ones. Here the matrices are not symmetric and nothing can be said about the approximation. It is clear then that if we wish to derive any kind of bounds another approach will be needed. This is done next.

B. Reactivity Bounds

Before proceeding we examine the matrices $\underset{\sim}{B}$ and $\underset{\sim}{G}^{i}$. In view of (2.6) and the orthogonality (2.8) it is concluded that $\underset{\sim}{B}$ is a diagonal matrix. Dividing every row of (2.5) by the corresponding element B_{jj} we arrive at the equation

$$\begin{bmatrix} \left(\frac{\rho}{\Lambda}\right)_0 & \frac{\langle \eta_0^* | \delta\underset{\sim}{\mathscr{A}} | \eta_1 \rangle}{\langle \eta_0^* | \eta_0 \rangle} & \cdots & \frac{\langle \eta_0^* | \delta\underset{\sim}{\mathscr{A}} | \eta_M \rangle}{\langle \eta_0^* | \eta_0 \rangle} \\ \frac{\langle \eta_1^* | \delta\underset{\sim}{\mathscr{A}} | \eta_0 \rangle}{\langle \eta_1^* | \eta_1 \rangle} & \left(\frac{\rho}{\Lambda}\right)_1 & & \frac{\langle \eta_1^* | \underset{\sim}{\mathscr{A}}_0 + \delta\underset{\sim}{\mathscr{A}} | \eta_M \rangle}{\langle \eta_1^* | \eta_1 \rangle} \\ \vdots & & \ddots & \vdots \\ \frac{\langle \eta_M^* | \delta\underset{\sim}{\mathscr{A}} | \eta_0 \rangle}{\langle \eta_M^* | \eta_M \rangle} & \frac{\langle \eta_M^* | \underset{\sim}{\mathscr{A}}_0 + \delta\underset{\sim}{\mathscr{A}} | \eta_1 \rangle}{\langle \eta_M^* | \eta_M \rangle} \cdots & & \left(\frac{\rho}{\Lambda}\right)_M \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_M \end{bmatrix} =$$

$$\begin{bmatrix} f_0(\omega) & \sum_{i=1}^{6} \left(\frac{\omega}{\omega+\lambda_i}\right) \frac{\langle \eta_0^* | \delta\underset{\sim}{M}_i | \eta_1 \rangle}{\langle \eta_0^* | \eta_0 \rangle} & \cdots & \sum_{i=1}^{6} \left(\frac{\omega}{\omega+\lambda_i}\right) \frac{\langle \eta_0^* | \delta\underset{\sim}{M}_i | \eta_M \rangle}{\langle \eta_0^* | \eta_0 \rangle} \\ \sum_{i=1}^{6} \left(\frac{\omega}{\omega+\lambda_i}\right) \frac{\langle \eta_1^* | \delta\underset{\sim}{M}_i | \eta_0 \rangle}{\langle \eta_1^* | \eta_1 \rangle} & f_1(\omega) & \cdots & \sum_{i=1}^{6} \left(\frac{\omega}{\omega+\lambda_i}\right) \frac{\langle \eta_1^* | \delta\underset{\sim}{M}_i | \eta_M \rangle}{\langle \eta_1^* | \eta_1 \rangle} \\ \vdots & & \ddots & \vdots \\ \sum_{i=1}^{6} \left(\frac{\omega}{\omega+\lambda_i}\right) \frac{\langle \eta_M^* | \delta\underset{\sim}{M}_i | \eta_0 \rangle}{\langle \eta_M^* | \eta_M \rangle} & \cdots & & f_M(\omega) \end{bmatrix} \underline{c} \tag{2.9}$$

where

$\left(\frac{\rho}{\Lambda}\right)_j$ = ratio of reactivity to generation time for the j^{th} mode,

$$f_j(\omega) = \omega + \sum_{i=1}^{6} \frac{\omega}{\omega+\lambda_i} \left(\frac{\beta_i}{\Lambda}\right)_j ,$$

$\left(\frac{\beta_i}{\Lambda}\right)_j$ = ratio of effective delayed neutron fraction to generation time for the j^{th} mode (see also Appendices A and B),

and $\underset{\sim}{\mathcal{K}}_0 \underline{\eta}_0 = 0$ has been used.

We observe that the matrix on the right-hand side of (2.9) will be diagonal if the multiplication operator does not change, i.e. $\delta\underset{\sim}{M}_i = 0$. In general, of course, $\delta\underset{\sim}{M}_i \neq 0$. However, as it is argued in Ref. 6, β_i and Λ are practically constants for every mode unless there is a drastic change in the multiplicative properties of the system. This means that with the choice made for the trial and weighting functions (inhour modes, which are very close to the eigenfunctions of the static neutron operator) the terms $\langle \underline{\eta}_j^* | \delta\underset{\sim}{M}_i | \underline{\eta}_j \rangle$ in the definition of $(\beta_i/\Lambda)_j$ are much smaller than $(\rho/\Lambda)_j$, which represents the difference of neutron production minus neutron destruction and, hence, is much more sensitive to changes in the reactor. But then the same reasoning leads to neglecting the terms $\langle \underline{\eta}_j^* | \delta\underset{\sim}{M}_i | \underline{\eta}_n \rangle$, $j \neq n$ and again the matrix on the right-hand side of (2.9) can be taken to be diagonal.

In matrix algebra there are many theorems which specify domains in the complex plane where the eigenvalues of a matrix are located. The most well known is Gershgorin's theorem[29]. Of the same type is Ostrowski's theorem[30] which we will now use. Since

in (2.9) the eigenvalues ω enter through the function $f_j(\omega)$, we modify the theorem a little and for completeness we give its proof in Appendix C.

Ostrowski's Theorem. Let there be two positive numbers p and q satisfying the relation

$$\frac{1}{p} + \frac{1}{q} = 1 \ .$$

Let

$$L_m^{(p)} = \left[\sum_{\substack{j=0 \\ j \neq m}}^{M} \left| \frac{\langle \underline{\eta}_m^* | \underset{\sim}{\mathcal{K}}_0 + \delta \underset{\sim}{\mathcal{K}} | \underline{\eta}_j \rangle}{\langle \underline{\eta}_m^* | \underline{\eta}_m \rangle} \right|^p \right]^{1/p}$$

Choose $M+1$ positive numbers $k_0, k_1, \ldots, k_M$ such that

$$\sum_{j=0}^{M} \frac{1}{k_j + 1} \leq 1 \tag{2.10}$$

then the eigenvalues ω of (2.9) are such that the numbers $f_j(\omega)$ are contained in at least one of the circles

$$\left| \left(\frac{\rho}{\Lambda} \right)_m - f_m \right| \leq k_m^{1/q} L_m^{(p)} \qquad m = 0, 1, \ldots, M \tag{2.11}$$

The theorem would be useless did we not have the following

Theorem[29]. Let $D_0, D_1, \ldots, D_\mu$ be the disjoint components of the Ostrowski circles (2.11). Let D_i be the union of n_i circles (so that $\sum_{i=0}^{\mu} n_i = M+1$). Then D_i contains exactly n_i numbers $f_j(\omega)$.

The last theorem is a consequence of the fact that the eigenvalues of a matrix are continuous functions of its elements. In the limiting case where all the off-diagonal terms are zero we will have $L_m^{(p)} \equiv 0$ and

$$\left(\frac{\rho}{\Lambda}\right)_m = f_m$$

This implies that if the first circle (2.11) were isolated from the others we would be sure that only $f_0(\omega)$ would be contained in it and hence a bound to $f_0(\omega)$ could be obtained. In fact this isolation can always be achieved, as we will describe later, so, for the moment, we assume it is isolated and we have, taking p=q=2

$$\left|\left(\frac{\rho}{\Lambda}\right)_0 - \omega - \sum_{i=1}^{6} \frac{\omega}{\omega+\lambda_i}\left(\frac{\beta_i}{\Lambda}\right)_0\right| \leq k_0^{1/2} L_0^{(2)} \qquad (2.12)$$

We recognize that if the left-hand side of (2.12) is set equal to zero, we will get the inhour equation for the fundamental mode.

A closer look at (2.12) reveals that in the present form it does not really say much, because we do not know how the ω appearing in the left-hand side is related to the true eigenvalue of the system and the terms of the sum $L_0^{(2)}$ cannot, in general, be calculated, since the inhour modes are difficult to find in the first place.

Recall that we are using as trial functions the inhour modes. To include all of them we let M in (2.9) tend to ∞. Equation (2.12) holds for every M hence we can take the limit $M \to \infty$. Then the eigenvalue will tend to a limit, say ω_{IM}. This will not be the true eigenvalue of the system, since the inhour modes do not span the whole space. However, it is a much better estimate than the one given by point kinetics, because in the latter case only one trial vector is used while in the former the whole subspace of the inhour modes is employed. It is generally accepted[31] that by increasing the number of linearly

independent trial vectors a better estimate for the quantity sought by the method of weighted residuals is found.

On physical grounds we can argue that for phenomena in which the perturbation does not excite significant energy and angular transients the ω_{IM} will be an excellent approximation to the true eigenvalue and we can expect that the bound will include the latter. Such a case occurs in unreflected systems and in large reactors.

The left-hand side of (2.12) thus gives

$$\lim_{M\to\infty} \left|\left(\frac{\rho}{\Lambda}\right)_0 - \omega - \sum_{i=1}^{6} \frac{\omega}{\omega+\lambda_i}\left(\frac{\beta_i}{\Lambda}\right)_0\right| = \left|\left(\frac{\rho}{\Lambda}\right)_0 - \left(\frac{\rho}{\Lambda}\right)_{IM}\right|$$

Next we prove that there exists a sequence of positive numbers k_j satisfying (2.10). It is known that

$$\sum_{j=1}^{\infty} \frac{1}{(2j)^2} = \frac{\pi^2}{24}$$

Choosing $k_0=1$, $k_j=(2j)^2-1$ we have

$$\sum_{j=0}^{\infty} \frac{1}{k_j+1} = \frac{12+\pi^2}{24} < 1$$

therefore the required sequence exists. Of course, other choices for k_j may be found and, furthermore, some of the k_j may be changed as long as the requirement (2.10) is fulfilled.

Finally the limit of $L_0^{(2)}$ as $M\to\infty$ must be calculated. Since

$$\underset{\sim}{\aleph}_0^* \underline{n}_0^* = 0$$

the sum becomes

$$\lim_{M\to\infty} L_0^{(2)} = \left[\sum_{j=1}^{M} \left| \frac{\langle \delta\underset{\sim}{\mathscr{K}}^* \underline{\eta}_0 | \underline{\eta}_j \rangle}{\langle \underline{\eta}_0^* | \eta_0 \rangle} \right|^2 \right]^{1/2} \equiv L_0$$

We assume that $\underline{\eta}_j$ and $\underline{\eta}_j^*$ are normalized, i.e.

$$\langle \underline{\eta}_j | \underline{\eta}_j \rangle = 1 \qquad \text{and} \qquad \langle \eta_j^* | \eta_j^* \rangle = 1$$

Define the cosine of the angle between the normalized vectors $\underline{\eta}_j^*$ and $\underline{\eta}_j$ as

$$\gamma_j = \langle \underline{\eta}_j^* | \underline{\eta}_j \rangle \leq 1$$

Then we see that

$$L_0 \leq \frac{1}{\gamma_0} \left[\sum_{j=1}^{\infty} \left| \frac{\langle \delta\underset{\sim}{\mathscr{K}}^* \underline{\eta}_0^* | \underline{\eta}_j \rangle}{\gamma_j} \right|^2 \right]^{1/2}$$

It is clear that if we consider the expansion

$$\delta\underset{\sim}{\mathscr{K}}^* \underline{\eta}_0^* = \sum_j \alpha_j \underline{\eta}_j^*$$

the coefficients α_j are given by

$$\alpha_j = \frac{\langle \delta\underset{\sim}{\mathscr{K}}^* \underline{\eta}_0^* | \underline{\eta}_j \rangle}{\gamma_j}$$

therefore

$$L_0 \leq \frac{1}{\gamma_0} \left[\sum_j |\alpha_j|^2 \right]^{1/2}$$

We now use the following lemma[25,32].

Lemma. Let $\underline{\xi}_1, \ldots, \underline{\xi}_n$ be a basis of unit vectors for an n-dimensional inner-product space. Let $\underline{\zeta} = \sum_i \tau_i \underline{\xi}_i$. Then

$$\sum_{i=1}^{n} |\tau_i|^2 \leq \frac{\|\underline{\zeta}\|^2}{m}$$

where m is the least eigenvalue of the Gram matrix $\{\langle \underline{\xi}_i | \underline{\xi}_j \rangle\}$.

Since the vectors $\underline{\xi}_i$ are linearly independent m lies between 0 and 1.

In our case we expand in the infinite set of linearly independent vectors $\underline{\eta}_j^*$. Courant and Hilbert ascribe to such a sequence the asymptotic dimension ∞. Furthermore, there exists a lower bound μ to the least eigenvalue m (measure of independence) between 0 and 1, otherwise the asymptotic dimension would be finite and the vectors $\underline{\eta}_j^*$ would be linearly dependent.

Using these results we derive the following bound

$$\left|\left(\frac{\rho}{\Lambda}\right)_0 - \left(\frac{\rho}{\Lambda}\right)_{IM}\right| \leq \frac{1}{\gamma_0}\left[\frac{\langle \delta\underset{\sim}{\mathscr{A}}^* \underline{\eta}_0^* | \delta\underset{\sim}{\mathscr{A}}^* \underline{\eta}_0^* \rangle}{\mu} - \left(\frac{\langle \underline{\eta}_0^* | \delta\underset{\sim}{\mathscr{A}} | \eta_0 \rangle}{\gamma_0}\right)^2\right]^{1/2} \qquad (2.13)$$

where $0<\mu\leq 1$ and $\gamma_0 = \langle \underline{\eta}_0^* | \eta_0 \rangle$.

A problem which arises now is how to estimate μ. If we take only M vectors $\underline{\eta}_j^*$ and form their Gram matrix, its least eigenvalue will be greater than the least eigenvalue of the infinite Gram matrix. We observe however that all the diagonal elements are 1 because the $\underline{\eta}_j^*$ are normalized. Also, for a homogeneous reactor the spatial part of $\underline{\eta}_j^*$ is determined solely by ∇^2 and the boundary conditions as we have

stated in Chapter I, hence

$$\langle \underline{n}_j^* | \underline{n}_i^* \rangle = \delta_{ji}$$

and we have that $\mu=m=1$. So for homogeneous systems there is no problem and μ is exactly unity.

For multiregion reactors this is not the case, since the off-diagonal elements of the Gram matrix are nonzero. But for large reflected reactors the spatial modes, especially those of higher harmonics, are very similar to the orthogonal Helmholtz modes and hence the off-diagonal elements will be expected to be orders of magnitude less than unity. In practice, one should calculate $\langle \underline{n}_0^* | \underline{n}_1^* \rangle$ and consider the Gram matrix

$$G_2 = \begin{bmatrix} 1 & \langle \underline{n}_0^* | \underline{n}_1^* \rangle \\ \langle \underline{n}_0^* | \underline{n}_1^* \rangle & 1 \end{bmatrix}$$

Ostrowski's theorem gives a lower bound for the least eigenvalue

$$\mu = 1 - | \langle \underline{n}_0^* | \underline{n}_1^* \rangle | \leq m$$

If indeed $| \langle \eta_0^* | \underline{n}_1^* \rangle | \ll 1$ then the above μ should be satisfactory. If this is not the case, the use of more vectors $\underline{n}_j^*$ may give a clue for a good estimate of μ. Of course, it may turn out that the off-diagonal elements are comparable to unity and by increasing the number of vectors $\underline{n}_j^*$ the least eigenvalue m decreases significantly. But this just means that μ is very small and the bound very large so the use of point kinetics is inadvisable. Such a case would occur in small reflected reactors where not even the inhour approximation would hold and one

would know from the beginning that point kinetics could not be used. Then there would no reason to try to find the bound in any case.

C. Improvements of the Bound

An essential assumption in developing relation (2.13) has been that the first Ostrowski disc is isolated from the others, because only then the bound makes any sense at all. Even if this disc is isolated, its radius can be reduced by using elementary theorems from matrix algebra[33]. To see this we go back to the matrix equation (2.9). The matrix on the left-hand side may be written as

$$\begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \left(\frac{\rho}{\Lambda}\right)_1^0 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \left(\frac{\rho}{\Lambda}\right)_M^0 \end{bmatrix} + \begin{bmatrix} \Delta_{00} & \Delta_{02} & \cdots & \Delta_{0M} \\ \Delta_{10} & \Delta_{11} & & \Delta_{1M} \\ \vdots & & \ddots & \vdots \\ \Delta_{M0} & \cdots & & \Delta_{MM} \end{bmatrix}$$

where

$$\Delta_{ij} = \frac{\langle \underline{\eta}_i^* | \underset{\sim}{\mathscr{A}}_0 + \delta \underset{\sim}{\mathscr{A}} | \underline{\eta}_j \rangle}{\langle \underline{\eta}_i^* | \underline{\eta}_i \rangle}$$

and $\left(\frac{\rho}{\Lambda}\right)_j^0$ is the reactivity of the unperturbed reactor corresponding to the j^{th} cluster. It is known that

$$0 > \left(\frac{\rho}{\Lambda}\right)_1^0 > \left(\frac{\rho}{\Lambda}\right)_2^0 \quad \text{etc.}$$

We now use the theorem that says that if we multiply the i^{th} row of any matrix by a number $\frac{1}{\lambda}$ and its i^{th} column by λ, the eigenvalues remain unaltered. We do that for the first row and column of

the above matrix and we get

$$\begin{bmatrix} 0 & & \\ & \left(\frac{\rho}{\Lambda}\right)_1^0 & \\ & & \ddots \\ & & & \left(\frac{\rho}{\Lambda}\right)_1^0 \end{bmatrix} + \begin{bmatrix} \Delta_{00} & \frac{\Delta_{02}}{\lambda} & \cdots & \frac{\Delta_{0M}}{\lambda} \\ \lambda\Delta_{10} & \Delta_{11} & & \vdots \\ \vdots & & \ddots & \vdots \\ \lambda\Delta_{M0} & \cdots & & \Delta_{MM} \end{bmatrix}$$

The first Ostrowski disc now has

$$\text{center at } \Delta_{00}$$

$$\text{radius} = r_0 = \frac{1}{\lambda}\left[\sum_{j=1}^{M} |\Delta_{0j}|^2\right]^{1/2}$$

Similarly the second disc has

$$\text{center at } \left(\frac{\rho}{\Lambda}\right)_1^0 + \Delta_{11}$$

$$\text{radius} = r_1 = \sqrt{k_1}\left[\lambda^2|\Delta_{10}|^2 + \sum_{j=2}^{M}|\Delta_{1j}|^2\right]^{1/2}$$

The number λ will be chosen to be big enough so that the above radius can be approximated by

$$r_1 \simeq \sqrt{k_1}\,\lambda|\Delta_{10}|$$

These two discs have the closest centers and it is our task not only to isolate the first one but to make r_1 as big as possible and r_0 as small as possible without overlapping. Ideally, the best we can do is

to make the circles tangent to each other, i.e.

$$r_0 + r_1 = |\Delta_{00}| + \left|\left(\frac{\rho}{\Lambda}\right)_1^0 + \Delta_{11}\right|$$

To avoid calculations of the exact r_1 and Δ_{11} we just use the approximate r_1 and we use the fact that $\left(\frac{\rho}{\Lambda}\right)_1^0$ is greater by at least two orders of magnitude than $|\Delta_{00}|$ and $|\Delta_{11}|$. So we choose λ to be

$$\lambda = \frac{k\left|\left(\frac{\rho}{\Lambda}\right)_1^0\right|}{\sqrt{k_1}\,|\Delta_{10}|}$$

where k is any number between 0 and 1. The reason we cannot use k=1 is that $\left(\frac{\rho}{\Lambda}\right)_1^0$ is not the exact distance between the two centers and an approximate r_1 was used.

As stated previously, k_1 can be 4-1=3. Simple calculations reveal that it can be decreased at the expense of k_2, k_3, etc. (as long as $\Sigma \frac{1}{k_j+1} \leq 1$) and we choose $k_1 = 1.4$.

The bound now becomes

$$\left|\left(\frac{\rho}{\Lambda}\right)_0 - \left(\frac{\rho}{\Lambda}\right)_{IM}\right| \leq \frac{1}{\lambda\gamma_0}\left[\frac{\langle \underset{\sim}{\delta\mathscr{K}}^* \underline{\eta}_0^* | \underset{\sim}{\delta\mathscr{K}}^* \underline{\eta}_0^* \rangle}{\mu} - \left(\frac{\langle \underline{\eta}_0^* | \underset{\sim}{\delta\mathscr{K}}^* | \underline{\eta}_0 \rangle}{\gamma_0}\right)^2\right]^{1/2} \qquad (2.14)$$

Of course now we need to know the unperturbed first harmonic eigenfunction.

The whole procedure seems quite complicated, while actually it is very simple. When solving a particular problem and one has the numbers $\left(\frac{\rho}{\Lambda}\right)_1^0$ and Δ_{10}, it is a simple matter to determine k. This point will be clarified in the examples.

D. Time-Dependent Perturbations

Thus far the nuclear properties of the reactor have been assumed to be independent of time. However, the results can be extended to the case where $\delta\underset{\sim}{\mathcal{A}}$ is time dependent. The most common situation is linear time dependence of $\delta\underset{\sim}{\mathcal{A}}$. The reactivity of point kinetics is now time dependent and in fact it depends on time in the same way that $\delta\underset{\sim}{\mathcal{A}}$ does, because the shape function used is fixed. The true reactivity is much more complicated since the true eigenfunction changes shape continuously as a result of the perturbation.

If the changes in the parameters of the system are fairly slow, a quasi-static approach can be used to define the true reactivity. At selected times t_j the shape function is derived as the solution of a static eigenvalue problem (Chapter I) with cross-sections at time t_j. This shape function is used for the definition of the true reactivity. The assumption of slow changes seems vague, but we can say that by slow we mean phenomena lasting at least several tens of milliseconds. For instance if $\delta\underset{\sim}{\mathcal{A}}$ is linear in time, the point kinetics reactivity can readily be found and its rate of insertion should be at most \$ 50/sec, which simulates the sodium voiding reactivity insertion in fast reactors[24].

Under these conditions the bound of (2.14) can be used. Now $\delta\underset{\sim}{\mathcal{A}}$ and Δ_{10} are functions of time, but nothing essential changes.

Examples

In the examples that are presented here step and ramp insertions of reactivity are considered in slab geometries.

The exact reactivity is found readily by a modal analysis which uses expansions of the neutron density into a truncated series of Helmholtz modes satisfying the boundary conditions[34]. Thus the k-eigenvalue problem, as defined in Chapter I, is solved; having the fundamental eigenvalues of two states of the reactor we can compute the difference in reactivity by

$$\rho = \frac{k-k'}{k'}$$

The composition of the slabs was typical of fast reactors and the various parameters were taken directly from the paper of Jackson and Kastenberg[24]. They are shown in Tables II.1 and II.2.

TABLE II.1

Two-Group Diffusion Parameters for Bare Cores

	v^{-1} (sec/cm$\times 10^9$)	χ	D(cm)
Group 1	0.445	0.548	3.32
Group 2	1.52	0.452	1.88
	$\Sigma_a(cm^{-1})$	$\nu\Sigma_f(cm^{-1})$	$\Sigma_{1\to 2}(cm^{-1})$
Group 1	0.000477	0.01408	0.04121
Group 2	0.003241	0.00534	—
Group Boundaries Group 1 : 1.4 MeV Group 2 : 0 - 1.4 MeV Critical Transverse Buckling B_T^2 (cm^{-2}) 200-cm Core : 0.00122026 300-cm Core : 0.00135692			

The point kinetics parameters are $\beta = 0.0039$, $\lambda = 0.073\ \text{sec}^{-1}$, and $\Lambda = 2.16 \times 10^{-7}$ sec.

TABLE II.2

Two-Group Diffusion Parameters for Blankets

	D(cm)	$\Sigma_a(\text{cm}^{-1})$	$\nu\Sigma_f(\text{cm}^{-1})$	$\Sigma_{1\to 2}(\text{cm}^{-1})$
Group 1	2.76	0.000637	0.017147	0.0497
Group 2	1.45	0.00537	0.000805	—

The core is a 300-cm slab with properties as in Table I and with 50-cm blankets on each side. The critical transverse buckling is then $B_T^2 = 0.00138043\ \text{cm}^{-2}$ and the point kinetics parameters are $\beta = 0.0036$, $\lambda = 0.073\ \text{sec}^{-1}$, and $\Lambda = 2.17 \times 10^{-7}$ sec.

Example 1

Perturbations of the absorption cross section Σ_{a2} are made in the first 20% of the 200-cm bare slab. Then the perturbing operator is

$$\delta\underset{\sim}{\mathcal{M}} = \begin{bmatrix} 0 & 0 \\ 0 & \delta\Sigma v_2 \end{bmatrix} \quad \text{for } 0 \le x \le 40\ \text{cm}$$

As a measure of the error of point kinetics we take the ratio

$$\epsilon = \frac{\rho' - \rho_0}{\rho'}$$

where ρ' is the "true" reactivity found as described before and ρ_0 the reactivity calculated using the unperturbed shape function (point kinetics). Since the reactor is homogeneous $\mu = 1$.

To show that the bound is indeed improved with the use of λ we consider a specific perturbation $\delta\Sigma = -2.21\times10^{-4}\ \mathrm{cm}^{-1}$. For the unperturbed system we find

$$\left(\frac{\rho}{\Lambda}\right)_1^0 = -1.06\times10^{6}\ \mathrm{sec}^{-1}$$

The radius of the first disc is $r_0 = 4.8\times10^{4}\ \mathrm{sec}^{-1}$ and the approximate radius of the second disc (corresponding to the first harmonic) is $r_1 \cong \sqrt{k_1}\,\Delta_{10} = 2.3\times10^{4}\ \mathrm{sec}^{-1}$. Clearly

$$r_0 + r_1 < \left|\left(\frac{\rho}{\Lambda}\right)_1^0\right|$$

so improvement can be achieved (Figure II.1). We estimate that a distance

$$k\left|\left(\frac{\rho}{\Lambda}\right)_1^0\right| = 9.6\times10^{5}\ \mathrm{sec}^{-1}$$

will be enough to avoid overlapping of the discs and we choose

$$\lambda = \frac{k\left|\left(\frac{\rho}{\Lambda}\right)_1^0\right|}{r_1} \doteq 43$$

Then the improved bound is

$$r_0' = \frac{r_0}{\lambda} = 0.111\times10^{4}$$

and we see it decreases more than an order of magnitude.

In this case point kinetics predicts

$$\left(\frac{\rho}{\Lambda}\right)_0 = 1.077\times10^{4}\ \mathrm{sec}^{-1}$$

and a static eigenvalue calculation yields the true

$$\left(\frac{\rho}{\Lambda}\right) = 1.15\times 10^{4}\ \text{sec}^{-1}$$

Our estimate is

$$\left(\frac{\rho}{\Lambda}\right)_b = \left(\frac{\rho}{\Lambda}\right)_0 + r'_0 = 1.188\times 10^{4}\ \text{sec}^{-1}$$

In Figure II.2 the error parameter ϵ and its bound are given as a function of the reactivity insertion for a series of step perturbations.

Example 2

The absorption cross section for the second group is decreased linearly in time in the first 60 cm of the 300-cm bare core. The perturbing operator is

$$\delta\underset{\sim}{\mathcal{H}} = \begin{bmatrix} 0 & 0 \\ 0 & -4.56\times 10^{-3} t v_2 \end{bmatrix} \qquad \begin{matrix} 0 \le x \le 60\ \text{cm} \\ 0 < t \end{matrix}$$

Point kinetics gives the reactivity as a linear function of time with insertion rate 7.93 dollars/sec. As shown in Figure II.3 there is no fixed rate of insertion for the actual reactivity due to the continuous flux tilting. The bound is shown in the same figure.

Example 3.

In this example we consider a ramp decrease of the absorption cross section of the second group in the blanketed core. The perturbation is made in the crosshatched region of Figure II.4

The cross section is being decreased at a rate of -1.98×10^{-3} cm^{-1}/sec which yields a reactivity insertion rate (point kinetics) of 8 dollars/sec.

Since the reactor is multiregion we must give an estimate of the lower bound μ to the least eigenvalue of the Gram matrix $\{\langle \underline{n}_j^* | \underline{n}_j^* \rangle\}$. We normalize $\underline{n}_j^*$ and we calculate

$$\langle \underline{n}_0^* | \underline{n}_1^* \rangle = -0.008$$

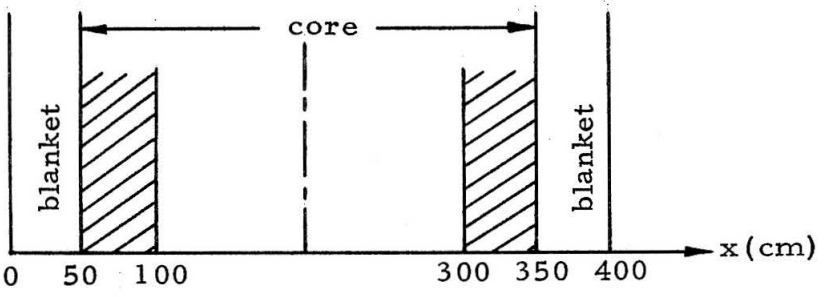

Figure II.4. Perturbation in Blanketed Core.

The 2×2 Gram matrix is then

$$G_2 = \begin{bmatrix} 1 & -0.008 \\ -0.008 & 1 \end{bmatrix}$$

which is strongly diagonal dominant. We do not expect that by increasing the dimension of G the eigenvalues will change much. Therefore we use as lower bound to the least eigenvalue the Ostrowski estimate

$$\mu = 1 - 0.008 = 0.992$$

In Figure II.5 the reactivity and the bound are shown as functions of time.

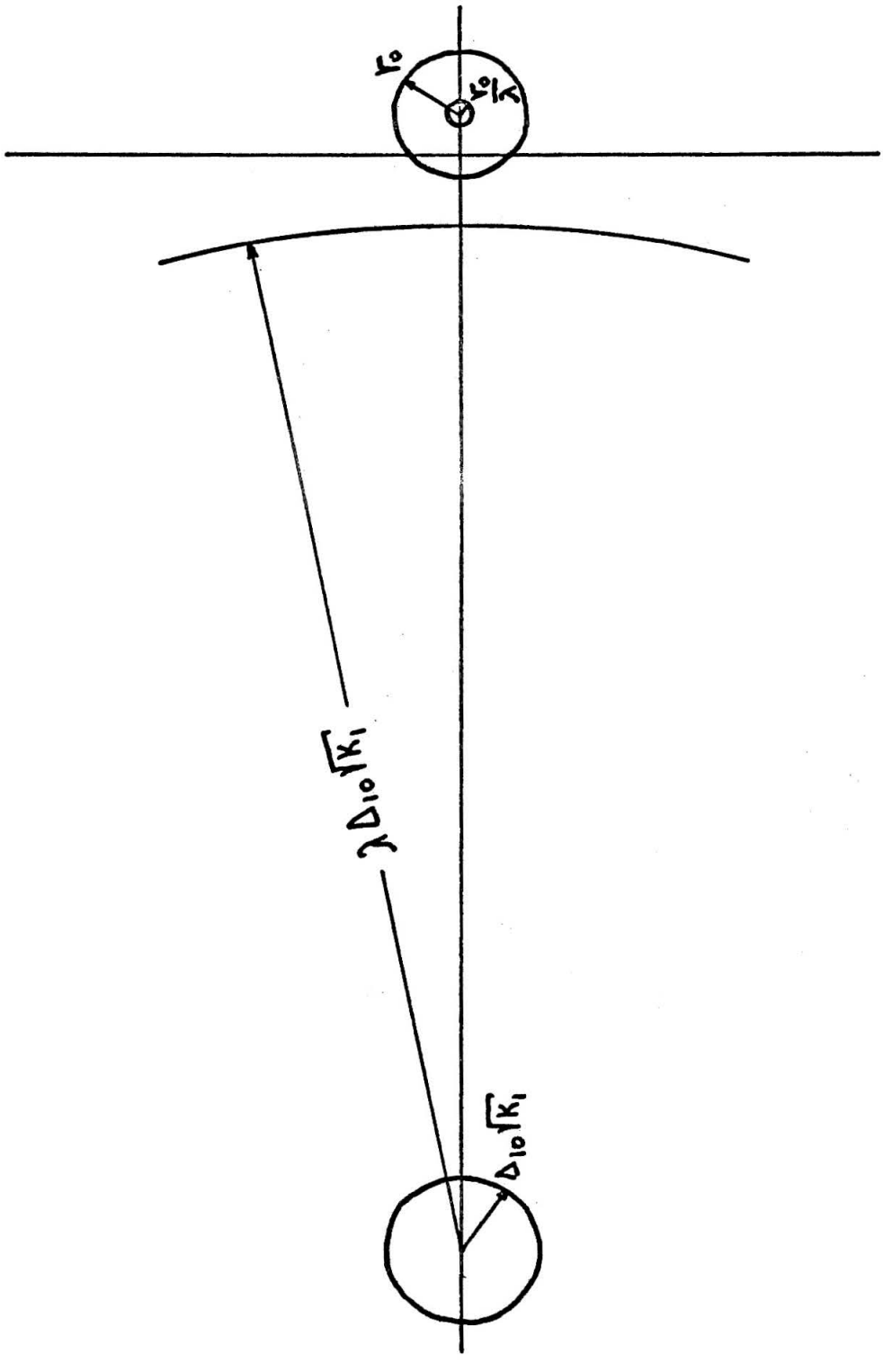

Figure II. 1. Bound Improvement.

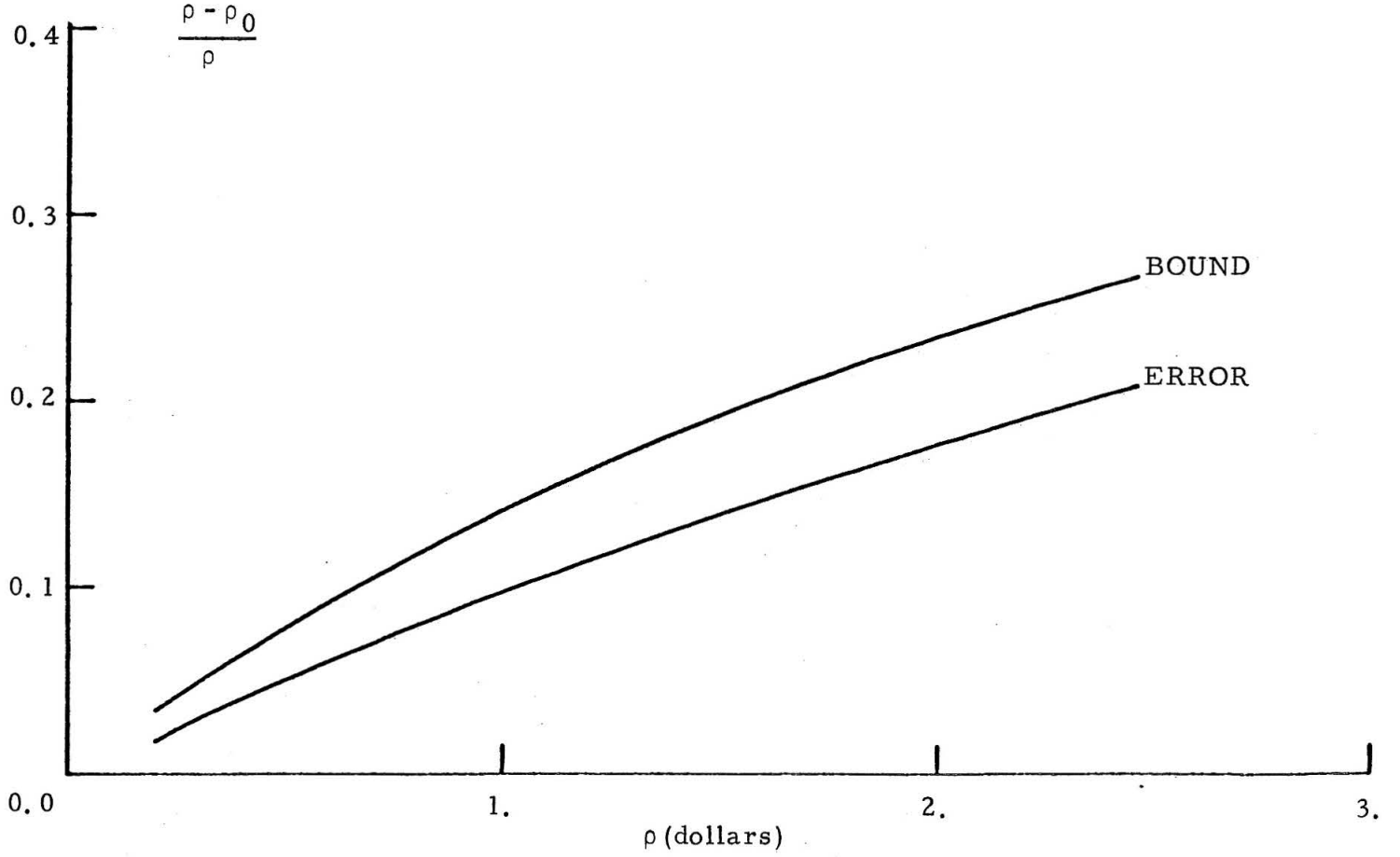

Figure II. 2. Step Insertion of Reactivity. 200 cm Bare Core.

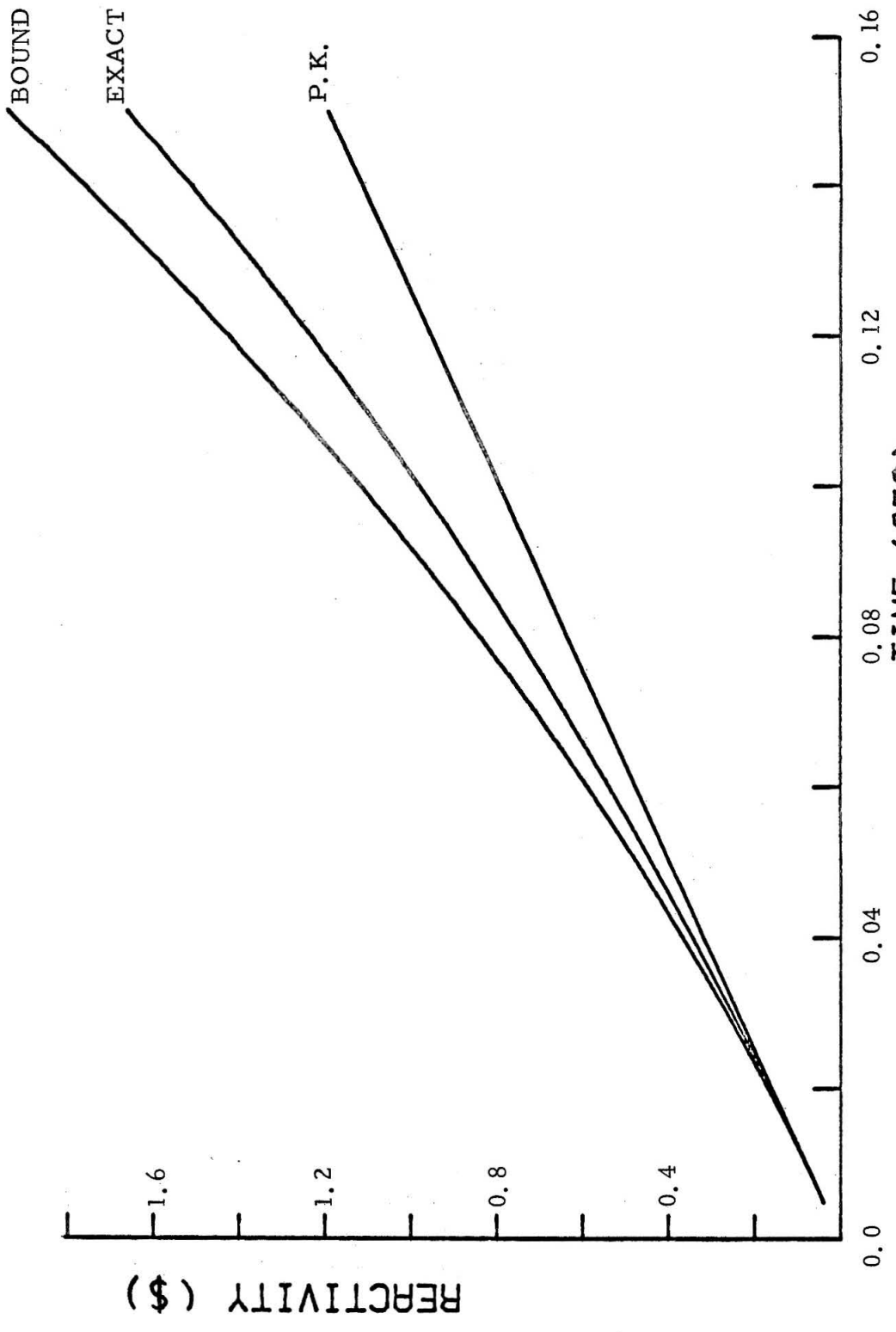

Figure II.3. Ramp Insertion of Reactivity. 300 cm Bare Core.

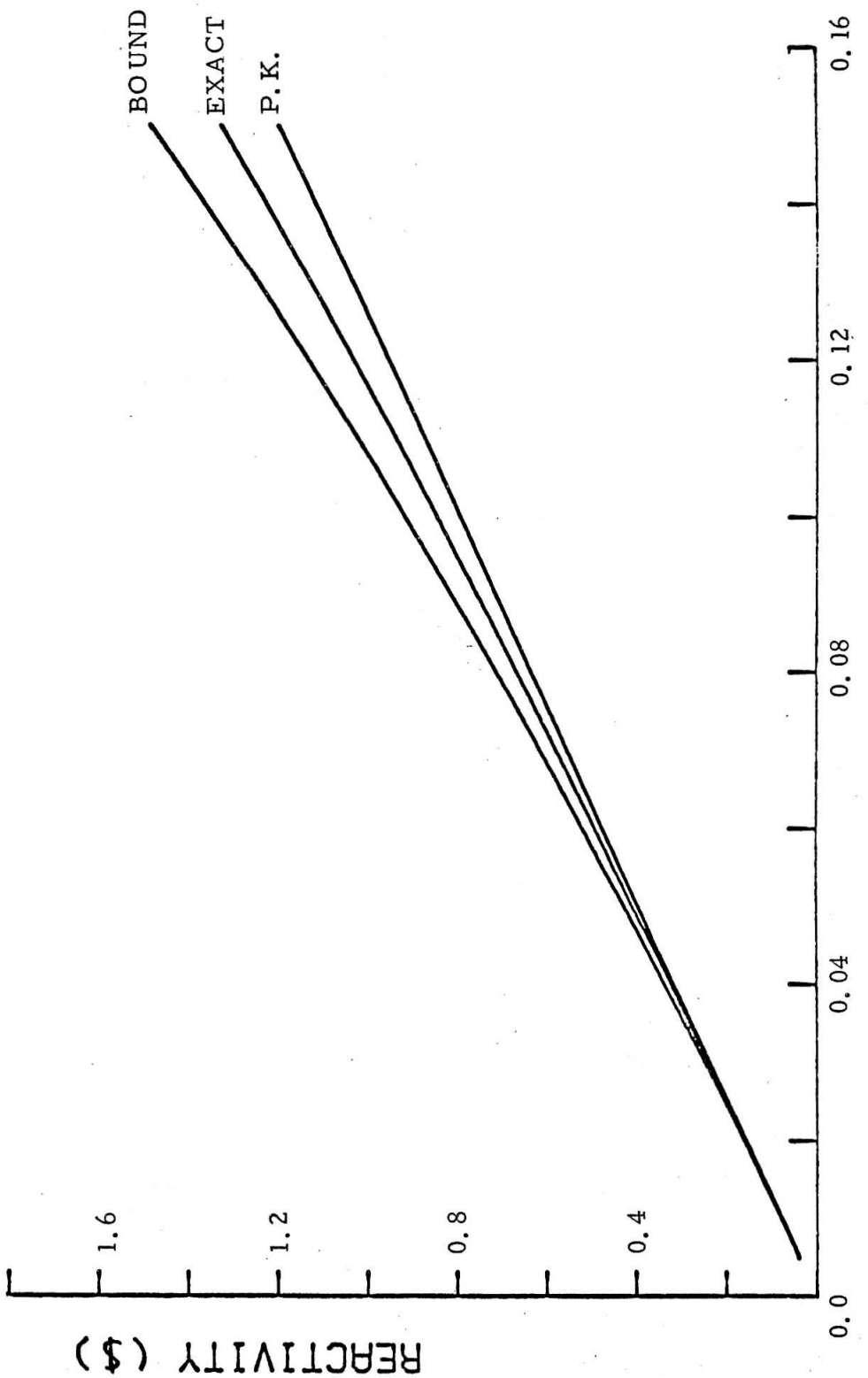

Figure II.5. Ramp Insertion of Reactivity, 300 cm Blanketed Core.

III. THE EIGENVALUES OF THE GROUP DIFFUSION EQUATIONS

In this chapter we focus our attention to the study of the effect of delayed neutrons on the eigenvalues of the group diffusion equations. Including delayed neutrons these equations are (Eqns. (1.3) and (1.4))

$$\underset{\sim}{O}\underline{\psi}_n = \omega_n \underline{\psi}_n$$

As stated in Chapter I, a consequence of the two time scales in the neutron multiplication is that the prompt neutrons excite modes with large $|\omega_n|$ (larger than 10^3 sec^{-1} for thermal reactors and 10^7 sec^{-1} for fast reactors) which are approximated by (Eqn. (1.7), no delayed neutrons)

$$\left((1-\beta)\underset{\sim}{M}_p - \underset{\sim}{L}\right)\underline{N}_n^{(p)} = \omega_n^{(p)}\underline{N}_n^{(p)} \tag{3.1}$$

On the other hand the delayed eigenvalues are of the order of magnitude of $|\lambda_i|$, i.e. less than 2 or 3 sec^{-1}. Their modes are approximately given by (Eqn. (1.11))

$$\underset{\sim}{L}\underline{N}_n = \frac{1}{k_n}\underset{\sim}{M}_p\underline{N}_n \tag{3.2}$$

It is our task to see how the prompt eigenvalues of (3.1) change when delayed neutrons are included and to predict the appearance of the delayed eigenvalues. A by-product of the method is a criterion for the accuracy of the inhour modes approximation.

A. The Perturbation Method

We write the operator $\underset{\sim}{O}$ in (1.3) as

$$\underset{\sim}{O} \equiv \underset{\sim}{O}_1 + \epsilon\underset{\sim}{P}$$

where

$$\underset{\sim}{O}_1 \equiv \begin{bmatrix} (1-\beta)\underset{\sim}{M}_p - \underset{\sim}{L} & \lambda_1 \underline{\chi}_1 \cdots \lambda_6 \underline{\chi}_6 \\ \begin{matrix} 0 \\ \vdots \\ 0 \ldots \end{matrix} & \begin{matrix} -\lambda_1 & & 0 \\ 0 & \ddots & \vdots \\ & \ddots & 0 \\ 0 \ldots & 0 & -\lambda_6 \end{matrix} \end{bmatrix}$$

and

$$\underset{\sim}{P} \equiv \begin{bmatrix} \underset{\sim}{O} & 0 \ldots 0 \\ \beta_1 \underline{F}^T & 0 \\ \vdots & \\ \beta_6 \underline{F}^T & 0 \ldots 0 \end{bmatrix}$$

The fractional yields β_i are of order 10^{-3} and this makes the norm of $\underset{\sim}{P}$ to be orders of magnitude (at least 10^2) less than the norm of $\underset{\sim}{O}_1$. This enables us to consider $\underset{\sim}{P}$ as a perturbation to $\underset{\sim}{O}_1$. To indicate this we use the parameter ϵ which eventually will be set equal to one.

Of course we recognize that $\underset{\sim}{O}_1$ does not describe any physical situation, but it is merely a convenient operator to work with.

We assume we know the prompt periods and modes (Eqn. (3.1)) and those of the adjoint problem. To proceed we need to know the eigenvalues and eigenfunctions of $\underset{\sim}{O}_1$.

It is clear that $\omega_n^{(p)}$ is an eigenvalue of $\underset{\sim}{O}_1$ with corresponding eigenfunction $\underline{\psi}_n^{(p)T} \equiv (\underline{N}_n^{(p)}, 0 \ldots 0)$. Each $\omega_n^{(p)}$ is a simple eigenvalue of $((1-\beta)\underset{\sim}{M}_p - \underset{\sim}{L})$ and also of $\underset{\sim}{O}_1$. Furthermore, each $-\lambda_i$ is an eigenvalue of $\underset{\sim}{O}_1$, because the corresponding row has zero off-diagonal elements.

Since there is no differential operator involved in the last six rows of $\underset{\sim}{O}_1$, each $-\lambda_i$ is an eigenvalue of infinite multiplicity with eigenfunctions

$$\underline{\psi}_{n,i}^{(d)T} \equiv \left(\Sigma a_j^{in} \underline{N}_j^{(p)}, \delta_{i1} u_n, \delta_{i2} u_n, \ldots, \delta_{i6} u_n\right) \qquad n = 1, 2, \ldots$$

where

$$\delta_{ij} \equiv \text{Kronecker's delta}$$

and u_n is the solution of the Helmholtz equation

$$\nabla^2 u_n + B_n^2 u_n = 0$$

in the region of the reactor with zero boundary conditions. This choice of functions for the last six components of $\underline{\psi}_{n,i}^{(d)}$ is possible because, as we said above, only the numbers $-\lambda_i$ appear in the diagonal entries of the last six rows of $\underset{\sim}{O}_1$. Therefore we can employ any set of functions complete in space and satisfying the boundary conditions to describe the spatial dependence. This will be needed later, when perturbations will be introduced.

The coefficients a_j^{in} can easily be found from

$$\underset{\sim}{O}_1 \underline{\psi}_n^{(d)} = -\lambda_i \underline{\psi}_n^{(d)}$$

Multiplying through the left with $(\underline{N}_j^{(p)*T}, 0 \ldots 0)$ and integrating over the volume of the reactor gives

$$a_j^{in} = -\frac{\langle \underline{N}_j^{(p)*} | \lambda_i \underline{\chi}_i u_n \rangle}{\langle \underline{N}_j^{(p)*} | \underline{N}_j^{(p)} \rangle \left(\omega_j^{(p)} + \lambda_i\right)} \tag{3.3}$$

In Table III.1 we show the eigenfunctions and eigenvalues of $\underset{\sim}{O}_1$.

TABLE III.1

Eigenvalues and Eigenfunctions of the Unperturbed Operator $\underset{\sim}{O}_1$

Eigenvalue	Multiplicity	Eigenfunction
$\omega_n^{(p)}$	simple	$\underline{\psi}_n^{(p)T} \equiv \left(\underline{N}_n^{(p)}, 0, \ldots 0\right)$
$-\lambda_i$	infinite	$\underline{\psi}_{n,i}^{d(T)} \equiv \left(\Sigma a_j^{in} \underline{N}_j^{(p)T}, \delta_{i1} u_n \cdots \delta_{i6} u_n\right)$

To use perturbation theory we also need the characteristic values and functions of the adjoint of $\underset{\sim}{O}_1$, which is

$$\underset{\sim}{O}_1^* = \begin{bmatrix} (1-\beta)\underset{\sim}{M}_p^T - \underset{\sim}{L}^T & 0 & \cdots & 0 \\ \lambda_1 \underline{\chi}_1^T & -\lambda_1 & & \vdots \\ \vdots & 0 & \ddots & \vdots \\ \lambda_6 \underline{\chi}_6^T & 0 & \cdots 0 & -\lambda_6 \end{bmatrix}$$

Again we observe that $-\lambda_i$ is an eigenvalue with infinite multiplicity and eigenfunctions

$$\underline{\psi}_n^{(d)*T} \equiv \left(\underline{0}, \delta_{i1} u_n, \ldots, \delta_{i6} u_n\right) \qquad , \qquad n = 1, 2, \ldots$$

Also $\overline{\omega}_n^{(p)}$ is a simple eigenvalue but now the corresponding eigenfunction is written in the form

$$\underline{\psi}_n^{(p)*T} \equiv \left(\underline{N}_n^{(p)*T}, \sum_{j=1}^{\infty} d_j^{1n} u_j, \ldots, \sum_{j=1}^{\infty} d_j^{6n} u_j\right)$$

where the coefficients d_j^{in} must be determined. As before we must have

$$\underset{\sim}{O}_1^* \underline{\psi}_n^{(p)*} = \overline{\omega}_n^{(p)} \underline{\psi}_n^{(p)*}$$

Multiply through the left with $(\underline{0}, \ldots 0, u_j, 0 \ldots)$ and integrate to get

$$d_j^{in} = \frac{\langle u_j \lambda_i \underline{\chi}_i | \underline{N}_n^{(p)*} \rangle}{\langle u_j | u_j \rangle \left(\overline{\omega}_n^{(p)} + \lambda_i \right)}$$

Table III. 2 shows the above functions

TABLE III. 2

Eigenvalues and Eigenfunctions of $\underset{\sim}{O}_1^*$

Eigenvalue	Multiplicity	Eigenfunction
$\overline{\omega}_n^{(p)}$	simple	$\underline{\psi}_n^{(p)*T} \equiv \left(\underline{N}_n^{(p)*T}, \sum_{j=1}^{\infty} d_j^{1n} u_j \ \ldots \ \sum_{j=1}^{\infty} d_j^{6n} u_j \right)$
$-\lambda_i$	infinite	$\underline{\psi}_{n,i}^{(d)*T} \equiv (\underline{0} \ldots \delta_{i1} u_n \ldots \delta_{i6} u_n)$

Among the eigenfunctions of Tables III. 1 and III. 2 the usual orthogonality properties hold, i.e.

$$\langle \underline{\psi}_n^{(p)*} | \underline{\psi}_m^{(p)} \rangle = 0 \quad , \quad n \neq m$$

$$\langle \underline{\psi}_n^{(p)*} | \underline{\psi}_{m,i}^{(d)} \rangle = 0 \quad , \quad \text{any } n, m, i$$

$$\langle \underline{\psi}_{n,i}^{(d)*} | \underline{\psi}_m^{(p)} \rangle = 0 \quad , \quad \text{any } n, m, i$$

$$\langle \underline{\psi}_{n,i}^{(d)*} | \underline{\psi}_{m,j}^{(d)} \rangle = 0 \quad , \quad n \neq m \text{ and/or } i \neq j$$

At this point one may wonder why we have to go through all this calculation. A natural approach would be to eliminate the precursor densities in (1.3) using the formulas

$$C_{i,n} = \frac{\beta_i}{\omega_n + \lambda_i} \langle \underline{F} | \underline{N}_n \rangle$$

and get the problem

$$\left(\left((1-\beta) \underset{\sim}{M}_p - \underset{\sim}{L} \right) + \sum_{i=1}^{6} \frac{\lambda_i \beta_i}{\omega_n + \lambda_i} |\underline{\chi}\rangle \langle \underline{F}| \right) \underline{N}_n = \omega_n \underline{N}_n \tag{3.4}$$

Then we could treat the prompt-neutron operator $((1-\beta)\underset{\sim}{M}_p - \underset{\sim}{L})$ as the unperturbed operator and the sum of dyadics in the left-hand side as the perturbation. But this is now a singular-perturbation problem because of the presence of the denominators $(\omega_n + \lambda_i)$. The eigenvalues of (3.4) will be the $\omega_n^{(p)}$ shifted due to the perturbation and a whole new set of infinite eigenvalues grouped around each $-\lambda_i$. Although the calculation of the shifting of $\omega_n^{(p)}$ is a relatively simple matter the calculation of these new eigenvalues is difficult. In our approach we use as unperturbed operator $\underset{\sim}{O}_1$, which already has $-\lambda_i$ as eigenvalues. All we have to do is estimate the shifting of the eigenvalues, as in ordinary perturbation theory, and we do not have to worry about the appearance of new eigenvalues.

To find the first order correction to the prompt eigenvalues we have as usual

$$\left(\underset{\sim}{O}_1 - \omega_n^{(p)} \right) \left(\sum_{\substack{j \\ j \neq n}} \gamma_j^n \underline{\psi}_j^{(p)} + \sum_{i=1}^{6} \sum_{m=1}^{\infty} \zeta_m^{in} \underline{\psi}_{m,i}^{(d)} \right) = -\underset{\sim}{P} \underline{\psi}_n^{(p)} + \delta\omega_n^{(p)} \underline{\psi}_n^{(p)}$$

The above equation is premultiplied by $\underline{\psi}_n^{(p)*}$ and integrated over the volume of the reactor. Using the orthogonality properties we get

$$\delta\omega_n^{(p)} = \sum_{i=1}^{6} \frac{\langle \underline{N}_n^{(p)*} \lambda_i \underline{\chi}_i \,|\, \beta_i \underline{F}^T \underline{N}_n^{(p)} \rangle}{\langle \underline{N}_n^{(p)*} \,|\, \underline{N}_n^{(p)} \rangle \left(\omega_n^{(p)} + \lambda_i \right)} \tag{3.5}$$

where we have also used the identity

$$\sum_j \frac{|u_j\rangle\langle u_j|}{\langle u_j | u_j \rangle} = \underset{\sim}{1}$$

The periods of the system are then

$$\omega_n = \omega_n^{(p)} + \delta\omega_n^{(p)}$$

Of course we can proceed and derive expressions for the expansion coefficients γ_j^n and ζ_m^{in}, but this is of no practical use and the algebra is quite involved, so we do not do it.

We now turn our attention to the delayed eigenvalues. Again we have

$$\left(\underset{\sim}{Q}_1 + \lambda_i \right) \left(\sum_j \eta_j^{n,i} \underline{\psi}_j^{(p)} + \sum_{i=1}^{6} \sum_{m=1}^{\infty} \theta_m^{n,i} \underline{\psi}_{m,i}^{(d)} \right) = -\underset{\sim}{P} \psi_{n,i}^{(d)} + \delta\omega_{n,i} \psi_{n,i}^{(d)}$$

We operate as before with $\underline{\psi}_{n,i}^{*(d)}$ and get

$$\delta\omega_{n,i} = -\sum_j \frac{\langle u_n \,|\, \beta_i \underline{F}^T \underline{N}_j^{(p)} \rangle \langle \underline{N}_j^{*(p)} \,|\, \lambda_i \underline{\chi}_i u_n \rangle}{\langle \underline{N}_j^{*(p)} \,|\, \underline{N}_j^{(p)} \rangle \left(\omega_j^{(p)} + \lambda_i \right)} \tag{3.6}$$

The delayed eigenvalues are then

$$\omega_{n,i} = -\lambda_i + \delta\omega_{n,i}$$

We observe that the result (3.5) is given in closed form while (3.6) involves an infinite number of terms. This is a consequence of the fact that $\omega_n^{(p)}$ are simple eigenvalues and the $-\lambda_i$ have infinite multiplicity. In the case of $\underline{F}$ being independent of position the terms $\langle \underline{\psi}_{m,i}^{*(d)} | \underset{\sim}{P} | \underline{\psi}_{n,i}^{(d)} \rangle$ are zero for $m \neq n$ and we need not use degenerate perturbation theory. Otherwise the $\delta\omega_{n,i}$'s are given by the eigenvalues of the infinite matrix $\{\langle \underline{\psi}_{m,i}^{*(d)} | \underset{\sim}{P} | \underline{\psi}_{n,i}^{(d)} \rangle\}$.

For a bare homogeneous reactor the $\underline{N}_j^{(p)}$ are separable in space and energy; the spatial part is given by the u_j, i.e.

$$\underline{N}_j^{(p)} = \underline{w}_j u_j(\underline{r})$$

Then the sum in (3.6) has a finite number of terms, only those which correspond to the same buckling B_n. In the general case we would expect that only the terms corresponding to the n^{th} spatial harmonic will be significant.

B. The Inhour Modes

In the inhour approximation it is assumed that some of the eigenfunctions of $\underset{\sim}{Q}$ are grouped in clusters of seven, all vectors of a cluster having the same neutron density vector. An indication for the accuracy of this assumption can be deduced from the results of the above analysis.

In Table III.1 we see that for every spatial mode there will be G vectors (G-group diffusion theory) $\underline{\psi}_n^{(p)}$ and six vectors $\underline{\psi}_{n,i}^{(d)}$. These

are correct within zero-order in β and they do not change much after the perturbation. Therefore, if we can have evidence that seven of these G+6 eigenvectors have very similar neutron-density vectors, then the inhour approximation will be justified.

But the neutron-density part of each $\psi_{n,i}^{(d)}$ is given in the form of a series

$$\sum_j a_j^{in} \underline{N}_j^{(p)}$$

where a_j^{in} is given by (3.3).

Since for every spatial harmonic there are G vectors $\psi_n^{(p)}$ we can change the notation to show this by writing

$$\underline{\psi}_n^{(p)} \rightarrow \underline{\psi}_{nk}^{(p)} \qquad k = 1, 2, \ldots G$$

where n indicates the spatial harmonic (buckling in a bare homogeneous reactor) and k indicates the G different vectors corresponding to the same spatial harmonic (i.e. the same number of zeros in the domain of the reactor, see Chapter I). Each $\underline{\psi}_{nk}^{(p)}$ has its own eigenvalue $\omega_{nk}^{(p)}$. The series expansions for $\psi_{n,i}^{(d)}$ can be written as

$$\sum_{jk} a_{jk}^{in} \underline{N}_{jk}^{(p)}$$

where

$$a_{jk}^{in} = - \frac{\langle \underline{N}_{jk}^{(p)*} | \lambda_i \underline{\chi}_i u_n \rangle}{\langle \underline{N}_{jk}^{(p)*} | \underline{N}_{jk}^{(p)} \rangle \left(\omega_{jk}^{(p)} + \lambda_i \right)}$$

But now it is clear that we must compare the various $|a_{jk}^{in}|$ for different j and k. If one of them is much greater than the rest, then the whole series will be dominated by the corresponding $\underline{N}_{jk}^{(p)}$ and hence the inhour approximation is valid. That such a case must be expected is justified by the fact that for j=n the eigenvalues $\omega_{nk}^{(p)}$, k=1, ..., G differ by orders of magnitude, as shown in Chapter I. Furthermore, for j≠n the numerator is expected to decrease rapidly as $j \to \infty$.

Example

The 200-cm bare core of Chapter II is again considered. The absorption cross section of the second group is perturbed stepwise and eigenvalues are calculated using the perturbation method. These are compared to the exact ones found by direct solution of the matrices. This can be done here relatively easily, because two-group theory with one delayed neutron group is used.

The results for two perturbations are shown in Tables III.3 and III.4.

TABLE III.3

Perturbed $\Sigma_{a2} = 0.003226$ cm^{-1}

Mode		Fundamental	1st Harmonic
ω_1 (sec^{-1})	Prompt	-8.969819×10^{7}	-9.510598×10^{7}
	Perturb.	-8.969818×10^{7}	-9.510597×10^{7}
	Exact	-8.969818×10^{7}	-9.510597×10^{7}

(continued)

TABLE III.3
(continued)

Mode		Fundamental	1st Harmonic
ω_2 (sec^{-1})	Prompt	-9.011826×10^{3}	-1.039302×10^{6}
	Perturb.	-9.011969×10^{3}	-1.039302×10^{6}
	Exact	-9.011969×10^{3}	-1.039302×10^{6}
ω_d (sec^{-1})	Perturb.	7.234895×10^{-2}	-7.174027×10^{-2}
	Exact	7.234663×10^{-2}	-7.174027×10^{-2}
$\lvert a_{n1}\rvert$		1.017	9.580×10^{-1}
$\lvert a_{n2}\rvert$		5.841×10^{3}	4.928×10^{-1}

TABLE III.4

Perturbed $\Sigma_{a2} = 0.003211\ cm^{-1}$

Mode		Fundamental	1st Harmonic
ω_1 (sec^{-1})	Prompt	-8.969793×10^{7}	-9.510574×10^{7}
	Perturb.	-8.969792×10^{7}	-9.510573×10^{7}
	Exact	-8.969792×10^{7}	-9.510573×10^{7}
ω_2 (sec^{-1})	Prompt	5.990528×10^{2}	-1.029664×10^{6}
	Perturb.	6.012380×10^{2}	-1.029669×10^{6}
	Exact	6.012302×10^{2}	-1.029669×10^{6}

TABLE III.4
(continued)

Mode		Fundamental	1st Harmonic
ω_d (sec^{-1})	Perturb.	-2.258365	-7.1729×10^{-2}
	Exact	-2.250452	-7.1729×10^{-2}
$\lvert a_{n1} \rvert$		1.017	9.58×10^{-1}
$\lvert a_{n2} \rvert$		8.785×10^{4}	4.97×10^{1}

Since we are using two energy groups there are two eigenvalues for each spatial mode, ω_1 and ω_2, corresponding to the prompt modes and one small eigenvalue ω_d corresponding to the delayed neutrons (one group of precursors).

We see that the results of the perturbation method are in good agreement with the exact solutions. Especially for the harmonics the agreement is excellent.

The remarkable result is the good prediction of the delayed eigenvalue, because the correction to ω_1 and ω_2 is insignificant and was expected.

Finally the coefficients a_{jk}^{in} are given. We see that for the fundamental the coefficient a_{n2} is greater than a_{n1} by a least a factor of 10^3. Hence we can claim that the inhour approximation is justified here, i.e. the delayed mode and the mode with eigenvalue ω_2 have approximately the same neutron density. For the harmonics the difference of the coefficients is not so pronounced and the approximation

becomes worse, as expected. Since the reactor is homogeneous we need only worry about the coefficients of the modes with the same buckling and since we use only one group of precursors (i=1) we have dropped the superscripts in a_{jk}^{in}. Thus the delayed eigenfunction for the n^{th} buckling is

$$\left(a_{n1}\underline{N}_{n1}^{(p)} + a_{n2}\underline{N}_{n2}^{(p)}\right)$$

APPENDIX A

The Point Kinetics Model

The neutron density is written as the product of a shape function $\underline{\eta}(\underline{r}, t)$ and an amplitude function P(t), i. e.

$$\underline{N}(\underline{r}, t) = \underline{\eta}(\underline{r}, t)P(t) \tag{A. 1}$$

The vector $\underline{\eta}(\underline{r}, t)$ of group densities implicitly contains the energy dependence of $\underline{N}$. It is assumed to vary slowly with time. The main time-dependence of $\underline{N}$ is given by P(t).

We now define a vector of weighting functions $\underline{w}_j(\underline{r})$, i. e.

$$\underline{w}(\underline{r}) = \left(w_1(\underline{r}), \ldots, w_N(\underline{r})\right)^T \tag{A. 2}$$

and using (A. 1) in (1. 1) and (1, 2) we premultiply by $\underline{w}^T(\underline{r})$ the first equation and by $\underline{w}^T(\underline{r})\underline{\chi}_i$ the second and we integrate over the volume of the reactor to get the point kinetics equations

$$\frac{dP(t)}{dt} = \frac{\left(\rho(t) - \overline{\beta}\right)}{\Lambda} P(t) + \sum_{i=1}^{6} \lambda_i \hat{C}_i(t) + \hat{Q}(t) \tag{A. 3}$$

$$\frac{d\hat{C}_i}{dt} = \frac{\overline{\beta}_i}{\Lambda} P(t) - \lambda_i \hat{C}_i(t) \tag{A. 4}$$

where we have defined:

reactivity:

$$\rho(t) \equiv \frac{\int d\underline{r}\, \underline{w}^T \left[\nabla \cdot \underset{\sim}{D}\nabla - \underset{\sim}{R}_a - \underset{\sim}{R}_s + \underset{\sim}{S} + (1-\beta)\underline{\chi}_p \underline{F}^T + \sum_{i=1}^{6} \beta_i \underline{\chi}_i \underline{F}^T\right] \underline{n}}{F} \tag{A.5}$$

effective delayed neutron fraction:

$$\overline{\beta}_i \equiv \frac{\beta_i}{F} \int d\underline{r}\, \underline{w}^T \underline{\chi}_i \underline{F}^T \underline{n} \tag{A.6}$$

$$\overline{\beta} \equiv \sum_{i=1}^{6} \overline{\beta}_i$$

effective concentration of delayed neutron precursors:

$$\hat{C}_i(t) \equiv \int d\underline{r}\, \underline{w}^T \underline{\chi}_i C_i \tag{A.7}$$

mean prompt generation time:

$$\Lambda \equiv \frac{1}{F} \int d\underline{r}\, \underline{w}^T \underline{n} \tag{A.8}$$

effective source:

$$\hat{Q}(t) \equiv \int d\underline{r}\, \underline{w}^T \underline{Q} \tag{A.9}$$

normalization factor:

$$F \equiv \int d\underline{r}\, \underline{w}^T \left[(1-\beta)\underline{\chi}_p \underline{F}^T + \sum_{i=1}^{6} \beta_i \underline{\chi}_i \underline{F}^T\right] \underline{n} \tag{A.10}$$

To insure uniqueness of the decomposition (A.1) a normalization condition is imposed on the shape function. Usually it is taken to be

$$\frac{\partial}{\partial t} \int d\underline{r}\, \underline{w}^T \underline{n} = 0 \tag{A.11}$$

It should be noted that no approximation has been involved now and that Equations (A.3) to (A.11) are completely equivalent to (1.1) and (1.2).

In conventional point kinetics the weight vector is taken to be the adjoint steady-state flux of a critical reactor with similar nuclear characteristics and of the same geometry (reference reactor). In this connection we define the operators

$$\underset{\sim}{L} \equiv -\nabla \cdot \underset{\sim}{D} \nabla + \underset{\sim}{R}_a + \underset{\sim}{R}_s - \underset{\sim}{S} \tag{A.12}$$

$$\underset{\sim}{M}_p \equiv \underline{\chi}_p \underline{F}^T \tag{A.13}$$

$$\underset{\sim}{M}_i \equiv \beta_i \underline{\chi}_i \underline{F}^T \quad , \quad i=1, \ldots, 6 \tag{A.14}$$

$$\underset{\sim}{H} \equiv (1-\beta)\underset{\sim}{M}_p - \underset{\sim}{L} \tag{A.15}$$

$$\underset{\sim}{\mathscr{A}} \equiv \underset{\sim}{H} + \sum_{i=1}^{6} \underset{\sim}{M}_i \tag{A.16}$$

where the operators appearing in the right-hand side of the equations have been defined in Chapter I.

If $\underset{\sim}{\mathscr{A}}_0$ is the operator for the reference reactor, then the weight function is the solution of

$$\underset{\sim}{\mathscr{A}}_0^* \underline{n}_0^* = 0 \tag{A.17}$$

with zero boundary conditions.

If a critical state of the reactor at hand is known then it is taken as the reference state.

Let $\delta \underset{\sim}{\mathscr{A}}$ denote the difference in the properties of the actual reactor and the fictitious one, i.e.

$$\delta\underset{\sim}{\mathcal{H}} \equiv \underset{\sim}{\mathcal{H}} - \underset{\sim}{\mathcal{H}}_0 \tag{A.18}$$

Then with the above choice of weight functions the reactivity is given by

$$\rho(t) \equiv \frac{1}{F} \int d\underline{r}\, \underline{\eta}_0^*(\underline{r}) \delta\underset{\sim}{\mathcal{H}}(t) \underline{\eta}(\underline{r},t) \tag{A.19}$$

The point kinetics equations cease to be exact when the shape function is approximated by a known function. Various choices can be found in the literature. The most widely used is the first-order approximation in which the steady-state distribution of the reference reactor is taken to represent $\underline{\eta}(\underline{r},t)$ at all times, i.e.

$$\underline{\eta}(\underline{r},t) \cong \underline{\eta}_0(\underline{r}) \tag{A.20}$$

Then the normalization condition (A.11) is automatically satisfied.

When this approximation is used the error in the parameters $\overline{\beta}$, Λ and F is of higher order than that in ρ and thus they are treated as constants while the reactivity is the only parameter that is affected by changes of the nuclear characteristics of the system. Furthermore changes in reactivity due to perturbations of the various cross sections are additive. However the greatest shortcoming of the first-order perturbation approximation is that changes of the shape function during a transient cannot be accounted for and this can lead to serious errors.

An in-depth analysis of the point kinetics equations is given in Chapter 2 of the book by Akcasu, Lellouche and Shotkin (Ref. 19).

APPENDIX B

The Inhour Modes

With the operator notation defined in Chapter I and Appendix A the natural modes are given by

$$\begin{bmatrix} \underset{\sim}{H} & \lambda_1 \underline{\chi}_1 & \cdots & \lambda_6 \underline{\chi}_6 \\ \beta_1 \underline{F}^T & -\lambda_i & 0 \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ \beta_6 \underline{F}^T & 0 & \cdots\ 0 & -\lambda_6 \end{bmatrix} \begin{bmatrix} \underline{N}_n \\ C_{1,n} \\ \vdots \\ C_{6,n} \end{bmatrix} = \omega_n \begin{bmatrix} \underline{N}_n \\ C_{1,n} \\ \vdots \\ C_{6,n} \end{bmatrix} \tag{B.1}$$

The adjoint problem is derived by taking the transpose of the operators in (B.1)

$$\begin{bmatrix} H^T & \beta_1 \underline{F} & \cdots & \beta_6 \underline{F} \\ \lambda_1 \underline{\chi}_1^T & -\lambda_1 & 0 \cdots & 0 \\ \vdots & 0 & \ddots & \\ \lambda_6 \underline{\chi}_6^T & 0 & \cdots & -\lambda_6 \end{bmatrix} \begin{bmatrix} \underline{N}_n^* \\ C_{1,n}^* \\ \vdots \\ C_{6,n}^* \end{bmatrix} = \overline{\omega}_n \begin{bmatrix} \underline{N}_n^* \\ C_{1,n}^* \\ \vdots \\ C_{6,n}^* \end{bmatrix} \tag{B.2}$$

We wish to derive an eigenvalue problem for the flux vector alone. This is easily done because the last six equations in (B.1) and (B.2) can be solved for the precursor concentrations yielding

$$C_{i,n} = \frac{\beta_i}{\omega_n + \lambda_i} \underline{F}^T \underline{N}_n \tag{B.3}$$

and

$$C^*_{i,n} = \frac{\lambda_i}{\omega_n + \lambda_i} \underline{\chi}_i^T \underline{N}^*_n \qquad (B.4)$$

Using these expressions and the definitions (A.13), (A.14) and (A.16) we get

$$\underset{\sim}{\mathcal{A}} \underline{N}_n = \left(\omega_n + \sum_{i=1}^{6} \frac{\omega_n}{\omega_n + \lambda_i} \underset{\sim}{M}_i \right) \underline{N}_n \qquad (B.5)$$

Multiplication of (B.5) from the left by $\underline{N}^{*T}_n$ and integration over the reactor volume gives the inhour equation for the eigenvalues[10].

$$\left(\frac{\rho}{\Lambda}\right)_n = \omega_n + \sum_{i=1}^{6} \left(\frac{\beta_i}{\Lambda}\right)_n \left(\frac{\omega_n}{\omega_n + \lambda_i}\right) \qquad (B.6)$$

with obvious extension of the definitions (A.5) to (A.10) for every mode.

For each n the inhour equation gives seven roots of which only one is an eigenvalue of the original problem, because the quantities $\left(\frac{\rho}{\Lambda}\right)_n$ and $\left(\frac{\beta_i}{\Lambda}\right)_n$ depend on ω_n implicitly through $\underline{N}_n$. This fact has been demonstrated by Henry[28].

However all seven roots of (B.6) are accepted as eigenvalues because, as argued in Reference 10, some of the natural modes come in clusters of seven all members of a cluster having the neutron density vector approximately the same. Thus in G-group diffusion theory with six delayed neutron groups, Equation (B.1) gives G+6 eigenvectors with similar spatial shapes. Of these the six vectors corresponding to $|\omega| \approx \lambda_i$ (delayed modes) have almost the same $\underline{N}$'s. To this cluster we

add a seventh vector, the one with the smallest $|\omega|$, which also will have an $\underline{N}$ very similar to the one of the cluster.

At this point it is convenient to adjust the notation by denoting all members of a cluster as $\underline{N}_{nj}$ where n corresponds to the spatial mode and j=0,...,6.

If we assume that the $\underline{N}_{nj}$ are exactly the same for any j we get the set of inhour modes. This assumption implies that

$$\underline{N}_{nj} = \underline{N}_{nk} \quad , \quad j,k = 0, \ldots, 6$$

Observe that eigenfunctions corresponding to higher energy modes are not included in the set of inhour modes.

It is proven in Reference 10 that the inhour modes span a subspace and they satisfy useful orthogonality properties

$$\langle \underline{N}^*_{nj} | \underline{N}_{mj} \rangle \sim \delta_{nm} \tag{B.7}$$

$$\langle \underline{N}^*_{nj} | \underline{\underline{M}}_i | \underline{N}_{mj} \rangle \sim \delta_{nm} \tag{B.8}$$

$$\langle \underline{N}^*_{nj} | \underline{\underline{H}} | \underline{N}_{mj} \rangle \sim \delta_{nm} \tag{B.9}$$

normalization:

$$\langle \underline{N}^*_{nj} | \underline{N}_{nj} \rangle + \sum_{i=1}^{6} \frac{\lambda_i \langle \underline{N}^*_{nj} | \underline{\underline{M}}_i | \underline{N}_{nj} \rangle}{(\lambda_i + \omega_n)^2} = 1 \tag{B.10}$$

or, using the definition of $\left(\frac{\beta_i}{\Lambda}\right)_n$

$$\langle \underline{N}^*_{nj} | \underline{N}_{nj} \rangle = \left[1 + \sum_{i=1}^{6} \left(\frac{\beta_i}{\Lambda}\right)_n \frac{\lambda_i}{(\lambda_i + \omega_n)^2} \right]^1 \tag{B.11}$$

The inhour modes form a set complete in space but not in energy, because of the omission of the higher energy transients. The latter have very large periods and as a result the inhour modes cannot describe very fast transients. They are suitable for transients in the range of tens of milliseconds to minutes[10]. Phenomena due to control rod motion or changes in the flow of coolant in large reactors can be described.

APPENDIX C

Ostrowski's Theorem

Consider the following eigenvalue problem

$$\underset{\sim}{A}\underline{x} = \underset{\sim}{F}\underline{x}$$

where $\underset{\sim}{A}$ is an n×n matrix and $\underset{\sim}{F}$ an n×n diagonal matrix with elements $f_j(\omega)$ functions of the eigenvalues ω.

Define two positive numbers p and q such that

$$\frac{1}{p} + \frac{1}{q} = 1$$

and

$$L_m^{(p)} = \left[\sum_{\substack{j=1 \\ j \neq m}}^{n} |a_{mj}|^p \right]^{1/p}$$

Then for every eigenvalue ω the following is true[30]

$$\sum_{m=1}^{n} \frac{1}{1 + \left| \frac{f_m(\omega) - a_{mm}}{L_m^{(p)}} \right|^q} \geq 1 \qquad \text{(C.1)}$$

(If for a row $L_m^{(p)} = 0$ then $f_m(\omega) = a_{mm}$ and the left-hand side is assigned the value 1.)

Proof

For every eigenvalue ω we have the system of equations

$$f_m(\omega)x_m = \sum_{j=1}^{n} a_{mj}x_j \qquad m = 1, \ldots, n \tag{C.2}$$

where at least one of the x_j is nonzero (we assume that no $L_m^{(p)}$ is zero).

We can normalize

$$\sum_{j=1}^{n} |x_j|^q = 1 \tag{C.3}$$

From (C.2) we have

$$|x_m||f_m(\omega) - a_{mm}| \le \sum_{\substack{j=1 \\ j\neq m}}^{n} |a_{mj}||x_j| \le L_m^{(p)}\left(1 - |x_m|^q\right)^{1/q}$$

where the last inequality follows by applying Hölder's inequality and using the definition of $L_m^{(p)}$ and the normalization (C.3).

Elevating to the q^{th} power and rearranging terms we get

$$|x_m|^q \le \frac{L_m^{(p)q}}{L_m^{(p)q} + |f_m(\omega) - a_{mm}|^q} \qquad m = 1, \ldots, n$$

Summing over m and using (C.3) the result (C.1) follows.

Suppose now that we have n positive numbers k_i such that

$$\sum_{i=1}^{n} \frac{1}{k_i + 1} \le 1 \tag{C.4}$$

From (C.4) and (C.1) we see that for every ω there exists at least one k_i, say k_m, such that

$$\frac{1}{1+\left|\frac{f_m(\omega)-a_{mm}}{L_m^{(p)}}\right|^q} \geq \frac{1}{k_m+1}$$

hence

$$|f_m(\omega)-a_{mm}| \leq k_m^{1/q} L_m^{(p)} \tag{C.5}$$

This last inequality means that for every eigenvalue ω there will be at least one $f_m(\omega)$ contained in a disc with radius $k_m^{1/q} L_m^{(p)}$ and center at a_{mm}.

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