ESSAYS IN OPTIMAL RESOURCE ALLOCATION UNDER UNCERTAINTY WITH CAPACITY CONSTRAINTS

Thesis by

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ABSTRACT

This thesis brings together four papers on optimal resource allocation under uncertainty with capacity constraints. The first is an extension of the Arrow-Debreu contingent claim model to a good subject to supply uncertainty for which delivery capacity has to be chosen before the uncertainty is resolved. The second compares an ex-ante contingent claims market to a dynamic market in which capacity is chosen ex-ante and output and consumption decisions are made ex-post. The third extends the analysis to a storable good subject to random supply. Finally, the fourth examines optimal allocation of water under an appropriative rights system.

CONTENTS

CHAPTER I

INTRODUCTION AND REVIEW OF THE LITERATURE

The four papers brought together in this thesis address a theoretical issue which draws together at least three different economic literatures: optimal resource allocation under uncertainty, peak load pricing and the effect on resource allocation of the legal doctrine of appropriative water rights. The issue is to identify conditions which allow competitive markets to produce ex-ante optimal outcomes when the good being allocated is subject to random supply and delivery or storage capacity has to be built before the random variable is observed. An ex-ante optimal allocation is defined as an allocation which maximizes consumer i's expected utility ex-ante, subject to the constraint that all other consumers' expected utilities are held constant ex-ante. Only ex-ante optimality is considered because the introduction of capacity constraints implies that ex-ante and ex-post optimality cannot necessarily be equated. The particular good analyzed specifically is Colorado River water.

The literature on optimal resource allocation under uncertainty can be dated from the original work by Arrow[1] and Debreu[2] on ex-ante contingent claim markets. They concluded that if all decisions were made ex-ante, contingent on the observation of the random variable, a competitive market in contingent claims would achieve the same allocation that would be achieved in a competitive market under certainty. Recognizing, however, that such a system of markets does not exist and probably would not come into being if only because of its informational inefficiencies, Arrow suggested that a market in firm securities should achieve that same result.

Since the seminal work by Arrow and Debreu this literature has developed along several different lines. These include extensions and qualifications of Arrow's and Debreu's conclusions (Dreze,[3] Starr,[4] Radner,[5] Nagatani[6]), rigorous analysis of the conditions which allow securities markets to achieve optimality (Diamond,[7] Leland,[8] Ekern and Wilson,[9] Radner,[10] Forsythe,[11]), and informational equilibrium models (Radner,[12] Rubinstein[13]).

The papers presented below extend this literature in two directions. First, ex-ante optimality conditions under capacity constraints are outlined and sufficient conditions for achieving those optimality conditions under a competitive ex-ante contingent claims market and a competitive securities market are explored. Second, two of the papers consider sufficient conditions for ex-ante optimality under a competitive market structure which has not been explored to any great extent in the literature. In this alternative to an ex-ante contingent claims market, capacity decisions are made before the random variable is observed, but marginal production and consumption decisions are made in competitive markets after.

The current American literature on peak load pricing under uncertainty (Joskow,[14] Johnson,[15] Crew and Kleindorfer[16]) owes its start to the work of French economists (Boiteux,[17] Dreze[18]) who were trying to devise pricing schemes which would both satisfy marginal optimality conditions and cover operating costs for the French national electricity network. The general approach has been to assume a random demand and then use consumer surplus analysis to devise optimal peak and off peak prices and optimal size generation facilities given a probability distribution over demands. While the papers presented below are not specifically concerned with optimal pricing schemes and the peak load pricing literature does not deal with competitive markets, the models used and the results obtained are similar. Basically, most of the optimality conditions presented in these papers should not be seen as new. Rather, they are rearrangements or reinterpretations of the optimal pricing conditions from the peak load pricing literature. Where these papers depart from tradition is in asking what conditions would allow ex-ante optimality to be achieved under competition.

A recent paper by McKay[19] has moved away from the emphasis on consumer surplus analysis. His welfare model is a weighted sum of individual utility functions and his analysis is similar to that used in the papers collected here, although the questions he addresses are different.

Recent papers by Burness and Quirk[20] on the appropriative water rights system form a natural introduction to the third and fourth papers presented below. Their papers consider two issues:

the effect on resource allocation of the legal doctrine of appropriative water rights with a prohibition against sale of those rights and optimal dam and reservoir size and release policy under that institution. The papers presented below extend their analysis in two directions. First, conditions for optimal storage size, storage, and releases are analyzed from a social welfare point of view, independent of legal impediments. Second, the effect on resource allocation of simply lifting the ban on the sale of appropriative water rights is explored.

Turning now to the four papers presented below, the first one is simply an extension of Arrow's and Debreu's contingent claim and securities model to a good subject to random supply for which delivery capacity has to be chosen before the random variable is observed. The second paper compares an ex-ante contingent claims market to a dynamic market in which capacity is chosen before the random variable is observed and consumption and marginal production decisions are made after. The third paper extends the analysis in the second paper to a storable good subject to random supply. Finally, the fourth paper applies the analysis in the third paper to water allocated under an appropriative water rights system.

FOOTNOTES

1. K. J. Arrow, "The Role of Securities in the Optimal Allocation of Risk-Bearing," Review of Economic Studies 31 (April 1964).

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3. J. H. Drèze (ed.), Allocation Under Uncertainty, Equilibrium and Optimality (New York: McMillan, 1974).

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5. R. Radner, "Existence of Equilibrium of Plans, Prices, and Price Expectations in a Sequence of Markets," Econometrica 40 (2) (March 1972).

6. K. Nagatoni, "On a Theorem of Arrow," Review of Economic Studies 41 (1974).

7. P. A. Diamond, "The Role of a Stock Market in a General Equilibrium Model with Technological Uncertainty," American Economic Review 57 (4) (September 1967).

8. H. E. Leland, "Production Theory and the Stock Market," The Bell Journal of Economics and Management Science 5 (1) (Spring 1974).

9. S. Ekern and R. Wilson, "On the Theory of the Firm in an Economy with Incomplete Markets," The Bell Journal of Economics and Management Science 5 (1) (Spring 1974).

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13. M. Rubinstein, "Securities Market Efficiency in an Arrow-Debreu Economy," The American Economic Review 65 (5) (December 1975).

14. P. Joskow, "Contributions to the Theory of Regulatory Economics," The Bell Journal of Economics and Management Science 7 (1) (Spring 1976)

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18. J. H. Drèze, "Some Postwar Contributions of French Economists to Theory and Public Policy," The American Economic Review 54 (3) (June 1964, Supplement).

19. D. McKay, "Two Essays on the Economics of Electricity Supply and Pricing. Part I: Has the Averch-Johnson Effect Been Empirically Verified? Part II: Electricity Pricing," Ph.D. Thesis, California Institute of Technology, 1978.

20. H. S. Burness and J. P. Quirk, "Appropriative Water Rights and the Efficient Allocation of Resources," California Institute of Technology, Social Science Working Paper No. 157, Revised March 1978. Forthcoming in The American Economic Review (December 1978). "Appropriative Water Rights and the Theory of the Dam." To appear in festshrift for E. T. Weiler, published by Purdue University Press, 1979.

CHAPTER II

SUMMARY AND DISCUSSION OF RESULTS

The four papers presented below show several results or conclusions which can be seen as either new or new extensions of previous results. The first three papers make clear the limitations of an ex-ante contingent claims market as a model for describing market decisions under uncertainty. The first paper shows that if all choices for all states of the world have to be made before any observation of the random variable, as would happen with complete contingent claims markets, then the introduction of fixed capacity constraints does not alter the Arrow-Debreu result that competitive contingent claims markets are ex-ante Pareto optimal. Arrow's contention that a market in securities achieves the same results as can be achieved with an ex-ante contingent claims market does not necessarily extend to the case where there are ex-ante fixed capacity constraints, however. Further, as the second and third papers show, when capacity or inventory choices have to be made ex-ante, but marginal production and consumption decisions can be made ex-post, then such noncontingent claims competitive markets will not necessarily allow ex-ante optimal capacity and inventory choices.

There are at least two general reasons why such noncontingent claims competitive markets might not be ex-ante Pareto optimal when capacity has to be chosen ex-ante. In the first place, a problem may occur because gains from trade exist among consumers when capacity and inventory choices are being made. Consider the following simple example. Suppose there are two individuals, Mr. A and Mr. B, who jointly own a river and wish to dam it, store water, and deliver it to their homes. Suppose in addition that Mr. A thinks there will be a great deal of water in both periods and Mr. B thinks there will only be a moderate amount of water in each period. Consequently, Mr. A thinks a large storage facility unnecessary, but wants to build a large diversion system. Mr. B, in contrast, wants a larger storage facility and a smaller diversion system. Assume larger capacity is more expensive in each case. Clearly, in the absence of transactions costs, Mr. A would be willing to bribe Mr. B to build a larger diversion system and Mr. B would be willing to bribe Mr. A to build a larger storage facility and store more water.

Now consider two market means of choosing capacity size and inventory policy. In the first case, firms producing capacity sell shares to consumers who own delivery rights. Consumers then both rent capacity shares and delivery rights to firms and use capacity shares and delivery rights to store water for future use. In this case the market in shares becomes the means by which consumers such as Mr. A bribe those such As Mr. B and visa versa.

Alternatively, suppose firms own delivery rights and produce capacity and store water on the basis of their own subjective probability distributions, discount rates and attitudes towards risk.

In this case, no mechanism exists to exhaust the gains from trade among consumers. Finding prices which allow an optimal allocation then resembles the problem of designing an optimal tax. However, if all consumers and firms owing capacity have the same probability distribution and discount rate and firms owning capacity are risk neutral (i.e., expected profit maximizing), then these gains from trade do not exist to begin with.

The second reason why noncontingent claims competitive markets might not be ex-ante Pareto optimal is related to the fact that more than one kind of uncertainty is introduced when some decisions are allowed after the random variable is observed. The first kind of uncertainty is uncertainty about capacity utilization because of supply uncertainty. The problem generated by that uncertainty can be overcome by giving consumers title to all risky assets, as discussed above. The second kind of uncertainty occurs because demand is uncertain when marginal consumption choices can be made ex-post. Since differences in available supply of one good imply differences in relative prices and incomes in different states of the world, final demand cannot be known with certainty.

Returning to the simple example discussed above, suppose Mr. A also thinks it highly probable that if there is a lot of water that prices and his income will be such that he will choose to purchase a great deal of that water if sufficient capacity is built. In this case, if Mr. A could guarantee his intuition, he would be willing to pay Mr. B even more to build a larger delivery capacity than if prices and his income were the same in all states of the world. Otherwise, the gains from trade created by this kind of uncertainty remain unexhausted. Only if such income effects do not exist to begin with can noncontingent claims competitive markets be ex-ante Pareto optimal when some choices are made ex-ante and others are made ex-post. In an analogy with the constant marginal utility of income requirement when consumer surplus analysis is used, these papers adopt the sufficient condition that expected marginal utilities of income be discounted constant functions of consumer subjective probability distributions. This condition eliminates the income effects of demand uncertainty, while still retaining all the effects on consumer choice of the supply uncertainty.

After outlining the conditions discussed above in the first three papers, the fourth paper shows that if all the sufficient conditions for an ex-ante optimal competitive allocation, when firms own capacity and inventories hold, then the appropriative rights system can be made efficient. The means for achieving efficiency discussed in that paper is to open a market in state dependent percentage shares of appropriative rights. Such a system would allow firms both to pool risks and thereby eliminate the monopoly element of the appropriative rights system and also to only purchase as many rights as are needed in any particular state of the world.

CHAPTER III

A FURTHER NOTE ON THE ROLE OF SECURITIES IN THE OPTIMAL ALLOCATION OF RISK-BEARING: OPTIMALITY UNDER CAPACITY CONSTRAINTS

INTRODUCTION

This paper develops ex-ante optimality conditions under uncertainty for a situation in which delivery capacity of one good has to be chosen before the random supply to be delivered is observed. Examples of goods which satisfy this condition are water delivered by aqueduct from a river of random flow and variable supply goods transported by rail or truck. In each case, the delivery capacity cannot be adjusted optimally in every state of the world, although other inputs can be adjusted optimally for a given capacity.

The model developed in this paper is an extension of the Arrow[1]-Debreu[2] ex-ante contingent claim model to an economy with a capacity constraint on one good. Assuming differentiable utility functions, ex-ante optimality is defined as an ex-ante allocation of contingent claims which maximizes consumer i's expected utility subject to the constraint that the expected utilities of all other consumers are held constant. The addition of the capacity constraint implies that even a market in ex-ante contingent claims no longer eliminates consumer uncertainty from the allocation decision ex-ante. Rather, when the capacity constraint is binding, the ex-ante optimal allocation equates expected marginal rates of substitution and expected price ratios. Thus, in this situation, ex-ante and ex-post optimality will not necessarily be equivalent even though for a given capacity larger then available delivery supply they are equivalent.

After developing ex-ante optimality conditions, this paper shows that an ex-ante optimal allocation can be achieved by an ex-ante contingent claims market even if firms hold title to delivery and capacity rights and consumers have differing probability judgments. If a stockmarket in shares of firms owning capacity is substituted for the contingent claims market, however, an optimal allocation of risk bearing cannot necessarily be achieved under the usual assumptions of stockholder unanimity models. Rather, additional restrictive assumptions are close to being necessary in order to achieve unanimity: Consumers have equal subjective probability distributions and constant marginal utilities of income.

REVIEW OF THE ARROW-DEBREU CONTINGENT CLAIM MARKET

The original paper by Arrow describes how an ex-post Pareto optimal allocation of resources can be achieved under uncertainty with a complete set of ex-ante contingent claim markets. While the specific optimality conditions he describes are not spelled out in the paper, he says that making each good in each state of the world a separate good makes the uncertainty case exactly analogous to the certainty case.

> Let $V_i(x_{i11}, \ldots, x_{i1C}, x_{i21}, \ldots, x_{iSC})$ be the utility of individual i if he is assigned claims of amount x_{isc} for

commodity c if state s occurs ($c = 1, \ldots, C$; $s = 1, \ldots, S$). This is exactly analogous to the utility function in the case of certainty except that the number of variables has increased from C to CS. We may therefore achieve any optimal allocation of risk-bearing by a competitive system. Let x^*_{isc} ($i = 1, \ldots, I$; $s = 1, \ldots, S$; $c = 1, \ldots, C$) be any optimal allocation; then there exist a set of money incomes y_i for individual i and prices $\bar{p}_{sc}$ for a unit claim on commodity c if state s occurs, such that if each individual i chose values of the variable x_{isc} ($s = 1, \ldots, S$; $c = 1, \ldots C$) subject to the restraint

$$\sum_{s=1}^{S} \sum_{c=1}^{C} \bar{p}_{sc} x_{isc} = y_i$$

taking prices as given, the chosen values of the x_{isc}'s will be the given optimal allocation

$$x^*_{isc} (i = 1, \ldots, I;\ s = 1, \ldots, S;\ c = 1, \ldots, C).$$

The argument is a trivial reformation of the usual one in welfare economics.[3]

The following problem is equivalent to the one Arrow is stating:

maximize: $\quad V_i(x_{i11}, \ldots, x_{iSC})$

subject to: $\quad \sum_{s=1}^{S} \sum_{c=1}^{C} \bar{p}_{sc} x_{isc} = y_i. \qquad$ (a)

Ex-ante, the first order condition is:

$$\frac{\dfrac{\partial V_i}{\partial x_{isc}}}{\dfrac{\partial V_i}{\partial x_{ird}}} = \frac{\bar{P}_{sc}}{\bar{P}_{rd}} \qquad \forall\, i, c, d, r, s.$$

For a given state of the world, this ex-ante condition reduces to:

$$\frac{\dfrac{\partial V_i}{\partial x_{isc}}}{\dfrac{\partial V_i}{\partial x_{isd}}} = \frac{\bar{P}_{sc}}{\bar{P}_{sd}} \qquad \forall\, i, c, d, s.$$

This is the same optimality condition that would prevail if the choice could be made after the random variable were observed (i.e. ex-post, under certainty). This implies that a complete set of ex-ante contingent claim markets will lead to the same ex-post optimality conditions under uncertainty as under certainty. Let us call this effect removing uncertainty from the allocation decision.

A CONTINGENT CLAIM MODEL WITH A CAPACITY CONSTRAINT

Turning now to the situation such that capacity has to be chosen before the random variable is observed, let x_{is} denote the amount of good x consumed by individual i in state of the world $s = 1, \ldots, S$. x_s is the amount of x which is delivered to consumers in state s. $\sum_{i=1}^{n} x_{is} \leq x_s$. Let z_s denote the supply of x available for delivery in state s. z_s is a discrete random variable which occurs with subjective probability α_{is}. We identify states of the world s with levels of supply of x, where $x_s \leq z_s$ for all s. There are n individuals in society, the i^{th} with strictly quasi-concave, differentiable utility function $U_i(x_{is}, y_{is}, L_{is})$,[4] where y_{is} is the amount of the composite good y consumed by i in state of the world s and L_{is} is the amount of labor offered by i in s. The delivery capacity for x_s is $\hat{x}$ and $x_s \leq \hat{x}$ in every state of the world. For a given capacity, x_s is delivered according to production function $x(L_{xs})$ and y_s is produced according to production function $y(L_{ys})$, where $\sum_{i=1}^{n} y_{is} \leq y_s$, $L_{xs} + L_{ys} \leq \sum_{i=1}^{n} L_{is}$. $\hat{x}$ is produced in period 0 according to production function $\hat{x}(L_{\hat{x}})$. Labor is supplied in period 0 according to i's utility function $U_i(L_{i0})$ and $L_{\hat{x}} \leq \sum_{i=1}^{n} L_{i0}$.[5]

Now we wish to define a Pareto optimal allocation of x_s and y_s. For this model we adopt the following definition of Pareto optimality. An allocation is said to be ex-ante Pareto optimal if, given consumer subjective probability distributions over states of the world, it is not possible to increase consumer i's expected utility ex-ante without reducing at least one other consumer's expected utility ex-ante. All trading and production are assumed to take place before the random variable is observed. Ex-post optimality will be considered later. Now, given this definition and assuming x and a composite good enter all appropriate utility functions, an ex-ante Pareto optimal state can be characterized as the maximum of a weighted sum of individual expected utility functions. Therefore, we can identify any arbitrary ex-ante Pareto optimum by solving the following problem:

$$\text{Max } W = \sum_{i=1}^{n} \beta_i [U_i(L_{i0}) + \sum_{s=1}^{S} \alpha_{is} U_i(x_{is}, y_{is}, L_{is})]$$

Subject to the following constraints:

		Multipliers
$\sum_{i=1}^{n} x_{is} \leq x(L_{xs})$	$\forall s$	λ_{1s}
$x(L_{xs}) \leq z_s$	$\forall s$	λ_{2s}
$\sum_{i=1}^{n} y_{is} \leq y_s$	$\forall s$	λ_{3s}
$y_s = y(L_{ys})$	$\forall s$	λ_{4s}
$\hat{x} = \hat{x}(L_{\hat{x}})$		γ_1
$x(L_{xs}) \leq \hat{x}$	$\forall s$	λ_{5s}
$L_{\hat{x}} \leq \sum_{i=1}^{n} L_{i0}$		γ_2
$L_{xs} + L_{ys} \leq \sum_{i=1}^{n} L_{is}$	$\forall s$	λ_{65}
$L_{is} \leq H$[6]	$\forall i, s$	λ_{75}

where:

W = the social welfare function, and

β_i = the weight given to i's utility function,

H = maximum time an individual can spend working per time period.

Forming a Lagrange expression and differentiating, the first order conditions are:

$$\beta_i \frac{\partial U_i}{\partial L_{i0}} + \gamma_2 = 0$$

$$\beta_i \alpha_{is} \frac{\partial U_i}{\partial x_{is}} - \lambda_{1s} = 0 \qquad \forall i, s$$

$$\beta_i \alpha_{is} \frac{\partial U_i}{\partial y_{is}} - \lambda_{3s} = 0 \qquad \forall i, s$$

$$\beta_i \alpha_{is} \frac{\partial U_i}{\partial L_{is}} + \lambda_{6s} = 0 \qquad \forall i, s$$

$$\frac{\partial x}{\partial L_{xs}}(\lambda_{1s} - \lambda_{2s} - \lambda_{5s}) - \lambda_{65} = 0 \qquad \forall s$$

$$\lambda_{3s} - \lambda_{5s} = 0 \qquad \forall s$$

$$\lambda_{5s} \frac{\partial y}{\partial L_{ys}} - \lambda_{6s} = 0 \qquad \forall s$$

$$\gamma_1 \frac{\partial \hat{x}}{\partial L_{\hat{x}}} - \gamma_2 = 0$$

$$-\gamma_1 + \sum_{s=1}^{S} \lambda_{5s} = 0$$

If we consider only effective constraints that are in fact binding, there are three cases:

1. If $z_s < \hat{x}$, then $\lambda_{5s} = 0$ and $\lambda_{2s} \neq 0$.

2. If $z_s = \hat{x}$, then $\lambda_{5s} = \lambda_{2s} = 0$.

3. If $z_s > \hat{x}$, then $\lambda_{5s} \neq 0$ and $\lambda_{2s} = 0$.

These observations allow us to reduce the first order conditions to the following equations:

Case 1 ($z_s < \hat{x}$; $\lambda_{5s} = 0$, $\lambda_{2s} \neq 0$):

$$\frac{\alpha_{is} \dfrac{\partial U_i}{\partial x_{is}}}{\alpha_{ir} \dfrac{\partial U_i}{\partial y_{ir}}} = \frac{\alpha_{hs} \dfrac{\partial U_h}{\partial x_{hs}}}{\alpha_{hr} \dfrac{\partial U_h}{\partial y_{hr}}} \qquad \forall\, h, i, s, r$$

$$\frac{\alpha_{is} \dfrac{\partial U_i}{\partial x_{is}}}{\alpha_{ir} \dfrac{\partial U_i}{\partial y_{ir}}} = \frac{-\beta_i \alpha_{is} \dfrac{\partial U_i/\partial L_{is}}{\partial x/\partial L_{xs}} + \lambda_{2s}}{-\beta_i \alpha_{ir} \dfrac{\partial U_i/\partial L_{ir}}{\partial y/\partial L_{yr}}} \qquad \forall\, i, r$$

Case 2 ($z_s = \hat{x}$; $\lambda_{5s} = \lambda_{2s} = 0$):

$$\frac{\alpha_{is} \dfrac{\partial U_i}{\partial x_{is}}}{\alpha_{ir} \dfrac{\partial U_i}{\partial y_{ir}}} = \frac{\alpha_{hs} \dfrac{\partial U_h}{\partial x_{hs}}}{\alpha_{hr} \dfrac{\partial U_h}{\partial y_{hr}}} \qquad \forall\, h, i, r$$

$$\frac{\dfrac{\partial U_i}{\partial x_{is}}}{\dfrac{\partial U_i}{\partial y_{ir}}} = \frac{\dfrac{\partial U_i/\partial L_{is}}{\partial x/\partial L_{xs}}}{\dfrac{\partial U_i/\partial L_{ir}}{\partial y/\partial L_{yr}}} \qquad \forall\, i, r$$

Case 3($z_s > \hat{x}$; $\lambda_{5s} \neq 0$, $\lambda_{2s} = 0$):

$$\frac{\sum\limits_{s>\hat{x}} \alpha_{is} \dfrac{\partial U_i}{\partial x_{is}}}{\alpha_{ir} \dfrac{\partial U_i}{\partial y_{ir}}} = \frac{\sum\limits_{s>\hat{x}} \alpha_{hs} \dfrac{\partial U_h}{\partial x_{hs}}}{\alpha_{hr} \dfrac{\partial U_h}{\partial y_{hr}}} \qquad \forall h, i, r$$

$$\frac{\sum\limits_{s>\hat{x}} \alpha_{is} \dfrac{\partial U_i}{\partial x_{is}}}{\alpha_{ir} \dfrac{\partial U_i}{\partial y_{ir}}} = \sum_{s>\hat{x}} \frac{\alpha_{is} \dfrac{\partial U_i/\partial L_{is}}{\partial x/\partial L_{xs}} + \dfrac{\partial U_i/\partial L_{i0}}{\partial \hat{x}/\partial L_{\hat{x}}}}{\alpha_{ir} \dfrac{\partial U_i/\partial L_{ir}}{\partial y/\partial L_{yr}}} \qquad \forall i, r$$

where:

$$-\beta_i \alpha_{is} \frac{\partial U_i}{\partial L_{is}} = -\beta_h \alpha_{hs} \frac{\partial U_h}{\partial L_{hs}} = \lambda_{6s} = \frac{\partial W}{\partial \left(\sum\limits_{i=1}^{n} L_{is} \right)}$$

is the marginal social value of labor in s;

$$-\beta_i \frac{\partial U_i}{\partial L_{i0}} = -\beta_h \frac{\partial U_h}{\partial L_{h0}} = \gamma_2 = \frac{\partial W}{\partial(\sum_{i=1}^{n} L_{i0})}$$

is the marginal social value of labor in period 0; $\lambda_{2s} = \frac{\partial W}{\partial z_s}$ is the marginal social value of z in s.[7]

Notice that in Cases 1 and 2 the ex-ante optimality conditions are the same as Arrow's. In Case 3, however, these first order conditions imply that at an ex-ante Pareto optimum, for $z_s > \hat{x}$, only the expected marginal rates of substitution will be the same across all individuals and equal to the expected marginal social values of x_s and y_s. This deviation from the Arrow-Debreu result that a contingent claim market transforms an uncertainty problem to a certainty one occurs because the capacity constraint makes it impossible for producers of x to optimize productive inputs in every state of the world. These results can be summarized as follows:

Proposition 1: If the supply of a good is a random variable and delivery capacity has to be chosen before the random variable is observed, then for those states of the world such that available supply exceeds delivery capacity, an ex-ante optimal allocation of contingent claims equates the expected marginal rates of substitution between the random good and all other goods across all individuals to the expected marginal social values of x and y. Uncertainty has not been removed from the allocation decision, ex-ante.

Proposition 2: If the supply of a good is a random variable and delivery capacity has to be chosen before the random variable is observed then for those states of the world such that capacity exceeds or is equal to available supply, an ex-ante optimal allocation of contingent claims removes uncertainty from the allocation decision.

The question now is whether a competitive equilibrium will satisfy these optimality conditions and if so, whether any special conditions will have to be imposed. First, complete contingent claim markets have to exist for labor and delivery rights concurrently with a contingent claim market for delivery of x. Otherwise, a contingent claim market for delivery of x will not correctly evaluate the expected relative values of x and y.

Consider now a competitive economy with n consumers and m + k + g firms. The first m firms produce $\hat{x}$ in period 0, the next k firms produce x and the last g firms produce y. Consumers hold title to delivery rights and capacity units once capacity has been built. Consumer i's problem is:

$$\text{Max } EU_i = U_i(L_{i0}) + \sum_{s=1}^{S} \alpha_{is} U_i(x_{is}, y_{is}, L_{is})$$

subject to:

$$p_{\hat{x}}\hat{x}_i + \sum_{s=1}^{S}[p_{xs}x_{is} + p_{ys}y_{is}] \leq w_0 L_{i0} + \sum_{j=1}^{m}\theta_{\hat{x}}^{ij}\pi_{\hat{x}}^{j} +$$

Multiplier: λ_i

$$\sum_{s=1}^{S}[p_{\hat{x}s}\hat{x}_i + w_s L_{is} + q_s\bar{r}_{is} + \sum_{j=m+1}^{m+k}\theta_{xs}^{ij}\pi_{xs}^{j} + \sum_{j=m+k+1}^{m+k+g}\theta_{ys}^{ij}\pi_{ys}^{j}]$$

where:

$\theta_{\hat{x}}^{ij}$ = percentage of firm $\hat{x}^j$'s profits going to i;

θ_{xs}^{ij} = percentage of firm x^j's profits going to i in s;

θ_{ys}^{ij} = percentage of firm y^j's profits going to i in s;

w_0 = competitive wage in period 0;

w_s = competitive contingent claim wage in state s;

$\pi_{\hat{x}}^{j}$ = profits of firm $\hat{x}^j$ in period 0;

π_{xs}^{j} = profits of firm x^j in state s;

π_{ys}^{j} = profits of firm y^j in state s;

$p_{\hat{x}}$ = competitive price of a unit of capacity purchased from the firms producing capacity in period 0.

p_{xs} = competitive contingent claim price of x in state s;

p_{ys} = competitive contingent claim price of y in state s;

$p_{\hat{x}s}$ = competitive contingent claim price of a unit of capacity in state s;

$\hat{x}_i$ = number of capacity rights owned by consumer i;[8]

q_s = competitive contingent claim price of a state s delivery right;[9]

$\bar{r}_{is}$ = initial allocation of state s delivery rights to i;

$$\sum_{i=1}^{n} \theta_{xs}^{ij} = 1;$$

$$\sum_{i=1}^{n} \theta_{ys}^{ij} = 1; \text{ and}$$

$$\sum_{i=1}^{n} \theta_{\hat{x}}^{ij} = 1.$$

Making the assumption of a decomposible profit function, firm $\hat{x}^j$'s problem is:

	Multipliers
$\text{Max } \pi_{\hat{x}}^j = p_{\hat{x}0}\hat{x}^j - w_0 L_{\hat{x}}^j$	
s.t. $\hat{x}^j \leq \hat{x}(L_{\hat{x}}^j)$	$\lambda_{\hat{x}}^j$

Firm x^j's problem is:

$$\text{Max } \pi_x^j = \sum_{s=1}^{S} [p_{xs}x_s^j - w_s L_{xs}^j - p_{\hat{x}s}\hat{x}_s^j - q_s r_s^j]$$

s.t. $x_s^j \leq x(L_{xs}^j)$	λ_{1s}^j
$x(L_{xs}^j) \leq \hat{x}_s^j$	λ_{2s}^j
$x(L_{xs}^j) \leq r_s^j$	λ_{3s}^j

where:

$\hat{x}_s^j$ = amount of capacity rented by firm x_j in state s, and

r_s^j = number of delivery rights rented by firm x_j in state s.

Firm y_j's problem is:

$$\text{Max } \pi_y^j = \sum_{s=1}^{S} [p_{ys} y_s^j - w_s L_{ys}^j] \qquad \underline{\text{Multipliers}}$$

$$\text{s.t.} \quad y_s^j \leq y(L_{ys}^j) \qquad \lambda_{4s}^j$$

The equilibrium conditions are:

$$\sum_{i=1}^{n} \hat{x}_i = \sum_{j=1}^{m} \hat{x}^j$$

$$\sum_{i=1}^{n} x_{is} = \sum_{j=m+1}^{m+k} x_s^j$$

$$\sum_{i=1}^{n} y_{is} = \sum_{j=m+k+1}^{m+k+g} y_s^j$$

$$\sum_{i=1}^{n} L_{i0} = \sum_{j=1}^{m} L_{\hat{x}}^j$$

$$\sum_{i=1}^{n} L_{is} = \sum_{j=m+1}^{m+k} L_{xs}^j + \sum_{j=m+k+1}^{m+k+g} L_{ys}^j$$

$$\sum_{i=1}^{n} \bar{r}_{is} = \sum_{j=m+1}^{m+k} r_s^j$$

Forming Lagrange expressions $L_i (i = 1, \ldots, n)$, $L^j_{\hat{x}} (j = 1, \ldots, m)$, $L^j_x (j = m+1, \ldots, m+k)$, and $L^j_y (j = m+k+1, \ldots, m+k+g)$ and differentiating, the first order conditions are:

$$\frac{\partial U_i}{\partial L_{i0}} + \lambda_i w_0 = 0 \qquad \forall i$$

$$\alpha_{is} \frac{\partial U_i}{\partial x_{is}} - \lambda_i p_{xs} = 0 \qquad \forall i, s$$

$$\alpha_{is} \frac{\partial U_i}{\partial y_{is}} - \lambda_i p_{ys} = 0 \qquad \forall i, s$$

$$\alpha_{is} \frac{\partial U_i}{\partial L_{is}} + \lambda_i w_s = 0 \qquad \forall i, s$$

$$\lambda_i [p_{\hat{x}} - \sum_{s=1}^{S} p_{\hat{x}s}] = 0 \qquad \forall i$$

$$p_{\hat{x}o} - \lambda^j_{\hat{x}} = 0 \qquad j = 1, \ldots, m$$

$$-w_0 + \lambda^j_x \frac{\partial \hat{x}}{\partial L^j_{\hat{x}}} = 0 \qquad j = 1, \ldots, m$$

$$p_{xs} - \lambda^j_{1s} = 0 \qquad \forall s,\ j = m+1, \ldots, m+k$$

$$-w_s + \frac{\partial x}{\partial L^j_{xs}} (\lambda^j_{1s} - \lambda^j_{2s} - \lambda^j_{3s}) = 0 \qquad \forall s,\ j = m+1, \ldots, m+k$$

$$-p_{\hat{x}s} + \lambda^j_{2s} = 0 \qquad \forall s,\ j = m+1, \ldots, m+k$$

$$-q_s + \lambda^j_{3s} = 0 \qquad \forall s,\ j = m+1, \ldots, m+k$$

$$p_{ys} - \lambda^j_{4s} = 0 \qquad \forall s,\ j = m+k+1, \ldots, m+k+g$$

$$-w_s + \lambda^j_{4s} \frac{\partial y}{\partial L^j_{ys}} = 0 \qquad s,\ j = m+k+1, \ldots, m+k+g$$

Considering only effective constraints which are binding, these first order conditions reduce to:

$$\frac{\dfrac{\partial U_i}{\partial L_{i0}}}{\alpha_{is} \dfrac{\partial U_i}{\partial L_{is}}} = \frac{w_0}{w_s} \qquad \forall i, s \qquad \text{(a)}$$

$$\frac{\alpha_{is} \dfrac{\partial U_i}{\partial L_{is}}}{\alpha_{ir} \dfrac{\partial U_i}{\partial L_{ir}}} = \frac{w_s}{w_r} \qquad \forall i, r, s \qquad \text{(b)}$$

$$\frac{\alpha_{is} \dfrac{\partial U_i}{\partial x_{is}}}{\alpha_{ir} \dfrac{\partial U_i}{\partial y_{ir}}} = \frac{p_{xs}}{p_{yr}} \qquad \forall i, r, s \qquad \text{(c)}$$

$$w_s = \frac{-\alpha_{is} \partial U_i / \partial L_{is}}{\lambda_i} \qquad \forall i, s \qquad \text{(d)}$$

Case 1 ($z_s < \hat{x}$):

$$\frac{p_{xs}}{p_{yr}} = \frac{\dfrac{w_s}{\partial x/\partial L^j_{xs}} + q_s}{\dfrac{w_r}{\partial y/\partial L^a_{yr}}} \qquad \forall j, a, r \qquad \text{(e)}$$

Case 2 ($z_s = \hat{x}$):

$$\frac{p_{xs}}{p_{yr}} = \frac{\dfrac{w_s}{\partial x/\partial L^j_{xs}}}{\dfrac{w_r}{\partial y/\partial L^a_{yr}}} \qquad \forall j, a, r \qquad \text{(f)}$$

Case 3 ($z_s > \hat{x}$):

$$\frac{p_{xs}}{p_{yr}} = \frac{\dfrac{w_0}{\partial \hat{x}/\partial L^j_{\hat{x}}} + \dfrac{w_s}{\partial x/\partial L^j_{xs}}}{\dfrac{w_r}{\partial y/\partial L^a_{ys}}} \qquad \forall j, a, r \qquad \text{(g)}$$

Now compare (c) with the production conditions (e), (f), and (g) and substitute (d) for w_s. The three cases reduce to:

Case 1 ($z_s < \hat{x}$):

$$\frac{\alpha_{is} \dfrac{\partial U_i}{\partial x_{is}}}{\alpha_{ir} \dfrac{\partial U_i}{\partial y_{ir}}} = \frac{\dfrac{-\alpha_{is} \partial U_i / \partial L_{is}}{\partial x / \partial L^j_{xs}} + \lambda_i q_s}{\dfrac{-\alpha_{ir} \partial U_i / \partial L_{ir}}{\partial y / \partial L^a_{yr}}} \qquad \forall i, a, j, r \qquad (h)$$

Case 2 ($z_s = \hat{x}$):

$$\frac{\dfrac{\partial U_i}{\partial x_{is}}}{\dfrac{\partial U_i}{\partial y_{ir}}} = \frac{\dfrac{\partial U_i / \partial L_{is}}{\partial x / \partial L^j_{xs}}}{\dfrac{\partial U_i / \partial L_{ir}}{\partial y / \partial L^a_{yr}}} \qquad \forall i, a, j, r \qquad (i)$$

Case 3 ($z_s > \hat{x}$):

$$\frac{\sum\limits_{s>\hat{x}} \alpha_{is} \dfrac{\partial U_i}{\partial x_{is}}}{\alpha_{ir} \dfrac{\partial U_i}{\partial y_{ir}}} = \frac{- \sum\limits_{s>\hat{x}} \alpha_{is} \dfrac{\partial U_i / \partial L_{is}}{\partial x / \partial L^j_{xs}} - \dfrac{\partial U_i / \partial L_{i0}}{\partial \hat{x} / \partial L^b_{\hat{x}}}}{- \alpha_{ir} \dfrac{\partial U_i / \partial L_{ir}}{\partial y / \partial L^a_{yr}}} \qquad \forall i, a, b, j, r \qquad (j)$$

Notice that equations (h)-(j) describe an ex-ante Pareto optimum with welfare weights $\beta_i = \frac{1}{\lambda_i}$. This result can be summarized as follows:

Proposition 3: If the supply of a good is a random variable and delivery capacity has to be chosen and produced before the random variable is observed, then a competitive contingent claims equilibrium is ex-ante Pareto optimal if consumers hold title to all resources.

A COMPETITIVE CONTINGENT CLAIMS MARKET IN WHICH FIRMS HOLD TITLE TO RESOURCES

Suppose now we consider a somewhat more likely market situation. In this case consumers participate in a contingent claims market for labor and output, but firms delivering x build and hold title to their own capacity and hold title to delivery rights.[10] Firms alone participate in a contingent claim market in delivery rights. The maximization problems now become:

$$\text{Consumer:}\quad \text{Max } U_i(L_{i0}) + \sum_{s=1}^{S} \alpha_{is} U_i(x_{is}, y_{is}, L_{is})$$

$$\text{s.t.}\quad \sum_{s=1}^{S} [p_{xs}x_{is} + p_{ys}y_{is}] \le w_0 L_{i0} + \sum_{s=1}^{S} [w_s L_{is} + \sum_{j=1}^{k} \theta_{xs}^{ij}\pi_{xs} + \sum_{j=k+1}^{k+g} \theta_{ys}^{ij}\pi_{ys}] \quad \forall i$$

$$\text{Firm } x_j:\quad \text{Max } -w_0 L_{\hat{x}}^j + \sum_{s=1}^{S} [p_{xs}x_s^j - w_s L_{xs}^j - q_s(r_s^j - \bar{r}_s^j)]^{11}$$

$$\text{s.t.}\quad x_s^j \le x(L_{xs}^j)$$

$$x(L_{xs}^j) \le \hat{x}^j$$

$$x(L_{xs}^j) \le r_s^j$$

$$\hat{x}^j \le \hat{x}(L_{\hat{x}}^j)$$

$$\text{Firm } y_j:\quad \text{Max } \sum_{s=1}^{S} [p_{ys}y_s^j - w_s L_{ys}^j]$$

$$\text{s.t.}\quad y_s^j \le y(L_{ys}^j)$$

The first order conditions for this problem reduce to:

Consumers: same as (a)-(d)

Firms:

Case 1 ($z_s < \hat{x}$): same as (f).

Case 2 ($z_s = \hat{x}$): same as (g).

Case 3 ($z_s > \hat{x}$):

$$\sum_{s>\hat{x}} \frac{p_{xs}}{p_{yr}} = \sum_{s>\hat{x}} \frac{\left(\dfrac{w_s}{\partial x/\partial L^j_{xs}}\right) + \dfrac{w_0}{\partial \hat{x}/\partial L^j_{\hat{x}}}}{\dfrac{w_r}{\partial y/\partial L^a_{yr}}} \qquad a, j, r \qquad \text{(k)}$$

The fact that Cases 1 and 2 are identical to the equations describing the allocation when consumers own delivery rights indicates that the conclusions of Proposition 3 generalize to the case such that firms own the state-dependent delivery rights. This result can be generalized as follows.

<u>Proposition 4</u>: If the supply of a good is a random variable and delivery capacity is not a binding constraint, then the distribution of delivery rights does not affect the allocation of final output if the market is organized as an ex-ante contingent claims market.

Notice that equation (k) describes an optimal allocation also. To prove this statement, substitute equation (a) for $\frac{p_{xs}}{p_{yr}}$,

(b) for $\frac{w_0}{w_r}$, (c) for $\frac{w_s}{w_r}$ and sum over $s > \hat{x}$:

$$\frac{\sum\limits_{s>\hat{x}} \alpha_{is} \frac{\partial U_i}{\partial x_{is}}}{\alpha_{ir} \frac{\partial U_i}{\partial y_{ir}}} = \frac{\sum\limits_{s>\hat{x}} \frac{\alpha_{is} \partial U_i / \partial L_{is}}{\partial x / \partial L^j_{xs}} + \frac{\partial U_i / \partial L_{i0}}{\partial \hat{x} / \partial L^j_{\hat{x}}}}{\alpha_{ir} \frac{\partial U_i / \partial L_{ir}}{\partial y / \partial L^a_{yr}}} \quad \forall\, i, j, a, r \tag{1}$$

This result can be summarized as follows:

<u>Proposition 5</u>: If the supply of a good is a random variable and delivery capacity has to be chosen before the random variable is observed, then the distribution of capacity rights does not affect the final allocation if the market is organized as an ex-ante contingent claims market.

A STOCKHOLDER MODEL WITH A CAPACITY CONSTRAINT

Since ex-ante contingent claims markets are unwieldy and don't allow consumers to adjust their portfolios of consumption claims after the random variable is observed, much of the literature on optimal risk bearing has focused on outlining the conditions under which a contingent claims allocation can be achieved by allocating risk through a securities market. In such a market consumers need only purchase securities ex-ante and they can alter their portfolios of securities at any time. In this literature, finding conditions which allow an optimal allocation of risk bearing is equivalent to finding conditions which allow consumers to readjust their portfolios of stocks to hedge against a firm's decision they disagree with. Under such conditions holders of a firm's stock could always agree on what decision the firm should make. It turns out, however, that when the decision is what size capacity a firm should build, unanimity cannot in general be achieved. Consider the following example.

Adapting Ekern and Wilson's model,[12] as interpreted by Forsythe,[13] let individual i have an undiscounted, additive utility function over certain current consumption and uncertain future consumption, such that

$$EU_i = U_i(x_{i0}, y_{i0}) + \sum_{s=1}^{S} \alpha_{is} U_i(x_{is}, y_{is}),^{14}$$

which he maximizes subject to budget constraints for current and future consumption. The current budget constraint is

$$x_{i0}P_{x0} + y_{i0}P_{y0} + \sum_{j=1}^{k} V_x^j S_{ix}^j + \sum_{j=k+1}^{k+g} V_y^j S_{iy}^j \leq w_0 L_{i0} + \sum_{j=1}^{k} V_x^j \bar{S}_{ix}^j + \sum_{j=k+1}^{k+g} V_y^j \bar{S}_{iy}^j,$$

$$\text{Multiplier: } \lambda_0^i$$

where:

V_x^j, V_y^j = value of firm $(x^j, y^j) = (N_x^j v_{xj}, N_y^j v_{yj})$;

N_x^j, N_y^j = number of (x^j, y^j) shares outstanding;

v_{xj}, v_{yj} = price per share of (x^j, y^j) stock;

S^j_{ix}, S^j_{iy} = (x^j, y^j) stocks owned by i;

$\bar{S}^j_{ix}, \bar{S}^j_{iy}$ = i's initial endowment of (x^j, y^j) shares.

The future budget constraints are:

$$x_{is}p_{xs} + y_{is}p_{ys} \leq w_s L_{is} + \sum_{j=1}^{k} \pi^j_{xs} S^j_{ix} + \sum_{j=k+1}^{k+g} \pi^j_{ys} S^j_{iy} \qquad \forall s.$$

Multiplier: λ^i_s

Forming a Lagrange expression and differentiating, first order conditions are

$$\frac{\partial U_i}{\partial x_{i0}} - \lambda^i_0 p_{x0} = 0$$

$$\frac{\partial U_i}{\partial y_{i0}} - \lambda^i_0 p_{y0} = 0$$

$$\alpha_{is} \frac{\partial U_i}{\partial x_{is}} - \lambda^i_s p_{xs} = 0$$

$$\alpha_{is} \frac{\partial U_i}{\partial y_{is}} - \lambda^i_s p_{ys} = 0$$

$$-\lambda^i_0 v^j_x + \sum_{s=1}^{S} \lambda^i_s \pi^j_{xs} = 0$$

$$-\lambda_0^i V_y^j + \sum_{s=1}^{S} \lambda_s^i \pi_{ys}^i = 0$$

These reduce to:

$$\frac{\dfrac{\partial U_i}{\partial x_{i0}}}{p_{x0}} V_x^j = \sum_{s=1}^{S} \left(\frac{\alpha_{is} \dfrac{\partial U_i}{\partial x_{is}}}{p_{xs}} \right) \pi_{xs}^j \tag{m}$$

$$\frac{\dfrac{\partial U_i}{\partial y_{i0}}}{p_{y0}} V_y^j = \sum_{s=1}^{S} \left(\frac{\alpha_{is} \dfrac{\partial U_i}{\partial y_{is}}}{p_{ys}} \right) \pi_{ys}^j \tag{n}$$

Now, let us consider the effect on person i's utility of a change in the amount of labor firm x^j devotes to capacity:

$$\frac{\partial U_i}{\partial L_{\hat{x}}^j} = \frac{\partial U_i}{\partial x_{i0}} \frac{\partial x_{i0}}{\partial L_{\hat{x}}^j} + \sum_{s=1}^{S} \alpha_{is} \frac{\partial U_i}{\partial x_{is}} \frac{\partial x_{is}}{\partial L_{\hat{x}}^j} \tag{o}$$

From the budget constraints:

$$x_{i0} = \frac{1}{p_{x0}} \left[w_0 L_{i0} + \sum_{j=1}^{k} V_x^j \bar{S}_{ix}^j + \sum_{j=k+1}^{k+g} V_y^j \bar{S}_{iy}^j - y_{i0} p_{y0} - \sum_{j=1}^{k} V_x^j S_{ix}^j - \sum_{j=k+1}^{k+g} V_y^j S_{iy}^j \right] \tag{p}$$

$$x_{is} = \frac{1}{p_{xs}} \left[w_s L_{is} + \sum_{j=1}^{k} \pi_{xs}^j S_{ix}^j + \sum_{j=k+1}^{k+g} \pi_{ys}^j S_{iy}^j - y_{is} p_{ys} \right] \tag{q}$$

Now making the usual assumptions that the state distribution of prices of outputs and inputs and the other firms' profits are independent of $L^j_{\hat{x}}$:

$$\frac{\partial U_i}{\partial L^j_{\hat{x}}} = \frac{\partial U_i}{\partial x_{i0}} \left[\sum_{m=1}^{k} \frac{\partial V^m_x}{\partial L^j_{\hat{x}}} \frac{(\bar{S}^m_{ix} - S^m_{ix})}{P_{x0}} + \sum_{m=k+1}^{k+g} \frac{\partial V^m_y}{\partial L^j_{\hat{x}}} \frac{(\bar{S}^m_{iy} - S^m_{iy})}{P_{x0}} \right]$$

$$+ \sum_{s=1}^{S} \alpha_{is} \frac{\partial U_i}{\partial x_{is}} \left[\frac{\partial \pi^j_{xs}}{\partial L^j_{\hat{x}}} \frac{S^j_{ix}}{P_{xs}} \right] \qquad \text{(r)}$$

From (m):

$$V^j_x = \sum_{s=1}^{S} \alpha_{is} \frac{\dfrac{\partial U_i}{\partial x_{is}}}{\dfrac{\partial U_i}{\partial x_{i0}}} \frac{P_{x0}}{P_{xs}} \pi^j_{xs} = \sum_{s=1}^{S} \frac{\lambda^i_s}{\lambda^i_0} \pi^j_{xs}$$

Forsythe interprets $\dfrac{\lambda^i_s}{\lambda^i_0}$ as a contingent claim price for consumption of x_{is}, independent of $L^j_{\hat{x}}$. This implies:

$$\frac{\partial V^m_y}{\partial L^j_{\hat{x}}} = 0 \qquad \forall\, m;$$

$$\frac{\partial V^m_x}{\partial L^j_{\hat{x}}} = 0 \qquad \text{for } m \neq j\text{; and}$$

$$= \sum_{s=1}^{S} \frac{\lambda^i_s}{\lambda^i_0} \frac{\partial \pi^j_{xs}}{\partial L^j_{\hat{x}}} \qquad \text{for } m = j \qquad \text{(s)}$$

Substituting (s) in (r) gives:

$$\frac{\partial U_i}{\partial L^j_{\hat{x}}} = \bar{S}^j_{ix} \sum_{s=1}^{S} \alpha_{is} \frac{\frac{\partial U_i}{\partial x_{is}}}{p_{xs}} \frac{\partial \pi^j_{xs}}{\partial L^j_{\hat{x}}} \tag{t}$$

At this point an assumption is usually made which is that the change being undertaken does not alter the feasible set of profits or the probability distribution of profits over states of the world. This is a sufficient condition for stockholder unanimity. Following Forsythe's spanning formulation:

$$\text{Let } \frac{\partial \pi^j_{xs}}{\partial L^j_{\hat{x}}} = \sum_{k=1}^{m} a_{jk} \pi^j_{ks}, \quad \text{where,}$$

the a_{jk}'s are a set of weights such that the change in capacity keeps the profits within the same linear subspace.

$$\text{Thus, } \frac{\partial U_i}{\partial L^j_{\hat{x}}} = \bar{S}^j_{ix} \sum_{k=1}^{N} a_{jk} \sum_{s=1}^{S} \alpha_{is} \frac{\partial U_i / \partial x_{is}}{p_{xs}} \pi^j_{ks}$$

$$= \bar{S}^j_{ix} \sum_{k=1}^{N} a_{jk} \frac{\partial U_i / \partial x_{i0}}{p_{x0}} V_k, \text{ by (m).} \tag{u}$$

Since $\partial U_i / \partial x_{i0} \geq 0$ by the assumption of quasi-concavity, (n) has the same sign for every i for a change in capacity such that $\sum_{k=1}^{m} a_{jk} V_k > 0$ (<0). Thus, stockholders would agree about the affect on their utilities of a change in capacity.

Such an assumption is not valid in this case, however, because the state distribution of profits depends upon the choice of capacity. Higher or lower capacity means higher or lower profits in all states of the world affected by the capacity change because the states of the world covered by the three cases change. In other words, the feasible set of profits is altered over certain states of the world if capacity is changed. Thus, while there may still exist special conditions which would allow stockholder unanimity, it cannot be proved as a general theorem by employing the usual assumptions of these models. This result can be summarized as follows:

Proposition 7: When firms hold title to capacity which must be built before the supply random variable is observed, then a stock market in shares of firms owning capacity does not necessarily allow stockholders to come to agreement on the choice of capacity, given the usual assumptions of stockholder unanimity models.

Suppose, however, that we impose the following more strict conditions on equation (t):

1. all consumers have the same subjective probability distribution; and

2. each consumer has a constant marginal utility of income and an expected marginal utility of income which is a constant function of his subjective probabilities (i.e. $\lambda_0^i = C_i$ and $\lambda_s^i = C_i\alpha_{is}$ $\forall i, s$).[15]

Under these conditions an optimal allocation of risk bearing can be achieved through a securities market. To prove this statement let:

$$\lambda_s^i = C_i\alpha_s \qquad \forall i, s, \text{ where} \tag{v}$$

α_s = true or agreed upon probability of being in s;

C_i = i's constant marginal utility of income.

This implies:

$$\alpha_s \frac{\partial U_i}{\partial x_{is}} = C_i\alpha_s p_{xs} \qquad \forall i, s \tag{w}$$

by the first order conditions.

Substituting (w) in (t):

$$\frac{\partial U_i}{\partial L_{\hat{x}}^j} = \bar{S}_{ix}^j C_i \sum_{s=1}^{S} \alpha_s \frac{\partial \pi_{xs}^j}{\partial L_{\hat{x}}^j} \tag{x}$$

Since $\bar{S}_{ix}^j > 0$ and $C_i > 0$ by quasi-concavity (i.e., $\frac{\partial U_i}{\partial M} > 0$), the sign of (x) is independent of i and all consumers will agree on the effect of the change. This result can be summarized as follows:

Proposition 8: If supply of a good is a random variable and delivery capacity is owned by firms and has to be chosen before the random variable is observed, then a market in shares of firms owning capacity allows consumers to come to agreement on firms' choices of capacity when the following sufficient conditions hold in addition to the usual assumptions of stockholder unanimity models:

1. consumers have the same probability distributions over states of the world; and

2. consumers' expected marginal utilities of income are constant functions of those subjective probabilities.

FOOTNOTES

1. K. J. Arrow, "The Role of Securities in the Optimal Allocation of Risk-Bearing," Review of Economic Studies 31 (April 1964).

2. Gerard Debreu, The Theory of Value: An Axiomatic Analysis of Economic Equilibrium, Cowles Foundation Monograph No. 17, Yale University Press.

3. K. J. Arrow, "The Role of Securities in the Optimal Allocation of Risk-Bearing," Review of Economic Studies 31 (April 1964).

4. Without loss of generality we assume that i's utility function is constant over states of the world. Further, utility functions are assumed to be neoclassical. That is, $\lim_{x_{is}\to 0} \frac{\partial U_i}{\partial x_{is}} = \lim_{y_{is}\to 0} \frac{\partial U_i}{\partial y_{is}} = +\infty$; $\lim_{x\to\infty} \frac{\partial U_i}{\partial x_{is}} > 0$; $\lim_{y\to\infty} \frac{\partial U_i}{\partial y_{is}} > 0$ and $\lim_{L_{is}\to 0} \frac{\partial U^i}{\partial L_{is}} < 0$ so that positive amounts of x and y are consumed in all states, positive amounts of labor are supplied in all states and there is no bliss point.

5. Without loss of generality we assume that no consumption takes place in period 0.

6. We assume $\lim_{L_{is}\to H} \frac{\partial U_i}{\partial L_{is}} = -\infty$ so that the constraint $L_{is} \leq H$ is never binding.

7. To prove this statement use the envelope theorem.

$$\text{Let } W = W(x) \tag{a}$$

$$\text{Subject to } g^j(x,L) \geq 0$$

$$\frac{dW}{dL} = \sum_{i=1}^{N} \frac{\partial W}{\partial x_i} \frac{dx_i}{dL} \tag{b}$$

$$\mathcal{L} = W + \sum_{j=1}^{m} \lambda^j g^j \tag{c}$$

$$\frac{\partial L}{\partial x_i} = \frac{\partial W}{\partial x_i} + \sum_{j=1}^{m} \lambda^j \frac{\partial g^i}{\partial x_i} = 0 \tag{d}$$

Now consider all effective constraints:

If $g^j(x,\bar{x}) = 0$, then

$$\sum_{i=1}^{k} \frac{\partial g^j}{\partial x_i} \frac{dx_i}{dL} = \frac{\partial g^j}{\partial L} \tag{e}$$

Substituting (e) and (d) in (b):

$$\frac{dW}{dL} = \sum_{i=1}^{n} \sum_{j=1}^{k} \lambda^j \frac{\partial g^j}{\partial x_i} \frac{dx_i}{dL} = - \sum_{j=1}^{k} \lambda^j \Big(\sum_{i=1}^{n} \frac{\partial g^j}{\partial x_i} \frac{dx_i}{dL} \Big) = \sum_{j=1}^{k} \lambda^j \frac{\partial g^j}{\partial L}$$

$= \lambda^j$ since L only enters the $j^{\underline{th}}$ constraint $\frac{\partial g^i}{\partial L} = 0$, $i \neq j$,

and $\frac{\partial g_j}{\partial \bar{x}} = 1$ when $\frac{\partial g^j}{\partial \bar{x}} \neq 0$.

The same proof applies for dw/dz.

8. A capacity right is a right to use one unit of capacity. Consumers own these rights and rent them to firms who use them to deliver x.

9. A delivery right in state s is a right to use one unit of z in state s. Consumers own such rights and rent them to firms who use them to deliver units of x in s.

10. The results of this model follow whether delivery firms produce their own capacity or a separate set of firms produce capacity and sell it to delivery firms.

11. Note that firms do not bear any risk in this model even though they own capacity. The reason there is no risk is that all trading takes place before there is any production. Therefore, firms know with certainty their total revenues and expenditures regardless of what state of the world occurs.

12. S. Ekern and R. Wilson, "On the Theory of the Firm in an Economy with Incomplete Markets," Bell Journal of Economics and Management Sciences 5 (1) (1974): 171-180.

13. R. E. Forsythe, "Unanimity and the Theory of the Firm Under Multiplicative Uncertainty," California Institute of Technology, Social Science Working Paper No. 90, February 1976.

14. This model assumes that some capacity exists and the firm might wish to change it. Without loss of generality we assume that labor does not enter an individual's utility function.

15. See footnote 7 for a proof of the statement that λ_s^i is the expected marginal utility of income in state s .

CHAPTER IV

A DYNAMIC PROGRAMMING MODEL OF THE CONDITIONS FOR OPTIMAL CAPACITY CHOICE UNDER SUPPLY UNCERTAINTY

INTRODUCTION

The previous paper developed ex-ante optimality conditions for the allocation of a good subject to supply uncertainty and capacity constraints. Then it showed that, although an ex-ante contingent claims market would achieve an optimal allocation regardless of who owned the capacity and delivery rights, a market simply in securities would not necessarily achieve an optimal allocation unless more strict assumptions were made than is customary in stockholder unanimity models. The particular additional sufficient conditions examined in that paper were:

1. consumers have identical subjective probability distributions over states of the world; and

2. consumers have expected marginal utilities of income which are constant functions of those subjective probabilities.

While contingent claim markets are interesting examples of solutions to problems caused by uncertainty, most economists would agree that it is unlikely they will ever develop as a means of allocating goods for final consumption. The information needed for consumer decision-making is too great. Further, the need to make all decisions for future time periods at the birth of an economy does not allow consumers to readjust their portfolios in light of new information. This problem is similar to the one noted by Starr:[1] namely that unless consumers have sufficiently similar probability distributions ex-ante contingent claim markets will not allow an efficient allocation between present and future consumption.

This paper considers an alternative competitive market structure under both distributions of capacity and delivery rights. In this market capacity is chosen before the random variable is observed, but purchases of goods and marginal production decisions are made via spot markets after the random variable is observed. The object of this exercise is to see if a normal market might allow an ex-ante optimal choice of capacity under uncertainty if less stringent conditions were imposed than were imposed on the securities market in the previous paper.

The technique used to model this market is expected utility maximization subject to random income constraints since consumers must plan purchases subject to receiving different incomes in different states of the world. This is mathematically equivalent to a dynamic programming formulation. It is used in this paper in preference to a standard dynamic programming formulation because it allows a more direct comparison between the results in this paper and those in the previous paper.

First, we consider conditions which apply when the capacity constraint is not binding. At a pure trade competitive spot market equilibrium, a necessary and sufficient condition for an ex-ante optimum when consumers have different incomes in different states of the world is for the ratios of individual expected marginal utilities of income across states of the world to be constant across consumers. That is $\frac{\lambda_{is}}{\lambda_{ir}}$ is independent of i, where λ_{is} is the multiplier associated with i's income in state s.

When production is introduced, a sufficient condition for ex-ante optimality of a competitive spot market equilibrium is for each consumer's expected marginal utility of income in each state of the world to be a constant function of his subjective probabilities (i.e. $\lambda_{is} = C_i \alpha_{is}$ $\forall i, s$, where α_{is} is i's subjective probability). This strong condition, which we imposed on the securities market, appears simply because consumers are planning purchases subject to different incomes in different states of the world.

When the capacity constraint is binding, the conditions for ex-ante optimality of the dynamic competitive equilibrium depend upon the structure of rights. If consumers hold title to capacity rights a sufficient condition for ex-ante optimality of a dynamic competitive equilibrium is the same as the sufficient condition when the capacity is not binding: for example, $\lambda_{is} = C_i \alpha_{is}$ $\forall i, s$.

If firms hold title to delivery capacity, the following sufficient conditions are close to being necessary: (1) firms owing capacity are risk neutral; (2) all consumers and firms have the same probability distribution; and (3) $\lambda_{is} = C_i \alpha_{is}$ $\forall i,s$. A simple example is used to show that even with similar probability distributions ex-ante optimality can fail.

In general, therefore, the results presented in this paper indicate that if firms own capacity, the conditions for ex-ante optimality are more strict in a competitive spot market than in a securities market.

REVIEW OF OPTIMALITY CONDITIONS UNDER UNCERTAINTY WITH CAPACITY CONSTRAINTS

As described in the previous paper, the ex-ante optimal allocation is identified by solving the following problem:

$$\text{Max } W = \sum_{i=1}^{n} \beta_i [U_i(L_{i0}) + \sum_{s=1}^{S} \alpha_{is} U_i(x_{is}, y_{is}, L_{is})].$$

Subject to the following constraints:

		Multipliers
$\sum_{i=1}^{n} x_{is} \leq x(L_{xs})$	$\forall s$	λ_{1s}
$x(L_{xs}) \leq z_s$	$\forall s$	λ_{2s}
$\sum_{i=1}^{n} y_{is} \leq y_s$	$\forall s$	λ_{3s}
$y_s = y(L_{ys})$	$\forall s$	λ_{4s}
$\hat{x} = \hat{x}(L_{\hat{x}})$		γ_1
$x(L_{xs}) \leq \hat{x}$	$\forall s$	λ_{5s}

Multipliers

$$L_{\hat{x}} \leq \sum_{i=1}^{n} L_{i0} \qquad \gamma_2$$

$$L_{xs} + L_{ys} \leq \sum_{i=1}^{n} L_{is} \quad \forall s \qquad \lambda_{6s}$$

where:

W = the social welfare function,

β_i = the weight given to i's utility function,

U_i = i's neo-classical utility function,

x_s = variable supply good produced in state s,

z_s = random supply of x available in s,

y_s = composite good produced in s,

$\hat{x}$ = delivery capacity for x.

The optimality conditions reduce to three cases:

1. If $z_s < \hat{x}$, then $\lambda_{5s} = 0$ and $\lambda_{2s} \neq 0$
2. If $z_s = \hat{x}$, then $\lambda_{5s} = \lambda_{2s} = 0$.
3. If $z_s > \hat{x}$, then $\lambda_{5s} \neq 0$ and $\lambda_{2s} = 0$.

These observations allow us to reduce the first order conditions to the following equations:

Case 1 ($z_s < \hat{x}$; $\lambda_{5s} = 0$, $\lambda_{2s} \neq 0$):

$$\frac{\alpha_{is} \dfrac{\partial U_i}{\partial x_{is}}}{\alpha_{ir} \dfrac{\partial U_i}{\partial y_{ir}}} = \frac{\alpha_{hs} \dfrac{\partial U_h}{\partial x_{hs}}}{\alpha_{hr} \dfrac{\partial U_h}{\partial y_{hr}}} \qquad \forall h, i, r$$

$$\frac{\alpha_{is} \dfrac{\partial U_i}{\partial x_{is}}}{\alpha_{ir} \dfrac{\partial U_i}{\partial y_{ir}}} = \frac{- \beta_i \alpha_{is} \dfrac{\partial U_i / \partial L_{is}}{\partial x / \partial L_{xs}} + \lambda_{2s}}{- \beta_i \alpha_{ir} \dfrac{\partial U_i / \partial L_{ir}}{\partial y / \partial L_{yr}}} \qquad \forall i, r$$

Case 2 ($z_s = \hat{x}$; $\lambda_{5s} = \lambda_{2s} = 0$):

$$\frac{\alpha_{is} \dfrac{\partial U_i}{\partial x_{is}}}{\alpha_{ir} \dfrac{\partial U_i}{\partial y_{ir}}} = \frac{\alpha_{hs} \dfrac{\partial U_h}{\partial x_{hs}}}{\alpha_{hr} \dfrac{\partial U_h}{\partial y_{hr}}} \qquad \forall h, i, r$$

$$\frac{\dfrac{\partial U_i}{\partial x_{is}}}{\dfrac{\partial U_i}{\partial y_{ir}}} = \frac{\dfrac{\partial U_i / \partial L_{is}}{\partial x / \partial L_{xs}}}{\dfrac{\partial U_i / \partial L_{ir}}{\partial y / \partial L_{yr}}} \qquad \forall i, r$$

Case 3 ($z_s > \hat{x}$; $\lambda_{5s} \neq 0$, $\lambda_{2s} = 0$):

$$\frac{\sum\limits_{s>\hat{x}} \alpha_{is} \dfrac{\partial U_i}{\partial x_{is}}}{\alpha_{ir} \dfrac{\partial U_i}{\partial y_{ir}}} = \frac{\sum\limits_{s>\hat{x}} \alpha_{hs} \dfrac{\partial U_h}{\partial x_{hs}}}{\alpha_{hr} \dfrac{\partial U_h}{\partial y_{hr}}} \qquad \forall h, i, r$$

$$\frac{\sum\limits_{s>\hat{x}} \alpha_{is} \frac{\partial U_i}{\partial x_{is}}}{\alpha_{ir} \frac{\partial U_i}{\partial y_{ir}}} = \frac{\sum\limits_{s>\hat{x}} \alpha_{is} \frac{\partial U_i/\partial L_{is}}{\partial x/\partial L_{xs}} + \frac{\partial U_i/\partial L_{i0}}{\partial \hat{x}/\partial L_{\hat{x}}}}{\alpha_{ir} \frac{\partial U_i/\partial L_{ir}}{\partial y/\partial L_{yr}}} \qquad \forall i, r$$

where:

$$- \beta_i \alpha_{is} \frac{\partial U_i}{\partial L_{is}} = - \beta_h \alpha_h \frac{\partial U_h}{\partial L_{hs}} = \lambda_{6s} = \frac{\partial W}{\partial (\sum\limits_{i=1}^{n} L_{is})}$$

is the marginal social value of labor in s;

$$- \beta_i \frac{\partial U_i}{\partial L_{i0}} = - \beta_h \frac{\partial U_h}{\partial L_{h0}} = \quad \gamma_2 = \frac{\partial W}{\partial (\sum\limits_{i=1}^{n} L_{i0})}$$

is the marginal social value of labor in period 0;

$\lambda_{2s} = \frac{\partial W}{\partial z_s}$ is the marginal social value of z in s.[2]

A DYNAMIC PROGRAMMING MODEL OF EX-ANTE CAPACITY CHOICE AND EX-POST SPOT MARKETS

1. Consumers Own Rights

Consider now a competitive economy with n consumers and m + k + g firms. The first m firms produce $\hat{x}$ in period 0, the next k firms produce x and the last g firms produce y. Consumers hold title to delivery rights and capacity units once capacity has been built. Consumption of i in s is limited by i's resources available in s because marginal transactions are made in spot markets. Consumer i's problem is:

Multipliers

$$\text{Max } EU_i = U_i(L_{i0}) + \sum_{s=1}^{S} \alpha_{is} U_i(x_{is}, y_{is}, L_{is})$$

$$\text{Subject to: } p_{\hat{x}0}\hat{x}_i + \sum_{s=1}^{S} q_s^0 r_{is} \leq \sum_{s=1}^{S} q_s^{0-} r_{is} + w_0 L_{i0} + \sum_{j=1}^{m} \theta_{\hat{x}}^{ij} \pi_{\hat{x}}^{j} \qquad \lambda_0^i$$

$$p_{xs}x_{is} + p_{ys}y_{is} \leq w_s L_{is} + q_s r_{is} + \sum_{j=m+1}^{m+k} \theta_{xs}^{ij} \pi_{xs}^{j} + \sum_{j=m+k+1}^{m+k+g} \theta_{ys}^{ij} \pi_{ys}^{j} + p_{\hat{x}s}\hat{x}_i \ \forall s \qquad \lambda_s^i$$

where:

$\theta_{\hat{x}}^{ij}$ = percentage of firm $\hat{x}^j$'s profits going to i;

θ_{xs}^{ij} = percentage of firm x^j's profits going to i in s;

θ_{ys}^{ij} = percentage of firm y^j's profits going to i in s;

w_0 = competitive wage in period 0;

w_s = competitive wage in state s;

$\pi_{\hat{x}}^j$ = profits of firm $\hat{x}^j$ in period 0;

π_{xs}^j = profits of firm x^j in state s;

π_{ys}^j = profits of firm y^j in state s;

$p_{\hat{x}0}$ = competitive price of a unit of capacity in period 0;

p_{xs} = competitive price of x in state s;

p_{ys} = competitive price of y in state s;

$p_{\hat{x}s}$ = competitive price of a unit of capacity in state s;

$\hat{x}_i$ = units of capacity owned by consumer i ;

q_s^0 = competitive price of a state s delivery right in period 0;

q_s = competitive price of a state s delivery right;

r_{is} = state s delivery rights owned by person i;

$\bar{r}_{is}$ = initial allocation of state s delivery rights to i.

Making the assumption of a decomposible profit function firm $\hat{x}^j$'s problem is:

Multipliers

$$\text{Max } (\pi^j_{\hat{x}}) = (p_{\hat{x}0}\hat{x}^j - w_0 L^j_{\hat{x}})$$

$$\text{s.t. } \hat{x}^j \leq \hat{x}\,(L^j_{\hat{x}}) \qquad \lambda^j_{\hat{x}}$$

Firm x^j's problem is:

$$\text{Max } E\phi^j_x(\pi^j_x) = \sum_{s=1}^{S} \alpha^j_{xs}\phi^j_x(p_{xs}x^j_s - w_s L^j_{xs} - p_{\hat{x}s}\hat{x}^j_s - q_s r^j_s)$$

$$\text{s.t. } x^j_s \leq x(L^j_{xs}) \qquad \forall s \qquad \lambda^j_{1s}$$

$$x(L^j_{xs}) \leq \hat{x}^j_s \qquad \forall s \qquad \lambda^j_{2s}$$

$$x(L^j_{xs}) \leq r^j_s \qquad \forall s \qquad \lambda^j_{3s}$$

where:

α^j_{xs} = firm x^j's subjective probability of being in s;

ϕ^j_x = firm x^j's differentiable utility function over profits;[3]

$\hat{x}^j_s$ = amount of capacity rented by j from consumers in state s;

r^j_s = number of state s capacity rights rented by j from consumers.

Firm y^j's problem is:

$$\text{Max} \quad E\phi^j_y(\pi^j_y) = \sum_{s=1}^{S} \alpha^j_{ys}\phi^j_y(p_{ys}y^j_s - w_s L^j_{ys})$$

$$\text{s.t.} \quad y^j_s \leq y\,(L_{ys}) \qquad \forall s \qquad\qquad \lambda^j_{4s}$$

The equilibrium conditions are:

$$\sum_{i=1}^{n} \hat{x}_i = \sum_{j=1}^{m} \hat{x}^j$$

$$\sum_{i=1}^{n} r_{is} = \sum_{i=1}^{n} \bar{r}_{is} = \sum_{j=m+1}^{m+k} r^j_s$$

$$\sum_{i=1}^{n} x_{is} = \sum_{j=m+1}^{m+k} x^j_s$$

$$\sum_{i=1}^{n} y_{is} = \sum_{j=m+k+1}^{m+k+g} y^j_s$$

$$\sum_{i=1}^{n} \theta_{xs}^{ij} = 1$$

$$\sum_{i=1}^{n} \theta_{ys}^{ij} = 1$$

$$\sum_{i=1}^{n} \theta_{\hat{x}}^{ij} = 1.$$

Forming Lagrange equations $L_i (i = 1, \ldots, n)$; $L_{\hat{x}}^{j} (j = 1, \ldots, m)$; $L_x^j (j = m+1, \ldots, m+k)$; and $L_y^j (j = m+k+1, \ldots, m+k+g)$, differentiating and considering only effective constraints which are binding, these equations imply the following marginal conditions.

$$w_0 = -\frac{\partial U_i / \partial L_{i0}}{\lambda_0^i} \qquad \forall i \qquad \text{(a)}$$

$$\frac{\alpha_{is} \dfrac{\partial U_i}{\partial x_{is}}}{\alpha_{ir} \dfrac{\partial U_i}{\partial y_{ir}}} = \frac{\lambda_s^i}{\lambda_r^i} \frac{p_{xs}}{p_{yr}} \qquad \forall i, s, r \qquad \text{(b)}$$

$$q_s = \frac{\lambda_0^i q_s^0}{\lambda_s^i} \qquad \forall i, s \qquad \text{(c)}$$

$$w_s = \frac{-\alpha_{is} \partial U_i / \partial L_{is}}{\lambda_s^i} \qquad \forall i, s \qquad \text{(d)}$$

Case 1 ($z_s < \hat{x}$):

$$\frac{p_{xs}}{p_{yr}} = \frac{\dfrac{w_s}{\partial x/\partial L^j_{xs}} + q_s}{\dfrac{w_r}{\partial y/\partial L^a_{yr}}} \qquad \forall j, a, r \qquad \text{(e)}$$

Case 2 ($z_s = \hat{x}$):

$$\frac{p_{xs}}{p_{yr}} = \frac{\dfrac{w_s}{\partial x/\partial L^j_{xs}}}{\dfrac{w_r}{\partial y/\partial L^a_{yr}}} \qquad \forall j, a, r \qquad \text{(f)}$$

Case 3 ($z_s > x$):

$$\frac{p_{xs}}{p_{yr}} = \frac{\dfrac{w_0}{\partial \hat{x}/\partial L^j_{\hat{x}}} + \dfrac{w_s}{\partial x/\partial L^j_{xs}}}{\dfrac{w_r}{\partial y/\partial L^a_{yr}}} \qquad \forall j, a, r \qquad \text{(g)}$$

Consider first equation (b) without any production equations. Notice that a necessary and sufficient condition for (b) to be ex-ante optimal is for $\frac{\lambda^i_s}{\lambda^i_r}$ to be constant across individuals, where λ^i_s is i's expected marginal utility of income in state s (see footnote 1). This is equivalent to saying that marginal rates of substitution for income in different states of the world are equal across consumers. This result can be summarized as follows:

<u>Proposition 1</u>: If consumers earn different incomes in different states of the world, then a necessary and sufficient condition for a pure trade competitive spot market equilibrium to be ex-ante Pareto optimal is for the ratios of consumer expected marginal utilities of income across states of the world to be constant across consumers.

Now compare (b) with the production conditions (e), (f), and (g). Assume now that $\lambda_s^i = C_i \alpha_{is}$ $\forall i, s$. This is the same sufficient condition used in the stock market model in the previous paper and it is equivalent to saying that each consumer's expected marginal utility of income is a constant function of his subjective probability distribution. Substituting (c) for q_s and (d) for w_s, the three cases reduce to:

Case 1 ($z_s < \hat{x}$):

$$\frac{\alpha_{is} \dfrac{\partial U_i}{\partial x_{is}}}{\alpha_{ir} \dfrac{\partial U_i}{\partial y_{ir}}} = \frac{\dfrac{-\alpha_{is} \partial U_i / \partial L_{is}}{\partial x / \partial L^j_{xs}} + C_i q^0_s}{\dfrac{-\alpha_{ir} \partial U_i / \partial L_{ir}}{\partial y / \partial L^a_{yr}}} \qquad \forall i, a, j, r \tag{h}$$

Case 2 ($z_s = \hat{x}$):

$$\frac{\dfrac{\partial U_i}{\partial x_{is}}}{\dfrac{\partial U_i}{\partial y_{ir}}} = \frac{\dfrac{\partial U_i / \partial L_{is}}{\partial x / \partial L^j_{xs}}}{\dfrac{\partial U_i / \partial L_{ir}}{\partial y / \partial L^a_{yr}}} \qquad \forall i, a, j, r \tag{i}$$

Case 3 ($z_s > \hat{x}$):

$$\frac{\sum_{s>\hat{x}} \alpha_{is} \dfrac{\partial U_i}{\partial x_{is}}}{\alpha_{ir} \dfrac{\partial U_i}{\partial y_{ir}}} = \frac{-\sum_{s>\hat{x}} \alpha_{is} \dfrac{\partial U_i/\partial L_{is}}{\partial x/\partial L^j_{xs}} - \dfrac{\partial U_i/\partial L_{i0}}{\partial x/\partial L^b_{\hat{x}}}}{-\alpha_{ir} \dfrac{\partial U_i/\partial L_{ir}}{\partial y/\partial L^a_{yr}}} \qquad \forall\, i, a, b, j, r \qquad \text{(j)}$$

where: C_i = i's constant multiple of his subjective probabilities.

Notice that equations (h)-(j) describe an ex-ante Pareto optimum with welfare weights $\beta_i = \frac{1}{C_i}$ and $q_s^0 = \lambda_{2s}$. This result can be summarized as follows:

<u>Proposition 2</u>: If the supply of a good is a random variable and delivery capacity has to be chosen before the random variable is observed, then a competitive spot market equilibrium is ex-ante Pareto optimal if the following conditions hold: (1) consumers hold title to all resources; and (2) consumers' expected marginal utilities of income are constant functions of their subjective probabilities.

<u>2. Firms Own Rights</u>

Suppose now we consider a somewhat more likely market situation. In this case consumers sell labor and purchase output in an ex-post spot market but firms delivering water build and hold title to their own capacity ex-ante and hold title to delivery rights.[4] Firms alone participate in a spot market in delivery rights. The maximization problems now become:

<u>Multipliers</u>

$$\text{Consumer:}\quad \text{Max } U_i(L_{i0}) + \sum_{s=1}^{S} \alpha_{is} U_i(x_{is}, y_{is}, L_{is})$$

$$\text{s.t.}\quad M_{i0} \leq w_0 L_{i0} \qquad \lambda_0^i$$

$$p_{xs}x_{is} + p_{ys}y_{is} \leq M_{i0} + w_s L_{is} + \sum_{j=1}^{k} \theta_{xs}^{ij}\pi_{xs} + \sum_{j=k+1}^{k+g} \theta_{ys}^{ij}\pi_{ys} \qquad \lambda_s^i$$

$$\text{Firm } x^j:\quad \text{Max } - w_0 L_{\hat{x}}^j + \sum_{s=1}^{S} \alpha_{xs}^j \phi_x^j [p_{xs}x_s^j - w_s L_{xs}^j - q_s(r_s^j - \bar{r}_s^j)]$$

$$\text{s.t.}\quad x_s^j \leq x(L_{xs}^j) \qquad \lambda_{1s}^j$$

$$x(L_{xs}^j) \leq \hat{x}^j \qquad \lambda_{2s}^j$$

$$x(L_{xs}^j) \leq r_s^j \qquad \lambda_{3s}^j$$

$$\hat{x}^j \leq \hat{x}(L_{\hat{x}}^j) \qquad \lambda_{4s}^j$$

$$\text{Firm } y^j:\quad \text{Max } \sum_{s=1}^{S} \alpha_{ys}^j \phi_y^j (p_{ys}y_s^j - w_s L_{ys}^j)$$

$$\text{s.t.}\quad y_s^j \leq y(L_{ys}^j). \qquad \lambda_{5s}^j$$

Where: M_{i0} = i's income earned in period 0 for purchase of consumption goods.

The equilibrium conditions are:

$$\sum_{i=1}^{n} L_{i0} = \sum_{j=1}^{k} L^{j}_{\hat{x}}$$

$$\sum_{i=1}^{n} L_{is} = \sum_{j=1}^{k} L^{j}_{xs} + \sum_{j=k+1}^{k+g} L^{j}_{ys}$$

$$\sum_{i=1}^{n} x_{is} = \sum_{j=1}^{k} x^{j}_{s}$$

$$\sum_{i=1}^{n} y_{is} = \sum_{j=k+1}^{k+g} y^{j}_{s}$$

$$\sum_{j=1}^{k} r^{j}_{s} = \sum_{j=1}^{k} \bar{r}^{j}_{s}$$

$$\sum_{i=1}^{n} \theta^{ij}_{xs} = 1$$

$$\sum_{i=1}^{n} \theta^{ij}_{ys} = 1$$

Forming Lagrange expressions $\mathcal{L}_i (i = 1, \ldots, n)$; $\mathcal{L}^j_x (j = 1, \ldots, k)$; $\mathcal{L}^j_y (j = k+1, \ldots, k+g)$, differentiating, and considering only effective constraints which are binding, these equations imply the following marginal conditions.

$$\frac{\alpha_{is}\dfrac{\partial U_i}{\partial x_{is}}}{\alpha_{ir}\dfrac{\partial U_i}{\partial y_{ir}}} = \frac{\lambda_s^i p_{xs}}{\lambda_r^i p_{yr}} \qquad \forall i, r \qquad \text{(k)}$$

Case 1 ($z_s < \hat{x}$):

$$\frac{p_{xs}}{p_{yr}} = \frac{\dfrac{w_s}{\partial x/\partial L_{xs}^j} + q_s}{\dfrac{w_r}{\partial y/\partial L_{yr}^a}} \qquad \forall a, j, r \qquad \text{(l)}$$

Case 2 ($z_s = \hat{x}$):

$$\frac{p_{xs}}{p_{yr}} = \frac{\dfrac{w_s}{\partial x/\partial L_{xs}^j}}{\dfrac{w_r}{\partial y/\partial L_{yr}^a}} \qquad \forall a, j, r \qquad \text{(m)}$$

Case 3 ($z_s > \hat{x}$):

$$\sum_{s>\hat{x}} \frac{\alpha_{xs}^j \phi_x'^j p_{xs}}{p_{yr}} = \frac{\sum_{s>\hat{x}} \alpha_{xs}^j \phi_x'^j \left(\dfrac{w_s}{\partial x/\partial L_{xs}^j}\right) + \dfrac{w_0}{\partial \hat{x}/\partial L_{\hat{x}}^j}}{\dfrac{w_r}{\partial y/\partial L_{yr}^a}} \qquad \forall a, j, r \qquad \text{(n)}$$

Notice that equations (k)-(m) are identical to equations (b), (e) and (f). This indicates that the conclusions of Proposition 1 generalize to the case such that firms own the state-dependent delivery rights. This result can be generalized as follows.

<u>Proposition 3</u>: If the supply of a good is a random variable and delivery capacity is not a binding constraint, then the distribution of delivery rights does not affect the allocation of final output when output is sold in an ex-post spot market.

It is clear from equation (n), however, that the distribution of capacity rights does affect the allocation of output for states of the world such that capacity is a binding constraint. In particular, when firms assume the risk of the returns from investment in capacity, then the subjective probabilities and relative risk aversion of firms enter the allocation decision. Notice that if the following three conditions hold, equations (k) and (n) describe an ex-ante Pareto optimum:

1. firms owning capacity are risk neutral;
2. all individuals and firms owning capacity have the same subjective probability distribution; and
3. $\lambda_{is} = C_i \alpha_{is} \quad \forall i, s.$

To prove this statement rewrite (n) as:

$$\phi_x^{\prime j} \sum_{s>\hat{x}} \frac{\alpha_s p_{xs}}{\alpha_r p_{yr}} = \frac{\phi_x^{\prime j} \sum_{s>\hat{x}} \alpha_s \left(\dfrac{w_s}{\partial x/\partial L_{xs}^j} \right) + \dfrac{w_0}{\partial \hat{x}/\partial L_{\hat{x}}^j}}{\dfrac{\alpha_r w_r}{\partial y/\partial L_{yr}^a}} \qquad \forall j, a, r \qquad (0)$$

where:

α_s = true or agreed upon probability of being in s; and

$\phi_x'^j$ = firm x^j's constant marginal utility of profits.

This reduces to:

$$\frac{\sum\limits_{s>\hat{x}} \alpha_s p_{xs}}{\alpha_r p_{yr}} = \frac{\sum\limits_{s>\hat{x}} \alpha_s \left(\dfrac{w_s}{\partial x/\partial L^j_{xs}}\right) + \dfrac{w_0}{\partial \hat{x}/\partial L^j_{\hat{x}}}}{\dfrac{\alpha_r w_r}{\partial y/\partial L^a_{yr}}} \qquad \forall j,\ a,\ r. \tag{p}$$

Now, let $\lambda^i_s = C_i \alpha_s$ $\forall$ i,s and consider all equations which have a particular $\alpha_r \dfrac{\partial U_i}{\partial y_{ir}}$ and a particular $\lambda^i_r p_{ys}$ in the denominators and and some $\alpha_s \dfrac{\partial U_i}{\partial x_{is}}$ and some $\lambda^i_s p_{xs}$ in each numerator. Canceling the C_i's in each equation and summing over $s > \hat{x}$ gives the following result:

$$\frac{\sum\limits_{s>\hat{x}} \alpha_s \partial U_i/\partial x_{is}}{\alpha_r \partial U_i/\partial y_{ir}} = \frac{\sum\limits_{s>\hat{x}} \alpha_s \partial U_h/\partial x_{hs}}{\alpha_r \partial U_h/\partial y_{hs}} = \frac{\sum\limits_{s>\hat{x}} \alpha_s p_{xs}}{\alpha_r p_{yr}} \qquad \forall h,\ i,\ r. \tag{q}$$

Substitution of (q) in (p), $\dfrac{\alpha_{is} \partial U_i/\partial L_{is}}{\lambda^i_s}$ for w_s, and $\dfrac{\partial U_i/\partial L_{i0}}{\lambda^i_0}$ for w_0 implies:

$$\frac{\sum\limits_{s>\hat{x}} \alpha_s \partial U_i/\partial x_{is}}{\alpha_r \partial U_i/\partial y_{ir}} = \frac{\sum\limits_{s>\hat{x}} \alpha_s \partial U_h/\partial x_{hs}}{\alpha_r \partial U_h/\partial y_{hr}} = \frac{-\sum\limits_{s>\hat{x}} \alpha_s \left(\dfrac{\partial U_i/\partial L_{is}}{\partial x/\partial L^j_{xs}}\right) - \dfrac{\partial U_i/\partial L_{i0}}{\partial \hat{x}/\partial L^j_{\hat{x}}}}{-\dfrac{\alpha_r \partial U_i/\partial L_{is}}{\partial y/\partial L^a_{yr}}} \tag{r}$$

$\forall$ h,i,j,a,r.

which is the desired result. This can be summarized as follows:

Proposition 4: If firms delivering a good subject to random supply are risk neutral, all consumers and firms owning capacity have the same subjective probability distribution over states of the world, and consumer expected marginal utilities of income are constant functions of their subjective probabilities, then a competitive spot market equilibrium is ex-ante Pareto optimal when firms own capacity and capacity is a binding constraint.

The obvious next question to ask is what happens if firms owning capacity are not risk neutral or agents have different probability distributions, or consumers have expected marginal utilities of income which are not constant functions of their subjective probabilities. The following example shows that it is easy to construct such situations which are not ex-ante Pareto optimal. Consider an economy with two consumers, one x firm and one y firm, and two states of the world.

Let $$EU_1 = U_1(L_{10}) + .5U_1(x_{11},y_{11},L_{11}) + .5U_1(x_{12},y_{12},L_{12})$$

$$EU_2 = U_2(L_{20}) + .25U_2(x_{21},y_{21},L_{21}) + .75U_2(x_{22},y_{22},L_{22})$$

$$E\phi_x = -w_0L_{\hat{x}} + .5\pi_{x1}^{1/2} + .5\pi_{x2}^{1/2}$$

$$E\phi_y = .5\pi_{y1} + .5\pi_{y2}$$

Thus, all agents but one have the same subjective probability distribution and one firm is risk neutral. The optimality conditions will be:

Case 1 ($z_s < \hat{x}$):

$$\frac{.5\,\frac{\partial U_1}{\partial x_{11}}}{.5\,\frac{\partial U_1}{\partial y_{12}}} = \frac{.25\,\frac{\partial U_2}{\partial x_{21}}}{.75\,\frac{\partial U_2}{\partial y_{22}}} = \frac{\frac{-\beta_1(.5\partial U_1/\partial L_{11})}{\partial x/\partial L_{x1}} + \lambda_{21}}{\frac{-\beta_1(.5\partial U_1/\partial L_{12})}{\partial y/\partial L_{y2}}} = \frac{\frac{-\beta_2(.25\partial U_2/\partial L_{21})}{\partial x/\partial L_{x1}} + \lambda_{21}}{\frac{-\beta_2(.75\partial U_2/\partial L_{22})}{\frac{\partial y}{\partial L_{y2}}}} \qquad \text{(s)}$$

Case 2 ($z_s = \hat{x}$):

$$\frac{.5\,\frac{\partial U_1}{\partial x_{11}}}{.5\,\frac{\partial U_1}{\partial y_{12}}} = \frac{.25\,\frac{\partial U_2}{\partial x_{21}}}{.75\,\frac{\partial U_2}{\partial y_{22}}} = \frac{\frac{.5\partial U_1/\partial L_{11}}{\partial x/\partial L_{x1}}}{\frac{.5\partial U_1/\partial L_{12}}{\partial y/\partial L_{y2}}} = \frac{\frac{.25\partial U_2/\partial L_{21}}{\partial x/\partial L_{x1}}}{\frac{.75\partial U_2/\partial L_{22}}{\partial y/\partial L_{y2}}} \qquad \text{(t)}$$

Case 3 ($z_s > \hat{x}$) -- Actual items summed depend upon the size of $\hat{x}$:

$$\frac{.5\,\frac{\partial U_1}{\partial x_{11}} + .5\,\frac{\partial U_1}{\partial x_{12}}}{.5\,\frac{\partial U_1}{\partial y_{12}}} = \frac{.25\,\frac{\partial U_2}{\partial x_{21}} + .75\,\frac{\partial U_2}{\partial x_{22}}}{.75\,\frac{\partial U_2}{\partial y_{22}}} =$$

$$\frac{\left(\frac{.5\partial U_1/\partial L_{11}}{\partial x/\partial L_{x1}} + \frac{.5\partial U_1/\partial L_{12}}{\partial x/\partial L_{x2}} + \frac{\partial U_1/\partial L_{10}}{\partial \hat{x}/\partial L_{\hat{x}}}\right)}{\frac{.5\partial U_1/\partial L_{12}}{\partial y/\partial L_{ys}}} =$$

$$\frac{\left(\frac{.25\partial U_2/\partial L_{21}}{\partial x/\partial L_{x1}} + \frac{.75\partial U_2/\partial L_{22}}{\partial x/\partial L_{x2}} + \frac{\partial U_2/\partial L_{20}}{\partial \hat{x}/\partial L_{\hat{x}}}\right)}{\frac{.75\partial U_2/\partial L_{22}}{\partial y/\partial L_{y2}}}.$$

Under competition, the marginal conditions will be:

$$\frac{.5\,\frac{\partial U_1}{\partial x_{11}}}{.5\,\frac{\partial U_1}{\partial y_{12}}} = \frac{\lambda_1^1}{\lambda_2^1}\,\frac{P_{x1}}{P_{y2}}$$

$$\frac{.25\,\frac{\partial U_2}{\partial x_{21}}}{.75\,\frac{\partial U_2}{\partial y_{22}}} = \frac{\lambda_1^2}{\lambda_2^2}\,\frac{P_{x1}}{P_{y2}} \tag{w}$$

Case 1 ($z_s < \hat{x}$):

$$\frac{P_{x1}}{P_{y2}} = \frac{\frac{w_1}{\partial x/\partial L_{x1}} + q_1}{\frac{w_2}{\partial y/\partial L_{y2}}} \tag{x}$$

Case 2 ($z_s = \hat{x}$):

$$\frac{P_{x1}}{P_{y2}} = \frac{\frac{w_1}{\partial x/\partial L_{x1}}}{\frac{w_2}{\partial y/\partial L_{y2}}} \tag{y}$$

Case 3 ($z_s > \hat{x}$):

$$\frac{.25(\pi_{x1}^{-1/2}p_{x1} + \pi_{x2}^{-1/2}p_{x2})}{p_{y2}} = \frac{.25\left(\frac{\pi_{x1}^{-1/2}w_1}{\partial x/\partial L_{x1}} + \frac{\pi_{x2}^{-1/2}w_2}{\partial x/\partial L_{x2}}\right) + \frac{w_0}{\partial \hat{x}/\partial L_{\hat{x}}}}{\frac{w_2}{\partial y/\partial L_{y2}}} \qquad (z)$$

Clearly, in order for (v) and (w) to satisfy (s) it is necessary to have:

$$\frac{\lambda_1^1 p_{x1}}{\lambda_2^1 p_{y2}} = \frac{\lambda_1^2 p_{x1}}{\lambda_2^2 p_{y2}}$$

This will be true iff:

$$\frac{\lambda_1^1}{\lambda_2^1} = \frac{\lambda_1^2}{\lambda_2^2}$$

However, even assuming this special form of the expected marginal utilities of income, (v), (w), and (z) would only satisfy (u) in general if $\lambda_1^1 = \lambda_1^2 = .25\,\pi_{x1}^{-1/2}$ and $\lambda_2^1 = \lambda_2^2 = .25\pi_{x2}^{-1/2}$. Even if firm x were risk neutral (u) would still not in general be satisfied without sufficiently similar probabilities. Thus, while necessary conditions may be difficult to find, the sufficient conditions outlined in Proposition 4 are close to being necessary when firms hold capacity rights.

This seemingly bizarre result occurs because consumers with differing probability distributions view the probability of having either unused or insufficient capacity differently. This does not present a problem when consumers themselves assume the risk of investing in capacity because the market in capacity rights allows consumers with differing probabilities to exhaust the gains from trade created by the differing probabilities. Neither does it cause difficulties in a full ex-ante contingent claims market because firms assume no risk in such a market. However, if firms do assume risks in providing capacity these gains from trade among consumers are not exhausted. In a sense, a free rider problem develops if firms assume the risks because consumers who place a low probability on the event that there will be insufficient capacity wish to pay less for x relative to y on the average when there is no shortage. If firms are risk neutral and consumers have identical probability distributions such gains from trade do not exist to begin with.

CONCLUSION

This paper has considered the effect on resource allocation under uncertainty of moving from a complete contingent claim market to a market in which capacity is built ex-ante but marginal production and consumption decisions are made ex-post. The result that the ex-ante efficiency of the competitive spot equilibrium under uncertainty depends upon the distribution of ex-ante capacity rights provides insight into the real world welfare effects of various production rigidities which firms typically impose upon themselves either because of technological rigidities or long lead times for construction, or to reduce output or input price or quantity demanded uncertainty. Future research may extend the analysis in this paper to a more general examination of the welfare effects of supply rigidities.

FOOTNOTES

1. R. M. Starr, "Optimal Production and Allocation Under Uncertainty," Quarterly Journal of Economics 87 (1973).

2. For a proof of this statement, see footnote 7 of the previous paper.

3. While the assumption that a firm possesses a utility function is controversial, the use of a firm utility function makes explicit the possibility that firm managers may have attitudes towards risks which affect the firm's decisions.

4. The results follow whether firms delivering x produce capacity or a separate set of firms produce capacity and rent it to x firms.

CHAPTER V

OPTIMAL CAPACITY CHOICE AND INVENTORY POLICY FOR A STORABLE GOOD SUBJECT TO RANDOM SUPPLY

INTRODUCTION

The previous two papers have considered ex-ante optimality conditions under uncertainty when capacity alone has to be chosen before the random variable is observed. The results were that ex-ante contingent claims markets led to ex-ante optimal allocations but stock markets in securities alone required strict conditions to achieve optimality. Sufficient conditions were for consumers to have identical subjective probability distributions and constant marginal utilities of income such that expected marginal utilities of income were constant functions of those subjective probabilities. If spot markets in labor and final output were allowed after the random variable was observed, optimality conditions depended on the structure of rights. If consumers owned capacity rights optimality could be achieved if expected marginal utilities of income were constant functions of consumer subjective probabilities. If firms owned capacity sufficient conditions included firm risk neutrality, equal subjective probability distributions of all agents and the above constant function expected marginal utilities of income.

A natural extension of the models developed in the previous papers is to a storable good subject to random supply. In this case, the optimal size of the storage facility and an optimal inventory policy have to be determined in addition to the optimal delivery capacity. Water is perhaps the best example of such a good since the storage facilities in the form of dams and reservoirs represent large, long-lived capital investments.

The welfare model developed in this paper is an extension of the model developed in the previous paper. In this case the delivery and storage capacities are built before any consumption takes place and consumption plans are made contingent on the observation of the random variable subject to supply constraints. Then, for every observation of the random variable in period 1 there is an ex-ante optimal storage and an optimal portfolio of consumption plans contingent on the observation of the random variable in period 2. The optimality conditions involve eight cases which describe the range of combinations of binding constraints and are too complicated to enumerate here. However, in general they involve the same basic rules developed in the previous papers. Whenever the delivery or capacity constraints are binding, expected marginal rates of substitution equal expected marginal resource values. In addition, whenever there is stored water which does not go to waste in the next period, the discounted expected marginal rate of substitution equals the discounted expected marginal resource value.

The competitive model used in this paper is an extension of the dynamic programming model developed in the previous paper. Delivery and storage capacity are built before any consumption takes place and then capacity rights are either sold to consumers or held by firms. Marginal production, consumption and storage decisions for period 1 are made after the random variable is observed. Then, given an amount of storage from period 1, marginal production and consumption decisions for period 2 are made after the second observation of the random variable.

The results on the effects of the distribution of delivery capacity rights generalize to the case of a storable good subject to capacity constraints. In this case, if consumers own delivery capacity, storage capacity, and storage rights then a competitive spot market equilibrium will be ex-ante Pareto optimal if consumers have expected marginal utilities of income which are discounted constant functions of their conditional subjective probabilities. If firms own any of these risky assets, however, additional sufficient conditions include identical subjective distributions and firm risk neutrality. Further, nonidentical discount factors among consumers and firms cause difficulties when firms assume risks.

A TWO PERIOD DYNAMIC PROGRAMMING MODEL WITH CAPACITY CONSTRAINTS AND A STORAGE FACILITY

Let X_{is}^{1} be i's consumption of x in period 1, state s and x_{irs}^{2} be i's consumption of x in period 2, state r, given s was observed in period 1. L_{xs}^{2} and L_{xrs}^{2} are labor devoted to the production of x in periods 1 and 2, states s and r, given s. $x(L_{xs}^{1})$ and $x(L_{xrs}^{2})$ are delivered x in periods 1 and 2, states s and r, given s. $\sum_{i=1}^{n} x_{is}^{1} \leq x(L_{xs}^{1})$ $\forall s$ and $\sum_{i=1}^{n} x_{irs}^{2} \leq x(L_{xrs}^{2})$ $\forall r, s$ z_s^1 and z_r^2 are the random supplies of x available in periods 1 and 2, states s and r. z_s^1 and z_r^2 are independently and identically distributed according to i's discrete subjective probabilities α_{is} and α_{ir}. z_s^1 can either be consumed as $x(L_{xs}^{1})$ or stored as X_s^1 in the storage facility F, subject to storage losses of $\theta\varepsilon(0, 1)$ per unit of storage. Delivery of $x(L_{xs}^{1})$ and $x(L_{xrs}^{2})$ are through delivery capacity $\hat{x}$, such that $x(L_{xs}^{1}) \leq \hat{x}$ $\forall s$ and $x(L_{xrs}^{2}) \leq \hat{x}$ $\forall r, s$. F is produced according to production function $F(L_F)$ in period 0, where L_F is labor used in the production of F. $\hat{x}$ is produced according to production function $\hat{x}(L_{\hat{x}})$ in period 0, where $L_{\hat{x}}$ is labor used in the production of $\hat{x}$. $L_F + L_{\hat{x}} \leq \sum_{i=1}^{n} L_{i0}$, where L_{i0} is the labor supplied by i in period 0.

Now, let y_{is}^{1} and y_{irs}^{2} be i's consumption of a composite good y in periods 1 and 2, states s and r, given s. y_s^1 and y_{rs}^2 are the amounts of y produced in periods 1 and 2, states s and r, given s. $\sum_{i=1}^{n} y_{is}^{1} \leq y_{s}^{1}$ $\forall s$ and $\sum_{i=1}^{n} y_{irs}^{2} \leq y_{rs}^{2}$ $\forall r, s.$ L_{ys}^{1} and L_{yrs}^{2} are labor devoted to the production of y in period 1 and 2, states s and r, given s. $y(L_{ys}^{2})$ and $y(L_{yrs}^{2})$ are the production functions for y in periods 1 and 2, states s and r, given s. L_{is}^{1} and L_{irs}^{2} are labor supplied by i in periods 1 and 2 and states s and r, given s. $L_{xs}^{1} + L_{ys}^{1} \leq \sum_{i=1}^{n} L_{is}^{1}$ $\forall s$ and $L_{xrs}^{2} + L_{yrs}^{2} \leq \sum_{i=1}^{n} L_{irs}^{2}$ $\forall i, s.$

Now assume that no consumption takes place in period 0. Consumer i supplies labor to the production of F and $\hat{x}$ according to differentiable utility function $U^{i}(L_{i0})$. In periods 1 and 2, states s and r, given s, consumer i has strictly quasi-concave differentiable utility functions $U^{i}(x_{is}^{1}, y_{is}^{1}, L_{is}^{1})$ and $U^{i}(x_{irs}^{2}, y_{irs}^{2}, L_{irs}^{2})$.

$$\lim_{x_{is}^{1} \to 0} \frac{\partial U^{i}}{\partial x_{is}^{1}} = \lim_{x_{irs}^{2} \to 0} \frac{\partial U^{i}}{\partial x_{irs}^{2}} = \lim_{y_{is}^{1} \to 0} \frac{\partial U^{i}}{\partial y_{is}^{1}} = \lim_{y_{irs}^{2} \to 0} \frac{\partial U^{i}}{\partial y_{irs}^{2}} = +\infty \quad \forall i, r, s$$

and $\lim_{L_{i0} \to H} \frac{\partial U^{i}}{\partial L_{i0}} = \lim_{L_{is}^{1} \to H} \frac{\partial U^{i}}{\partial L_{is}^{1}} = \lim_{L_{irs}^{2} \to H} \frac{\partial U^{i}}{\partial L_{irs}^{2}} = -\infty$ $\forall i, r, s,$ where H is the maximum time available for work in all periods. $\lim_{x_{is}^{1} \to \infty} \frac{\partial U^{i}}{\partial x_{is}^{1}} > 0$

$\forall i, s;$ $\lim_{x_{irs}^{2} \to \infty} \frac{\partial U^{i}}{\partial x_{irs}^{2}} > 0$ $\forall i, r, s;$ $\lim_{y_{is}^{1} \to \infty} \frac{\partial U^{i}}{\partial y_{is}^{1}} > 0$ $\forall i, s;$

$\lim_{y_{irs}^2 \to \infty} \frac{\partial U^i}{\partial y_{irs}^2} > 0 \quad \forall i, r, s$; $\lim_{L_{i0} \to 0} \frac{\partial U^i}{\partial L_{i0}} < 0 \quad \forall i$; $\lim_{L_{is}^1 \to 0} \frac{\partial U^i}{\partial L_{is}^1} < 0 \quad \forall i, s$;

and $\lim_{L_{irs}^2 \to 0} \frac{\partial U^i}{\partial L_{irs}^2} < 0 \quad \forall i, r, s.$

Now, since x_{is}^1, x_{irs}^2, y_{is}^1, and y_{irs}^2 are always consumed in positive quantities by all consumers, we can express an ex-ante Pareto optimal allocation of x_s^1, x_{rs}^2, y_s^1, and y_{rs}^2 by maximizing a weighted sum of individual utility functions. Let

$$W = \sum_{i=1}^{n} \beta_i \{U^i(L_{i0}) + \sum_{s=1}^{S} \alpha_{is} [U^i(x_{is}^1, y_{is}^1, L_{is}^1) +$$

$$\delta_i \sum_{r=1}^{R} \alpha_{ir} U^i(x_{irs}^2, y_{irs}^2, L_{irs}^2)\},$$

where:

β_i = welfare weight assigned to i, and

δ_i = i's discount rate.

The welfare problem is to maximize W subject to the following constraints:

	Multipliers
$F = F(L_F)$	γ_1
$\hat{x} = \hat{x}(L_{\hat{x}})$	γ_2

		Multipliers
$\sum_{i=1}^{n} x_{is}^1 \leq x(L_{xs}^1)$	$\forall s$	λ_{1s}^1
$\sum_{i=1}^{n} x_{irs}^2 \leq x(L_{xrs}^2)$	$\forall r, s$	λ_{1rs}^2
$x(L_{xs}^1) \leq z_s^1 - X_s^1$	$\forall s$	λ_{2s}^1
$x(L_{xrs}^2) \leq X_s^1(1-\theta) + z_r^2$	$\forall s, r$	λ_{2rs}^2
$x(L_{xs}^1) \leq \hat{x}$	$\forall s$	λ_{3s}^1
$x(L_{xrs}^2) \leq \hat{x}$	$\forall s, r$	λ_{3rs}^2
$X_s^1 \leq F$	$\forall s$	λ_{4s}^1
$\sum_{i=1}^{n} y_{is}^1 \leq y(L_{ys}^1)$	$\forall s$	λ_{5s}^1
$\sum_{i=1}^{n} y_{irs}^2 \leq y(L_{yrs}^2)$	$\forall r, s$	λ_{5rs}^2
$L_F + L_{\hat{x}} \leq \sum_{i=1}^{n} L_{i0}$		γ_3
$L_{xs}^1 + L_{ys}^1 \leq \sum_{i=1}^{n} L_{is}^1$	$\forall s$	λ_{6s}^1

Multipliers

$$L^2_{xrs} + L^2_{yrs} \leq \sum_{i=1}^{n} L^2_{irs} \qquad \forall r, s \qquad \lambda^2_{6rs}$$

The first order conditions are:

$$\beta_i \frac{\partial U^i}{\partial L_{i0}} + \gamma_3 = 0 \qquad \forall i$$

$$\beta_i \alpha_{is} \frac{\partial U^i}{\partial L^1_{is}} + \lambda^1_{6s} = 0 \qquad \forall i, s$$

$$\beta_i \delta_i \alpha_{is} \alpha_{ir} \frac{\partial U^i}{\partial L^2_{irs}} + \lambda^2_{6rs} = 0 \qquad \forall i, r, s$$

$$\beta_i \alpha_{is} \frac{\partial U^i}{\partial x^1_{is}} - \lambda^1_{1s} = 0 \qquad \forall i, s$$

$$\beta_i \delta_i \alpha_{is} \alpha_{ir} \frac{\partial U^i}{\partial x^2_{irs}} - \lambda^2_{1rs} = 0 \qquad \forall i, r, s$$

$$\beta_i \alpha_{is} \frac{\partial U^i}{\partial y^1_{is}} - \lambda^1_{5s} = 0 \qquad \forall i, s$$

$$\beta_i \delta_i \alpha_{is} \alpha_{ir} \frac{\partial U^i}{\partial y^2_{irs}} - \lambda^2_{5rs} = 0 \qquad \forall i, r, s$$

$$\frac{\partial x}{\partial L^1_{xs}} (\lambda^1_{1s} - \lambda^1_{2s} - \lambda^1_{3s}) - \lambda^1_{6s} = 0 \qquad \forall s$$

$$\frac{\partial x}{\partial L^2_{xrs}} (\lambda^2_{1rs} - \lambda^2_{2rs} - \lambda^2_{3rs}) - \lambda^2_{6rs} = 0 \qquad \forall r, s$$

$$-\lambda^1_{2s} + \sum_{r \ni x(L^2_{xrs}) = X^1_s(1-\theta) + z^2_r} \lambda^2_{2rs}(1-\theta) - \lambda_{4s} = 0 \qquad \forall s$$

$$-\gamma_1 + \sum_{s \ni X^1_s = F} \lambda^1_{4s} = 0$$

$$\gamma_1 \frac{\partial F}{\partial L_F} - \gamma_3 = 0$$

$$-\gamma_2 + \sum_{s \ni x(L^1_{xs}) = \hat{x}} \lambda^1_{3s} + \sum_{s,r \ni x(L^2_{xrs}) = \hat{x}} \lambda^2_{3rs} = 0$$

$$\gamma_2 \frac{\partial \hat{x}}{\partial L_{\hat{x}}} - \gamma_3 = 0$$

$$\lambda^1_{5s} \frac{\partial y}{\partial L^1_{ys}} - \lambda^1_{6s} = 0 \qquad \forall s$$

$$\lambda^2_{5rs} \frac{\partial y}{\partial L^2_{yrs}} - \lambda^2_{6rs} = 0 \qquad \forall r, s$$

Now if we consider only effective constraints which are binding, these first order conditions reduce to nine cases:

Case 1 ($z_s^1 - X_s^1 = \hat{x}$ and $X_s^1(1-\theta) + z_r^2 = \hat{x} \Rightarrow \lambda_{2s}^1 = \lambda_{3s}^1 = \lambda_{2rs}^2 = \lambda_{3rs}^2 = 0$):

$$\frac{\alpha_{is} \dfrac{\partial U^i}{\partial x_{is}^1}}{\left[\text{either } \alpha_{iq} \dfrac{\partial U^i}{\partial y_{iq}^1} \text{ or } \delta_i \alpha_{iq} \alpha_{ir} \dfrac{\partial U^i}{\partial y_{irq}^2}\right]} =$$

$$\frac{-\beta_i \alpha_{is} \dfrac{\partial U^i / \partial L_{is}^1}{\partial x / \partial L_{xs}^1}}{\left[\text{either } -\beta_i \alpha_{iq} \dfrac{\partial U^i / \partial L_{iq}^1}{\partial y / \partial L_{yq}^1} \text{ or } -\beta_i \delta_i \alpha_{iq} \alpha_{ir} \dfrac{\partial U^i / \partial L_{irq}^2}{\partial y / \partial L_{yrq}^2}\right]} =$$

$$\frac{\dfrac{\lambda_{6s}^1}{\partial x / \partial L_{xs}^1}}{\left[\text{either } \dfrac{\lambda_{6q}^1}{\partial y / \partial L_{yq}^1} \text{ or } \dfrac{\lambda_{6rq}^2}{\partial y / \partial L_{yrq}^2}\right]} \qquad \forall\, i, q, r \qquad (1)$$

and

$$\frac{\delta_i \alpha_{is} \alpha_{ir} \dfrac{\partial U^i}{\partial x_{irs}^2}}{\left[\text{either } \alpha_{ip} \dfrac{\partial U^i}{\partial y_{ip}^1} \text{ or } \delta_i \alpha_{ip} \alpha_{iq} \dfrac{\partial U^i}{\partial y_{iqp}^2}\right]} =$$

$$\frac{-\beta_i \delta_i \alpha_{is} \alpha_{ir} \dfrac{\partial U^i / \partial L^2_{irs}}{\partial x / \partial L^1_{xrs}}}{[\text{either } -\beta_i \alpha_{ip} \dfrac{\partial U^i / \partial L^1_{ip}}{\partial y / \partial L^1_{yp}} \text{ or } -\beta_i \delta_i \alpha_{ip} \alpha_{iq} \dfrac{\partial U^i / \partial L^2_{iqp}}{\partial y / \partial L^2_{yqp}}]} =$$

$$\frac{\dfrac{\lambda^2_{6rs}}{\partial x / \partial L^2_{xrs}}}{[\text{either } \dfrac{\lambda^2_{6p}}{\partial y / \partial L^1_{yp}} \text{ or } \dfrac{\lambda^2_{6qp}}{\partial y / \partial L^2_{yqp}}]} \qquad \forall\, i,\, p,\, q \qquad (2)$$

Case 2 ($z^1_s - X^1_s = \hat{x}$ and $X^1_s(1-\theta) + z^2_r > \hat{x}$ or $z^1_s - X^1_s < \hat{x}$, $X^1_s < F$, and $X^1_s(1-\theta) + z^2_r > \hat{x} \Rightarrow \lambda^1_{2s} = \lambda^1_{3s} = \lambda^2_{2rs} = 0$):

(1) and

$$\frac{\delta_i \sum_{s,r \ni x(L^2_{xrs}) = \hat{x}} \alpha_{is} \alpha_{ir} \dfrac{\partial U^i}{\partial x^2_{irs}}}{[\text{either } \alpha_{ip} \dfrac{\partial U^i}{\partial y^1_{ip}} \text{ or } \delta_i \alpha_{ip} \alpha_{iq} \dfrac{\partial U^i}{\partial y^2_{iqp}}]} =$$

$$\frac{-\beta_i \delta_i \sum_{s,r \ni x(L^2_{xrs}) = \hat{x}} \alpha_{is} \alpha_{ir} \dfrac{\partial U^i / \partial L^2_{irs}}{\partial x / \partial L^2_{xrs}} - \beta_i \dfrac{\partial U^i / \partial L_{i0}}{\partial \hat{x} / \partial L_{\hat{x}}}}{[\text{either } -\beta_i \alpha_{ip} \dfrac{\partial U^i / \partial L^1_{ip}}{\partial y / \partial L^1_{yp}} \text{ or } -\beta_i \delta_i \alpha_{ip} \alpha_{iq} \dfrac{\partial U^i / \partial L^2_{iqp}}{\partial y / \partial L^2_{yqp}}]} =$$

$$\frac{\sum_{s,r \ni x(L^2_{xrs}) = \hat{x}} \frac{\lambda^2_{6rs}}{\partial x/\partial L^2_{xrs}} + \frac{\gamma_3}{\partial \hat{x}/\partial L_{\hat{x}}}}{[\text{either } \frac{\lambda^1_{6p}}{\partial y/\partial L^1_{yp}} \text{ or } \frac{\lambda^2_{6qp}}{\partial y/\partial L^2_{yqp}}]} \qquad \forall i, q, p \qquad (3)$$

<u>Case 3 ($z^1_s - X^1_s > \hat{x}$ and $X^1_s(1-\theta) + z^2_r = \hat{x} \Rightarrow \lambda^1_{2s} = \lambda^2_{2sr} = \lambda^2_{3sr} = 0$):</u>

(2) and

$$\frac{\sum_{s \ni x(L^1_{xs}) = \hat{x}} \alpha_{is} \frac{\partial U^i}{\partial x^1_{is}}}{[\text{either } \alpha_{ip} \frac{\partial U^i}{\partial y^1_{ip}} \text{ or } \delta_i \alpha_{ip} \alpha_{iq} \frac{\partial U^i}{\partial y^2_{iqp}}]} =$$

$$\frac{-\beta_i \sum_{s \ni x(L^1_{xs}) = \hat{x}} \alpha_{is} \frac{\partial U^i/\partial L^1_{is}}{\partial x/\partial L^1_{xs}} - \beta_i \frac{\partial U^i/\partial L_{i0}}{\partial \hat{x}/\partial L_{\hat{x}}}}{[\text{either } -\beta_i \alpha_{ip} \frac{\partial U^i/\partial L^1_{ip}}{\partial y/\partial L^1_{yp}} \text{ or } -\beta_i \delta_i \alpha_{ip} \alpha_{iq} \frac{\partial U^i/\partial L^2_{iqp}}{\partial y/\partial L^2_{yqp}}]} =$$

$$\frac{\sum_{s \ni x(L^1_{xs}) = \hat{x}} \frac{\lambda^1_{6s}}{\partial x/\partial L^1_{xs}} + \frac{\gamma_3}{\partial \hat{x}/\partial L_{\hat{x}}}}{[\text{either } \frac{\lambda^1_{6p}}{\partial y/\partial L^1_{yp}} \text{ or } \frac{\lambda^2_{6qp}}{\partial y/\partial L^2_{yqp}}]} \qquad \forall i, p, q \qquad (4)$$

Case 4 ($z_s^1 - X_s^1 > \hat{x}$ and $X_s^1(1-\theta) + z_r^2 > \hat{x} \Rightarrow \lambda_{2s}^1 = \lambda_{2sr}^2 = 0$):

$$\frac{\sum_{s \ni x(L_{xs}^1) = \hat{x}} \alpha_{is} \frac{\partial U^i}{\partial x_{is}^1} + \delta_i \sum_{s,r \ni x(L_{xrs}^2) = \hat{x}} \alpha_{is}\alpha_{ir} \frac{\partial U^i}{\partial x_{irs}^2}}{[\text{either } \alpha_{ip} \frac{\partial U^i}{\partial y_{ip}^1} \text{ or } \delta_i \alpha_{ip}\alpha_{iq} \frac{\partial U^i}{\partial y_{iqp}^2}]} =$$

$$\frac{-\beta_i \frac{\partial U^i/\partial L_{i0}}{\partial \hat{x}/\partial L_{\hat{x}}} - \beta_i \sum_{s \ni x(L_{xs}^1)=\hat{x}} \alpha_{is} \frac{\partial U^i/\partial L_{is}^1}{\partial x/\partial L_{xs}^1} - \beta_i \delta_i \sum_{s,r \ni x(L_{xrs}^2)=\hat{x}} \alpha_{is}\alpha_{ir} \frac{\partial U_i/\partial L_{irs}^2}{\partial x/\partial L_{xrs}^2}}{[\text{either } -\beta_i \alpha_{ip} \frac{\partial U^i/\partial L_{ip}^1}{\partial y/\partial L_{yp}^1} \text{ or } -\beta_i \delta_i \alpha_{ip}\alpha_{iq} \frac{\partial U^i/\partial L_{iqp}^2}{\partial y/\partial L_{yqp}^2}]} =$$

$$\frac{\frac{\gamma_3}{\partial \hat{x}/\partial L_{\hat{x}}} + \sum_{s \ni x(L_{xs}^1) = \hat{x}} \frac{\lambda_{6s}^1}{\partial x/\partial L_{xs}^1} + \sum_{s,r \ni x(L_{xrs}^2) = \hat{x}} \frac{\lambda_{6rs}}{\partial x/\partial L_{xrs}^2}}{[\text{either } \frac{\lambda_{6p}^1}{\partial y/\partial L_{yp}^1} \text{ or } \frac{\lambda_{6qp}^2}{\partial y/\partial L_{yqp}^2}]} \quad \forall i, p, q. \qquad (5)$$

Case 5 ($z_s^1 - X_s^1 < \hat{x}$, $X_s^1 < F$, and $X_s^1(1-\theta) + z_r^2 < \hat{x} \Rightarrow \lambda_{3s}^1 = \lambda_{3rs}^2 = \lambda_{4s}^1 = 0$):

$$\frac{\alpha_{is} \frac{\partial U^i}{\partial x_{is}^1} - \delta_i(1-\theta) \sum_{r \ni x(L_{xrs}^2) = X_s^1(1-\theta) + z_r^2} \alpha_{is}\alpha_{ir} \frac{\partial U^i}{\partial x_{irs}^2}}{[\text{either } \alpha_{ip} \frac{\partial U^i}{\partial y_{ip}^1} \text{ or } \delta_i \alpha_{ip}\alpha_{iq} \frac{\partial U^i}{\partial y_{iqp}^2}]} =$$

$$\frac{-\beta_i\alpha_{is}\dfrac{\partial U^i/\partial L^1_{is}}{\partial x/\partial L^1_{xs}} + \beta_i\delta_i(1-\theta)\displaystyle\sum_{r\ni x(L^2_{xrs})=X^1_s(1-\theta)+z^2_r}\alpha_{is}\alpha_{ir}\dfrac{\partial U^i/\partial L^2_{irs}}{\partial x/\partial L^2_{xrs}}}{\left[\text{either } -\beta_i\alpha_{ip}\dfrac{\partial U^i/\partial L^1_{ip}}{\partial y/\partial L^1_{yp}} \text{ or } -\beta_i\delta_i\alpha_{ip}\alpha_{iq}\dfrac{\partial U^i/\partial L^2_{iqp}}{\partial y/\partial L^2_{yqp}}\right]} =$$

$$\frac{\dfrac{\lambda^1_{6s}}{\partial x/\partial L^1_{xs}} - (1-\theta)\displaystyle\sum_{r\ni x(L^2_{xrs}) = X^1_s(1-\theta)+z^2_r}\dfrac{\lambda^2_{6rs}}{\partial x/\partial L^2_{xrs}}}{\left[\text{either } \dfrac{\lambda^1_{6p}}{\partial y/\partial L^1_{yp}} \text{ or } \dfrac{\lambda^2_{6qp}}{\partial y/\partial L^2_{yqp}}\right]} \quad \forall i, p, q. \qquad (6)$$

<u>Case 6 ($z^1_s - X^1_s < \hat{x}$, $X^1_s = F$ and $X^1_s(1-\theta) + z^2_r < \hat{x} \Rightarrow \lambda^1_{3s} = \lambda^2_{3rs} = 0$):</u>

$$\frac{\displaystyle\sum_{s\ni X^1_s = F}\alpha_{is}\left[\dfrac{\partial U^i}{\partial x^1_{is}} - \delta_i(1-\theta)\displaystyle\sum_{r\ni x(L^2_{xrs}) = X^1_s(1-\theta)+z^2_r}\alpha_{ir}\dfrac{\partial U^i}{\partial x^2_{irs}}\right]}{\left[\text{either } \alpha_{ip}\dfrac{\partial U^i}{\partial y^1_{ip}} \text{ or } \delta_i\alpha_{ip}\alpha_{iq}\dfrac{\partial U^i}{\partial y^2_{iqp}}\right]} =$$

$$\frac{-\beta_i\dfrac{\partial U^i/\partial L_{i0}}{\partial F/\partial L_F} - \displaystyle\sum_{s\ni X^1_s=F}\left[\beta_i\alpha_{is}\dfrac{\partial U^i/\partial L^1_{is}}{\partial x/\partial L^1_{xs}} - (1-\theta)\displaystyle\sum_{\substack{r\ni x(L^2_{xrs})=\\ X^1_s(1-\theta)+z^2_r}}\beta_i\delta_i\alpha_{is}\alpha_{ir}\dfrac{\partial U^i/\partial L^2_{irs}}{\partial x/\partial L^2_{xrs}}\right]}{\left[\text{either } -\beta_i\alpha_{ip}\dfrac{\partial U^i/\partial L^1_{ip}}{\partial y/\partial L^1_{yp}} \text{ or } -\beta_i\delta_i\alpha_{ip}\alpha_{iq}\dfrac{\partial U^i/\partial L^2_{iqp}}{\partial y/\partial L^2_{yqp}}\right]}$$

$$= \frac{\sum_{s \ni X_s^1 = F} \left[\frac{\lambda_{6s}^1}{\partial x/\partial L_{xs}^1} - (1-\theta) \sum_{\substack{r \ni x(L_{xrs}^2) = \\ X_s^1(1-\theta)+z_r^2}} \frac{\lambda_{6rs}^2}{\partial x/\partial L_{xrs}^2} \right] + \frac{\gamma_3}{\partial F/\partial L_F}}{\left[\text{either } \frac{\lambda_{6p}^1}{\partial y/\partial L_{yp}^1} \text{ or } \frac{\lambda_{6qp}^2}{\partial y/\partial L_{yqp}^2} \right]} \quad \forall\, i, p, q \quad (7)$$

Case 7 $(z_s^1 - X_s^1 = \hat{x},\ X_s^1 = F,\ X_s^1(1-\theta) + z_r^2 < \hat{x} \Rightarrow \lambda_{3s}^1 = \lambda_{3rs}^2 = \lambda_{2s}^1 = 0)$:

(1) and

$$\frac{\delta_i (1-\theta) \sum_{s \ni X_s^1 = F} \alpha_{is} \sum_{\substack{r \ni x(L_{xrs}^2) = \\ X_s^1(1-\theta)+z_r^2}} \alpha_{ir} \frac{\partial U^i}{\partial x_{irs}^2}}{\left[\text{either } \alpha_{ip} \frac{\partial U^i}{\partial y_{ip}^1} \text{ or } \delta_i \alpha_{ip} \alpha_{iq} \frac{\partial U^i}{\partial y_{iqp}^2}\right]} =$$

$$\frac{-\beta_i \frac{\partial U^i/\partial L_{i0}}{\partial F/\partial L_F} - \beta_i \delta_i (1-\theta) \sum_{(s \ni X_s^1 = F)} \alpha_{is} \sum_{\substack{(r \ni x(L_{xrs}^2) = \\ X_s^1(1-\theta)+z_r^2)}} \alpha_{ir} \frac{\partial U^i/\partial L_{irs}^2}{\partial x/\partial L_{xrs}^2}}{\left[\text{either } -\beta_i \alpha_{ip} \frac{\partial U^i/\partial L_{ip}^1}{\partial y/\partial L_{yp}^1} \text{ or } -\beta_i \delta_i \alpha_{ip} \alpha_{iq} \frac{\partial U^i/\partial L_{iqp}^2}{\partial x/\partial L_{yqp}^2}\right]} =$$

$$\frac{\frac{\gamma_3}{\partial F/\partial L_F} + (1-\theta) \sum_{(s \ni X_s^1 = F)} \sum_{(r \ni x(L_{xrs}^2) = X_s^1(1-\theta)+z_r^2)} \frac{\lambda_{6rs}^2}{\partial x/\partial L_{xr}^2}}{\left[\text{either } \frac{\lambda_{6p}^1}{\partial y/\partial L_{yp}^1} \text{ or } \frac{\lambda_{6qp}^2}{\partial y/\partial L_{yqp}^2}\right]} \quad \forall\, i, p, q \quad (8)$$

Case 8 ($z_s^1 - X_s^1 > \hat{x}$, $X_s^1 = F$, $X_s^1(1-\theta) + z_r^2 < \hat{x} \Rightarrow \lambda_{2s}^1 = \lambda_{3rs}^2 = 0$):

(4) and (8).

Case 9 ($z_s^1 - X_s^1 < \hat{x}$, $X_s^1 = F$ and $X_s^1(1-\theta) + z_r^2 > \hat{x} \Rightarrow \lambda_{3s}^1 = \lambda_{2rs}^2 = 0$):

(3) and

$$\frac{\sum_{s \ni X_s^1 = F} \alpha_{is} \dfrac{\partial U^i}{\partial x_{is}^1}}{[\text{either } \alpha_{ip} \dfrac{\partial U^i}{\partial y_{ip}^1} \text{ or } \delta_i \alpha_{ip} \alpha_{iq} \dfrac{\partial U^i}{\partial y_{iqp}^2}]} =$$

$$\frac{-\beta_i \sum_{s \ni X_s^1 = F} \alpha_{is} \dfrac{\partial U^i / \partial L_{is}^1}{\partial x / \partial L_{xs}^1} - \beta_i \dfrac{\partial U^i / \partial L_{i0}}{\partial F / \partial L_F}}{[\text{either } -\beta_i \alpha_{ip} \dfrac{\partial U^i / \partial L_{ip}^1}{\partial y / \partial L_{yp}^1} \text{ or } -\beta_i \delta_i \alpha_{ip} \alpha_{iq} \dfrac{\partial U^i / \partial L_{iqp}^2}{\partial y / \partial L_{yqp}^2}]} =$$

$$\frac{\sum_{s \ni X_s^1 = F} \dfrac{\lambda_{6s}^1}{\partial x / \partial L_{xs}^1} + \dfrac{\gamma_3}{\partial F / \partial L_F}}{[\text{either } \dfrac{\lambda_{6p}^1}{\partial y / \partial L_{yp}^1} \text{ or } \dfrac{\lambda_{6qp}^2}{\partial y / \partial L_{yqp}^2}]} \qquad \forall\, i,\ p,\ q \qquad (9)$$

A COMPETITIVE MARKET

1. Consumers Own Rights

Now let us consider a competitive market in which consumers hold title to delivery capacity, storage capacity, and storage rights. Firms $\hat{x}^a$, $a = 1, \ldots, A$ produce delivery capacity and sell capacity rights to consumers. Firms F^b, $b = 1, \ldots, B$ produce storage capacity and sell rights to consumers. Consumers own delivery rights which they can use either to rent to firms or to store water in the storage facility. Each consumer can store as many units of water as the number of storage units he or she owns. All consumers' assets are liquidated during period 2.

Consumer i's problem is:

$$\text{Max } U^i(L_{i0}) + \sum_{s=1}^{S} \alpha_{is}[U^i(x^1_{is}, y^1_{is}, L^1_{is}) + \delta_i \sum_{r=1}^{R} \alpha_{ir} U^i(x^2_{irs}, y^2_{irs}, L^2_{irs})]$$

subject to:

Multipliers

$$p^0_{\hat{x}}\hat{x}_i + p^0_F F^0_i + \sum_{s=1}^{S} q^0_s r^0_{is} \leq w_0 L_{i0} + \sum_{s=1}^{S} q^0_s \bar{r}_{is} + \sum_{a=1}^{A} \theta^{ia}_{\hat{x}} \pi^a_{\hat{x}} + \sum_{b=1}^{B} \theta^{ib}_F \pi^b_F. \qquad \lambda^0_i$$

$$p^1_{xs} x^1_{is} + p^1_{ys} y^1_{is} + p^1_{Fs} F^1_{is} + \sum_{r=1}^{R} q^1_{rs} r^1_{irs} \leq$$

$$w^1_s L^1_{is} + C^1_{\hat{x}s} \hat{x}_i + \sum_{r=1}^{R} q^1_{rs} r^0_{ir} + C^1_{qs}(r^1_{iss} - X^1_{is}) +$$

Multipliers

$$p^1_{Fs}F^0_i + \sum_{j=1}^{m} \theta^{ij}_{xs}\pi^{1j}_{xs} + \sum_{j=m+1}^{m+k} \theta^{ij}_{ys}\pi^{1j}_{ys}. \qquad \forall s \qquad \lambda^1_{is}$$

$$p^2_{xrs}x^2_{irs} + p^2_{yrs}y^2_{irs} \leq w^2_{rs}L^2_{irs} + p^2_{\hat{x}rs}\hat{x}_i + q^2_{rs}(X^1_{is}(1-\theta) + r^1_{irs})$$

$$+ \sum_{j=1}^{m} \theta^{ij}_{xr}\pi^{2j}_{xrs} + \sum_{j=m+1}^{m+k} \theta^{ij}_{yr}\pi^{2j}_{yrs}. \qquad \forall r, s \qquad \lambda^2_{irs}$$

$$X^1_{is} \leq F^1_{is} \qquad \forall s \qquad \gamma_{is}$$

Where:

$p^0_{\hat{x}}, p^2_{\hat{x}rs}$ = selling price for a unit of delivery capacity in period 0, and period 2, state r, given s in period 1;

$\hat{x}_i$ = units of delivery capacity owned by i;

p^0_F, p^1_{Fs} = price of a unit of storage in period 0 and period 1, state s;

F^0_i, F^1_{is} = units of storage capacity owned by i in period 0 and period 1, state s;

$q^0_r, q^1_{rs}, q^2_{rs}$ = selling price for a state r delivery right in period 0, period 1, given s in period 1 and in period 2, given s in period 1;

r^0_{is}, r^1_{irs} = state s delivery right purchased by i in period 0, and state r delivery right purchased in period 1, given s in period 1;

$\bar{r}_{is}$ = i's initial endowment of state s delivery rights;

$\theta^{ia}_{\hat{x}}$ = percentage of firm $\hat{x}^a$'s profits going to i;

$\pi^a_{\hat{x}}$ = firm $\hat{x}^a$'s profits;

θ^{ib}_F = percentage of firm F^b's profits going to i;

π^b_F = firm F^b's profits;

w_0, w^1_s, w^2_{rs} = wage rate in period 0, period 1, state s and period 2, state r, given s;

p^1_{xs}, p^2_{xrs} = price of x in period 1, state s and period 2 state r, given s;

p^1_{ys}, p^2_{yrs} = price of y in period 1, state s and period 2, state r, given s;

$c^1_{\hat{x}s}$ = rental rate for a unit of capacity in period 1, state s;[1]

c^1_{qs} = rental rate for a state s delivery right in period 1;

X^1_{is} = amount stored by i in period 1 and state s;

θ^{ij}_{xs} = percentage of x^j's profits going to i in state s;

$\pi^{1j}_{xs}, \pi^{2j}_{xrs}$ = profits of firm x^j in period 1, state s and period 2, state r, given s;

θ^{ij}_{ys} = percentage of y^j's profits going to i in state s;

$\pi^{1j}_{ys}, \pi^{2j}_{yrs}$ = profits of firm y^j in period 1, state s and period 2, state r, given s.

Firm $\hat{x}^a$'s problem is:

<u>Multipliers</u>

$$\text{Max} \quad p^0_{\hat{x}}\hat{x}^a - w_0 L^a_{\hat{x}}$$

$$\text{s.t.} \quad \hat{x}^a = \hat{x}(L^a_{\hat{x}}) \qquad \lambda^a_{\hat{x}}$$

where:

$\hat{x}^a$ = output of firm $\hat{x}^a$;

$L^a_{\hat{x}}$ = labor hired by firm $\hat{x}^a$.

Multipliers

Firm F^b's problem is:

$$\text{Max} \quad p_F^0 F^b - w_0 L_F^b$$

$$\text{s.t.} \quad F^b = F(L_F^b) \qquad \lambda_F^b$$

where:

F^b = output of firm F_b

L_F^b = labor hired by firm F^b

Firm x^j's problem is:

$$\text{Max} \sum_{s=1}^{S} \alpha_{xs}^j [\phi_x^j (p_{xs}^1 x_s^{1j} - w_s^1 L_{xs}^{1j} - c_{qs}^1 r_s^{1j} - c_{\hat{x}s}^1 \hat{x}_s^{1j}) +$$

$$\delta_i \sum_{r=1}^{R} \alpha_{xr}^j \phi_x^j (p_{xrs}^2 x_{rs}^{2j} - w_{rs}^2 L_{xrs}^{2j} - q_{rs}^2 r_{rs}^{2j} - p_{\hat{x}rs}^2 \hat{x}_{rs}^{2j})]$$

subject to:

Multipliers

$$x_s^{1j} \leq x(L_{xs}^{1j}) \qquad \forall s \qquad \lambda_{1s}^{1j}$$

$$x_{rs}^{2j} \leq x(L_{xrs}^{2j}) \qquad \forall r, s \qquad \lambda_{1rs}^{2j}$$

		Multipliers
$x(L^{1j}_{xs}) \leq r^{1j}_{s}$	$\forall$ s	λ^{1j}_{2s}
$x(L^{2j}_{xrs}) \leq r^{2j}_{rs}$	$\forall$ s, r	λ^{2j}_{2rs}
$x(L^{1j}_{xs}) \leq \hat{x}^{1j}_{s}$	$\forall$ s	λ^{1j}_{3s}
$x(L^{2j}_{xrs}) \leq \hat{x}^{2j}_{rs}$	$\forall$ r, s	λ^{2j}_{3rs}

where:

α^{j}_{xs} = x^j's subjective probability of being in s;

ϕ^{j}_{x} = firm x^j's differentiable utility function over profits;

x^{1j}_{s}, x^{2j}_{rs} = output of x^j in period 1, state s and period 2, state r, given s;

$L^{1j}_{xs}, L^{2j}_{xrs}$ = labor hired by x^j in period 1, state s and period 2, state r, given s;

r^{1j}_{s}, r^{2j}_{rs} = state s delivery rights rented by x^j in period 1 and state r rights purchased in period 2, given s in period 1;

$\hat{x}^{1j}_{s}, \hat{x}^{2j}_{rs}$ = capacity rights rented by x^j in period 1, state s and purchased in period 2, state r, given s.

Firm y^j's problem is:

$$\text{Max} \sum_{s=1}^{S} \alpha_{ys}^{j}[\phi_y^j(p_{ys}^1 y_s^{1j} - w_s L_{ys}^{1j}) +$$

$$\delta_j \sum_{r=1}^{R} \alpha_{yr}^{j} \phi_y^j (p_{yrs}^2 y_{rs}^{2j} - w_{rs}^2 L_{yrs}^{2j})]$$

subject to:

		Multipliers
$y_s^{1j} \leq y(L_{ys}^{1j})$	$\forall s$	λ_{ys}^{1j}
$y_{rs}^{2j} \leq y(L_{yrs}^{2j})$	$\forall s, r$	λ_{yrs}^{2j}

where:

α_{ys}^{j} = firm y^j's subjective probability of being in s;

ϕ_y^j = firm y^j's differentiable utility function over profits;

y_s^{1j}, y_{rs}^{2j} = output of firm y^j in period 1, state s and period 2, state r, given s;

$L_{ys}^{1j}, L_{yrs}^{2j}$ = labor hired by y^j in period 1, state s and period 2, state r, given s.

The equilibrium conditions are:

$$\sum_{i=1}^{n} x^{1}_{is} = \sum_{j=1}^{m} x^{1j}_{s}, \quad \sum_{i=1}^{n} x^{2}_{irs} = \sum_{j=1}^{m} x^{2j}_{rs} \qquad \forall r, s \tag{10}$$

$$\sum_{i=1}^{n} y^{1}_{is} = \sum_{j=m+1}^{m+k} y^{2j}_{s}, \quad \sum_{i=1}^{m} y^{2}_{irs} = \sum_{j=m+1}^{m+k} y^{2j}_{rs} \qquad \forall r, s \tag{11}$$

$$\sum_{a=1}^{A} L^{a}_{\hat{x}} + \sum_{b=1}^{B} L^{b}_{F} = \sum_{i=1}^{n} L_{i0} \tag{12}$$

$$\sum_{j=1}^{m} L^{1j}_{xs} + \sum_{j=m+1}^{m+k} L^{1j}_{ys} = \sum_{i=1}^{n} L^{1}_{is}, \quad \sum_{j=1}^{m} L^{2j}_{xrs} + \sum_{j=m+1}^{m+k} L^{2j}_{yrs} = \sum_{i=1}^{n} L^{2}_{irs} \qquad \forall r, s \tag{13}$$

$$\sum_{i=1}^{n} \hat{x}_{i} = \sum_{a=1}^{A} \hat{x}^{a} \tag{14}$$

$$\sum_{j=1}^{m} \hat{x}^{1j}_{s} = \sum_{i=1}^{n} \hat{x}_{i}, \quad \sum_{j=1}^{m} \hat{x}^{2j}_{rs} = \sum_{i=1}^{n} \hat{x}_{i} \qquad \forall r, s \tag{15}$$

$$\sum_{i=1}^{n} F^{0}_{i} = \sum_{b=1}^{B} F^{b} \qquad \forall s \tag{16}$$

$$\sum_{i=1}^{n} F_{is} = \sum_{i=1}^{n} F_{i} \qquad \forall s \tag{17}$$

$$\sum_{i=1}^{n} \theta_{\hat{x}}^{ia} = 1 \qquad \forall a \qquad (18)$$

$$\sum_{i=1}^{n} \theta_{F}^{ib} = 1 \qquad \forall b \qquad (19)$$

$$\sum_{i=1}^{n} \theta_{xs}^{ij} = 1 \qquad \forall s,\ j = 1, \ldots, m \qquad (20)$$

$$\sum_{i=1}^{n} \theta_{ys}^{ij} = 1 \qquad \forall s,\ j = m+1, \ldots, m+k \qquad (21)$$

$$\sum_{i=1}^{n} \bar{r}_{ir} = \sum_{i=1}^{n} r_{ir}^{0} = \sum_{i=1}^{n} r_{irs}^{1} \qquad \forall r, s \qquad (22)$$

$$\sum_{j=1}^{m} r_{s}^{1j} = \sum_{i=1}^{n} (r_{iss}^{1} - X_{is}^{1}) \qquad \forall s \qquad (23)$$

$$\sum_{j=1}^{m} r_{rs}^{2j} = \sum_{i=1}^{n} [X_{is}^{1}(1-\theta) + r_{irs}^{1}] \qquad \forall s, r \qquad (24)$$

Forming Lagrange functions $L_i (i = 1, \ldots, n)$, $L_{\hat{x}}^{a} (a = 1, \ldots, A)$, $L_{F}^{b} (b = 1, \ldots, B)$, $L_{x}^{j} (j = 1, \ldots, m)$, and $L_{y}^{j} (j = m+1, \ldots, m+k)$, the first order conditions are:

$$-\lambda_{is}^{1}p_{Fs}^{1} + \gamma_{is} = 0 \qquad \forall i, s \qquad (25)$$

$$\frac{\partial U^{i}}{\partial L_{i0}} + \lambda_{i}^{0}w^{0} = 0 \qquad \forall i \qquad (26)$$

$$\alpha_{is}\frac{\partial U^{i}}{\partial L_{is}^{1}} + \lambda_{is}^{1}w_{s}^{1} = 0 \qquad \forall i, s \qquad (27)$$

$$\delta_{i}\alpha_{is}\alpha_{ir}\frac{\partial U^{i}}{\partial L_{irs}^{2}} + \lambda_{irs}^{2}w_{rs}^{2} = 0 \qquad \forall i, r, s \qquad (28)$$

$$\alpha_{is}\frac{\partial U^{i}}{\partial x_{is}^{1}} - \lambda_{is}^{1}p_{xs}^{1} = 0 \qquad \forall i, s \qquad (29)$$

$$\delta_{i}\alpha_{is}\alpha_{ir}\frac{\partial U^{i}}{\partial x_{irs}^{2}} - \lambda_{irs}^{2}p_{xrs}^{2} = 0 \qquad \forall i, r, s \qquad (30)$$

$$\alpha_{is}\frac{\partial U^{i}}{\partial y_{is}^{1}} - \lambda_{is}^{1}p_{ys}^{1} = 0 \qquad \forall i, s \qquad (31)$$

$$\delta_{i}\alpha_{is}\alpha_{ir}\frac{\partial U^{i}}{\partial y_{irs}^{2}} - \lambda_{irs}^{2}p_{yrs}^{2} \qquad \forall i, r, s \qquad (32)$$

$$-\lambda_{i}^{0}p_{F}^{0} + \sum_{s \ni p_{Fs}^{1} > 0}\lambda_{is}^{1}p_{Fs}^{1} = 0 \qquad \forall i \qquad (33)$$

$$-\lambda_{i}^{0}p_{\hat{x}}^{0} + \sum_{s \ni c_{\hat{x}s}^{1} > 0}\lambda_{is}^{1}p_{\hat{x}s} + \sum_{s,r \ni p_{\hat{x}rs}^{2} > 0}\lambda_{irs}^{2}p_{\hat{x}rs}^{2} = 0 \qquad \forall i \qquad (34)$$

$$-\lambda_i^0 q_s^0 + \sum_{p \ni q_{sp}^1 > 0} \lambda_{ip}^1 q_{sp}^1 = 0 \qquad \forall i,\ s \qquad (35)$$

$$-\lambda_{is}^1 q_{rs}^1 + \lambda_{irs}^2 q_{rs}^2 = 0 \qquad \forall i,\ s \neq r \qquad (36)$$

$$-\lambda_{is}^1 (q_{ss}^1 - c_{qs}^1) + \lambda_{iss}^2 q_{ss}^2 = 0 \qquad \forall i,\ s \qquad (37)$$

$$-\lambda_{is}^1 c_{qs}^1 + (1-\theta) \sum_{r \ni q_{rs}^2 > 0} \lambda_{irs}^2 q_{rs}^2 - \gamma_{is} = 0 \qquad \forall i,\ s \qquad (38)$$

$$p_{\hat{x}}^0 - \lambda_{\hat{x}}^a = 0 \qquad \forall a \qquad (39)$$

$$-w_0 + \lambda_{\hat{x}}^a \frac{\partial \hat{x}}{\partial L_{\hat{x}}^a} = 0 \qquad \forall a \qquad (40)$$

$$p_F^0 - \lambda_F^b = 0 \qquad \forall b \qquad (41)$$

$$-w_0 + \lambda_F^b \frac{\partial F}{\partial L_F^b} = 0 \qquad \forall b \qquad (42)$$

$$\alpha_{xs}^j \phi_x^{j'} p_{xs}^1 - \lambda_{1s}^{1j} = 0 \qquad \forall s,\ j = 1, \ldots, m \qquad (43)$$

$$\delta_j \alpha_{xs}^j \alpha_{xr}^j \phi_x^{j'} p_{xrs}^2 - \lambda_{1rs}^{2j} = 0 \qquad \forall r,\ s,\ j = 1, \ldots, m \qquad (44)$$

$$-\alpha_{xs}^j \phi_x^{j'} w_s^1 + \frac{\partial x}{\partial L_{xs}^{1j}} (\lambda_{1s}^{1j} - \lambda_{2s}^{1j} - \lambda_{3s}^{1j}) = 0 \qquad \forall s,\ j = 1, \ldots, m \qquad (45)$$

$$-\delta_j \alpha_{xs}^j \alpha_{xr}^j \phi_x^{j'} w_{rs}^2 + \frac{\partial x}{\partial L_{xrs}^{2j}} (\lambda_{1rs}^{2j} - \lambda_{2rs}^{2j} - \lambda_{3rs}^{2j}) = 0$$

$$\forall r,\ s,\ j = 1, \ldots, m \qquad (46)$$

$$-\alpha^j_{xs}\phi^{j'}_x c^1_{qs} + \lambda^{1j}_{2s} = 0 \qquad \forall s;\ j = 1, \ldots, m \qquad (47)$$

$$-\delta_j \alpha^j_{xs}\alpha^j_{xr}\phi^{j'}_x q^1_{rs} + \lambda^{2j}_{2rs} = 0 \qquad \forall r,\ s;\ j = 1, \ldots, m \qquad (48)$$

$$-\alpha^j_{xs}\phi^{j'}_x c^1_{\hat{x}s} + \lambda^{1j}_{3s} = 0 \qquad \forall s;\ j = 1, \ldots, m \qquad (49)$$

$$-\delta_j \alpha^j_{xs}\alpha^j_{xr}\phi^{j'}_x p^2_{\hat{x}rs} + \lambda^{2j}_{3rs} = 0 \qquad \forall r,\ s;\ j = 1, \ldots, m \qquad (50)$$

$$\alpha^j_{ys}\phi^{j'}_y p^1_{ys} - \lambda^{1j}_{ys} = 0 \qquad \forall s,\ j = m+1, \ldots, m+k \qquad (51)$$

$$\delta_j \alpha^j_{ys}\alpha^j_{yr}\phi^{j'}_y p^2_{yrs} - \lambda^{2j}_{yrs} = 0 \qquad \forall r, s, j = m+1, \ldots, m+k \qquad (52)$$

$$-\alpha^j_{ys}\phi^{j'}_y w^1_s + \lambda^{1j}_{ys} \frac{\partial y}{\partial L^{1j}_{ys}} = 0 \qquad \forall s, j = m+1, \ldots, m+k \qquad (53)$$

$$-\delta_j \alpha^j_{ys}\alpha^j_{yr}\phi^{j'}_y w^2_{rs} + \lambda^{2j}_{yrs} \frac{\partial y}{\partial L^{2j}_{yrs}} = 0 \qquad \forall r, s, j = m+1, \ldots, m+k \qquad (54)$$

Now if we consider only cases such that prices are positive and we let $\lambda^1_{is} = C_i\alpha_{is}$ $\forall i$, s and $\lambda^2_{irs} = \delta_i C_i \alpha_{is}\alpha_{ir}$ $\forall i$, r, s, these first order conditions reduce to nine cases, which are identical to equations (1)-(9), given the equilibrium conditions (10)-(24). This result can be summarized as follows.

<u>Proposition 1</u>: If consumers hold title to delivery, capacity, storage capacity, and storage rights and each consumer's discounted expected marginal utility of income in each state of the world is a

constant function of his conditional subjective probabilities then a competitive spot market equilibrium is ex-ante Pareto optimal when storage capacity and delivery capacity constraints are binding.

2. Producers Own Rights

If producers own capacity, storage, and storage facility rights instead of consumers the competitive market can be described as follows:

$$\text{Consumer } i: \quad \text{Max } U^i(L_{i0}) + \sum_{s=1}^{S} \alpha_{is}[U^i(x^1_{is}, y^1_{is}, L^1_{is}) +$$

$$\sum_{r=1}^{R} \delta_i \alpha_{ir} U^i(x^2_{irs}, y^2_{irs}, L^2_{irs})].$$

Multipliers

$$\text{s. t.} \quad M_{i0} \leq w_0 L_{i0} \qquad \lambda^0_i$$

$$p^1_{xs} x^1_{is} + p^1_{ys} y^1_{is} \leq M_{i0} + w^1_s L^1_{is} + \sum_{j=1}^{m} \theta^{ij}_{xs} \pi^{1j}_{xs} + \sum_{j=m+1}^{m+k} \theta^{ij}_{ys} \pi^{1j}_{ys} \qquad \lambda^1_{is}$$

$$p^2_{xrs} x^2_{irs} + p^2_{yrs} y^2_{irs} \leq w^2_{rs} L^2_{irs} + \sum_{j=1}^{m} \theta^{ij}_{xr} \pi^{2j}_{xrs} + \sum_{j=m+1}^{m+k} \theta^{ij}_{yr} \pi^{2j}_{yrs} \qquad \lambda^2_{irs}$$

$$\text{Firm } x^j: \quad \text{Max} - w_0(L^j_{\hat{x}} + L^j_F) - \sum_{s=1}^{S} q^0_s(r^{0j}_s - \bar{r}^j_s) + \sum_{s=1}^{S} \alpha^j_{xs}[\phi^j_x(p^1_{xs}x^{1j}_s -$$

$$w^1_s L^{1j}_{xs} - \sum_{r=1}^{R} q^1_{rs}(r^{1j}_{rs} - r^{0j}_r)) + \delta_j \sum_{r=1}^{R} \alpha^j_{xr}\phi^j_x(p^2_{xrs}x^{2j}_{rs} - w^2_{rs}L^{2j}_{xrs} - q^2_{rs}(r^{2j}_{rs} - r^{1j}_{rs}))]$$

			Multipliers
s.t.	$x^{1j}_s \leq x(L^{1j}_{xs})$	$\forall s$	λ^{1j}_{1s}
	$x^{2j}_{rs} \leq x(L^{2j}_{xrs})$	$\forall r, s$	λ^{2j}_{1rs}
	$x(L^{1j}_{xs}) \leq \hat{x}^j$	$\forall s$	λ^{1j}_{2s}
	$x(L^{2j}_{xrs}) \leq \hat{x}^j$	$\forall r, s$	λ^{2j}_{2rs}
	$x(L^{1j}_{xs}) \leq r^{1j}_{ss} - X^{1j}_s$	$\forall s$	λ^{1j}_{3s}
	$x(L^{2j}_{xrs}) \leq X^{1j}_s(1-\theta) + r^{2j}_{rs}$	$\forall r, s$	λ^{2j}_{3rs}
	$X^{1j}_s \leq F^j$	$\forall s$	λ^{1j}_{4s}
	$\hat{x}^j \leq \hat{x}(L^j_{\hat{x}})$		γ^j_1
	$F^j \leq F(L^j_F)$		γ^j_2

where:

$\bar{r}_s^j$ = j's initial endowment of state s delivery rights;

$r_r^{0j}, r_{rs}^{1j}, r_{rs}^{2j}$ = state r delivery rights purchased by j in period 0, period 1, given s in period 1, and period 2, given s in period 1;

$\hat{x}^j$ = delivery capacity owned by j;

F^j = storage capacity owned by j;

$L_{\hat{x}}^j$ = labor hired by j to build delivery capacity;

L_F^j = labor hired by j to build storage capacity;

X_s^{1j} = amount of x stored by j, given s in period 1.

$$\text{Firm } y^j: \quad \text{Max} \sum_{s=1}^{S} \alpha_{ys}^j [\phi_y^j (p_{ys}^1 y_s^{1j} - w_s^1 L_{ys}^{1j}) +$$

$$\delta_j \sum_{r=1}^{R} \alpha_{yr}^j \phi_y^j (p_{yrs}^2 y_{rs}^{2j} - w_{rs}^2 L_{yrs}^{2j})]$$

		Multipliers
s.t.	$y_s^{1j} \leq y(L_{ys}^{1j})$	λ_{5s}^{1j}
	$y_{rs}^{2j} \leq y(L_{yrs}^{2j})$	λ_{5rs}^{2j}

The equilibrium conditions are:

$$\sum_{i=1}^{n} L_{i0} = \sum_{j=1}^{m} (L_{\hat{x}}^{j} + L_{F}^{j})$$

$$\sum_{i=1}^{n} L_{is}^{1} = \sum_{j=1}^{m} L_{xs}^{1j} + \sum_{j=m+1}^{m+k} L_{ys}^{1j} \qquad \forall s$$

$$\sum_{i=1}^{n} L_{irs}^{2} = \sum_{i=1}^{m} L_{xrs}^{2j} + \sum_{j=m+1}^{m+k} L_{yrs}^{2j} \qquad \forall r, s$$

$$\sum_{i=1}^{n} x_{is}^{1} = \sum_{j=1}^{m} x_{s}^{1j} \qquad \forall s$$

$$\sum_{i=1}^{n} x_{irs}^{2} = \sum_{i=1}^{m} x_{rs}^{2j} \qquad \forall r, s$$

$$\sum_{i=1}^{n} y_{is}^{1} = \sum_{j=m+1}^{m+k} y_{s}^{1j} \qquad \forall s$$

$$\sum_{i=1}^{m} y_{irs}^{2} = \sum_{j=m+1}^{m+k} y_{rs}^{2j} \qquad \forall r, s$$

$$\sum_{j=1}^{m} r_{rs}^{1j} = \sum_{j=1}^{m} r_{rs}^{2j} = \sum_{j=1}^{m} \bar{r}_{r}^{j} = \sum_{j=1}^{m} r_{r}^{0j} \qquad \forall r, s$$

Forming Lagrange equations $L_i (i = 1, \ldots, n)$, $L_x^j (j = 1, \ldots, m)$; and $L_y^j (j = m + 1, \ldots, m + K)$ and differentiating, the first order conditions are:

$$\frac{\partial U^i}{\partial L_{i0}} + \lambda_i^0 w_0 = 0 \qquad \forall i \qquad (52)$$

$$\alpha_{is} \frac{\partial U^i}{\partial L_{is}^1} + \lambda_{is}^1 w_s^1 = 0 \qquad \forall i, s \qquad (53)$$

$$\delta_i \alpha_{is} \alpha_{ir} \frac{\partial U^i}{\partial L_{irs}^2} + \lambda_{irs}^2 w_{rs}^2 = 0 \qquad \forall i, r, s \qquad (54)$$

$$\alpha_{is} \frac{\partial U^i}{\partial x_{is}^1} - \lambda_{is}^1 p_{xs}^1 = 0 \qquad \forall i, s \qquad (55)$$

$$\delta_i \alpha_{is} \alpha_{ir} \frac{\partial U^i}{\partial x_{irs}^2} - \lambda_{irs}^2 p_{xrs}^2 = 0 \qquad \forall i, r, s \qquad (56)$$

$$\alpha_{is} \frac{\partial U^i}{\partial y_{is}^1} - \lambda_{is}^1 p_{ys}^1 = 0 \qquad \forall i, s \qquad (57)$$

$$\delta_i \alpha_{is} \alpha_{ir} \frac{\partial U^i}{\partial y_{irs}^2} - \lambda_{irs}^2 p_{yrs}^2 = 0 \qquad \forall i, r, s \qquad (58)$$

$$-w_0 + \gamma_1^j \frac{\partial \hat{x}}{\partial L_{\hat{x}}^j} = 0 \qquad \forall j = 1, \ldots, m \qquad (59)$$

$$-w_0 + \gamma_2^j \frac{\partial F}{\partial L_F^j} = 0 \qquad \forall j = 1, \ldots, m \qquad (60)$$

$$\alpha_{xs}^j \phi_x^{j'} p_{xs}^1 - \lambda_{1s}^{1j} = 0 \qquad \forall s;\ j = 1, \ldots, m \qquad (61)$$

$$\delta_j \alpha^j_{xs} \alpha^j_{xr} \phi^{j'}_x p^2_{xrs} - \lambda^{2j}_{1rs} = 0 \qquad \forall r, s; j = 1, \ldots, m \qquad (62)$$

$$-\alpha^j_{xs} \phi^{j'}_x w^1_s + \frac{\partial x}{\partial L^{1j}_{xs}} (\lambda^{1j}_{1s} - \lambda^{1j}_{2s} - \lambda^{1j}_{3s}) = 0 \qquad \forall s; j = 1, \ldots, m \qquad (63)$$

$$-\delta_j \alpha^j_{xs} \alpha^j_{xr} w^2_{rs} + \frac{\partial x}{\partial L^{2j}_{xrs}} (\lambda^{2j}_{1rs} - \lambda^{2j}_{2rs} - \lambda^{2j}_{3rs}) = 0 \quad \forall r, s; j = 1, \ldots, m \quad (64)$$

$$-q^0_s + \sum_{p=1}^{P} \alpha^j_{xp} \phi^{j'}_x q^1_{sp} = 0 \qquad \forall s; j = 1, \ldots, m \qquad (65)$$

$$-\alpha^j_{xs} [\phi^{j'}_x q^1_{rs} - \delta_j \alpha^j_{xr} \phi^{j'}_x q^2_{rs}] = 0 \qquad \forall r \neq s; j = 1, \ldots, m \qquad (66)$$

$$-\alpha^j_{xs} [\phi^{j'}_x q^1_{ss} - \delta_j \alpha^j_{xs} \phi^{j'}_x q^2_{ss}] + \lambda^{1j}_{3s} = 0 \qquad \forall s; j = 1, \ldots, m \qquad (67)$$

$$-\delta_j \alpha^j_{xs} \alpha^j_{xr} \phi^{j'}_x q^2_{rs} + \lambda^{2j}_{3rs} = 0 \qquad \forall r, s; j = 1, \ldots, m \qquad (68)$$

$$\sum_{s \ni x(L^{1j}_{xs}) = \hat{x}^j} \lambda^{1j}_{2s} + \sum_{r, s \ni x(L^{2j}_{xrs}) = \hat{x}^j} \lambda^{2j}_{2rs} - \gamma^j_1 = 0 \qquad (69)$$

$$j = 1, \ldots, m$$

$$-\lambda^{1j}_{3s} + \sum_{r, s \ni x(L^{2j}_{xrs}) = X^{1j}_s (1 - \theta) + r^{2j}_{rs}} \lambda^{2j}_{3rs} = 0 \qquad (70)$$

$$\forall s; j = 1, \ldots, m$$

$$\sum_{s \ni X^{1j}_s = F^j} \lambda^{1j}_{4s} - \gamma^j_2 = 0 \qquad j = 1, \ldots, m \qquad (71)$$

$$\alpha^j_{ys}\phi^{j'}_y p^1_{ys} - \lambda^{1j}_{5s} = 0 \qquad \forall s;\ j = m+1, \ldots, m+k \qquad (72)$$

$$\delta_j \alpha^j_{ys}\alpha^j_{yr}\phi^{j'}_y p^2_{yrs} - \lambda^{2j}_{5rs} = 0 \qquad \forall s;\ j = m+1, \ldots, m+k \qquad (73)$$

$$-\alpha^j_{ys}\phi^{j'}_y w^1_s + \lambda^{1j}_{5s}\frac{\partial y}{\partial L^{1j}_{ys}} = 0 \qquad \forall s;\ j = m+1, \ldots, m+k \qquad (74)$$

$$-\delta_j \alpha^j_{ys}\alpha^j_{yr}\phi^{j'}_y w^2_{rs} + \lambda^{2j}_{5rs}\frac{\partial y}{\partial L^2_{yrs}} = 0 \qquad \forall r, s;\ j = m+1, \ldots, m+k \qquad (75)$$

If we compare these equations with equations (25)-(51) which we know describe an ex-ante Pareto optimum, it is clear that the same difficulties arise in this problem as arise in the problem in the previous paper. In particular, equations (69)-(71), which describe a firm's capacity choice and inventory policy, ensure that when equations (61) and (62) are evaluated firms owning capacity and inventory will evaluate expected prices and returns for states of the world such that capacity and storage constraints are binding on the basis of their own discount rates, utility functions, and subjective probability distributions. An optimal allocation uses consumer discounted expected utilities to evaluate the returns from capacity choices and inventory policies. On the other hand, if firms are risk neutral, consumers and firms owning capacity have the same discount rates and subjective probability distributions, $\lambda^1_{is} = C_i\alpha_{is}\ \forall i, s$ and $\lambda^2_{irs} = \delta_i C_i \alpha_{is}\alpha_{ir}$

$\forall i$, r, s (i.e. expected marginal utilities of income are discounted constant functions of consumer conditional subjective probabilities), firms and consumers will agree about the expected utility of different capacity and inventory policies. To prove this statement let $\lambda_i = C_i$, $\lambda^1_{is} = C_i\alpha_s$ $\forall i, s$, $\lambda^2_{irs} = \delta C_i\alpha_s\alpha_r$ $\forall i, r, s$, $\phi^{j'}_x = C_j$, $j = 1, \ldots, m$; $\alpha_{is} = \alpha^j_{xs} = \alpha_s$ $\forall i, s; j = 1, \ldots, m$; and $\delta_i = \delta_j = \delta$ $\forall i, j = 1, \ldots, m$. The first order conditions (52)-(68) which have consumer and x firm probability distributions, now can be rewritten as follows.

$$\frac{\partial U^i}{\partial L_{i0}} + C_i w_0 = 0 \qquad \forall i \tag{76}$$

$$\alpha_s \frac{\partial U^i}{\partial L^1_{is}} + C_i\alpha_s w^1_s = 0 \qquad \forall i, s \tag{77}$$

$$\delta\alpha_s\alpha_r \frac{\partial U^i}{\partial L^2_{irs}} + \delta C_i\alpha_s\alpha_r w^2_{rs} = 0 \qquad \forall i, r, s \tag{78}$$

$$\alpha_s \frac{\partial U^i}{\partial x^1_{is}} - C_i\alpha_s p^1_{xs} = 0 \qquad \forall i, s \tag{79}$$

$$\delta\alpha_s\alpha_r \frac{\partial U^i}{\partial x^2_{irs}} - \delta C_i\alpha_s\alpha_r p^2_{xrs} = 0 \qquad \forall i, r, s \tag{80}$$

$$\alpha_s \frac{\partial U^i}{\partial y^1_{is}} - C_i\alpha_s p^1_{ys} = 0 \qquad \forall i, s \tag{81}$$

$$\delta\alpha_s\alpha_r \frac{\partial U^i}{\partial y^2_{irs}} - \delta C_i\alpha_s\alpha_r p^2_{yrs} = 0 \qquad \forall i,\ r,\ s \tag{82}$$

$$-w_0 + \gamma^j_1 \frac{\partial \hat{x}}{\partial L^j_{\hat{x}}} = 0 \qquad j = 1, \ldots, m \tag{83}$$

$$-w_0 + \gamma^j_2 \frac{\partial F}{\partial L^j_F} = 0 \qquad j = 1, \ldots, m \tag{84}$$

$$\alpha_s p^1_{xs} - \lambda^{1j}_{1s} = 0 \qquad \forall s;\ j = 1, \ldots, m \tag{85}$$

$$\delta\alpha_s\alpha_r p^2_{xrs} - \lambda^{2j}_{1rs} = 0 \qquad \forall r,\ s;\ j = 1, \ldots, m \tag{86}$$

$$-\alpha_s w^1_s + \frac{\partial x}{\partial L^{1j}_{xs}} (\lambda^{1j}_{1s} - \lambda^{1j}_{2s} - \lambda^{1j}_{3s}) = 0 \qquad \forall s;\ j = 1, \ldots, m \tag{87}$$

$$-\delta\alpha_s\alpha_r w^2_{rs} + \frac{\partial x}{\partial L^{2j}_{xrs}} (\lambda^{2j}_{1rs} - \lambda^{2j}_{2rs} - \lambda^{2j}_{3rs}) = 0 \qquad \forall r,\ s;\ j = 1, \ldots, m \tag{88}$$

$$-q^0_s + \sum_{p=1}^{P} \alpha_p q^1_{sp} = 0 \qquad \forall s \tag{89}$$

$$-\alpha_s[q^1_{rs} - \delta\alpha_r q^2_{rs}] = 0 \qquad \forall r \neq s \tag{90}$$

$$-\alpha_s[q^1_{ss} - \delta\alpha_s q^2_{ss}] + \lambda^{1j}_{3s} = 0 \qquad \forall s;\ j = 1, \ldots, m \tag{91}$$

$$-\delta\alpha_s\alpha_r q^2_{rs} + \lambda^{2j}_{3rs} = 0 \qquad \forall r,\ s;\ j = 1, \ldots, m \tag{92}$$

Equations (76)-(92) together with equations (69)-(75), reduce to conditions which are equivalent to the ex-ante Pareto optimal conditions. This result can be summarized as follows.

Proposition 2: If delivery and storage capacity have to be chosen before the random variable is observed, the planned life of those facilities exceeds one year, and firms hold title to delivery, storage capacity, and inventory, then the following conditions are sufficient to guarantee an ex-ante optimal choice of capacity.

1. Consumers expected marginal utilities of income are discounted constant functions of their conditional subjective probability distributions.

2. Consumers and firms owning capacity have identical probability distributions and discount rates.

3. Firms owning capacity are risk neutral.

CONCLUSION

This paper has extended the analysis of the welfare effects of supply rigidities to the case of a storable good subject to random supply. The important result of this analysis is that the ownership of rights affects ex-ante resource allocation not only when firms own capacity, but when they have to hold inventory as well. This suggests that any risk-taking by firms makes it difficult to achieve an ex-ante Pareto optimal allocation of resources.

FOOTNOTES

1. Note that consumers can only rent out delivery capacity in period 1. This restriction is imposed to keep this model consistent with the welfare model since the welfare model does not allow any adjustments in capacity after the observation of the random variable in period 1. A more general model would allow delivery capacity to be changed in period 1 for use in period 2. The results presented in this paper generalize to that model.

CHAPTER VI

OPTIMAL RESOURCE ALLOCATION UNDER APPROPRIATIVE WATER RIGHTS

INTRODUCTION

The previous three papers have developed conditions which allow competitive markets to make ex-ante Pareto optimal choices of capacity and inventory policies for a storable good subject to random supply. The result was that, under the normal condition that firms own storage and delivery capacity and make inventory decisions, sufficient conditions for optimality were quite strict. In addition to the general sufficiency condition whenever there are ex-post spot markets (namely that consumers have constant marginal utilities of income such that expected marginal utilities of income are those constants, discounted, times the subjective probabilities), sufficient conditions included identical discount rates and subjective probability distributions among consumers and firms owning capacity and risk neutrality for firms owning capacity.

This paper analyzes the allocation of one such good, water. The focus is on the allocation of Colorado River Water. The object is to determine whether it is possible under the current structure of property rights in water to reach an optimal allocation.

In brief, the current system of appropriative water rights works as follows. Each firm has a legal right to a certain flow of water per year, subject to the constraint that senior claimants have absolute priority over the available flow. The water right is obtained by diverting water and putting it to "beneficial consumptive use." Seniority (priority) is based on the chronological order of appropriation: "first in time means first in right."

Under current Bureau of Reclamation policy, water rights can not be sold by firms owing money to the Bureau of Reclamation for capital projects, such as diversion works. This creates inefficiencies both because prices do not reflect the scarcity value of water and because risks are shared unequally among appropriators.[1] The question is whether the introduction of a market in state dependent percentage shares of appropriative water rights can correct these inefficiencies if the general conditions for optimality under firm ownership of rights are already satisfied.

REVIEW OF APPROPRIATE OPTIMALITY CONDITIONS

Recall from the previous paper that an ex-ante optimal allocation of water can be described by solving the following welfare maximization problem.

$$\text{Max } W = \sum_{i=1}^{n} \beta_i \{U^i(L_{i0}) + \sum_{s=1}^{S} \alpha_{is}[U^i(x^1_{is}, y^1_{is}, L^1_{is}) + \sum_{r=1}^{R} \alpha_{ir} U^i(x^2_{irs}, y^2_{irs}, L^2_{irs})]\}$$

$$\text{s.t.} \quad F = F(L_F)$$

$$\hat{x} = \hat{x}(L_{\hat{x}})$$

$$\sum_{i=1}^{n} x_{is}^{1} \le x(L_{xs}^{1}) \qquad \forall s$$

$$\sum_{i=1}^{n} x_{irs}^{2} \le x(L_{xrs}^{2}) \qquad \forall r, s$$

$$x(L_{xs}^{1}) \le z_s^1 - X_s^1 \qquad \forall s$$

$$x(L_{xrs}^{2}) \le X_s^1(1 - \theta) + z_r^2 \qquad \forall s, r$$

$$x(L_{xs}^{1}) \le \hat{x} \qquad \forall s$$

$$x(L_{xrs}^{2}) \le \hat{x} \qquad \forall r, s$$

$$X_s^1 \le F \qquad \forall s$$

$$\sum_{i=1}^{n} y_{is}^{1} \le y(L_{ys}^{1}) \qquad \forall s$$

$$\sum_{i=1}^{n} y_{irs}^{2} \le y(L_{yrs}^{2}) \qquad \forall r, s$$

$$L_F + L_{\hat{x}} \le \sum_{i=1}^{n} L_{i0}$$

$$L^1_{xs} + L^1_{ys} \leq \sum_{i=1}^{n} L^1_{is} \qquad \forall s$$

$$L^2_{xrs} + L^2_{yrs} \leq \sum_{i=1}^{n} L^2_{irs} \qquad \forall r,\ s$$

where:

β_i = i's social welfare weight;

U^i = i's neo-classical utility function;

L_{i0} = labor supplied by i in period 0;

δ_i = i's discount rate;

α_{is}, α_{ir} = i's subjective probability of being in states s and r;

x^1_{is}, x^2_{irs} = i's consumption of water in period 1 state s, and period 2, state r, given s;

y^1_{is}, y^2_{irs} = i's consumption of a composite good in period 1 state s, and period 2, state r, given s;

$L^1_{is}\ L^2_{irs}$ = labor supplied by i period 1 in state s, and period 2, state r, given s;

F = size of water storage capacity

$F(L_F)$ = production function for storage capacity;

L_F = labor devoted to building storage capacity;

$\hat{x}$ = size of water delivery capacity;

$\hat{x}(L_{\hat{x}})$ = production function for delivery capacity;

$L_{\hat{x}}$ = labor devoted to building delivery capacity;

$x(L^1_{xs}), x(L^2_{xrs})$ = production function for delivery of water in period 1, state s, and period 2, state r, given s;

L^1_{xs}, L^2_{xrs} = labor devoted to delivery of water in period 1, state s, and period 2, state r, given s;

z^1_s, z^2_r = random supply of water available in period 1, state s, and period 2, state r;

X^1_s = amount of water stored after period 1, given state s in period 1.

$y(L^1_{ys}), y(L^2_{yrs})$ = production function for the composite good in period 1, state s, and period 2, state r, given s;

L^1_{ys}, L^2_{yrs} = labor devoted to the production of the composite good in period 1, state s, and period 2, state r, given s; and

θ = rate of evaporation loss.[2]

Since we are assuming for the purposes of this paper that all the general conditions for ex-ante optimality are satisfied by this market, we need only focus on the first order conditions which describe the allocation of water rights when capacity constraints are not binding. The appropriate condition is case 5 from the previous paper. This is a situation such that all water flowing down the river in the two time periods is used, but neither the delivery capacity nor the storage capacity is fully utilized. Assuming consumers and firms delivering water have identical discount rates and subjective probability distributions, the ex-ante optimality condition is as follows:

$$\frac{\alpha_s \dfrac{\partial U^i}{\partial x^1_{is}} - (1-\theta) \displaystyle\sum_{r \ni x(L^2_{xrs}) = X^1_s(1-\theta) + z^2_r} \alpha_s \alpha_r \dfrac{\partial U^i}{\partial x^2_{irs}}}{\left[\text{either } \alpha_p \dfrac{\partial U^i}{\partial y^1_{ip}} \text{ or } \delta \alpha_p \alpha_q \dfrac{\partial U^i}{\partial y^2_{iqp}}\right]} =$$

$$\frac{-\beta_i \alpha_s \dfrac{\partial U^i / \partial L^1_{is}}{\partial x / \partial L^1_{xs}} + \beta_i (1-\theta) \displaystyle\sum_{r \ni x(L^2_{xrs}) = X^1_s(1-\theta) + z^2_r} \alpha_s \alpha_r \dfrac{\partial U^i / \partial L^2_{irs}}{\partial x / \partial L^2_{xrs}}}{\left[\text{either } -\beta_i \alpha_p \dfrac{\partial U^i / \partial L^1_{ip}}{\partial y / \partial L^1_{yp}} \text{ or } -\beta_i \delta \alpha_p \alpha_q \dfrac{\partial U^i / \partial L^2_{ipq}}{\partial y / \partial L^2_{ypq}}\right]}$$

$$\forall i, p, q \quad (1)$$

ALLOCATION UNDER APPROPRIATIVE WATER RIGHTS

The current structure of appropriative water rights is clearly inefficient. First, the restriction on the sale of rights does not allow a means of equilibrating the marginal scarcity value of water across appropriators. Further the distinction between senior and junior appropriators creates monopoly rights in segments of the probability distribution. If sale of rights is prohibited, this situation results in an unequal sharing of risk among otherwise identical firms. Even if rights could be sold, some sort of monopoly power would remain as long as rights were defined over segments of the probability distribution. This problem would persist because prices for water rights would reflect differing probabilities of receiving water. Therefore, a particular priority water right would be a unique commodity owned by one firm.

If institutions for pooling risk could be developed at the same time as a market in water rights, however, it might be possible for prices of marginal water rights to reflect marginal scarcity values even though lump sum monopoly profits would be earned during the first round of trading. Consider the following institutional arrangement. Suppose firms delivering water could purchase percentage shares of other firms' appropriative rights and the shares could vary according to what state of the world occurred. Now, if we number states of the world in order of increasing river flows and segment the probability distribution into $m+1$ blocks, each of the first m corresponding to a firm's original appropriative right and the $m+1^{st}$ being the unappropriated portion of the distribution, the decision problem under appropriative rights of the j^{th} firm delivering water can be stated similarly to that under state-dependent rights. Assuming risk neutrality and a known discount rate and probability distribution, firm x^j's problem is:

$$\text{Max } -w_0[L^j_{\hat{x}} + L^j_F] - \sum_{\ell=1}^{m} \sum_{s=s_{\ell-1}}^{s_\ell} [\beta^{0j}_{\ell s} - \bar{\beta}^j_{\ell s}]R_\ell q^0_{\ell s}] +$$

$$\sum_{\ell=1}^{m+1} \sum_{s=s_{\ell-1}}^{s_\ell} \alpha_s\{p^1_{xs}x^{1j}_s - w^1_s L^{1j}_{xs} - \sum_{k=1}^{m} \sum_{r=r_{k-1}}^{r_k} [\beta^{1j}_{krs} - \beta^{0j}_{kr}]R_k q^1_{krs} +$$

$$\delta \sum_{q=1}^{m+1} \sum_{r=r_{q-1}}^{r_q} \alpha_r[p^2_{xrs}x^{2j}_{rs} - w^2_{rs} L^{2j}_{xrs} - \sum_{k=1}^{q\neq m+1} [\beta^{2j}_{krs} - \beta^{1j}_{krs}]R_k q^2_{krs}]\} \quad (2)$$

<u>Multipliers</u>

s.t. $x^{1j}_s \leq x(L^{1j}_{xs})$	$\forall$ s	λ^{1j}_{1s}
$x^{2j}_{rs} \leq x(L^{2j}_{xrs})$	$\forall$ r, s	λ^{2j}_{1rs}
$x(L^{1j}_{xs}) \leq \hat{x}^j$	$\forall$s	λ^{1j}_{2s}
$x(L^{2j}_{xrs}) \leq \hat{x}^j$	$\forall$r, s	λ^{2j}_{2s}

Multipliers

$$x(L^{1j}_{xs_\ell}) \leq \sum_{k=1}^{\ell} \beta^{1j}_{kss} R_k - X^{1j}_s \qquad \forall s \qquad \lambda^{1j}_{3s}$$

$$x(L^{2j}_{xr_q s}) \leq X^{1j}_s(1-\theta) + \sum_{k=1}^{q} \beta^{2j}_{kr_q s} R_k \qquad \forall r, s \qquad \lambda^{2j}_{3rs}$$

$$X^{1j}_s \leq F^j \qquad \forall s \qquad \lambda^{1j}_{4s}$$

$$\hat{x}^j = \hat{x}(L^j_{\hat{x}}) \qquad \gamma^j_1$$

$$F^j = F(L^j_F) \qquad \gamma^j_2$$

where:

$[s_{\ell-1}, s_\ell)$ = interval in the cumulative probability distribution encompassing firm ℓ's original appropriative rights;

$\bar{\beta}^j_{kr}, \beta^{0j}_{kr}, \beta^{1j}_{krs}, \beta^{2j}_{krs}$ = state r dependent percentage of firm k's original appropriative right owned by j as an initial endowment, in period 0, in period 1, given s in period 1, and in period 2, given s in period 1;

R_k = acre-feet of water in firm k's original appropriative right;

$q^0_{kr}, q^1_{krs}, q^2_{krs}$ = state r dependent price per acre-foot for water from k's original appropriative right in period 0, period 1, state s and period 2, given s in period 1.

Equation (2) essentially says that if state of the world s occurs it implies that priority rights 1 to $\ell - 1$ are satisfied and ℓ can be at least partially satisfied. In order to deliver x^{1j}_s water in state s and store X^{1j}_s, firm j has to amass enough shares in rights $1 - \ell$: i.e. shares such that $x^{1j}_{s_\ell} + X^{1j}_{s_\ell} \leq \sum_{k=1}^{\ell} \beta^{1j}_{ks_\ell s} R_k$.

Turning now to consumers and firms producing the composite good, the remainder of the competitive market can be described as follows:

$$\text{Consumer:} \quad \text{Max } U^i(L_{i0}) + \sum_{s=1}^{S} \alpha_s [U^i(x^1_{is}, y^1_{is}, L^1_{is}) +$$

$$\delta \sum_{r=1}^{R} \alpha_r U^i(x^2_{irs}, y^2_{irs}, L^2_{irs})]$$

<u>Multipliers</u>

$$\text{s.t.} \quad M_{i0} \leq w_0 L_{i0} \qquad \lambda^0_i$$

$$p^1_{xs} x^1_{is} + p^1_{ys} y^1_{is} \leq w^1_s L^1_{is} + \sum_{j=1}^{m} \theta^{ij}_{xs} \pi^{1j}_{xs} +$$

$$\sum_{j=m+1}^{m+k} \theta^{ij}_{ys} \pi^{1j}_{ys} + M_{i0} \qquad \forall s \qquad \lambda^1_{is}$$

Multipliers

$$p^2_{xrs}x^2_{irs} + p^2_{yrs}y^2_{irs} \leq w^2_{rs}L^2_{irs} + \sum_{j=1}^{m} \theta^{ij}_{xr}\pi^{2j}_{xrs} +$$

$$\sum_{j=m+1}^{m+k} \theta^{ij}_{yr}\pi^{2j}_{yrs} \qquad \forall r, s \qquad \lambda^2_{irs}$$

Firm y^j: $$\text{Max} \sum_{s=1}^{S} \alpha^j_{ys}\{\phi^j_y(p^1_{ys}y^{1j}_s - w^1_s L^{1j}_{ys}) +$$

$$\delta_j \sum_{r=1}^{R} \alpha^j_{yr}[\phi^j_y(p^2_{yrs}y^{2j}_{rs} - w^2_{rs}L^{2j}_{yrs})]\}$$

$$\text{s.t.} \qquad y^{1j}_s \leq y(L^{1j}_{ys}) \qquad \forall s \qquad \lambda^{1j}_{5s}$$

$$y^{2j}_{rs} \leq y(L^{2j}_{yrs}) \qquad \forall r, s \qquad \lambda^{2j}_{5rs}$$

Where:

M_{i0} = Income earned by i in period 0;

θ^{ij}_{xs} = i's percentage of x^j's profits in state s;

$\pi^{1j}_{xs}, \pi^{2j}_{xrs}$ = x^j's profits in period 1, state s, and period 2, state r, given s;

θ^{ij}_{ys} = i's percentage of y^j's profits in s;

$\pi^{1j}_{ys}, \pi^{2j}_{yrs}$ = y^j's profits in period 1, state s, and period 2, state r, given s;

α^j_{ys} = y^j's subjective probability of being in s; and

ϕ^j_y = y^j's differentiable utility function over profits.

The equilibrium conditions are:

$$\sum_{i=1}^{n} x^1_{is} = \sum_{j=1}^{m} x^{1j}_s, \quad \sum_{i=1}^{n} x^2_{irs} = \sum_{j=1}^{m} x^{2j}_{rs} \qquad \forall r, s$$

$$\sum_{i=1}^{n} y^1_{is} = \sum_{j=m+1}^{m+k} y^{1j}_s, \quad \sum_{1=1}^{n} y^2_{irs} = \sum_{j=m+1}^{m+k} y^{2j}_{rs} \qquad \forall r, s$$

$$\sum_{j=1}^{m} (L^j_{\hat{x}} + L^j_F) = \sum_{i=1}^{n} L_{i0}$$

$$\sum_{j=1}^{m} L^{1j}_{xs} + \sum_{j=m+1}^{m+k} L^{1j}_{ys} = \sum_{i=1}^{n} L^1_{is}, \quad \sum_{j=1}^{m} L^{2j}_{xrs} + \sum_{j=m+1}^{m+k} L^{2j}_{yrs} = \sum_{i=1}^{n} L^2_{irs} \qquad \forall r, s$$

$$\sum_{i=1}^{n} \theta^{ij}_{xs} = 1 \qquad \forall s$$

$$\sum_{i=1}^{n} \theta^{ij}_{ys} = 1 \qquad \forall s$$

$$\sum_{j=1}^{m} \bar{\beta}^j_{kr} = \sum_{j=1}^{m} \beta^{0j}_{kr} = \sum_{j=1}^{m} \beta^{1j}_{krs} = \sum_{j=1}^{m} \beta^{2j}_{krs} = 1 \qquad \forall k, r, s$$

Forming Lagrange functions $L_i (i = 1, \ldots, n)$; $L_x^j (j = 1, \ldots, m)$; and $L_y^j (j = m+1, \ldots, m+k)$ and differentiating, the first order conditions for this problem are:

$$\frac{\partial U^i}{\partial L_{i0}} + \lambda_i^0 w_0 = 0 \qquad \forall\, i \qquad (3)$$

$$\alpha_s \frac{\partial U^i}{\partial x_{is}^1} - \lambda_{is}^1 p_{xs}^1 = 0 \qquad \forall\, i,\ s \qquad (4)$$

$$\delta\alpha_s\alpha_r \frac{\partial U^i}{\partial x_{irs}^2} - \lambda_{irs}^2 p_{xrs}^2 = 0 \qquad \forall\, i,\ r,\ s \qquad (5)$$

$$\alpha_s \frac{\partial U^i}{\partial y_{is}^1} - \lambda_{is}^1 p_{ys}^1 = 0 \qquad \forall\, i,\ s \qquad (6)$$

$$\delta\alpha_s\alpha_r \frac{\partial U^i}{\partial y_{irs}^2} - \lambda_{irs}^2 p_{yrs}^2 = 0 \qquad \forall\, i,\ r,\ s \qquad (7)$$

$$\alpha_s \frac{\partial U^i}{\partial L_{is}^1} + \lambda_{is}^1 w_s^1 = 0 \qquad \forall\, i,\ s \qquad (8)$$

$$\delta\alpha_s\alpha_r \frac{\partial U^i}{\partial L_{irs}^2} + \lambda_{irs}^2 w_{rs}^2 = 0 \qquad \forall\, i,\ r,\ s \qquad (9)$$

$$-w_0 + \gamma_1^j \frac{\partial \hat{x}}{\partial L_{\hat{x}}^j} = 0 \qquad j = 1, \ldots, m \qquad (10)$$

$$-w_0 + \gamma_2^j \frac{\partial F}{\partial L_F^j} = 0 \qquad j = 1; \ldots, m \qquad (11)$$

$$\alpha_s p_{xs}^1 - \lambda_{1s}^{1j} = 0 \qquad \forall s;\ j = 1, \ldots, m \qquad (12)$$

$$\delta\alpha_s \alpha_r p_{xrs}^2 - \lambda_{1rs}^{2j} = 0 \qquad \forall r, s;\ j = 1, \ldots, m \qquad (13)$$

$$-\underset{s}{\alpha} w_s + \frac{\partial x}{\partial L_{xs}^{1j}} (\lambda_{1s}^{1j} - \lambda_{2s}^{1j} - \lambda_{3s}^{1j}) = 0 \qquad \forall s;\ j = 1, \ldots, m \qquad (14)$$

$$-\delta\alpha_s \alpha_r w_{rs}^2 + \frac{\partial x}{\partial L_{xrs}^{2j}} (\lambda_{1rs}^{2j} - \lambda_{2rs}^{2j} - \lambda_{3rs}^{2j}) = 0 \qquad \forall r, s;\ j = 1, \ldots, m \qquad (15)$$

$$-R_k q_{kr}^0 + \sum_{\ell=1}^{m} \sum_{s=s_{\ell-1}}^{s\ell} \alpha_s R_k q_{krs}^1 = 0 \qquad \forall k, r \qquad (16)$$

$$-\alpha_s R_k q_{krs}^1 + \delta\alpha_s \alpha_r R_k q_{krs}^2 = 0 \qquad \forall k, r \neq s \qquad (17)$$

$$-\alpha_s R_k q_{kss}^1 + \delta\alpha_s \alpha_s R_k q_{kss}^2 + R_k \lambda_{3s}^{1j} = 0 \qquad \forall k, s;\ j = 1, \ldots, m \qquad (18)$$

$$-\delta\alpha_s \alpha_r R_k q_{krs}^2 + R_k \lambda_{3rs}^{2j} = 0 \qquad \forall k, r, s; j=1, \ldots, m \qquad (19)$$

$$\sum_{s \ni x(L_{xs}^{1j}) = \hat{x}^j} \lambda_{2s}^{1j} + \sum_{s, r \ni x(L_{xrs}^{2j}) = \hat{x}^j} \lambda_{2rs}^{2j} - \gamma_1^j = 0 \qquad j = 1, \ldots, m \qquad (20)$$

$$-\lambda_{3s}^{1j} + (1-\theta) \sum_{\left\{ \begin{array}{c} r \ni x(L_{xr_q s}^{2j}) = X_s^{1j}(1-\theta) + \\ \sum_{k=1}^{q} \beta_{krs}^{2j} R_k \end{array} \right\}} \lambda_{3rs}^{2j} - \lambda_{4s}^{1j} = 0 \qquad \forall s;\ j = 1, \ldots, m \qquad (21)$$

$$\sum_{s \ni X_s^{1j} = F^j} \lambda_{4s}^{1j} - \gamma_2^j = 0 \qquad j = 1, \ldots, m \tag{22}$$

Consider first equations (18) and (19). Rearranging terms:

$$\lambda_{3s}^{1j} = \alpha_s [q_{kss}^1 + \delta\alpha_s q_{kss}^2] \qquad \forall k, j = 1, \ldots, m \tag{23}$$

$$\lambda_{3rs}^{2j} = \delta\alpha_s \alpha_r q_{krs}^2 \qquad \forall k, j = 1, \ldots, m \tag{24}$$

(23) and (24) imply that given a state of the world s, or r, given s, all water rights which can be satisfied will be priced at the same marginal cost. Thus, despite possible early monopoly profits, a competitive shares market will develop.

Now, if we assume that $\lambda_{is}^1 = C_i \alpha_{is}$ $\forall i$, s and $\lambda_{irs}^2 = \delta C_i \alpha_{is} \alpha_{ir}$ $\forall i$, r, s and that capacity constraints are not binding, equations (4)-(9), (12)-(15), (21) and (23)-(24) reduce to:

$$\frac{\alpha_s \dfrac{\partial U^i}{\partial x_{is}^1} - \delta(1-\theta) \displaystyle\sum_{r \ni x(L_{xr_q s}^{2j}) = X_s^{1j}(1-\theta) + \sum_{k=1}^{q} \alpha_s \beta_{kr_q s}^{2j} R_k} \alpha_r \dfrac{\partial U^i}{\partial x_{ir_q s}^2}}{[\text{either } \alpha_p \dfrac{\partial U^i}{\partial y_{ip}^1} \text{ or } \delta\alpha_p \alpha_v \dfrac{\partial U^i}{\partial y_{ipv}^2}]} =$$

$$\frac{\alpha_s \dfrac{\partial U^i / \partial L_{is}^1}{\partial x / \partial L_{xs}^1} - \delta(1-\theta) \displaystyle\sum_{r \ni x(L_{xr_q s}^{2j}) = X_s^{1j}(1-\theta) + \sum_{k=1}^{q} \alpha_s \beta_{kr_q s}^{2j} R_k} \alpha_r \dfrac{\partial U^i / \partial L_{ir_q s}^2}{\partial x / \partial L_{xr_q s}^{2j}}}{[\text{either } \alpha_p \dfrac{\partial U^i / \partial L_{ip}^1}{\partial y / \partial L_{yp}^{1\ell}} \text{ or } \delta\alpha_p \alpha_v \dfrac{\partial U^i / \partial L_{ipv}^2}{\partial y / \partial L_{ypv}^{2\ell}}]}$$

$$\forall i, j, \ell, p, v \tag{25}$$

Since (25) is the same as (1), given $\sum_{j=1}^{m} \beta_{krs}^{2j} = 1$, the resulting allocation is ex-ante Pareto optimal.

CONCLUSION

This paper has analyzed the effect of opening a market in percentage shares of appropriative water rights, assuming that the general conditions for ex-ante Pareto optimality are already satisfied. Depending on one's point of view the results could be seen as either encouraging or discouraging about the possibilities for improving the efficiency of the market for Colorado River water. On the one hand, it is reasonable to expect that a percentage share market in water rights might develop if the prohibition against sale were lifted. However, a state dependent share market is perhaps more unlikely because of the information costs involved. On the other hand, the general optimality conditions are quite restrictive and competition among water companies is very limited because of the locationally fixed nature of distribution systems. Consumers can only change water companies by moving and prices are regulated by city governments or public utilities commissions. Despite these caveats, however, the results presented in this paper do suggest that the appropriative rights system can at least be made a better allocative mechanism with little tampering with existing legal structures.

FOOTNOTES

1. H. S. Burness and J. P. Quirk, "Appropriative Water Rights and the Efficient Allocation of Resources," Social Science Working Paper No. 157, California Institute of Technology, Revised March 1978. Forthcoming in The American Economic Review. "Appropriative Water Rights and the Theory of the Dam," December 1977. To appear in Festshrift for E. T. Weiler, published by Purdue University Press, 1979.

2. We recognize that this model does not exactly describe the allocation of water since it is both a consumption good and an intermediate good. However, even if another final good which used water were included, the marginal conditions describing the allocation of water for final consumption would not change. To prove this statement, consider the following simple welfare model with one consumer:

$$\text{Max } U(x_c, y, z, L)$$

	Multipliers
s.t. $x = x(L_x)$	λ_1
$y = y(L_y)$	λ_2
$z = z(x_z)$	λ_3
$x_c + x_z \leq x$	λ_4
$L_x + L_y \leq L$	λ_5

The first order conditions are:

$$\frac{\partial U}{\partial x_c} - \lambda_4 = 0 \qquad (1)$$

$$\frac{\partial U}{\partial y} - \lambda_2 = 0 \qquad (2)$$

$$\frac{\partial U}{\partial z} - \lambda_3 = 0 \tag{3}$$

$$\frac{\partial U}{\partial L} + \lambda_5 = 0 \tag{4}$$

$$\lambda_1 \frac{\partial x}{\partial L_x} - \lambda_5 = 0 \tag{5}$$

$$\lambda_2 \frac{\partial y}{\partial L_y} - \lambda_5 = 0 \tag{6}$$

$$-\lambda_1 + \lambda_4 = 0 \tag{7}$$

$$\lambda_3 \frac{\partial z}{\partial x_z} - \lambda_4 = 0 \tag{8}$$

These reduce to:

$$\frac{\frac{\partial U}{\partial x_c}}{\frac{\partial U}{\partial y}} = \frac{\frac{\partial y}{\partial L_y}}{\frac{\partial x}{\partial L_x}} = \frac{\frac{\partial U}{\partial z}\,\frac{\partial z}{\partial x_z}}{\frac{\partial U}{\partial y}} \tag{9}$$

It is clear from (9) that the addition of good z has not affected the marginal rate of substitution or the price ratio between x and y.