Three-Dimensional Superconformal Field Theory, Chern-Simons Theory, and Their Correspondence

Thesis by

Hee Joong Chung

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Abstract

In this thesis, we discuss 3d-3d correspondence between Chern-Simons theory and three-dimensional $\mathcal{N} = 2$ superconformal field theory. In the 3d-3d correspondence proposed by Dimofte-Gaiotto-Gukov information of abelian flat connection in Chern-Simons theory was not captured. However, considering M-theory configuration giving the 3d-3d correspondence and also other several developments, the abelian flat connection should be taken into account in 3d-3d correspondence. With help of the homological knot invariants, we construct 3d $\mathcal{N} = 2$ theories on knot complement in 3-sphere S^3 for several simple knots. Previous theories obtained by Dimofte-Gaiotto-Gukov can be obtained by Higgsing of the full theories. We also discuss the importance of all flat connections in the 3d-3d correspondence by considering boundary conditions in 3d $\mathcal{N} = 2$ theories and 3-manifold.

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Contents

Chapter 1

Introduction

In M-theory, there are M2-brane and M5-brane. The worldvolume theory of multiplet M2-branes is relatively well-understood by Bagger-Lambert and Aharony-Bergman-Jafferis-Maldacena. However, the worldvolume theory of multiple M5-brane is still very mysterious. From the string/M-theory argument, it is known that there exist 6d $(2,0)$ superconformal field theory, though people don't know a proper description of this theory yet.

Instead, physicsists have tried to compactify 6d $(2,0)$ theory on manifolds whose dimension is less than 6. For example, if one compactifies 6d $(2,0)$ theory on a circle, one obtains 5d maximally supersymmetric field theory. Or if one compactify 6d $(2,0)$ theory on the torus T^2, 4d $\mathcal{N}=4$ superconformal theory is obtained.

This program of compactification on general punctured Riemann surfaces C was pioneered by Gaiotto. He constructed 4d $\mathcal{N}=2$ superconformal field theory labelled by Riemann surface C. Information of Riemann surface is realized in 4d gauge theory side, for example, mapping class group of C is identified as S-duality group of 4d $\mathcal{N}=2$. More surprisingly, it turned out that instanton partition function pioneered by Nekrasov is actually identified with the conformal block of Liouville theory (or more generally Toda theory) on Riemann surface C. This surprising relation was found by Alday-Gaiotto-Tachikawa (AGT) and often called AGT correspondence or 2d-4d correspondence.

Therefore from 2d-4d correspondence, one might expect that compactification of 6d $(2,0)$ theory on a certain d-dimensional manifold M_d would give a certain $6-d$-dimensional superconformal field theory.

About two years later, Dimofte-Gaiotto-Gukov (DGG) showed that it also holds for $d=3$ by using gluing ideal hyperbolic tetrahedra. The 3d-3d correspondence states that when M5 branes are wrapped on a 3-manifold $-$ M_3 $-$ we have 3d $\mathcal{N}=2$ superconformal field theories (SCFT) $-$ $T[M_3]$ $-$ described as the IR fixed points of abelian Chern-Simons-matter theories determined by M_3 and other extra information. At the same time, physical quantities such as partition functions of non-supersymmetric complex Chern-Simons (CS) theory on M_3 are matched with those of 3d $\mathcal{N}=2$ abelian superconformal theory.

Though it seemed successful, actually it was not a complete correspondence as also mentioned in the original papers. In the correspondence of DGG, abelian branch of flat connections in Chern-Simons theory is lost. However, considering M-theory point of view and several developments, the 3d-3d correspondence should capture all branches of flat connections.

In author's work with T. Dimofte, S. Gukov, P. Sułkowski [1], we found 3d $\mathcal{N}=2$ theories corresponding to knot complement in S^3 for several simple knots whose partition functions are equal to the homological knot invariants. Our examples give a full complete 3d-3d correspondence in the sense that they capture all flat connection without anything lost. It is also possible to reproduce theories constructed by DGG by Higgs mechanism from our theories. In addition, we see the importance of abelian flat connections by considering boundaries of 3d $\mathcal{N}=2$ theories and 3-manifold.

This thesis has following structure;

In Chapter 2, we review the basic aspects of 3d $\mathcal{N}=2$ supersymmetric theories with focus on the necessary elements in 3d-3d correspondence.

In Chapter 3, we review the analytic continuation of Chern-Simons theory with focus on the relevant materials for our purpose.

In Chapter 4, we summarize the 3d-3d correspondence by Dimofte-Gaiotto-Gukov obtained by gluing ideal hyperbolic tetrahedra.

In Chapter 5, we engineer the 3d $\mathcal{N}=2$ theories corresponding to a knot complement in S^3 for unknot, trefoil knot, and figure-eight knot. We discuss Higgs mechanism to reproduce knot polynomials and the previous theories of DGG. We also see the importance of including all flat connection in 3d-3d correspondence by considering boundaries of 3d $\mathcal{N}=2$ theories and those of 3-manifold.

Chapter 2

3d $\mathcal{N} = 2$ Superconformal Theories

In this chapter, we review topics in 3d $\mathcal{N} = 2$ theories with focus on the 3d-3d correspondence.

In section 2.1, we summarize several aspects of 3d $\mathcal{N} = 2$ theories including supersymmetry algebra, supersymmetric action, parity anomaly, monopole operators, vortices, and 3d $\mathcal{N} = 2$ mirror symmetry. This section is mostly for setting up basic terminology or background for the rest of thesis.

In section 2.2, we review aspects of compacitification of 3d $\mathcal{N} = 2$ on S^1. The resulting theory becomes effective 2d $\mathcal{N} = (2, 2)$ theory which is characterized by twisted superpotential.

In section 2.3, we summarize results on the exact calculation for partition functions on the squashed 3-sphere S_b^3 and superconformal index on $S^2 \times_q S^1$ of 3d $\mathcal{N} = 2$ theories.

In section 2.4, we review aspects of holomorphic block in 3d $\mathcal{N} = 2$ theories with discrete, massive, sueprsymmetric vacua. A partition function on the squashed 3-sphere S_b^3 and superconformal index $S^2 \times_q S^1$ can be understood as appropriate fusion of holomorphic and anti-holomorphic block. We mainly discuss formal aspects, so discussion is a bit abstract. Examples are deferred to section 4.

In section 2.5, we summarize the $Sp(2L, \mathbb{Z})$ action on 3d (super)conformal field theories. This will be eventually related to a certain similar symplectic action in 3-manifold side.

2.1 Aspects of 3d $\mathcal{N} = 2$ supersymmetric gauge theories

In this section, we review some basic aspects of 3d $\mathcal{N} = 2$ supersymmetric gauge theories [2, 3].

2.1.1 3d $\mathcal{N}=2$ supersymmetry algebra and its irreducible representations

2.1.1.1 3d $\mathcal{N}=2$ supersymmetry algebra

3d $\mathcal{N}=2$ supersymmetry algebra is obtained from the dimensional reduction of 4d $\mathcal{N}=1$ supersymmetry algebra, and it is given by

$$\{Q_\alpha, Q_\beta\} = \{\overline{Q}_\alpha, \overline{Q}_\beta\} = 0, \qquad \{Q_\alpha, \overline{Q}_\beta\} = 2\sigma^\mu_{\alpha\beta} P_\mu + 2i\epsilon_{\alpha\beta} Z. \tag{2.1}$$

Here, α and β are indices for spinor which go from 1 to 2, and Z is a central charge which can be thought as the fourth component of momentum P in 4d. There is a $U(1)_R$ symmetry which rotates Q and $\overline{Q}$ in opposite phase, more explicitly we choose convention that Q is charged -1 and $\overline{Q}$ is charged $+1$ under $U(1)_R$.

2.1.1.2 Irreducible representations

The irreducible representation of 3d $\mathcal{N}=2$ superalgebra contains vector multiplet, chiral multiplet, which can also be obtained from dimensional reduction of 4d $\mathcal{N}=1$ multiplets. 3d $\mathcal{N}=2$ vector multiplet V consists of a gauge field A_μ, a complex Dirac spinor λ, a real scalar field σ, and a real auxiliary scalar D. In superspace with Wess-Zumino gauge, vector multiplet V is given by

$$V = -\theta\sigma^\mu\overline{\theta}A_\mu(x) - \theta\overline{\theta}\sigma + i\theta\theta\overline{\theta}\overline{\lambda}(x) - i\overline{\theta}\overline{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\overline{\theta}\overline{\theta}D(x) \tag{2.2}$$

where σ can be thought as the fourth component of 4d gauge field.

As in 4d, we also have chiral/anti-chiral field strength multiplets,

$$W_\alpha = -\frac{1}{4}\overline{D}\overline{D}e^{-V}D_\alpha e^V, \qquad \overline{W}_\alpha = -\frac{1}{4}DDe^{-V}\overline{D}_\alpha e^V \tag{2.3}$$

The lowest component of a field strength mutliplet is a gaugino.

It is also possible to define a multiplet whose lowest component is scalar field. This is called as linear multiplet, which is defined as

$$\Sigma = \overline{D}^\alpha D_\alpha V, \tag{2.4}$$

satisfying

$$D^\alpha D_\alpha \Sigma = \overline{D}^\alpha \overline{D}_\alpha \Sigma = 0, \qquad \Sigma = \Sigma^\dagger \tag{2.5}$$

3d $\mathcal{N}=2$ chiral multiplet Φ consists of a complex Dirac spinor ψ, a real scalar field ϕ, and a real auxiliary scalar field F;

$$\Phi = \phi + \sqrt{2}\theta\psi + \theta\theta D, \tag{2.6}$$

and similarly for anti-chiral multiplet.

2.1.2 Supersymmetric actions

The classical Yang-Mills kinetic terms for vector multiplet can be written in terms of superfield;

$$\mathcal{L}_{kin} = \frac{1}{g^2}\int d^2\theta \mathrm{Tr} W_\alpha^2 + h.c. \tag{2.7}$$

where trace is over fundamental representation in this subsection. This can be also written in terms of linear multiplet;

$$\mathcal{L}_{kin} = \frac{1}{g^2}\int d^2\theta d^2\overline{\theta}\ \mathrm{Tr}\frac{1}{4}\Sigma^2. \tag{2.8}$$

The classical kinetic terms for chiral multiplets Φ in representation of gauge group is written as

$$\mathcal{L}_{kin,matter} = \sum_i \int d^2\theta d^2\overline{\theta}\ \Phi_i^\dagger e^V \Phi_i \tag{2.9}$$

We can get mass terms in Lagrangian from non-zero vacuum expectation value (vev) of scalar component of background vector multiplet. If a background vector multiplet contains $m\theta\overline{\theta}$ from a nonzero vev of an adjoint scalar and other components are turned off, we have

$$\mathcal{L}_{mass,matter} = \sum_i \int d^2\theta d^2\overline{\theta}\ \Phi_i^\dagger e^{m\theta\overline{\theta}} \Phi_i, \tag{2.10}$$

which gives a mass to matter multiplets. This mass is called as “real mass”.

We also have Chern-Simons therm, which is given by

$$\mathcal{L}_{CS} = \frac{k}{4\pi}\int d^2\theta d^2\overline{\theta}\, \mathrm{Tr}\, \Sigma\, V \tag{2.11}$$

where $k \in \mathbb{Z}$ is a Chern-Simons level.

For $U(1)$ factor of gauge group, we can also add Fayet-Iliopoulos (FI) term;

$$\mathcal{L}_{FI} = \int d^2\theta d^2\overline{\theta}\ \zeta\, V \tag{2.12}$$

where ζ is an FI parameter. This term can also be thought as

$$\mathcal{L}_{FI} = \int d^2\theta d^2\overline{\theta}\ \Sigma V \tag{2.13}$$

where a linear multiplet Σ has a scalar component $\sigma = \zeta$ and the rest components are turned off.

For abelian vector multiplets, more generally, CS term and FI term can be thought as a special case of

$$\mathcal{L}_{BF} = \frac{k_{ij}}{4\pi} \int d^2\theta d^2\overline{\theta}\ \Sigma_i V_j. \tag{2.14}$$

For $i \neq j$, it is called as BF term in literature. In this thesis, we call it as mixed Chern-Simons term.

2.1.3 Parity anomaly

Unlike 4d gauge theory, in 3d there is no local anomaly. However, there is a parity anomaly, due to the generation of Chern-Simons term upon integrating out massive charged fermions [4, 5, 6]. More explicitly, when integrating out charged fermions, the Chern-Simons term is generated at the one loop, and for the case of abelian gauge group (also abelian global symmetry group), the effective Chern-Simons levels in low energy become,

$$(k_{ij})_{eff} = k_{ij} + \frac{1}{2}(q_f)_i(q_f)_j \,\mathrm{sign}(M_f) \tag{2.15}$$

where f labels fermions, i, j denote $U(1)_i$ gauge or global symmetries, $(q_f)_i$ denotes the charge of f-th fermions charged under $U(1)_i$ symmetry, and M_f is the mass of f-th fermions. One can do similarly for the nonabelian global symmetry [2].

Due to the requirement that the path integral is invariant under large gauge transformation, the effective Chern-Simons level at the IR should be integer.[1]

2.1.4 Monopole operators and vortices

Monopole operator [7, 8, 9] in 3d theory is defined as a prescription of singular behavior of gauge field;

$$F \sim q_J \star d\frac{1}{|\vec{x} - \vec{x}_0|} \tag{2.16}$$

In other words, insertion of a monopole operator at $\vec{x}_0$ creates a monopole source at that point with flux $q_J = \int_{S^2} F$ where S^2 encloses the insertion point $\vec{x}_0$.

[1] This integer condition is for dynamical gauge field. However, if we want to gauge the global symmetries, then the integer condition also needs to be imposed on them.

A chiral monopole operator in 3d $\mathcal{N}=2$ theories is expressed in terms of a linear mutliplet Σ;

$$D^2\Sigma = 0, \qquad \overline{D}^2 = q_J 2\pi\delta^{(3)}(\vec{x}-\vec{x}_0)\theta^2, \tag{2.17}$$

and similarly for anti-chiral monopole operator in obvious way. As seen in the definition, this is the UV operator, which is independent of the choice of the vacua.

Though the monopole operator is a UV operator, it is related to (non-compact) Coulomb branch of the theory; Coulomb branch exists when the monopole operator is gauge neutral. In Coulomb branch, the linear multiplet Σ is massless and can be dualized to a chiral multiplet X, and the scalar component of X parametrize the (non-compact) Coulomb branch. X is charged +1 under $U(1)_J$ topological symmetry. Semi-classically, for large scalar component ϕ of the chiral multiplet Φ, the Coulomb branch modulus X_j where j labels the basis of simple roots β_j of gauge group G is asymptotically $X_j \sim e^{\chi\cdot\beta_j/g^2}$ where $\chi = \phi + i\gamma$ with γ being a dual photon.[2] This can be regarded as a semi-classical expression of a monopole operator.

Meanwhile, there is a vortex solution in Higgs branch of the theory. For 3d $\mathcal{N}=2$ theories, one can have a BPS vortices under certain condition. For example, in the abelian Higgs model with an FI parameter ζ, there is a vortex configuration;

$$\phi \sim \zeta^{1/2}e^{\pm\theta}, \qquad A_\theta \sim \pm\frac{1}{r} \tag{2.18}$$

with BPS mass given by $m = |Z| = |q_J\zeta|$. As $\zeta \to 0$ where the Coulomb branch and Higgs branch meet, one can smoothly move onto Coulomb branch where $U(1)_J$ is explicitly broken by the Coulomb branch modulus charged under $U(1)_J$.

2.1.5 3d $\mathcal{N}=2$ mirror symmetry

There is a "mirror symmetry" in 3d $\mathcal{N}=2$ theories. Since there is no invariant way to distinct Higgs branch and Coulomb branch, which are exchanged under mirror symmetry, it is a misnomer. But since at least for known examples in SQED and SQCD they are from 3d $\mathcal{N}=4$ where there is actual mirror symmetry[3], same term is usually used also for 3d $\mathcal{N}=2$. The simplest example of 3d $\mathcal{N}=2$ mirror symmetry is the mirror symmetry between SQED with one flavor and XYZ model. This will be discussed in section 4 a bit more detail.

[2]For abelian vector fields, it is possible to dualize it to scalar (dual photon) via $F_{\mu\nu} = \epsilon_{\mu\nu\rho}\partial^\rho\gamma$. Charge quantization condition constraints the dual photon γ to live on a circle whose radius is proportional to g^2 where g is a coupling constant of the gauge theory. The current $J^\rho = \epsilon^{\mu\nu\rho}F_{\mu\nu}$ generates $U(1)_J$ symmetry which shifts γ by constant.

[3]For example, mirror symmetry in 3d $\mathcal{N}=4$ theories whose R-symmetry group is $SU(2)_{R_1} \times SU(2)_{R_2}$ exchange $SU(2)_{R_1}$ and $SU(2)_{R_2}$ symmetries, Higgs branch and Coulomb branch, and mass term and FI term.

2.2 Compactification of 3d $\mathcal{N} = 2$ gauge theories on S^1

In this section, we review reduction of 3d $\mathcal{N} = 2$ theories on S^1 [10, 11, 12, 13]. Reduction of 3d $\mathcal{N} = 2$ theories on S^1 is described by the effective 2d $\mathcal{N} = (2,2)$ theories. This effective 2d supersymmetric theory is characterized by the twisted superpotential. Here, we only consider the case that all symmetries are abelian.

Upon reducing 3d theory on S^1 with radius R, one can include a Wilson line for the (linear combination of) $U(1)$ global symmetry and they complexity the (linear combination of) real mass parameter of 3d associated to the (linear combination of) global $U(1)$ symmetry;

$$m_i = m_i^{3d} + \frac{i}{R}\oint_{S^1} A_i \tag{2.19}$$

where m_i denotes (complexified) twisted mass or complexified FI term in 2d, which can be regarded as the scalar component of the background twisted chiral mutliplet M_i in 2d. Similarly, the scalar component σ_a of the 3d linear multiplet Σ_a are also complexified;

$$\sigma_a = \sigma_a^{3d} + \frac{i}{R}\oint_{S^1} A_a \tag{2.20}$$

where a runs from 1 to r, the rank of gauge group. If the abelian symmetry is compact, the invariance under large gauge transformation of 3d theory leads to the periodicity of σ_a and m_i; $\sigma_a \sim \sigma_a + \frac{2\pi i}{R}$, $m_i \sim m_i + \frac{2\pi i}{R}$. This is not an intrinsic property in 2d supersymmetric effective theory itself, but rather it is a property of the effective 2d supersymmetric theory obtained from the reduction of 3d theory.

Upon reduction on S^1, the chiral multiplet Φ in 3d gives a tower of Kaluza-Klein modes in 2d. If Φ is charged under overall $U(1)$ whose overall real mass parameter is m_Φ^{3d}, the KK mode Φ_n with momentum n on S^1 has a twisted mass $m_{\Phi_n} = m_\Phi + \frac{2\pi i n}{R}$. Due to the periodicity mentioned above, all KK modes should be included in the effective 2d theory.

The twisted superpotential gets one-loop correction from integrating out the massive charged chiral multiplet. Taking into account all KK modes, one obtains [10, 11, 12, 13]

$$\widetilde{\mathcal{W}}(M_\Phi) = \frac{1}{R}\left(\frac{1}{4}M_\Phi^2 + \mathrm{Li}_2(-e^{-M_\Phi})\right) \tag{2.21}$$

There are also contributions from tree-level mixed Chern-Simons terms, which is given by

$$\widetilde{\mathcal{W}}_{CS}(\Sigma_a, M_i) = \frac{1}{R}\left(\frac{1}{2}k_{ab}\Sigma_a\Sigma_b + k_{ai}\Sigma_a M_i + \frac{1}{2}k_{ij}M_iM_j\right), \tag{2.22}$$

where in (2.21) and (2.22), we have absorbed R into M_Φ and Σ_a to make them dimensionless.[4]

Given the twisted superpotential $\widetilde{\mathcal{W}}$, the condition for supersymmetric vacua is given by

$$\frac{\partial \widetilde{\mathcal{W}}}{\partial \sigma_a} = 2\pi i n_a, \qquad n_a \in \mathbb{Z}. \tag{2.24}$$

If one uses expontentiated variables valued in $\mathbb{C}^*$, $s_a = e^{\sigma_a}$ and $x_i = e^{m_i}$, we get

$$\exp\left(s_a \frac{\partial \widetilde{\mathcal{W}}}{\partial s_a}\right) = 1. \tag{2.25}$$

If one gauged $U(1)_x$ symmetry associated to x_i weakly and y_i is the effective FI parameter for $U(1)_x$, one can also consider the supersymmetric conditions for x_i and y_i, and they are given by

$$\exp\left(x_i \frac{\partial \widetilde{\mathcal{W}}}{\partial x_i}\right) = y_i. \tag{2.26}$$

Here, $\widetilde{\mathcal{W}}$ is the twisted superpotential which was solved from (2.25), *i.e.* we solve (2.25) and put the solution s_a back to $\widetilde{\mathcal{W}}$, and the resulting twisted superpotential is the one used in (2.26). We will call the moduli space of (2.26) as the moduli space of supersymmetric vacua or supersymmetric parameter space.

2.3 The exact calculations in 3d $\mathcal{N} = 2$ superconformal theories

Since the pioneering work of Pestun [14], there has been many developments in exact calculations in various dimensions and with various operators or defects. In this section, we quote result from the recent development in exact calculations in 3d $\mathcal{N} = 2$ theories [15, 16, 17, 18, 19, 20, 21].

[4] Expressions (2.21) and (2.22) are not single-valued under large gauge transformation, *i.e.* under the periodicity of σ_a and m_i. In order to remedy this, one adds

$$\sum_{n_a \in \mathbb{Z}} \exp\left[-2\pi i n_a \int d^2\theta \Sigma_a\right]. \tag{2.23}$$

into path integral, which ensures that $\int F_a/2\pi \in \mathbb{Z}$ that is required for the path integral to be well-defined. By adding this term, the resulting action becomes single-valued.

2.3.1 Supersymmetric partition function on the squashed 3-sphere

Our main interest is the partition function on squashed 3-sphere S_b^3 preserving $U(1)\times U(1)$ symmetry [18]. Metric for such squashed 3-sphere S_b^3 is

$$ds^2 = b^2(dx_0^2 + dx_1^2) + b^{-2}(dx_2^2 + dx_3^2), \qquad \text{with} \quad x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1 \tag{2.27}$$

The partition function on S_b^3 takes the form

$$\mathcal{Z}_{S_b^3} = \frac{1}{|\mathcal{W}|} \int d^r\sigma \, Z_{\text{classical}} Z_{\text{one-loop}} \tag{2.28}$$

where $|\mathcal{W}|$ is the order of Weyl group and r is the rank of gauge group.

If chiral multiplet has R-charge R and gauge charge ρ_a under the Cartan subgroup H_a of gauge group G, then the one-loop determinant from chiral multiplet is given by

$$s_b\left(\frac{iQ}{2}(1-R) - \rho_a\sigma_a\right) \tag{2.29}$$

where σ_a is (scaled) scalar component of vector multiplet and $Q = b + b^{-1}$. $s_b(x)$ is sine double function and is given by

$$s_b(x) = \prod_{m,n\in\mathbb{Z}_{\geq 0}} \frac{mb + nb^{-1} + \frac{Q}{2} - ix}{mb + nb^{-1} + \frac{Q}{2} + ix} = e^{-\frac{i\pi}{2}x^2} \prod_{j=1}^{\infty} \frac{1 + e^{2\pi bx + 2\pi i b^2(j-\frac{1}{2})}}{1 + e^{2\pi b^{-1}x + 2\pi i b^{-2}(\frac{1}{2}-j)}} \tag{2.30}$$

which is a variant of non-compact quantum dilogarithm function.

One-loop determinant from vector multiplet is trivial for abelian gauge group, but for nonabelian gauge group G it is given by

$$\prod_{\alpha\in\Delta_+} \frac{\sinh(\pi b\sigma\alpha)\sinh(\pi b^{-1}\sigma\alpha)}{\pi\sigma\alpha} \tag{2.31}$$

Here, α takes positive roots and $\sigma\alpha$ denotes $\sigma\alpha = \sum_{a=1}^{r}\sigma_a\alpha_a$.

There are non-zero contributions from classical action of Chern-Simons term and FI term at the saddle point. For Chern-Simons term with level k and FI term for $U(1)$ gauge group with FI parameter ζ, they are given by

$$\exp(-ik\pi\sigma_a\sigma_a) \tag{2.32}$$

and

$$\exp(4\pi i\zeta\sigma), \tag{2.33}$$

respectively.

Assuming that R-symmetry at the IR fixed point is not mixed with the accidental symmetry, it is known that the IR superconformal R-charges are fixed by maximizing $F = -\log|Z_{S^3_{b=1}}|$ [16, 22, 23].

2.3.2 Superconformal index

Superconformal index counts certain BPS operators in superconformal field theory on $S^2 \times_q S^1$ [19, 20, 21]. More explicitly it is given by

$$\mathcal{I}(t_i; q) = \mathrm{tr}\left[(-1)^F e^{\beta H} q^{\Delta + j_3} \prod_a t_i^{F_i}\right] \tag{2.34}$$

where Δ is energy, R is R-charge, j_3 is angular momentum on S^2, F_i are generator for the global (or flavor) symmetry, and H is given by

$$H = \{Q, Q^\dagger\} = \Delta - R - j_3 \tag{2.35}$$

where Q is a certain supercharge in 3d $\mathcal{N} = 2$ superconformal algebra. Since only Q and $Q^\dagger$-invariant states contribute to the index, index (2.34) doesn't depend on β, but only depends on fugacities, q, t_i.

From the localization technique, one can obtain the exact result for the index. But if one incorporates the magnetic fluxes and Wilson line for flavor symmetries, then the resulting index has extra discrete and continuous parameters. This is called the generalized index. It is given by

$$\mathcal{I}(t_i, n_i; q) = \sum_s \frac{1}{Sym} \int \prod_a \frac{dz_a}{2\pi z_a} e^{-S_{CS}(h,m)} Z_{gauge}(z_a, m_a; q) \prod_\Phi Z_\Phi(z_a, m_a; t_i, n_i; q) \tag{2.36}$$

where

$$Z_{gauge}(z_a = e^{ih_a}, m_a; q) = \prod_{\alpha\in\Delta_+} q^{-\frac{1}{2}|\alpha(m)|}\left(1 - e^{i\alpha(h)} q^{|\alpha(m)|}\right) \tag{2.37}$$

$$Z_\Phi(z_a = e^{ih_a}, m_a; t_i, n_i, q) = \prod_{\rho\in R_\Phi}\left(q^{\frac{1}{2}(1-\Delta_\Phi)} \prod_a e^{-i\rho(h)} \prod_j t_j^{-f_j(\Phi)}\right)^{\frac{1}{2}|\rho(s)+\sum_j f_j(\Phi) n_j|} \tag{2.38}$$

$$\times \frac{(e^{-i\rho(h)} t_j^{-f_j(\Phi)} q^{|\rho(s)+\sum_j f_j(\Phi) n_j| + 1 - \frac{\Delta_\Phi}{2}}; q)_\infty}{(e^{i\rho(h)} t_j^{f_j(\Phi)} q^{|\rho(s)+\sum_j f_j(\Phi) n_j| + \frac{\Delta_\Phi}{2}}; q)_\infty} \tag{2.39}$$

and Sym is a symmetric factor arising from non-abelian gauge group generically broken by monopole, *i.e.*, if gauge group G is broken to its subgroup $\otimes_k G_k$ by monopole then $Sym = \prod_{k=1} \text{Rank}(G_k)!$.

Here, α is positive roots of gauge group G, h whose component is h_a denotes maximal torus of the gauge group which parametrize the Wilson line of the gauge group, and s whose component is m_a denotes the Cartan generator of the gauge group which parametrize the GNO charge of magnetic monopole configuration of gauge field. a runs over the rank of the gauge group, and takes integer value.

If chiral multiplet Φ is in representation R of gauge group, ρ denotes a weight vector of representation R, R denotes R-charge of chiral multiplet, t_j denotes the maximal torus of the global symmetry under which chiral multiplets Φ are charged, n_j denotes the Cartan generator of the global symmetry group which parametrize the GNO charge of magnetic monopole configuration of background gauge field for global symmetry, and $f_j(\Phi)$ denotes the charge of chiral multiplet Φ under $U(1)$ subgroup of maximal torus associated to t_j. j runs over the rank of the global symmetry group, and takes integer value.

Contribution from Chern-Simons term is given by $e^{ikS_{CS}(h,s)} = \exp(ik\sum_a h_a m_a) = \prod_a z_a^{km_a}$. For the mixed Chern-Simons term with level $k_{12} = k_{21}$ between two abelian symmetries $U(1)$ with Wilson line and magnetic flux being (h_1, m_1) and (h_2, m_2) is given by $\left(z_1^{m_2} z_2^{m_1}\right)^{k_{12}}$.

2.4 Holomorphic block in 3d $\mathcal{N} = 2$ theories

It turns out that above partition functions on squashed 3-sphere S_b^3 $-$ $Z_{S_b^3}$ $-$ and superconformal index on $S^2 \times_q S^1$ $-$ $I_{S^2\times_q S^1}$ $-$ are factorized [24, 25], and this was systematically studied in [13].

The basic ingredients for $Z_{S_b^3}$ and $I_{S^2\times_q S^1}$ in 3d $\mathcal{N} = 2$ theory which have discrete, massive, supersymmetric vacua (*i.e.* there should be enough flavor symmetries so that it completely lifts all flat directions of the moduli space) are called as holomorphic block. Holomorphic block is defined as a partition function on $D^2 \times_q S^1$ which is topologically a solid torus and whose metric is given by

$$ds^2 = dr^2 + f(r)^2(d\varphi + \epsilon\beta d\theta)^2 + \beta^2 d\theta^2 \tag{2.40}$$

where local coordinate is (r, φ, θ) with $r \in [0, \infty)$, $\varphi \sim \varphi + 2\pi$, and $\theta \sim \theta + 2\pi$. $f(r)$ approaches to 0 as $r \to 0$ and ρ as $r \to \infty$. The geometry $D^2 \times_q S^1$ can be regarded as cigar geometry parametrized by (r, φ) is fibered over S^1 parametrized by θ in such a way that $(z, \theta) \sim (q^{-1}z, \theta + 2\pi)$ if we set $z = re^{i\varphi}$ and

$$q = e^{2\pi i\epsilon\beta} = e^{\hbar}. \tag{2.41}$$

Since it is curved, in order to preserve supersymmetry, appropriate twisting is necessary. There are two choices of twisting, one is “topological” and another is “anti-topological”. They preserve two different sets of two supercharges with different BPS conditions ($P^0 = Z$ for the former and $P^0 = -Z$ for the latter).

2.4.1 Holomorphic block as the BPS index

Since the theory is twisted, partition function doesn't depend on the size of ρ. So one can equivalently think of the theory on $\mathbb{R}^2 \times_q S^1$. In this setup, one can associate supersymmetric state at the origin of $\mathbb{R}^2$ and also supersymmetric state $|\alpha\rangle$ on T^2 at infinity (since asymptotically $\mathbb{R}^2 \times_q S^1$ approaches to $\mathbb{T}^2 \times \mathbb{R}$) which one-to-one corresponds to discrete, massive, supersymmetric vacua α of 3d $\mathcal{N} = 2$ theories. Roughly speaking, their overlap give the BPS index. If we deform superconformal field theory away from the superconformal fixed point, we can obtain massive theory, and the BPS index counts BPS states in the vacuum of the massive theory. Given this setup, one can consider holomorphic block as BPS index for each vacua α;

$$B^\alpha(x;q) \simeq \begin{cases} Z^\alpha_{\overline{\text{BPS}}}(x;q) & |q| < 1 \\ Z^\alpha_{\text{BPS}}(x;q) & |q| > 1 \end{cases} \tag{2.42}$$

with

$$\begin{aligned} Z^\alpha_{\overline{\text{BPS}}}(x;q) &= \text{Tr}_{\mathcal{H}_\alpha}(-1)^R e^{-\beta H} q^{-J_3+\frac{R}{2}} x^{-e} \qquad && |q| < 1 \qquad \text{(anti-topological)} \\ Z^\alpha_{\text{BPS}}(\widetilde{x};\widetilde{q}) &= \text{Tr}_{\mathcal{H}_\alpha}(-1)^R e^{-\beta \widetilde{H}} \widetilde{q}^{-J_3-\frac{R}{2}} \widetilde{x}^{-e} \qquad && |\widetilde{q}| > 1 \qquad \text{(topological)} \end{aligned} \tag{2.43}$$

where $H = 2(P^0 + Z)$, $\widetilde{H} = 2(P^0 - Z)$, and x and $\widetilde{x}$ are certain exponential of complexfied twisted mass parameter associated to $U(1)_x$ global symmetry. In the BPS indices, $(-1)^R$ was chosen instead of usual choice $(-1)^F$, but both give essentially equivalent information.

As varying x, there is an interesting stoke phenomena, which we don't discuss here. Holomorphic block is unique modulo the overall factor of an elliptic ratio of theta functions or $\exp\left(-\frac{1}{24}(\hbar - \frac{4\pi^2}{\hbar})\right)$.

2.4.2 Fusion of holomorphic and anti-holomorphic block

In analogue of topological/anti-topological fusion in 2d $\mathcal{N} = 2$ theories [26], one can consider fusion of two copies of $D^2 \times_q S^1$. Fusion is possible when Hilbert spaces on the each asymptotic regions are dual [13]. Considering two infinitely long cigar geometries where $D^2 \times_q S^1$ is asymptotically $T^2 \times \mathbb{R}$, we would like to perform fusion by gluing the two tori with opposite orientation via $SL(2,\mathbb{Z})$ action on the torus, which also induces modular action (combined with reflection) on the parameter q. A bit more specifically, we would like to consider fusion such that if the complex structure

$\tau = \epsilon\beta + i\beta\rho^{-1}$ of T^2 in infinitely long cigar geometry for anti-holomorphic block is related to an infinitely long cigar geometry with complex structure $\widetilde{\tau}$ by $\tau \to \widetilde{\tau} = -\overline{g \cdot \tau}$ for holomorphic block then $\beta\epsilon \to \widetilde{\beta}\widetilde{\epsilon} = -\overline{g \cdot (\beta\epsilon)}$.

Then for S-fusion one obtains $q = \exp(2\pi i b^2)$, $\widetilde{q} = \exp(2\pi i b^{-1})$ and $x =: \exp(X) = \exp(2\pi b\mu)$, $\widetilde{x} =: \exp(\widetilde{X}) = \exp(2\pi b^{-1}\mu)$ where b is squashing parameter of squashed 3-sphere S_b^3 mentioned above and μ are complexified mass parameter. For id-fusion one obtains $q = \widetilde{q}^{-1}$ and $x =: \exp(X) = q^{\frac{m}{2}}\zeta$, $\widetilde{x} =: \exp(\widetilde{X}) = q^{\frac{m}{2}}\zeta^{-1}$ where $q = \exp(2\pi\epsilon\beta)$ and m and ζ are magnetic flux and fugacity of the index associated to $U(1)$ global symmetry. Along the way of obtaining these, in order to have exact same holomorphic block for both S_b^3 partition function and the index, R-charges of chiral multiplets need to be all integers.

In sum, via fusion, one obtains

$$\mathcal{Z}^{[g]}_{\text{fusion}} = \sum_{\alpha} B^{\alpha}(x;q)B^{\alpha}(\widetilde{x},\widetilde{q}) = \left\|B^{\alpha}(x;q)\right\|_g^2 \tag{2.44}$$

where we are interested $\mathcal{Z}^{[S]}_{\text{fusion}}$ or $\mathcal{Z}^{[id]}_{\text{fusion}}$, which give the partition function on S_b^3 and the index on $S^2 \times_q S^1$.

2.4.3 Holomorphic block from supersymmetric quantum mechanics

The cigar geometry $D^2 \times_q S^1$ can be considered as torus fiberation over half-line $\mathbb{R}_+ = \{t \in [0,\infty)\}$ with fixed area and complex structure of torus as $t \to \infty$. Thus, macroscopically 3d $\mathcal{N} = 2$ theory can be seen as supersymmetric quantum mechanics on $\mathbb{R}_+$.

From the point of view of supersymmetric quantum mechanics, one can see that path integral on $D^2 \times_q S^1$ becomes a finite-dimensional contour integral with the integrand determined perturbatively all order of $\hbar$ [13]. The result is that for a discrete, massive, supersymmetric vacua labelled by α, there is an associated integration cycle Γ_α, and contour integral with integrand from twisted superpotential gives holomorphic block B^α. A bit more specifically, it turns out that the path integral of supersymmetric quantum mechanics become a contour integral

$$B^{\alpha}(x;q) = Z^{\alpha}_{QM} \simeq \int_{\Gamma_\alpha} \frac{ds_1}{2\pi i s_1} \cdots \frac{ds_r}{2\pi i s_r} \exp\left(\frac{1}{\hbar}\widetilde{\mathcal{W}}_{\hbar}(s_a, x_i;\hbar)\right) \tag{2.45}$$

in complex space $\mathbb{C}^{*r}$ parametrized by s_a where we use the notations in section 2.2. The integration cycle Γ_α should be convergent and determined by gradient flow equation with respect to $\mathrm{Im}\mathcal{W}^{QM}$ given a choice of asymptotic boundary condition at $t \to \infty$ labelled by the massive vacua α. Here, superpotential $\mathcal{W}^{QM}$ of quantum mechanics is given by

$$\mathcal{W}^{QM}(\Sigma_a, M_i) = 2\pi\rho\widetilde{\mathcal{W}}(\Sigma_a, M_i) = \frac{i}{\hbar}\left[\sum_{\Phi}\left(\frac{1}{4}M_\Phi^2 + \mathrm{Li}_2(-e^{-M_\Phi})\right) + \frac{1}{2}k_{ab}\Sigma_a\Sigma_b + k_{ai}\Sigma_a M_i + \frac{1}{2}k_{ij}M_iM_j\right] \tag{2.46}$$

which is obtained from further compactification on S^1_ρ of the twisted superpotential obtained in section 2.2 modulo a certain subtlety. The term $\widetilde{\mathcal{W}}_\hbar(s_a, m_i; \hbar)$ in the integrand is the "quantum-corrected superpotential"

$$\widetilde{\mathcal{W}}_\hbar(s_a, x_i; \hbar) = \sum_\Phi\left(\frac{1}{4}m_\Phi^2 + \mathrm{Li}_2(-e^{-m_\Phi - \frac{\hbar}{2}}; \hbar)\right) + \frac{1}{2}k_{ab}\sigma_a\sigma_b + k_{ai}\sigma_a m_i + \frac{1}{2}k_{ij}m_im_j \tag{2.47}$$

where $\mathrm{Li}_2(x;\hbar) = \sum_{n=0}^{\infty}\frac{B_n\hbar^n}{n!}$ with Bernoulli number B_n. The R-charge of chiral multiplet is set to one here. It is also possible to have general R-charge R_Φ by shifting, $m_\Phi \to m_\Phi + (R_\Phi - 1)(i\pi + \hbar/2)$. If we take $\hbar \to 0$ limit, quantum-corrected superpotential $\widetilde{\mathcal{W}}_\hbar(s_a, x_i; \hbar)$ becomes $\frac{\hbar}{i}\mathcal{W}^{QM}(\sigma_a, m_i)$.

Though above analysis from supersymmetric quantum mechanics gives rough idea that the holomorphic block is expressed as certain finite-dimensional contour integral, as stated earlier this is a perturbative analysis for both integrand and the integration cycle. With help of Ward identity of Wilson and 't Hooft loop operators, the exact non-perturbative integrand can be obtained. However, since the exact superpotential whose only critical points correspond to the true vacua of the theory[5], which is used in gradient flow equation to determine exact Γ_α, is still unknown, one can just use approximated Γ_α from above perturbative argument [13]. With several conditions imposed on the Γ_α in perturbative analysis, one can obtain exact non-perturbative holomorphic block. Those conditions are that if one shifts entire Γ_α by q, one should be able to deform smoothly it back to original Γ before shift. This means that the contour should be closed or end asymptotically at 0 or ∞ in each $\mathbb{C}^{*r}$. Also, Γ_α should be far away at least by q from the poles of the integrand. In addition, Γ_α is not allowed to path line of poles but is allowed to cross or lie on the line of zeroes.

2.5 $Sp(2L, \mathbb{Z})$ action on 3d conformal field theories

For 3d conformal field theories with $U(1)$ symmetry which one can couple to $U(1)$ background gauge symmetry, there is an $SL(2,\mathbb{Z})$ action on them. This action is generated by $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

[5] A non-perturbative integrand Υ obtained with help of Ward identity of line operators has too many critical points including the true vacua.

The generator T acts on the theory in a way that it adds Chern-Simons coupling with level 1 for background gauge field A;

$$\mathcal{L} \to \mathcal{L} + \frac{1}{4\pi} A \wedge dA. \tag{2.48}$$

The generator S acts on the theory in a way that it gauges $U(1)$ symmetry and introduces a new background $U(1)_{\text{new}}$ gauge field A_{new} by coupling A_{new} with the current that is a Hodge star of field strength for $U(1)$ symmetry which is now gauged. In terms of Lagrangian, we add a mixed Chern-Simons term of $U(1)$ symmetry which is now gauged and newly introduced $U(1)_{\text{new}}$ background gauge symmetry;

$$\mathcal{L} \to \mathcal{L} + \frac{1}{2\pi} A_{\text{new}} \wedge dA \tag{2.49}$$

where A is dynamical gauge field of $U(1)$ symmetry which is now gauged. A new $U(1)$ background gauge symmetry whose gauge field is A_{new} is usually called as topological symmetry with respect to A, because monopole operator defined from A is charged under this new $U(1)$ symmetry.

When 3d conformal field theory is coupled to 4d free abelian theory as a boundary theory, this $SL(2,\mathbb{Z})$ action can be thought as electro-magnetic duality on a space of conformally invariant boundary condition of a 4d free abelian theory.

If there are more than one $U(1)$ symmetry, say L $U(1)$ global symmetries, $SL(2,\mathbb{Z})$ action can be generalized to $Sp(2L,\mathbb{Z})$ action on the 3d conformal field theory with $U(1)^L$ global symmetries.

Let $\vec{A} = (A_1, A_2, \cdots, A_L)$ be gauge fields for $U(1)^L$ global symmetries. For $Sp(2L,\mathbb{Z})$ action, there are three types of generators, which are called as T-type, S-type, and GL-type. More explicitly, by representing them as $L \times L$ block matrices, they are expressed and act on Lagrangian as follows;

$$T\text{-type} \quad \rightsquigarrow \quad g_T = \begin{pmatrix} I & 0 \\ B & I \end{pmatrix} : \quad \mathcal{L}[\vec{A}] \to \mathcal{L}[\vec{A}_{\text{new}}] + \frac{1}{4\pi} \vec{A}_{\text{new}} \cdot B \, d\vec{A}_{\text{new}} \tag{2.50}$$

where B is a symmetric L by L matrices;

$$S\text{-type} \quad \rightsquigarrow \quad g_S = \begin{pmatrix} I-J & -J \\ J & I-J \end{pmatrix} : \quad \mathcal{L}[\vec{A}] \to \mathcal{L}[\vec{A}] + \frac{1}{2\pi} \vec{A}_{\text{new}} \cdot J \, d\vec{A} \tag{2.51}$$

where J is a diagonal matrix $J = \text{diag}(j_1, j_2, \cdots, j_L)$ with j_i is 0 or 1. When $j_i = 1$, we gauge $U(1)$ symmetry whose gauge field is A_i and introduce $U(1)$ topological symmetry whose gauge field is

$A_{\text{new},i}$;

$$GL\text{-type} \quad \rightsquigarrow \quad g_U = \begin{pmatrix} U & 0 \\ 0 & U^{-1\ t} \end{pmatrix} : \quad \mathcal{L}[\vec{A}] \to \mathcal{L}[U^{-1}\vec{A}_{\text{new}}] \tag{2.52}$$

where $U \in GL(N, \mathbb{Z})$ is invertible, which redefines global symmetries.

Above argument can also be extended to supersymmetric case. More explicitly, for vector multiplet $\vec{V}$ and linear multiplet $\vec{\Sigma}$ we have

$$T\text{-type} \quad \rightsquigarrow \quad g_T = \begin{pmatrix} I & 0 \\ B & I \end{pmatrix} : \quad \mathcal{L}[\vec{V}] \to \mathcal{L}[\vec{V}_{\text{new}}] + \frac{1}{4\pi} \int d^4\theta\, \vec{\Sigma}_{\text{new}} \cdot B\, d\vec{V}_{\text{new}} \tag{2.53}$$

$$S\text{-type} \quad \rightsquigarrow \quad g_S = \begin{pmatrix} I - J & -J \\ J & I - J \end{pmatrix} : \quad \mathcal{L}[\vec{V}] \to \mathcal{L}[\vec{V}_{\text{new}}] + \frac{1}{2\pi} \int d^4\theta\, \vec{\Sigma}_{\text{new}} \cdot J\, d\vec{V} \tag{2.54}$$

$$GL\text{-type} \quad \rightsquigarrow \quad g_U = \begin{pmatrix} I & 0 \\ B & I \end{pmatrix} : \quad \mathcal{L}[\vec{V}] \to \mathcal{L}[U^{-1}\vec{V}_{\text{new}}] \tag{2.55}$$

We will see in Chapter 4 that how $Sp(2L, \mathbb{Z})$ will be related to the operation in Chern-Simons theory.

Chapter 3

Chern-Simons Theory

In this chapter, we review several aspects of Chern-Simons theory. We follow closely [27, 28, 29, 30, 31, 32]. Before we review analytically continued Chern-Simons theory, we briefly summarize Chern-Simons theory which is not analytically continued.

When the gauge group is G with Lie algebra $\mathfrak{g}$, gauge field A is a connection on G-bundle E on 3-manifold M_3, and Chern-Simons action is given by

$$I(A) = \frac{k}{4\pi}\int_{M_3} \mathrm{Tr}\ A \wedge dA + \frac{2}{3} A \wedge A \wedge A \tag{3.1}$$

where Tr is a trace on fundamental representation of the gauge group. If we take a proper normalization for the gauge group generator $\mathrm{Tr}(T_a T_b) = \delta_{ab}$, due to the requirement of invariance of the path integral

$$\mathcal{Z}(M_3) = \int DA \exp(iI(A)) \tag{3.2}$$

under large gauge transformation, Chern-Simons level k should be an integer, $k \in \mathbb{Z}$.

The observable in Chern-Simons theory is Wilson loop operator which depends on the representation of gauge group and is supported on a knotted curve (or more generally link) in M_3. In the presence of Wilson loop operators, the partition function is

$$\mathcal{Z}(M_3; \gamma_i, R_i) = \int DA \exp(iI(A)) \prod_{i=1}^{r} W_{R_i}(\gamma_i) \tag{3.3}$$

where Wilson loop operator $W_{R_i}(\gamma_i)$ is given by

$$W_{R_i}(\gamma_i) = \mathrm{Tr}_{R_i}\, P\ \exp\left(i \oint_{\gamma_i} A\right). \tag{3.4}$$

where R_i is representation of gauge group G for each i-th knot in link. It turns out that Wilson loop operator expectation value gives Jones polynomial of knot γ in M_3. As a simple example, for unknot ($\mathbf{0_1}$) in S^3, the expectation value of Wilson loop operator in r-symmetric representation of $G = SU(2)$ is given by

$$\langle W_{S^r}(\mathbf{0_1})\rangle = \sqrt{\frac{2}{k+2}} \sin\left(\frac{(r+1)\pi}{2}\right), \tag{3.5}$$

which agrees with Jones polynomial of unknot. Expectation value of Wilson loop operator on more complicated knots can be calculated via surgery of 3-manifold by calculating corresponding matrix elements in WZW model where the Hilbert space of Chern-Simons theory is equivalent to the space of conformal block of WZW model whose level is k [31].

Later, Chern-Simons theory with complex gauge group was investigated [33], and analytic continuation of Chern-Simons theory with complex gauge group was studied in [27] and also in [30, 28, 29]. These and further developments will be a main review topic of this chapter.

In section 3.1, we review the structure of analytically continued Chern-Simons theory with complex gauge group $G_{\mathbb{C}}$ and its real form $G_{\mathbb{R}}$ and G. Mostly, we take $G = SU(2)$, so $G_{\mathbb{R}} = SL(2,\mathbb{R})$ and $G_{\mathbb{C}} = SL(2,\mathbb{C})$. We also briefly mention some aspects of Wilson loop.

In section 3.2, we review classical moduli space of Chern-Simons theory on a knot complement, which is known as A-polynomial. This will be related to the moduli space of supersymmetric vacua or supersymmetric parameter space of effective 3d $\mathcal{N} = 2$ theories on S^1 corresponding to knot complement. Also, we discuss briefly quantum $\widehat{A}$-polynomial, which gives recursion relation for knot polynomials.

In section 3.3, we review some aspects of Chern-Simons theory on 3-manifold obtained from gluing tetrahedra. An ideal tetrahedron is a basic building block of 3d-3d correspondence of Dimofte-Gaiotto-Gukov, so we review some properties of an ideal tetrahedron and gluing procedure for classical moduli space, quantum operator, and partition function.

3.1 Analytic continuation of Chern-Simons theory

Given gauge group $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}_{\mathbb{C}}$, let the gauge field $\mathcal{A}$ be a connection on $G_{\mathbb{C}}$ bundle $E_{\mathbb{C}}$ on three manifold M_3. We first consider a complex Chern-Simons action,

$$I(t,\widetilde{t}) = \frac{t}{8\pi}\int_{M_3} \mathrm{Tr}(\mathcal{A}\wedge d\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}) + \frac{\widetilde{t}}{8\pi}\int_{M_3} \mathrm{Tr}(\overline{\mathcal{A}}\wedge d\overline{\mathcal{A}} + \frac{2}{3}\overline{\mathcal{A}}\wedge\overline{\mathcal{A}}\wedge\overline{\mathcal{A}}) \tag{3.6}$$

where $\overline{\mathcal{A}}$ is a complex conjugate of $\mathcal{A}$ and in order for it to be real t and $\widetilde{t}$ should be complex conjugate. If we denote $t = k + is$ and $\widetilde{t} = k - is$ and require invariance of the path integral under the large gauge transformation then k should be integer $k \in \mathbb{Z}$. In addition, unitarity requires s to be real or imaginary for Euclidean action [33].

From the above action, we can analytically continue both t and $\widetilde{t}$ and regard them separate and independent variables. At the same time, we also analytically continue $\mathcal{A}$. This leads to $(\mathfrak{g}_{\mathbb{C}})_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}} \oplus \mathfrak{g}_{\mathbb{C}}$. So we have two copies of gauge connection, which are regarded as independent. We denote them as $\mathcal{A}$ and $\widetilde{\mathcal{A}}$. Then the action for analytically continued complex Chern-Simons theory is given by

$$I_{\mathbb{C}}(\mathcal{A}, \widetilde{\mathcal{A}}; t, \widetilde{t}) = \frac{t}{8\pi} \int_{M_3} \mathrm{Tr}(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}) + \frac{\widetilde{t}}{8\pi} \int_{M_3} \mathrm{Tr}(\widetilde{\mathcal{A}} \wedge d\widetilde{\mathcal{A}} + \frac{2}{3} \widetilde{\mathcal{A}} \wedge \widetilde{\mathcal{A}} \wedge \widetilde{\mathcal{A}}) \tag{3.7}$$

So we can think of analytically continued complex Chern-Simons theory as sum of two independent Chern-Simons theory whose levels are $k_1 = \frac{t}{2}$ and $k_2 = \frac{\widetilde{t}}{2}$.

We can also consider the case of compact group G and non-compact real group $G_{\mathbb{R}}$. From (3.1), we analytically continue level k to arbitrary complex number t. At the same time, we also analytically continue the gauge field A to $G_{\mathbb{C}}$ gauge field $\mathcal{A}$. The resulting action becomes

$$I(\mathcal{A}; t) = \frac{t}{8\pi} \int_{M_3} \mathrm{Tr} \mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \tag{3.8}$$

where we took 1/2 normalization to the action. We can regard (3.8) as "holomorphic half" of (3.7). Similar argument is also applied to the case of non-compact real form $G_{\mathbb{R}}$ of $G_{\mathbb{C}}$.

3.1.1 Structure of Chern-Simons partition functions

The partition function of Chern-Simons theory is obtained from path integral

$$\mathcal{Z}_{\mathbb{C}}(M_3) = \int_{\mathcal{C}} \mathcal{D}\mathcal{A} \mathcal{D}\widetilde{\mathcal{A}} \exp(i I_{\mathbb{C}}(\mathcal{A}, \widetilde{\mathcal{A}})). \tag{3.9}$$

If s is not real, the path integral is generically not convergent, so it is desirable to find a well-defined integration cycle such that the path integral is well-defined. In addition, when s is real such integration cycle should be the middle dimensional cycle such that $\widetilde{\mathcal{A}} = \overline{\mathcal{A}}$. In [30], such integration cycle $\mathcal{C}$ was analyzed via Morse theory and the steepest descent method.

Briefly summarizing, integration cycle is associated to each set of critical points of the action

$I(\mathcal{A}, \widetilde{\mathcal{A}})$. The critical points $\mathcal{A}_\alpha$ of the action are given by flat $G_\mathbb{C}$ connections on M_3,

$$\mathcal{F} = \widetilde{\mathcal{F}} = 0, \tag{3.10}$$

where $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ and $\widetilde{\mathcal{F}} = d\widetilde{\mathcal{A}} + \widetilde{\mathcal{A}} \wedge \widetilde{\mathcal{A}}$. It is known that there are finite number of critical points in many 3-manifolds including knot complement in 3-manifold with appropriate boundary conditions along the knot complement and without closed incompressible surface [34, 29]. Given a critical point, integration cycle is given by a downward gradient flow from it with Morse function from $G_\mathbb{C}$ Chern-Simons functional. This integration cycle $\mathcal{J}_\alpha$ associated to a critical point $\mathcal{A}_\alpha$ is called Lefschetz thimbles modulo some subtleties. Then the convergent integration cycle $\mathcal{C}$ is given by $\mathcal{C} = \sum_\alpha \mathfrak{n}_\alpha \mathcal{J}_\alpha$ where $\mathfrak{n}_\alpha \in \mathbb{Z}$, which is calculated in Morse theory. Since Morse function depends on $G_\mathbb{C}$ Chern-Simons functional which depends on s, Lefschetz thimble $\mathcal{J}_\alpha$ depends on the value of s.

Similar argument applies for the analytic continuation of Chern-Simons theory for compact real form G and also for non-compact real form $G_\mathbb{R}$. The integration cycle $\mathcal{C}_{compact}$ for analytically continued Chern-Simons theory for compact group G is given by $\mathcal{C}_{compact} = \sum_\alpha \mathfrak{n}_\alpha \mathcal{J}_\alpha$. For the non-compact real form $G_\mathbb{R}$, the integration cycle $\mathcal{C}^{'}$ is also given by linear combination of $\mathcal{J}_\alpha$ over $\mathbb{Z}$ for each critical point which corresponds to critical point $\mathcal{A}_\alpha$.

In sum, for any $G_\mathbb{C}$, $G_\mathbb{R}$, and G, the downward flow cycle $\mathcal{J}_\alpha$ corresponds to critical point $\mathcal{A}_\alpha$ of flat $G_\mathbb{C}$ connection and they form a vector space over $\mathbb{Z}$. But the integer coefficient $\mathfrak{n}_{\alpha\widetilde{\alpha}}$, $\mathfrak{n}'_\alpha$, and $\mathfrak{n}_\alpha$ depends on a specific integration cycle in a particular Chern-Simons theory under consideration.

Thus, given the integration cycle $\mathcal{J}_\alpha$, the form of partition function of $SL(2,\mathbb{C})$, $SL(2,\mathbb{R})$, and $SU(2)$ Chern-Simons theory has the following form

$$\mathcal{Z}_{SU(2)}(M_3) = \sum_\alpha \mathfrak{n}_\alpha Z_\alpha(M_3) \tag{3.11}$$

$$\mathcal{Z}_{SL(2,\mathbb{R})}(M_3) = \sum_\alpha \mathfrak{n}'_\alpha Z_\alpha(M_3) \tag{3.12}$$

$$\mathcal{Z}_{SL(2,\mathbb{C})}(M_3) = \sum_{\alpha,\widetilde{\alpha}} \mathfrak{n}_{\alpha,\widetilde{\alpha}} Z_\alpha(M_3) \widetilde{Z}_{\widetilde{\alpha}}(M_3) \tag{3.13}$$

where

$$Z_\alpha(M_3) = \int_{\mathcal{J}_\alpha} \exp(iI(\mathcal{A})). \tag{3.14}$$

In (3.13), $\mathfrak{n}_{\alpha,\widetilde{\alpha}}$ is diagonal for non-anlytically-continued Chern-Simons theory where integration cycles satisfy $\overline{\mathcal{A}} = \widetilde{\mathcal{A}}$, but in general it can be arbitrary.

3.1.2 Wilson loop operator

Wilson loop operator is an observable in Chern-Simons theory. It is labeled by a knotted loop on which the Wilson loop is supported and the representation of gauge group. We can also have Wilson loop in analytically continued Chern-Simons theory,

$$W_R(\mathcal{A};\gamma) = \mathrm{Tr}_R\, P\, \exp\left(i \oint_\gamma \mathcal{A}\right) \tag{3.15}$$

There are two ways to think of Wilson loop operator. We can regard it as a path-ordered integral of gauge field on 1-dimensional curve γ as above. However, we can also consider it as a particle moving around γ. In this case, the particle creates defect of cusp along the trajectory and inserting Wilson loop operator in 3-manifold can be interpreted as giving a boundary condition for gauge field so that one has holonomy around γ. In other words, we can consider it as 't Hooft operator, which we also call as monodromy defect. Actually, in 3d, it was shown that 't Hooft operator is equivalent to Wilson operator [32]. For example, for r-dimensional irreducible representation of $SU(2)$, the monodromy defect gives the boundary condition for gauge field such that it has holomony

$$\begin{pmatrix} e^{\pi i r/k} & 0 \\ 0 & e^{-\pi i r/k} \end{pmatrix} \tag{3.16}$$

where k is a renormalized level here. This alternative interpretation is useful. For compact gauge group, say $SU(2)$, we can use both interpretation interchangeably. However, considering the infinite dimensional representation of complex gauge group, for example $SL(2,\mathbb{C})$, it is more natural to consider monodromy defect.[1] Detail discussion on infinite dimensional representation is summarized in Chapter 7 of [35].

3.2 Classical moduli space, A-polynomial and quantum $\widehat{A}$-polynomial of knot

In this section, we review the classical vacuum moduli space of Chern-Simons theory on a knot complement in three sphere S^3, and quantization of it.

3.2.1 Classical moduli space and A-polynomial

Consider knot complement in 3-manifold, M_3, for example, $S^3\backslash K$ where K is a knot. The boundary of a knot is a torus, T^2, so there are two curves γ_m and γ_l corresponding to meridian and longitude curve, respectively.

[1] For example, it is unclear what would be Tr in above equation for infinite dimensional representation.

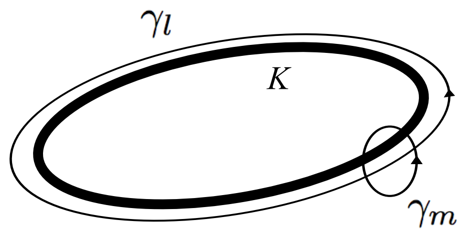


Figure 3.1: Meridian cycle γ_m and longitude cycle γ_l on the boundary of a knot complement

Holonomies around γ_m and γ_l generate the peripheral subgroup of M_3, which is $\pi_1(T^2) = \mathbb{Z}\times\mathbb{Z}$. So $SL(2,\mathbb{C})$ holonomies around γ_m and γ_l can be simultaneously taken in upper triangular matrices

$$\rho_{\gamma_m} = \begin{pmatrix} m & * \\ 0 & m^{-1} \end{pmatrix}, \qquad \rho_{\gamma_l} = \begin{pmatrix} l & * \\ 0 & l^{-1} \end{pmatrix} \tag{3.17}$$

with eigenvalues $l, m \in \mathbb{C}^*$. The action of Weyl group $\mathbb{Z}_2$ of $SL(2,\mathbb{C})$ on (3.17) exchanges (l,m) and (l^{-1}, m^{-1}). Thus, the classical phase space for M_3 with a single boundary given by a knot is

$$\mathcal{P}_{T^2} = (\mathbb{C}^* \times \mathbb{C}^*)/\mathbb{Z}_2 \tag{3.18}$$

We are interested in the moduli space of flat connection, and it is given by the homomorphism from fundamental group of M_3 to $SL(2,\mathbb{C})$,

$$\mathcal{M}_{\text{flat}}(SL(2,\mathbb{C}), M_3) = \text{Hom}(\pi_1(M_3), SL(2,\mathbb{C}))/\text{conj.} =: \mathcal{L} \tag{3.19}$$

It is known that for a single torus boundary of M_3, the complex dimension of $\mathcal{L}$ is 1. Two generators ρ_{γ_m} and ρ_{γ_l} of peripheral subgroup determines embedding of $\mathcal{L}$ into $\mathcal{P}_{T^2}$, and it can be shown that $\mathcal{L}$ is characterized by the zero locus of a single polynomial,

$$\mathcal{L} = \{(l,m) \in \mathcal{P}_{T^2} | A(l,m) = 0\} \tag{3.20}$$

This also can be thought as the condition that the flat connection on T^2 is extended to the flat connection on the bulk M_3. This polynomial $A(l,m)$ is called as A-polynomial of a knot [34].

For every knot complement, there is always $(l-1)$ factor. This corresponds to the abelian sector of $\mathcal{L}$ where $SL(2,\mathbb{C})$ flat connection are simultaneously diagonalized. Since $H_1(M_3,\mathbb{Z}) = \mathbb{Z}$ for any knot complement, there is always abelian representation in $\mathcal{L}$ which is generated by the holonomy around meridian curve and holonomy around longitude curve is trivial implying that $l-1=0$.

As an example, A-polynomial of unknot, trefoil knot, and figure-eight knot are

$$A_{0_1}(l,m) = (l-1) \tag{3.21}$$

$$A_{3_1}(l,m) = (l-1)(l+m^6) \tag{3.22}$$

$$A_{4_1}(l,m) = (l-1)(l^2-(m^4+m^{-4}-m^2-m^{-2}-2)l+1) \tag{3.23}$$

In classical phase space and also in quantum phase space, we can only specify one of l or m, but not both. In the following, we will specify m. For a given m, *i.e.* the boundary condition on the holonomy of flat $SL(2,\mathbb{C})$ connection around meridian cycle on the torus boundary, the order of l in A-polynomial is then the number of flat connection.

3.2.2 Quantization and quantum $\widehat{A}$-polynomial

In analytically continued Chern-Simons theory, the classical phase space is given by $\mathcal{P}_{T^2} = (\mathbb{C}^* \times \mathbb{C}^*)/\mathbb{Z}_2$ and (semi)-classical state is given by algebraic curve $\mathcal{L} \subset \mathcal{P}_{T^2}$. Considering the Hamiltonian approach in [33], one can quantize analytically continued Chern-Simons theory [27].

The holomorphic symplectic structure on the (semi)-classical phase space $\mathcal{P}_{T^2}$ is induced from the holomorphic Chern-Simons term

$$\omega_{T^2} = \frac{2}{\hbar} d\log l \wedge d\log m, \tag{3.24}$$

where $\hbar = \frac{2\pi i}{k}$. If we introduce logarithmic variables $v = \log l, u = \log m$ for l and m, $\omega_{T^2} = \frac{2}{\hbar} dv \wedge du$. It can be shown that the algebraic variety $\mathcal{L}$ given by $\{A(l,m) = 0\}$ is a Lagrangian submanifold in phase space $(\mathcal{P}_{T^2}, \omega_{T^2})$.

Also from ω_{T^2}, commutation relation is given by $[\widehat{v}, \widehat{u}] = \hbar/2$, and this leads to

$$\widehat{l}\widehat{m} = q^{1/2}\widehat{m}\widehat{l}, \tag{3.25}$$

where $\widehat{l} = e^{\widehat{v}}, \widehat{m} = e^{\widehat{u}}$, and $q = e^{\hbar}$. Correspondingly, under certain conditions, classical A-polynomial can be quantized to quantum $\widehat{A}$-polynomial [27, 29, 36, 37, 38],

$$A(l,m) \rightsquigarrow \widehat{A}(\widehat{l}, \widehat{m}; q) \tag{3.26}$$

Upon quantization, the classical state $\mathcal{L} = \{A(l,m) = 0\}$ becomes the wavefunction in the Hilbert space $\mathcal{H}_{T^2}$. A wavefunction in Hilbert space is a Chern-Simons partition function (3.14), and this is annihilated by $\widehat{A}(\widehat{l},\widehat{m};q)$. $\widehat{u}$ and $\widehat{v}$ act on the wave function $f(u)$;

$$\widehat{v}f(u)=\frac{\hbar}{2}\partial_u f(u),\qquad \widehat{u}f(u)=uf(u). \tag{3.27}$$

As classical A-polynomial $A(l,m)$ always have $(l-1)$ factor, quantum $\widehat{A}$-polynomial $\widehat{A}(\widehat{l},\widehat{m};q)$ also always have $(\widehat{l}-1)$ factor.

3.2.3 Recursion relation for Jones polynomial

Above discussion leads to the recursion relation of Jones polynomial. A colored Jones polynomial $J_r(K;q)$ is obtained from $SU(2)$ Chern-Simons partition function on knot complement in S^3 and r denotes the dimension of the irreducible representation of $SU(2)$. Then the holonomy around the meridian cycle is conjugated to (3.16). So we have appropriate change of variables; $u=i\pi r/k$ with $\hbar=2\pi i/k$. Thus,

$$\widehat{u}J_r(K;q)=q^{r/2}J_r(K;q),\qquad \widehat{v}J_r(K;q)=J_{r+1}(K;q) \tag{3.28}$$

In general, full Chern-Simons partition function such as Jones polynomial is given by linear combination of partition function as in (3.11) for both abelian and non-abelian flat $SL(2,\mathbb{C})$ connections. In other words,

$$J_r(K;q)=\sum_\alpha \mathfrak{n}_\alpha Z_\alpha \tag{3.29}$$

where α contains both abelian and non-abelian holonomy of $SL(2,\mathbb{C})$ flat connections. Upon quantization, this full partition function is annihilated by proper quantum $\widehat{A}$-polynomial which always include $(\widehat{l}-1)$ factor in the left of everything else, and there are also nontrivial additional factor corresponding to non-abelian flat $SL(2,\mathbb{C})$ connections. If we take off $(\widehat{l}-1)$ part and call the rest of full quantum $\widehat{A}$-polynomial as $\widehat{A}^{nab}$, $\widehat{A}^{nab}$ cannot annihilate the full partition function. More explicitly, for Jones polynomial,

$$A^{nab}(\widehat{l},\widehat{m})J_r(K;q)=B(m;q)\neq 0 \tag{3.30}$$

After diving by $B(m;q)$ for both sides, multiplying $(\widehat{l}-1)$ gives RHS to be zero. After some calculation, we obtain

$$\Big((B(\widehat{m};q)\widehat{l}-B(q^{1/2}\widehat{m};q))\widehat{A}^{nab}(\widehat{l},\widehat{m};q)\Big)\,J_r(K;q)=0, \tag{3.31}$$

and the LHS is what we mean by full quantum $\widehat{A}$-polynomial. For example, $\widehat{A}^{nab}$ for non-abelian flat connections for trefoil knot, and figure-eight knot in S^3 are

$$\widehat{A}^{nab}_{3_1}(\widehat{l},\widehat{m};q) = \widehat{l} + q^{3/2}\widehat{m}^6 \tag{3.32}$$

$$\widehat{A}^{nab}_{4_1}(\widehat{l},\widehat{m};q) = q^{5/2}(1-q\widehat{m}^4)\widehat{m}^4\widehat{l}^2 - (1-q^2\widehat{m}^2)(1-q\widehat{m}-(q+q^3)\widehat{m}^4 - q^3\widehat{m}^6 + q^4\widehat{m}^8)\widehat{l} + q^{3/2}(1-q^3\widehat{m}^4)\widehat{m}^4 \tag{3.33}$$

By taking $q \to 1$ (or $\hbar \to 0$), above $\widehat{A}^{nab}(\widehat{l},\widehat{m};q)$ becomes classical A-polynomial for non-abelian flat connection.

There are commutative deformations of classical and quantum A-polynomials. Such quantum $\widehat{A}$-polynomials annihilate or give recursion relation for the corresponding Poincaré polynomials of the $sl(N)$ knot homology or the HOMFLY homology. This will be discussed in Chapter 5.

For above two examples, it can be obtained via gluing ideal tetrahedra, which is a next topic.

3.3 Ideal hyperbolic tetrahedra and their gluing

In [29], Chern-Simons theory on 3-manifolds M_3 which admits tetrahedra triangulation were considered. The purpose of this section is to give a general idea on ideal tetrahedra and gluing them. Detail construction and relevant subtleties are explained in [29, 39, 28]. In this section, we only consider $\mathfrak{g} = su(2)$.

3.3.1 An ideal tetrahedron, boundary phase space, and Lagrangian submanifold

It is known that one can describe flat $SL(2,\mathbb{C})$ structure in terms of hyperbolic geometry.

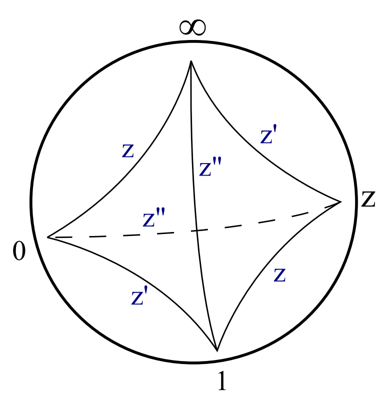


Figure 3.2: An ideal hyperbolic tetrahedron

A hyperbolic 3-manifold $\mathbb{H}^3$ is a 3-manifold if it admits a hyperbolic metric, which is a metric whose constant curvature -1, of finite volume. A hyperbolic 3-manifold can be regarded as inside of 3-ball B^3 with boundary $\partial\mathbb{H}^3 = \mathbb{C} \cup \{\infty\}$ which is a Riemann sphere. An ideal hyperbolic tetrahedron is a tetrahedron whose vertices are on the boundary of hyperbolic space $\partial\mathbb{H}^3$ and whose faces are geodesic surface in $\mathbb{H}^3$.

Its full hyperbolic structure is determined by a single cross ratio of the position of the vertices on $\partial\mathbb{H}^3$, which is called the shape parameter. By using isometry group $PSL(2, \mathbb{C})$ of hyperbolic 3-space which acts on the boundary as Möbius transformation, it is possible to fix three vertices at $0, 1$, and ∞. Let the position of remaining vertex be $z \in \mathbb{C}^*\backslash\{1\}$. Then the opposite edge of the ideal tetrahedron has dihedral angle $\arg(z)$, so we label those two edges as z.

Let other two edge parameters as z' and z''. These edge parameters are related to z;

$$zz'z'' = -1 \tag{3.34}$$

and any one of three equivalent forms;

$$z + z'^{-1} - 1 = 0 \tag{3.35}$$

$$z' + z''^{-1} - 1 = 0 \tag{3.36}$$

$$z'' + z^{-1} - 1 = 0 \tag{3.37}$$

where we will choose the last one. Or in logarithmic variables, $Z = \log z$ and similarly for z' and z'', we have

$$Z + Z' + Z'' = i\pi, \qquad e^{Z''} + e^{-Z} - 1 = 0. \tag{3.38}$$

The boundary phase space is given by

$$\mathcal{P}_{\partial\Delta} = \{(Z, Z', Z'') \in (\mathbb{C}\backslash 2\pi i\mathbb{Z})^3 | Z + Z' + Z'' = i\pi\}, \tag{3.39}$$

and this affine linear space is equipped with symplectic form

$$\omega_{\partial\Delta} = \frac{1}{\hbar} dZ'' \wedge dZ \tag{3.40}$$

where $\hbar$ is chosen as a normalization for the symplectic form. Or equivalently we have Poisson brackets;

$$\{Z, Z'\} = \{Z', Z''\} = \{Z'', Z\} = \hbar. \tag{3.41}$$

A polarization $\Pi_{\partial\Delta}$ for the boundary phase space $\mathcal{P}_{\partial\Delta}$ is about what we choose for position and conjugate momentum with respect to symplectic form $\omega_{\partial\Delta}$. For a tetrahedron, we have three sets of position X and conjugate momentum P;

$$(X,P) \quad : \quad \Pi_Z = (Z, Z''), \quad \Pi'_Z = (Z', Z), \quad \Pi''_Z = (Z'', Z'). \tag{3.42}$$

They are related by the action of affine symplectic transformation $Sp(2,\mathbb{Z}) \ltimes (i\pi\mathbb{Z})^2$. More explicitly, for example, from the relation above, one can map Π_Z to Π'_Z via

$$\begin{pmatrix} Z' \\ Z \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} Z \\ Z'' \end{pmatrix} + \begin{pmatrix} i\pi \\ 0 \end{pmatrix}. \tag{3.43}$$

Above matrix is expressed as $ST \in Sp(2,\mathbb{Z}) \simeq SL(2,\mathbb{Z})$ where $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Since $(ST)^3 = I$, one polarization gets back the original polarization after the action of ST three times.

Lagrangian submanifold of $\mathcal{P}_{\partial\Delta}$ of a tetrahedron is

$$\mathcal{L}_\Delta = \{e^{Z''} + e^{-Z} - 1 = 0\} \subset \mathcal{P}_{\partial\Delta}. \tag{3.44}$$

We have seen a boundary phase space and Lagrangian submanifold of a single tetrahedron, and we would like to see what they become upon gluing a number of tetrahedra.

3.3.2 Gluing tetrahedra and boundaries of 3-manifold

Before actually mentioning the gluing procedure, we consider the boundaries of 3-manifold from gluing tetrahedra.

If a 3-manifold M_3 obtained by gluing ideal tetrahedra, there are two kinds of boundaries, which are geodesic boundary and cusp boundary. The geodesic boundary is a geodesic surface of any genus possibly with punctures, and there is an induced 2d hyperbolic metric on it. 3d triangulation determines 2d triangulation on the geodesic boundary. This comes from the unglued faces of tetrahedra.

The cusp boundaries don't allow such triangulation. They are knotted loci in M_3 along which hyperbolic metric has a cone angle, or around which the flat $SL(2,\mathbb{C})$ connection has a monodromy. If one replaces ideal tetrahedra participating in gluing by the ideal tetrahedra whose four vertices are truncated or regularized, the loci are resolved to the boundary of M_3 with topology of tori T^2 or annuli $S^1 \times I$ where I denotes interval. The latter connects the punctures on the geodesic boundary of M_3. An induced metric on those boundaries are Euclidean.

Going back to gluing, we want to see how we get the boundary phase space $\mathcal{P}_{\partial M_3}$ and Lagrangian submanifold $\mathcal{L}_{M_3}$ for M_3 obtained from gluing ideal tetrahedra. Given a collection of a number of tetrahedra, what we mean by gluing is that the faces of tetrahedra are glued such that three edges of the each faces are also glued.

If we want to make hyperbolic metric on M_3 to be smooth on the internal edge from the gluing, the total dihedral angles around the internal edge should be 2π with zero hyperbolic torsion. More explicitly, if C_I is the internal edge from the gluing, the following should hold

$$C_I = \sum_{i=1}^{L} (n(I,i)Z_i + n'(I,i)Z'_i + n''(I,i)Z''_i) = 2\pi i \tag{3.45}$$

where L is the total number of tetrahedra and $n(I,i)$ denotes how many edges are glued to I-th internal edge. So $n(I,i)$ takes value 0,1,2,[2] and similar for $n'(I,i)$ and $n''(I,i)$.

We are interested in the boundary phase space $\mathcal{P}_{\partial M_3}$ for three-manifold M_3 made of gluing tetrahedra. For this purpose, one first has the product of phase space of each tetrahedra Δ_i;

$$\mathcal{P}_{\{\partial\Delta_i\}} = \prod_{i=1}^{L} \mathcal{P}_{\partial\Delta_i} \tag{3.46}$$

and Poisson brackets

$$\{Z_i, Z'_j\} = \{Z'_i, Z''_j\} = \{Z''_i, Z_j\} = \hbar\delta_{ij}. \tag{3.47}$$

It is known that the parameter for the internal edges $-C_I-$ in (3.45) commute each other [40]. Among all edges in $\mathcal{P}_{\{\partial\Delta_i\}}$, the remaining linear combination of edge parameters which commute with C_I parametrize the boundary phase space $\mathcal{P}_{\partial M_3}$ of M_3. Also, it is known that the map between positions and momenta (polarization $\{\Pi_i\}$) of product phase space $\mathcal{P}_{\partial\Delta_i}$ and polarization $\widetilde{\Pi}$ of $\mathcal{P}_{\partial M_3}$ which we choose is the affine $Sp(2L,\mathbb{C})$ transformation.[3]

One can regard the process to obtain $\mathcal{P}_{\partial M_3}$ mentioned above as a symplectic quotient of $\mathcal{P}_{\{\partial\Delta_i\}}$ with moment map C_I;

$$\mathcal{P}_{\partial M} = \left(\prod_{i=1}^{L} \mathcal{P}_{\partial\Delta_i}\right) // \left(C_I = 2\pi i\right). \tag{3.48}$$

[2]Note that there are two edge parameters in a single tetrahedron.

[3]More precisely, $Sp(2L,\mathbb{Q})$ with translations by rational multiples of $i\pi$.

Regarding Lagrangian submanifold, one can do similarly. After taking products of each Lagrangian submanifolds, one eliminates algebraically the edge parameters which don't commute with C_I, then takes $C_I = 2\pi i$. Then one obtains the Lagrangian submanifold $\mathcal{L}_{M_3}$ of $\mathcal{P}_{\partial M_3}$.

One can apply above techniques similarly to 3-manifold with the geodesic surfaces boundary or the cusp boundary. If one consider a knot complement, one obtains $\mathcal{P}_{T^2}$ as a boundary phase space and one can show that from $\mathcal{L}_{M_3}$ constructed via gluing procedure gives non-abelian branch of A-polynomial mentioned in section 3.2.

3.3.3 2-3 Pachner move

We have discussed how boundary phase space $\mathcal{P}_{\partial M_3}$ and Lagrangian submanifold $\mathcal{L}_{M_3}$ are obtained upon gluing tetrahedra. It is desirable that these quantities are independent of choice of triangulations of M_3. Meanwhile, it is known that any two triangulations of M_3 which have same triangulations on the boundary surface of M_3 are related by a sequence of 2-3 Pachner moves. 2-3 Pachner move states that two tetrahedra glued along a common face is interchangeable with three tetrahedra glued along three each faces with a common internal edge.

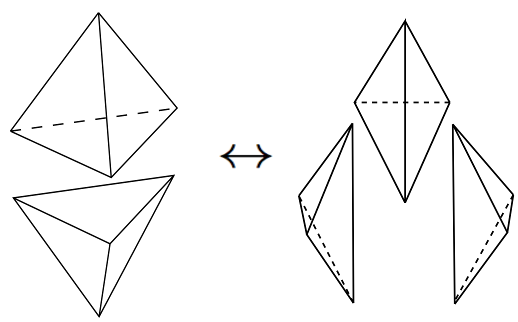


Figure 3.3: 2-3 Pachner move

Therefore, if one can show that $\mathcal{P}_{\partial M_3}$ and $\mathcal{L}_{M_3}$ stay same under 2-3 Pachner move, then given a fixed triangulation at the boundary surface of M_3, $\mathcal{P}_{\partial M_3}$ and $\mathcal{L}_{M_3}$ are independent with specific triangulation of M_3. This was actually shown in [29]

3.3.4 Quantization, wavefunction, and partition function

One can quantize above classical analysis systematically. [4] We just quote the results here.

[4]Actually, what we mean by quantization here is holomorphic quantization, so Hilbert space in this subsection is not an honest Hilbert space which is from quantization of phase space with real polarization and real symplectic form. Chern-Simons wavefunction or holomorphic block lives in the vector space of holomorphic functions which forms representation of operator algebra. But we will just use terminology "Hilbert space".

Upon quantization, edge parameters Z, Z', Z'' are promoted to quantum operator $\widehat{Z}, \widehat{Z}', \widehat{Z}''$,

$$[\widehat{Z}, \widehat{Z}'] = [\widehat{Z}', \widehat{Z}''] = [\widehat{Z}'', \widehat{Z}] = \hbar; \tag{3.49}$$

and the equation for the boundary phase space becomes

$$\widehat{Z} + \widehat{Z}' + \widehat{Z}'' = i\pi + \frac{\hbar}{2} \tag{3.50}$$

where the $\hbar$ term is regarded as a quantum correction which is determined by requiring topological invariance of combinatorial construction of $\widehat{\mathcal{L}}$ and S-duality of the operator algebra.

The Lagrangian submanifold becomes a quantum operator on Hilbert space,

$$\widehat{\mathcal{L}}_\Delta = \widehat{z}'' + \widehat{z}^{-1} - 1 \simeq 0 \tag{3.51}$$

where $\simeq$ is understood in the sense that it annihilates a certain wavefunction. An operator algebra act on the space of locally holomorphic function in a way that

$$\widehat{Z} \simeq \hbar Z, \quad \widehat{Z}' \simeq i\pi + \frac{\hbar}{2} - \hbar\partial_Z - Z, \quad \widehat{Z}'' \simeq \partial_Z, \tag{3.52}$$

Then up to normalization ambiguity, a wavefunction on a tetrahedron $\psi(\Delta; Z) := \psi_\Delta(Z)$ is

$$\psi_\Delta(Z) = \prod_{r=1}^{\infty}(1 - q^r z^{-1}), \tag{3.53}$$

where one can check $\widehat{\mathcal{L}}_\Delta\, \psi_\Delta(Z) = 0$.

One can perform quantum gluing to obtain $\widehat{\mathcal{L}}_{M_3}$ from the product of a number of $\widehat{\mathcal{L}}_\Delta$ as done in classical case. A bit more specifically, the internal edge parameter C_I is lifted to operator $\widehat{C}_I$, and from the product of $\widehat{\mathcal{L}}_\Delta$'s, one removes all elements which do not commute with the $\widehat{C}_I$, and then takes $\widehat{C}_I = 2\pi i + \hbar$. But it is more complicated due to noncommutativity. The coefficients for $\hbar$ in $\widehat{C}_I$ is fixed from the consideration of topological invariance of combinatorial construction of $\widehat{\mathcal{L}}_{M_3}$.

As mentioned above, in the gluing procedure, there is a map from the positions and conjugate momenta in polarization Π of the product of phase space of each tetrahedra to the positions and conjugate momenta in polarization $\widetilde{\Pi}$ of the phase space $\mathcal{P}_{\partial M_3}$, and this map is given by the affine symplectic group $ISp(2L, \mathbb{Q})$ (semiproduct of $Sp(2L, \mathbb{Q})$ and affine shifts). At the quantum level, it is similar, but there is $\hbar$ correction in affine shift. We are interested in what the wavefunction will be upon change of polarization from Π to $\widetilde{\Pi}$.

We are interested in using

$$\Psi_\Delta(Z) = \Phi_{\hbar/2}(-Z + i\pi + \hbar/2), \tag{3.54}$$

for a single tetrahedron as a "wavefunction", and how one can obtain the corresponding quantity on M_3 via gluing procedure from the product of $\Psi_\Delta(Z)$'s. Here, $\Phi_{\hbar/2}$ is the non-compact quantum dilogarithm [41]

$$\Phi_{\hbar/2}(p) = \begin{cases} \prod_{j=1}^{\infty} \frac{1+q^{j-1/2}e^{p}}{1+\widetilde{q}^{-j+1/2}e^{\widetilde{p}}} & |q| < 1 \\ \prod_{j=1}^{\infty} \frac{1+\widetilde{q}^{j-1/2}e^{\widetilde{p}}}{1+q^{-j+1/2}e^{p}} & |q| > 1 \end{cases} \tag{3.55}$$

where

$$\widetilde{p} = \frac{2\pi i}{\hbar} p, \qquad \widetilde{q} = e^{-\frac{4\pi^2}{\hbar}}. \tag{3.56}$$

So (3.54) can be thought as doubling of wavefunction in (3.53), which is annihilated by $\widehat{\mathcal{L}}_\Delta = \widehat{z}'' + \widehat{z}^{-1} - 1$ and $\widehat{\widetilde{\mathcal{L}}} = \widehat{\widetilde{z}}'' + \widehat{\widetilde{z}}^{-1} - 1$ where

$$\widehat{\widetilde{z}} = \exp(\widehat{\widetilde{Z}}), \quad \widehat{\widetilde{z}}'' = \exp(\widehat{\widetilde{Z}}''), \qquad \text{with } \widehat{\widetilde{Z}} = \frac{2\pi i}{\hbar}\widehat{Z}, \quad \widehat{\widetilde{Z}}'' = \frac{2\pi i}{\hbar}\widehat{Z}'' \tag{3.57}$$

and original variables (without tilde) and dual variables (with tilde) commute each other. Dual variables satisfy $\widehat{\widetilde{z}}''\widehat{\widetilde{z}} = \widetilde{q}\widehat{\widetilde{z}}\widehat{\widetilde{z}}''$.

This "wavefunction" $\Psi_\Delta(Z)$ on a single tetrahedron is actually analytically continued $SL(2,\mathbb{R})$ Chern-Simons partition function on a single tetrahedron. Modulo some subtleties, formally it is in Weil representation of $ISp(2L,\mathbb{C})$ [42, 43, 29].

More explicitly, let φ be affine symplectic $ISp(2L,\mathbb{C})$ action on wavefunctions

$$\Psi(Z_1, \cdots, Z_L) \overset{\varphi}{\mapsto} \Psi(\underbrace{E_1, \cdots, C_1, \cdots}_{L}) \tag{3.58}$$

induced from the change of symplectic basis;

$$(\widehat{Z}_1, \cdots, \widehat{Z}_L, \widehat{Z}_1'', \cdots, \widehat{Z}_L'') \overset{\varphi_*}{\longmapsto} (\underbrace{\widehat{E}_1, \cdots, \widehat{C}_1, \cdots}_{L}, \underbrace{\widehat{\Gamma}_{E,1}, \cdots, \widehat{\Gamma}_{C,1}, \cdots}_{L}) \tag{3.59}$$

where $\widehat{E}_K$, $K = 1, \cdots$ denote the operator for external edge (or cusp holonomy) parameters which are "positions" in polarization $\Pi \subset \widetilde{\Pi}$ and commute with $\widehat{C}_I$, and Γ's are corresponding conjugate operators. For convenience, we call the former set as $(\vec{\widehat{Z}}, \vec{\widehat{Z}}'')$ and the latter set as $(\vec{\widehat{X}}, \vec{\widehat{Y}})$.

Symplectic group $Sp(2L,\mathbb{C})$ is generated by three generators and acts on the column vector

$(\vec{\hat{Z}}, \vec{\hat{Z}''})^t$ as a form of L by L block matrices;

$$\varphi_{*,T} = \begin{pmatrix} I & 0 \\ B & I \end{pmatrix}, \quad \varphi_{*,S} = \begin{pmatrix} I-J & -J \\ J & I-J \end{pmatrix}, \quad \varphi_{*,U} = \begin{pmatrix} U & 0 \\ 0 & U^{-1\,t} \end{pmatrix} \tag{3.60}$$

where I is the identity matrix, B is symmetric matrix, J is diagonal matrices whose entries are 0 or 1, and U is invertible. Then wavefunction $\Psi(\vec{Z})$ transform under above φ_* as

$$\Psi(\vec{Z}) \xmapsto{\varphi_T} \Psi'(\vec{X}) = \Psi(\vec{X})\, e^{\frac{1}{2\hbar}\vec{X}^t \cdot B \cdot \vec{X}} \tag{3.61}$$

$$\Psi(\vec{Z}) \xmapsto{\varphi_S} \Psi'(\vec{X}) \simeq \int d\vec{Z}\, \Psi(\vec{Z})\, e^{\frac{1}{\hbar}\vec{X}\cdot J\vec{Z}} \tag{3.62}$$

$$\Psi(\vec{Z}) \xmapsto{\varphi_U} \Psi'(\vec{X}) \simeq \Psi(U^{-1} \cdot \vec{X}) \tag{3.63}$$

up to overall normalization. For the affine transformation

$$(\vec{\hat{Z}},\, \vec{\hat{Z}''}) \xmapsto{\varphi_{*,\mathrm{pos.}}} (\vec{\hat{Z}} + \vec{s},\, \vec{\hat{Z}''}) \tag{3.64}$$

$$(\vec{\hat{Z}},\, \vec{\hat{Z}''}) \xmapsto{\varphi_{*,\mathrm{mom.}}} (\vec{\hat{Z}},\, \vec{\hat{Z}''} + \vec{t}), \tag{3.65}$$

the wavefunction $f(Z)$ transform as

$$\Psi(Z) \xmapsto{\varphi_{\mathrm{pos.}}} \Psi'(\vec{X}) = \Psi(\vec{X} - \vec{s}) \tag{3.66}$$

$$\Psi(Z) \xmapsto{\varphi_{\mathrm{mom.}}} \Psi'(\vec{X}) = \Psi(\vec{X}) e^{\frac{1}{\hbar}\vec{t}\cdot\vec{X}} \tag{3.67}$$

where elements in $\vec{s}$ or $\vec{t}$ are rational multiple of $i\pi$ and $\hbar$.

This looks similar $Sp(2L, \mathbb{Z})$ action on 3d conformal field theories with abelian symmetries discussed in Chapter 2.

Chapter 4

3d-3d Correspondence from Gluing Tetrahedra

In similar spirit of the correspondence between 4d $\mathcal{N}=2$ superconformal field theories and Liouville (or Toda) theories on a Riemann surface [44], when 6d (2,0) theory with Lie algebra $\mathfrak{g}=\text{Lie}(G)$ of ADE type is wrapped on three manifold $-$ M_3 $-$ with partial twisting, we have 3d $\mathcal{N}=2$ superconformal field theories $-$ $T[M_3;G]$ $-$ described as the IR fixed points of abelian Chern-Simons-matter theories determined by M_3, and if M_3 has a boundary they are also determined by a chosen polarization $\widetilde{\Pi}$ of complex phase space $\mathcal{P}_{\partial M_3}$ of flat $G_{\mathbb{C}}$ connection on ∂M_3;

$$(M_3\,,\widetilde{\Pi}\,,G) \qquad \rightsquigarrow \qquad T[M_3,\Pi;G], \tag{4.1}$$

and the 3d-3d correspondence states that physical quantities such as partition functions and classical vacua of non-supersymmetric complex Chern-Simons theory on M_3 are matched with those of 3d $\mathcal{N}=2$ abelian SCFT [39, 25, 45, 13, 46, 47, 48].

In this thesis, we are only interested in the case the number of brane is 2, *i.e.* the case $G_{\mathbb{C}} = SL(2,\mathbb{C})$, and for simplification $T[M_3,\widetilde{\Pi};G]$ is denoted as $T[M_3]$. The case for higher rank of A-type G was considered in [45].

Original 3d-3d correspondence in [39] was constructed from gluing ideal tetrahedra. As we discussed in previous section, one can consider Chern-Simons theory on 3-manifold obtained from gluing tetrahedra. One can also calculate classical moduli space of flat $SL(2,\mathbb{C})$ vacua $\mathcal{L}_{M_3}$, and quantization $\widehat{\mathcal{L}}_{M_3}$ which annihilates non-perturbative Chern-Simons wavefunction or analytically continued $SL(2,\mathbb{R})$ partition functions. If one can find a 3d $\mathcal{N}=2$ gauge theory corresponding to Chern-Simons theory on an ideal tetrahedron and also a proper gauge-theoretic interpretation of gluing procedure in Chern-Simons theory side, the 3d-3d correspondence is built in the construction. This is a basic idea in [39].

However, this construction doesn't give a complete 3d-3d correspondence. This is because Chern-

Simons theory on three-manifold from gluing tetrahedra always missing information of abelian flat connection. Examples which captures all flat $SL(2,\mathbb{C})$ connections will be discussed in Chapter 5.

In this chapter, we summarize the result of the 3d-3d correspondence of [39, 25, 13];

In section 4.1, we review motivations of 3d-3d correspondence from 2d-4d correspondence and vortex partition function.

In section 4.2, we summarize dictionary between Chern-Simons theory and 3d $\mathcal{N}=2$ theories.

In section 4.3, we discuss known examples for 3d-3d correspondence.

4.1 Motivation for 3d-3d correspondence

If M5 branes are compactified on a certain d-dimensional manifold M_d with an appropriate twisting on it, one obtains $6-d$ dimensional supersymmetric field theory. In this perspective, when considering a Riemann surface with some number of punctures, one obtains 4d $\mathcal{N}=2$ superconformal field theories characterized by the Riemann surface on which M5-branes are wrapped. Interestingly it was found in [44] that when the number of M5-branes is 2 the instanton partition function of 4d $\mathcal{N}=2$ SCFT is matched with conformal block of Liouville CFT (or Toda theory for N M5-branes). Such *correspondence* was extended to more general gauge group, also for superconformal index on $S^1\times S^3$, and with insertion of extended object such as Wilson loop, 't Hooft loop, or surface operators.

Also, the domain wall in 4d $\mathcal{N}=4$ or 2^* theory were considered in [49, 50, 51, 11]. In geometry side, this corresponds to the mapping cylinder $M_3 = C\times_\varphi I$ where the copy of a once-punctured Riemann surface C on the one side is related to the copy of a once-punctured Riemann surface C on the another side via mapping class group twist φ. In this setup, the duality kernel $\mathcal{Z}_\varphi^{\mathcal{N}=2}(\epsilon_1,\epsilon_2)$ can be interpreted as a partition function $\mathcal{Z}_\varphi^{3d}(\epsilon_1/\epsilon_2)$ of certain 3d $\mathcal{N}=2$ theory on three-sphere S^3. In addition, the 2d-4d correspondence tells us that

$$\mathcal{Z}_\varphi^{\mathcal{N}=2}(\epsilon_1,\epsilon_2)=\mathcal{Z}_\varphi^{Liouville}(b) \tag{4.2}$$

where the LHS is a partition function on the domain wall and the RHS is so called Moore-Seiberg kernel in Liouville theory with $b^2=\epsilon_1/\epsilon_2$.

In literature, it has been well-known that $\mathcal{Z}_\varphi^{Liouville}(b)=\mathcal{Z}_\varphi^{Teich}(\epsilon_1/\epsilon_2)=\mathcal{Z}^{CS}(M_\varphi;\epsilon_1/\epsilon_2)$ where $\mathcal{Z}_\varphi^{Teich}(\epsilon_1/\epsilon_2)$ is a kernel in quantum Teichmüler theory associated to mapping class group φ and $\mathcal{Z}^{CS}(M_\varphi;\epsilon_1/\epsilon_2)$ is Chern-Simons partition function on mapping cylinder M_φ with analytically level $k=\epsilon_2/\epsilon_1$.

Therefore, one can expect with above relations,

$$\mathcal{Z}^{CS}(M_\varphi;\epsilon_1/\epsilon_2)=\mathcal{Z}^{3d}_\varphi(\epsilon_1/\epsilon_2). \tag{4.3}$$

In other words, one can expect that there is a correspondence at least for Chern-Simons theory on mapping cylinder M_φ and 3d $\mathcal{N}=2$ gauge theory on S^3.

In addition, from the "classical" limit ($\hbar\to 0$) of vortex partition function of effective 2d $\mathcal{N}=(2,2)$ theory [52], it was expected that (complex) Chern-Simons theory corresponds to 3d $\mathcal{N}=2$ theories in some way. For example, in *loc. cit.*, it was discussed that moduli space of complex flat connection in Chern-Simons theory on 3-manifold is equal to the moduli space of supersymmetric vacua of effective 2d $\mathcal{N}=(2,2)$ theories. So these make us to expect that there is a correspondence between Chern-Simons theory on 3-manifold and 3d $\mathcal{N}=2$ theories.

4.2 3d-3d correspondence of Dimofte-Gaiotto-Gukov

In this section, we summarize the dictionary between 3d $\mathcal{N}=2$ theories and Chern-Simons theory on M_3 admitting tetrahedra triangulation [39, 25, 13].

4.2.1 3d-3d correspondence for a tetrahedron

We first consider a 3d $\mathcal{N}=2$ theory for a single tetrahedra. Regarding partition functions, we saw that sine double function appeared in partition function on S^3_b of 3d $\mathcal{N}=2$ theories. At the same time, as mentioned in section 3, the non-compact quantum dilogarithm function, which is closely related to sine double function, appeared in a partition function of $SL(2,\mathbb{R})$ Chern-Simons theory. From this observation, one can read off field contents and Chern-Simons coupling corresponding to a single tetrahedron by matching variables between two such as a certain quantity of a chiral multiplet and an edge parameter. At the same time, one can also check with moduli space of suerpsymmetric vacua (or supersymmetric parameter space) of 3d $\mathcal{N}=2$ theories and the moduli space of flat $SL(2,\mathbb{C})$ connection of Chern-Simons theory on a single tetrahedron.

In sum, the 3d-3d correspondence for a single tetrahedron $M_3=\Delta$,

- A 3d $\mathcal{N}=2$ theory $T[\Delta;\Pi]$ associated to a single tetrahedron Δ and polarization Π of $\mathcal{P}_{\partial\Delta}$ is a free chiral multiplet charged $+1$ under background $U(1)$ gauge symmetry (*i.e.* flavor or global symmetry) with Chern-simons level $-1/2$.

  Given a tetrahedron Δ, if choosing a polarization as $\Pi_Z=(Z,Z'')$, a free chiral Φ with $U(1)_Z$ corresponds to an edge parameter Z of Δ, which we denote such chiral as Φ_Z. Then $\mathrm{Re}(Z)$ and

$\mathrm{Im}(Z)/\pi$ corresponds to the twisted mass parameter of $U(1)_Z$ background gauge symmetry or global symmetry and R-charge of Φ_Z, respectively.

- Lagrangian submanifold $\mathcal{L}_\Delta$ in classical phase space $\mathcal{P}_{\partial\Delta}$, which is classical moduli space of flat $SL(2,\mathbb{C})$ connections, corresponds to supersymmetric parameter space of $T[\Delta]$ on $\mathbb{R}^2 \times S^1$;

$$\mathcal{M}_{\text{flat}}(\Delta; SL(2,\mathbb{C})) = \{z'' + z^{-1} - 1 = 0\} = \mathcal{M}_{\text{SUSY}}(\Delta) \tag{4.4}$$

- Partition function on squashed 3-sphere S^3_b, superconformal index on $S^2 \times_q S^1$, and holomorphic block $D^2 \times_q S^1$ of $T[\Delta]$ are equal to partition function of $SL(2,\mathbb{R})$ Chern-Simons theory, partition function of $SL(2,\mathbb{C})$ Chern-Simons theory, and analytically continued Chern-Simons wavefunction on Δ, respectively;

$$\begin{array}{rcccl} \mathcal{Z}_{S^3_b}(T[\Delta]) & = & \prod_{r=0}^{\infty} \frac{1-q^{r+1}z^{-1}}{1-({}^L q)^r({}^L z)^{-1}} & = & \mathcal{Z}_{CS}(\Delta; SL(2,\mathbb{R})) \\ \mathcal{Z}_{S^2\times_q S^1}(T[\Delta]) & = & \prod_{r=0}^{\infty} \frac{1-q^{r+1}z^{-1}}{1-q^r\bar{z}} & = & \mathcal{Z}_{CS}(\Delta; SL(2,\mathbb{C})) \\ \mathcal{Z}_{D^2\times_q S^1}(T[\Delta]) & = & \prod_{r=0}^{\infty}(1-q^{r+1}z^{-1}) & = & \psi_{CS}(\Delta) \end{array} \tag{4.5}$$

where $q = e^{\hbar}$. For the S^3_b partition function, $\hbar = 2\pi i b$, ${}^L\hbar = 2\pi i b^{-1}$ with ${}^L q = e^{{}^L\hbar}$, and b is a squash parameter of S^3_b; ${}^L z = z^{2\pi i/\hbar}$ with $z = e^Z$ and $Z = 2\pi b m_Z + \left(i\pi + \frac{\hbar}{2}\right) R_Z$ where m_Z and R_Z are real mass and R-charge of Φ_Z. For the index, $z = q^{m/2}\zeta$, $\bar{z} = q^{m/2}\zeta^{-1}$ with m and ζ being magnetic flux and fugacity (also called as chemical potential) for $U(1)_Z$ symmetry.

In addition, quantum operator $\widehat{\mathcal{L}}_\Delta$ discussed in section 3.3 is interpreted as Ward identity for line operators in a 4d theory coupled to the boundary theory $T[\Delta]$. A bit more explicitly, if M_3 has geodesic boundary $\mathcal{C} = \partial M_3$, $T[M_3]$ provide a half-BPS boundary condition of $\mathcal{N} = 2$ abelian theory on Coulomb branch of 4d $\mathcal{N} = 2$ $T[\mathcal{C}, \mathfrak{su}(2)]$ theory, *i.e.* Seiberg-Witten theory of $T[\mathcal{C}, \mathfrak{su}(2)]$. Then line operators in 4d $\mathcal{N} = 2$ abelian theory, which are brought into the 3d boundary where $T[M_3]$ live, satisfy Ward identity. For example, when $M_3 = \Delta$ with polarization Π_Z, $z'' + z^{-1} - 1 \simeq 0$ is interpreted as Ward identity $H + W^{-1} - 1 \simeq 0$ of Wilson (W) and 't Hooft (H) line operator satisfy where z, z', and z'' correspond to Wilson, Wilson-'t Hooft, and 't Hooft line operator, respectively [39].

We have seen how 3d-3d correspondence works for a single tetrahedra. In order to have the correspondence for general 3-manifold M_3 obtained by gluing tetrahedra, we also need to match gluing procedures of both sides. This is successful [39, 25, 13] as we will review in next subsection, so 3d-3d correspondence holds for general 3-manifold obtained by gluing tetrahedra.

4.2.2 Gluing procedures

Given a correspondence above, one can expect that (affine) $SL(2,\mathbb{Z})$ action on 3d $\mathcal{N}=2$ superconformal field theories with an abelian symmetry correspond to (affine) $SL(2,\mathbb{Z})$ action on the polarization Π of $\mathcal{P}_{\partial\Delta}$. More generally, (affine) $Sp(2L,\mathbb{Z})$ action on 3d $\mathcal{N}=2$ superconformal field theories with $U(1)^L$ symmetries is expected to correspond to (affine) $Sp(2L,\mathbb{Z})$ action which maps the polarization $\Pi_1\times\cdots\times\Pi_L$ of $\mathcal{P}_{\{\partial\Delta_i\}}$ to polarization $\widetilde{\Pi}$ of $\mathcal{P}_{\partial M_3}$. It turns out that one can match each other. More specifically, gluing procedure for 3d $\mathcal{N}=2$ theories $T[\Delta_i;\Pi]$, $i=1,\cdots L$ for L tetrahedra is constructed as follows;

1. One first takes a product of theories $T[\Delta_i,\Pi_i]$; $T[\{\Delta_i\},\{\Pi_i\}]=T[\Delta_1,\Pi_1]\otimes\cdots\otimes T[\Delta_L,\Pi_L]$. The resulting theory $T[\{\Delta_i\},\{\Pi_i\}]$ consists of L chiral multiplets Φ_i corresponding to edge parameter Z_i of product polarization $\Pi_1\times\cdots\Pi_L$ of product boundary phase space $\mathcal{P}_{\partial\Delta_1}\times\cdots\times\mathcal{P}_{\partial\Delta_L}$, and each of which are charged $+1$ under $U(1)^L:=\underbrace{U(1)\times\cdots\times U(1)}_{L}$ global symmetries. There are also Chern-Simons term with level $-\frac{1}{2}$ for each $U(1)^L$.

2. One chooses a new polarization $\widetilde{\Pi}$ for the boundary phase space $\mathcal{P}_{\partial M_3}$ of M_3 such that positions and momenta in $\Pi=\Pi_1\times\cdots\times\Pi_L$ maps to positions and momenta in $\widetilde{\Pi}$ and in a new polarization the internal edge C_I is given by linear combinations of positions of $\widetilde{\Pi}$. Quantization of positions and momenta in $\widetilde{\Pi}$ are related to those in $\Pi_1\times\cdots\times\Pi_L$ by affine symplectic transformation $Sp(2L,\mathbb{Z})\ltimes\left[\left((i\pi+\frac{\hbar}{2})\mathbb{Z}\right)^{2L}\right]$ also acting on $2L$ column vector $(\widehat{Z}_1,\cdots,\widehat{Z}_L,\widehat{Z}''_1,\cdots\widehat{Z}''_L)^T$ of $\Pi_1\times\cdots\times\Pi_L$.

3. One applies the affine symplectic action $g\in Sp(2L,\mathbb{Z})\ltimes\left[\left((i\pi+\frac{\hbar}{2})\mathbb{Z}\right)^{2L}\right]$ on product theories $T[\Delta_i]$, which also appeared in Chern-Simons theory as Weil transformation

   3a. GL-type action $g_U=\begin{pmatrix}U & 0\\ 0 & U^{-1\ t}\end{pmatrix}$ with invertible $U\in GL(L,\mathbb{Z})$ act on $U(1)^L$ vector multiplets by linear transformation so that it refines $U(1)^L$ global symmetries.

   3b. T-type action $g_T=\begin{pmatrix}I & 0\\ B & I\end{pmatrix}$ with symmetric matrix B add (mixed) Chern-Simons term for background vector multiplet with level $k_{ij}=B_{ij}$.

   3c. S-type action $g_S=\begin{pmatrix}I-J & -J\\ J & I-J\end{pmatrix}$ with a diagonal matrix $J=\mathrm{diag}(j_1,\cdots,j_L)$ with j_i is 0 or 1 gauge background $U(1)_i$ symmetry and introduce topological $U(1)$ symmetry with respect to $U(1)_i$ for i such that $j_i=1$.

   3d. Affine shifts are relevant for the theories on compactified spaces such as $D^2\times_q S^1$, S^3_b, or $S^2\times_q S^1$, but not on $\mathbb{R}^3$. Affine shifts on the positions add a unit of flavor symmetry to

R-symmetry. In terms of holomorphic block, it shifts Wilson line for flavor symmetry by $-i\pi - \frac{\hbar}{2} = \log(-q^{\frac{1}{2}})$.

Affine shifts on the momenta add mixed Chern-Simons term between the background $U(1)_R$ symmetry and global symmetry.

4. We add a superpotential $W = \sum_I \mathcal{O}_I$ where each operator $\mathcal{O}_{C_I}$ is a product of chiral fields, which comes from the internal edge parameter C_I given by linear combination of edge parameters for positions. This breaks all $U(1)$ global symmetries associated to C_I for each I, and $U(1)^{\frac{1}{2}\dim\mathcal{P}_{\partial M}}$ are left.

Thus from above gluing procedure, one can construct 3d $\mathcal{N}=2$ theories corresponding to Chern-Simons theory on M_3 admitting tetrahedra triangulations. Also, as the quantities are matched in the case of a tetrahedron and gluing procedure is identified, one can calculate a supersymmetric parameter space, Ward identity of line operators, partition function on S^3_b, index on $S^2 \times_q S^1$, and holomorphic block of $T[M_3]$ via gluing procedure as done in section 3.3 for Chern-Simons theory[1] and by construction they correspond, respectively, to classical moduli space of flat $SL(2,\mathbb{C})$ connections $\mathcal{L}_{M_3}$, quantum operator $\widehat{\mathcal{L}}_{M_3}$ obtained by quantization of $\mathcal{L}_{M_3}$, $SL(2,\mathbb{R})$ partition function $\mathcal{Z}_{CS}(M_3; SL(2,\mathbb{R}))$, full $SL(2,\mathbb{C})$ partition function $\mathcal{Z}_{CS}(M_3; SL(2,\mathbb{C}))$, and analytically continued Chern-Simons wavefunction $\psi_{CS}(M_3)$.

4.3 Examples

We would like to provide briefly some known examples from [39, 25, 13].

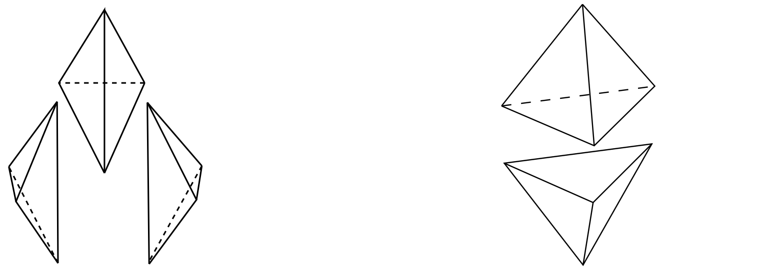
4.3.1 SQED with $N_f = 1$, XYZ model, and 3d $\mathcal{N}=2$ mirror symmetry

The SQED with one flavor ($N_f = 1$) in 3d $\mathcal{N} = 2$ theory is $U(1)$ gauge theory with two chiral mulitplets charged oppositely under gauge group. They have same charge under axial $U(1)$ global symmetry. For the XYZ model, there is no gauge symmetry, and there are three chiral multiplets with superpotential given by product of three chirals. Firstly, we would like to see how they arise from gluing tetrahedra.

4.3.1.1 SQED $N_f = 1$ and XYZ model from gluing tetrahedra

XYZ model is made from gluing three tetrahedra with two adjacent faces of each three tetrahedra are glued in a way that they have a common internal edge. One choose a polarization

[1]The full $SL(2,\mathbb{C})$ Chern-Simons partition function on M_3 via gluing procedure was not discussed in section 3.3, but it can be done similarly and was discussed in detail in section 6.2 of [25].



(a) XYZ model: gluing 3 tetrahedra (b) SQED $N_f = 1$: gluing 2 tetrahedra

Figure 4.1: Gluing tetrahedra

$\widetilde{\Pi} = (X_1, X_2, C; P_1, P_2, \Gamma)$ which are related to position-momentum (Z, Z''), (W, W''), and (Y, Y'') of $\Pi_1 \times \Pi_2 \times \Pi_3$ by

$$\begin{pmatrix} X_1 \\ X_2 \\ C \\ P_1 \\ P_2 \\ \Gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} Z \\ W \\ Y \\ Z'' \\ W \\ Y'' \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ i\pi + \frac{\hbar}{2} \\ i\pi + \frac{\hbar}{2} \\ -i\pi - \frac{\hbar}{2} \end{pmatrix} \tag{4.6}$$

where the matrix on the right is GL-type and the one on the left is T-type. From the product of three Lagrangian (in flat spacetime)

$$\mathcal{L}_{\{\Pi_i\}}[V_Z, V_W, V_Y] = \frac{1}{4\pi} \int d^4\theta \left(-\frac{1}{2}\Sigma_Z V_Z - \frac{1}{2}\Sigma_W V_W - \frac{1}{2}\Sigma_Y V_Y \right) + \int d^4\theta \left(\Phi_Z^\dagger e^{V_Z} \Phi_Z + \Phi_W^\dagger e^{V_W} \Phi_W + \Phi_Y^\dagger e^{V_Y} \Phi_Y \right) \tag{4.7}$$

By applying the dictionary, adding Chern-Simons term for V_Y (T-type) and redefining $V_Z = V_{X_1}$, $V_W = V_{X_2}$, and $V_Y = V_C - V_{X_1} - V_{X_2}$, one obtains

$$\mathcal{L}_{\widetilde{\Pi}}[V_{X_1}, V_{X_2}, V_C] = \mathcal{L}_{\{\Pi_i\}}[V_{X_1}, V_{X_2}, V_C - V_{X_1} - V_{X_2}] + \frac{1}{4\pi} \int d^4\theta (\Sigma_C - \Sigma_{X_1} - \Sigma_{X_2})(V_C - V_{X_1} - V_{X_2}) \tag{4.8}$$

From $V_Y = V_C - V_{X_1} - V_{X_2}$, there is a superpotential $W = \mu \Phi_X \Phi_Y \Phi_Z$.

One can do similarly for SQED with $N_f = 1$ which is obtained from gluing two tetrahedra glued along one face, but we don't discuss it further.

Summarizing the field contents and Chern-Simons levels, with relabeling parameters for later use

[13], theory T_{XYZ} of XYZ model is

charges :

	Φ_1	Φ_2	Φ_3
$U(1)_x$	1	0	-1
$U(1)_y$	0	1	-1
$U(1)_R$	0	0	2

CS levels :

	$U(1)_x$	$U(1)_y$	$U(1)_R$
$U(1)_x$	0	1/2	0
$U(1)_y$	1/2	0	0
$U(1)_R$	0	0	$-1/2$

(4.9)

with a superpotential

$$W = \mu\, \Phi_1 \Phi_2 \Phi_3. \tag{4.10}$$

$U(1)_x$ and $U(1)_y$ are flavor symmetries whose complexified twisted mass parameters are $X = \log x$ and $Y = \log y$, respectively. Above Chern-Simons level above are UV Chern-Simons level.

For SQED with $N_f = 1$,

charges :

	φ_1	φ_2	v_+	v_-
$U(1)_s$	1	-1	0	0
$U(1)_x$	0	1	0	-1
$U(1)_y$	0	0	1	-1
$U(1)_R$	0	0	0	2

CS levels :

	$U(1)_s$	$U(1)_x$	$U(1)_y$	$U(1)_R$
$U(1)_s$	0	1/2	1	-1
$U(1)_x$	1/2	$-1/2$	0	1/2
$U(1)_y$	1	0	0	0
$U(1)_R$	-1	1/2	0	-1

(4.11)

without superpotential. $U(1)_s$ denotes gauge symmetry, and $U(1)_x$ is flavor symmetry and $U(1)_y$ is a topological symmetry where this notation with label x and y is chosen because they are related to those of XYZ model via 3d $\mathcal{N} = 2$ mirror symmetry. Here, v_+ and v_- are gauge invariant monopole operators.

4.3.1.2 Holomorphic block, partition function on S_b^3, and index

For general R-charge of chiral multiplet, a tetrahedron theory T_Δ^R charged under $U(1)_y$ global flavor symmetry has the following charges and Chern-Simons level;

charges :

	Φ
$U(1)_y$	$+1$
$U(1)_R$	R

	$U(1)_y$	$U(1)_R$
$U(1)_y$	$-1/2$	$(1-R)/2$
$U(1)_R$	$(1-R)/2$	$-(1-R)^2/2$

(4.12)

Holomorphic block of T^R_Δ is given by

$$B^{(R)}_\Delta(y;q) = ((-q^{1/2})^{2-R}y^{-1};q)_\infty \tag{4.13}$$

Chern-Simons term of $U(1)_y$ gauge multiplet with level -1 are encoded in theta function

$$\theta(x;q) = (-q^{1/2}x;q)_\infty(-q^{1/2}x^{-1};q)_\infty. \tag{4.14}$$

More generally, if we have theory with Chern-Simons level matrix $k_{ij},\ i,j = 1,\cdots p$ of $U(1)^p$ gauge/global symmetries and mixed Chern-Simons level $\sigma_i,\ i = 1,\cdots p$ with $U(1)^p$ symmetries and $U(1)_R$ symmetry, then mixed CS levels k_{ij} and σ_i are encoded in product of theta functions;

$$\prod_h \theta\left((-q^{1/2})^{b_h}\vec{y}^{\vec{a}_h};q\right)^{n_h} \tag{4.15}$$

where b_h and n_h are integers for each h, $\vec{y} = (y_1,\cdots,y_p)$, and $\vec{a}_h = (a_{h,1},\cdots,a_{h,p})$ are vectors of p integers for each h.

When performing fusion, for S-fusion

$$\left\|\theta((-q^{1/2})^b y^a;q)\right\|^2_S = i^\sharp C^\flat \exp\left[-\frac{1}{2\hbar}\left((a\cdot X)^2 + (i\pi+\frac{\hbar}{2})\,b\,(a\cdot X)\right)\right] \tag{4.16}$$

where $x = \exp X$, $\widetilde{x} = \exp\frac{2\pi i}{\hbar}X$ and $\sharp$ and $\flat$ are certain number. For id-fusion,

$$\left\|\theta((-q^{1/2})^b y^a;q)\right\|^2_{id} = (-q^{1/2})^{-(a\cdot m)b}\zeta^{-(a\cdot m)a} \tag{4.17}$$

where $x = q^{m/2}\zeta$ and $\widetilde{x} = q^{m/2}\zeta^{-1}$.

From this information, holomorphic block of T_{XYZ} is given by

$$B_{XYZ}(x,y;q) = \frac{(qx^{-1};q)_\infty(qy^{-1};q)_\infty(xy;q)_\infty}{\theta(-q^{-1/2}xy;q)} \tag{4.18}$$

Holomorphic block of SQED with $N_f = 1$ is

$$B_{SQED}(x,y;q) = \int_\Gamma \frac{ds}{2\pi is}\Upsilon_{SQED}(s,x,y;q) \tag{4.19}$$

$$= \int_\Gamma \frac{ds}{2\pi is}\frac{\theta((-q^{-1/2})y;q)}{\theta((-q^{-1/2})sy;q)}(qs^{-1};q)_\infty(qx^{-1}s;q)_\infty \tag{4.20}$$

where Γ is an integration cycle we would like to find. There is only one critical point of twisted superpotential $\hbar \log \Upsilon_{SQED}(s, x, y; q)$, which is $s^{(1)} = (y - x^{-1})/(y - 1)$.

Appropriate integration cycle Γ depends on the sign of $\hbar$ and value of x and y. We don't cover all possibilities here, but for example when $\hbar > 0$ (*i.e.* $|q| > 1$) and for certain value of x and y, we can choose a contour enclosing poles from $(qs^{-1}; q)_\infty$ or $(qx^{-1}s; q)_\infty$. [2]

For either $|\hbar| > 0$ or $|\hbar| < 0$, it can be shown that exactly or numerically [13] that two holomorphic blocks from XYZ model and SQED with $N_f = 1$ are same up to overall q-dependent factors. Modulo q-dependent factor for S-fusion, partition function on S^3_b or index agree;

$$\|B_{XYZ}\|^2 = \int_{\mathbb{R}} dS \|\Upsilon_{SQED}\|^2_S = \|B_{SQED}\|^2_S \tag{4.21}$$

$$\|B_{XYZ}\|^2 = \int_{S^1} \frac{d\sigma}{2\pi i\sigma} \|\Upsilon_{SQED}\|^2_{id} = \|B_{SQED}\|^2_{id} \tag{4.22}$$

where $S = \log s$ for S-fusion and $s = q^{m/2}\zeta$ for id-fusion. This checks that XYZ model and SQED with $N_f = 1$ are 3d $\mathcal{N} = 2$ mirror dual.

4.3.1.3 Supersymmetric parameter space

After solving $\partial\widetilde{\mathcal{W}}_{SQED}/\partial\sigma = 0$ for σ, putting the solution $\sigma = \sigma^{(1)}$ back to $\widetilde{\mathcal{W}}_{SQED}$, and by solving $P_1 = \partial\widetilde{\mathcal{W}}_{SQED}(\sigma^{(1)})/\partial X_1$, $P_2 = \partial\widetilde{\mathcal{W}}_{SQED}(\sigma^{(1)})/\partial X_2$, we obtain the supersymmetric parameter space

$$\left(p_1 + \frac{p_2}{x_1} - 1\right) = 0, \qquad \left(p_2 + \frac{p_1}{x_2} - 1\right) = 0 \tag{4.23}$$

Meanwhile, for XYZ model with the twisted superpotential $\widetilde{\mathcal{W}}_{XYZ}$, one obtains

$$\gamma p_1 + \frac{1}{x_1} - 1 = 0, \qquad \gamma p_2 + \frac{1}{x_2} - 1 = 0, \qquad -\frac{\gamma x_1 x_2}{c} + \frac{x_1 x_2}{c} - 1 = 0 \tag{4.24}$$

from $P_1 = \partial\widetilde{\mathcal{W}}_{XYZ}/\partial X_1$, $P_2 = \partial\widetilde{\mathcal{W}}_{XYZ}/\partial X_2$, and $P_\gamma = \partial\widetilde{\mathcal{W}}_{XYZ}/\partial\Gamma$ where upper case letter is log of lower case letter and P_1, P_2, and Γ are interpreted as effective FI parameters with respect to $U(1)_{X_1}$, $U(1)_{X_2}$, and $U(1)_C$. The condition $C = 2\pi i$ gives $c = 1$. Eliminating γ, we obtain the moduli space of supersymmetric vacua

$$(x_1 - 1)\left(p_1 + \frac{p_2}{x_1} - 1\right) = 0, \qquad (x_2 - 1)\left(p_2 + \frac{p_1}{x_2} - 1\right) = 0. \tag{4.25}$$

Here, $x_1 = 1$ or $x_2 = 1$ *i.e.* $X_1 = 0$ or $X_2 = 0$ corresponds to singular locus since they mean that the mass of chiral multiplets are zero, which we should not integrate out. So for $x_1, x_2 \neq 1$, they are equivalent.

[2] Note that these expression is valid for $|q| < 1$. For $|q| > 1$ one has $1/(s^{-1}; q^{-1})_\infty$ and $1/(x^{-1}s; q^{-1})_\infty$, respectively.

4.3.2 Trefoil knot and figure-eight knot

In this subsection, we quote results from trefoil knot ($\mathbf{3_1}$) and figure-eight knot ($\mathbf{4_1}$) complement in S^3 [39, 25, 13].

One can glue two tetrahedra to make trefoil knot complement in S^3. By similar way above, one obtains $T^{DGG}[\mathbf{3_1}]$

$T^{DGG}[\mathbf{3_1}]$:

	Φ_1	Φ_2
$U(1)_s$	0	0
$U(1)_x$	1	-1
$U(1)_R$	2	0

, CS :

	$U(1)_x$	$U(1)_R$
$U(1)_x$	3	3
$U(1)_R$	3	$*$

. (4.26)

Thus, holomorphic block of $T^{DGG}[\mathbf{3_1}]$ is simple and is given by $\theta(x;q)^{-3}$. So partition functions on S_b^3 and index are readily calculated. The moduli space of supersymmetric vacua is

$$y + x^3 = 0 \tag{4.27}$$

which is A-polynomial of trefoil knot for irreducible flat $SL(2,\mathbb{C})$ connection.[3]

Actually, above $T^{DGG}[\mathbf{3_1}]$ is simplified theory, because we don't have enough superpotential to break a flavor symmetry, which is turned off by hand in above gluing ([25, Section 4.3] for detail). But since partition functions on S_b^3, index, and moduli space of supersymmetric vacua are insensitive to superpotential, above theory is good enough for calculating them.

For figure-eight complement,

$T^{DGG}[\mathbf{4_1}]$:

	Φ_1	Φ_2
$U(1)_s$	1	1
$U(1)_x$	1	-1
$U(1)_R$	0	0

. (4.29)

and there is no UV Chern-Simons term. This is also a simplified theory, but there is refined triangulations with 6 tetrahedra, which have enough superperpotential to break flavor symmetries ([39, Section 4.6] for detail).

One can calculate holomorphic block, partition function on S_b^3, and superconformal index, which

[3] Notation l and m for longitude and meridian eigenvalues are related to y and x as

$$l \leftrightarrow y\,, \qquad m^2 \leftrightarrow x \tag{4.28}$$

we don't calculate here. The moduli space of supersymmetric vacua is

$$y^2 - (x^2 - x - 2 - x^{-1} - x^{-2})y + 1 = 0, \tag{4.30}$$

which is A-polynomial of $\mathbf{4_1}$ for non-abelian flat $SL(2, \mathbb{C})$ connection.

Chapter 5

Toward a Complete 3d-3d Correspondence

5.1 3d-3d correspondence revisited

So far we have summarized each side of the correspondence and discussed dictionary between Chern-Simons theory and 3d $\mathcal{N}=2$ theory. However, as we have mentioned, Chern-Simons theories on 3-manifold obtained from gluing ideal hyperbolic tetrahedra do not capture abelian branch. Thus, the corresponding 3d $\mathcal{N}=2$ theories corresponding to gluing tetrahedra do not capture such branch. This can be seen in supersymmetric parameter space, partition functions, or superconformal index.

5.1.1 M-theory perspective

To better understand the problem, it is useful to consider M5-brane systems giving 3d-3d correspondence. When N M5 branes are wrapped on 3-manifold M_3, we obtain the theory $T[M_3]$ on $\mathbb{R}^3$ part (or $S^2\times\mathbb{R}$, $D^2\times\mathbb{R}$) of M5 branes worldvolume in the following M5-brane system;

$$\begin{array}{lccc} \text{space-time:} & \mathbb{R}^5 & \times & CY_3 \\ & \cup & & \cup \\ N\ \text{M5-branes:} & \mathbb{R}^3 & \times & M_3 \end{array} \tag{5.1}$$

where M_3 is embedded in a Calabi-Yau 3-fold CY_3 as a special Lagrangian submanifold. In order to make $T[M_3]$ be supersymmetric, there should be appropriate topological twisting on M_3.

There are usually two choices on Calabi-yau 3-fold. One is a cotangent bundle of M_3, T^*M_3, and another is resolved conifold X which is $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$ bundle over $\mathbb{C}\mathbf{P}^1$. The latter is relevant to physical realization of knot homologies. In the study of knot homologies for knot complement in

S^3, M5-brane system is

$$\begin{array}{rccc} \text{space-time:} & \mathbb{R}^5 & \times & X \\ & \cup & & \cup \\ \text{M5-brane:} & \mathbb{R}^3 & \times & \widetilde{L}_K \end{array} \tag{5.2}$$

Here, $\widetilde{L}_K$ is a Lagrangian submanifold in resolved conifold X.

In the standard 3d-3d correspondence, we are more interested in the case where Calabi-Yau 3-fold is a cotangent bundle T^*M_3 of M_3 in (5.1). More specifically, in this thesis we are interested in a knot complement in S^3 – $S^3 \backslash K$ – and the number of M5-branes is two. Therefore we take M_3 in as 3-sphere S^3, so we consider $CY_3 = T^*S^3$. In addition, in order to incorporate knot K, we also introduce a single M5-brane. The resulting M5-brane system is

$$\begin{array}{rccc} \text{space-time:} & \mathbb{R}^5 & \times & T^*S^3 \\ & \cup & & \cup \\ N \text{ M5-branes:} & \mathbb{R}^3 & \times & S^3 \\ \text{M5-brane:} & \mathbb{R}^3 & \times & L_K \end{array} \tag{5.3}$$

where L_K is a conormal bundle to the knot K [53], which is a lagrangian submanifold in T^*S^3 such that

$$L_K \cap S^3 = K \tag{5.4}$$

Here, a single M5-brane corresponds to knot in S^3, and we can regard that L_K creates a monodromy defect along K in S^3. So in this setup 3d-3d correspondence is about $SL(2,\mathbb{C})$ Chern-Simons theory on a knot complement in S^3 – $S^3 \backslash K$ – and 3d $\mathcal{N} = 2$ theory $T[S^3 \backslash K]$. For partition function on squashed 3-sphere S^3_b, superconformal index on $S^2 \times_q S^1$, and holomorphic block on $\mathbb{R}^2_q \times S^1$ of $T[S^3 \backslash K]$, $\mathbb{R}^3$ in above M5-brane system can be taken as S^3_b, $S^2 \times_q S^1$, or $\mathbb{R}^2_q \times S^1$.

As a side remark, M5-brane system for knot homologies in (5.2) can be thought as large N dual of M5-brane system in (5.3); for example, under geometric transition, Lagrangian submanifold L_K in T^*S^3 maps to Lagrangian submanifold $\widetilde{L}_K$ in a resolve conifold X [54].

5.1.1.1 Symmetries of $T[M_3]$ from the perspective of M5-brane system

Back to the most general M-theory system, we would like to symmetries in (5.1). There are at least three $U(1)$ symmetries which are independent of the specific choice of M_3. One is a Cartan subgroup of the $SO(3)$ rotation symmetry of $\mathbb{R}^3$. Another is a rotation symmetry in two-dimensional transverse space of $\mathbb{R}^3$ in $\mathbb{R}^5$. We also have $U(1)$ R-symmetry. Certain linear combinations of these three $U(1)$ symmetries give three conserved charges of the gauge theories. This is familiar to the case of surface operator in 4d $\mathcal{N}=2$ gauge theory where there are three fugacities in the superconformal index in the presence of surface operator whose nature is independent of specific choice of surface operators and the theory that surface operator is present.

These three $U(1)$ symmetries should appear in 3d theory $T[M_3]$ as $T[M_3]$ should exhibit all properties the M5-brane system. Firstly, $U(1)$ rotational symmetry on $\mathbb{R}^3$ is obviously a part of Lorentz symmetry. And certain linear combination of other two $U(1)$ symmetries become $U(1)_R$ symmetry of $T[M_3]$. Another linear combination gives non-R global symmetry which we called as $U(1)_t$. This $U(1)_t$ symmetry has not appeared in the previous 3d-3d correspondence, or usual 3d $\mathcal{N}=2$ theories. However, since this is a symmetry of M5-brane system (5.1), theories $T[M_3]$ should have it as well. Actually, this $U(1)_t$ symmetry plays a key role in the 3d-3d correspondence as we will see soon.

5.1.1.2 Supersymmetric parameter space

As mentioned in section 4.1, in [52], by compactifying M5-brane system on M-theory circle of $\mathbb{R}^2_q \times S^1$ (instead of "$\mathbb{R}^3$" and similarly for $\mathbb{R}^5$ in (5.1)), it was shown that with appropriate twist on M_3 the supersymmetric parameter space of effective 3d $\mathcal{N}=2$ theory which is regarded as 2d $\mathcal{N}=(2,2)$ theory on $\mathbb{R}^2_q$ is the moduli space of $G_\mathbb{C}$ flat connection on M_3;

$$\mathcal{M}_{\text{SUSY}}(T[M_3;G]) \;=\; \mathcal{M}_{\text{flat}}(M_3;G_\mathbb{C}) \tag{5.5}$$

What we have seen above for the previous 3d-3d correspondence by Dimofte-Gaiotto-Gukov was that

$$\mathcal{M}_{\text{SUSY}}(T_{DGG}[M_3;G]) \;\neq\; \mathcal{M}_{\text{flat}}(M_3;G_\mathbb{C}) \tag{5.6}$$

That is because abelian flat $G_\mathbb{C}$ connection on M_3 is captured via gluing tetrahedra. So, more precisely,

$$\mathcal{M}^{DGG}_{\text{flat}}(M_3;G_\mathbb{C}) \neq \mathcal{M}_{\text{flat}}(M_3;G_\mathbb{C}) = \mathcal{M}_{\text{SUSY}}(T[M_3;G]) \neq \mathcal{M}^{DGG}_{\text{SUSY}}(T[M_3;G]) \tag{5.7}$$

where $\mathcal{M}^{DGG}_{\text{flat}}(M_3;G_\mathbb{C}) = \mathcal{M}^{DGG}_{\text{SUSY}}(T[M_3;G])$. What we would like to do is to find several simple examples such that (5.5) actually hold for $G=SU(2)$.

5.1.2 Brief comments on recent developments in 3d-3d correspondence

There also have been some number of developments in 3d-3d correspondence, and they all imply that the gauge theory $T[M_3]$ should realize all $G_\mathbb{C}$ flat connections on M_3. They include

- The recent proof [47, 48] of the 3d-3d correspondence indicates that *all* complex flat connections on M_3 should be treated democratically and, therefore, no one should be left behind.
- Various deformations / refinements of (5.5) necessarily require taking a proper account of *all* branches [37, 55, 56, 57], and can serve as a useful tool in identifying $T[M_3]$ even in the undeformed case. This will be our approach in this thesis.
- The correspondence [58] between 4-manifolds and 2d $\mathcal{N}=(0,2)$ theories $T[M_4;G]$ represents gluing 4-manifolds along M_3 as a sequence of domain walls and boundary conditions in 3d $\mathcal{N}=2$ theory $T[M_3;G]$. Much like the other developments, it works only if *all* $G_{\mathbb{C}}$ flat connections on M_3 are realized in $T[M_3;G]$.

5.2 Goal and strategy

5.2.1 $T[M_3]$ from homological knot invariants and Higssing

We would like to construct some theories $T[M_3;G]$ with all expected flavor symmetries and with vacua corresponding to all flat connections on M_3, and to investigate their relation to theories $T_{DGG}[M_3;G]$. We will mainly focus on the case $G=SU(2)$, and on knot complements $M_3=S^3\backslash K$. A knot-complement theory $T[M_3]:=T[M_3;SU(2)]$ is defined by compactification of the 6d (2,0) theory on S^3 with a codimension-two defect wrapping the knot $K\subset S^3$. In this case $T[M_3]$ should gain a $U(1)_x$ flavor symmetry, part of the $SU(2)_x$ flavor symmetry of the defect, in addition to $U(1)_t$ and $U(1)_R$. What we find can be then summarized by the following diagram:

$$\begin{array}{ccc} & T[M_3] & \\ {}^{\langle\partial_r\mathcal{O}_x\rangle\neq 0}\swarrow & & \searrow^{\langle\mathcal{O}_t\rangle\neq 0} \\ T_{\text{poly}}[M_3;r]^{\not{U(1)}_x} & & T_{DGG}[M_3]^{\not{U(1)}_t} \end{array} \tag{5.8}$$

In particular, the theory $T_{DGG}[M_3]$ is a particular subsector of $T[M_3]$ obtained by Higgsing the $U(1)_t$ symmetry.

The left-hand side of the diagram (5.8) indicates an expected relation between $T[M_3]$ and a theory $T_{\text{poly}}[M_3;r]$ whose partition functions compute the Poincaré polynomials of r-colored $SU(2)$ knot homology for K. Indeed, our practical approach to constructing $T[M_3]$ will be to identify a 3d $\mathcal{N}=2$ theory with $U(1)_x\times U(1)_t$ symmetry whose partition functions reduce to the desired Poincaré polynomials in a special limit. Physically this limit corresponds to another Higgsing procedure, this time breaking the $U(1)_x$ symmetry of $T[M_3]$ while creating a line defect or vortex, similar to scenarios studied in [59, 60, 61].

An important feature of (5.8) is that the two arrows corresponding to Higgsing do not commute. In particular, while it is easy to obtain Jones polynomials of knots from the Poincaré polynomials on the left-hand side by ignoring $U(1)_t$ fugacities, it is (seemingly) impossible to do this from $T_{DGG}[M_3]$ on the right-hand side. Jones polynomials include a crucial contribution from the abelian flat connection on a knot complement M_3 as discussed in section 3, and vacua corresponding to the abelian flat connection are lost during the Higgsing of $U(1)_t$.

5.2.2 Boundary of M_3 and $T[M_3]$

Later, in section 5.6 we discuss gluing of knot and link-complement theories to form closed M_3, in particular 3d $\mathcal{N}=2$ theories for lens spaces, and Brieskorn spheres. The importance of such gluing or surgery operations is two-fold. First, it will give us another clear illustration why all flat connections need to be accounted by 3d $\mathcal{N}=2$ theories $T[M_3]$ in order for cutting and gluing operations to work. Moreover, it will help us to understand half-BPS boundary conditions that one needs to choose in order to compute the half-index of $T[M_3]$. As explained in [58], a large class ("class $\mathcal{H}$") of boundary conditions can be associated to 4-manifolds bounded by M_3,

$$\boxed{\begin{array}{c}\text{4-manifold } M_4\\ \text{bounded by } M_3\end{array}} \quad \rightsquigarrow \quad \boxed{\begin{array}{c}\text{boundary condition for}\\ \text{3d } \mathcal{N}=2 \text{ theory } T[M_3]\end{array}} \tag{5.9}$$

therefore making the half-index of $T[M_3]$ naturally labeled by 4-manifolds.

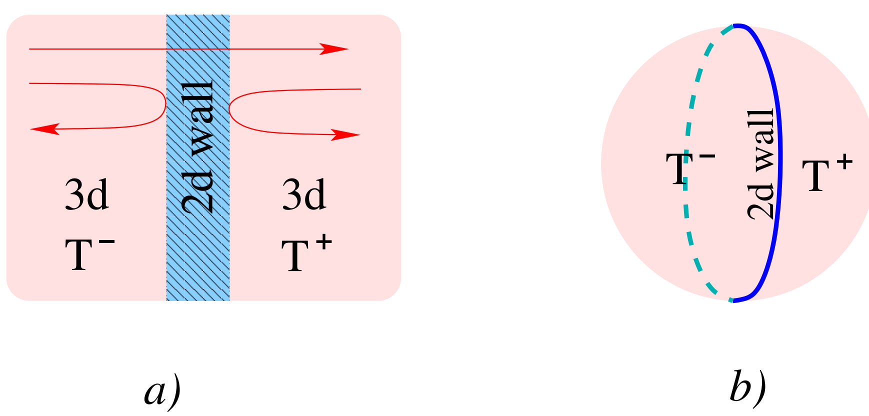


Figure 5.1: The index of 3d $\mathcal{N}=2$ theories can be generalized to include domain walls and boundary conditions [62]. It is obtained from two copies of the half-index $\mathcal{I}_{S^1\times_q D^\pm}(T^\pm) \simeq Z_{\text{vortex}}(T^\pm)$ convoluted via the index (flavored elliptic genus) of the wall supported on $S^1 \times S^1_{\text{eq}}$, where $D^\pm$ is the disk covering right (resp. left) hemisphere of the S^2 and $S^1_{\text{eq}} := \partial D^+ = -\partial D^-$ is the equator of the S^2.

5.3 Contour integrals for Poincaré polynomials

Even though our main goal is to identify all symmetries and flat connections in the 3d $\mathcal{N}=2$ theory $T[M_3]$, one of the intermediate steps is of mathematical value on its own. Namely, the point of this section will be to show that Poincaré polynomials of homological link invariants can be expressed as contour integrals

$$P_K(q,t\ldots) \;=\; \int_\Gamma \frac{ds}{2\pi i s}\Upsilon(s,q,t,\ldots) \tag{5.10}$$

in complex space $\mathbb{C}^{*m}$ parametrized by (multi-)variable s. Here, $P_K(q,t\ldots)$ stands for the Poincaré polynomial of a doubly-graded [63, 64, 65, 66] or triply-graded [67, 68, 69] homology theory $\mathcal{H}(K)$ of a link K:

$$P_K(q,t\ldots) \;=\; \sum_{i,j,\ldots} q^i t^j \ldots \dim \mathcal{H}^{i,j}(K) \tag{5.11}$$

that categorifies either quantum $sl(N)$ invariant [31] or colored HOMFLY polynomial [70], respectively. Depending on the context and the homology theory in question, the sum runs over all available gradings, among which two universal ones — manifest in (5.11) — are the homological grading and the so-called q-grading. In the case of HOMFLY homology, there is at least one extra grading and, correspondingly, the Poincaré polynomial depends on one extra variable a, whose specialization to $a=q^N$ makes contact with $sl(N)$ invariants. The Poincaré polynomials of triply-graded HOMFLY homology theories are often called superpolynomials. In general, such invariants are also labeled by a representation / Young diagram R and referred to as *colored*, unless $R=\square$ in which case the adjective 'colored' is often omitted.

In this section we will write the Poincaré polynomials of colored knot homologies in the form (5.10) of contour integrals, whose physical interpretation will be discussed in the later sections. Our basic examples here and the rest part of this thesis will be the unknot, trefoil, and figure-eight knot complements.

In general, superpolynomials or Poincaré polynomials are expressed as finite sums of products of q-Pochhammer symbols

$$(z;q)_n := \prod_{i=0}^{n-1}(1-q^i z) \tag{5.12}$$

and monomials. For instance, the *unnormalized* superpolynomial[1] of the trefoil $\mathbf{3_1}$ is [38] (see also [69, 71, 72, 73]):

$$\overline{\mathcal{P}}^{S^r}_{\mathbf{3_1}}(a,q,t) = \sum_{k=0}^{r} \frac{(a(-t)^3;q)_r(-aq^{-1}t;q)_k}{(q;q)_k(q;q)_{r-k}} a^{\frac{r}{2}} q^{-\frac{r}{2}} q^{(r+1)k}(-t)^{2k-\frac{3r}{2}}\,. \tag{5.13}$$

[1] We can proceed with normalized superpolynomial, which is obtained from dividing unnormalized superpolynomial by superpolynomial of unknot. However, since physical interpretation of unnormalized superpolynomial is clearer than than normalized one, so we focus on unnormalized superpolynomial.

This is the Poincaré polynomial of the HOMFLY homology (5.11) colored by the r-th symmetric power of the fundamental representation of $SU(N)$ or, in the language of Young diagrams, by a Young tableau with a single row and r boxes. For our applications here, we specialize to $SU(2)$ homology[2] by setting $a = q^2$. It is further convenient to renormalize the $SU(2)$ polynomial by a factor $(-1)^r$, defining[3]

$$\begin{aligned} P^r_{\mathbf{3_1}}(t;q) &:= (-1)^r\, \overline{\mathcal{P}}^{S^r}_{\mathbf{3_1}}(a=q^2,q,t) \\ &= \sum_{k=0}^{r} \frac{(q^2(-t)^3;q)_r (q(-t);q)_k}{(q;q)_k (q;q)_{r-k}} (-q^{\frac{1}{2}})^{2rk+2k+r}(-t)^{2k-2r}\,. \end{aligned} \tag{5.14}$$

We remark that the following steps could also be carried out for generic a, though for our applications we specialize from $SU(N)$ to $SU(2)$.

Let us suppose that $|q| > 1$ (for reasons that will become clear momentarily), and define

$$(z)^-_\infty := (z;q^{-1})_\infty = \prod_{i=0}^{\infty}(1-q^{-i}z)\,, \qquad \theta^-(z) := \theta(z;q^{-1}) = (-q^{-\frac{1}{2}}z)^-_\infty(-q^{-\frac{1}{2}}z^{-1})^-_\infty\,, \tag{5.15}$$

as well as

$$\theta^-(z_1,...,z_n) := \theta^-(z_1)\cdots\theta^-(z_n)\,. \tag{5.16}$$

Then, by using identities such as $(q^r z)^-_\infty/(z)^-_\infty = (qz;q)_r = (-1)^r q^{\frac{r(r+1)}{2}} z^r (q^{-1}z^{-1};q^{-1})_r$ and $\theta^-(q^n z)/\theta^-(z) = q^{\frac{n^2}{2}} z^n$, we may rewrite

$$\begin{aligned} P^r_{\mathbf{3_1}}(t;q) &= \frac{(-1/(q^2t^3))^-_\infty(-1/(qt))^-_\infty}{(q^{-1})^-_\infty(-1/(q^2xt^3))^-_\infty} \sum_{k=0}^{\infty} \frac{(s/(qx))^-_\infty}{(q;q)_k(-1/(qst))^-_\infty} \frac{\theta^-(q^{\frac{3}{2}}sxt^3, -q^{\frac{1}{2}}x, -q^{\frac{3}{2}}x(-t)^{\frac{3}{2}},1)}{\theta^-(q^{\frac{3}{2}}xt^3, -q^{\frac{1}{2}}x/s, -q^{\frac{3}{2}}(-t)^{\frac{3}{2}},x)}\Bigg|_{x=q^r,\, s=q^k} \\ &=: \sum_{k=0}^{\infty} \frac{1}{(q;q)_k(q^{-1})^-_\infty} \Upsilon^{(0)}_{\mathbf{3_1}}(s,x,t;q)\big|_{x=q^r,\, s=q^k}\,. \end{aligned} \tag{5.17}$$

Note in particular that upon setting $x = q^r$ and $s = q^k$ the term $(s/(qx))^-_\infty$ in the numerator on the LHS vanishes unless $k \le r$. Thus the sum naturally truncates to the one in (5.14).

Going further, we observe that the sum in (5.17) may be rewritten as a sum of residues

$$P^r_{\mathbf{3_1}}(t;q) = \left[\sum_{k=0}^{\infty} \mathrm{Res}_{s=q^k} \frac{1}{2\pi i s}\frac{1}{(s)^-_\infty}\Upsilon^{(0)}_{\mathbf{3_1}}(s,x,t;q)\right]_{x=q^r}\,, \tag{5.18}$$

since $\Upsilon^{(0)}_{\mathbf{3_1}}$ is smooth at $s = q^k$, while the residue of $1/[2\pi i s(s)^-_\infty]$ at $s = q^k$ is precisely $1/[(q;q)_k(q^{-1})^-_\infty]$. It was the initial choice $|q| > 1$ that allowed us to write the sum as residues like this. Therefore, at

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[2]Specialization $a = q^2$ leads to Poincaré polynomials of colored $SU(2)$ knot homologies for a certain class of knots, which include unknot, trefoil, and figure-8 knot considered in this thesis. In general and for more complicated knots, specialization of superpolynomials to Poincaré polynomials of $SU(N)$ knot homologies requires taking into account a nontrivial action of differentials [67, 69, 37].

[3]In the next section, the rescaling by $(-1)^r$ leads to a convenient choice of fermion-number twist when identifying $P^r_{\mathbf{3_1}}(t;q)$ with a partition function of $T[\mathbf{3_1}]$ on $\mathbb{R}^2 \times_q S^1$.

least formally,

$$P^r_{\mathbf{3_1}}(t;q) = \int_{\Gamma_I} \frac{ds}{2\pi i s} \Upsilon_{\mathbf{3_1}}(s,x,t;q)\bigg|_{x=q^r} \tag{5.19}$$

with

$$\begin{aligned} \Upsilon_{\mathbf{3_1}}(s,x,t;q) &:= \frac{1}{(s)^-_\infty}\Upsilon^{(0)}_{\mathbf{3_1}}(s,x,t;q) \\ &= \frac{\theta^-(-q^{\frac{1}{2}}x, -q^{\frac{3}{2}}x(-t)^{\frac{3}{2}},1)}{\theta^-(q^{\frac{3}{2}}xt^3, -q^{\frac{3}{2}}(-t)^{\frac{3}{2}},x)} \frac{(-1/(q^2t^3))^-_\infty(-1/(qt))^-_\infty}{(-1/(q^2xt^3))^-_\infty} \times \frac{\theta^-(q^{\frac{3}{2}}sxt^3)}{(s)^-_\infty(-1/(qst))^-_\infty(x/s)^-_\infty}\,, \end{aligned} \tag{5.20}$$

where the contour Γ_I is shown in Figure 5.2. (We have put all s-dependent terms in $\Upsilon_{\mathbf{3_1}}$ on the right.) This is now the form of a contour integral (5.10).

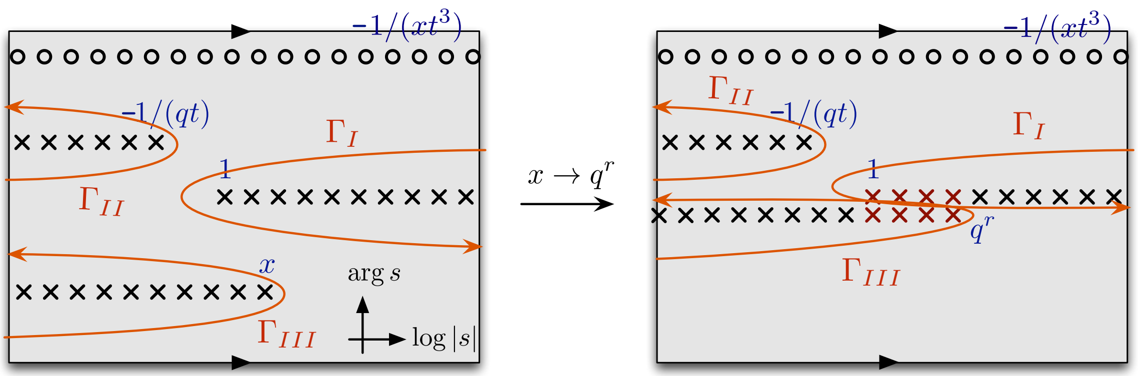

Figure 5.2: Possible integration contours for the trefoil, drawn on the cylinder parametrized by $\log s$. There are three half-lines of poles in the integrand $\Upsilon_{\mathbf{3_1}}(s,x,t;q)$, coming from $(s)^-_\infty$, $(-1/(qst))^-_\infty$, $(x/s)^-_\infty$ in the denominator; and a full line of zeroes from $\theta^-(q^{\frac{3}{2}}sxt^3)$ in the numerator. On the right, we demonstrate a pinching of contours as $x \to q^r$.

Note that the three terms $(s)^-_\infty$, $(-1/(qst))^-_\infty$, and $(x/s)^-_\infty$ each contribute a half-line of poles to $\Upsilon_{\mathbf{3_1}}$. If we take $q > 1$ to be real, then the asymptotics of the integrand are given by

$$\Upsilon_{\mathbf{3_1}} \sim \begin{cases} \exp \frac{1}{\log q}\big[(\log x + 3\log(-t))\log s + \dots\big] & \log|s| \to \infty \\ \exp \frac{1}{\log q}\big[(-\frac{1}{2}(\log s)^2 + \dots\big] & \log|s| \to -\infty\,, \end{cases} \tag{5.21}$$

so the integral along Γ_I does converge in a suitable range of x and t (namely, if $|xt^3| < 1$). In contrast, the integrals along the other obvious cycles here, Γ_{II} and Γ_{III}, *always* converge. Moreover, a little thought shows that upon setting $x = q^r$ the integral along Γ_I must equal the integral along Γ_{III}; indeed, as $x \to q^r$, some $r+1$ pairs of poles in the lines surrounded by Γ_I and Γ_{III} collide, and all contributions to the integrals along either Γ_I and Γ_{III} come from the $r+1$ points where the contours get pinched by colliding poles. (Such pinching would usually cause integrals to diverge, but here the divergence is cancelled by one of the s-independent theta-functions in $\Upsilon_{\mathbf{3_1}}$.) Therefore, letting

$$B^{\mathbf{3_1}}_*(x,t;q) := \int_{\Gamma_*} \frac{ds}{2\pi i s} \Upsilon_{\mathbf{3_1}}(s,x,t;q)\,, \tag{5.22}$$

(with the obvious relation $B_I + B_{II} + B_{III} = 0$), we find

$$P^r_{\mathbf{3_1}}(t;q) = B^{\mathbf{3_1}}_I(x,t;q)\big|_{x=q^r} = B^{\mathbf{3_1}}_{III}(x,t;q)\big|_{x=q^r}\,. \tag{5.23}$$

We can repeat the analysis for the unknot $U = \mathbf{0_1}$ and figure-eight knot $\mathbf{4_1}$. The superpolynomials of these knots are given by [74, 72, 75, 38]:

$$\overline{\mathcal{P}}^{S^r}_{\mathbf{0_1}}(a,q,t) = a^{-\frac{r}{2}} q^{\frac{r}{2}} (-t)^{-\frac{3}{2}r} \frac{(a(-t)^3;q)_r}{(q;q)_r} \tag{5.24}$$

$$\overline{\mathcal{P}}^{S^r}_{\mathbf{4_1}}(a,q,t) = \sum_{k=0}^{r} \frac{(a(-t)^3;q)_r}{(q;q)_k (q;q)_{r-k}} (aq^{-1}(-t);q)_k (aq^r(-t)^3;q)_k a^{-k-\frac{r}{2}} q^{\frac{r}{2}+k(1-r)} (-t)^{-2k-\frac{3}{2}r}\,, \tag{5.25}$$

and Poincaré polynomials for $G = SU(2)$, *i.e.* specializations to $a = q^2$, normalized by $(-1)^r$, are given by

$$P^r_{\mathbf{0_1}}(t;q) = (-q^{\frac{1}{2}})^{-r} (-t)^{-\frac{3}{2}r} \frac{(q^2(-t)^3;q)_r}{(q;q)_r} \tag{5.26}$$

$$P^r_{\mathbf{4_1}}(t;q) = \sum_{k=0}^{r} \frac{(q^2(-t)^3;q)_r}{(q;q)_k (q;q)_{r-k}} (q(-t);q)_k (q^2 q^r (-t)^3;q)_k (-q^{\frac{1}{2}})^{-r-2k(1+r)} (-t)^{-2k-\frac{3}{2}r}\,, \tag{5.27}$$

respectively. Repeating the above procedure, we find

$$P^r_{\mathbf{0_1}}(t;q) = B^{\mathbf{0_1}}(x,t;q)\big|_{x=q^r}\,, \qquad B^{\mathbf{0_1}}(x,t;q) := \frac{\theta^-(1,-q^{\frac{1}{2}}x(-t)^{\frac{3}{2}})}{\theta^-(x,-q^{\frac{1}{2}}(-t)^{\frac{3}{2}})} \frac{(q^{-1}/x)^-_\infty (-q^{-2}/t^3)^-_\infty}{(q^{-1})^-_\infty (-q^{-2}/(xt^3))^-_\infty} \tag{5.28}$$

for the unknot, and

$$\Upsilon_{\mathbf{4_1}}(s,x,t;q) := \frac{\theta^-(-q^{\frac{1}{2}}x, q^{\frac{1}{2}}tx, (-t)^{-\frac{1}{2}})}{\theta^-(q,t^2,q^{\frac{1}{2}}t, x(-t)^{-\frac{1}{2}})} \theta^-(qs,t^2 s) \frac{(-1/(q^2t^3))^-_\infty (-1/(qt))^-_\infty}{(s)^-_\infty (-1/(qts))^-_\infty (x/s)^-_\infty (-1/(q^2xt^3s))^-_\infty}\,. \tag{5.29}$$

for the figure-eight knot. In the latter case, the integrand $\Upsilon_{\mathbf{4_1}}$ has four half-lines of poles in the s-plane, coming from the four factors $(s)^-_\infty$, $(-1/(qts))^-_\infty$, $(x/s)^-_\infty$, $(-1/(q^2xt^3s))^-_\infty$ in the denominator of (5.29). Let $\Gamma_I, \Gamma_{II}, \Gamma_{III}, \Gamma_{IV}$ be contours encircling these respective half-lines of poles. A formal sum of residues along poles in the first half-line, evaluated at $x = q^r$, most directly gives $P^r_{\mathbf{4_1}}(t;q)$; but the actual integral along Γ_I does not converge for generic x. In contrast, the integrals along $\Gamma_{II}, \Gamma_{III}, \Gamma_{IV}$ always converge, and

$$P^r_{\mathbf{4_1}}(t;q) = B^{\mathbf{4_1}}_{III}(x,t;q)\big|_{x=q^r} = \text{“}B^{\mathbf{4_1}}_{I}(x,t;q)\big|_{x=q^r}\text{”}\,, \tag{5.30}$$

where

$$B^{\mathbf{4_1}}_{*}(x,t;q) := \int_{\Gamma_*} \frac{ds}{2\pi i s} \Upsilon_{\mathbf{4_1}}(s,x,t;q)\,. \tag{5.31}$$

These examples indicate how the analysis may be extended to other knots and links (for example, those whose superpolynomials are found in [56, 57]), and to Poincaré polynomials of other homological invariants. In general, the required integrals will not be one-dimensional, but will require higher-dimensional integration cycles. Generalizations of some results in this chapter to other knots and links are also discussed in section 5.5.4.

5.4 Knot polynomials as partition functions of $T[M_3]$

In this section, we construct some examples of 3d $\mathcal{N}=2$ theories $T[M_3]$ for knot complements M_3 (and $G = SU(2)$) with the properties outlined above. In particular, we would like the vacua of $T[M_3]$ on $\mathbb{R}^2 \times S^1$ to match all flat connections on M_3.

Although our strategy will be a little indirect, it is based on a simple key observation: the contour integral (5.10) for colored Poincaré polynomials has the form of localization integrals in supersymmetric 3d $\mathcal{N}=2$ theories as well as in Chern-Simons theory on certain 3-manifolds. Indeed, powerful localization techniques reduce the computation of Chern-Simons partition functions to finite dimensional integrals of the form (5.10), where the choice of the contour is related to the choice of the classical vacuum [76, 77, 78, 79, 80], as we briefly reviewed in section 5.6.

Similar — and, in fact, closer to our immediate interest — contour integrals of the form (5.10) appear as a result of localization in supersymmetric partition functions of 3d $\mathcal{N}=2$ theories, such as the (squashed) sphere partition function [15, 18], the index [19, 20, 21], and the vortex partition function [39] or the half-index [25]. Since in the last case the space-time is non-compact it requires a choice of the asymptotic boundary condition or vacuum of the theory on $\mathbb{R}^2 \times_q S^1$, which manifests as a choice of the integration contour in the localization calculation. (The integrand is completely determined by the Lagrangian of 3d $\mathcal{N}=2$ theory.) This has to be compared with the first two cases, where localization of 3-sphere partition function and index lead to a contour integral with canonical choices of the integration contour.

Therefore, in order to interpret (5.10) as a suitable partition function of 3d $\mathcal{N}=2$ theory in this thesis we mainly focus on half-indices, vortex partition functions, and their IR variants called holomorphic blocks [13] that do not necessarily come from localization. This gives us enough flexibility to interpret (5.10) and we generically expect that the *full* set of blocks for $T[M_3]$, labelled by a full set of vacua, corresponds to a complete basis of independent convergent contours for the integrals of section 5.3. On the other hand, we also expect that a basis of convergent contours is in 1–1 correspondence with flat connections on M_3:

$$\text{vacua of } T[M_3] \;\leftrightarrow\; \text{hol'c blocks} \;\leftrightarrow\; \text{convergent contours} \;\leftrightarrow\; \text{flat conn's}\,. \tag{5.32}$$

The reason to expect these correspondences to hold was discussed in the case without $U(1)_t$ in [39, 25, 13]. Similar idea is expected to hold with $U(1)_t$ symmetry and is outlined more carefully in section 5.4.1. In order to capture *all* flat connections, it turns out to be crucial that we start with Poincaré polynomials for knot homology rather than unrefined Jones polynomials. In section 5.4.2 we then demonstrate the construction of $T[M_3]$ in a few examples.

In section 5.4.3 we examine the physical meaning of the limit $x \to q^r$ that recovers Poincaré polynomials from $T[M_3]$. We argue that it is a combination of Higgsing and creation of a line operator in $T[M_3]$, as on the left-hand side of (5.8). We also show that Poincaré polynomials can be obtained by directly taking residues of $S^2 \times_q S^1$ indices and S^3_b partition functions of $T[M_3]$, bypassing holomorphic blocks.

5.4.1 Recursion relations for Poincaré polynomials

One understanding of why contour integrals as in section 5.3 should capture all flat connections on a knot complement follows from looking at the q-difference relations that the integrals satisfy.

Let us start with the Poincaré polynomials $P^r_K(t;q)$ for colored $SU(2)$ knot homology of a knot K. As found in [37, 38, 56], the sequence of Poincaré polynomials obeys a recursion relation of the form

$$\widehat{A}^{\text{ref}}(\widehat{x},\widehat{y};t;q)\cdot P^r_K(t;q) = 0\,, \tag{5.33}$$

where $\widehat{A}^{\text{ref}}(\widehat{x},\widehat{y};t;q)$ is a polynomial operator in which $\widehat{x},\widehat{y}$ act as $\widehat{x}P^r_K = q^r P^r_K$ and $\widehat{y}P^r_K = P^{r+1}_K$.[4] The limit $q \to 1$ of $\widehat{A}^{\text{ref}}(\widehat{x},\widehat{y};t;q)$ is a classical polynomial $A^{\text{ref}}(x,y;t)$, whose subsequent $t \to -1$ limit contains the classical A-polynomial of K [34] as a factor,

$$\widehat{A}^{\text{ref}}(\widehat{x},\widehat{y};t;q) \;\overset{q\to 1}{\to}\; A^{\text{ref}}(x,y;t) \;\overset{t\to -1}{\to}\; A(x,y)\,. \tag{5.35}$$

The physical interpretation of the classical A-polynomial $A(x,y)$ goes back to [27]. Its roots at fixed

[4] As mentioned before, notation l and m for longitude and meridian eigenvalues are related to y and x as

$$l \leftrightarrow y\,, \qquad m^2 \leftrightarrow x \tag{5.34}$$

and similarly at the level of operator.

x are in 1-1 correspondence with all flat connections on M (with fixed boundary conditions at K), but the root corresponding to the abelian flat connection is distinguished because it comes from a universal factor $(y-1)$ in $A(x,y)$ as discussed in section 3.2. However, the t-deformed polynomial $A^{\mathrm{ref}}(x,y;t)$ is irreducible (at least in simple examples[5]) in the sense that $(y-1)$ is not factorized, and none of its roots is more or less important than the others.

Alternatively, note that the $t\to -1$ limit of $A^{\mathrm{ref}}(\widehat{x},\widehat{y};t;q)$ leads to a shift operator known as the quantum A-polynomial, $\widehat{A}(\widehat{x},\widehat{y};q)$, which annihilates colored Jones polynomials [27, 81] as discussed in section 3.2. One can also consider a–deformations of these shift operators. Such a deformation of the quantum A-polynomial was called Q-deformed A-polynomial in [82], and it agrees with the mathematically defined augmentation polynomial of [83, 84]. More generally, one can consider shift operators $\widehat{A}^{\mathrm{super}}(\widehat{x},\widehat{y};a;t;q)$ depending on both a and t, which annihilate colored superpolynomials, and which were called super-A-polynomials in [38] (for a concise review see [85]). However, as mentioned above, we are only interested here in $a=q^2$ specializations.

Now, in section 5.3 we expressed

$$P_K^r(t;q)=\left[\int_{\Gamma_P}\frac{ds}{2\pi i s}\Upsilon_K(s,x,t;q)\right]_{x=q^r}=B_P(x,t;q)\big|_{x=q^r} \tag{5.36}$$

for a suitable integrand Υ_K and a choice of integration contour Γ_P. It is easy to see that $B_P(x,t;q)$ satisfies a q-difference equation

$$\widehat{A}^{\mathrm{ref}}(\widehat{x},\widehat{y};t;q)\cdot B_P(x,t;q)=0 \tag{5.37}$$

even before setting $x=q^r$, with $\widehat{x},\widehat{y}$ acting as $\widehat{x}B_P(x,...)=xB_P(x,...)$ and $\widehat{y}B_P(x,...)=B_P(qx,...)$. More so, the integral $B_\alpha=\int_{\Gamma_\alpha}ds/s\,\Upsilon_K$ for *any* convergent integration contour Γ_α (that stays sufficiently far away from poles) should provide a solution to the q-difference equation $\widehat{A}^{\mathrm{ref}}\cdot B=0$, and one generally expects that a maximal independent set of integration contours generates the full vector space of solutions.[6]

If we fix the values of x, t, and q, the convergent integration cycles Γ_α can be labelled by the roots $y^{(\alpha)}(x,t)$ of the classical equation $A^{\mathrm{ref}}(x,y;t)=0$ — *i.e.* by the flat connections on M_3 with boundary conditions (meridian holonomy) fixed by x. The correspondence follows roughly by identifying the solutions to $A^{\mathrm{ref}}(x,y;t)=0$ with critical points of the integrand $\Upsilon_K(s,x,t;q)$ at $q\approx 1$, then using downward gradient flow with respect to $\log|\Upsilon_K(s,x,t;q)|$ to extend the critical

[5]To be more precise, both $A^{\mathrm{ref}}(x,y;t)$ and $A(x,y)$, obtained as appropriate limits of super-A-polynomials, may contain some additional factors. As explained in [29, 36] (for $t=-1$) and [37, 38] (for general t), these factors are necessary for quantization but are not associated to classical flat connections. For knots considered in this thesis these factors are independent of y, they do not affect the structure of roots of $A^{\mathrm{ref}}(x,y;t)$ or $A(x,y)$ at generic fixed x, and therefore they do not modify our discussion here.

[6]This is technically a vector space over modular elliptic functions of (x,q), on which q-shifts acts trivially, as discussed in greater detail in [13].

points into integration cycles Γ_α.

We have claimed that by writing one solution of $\widehat{A}^{\text{ref}} \cdot B = 0$ as a contour integral (5.36), we can actually reproduce all other solutions from integrals on a full basis of contours Γ_α. This reasoning relies on an important assumption: that the quantum $\widehat{A}^{\text{ref}}$ (and hence the classical A^{ref}) is irreducible. Otherwise, we may only get solutions corresponding to one irreducible component. For this reason, it is crucial that we use *refined* knot polynomials and recursion relations rather than Jones polynomials and the quantum A-polynomial. See [86, 85] for further details as well as pedagogical introduction.

To complete the chain of correspondences (5.32), we simply use [52, 51, 87, 11, 39, 46, 37, 56, 13, 88, 45] to translate the above observations to the language of gauge theory. Momentarily we will engineer gauge theories $T[M_3]$ for which the integrals $\int_* ds/s\, \Upsilon_K$ compute various partition functions on $\mathbb{R}^2 \times S^1$ annihilated by $\widehat{A}^{\text{ref}}$ and labelled by vacua of $T[M_3]$ on $\mathbb{R}^2 \times S^1$, *i.e.* classical solutions of $A(x, y; t) = 0$.

5.4.2 3d $\mathcal{N} = 2$ gauge theories for unknot, trefoil knot, and figure-eight knot

Having rewritten the Poincaré polynomials of colored $SU(2)$ knot homologies as special values of a contour integral, we try to engineer $T[M_3]$ so that the contour integral computes its partition function. In particular, by examining the integrand Υ_K and from discussion in Chapter 2 and 4 we associate

$$\begin{array}{ccc} \text{fugacities } x, (-t), q & \rightsquigarrow & \text{flavor and } R \text{ symmetries} \\ \text{fugacity } s & \rightsquigarrow & U(1)_s \text{ gauge symmetry} \\ (*)^-_\infty \text{ factors} & \rightsquigarrow & \text{chiral multiplets} \\ \theta^- \text{ functions} & \rightsquigarrow & \text{(mixed) Chern-Simons couplings} \end{array} \tag{5.38}$$

Then we can construct a putative UV description for $T[M_3]$ as an abelian Chern-Simons-matter theory.

This approach is almost successful, and good enough for our present purposes, though we should mention an important caveat. In general, one must also specify relevant superpotential couplings for a UV description of $T[M_3]$, which are crucial for attaining the right superconformal theory in the IR; but it is very difficult to specify such couplings just by looking at partition functions. At the very least one would like to find superpotential couplings that break all “extraneous” flavor symmetries whose fugacities don’t appear in supersymmetric partition functions, and are not expected for the true $T[M_3]$. Even this is difficult, because the naive prescription (5.38) leads to theories that simply don’t have chiral operators charged only under the extraneous symmetries. This problem was briefly discussed in Chapter 4, and solved by finding “resolved” theories with the same partition functions as the naive ones, but with all necessary symmetry-breaking operators present [39, Section 4].

We also note that, while it is possible to construct the space of holomorphic blocks for a number of examples considered here, the construction is not always systematic, even in the simplest 3d $\mathcal{N}=2$ theories such as pure super-Chern-Simons theory. For this reason, it may be more convenient to work with other partition functions of theories $T[M_3]$ that include half-indices, *i.e.* UV counterparts of holomorphic blocks that are labeled by boundary conditions on $\mathbb{R}^2 \times S^1$. A large class of boundary conditions in 3d $\mathcal{N}=2$ theories comes from 4-manifolds and will be discussed in section 5.6. It is expected that all holomorphic blocks can be reproduced (via RG flow) from suitable choice of boundary conditions in the UV. Other prominent examples of partition functions include the index (or, $S^2 \times_q S^1$ partition function) [19, 20, 21] and the S^3_b partition function [15, 18], both of which will be discussed in section 5.4.3.

Presently, we will follow the naive approach to obtain simple UV descriptions for putative $T[M_3]$'s, where some but not all symmetry-breaking superpotential couplings are present. We expect that these theories are limits of the "true" superconformal knot-complement theories $T[M_3]$, where some marginal couplings have been sent to infinity. Thus, any observables of $T[M_3]$ that are insensitive to marginal deformations — such as holomorphic blocks, supersymmetric indices, massive vacua on S^1, *etc.* — can be calculated just as well in our naive descriptions as in the true theories, as long as masses or fugacities corresponding to extraneous flavor symmetries are turned off by hand. This is sufficient for testing many of the properties we are interested in.

Theory for unknot, $T[\mathbf{0_1}]$

The theory for the unknot that gives (5.28) was already discussed in [38] and has four chirals Φ_i, corresponding to the terms $(q^{-1}/x)^-_\infty$, $(q^{-2}/t^3)^-_\infty$, $(q^{-1})^-_\infty$, $(q^{-2}/(xt^3))^-_\infty$. Letting x and $(-t)$ be fugacities for flavor symmetries $U(1)_x$ and $U(1)_t$, we use the rules discussed in section 4 of [52, 51, 87, 11, 39, 46, 37, 56, 13, 88, 45] to read off the precise charge assignments and levels of (mixed) background Chern-Simons couplings

$T[\mathbf{0_1}]$:

	Φ_1	Φ_2	Φ_3	Φ_4
$U(1)_x$	-1	0	0	1
$U(1)_t$	0	-3	0	3
$U(1)_R$	0	-2	2	4

CS:

	$U(1)_x$	$U(1)_t$	$U(1)_R$
$U(1)_x$	0	0	0
$U(1)_t$	0	0	0
$U(1)_R$	0	0	0

(5.39)

(Here all background Chern-Simons couplings simply vanish.[7]). In this case, we can add an obvious superpotential

$$W_{\mathbf{0_1}} = \mu\, \Phi_1 \Phi_2 \Phi_4 \tag{5.40}$$

that breaks most extraneous flavor symmetries and preserves $U(1)_x, U(1)_t$, and $U(1)_R$ (note that the operator in (5.40) has R-charge two). The chiral Φ_3 is completely decoupled from the rest of the theory and rotated by an extraneous $U(1)$ symmetry. We could break this $U(1)$ by adding Φ_3 to the superpotential (5.40), but prefer not to do this as it would forbid $T[\mathbf{0_1}]$ from having a supersymmetric vacuum. Ignoring the Φ_3 sector, the putative unknot theory looks just like the 3d $\mathcal{N} = 2$ XYZ model.

Similarly, if we follow [37] and compactify $T[\mathbf{0_1}]$ on a circle turning on masses (*i.e.* complexified scalars in background gauge multiplets) for $U(1)_x$ and $U(1)_t$, we find that the theory is governed by an effective twisted superpotential

$$\widetilde{W}_{\mathbf{0_1}} = \mathrm{Li}_2(x) + \mathrm{Li}_2(-t^3) + \mathrm{Li}_2(-x^{-1}t^{-3}) + \frac{1}{2}\big[(\log x)^2 + 3\log x \log(-t) + 9(\log(-t))^2\big]\,. \tag{5.41}$$

(We have removed from $\widetilde{W}_{\mathbf{0_1}}$ an infinite contribution from the massless Φ_3; this could be regularized by turning on a mass for the $U(1)$ symmetry rotating Φ_3.) The equation for the supersymmetric parameter space,[8]

$$\exp\left(x \frac{\partial \widetilde{W}_{\mathbf{0_1}}}{\partial x}\right) = -y\,, \tag{5.42}$$

becomes the refined A-polynomial equation

$$(-t)^{\frac{3}{2}} y = \frac{1+t^3 x}{1-x}\,, \tag{5.43}$$

which further reduces to the unknot A-polynomial $y - 1 = 0$ at $t \to -1$. Equation (5.43) has a unique solution in y at generic fixed x, t, corresponding to the unique, abelian flat connection on the unknot complement (with fixed holonomy eigenvalue x on a cycle linking the unknot).

[7] We can multiply an extra normalization factor to $SU(2)$ Poincaré polynomials to make the mixed IR CS levels for $U(1)_t$ to be integers, but we will work formally without such an extra normalization.

[8] On the RHS we define an effective FI parameter as $-y$ rather than y in order to match the knot-theoretic A-polynomial below. This is correlated with the renormalization of Poincaré polynomials above by $(-1)^r$.

Theory for trefoil knot, $T[\mathbf{3_1}]$

In this case, the integrand (5.20) suggests a theory with six chirals, with charges and Chern-Simons levels

$$T[\mathbf{3_1}]: \quad \begin{array}{c|cccccc|c} & \Phi_1 & \Phi_2 & \Phi_3 & \Phi_4 & \Phi_5 & \Phi_6 & V_- \\ \hline U(1)_s & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ U(1)_x & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ U(1)_t & 0 & 0 & 1 & -1 & -3 & 3 & -3 \\ \hline U(1)_R & 0 & 0 & 2 & 0 & -2 & 4 & -2 \end{array}\,, \qquad \text{CS}: \quad \begin{array}{c|c|cc|c} & U(1)_s & U(1)_x & U(1)_t & U(1)_R \\ \hline U(1)_s & -1/2 & 3/2 & 5/2 & 5/2 \\ \hline U(1)_x & 3/2 & 0 & 0 & 2 \\ U(1)_t & 5/2 & 0 & 0 & 0 \\ \hline U(1)_R & 5/2 & 2 & 0 & 0 \end{array}\,. \tag{5.44}$$

This is now a gauge symmetry with a dynamical $U(1)_s$ symmetry in addition to $U(1)_x$ and $U(1)_t$ flavor symmetries. Standard analysis of [2] shows that this theory has a gauge-invariant anti-monopole operator V_- formed from the dual photon, with charges as indicated in the table. Altogether we can write a superpotential

$$W_{\mathbf{3_1}} = \mu_1\,\Phi_1\Phi_2\Phi_5\Phi_6 + \mu_2\,\Phi_1\Phi_3\Phi_4 + \mu_3\,\Phi_6 V_- \tag{5.45}$$

that preserves all symmetries we want to keep, and breaks almost all other flavor symmetry. There remains a single extraneous $U(1)$, just like in the unknot theory, which plays (roughly) the role of a topological symmetry dual to $U(1)_s$.

When compactifying the theory on a circle with generic twisted masses x and $(-t)$ for $U(1)_x$ and $U(1)_t$, and scalar s in the $U(1)_s$ gauge multiplet, we obtain the effective twisted superpotential

$$\begin{aligned} \widetilde{W}_{\mathbf{3_1}} &= \mathrm{Li}_2(s) + \mathrm{Li}_2(-1/(st)) + \mathrm{Li}_2(x/s) + \mathrm{Li}_2(-t^3) + \mathrm{Li}_2(-t) + \mathrm{Li}_2(-1/(t^3x)) \\ &\quad + \tfrac{1}{2}\big((\log s)^2 + \log s(6\log t + 2\log x) + \log x(\log x + 3\log(-t)) + 10(\log t)^2\big)\,. \end{aligned} \tag{5.46}$$

The critical-point equation $\exp\big(s\,\partial\widetilde{W}_{\mathbf{3_1}}/\partial s\big) = 1$, namely

$$\frac{t^2(1+st)(s-x)x}{s(1-s)} = 1 \tag{5.47}$$

determines two solutions in s at generic values of x and t; plugging these into the SUSY-parameter-space equation

$$-y = \exp\big(x\,\partial\widetilde{W}_{\mathbf{3_1}}/\partial x\big) = -s^2(-t)^{-3/2}\frac{1+t^3x}{s-x} \tag{5.48}$$

then determines two values of y. More directly, they are solutions of the quadratic

$$A^{\mathrm{ref}}_{\mathbf{3_1}}(x,y;t) = (1-x)t^2y^2 - (1 - t^2x + 2t^2x^2 + 2t^3x^2 + t^5x^3 + t^6x^4)(-t)^{\frac{1}{2}}y + t^3(x^3 + t^3x^4) = 0\,,$$

which collapses to the $A^{\mathrm{ref}}_{\mathbf{3_1}}(x,y;-1) = (x-1)(y-1)(y+x^3)$, the trefoil's A-polynomial (with an extra $(x-1)$ factor) as $t \to -1$. Thus $T[\mathbf{3_1}]$ has vacua corresponding to *both* of the flat $SL(2,\mathbb{C})$ connections on the trefoil complement, one irreducible, and one abelian. The two independent holomorphic blocks B_{II} and B_{III} of (5.22) (or, more precisely, some linear combinations of these blocks) are in 1-1 correspondence with the two flat connections.

Theory for figure-eight knot, $T[\mathbf{4_1}]$

Finally, for the figure-eight knot, the integrand (5.29) suggests a theory with $U(1)_s$ gauge symmetry, $U(1)_x \times U(1)_t$ flavor symmetry, and six chirals of charges

$T[\mathbf{4_1}]$: (5.49)

	Φ_1	Φ_2	Φ_3	Φ_4	Φ_5	Φ_6
$U(1)_s$	-1	1	1	0	1	0
$U(1)_x$	0	-1	0	0	1	0
$U(1)_t$	0	0	1	-1	3	-3
$U(1)_R$	0	0	2	0	4	-2

The net Chern-Simons couplings all turn out to vanish. This particular theory does not admit gauge-invariant monopole or anti-monopole operators. We can introduce a superpotential

$$W_{4_1} = \mu_1\, \Phi_1\Phi_3\Phi_4 + \mu_2\, \Phi_1^2\Phi_2\Phi_5\Phi_6\,, \tag{5.50}$$

which breaks flavor symmetry to $U(1)^4$, including $U(1)_x \times U(1)_t$. Thus there are two extraneous $U(1)$'s, including the topological symmetry of the theory.

As before, we can find an effective twisted superpotential on $\mathbb{R}^2 \times S^1$ of the form

$$\widetilde{W}_{\mathbf{4_1}} = \mathrm{Li}_2(s) + \mathrm{Li}_2(x/s) + \mathrm{Li}_2(-1/(st)) + \mathrm{Li}_2(-1/(sxt^3)) + \mathrm{Li}_2(-t) + \mathrm{Li}_2(-t^3) + \log\text{'s}\,, \tag{5.51}$$

whose critical point equation

$$\exp\left(s\frac{\partial \widetilde{W}_{\mathbf{4_1}}}{\partial s}\right) = \frac{(1+st)(s-x)(1+st^3x)}{(1-s)st^2x} = 1 \tag{5.52}$$

generically has three solutions in s — which in turn determine

$$y = -\exp\left(x\,\partial \widetilde{W}_{\mathbf{4_1}}/\partial x\right) \sim \frac{1+st^3x}{s-x}. \tag{5.53}$$

More directly, the solutions in y are roots of the cubic

$$A^{\rm ref}_{\mathbf{4_1}} = (x^3-x^2)(-t)^{\frac{9}{2}}y^3 - (1+tx-t^2x+2t^2x^2+2t^3x^2+2t^4x^3+2t^5x^3-t^5x^4+t^6x^4+t^7x^5)ty^2$$
$$+(-1-tx+t^2x-2t^3x^2-2t^4x^2+2t^4x^3+2t^5x^3-t^6x^4+t^7x^4+t^8x^5)(-t)^{\frac{1}{2}}y-(x^2+t^3x)t^3\,,$$

which deforms the standard figure-eight A-polynomial $A^{\rm ref}_{\mathbf{4_1}}(x,y;t=-1) = (x-1)(y-1)(x^2-(1-x-2x^2-x^3+x^4)y+x^2y^2)$. Thus $T[\mathbf{4_1}]$ has massive vacua on S^1 corresponding to all three flat $SL(2,\mathbb{C})$ connections on the figure-eight knot complement, two irreducible and one abelian. Again, these flat connections label linear combinations of the three independent holomorphic blocks $B^{\mathbf{4_1}}_{II}, B^{\mathbf{4_1}}_{III}, B^{\mathbf{4_1}}_{IV}$ in (5.31).

5.4.3 Vortices in $S^2\times_q S^1$ and S^3_b

Having obtained a theory $T[M_3]$ whose vacua on $\mathbb{R}^2\times S^1$ match flat connections on the knot complement M_3, it is interesting to probe its other protected observables. Here we focus on the $S^2\times_q S^1$ indices of $T[M_3]$, and make some preliminary observations as to the nature of the "Poincaré polynomial theories" $T_{\rm poly}[M_3;r]$ on the left-hand side of the flow diagram (5.8).

The 3d index [19, 20, 21] of a knot-complement theory, or equivalently a partition function on $S^2\times_q S^1$, depends on three fugacities q,ξ,τ and two integer monopole numbers n,p:

fugacity	monopole #	symmetry
q	–	combo of $U(1)_J\subset SO(3)_{\rm Lorentz}$ and $U(1)_R$
ξ	n	$U(1)_x$
τ	p	$U(1)_t$

(5.54)

We'll consider "twisted" indices $\mathcal{I}(\zeta,n;\tau,p;q) = {\rm Tr}_{\,\mathcal{H}_{n,p}(S^2)}e^{i\pi R}q^{\frac{R}{2}-J}\zeta^{e_x}\tau^{e_p}$ as in [25, 13], in which case it's convenient to regroup fugacities into pairs of holomorphic and anti-holomorphic variables

$$q=q\,,\quad \widetilde{q}=q^{-1}\,;\qquad x=q^{\frac{n}{2}}\xi\,,\quad \widetilde{x}=q^{\frac{n}{2}}\xi^{-1}\,;\qquad -t=q^{\frac{p}{2}}\tau\,,\quad -\widetilde{t}=q^{\frac{p}{2}}\tau^{-1}\,. \tag{5.55}$$

Then we find in examples below that the indices $\mathcal{I}[M_3]$ of $T[M_3]$ develop poles at $n=r$ and $\xi=q^{\frac{r}{2}}$, or $(x,\widetilde{x})=(q^r,1)$, whose (logarithmic) residue is the r-th Poincaré polynomial of the colored $SU(2)$ knot homology,

$$\boxed{{\rm Res}_{(x,\widetilde{x})\to(q^r,1)}\mathcal{I}[M_3] \;=\; \lim_{\xi\to q^{r/2}}(1-q^{\frac{r}{2}}\xi^{-1})\cdot\mathcal{I}[M_3](\xi,n;\tau,p;q)\Big|_{n=r} \;=\; P^r_K(t;q)}\,. \tag{5.56}$$

A similar statement holds for S^3_b partition functions. The S^3_b partition function $\mathcal{Z}_b$ [15, 18] of a knot-complement theory depends on the ellipsoid deformation b as well as two dimensionless complexified masses m_x, m_t for $U(1)_x, U(1)_t$, which are conveniently grouped into holomorphic and anti-holomorphic parameters

$$q = e^{2\pi i b^2}\,,\ \widetilde{q} = e^{2\pi i/b^2}\,;\quad x = e^{2\pi b m_x}\,,\ \widetilde{x} = e^{2\pi m_x/b}\,;\quad -t = e^{2\pi b m_t}\,,\ -\widetilde{t} = e^{2\pi m_t/b}\,. \tag{5.57}$$

Then the S_b^3 partition function has poles at $m_x = ibr$, or $(x,\widetilde{x}) = (q^r,1)$, with

$$\boxed{\mathrm{Res}_{(x,\widetilde{x})\to(q^r,1)}\mathcal{Z}_b[M_3] \;=\; \lim_{m_x\to ibr}(m_x - ibr)\cdot\mathcal{Z}_b[M_3](m_x,m_t;b) \;=\; P_K^r(t;q)}\,. \tag{5.58}$$

As discussed in section 2.4, both $\mathcal{I}[M_3]$ and $\mathcal{Z}_b[M_3]$ take the form of a sum of products of holomorphic blocks [13] ,

$$\mathcal{I}[M_3],\, \mathcal{Z}_b[M_3] \sim \sum_\alpha B_\alpha(x,t;q)\widetilde{B}_\alpha(\widetilde{x},\widetilde{t};\widetilde{q})\,, \tag{5.59}$$

and our theory $T[M_3]$ was engineered so that the $x \to q^r$ specialization of a specific linear combination of blocks B_P would reproduce Poincaré polynomials. Below we will choose a convenient basis of blocks so that B_P is one of the B_α's, and manifestly gives the only contribution to the residues (5.56), (5.58). (Nevertheless, in the natural basis of blocks labelled by flat connections at fixed $(x, t \approx -1, q = 1)$, B_P may easily correspond to a sum over multiple flat connections, including the abelian one.)

Taking the residue of a pole in an index such as (5.56) has an important physical interpretation, which was discussed in [59] in the context of 4d indices and, closer to our present subject, in [60, 61] in the context of 3d indices. Let us suppose that $\mathcal{I}[M_3]$ is a superconformal index — *i.e.* that we have adjusted R-charges to take their superconformal values. Then the index counts chiral operators at the origin in $\mathbb{R}^3$, and a pole signals the presence of an unconstrained operator $\mathcal{O}$ whose vev can parametrize a flat direction in the moduli space of $T[M_3]$. Taking the residue of the pole is equivalent to giving a large vev to $\mathcal{O}$, thus Higgsing any flavor symmetries under which $\mathcal{O}$ transforms, and decoupling massless excitations of $T[M_3]$ around this vev.

Consider, for example, the pole at $(x,\widetilde{x}) = (1,1)$, or $(\xi,n) = (1,0)$. The pole suggests the presence of an operator $\mathcal{O}_x$, of charge $+1$ under $U(1)_x$, in the zero-th $U(1)_x$ monopole sector. The contribution of this operator and its powers to the index is

$$(1+\xi+\xi^2+\ldots)\times\mathcal{I}' = \frac{1}{1-\xi}\times\mathcal{I}'\,. \tag{5.60}$$

Taking the residue $\mathcal{I}'$ amounts to giving a vev to $\mathcal{O}_x$ and decoupling massless excitations around it, thereby Higgsing $U(1)_x$ symmetry. One can interpret $\mathcal{I}'$ as the index of a new superconformal theory, the IR fixed point of a flow triggered by the vev $\langle\mathcal{O}_x\rangle$.

More generally, taking a residue at $(x,\widetilde{x}) = (q^r,1)$ or $(\xi,n) = (q^{\frac{r}{2}},r)$ gives a space-dependent vev (with nontrivial spin) to an operator in the r-th monopole sector. This not only Higgses the $U(1)_x$ symmetry of $T[M_3]$ but creates a vortex defect. We therefore expect that the residue of $\mathcal{I}[M_3]$ at $(x,\widetilde{x}) = (q^r,1)$ is the index of a new 3d theory $T_{\text{poly}}[M_3,r]$ in the presence of a (complicated) line operator.

In the context of 4d theories $T[C;G]$ coming from compactification of the 6d $(2,0)$ theory on a punctured Riemann surface C, taking the residue at a pole in the index amounted to removing a puncture from C — or more generally replacing the codimension-two defect at the puncture by a dimension-two defect in a finite-dimensional representation of G. Similarly, we expect here that taking a residue replaces the codimension-two defect along a knot $K \subset M$ by a dimension-two defect in the $(r+1)$-dimensional representation of $SU(2)$. We hope to elucidate this interpretation in future work.

We proceed to examples of (5.56). Our conventions for indices follow [25, 13]. Below, all indices depend on fugacities from (5.55) as well as the pair

$$s = q^{\frac{k}{2}}\sigma\,, \quad \widetilde{s} = q^{\frac{k}{2}}\sigma^{-1}\,, \tag{5.61}$$

which is used for summations/integrations. We assume $|q| < 1$, as is physically sensible for the index. Thus, the convergent q-Pochhammer symbols are

$$(z)_\infty := (z;q)_\infty = \textstyle\prod_{i=1}^{\infty}(1-q^i z)\,, \tag{5.62}$$

and theta-functions are

$$\theta(z_1,...,z_n) := \theta(z_1;q)\cdots\theta(z_n;q)\,, \qquad \theta(z;q) := (-q^{\frac{1}{2}}z)_\infty(-q^{\frac{1}{2}}z^{-1})_\infty\,. \tag{5.63}$$

Since we do our calculations while maintaining a manifest factorization into holomorphic blocks, results for S^3_b follow immediately from their index analogues, by reinterpreting the meaning of $\widetilde{x}, \widetilde{t}$, *etc.*

Unknot

The index of the unknot theory $T[\mathbf{0_1}]$ from (5.39) is given equivalently by

$$\begin{aligned}\mathcal{I}[\mathbf{0_1}] &= (-q^{\frac{1}{2}})^n \xi^{\frac{3}{2}p} \tau^{\frac{3}{2}n} \frac{(q/\widetilde{x})_\infty (-q^2/\widetilde{t}^3)_\infty (-1/(qxt^3))_\infty}{(x^{-1})_\infty (-q^{-1}/t^3)_\infty (-q^2/(\widetilde{x}\widetilde{t}^3))_\infty} \\ &= \left\| \frac{\theta(x, -q^{\frac{1}{2}}(-t)^{\frac{3}{2}})}{\theta(1, -q^{\frac{1}{2}}x(-t)^{\frac{3}{2}})} \frac{(-1/(qxt^3))_\infty}{(x^{-1})_\infty (-1/(qt^3))_\infty} \right\|^2_{\mathrm{id}} \\ &= \frac{\theta(x, -q^{\frac{1}{2}}(-t)^{\frac{3}{2}}, -q^{-\frac{1}{2}}\widetilde{x}(-\widetilde{t})^{\frac{3}{2}})}{\theta(\widetilde{x}, -q^{-\frac{1}{2}}(-\widetilde{t})^{\frac{3}{2}}, -q^{\frac{1}{2}}x(-t)^{\frac{3}{2}})} \times \frac{(q/\widetilde{x})_\infty (-q^2/\widetilde{t}^3)_\infty (-1/(qxt^3))_\infty}{(x^{-1})_\infty (-q^{-1}/t^3)_\infty (-q^2/(\widetilde{x}\widetilde{t}^3))_\infty}\,.\end{aligned} \tag{5.64}$$

In the first line, we simply write down the index as defined by the theory — with the massless chiral Φ_3 decoupled in order to remove an otherwise infinite factor. In the second line, we show that this index comes from a fusion norm $\|B^{\mathbf{0_1}}(x,t;q)\|^2_{\mathrm{id}}$ of the holomorphic block (5.28), with $(q^{-1})^-_\infty$ removed. Since we are working at $|q|<1$, we replace all q-Pochhammer symbols and theta-functions

$$(z)^-_\infty \to \frac{1}{(q^{-1}z)_\infty}\,, \qquad \theta^-(z) \to \frac{1}{\theta(z)} \tag{5.65}$$

in the definition of the block. In the third line, we explicitly write out what the fusion product means, following [13].

We could take the limit $(\xi, n) \to (q^{\frac{r}{2}}, r)$ in the first line of (5.64); after setting $n = r$, we would find a pole at $\xi \to q^{\frac{r}{2}}$ whose residue is the Poincare polynomial $P^r_U(t,q)$. But it is more illustrative to take the equivalent limit $(x, \widetilde{x}) \to (q^r, 1)$ in the factorized expression on the last line. Setting $\widetilde{x} = 1$ produces no divergence. The pole we are looking for comes from $(x^{-1})_\infty$ in the denominator. We get

$$\begin{aligned}\lim_{(x,\widetilde{x})\to(q^r,1)} (1-q^{-r}x)\,\mathcal{I}[U] &= \frac{\theta(q^r, -q^{\frac{1}{2}}(-t)^{\frac{3}{2}}, -q^{-\frac{1}{2}}(-\widetilde{t})^{\frac{3}{2}})}{\theta(1, -q^{-\frac{1}{2}}(-\widetilde{t})^{\frac{3}{2}}, -q^{\frac{1}{2}+r}(-t)^{\frac{3}{2}})} \times \frac{(q)_\infty (-q^2/\widetilde{t}^3)_\infty (-q^{-r-1}/t^3)_\infty}{(q^{-1};q^{-1})_r (q)_\infty (-q^{-1}/t^3)_\infty (-q^2/\widetilde{t}^3)_\infty} \\ &= (-q^{\frac{1}{2}})^{-r} (-t)^{-\frac{3r}{2}} \frac{(-q^2t^3)_r}{(q)_r} = P^r_U(t;q)\,.\end{aligned} \tag{5.66}$$

Note how the $\widetilde{t}$ dependence completely cancelled out of the problem. If we had taken a more general limit $(x,\widetilde{x}) \to (q^r, q^{r'})$, we would have found a similar pole, with residue $P^r_U(t;q) P^{r'}_U(\widetilde{t};q^{-1})$. The fact that the $\widetilde{t}$ dependence cancels out follows from the simple identity $P^{r'=0}_U(\widetilde{t};q^{-1}) = 1$.

Trefoil

For the trefoil, the theory $T[\mathbf{3_1}]$ of (5.44) leads to an integral formula for the index,

$$\mathcal{I}[\mathbf{3_1}] = \mathcal{I}_0 \sum_{k\in\mathbb{Z}} \oint \frac{d\sigma}{2\pi i\sigma} \frac{\theta(-q^{-\frac{3}{2}}\widetilde{s}\widetilde{x}(-\widetilde{t})^3)}{\theta(-q^{\frac{3}{2}}sx(-t)^3)} \frac{(qs)_\infty (1/(-st))_\infty (qx/s)_\infty}{(\widetilde{s})_\infty (q/(-\widetilde{s}\widetilde{t}))_\infty (\widetilde{x}/\widetilde{s})_\infty}\,, \tag{5.67}$$

where

$$\mathcal{I}_0 = \frac{\theta(-q^{-\frac{1}{2}}\widetilde{x}, -q^{-\frac{3}{2}}\widetilde{x}(-\widetilde{t})^{\frac{3}{2}}, -q^{\frac{3}{2}}x(-t)^3, -q^{\frac{3}{2}}(-t)^{\frac{3}{2}}, x)}{\theta(-q^{\frac{1}{2}}x, -q^{\frac{3}{2}}x(-t)^{\frac{3}{2}}, -q^{-\frac{3}{2}}\widetilde{x}(-\widetilde{t})^3, -q^{-\frac{3}{2}}(-\widetilde{t})^{\frac{3}{2}}, \widetilde{x})} \times \frac{(-1/(qxt^3))_\infty(-q^2/\widetilde{t}^3)_\infty(-q/\widetilde{t})_\infty}{(-q^2/(\widetilde{x}\widetilde{t}^3))_\infty(-1/(qt^3))_\infty(-1/t)_\infty} \tag{5.68}$$

Again, we have chosen to regroup Chern-Simons contributions into ratios of theta-functions, separating out the x and $\widetilde{x}$ dependence. The integrand in (5.67) has three pairs of half-lines of zeroes and poles in the σ-plane, coming from the three terms $(\,)_\infty/(\,)_\infty$. They lie at

	I $(qs)_\infty/(\widetilde{s})_\infty$	II $(-1/st)_\infty/(-q/\widetilde{s}\widetilde{t})_\infty$	III $(qx/s)_\infty/(\widetilde{x}/\widetilde{s})_\infty$
zeroes	$\sigma = q^{-\frac{k}{2}-1-m}$	$\sigma = q^{-\frac{k+p}{2}+m}\tau^{-1}$	$\sigma = q^{-\frac{k-n}{2}+1+m}\xi$
poles	$\sigma = q^{\frac{k}{2}+m}$	$\sigma = q^{\frac{k+p}{2}-1-m}\tau^{-1}$	$\sigma = q^{\frac{k-n}{2}-m}\xi$
	$m \geq \max(-k,0)$	$m \geq \max(k+p,0)$	$m \geq \max(k-n,0)$

(5.69)

The real, physical contour in (5.67) should lie on or around the unit circle, separating each half-line of zeroes from its corresponding half-line of poles.

We also observe that the integrand of (5.67) vanishes as $|\sigma| \to \infty$, if we stay away from half-lines of poles. Thus we can attempt to deform the contour outwards, closing it around $\sigma = \infty$. We pick up the poles in lines II and III, obtaining an expression of the form

$$\mathcal{I}[\mathbf{3_1}] = \mathcal{I}_0 \left(\|B_{II}\|^2_{\mathrm{id}} + \|B_{III}\|^2_{\mathrm{id}}\right), \tag{5.70}$$

where[9]

$$\|B_{II}\|^2_{\mathrm{id}} = \sum_{k,m\geq 0} \frac{\theta(-q^{-\frac{1}{2}+m}\widetilde{x}\widetilde{t}^2)}{\theta(-q^{\frac{1}{2}-k}xt^2)} \frac{1}{(q)_k(q^{-1};q^{-1})_m} \frac{(-q^{-k}t^{-1})_\infty(-q^{2+k}tx)_\infty}{(-q^{m+1}\widetilde{t}^{-1})_\infty(-q^{-1-m}\widetilde{t}\widetilde{x})_\infty}\,, \tag{5.71a}$$

$$\|B_{III}\|^2_{\mathrm{id}} = \sum_{k,m\geq 0} \frac{\theta(q^{-\frac{3}{2}+m}\widetilde{x}^2\widetilde{t}^3)}{\theta(q^{\frac{3}{2}-k}x^2t^3)} \frac{1}{(q)_k(q^{-1};q^{-1})_m} \frac{(q^{1-k}x)_\infty(-q^k/(xt))_\infty}{(q^m\widetilde{x})_\infty(-q^{1-m}/(\widetilde{x}\widetilde{t}))_\infty}\,. \tag{5.71b}$$

The holomorphic blocks B_{II} and B_{III} here correspond to integrals along contours Γ_{II} and Γ_{III} in Figure 5.2, with substitutions of the form $(x)^-_\infty \to 1/(qx)_\infty$ to account for $|q| < 1$.

Now, if we send $(x, \widetilde{x}) \to (q^r, 1)$, the leading pole in line I can collide with the leading pole in line III, pinching the integration contour in the σ-plane, and leading to a divergence of the the index. We see this explicitly in the evaluated expression (5.70): while the prefactor $\mathcal{I}_0$ and the blocks $\|B_{II}\|^2_{\mathrm{id}}$ are finite in this limit, the blocks $\|B_{III}\|^2_{\mathrm{id}}$ have the expected divergence. It comes from the denominator $(q^m\widetilde{x})_\infty$ in (5.71b), and occurs only for $m = 0$. The related factor $(q^{1-k}x)_\infty$

[9] A redefinition of summation indices turns the sum over $k \in \mathbb{Z}$ into sums $k \geq 0$.

in the numerator vanishes as $x = q^r$ unless $k \le r$. Therefore, we find a residue

$$
\begin{aligned}
\lim_{(x,\widetilde{x})\to(q^r,1)} (1-\widetilde{x})\,\mathcal{I}[\mathbf{3_1}] &= \lim_{(x,\widetilde{x})\to(q^r,1)} (1-\widetilde{x})\,\mathcal{I}_0\,\|B_{III}\|^2_{\text{id}} \\
&= \mathcal{I}_0(x=q^r,\widetilde{x}=1;t,\widetilde{t};q)\sum_{k=0}^{r}\frac{\theta(q^{-\frac{3}{2}}\widetilde{t}^3)}{\theta(q^{\frac{3}{2}-k}t^3)}\frac{(q^{1-k+r})_\infty(-q^{k-r}t^{-1})_\infty}{(q)_k(q)_\infty(-q/\widetilde{t})_\infty} \\
&= P^r_{3_1}(t;q)\;P^0_{3_1}(\widetilde{t};q^{-1}) = P^r_{3_1}(t;q)\,, \qquad (5.72)
\end{aligned}
$$

reproducing the superpolynomial after some straightforward manipulations.

Figure-eight knot

The setup for the figure-eight knot is almost identical to that for the trefoil. Now the index is given by

$$
\mathcal{I}[\mathbf{4_1}] = \mathcal{I}_0\sum_{k\in\mathbb{Z}}\oint\frac{d\sigma}{2\pi i\sigma}\frac{\theta(q^{-1}\widetilde{s},\widetilde{t^2}\widetilde{s})}{\theta(qs,t^2s)}\frac{(qs)_\infty(-1/(ts))_\infty(qx/s)_\infty(-1/(qxt^3s))_\infty}{(\widetilde{s})_\infty(-q/(\widetilde{ts}))_\infty(\widetilde{x}/\widetilde{s})_\infty(-q^2/(\widetilde{x}\widetilde{t^3}\widetilde{s}))_\infty}\,, \qquad (5.73)
$$

with

$$
\mathcal{I}_0 = \frac{\theta(t^2,q^{\frac{1}{2}}t,x(-t)^{-\frac{1}{2}},-q^{-\frac{1}{2}}\widetilde{x},q^{-\frac{1}{2}}\widetilde{t}\widetilde{x},(-\widetilde{t})^{-\frac{1}{2}})}{\theta(\widetilde{t}^2,q^{-\frac{1}{2}}\widetilde{t},\widetilde{x}(-\widetilde{t})^{-\frac{1}{2}},-q^{\frac{1}{2}}x,q^{\frac{1}{2}}tx,(-t)^{-\frac{1}{2}})}\times\frac{(-q^2/\widetilde{t}^3)_\infty(-q/\widetilde{t})_\infty}{(-1/qt^3)_\infty(-1/t)_\infty}\,. \qquad (5.74)
$$

There are four pairs of half-lines of zeroes and poles in the integrand; three are identical to those in the trefoil integrand above, which we denote I, II, III as in (5.69), and there is one new pair

$$
\text{IV}:\qquad \begin{array}{r} \text{zeroes } \sigma = q^{-\frac{k+n+3p}{2}-1+s}\xi^{-1}\tau^{-3} \\ \text{poles } \sigma = q^{\frac{k+n+3p}{2}-2+s}\xi^{-1}\tau^{-3} \end{array}\,,\quad \text{for } m \ge \max(k+n+3p,0)\,. \qquad (5.75)
$$

We close the contour around $\sigma = \infty$ (where the integrand generically vanishes), picking up the poles in lines II, III, and IV, to give

$$
\mathcal{I}[\mathbf{4_1}] = \mathcal{I}_0\left(\|B_{II}\|^2_{\text{id}} + \|B_{III}\|^2_{\text{id}} + \|B_{IV}\|^2_{\text{id}}\right)\,, \qquad (5.76)
$$

with

$$\|B_{II}\|^2_{\rm id} = \sum_{k,m\geq 0} \frac{\theta(-q^m/\widetilde{t}, -q^{m+1}\widetilde{t})}{\theta(-q^{-k}/t, -q^{-k-1}t)} \frac{1}{(q)_k(q^{-1},q^{-1})_m} \frac{(-q^{-k}/t)_\infty(-q^{2+k}xt)_\infty(q^k/(xt^2))_\infty}{(-q^{m+1}/\widetilde{t})_\infty(-q^{-m-1}\widetilde{x}\widetilde{t})_\infty(q^{1-m}/(\widetilde{x}\widetilde{t}^2))_\infty}\,, \tag{5.77a}$$

$$\|B_{III}\|^2_{\rm id} = \sum_{k,m\geq 0} \frac{\theta(q^{m-1}\widetilde{x}, q^m\widetilde{t}^2\widetilde{x})}{\theta(q^{1-k}x, q^{-k}t^2x)} \frac{1}{(q)_k(q^{-1},q^{-1})_m} \frac{(q^{1-k}x)_\infty(-q^k/(xt))_\infty(-q^{k-1}/(x^2t^3))_\infty}{(q^m\widetilde{x})_\infty(-q^{1-m}/(\widetilde{t}\widetilde{x}))_\infty(-q^{2-m}/(\widetilde{x}^2\widetilde{t}^3))_\infty}\,, \tag{5.77b}$$

$$\|B_{IV}\|^2_{\rm id} = \sum_{k,m\geq 0} \frac{\theta\Big(\frac{-q^{m+1}}{\widetilde{x}\widetilde{t}^3}, \frac{-q^{m+2}}{\widetilde{x}\widetilde{t}}\Big)}{\theta\Big(\frac{-q^{-k-1}}{xt^3}, \frac{-q^{-k-2}}{xt}\Big)} \frac{1}{(q)_k(q^{-1},q^{-1})_m} \frac{(-q^{-k-1}/(xt^3))_\infty(q^{2+k}xt^2)_\infty(-q^{k+3}x^2t^3)_\infty}{(-q^{m+2}/(\widetilde{x}\widetilde{t}^3))_\infty(q^{-m-1}\widetilde{x}\widetilde{t}^2)_\infty(-q^{-m-2}\widetilde{x}^2\widetilde{t}^3)_\infty}\,, \tag{5.77c}$$

The holomorphic blocks in these expressions correspond to the integration cycles discussed above (5.31) (with the usual translation from $|q|>1$ to $|q|<1$).

Now as $(x,\widetilde{x})\to(q^r,1)$, the prefactor $\mathcal{I}_0$ along with $\|B_{II}\|^2_{\rm id}$ and $\|B_{VI}\|^2_{\rm id}$ all have finite limits; while $\|B_{III}\|^2_{\rm id}$ has a pole due $1/(q^m\widetilde{x})_\infty$ at $m=0$, and is nonvanishing for $k\leq r$. As in the case of the trefoil, the divergence can be attributed to the poles of lines I and III pinching the contour of the integrand (5.73). We then find

$$\begin{aligned}\lim_{(x,\widetilde{x})\to(q^r,1)} (1-\widetilde{x})\,\mathcal{I}[\mathbf{4_1}] &= \lim_{(x,\widetilde{x})\to(q^r,1)} (1-\widetilde{x})\,\mathcal{I}_0\,\|B_{III}\|^2_{\rm id} \\ &= P^r_{4_1}(t;q)\,P^0_{4_1}(\widetilde{t};q^{-1}) = P^r_{4_1}(t;q)\,.\end{aligned} \tag{5.78}$$

5.5 The $t=-1$ limit and DGG theories

Above, we saw that sending $x\to q^r$ in partition functions of $T[M_3]$ (and perhaps discarding an overall divergence) produced finite Poincaré polynomials of colored $SU(2)$ knot homologies. Once the Poincaré polynomials are obtained, we are free to send $t\to -1$ to directly recover the colored Jones polynomials. No further divergences are encountered. Physically, we proposed an identification of the regularized $x\to q^r$ limit with a physical "Higgsing" process, by which an operator in $T[M_3]$ charged under $U(1)_x$ is given a space-dependent vev, initiating an RG flow to a new theory in the presence of a line defect. Subsequently sending $t\to -1$ should not correspond to any further flow.

One may wonder what would happen if we sent $t\to -1$ *before* $x\to q^r$. We present evidence in this section that this initiates a *different* RG flow in $T[M_3]$, which ends at a DGG theory $T_{DGG}[M_3]$. In particular, an operator $\mathcal{O}_t$ is given a (constant) vev, breaking the $U(1)_t$ symmetry characteristic of $T[M_3]$. Moreover, vacua of $T[M_3]$ on $\mathbb{R}^2\times S^1$ that correspond to abelian or reducible flat connections on M_3 are lost.

As above, our analysis will be largely example-driven. In section 5.5.1 we examine how the trefoil and figure-eight knot theories of section 5.4.2 flow to DGG theories. We verify in section 5.5.2 that $t \to -1$ limits induce divergences in $S^2 \times_q S^1$ indices, indicative of Higgsing. Then in section 5.5.3 we use effective twisted superpotentials on $\mathbb{R}^2 \times S^1$ to better understand how vacua corresponding to abelian flat connections decouple.

5.5.1 The DGG theories

We can see an explicit example of the proposed DGG flow by considering the trefoil theory $T[\mathbf{3_1}]$ of (5.44). If we turn off the real mass for the flavor symmetry $U(1)_t$, then the chiral operator $\mathcal{O}_t = \Phi_4$ can get a vev,

$$\langle \Phi_4 \rangle = \Lambda \,. \tag{5.79}$$

The vev breaks $U(1)_t$, but no other symmetries. Moreover, it induces a complex mass for Φ_1 and Φ_3 due to the superpotential

$$W_{\mathbf{3_1}} = \mu_1\, \Phi_1\Phi_2\Phi_5\Phi_6 + \mu_2\Lambda\, \Phi_1\Phi_3 + \mu_3\Phi_6 V_- \,. \tag{5.80}$$

Therefore, taking $\Lambda \to \infty$, we may decouple fluctuations of Φ_4 and integrate out Φ_1 and Φ_3, arriving at

$$T'[\mathbf{3_1}]: \quad \begin{array}{c|ccc|c} & \Phi_2 & \Phi_5 & \Phi_6 & V_- \\ \hline U(1)_s & 1 & 0 & 0 & 0 \\ U(1)_x & -1 & 0 & 1 & -1 \\ \hline U(1)_R & 0 & -2 & 4 & -2 \end{array}\,, \qquad \text{CS}: \begin{array}{c|c|c|c} & U(1)_s & U(1)_x & U(1)_R \\ \hline U(1)_s & -1/2 & 3/2 & 5/2 \\ \hline U(1)_x & 3/2 & 0 & 2 \\ \hline U(1)_R & 5/2 & 2 & 0 \end{array}\,. \tag{5.81}$$

with superpotential

$$W'_{\mathbf{3_1}} = \mu'_3\, \Phi_6 V_- \,. \tag{5.82}$$

At this point, we observe that $T[\mathbf{3_1}]$ has a sector containing a $U(1)_s$ gauge theory with a single charged chiral Φ_2, together with minus half a unit of background Chern-Simons coupling. This sector can be dualized to an ungauged chiral φ as in [39, Sec 3.3], a consequence of a basic 3d mirror symmetry [89, 3]. Indeed, the dual ungauged chiral is identified with the (anti-)monopole operator $\varphi = V_-$ of $U(1)_s$! Thus, $T'[\mathbf{3_1}]$ is dual to

$$T''[\mathbf{3_1}]: \quad \begin{array}{c|ccc} & \Phi_5 & \Phi_6 & \varphi \\ \hline U(1)_s & 0 & 0 & 0 \\ U(1)_x & 0 & 1 & -1 \\ \hline U(1)_R & -2 & 4 & -2 \end{array}\,, \qquad \text{CS}: \begin{array}{c|c|c} & U(1)_x & U(1)_R \\ \hline U(1)_x & 3 & 6 \\ \hline U(1)_R & 6 & * \end{array}\,. \tag{5.83}$$

with $W''_{\mathbf{3_1}} = \mu''_3\,\Phi_6\varphi$. The superpotential lets us integrate out Φ_6 and φ, leaving behind

$$T''[\mathbf{3_1}] \;\rightsquigarrow\; T_{DGG}[\mathbf{3_1}] \otimes T_{\Phi_5}\,. \tag{5.84}$$

Here Φ_5 is a fully decoupled free chiral, while $T_{DGG}[\mathbf{3_1}]$ is a slightly degenerate description of the DGG trefoil theory.

Namely, $T_{DGG}[\mathbf{3_1}]$ here is a "theory" consisting only of a background Chern-Simons coupling at level 3 for the flavor symmetry $U(1)_x$, and some flavor-R contact terms given by the matrix on the RHS of (5.83). A similar "theory" was obtained by DGG methods in section 4.3 ([25, Section 4.3] for detail), using a degenerate triangulation of the trefoil knot complement into two ideal tetrahedra. It was interpreted as an extreme limit of the true DGG theory $T_{DGG}[\mathbf{3_1}]$ in marginal parameter space. It is not surprising that we have hit such a limit, since, as discussed at the beginning of section 5.4.2, we are ignoring some marginal deformations.

Our $T_{DGG}[\mathbf{3_1}]$ becomes identical to that in section 4.3 upon shifting R-charges by minus two units of $U(1)_x$ charge. The shift is due to difference of conventions: we initially set $x = q^r$ in Poincaré polynomials whereas the equivalent choice for [39, 25] would be $x = q^{r+1}$.

We can repeat this exercise for the figure-eight knot. The theory $T[\mathbf{4_1}]$ of (5.49) again has a chiral operator $\mathcal{O}_t = \Phi_4$ that is charged only under $U(1)_t$, and can get a vev when the real mass corresponding to $U(1)_t$ is turned off,

$$\langle \Phi_4 \rangle = \Lambda\,. \tag{5.85}$$

Then the effective superpotential

$$W_{\mathbf{4_1}} = \mu_1 \Lambda\, \Phi_1\Phi_3 + \mu_2\, \Phi_1^2\Phi_2\Phi_5\Phi_6 \tag{5.86}$$

lets us integrate out Φ_1 and Φ_3. We flow directly to a theory

$$T[\mathbf{4_1}] \;\rightsquigarrow\; T_{DGG}[\mathbf{4_1}] \otimes T_{\Phi_6}\,, \tag{5.87}$$

where Φ_6 is a decoupled chiral and

$T_{DGG}[\mathbf{4_1}]$:

	Φ_2	Φ_5
$U(1)_s$	1	1
$U(1)_x$	-1	1
$U(1)_R$	0	4

, (CS vanishing) (5.88)

is basically the GLSM description of the $\mathbb{CP}^1$ sigma-model. It is equivalent (after shifting R-charges by minus two units of $U(1)_x$ charge) to the DGG theory in section 4.3 obtained from a triangulation of the figure-eight knot complement into two tetrahedra.[10] Again, this triangulation is a little degenerate as discussed in section 4.3 ([39, Section 4.6] for detail), so (5.88) should be viewed as a limit of the true $T_{DGG}[\mathbf{4_1}]$, which has the same protected partition functions (index, half-indices, and holomorphic blocks).

5.5.2 Indices and residues

The $S^2 \times_q S^1$ indices of theories $T[M_3]$ help us to further illustrate the breaking of $U(1)_t$ by "Higgsing" and the flow to $T_{DGG}[M_3]$. As discussed in section 5.4.3, Higgsing corresponds to taking residues in an index. In particular, we expect here to find the indices $\mathcal{I}_{DGG}[M_3]$ of DGG theories as residues of $\mathcal{I}[M_3]$ at $(t,\widetilde{t}) \to (-1,-1)$.

Consider, for example, the index $\mathcal{I}[\mathbf{3_1}]$ of the trefoil theory as given by (5.70). Sending $t \to -1$, the prefactor $\mathcal{I}_0$ develops a pole due to the factor $1/(-1/t)_\infty$. This factor comes directly from the chiral Φ_4 in $T[\mathbf{3_1}]$. (The factor $1/(-1/(qt^3))_\infty$ in $\mathcal{I}_0$, coming from the chiral Φ_4, also develops a pole, but it is not relevant for the Higgsing we want to do.) In addition, we see that $\|B_{III}\|^2_{\text{id}}$ has a finite limit as $(t,\widetilde{t}) \to (-1,-1)$, whereas $\|B_{III}\|^2_{\text{id}}$ *vanishes* due to $(-q^{-k}t^{-1})_\infty$ in the numerator. One way to understand this vanishing is to observe that the zeroes in line I of the index integrand perfectly cancel all poles in line II when $(t,\widetilde{t}) = (-1,-1)$. Therefore,

$$\begin{aligned}
\lim_{t,\widetilde{t}\to -1} (1-t)\mathcal{I}[\mathbf{3_1}] &= \lim_{t,\widetilde{t}\to -1} (1-t)\,\mathcal{I}_0\,\|B_{III}\|^2_{\text{id}} \qquad (5.89)\\
&= \text{“}\frac{(-q^2/\widetilde{t}^3)_\infty}{(-1/(qt^3))_\infty}\text{”}\,\frac{\theta(-q^{-\frac12}\widetilde{x},x)(1/(qx))_\infty}{\theta(-q^{\frac12}x,\widetilde{x})(q^2/\widetilde{x})_\infty}\sum_{k,m\geq 0}\frac{1}{(q)_k(q^{-1};q^{-1})_m}\frac{\theta(-q^{m-\frac32}\widetilde{x}^2,-q^{\frac12-k}x)}{\theta(-q^{\frac32-k}x^2,-q^{m-\frac12}\widetilde{x})}\\
&= \text{“}\frac{(-q^2/\widetilde{t}^3)_\infty}{(-1/(qt^3))_\infty}\text{”}\,\frac{\theta(x,q^{-\frac32}\widetilde{x}^2)}{\theta(\widetilde{x},-q^{\frac32}x^2)} = \text{“}\frac{(-q^2/\widetilde{t}^3)_\infty}{(-1/(qt^3))_\infty}\text{”}\,q^{3n}\xi^{3n}\\
&= \text{“}\frac{(-q^2/\widetilde{t}^3)_\infty}{(-1/(qt^3))_\infty}\text{”}\,\mathcal{I}_{DGG}[\mathbf{3_1}]\,.
\end{aligned}$$

The resummation in the third line captures the duality between a charged chiral (Φ_2) and a free chiral ($\varphi = V_-$) discussed in section 5.5.1. Then the expression $q^{3n}\xi^{3n}$ matches the DGG trefoil index of [25], modulo a redefinition of R-charges $\xi \to q^{-1}\xi$. The infinite prefactor $(-q^2/\widetilde{t}^3)_\infty/(-1/(qt^3))_\infty \to (q^2)_\infty/(q^{-1})_\infty$ is the contribution of the decoupled chiral Φ_5.

When considering the $t,\widetilde{t} \to -1$ limit of the figure-eight index $\mathcal{I}[\mathbf{4_1}]$ from (5.76), the prefactor $\mathcal{I}_0$ has the same divergent term $(-1/t)^{-1}_\infty$ that appeared for the trefoil. Moreover, the contribution $\|B_{II}\|^2_{\text{id}}$ to the figure-eight index vanishes, because poles of the index integrand in line II are cancelled

[10] The equivalence is most directly seen using the polarization discussed in Appendix B and section 6.3 of [13]. The "degenerate" DGG theory for the figure-eight knot, *a.k.a.* the $\mathbb{CP}^1$ sigma-model, has three standard duality frames that are analyzed in section 5.1 of [13], and the most symmetric of these duality frames agrees with (5.88). Another frame matches section 4.6 of [39].

by zeroes in line I. Thus, following a short calculation, the figure-eight index takes the form

$$\begin{aligned}\lim_{t,\widetilde{t}\to -1}(1-t)\mathcal{I}[\mathbf{4_1}] &= \lim_{t,\widetilde{t}\to -1}(1-t)\,\mathcal{I}_0\left(\|B_{III}\|^2_{\rm id}+\|B_{IV}\|^2_{\rm id}\right) \\ &= \text{``}\frac{(-q^2/\widetilde{t}^3)_\infty}{(-1/(qt^3))_\infty}\text{''}(q\xi)^{2n}\Big[(-q^{\frac12})^n\sum_{k,m\geq 0}\frac{(qx)^k(q^{-1}\widetilde{x})^m}{(q^{-1};q^{-1})_k(q)_m}\frac{(q^{k+1}(qx)^2)_\infty}{(q^{-m}(q^{-1}\widetilde{x})^2)_\infty} \\ &\qquad\qquad + (n,q\xi)\leftrightarrow(-n,1/(q\xi))\Big] \\ &= \text{``}\frac{(-q^2/\widetilde{t}^3)_\infty}{(-1/(qt^3))_\infty}\text{''}\,\mathcal{I}_{DGG}[\mathbf{4_1}]\,.\end{aligned} \tag{5.90}$$

We recognize in this the DGG index of the figure-eight knot, already split into two holomorphic blocks. For proper comparison to [25] or [13], we should again rescale $\xi\to q^{-1}\xi$, or $(x,\widetilde{x})\to(q^{-1}x,q\widetilde{x})$.

5.5.3 Critical points and missing vacua

We saw in section 5.5.2 that in the limit $t,\widetilde{t}\to -1$, some parts of indices $\mathcal{I}[M_3]$ vanished, while others contributed to $\mathcal{I}_{DGG}[M_3]$. This is a reflection of the fact that the DGG theories $T_{DGG}[M_3]$ don't capture all information about flat connections on M_3, and in particular don't have massive vacua on $\mathbb{R}^2\times S^1$ corresponding to abelian or reducible flat $SL(2,\mathbb{C})$ connections.

We can make this idea much more precise by considering the effective twisted superpotentials that govern theories $T[M_3]$ on $\mathbb{R}^2\times S^1$. For example, for the trefoil, this was given by (5.46):

$$\begin{aligned}\widetilde{W}_{\mathbf{3_1}}(s;x,t) &= \mathrm{Li}_2(s)+\mathrm{Li}_2(-1/(st))+\mathrm{Li}_2(x/s)+\mathrm{Li}_2(-t^3)+\mathrm{Li}_2(-t)+\mathrm{Li}_2(-1/(t^3x)) \\ &\quad+\tfrac12\big((\log s)^2+\log s(6\log t+2\log x)+\log x(\log x+3\log(-t))+10(\log t)^2\big)\,.\end{aligned} \tag{5.91}$$

It is important to note that this function on $\mathbb{C}^*$ (parametrized by the dynamic variable s) has branch cuts coming from integrating out chiral matter that at some points in the s-plane becomes massless. In particular, each term $\mathrm{Li}_2(f(s))$ has a cut along a half-line starting at the branch point $f(s)=1$ and running to zero or infinity. Such cuts and their consequences have been discussed from various perspectives in *e.g.* [12, 10, 90, 62]. Often one writes the vacuum or critical-point equations as

$$\exp\left(s\,\partial\widetilde{W}_{\mathbf{3_1}}/\partial s\right)=1\,, \tag{5.92}$$

because in this form they are algebraic in s. However, when analyzing vacua of $T[M_3]$ on $\mathbb{R}^2\times S^1$, one must remember to lift solutions of (5.92) back to the cover of the s-plane defined by $\widetilde{W}$ — and to make sure they are actual critical points on some sheets of the cover.

Now consider what happens if we send $t\to -1$. The branch points of $\mathrm{Li}_2(s)$ and $\mathrm{Li}_2(-1/(st))$, located at $s = 1$ and $s = -1/t$, collide. (These branch points came directly from the chirals Φ_1 and Φ_3, which we integrated out of $T[\mathbf{3_1}]$ in (5.81).) In the process, the half-line cuts originating at these branch points coalesce into a full cut running from $s = 0$ to $s = \infty$; this is easy to see from the inversion formula

$$\mathrm{Li}_2(s) + \mathrm{Li}_2(1/s) = -\tfrac{\pi^2}{6} - \tfrac{1}{2}\log(-s)^2 \qquad (s \notin [0,1)\ ) \,. \tag{5.93}$$

Moreover, one of the solutions s_* to (5.92), or rather its lift(s) to the covering of the s-plane, gets trapped between the colliding branch points and ceases to be a critical point as $t \to -1$. One can see this from the explicit form of the critical-point equations (5.47), which are reduced from quadratic to linear order in s by a cancellation at $t = -1$. However, to properly interpret this limit, it is helpful to think about the branched cover of the s-plane as we have done.

Physically, each solution of (5.92) is a vacuum of $T[M_3]$ on $\mathbb{R}^2 \times S^1$. As $t \to -1$, the vacuum at s_* is lost. This is possible precisely because the $t \to -1$ limit is singular. Indeed, we know that $t \to -1$ corresponds to making $T[M_3]$ massless, so that the reduction on $\mathbb{R}^2 \times S^1$ is no longer fully described by an effective twisted superpotential. The specialized superpotential $\widetilde{W}(s; x, t = -1)$ does not describe $T[M_3]$ itself at the massless point, but rather the Higgsed $T_{DGG}[M]$ as found in section 5.5.1.

In the case of the trefoil, the vacuum at s_* close to $t = -1$ is labelled (via the 3d-3d correspondence) by the abelian flat connection on $M_3 = S^3 \backslash K$. Indeed, if we substitute the limiting $t \to -1$ value of s_* (namely $s_* = 1$) into the SUSY-parameter-space equation $\exp\big(x\, \partial\widetilde{W}/\partial x\big) = y$, we find

$$y_* := \exp\big(x\, \partial\widetilde{W}/\partial x\big)\Big|_{s_*} = 1 \qquad \text{at } t = -1\,, \tag{5.94}$$

corresponding to the abelian factor $y - 1 = 0$ of the trefoil's classical A-polynomial. Thus we see explicitly that the DGG theory $T_{DGG}[\mathbf{3_1}]$ loses a vacuum corresponding to the abelian flat connection.

We may also perform this analysis at the level of holomorphic blocks. Holomorphic blocks are labelled by (q-deformed) critical points of $\widetilde{W}$ — or more precisely by integration cycles Γ_α obtained by starting at a critical point of $\widetilde{W}$ and approximately following gradient flow with respect to $\mathrm{Re}\, \frac{1}{\log q}\widetilde{W}$. For the trefoil we can choose a basis of integration cycles given by Γ_{II} and Γ_{III} in Figure 5.2. The precise correspondence with critical points depends on x, t, q. Close to $t = -1$, however, it is clear that Γ_{II} corresponds to the "abelian" critical point s_*. As $t \to -1$, the contour Γ_{II} gets trapped crossing a full line of poles (resolutions of the classical branch cuts described above), and ceases to be a good holomorphic-block integration cycle in the sense of [13].[11] Most importantly, it

[11] Of course, Γ_{II} *is* still a reasonable integration cycle, mathematically, at $t = -1$ and any finite q. The integral along it does reproduce a Jones polynomial as $x \to q^r$. It is tempting to wonder whether one could engineer such a

no longer flows from any classical critical point. Beautifully, the remaining contour Γ_{III} is isolated away from the point s_* where half-lines of poles merge. The $t \to -1$ limit of the corresponding block $B_{III}(x,t;q)$ is precisely the holomorphic block of $T_{DGG}[\mathbf{3_1}]$, labelled by the irreducible flat $SL(2,\mathbb{C})$ connection, and contributing to the index (5.89).

Analogous remarks apply to the figure-eight example. The 3d Higgsing and integrating out of Φ_1, Φ_3 in $T[\mathbf{4_1}]$ translates on $\mathbb{R}^2 \times S^1$ to branch points of $\mathrm{Li}_2(s)$ and $\mathrm{Li}_2(-1/(st))$ colliding in (5.51), and trapping a critical point between them. Thus, as $t \to -1$, $T[\mathbf{4_1}]$ looses one of its three massive vacua on $\mathbb{R}^2 \times S^1$ — the one labeled by the abelian connection on the figure-eight knot complement. The $T_{DGG}[\mathbf{4_1}]$ only has two massive vacua, labelled by irreducible flat connections. The remaining vacua correspond to the holomorphic blocks B_{III} and B_{IV}, which at $t \to -1$ become the holomorphic blocks of $T_{DGG}[\mathbf{4_1}]$.

5.5.4 Relation to colored differentials

We expect that the Higgsing procedure found to relate $T[M_3]$ to $T_{DGG}[M_3]$ in the examples above holds much more generally. We can actually recognize some key signatures of the reduction in a much larger family of examples, which include so-called thin knots. The phenomena described above follow from the structure of colored Poincaré polynomials for these knots. The structure of the Poincaré polynomials is highly constrained by the properties of colored differentials whose existence in S^r-colored homologies was postulated in [69, 91], as well as by the so-called exponential growth. Using these properties, in [56] colored Poincaré polynomials of many thin knots, including the infinite series of $(2, 2p+1)$ torus knots and twist knots with $2n+2$ crossings, were uniquely determined.

More precisely, colored differentials enable transitions between homology theories labeled by the r-th and k-th symmetric-power representations S^r and S^k. The existence of these differentials implies that Poincaré polynomials take the form of a summation (over $k = 0, \ldots, r$), with the summand involving a factor $(-aq^{-1}t;q)_k$. On the other hand, the exponential growth is the statement that for $q=1$ (normalized) colored Poincaré polynomials (superpolynomials) satisfy the relation

$$\mathcal{P}_K^{S^r}(a, q=1, t) = \left(\mathcal{P}_K^{S^1}(a, q=1, t)\right)^r . \tag{5.95}$$

If the uncolored superpolynomial on the right hand side is a sum of a few terms, its r'th power can be written as a (multiple) summation involving Newton binomials, which for arbitrary q turn out to be replaced by q-binomials [56, 57]. This structure can be clearly seen in the example of $(2, 2p+1)$

"Jones" cycle starting directly with $T_{DGG}[M_3]$, with no prior knowledge of the full $T[M_3]$ — and what the physical meaning of this cycle might be.

torus knots considered in [56, 57], whose (normalized) colored superpolynomials take the form

$$\mathcal{P}^{S^r}_{T^{2,2p+1}}(a,q,t) \;=\; a^{pr}q^{-pr}\sum_{0\le k_p\le\ldots\le k_2\le k_1\le r}\begin{bmatrix} r \\ k_1\end{bmatrix}\begin{bmatrix} k_1 \\ k_2\end{bmatrix}\cdots\begin{bmatrix} k_{p-1} \\ k_p\end{bmatrix}\times \qquad (5.96)$$

$$\times\; q^{(2r+1)(k_1+k_2+\ldots+k_p)-\Sigma_{i=1}^p k_{i-1}k_i}t^{2(k_1+k_2+\ldots+k_p)}\prod_{i=1}^{k_1}(1+aq^{i-2}t).$$

Here the last product originating from the structure of differentials, as well as a series of q-binomials originating from the exponential growth, are manifest (in this formula $k_0 = r$). Poincaré polynomials for infinite families of twist knots derived in [56, 57] share analogous features.

It becomes clear now that various properties of trefoil and figure-8 knots, discussed earlier, should also be present for other knots, such as thin knots discussed above. For example, as discussed in section 5.5.2, the divergence at $t \to -1$ in the trefoil and figure-8 indices, $\mathcal{I}[\mathbf{3_1}]$ and $\mathcal{I}[\mathbf{4_1}]$, is a manifestation of a pole due to the factor $1/(-1/t)_\infty$. This factor originates from the q-Pochhammer symbol $(-aq^{-1}t;q)_k$ in corresponding Poincaré polynomials (5.13) and (5.25), after setting $a = q^2$ and rewriting this term in the denominator. As follows from the discussion above, such a factor is present in general for other thin knots (and represents the action of colored differentials), so for such knots an analogous pole at $t \to -1$ should develop. We postulate that the residue at this pole in general reproduces indices $\mathcal{I}_{DGG}[M]$ for theories dual to other (thin) knots.

Similarly, a decoupling of the abelian branch for more general knots is a consequence of the structure of superpolynomials described above. From this perspective, let us recall once more how this works for trefoil and figure-8 knot, just on the level of critical point equations (5.47) and (5.48), or (5.52) and (5.53). If we set $t = -1$ in (5.47) or (5.52), the ratio $\frac{1+st}{1-s}$ on the left hand side drops out of the equation (this is a manifestation of the cancelation (5.93) at the level of twisted superpotential). In this ratio the numerator $1 + st$ has its origin in the $(-aq^{-1}t;q)_k$ term in superpolynomials (5.13) and (5.25), while the denominator $1 - s$ originates from q-Pochhamer $(q;q)_k$ being a part of the q-binomial in those superpolynomials. As explained above, such terms appear universally in superpolynomials for thin knots. Similarly, for $t = -1$ the equations (5.48) and (5.53) reduce to $y = 1$ (which represents the abelian branch that drops out when $t \to -1$ is set first) due to a cancellation between the term in their numerator and $s - x$ in denominator. The terms in numerator have the origin in $(a(-t)^3;q)_r$ from unknot normalization (5.24), possibly combined with another term $(aq^r(-t)^3;q)_k$ representing colored differentials for figure-8 knot (5.25). The term $s - x$ in denominator has its origin in $(q;q)_{r-k}$ ingredient of q-binomial. Analogous terms, responsible for cancellations, are also universally present in superpolynomials for other knots. The analysis is slightly more involved if Poincaré polynomials include multiple summations — *e.g.* for $(2, 2p + 1)$ torus knots (5.96) — however one can check that similar cancellations between "universal" terms decrease the degree of saddle equations and result in the decoupling of the abelian branch.

5.6 Boundaries in three dimensions

In this section we discuss the gluing along boundaries of M_3 and the boundary conditions in 3d $\mathcal{N} = 2$ theories $T[M_3]$.

In particular, understanding the operations of cutting and gluing M_3 along a Riemann surface C opens a new window into the world of closed 3-manifolds. The basic idea of how such operations should manifest in 3d $\mathcal{N} = 2$ theory $T[M_3]$ was already discussed *e.g.* in [51, 11] and will be reviewed below. The details, however, cannot work unless $T[M_3]$ accounts for all flat connections on M_3. This was recently emphasized in [58] where the general method of building $T[M_3]$ via gluing was carried out for certain homology spheres.

After constructing 3d $\mathcal{N} = 2$ theories $T[M_3]$ for certain homology spheres, we turn our attention to boundary conditions in such theories. Incorporating boundary conditions and domain walls in general 3d $\mathcal{N} = 2$ theories was discussed in [62] and involves the contribution of the 2d index of the theory on the boundary / wall that is a "flavored" generalization of the elliptic genus. For theories of class $\mathcal{R}$ that come from 3-manifolds, many such boundary conditions come from 4-manifolds as illustrated in (5.9). In this case, the flavored elliptic genus of a boundary condition / domain wall is equal to the Vafa-Witten partition function of the corresponding 4-manifold [58].

5.6.1 Cutting and gluing along boundaries of M_3

It is believed that a 3-manifold with boundary C gives rise to a boundary condition in 4d $\mathcal{N} = 2$ theory of class $\mathcal{S}$, see Figure 2 in [39] or Figure 6 in [58]. This system can be understood as a result of 6d $(2,0)$ theory compactified on a 3-manifold with cylindrical end $\mathbb{R}_+ \times C$ and to some extent was studied previously.[12] For example, when $C = T^2$ is a 2-torus (with puncture) the corresponding 4d $\mathcal{N} = 2$ theory is actually $\mathcal{N} = 4$ super-Yang Mills (resp. $\mathcal{N} = 2^*$ theory).

A simple class of 3-manifolds bounded by C includes *handlebodies*, which for a genus-g Riemann surface C is determined by a choice of g pairwise disjoint simple closed curves on C (that are contractible in the handlebody 3-manifold). For example, if $C = T^2$, then the corresponding handlebody is a solid torus:

$$M_3 \;\cong\; S^1 \times D^2 \,. \tag{5.97}$$

It is labeled by a choice (p,q) of the 1-cycle that becomes contractible in M_3. In the basic case of $(p,q) = (0,1)$ the Chern-Simons path integral on M_3 defines a state (in the Hilbert space $\mathcal{H}_{T^2}$) that is usually denoted $|0\rangle$, so that we conclude

$$|0\rangle \;=\; |\text{solid torus}\rangle \tag{5.98}$$

[12] See *e.g.* [51, 87, 11, 39, 92] for a sample of earlier work; unfortunately the methods of these papers cannot be used to recover all flat connections for general 3-manifolds, even in the simplest cases of knot complements.

It was proposed in [58] that the corresponding boundary condition in 4d theory $T[C]$ is Nahm pole boundary condition [93, 94] that can be described by a system of D3-branes ending on D5-branes[13]

$$|0\rangle \;=\; |\text{Nahm}\rangle \;=\; |\text{D5}\rangle \tag{5.99}$$

More generally, for $M_3 \cong S^1 \times D^2$ obtained by filling in the cycle in homology class (p, q) the corresponding boundary condition is defined by a system of D3-branes ending on IIB five-branes of type (p, q).

This class of boundary conditions can be easily generalized to other Riemann surfaces C and 3-manifolds with several boundary components. The latter correspond to domain walls in 4d $\mathcal{N} = 2$ theories $T[C]$, see *e.g.* [39, 58, 92] for details. For example, each element ϕ of the mapping class group of C corresponds, on the one hand, to a mapping cylinder M_3 (with two boundary components identified via ϕ) and, on the other hand, to a duality wall of type ϕ in the 4d theory $T[C]$. In the case $C = T^2$ we have the familiar walls that correspond to the generators $\phi = S$ and $\phi = T$ of the $SL(2, \mathbb{Z})$ duality group of $\mathcal{N} = 4$ super-Yang-Mills, and the general "solid torus boundary condition" described above can be viewed as the IR limit of a concatenation of S- and T-walls with the basic Nahm pole boundary condition, see [58, pp.20-21] for details. For instance,

$$S|0\rangle \;=\; |\text{Neumann}\rangle \;=\; |\text{NS5}\rangle \tag{5.100}$$

Clearly, there are still many details to work out, but we have outlined the key elements necessary to glue 3-manifolds along a common boundary and, in particular, to illustrate why (5.5) must hold in a proper 3d $\mathcal{N} = 2$ theory $T[M_3]$. Suppose $C = \pm\partial M_3^{\pm}$ is a common boundary component of 3-manifolds M_3^+ and M_3^-, which in general may have other boundary components, besides C. As we reviewed earlier, appropriately defined 3d $\mathcal{N} = 2$ theories $T[M_3^+]$ and $T[M_3^-]$ naturally couple to a 4d $\mathcal{N} = 2$ theory $T[C]$, which becomes dynamical upon the gluing process

$$M_3 \;=\; M_3^- \cup_\phi M_3^+ \tag{5.101}$$

Note, in the identification of the two boundaries here we included an element ϕ of the mapping class group of C that corresponds to duality wall in $T[C]$. Hence, the resulting theory $T[M_3]$ consists of a ϕ-duality wall in 4d $\mathcal{N} = 2$ theory $T[C]$ sandwiched between $T[M_3^+]$ and $T[M_3^-]$. At the level of partition functions,

$$Z_{T[M_3]} \;=\; Z_{CS}(M_3) \;=\; \langle M_3^- |\phi| M_3^+ \rangle \tag{5.102}$$

A particularly simple and useful operation that involves (re)gluing solid tori *a la* (5.97)–(5.101)

[13] Whether we identify the state $|0\rangle$ with D5 or NS5 is a matter of conventions. Here we follow the conventions of [58, 62].

is called surgery. In fact, it is also the most general one in a sense that, according to a theorem of Lickorish and Wallace, every closed oriented 3-manifold can be represented by (integral) surgery along a link $K \subset S^3$. Since the operation is defined in the same way on any component of the link L it suffices to explain it in the case when K has only one component, *i.e.* when K is a knot. Then, for a pair of relatively prime integers $p, q \in \mathbb{Z}$, the result of q/p Dehn surgery along K is the 3-manifold:

$$S^3_{q/p}(K) \; := \; (S^3 - N(K)) \cup_\phi (S^1 \times D^2) \tag{5.103}$$

where $N(K)$ is the tubular neighborhood of the knot, and $S^1 \times D^2$ is attached to its boundary by a diffeomorphism $\phi : S^1 \times \partial D^2 \to \partial N(K)$ that takes the meridian μ of the knot to a curve in the homology class

$$q[\mu] + p[\lambda] \tag{5.104}$$

The ratio $q/p \in \mathbb{Q} \cup \{\infty\}$ is called the surgery coefficient.

In what follows we discuss various aspects of cutting, gluing, and surgery operations. In particular, we shall see how the operations (5.102) and (5.103) manifest at various levels in 3d $\mathcal{N} = 2$ theory $T[M_3]$ — at the level of SUSY vacua, at the level of twisted superpotential, and at the level of quantum partition functions — thereby illustrating the important role of abelian flat connections. Needless to say, there are many directions in which one could extend this analysis, *e.g.* to various classes of 3-manifolds not considered in this thesis, as well as more detailed analysis of the ones presented here, to higher rank groups G and to relation with known properties of homological knot invariants.

5.6.1.1 Compactification on S^1 and branes on the Hitchin moduli space

A useful perspective on our 3d-4d system can be obtained by compactification on S^1 and studying the space of SUSY vacua. Thus, a compactification of 4d $\mathcal{N} = 2$ theory $T[C]$ on a circle yields a 3d $\mathcal{N} = 4$ sigma-model whose target is the hyper-Kähler manifold

$$\mathcal{M}_{SUSY}(T[C], G) = \mathcal{M}_H(G, C) \tag{5.105}$$

while a 3-manifold bounded by C defines a half-BPS boundary condition, *i.e.* a brane in the sigma-model language.

More precisely, a 3-manifold M_3 with $C = \partial M_3$ gives rise to a brane of type (A, B, A) with respect to the hyper-Kähler structure on $\mathcal{M}_H(G, C)$. It is supported on a mid-dimensional submanifold of $\mathcal{M}_H(G, C)$ which can be identified with the moduli space of flat $G_{\mathbb{C}}$ connections on M_3:

$$\mathcal{M}_{\text{flat}}(M_3, G_{\mathbb{C}}) \; \subset \; \mathcal{M}_H(G, C) \tag{5.106}$$

Note, according to (5.5), the space of flat $G_{\mathbb{C}}$ connections on M_3 is precisely the space of SUSY vacua (parameters) of the 3d $\mathcal{N}=2$ theory $T[M_3]$ on a circle. When combined with (5.105) this gives

$$\mathcal{M}_{\text{SUSY}}(T[M_3],G) \subset \mathcal{M}_{\text{SUSY}}(T[C],G) \tag{5.107}$$

In this description, the mapping class group of the Riemann surface C (which we already identified with the duality group of $T[C]$) acts by autoequivalences on branes in the sigma-model with the target space $\mathcal{M}_H(G,C)$. See [95, 11] for various examples of the mapping class group action on (A,B,A) branes in the Hitchin moduli space.

In particular, when $G=SU(2)$ and $C=T^2$ is a 2-torus, the Hitchin moduli space is a flat hyper-Kähler space $\mathcal{M}_H(G,C)\cong(\mathbb{C}^*\times\mathbb{C}^*)/\mathbb{Z}_2$ parametrized by $\mathbb{C}^*$-valued holonomy eigenvalues x and y modulo the Weyl group action. This is also the space of vacua of $T[C,G]$ after dimensional reduction on a circle. Each 3-manifold with a toral boundary defines a middle-dimensional submanifold or an (A,B,A) brane. Thus, when translated to language of geometry, the boundary conditions (5.99) and (5.100) correspond to (A,B,A) branes supported on $x=1$ and $y=1$, respectively:

$$\begin{aligned} |x=1\rangle &= |\text{D5}\rangle \\ |y=1\rangle &= |\text{NS5}\rangle \end{aligned} \tag{5.108}$$

Similarly, the duality wall of type $\phi=S$ is a "correspondence" $\mathcal{M}_{\text{flat}}(M_3,G_{\mathbb{C}})\subset\mathcal{M}_H(G,C)\times\mathcal{M}_H(G,C)$ associated with the mapping cylinder $M_3\cong C\times I$,

$$x+\frac{1}{x} = y'+\frac{1}{y'} \qquad , \qquad y+\frac{1}{y} = x'+\frac{1}{x'} \tag{5.109}$$

that exchanges the $SL(2,\mathbb{C})$ holonomies on a- and b-cycles of $C=T^2$. Note, these relations are deformed in $\mathcal{M}_{\text{SUSY}}(T[M_3],G)\subset\mathcal{M}_{\text{SUSY}}(T[C],G)\times\mathcal{M}_{\text{SUSY}}(T[C],G)$ for a generic value of the fugacity t.

5.6.1.2 Lens space theories and matrix models

In the above discussion we used the solid torus (5.97)–(5.98) as a simple example of a handlebody, in this case bounded by $C=T^2$. Likewise, the simplest example of a closed 3-manifold obtained by gluing two solid tori is the Lens space

$$L(p,1) \;=\; \langle 0|ST^pS|0\rangle \;\cong\; S^3/\mathbb{Z}_p \tag{5.110}$$

Using the dictionary (5.99) and (5.100), we can identify the corresponding 3d $\mathcal{N}=2$ theory $T[L(p,1)]$ as the theory on D3-branes suspended between a NS5-brane and a $(p,1)$-fivebrane:

$$T[L(p,1);G] \; = \; \text{SUSY } G_{-p} \text{ Chern-Simons theory} \tag{5.111}$$

Following [58], here we assumed that the gauge group G is of Cartan type A, *i.e.* $G=U(N)$ or $G=SU(N)$. It would be interesting, however, to test the conjecture (5.111) for other groups G.

Now, let us discuss this gluing more carefully, first from the viewpoint of flat connections (= SUSY vacua) and then from the viewpoint of partition functions. According to (5.100) and (5.108), the solid torus boundary condition $S|0\rangle$ in $\mathcal{N}=4$ super-Yang-Mills $T[C]$ imposes a Neumann boundary condition on x and a Dirichlet boundary condition on y. In fact, the solid torus theory here is basically the theory of the unknot, $T[\mathbf{0_1}]$, discussed in section 5.4.2. Its supersymmetric parameter space (5.43) is a linear subspace of $\mathcal{M}_{\text{SUSY}}(T[C],G)$ defined by $y=1$. Note, the equation $y-1=0$ is precisely the defining equation of the abelian branch, which in our present example is the entire moduli space $\mathcal{M}_{\text{flat}}(M_3,G_{\mathbb{C}})=\mathcal{M}_{\text{SUSY}}(T[M_3],G)$. Therefore, had we ignored this component, the space of SUSY vacua would be completely empty, both for the solid torus theory $T[S^1\times D^2]$ and for everything else that can be obtained from it by gluing!

A concatenation of the T^p duality wall with this boundary condition adds a supersymmetric Chern-Simons term at level p for the global $U(1)_x$ symmetry of the theory $T[\mathbf{0_1}]$. If we are only interested in SUSY vacua and parameters of a theory $T[M_3]$ (= flat connections on M_3) we need to know how this operation affects the effective twisted superpotential, which for a general theory $T[M_3]$ has a simple form:

$$T^p: \quad \widetilde{W} \to \widetilde{W} + \frac{p}{2}(\log x)^2 \tag{5.112}$$

For the case at hand, the result of this operation modifies the space of SUSY parameters from $y=1$ to $y=x^p$. Finally, gluing $\langle 0|S$ and $T^pS|0\rangle$ in (5.110) means sandwiching $\mathcal{N}=4$ super-Yang-Mills between the corresponding boundary conditions. In our IR description of the boundary conditions, this makes $U(1)_x$ dynamical, so that all critical points of the effective twisted superpotential [58]:

$$\widetilde{\mathcal{W}}_{T[M_3]} \; = \; \widetilde{\mathcal{W}}_{T[M_3^-]} - \widetilde{\mathcal{W}}_{\phi\,\circ\, T[M_3^+]} \tag{5.113}$$

become SUSY vacua (= flat connections) of the theory $T[M_3]$ associated with the gluing (5.101). For the Lens space (5.110), we get a set of points $\{y=1\}\cup\{y=x^p\}$ in $(\mathbb{C}^*\times\mathbb{C}^*)/\mathbb{Z}_2$, which are in one-to-one correspondence with massive SUSY vacua of the Lens space theory $T[L(p,1);SU(2)]$.[14]

More generally, for $G=U(N)$ the flat connections on $L(p,1)$ or, equivalently, the SUSY vacua of (5.111) are labeled by Young diagrams ρ with at most $p-1$ rows and N columns, *i.e.* Young

[14] In section 5.6.1.3, we will take a look at a more complicated gluing operation where this simple analysis of vacua encounters some subtleties.

diagrams that fit in a rectangle of size $N \times (p-1)$. Note, these are in one-to-one correspondence with integrable representations of $\widehat{\mathfrak{su}}(p)_N$ (equivalently, of $\widehat{\mathfrak{u}}(N)_p$), the fact that plays an important role [96, 97, 98, 99] in the study of Vafa-Witten partition function on ALE spaces bounded by $L(p,1)$.

Next, let us consider the gluing (5.110) at the level of partition functions. The partition function of the theory (5.111) on the ellipsoid S^3_b is given by (see *e.g.* [100]):

$$\begin{aligned} Z_{S^3_b} &= \frac{1}{|\mathcal{W}|} \int \prod_{i=1}^{r} d\sigma_i \; e^{-i\pi p \sigma \cdot \sigma} \prod_{\alpha \in \Lambda^+_{\text{rt}}} 4 \sinh(\pi b \sigma \cdot \alpha) \sinh(\pi b^{-1} \sigma \cdot \alpha) \\ &= \exp\left(\frac{i\pi}{4} \dim G - \frac{i\pi}{12p}(b^2 + b^{-2}) h \dim G \right) p^{r/2} \prod_{\alpha \in \Lambda^+_{\text{rt}}} 2 \sin\left(\frac{\pi \alpha \cdot \rho}{p} \right) \end{aligned} \tag{5.114}$$

where $r = \text{rank}(G)$, $\mathcal{W}$ is the Weyl group of G, h is the dual Coxeter number of G, Λ^+_{rt} is the set of positive roots of G, and ρ is the Weyl vector (half the sum of the positive roots). Furthermore, turning on a FI parameter ζ contributes an extra term $e^{4\pi i \zeta \text{Tr}\, \sigma}$ into the integral (5.114). As usual, the S^3_b partition function of $T[L(p,1);G]$ should have the following structure

$$Z_{S^3_b} = \sum_{\rho} \| B^{\rho}(q) \|^2_S \tag{5.115}$$

where $q = e^{\hbar} = e^{2\pi i b^2}$ and each block $B^{\rho}(q) \sim Z^{\rho}_{CS}(L(p,1);G)$ is expected to represent the Chern-Simons partition function computed in the background of a flat connection labeled by ρ. (Recall from our earlier discussion that classical solutions in Chern-Simons theory on $M_3 = L(p,1)$ are labeled by certain Young diagrams ρ.)

Unfortunately, there is no systematic algorithm to define holomorphic blocks in general 3d $\mathcal{N}=2$ theories and, as a result, the factorization (5.115) is not known at present for the $\mathcal{N}=2$ super-Chern-Simons theory (5.111). However, the integral form of the partition function (5.114) does share many key features with the Chern-Simons partition function on the Lens space that will be discussed in the next section (and extended to more general Seifert manifolds). Here, let us just note that $\log Z_{S^3_b}$ in (5.114) has the form of a power series in $\hbar = 2\pi i b^2$ that starts with the leading $\frac{1}{\hbar}$ term and terminates at the order $\mathcal{O}(\hbar)$. This is indeed the property of Chern-Simons partition function on $L(p,1)$: according to a famous result of Lawrence and Rozansky [101], higher loop corrections to $Z^{\rho}_{CS}(L(p,1);G)$ all vanish.

Finally, we propose a "lift" of the gluing formula (5.113) to a similar formula at the level of partition functions, *cf.* (5.102):

$$Z_{T[M_3]} = \int [dU(x)] \; Z_{T[M_3^-]}(x) \cdot Z_{\phi \circ T[M_3^+]}(x^{-1}) \tag{5.116}$$

where the integration measure $[dU] = Z_{T[C]} dx$ is determined by the 4d $\mathcal{N}=2$ theory $T[C;G]$

associated with the Riemann surface $C = \partial M_3^+ = -\partial M_3^-$. It would be interesting to test this gluing formula in concrete examples, including the Lens spaces and Seifert manifolds discussed here. Note that with the t-variable that keeps track of homological grading, (5.116) basically is a surgery formula for homological knot invariants. Such formulas are indeed known in the context of knot Floer homology and its version for general 3-manifolds, the Heegaard Floer homology.

As explained around (5.101), we can construct closed 3-manifolds by gluing open 3-manifolds along their boundaries. The Chern-Simons partition functions on manifolds with torus boundary depend on a parameter x, which should be integrated out upon gluing. For a particular class of 3-manifolds, the resulting Chern-Simons partition functions can be represented as matrix integrals, much like (5.114), where the integration measure is responsible for integrating out the parameters x. The integrands of such matrix models take the form

$$\exp\Big(-\frac{1}{\hbar}V(x)\Big), \tag{5.117}$$

where $V(x)$ is usually called potential and $\frac{2\pi i}{\hbar} = \frac{2\pi i}{\log q}$ is called the "level". Let us note that in the case of 3-manifolds with boundary, when the parameters x are not integrated out, the same representation of partition functions $Z_{CS} \sim \exp(\frac{1}{\hbar}\widetilde{W} + \ldots)$ was used to read off the twisted superpotentials of dual $\mathcal{N} = 2$ theories, such as (5.46) or (5.51). One is therefore tempted to postulate, that a matrix model potential $V(x)$ might encode information about the twisted superpotential and field content of the dual $\mathcal{N} = 2$ theory $T[M_3]$ associated to a closed 3-manifold M_3. Let us demonstrate that this is indeed the case.

For non-abelian Chern-Simons theories it is convenient to a write matrix model representation of their partition functions in terms of eigenvalues $\sigma_i = \log x_i$. A very well known example is a matrix model representation of the $U(N)$ Chern-Simons partition function on $M_3 = S^3$ [76, 102], whose measure takes the form of a trigonometric deformation of the Vandermonde determinant, and the potential $V(\sigma) = \sigma^2/2$ is Gaussian in $\sigma = \log x$. More generally, the matrix model potential for $M_3 = L(p, 1)$ and $G = U(N)$ takes the form $V(\sigma) = p\sigma^2/2$. More involved integral representations of Chern-Simons partition functions on other Lens spaces and Seifert homology spheres can be found in [101, 76, 102]. Various other matrix integral representations of Chern-Simons or related topological string partition functions, including the refined setting, were constructed in [103, 104, 105, 106, 107, 72, 108, 109].

Let us now consider more seriously the proposal that the potential of a Chern-Simons matrix model determines the dual 3d $\mathcal{N} = 2$ theory $T[M_3]$. For example, as reviewed above, the potential for a theory of the Lens space $L(p, 1)$ takes the form $V(\sigma) = p\sigma^2/2$. Taking into account a minus sign in (5.117), and using by now familiar 3d-3d dictionary, we might conclude that the dual theory is $\mathcal{N} = 2$ theory at level $-p$, at least in the abelian case. Due to the universal form of the matrix integral, we might also be tempted to declare that in the nonabelian case the dual theory is $U(N)$ theory at level $-p$. This is precisely the dual theory (5.111) which was originally constructed by other means. We also emphasize that the form of the matrix model reflects the structure of the gluing (5.101), namely the fact that the resulting Lens space (5.110) is constructed from two solid tori (unknot complements), glued with a suitable $SL(2,\mathbb{Z})$ twist ϕ. Indeed, in this case the potential factor (5.117) represents the gluing $SL(2,\mathbb{Z})$ element ϕ, while the information about two solid tori is encoded in the matrix model measure. This construction is discussed in detail *e.g.* in [102].

We can do similarly for Seifert manifold M_3 at least $G = U(1)$ [1].

5.6.1.3 Dehn surgery

As a final simple illustration of the necessity of accounting for all flat connections, we return to the basic Dehn surgery operation (5.103). Suppose that the knot $K = \mathbf{3_1}$ is the trefoil. As we know well from section 5.4.2, the A-polynomial[15] for the trefoil, parametrizing $\mathcal{M}_{\text{SUSY}}(T[\mathbf{3_1}], SU(2))$ for the full trefoil-complement theory $T[\mathbf{3_1}, SU(2)]$, is

$$A(x,y) = (y-1)(y+x^6) \ \subset (\mathbb{C}^* \times \mathbb{C}^*)/\mathbb{Z}_2\,. \tag{5.118}$$

Here x and y are the $\mathbb{C}^*$-valued eigenvalues of longitude and meridian $SL(2,\mathbb{C})$ holonomies on the torus boundary of the knot complement, well defined up to the Weyl-group action $(x,y) \mapsto (x^{-1}, y^{-1})$. We recall that the $(y-1)$ component of the A-polynomial corresponds to an abelian flat connection on the knot complement, while the $(y+x^6)$ component corresponds to an irreducible flat connection.

Suppose that we perform Dehn surgery with $q/p = \pm 1$ on the trefoil knot complement. The result is a Brieskorn sphere $\Sigma[2,3,5]$ and $\Sigma[2,3,7]$;

$$S^3_{p/q}(\mathbf{3_1}) = \begin{cases} \Sigma[2,3,5] & p/q = +1 \\ \Sigma[2,3,7] & p/q = -1 \end{cases} \tag{5.119}$$

The Brieskorn sphere is a closed 3-manifold and is defined as

$$\Sigma[a,b,c] := \{(x,y,z) \in \mathbb{C}^3 | x^a + y^b + z^c = 0\} \cap S^5 \tag{5.120}$$

where a, b, and c are coprime.

In each case, the moduli space of flat $SL(2,\mathbb{C})$ connections on $S^3_{p/q}(\mathbf{3_1})$, consists of isolated

[15] In contrast to the rest in this chapter, we take care in this section to write A-polynomials in terms of actual $SL(2,\mathbb{C})$ meridian and longitude eigenvalues rather than their squares. Thus, for the trefoil, the non-abelian A-polynomial is written as $y + x^6$ rather than $y + x^3$. The distinction is important for consistently counting $SL(2,\mathbb{C})$ (as opposed to $PSL(2,\mathbb{C})$, etc.) flat connections resulting from surgery.

points. It is easy to count them directly from a presentation of the fundamental groups of the Brieskorn spheres,

$$\pi_1(\Sigma[2,3,5]) = \langle a, b \,|\, a^3 = b^5 = (ab)^2 \rangle\,, \qquad \pi_1(\Sigma[2,3,7]) = \langle a, b \,|\, a^3 = b^7 = (ab)^2 \rangle\,. \tag{5.121}$$

We find $|\mathcal{M}_{\text{flat}}(S^3_{+1}(\mathbf{3_1}), SL(2,\mathbb{C}))| = 3$ and $|\mathcal{M}_{\text{flat}}(S^3_{-1}(\mathbf{3_1}), SL(2,\mathbb{C}))| = 4$. These counts must equal the numbers of isolated vacua of the theories $T[S^3_{\pm 1}(\mathbf{3_1}), SU(2)]$ on $\mathbb{R}^2 \times S^1$.

Now compare the count of flat connections on the Brieskorn spheres with the intersection points of the varieties

$$(x^p y^q = 1) \cap (A(x,y) = 0) \;=\; \begin{cases} 4 \text{ points} & p/q = 1 \\ 5 \text{ points} & p/q = -1\,. \end{cases} \tag{5.122}$$

This does not quite match the count of flat connections on the Brieskorn spheres: in each case, there is one extra intersection point in (5.122). In particular, in each case, the intersection point $(x,y) = (-1,-1)$ corresponds to flat connections on the knot complement $S^3 \backslash \mathbf{3_1}$ and the solid surgery torus whose *eigenvalues* match at the T^2 surgery interface, but whose full *holonomies* do not. Namely, the flat connection on the solid surgery torus with eigenvalues $(-1,-1)$ is trivial, while the flat connection on the trefoil knot complement with eigenvalues $(-1,-1)$ is parabolic, meaning the full holonomy matrix is $\left(\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right)$. This is not unexpected, since $(x,y) = (-1,-1)$ lies on the nonabelian branch $y + x^6 = 0$ of the trefoil's A-polynomial. After subtracting the "false" intersection point from the counts in (5.122), we recover the expected number of flat connections on $S^3_{+1}(\mathbf{3_1})$ and $S^3_{-1}(\mathbf{3_1})$.

Physically, (5.122) is the (naively) expected count of vacua when gluing the trefoil theory to an unknot theory with the appropriate element $\phi \in SL(2,\mathbb{Z})$ corresponding to the Dehn surgery. The presence of a "false" intersection point $(x,y) = (-1,-1)$ suggests that the corresponding vacuum in the glued theory must be lifted. It would be interesting to uncover the mechanism behind this. The remaining vacua match the count of flat connections on the Brieskorn spheres (*i.e.* vacua of $T[S^3_{\pm 1}(\mathbf{3_1}), SU(2)]$), as they should. Crucially the vacuum corresponding to the intersection point $(x,y) = (1,1)$ must be included in order for the count to work out; this intersection point sits on the abelian branch $(y-1)$ of the trefoil A-polynomial, and labels the trivial flat connection on $S^3_{\pm 1}$.

A similar phenomenon occurs when considering simple surgeries on the figure-eight knot complement $S^3 \backslash \mathbf{4_1}$. For example, the Brieskorn sphere $\Sigma[2,3,7]$ may be constructed from $+1$ or -1 surgeries on $S^3 \backslash \mathbf{4_1}$. (The two different surgeries produce opposite orientations on $\Sigma[2,3,7]$.) The intersection of the full figure-eight A-polynomial $A(x,y) = (y-1)\big(x^4 - (1 - x^2 - 2x^4 - x^6 + x^8)y + x^4 y^2\big)$ with the surgery conditions $xy^{\pm 1} = 1$ yield

$$(xy^{\pm 1} = 1) \cap (A(x,y) = 0) \;= 5 \text{ points}\,. \tag{5.123}$$

Four of these five intersection points, including the point on the abelian branch $y-1=0$, correspond to the expected flat $SL(2,\mathbb{C})$ connections on $\Sigma[2,3,7]$. The fifth intersection point, at $(x,y) = (-1,-1)$, does not correspond to any flat connection on $\Sigma[2,3,7]$, because the connection with eigenvalues $(x,y)=(-1,-1)$ on the knot complement is parabolic, while on the solid surgery torus it would have to be trivial. Explicitly, the meridian and longitude holonomies of the connections on $S^3\backslash\mathbf{4_1}$ with $(x,y)=(-1,-1)$ are conjugate to $\mu = \left(\begin{smallmatrix}-1 & 1\\ 0 & -1\end{smallmatrix}\right)$, $\lambda = \left(\begin{smallmatrix}-1 & \pm 2i\sqrt{3}\\ 0 & -1\end{smallmatrix}\right)$, which will never satisfy $\mu^p\lambda^q = I$ for any p, q.

5.6.2 Boundary conditions in 3d $\mathcal{N}=2$ theories

So far we discussed what happens when 3-manifolds have boundaries, along which they can be glued, *cf.* (5.101). Now let us briefly discuss what happens when the space-time of 3d $\mathcal{N}=2$ theory $T[M_3;G]$ has a boundary.

(0,2) multiplet	**contribution to half-index**
chiral	$\theta(-q^{\frac{R-1}{2}}x;q)^{-1}$
Fermi	$\theta(-q^{\frac{R}{2}}x;q)$
$U(N)$ gauge	$(q;q)_\infty^{2N}\prod_{i\neq j}\theta(-q^{-\frac{1}{2}}\sigma_i/\sigma_j;q)$

Table 5.1: Building blocks of 2d boundary theories and their contributions to the half-index.

Much like in Chern-Simons theory on M_3 the presence of non-trivial boundary requires specifying boundary conditions, the same is true in the case of 3d $\mathcal{N}=2$ theories. One important novelty, though, is that some boundary conditions are now distinguished if they preserve part of supersymmetry, such as half-BPS boundary conditions that preserve $\mathcal{N}=(0,2)$ supersymmetry on the boundary. These "B-type" boundary conditions have been studied only recently in [62] and then in [110].

In the presence of a boundary (or, more generally, a domain wall) one can define a generalization of the index as a partition function on $S^1\times_q D$ with a prescribed B-type boundary condition on the boundary torus $S^1\times_q S^1\cong T^2$ of modulus τ, as illustrated in Figure 5.1. The resulting half-index $\mathcal{I}_{S^1\times_q D}$ is essentially a convolution of the flavored elliptic genus of the 2d $\mathcal{N}=(0,2)$ boundary theory with the index of a 3d $\mathcal{N}=2$ theory on $S^1\times_q D$. The contribution of $(0,2)$ boundary degrees of freedom is summarized in Table 5.1 where, as usual, gauge symmetries result in integrals over the corresponding variables σ_i.

The half-index $\mathcal{I}_{S^1\times_q D}$ labeled by a particular choice of the boundary condition can be viewed as a UV counterpart of a holomorphic block labeled by a choice of the massive vacuum in the IR.

Moreover, since the half-index is invariant under the RG flow, it makes sense to identify some of massive vacua and integration contours in the IR theory with specific boundary conditions in the UV. The latter, in turn, can sometimes be identified with 4-manifolds via (5.9), which altogether leads to an interesting correspondence between certain holomorphic blocks and 4-manifolds.

Note, that for theories $T[M_3;G]$ labeled by closed 3-manifolds, supersymmetric vacua $\rho \in \mathcal{M}_{\text{SUSY}}(T[M_3;G])$ specify boundary conditions for the Vafa-Witten topological gauge theory on a 4-manifold bounded by M_3. Therefore, had we missed any of the vacua in constructing $T[M_3;G]$ there would be no hope to relate supersymmetric boundary and 4-manifolds in (5.9).

For instance, let us consider one of the simplest 3d $\mathcal{N}=2$ theories, namely the super-Chern-Simons theory with gauge group $G=U(N)$ that in (5.111) we identified with the Lens space theory. As we mentioned earlier, the holomorphic blocks for this theory are not known. However, their UV counterparts $\mathcal{I}_{S^1\times_q D}$ are easy to write down by choosing various B-type boundary conditions constructed in [62, 58, 110]. Thus, a simple boundary condition involves pN Fermi multiplets on the boundary. According to the rules in Table 5.1, its flavored elliptic genus can be interpreted as the half-index of 3d $\mathcal{N}=2$ super-Chern-Simons theory (5.111) with gauge group $G=U(N)$:

$$\mathcal{I}_{S^1\times_q D} \; = \; q^{-\frac{pN}{24}} \prod_{i=1}^{p}\prod_{j=1}^{N} \theta(x_i z_j;q) \tag{5.124}$$

Moreover, it can be identified with the Vafa-Witten partition function of the ALE space

$$A_{p-1} = M_4(\mathfrak{su}(p)) = M_4(\underbrace{\overset{-2}{\bullet}\!\!-\cdots-\!\!\overset{-2}{\bullet}}_{p-1}) \tag{5.125}$$

written in the “continuous basis”

$$Z_{\text{VW}}^{U(N)}[A_{p-1}](q,x|z) \; := \; \sum_{\rho} \chi_{\rho^t}^{\widehat{\mathfrak{u}}(N)_p}(q,z)\, Z_{\text{VW}}^{U(N)}[A_{p-1}]_\rho(q,x) \tag{5.126}$$

where

$$Z_{\text{VW}}^{U(N)}[A_{p-1}]_\rho(q,x) \; = \; \chi_\rho^{\widehat{\mathfrak{su}}(p)_N}(q,x) \tag{5.127}$$

is the well known form of the Vafa-Witten partition function on the ALE space (5.125) written in the “discrete basis” [96, 97, 98, 99]. Here, ρ is a Young diagram with at most $p-1$ rows and N columns that in the previous section we identified with the choice of flat connection on $M_3 = \partial M_4 = L(p,1)$.

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