On the local Tamagawa number conjecture for Tate motives

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Abstract

There is a wonderful conjecture of Bloch and Kato ([BK90]) that generalizes both the analytic Class Number Formula and the Birch and Swinnerton-Dyer conjecture. The conjecture itself was generalized by [FK06] to an equivariant formulation. In this thesis, I provide a new proof for the Equivariant local Tamagawa number conjecture in the case of Tate motives for unramified fields, using Iwasawa theory and (ϕ, Γ)-modules, and provide some work towards extending the proof to tamely ramified fields.

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Chapter 1

Introduction

1.1 Preliminaries

1.1.1 Basic definitions

Let p be an odd prime, and let K be a finite extension of $\mathbb{Q}_p$ with maximal unramified subextension F. We set $[K:F]=e$ and $[F:\mathbb{Q}_p]=f$, and throughout this document we assume p does not divide ef and $(e,p-1)=1$. We assume $K/\mathbb{Q}_p$ is Galois, and thus $e|p^f-1$ and $K=F(\sqrt[e]{p})$.

We have Galois groups $G=\mathrm{Gal}(K/\mathbb{Q}_p)$, $\Delta_e=\mathrm{Gal}(K/F), \Sigma=\mathrm{Gal}(F/\mathbb{Q}_p)$, K has residue field $\mathcal{O}_K/\sqrt[e]{p}\mathcal{O}_K=\mathcal{O}_F/p\mathcal{O}_F=k$. Then $\Delta_e\cong\mathbb{Z}/e\mathbb{Z}$ and $\Sigma\cong\mathbb{Z}/f\mathbb{Z}$ are cyclic, and we can fix as generators a Frobenius element $\sigma\in\Sigma$ and some $\delta_e\in\Delta_e$. Let $G_K=\mathrm{Gal}(\overline{K}/K)$, let $\chi^{\mathrm{cyclo}}:G_K\to\mathbb{Z}_p^\times$ be the cyclotomic character, and set $H_K=\ker\chi^{\mathrm{cyclo}}$ and $\Gamma_K=G_K/H_K$.

Set $\tilde{E}$ to be the set of sequences $(x^{(0)},x^{(1)},\dots)$ of elements of $\mathbb{C}_p$ satisfying $(x^{(n+1)})^p=x^{(n)}$, with addition given by $(x+y)^{(n)}=\lim_m(x^{(n+m)}+y^{(n+m)})^{p^m}$ and $(xy)^{(n)}=x^{(n)}y^{(n)}$. Then $\tilde{E}$ is a complete, algebraically closed field of characteristic p. We have obvious actions of the frobenius element ϕ and of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ on $\tilde{E}$.

Fix a compatible system of roots of unity $(\zeta_{p^m})_{m\in\mathbb{N}}$ where $(\zeta_{p^{m+1}})^p=\zeta_{p^m}$, and write $K_n=K(\zeta_{p^n})$. We set $\epsilon=(1,\zeta_p,\zeta_{p^2},\dots)\in\tilde{E}$, and denote by $E_{\mathbb{Q}_p}$ the subfield of $\tilde{E}$ given by $\mathbb{F}_p((\epsilon-1))$. We write E for its separable closure and note that $\tilde{E}$ is the completion of the algebraic closure. We write $E_K=E^{H_K}$ for the subfield of E fixed by H_K.

We take $\tilde{A}=W(\tilde{E})$ the ring of Witt vectors over $\tilde{E}$, and $\tilde{B}=\tilde{A}[1/p]=\mathrm{Frac}(\tilde{A})$. This is a complete discrete valuation field with residue field $\tilde{E}$. We can write $x\in\tilde{A}$ as $\sum_{k=0}^{\infty}p^k[x_k]$ where $x_k\in\tilde{E}$ and $[\cdot]$ is the Teichmuller lift.

We write $\pi=[\epsilon]-1$, and $A_{\mathbb{Q}_p}$ for the closure of $\mathbb{Z}_p[\pi,\pi^{-1}]$ in $\tilde{A}$; it is a complete discrete valuation ring with residue field $E_{\mathbb{Q}_p}$. We write $B_{\mathbb{Q}_p}=\mathrm{Frac}(A_{\mathbb{Q}_p})=A_{\mathbb{Q}_p}[1/p]$. We have actions of ϕ and G on $\tilde{B}$, given by $\phi(\pi)=(1+\pi)^p-1$ and $g(\pi)=(1+\pi)^{\chi^{\mathrm{cyclo}}(g)}-1$.

We write B for the closure of the maximal unramified extension of $B_{\mathbb{Q}_p}$ in $\tilde{B}$ and $A=B\cap\tilde{A}$

such that $A[1/p] = B$. We have $B_K = B^{H_K}$ and $A_K = A^{H_K}$. We observe that

$$B_K = \left\{ \sum_{n \in \mathbb{Z}} a_n \pi_K^n : a_n \in F, \lim_{n \to -\infty} a_n = 0 \right\}$$

where π_K is eth root of π and a uniformizer of B_K. We likewise have $A_K = \left\{ \sum_{n \in \mathbb{Z}} a_n \pi_K^n \in B_K : a_n \in \mathcal{O}_F \ \forall n \right\}$.

The actions of ϕ and G on $\tilde{B}$ restrict to endomorphisms of B_K and A_K for each K, and we further have the following operators, defined on B and on each B_K:

Definition 1.1.1. Let $f \in B$. Then we define

1. $\psi(f) = p^{-1}\phi^{-1} \operatorname{Tr}_{B/\phi B}(f)$.
2. $\mathcal{N}(f) = \phi^{-1} N_{B/\phi B}(f)$.

We observe that if $f \in B$, then $\psi(\phi(f)) = f$; thus ψ is an additive left inverse of ϕ.

1.1.2 de Rham and crystalline representations

In this section, we recall the definitions of the Fontaine rings of periods. We follow the presentation in [CC99] III.1.

Recall $\tilde{E}$ is the set of sequences $(x^{(0)}, x^{(1)}, \dots)$ of elements of $\mathbb{C}_p$ satisfying $(x^{(n+1)})^p = x^{(n)}$; take $\tilde{E}^+$ to be its ring of integers. We write $A_{inf} = W(\tilde{E}^+)$ for the ring of Witt vectors with coefficients in $\tilde{E}^+$ (see also our definition $\tilde{A} = W(\tilde{E})$ above), and if $x \in \tilde{E}^+$ we write $[x]$ for its Teichmüller representative in A_{inf}.

Any element of A_{inf} can be written $\sum_{n \geq 0} p^n [x_n]$ for some sequence of elements $x_n \in \tilde{E}^+$. The map $\theta : A_{inf} \to \mathcal{O}_{\mathbb{C}_p}$ given by $\sum_{n \geq 0} p^n [x_n] \mapsto \sum_{n \geq 0} p^n x_n^{(0)}$ is surjective, and its kernel is the ideal generated by $\omega = \frac{\pi}{\phi^{-1}(\pi)}$.

Define $B_{inf}^+ = A_{inf}[1/p]$, and we can extend θ to a map $B_{inf}^+ \twoheadrightarrow \mathbb{C}_p$. We write $B_{dR}^+ = \varprojlim B_{inf}^+/(\ker \theta)^n$, and by abuse of notation we have a map $\theta : B_{dR}^+ \to \mathbb{C}_p$. Then B_{dR}^+ is a discrete valuation ring with residue field $\mathbb{C}_p$, and the Galois action of $G_{\mathbb{Q}_p}$ on B_{inf}^+ gives rise to an action of $G_{\mathbb{Q}_p}$ on B_{dR}^+.

We see that $\log([\epsilon]) = \sum_{n \geq 1} (-1)^{n-1} \pi^n / n$ converges in B_{dR}^+; we write $\log([\epsilon]) = t$, sometimes referred to as "the p-adic analogue of $2\pi i$." We write $B_{dR} = B_{dR}^+[1/t]$; this is a field, and comes with a decreasing filtration given by $\operatorname{Fil}^i(B_{dR}) = t^i B_{dR}^+$. We can see that $\sigma(t) = \chi^{\text{cyclo}}(\sigma) \cdot t$ for any $\sigma \in G$, and thus this filtration is stable under the action of $G_{\mathbb{Q}_p}$.

Suppose V is a representation of G_K over $\mathbb{Q}_p$, that is, a finite-dimensional $\mathbb{Q}_p$-vector space with a continuous linear G_K-action. Then the K-vector space $D_{dR}(V) = (B_{dR} \otimes V)^{G_K}$ has dimension less than $\dim_{\mathbb{Q}_p} V$, and comes with a decreasing filtration induced by the filtration on B_{dR}; in particular, $\operatorname{Fil}^i D_{dR}(V) = D_{dR}(V)$ if $i \ll 0$ and $\operatorname{Fil}^i D_{dR}(V) = 0$ if $i \gg 0$.

Let B^+_{cris} be the set of elements $x \in B^+_{dR}$ such that $x = \sum_{n\geq 0} a_n \omega^n/n!$ for some elements $a_n \in B^+_{inf}$ with $\lim_n a_n = 0$. Then B^+_{cris} is a subring of B^+_{dR}, which is stable under the action of $G_{\mathbb{Q}_p}$; it clearly contains t, and we write $B_{cris} = B^+_{cris}[1/t]$. The action of ϕ on B^+_{inf} extends to an action on B^+_{cris}, and we have $\phi(t) = pt$. (Note that ϕ does not in fact have a nice extension to B^+_{dR}; B^+_{cris} is essentially the subring to which ϕ extends nicely).

As with B_{dR}, given a representation V of G_K over $\mathbb{Q}_p$, we define a K-vector space $D_{cris}(V) = (B_{cris} \otimes V)^{G_K}$. The action of ϕ on B_{cris} commutes with the action of G_K, so we have a semi-linear action of ϕ on $D_{cris}(V)$.

We have that $\dim_K D_{cris}(V) \leq \dim_K D_{dR}(V) \leq \dim_{\mathbb{Q}_p} V$. If $\dim_K D_{dR}(V) = \dim_{\mathbb{Q}_p}(V)$ we say that V is de Rham. If $\dim_K D_{cris}(V) = \dim_{\mathbb{Q}_p}(V)$ then we say V is crystalline. (Thus every crystalline representation is also de Rham.)

Let $k \in \mathbb{Z}$. We define the G_K representation $\mathbb{Q}_p(k)$, called the kth Tate twist of $\mathbb{Q}_p$, by the action $g \cdot s = \chi^{\text{cyclo}}(g)^k \cdot s$. Then for any G_K-representation V, we define the Tate twist of V to be $V(k) = V \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(k)$; that is, G_K acts on $V(k)$ with the given action on V times the kth power of the cyclotomic character. Then if V is de Rham (resp. crystalline) we have that $V(k)$ is de Rham (resp crystalline), and $D_{dR}(V(k)) = t^{-k} D_{dR}(V)$ (resp. $D_{cris}(V(k)) = t^{-k} D_{cris}(V)$).

1.1.3 The Bloch-Kato exponential map

We now recall the definition of the Bloch-Kato exponential (see [BK90] and [Ber03]). There is a short exact sequence, called the fundamental exact sequence:

$$0 \to \mathbb{Q}_p \xrightarrow{\alpha} B^{\phi=1}_{cris} \xrightarrow{\beta} B_{dR}/B^+_{dR} \to 0, \tag{1.1}$$

where α is the inclusion of $\mathbb{Q}_p \hookrightarrow B^{\phi=1}_{cris}$ and β is given by the inclusion $B^{\phi=1}_{cris} \hookrightarrow B_{dR}$. (See [BK90] Proposition 1.17).

Let V be a $\mathbb{Q}_p$-representation of G_K. We may tensor equation (1.1) with V and then take cohomology; we get a long exact sequence:

$$0 \to H^0(K, V) \to H^0(K, B^{\phi=1}_{cris} \otimes V) \to H^0(K, B_{dR}/\operatorname{Fil}^0 B_{dR} \otimes V) \to H^1(K, V) \to \dots$$

Recall that by definition, $D_{cris}(V) \cong (B_{cris} \otimes V)^{G_K} \cong H^0(K, B_{cris} \otimes V)$, and similarly $D_{dR}(V) \cong H^0(K, B_{dR}(V))$. Since ϕ commutes with the action of G_K, we have

$$0 \to V^{G_K} \to D_{cris}(V)^{\phi=1} \to D_{dR}(V)/\operatorname{Fil}^0 D_{dR}(V) \xrightarrow{\exp_V} H^1(K, V) \to \dots \tag{1.2}$$

and we may write the following definition:

Definition 1.1.2. Let V be a $\mathbb{Q}_p$-representation of G_K. Then we define the exponential map

$$\exp_V : D_{dR}(V)/\operatorname{Fil}^0 D_{dR}(V) \to H^1(K, V)$$

to be the connecting homomorphism in (1.2).

In particular, [BK90] show that if we take $V = \mathbb{Q}_p(r)$ for $r \geq 2$, then $\exp_{\mathbb{Q}_p(r)}$ is an isomorphism from K to $H^1(K, \mathbb{Q}_p(r))$.

1.2 The Tamagawa number conjecture for Tate motives

In this section we will describe the Tamagawa number conjecture of [BK90], and then a refinement due to [FK06]. In subsection 1.2.1 we will state the original conjecture, and the theorem proved in [BK90] as evidence. In subsection 1.2.2 we will develop some maps that will allow us to work in the group ring $\overline{\mathbb{Q}}_p[G]$, and in subsection 1.2.3 we will study the structure of this group ring. Finally, in subsection 1.2.4 we will use these tools to state a refined "equivariant" version of the Tamagawa number conjecture.

1.2.1 The conjecture of Bloch and Kato

For a motive M, Bloch and Kato ([BK90]) define abelian groups $A(\mathbb{Q}_p)$ for all p such that if we take

$$L(M, s) = \prod_{p \text{ good}} P_p(M, p^{-s})^{-1}$$

then $\mu_p(A(\mathbb{Q}_p)) = P_p(M, 1)$ for almost all p. They further postulate the existence of an abelian group $A(\mathbb{Q})$ that maps to every $A(\mathbb{Q}_p)$, and define

$$\operatorname{Tam}(M) = \mu\left(\prod_{p \leq \infty} A(\mathbb{Q}_p)/A(\mathbb{Q})\right).$$

Note that we have $\operatorname{Tam}(M) = L(M, 1) \cdot \prod_l c_l$ for finitely many primes l. They define a group $Ш(M)$, which they conjecture to be finite, and further conjecture:

$$\operatorname{Tam}(M) = \#H^0(\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), M^\vee \otimes \mathbb{Q}/\mathbb{Z}(1)) \cdot (\#Ш(M))^{-1}. \tag{1.3}$$

In particular, in the case where $M = h^0(\operatorname{Spec} L)(0)$ for a number field L, then $L(M, s) = \zeta_L(s)$, and equation (3.1) reduces to the analytic class number formula. If $M = h^1(A)(1)$ for some abelian variety A over a number field L, then $L(M, s-1)$ is the Hasse-Weil L-function of A. In particular, if A is an elliptic curve over $\mathbb{Q}$ then (3.1) reduces to the Birch and Swinnerton-Dyer conjecture.

[BK90] provide further evidence of their conjecture by showing that it holds for the Tate motive $\mathbb{Q}(r)$. This can be checked locally, by computing $\mu(H^1(\mathbb{Q}_p, \mathbb{Z}_p(r))$ for each p, and thus they prove the following theorem:

Theorem 1.2.1 (Bloch, Kato)**.** *Let $K/\mathbb{Q}_p$ be a finite unramified extension and let $r \geq 2$. Let $H^1(K, \mathbb{Z}_p(r))$ have the Haar measure induced by the isomorphism $\exp_{\mathbb{Q}_p(r)} : K \xrightarrow{\sim} H^1(K, \mathbb{Q}_p(r))$ from the Haar measure μ on K with $\mu(\mathcal{O}_K) = 1$. Then*

$$\mu\left(H^1(K, \mathbb{Z}_p(r))\right) = |1 - q^{-r}|_p^{-1} \cdot |(r-1)!|_K \cdot \#H^0(K, \mathbb{Q}_p/\mathbb{Z}_p(1-r)).$$

In [BK90] it is assumed that K is unramified; we wish to relax this assumption and extend these results to cases where K is tamely ramified. But first we will strengthen and make more precise the statement of the conjecture.

1.2.2 The dual of exp

We begin by observing that that there's a natural isomorphism $D_{dR}(\mathbb{Q}_p(1)) = t^{-1}K \xrightarrow{\sim} K$. For any G_K-representation V, we can naturally identify $D_{dR}(V^*(1))$ with the dual of $D_{dR}(V)$: there is a non-degenerate pairing given by

$$D_{dR}(V) \otimes D_{dR}(V^*(1)) \cong D_{dR}(V \otimes V^*(1)) \to D_{dR}(\mathbb{Q}_p(1)) \cong K \xrightarrow{\mathrm{Tr}_{K/\mathbb{Q}_p}} \mathbb{Q}_p.$$

Similarly, we can identify $H^1(K, V^*(1))$ with the dual of $H^1(K, V)$ by Tate duality:

$$H^1(K, V) \times H^1(K, V^*(1)) \to H^2(K, \mathbb{Q}_p(1)) = \mathbb{Q}_p.$$

We define the dual map

$$\exp^*_{V^*(1)} : H^1(K, V) \to D_{dR}(V)$$

to be the transpose of $\exp_{V^*(1)} : D_{dR}(V^*(1)) \to H^1(K, V^*(1))$, via the diagram

$$\begin{array}{ccccc}
H^1(K,V) & \times & H^1(K,V^*(1)) & \longrightarrow & H^2(K,\mathbb{Q}_p(1)) \xrightarrow{\cong} \mathbb{Q}_p \\
{\scriptstyle \exp_V}\uparrow & & \downarrow{\scriptstyle \exp^*_{V^*(1)}} & & \downarrow{\scriptstyle id} \\
D_{dR}(V) & \otimes & D_{dR}(V^*(1)) & \longrightarrow & D_{dR}(\mathbb{Q}_p(1)) \xrightarrow{\cong} \mathbb{Q}_p
\end{array}$$

In particular, since $(\mathbb{Q}_p(r))^* = \mathbb{Q}_p(-r)$, we have $(\mathbb{Q}_p(r))^*(1) = \mathbb{Q}_p(1-r)$, and thus we have a map $\exp^*_{\mathbb{Q}_p(1-r)} : H^1(K, \mathbb{Q}_p(r)) \xrightarrow{\sim} K$.

Using the dual exponential, we can actually make a finer statement than the conjecture of Bloch and Kato. Observe that since $H^1(K, \mathbb{Q}_p(r)) = H^1(K, \mathbb{Z}_p(r)) \otimes_{\mathbb{Z}} \mathbb{Q}$, we have the following diagram,

with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(K,\mathbb{Z}_p(r))_{\text{tors}} & \hookrightarrow & H^1(K,\mathbb{Z}_p(r)) & \longrightarrow & H^1(K,\mathbb{Q}_p(r)) \\ & & & & & & \cong \downarrow \exp^*_{\mathbb{Q}_p(1-r)} \\ 0 & \longrightarrow & & & \mathcal{O}_K & \longrightarrow & K \end{array}$$

where $H^1(K,\mathbb{Z}_p(r))_{\text{tors}}$ is the rational torsion of $H^1(K,\mathbb{Z}_p(r))$; we will write $H^1(K,\mathbb{Z}_p(r))_{\text{tf}} = H^1(K,\mathbb{Z}_p(r))/H^1(K,\mathbb{Z}_p(r))_{\text{tors}}$. Then we can view $\mathcal{O}_K$ and $\exp^*_{\mathbb{Q}_p(1-r)} H^1(K,\mathbb{Z}_p(r))_{\text{tf}}$ as two integral lattices in K, and there exists a $\beta \in H^1(K,\mathbb{Z}_p(r))$ such that $H^1(K,\mathbb{Z}_p(r))_{\text{tf}} = \mathbb{Z}_p[G]\cdot\beta$.

In fact, the situation becomes simpler if we pass to the group ring via the period isomorphism

$$\text{per} : K \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}_p} \xrightarrow{\sim} \left(\text{Ind}_{G_K}^{G_{\mathbb{Q}_p}} \mathbb{Q}_p\right) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p.$$

We have that $\text{Ind}_{G_K}^{G_{\mathbb{Q}_p}} \mathbb{Z}_p$ is a free left-$\mathbb{Z}_p[G]$-module of rank one, and thus after choosing an embedding $K \hookrightarrow \overline{\mathbb{Q}}_p$ we get an isomorphism $\psi : G_{\mathbb{Q}_p}/G_K \xrightarrow{\sim} G$ and an identification of $\text{Ind}_{G_K}^{G_{\mathbb{Q}_p}} \mathbb{Z}_p$ with $\mathbb{Z}_p[G]$, which gives us a left Γ-action determined by $\gamma x = x\psi(\gamma^{-1})$.

Lemma 1.2.2. *For a fixed choice of ψ, the period isomorphism is given by*

$$\text{per}(x \otimes 1) = \sum_{g\in G} g(x)\cdot g^{-1} \in \overline{\mathbb{Q}_p}[G]$$

With the period isomorphism established, we turn our attention to studying the structure of the group ring.

1.2.3 The group ring $\overline{\mathbb{Q}_p}[G]$

In this subsection, we will develop some character theory for $\overline{\mathbb{Q}_p}[G]$, where G is any group. We follow much of the presentation in [Lan02] XVIII.2-5.

Let $\hat{G}$ be the space of irreducible characters of G over $\overline{\mathbb{Q}_p}$. Then $\overline{\mathbb{Q}_p}[G]$ is semisimple, and decomposes into simple modules as $\overline{\mathbb{Q}_p}[G] = \prod_{\chi\in\hat{G}} R_\chi$ where each R_χ is a simple ring and thus a matrix algebra over $\overline{\mathbb{Q}_p}$. If G is abelian we have $R_\chi \cong \overline{\mathbb{Q}_p}$ for each χ.

We write ρ_χ for the representation of G on $\overline{\mathbb{Q}_p}[G]$ induced by χ. If $\alpha \in R_\eta$ and $\eta \neq \chi$, we have $\rho_\chi(\alpha) = 0$, and ρ_χ is in fact a representation $G \to R_\chi$. For any $g \in G$ we have that $\rho_\chi(g)$ is an invertible matrix in R_χ, and thus $\det\rho_\chi(g) \in \overline{\mathbb{Q}_p}^\times$. If G is abelian then $\det\rho_\chi(g) = \rho_\chi(g)$.

For each χ, we set e_χ to be the unit of R_χ. We can thus write $1 = \sum_{\chi\in\hat{G}} e_\chi$, and we observe that $e_\chi e_\eta = 0$ if $\chi \neq \eta$. We can further calculate that

$$e_\chi = \frac{d_\chi}{\#G}\sum_{g\in G}\chi(g^{-1})g,$$

where d_χ^2 is the dimension of the matrix algebra R_χ over $\overline{\mathbb{Q}}_p$. In particular, the trivial character $1 : G \to \overline{\mathbb{Q}_p}$ is a simple character of G, and we have

$$e_1 = \frac{1}{\#G} \sum_{\sigma \in G} \sigma.$$

Note that $ge_1 = e_1$ for any $g \in G$, and more generally, we have

$$\begin{aligned} h \cdot e_\chi &= \sum_{g \in G} \chi(g^{-1}) hg \\ &= \sum_{g \in G} \chi(g^{-1}h) g \\ &= \chi(h) \sum_{g \in G} \chi(g^{-1}) g \\ &= \chi(h) e_\chi, \end{aligned}$$

and thus the e_χ are eigenvectors for the action of G.

1.2.4 The equivariant Tamagawa number conjecture

We are now ready to state the equivariant Tamagawa number conjecture for tamely ramified extensions $K/\mathbb{Q}_p$. The discussion in this subsection draws on [Fla14], developing ideas from [FK06]. Recall that we have $\beta \in H^1(K, \mathbb{Z}_p(r))$ such that $H^1(K, \mathbb{Z}_p(r))_{\mathrm{tf}} = \mathbb{Z}_p[G] \cdot \beta$. We can restate our conjecture:

Conjecture 1.2.3 (Equivariant Tamagawa number conjecture)**.** *Let K be a tamely ramified extension of $\mathbb{Q}_p$, and let $p \nmid f$ and $(e, p-1) = 1$. Let b be a $\mathbb{Z}_p[G]$ basis of the inverse different $(\sqrt[e]{p_0})^{1-e}\mathcal{O}_K$, and set*

$$C_\beta = 1 - e_1 + \frac{\chi^{\mathrm{cyclo}}(\gamma)^{r-1} - 1}{\chi^{\mathrm{cyclo}}(\gamma)^r - 1} e_1.$$

Then for each $\chi \in \hat{G}$, we set $c(\chi)$ to be the conductor of χ, and we have

$$(r-1)! \cdot p^{(r-1)c(\chi)} \cdot \frac{\det \rho_\chi(\mathrm{per}(\exp^*_{\mathbb{Q}_p(1-r)}(\beta) \otimes 1))}{\det \rho_\chi(\mathrm{per}(b \otimes 1))} \cdot \det \rho_\chi(C_\beta) \cdot \frac{1 - p^{r-1} \det \rho_\chi(\sigma)}{1 - p^{-r} \det \rho_\chi(\sigma)^{-1}} \in \mathbb{Z}_p^{ur,\times}. \quad (1.4)$$

Note that since $p \nmid |G|$, we know that $\hat{G}$ coincides with the set of irreducible characters of G over $\mathbb{Q}_p^{ur}$, and thus the element given in (1.4) is always a unit of $\mathbb{Q}_p^{ur}$. Our aim is to prove that it is a unit of $\mathbb{Z}_p^{ur}$.

Let $\eta : \Delta_e \to \overline{\mathbb{Q}}_p^\times$ be a character, and set

$$\Sigma_\eta = \{\sigma \in \Sigma : \eta(\sigma \delta_e \sigma^{-1}) = \eta(\delta_e)\}.$$

Then we have a subgroup $\Sigma \ltimes \Delta_e \hookleftarrow \Sigma_\eta \times \Delta_e$, and for each $\eta' : \Sigma_\eta \to \overline{\mathbb{Q}}_p^\times$ we have a homomorphism

$$G_\eta = \Sigma_\eta \ltimes \Delta_e \xrightarrow{\eta' \cdot \eta} \overline{\mathbb{Q}}_p^\times,$$

and we can define a character $\chi = \mathrm{Ind}_{G_\eta}^G \eta' \eta$ of G. In fact every character $\chi \in \hat{G}$ arises this way. Thus it suffices to prove Conjecture 1.2.3 for $\chi = \mathrm{Ind}_{G_\eta}^G \eta' \cdot \eta$. In this case we have $c(\chi) = f_\eta = \#\Sigma/\Sigma_\eta$.

In the remainder of this thesis, we will prove Conjecture 1.2.3 for unramified extensions of $\mathbb{Q}_p$, and discuss some work towards proving the conjecture for tamely ramified extensions. Chapter 2 develops tools that we will need to study this problem, and Chapter 3 will apply them to the Λ_K-module $A_K^{\psi=1}(1)$. In Chapter 4 we will prove Conjecture 1.2.3 for unramified extensions, and in Chapter 5 we will discuss our work on tamely ramified extensions.

Chapter 2

Iwasawa theory and (ϕ, Γ_K)-modules

In this chapter we will develop a number of tools that we can use to study the equivariant Tamagawa number conjecture. In section 2.1 we will define Iwasawa cohomology and the Iwasawa algebra, and prove some results about the Iwasawa cohomology of the Tate motive. In section 2.2 we will introduce the theory of (ϕ, Γ_K)-modules and state an explicit reciprocity law for the map $\exp^*_{\mathbb{Q}_p(1-r)}$.

2.1 Iwasawa cohomology and the Iwasawa algebra

In this section we introduce some Iwasawa theory, define Iwasawa cohomology, and show that we can realize the group $H^1(K, \mathbb{Z}_p(r))$ of Theorem 1.2.1 and Conjecture 1.2.3 as a quotient of the Iwasawa cohomology $H^1_{Iw}(K, \mathbb{Z}_p(1))$.

2.1.1 Iwasawa cohomology

We begin by defining some basic objects of Iwasawa theory. Recall we have a compatible system of p^nth roots of unity ζ_{p^n}, and for each n we set $K_n = K(\zeta_{p^n})$. Then we have an ascending tower of p-adic extensions $K = K_0 \subset K_1 \subset K_2 \subset \dots$ where $[K_1 : K_0] = p - 1$ and $[K_{i+1} : K_i] = p$ for $i \geq 1$. We write $K_\infty = \bigcup_{n \in \mathbb{N}} K_n$.

Recall that $G_K = \mathrm{Gal}(\overline{K}/K)$ and H_K is the kernel of the cyclotomic character $\chi^{\mathrm{cyclo}} : G_K \to \mathbb{Z}_p^\times$. We note that K_∞ is precisely the fixed field of H_K and so $\Gamma_K = G_K/H_K = \mathrm{Gal}(K_\infty/K)$. If $p > 2$, then Γ_K is topologically cyclic with generator γ, and isomorphic to $\mathbb{Z}_p^\times$ as a multiplicative group. We write $\Gamma_{K_n} = \mathrm{Gal}(K_\infty/K_n) \subset \Gamma_K$; for $n \neq 0$, we have $\Gamma_{K_n} \cong \mathbb{Z}_p$ as an additive group, with topological generator γ_n. We note that in particular $\Gamma_{K_1} \cong \mathbb{Z}_p$, and $\Gamma_K/\Gamma_{K_1} = \mathrm{Gal}(K_1/K) \cong \mu_{p-1}$. We write $\Delta = \mathrm{Gal}(K_1/K)$, and observe that K and F_1 are linearly disjoint over F since $(e, p-1) = 1$.

We wish to study modules over the group Γ_K. We start by defining:

Definition 2.1.1. The Iwasawa algebra is the completed group ring $\Lambda = \Lambda(\Gamma_K) = \mathbb{Z}_p[\Delta][\![T]\!]$, with $T = \gamma_1 - 1$. We can also write $\Lambda = \varprojlim_n \mathbb{Z}_p[\Gamma_K/\Gamma_{K_n}]$.

Recall that $G = \mathrm{Gal}(K/\mathbb{Q}_p)$. We define the Iwasawa algebra over K to be $\Lambda_K = \mathbb{Z}_p[G][\![\Gamma_K]\!]$.

Remark 2.1.2. We see that $\Lambda_K \cong \mathcal{O}_K \otimes_{\mathbb{Z}_p} \Lambda$. Indeed, by the normal basis theorem there is a $\xi \in K$ such that $G \cdot \xi$ is a basis of K over $\mathbb{Q}_p$, and since $K/\mathbb{Q}_p$ is tamely ramified we can pick ξ such that $G \cdot \xi$ is also a basis for $\mathcal{O}_K$ over $\mathbb{Z}_p$. Then the map induced by

$$g \cdot \gamma^i \mapsto g(\xi) \otimes \gamma^i$$

is an isomorphism.

Now we recall the definition of Iwasawa cohomology:

Definition 2.1.3. Let T be a finitely-generated $\mathbb{Z}_p$-module with a continuous linear G_K-action. Then we write $H^m_{Iw}(K, T)$ for the projective limit $\varprojlim H^m(K_n, T)$ relative to the corestriction maps.

Let V be a finite-dimensional $\mathbb{Q}_p$-vector space with a continuous linear G_K-action and a G_K-stable lattice T. Then we write

$$H^m_{Iw}(K, V) = H^m_{Iw}(K, T). \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

Remark 2.1.4. The group $H^1_{Iw}(K, T)$ is equipped with a Λ action. Each $H^1(K_n, T)$ has a left Γ_K/Γ_{K_n} action, and thus is a $\mathbb{Z}_p[\Gamma_K/\Gamma_{K_n}]$-module. Then the projective limit is a $\varprojlim \mathbb{Z}_p[\Gamma_K/\Gamma_{K_n}]$-module, and $\varprojlim \mathbb{Z}_p[\Gamma_K/\Gamma_{K_n}] = \mathbb{Z}_p[\![\Gamma_K]\!] = \Lambda$.

2.1.2 Iwasawa cohomology of $\mathbb{Q}_p(1)$

For the rest of this section, we will study the Iwasawa cohomology group $H^1_{Iw}(K, \mathbb{Q}_p(1))$. Recall that we have

$$H^1_{Iw}(K, \mathbb{Q}_p(1)) = \varprojlim_n H^1(\Gamma_{K_n}, \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

By Shapiro's lemma we know that $H^1(\Gamma, \mathrm{Ind}^{\Gamma}_{\Gamma_{K_n}} \mathbb{Z}_p(1)) \cong H^1(\Gamma_{K_n}, \mathbb{Z}_p(1))$. We see that

$$\mathrm{Ind}^{\Gamma_K}_{\Gamma_{K_n}}(\mathbb{Z}_p(1)) = \mathbb{Z}_p[\Gamma_K] \otimes_{\mathbb{Z}_p[\Gamma_{K_n}]} \mathbb{Z}_p(1) \cong \mathbb{Z}_p[\Gamma_K/\Gamma_{K_n}] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1),$$

and so we have

$$H^1_{Iw}(K, \mathbb{Z}_p(1)) \cong \varprojlim_n H^1(K, \mathbb{Z}_p[\Gamma_K/\Gamma_{K_n}] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)).$$

But $\varprojlim H^1(K, G_n) \cong H^1(K, \varprojlim G_n)$ (see [NSW00] corollary 2.3.5), and we have defined the ring $\Lambda = \varprojlim_n \mathbb{Z}_p[\Gamma_K/\Gamma_{K_n}]$. Set $T = \Lambda \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)$, and we have

$$H^1_{Iw}(K, \mathbb{Z}_p(1)) = H^1(K, T). \tag{2.1}$$

(See also [CC99] II.1.2). We also observe that since the inverse limit does not depend on the first terms, for each $m \geq 0$ we have $H^1_{Iw}(K, \mathbb{Q}_p(1)) \cong H^1_{Iw}(K_m, \mathbb{Q}_p(1))$, and thus $H^1(K, T) \cong H^1(K_m, T)$, as groups.

Further, we note that T has a left action of Λ as described in Remark 2.1.4 and a right action of G_K given by $g \cdot \sum a_i \gamma^i \otimes n = \sum a_i \gamma^{i-m} \otimes \chi^{\text{cyclo}}(\gamma)^m(n)$, where $g = \gamma^m \cdot h$ for $\chi^{\text{cyclo}}(h) = 0$. It is this latter action by which it is a G_K-module, and by which $H^1(K, T)$ is defined; thus $H^1(K, T)$ is itself a left Λ-module. In order to properly analyze this module, we will need a brief digression into some results from the Iwasawa theory of p-adic local fields.

First, recall that we say a group H is pro-p if it is the inverse limit of an inverse system of discrete finite p-groups. If H is an abelian group, we can define the pro-p completion of H, written $\widehat{H}$, to be the inverse limit $\varprojlim_m H/H^{p^m}$. In particular, we note that $\mathbb{Z}_p$ is the pro-p completion of $\mathbb{Z}$.

Now recall that $\text{Gal}(K_\infty/K) \cong \mathbb{Z}_p^\times$, and in particular we have $\Gamma_{K_1} = \text{Gal}(K_\infty/K_1) \cong \mathbb{Z}_p$ and $\Delta = \text{Gal}(K_1/K) \cong \mu_{p-1}$. Following [NSW00] XI.2, we set

$$A(K_\infty) = \varprojlim_n \widehat{K_n} = \varprojlim_{n,m} K_n^\times/(K_n^\times)^{p^m}.$$

This is a finitely-generated Λ-module. We set

$$\widehat{U} = \varprojlim_n \widehat{\mathcal{O}_{K_n}} = \varprojlim_{n,m} \mathcal{O}_{K_n}^\times/(\mathcal{O}_{K_n}^\times)^{p^m}$$

the pro-p completion of the group U due to Coleman (see also [BK90] section 2). Then if $N_n : \widehat{K_n^\times} \to \widehat{K_{n-1}^\times}$ is the norm map, we have a commutative diagram for each $n \geq 1$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widehat{\mathcal{O}_{K_n}^\times} & \longrightarrow & \widehat{K_n^\times} & \xrightarrow{v_{K_n}} & \mathbb{Z}_p & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle N_n} & & \downarrow{\scriptstyle N_n} & & \downarrow{\scriptstyle id} & & \\
0 & \longrightarrow & \widehat{\mathcal{O}_{K_{n-1}}^\times} & \longrightarrow & \widehat{K_{n-1}^\times} & \xrightarrow{v_{K_{n-1}}} & \mathbb{Z}_p & \longrightarrow & 0
\end{array}$$

which induces a map $v : A(K_\infty) \to \mathbb{Z}_p$. As the kernel of v_{K_n} is precisely $\mathcal{O}_{K_n}^\times$, we have that $\widehat{U} = \ker v$. Thus we have a short exact sequence

$$0 \to \widehat{U} \to A(K_\infty) \xrightarrow{v} \mathbb{Z}_p \to 0.$$

We now have the following proposition:

Proposition 2.1.5. *We have the following isomorphisms of Λ_K-modules:*

1. $A(K_\infty) \cong \Lambda_K \oplus \mathbb{Z}_p(1)$.
2. $\widehat{U} \cong \Lambda_K \oplus \mathbb{Z}_p(1)$.

Proof. 1. Observe that $\mathrm{Gal}(K_\infty/\mathbb{Q}_p) = \Gamma_{K_1} \times \mathrm{Gal}(K_1/\mathbb{Q}_p) \cong \mathbb{Z}_p \times \mathrm{Gal}(K_1 \times \mathbb{Q}_p)$ and $\Lambda_K = \mathbb{Z}_p[\![\mathrm{Gal}(K_\infty/\mathbb{Q}_p)]\!]$, and see [NSW00] Theorem 11.2.4.

2. We observe that since $\widehat{U}$ is the kernel of the valuation map onto $\mathbb{Z}_p$, it is the augmentation ideal of $A(K_\infty)$. We can write $U = I \oplus J$ where $I \subset \Lambda_K$ and $J \subset \mathbb{Z}_p(1)$; then I is the augmentation ideal of Λ_K, and by [NSW00] Lemma 11.2.2. we have $I \cong \Lambda_K$.

   To show that $J \cong \mathbb{Z}_p(1)$, note that any additive homomorphism $\mathbb{Z}_p(1) \to \mathbb{Z}_p$ is either an embedding or the zero map. But $\gamma - 1$ is the zero map on $\mathbb{Z}_p$ and $(\gamma-1)(1) = \chi^{\mathrm{cyclo}}(\gamma) - 1 \neq 0$ in $\mathbb{Z}_p(1)$; thus $\ker v$ is non-trivial, and we must have $\ker v = \mathbb{Z}_p(1)$.

□

We can now describe the Λ-module structure of $H^1_{Iw}(K, \mathbb{Z}_p(1))$.

Lemma 2.1.6. *The Iwasawa cohomology group $H^1_{Iw}(K, \mathbb{Z}_p(1))$ is isomorphic to $A(K_\infty)$ as a Λ-module.*

Proof. Recall that by Kummer theory, we have $H^1(K_n, \mu_{p^m}) = K_n^\times/(K_n^\times)^{p^m}$. Then we have

$$\begin{aligned} H^1_{Iw}(K, \mathbb{Z}_p(1)) &\cong \varprojlim_n H^1(K_n, \mathbb{Z}_p(1)) \\ &= \varprojlim_n H^1(K_n, \varprojlim_m \mu_{p^m}) \\ &\cong \varprojlim_{m,n} H^1(K_n, \mu_{p^m}) \\ &\cong \varprojlim_{m,n} K_n^\times/(K_n^\times)^{p^m} \\ &= A(K_\infty). \end{aligned}$$

□

Corollary 2.1.7. *The cohomology group $H^1(K, T) \cong H^1_{Iw}(K, \mathbb{Z}_p(1))$ is a rank-1 Λ_K-module, and is isomorphic to $\Lambda_K \oplus \Lambda_K/(\sigma - 1, \delta_e - 1, \gamma - \chi^{\mathrm{cyclo}}(\gamma))\Lambda_K \cong \Lambda_K \oplus \mathbb{Z}_p(1)$.*

Proof. A simple calculation shows that $\Lambda_K/(\sigma - 1, \delta_e - 1, \gamma - \chi(\gamma))\Lambda_K \cong \mathbb{Z}_p(1)$, and the rest follows from Lemma 2.1.6. □

2.1.3 Certain quotients of $H^1_{Iw}(K, \mathbb{Z}_p(r))$

We now wish to realize the cohomology group $H^1(K, \mathbb{Z}_p(r))$ as a quotient of $H^1(K, T)$. We start with a lemma:

Lemma 2.1.8. *Let $T = \Lambda \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)$ and let γ be a topological generator for Γ. Then we have exact sequences:*

$$0 \longrightarrow T \xrightarrow{\gamma - \chi^{\text{cyclo}}(\gamma)^{1-r}} T \longrightarrow \mathbb{Z}_p(r) \longrightarrow 0$$

$$0 \longrightarrow T \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\gamma - \chi^{\text{cyclo}}(\gamma)^{1-r}} T \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathbb{Q}_p(r) \longrightarrow 0$$

Proof. We prove exactness of the first sequence; the proof for the second sequence is similar. We will define a map $T \twoheadrightarrow \mathbb{Z}_p(r)$ of G_K-modules, whose kernel is $(\gamma - \chi^{\text{cyclo}}(\gamma)^{1-r})T$, and show that it is G_K-equivariant.

Define $\theta : T \to \mathbb{Z}_p(r)$ by $\theta(\sum a_i \gamma^i \otimes n) = n \cdot \sum a^i \chi^{\text{cyclo}}(\gamma)^{i(1-r)}$ for any $\sum a_i \gamma^i \otimes n \in T$. This is clearly a group homomorphism; since the map is induced by $\gamma \mapsto \chi^{\text{cyclo}}(\gamma)^{1-r}$ it has the desired kernel. We wish to show that it is G_K-equivariant, so let $g \in G_K$. Then $g = \gamma^m \cdot h$ where $\chi^{\text{cyclo}}(h) = 1$. We have

$$\begin{aligned}
\theta\left(\left(\sum a_i \gamma^i \otimes n\right) \cdot g\right) &= \theta\left(\sum a_i \gamma^{i-m} \otimes \chi^{\text{cyclo}}(\gamma)^m n\right) \\
&= \chi^{\text{cyclo}}(\gamma)^m n \cdot \sum a_i \chi^{\text{cyclo}}(\gamma)^{(i-m)(1-r)} \\
&= \chi^{\text{cyclo}}(\gamma)^{mr} n \cdot \sum a_i \chi^{\text{cyclo}}(\gamma)^{i(1-r)} \\
&= \chi^{\text{cyclo}}(\gamma)^{mr} \theta\left(\sum a_i \gamma^i \otimes n\right) \\
&= g \cdot \theta\left(\sum a_i \gamma^i \otimes n\right)
\end{aligned}$$

as desired. □

Fix $m \geq 0$. Then the short exact sequence of this lemma induces a long exact sequence of cohomology groups:

$$\begin{array}{l}
0 \longrightarrow H^0(K_m, T) \xrightarrow[\gamma - \chi^{\text{cyclo}}(\gamma)^{1-r}]{} H^0(K_m, T) \longrightarrow H^0(K_m, \mathbb{Z}_p(r)) \\
\qquad\qquad \swarrow \\
H^1(K_m, T) \xrightarrow[\gamma - \chi^{\text{cyclo}}(\gamma)^{1-r}]{} H^1(K_m, T) \xrightarrow{pr_{m,r}} H^1(K_m, \mathbb{Z}_p(r)) \\
\qquad\qquad \swarrow \\
H^2(K_m, T) \xrightarrow[\gamma - \chi^{\text{cyclo}}(\gamma)^{1-r}]{} H^2(K_m, T) \longrightarrow H^2(K_m, \mathbb{Z}_p(r)) \longrightarrow H^3(K_m, T) \cong 0
\end{array}$$

We are interested in the map $pr_{m,r}$ above, and in particular its image when $m = 0$.

Lemma 2.1.9. *If $r \geq 2$, then the map $pr_{0,r}$ is surjective onto $H^1(K, \mathbb{Z}_p(r))$, which is isomorphic to $\mathbb{Z}_p[G](1-r) \oplus \mathbb{Z}/(\#H^0(K, \mathbb{Q}_p/\mathbb{Z}_p(r)))$.*

Proof. First, we claim that the map $pr_{0,r}$ is a surjection. Note that $H^2(K,T) \cong H^2_{Iw}(K, \mathbb{Z}_p(1)) \cong \varprojlim H^2(K_n, \mathbb{Z}_p(1))$. But we have that $H^2(K_n, \mathbb{Z}_p(1)) \cong \mathbb{Z}_p$, and the transition map is constant, so $H^2(K,T) \cong \mathbb{Z}_p$ (see [NSW00] 7.3.10). The map $H^2(K,T) \to H^2(K,T)$ is induced by $\gamma - \chi^{\text{cyclo}}(\gamma)^{1-r} = 1 - \chi^{\text{cyclo}}{}^{1-r}(\gamma)$ since the action of γ on $\mathbb{Z}_p$ is trivial. Since $\gamma \notin \ker(\chi^{\text{cyclo}})$ this gives us that $1 - \chi^{\text{cyclo}}(\gamma)^{1-r}$ is non-zero, and it has empty kernel on $\mathbb{Z}_p$. This is sufficient to establish that $pr_{0,r}$ is surjective onto $H^1(K, \mathbb{Z}_p(r))$.

From Corollary 2.1.7 we know that $H^1(K,T) \cong \Lambda_K \oplus \mathbb{Z}_p(1)$. Thus we have a right exact sequence

$$\Lambda_K \oplus \mathbb{Z}_p(1) \to \Lambda_K \oplus \mathbb{Z}_p(1) \to H^1(K, \mathbb{Z}_p(r)) \to 0.$$

In particular, we have

$$\begin{aligned} H^1(K, \mathbb{Z}_p(r)) &\cong (\Lambda_K \oplus \mathbb{Z}_p(1)))\,/(\gamma - \chi^{\text{cyclo}}(\gamma)^{1-r})\,(\Lambda_K \oplus \mathbb{Z}_p(1)) \\ &\cong \Lambda_K/(\gamma - \chi^{\text{cyclo}}(\gamma)^{1-r}) \oplus \mathbb{Z}_p(1)/(\gamma - \chi^{\text{cyclo}}(\gamma)^{1-r}) \\ &\cong \mathbb{Z}_p[G](1-r) \oplus \mathbb{Z}_p/(1 - \chi^{\text{cyclo}}(\gamma)^{-r}) \\ &\cong \mathbb{Z}_p[G](1-r) \oplus \mathbb{Z}/|1 - \chi^{\text{cyclo}}(\gamma)^{-r}|_p^{-1}\mathbb{Z}. \end{aligned}$$

Thus all that's left is the claim that $\#H^0(K, \mathbb{Q}_p/\mathbb{Z}_p(r)) = |1 - \chi^{\text{cyclo}}(\gamma)^{-r}|_p^{-1}$, which is the content of lemma 2.1.10 below.

□

Lemma 2.1.10. *Let $r \in \mathbb{Z}$. Then $\#H^0(K, \mathbb{Q}_p/\mathbb{Z}_p(r)) = |\chi^{\text{cyclo}}(\gamma)^r - 1|_p^{-1} = |\chi^{\text{cyclo}}(\gamma)^{-r} - 1|_p^{-1}$.*

Proof. We see that

$$\begin{aligned} H^0(K, \mathbb{Q}_p/\mathbb{Z}_p(r)) &= \{x \in \mathbb{Q}_p/\mathbb{Z}_p : gx = x \forall g \in \text{Gal}(\overline{\mathbb{Q}}_p/K\} \\ &= \{x \in \mathbb{Q}_p/\mathbb{Z}_p : \chi^{\text{cyclo}}(\gamma)^r x = x\} \\ &= \{x \in \mathbb{Q}_p/\mathbb{Z}_p : (\chi^{\text{cyclo}}(\gamma)^r - 1)x \in \mathbb{Z}_p\} \\ &= \{t/p^l \in \mathbb{Q}_p/\mathbb{Z}_p : t \in \mathbb{Z}_p^\times, (\chi^{\text{cyclo}}(\gamma)^r - 1)/p^l \in \mathbb{Z}_p\} \\ &= \left\{ \frac{a_0 + a_1 p + \cdots + a_{l-1}p^{l-1}}{p^l} : a_i \in \mathbb{Z}/p\mathbb{Z}, l = v_p(\chi^{\text{cyclo}}(\gamma)^r - 1) \right\} \\ &\cong \mathbb{Z}/p^{v_p(\chi^{\text{cyclo}}(\gamma)^r - 1)}\mathbb{Z} \end{aligned}$$

which has order $p^{v_p(\chi^{\text{cyclo}}(\gamma)^r - 1)} = |\chi^{\text{cyclo}}(\gamma)^r - 1|_p^{-1}$.

The second equality follows because G_K acts on $\mathbb{Q}_p/\mathbb{Z}_p(r)$ via χ^{cyclo} and $G_K/\ker(\chi^{\text{cyclo}}) = \Gamma$ is topologically generated by γ. The third equality follows since $|\chi^{\text{cyclo}}(\gamma_k)^r|_p = 1$. □

2.2 (ϕ, Γ_K)-modules and the reciprocity law of Cherbonnier and Colmez

In this section, we develop some ideas from [CC99]. In particular, we define the Cherbonnier-Colmez dual exponential, and recall a reciprocity law for it in the case of the Tate twist.

2.2.1 (ϕ, Γ_K)-modules and the Cherbonnier-Colmez dual exponential map

We begin with a definition.

Definition 2.2.1. If K is a finite extension of $\mathbb{Q}_p$, then we define a (ϕ, Γ_K)-module over A_K (resp. B_K) to be a finite-rank A_K-module (resp. a finite-dimensional B_K-vector space) with an action of Γ_K and an action of ϕ which commute.

If V is a $\mathbb{Z}_p$-module or a $\mathbb{Q}_p$-vector space with a continuous action of G_K, we set $D(V) = (A \otimes_{\mathbb{Z}_p} V)^{H_K}$. Then since the action of ϕ on A_K commutes with the action of G_K, we see that $D(V)$ has an action of ϕ, and an action of $G_K/H_K = \Gamma_K$ which commutes with ϕ. Thus $D(V)$ is a (ϕ, Γ_K)-module over A_K or B_K.

In fact, this induces an equivalence of categories between étale (ϕ, Γ_K)-modules and continuous G_K-representations. For if T is a (ϕ, Γ_K)-module, then $(A \otimes_{A_K} T)^{\phi=1}$ is a representation of G_K over $\mathbb{Q}_p$, and further, $(A \otimes_{A_K} D(V))^{\phi=1}$ is canonically isomorphic to V as a representation of G_K.

We may now define the Cherbonnier-Colmez dual exponential. Let V be a $\mathbb{Z}_p$-representation of G_K. From [CC99] Theorem II.1.3, we know that for each n we have a map $\iota_{\psi,n} : D(V)^{\psi=1} \to H^1(K_n, V)$. In fact, these maps are compatible with the corestriction maps, and thus we have a map $\text{Log}^*_{V^*(1)} : D(V)^{\psi=1} \to H^1_{Iw}(K, V)$ given by $y \mapsto (\dots, \iota_{\psi,n} y, \dots)$. This map is an isomorphism, and we denote its inverse, the Cherbonnier-Colmez dual exponential, by $\text{Exp}^*_{V^*(1)} : H^1_{Iw}(K, V) \xrightarrow{\sim} D(V)^{\psi=1}$.

In particular, if we take $V = \mathbb{Q}_p(1)$, then we have $\text{Exp}^*_{\mathbb{Q}_p} : H^1_{Iw}(K, \mathbb{Q}_p(1)) \xrightarrow{\sim} B_K(1)^{\psi=1}$, and if we take $V = \mathbb{Z}_p(1)$, we have $\text{Exp}^*_{\mathbb{Z}_p} : H^1_{Iw}(K, \mathbb{Z}_p(1)) \xrightarrow{\sim} A_K(1)^{\psi=1}$.

2.2.2 The reciprocity law

We define a family of operators $T_m : B_{dR}^{H_K} \to K_m((t))$. First, we note that if $x \in K_\infty$, then $x \in K_n$ for some n, and $p^{-n} \operatorname{Tr}_{K_n/K_m}$ doesn't depend on n as long as $x \in K_n$ and $n \geq m$. Thus we have a well-defined function, which we write $T_m : K_\infty \to K_m$. We extend T_m linearly to a

function $K_\infty((t)) \to K_m((t))$. But since $K_\infty((t))$ is dense in $B_{dR}^{H_K}$, then T_m extends continuously to a function $T_m : B_{dR}^{H_K} \to K_n((t))$, as desired.

Recall that every element $x \in \widetilde{B}$ can be written $\sum_{k \gg -\infty} p^k[x_k]$ for some $x_k \in \widetilde{E}$. Then each term of this sum is also an element of B_{dR}^+, and the sum converges if and only if $\sum_{k \gg -\infty} p^k x_k^{(0)}$ converges in $\mathbb{C}_p$, if and only if $\lim_{k \to +\infty} k + v_E(x_k) = +\infty$. Thus $\phi^{-n}(x)$ converges if and only if $\lim_{k \to +\infty} k + p^{-n} v_E(x_k) = +\infty$.

If V is a $\mathbb{Q}_p$-vector space with a continuous G_K action, then there exists some $n \in \mathbb{N}$ such that $\phi^{-n}(x)$ converges for every $x \in D(V)^{\psi=1}$, and thus ϕ^{-n} gives us a map $\phi^{-n} : D(V)^{\psi=1} \to B_{dR}^+$. In fact, $\phi^{-n}(x) \in K_n[\![t]\!]$ (see [CC99] III.3.2 and III.2.1). We can easily calculate this map after seeing that $\phi^{-n}(\pi) = \zeta_{p^n} e^{t/p^n} - 1$.

Now we are ready to state the reciprocity law.

Theorem 2.2.2 (Cherbonnier-Colmez IV.2.1)**.** *Let V be a de Rham representation of G_K, let $\mu \in H^1_{Iw}(K, V)$, and let $m \in \mathbb{N}$. If n is large enough, then we have*

$$\phi^{-n}\left(\mathrm{Exp}^*_{V^*(1)}(\mu)\right) \in B_{dR} \otimes_K V,$$

given by $\phi^{-n}(\pi) = \zeta_{p^n} e^{tp^{-n}} - 1$, and the element

$$T_m(\phi^{-n}(\mathrm{Exp}^*_{V^*(1)}(\mu))) \in K_m((t)) \otimes_K D_{dR}(V)$$

is independent of n. Further, we have the equality

$$T_m(\phi^{-n}(\mathrm{Exp}^*_{V^*(1)}(\mu))) = \sum_{k \in \mathbb{Z}} \exp^*_{V^*(1+k)}(pr_{m,-k}\mu) \otimes t^{-k}. \tag{2.2}$$

Recall that the p-adic realization of the Tate motive $\mathbb{Q}_p(r)$ is given a G_K action by $g \cdot s = \chi^{\mathrm{cyclo}}(g)^r \cdot s$, and that for any G_K-representation V we have $V(r) = V \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(r)$. In this paper, we wish to study the representations $\mathbb{Q}_p(1)$ and $\mathbb{Q}_p(r)$ for $r \geq 2$.

We compute that $D(\mathbb{Q}_p(1)) = (A \otimes_{\mathbb{Z}_p} \mathbb{Q}_p(1))^{H_K} \cong A_K \otimes_{\mathbb{Z}_p} \mathbb{Q}_p(1) \cong B_K(1)$. Further, recall that $(\mathbb{Q}_p(r))^* = \mathbb{Q}_p(-r)$. So taking $V = \mathbb{Q}_p(1)$, we have, after reindexing the right-hand sum:

$$T_m(\phi^{-n}(\mathrm{Exp}^*_{\mathbb{Q}_p}(\mu))) = \sum_{r \in \mathbb{Z}} \exp^*_{\mathbb{Q}_p(1-r)}(pr_{m,r}(\mu)) \otimes t^{r-1}. \tag{2.3}$$

Chapter 3

The Λ_K-module $A_K^{\psi=1}(1)$

In this chapter, we will study the submodule of A_K on which the operator $\psi = \frac{1}{p}\phi^{-1}\operatorname{Tr}_{B/\phi(B)}$ is trivial. In section 3.1, we define a useful derivative operator on B that restricts nicely to A_K, and use it to study the map $\frac{d^{r-1}}{dt^{r-1}}T_m \circ \phi^{-n}$ discussed in section 2.2. In section 3.2, we develop relationships between $A_K^{\psi=1}(1)$ and several other rank-1 Λ_K-modules.

3.1 The map $\frac{d^{r-1}}{dt^{r-1}}T_m \circ \phi^{-n}$

The goal of this section is to understand the map $\frac{d^{r-1}}{dt^{r-1}}T_m \circ \phi^{-n}$ on $A_K^{\psi=1}(1)$. It is convenient to begin by defining a derivative operator on A_K, and then consider $T_m \circ \phi^{-n}$ separately.

3.1.1 The operator ∇

We wish to define a derivative operator on elements of B, and hence on A_K. We write $\nabla = (1+\pi)\frac{d}{d\pi} : B \to B$ to be the derivation given by setting $\nabla\pi = 1+\pi$. If we define log by the usual power series, then we have that $\nabla \log \alpha = \frac{\nabla\alpha}{\alpha}$ for every α in B for which log converges. We further compute:

Lemma 3.1.1. *1.* $\nabla \log \pi = \frac{1+\pi}{\pi}$

2. $\nabla \pi_K = \frac{(1+\pi)\pi_K}{e\pi}$.

3. $\nabla \log \pi_K = \frac{1+\pi}{e\pi}$.

Proof. 1.

$$\begin{aligned}\nabla \log \pi &= \frac{\nabla\pi}{\pi} \\ &= \frac{1+\pi}{\pi}\end{aligned}$$

2.

$$\begin{aligned}\nabla(\pi_K) &= \nabla(\pi^{1/e}) \\ &= \frac{1}{e}\pi^{1/e-1}(1+\pi) \\ &= \frac{(1+\pi)\pi_K}{e\pi}.\end{aligned}$$

3.

$$\begin{aligned}\nabla \log(\pi_K) &= \frac{\nabla \pi_K}{\pi_K} \\ &= \frac{1+\pi}{e\pi}.\end{aligned}$$

□

Recall that if $f \in B_K^{\psi=1}$, then there exists some n such that $\phi^{-n} f \in B_{dR}$. Then we can prove:

Lemma 3.1.2. *Suppose $\phi^{-n} f$ converges in B_{dR}. Then*

$$\phi^{-n}(\nabla f) = p^n \frac{d}{dt}(\phi^{-n}(f)).$$

Proof. It's enough to check that $\phi^{-n} \circ \nabla$ and $p^n \frac{d}{dt} \circ \phi^{-n}$ both agree on $1+\pi$, since they are both derivations. We see that

$$\begin{aligned}\phi^{-n}\nabla(1+\pi) = \phi^{-n}(1+\pi) &= \zeta_{p^n} e^{t/p^n} \\ p^n \frac{d}{dt}\phi^{-n}(1+\pi) = p^n \frac{d}{dt}\zeta_{p^n} e^{t/p^n} &= \zeta_{p^n} e^{t/p^n}.\end{aligned}$$

□

Finally, we prove that ∇ is compatible with other operators that we wish to use.

Lemma 3.1.3. *Let $f \in B_K$. Then we have*

(a) $\nabla \gamma f = \chi^{\mathrm{cyclo}}(\gamma) \cdot \gamma \nabla f$.

(b) $\nabla \phi f = p \cdot \phi \nabla f$.

(c) $\nabla \operatorname{Tr}_{B/\phi B} f = \operatorname{Tr}_{B/\phi B} \nabla f$.

(d) $\nabla \psi f = p^{-1} \cdot \psi \nabla f$.

Proof. (a)

$$\begin{aligned}
\nabla\gamma\sum_{n\in\mathbb{Z}} a_n\pi_K^n &= \nabla\sum_{n\in\mathbb{Z}}\gamma(a_n)(\gamma\pi)^{n/e}\\
&= \nabla\sum_{n\in\mathbb{Z}}\gamma(a_n)((1+\pi)^{\chi^{\mathrm{cyclo}}(\gamma)}-1)^{n/e}\\
&= \sum_{n\in\mathbb{Z}\setminus 0}\frac{n}{e}\gamma(a_n)((1+\pi)^{\chi^{\mathrm{cyclo}}(\gamma)}-1)^{n/e-1}\chi^{\mathrm{cyclo}}(\gamma)(1+\pi)^{\chi^{\mathrm{cyclo}}(\gamma)}\\
&= \chi^{\mathrm{cyclo}}(\gamma)\gamma\sum_{n\in\mathbb{Z}\setminus 0}\frac{n}{e}a_n\pi^{n/e-1}(1+\pi)\\
&= \chi^{\mathrm{cyclo}}(\gamma)\cdot\gamma\nabla\sum_{n\in\mathbb{Z}}a_n\pi_K^n.
\end{aligned}$$

(b)

$$\begin{aligned}
\nabla\phi\sum_{n\in\mathbb{Z}} a_n\pi_K^n &= \nabla\sum_{n\in\mathbb{Z}}\phi(a_n)((1+\pi)^{p}-1)^{n/e}\\
&= \sum_{n\in\mathbb{Z}\setminus 0}\frac{n}{e}\phi(a_n)((1+\pi)^{p}-1)^{n/e-1}p(1+\pi)^{p}\\
&= p\phi\sum_{n\in\mathbb{Z}\setminus 0}\frac{n}{e}a_n\pi^{n/e-1}(1+\pi)\\
&= p\cdot\phi\nabla\sum_{n\in\mathbb{Z}}a_n\pi_K^n.
\end{aligned}$$

(c) We compute that $\{(\zeta(1+\pi)-1)^{1/e} : \zeta\in\mu_p\}$ is the set of Galois conjugates of π in B over $\phi(B)$. Thus $\mathrm{Tr}_{B/\phi(B)} f(\pi) = \sum_{\zeta\in\mu_p} f(((1+\pi)\zeta-1)^{1/e})$, and we have

$$\begin{aligned}
\nabla\,\mathrm{Tr}_{B/\phi(B)} f(\pi_K) &= \nabla\sum_{\zeta\in\mu_p} f(((1+\pi)\zeta-1)^{1/e})\\
&= \sum_{\zeta\in\mu_p} f'(((1+\pi)\zeta-1)^{1/e})\cdot\frac{1}{e}((1+\pi)\zeta-1)^{1/e-1}\cdot\zeta(1+\pi)\\
&= \mathrm{Tr}_{B/\phi(B)}\left(f'(\pi_K)\frac{1}{e}\pi^{1/e-1}(1+\pi)\right)\\
&= \mathrm{Tr}_{B/\phi(B)}\,\nabla f(\pi_K).
\end{aligned}$$

(d) From (c) and Lemma 3.1.2, we have

$$\begin{aligned}\nabla\psi &= p^{-1}\nabla\phi^{-1}\operatorname{Tr}_{B/\phi B}\\ &= p^{-2}\phi^{-1}\operatorname{Tr}_{B/\phi B}\nabla\\ &= p^{-1}\psi\nabla.\end{aligned}$$

□

3.1.2 The map $T_m\phi^{-n}$ on $A_K^{\psi=p^{r-1}}$

Recall that we wish to understand the map $\frac{d^{r-1}}{dt^{r-1}}T_m\circ\phi^{-n}$. From Lemmas 3.1.2 and 3.1.3 we see that

$$\frac{d^{r-1}}{dt^{r-1}}T_m\phi^{-n} = p^{-n(r-1)}T_m\phi^{-n}\nabla^{r-1}$$

and thus we can study the action of $T_m\phi^{-n}$ on $\nabla^{r-1}A_K^{\psi=1}$. But since $\psi\nabla x = p\nabla\psi x$, we see that $\nabla^{r-1}A_K^{\psi=1} = A_K^{\psi=p^{r-1}}$, and so we wish to study $T_m\phi^{-n}$ on $A_K^{\psi=p^{r-1}}$.

Lemma 3.1.4. *Let $P\in A_K^{\psi=p^{r-1}}$. Then if $1\le m\le n$ and $(\phi^{-m}P)(0)$ converges, we have*

$$T_m(\phi^{-n}P)(0) = p^{(r-1)(n-m)}(\phi^{-m}P)(0).$$

Further, if $\phi^{-0}(P)$ converges in B_{dR}, then we have

$$T_0(\phi^{-n}P)(0) = p^{(r-1)n}(1-p^{-r}\sigma^{-1})(\phi^{-0}P)(0).$$

Proof. Since $P\in A_K^{\psi=p^{r-1}}$, we know that $\psi(P)=p^{r-1}P$ and thus that $p^{-r}\operatorname{Tr}_{B/\phi B}(P)=\phi(P)$. But $\{((1+\pi)\zeta-1)^{1/e}:\zeta\in\mu_p\}$ is the set of conjugates of π over $\phi(B)$ in B, so this gives us

$$p^{-r}\sum_{\zeta\in\mu_p}P(((1+\pi)\zeta-1)^{1/e}) = P^{\sigma}(((1+\pi)^p-1)^{1/e}).$$

Since P is just a power series, for any $l\in\mathbb{N}$ we can set $\pi=\zeta_{p^{l+1}}-1$, and apply $\sigma^{-(l+1)}$ to each coefficient; this corresponds to the operator $\phi^{-(l+1)}P|_{t=0}$ whenever $\phi^{-(l+1)}$ is defined. We get

$$p^{-r}\sum_{\zeta\in\mu_p}P^{\sigma^{-(l+1)}}((\zeta\cdot\zeta_{p^{l+1}}-1)^{1/e}) = P^{\sigma^{-l}}((\zeta_{p^l}-1)^{1/e}). \tag{3.1}$$

If $l\ge 1$, the left hand side of equation (3.1) is just

$$p^{-r}\operatorname{Tr}_{K_{l+1}/K_l}P^{\sigma^{-(l+1)}}((\zeta_{p^{l+1}}-1)^{1/e}) = p^{1-r}T_lP^{\sigma^{-(l+1)}}((\zeta_{p^{l+1}}-1)^{1/e}),$$

and thus we have

$$p^{1-r}T_l P^{\sigma^{-(l+1)}}((\zeta_{p^{l+1}}-1)^{1/e}) = P^{\sigma^{-l}}((\zeta_{p^l}-1)^{1/e}).$$

By a simple induction, we see that for any $1 \leq m < n$, we have

$$p^{(1-r)(n-m)}T_m P^{\sigma^{-n}}((\zeta_{p^n}-1)^{1/e}) = P^{\sigma^{-m}}((\zeta_{p^m}-1)^{1/e}). \tag{3.2}$$

Note that $P^{\sigma^{-n}}((\zeta_{p^n}-1)^{1/e}) = \phi^{-n}P(0)$, so the first statement is proven.

If $l = 0$ then equation (3.1) becomes

$$p^{-r}\sum_{\zeta\in\mu_p} P^{\sigma^{-1}}((\zeta\cdot\zeta_p - 1)^{1/e}) = P(0).$$

The left hand side is now equal to

$$p^{-r}\left(P^{\sigma^{-1}}(0) + \mathrm{Tr}_{K(\zeta_p)/K}\, P^{\sigma^{-1}}((\zeta_p-1)^{1/e})\right) = p^{-r}P^{\sigma^{-1}}(0) + p^{1-r}T_0 P^{\sigma^{-1}}((\zeta_p-1)^{1/e}).$$

Thus we have $T_0(\phi^{-0}P)((\zeta_p-1)^{1/e}) = (p^{r-1} - p^{-1}\sigma^{-1})P(0)$. Combining this with (3.2) gives us

$$\begin{aligned}(p^{r-1}-p^{-1}\sigma^{-1})P(0) &= T_0\left(p^{(1-r)n}T_1 P^{\sigma^{-(n+1)}}((\zeta_{p^{n+1}}-1)^{1/e})\right)\\ &= T_0 p^{(1-r)(n-1)}P^{\sigma^{-n}}((\zeta_{p^n}-1)^{1/e})\\ &= p^{(1-r)(n-1)}T_0(\phi^{-n}P)(0)\end{aligned}$$

and we have (b). □

Corollary 3.1.5. *If $P \in A_K^{\psi=1}$, then if $1 \leq m < n$ we have*

(a)

$$T_m(\phi^{-n}P)(0) = (\phi^{-m}P)(0),$$

(b)

$$T_0(\phi^{-n}P)(0) = (1-p^{-r}\sigma^{-1})P(0).$$

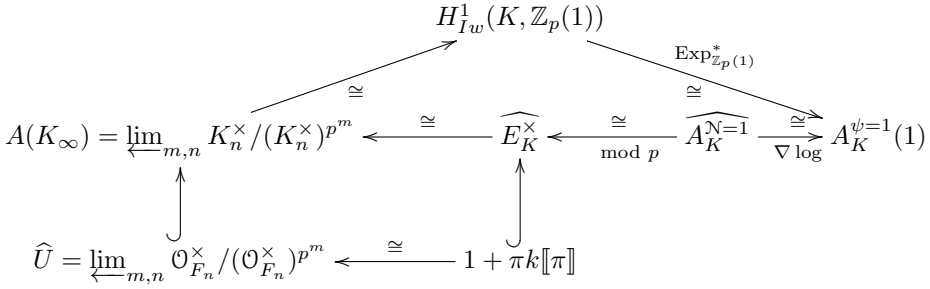
Remark 3.1.6. If we take $r = 1$, then we have found that $T_m\phi^{-n}P$ is invariant for large enough n—in particular for $n \geq 1$—if $P \in A_K^{\psi=1}$. The exact same proof works for $B_K^{\psi=1} = D(\mathbb{Q}_p)^{\psi=1}$, and thus we have recaptured a special case of the statement in 2.2.2 that $T_m(\phi^{-n}(\mathrm{Exp}^*_{V^*(1)}(\mu)))$ is invariant for large enough n.

3.2 $A_K^{\psi=1}(1)$ and $A(K_\infty)$

In this section, we will prove that $A_K^{\psi=1}$ is a rank-1 Λ_K module and relate it to several other rank-1 Λ_K-modules.

3.2.1 $A_K^{\psi=1}$ as an Iwasawa module

The purpose of this subsection is to establish the following commutative diagram:



Recall that the groups $A(K_\infty)$ and $\widehat{U}$ were studied in section 2.1.2, and are both isomorphic to $\Lambda_K \oplus \mathbb{Z}_p(1)$ by Proposition 2.1.5. Recall that for any abelian group H, we write $\widehat{H}$ for the pro-p completion, again defined in 2.1.2.

Since $\log \mathcal{N}(x) = p\psi \log(x)$, we know that $\nabla \log \mathcal{N}(x) = \psi \nabla \log(x)$, and thus $\mathcal{N}(x) = x$ if and only if $\psi(\nabla \log(x)) = \nabla \log x$. Thus $\nabla \log$ defines a map from $A_K^{\mathcal{N}=1}$ considered multiplicatively to $A_K^{\psi=1}(1)$ considered additively (the twist makes this map Γ-equivariant). We claim that $A_K^{\mathcal{N}=1} \xrightarrow{\sim} A_K^{\psi=1}(1)$, and the rest of this subsection will work towards a proof of this fact.

Recall that $\tilde{E} = \varprojlim \mathbb{C}_p$, $E_{\mathbb{Q}_p} = \mathbb{F}_p((\epsilon - 1))$, and E is the separable closure of $E_{\mathbb{Q}_p}$ in $\tilde{E}$, with $E_K = E^{H_K} = k((\pi_K)) = A_K/pA_K$. We can now state the following lemma:

Lemma 3.2.1. *The map $\widehat{A^{\mathcal{N}=1}} \to \widehat{E^\times}$ given by reduction modulo p is an isomorphism. Further, for any finite extension $K/\mathbb{Q}_p$ the map $\widehat{A_K^{\mathcal{N}=1}} \to \widehat{E_K^\times}$ is an isomorphism.*

Proof. We will prove this result for $A^{\mathcal{N}=1} \xrightarrow{\sim} E^\times$ and $A_K^{\mathcal{N}=1} \xrightarrow{\sim} E_K^\times$; the result will naturally extend to the pro-p completions.

We produce an inverse map. Let $x \in E^\times$. From [CC99] Lemma V.1.1, we see that if $f \in A$, then the sequence $\mathcal{N}^m f$ converges to some f^∞, and the images modulo p of f and f^∞ are the same. Thus, in particular, there is a unique $\hat{x} \in A$ with $\hat{x} \mod p = x$ and $\mathcal{N}(\hat{x}) = \hat{x}$. Further, since this element is unique, if $x, y \in E^\times$ then $\widehat{xy} = \hat{x}\hat{y}$.

Thus the map $x \mapsto \hat{x}$ is a homomorphism of multiplicative groups $E^\times \to A^{\mathcal{N}=1}$. Then $\hat{x} \mod p = x$; and if $f \in A^{\mathcal{N}=1}$, and x is its image modulo p in $E^\times$, then f is a lift of x, and $\mathcal{N}(f) = f$; and thus $f = \hat{x}$. So the map $x \mapsto \hat{x}$ is an inverse for the map $f \mapsto f \mod p$.

In particular, we claim that if $x \in E_K^\times$, then $\hat{x} \in A_K^{\mathcal{N}=1}$. For there is clearly a lift $\tilde{x} \in A_K$ of x, and the sequence $\mathcal{N}^m(\tilde{x})$ will converge to $\hat{x} \in A^{\mathcal{N}=1}$. But if $\tilde{x} \in A_K$, then $\mathcal{N}(\tilde{x}) \in A_K$, so $\hat{x} \in A_K \cap A^{\mathcal{N}=1} = A_K^{\mathcal{N}=1}$ as desired. □

Lemma 3.2.2. *There is an isomorphism $1 + \pi_K k[\![\pi_K]\!] \xrightarrow{\sim} \widehat{U}$, which extends to an isomorphism $\widehat{E_K^\times} \xrightarrow{\sim} A(K_\infty)$.*

Proof. There is an isomorphism $\iota_K : \mathcal{O}_{E_K} \to \varprojlim \mathcal{O}_{K_n}$ defined in [CC99] Proposition I.1.1. Thus we also have an isomorphism $\widehat{\mathcal{O}_{E_K}^\times} \xrightarrow{\sim} \widehat{\varprojlim \mathcal{O}_{K_n}^\times} = \widehat{U}$. But $\mathcal{O}_{E_K} = k[\![\pi_K]\!]$, and thus $\widehat{\mathcal{O}_{E_K}^\times} = \widehat{k[\![\pi_K]\!]^\times} = 1 + \pi_K k[\![\pi_K]\!]$, and we have the first isomorphism.

Our isomorphism $\mathcal{O}_{E_K} \to \varprojlim \mathcal{O}_{K_n}$ lifts to an isomorphism $E_K \xrightarrow{\sim} \varprojlim K_n$, simply by taking fields of fractions on each side. Thus we have an isomorphism $E_K^\times \xrightarrow{\sim} \varprojlim E_K^\times$, and taking pro-$p$ completions on both sides completes the proof. □

Finally, we note that $A(K_\infty) \cong H^1_{Iw}(K, \mathbb{Z}_p(1))$ by Lemma 2.1.6, and $H^1_{Iw}(K, \mathbb{Z}_p(1)) \cong A_K^{\psi=1}(1)$ by the Cherbonnier-Colmez dual exponential $\mathrm{Exp}^*_{\mathbb{Q}_p(1)}$, so we have

$$\widehat{A_K^{\mathcal{N}=1}} \cong \widehat{E_K^\times} \cong A(K_\infty) \cong H^1_{Iw}(K, \mathbb{Z}_p(1)) \cong A_K^{\psi=1}(1).$$

Now it only remains to show that our isomorphism $\widehat{A_K^{\mathcal{N}=1}} \xrightarrow{\sim} A_K^{\psi=1}(1)$ is given by $\nabla \log$. But this is a consequence of [CC99] Proposition V.3.2 (iii).

3.2.2 $\widehat{U}$ as a Λ_K-submodule of $A(K_\infty)$

Recall that $\widehat{U}$ and $A(K_\infty)$ are both isomorphic to $\Lambda_K \oplus \mathbb{Z}_p(1)$ (Proposition 2.1.5), and thus isomorphic to each other, but are not identical. In this subsection, we will describe $\widehat{U}$ as a submodule of $A(K_\infty)$.

If L is a Λ_K-module, we write L_{tors} for the Λ_K-torsion of L, and $L_{\mathrm{tf}} = L/L_{\mathrm{tors}}$ for the torsion-free part of L. Then we have $A(K_\infty)_{\mathrm{tf}} \cong \widehat{U}_{\mathrm{tf}} \cong \Lambda_K$, and thus we can view $A(K_\infty)_{\mathrm{tf}}$ and $\widehat{U}_{\mathrm{tf}}$ as rings. In this subsection, we will show that $\widehat{U}_{\mathrm{tf}}$ is a principal ideal inside $A(K_\infty)_{\mathrm{tf}}$. Recall that $G = \mathrm{Gal}(K/\mathbb{Q}_p) = \Sigma \ltimes \Delta_e$, where σ is cyclic and generated by σ, and Δ_e is cyclic and generated by δ_e.

Lemma 3.2.3. *The restriction of the valuation map $v : A(K_\infty)_{\mathrm{tf}} \to \mathbb{Z}_p$ is isomorphic to the map given by taking the quotient $A(K_\infty)_{\mathrm{tf}}/(\sigma - 1, \delta_e - 1, \gamma - 1)A(K_\infty)_{\mathrm{tf}}$. Thus in particular, $\widehat{U}_{\mathrm{tf}} = (\sigma - 1, \delta_e - 1, \gamma - 1)A(K_\infty)_{\mathrm{tf}}$.*

Proof. The image $\mathbb{Z}_p$ of the valuation map has trivial action from both Γ_K and Σ, so it must factor through the quotient $A(K_\infty)_{\mathrm{tf}}/(\sigma - 1, \delta_e - 1, \gamma - 1)A(K_\infty)_{\mathrm{tf}}$. But by lemma 2.1.7 we know that

$$A(K_\infty)_{\mathrm{tf}} \cong \Lambda_K \cong \mathbb{Z}_p[G][\![\Gamma_K]\!],$$

so

$$A(K_\infty)_{\mathrm{tf}}/(\sigma-1,\delta_e-1,\gamma-1)A(K_\infty)_{\mathrm{tf}} \cong \mathbb{Z}_p.$$

Thus the valuation map induces an isomorphism

$$A(K_\infty)_{\mathrm{tf}}/(\sigma-1,\delta_e-1,\gamma-1)A(K_\infty)_{\mathrm{tf}} \xrightarrow{\sim} \mathbb{Z}_p$$

and must have kernel $(\sigma-1,\delta_e-1,\gamma-1)A(K_\infty)_{\mathrm{tf}}$.

□

This lemma suggests that the ideal $(\sigma-1,\delta_e-1,\gamma-1)$ is a principal ideal of Λ_K, since we know that $\widehat{U}_{\mathrm{tf}} \cong \Lambda_K$ and thus is generated over Λ_K by a single element. In the following proposition, we compute a generator for this ideal.

Proposition 3.2.4. *The ideal* $(\sigma-1,\delta_e-1,\chi^{\mathrm{cyclo}}(\gamma)\gamma-1)$ *of* Λ_K *is principal, and generated by the element* $(1-e_1)+(\chi^{\mathrm{cyclo}}(\gamma)\gamma-1)e_1$.

Proof. We first wish to prove that $(1-e_1)+(\chi^{\mathrm{cyclo}}(\gamma)\gamma-1)e_1 \in (\sigma-1,\delta_e-1,\chi^{\mathrm{cyclo}}(\gamma)\gamma-1)$. It is clear that $(\chi^{\mathrm{cyclo}}(\gamma)\gamma-1)e_1 \in (\chi^{\mathrm{cyclo}}(\gamma)\gamma-1)$, so it is enough to show that $1-e_1 \in (\sigma-1,\delta_e-1)$.

Let $\chi_1,\ldots,\chi_{ef}$ be the $\mathbb{Q}_p$-rational characters of $\overline{\mathbb{Q}_p}[G]$. Then let $\chi_1,\ldots,\chi_e$ be the characters which are trivial on Σ. We have

$$(\delta_e-1)\sum_{i=1}^{e} e_i + (\sigma-1)\sum_{e+1}^{ef} e_i = \sum_{i=2}^{e}(\chi_i(\delta_e)-1) + \sum_{e+1}^{ef}(\chi_i(\sigma)-1)e_i.$$

But $\chi_i(\sigma)$ is a fth root of unity, and thus in $\mathbb{Z}_p^\times$, and $\chi_i(\delta_e)$ is an eth root of unity, and thus in $\mathbb{Z}_p^\times$ (since p does not divide e or f). Then, if $2 \le i \le e$, we have $(\chi_i(\delta)-1) \in \mathbb{Z}_p^\times$, since the only i for which $\chi_i(\delta_e)=\chi_i(\sigma)=1$ is $i=1$. Similarly, if $e+1 \le i \le ef$ then $(\chi_i(\sigma)-1) \in \mathbb{Z}_p^\times$. So we set

$$\lambda = \sum_{i=2}^{e}(\chi_i(\delta)-1)^{-1}e_i + \sum_{i=e+1}^{ef}(\chi_i(\sigma)-1)^{-1}e_i$$

and we have that

$$\begin{aligned}\lambda\cdot\left((\delta_e-1)\sum_{i=1}^{e} e_i + (\sigma-1)\sum_{e+1}^{ef} e_i\right) &= \sum_{i=2}^{ef} e_i \\ &= 1-e_1\end{aligned}$$

and thus $1-e$ is in the ideal generated by $(\sigma-1,\delta_e-1)$.

Now we wish to show that $(\sigma-1),(\delta_e-1),(\chi^{\mathrm{cyclo}}(\gamma)\gamma-1) \in ((1-e_1)+(\chi^{\mathrm{cyclo}}(\gamma)\gamma-1)e_1)$. But

we know that $e_1^2 = e_1, (1-e_1)^2 = 1-e_1$, and $e_1(1-e_1) = 0$, so we can write

$$\begin{aligned}
\sigma - 1 &= (\sigma - 1)(1 - e_1) \\
&= (\sigma - 1)(1 - e_1)((1 - e_1) + (\chi^{\text{cyclo}}(\gamma)\gamma - 1)e_1), \\
\delta_e - 1 &= (\delta_e - 1)(1 - e_1) \\
&= (\delta_e - 1)(1 - e_1)((1 - e_1) + (\chi^{\text{cyclo}}(\gamma)\gamma - 1)e_1) \\
\chi^{\text{cyclo}}(\gamma)\gamma - 1 &= (\chi^{\text{cyclo}}(\gamma)\gamma - 1)(1 - e_1) + (\chi^{\text{cyclo}}(\gamma)\gamma - 1)e_1 \\
&= \left((\chi^{\text{cyclo}}(\gamma)\gamma - 1)(1 - e_1) + e_1\right)\left((1 - e_1) + (\chi^{\text{cyclo}}(\gamma)\gamma - 1)e_1\right).
\end{aligned}$$

□

Chapter 4

The Tamagawa number conjecture for Tate motives over unramified extensions $F/\mathbb{Q}_p$

In this chapter, we will prove Conjecture 1.2.3 for unramified extensions $F/\mathbb{Q}_p$. Recall that we are studying the image of the map $\exp^*_{\mathbb{Q}_p(1-r)} : H^1(F, \mathbb{Z}_p(r)) \to F$. Using Theorem 2.2.2 we can answer this question by composing the Cherbonnier-Colmez dual exponential $\mathrm{Exp}^*_{\mathbb{Q}_p(r)} : H^1_{Iw}(F, \mathbb{Z}_p(1)) \to A_F^{\psi=1}(1)$ with the map $T_0 \circ \phi^{-n}$. In section 4.1, we will study the structure of $A_F^{\psi=1}$ and find an element that generates $A_F^{\psi=1}(1)_{\mathrm{tf}}$. In section 4.2, we will study the map $T_0 \circ \phi^{-n}$ and prove the conjecture for unramified extensions of $\mathbb{Q}_p$.

4.1 A basis for $A_F^{\psi=1}(1)$

In this section, we investigate the structure of $A_F^{\psi=1}(1)$ as a Λ_F-module. In subsection 4.1.1, we describe a useful Λ_F-module $\mathcal{P}_{F,\log}^{\psi=p^{-1}}$, and in subsection 4.1.2, we show that it can be viewed as a submodule of $A_F^{\psi=1}(1)$. In subsection 4.1.3, we develop another Λ_F-module $\mathcal{O}_F[\![\pi]\!]^{\psi=0}$ which has an easily-describable basis, and relate the modules $\mathcal{P}_{F,\log}^{\psi=p^{-1}}$ and $\mathcal{O}_F[\![\pi]\!]^{\psi=0}$.

4.1.1 The module $\mathcal{P}_F$

We begin by recalling some results of [PR90]. There is an obvious short exact sequence of additive groups

$$0 \longrightarrow p\mathcal{O}_F[\![\pi]\!] \longrightarrow \mathcal{O}_F[\![\pi]\!] \longrightarrow k[\![\pi]\!] \longrightarrow 0$$

and a corresponding short exact sequence of multiplicative groups

$$1 \longrightarrow 1 + p\mathcal{O}_F[\![\pi]\!] \longrightarrow \mathcal{O}_F[\![\pi]\!]^\times \longrightarrow k[\![\pi]\!]^\times \longrightarrow 1.$$

Consider the formal multiplicative group $\widehat{\mathbb{G}}(k[\![\pi]\!]) = 1 + \pi k[\![\pi]\!] \subset k((\pi))$. We write

$$\mathcal{O}_F[\![\pi]\!]_{\log} = \{f \in \mathcal{O}_F[\![\pi]\!]^\times : f \mod p\mathcal{O}_F[\![\pi]\!] \in 1 + \pi k[\![\pi]\!]\},$$

for the preimage in $\mathcal{O}_F[\![\pi]\!]^\times$ of $1 + \pi k[\![\pi]\!]$. We have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & 1 & \longrightarrow & k^\times & \xrightarrow{id} & k^\times & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & 1 + p\mathcal{O}_F[\![\pi]\!] & \longrightarrow & \mathcal{O}_F[\![\pi]\!]^\times & \longrightarrow & k^\times \times (1 + \pi k[\![\pi]\!]) & \longrightarrow & 1 \\
 & & \uparrow{\scriptstyle id} & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & 1 + p\mathcal{O}_F[\![\pi]\!] & \longrightarrow & \mathcal{O}_F[\![\pi]\!]_{\log} & \longrightarrow & 1 + \pi k[\![\pi]\!] & \longrightarrow & 1.
\end{array}$$

Recall that the logarithm map is given by $\log(1 + x) = \sum_{n \geq 0} \frac{x^n}{n}$, and thus converges as long as $x \in p\mathcal{O}_f[\![\pi]\!] + \pi\mathcal{O}_F[\![\pi]\!]$; thus, log converges everywhere on $\mathcal{O}_F^\times[\![\pi]\!]_{\log}$, since we can write $x \in \mathcal{O}_F[\![\pi]\!]_{\log}$ as $x = 1 + p\alpha + \pi\beta$ for $\alpha, \beta \in \mathcal{O}_F[\![\pi]\!]$. Further, we see that $\log : 1 + p\mathcal{O}_F[\![\pi]\!] \xrightarrow{\sim} p\mathcal{O}_F[\![\pi]\!]$ is an isomorphism, with inverse given by the usual power series for exp. We wish to extend this isomorphism to give an isomorphism from $\mathcal{O}_F[\![\pi]\!]_{\log}$ to a subgroup of $F[\![\pi]\!]$, which motivates the following discussion.

We define:

Definition 4.1.1. The space

$$\mathcal{P}_F = \left\{ \sum_{n \geq 0} a_n \pi^n \in F[\![\pi]\!] : na_n \in \mathcal{O}_F \right\}$$

is the space of power series in F whose derivative with respect to π lies in $\mathcal{O}_F[\![\pi]\!]$.

Since the map $d\log$ is given by an integral power series, we know that $\log \mathcal{O}_F[\![\pi]\!]_{\log} \subseteq \mathcal{P}_F$. We write $\overline{\mathcal{P}}_F = \mathcal{P}_F / p\mathcal{O}_F[\![\pi]\!]$. We define an automorphism V of $\overline{\mathcal{P}}_F$ by

$$V\left(\sum_n a_n \pi^n\right) = \sum_n p\sigma^{-1}(a_{np})\pi^n,$$

and observe that $p\psi \equiv V \mod p\mathcal{O}_F[\![\pi]\!]$ (see [PR90] 1.3, noting that our definition of ψ differs by a factor of p). Then we set

$$\overline{\mathcal{P}}_{F,\log} = \{f \in \overline{\mathcal{P}}_F : (p - \phi)(f) = 0, V(f) = f\}$$
$$\mathcal{P}_{F,\log} = \{f \in \mathcal{P}_F : (p - \phi)(f) \in p\mathcal{O}_F[\![\pi]\!]\}.$$

We have the following lemmas:

Lemma 4.1.2. *Let $f \in 1 + \pi k[\![\pi]\!]$ and let $\hat{f}$ be any lift of f to $\mathcal{O}_F[\![\pi]\!]_{\log}$. Then* $\log(\hat{f}) \mod p\mathcal{O}_F[\![\pi]\!]$

as an element of $\overline{\mathcal{P}}_{F,\log}$ *does not depend on the choice of* $\hat{f}$*, and the map* $f \mapsto \log(\hat{f}) \mod p\mathcal{O}_F[\![\pi]\!]$ *is an isomorphism* $1 + \pi k[\![\pi]\!] \xrightarrow{\sim} \overline{\mathcal{P}_{F,\log}}$.

Proof. See [PR90] Lemma 2.1. □

Lemma 4.1.3. *Let* $f \in \mathcal{P}_F$*. Then the sequence* $p^m\psi^m(f)$ *converges to a limit* $f^\infty \in \mathcal{P}_{F,\log}$*, and we have:*

1. $f^\infty \equiv f \mod p\mathcal{O}_F[\![\pi]\!]$
2. $\psi(f^\infty) = p^{-1}f^\infty$
3. $(1 - p^{-1}\phi)f^\infty \in \mathcal{O}_F[\![\pi]\!]$*, i.e.* $f^\infty \in \mathcal{P}_{F,\log}$
4. $f^\infty = 0$ *if* $f \in \mathcal{O}_F[\![\pi]\!]$
5. $f^\infty = g^\infty$ *if* $f \equiv g \mod p\mathcal{O}_F[\![\pi]\!]$.

Proof. See [PR90] Lemma 2.2. □

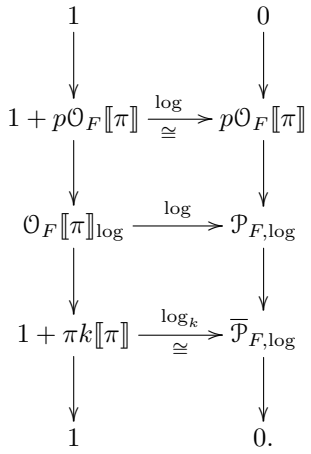
Proposition 4.1.4. *With the defintions above, we have:*

1. *There is a well-defined map* $\rho : \overline{\mathcal{P}}_{F,\log} \to \mathcal{P}_{F,\log}^{\psi=p^{-1}} \subset \mathcal{P}_{F,\log}$.
2. $\overline{\mathcal{P}}_{F,\log} = \overline{\mathcal{P}}_F^{\phi=p}$.
3. $\mathcal{P}_{F,\log}/p\mathcal{O}_F[\![\pi]\!] = \overline{\mathcal{P}}_{F,\log}$.
4. *The map* $\log : \mathcal{O}_F[\![\pi]\!]_{\log} \to \mathcal{P}_{F,\log}$ *is an isomorphism.*

Proof.

1. Let $f = \sum_n a_n\pi^n \in \mathcal{P}_{F,\log}$, where the $a_n \in k$. Then each a_n has a lift $\hat{a}_n \in \mathcal{O}_F$, and if we set $\hat{f} = \sum \hat{a}_n\pi^n \in \mathcal{P}_F$, then $\hat{f} \mod p\mathcal{O}_F[\![\pi]\!] = f$. Let $\hat{g}$ be another lift of f to $\mathcal{P}_F$. Then $\hat{f}^\infty, \hat{g}^\infty \in \mathcal{P}_{F,\log}$, and by Lemma 4.1.3 we have that $\hat{g}^\infty = \hat{f}^\infty$. Thus the function defined by $\rho(f) = \hat{f}^\infty$ is a well-defined homomorphism, and since $\psi(\hat{f}^\infty) = f^\infty/p$, its image is in $\mathcal{P}_{F,\log}^{\psi=p^{-1}}$.
2. It's clear that $\overline{\mathcal{P}}_{F,\log} \subset \overline{\mathcal{P}}_F^{\phi=p}$. Let $f \in \overline{\mathcal{P}}_F^{\phi=p}$, and let $\hat{f}$ be a lift to $\mathcal{P}_F$, so that $\hat{f}^\infty \in \mathcal{P}_{F,\log}$, with $\hat{f}^\infty \mod p\mathcal{O}_F[\![\pi]\!] \in \overline{\mathcal{P}}_{F,\log}$. But $\hat{f}^\infty \mod p\mathcal{O}_F[\![\pi]\!] = \hat{f} \mod p\mathcal{O}_F[\![\pi]\!] = f$, so $\psi(\hat{f}^\infty) = \hat{f}^\infty/p$ and thus $V(f) = f/p$.
3. It's trivial that $\mathcal{P}_{F,\log}/p\mathcal{O}_F[\![\pi]\!] \subset \overline{\mathcal{P}}_F^{\phi=p} = \overline{\mathcal{P}}_{F,\log}$. Let $f \in \overline{\mathcal{P}}_{F,\log}$. Then $\hat{f}^\infty \in \mathcal{P}_{F,\log}$, and its quotient mod $p\mathcal{O}_F[\![\pi]\!]$ is f.

4. We have a commutative diagram:



By the five lemma, the middle arrow is an isomorphism.

□

We recall the definition of the normalized norm map $\mathcal{N} = \phi^{-1} N_{B/\phi B}$. We can compute that $\log \mathcal{N}(x) = p\psi \log(x)$, so we have an isomorphism $\log : \mathcal{O}_F[\![\pi]\!]_{\log}^{\mathcal{N}=1} \xrightarrow{\sim} \mathcal{P}_{F,\log}^{\psi=p^{-1}}$.

Proposition 4.1.5. *The function $\rho : \overline{\mathcal{P}}_{F,\log} \to \mathcal{P}_{F,\log}^{\psi=p^{-1}}$ is an isomorphism.*

Proof. It's clear that ρ is injective. Let $f \in \mathcal{P}_{F,\log}^{\psi=p^{-1}}$, and let $\overline{f}$ be its image mod $p\mathcal{O}_F[\![\pi]\!]$ in $\overline{\mathcal{P}}_{F,\ log}$. We know that $\rho(\overline{f}) \in \mathcal{P}_{F,\log}^{\psi=p^{-1}}$ and that the image of $\rho(\overline{f})$ mod $p\mathcal{O}_F[\![\pi]\!]$ is $\overline{f}$, so ρ is surjective and thus an isomorphism. □

Corollary 4.1.6. *We have a chain of isomorphisms of Λ_F-modules:*

$$1 + \pi k[\![\pi]\!] \cong \overline{\mathcal{P}}_{F,\log} \cong \mathcal{P}_{F,\log}^{\psi=p^{-1}} \cong \mathcal{O}_F[\![\pi]\!]_{\log}^{\mathcal{N}=1}.$$

Proof. It is clear from the preceeding discussion that these are isomorphic as groups. Recall that $\Lambda_F = \mathbb{Z}_p[\Sigma][\![\Gamma_F]\!]$; we have obvious actions of $\mathbb{Z}_p$ and Σ and Γ_F. It is easy to see that our isomorphisms respect these actions.

Since log is given by a power series, it respects ring homomorphism and thus commutes with ϕ and Γ_F □

4.1.2 $\mathcal{P}_{F,\log}^{\psi=p^{-1}}$ as a submodule of $A_F^{\psi=1}$

In this subsection, we wish to embed $\mathcal{P}_{F,\log}^{\psi=p^{-1}}$ as a sub-Λ_F-module of $A_F^{\psi=1}(1)$. We begin by establishing a commutative diagram:

$$\begin{array}{ccccccc}
A(F_\infty) = \varprojlim_{m,n} F_n^\times/(F_n^\times)^{p^m} & \xleftarrow{\cong} & \widehat{E_F^\times} & \xleftarrow[\text{mod } p]{\cong} & \widehat{A_F^{\mathcal{N}=1}} & \xrightarrow[\nabla\log]{\cong} & A_F^{\psi=1}(1) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \nabla \\
\widehat{U} = \varprojlim_{m,n} \mathcal{O}_{F_n}^\times/(\mathcal{O}_{F_n}^\times)^{p^m} & \xleftarrow{\cong} & 1+\pi k[\![\pi]\!] & \xleftarrow[\text{mod } p]{\cong} & \mathcal{O}_F[\![\pi]\!]_{\log}^{\mathcal{N}=1} & \xrightarrow[\log]{\cong} & \mathcal{P}_{F,\log}^{\psi=p^{-1}}
\end{array}$$

The top row of this diagram was established in section 4.1.2, as was the leftmost arrow on the bottom row. The other two isomorphisms on the bottom row are given in corollary 4.1.6 of the previous section.

Of the vertical maps in the diagram, the left three are the obvious injections. The rightmost vertical arrow is clear, since $A_F^{\psi=1}(1)$ is the image of $\nabla\log$ and $\mathcal{P}_{F,\log}^{\psi=p^{-1}}$ is the image of log. Thus the vertical maps exist as we wish. Further, $\nabla : \mathcal{P}_{F,\log}^{\psi=p^{-1}} \to A_F^{\psi=1}(1)$ is Γ-equivariant because $\nabla\gamma f = \chi^{\text{cyclo}}(\gamma)\gamma\nabla f$, and thus we have a commutative diagram of Λ_F-modules.

Now recall that we showed in Lemma 3.2.3 that $\widehat{U}_{\text{tf}} = (\sigma-1,\gamma-1)A(F_\infty)_{\text{tf}}$ (since $\delta_e = 1$ when $K = F$), and thus that

$$\nabla\left(\mathcal{P}_{F,\log}^{\psi=p^{-1}}\right)_{\text{tf}} = (\sigma-1,\gamma-1)A_F^{\psi=1}(1)_{\text{tf}}.$$

We can then prove the following lemma:

Lemma 4.1.7. *There are elements $\alpha \in A_F^{\psi=1}(1)$, $\beta \in \mathcal{P}_{F,\log}^{\psi=p^{-1}}$ such that*

1. $A_F^{\psi=1} = \Lambda_F\cdot\alpha \oplus \mathbb{Z}_p(1)\cdot 1$,
2. $\mathcal{P}_{F,\log}^{\psi=p^{-1}} = \Lambda_F\cdot\beta \oplus \mathbb{Z}_p\cdot\log(1+\pi)$,
3. *As ideals,* $(\nabla\beta) = (((1-e_1)+(\chi^{\text{cyclo}}(\gamma)\gamma-1))\alpha)$.

Proof. All we need to check for (1) and (2) is that $\mathbb{Z}_p(1)\cdot 1 \cong \mathbb{Z}_p(1)$ and that $\mathbb{Z}_p(1) \cong \mathbb{Z}_p\cdot\log(1+\pi)$. The first is clear; the second follows from the fact that

$$\gamma\log(1+\pi) = \log(\gamma(1+\pi)) = \log((1+\pi)^{\chi^{\text{cyclo}}(\gamma)}) = \chi^{\text{cyclo}}(\gamma)\log(1+\pi).$$

For (3), we confirm that $\nabla(\log(1+\pi)) = 1$, and then since $\nabla\left(\mathcal{P}_{F,\log}^{\psi=p^{-1}}\right)_{\text{tf}} = (\sigma-1,\gamma-1)A_F^{\psi=1}(1)_{\text{tf}}$, we must have $(\nabla\beta) = (\sigma-1,\chi^{\text{cyclo}}(\gamma)\gamma-1)\alpha$. But in Proposition 3.2.4 we showed that $(\sigma-1,\chi^{\text{cyclo}}(\gamma)\gamma-1)$ is principal and equal to $((1-e_1)+(\chi^{\text{cyclo}}(\gamma)\gamma-1))\alpha$. (Note the extra factor of $\chi^{\text{cyclo}}(\gamma)$, which appears because $A_F^{\psi=1}$ is twisted by 1.)

□

4.1.3 The module $\mathcal{O}_F[\![\pi]\!]^{\psi=0}$

Recall that we wish to find a generator of $\left(\mathcal{P}_{F,\log}^{\psi=p^{-1}}\right)_{\text{tf}}$. We will show that there is an easily computable generator of $\mathcal{O}_F[\![\pi]\!]^{\psi=0}$, and then relate this to $\left(\mathcal{P}_{F,\log}^{\psi=p^{-1}}\right)_{\text{tf}}$. Much of this subsection is

drawn from [PR90] sections 1 and 2.

Proposition 4.1.8. *Let $\xi \in \mathcal{O}_F^\times$ such that $\Sigma\xi$ is a basis for $\mathcal{O}_F$ over $\mathbb{Z}_p$. Then $\mathcal{O}_F[\![\pi]\!]^{\psi=0}$ is a free Λ_F-module of rank 1 generated by $\xi(1+\pi)$.*

Proof. This is Lemma 1.5 in [PR90]. □

Now that we have this basis, we wish to find a useful map $\mathcal{P}_{F,\log}^{\psi=p^{-1}} \to \mathcal{O}_F[\![\pi]\!]$. We first need a lemma:

Lemma 4.1.9. *1. Let $f(\pi) \in \pi^2\mathcal{O}_F[\![\pi]\!]$. Then*

$$g(\pi) = \sum_{n\geq 0} p^{-n}\phi^n f$$

converges in $\mathcal{P}_F$, and we have $(1-\phi/p)g = f$. Further, $\psi(g) = g/p$ if and only if $\psi(h) = 0$.

2. Let $f(\pi) \in \pi\mathcal{O}_F[\![\pi]\!]$. Then

$$g(\pi) = \sum_{n\geq 0} \phi^n f$$

converges in $\mathcal{O}_F[\![\pi]\!]$, and we have $(1-\phi)g = f$. Further, $\psi(g) = g$ if and only if $\psi(h) = 0$.

Proof. The first part is Lemma 2.4 in [PR90]. The second part is similar. □

Proposition 4.1.10. *There is a short exact sequence*

$$0 \to \mathbb{Z}_p \log(1+\pi) \to \mathcal{P}_{F,\log}^{\psi=p^{-1}} \xrightarrow{1-\phi/p} \mathcal{O}_F[\![\pi]\!]^{\psi=0} \to \mathbb{Z}_p(1) \to 0.$$

Remark 4.1.11. This is Theorem 2.3' in [PR90], and is a generalization of an earlier result in [Col79]. Note also (as shown in the proof of Lemma 4.1.7) that $\mathbb{Z}_p \log(1+\pi) \cong \mathbb{Z}_p(1)$ as Λ_F-modules.

In fact, $\mathbb{Z}_P \log(1+\pi) = \left(\mathcal{P}_{F,\log}^{\psi=p^{-1}}\right)_{\text{tors}}$, and thus we have a short exact sequence

$$0 \to \left(\mathcal{P}_{F,\log}^{\psi=p^{-1}}\right)_{\text{tf}} \xrightarrow{1-\phi/p} \mathcal{O}_F[\![\pi]\!]^{\psi=0} \to \mathbb{Z}_p(1) \to 0.$$

Proof. Recall that $\mathcal{P}_{F,\log} = \{f \in \mathcal{P}_F : (p-\phi)f \in p\mathcal{O}_F[\![\pi]\!]\}$. So it's clear that $(1-\phi/p)(\mathcal{P}_{F,\log}) \subset \mathcal{O}_F[\![\pi]\!]$. Then $\psi(1-\phi/p)f = \psi f - f/p = 0$ if and only if $\psi f = f/p$, and thus $(1-\phi/p)(\mathcal{P}_{F,\log}^{\psi=p^{-1}}) \subset \mathcal{O}_F[\![\pi]\!]^{\psi=0}$.

It's clear that $\mathbb{Z}_p \cdot \log(1+\pi)$ is in the kernel of $(1-\phi/p)$, since $\phi(\log(1+\pi)) = p\log(1+\pi)$. Let $f \in \mathcal{P}_{F,\log}^{\psi=p^{-1}}$, and suppose $(1-\phi/p)(f) = 0$, i.e. that $\phi f = pf$. Suppose further that $f(\pi) = \sum_{n\geq k} a_n\pi^n$ with $k \geq 2$ and $a_k \neq 0$. Then since $\phi f = pf$, we have that

$$pf(\pi) = \sum_{n\geq k} \sigma(a_n)((1+\pi)^p - 1)^n$$

and thus in particular $pa_k = p^k\sigma(a_k)$, which is false for any $a_k \in F$.

Now suppose $f = \sum_{n\geq 0} a_n\pi^n$ is any element of $\ker(1-\phi/p)$. We see that $a_0 = 0$ since $\psi(a_0) = \sigma^{-1}(a_0) \neq a_0/p$, and so we must have $a_1 \neq 0$. Further, the same argument shows that $a_1 = \sigma(a_1)$, and thus $a_1 \in \mathbb{Z}_p$. Finally, $f - a_1\log(1+\pi) \in \ker(1-\phi/p) \cap \pi^2\mathcal{P}_{F,\log}^{\psi=p^{-1}}$, and we've already shown that this intersection is zero, so $f = a_1\log(1+\pi)$ for some $a_1 \in \mathbb{Z}_p$.

Finally, we compute the cokernel of $1-\phi/p$. Let $h(\pi) = h_0 + h_1\pi + h_2(\pi)$ where $h_2(\pi) \in \pi^2\mathcal{O}_F[\![\pi]\!]^{\psi=0}$. Suppose there is a $g \in \mathcal{P}_{F,\log}^{\psi=p^{-1}}$ such that $(1-\phi/p)g = h$, and set $g = g_0 + g_1\pi + g_2(\pi)$ for $g_2 \in \pi^2\mathcal{P}_{F,\log}^{\psi=p^{-1}}$. Then we must have

$$g_0 - p^{-1}\sigma(g_0) = h_0, \tag{4.1}$$

$$g_1 - \sigma(g_1) = h_1. \tag{4.2}$$

Note that if these conditions are satisfied, then $(1-\phi/p)(g_0+\pi g_1) \in \pi^2\mathcal{O}_F[\![\pi]\!]^{\psi=0}$ and thus has a preimage in $\mathcal{P}_{F,\log}^{\psi=p^{-1}}$ (by lemma 4.1.9); thus (4.1) and (4.2) are both necessary and sufficient for h to lie in the image of $(1-\phi/p)$.

Equation (4.1) is satisfied if we set $g_0 = \sum_{n\geq 1} p^n\sigma^{-n}(h_0)$, so $h \in \mathrm{Im}(1-\phi/p)$ if and only if there is some g_1 such that $g_1 - \sigma(g_1) = h_1$. Thus the cokernel of $(1-\phi/p)$ is precisely $(\mathcal{O}_F/(\sigma-1)\mathcal{O}_F)\pi \cong \mathbb{Z}_p\cdot\pi$. Finally, we see that

$$\begin{aligned}\gamma\cdot\sum a_i\pi^i &= \sum a_i((1+\pi)^{\chi^{\mathrm{cyclo}}(\gamma)} - 1)^i \\ &= \chi^{\mathrm{cyclo}}(\gamma)a_1\pi + \left(a_1\binom{\chi^{\mathrm{cyclo}}(\gamma)}{2} + a_2\chi^{\mathrm{cyclo}}(\gamma)^2\right)\pi^2 + \dots,\end{aligned}$$

so γ acts on the cokernel by $\gamma\cdot(a_1\pi) = \chi^{\mathrm{cyclo}}(\gamma)a_1\pi$. Thus our cokernel is isomorphic to $\mathbb{Z}_p(1)$ as a Γ-module. □

In particular, the cokernel of $(1-\phi/p)$ is $(\Lambda_F/(\sigma-1, \gamma-\chi^{\mathrm{cyclo}}(\gamma)))\cdot\xi(1+\pi)$, so the image must be the ideal $(\sigma-1, \gamma-\chi^{\mathrm{cyclo}}(\gamma))\cdot\xi(1+\pi)$. As in proposition 3.2.4, we can show that this ideal is principal:

Proposition 4.1.12. *The ideal $(\sigma-1, \gamma-\chi^{\mathrm{cyclo}}(\gamma))$ of Λ_F is principal, and generated by the element $(1-e_1) + (\gamma-\chi^{\mathrm{cyclo}}(\gamma))e_1$.*

Proof. See proof of 3.2.4. □

4.2 The map $A_F^{\psi=1} \to F$

In this section, we will compute $\exp^*_{\mathbb{Q}_p(r)}(H^1(F, \mathbb{Z}_p(r)))$ explicitly, and use this explicit computation to prove Conjecture 1.2.3 for F.

Recall the reciprocity law of Cherbonnier and Colmez for $\mathbb{Q}_p(1)$ (equation (2.3)): for large enough n, we have that

$$T_0\phi^{-n}\operatorname{Exp}^*_{\mathbb{Q}_p}(\mu) = \sum_{r\geq 1} \exp^*_{\mathbb{Q}_p(1-r)} pr_{0,r}(\mu)\cdot t^{r-1},$$

and thus in particular that

$$\frac{d^{r-1}}{dt^{r-1}} T_0\phi^{-n}\operatorname{Exp}^*_{\mathbb{Q}_p}(\mu)|_{t=0} = (r-1)!\exp^*_{\mathbb{Q}_p(r)}(pr_{0,r}(\mu)) \in F. \tag{4.3}$$

In particular, conjecture 1.2.3 asks about $\exp^*_{\mathbb{Q}_p(1-r)}\beta$ for some $\beta \in H^1(F, \mathbb{Z}_p(r))$ such that $\mathbb{Z}_p[\Sigma]\cdot\beta = H^1(F, \mathbb{Z}_p(r))_{\mathrm{tf}}$. Since the element $\alpha \in A_F^{\psi=1}(1)$ given in Lemma 4.1.7 generates $A_F^{\psi=1}(1)_{\mathrm{tf}}$, then it's enough to compute the image of α under $\frac{d^{r-1}}{dt^{r-1}}T_0\phi^{-n}$.

Lemma 4.2.1. *Let α be a generator of $A_F^{\psi=1}(1)_{\mathrm{tf}}$, as in Lemma 4.1.7. Then*

$$\frac{d^{r-1}}{dt^{r-1}} T_0\phi^{-n}\alpha = (1-p^{-r}\sigma^{-1})(\nabla^{r-1}\alpha)(0).$$

Proof. Recall from subsection 3.1.2 that

$$\frac{d^{r-1}}{dt^{r-1}} T_0\phi^{-n}\alpha = p^{(-n)(r-1)} T_0\phi^{-n}\nabla^{r-1}\alpha.$$

Further, since $\alpha \in A_K^{\psi=1}(1)_{\mathrm{tf}}$, we have that $\psi\nabla^{r-1}\alpha = p^{r-1}\nabla^{r-1}\psi\alpha = p^{r-1}\nabla^{r-1}\alpha$, and thus $\nabla^{r-1}\alpha \in A_K^{\psi=p^{r-1}}$. Thus from Corollary 3.1.5, we have

$$\begin{aligned}\frac{d^{r-1}}{dt^{r-1}} T_0\phi^{-n}\alpha\,|_{t=0} &= p^{(-n)(r-1)}\cdot p^{(r-1)n}(1-p^{-r}\sigma^{-1})(\nabla^{r-1}\alpha)(0)\\ &= (1-p^{-r}\sigma^{-1})(\nabla^{r-1}\alpha)(0).\end{aligned}$$

□

4.2.1 Computing ∇ on $A_F^{\psi=1}$

We've now reduced our problem to computing$(\phi^{-m}\nabla^{r-1}\alpha)(0)$. For convenience, we set

$$\begin{aligned}\rho_r &= ((1-e_1) + (\chi^{\mathrm{cyclo}}(\gamma)^r\gamma - 1)e_1),\\ \tau_r &= ((1-e_1) + (\chi^{\mathrm{cyclo}}(\gamma)^r\gamma - \chi^{\mathrm{cyclo}}(\gamma))e_1).\end{aligned}$$

From Lemma 3.1.3 we can see that $\nabla\rho_r = \rho_{r+1}\nabla$ and $\nabla\tau_r = \tau_{r+1}\nabla$. Then we may compute:

Lemma 4.2.2. *We have*

$$(\nabla^{r-1}\alpha)(0) = (1-p^{r-1}\sigma)^{-1}\frac{\tau_r|_{\gamma=1}}{\rho_r|_{\gamma=1}}\cdot\xi.$$

Proof. Recall that

$$\begin{aligned}\nabla\left(\mathcal{P}_F^{\psi=p^{-1}}\right)_{\mathrm{tf}} &= \Lambda_F\cdot((1-e_1)+(\chi^{\mathrm{cyclo}}(\gamma)\gamma-1)e_1)\alpha\\ &= \Lambda_F\cdot\rho_1\cdot\alpha\subset A_F^{\psi=1}(1)\end{aligned}$$

and

$$\begin{aligned}(1-\phi/p)\left(\mathcal{P}_F^{\psi=p^{-1}}\right)_{\mathrm{tf}} &= \Lambda_F\cdot((1-e_1)+(\gamma-\chi^{\mathrm{cyclo}}(\gamma))e_1)\xi(1+\pi)\\ &= \Lambda_F\cdot\tau_0\cdot\xi(1+\pi)\subset\mathcal{O}_F[\![\pi]\!]^{\psi=0}.\end{aligned}$$

By Lemma 3.1.3, we know that $(1-\phi)\nabla = \nabla(1-\phi/p)$ on $\mathcal{P}_F^{\psi=p^{-1}}$, and thus

$$\Lambda_F\cdot\tau_0\xi(1+\pi) = \Lambda_F\cdot(1-\phi)\rho_1\alpha.$$

We can then pick our generator α so that

$$\nabla(\tau_0\xi(1+\pi)) = (1-\phi)\rho_1\alpha. \tag{4.4}$$

First we look at the left-hand side of (4.4). Applying ∇^{r-1} and evaluating at $\pi=0$ gives us:

$$\begin{aligned}&\left(\nabla^r(\tau_0\xi(1+\pi))\right)|_{\pi=0}\\ =&\left(\tau_r\cdot\xi(1+\pi)\right)|_{\pi=0}\\ =&\tau_r|_{\gamma=1}\cdot\xi.\end{aligned}$$

Applying the same calculation to the right-hand side of (4.4) gives us

$$\begin{aligned}&\left(\nabla^{r-1}(1-\phi)\rho_1\alpha\right)|_{\pi=0}\\ =&\left((1-p^{r-1}\phi)\rho_r\nabla^{r-1}\alpha\right)|_{\pi=0}\\ =&\rho_r|_{\gamma=1}\left((\nabla^{r-1}\alpha)(0)-p^{r-1}(\phi\nabla\alpha)(0)\right)\\ =&\,\rho_r|_{\gamma=1}(1-p^{r-1}\sigma)(\nabla^{r-1}\alpha)(0).\end{aligned}$$

Since $(\chi^{\mathrm{cyclo}}(\gamma)^r - 1) \in F^\times$, it's easy to see that $\rho_r|_{\gamma=1} \in F[\Sigma]^\times$, and thus we have

$$\tau_r|_{\gamma=1} \cdot \xi = \rho_r|_{\gamma=1}(1 - p^{r-1}\sigma)(\nabla^{r-1}\alpha)(0)$$
$$(\nabla^{r-1}\alpha)(0) = \frac{\tau_r|_{\gamma=1}}{\rho_r|_{\gamma=1}}(1 - p^{r-1}\sigma)^{-1} \cdot \xi.$$

□

Lemma 4.2.2 and corollary 3.1.5 then combine to give us the explicit computation we wanted:

Proposition 4.2.3. *We have*

$$\left(\frac{d^{r-1}}{dt^{r-1}} T_0 \phi^{-n} \alpha\right)(0) = \frac{1 - p^{-r}\sigma^{-1}}{1 - p^{r-1}\sigma} \frac{\rho_r|_{\gamma=1}}{\tau_r|_{\gamma=1}} \xi. \tag{4.5}$$

Proof. This follows from Lemmas 4.2.1 and 4.2.2. □

4.2.2 Proof of the conjecture for unramified extensions

We are now ready to prove Conjecture 1.2.3 in the case where $K = F$ is ramified and $G = \Sigma$. We recall that the reciprocity law of equation (2.3) gives us the following commutative diagram:

$$\begin{array}{ccccc}
H^1_{Iw}(F, \mathbb{Z}_p(1))_{\mathrm{tf}} & \xrightarrow{i} & H^1_{Iw}(F, \mathbb{Q}_p(1)) & \xrightarrow{\bigoplus_{r\in\mathbb{Z}} pr_{0,r}} & \bigoplus_{r\in\mathbb{Z}} H^1(F, \mathbb{Q}_p(r)) \\
\cong \downarrow \mathrm{Exp}^*_{\mathbb{Z}_p} & & \cong \downarrow \mathrm{Exp}^*_{\mathbb{Q}_p} & & \cong \downarrow \sum_{r\in\mathbb{Z}} \exp^*_{\mathbb{Q}_p(1-r)} \otimes t^{r-1} \\
A_F^{\psi=1}(1)_{\mathrm{tf}} & \longrightarrow & B_F^{\psi=1}(1) & \xrightarrow{T_0\phi^{-n}} & F((t))
\end{array}$$

Since we are studying $H^1(F, \mathbb{Q}_p(r))$, by Theorem 2.2.2 we can take only the rth projection on the top line; this corresponds to taking the coefficient of t^{r-1} on the bottom line, or in other words applying the map $\frac{1}{(r-1)!}\frac{d^{r-1}}{dt^{r-1}}|_{t=0}$. Thus we have a commutative diagram:

$$\begin{array}{ccccc}
H^1_{Iw}(F, \mathbb{Z}_p(1))_{\mathrm{tf}} & \xrightarrow{i} & H^1_{Iw}(F, \mathbb{Q}_p(1)) & \xrightarrow{pr_{0,r}} & H^1(F, \mathbb{Q}_p(r)) \\
\cong \downarrow \mathrm{Exp}^*_{\mathbb{Z}_p} & & \cong \downarrow \mathrm{Exp}^*_{\mathbb{Q}_p} & & \cong \downarrow \exp^*_{\mathbb{Q}_p(1-r)} \\
A_F^{\psi=1}(1)_{\mathrm{tf}} & \longrightarrow & B_F^{\psi=1}(1) & \xrightarrow{\frac{d^{r-1}}{dt^{r-1}} \frac{T_0\phi^{-n}}{(r-1)!}|_{t=0}} & F
\end{array}$$

Therefore, we see that

$$\exp^*_{\mathbb{Q}_p(1-r)}(H^1(F, \mathbb{Z}_p(r)))_{\mathrm{tf}} = \left(\frac{d^{r-1}}{dt^{r-1}} \frac{T_0\phi^{-n}}{(r-1)!}|_{t=0}\right)(A_F^{\psi=1}(1))_{\mathrm{tf}}.$$

But $A_F^{\psi=1}(1)_{\mathrm{tf}}$ is generated by the element α, so there is some $\beta \in H^1(F, \mathbb{Z}_p(r))_{\mathrm{tf}}$ such that $\mathbb{Z}_p[\Sigma]\cdot\beta =$

$H^1(F, \mathbb{Z}_p(r))_{\mathrm{tf}}$, and by proposition 4.2.3 we see that

$$\begin{aligned}\exp^*_{\mathbb{Q}_p(1-r)} \beta &= \left(\frac{d^{r-1}}{dt^{r-1}} \frac{T_0 \phi^{-n}}{(r-1)!} \alpha|_{t=0} \right) \\ &= \frac{1 - p^{-r}\sigma^{-1}}{(r-1)!(1-p^{r-1}\sigma)} \frac{\rho_r|_{\gamma=1}}{\tau_r|_{\gamma=1}} \xi.\end{aligned}$$

So let $\chi \in \hat{\Sigma}$. Since χ is unramified, we have $c(\chi) = 0$; and since $e = 1$, we have that the inverse different $(\sqrt[e]{p_0})^{1-e}\mathcal{O}_F = \mathcal{O}_F$, and thus we can take $b = \xi$ to be a basis for the inverse different. Recall Σ is abelian, and thus $\det \rho_\chi = \rho_\chi$. Then equation (1.4) reduces to:

$$(r-1)! \cdot 1 \cdot \frac{\rho_\chi(\mathrm{per}(\exp^*(\beta) \otimes 1))}{\rho_\chi(\mathrm{per}(\xi \otimes 1))} \cdot \rho_\chi(C_\beta) \cdot \frac{1 - p^{r-1}\rho_\chi(\sigma)}{1 - p^{-r}\rho_\chi(\sigma)^{-1}} \in \mathbb{Z}_p^{ur,\times}. \tag{4.6}$$

We have that $\mathrm{per}(\xi \otimes 1) = \sum_{\tau \in \Sigma} \tau(\xi)\tau^{-1}$, and thus $\rho_\chi \,\mathrm{per}(\xi \otimes 1) = \sum_{\tau \in \Sigma} \tau(\xi)\rho_\chi(\tau^{-1}) = e_\chi \cdot \xi$. Similarly,

$$\begin{aligned}\rho_\chi \,\mathrm{per}(\exp^*(\beta) \otimes 1) &= \rho_\chi \,\mathrm{per}\left(\frac{1 - p^{-r}\sigma^{-1}}{(r-1)!(1-p^{r-1}\sigma)} \cdot \rho_\chi \left(\frac{\rho_r|_{\gamma=1}}{\tau_r|_{\gamma=1}} \right) \xi \otimes 1 \right) \\ &= \rho_\chi \left(\sum_{\tau \in \Sigma} \frac{1 - p^{-r}\sigma^{-1}}{(r-1)!(1-p^{r-1}\sigma)} \cdot \rho_\chi \left(\frac{\rho_r|_{\gamma=1}}{\tau_r|_{\gamma=1}} \right) \tau(\xi) \cdot \tau^{-1} \right) \\ &= \sum_{\tau \in \Sigma} \frac{1 - p^{-r}\rho_\chi(\sigma^{-1})}{(r-1)!(1-p^{r-1}\rho_\chi(\sigma))} \cdot \rho_\chi \left(\frac{\rho_r|_{\gamma=1}}{\tau_r|_{\gamma=1}} \right) \tau(\xi) \cdot \rho_\chi(\tau)^{-1} \\ &= \left(\frac{1 - p^{-r}\rho_\chi(\sigma^{-1})}{(r-1)!(1-p^{r-1}\rho_\chi(\sigma))} \cdot \rho_\chi \left(\frac{\rho_r|_{\gamma=1}}{\tau_r|_{\gamma=1}} \right) \right) \sum_{\tau \in \Sigma} \tau(\xi) \cdot \rho_\chi(\tau)^{-1} \\ &= \left(\frac{1 - p^{-r}\rho_\chi(\sigma^{-1})}{(r-1)!(1-p^{r-1}\rho_\chi(\sigma))} \cdot \rho_\chi \left(\frac{\rho_r|_{\gamma=1}}{\tau_r|_{\gamma=1}} \right) \right) e_\chi \cdot \xi.\end{aligned}$$

Since $C_\beta = \frac{\rho_r|_{\gamma=1}}{\tau_r|_{\gamma=1}}$, equation (4.6) reduces to the statement that $1 \in \mathbb{Z}_p^{ur,\times}$, which is of course true. Thus Conjecture 1.2.3 is proven in the case where $K = F$ is unramified.

Chapter 5

The Tamagawa number conjecture for Tate motives over tamely ramified extensions $K/\mathbb{Q}_p$

In this chapter, we will study Conjecture 1.2.3 for tamely ramified extensions $K/\mathbb{Q}_p$. Though we do not generate a complete answer, we show some work towards the conjecture and compute formulas for various actions of G and Γ_K.

5.1 Finding a generator for $A_K^{\psi=1}(1)$

The goal of this section is to find a generator for the free part of $A_K^{\psi=1}(1)$. We begin with a commutative diagram:

$$\begin{array}{ccccc}
 & & \widehat{A_K^{\mathrm{N}=1}} & \xrightarrow{\nabla\log} & A_K^{\psi=1}(1) \\
 & & \cong\downarrow \mod p & & \downarrow \mod p \\
A(K_\infty)=\varprojlim_{m,n} K_n^\times/(K_n^\times)^{p^m} & \xleftarrow[\cong]{} & \widehat{E_K^\times} & \xrightarrow{\nabla\log} & \widetilde{V} \\
\uparrow & & \uparrow & & \uparrow \\
\widehat{U}=\varprojlim_{m,n} \mathcal{O}_{K_n}^\times/(\mathcal{O}_{K_n}^\times)^{p^m} & \xleftarrow[\cong]{} & 1+\pi_K k[\![\pi_K]\!] & \xrightarrow{\nabla\log} & V
\end{array}$$

Except for the spaces $\widetilde{V}$ and V, this is the diagram of subsection 4.1.2. We define $V = \nabla\log(1+\pi_K k[\![\pi_K]\!])$ and $\widetilde{V} = \nabla\log(\widehat{E_K^\times})$, and thus the diagram commutes. We note that for $x \in \widehat{E_K^\times}$, we have that $\nabla\log(x) = 0$ if and only if $x = y^p$ for some $y \in \widehat{E}_K^\times$. Thus we have $\widetilde{V} \cong \widehat{E_K^\times}\Big/\left(\widehat{E_K^\times}\right)^p$ and $V \cong (1+\pi_K k[\![\pi_K]\!])/(1+\pi_K k[\![\pi_K]\!])^p$.

By Nakayama's Lemma, we can lift a basis of $\widetilde{V}$ to a basis of $A_K^{\psi=1}(1) \bmod p$. Recall that $\widehat{U}_{\mathrm{tf}} = (\sigma-1, \delta_e-1, \chi^{\mathrm{cyclo}}(\gamma)\gamma-1)A(K_\infty)_{\mathrm{tf}}$ by Lemma 3.2.3, and thus equal to $((1-e_1)+(\chi^{\mathrm{cyclo}}(\gamma)\gamma-1)e_1)A(K_\infty)_{\mathrm{tf}}$ by Proposition 3.2.4. Thus $V_{\mathrm{tf}} = ((1-e_1)+(\chi^{\mathrm{cyclo}}(\gamma)\gamma-1)e_1)\widetilde{V}_{\mathrm{tf}}$, so it will be enough

to find a generator of V_{tf}.

5.1.1 The vector space $V/(\gamma_1 - 1)V$

In this section, we will study the quotient $V/(\gamma_1 - 1)V$ as a vector space, and compute a basis. Since $\widehat{E_K^\times} \cong (1 + \pi_K k[\![\pi_K]\!]) \cong \Lambda_K \oplus \mathbb{Z}_p(1)$, we have $V \cong (\Lambda_K \oplus \mathbb{Z}_p(1))/(p) = \mathbb{F}_p[G][\![\Gamma_K]\!] \oplus \mathbb{F}_p$. Thus $V/(\gamma_1 - 1) \cong \mathbb{F}_p[G][\Delta] \oplus \mathbb{F}_p$ is a $ef(p-1)+1$-dimension $\mathbb{F}_p$ vector space. We wish to find a basis, which first requires studying the action of $\gamma_1 - 1$.

Lemma 5.1.1. *1. We can choose γ_1 such that $\gamma_1(\pi) \equiv (\pi + \pi^p + \pi^{p+1}) \mod p$.*

2. Suppose $p \nmid \ell$. Then

$$(\gamma_1 - 1)\pi_K^{p^r \ell} = \frac{\ell}{e}\pi_K^{p^r(\ell + e(p-1))} + \pi_K^{p^r(\ell+e(p-1))+1} f(\pi_K)$$

for some $f \in k[\![T]\!]$.

Proof. 1. γ_1 is just an element such that $\chi^{\mathrm{cyclo}}(\gamma_1)$ is a multiplicative generator of $1 + p\mathbb{Z}_p$, so we can choose that $\chi^{\mathrm{cyclo}}(\gamma_1) = 1 + p$. Then

$$\begin{aligned}
\gamma_1(\pi) &= (1+\pi)^{\chi^{\mathrm{cyclo}}(\gamma_1)} - 1 \\
&= (1+\pi)^{1+p} - 1 \\
&\equiv (1+\pi)(1+\pi^p) - 1 \mod p \\
&\equiv \pi + \pi^p + \pi^{p+1} \mod p.
\end{aligned}$$

2. We calculate that

$$\begin{aligned}
\gamma_1(\pi_K^{p^r\ell}) &= \gamma_1(\pi^{p^r\ell})^{1/e} \\
&= (\pi + \pi^p + \pi^{p+1})^{p^r\ell/e} \\
&= (\pi^{p^r} + \pi^{p^{r+1}} + \pi^{(p^{r+1}+p^r)})^{\ell/e} \\
&= \pi_K^{p^r\ell}(1 + \pi^{p^r(p-1)} + \pi^{p^{r+1}})^{\ell/e} \\
&= \pi_K^{p^r\ell}(1 + \tfrac{\ell}{e}\pi^{p^r(p-1)} + \pi^{p^r(p-1)+1} f(\pi_K))
\end{aligned}$$

for some $f \in k[\![T]\!]$. Thus

$$\begin{aligned}
(\gamma_1 - 1)(\pi_K^{p^r\cdot\ell}) &= \tfrac{\ell}{e}\pi_K^{p^r\ell}\pi^{p^r(p-1)} + \pi_K^{p^r\ell}\pi^{p^r(p-1)+1} f(\pi) \\
&= \pi_K^{p^r(\ell+e(p-1))} + \pi_K^{p^r(\ell+e(p-1))+1} f(\pi_K).
\end{aligned}$$

□

Now we are prepared to produce a basis for V.

Lemma 5.1.2. *The set*

$$B' = \{1 + \tau(\xi)\pi_K^\ell : \tau \in \Sigma, p \nmid \ell \geq 1\}$$

is a topological basis for $1 + \pi_K k[\![\pi_K]\!]$ *as a multiplicative* $\mathbb{Z}_p$*-module.*

Proof. Let $f = 1 + \sum_{m>0} a_m \pi_K^m \in 1 + \pi_K k[\![\pi_K]\!]$. For any $n > 0$, we will find a finite combination

$$g_n = \prod_{\tau\in\Sigma, 1\leq i\leq n} (1 + \tau(\xi)\pi_K^i)^{i_j}$$

such that $g_n \equiv f \mod \pi_K^n$, which is enough to prove that f is in the closure of the span of B'.

We proceed by induction. For $n = 0$, it is clear that $g_0 = 1 + \xi\pi_K \equiv f \mod \pi_K$.

Now let $n > 0$. By induction, we have some g_{n-1} such that $g_{n-1} \equiv f \mod \pi_K^n$. So suppose $f - g_{n-1} \equiv b_n \pi_K^n \mod \pi_K^{n+1}$, where $b_n \in k$. Then we can write $b_n = \sum_{\tau\in\Sigma} b_{n,\tau}\tau(\xi)$ for $b_{n,\tau} \in \mathbb{F}_p$, since $\Sigma \cdot \xi$ spans k over $\mathbb{F}_p$. If we take

$$\begin{aligned} g_n &= g_{n-1} \cdot \prod_{\tau\in\Sigma} (1 + a_{n,\tau}\tau(\xi)\pi_K^n) \\ &= g_{n-1} \cdot \prod_{\tau\in\Sigma} (1 + \tau(\xi)\pi_K^n)^{a_{n,\tau}} \\ &\equiv g_{n-1}(1 + b_n\pi_K^n) \mod \pi_K^{n+1}, \end{aligned}$$

we have

$$g_{n-1}(1 + b_n\pi_K^n) \mod \pi_K^{n+1} = g_{n-1} + b_n\pi_K^n + \pi_K^{n+1} \cdot h$$

for some $h \in k[\![\pi_K]\!]$ and thus $g_n \equiv f \mod \pi_K^{n+1}$, as desired. □

Thus the image of B' under $\nabla \log$ is a topological $\mathbb{F}_p$-basis for V. We can easily compute this image:

Lemma 5.1.3. $\nabla \log(1 + \tau(\xi)\pi_K^\ell) = \frac{\tau(\xi)\ell}{e} \frac{1+\pi}{1+\tau(\xi)\pi_K^\ell} \pi_K^{\ell-e}$.

Proof.

$$\begin{aligned}\nabla \log(1+\tau(\xi)\pi_K^\ell) &= \frac{\nabla\tau(\xi)\pi_K^\ell}{1+\tau(\xi)\pi_K^\ell} \\ &= \frac{\tau(\xi)}{1+\tau(\xi)\pi_K^\ell}\ell\pi_K^{\ell-1}\nabla\pi_K \\ &= \tau(\xi)\ell\frac{\pi_K^{\ell-1}}{1+\tau(\xi)\pi_K^\ell}\frac{(1+\pi)\pi_K}{e\pi} \\ &= \frac{\tau(\xi)\ell}{e}\frac{1+\pi}{1\tau(\xi)\pi_K^\ell}\pi_K^{\ell-e}\end{aligned}$$

□

Corollary 5.1.4. *The space V has a $\mathbb{F}_p$-basis*

$$B = \left\{ \frac{\tau(\xi)\ell}{e}\frac{1+\pi}{1+\tau(\xi)\pi_K^\ell}\pi_K^{\ell-e} : \tau\in\Sigma, p\nmid\ell\geq 0 \right\}.$$

We can now find a basis for the finite-dimensional vector space $V/(\gamma_1-1)V$:

Lemma 5.1.5. *$V/(\gamma_1-1)V \cong \mathbb{F}_p[G][\Delta]\oplus\mathbb{F}_p$ is an $ef(p-1)+1$-dimensional $\mathbb{F}_p$-vector space, and there exists some $f'_K\in k[\![T]\!]$ such that $C'\cup D'\cup\{\pi_K^{1+p(e-2)}f'_K(\pi_K\}$ is a basis for $V/(\gamma_1-1)V$, where*

$$C' = \left\{ \frac{\tau(\xi)\ell}{e}\frac{1+\pi}{1+\tau(\xi)\pi_K^\ell}\pi_K^{\ell-e} : 1\leq\ell\leq e(p-1), p\nmid\ell, \tau\in\Sigma \right\}$$

and

$$D' = \left\{ \frac{\tau(\xi)(\ell+e(p-1))}{e}\frac{1+\pi}{1+\tau(\xi)\pi_K^{\ell+e(p-1)}}\pi_K^{\ell+e(p-2)} : 1\leq\ell\leq e(p-1), p\mid\ell, \tau\in\Sigma \right\}$$

Proof. We wish to show that the elements of $C'\cup D'$ are linearly independent. We first will show that all the elements are non-zero—that is, not in the image of γ_1-1. But we know that $(\gamma_1-1)V\subset(\pi_K^{1+e(p-1)})V$ by Lemma 5.1.1, so no element of C' is in the image of (γ_1-1). Similarly, for each $g(\pi_K)\in D'$ we have that the lowest degree term of g is $\pi_K^{\ell+e(p-2)}$, and $\ell+e(p-1)$ is not divisible by p since p divides ℓ. Thus if $(\gamma_1-1)h(\pi_K)=g(\pi_K)$, we must have that the lowest degree of $h(\pi_K)$ is $\pi_K^{\ell-e}$; but there is no such element of V, since $\nabla\log(1+\pi_K^\ell)=0$ if p divides ℓ; in fact, C' spans the set of all elements of V with minimal degree $\leq e(p-2)$, and C' is linearly dependent. Thus $D'\cap(\gamma_1-1)V=\varnothing$.

Now suppose there are power series $g_{\tau,n} \in \mathbb{F}_p[\![T]\!]$ such that

$$0 = \sum_{\tau\in\Sigma, 1\le\ell\le e(p-1), p\nmid\ell} g_{\tau,\ell-e}(\gamma_1-1)\cdot\frac{\tau(\xi)\ell}{e}\frac{1+\pi}{1+\tau(\xi)\pi_K^{\ell}}\pi_K^{\ell-e}$$
$$+\sum_{\tau\in\Sigma, 1\le\ell\le e(p-1), p\mid\ell} g_{\tau,\ell+e(p-2)}(\gamma_1-1)\frac{\tau(\xi)(\ell+e(p-1))}{e}\frac{1+\pi}{1+\tau(\xi)\pi_K^{\ell+e(p-1)}}\pi_K^{\ell+e(p-2)}$$

Then since the set $\Sigma\xi$ is linearly independent, we can induct on ℓ and check the coefficient of $\pi_K^{\ell-e}$ to determine that $g_{\tau,\ell-e}(0) = 0$ for $\tau \in \Sigma, 1 \le \ell \le e(p-1), p \nmid \ell$. Now checking the coefficient for $\pi_K^{\ell+e(p-2)}$ tells us that $g_{\tau,\ell+e(p-2)}(0) = 0$ for $\tau \in \Sigma, 1 \le \ell \le e(p-1), p \mid \ell$.

Thus we can factor a $\gamma_1 - 1$ out of all the $g_{\tau,i}$, and we have a collection of $h_{\tau,i}$ such that

$$0 =(\gamma_1-1)\left(\sum_{\tau\in\Sigma, 1\le\ell\le e(p-1), p\nmid\ell}\frac{\tau(\xi)\ell}{e}h_{\tau,\ell-e}(\gamma_1-1)\frac{1+\pi}{1+\tau(\xi)\pi_K^{\ell}}\pi_K^{\ell-e}\right.$$
$$\left.+\sum_{\tau\in\Sigma, 1\le\ell\le e(p-1), p\mid\ell} h_{\tau,\ell+e(p-2)}(\gamma_1-1)\frac{\tau(\xi)(\ell+e(p-1))}{e}\frac{1+\pi}{1+\tau(\xi)\pi_K^{\ell+e(p-1)}}\pi_K^{\ell+e(p-2)}\right)$$

and thus our sum is in the ideal generated by $(\gamma_1 - 1)$ as desired.

Thus there is some basis of V containing $C' \cup D'$; take some $f'_K \in V$ such that $C' \cup \{f'_K\}$ to a basis for $V/(\gamma_1 - 1)V$. Since linear combinations of the elements of C' can produce elements with arbitrary coefficients of π_K in degrees $\ell - e$ for $1 \le \ell \le e(p-1), p \nmid \ell$, and since no element of V may have minimal degree $\ell - e$ for $\ell \le e(p-1), p \mid \ell$, we may assume that f'_K has no non-zero coefficients in degrees less than $e(p-2)+1$.

□

5.1.2 $V/(\gamma_1 - 1)V$ as a $\mathbb{F}_p[G][\Delta]$-module

The previous section gives us a basis of $V/(\gamma_1 - 1)V$ as a vector space, but we want to find a basis as a $\mathbb{F}_p[G][\Delta]$-module. If we have such a basis, we can use Nakayama's lemma to lift it to a basis of V as a $\mathbb{F}_p[G][\Delta][\![\gamma_1 - 1]\!] = \mathbb{F}_p[G][\![\Gamma_K]\!]$-module. But establishing this basis will require some computation.

Recalling our discussion of character theory from subsection 1.2.3, we let $\omega_i : \Delta \to \overline{\mathbb{Q}_p}^\times$ be the simple characters of Δ. Then we set

$$\varepsilon_i = \frac{1}{p-1}\sum_{\delta\in\Delta}\omega_i(\delta^{-1})\delta$$

to be the orthogonal idempotents of $\overline{\mathbb{Q}_p}[\Delta]$. In fact, we can fix a generator δ of Δ, and $\omega_i(\delta)$ is a $p-1$st root of 1 for each i, and thus $\omega_i(\delta) \in \mathbb{Q}_p$. Fix some primitive $p-1$st root of unity $\omega(\delta)$, and

we have

$$\varepsilon_i = \frac{1}{p-1}\sum_{j=0}^{p-2}\omega(\delta)^{-ij}\delta^j.$$

Lemma 5.1.6. *Fix some integer ℓ with $1 \le \ell \le e$. Then we have*

$$(p-1)\varepsilon_i(\pi_K^\ell) = \sum_{r\ge 0}\left(\sum_{m=0}^{r}\binom{\ell/e}{r-m}\sum_{j=0}^{p-2} d_{m,r-m,j}\omega(\delta)^{j\ell/e-ij}\right)\pi_K^\ell\pi^r$$

where $d_{0,s,j} = (\omega(\delta)^j - 1)^s/2^s$ and

$$d_{m,s,j} = \frac{2}{m(\omega(\delta)^j-1)}\sum_{k=1}^{m}(ks-m+k)\binom{\omega(\delta)^j-1}{k+1}\frac{d_{m-k,s}}{k+2}.$$

Proof. We compute:

$$\begin{aligned}
(p-1)\varepsilon_i(\pi_K) &= \sum_{j=0}^{p-2}\omega(\delta)^{-ij}\delta^j(\pi_K^\ell)\\
&= \sum_{j=0}^{p-2}\omega(\delta)^{-ij}((1+\pi)^{\omega(\delta)^j}-1)^{\ell/e}\\
&= \sum_{j=0}^{p-2}\omega(\delta)^{-ij}\left(\sum_{r\ge 1}\binom{\omega(\delta)^j}{r}\pi^r\right)^{\ell/e}\\
&= \sum_{j=0}^{p-2}\omega(\delta)^{-ij}\left(\pi\omega(\delta)^j\right)^{\ell/e}\left(\sum_{r\ge 0}\binom{\omega(\delta)^j}{r+1}\omega(\delta)^{-j}\pi^r\right)^{\ell/e}\\
&= \pi_K^\ell\sum_{j=0}^{p-2}\omega(\delta)^{j\ell/e-ij}\left(1+\sum_{r\ge 1}\binom{\omega(\delta)^j-1}{r}\frac{\pi^r}{r+1}\right)^{\ell/e}\\
&= \pi_K^\ell\sum_{j=0}^{p-2}\omega(\delta)^{j\ell/e-ij}\left(\sum_{s\ge 0}\binom{\ell/e}{s}\left(\sum_{r\ge 1}\binom{\omega(\delta)^j-1}{r}\frac{\pi^r}{r+1}\right)^s\right)
\end{aligned}$$

Set

$$b_{r-1,j} = \binom{\omega(\delta)^j-1}{r}\frac{1}{r+1}.$$

Then we have

$$\begin{aligned}
\left(\sum_{r\ge 1} b_{r-1,j}\pi^r\right)^s &= \pi^s\left(\sum_{r\ge 0} b_{r,j}\pi^r\right)^s\\
&= \pi^s\sum_{m\ge 0} d_{m,s,j}\pi^m
\end{aligned}$$

where $d_{0,s,j} = b_{0,j}^s = (\omega(\delta)^j - 1)^s/2^s$ and

$$d_{m,s,j} = \frac{2}{m(\omega(\delta)^j - 1)} \sum_{k=1}^{m} (ks - m + k) \binom{\omega(\delta)^j - 1}{k+1} \frac{d_{m-k,s}}{k+2}.$$

Thus we have

$$\begin{aligned}
(p-1)\varepsilon_i(\pi_K) &= \pi_K^\ell \sum_{j=0}^{p-2} \omega(\delta)^{j\ell/e - ij} \left(\sum_{s\geq 0} \binom{\ell/e}{s} \left(\sum_{r\geq 1} \binom{\omega(\delta)^j - 1}{r} \frac{\pi^r}{r+1} \right)^s \right) \\
&= \pi_K^\ell \sum_{j=0}^{p-2} \omega(\delta)^{j\ell/e - ij} \left(\sum_{s\geq 0} \binom{\ell/e}{s} \left(\sum_{m\geq 0} d_{m,s,j} \pi^m \right) \pi^s \right) \\
&= \pi_K^\ell \sum_{s\geq 0} \sum_{m\geq 0} \left(\sum_{j=0}^{p-2} \omega(\delta)^{j\ell/e - ij} \binom{\ell/e}{s} d_{m,s,j} \right) \pi^{m+s} \\
&= \pi_K^\ell \sum_{s\geq 0} \sum_{m\geq 0} \left(\binom{\ell/e}{s} d_{m,s,j} \sum_{j=0}^{p-2} \omega(\delta)^{j\ell/e - ij} \right) \pi^{m+s} \\
&= \sum_{r\geq 0} \left(\sum_{m=0}^{r} \binom{\ell/e}{r-m} \sum_{j=0}^{p-2} d_{m,r-m,j} \omega(\delta)^{j\ell/e - ij} \right) \pi_K^\ell \pi^r.
\end{aligned}$$

□

Lemma 5.1.7. *For every $m \geq 1$ and for every j, we have that $d_{m,0,j} = 0$.*

Proof. We prove this by induction. If $m = 1$, then the sum is over one term and $ks - m + k = 0 - 1 + 1 = 0$, and thus $d_{1,0,j} = 0$.

Now suppose $d_{m,0,j} = 0$ for $1 \leq m < n$. Then

$$\begin{aligned}
d_{n,0,j} &= \frac{2}{n(\omega(\delta)^j - 1)} \sum_{k=1}^{n} (k-n) \binom{\omega(\delta)^j - 1}{k+1} \frac{d_{n-k,0}}{k+2} \\
&= \frac{2}{n(\omega(\delta)^j - 1)} \left(\sum_{k=1}^{n-1} (k-n) \binom{\omega(\delta)^j - 1}{k+1} \frac{0}{k+2} + 0 \binom{\omega(\delta)^j - 1}{n+1} \frac{1}{n+2} \right) \\
&= 0
\end{aligned}$$

since $n - k < n$. □

Lemma 5.1.8. *If $p|\ell$, then for each $i \in \mathbb{Z}/(p-1)\mathbb{Z}$, we have*

$$\epsilon_i(\pi_K^\ell) \equiv \frac{\pi_K^\ell}{p-1} \sum_{j=0}^{p-2} \omega(\delta)^{j(\ell/e - i)} \mod p$$

Proof. First, note that if $p \mid \ell$, then $\binom{\ell/e}{s} \equiv 0 \mod p$ if $s > 0$ and $\binom{\ell/e}{0} = 1$. Then we have

$$\begin{aligned}
(p-1)\epsilon_i(\pi_K^\ell) &= \sum_{r\geq 0}\left(\sum_{m=0}^{r}\binom{\ell/e}{r-m}\sum_{j=0}^{p-2} d_{m,r-m,j}\omega(\delta)^{j\ell/e-ij}\right)\pi_K^\ell\pi^r \\
&\equiv \sum_{r\geq 0}\left(\sum_{j=0}^{p-2} d_{r,0,j}\omega(\delta)^{j\ell/e-ij}\right)\pi_K^\ell\pi^r \mod p \\
&\equiv \left(\sum_{j=0}^{p-2}\omega(\delta)^{j\ell/e-ij}\right)\pi_K^\ell + \sum_{r\geq 1}\left(\sum_{j=0}^{p-2} 0\cdot\omega(\delta)^{j\ell/e-ij}\right)\pi_K^\ell\pi^r \mod p \\
&\equiv \pi_K^\ell\sum_{j=0}^{p-2}\omega(\delta)^{j\ell/e-ij} \mod p.
\end{aligned}$$

□

Thus we see that the action of the ϵ_i, while computable, is complicated and not terribly tractable. In principle, however, we should be able to find a basis for $V/(\gamma_1 - 1)V$ as an $\mathbb{F}_p[G][\Delta]$-module, or at least as an $\mathbb{F}_p[G]$-module.

5.2 $T_0\phi^{-n}$ on $A_K^{\psi=1}$

Recall that Conjecture 1.2.3 asks us to study $\exp^*_{\mathbb{Q}_p(1-r)}(\beta)$ where β is a generator for $H^1(K, \mathbb{Z}_p(r))$, and that Theorem 2.2.2 tells us that $\exp^*_{\mathbb{Q}_p(1-r)}(\beta)$ is the coefficient of t^{r-1} in $T_m\phi^{-n}\alpha$ where α is a Λ_K generator of $A_K^{\psi=1}(1)_{\mathrm{tf}}$. In the previous section, we discussed identifying such an α, and in this section, we will develop methods to compute $\exp^*_{\mathbb{Q}_p(1-r)}(\beta)$ once we have done so.

We begin by computing the behavior of ϕ^{-m} on π_K^ℓ.

Lemma 5.2.1. *If $\phi^{-m}(\pi_K^\ell)$ converges in B_{dR}, then we have*

$$\phi^{-m}(\pi_K^\ell) = \sum_{r\geq 0}(\zeta_{p^m}-1)^{\ell/e}\left(1+\sum_{s=1}^{r}\binom{\ell/e}{s}c_{r-s,s,m}\right)t^r$$

where $c_{0,s,m} = p^{-ms}\zeta_{p^m}^s(\zeta_{p^m}-1)^{-s}$ and

$$c_{i,s,m} = \frac{p^m(\zeta_{p^m}-1)}{\zeta_{p^m}i}\sum_{k=1}^{i}(ks-i+k)a_k c_{i-k,s}.$$

Proof.

$$\begin{aligned}
\phi^{-m}(\pi_K^\ell) &= (\phi^{-m}(\pi))^{\ell/e} \\
&= \left(\zeta_{p^m} e^{t/p^m} - 1\right)^{\ell/e} \\
&= \left(\zeta_{p^m} - 1 + \sum_{r\geq 1} \frac{t^r}{p^{rm} r!}\right)^{\ell/e} \\
&= (\zeta_{p^m} - 1)^{\ell/e} \left(1 + \sum_{r\geq 1} \frac{t^r}{p^{rm} r! (\zeta_{p^m} - 1)}\right)^{\ell/e} \\
&= (\zeta_{p^m} - 1)^{\ell/e} \left(1 + \sum_{s\geq 1} \binom{\frac{\ell}{e}}{s} \left(\sum_{r\geq 1} \frac{t^r}{p^{rm} r! (\zeta_{p^m} - 1)}\right)^s\right)
\end{aligned}$$

So set $a_{r-1,m} = \frac{\zeta_{p^m}}{p^{rm} r! (\zeta_{p^m} - 1)}$. Then

$$\left(\sum_{r\geq 0} a_{r,m} t^{r+1}\right)^s = t^s \sum_{i=0}^{\infty} c_{i,s,m} t^i$$

where

$$\begin{aligned}
c_{0,s,m} &= a_{0,m}^s \\
&= \left(\frac{\zeta_{p^m}}{p^m(\zeta_{p^m} - 1)}\right)^s \\
&= p^{-ms} \zeta_{p^m}^s (\zeta_{p^m} - 1)^{-s}
\end{aligned}$$

and

$$c_{i,s,m} = \frac{p^m(\zeta_{p^m} - 1)}{\zeta_{p^m} i} \sum_{k=1}^{i} (ks - i + k) a_k c_{i-k,s,m}.$$

Thus

$$\begin{aligned}
\phi^{-m}(\pi_K^\ell) &= (\zeta_{p^m}-1)^{\ell/e}\left(1+\sum_{s\geq 1}\binom{\frac{\ell}{e}}{s}\left(\sum_{r\geq 1}a_{r-1,m}t^r\right)^s\right)\\
&= (\zeta_{p^m}-1)^{\ell/e}\left(1+\sum_{s\geq 1}\binom{\frac{\ell}{e}}{s}t^s\sum_{i\geq 0}c_{i,s,m}t^i\right)\\
&= (\zeta_{p^m}-1)^{\ell/e}\left(1+\sum_{s\geq 1,i\geq 0}\binom{\frac{\ell}{e}}{s}c_{i,s,m}t^{i+s}\right)\\
&= \sum_{r\geq 0}(\zeta_{p^m}-1)^{\ell/e}\left(1+\sum_{s=1}^{r}\binom{\ell/e}{s}c_{r-s,s,m}\right)t^r.
\end{aligned}$$

□

Corollary 5.2.2. *The coefficient of t^r in $T_0\phi^{-1}(\pi_K^\ell)$ is*

$$\left(\sum_{i=1}^{p-1}(\zeta_p^i-1)^{\ell/e}\right)\left(1+\sum_{s=1}^{r}\binom{\ell/e}{s}c_{r-s,s,1}\right)$$

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