Control of Uncertain Systems:

State-Space Characterizations

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谨 以 此 论 文

献 给 我 敬 爱 的 父 亲

To my father

who could not live to see this thesis

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Abstract

A central issue in control system design has been to deal with uncertainty and nonlinearity in the systems. In this dissertation, an integrated treatment for both uncertainty and nonlinearity is proposed. This dissertation consists of two relatively independent parts. The first part deals with uncertain linear systems, while the second part treats uncertain nonlinear systems.

In the first part, the problem of control synthesis of uncertain linear systems is considered. A linear fractional transformation (LFT) framework is proposed for robust control design of uncertain linear control systems with structured uncertainty. Linear parameter-varying systems whose coefficients depend on some time-invariant unknown parameters are treated in a general algebraic framework; both the stabilization and the $\mathcal{H}_\infty$-control problems are considered. For uncertain linear systems under structured perturbations, robustness synthesis problems are characterized in terms of linear matrix inequalities (LMIs) in the LFT framework. A generalized PBH test is also used to characterize the robustness synthesis problems. Moreover, a separation principle for the control synthesis of uncertain linear systems is revealed. The machinery also streamlines a number of results concerning the analysis and synthesis of multidimensional systems.

In the second part, the problem of control synthesis for nonlinear systems is addressed; stabilization, $\mathcal{L}^1$-control, $\mathcal{H}_\infty$-control, robustness analysis, and robustness synthesis problems for nonlinear systems are examined in detail. In particular, locally and globally stabilizing controller parameterizations for nonlinear systems are derived; the formulae generalize the celebrated Youla-parameterization for linear systems. Both nonlinear $\mathcal{L}^1$-control and nonlinear $\mathcal{H}_\infty$-control are also considered for dealing with disturbance attenuation problems for nonlinear systems. The $\mathcal{L}^1$-performance and $\mathcal{L}^1$-control of nonlinear systems are characterized in terms of certain invariance sets of the state space; in addition, the relation between the $\mathcal{L}^1$-control of a continuous-time system and the ℓ^1-control of the related Euler

approximated discrete-time systems is established. A systematic treatment for $\mathcal{H}_\infty$-control synthesis of nonlinear systems is provided; the nonlinear $\mathcal{H}_\infty$-control problem is characterized in terms of Hamilton-Jacobi Inequalities (HJIs) and nonlinear matrix inequalities (NLMIs); a class of $\mathcal{H}_\infty$-controllers are parameterized as a fractional transformation of contractive stable parameters. Finally, the problems of stability and performance robustness analysis and synthesis for uncertain nonlinear systems subject to structured perturbations with bounded $\mathcal{L}_2$-gains are introduced; they are characterized in terms of HJIs and NLMIs as well. Computational issues are also addressed; it is confirmed that the computation needed for robustness analysis and synthesis of nonlinear systems is of equivalent difficulty to that for checking Lyapunov stability.

Preface

Control Science as Science

A science, as an organic whole, is a coherent system of principles and methodologies, a more or less verified or established explanation accounting for known facts or phenomena, and an internal mechanism which unifies the knowledge as an identity and reconciles practical ends with new ideas. A successful new idea typically alters and extends the existing body of the science to allow for observational facts that could not previously be understood or incorporated. It also makes possible new predictions that can some day be tested. In this sense, the vitality of a science depends on its well-established infra-structure as a rigorous discipline and the dynamics of its methodologies and boundaries.

In retrospect, the development of control science follows this pattern. The essential principle for control science is feedback, i.e., the property of being able to adjust future conduct by past performance[1]. Control and feedback are two inseparable terms.

Recorded control practice can be traced back to at least the first half of the 3rd century B.C., when level regulating devices (float valves), based on the feedback principle, were used in water clocks[2]. Yet as a science in the modern sense, control only emerges much later. James Watt's use of the governor (1788) can be taken as the starting point for the development of control as a science. However, it is generally thought that it was J. Clerk Maxwell who founded the theory of automatic control systems with his 1868 paper *On Governor*, in which the method of regulation received a thorough mathematical analysis[3]. With feedback control devices being broadly used during the (First) Industrialization, the

[1]Nobert Wiener, *The Human Use of Human Beings: Cybernetics and Society* (2nd edition), New York: Garden City, 1954.

[2]Otto Mayr, *The Origins of Feedback Control*, The MIT Press, 1970.

[3]See for example, A.T. Fuller, "The Early Development of Control Theory II," *Journal of Dynamic Systems, Measurement, and Control*, Vol.98, pp.224–235, 1976

growing importance of automatic control was marked by the awarding of the Nobel Prize in Physics in 1912 to Nils Gustal Dalen, a Swedish scientist, "for his invention of automatic regulators ······"[4]

Nonetheless, the first methodological breakthrough of control science dates as late as the 1930's with the development of the feedback amplifier and the genesis of frequency-response graphical techniques[5]. Control science entered the classical epoch linked with the names Bode, Nyquist, and Nichols. Those techniques deal with single-input-single-output systems, and emphasize design trade-offs, such as the effect of uncertainty. Feedback was used to desensitize control systems to changes in processes as well as to stabilize unstable systems. The limitation of the classical techniques, however, was soon revealed; the classical control theory was overwhelmed by the need for more sophisticated control systems with more precise specification requirements.

Modern control theory originated under such circumstances. The turning point was marked by the publication of two celebrated books: *Cybernetics*[6] by Nobert Wiener in 1948 and *Engineering Cybernetics*[7] by Hsue-Shen Tsien in 1954. The techniques of prediction, filtering, smoothing, and optimization symbolize this period via the introduction of rigorous mathematical methods. The methodological revolution led to a great leap forward for control science, allowing for the treatment of a broader class of problems. The notable technical breakthroughs are Wiener's cybernetics (filtering, prediction, and smoothing), linear and nonlinear programming, Kolmogorov's stochastic process theory, Pontryagin's maximum principle, Bellman's dynamic programming, Kalman's filtering theory, the notions of controllability and observability, the state-space theory of pole assignment and stabilization, as well as special methods such as LQR, LQG/LTR, geometrical methods, and algebraic methods. The most comprehensive and significant application of modern control theory was to the Apollo project in the 1970's. However, with a few notable exceptions, the techniques in this period did not have much impact on feedback design because they neglected fundamental feedback tradeoffs (effect of plant uncertainty); the robustness of control systems to some level of uncertainty is guaranteed only *a posteriori*.

It was a time for change, Industrialization was quickly giving way to Post-industria-

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[4] N. Heathcote, *Nobel Prize Winners in Physics 1901–1950.* New York: Schuman, 1953.

[5] A.G.J. MacFarlane, "The Development of Frequency-Response Methods in Automatic Control," *IEEE Transactions on Automatic Control*, Vol.AC-24, pp.250–265, 1979.

[6] Nobert Wiener, *Cybernetics, or Control and Communication in the Animal and the Machine*, New York: John Wiley & Sons, Inc., 1948 (2nd edition, 1961).

[7] Hsue-Shen Tsien, *Engineering Cybernetics*, New York: McGraw-Hill Book Company, Inc., 1954. In this book, control science was systemized as a rigorous discipline for the first time. Dr. Tsien was then a professor at Caltech.

lization. Post-modernism[8] sprang up in many areas, such as art, literature, and architecture. Post-modernism can be taken as a cultural movement and historical epoch; it is both a continuation of modernism and its transcendence. The post-modern age is a time of incessant choosing; it is an era when no orthodoxy can be adopted without self-consciousness and irony, because all traditions seem to have some validity. Post-modernism is fundamentally the eclectic mixture and synthesis of any tradition with the immediate past. Its best works are characteristically doubly-coded; they are the inventive combination of modern techniques with something traditional in order to communicate with the public and a concerned minority; they are also making a feature of the wide choice, conflict, and discontinuity of traditions, because this heterogeneity most clearly captures pluralism and freedoms of our culture.

The post-modern period is also a time for control scientists to relocate the control science and themselves. Control science as an expression of the human mind reflects the active will and the restless desire for perfection. Like many other disciplines of human knowledge, all development of control science has its psychological roots in practical requirements. But once started under the pressure of necessary applications, it inevitably gains momentum in itself and transcends the confines of immediate utility.

Post-modern control theory[9] emphasizes control as a strategy for dealing with complexity due to uncertainty and nonlinearity. Although such efforts have been beset from the beginning with some difficulties, interest in them has been maintained. The main motive for post-modern control theory is the inadequacy of modern control theory in dealing with uncertainty and nonlinearity, although there is no vivid "death" for modern control theory. The internal force to enlarge the boundary of control science and its ideology of progress which offered technical solutions to many new problems were seen vividly by every well-motivated control scientist; there is a great desire for control scientists to seek new tools to deal with these new problems. Today's control scientists are committed to using new mathematical tools and contemporary technology as well as facing current reality. So far, many advanced mathematical techniques and technological tools have been introduced into control science[10]. The marriage between new techniques and some classical ideas generates new

[8]Charles Jencks, *What Is Post-Modernism?* London: Academy Editions/ New York: St. Martine's Press, 1986. Post-modernism varies for each area, both in its motives and time-frame.

[9]The term, post-modern control, was coined by John C. Doyle in the late 80s. In a lecture at University of Minnesota in May 1991, John C. Doyle gave the tenets of post-modern control theory, including i) Theoreticians do not design, control systems engineers do, ii) The application of any methodology to real problems will require some leap of faith on the part of engineers and some *ad hoc* fixes, iii) The goal of the theoreticians should be to make this leap smaller and the *ad hoc* fixes less dominant.

[10]For example, George Zames, "Functional Analysis Applied to Nonlinear Feedback Systems," *IEEE Transactions on Circuit Theory*, Vol.CT-10, pp.392–404, 1963; Roger W. Brockett, "Nonlinear Systems and

powerful tools to cope with more complicated control systems. For example, the $\mathcal{H}_\infty$-control problem formulated by Zames is strongly rooted in classical techniques and has motivated a powerful paradigm for control system design with uncertainty. In this paradigm, uncertainty is incorporated *a priori* in the design. In addition, the state-space solution of the $\mathcal{H}_\infty$-control problem establishes a strong connection between $\mathcal{H}_\infty$-control and the classical LQG optimization. The robustness problem under structured perturbation, formulated by Doyle, has highlighted the differences as well as technical relations between multiple-input-multiple-output systems and classical single-input-single-output systems, and has made it possible to deal with multivariable (uncertain) systems through structured singular value analysis and synthesis in an elegant way. Thus, the post-modern period of control science not only has been a period of new advances, but also has been characterized by a successful return to the classical ideal of coping with uncertainty and nonlinearity.

Control Science as Artificial Science

Science originates from experience. The vitality of science essentially depends on whether or not the empirical world keeps providing problems which stimulate the generation of new methodologies and the extension of its boundary. Based on the development of experience, there are two kinds of science about the objects and phenomena in the empirical world. Together, they should encompass and reveal the individuals and behaviors which represent natural rules and human objectives. On one hand, they should disclose the laws and rules for the system of individual objects, while on the other hand, they should alter the laws and rules to satisfy some prescribed objectives by changing the system. We call the former process analysis, and the latter synthesis. Correspondingly, there are two kinds of empirical sciences: natural science and artificial science.

Natural science is knowledge about natural objects and phenomena. What do we mean by natural? Epistemologically speaking, nature is what we observe in perception through the senses. In this sense-perception, we are aware of something which is self-contained (with its intrinsic goal). It means that nature (or a natural system) can be thought of as a closed system whose mutual relations do not require the expression of the fact that they are thought about[11]. This property of being self-contained for thought lies at the base of natural science. Natural science is a descriptive science, and its methodology is analytic.

However, we are oftentimes merely concerned about a particular part of a natural system which could not be self-contained with the intrinsic goal, or we need to replace its intrinsic

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Differential Geometry," *Proceedings of IEEE*, Vol.64, pp.61–72, 1976.

[11] A.N. Whitehead, *The Concept of Nature* (Chapter 1), Cambridge University Press, 1920.

goal with some prescribed alternative with which the system cannot be self-contained either. In either case, the system needs to be artificially changed such that it evolves according to some required laws and rules for the new artificial object. The science of the knowledge about artificial objects and phenomena and how to synthesize such artificial objects, or artifacts, is called artificial science. Artificial science is a imperative science, and its methodology is mainly synthetic.

We can now identify four indicia that distinguish the artificial from natural; hence we can set the boundaries for artificial sciences[12]:

- Artificial things are synthesized (though not always or usually with full forethought) by man.
- Artificial things may imitate appearances in natural things while lacking, in one or many respects, the reality of the latter.
- Artificial things can be characterized in terms of functions, goals, and adaptation.
- Artificial things are often discussed, particularly when they are being designed, in terms of imperatives as well as descriptives.

In the following, let us look a little more closely at the functional or purposeful aspects of artifacts. The fulfillment of purpose or adaptation to a goal involves a relation among three terms: the purpose or goal, the character of the artifact, and the environment in which the artifact performs. Natural science impinges on an artifact through two of the three terms of the relation that characterizes it: the (internal) structure of the artifact itself and the environment in which it performs. The artificial sciences are about neither the internal structure of the artifact itself nor its environment, but the relation between these two terms such that a specified goal is achieved. On the other hand, the fulfillment of purpose or adaptation to a goal can also be represented by the interface between the inner environment, which is the substance and organization of the artifact itself, and an outer environment, which is the surroundings in which it operates. If the inner environment and outer environment are appropriate to each other (with respect to the goal), then the artifact will serve its intended purpose. Thus we can often predict behavior from knowledge of the system's goals and its outer environment, with only minimal assumptions about the inner environment.

While complexity is a key issue to the natural systems, to some extent, we can use simple models to understand and describe them. This is due to the characteristics of the goals and

[12]This summary is due to H.A. Simon, in his *The Sciences of the Artificials*, The MIT press, 1981 (first edition 1969).

methodology. Since natural sciences are the studies on self-contained systems, the most important premise on which the investigations are carried out is that the natural systems are well ordered. The duty for a natural scientist is to explore the connections between the simple and the complex, between the universal and the individual, and between the basic laws of the nature and the manifestations of nature's complexity[13]. In this sense, the criteria for natural science have some aesthetic flavor. Nevertheless, the implications of complexity of artificial systems are totally different from those for natural systems. The complexity of the artificial systems are due to the uncertainty and complexity of human behaviors and their interactions with other elements of the artificial systems. Artificial sciences are about synthesis of the external behaviors of artificial systems, more precisely, how to exhibit the macro stable functions in terms of micro accidental complex events and forms. In this sense, the criteria for artificial science are mainly pragmatic.

Control science as we discussed in the last section is a typical artificial science. The goal for a control scientist is to synthesize control systems, which are dynamical artificial systems, such that the control systems adapt to its environment and achieve some intended performances. In control theory, the concept of feedback plays a key role; the implementation of such control systems is via feedback. To conclude the discussion, we will next examine some examples to reveal the essential difference between a natural dynamical system and a control system which achieves some optimal performance.

Typical systems in the natural world are mechanical systems. A mechanical system is natural if it satisfies Galileo's principle of relativity and Newton's principle of determinacy (i.e., their evolutions are described by Newton's differential equations); in this case, it also has the property that the Hamiltonian, which is the sum of kinetic and potential energy, is constant during the evolution, i.e., the well-known law of conservation of energy holds, and Hamilton's principle of least action is satisfied[14]. Therefore, a (mechanical) system is natural if and only if it satisfies the law of conservation of energy and the principle of least action. In this sense, the notion of naturality can be generalized to include a much broader class of systems which are not necessarily mechanical by suitably defining the Hamiltonians. In addition, the least action is an intrinsic performance for a natural system; it is in this

[13]M. Gell-Mann, *The Quark and the Jaguar: Adventures in the Simple and the Complex*, New York: W.H. Freeman and Company, 1994.

[14]The principle of least action says that the motion of a (natural) mechanical system coincides with the minimals of the functional

$$\phi(\gamma) = \int_{t_0}^{t_1} L dt$$

where L is the Lagrangian, which is the difference between the kinetic energy and potential energy. Note that there is a unique correspondence between the Hamiltonian H and the Lagrangian L. See V.I. Arnold, *Mathematical Methods of Classical Mechanics*, New York: Springer-Verlag, 1978.

sense that the natural system is an optimal system. This observation is also reflected in the synthesis of control systems.

A dynamical system is not natural if it doesn't satisfy the law of conservation of energy and the principle of least action. Such examples include the individual parts of (interconnected) natural systems, where there is an exchange of energy between different parts, and systems which have enforced artificial performances different from the intrinsic performances. In these cases, the objective performances are not the inherent (least action) performances. Motivated by the observation of natural systems, we need to redefine the Hamiltonian in terms of the intended performances, and synthesize the system via feedback techniques such that the Hamiltonian, which is thus the newly-defined energy, is constant. The resultant system, which is thus optimal, is a control system whose optimum is achieved through feedback.

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Control as a science reflects the human beings' active will and the restless desire for perfection. By all standards, the achievements in control science so far have been spectacular. The influence and impact brought about by the advances in control science upon other sciences, technologies, and human affairs have been important beyond description. But it is not in these influences and impacts that the glory of control science and the heart of a control scientist lies. It is not even in the continued enlargement of the domain of control science, as important as this is, that the control scientist takes the greatest pride and satisfaction. What makes control science so unique as an intellectual endeavor lies in the enhanced possibility of human beings' pursuit of perfection through the provided ideas and techniques which establish the bridge between the ideal and the reality.

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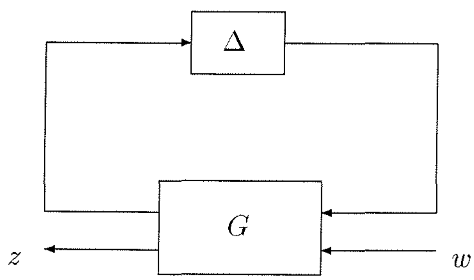
Chapter 1

Introduction

Control systems are synthesized via feedback so that some of the design goals are achieved. Hindering the achievement of the goals are the uncertainty and nonlinearity of the control systems. Therefore, dealing with uncertainty and nonlinearity is a central issue in control system analysis and synthesis. In this dissertation, we will emphasize on an integrated treatment for both uncertainty and nonlinearity; two problems will be addressed, i.e., can stability and/or a target performance for control systems be achieved in the presence of uncertainty? and how to develop a systematic procedure for designing control systems that achieve the stability and/or performance limitations in the presence of uncertainty? The first problem is robustness analysis issue, while the second one is robustness synthesis issue.

A standard representation for an uncertain system can be depicted as in the following block diagram,



where G is the nominal system which is linear or nonlinear, Δ is the uncertainty, w is external disturbance signals, and z is the regulated signals. The robustness analysis is to determine that under what conditions for the nominal system G, the uncertain system is stable and/or satisfies some performance for all admissible uncertainty Δ; while the robustness synthesis problem is to decide under what conditions there is a feedback control law for the uncertain system such that the closed loop uncertain system has the required robustness, and then how to design the control law.

1.1 Overview of Some Related Work

This section is not intended as a comprehensive historical review of the literature in robustness analysis and synthesis for uncertain systems, but rather an attempt to outline some of the work most relevant to this research so as to provide some motivations of the research as well as to put the present work in context. Because of the diversity of different issues, in what follows the review is conducted in different categories.

1.1.1 Robustness Analysis: Small-Gain Theorems

The systematic treatment of robustness analysis for uncertain systems can be traced back to at least the 60's[1]; the basic tool for robustness analysis of such uncertain systems is the small gain theorem introduced to control by Zames [202, 203] and Sandberg [154] (see also [195, 50]). A general sufficient condition for robust stability analysis against uncertainty with $\mathcal{L}_2$-gains less than or equal to 1 is that the $\mathcal{L}_2$-gain of the nominal system G is less than 1 [152, 150, 62, 36]. In the case that the uncertainty set are the causal unstructured operators, Doyle-Stein and Chen-Desoer proved that the small-gain condition, i.e., the $\mathcal{L}_2$-gain of the plant is less than 1, is also necessary for robust stability if both the uncertainty and the nominal system are linear [62, 36]. Shamma further showed that the necessity is still true if the uncertainty is linear time-varying or nonlinear, and the nominal plant is linear or nonlinear with fading memory [156, 166].

However, it is often the case in practical situations that the uncertainty Δ is structured, i.e., it consists of multiple uncertainty blocks, $\Delta = \mathrm{DIAG}\{\Delta_1, \Delta_2, \cdots, \Delta_N\}$; the small-gain test as a sufficient condition could be arbitrarily conservative for robust stability in such cases. Doyle (1982) in the seminal paper [53] introduced the notion of structured singular value (μ) to formulate the robustness problem in this setting. With the aid of this new notion, it is shown that in the case where both the uncertainty and the nominal system are linear time-invariant, a necessary and sufficient condition for robust stability is that the nominal system has small **structured gain**, i.e. its structured singular value $\mu_\Delta(G) < 1$ [53]. However, the computation for μ is usually difficult [31]. In the general case, by scaling

[1] However, the robustness issue was touched much earlier; see for example, Peter Dorato, "A Historical Review of Robust Control," *IEEE Control Systems Magazine*, April 1987.

treatment, a sufficient condition for robust stability in this case is that the nominal system has **scaled** small gain: $\|DGD^{-1}\|_{\mathcal{L}_2} \leq 1$ for some scale D commuting with Δ (see for example [53, 151]). This condition is also necessary for certain simple classes of uncertainty structures [53, 132]. In addition, it is confirmed by Shamma (1993), Megretski (1993), and Savkin-Petersen (1993) that this condition is still necessary for certain class of general uncertainty structures in such cases as the nominal plant G is linear time-invariant and the uncertainty is allowed to be time-varying or nonlinear [159, 123, 124], or the nominal system is nonlinear with fading memory and the uncertainty structure is linear time-varying or nonlinear [155]. The scaling treatment not only reduces the conservatism, but also yields a convex characterization, which facilitates computations. It is remarked that the scaling treatment for reducing the conservatism arising from the constrained structure was used earlier in conjunction with so called "interconnected systems stability theory" (see for example [4, 127]). Some extensions on the robustness analysis in the ℓ_1 sense are made by Khammash *et al.* [99, 41, 40].

As for the robust performance analysis problem, Doyle *et al.* (1982) confirmed that the performance robustness analysis problem can be treated as a robust stability problem against the structured uncertainty with an extra "uncertainty" block [63]. Therefore, the small-gain arguments still apply in this case. The detailed treatments for different cases appeared in [63, 132, 142, 99, 41, 40] and references therein.

The computational issue has been a major concern in robustness analysis. Some of the related work is summarized in [200, Section 1.1]. It is noted that the computation of certain robustness conditions can be converted to the solution of finite dimensional **linear matrix inequalities** (LMIs), which have some attractive computational properties. A review of LMIs and their use will be given in Section 1.1.3.

1.1.2 Robustness Synthesis: Linear $\mathcal{H}_\infty$-Control Theory

The analysis of robust stability and performance for uncertain systems, which are treated in the input/output setting, are essentially reduced to system gain analysis. For the linear systems, both the robust stabilization and the robust performance synthesis problems can be cast as $\mathcal{H}_\infty$-control problems for some nominal systems. Zames in 1981 first formulated the $\mathcal{H}_\infty$-control problem in an input-output setting with emphasis on disturbance attenuation [204]. This problem, which re-established the link to the classical techniques of Bode and others, was later solved by rigorous use of operator-theoretic techniques [67]. An earlier state-space results in this context was streamlined by Doyle in 1984 [55]. Detailed accounts on the development of the $\mathcal{H}_\infty$-control theory up to this point were given in [67, 60].

An alternative view of the $\mathcal{H}_\infty$-control problem is game-theoretic. The $\mathcal{H}_\infty$-control synthesis is a worst-case design. The control inputs can be viewed as signals trying to satisfy the performance objective. The exogenous disturbances and plant perturbations are regarded as adversaries which are trying to maximize the errors. This linear-quadratic differential game problem was considered in the 1970's by Mageirou-Ho [121]. The state feedback $\mathcal{H}_\infty$-control solution is characterized in terms of algebraic Riccati equations [121]. In the late 80's, Petersen, Zhou, and Khargonekar [140, 208, 100] rediscovered this fact using the (bounded-real) KYP-lemma, and showed that for the state-feedback $\mathcal{H}_\infty$-control one can choose a constant gain as a sub-optimal $\mathcal{H}_\infty$-controller. However, a complete state-space treatment of the $\mathcal{H}_\infty$-control problem via output feedback was provided by Doyle *et al.* in the celebrated DGKF paper [60] as well as [74, 75]. They showed that a sub-optimal $\mathcal{H}_\infty$-controller exists if and only if the unique stabilizing solutions to two algebraic Riccati Equations are positive definite and satisfy certain coupling condition. Under these conditions, a parameterization of all controllers solving the sub-optimal $\mathcal{H}_\infty$-control problem is given as a linear fractional transformation on a contractive, stable free parameter. The controller has a separation structure reminiscent of classical LQG theory. Some other approaches which lead to the same results or extensions were developed afterwards, for example, the maximal principle used by Tadmore [178], the differential game-theory used by Basar and Bernhard [16], and the bounded-real lemma used by Petersen *et al.* [141]. It is noted that the state-space treatment for the $\mathcal{H}_\infty$-control problem not only is conceptually simple, but also establishes clear connections with traditional methods in optimal control.

More recently, some alternative characterizations to the state-space $\mathcal{H}_\infty$-control which have computationally attractive properties have been derived. The most notable results are linear matrix inequality (LMI) characterizations [119, 133, 69], whose use is reviewed in the next subsection.

1.1.3 Robustness and Linear Matrix Inequalities

The story of LMIs in control theory begins in about 1890 when Lyapunov stability theory was born[2]. Therefore, the first class of LMIs is that of Lyapunov inequalities where the existence of positive definite solutions insures the stability of the relevant systems. However,

[2]For a detailed account of the history of LMIs in control theory, one is referred to Boyd *et al.* [29, Section 1.2].

the important roles of LMIs in robust control theory was not revealed until the 1960's when the solution to the classical problem of Lur'e was reduced to the solutions of some LMIs by the use of Kalman-Yakubovich-Popov (KYP) lemma. This resulted in the celebrated Popov Criterion, Circle Criterion, and many other variations. The KYP lemma and extensions were intensively studied in the latter half of the 1960's, and were found to be related to the ideas of passivity, bounded-realness, and quadratic optimal control. It was known then that the LMI appearing in the KYP lemma could be solved by solving a certain algebraic Riccati Equation. In the early 1970's, Willems systematically examined the dissipativity of general dynamic systems which encompasses the ideas of passivity, bounded-realness, and quadratic optimization [197]. He thus proposed a characterization of linear quadratic optimal control in terms of LMIs [196, 197].

While the connection of robustness analysis to LMIs is through the small-gain theorem and the KYP lemma, the characterization of robustness synthesis in terms of LMIs is through the KYP lemma (LMI version) and Parrott's Theorem or its variations in operator dilation theory [139, 45, 55]. The earliest form of LMIs for robustness synthesis was developed by Doyle-Chu in 1985 [57]. The concrete use of the characterization in the robustness synthesis of discrete-time systems appeared in [119] to deal with robust stabilization, and in [133, 131] to deal with robust performance synthesis. The other work in this context appeared in [134, 111, 132]. A variation of Parrott's Theorem, which is known as Finsler's Theorem, was used to deal with continuous-time systems later [69, 3, 94]. In [29, Chapter 7], there is a detailed discussion about the roles of LMIs in different robustness problems.

One of the advantages of LMI characterization is that it has attractive computational properties. Some feasible algorithms are available to deal with the computation of LMIs, such as the ellipsoid method and the interior-point method [128, 29, 70]. It is remarked that the nonlinear extensions of LMIs for robustness analysis and synthesis of uncertain nonlinear systems were given in [113, 115], they are known as nonlinear matrix inequalities (NLMIs); however, the computational implications of the NLMIs are far less straightforward than those of LMIs.

1.1.4 Nonlinear Robustness Analysis and Synthesis

As pointed out in Section 1.1.1, the small-gain theorem provides a test for robust stability and robust performance of uncertain nonlinear systems [202, 154, 203, 156, 166, 155]. Therefore, the study of robustness is reduced to gain analysis [115]. The research on general dynamical systems which have finite $\mathcal{L}_2$-gains can be traced back to at least the early 70's. The $\mathcal{L}_2$-gain and the dissipativity for a dynamical system are closely related. A systematic exploration of general dissipative systems was performed by Willems in 1972 [197]; some extensions appeared in [126, 84]. In the context of $\mathcal{L}_2$-gain analysis, van der Schaft [183] extensively studied the $\mathcal{L}_2$-gain of input-affine nonlinear time-invariant (NLTI) systems by using dissipation theory. He reconfirmed that the $\mathcal{L}_2$-gains for a class of systems can be characterized by Hamilton-Jacobi equations (HJEs) or inequalities (HJIs). Recently, Lu and Doyle reformulated the $\mathcal{L}_2$-gain characterization in terms of nonlinear matrix inequalities (NLMIs), which result in convex feasibility problems [113, 115].

The robustness synthesis for a nonlinear system involves the design of controllers such that the closed-loop system has small $\mathcal{L}_2$-gain or small scaled $\mathcal{L}_2$-gain. We will use the accepted but unfortunate misnomer "nonlinear $\mathcal{H}_\infty$" to describe this research direction. The nonlinear $\mathcal{H}_\infty$-control problem was first recognized and partially solved in the context of operator theory by Ball-Helton [11], Foias-Tannenbaum [66], and de Figueiredo-Chen [46][3]. We are mainly concerned with the state-space theory in the following review.

The nonlinear generalization of state-space linear $\mathcal{H}_\infty$-control theory is motivated by its simplicity of the characterization as well as its clear connection with traditional methods in optimal control [60]. As a result, the initial efforts were concentrated on relating of the nonlinear $\mathcal{H}_\infty$-control to differential game theory by several researchers [12, 16], see also [52]. Using the $\mathcal{L}_2$-gain analysis result, van der Schaft derived some elegant HJE characterizations to the $\mathcal{H}_\infty$-control problem for the class of input-affine NLTI systems [183]. He showed that a sufficient condition for the state feedback $\mathcal{H}_\infty$-control problem to be solvable is that the corresponding HJE has a positive solution. In the output feedback case, he asserted that the $\mathcal{H}_\infty$-control problem is locally solvable if it is solvable for the linearized system. But this assertion requires that the equilibrium point of the related Hamiltonian vector field be hyperbolic. Isidori and Astolfi [93, 91] developed other less conservative sufficient conditions for the output feedback $\mathcal{H}_\infty$-control problem to be solvable. They showed that the solution to the $\mathcal{H}_\infty$-control problem requires the existence of positive definite solutions of two hierarchically coupled HJIs. In [112], Lu-Doyle developed a complete nonlinear $\mathcal{H}_\infty$-control theory parallel to the work [60] of Doyle *et al.*; in particular, a class of local

[3] The early development of nonlinear $\mathcal{H}_\infty$-control theory, in particular the operator-theoretical approach, was reviewed by J.A. Ball and J.W. Helton in "Nonlinear $\mathcal{H}_\infty$-Control Theory: A Literature Survey," *Robust Control of Linear Systems and Nonlinear Control* (M.A. Kaashoek *et al.* eds), pp.1–12, Boston: Birkhauser, 1990.

$\mathcal{H}_\infty$-controllers are parameterized as a fractional transformation of stable and contractive parameters. Some other notable related work includes the global solutions [82, 110], the finite-time horizon solutions [107, 102], and the solutions in discrete-time case [76].

Ball-Helton-Walker [15] examined the nonlinear $\mathcal{H}_\infty$-control problem from another perspective. They derived the necessary conditions for the existence of an output feedback controller such that the HJI related to the closed loop system has a positive smooth solution (specifically, the $\mathcal{H}_\infty$-control problem is solvable). These conditions are that two HJIs have positive solutions and the solutions are coupled locally. They confirmed the separation principle for the nonlinear $\mathcal{H}_\infty$-control system, and also provided a recipe to construct the controllers. Similar necessary conditions were also obtained by Van der Schaft and Isidori [185, 92]. Recently, Lu-Doyle considered a stronger $\mathcal{H}_\infty$-control problem, where necessary conditions in terms of NLMIs were developed [114].

Basically, the premise on which those generalizations are carried out is to assume that the (dynamic) output feedback $\mathcal{H}_\infty$-controllers have some separation structures; under this assumption, some necessary or sufficient conditions for the $\mathcal{H}_\infty$-control problem to be (locally or globally) solvable are characterized in terms of some Hamilton-Jacobi equations or inequalities which are partial differential equations or inequalities. Whence, one of the major concerns in the state-space nonlinear $\mathcal{H}_\infty$-control theory is the computation issue to solve these Hamilton-Jacobi partial differential equations or inequalities, and progress along this line would be beneficial to applications of nonlinear $\mathcal{H}_\infty$-control theory. For example, Huang and Lin proposed a systematic procedure to find Taylor series approximations to the solutions of the HJEs [86] (see also [120, 183]). In addition, van der Schaft [183] considered the properties of solutions of HJEs in depth in terms of the related Hamiltonian vector field. Recently, Lu-Doyle (1994) proposed the characterization of a class of nonlinear $\mathcal{H}_\infty$-control solutions in terms of nonlinear matrix inequalities [114, 115], which result in convex optimization problems and potentially have some nice computational properties.

1.2 Some General Considerations

Both robustness analysis and synthesis problems for a class of uncertain systems are addressed in this thesis; the emphasis is on the development of systematic state-space design techniques with attractive computational properties. The following are some considerations which help to accomplish this goal.

1.2.1 Uncertainty in Control Systems

A physical control system is an artificial system (*cf.* Preface). Its description should capture its three functional elements: its goal, its internal character, and its outer environment. It is convenient and often necessary to conduct analysis and synthesis on the mathematical model which abstracts, simplifies, and approximates the real behavior of the physical system. There therefore exists a gap between the model and the real system. This gap is assessed by uncertainty which is supposed to capture the information lost in the modeling process. This uncertainty is usually small, and is ignored in the traditional control analysis and design. However, the need for very sophisticated control systems with high performance makes one take the uncertainty into account in the design.

Nonetheless, the accurate information about the uncertainty usually is not available *a priori*, i.e., it can not be decided until the real system is in operation, or even in many situations, it is not verifiable *a posteriori*. Therefore, the description of the uncertainty does not exactly measure the real gap; on one hand, it reflects the measurement of the designer's lack of confidence in the model, on the other hand, it is also a trade-off between the mathematical maneuverability of the model and the desired accuracy on the performance of the control system once implemented.

The model of a control system has two facets: the nominal model which approximates the internal structure of the real system, and the uncertainty which assesses the gap between the nominal model and the real system. Since the control system is an artificial system, its model should also reflects some of its functional aspects, i.e., the internal structure of the system, and its environment. Therefore, the uncertainty in the model correspondingly comes from two sources.

Internal Perturbations

- **Internal Model Errors.** The usual way of deriving a model of the internal structure is by applying the physical laws. However, the general laws and the special reality usually can not match perfectly, this results in errors. In addition, due to the consideration of mathematical maneuverability, one tends to obtain a model for this part as simple as possible. For example, the model from the physical laws usually is nonlinear, its linear approximation some times is preferred because the well-developed linear design techniques are available; therefore, an error occurs between the nonlinear physical model and the accepted linearized model. The existing model usually has very high order, the reduction of the order leads to some error between the original high-order model and the reduced model. The approximation of a time-varying model by a linear time-invariant model also leads to model errors. The high frequency dynamics is usually represented as an additive perturbation with bounded frequency response.

- **Parametric Perturbation.** In many situations, some specific parameters of a process are not fixed during operation, such as a spring constant in a mechanical system, a resistance in a circuit, or the temperature in a chemical process. The perturbation occurs when one takes some fixed values of the parameters to model the process.

Perturbations from Outer Environment

- A control system (as an artificial system) can not be independent of its outer environment. It is however difficult to mathematically model the dynamical interaction of the internal structure with its outer environment due to its unpredictability. Usually, the interaction is weak, and it is ignored when modeling the internal structure. This results in the unmodeled dynamics from the environment.

To conclude this subsection, we remark that besides the dynamical interactions with internal structure, the outer environment influences the internal structure by imposing some exogenous input signals, which include some undesired disturbances. The robust control design, roughly speaking, is to attenuate the external disturbances and uncertainty. This consideration reflects the other element of an artificial systems: the goal.

1.2.2 State-Space Treatment

The robustness problems are generally formulated in terms of input-output properties of the relevant control systems. For example, the performances are measured by input-output (I/O) relations; the uncertainty is modeled as a bounded causal I/O operator. However, the KYP Lemma establishes the connection between the input-output consideration and the state-space characterizations; in this thesis, robustness analysis and synthesis are conducted in state-space setting. There are several advantages for the state-space treatment:

- In the I/O setting, the internal behaviors of the systems, such as asymptotic stability, are generally hard to capture during the analysis and synthesis. This kind of internal issues are naturally and directly treated by state-space techniques. In particular, for the nonlinear systems, the relation between I/O properties and internal properties is far less clear, the state-space techniques are thus especially suitable to deal with the internal issues.

- The characterizations of robustness analysis and synthesis in this thesis are reduced to gain conditions which are I/O relations. For the linear systems, the state-space treatment reduces system analysis and synthesis to matrix computations; the matrix manipulations are inherently more numerically stable than symbolic or (I/O) transfer function manipulations. On the other hand, for the nonlinear systems, the computational implications for those characterizations in the I/O setting are far from clear, however, it will be seen that by the use of state-space techniques, the computation needed for robustness analysis and synthesis is of equivalent difficulty to the one for checking the Lyapunov stability; in other words, the provided approach reduces the robustness analysis and synthesis problems to the possible easiest problems.

1.2.3 Fractional Transformation

Fractional transformations formally represents feedback structures. In this dissertation, the uncertainty is considered to enter the control systems in feedback fashion, fractional transformations are natural tools to describe the uncertain control systems. For the linear systems, one of the advantages of the use of linear fractional transformation (LFT) representations is that it facilitates manipulation using state-space-like machinery which could provide some convenience in computation (see Section 1.2.2). Moreover, the μ framework taken in the linear robustness analysis has an intimate relation with the LFT machinery [132, 143, 144].

In addition, the adoption of LFT machinery in linear robustness synthesis simplifies the design procedure. Because of the nature of a uncertain linear system, whose coefficients are functions of uncertainty, a controller depending on the uncertainty, instead of a single linear time-invariant controller, is required to achieve the desired specifications. Such controllers can be implemented for real-time operation in terms of gain/dynamic scheduling or adaptive/supervisory control schemes. The design of the scheduling schemes is conducted in a LFT framework in this thesis. The type of scheduling that results from this LFT approach avoids some potential hazards arising from conventional scheduling [164, 149]. The need to select (slow) scheduling variables and scheduling procedures, which is addressed in [149], disappears in this LFT setting. The uncertainties are allowed to vary arbitrarily fast and the design procedure guarantees stability, which is not guaranteed in the conventional setting [164]. The conventional *ad hoc* point-wise controller design and curve-fitting procedures (see [163, 149]) are also avoided, since the control law is designed analytically. Although the results might be potentially conservative for slowly varying parameters, the methods can at least be used as a possible aid in conventional scheduling design (rather than a replacement).

1.2.4 Convex Characterizations: LMIs and NLMIs

In this thesis, most of the results on robustness analysis and control synthesis are characterized in terms of the elegant **linear matrix inequalities** (LMIs) for linear systems or **nonlinear matrix inequalities** (NLMIs) for nonlinear systems. However, the characterizations are not for the sake of elegance, but rather that they lead to the reduction of robustness analysis and synthesis problems to a handful of standard convex optimization problems.

For the LMIs, they can be numerically solved in polynomial-time (*cf.* Boyd *et al.* [29]); this shows that the computation of solving LMIs is tractable. Many algorithms, such as the ellipsoid algorithm and interior-point method, have recently been developed for these problems, and have been found to be efficient in practice [29, 70].

An NLMI is in fact a linear partial differential inequality; it can be decomposed into a state-dependent LMI and a partial differential equation. Although in general, the state-dependent LMI is unfortunately an infinite-dimensional LMI, it will be shown that the computation needed for solving robustness problems is not harder than the one for checking Lyapunov stability; in other words, the provided approach reduces the robustness analysis and synthesis problems to the possible easiest problems. Moreover, the solution of NLMIs can be reduced to the solutions of finite many LMIs if the state set of interest is bounded. Therefore, the existing convex optimization methods for solving LMIs can be used in practical computation for solving the NLMIs.

1.3 Overview of This Thesis

This dissertation is divided into two parts. The first part (Chapters 2 – 4) deals with the problems of control of uncertain linear systems where the nominal systems are linear time-invariant, while the second part (Chapters 5 – 9) tackles the problems of control of uncertain nonlinear systems where the nominal systems are nonlinear. Each chapter treats different control problems, and the material for each chapter is made as self-contained as possible.

1.3.1 Control of Uncertain Linear Systems

Chapter 2: Linear Parameter-Varying Systems

The control problems of linear parameter-varying (LPV) systems are considered. Those systems are defined in an algebra of continuous functions. Both stabilization and $\mathcal{H}_\infty$-control problems are considered in this algebra. The key to the treatment is the employment of the LMI characterizations of stabilization and $\mathcal{H}_\infty$-control of LTI systems.

Chapter 3: Linear Systems with LTI Uncertainty

The control problems of uncertain linear systems whose coefficients are LFT of uncertainties, which are linear time-invariant, are considered. This class of uncertain linear systems are thus defined in the LFT algebra. Both stabilization and $\mathcal{H}_\infty$-control problems are considered in the LFT algebra, and some necessary and sufficient conditions for the solvability in the LFT algebra are presented. In particular, the robust stabilization under real perturbation is characterized as a PBH-like test.

Chapter 4: Linear Systems with LTV Uncertainty

This chapter is a continuation of the discussion in the previous chapter. However, the uncertainty is allowed to be linear time-varying or nonlinear. This results in the $\mathcal{Q}$-stabilization problem, and some necessary and sufficient conditions for the $\mathcal{Q}$-stabilization are presented. The techniques in this chapter are motivated by those in [60]. In particular, the properties of $\mathcal{Q}$-stability, stabilizability, and detectability are characterized in terms of LMIs. The output feedback problem is solved via separation arguments; the stabilizing controllers are parameterized as LFT on some $\mathcal{Q}$-stable systems. The structure of the parameterized closed-loop map is examined. It is remarked that the parameterization formula reduces to the celebrated Youla-parameterizations for the conventional one-dimensional linear systems.

1.3.2 Control of Uncertain Nonlinear Systems

Chapter 5: Nonlinear Stabilizing Controller Parameterization

In this chapter, a parameterization formula of stabilizing controllers is derived for nonlinear control systems directly in the state space without using the coprime factorization. In the resulting parameterization formula, the asymptotically stabilizing controllers are characterized as fractional transformations of some asymptotically stable parameters. From the state-space point of view, a parameterized controller is structured as an observer which estimates the state of the plant with zero input, a state feedback which uses the estimated state, and a free stable parameter.

Chapter 6: Nonlinear $\mathcal{L}^1$-Control

A version of nonlinear generalization of the $\mathcal{L}^1$-control problem, which deals with the attenuation of persistent bounded disturbances in $\mathcal{L}_\infty$-sense, is investigated in this paper. The main idea in the $\mathcal{L}^1$-performance analysis and synthesis is to construct a certain invariant subset of the state-space such that achieving disturbance rejection is equivalent to restricting the state-dynamics to this set. The concepts from viability theory, nonsmooth analysis, and set-valued analysis play important roles. In particular, the $\mathcal{L}^1$-performance for a nonlinear system is characterized in terms of $\mathcal{L}^1$-performance domains, which are invariance state sets. A continuous static state-feedback $\mathcal{L}^1$-controller is constructed based on controlled $\mathcal{L}^1$-performance domains. In addition, the relation between the $\mathcal{L}^1$-control of a continuous-time system and the ℓ^1-control of its Euler approximated discrete-time systems is established.

Chapter 7: Nonlinear $\mathcal{H}_\infty$-Control: Sufficiency

Our goal in this chapter is to systematically examine the nonlinear $\mathcal{H}_\infty$-control problem in state space for a class of nonlinear systems and to obtain an $\mathcal{H}_\infty$ controller parameterization. This investigation is carried out in a methodical fashion. We follow similar techniques used in the linear case [60]. We first consider the $\mathcal{H}_\infty$-control problem for four special structures (FI, FC, DF, and OE). The solutions to the output feedback $\mathcal{H}_\infty$-control problem, are constructed by the solutions to the special problems. The solvability of this problem requires the coupled positive definite solutions to two decoupled HJIs. The standard separation principle in this case is re-examined and a class of $\mathcal{H}_\infty$-controllers are parameterized as a nonlinear fractional transformation on contractive, stable free nonlinear operators.

Chapter 8: Nonlinear $\mathcal{H}_\infty$-Control: Necessity

In this chapter, we propose an approach with promising computational properties to the state-space nonlinear $\mathcal{H}_\infty$-control problem, and characterize the solutions in terms of convex conditions instead of the Hamilton-Jacobi equations or inequalities. In particular, the $\mathcal{L}_2$-gains are characterized in terms of NLMIs. The output feedback $\mathcal{H}_\infty$-control are treated without the separation assumption, and the solutions are characterized by three NLMIs. Furthermore. under some (weak) separation structure assumptions, the solvability of the output-feedback problem implies the solvability by static state-feedback. Some issues related to the solutions to these NLMIs which yield the $\mathcal{H}_\infty$-control solutions are addressed.

Chapter 9: Nonlinear Robustness Analysis and Synthesis

In this chapter, we give state-space NLMI characterizations of stability and performance robustness for nonlinear uncertain systems with structured perturbations, and consider both analysis and synthesis problems. The robust stability, robust performance, and robustness synthesis are characterized in terms of NLMIs. From the characterizations, it is concluded that the computation needed for robustness analysis and synthesis of nonlinear uncertain systems is not more difficult than that for checking Lyapunov stability of nonlinear systems; in other words, the provided approach reduces the robustness analysis and synthesis problems to the possible easiest problems. Some computational issues for robustness analysis and synthesis are addressed.

1.4 Conventions

The following conventions are made in this paper.

Acronyms

AMI	Affine Matrix Inequality.
ARE	Algebraic Riccati Equation.
BIBO	Bounded-Input Bounded-Output.
BIBS	Bounded-Input Bounded-State.
DDI	Differential Dissipation Inequality.
DF	Disturbance Feedforward.
DGKF	Doyle-Glover-Khargonekar-Francis.
FC	Full Control.
FI	Full Information.
HJE	Hamilton-Jacobi Equation.
HJI	Hamilton-Jacobi Inequality.

IDI	Integral Dissipation Inequality.
I/O	Input-Output.
I/S	Input-to-State.
KYP	Kalman-Yakubovich-Popov.
LFT	Linear Fractional Transformation.
LMI	Linear Matrix Inequality.
LPV	Linear Parameter-Varying.
LQG	Linear Quadratic Gaussian.
LQR	Linear Quadratic Regulator.
LSC	Lower Semi-Continuous.
LSI	Linear Shift-Invariant.
LTI	Linear Time-Invariant.
LTV	Linear Time-Varying.
MIMO	Multiple-Input Multiple-Output.
NLMI	Nonlinear Matrix Inequality.
NLTI	Nonlinear Time-Invariant.
NLTV	Nonlinear Time-Varying.
OE	Output Estimation.
OF	Output Feedback.
PBH	Popov-Belevitch-Hautus.
SISO	Single-Input Single-Output.
SSV	Structured Singular Value.
USC	Upper Semi-Continuous.

Notations

$\mathbf{Z}$	Set of all integers.
$\mathbf{Z}^+$	Set of all nonnegative integers.
$\mathbf{R}$	Field of real numbers, i.e., $\mathbf{R} := (-\infty, \infty)$.
$\mathbf{R}^+$	Set of nonnegative real numbers, i.e., $\mathbf{R}^+ := [0, \infty) \subset \mathbf{R}$.
$\mathbf{C}$	Field of complex numbers.
$\mathbf{C}^+(\bar{\mathbf{C}}^+)$	Open (closed) right half complex plane.
$\mathbf{F}$	Field of real or complex numbers when the distinction is un-necessary.
$\mathbf{R}^n$	n-dimensional real Euclidean space.

$\mathbf{BR}^n$	$\mathbf{BR}^n := \{v \in \mathbf{R}^n \mid \|v\| \leq 1\}$.
$\mathbf{F}^n$	n-dimensional $\mathbf{F}$-valued vector space.
$\mathbf{R}^{n\times m}$	Set of real $n \times m$ matrices.
$\mathbf{F}^{n\times m}$	Set of $n \times m$ $\mathbf{F}$-valued matrices.
$\mathcal{B}_r$	Open ball in some Euclidean space centered at the origin and with radius $r > 0$ which is measured by Euclidean norm.
$\mathbf{X}$ (or $\mathbf{X}_o$)	State set which is a convex open subset of some Euclidean space and contains the origin.
$\mathbf{B\Delta}$	Elements of $\mathbf{\Delta}$ with norm less than one.
$\mathrm{Co}(S)$	Convex hull of set S.
$\mathrm{SPAN}(M)$	Space spanned by the column vectors of matrix M (or set M).
$\mathcal{N}(M)$	Null space annihilating the row vectors of matrix M.
$\mathrm{INT}(\mathbf{\Omega})$	Interior points of set $\mathbf{\Omega}$.
$\mathrm{DIAG}[D_1, \cdots, D_r]$	Diagonal or block-diagonal matrix with the designated diagonal elements.
A^T	Transpose of matrix A; if $A \in \mathbf{F}^{n\times m}$, then $A^T \in \mathbf{F}^{m\times n}$.
A^*	Conjugate transpose of matrix A; if $A \in \mathbf{F}^{n\times m}$, then $A^* \in \mathbf{F}^{m\times n}$.
A^{-1}	Inverse of square matrix $A \in \mathbf{F}^{n\times n}$.
A^+	Pseudo-inverse of matrix $A \in \mathbf{F}^{m\times n}$.
$\|u\|$	Euclidean norm of vector $u \in \mathbf{F}^n$; $\|u\| := \sqrt{u^*u}$.
$\rho(A)$	Spectral radius of square matrix $A \in \mathbf{F}^{n\times n}$.
$\bar{\sigma}(A)$	Largest singular value of matrix $A \in \mathbf{F}^{n\times m}$.
$\mu_\Delta(M)$	Structured singular value of square matrix $M \in \mathbf{F}^{n\times n}$ w.r.t. Δ.
$\mathcal{Q}_\Delta(M)$	$\mathcal{Q}$-value of square matrix $M \in \mathbf{F}^{n\times n}$ w.r.t. Δ.
$\mu_\Delta(M, N)$	Generalized structured singular value of ordered matrix pair $(M, N) \in \mathbf{F}^{n\times m} \times \mathbf{R}^{n\times m}$ w.r.t. Δ.
$\mathrm{TR}(A)$	Trace of square matrix $A \in \mathbf{F}^{n\times n}$.
$\mathrm{DET}(A)$	Determinant of square matrix $A \in \mathbf{F}^{n\times n}$.
$\mathrm{RANK}(A)$	Rank of matrix $A \in \mathbf{F}^{n\times m}$.
$\mathrm{KER}(A)$	Kernel of matrix $A \in \mathbf{F}^{n\times m}$.
$\mathrm{RE}(x)$	Real part of complex number $x \in \mathbf{C}$.
$\mathrm{IM}(x)$	Imaginary part of complex number $x \in \mathbf{C}$.
$\mathrm{SUPP}(\psi)$	Support of function ψ.
$\mathrm{DOM}(F)$	Domain of map F.
$\mathrm{GRAPH}(F)$	Graph of map F.

$T_K(x)$	Contingent cone to set K at $x \in K$.
$\mathcal{C}([0,\infty), X)$	Class of continuous maps: $[0,\infty) \rightarrow X$.
$\mathcal{C}^0(\mathbf{\Omega}, \mathbf{F})$	Class of continuous functions: $\mathbf{\Omega} \rightarrow \mathbf{F}$.
$\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$	Class of functions: $\mathbf{\Omega} \rightarrow \mathbf{F}$, which are continuously differentiable k times.
$\mathcal{F}(\mathbf{\Delta}, \mathbf{F})$	Class of linear fractional transformations: $\mathbf{\Delta} \rightarrow \mathbf{F}$.
$\mathcal{K}$	Class of continuous and strictly increasing functions $\gamma : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ with $\gamma(0) = 0$.
$\mathcal{K}_\infty$	Class of functions γ in $\mathcal{K}$ with $\gamma(s) \rightarrow \infty$ as $s \rightarrow \infty$.
$\mathcal{KL}$	Class of functions $\beta : \mathbf{R}^+ \times \mathbf{R}^+ \rightarrow \mathbf{R}^+$ such that for each fixed t, the mapping $\beta(\cdot, t)$ is in $\mathcal{K}$ and for each s, $\beta(s,t)$ is decreasing to 0 as $t \rightarrow \infty$.
$\mathcal{RH}_\infty$	Class of real rational matrix-valued functions on $\mathbf{C}$ analytic in $\mathbf{C}^+$ (or $\|z\| < 1$).
$\mathcal{C}([0,\infty), X)$	Class of continuous maps: $[0,\infty) \rightarrow X$.
$\mathcal{L}_2[0,T]$	Lebesgue space of all measurable (vector-valued) functions u : $[0,T] \rightarrow \mathbf{R}^m$ such that $\int_o^T \|u(t)\|^2 \, dt < \infty$.
$\mathcal{L}_2(\mathbf{R}^+)$ $(\mathcal{L}_2^p(\mathbf{R}^+))$	Lebesgue space of all measurable real (p-dimensional) vector-valued functions $u : \mathbf{R}^+ \rightarrow \mathbf{R}^m$ such that $\int_{\mathbf{R}^+} \|u(t)\|^2 \, dt < \infty$.
$\mathcal{L}_2^e[0,\infty)$	Extended space of $\mathcal{L}_2[0,\infty)$, i.e., set of measurable vector-valued functions $u(t)$ on $\mathbf{R}^+$ such that $P_T u(t) \in \mathcal{L}_2(\mathbf{R}^+)$ for all $T \in \mathbf{R}^+$.
$\mathcal{L}_\infty[0,\infty)$	Space of vector-valued functions $u : \mathbf{R}^+ \rightarrow \mathbf{R}^p$ which are measurable and essentially bounded.
$\mathcal{BL}_\infty[0,\infty)$	$\mathcal{BL}_\infty[0,\infty) := \{w \in \mathcal{L}_\infty[0,\infty) \| \|w\|_\infty \leq 1\}$.
$\mathcal{L}_\infty^e[0,\infty)$	Extended space of $\mathcal{L}_\infty[0,\infty)$.
$\|u\|_\infty$	Infinity norm of vector-valued function $u \in \mathcal{L}_\infty[0,\infty)$, i.e., $\|u\|_\infty :=$ess-sup$\{\|u(t)\| : t \in \mathbf{R}^+\}$.
P_T	Truncation operator for any $T \geq 0$.
Q_T	$Q_T := I - P_T$.
$\mathcal{F}_l(M, Q)$	Fractional transformation of operator M on operator Q [144, 73].
$\mathcal{S}(M_1, M_2)$	Redheffer product of operators M_1 and M_2 [144].
$\mathcal{SP}_{af}$	Class of input-affine locally stable nonlinear systems.
$\mathcal{SP}_{loc}$	Class of locally stable nonlinear systems.
$\mathcal{SP}_{I/S}$	Class of input-to-state stable nonlinear systems.
$\mathcal{FG}$	Class of input-affine stable nonlinear systems having $\mathcal{L}_2$-gains ≤ 1 with smooth storage functions.
$\frac{\partial V}{\partial x}(x)$	Gradient of $\mathcal{C}^1$ function $V : \mathbf{R}^n \rightarrow \mathbf{R}$: $\frac{\partial V}{\partial x}(x) = [\frac{\partial V}{\partial x_1}(x), \cdots, \frac{\partial V}{\partial x_n}(x)]$.

PART I

Uncertain Linear Systems

Chapter 2

Control of Linear Parameter-Varying Systems

2.1 Introduction

In many applications, the structure of a control system is known to be fixed, but some of the coefficients of its mathematical model can not be exactly estimated or determined until the system is in operation. These kinds of systems, whose coefficients are functions of parameters, are called **linear parameter-varying** (LPV) systems. The control problem of interest is to design controllers for such a system such that the closed loop system satisfies a set of performance specifications for all possible values of the parameters. Two problems to be treated in this thesis are stabilization and $\mathcal{H}_\infty$-control. In such cases, a designed controller which also depends on the parameters, instead of a single linear time-invariant controller, is usually required to achieve the required specifications. Such controllers can be implemented for real-time operation in terms of gain/dynamical scheduling [162, 163, 164, 149, 145] or adaptive/supervisory control schemes [176, 125].

In this chapter, we are concerned with the linear systems whose coefficients are continuous and smooth functions of (unknown) parameters. The systems in these classes will be viewed as linear systems defined over some (commutative) algebras of scalar-valued functions. An important special case in which a system has coefficients depending on the unknowns in a linear fractional transformation fashion will be treated in the next two chapters.

Let $\mathbf{F}$ be the field $\mathbf{R}$ of real numbers or the field $\mathbf{C}$ of complex numbers. $\mathbf{F}^N$ for some positive integer N denotes the space of N-element vectors with entries in the field $\mathbf{F}$. The $\|x\|$ will be the norm of $x \in \mathbf{F}^N$. $\mathbf{F}^{n\times m}$ for some positive integers n and m denotes the space of $n \times m$ matrices with entries in the field $\mathbf{F}$.

Suppose $\mathbf{\Omega} \subset \mathbf{F}^N$. Let $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$ (for some integer $k \geq 0$) denote the set of all $\mathbf{F}$-valued functions which is continuous on $\mathbf{\Omega}$ and $\mathcal{C}^k$ on $\textsc{Int}(\mathbf{\Omega})$. Specially, $\mathcal{C}(\mathbf{\Omega}, \mathbf{F})$ and $\mathcal{C}^\infty(\mathbf{\Omega}, \mathbf{F})$ are two sets of continuous functions on $\mathbf{\Omega}$, and the latter further requires those functions to be $\mathcal{C}^\infty$ on $\textsc{Int}(\mathbf{\Omega})$.

The set $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$ is a commutative ring with the pointwise operations

$$(f+g)(\omega) = f(\omega) + g(\omega)$$
$$(fg)(\omega) = f(\omega)g(\omega),$$

where $f, g \in \mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$, $\omega \in \mathbf{\Omega}$. With these pointwise operations and the following scalar multiplication

$$(\alpha f)(\omega) = \alpha f(\omega), \alpha \in \mathbf{F},$$

$\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$ is a **commutative algebra** (over $\mathbf{F}$).

Given fixed positive integers $m, n,$ and p, an m-input, p-output, and an n-th order linear time invariant (discrete-time or continuous time) system Σ over $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$ is denoted by the following dynamical equations.

$$\left\{ \begin{array}{lcl} x(t+1) & = & A(\omega)x(t) + B(\omega)u(t) \\ y(t) & = & C(\omega)x(t) + D(\omega)u(t) \end{array} \right. \tag{2.1}$$

with $t \in \mathbf{Z}^+$; or

$$\left\{ \begin{array}{lcl} \dot{x}(t) & = & A(\omega)x(t) + B(\omega)u(t) \\ y(t) & = & C(\omega)x(t) + D(\omega)u(t) \end{array} \right. \tag{2.2}$$

with $t \in \mathbf{R}^+$; where the matrices $A, B, C,$ and D have their entries in $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$.

In the following, we mainly emphasize system (2.1) over $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$ for some $k \in \mathbf{Z}^+$. Some necessary and sufficient conditions are provided for the system to have the controllers which solve the stabilization and $\mathcal{H}_\infty$-control problem over $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$, i.e., the coefficients of the controllers have entries in $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$. The results can be correspondingly provided for the continuous system (2.2) over $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$ for some $k \in \mathbf{Z}^+$. To this end, we first review some results on stabilization and $\mathcal{H}_\infty$-control of LTI systems in the next section to develop some tools.

2.2 Stabilization and $\mathcal{H}_\infty$-Control of LTI Systems

In this section, we will review some characterizations of the basic control properties for linear time-invariant systems, such as stabilizability, detectability, and solvability of $\mathcal{H}_\infty$-control problem. These characterizations are in terms of **linear matrix-inequalities** (LMIs).

2.2.1 Stabilization of LTI Systems

Two well known characterizations of stabilizability and detectability are PBH tests [95, p. 366], and Riccati equations/inequalities. In this subsection, we will give some characterizations for stabilizability and detectability in terms of Lyapunov inequalities, which are LMIs and are more computable. The feedback laws can be constructed by the solutions of the corresponding LMIs.

Stabilization of Discrete-Time Systems

In this section, the following LTI discrete-time system over $\mathbf{F}$ is considered.

$$\begin{cases} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{cases} \tag{2.3}$$

with $t \in \mathbf{Z}^+$, where x, u, y denote the n, p, q-dimensional state vector, the input vector, and the output vector, respectively.

It is known that system (2.3) (with $u(t) = 0$) is **stable** if and only if $\rho(A) < 1$, or $\bar{\sigma}(D^{-1}AD) < 1$ for some invertible matrix $D \in \mathbf{F}^{n\times n}$, i.e., there exists a positive definite matrix $P \in \mathbf{F}^{n\times n}$ such that it satisfies the following Lyapunov inequality.

$$APA^* - P < 0$$

which is equivalent to $\bar{\sigma}(D^{-1}AD) < 1$ for some positive definite matrix $D = P^{\frac{1}{2}} \in \mathbf{F}^{n\times n}$. It is **stabilizable** if there is a state feedback $F \in \mathbf{F}^{p\times n}$ such that the closed loop system, $x(t+1) = (A+BF)x(t)$, is stable, i.e., $\rho(A+BF) < 1$. It is **detectable** if there is an output injection $L \in \mathbf{F}^{n\times q}$ such that system $x(t+1) = (A+LC)x(t)$ is stable, i.e., $\rho(A+LC) < 1$. Stabilizability and detectability are dual concepts. In the following, only stabilizability is discussed in detail. The results for detectability can be derived by duality argument.

We first summarize the following properties for stabilizability.

Lemma 2.2.1 *Consider system (2.3), then the following statements are equivalent.*

(i) System (2.3) is stabilizable.

(ii) $\text{RANK}\begin{bmatrix} zI - A & B \end{bmatrix} = n$ *for all* $z \in \mathbf{C}$ *and* $|z| \geq 1$.

(iii) There exists a positive definite $\mathbf{F}$*-valued matrix* P *which satisfies the following Riccati equation,*

$$M + A^*PA - P - A^*PB(N + B^*PB)^{-1}B^*PA = 0 \tag{2.4}$$

for some positive definite matrices $M \in \mathbf{F}^{n\times n}$ *and* $N \in \mathbf{F}^{p\times p}$.

The following result is from [97, Theorem 5.6].

Lemma 2.2.2 *Suppose system (2.3) is a stabilizable over* $\mathbf{F}$*. Given some positive definite matrices* $M \in \mathbf{F}^{n\times n}$ *and* $N \in \mathbf{F}^{p\times p}$*, define a Riccati difference equation:*

$$P_{k+1} = M + A^*P_kA - A^*P_kB(B^*P_kB + N)^{-1}B^*P_kA, \quad k \in \mathbf{Z}^+ \tag{2.5}$$

with initial condition $P_0 = I$*, and*

$$F_k := (B^*P_kB + N)^{-1}B^*P_kA, \quad k \in \mathbf{Z}^+. \tag{2.6}$$

Then there exist a positive integer k_a *such that* F_k *is a stabilizing feedback for all* $k \geq k_a$.

Suppose $\text{RANK}(B) = p < n$ and $\text{RANK}(C) = q < n$. Let $B_\perp \in \mathbf{F}^{n\times(n-p)}$ be such that $B^*B_\perp = 0$ and $\text{RANK}\begin{bmatrix} B & B_\perp \end{bmatrix} = n$, and $C_\perp \in \mathbf{F}^{(n-q)\times n}$ be such that $CC_\perp^* = 0$ and $\text{RANK}\begin{bmatrix} C^* & C_\perp^* \end{bmatrix} = n$. We have the following characterizations.

Theorem 2.2.3 *Consider system (2.3), then the following statements are equivalent.*

(i) System (2.3) is stabilizable.

(ii) There exists a positive definite matrix $P \in \mathbf{F}^{n\times n}$ *such that the following LMI is satisfied*

$$B_\perp^*(APA^* - P)B_\perp < 0. \tag{2.7}$$

(iii) There exists a positive definite matrix $P \in \mathbf{F}^{n\times n}$ *such that the following LMI is satisfied*

$$APA^* - P - BB^* < 0. \tag{2.8}$$

Moreover, if $P = P^* > 0$ *satisfies any of the above inequalities, then a stabilizing static state feedback matrix can be chosen as*

$$F = -(B^*P^{-1}B)^{-1}B^*P^{-1}A. \tag{2.9}$$

The following lemma is used in the proof of this theorem.

Lemma 2.2.4 *Assume* $(A,B) \in \mathbf{F}^{n\times n} \times \mathbf{F}^{n\times p}$ *and* $\mathrm{RANK}(B) = p < n$. *Let* $B_\perp \in \mathbf{F}^{n\times(n-p)}$ *and* $B_0 \in \mathbf{F}^{p\times n}$ *be such that* $B_\perp^* B = 0$ *and* $\begin{bmatrix} B_0 & B_\perp \end{bmatrix}$ *is unitary. Then*

$$\inf_{F\in\mathbf{F}^{p\times n}} \bar{\sigma}(A+BF) = \bar{\sigma}(B_\perp^* A)$$

and the infimum is attained by $F = -(B_0^* B)^{-1} B_0^* A$.

Proof. Since $U := \begin{bmatrix} B_0 & B_\perp \end{bmatrix}$ is unitary,

$$\inf_{F\in\mathbf{F}^{p\times n}} \bar{\sigma}(A+BF) = \inf_{F\in\mathbf{F}^{p\times n}} \bar{\sigma}(U^*(A+BF))$$

$$= \inf_{F\in\mathbf{F}^{p\times n}} \bar{\sigma}\left(\begin{bmatrix} B_0^*A + B_0^*BF \\ B_\perp^* A \end{bmatrix}\right) = \bar{\sigma}\left(\begin{bmatrix} 0 \\ B_\perp^* A \end{bmatrix}\right) = \bar{\sigma}(B_\perp^* A).$$

Moreover the infimum is attained if $B_0^*A + B_0^*BF = 0$ or $F = -(B_0^*B)^{-1}B_0^*A$. □

Proof. **[Theorem 2.2.3]** We will show that **(i)** $\Leftrightarrow$ **(ii)** and **(ii)**$\Leftrightarrow$ **(iii)**.

(i) $\Leftrightarrow$ **(ii):** The given system is stabilizable if and only if there is a positive definite matrix $D \in \mathbf{F}^{n\times n}$ such that

$$1 > \inf_{F\in\mathbf{F}^{p\times n}} \bar{\sigma}(D(A+BF)D^{-1}) = \inf_{F\in\mathbf{F}^{p\times n}} \bar{\sigma}(DAD^{-1} + DBFD^{-1}).$$

Let $V_\perp^* = (B_\perp^*(D^*D)^{-1}B_\perp)^{-\frac{1}{2}} B_\perp^* D^{-1}$, it is easy to check that $V_\perp^* V_\perp = I$ and $V_\perp^*(DB) = 0$. By Lemma 2.2.4, we have

$$1 > \inf_{F\in\mathbf{F}^{p\times n}} \bar{\sigma}(D(A+BF)D^{-1}) = \bar{\sigma}(V_\perp^* DAD^{-1})$$

or equivalently,

$$(V_\perp^* DAD^{-1})(V_\perp^* DAD^{-1})^* < I. \tag{2.10}$$

Take $P = (D^*D)^{-1}$, then $P \in \mathbf{F}^{n\times n}$ and $P = P^* > 0$, hence (2.10) is equivalent to the following inequality.

$$(B_\perp^* P B_\perp)^{-\frac{1}{2}} B_\perp^* A P A^* B_\perp (B_\perp^* P B_\perp)^{-\frac{1}{2}} - I < 0,$$

or

$$B_\perp^*(APA^* - P)B_\perp < 0.$$

Moreover, if some $P \in \mathbf{F}^{n\times n}$ with $P = P^* > 0$ satisfies the above inequality, then we can construct a constant state feedback matrix F via Lemma 2.2.4 such that $\rho(A+BF) < 1$. Let $V_0^* = (B^*(D^*D)B)^{-1/2}B^*D^*$, then $[V_0, V_\perp]$ unitary; thus $FD^{-1} = -(V_0^*DB)^{-1}V_0^*DAD^{-1}$ by the preceding lemma. Therefore,

$$F = -(V_0^*DB)^{-1}V_0^*DA = -(B^*P^{-1}B)^{-1}B^*P^{-1}A.$$

(ii) ⇒ (iii): There exists a positive definite matrix $Q \in \mathbf{F}^{n\times n}$ such that

$$AQA^* - Q - BB^* < 0,$$

which is equivalent to

$$\begin{bmatrix} B_\perp^* \\ B^* \end{bmatrix} (AQA^* - Q - BB^*) \begin{bmatrix} B_\perp & B \end{bmatrix} < 0, \tag{2.11}$$

i.e.,

$$\begin{bmatrix} B_\perp^*(AQA^* - Q)B_\perp & B_\perp^*(AQA^* - Q)B \\ B^*(AQA^* - Q)B_\perp & -(B^*B)^2 \end{bmatrix} < 0.$$

By Schur complement argument, it follows that the above inequality holds if and only of

$$B_\perp^*(AQA^* - Q)B_\perp + (B_\perp^*(AQA^* - Q)B)(B^*B)^{-2}(B^*(AQA^* - Q)B_\perp) < 0, \tag{2.12}$$

which implies

$$B_\perp^*(AQA^* - Q)B_\perp < 0,$$

which is exactly (2.7) with $P = Q$ positive definite.

(iii) ⇒ (ii): Conversely, as $(B_\perp^*(APA^* - P)B)(B^*B)^{-2}(B^*(APA^* - P)B_\perp)$ is a symmetric matrix, if a positive definite matrix P satisfies (2.7), then there is a $\alpha > 0$, such that

$$B_\perp^*(APA^* - P)B_\perp + \alpha(B_\perp^*(APA^* - P)B)(B^*B)^{-2}(B^*(APA^* - P)B_\perp) < 0.$$

Let $Q = \alpha P$ which is positive definite, then the above inequality is equivalent to inequality (2.12), which is equivalent to (2.11) or (2.8). □

Remark 2.2.5 *The above LMI characterization (2.8) is derived directly from the definition of stabilizability. It can also be derived from Riccati characterization*[1]*. In fact, system (2.3)*

[1]This observation is provided by Zhou [206].

is stabilizable if and only if there exists a positive definite matrix $Q \in \mathbf{F}^{n\times n}$ which satisfies the following Riccati inequality:

$$A^*QA - Q - A^*QB(I + B^*QB)^{-1}B^*QA < 0, \tag{2.13}$$

which is equivalent to

$$A^*(I + QBB^*)^{-1}QA - Q < 0,$$

or

$$A^*Q^{\frac{1}{2}}(I + Q^{\frac{1}{2}}BB^*Q^{\frac{1}{2}})^{-1}Q^{\frac{1}{2}}A - Q < 0.$$

By Schur complement argument, the above inequality is equivalent to

$$\begin{bmatrix} Q & A^*Q^{\frac{1}{2}} \\ Q^{\frac{1}{2}}A & I + Q^{\frac{1}{2}}BB^*Q^{\frac{1}{2}} \end{bmatrix} > 0.$$

Using Schur complement argument again,

$$I + Q^{\frac{1}{2}}BB^*Q^{\frac{1}{2}} - Q^{\frac{1}{2}}AQ^{-1}A^*Q^{\frac{1}{2}} > 0.$$

Now take $P = Q^{-1}$, it follows that

$$APA^* - P - BB^* < 0.$$

It is noted that the above characterizations of stabilizability have a direct application to the determination of the stability degree for the closed loop system.

Corollary 2.2.6 *Consider system (2.3), there exists a state feedback $F \in F^{p\times n}$ such that $\rho(A + BF) < \alpha^{-1}$ for some $\alpha \geq 1$ if and only if there exists a positive definite matrix $P \in \mathbf{F}^{n\times n}$ such that the following LMI is satisfied*

$$B_\perp^*(\alpha^2APA^* - P)B_\perp < 0, \tag{2.14}$$

or the following LMI is satisfied

$$\alpha^2APA^* - P - BB^* < 0. \tag{2.15}$$

Moreover, if $P = P^* > 0$ satisfies any of the above inequalities, then a stabilizing static state feedback matrix which achieves the stability degree can be chosen as

$$F = -(B^*P^{-1}B)^{-1}B^*P^{-1}A. \tag{2.16}$$

Dually, we have the following results about detectability.

Theorem 2.2.7 *Consider system (2.3), then the following statements are equivalent.*

(i) System (2.3) is detectable.

(ii) There exists a positive definite matrix $X \in \mathbf{F}^{n\times n}$ *such that the following LMI is satisfied*

$$C_\perp A^* X A C_\perp^* - C_\perp X C_\perp^* < 0. \tag{2.17}$$

(iii) There exists a positive definite matrix $X \in \mathbf{F}^{n\times n}$ *such that the following LMI is satisfied*

$$A^* X A - X - C^* C < 0. \tag{2.18}$$

Moreover, if $X = X^* > 0$ *satisfies any of the above inequalities, then a stabilizing static state feedback matrix can be chosen as*

$$L = -AX^{-1}C^*(CX^{-1}C^*)^{-1}. \tag{2.19}$$

Stabilization of Continuous-Time Systems

In this section, we will consider the following LTI (continuous time) system over F

$$\left\{ \begin{array}{lcl} \dot{x}(t) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) + Du(t) \end{array} \right. \tag{2.20}$$

with $t \in \mathbf{R}^+$, where x, u, y are n, p, q-dimensional state vector, input vector, and output vector, respectively.

It is known that system (2.20) (with $u(t) = 0$) is **stable** if and only if there exists a positive definite matrix $P \in \mathbf{F}^{n\times n}$ such that it satisfies the Lyapunov inequality $A^*P + PA < 0$.

System (2.20) is **stabilizable** if there is a state feedback $F \in \mathbf{F}^{p\times n}$ such that system $\dot{x} = (A + BF)x$ is stable. It is **detectable** if there is a output injection $L \in \mathbf{F}^{n\times q}$ such that system $\dot{x} = (A + LC)x$ is stable.

Suppose $\textsc{Rank}(B) = p < n$ and $\textsc{Rank}(C) = q < n$. Let $B_\perp \in \mathbf{F}^{n\times(n-p)}$ be such that $B^* B_\perp = 0$ and $\textsc{Rank}\begin{bmatrix} B & B_\perp \end{bmatrix} = n$, and $C_\perp \in \mathbf{F}^{(n-q)\times n}$ be such that $CC_\perp^* = 0$ and $\textsc{Rank}\begin{bmatrix} C^* & C_\perp^* \end{bmatrix} = n$. We have the following characterizations.

Theorem 2.2.8 *Consider system (2.20), then the following statements are equivalent.*

(i) System (2.20) is stabilizable.

(ii) There exists a solution X in the class of positive definite **F***-valued matrices to the following LMI,*

$$B_\perp^*(XA^* + AX)B_\perp < 0. \tag{2.21}$$

(iii) There exists a positive definite **F***-valued matrix X such that the following LMI is satisfied,*

$$XA^* + AX - BB^* < 0. \tag{2.22}$$

Moreover, a state feedback which yields the stability of the closed loop system is given by $F = -\frac{1}{2}B^*X^{-1}$.

Proof. We will show that **(i)** $\Rightarrow$ **(ii)** $\Rightarrow$ **(iii)** $\Rightarrow$ **(i)**.
(i) $\Rightarrow$ **(ii)**: Let F be such that $\dot{x} = (A+BF)x$ is stable, then there exists a positive definite matrix $X \in \mathbf{F}^{n \times n}$ such that

$$X(A+BF)^* + (A+BF)X < 0,$$

which implies

$$B_\perp^*(X(A+BF)^* + (A+BF)X)B_\perp < 0,$$

which in turn implies (2.21).
(ii) $\Rightarrow$ **(iii)**: Follows from the similar arguments in the proof ((**ii**) $\Leftrightarrow$ (**iii**)) of Theorem 2.2.3.
(iii) $\Rightarrow$ **(i)**: Let $X \in \mathbf{F}^{n \times n}$ be positive definite and satisfies

$$XA^* + AX - BB^* < 0,$$

or

$$X(A - \frac{1}{2}BB^*X^{-1})^* + (A - \frac{1}{2}BB^*X^{-1})X < 0,$$

which implies (i).

$\square$

Remark 2.2.9 *Statement (iii) can actually derived from Riccati equation characterizations. In fact, system (2.20) is stabilizable if and only if there exists a unique positive definite matrix $P \in \mathbf{F}^{n \times n}$ such that the following Riccati equation is satisfied,*

$$PA + A^*P - PBN^{-1}B^*P + M = 0 \tag{2.23}$$

for some $0 < M \in \mathbf{F}^{n\times n}$ *and* $0 < N \in \mathbf{F}^{p\times p}$*; or equivalently, there exists a solution* P *in the class of positive semi-definite matrices to the following Riccati inequality,*

$$PA + A^*P - PBB^*P < 0. \tag{2.24}$$

Now define $Y = P^{-1}$*, (2.24) is equivalent to (2.22).*

Similarly, we have the following results on determining the stability degree for the closed loop system.

Corollary 2.2.10 *Consider system (2.20), given* $\beta \in \mathbf{R}^+$ *the following statements are equivalent. There exists a state feedback* $F \in \mathbf{F}^{p\times n}$ *such that* $\mathrm{Re}(\lambda) < -\beta$ *for all* $\lambda \in \Lambda(A + BF)$ *if and only if there exists a positive definite matrix* $X \in \mathbf{F}^{n\times n}$ *which satisfies the following LMI,*

$$B_\perp^*(X(A + \beta I)^* + (A + \beta I)X)B_\perp < 0, \tag{2.25}$$

or the following LMI.

$$X(A + \beta I)^* + (A + \beta I)X - BB^* < 0. \tag{2.26}$$

Moreover, a state feedback which yields the required stability degree of the closed loop system is given by $F = -\frac{1}{2}B^*X^{-1}$ *with* $X > 0$ *satisfies (2.26).*

Dually, we have the following results for detectability.

Theorem 2.2.11 *Consider system (2.20), then the following statements are equivalent.*

(i) System (2.20) is detectable.

(ii) There exists a positive definite matrix $X \in \mathbf{F}^{n\times n}$ *such that it satisfies the following LMI,*

$$C_\perp(XA + A^*X)C_\perp^* < 0, \tag{2.27}$$

or the following LMI

$$XA + A^*X - C^*C < 0. \tag{2.28}$$

Moreover, a state feedback which yields that the stability of the closed loop system is given by $L = -\frac{1}{2}X^{-1}C^*$ *with* $X > 0$ *satisfying (2.28).*

2.2.2 $\mathcal{H}_\infty$-Control of LTI Systems

The solutions of linear $\mathcal{H}_\infty$-control problems are characterized in terms of Riccati equations [60]. However, in the following, we only review the characterization in terms of LMIs, which were first appeared in [119, 133]. In the following review, the results are taken from [69]. The following system is considered.

$$\begin{bmatrix} z(t) \\ y(t) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \begin{bmatrix} w(t) \\ u(t) \end{bmatrix} \tag{2.29}$$

where $t \in \mathbf{Z}^+$ for a discrete-time system and $t \in \mathbf{R}^+$ for a continuous-time system; w, u, z, y are p, p_2, p, q_2-dimensional disturbance input, control input, regulated output, and measured output vectors, respectively. It is assumed that $n + p > q_2$ and $n + p > p_2$. Define

$$\bar{A} := \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix}, \quad B := \begin{bmatrix} B_2 \\ D_{12} \end{bmatrix}^*, \quad C := \begin{bmatrix} C_2 & D_{21} \end{bmatrix}.$$

Suppose $\textsc{Rank}(B) = p_2$ and $\textsc{Rank}(C) = q_2$.

The system (2.29) is said to have $\mathcal{H}_\infty$**-control solution** if for each fixed $\omega \in \mathbf{\Omega}$, there exists a output feedback controller

$$u(t) = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right] y(t) \tag{2.30}$$

with the coefficient matrices having entries in $\mathbf{F}$, such that the closed loop system is well-posed and stable, and has $\mathcal{H}_\infty$-norm < 1. Moreover, the problem is said to have state feedback solutions if the above measurement output y, based on which the control action is produced by (2.30), is all states; the problem is said to accept full-information feedback solutions if both states and disturbances are available to the measurement output y.

Discrete-Time Systems

We first have the following result for state-feedback solutions.

Theorem 2.2.12 *Consider the system (2.29) with $t \in \mathbf{Z}^+$. The following statements are equivalent.*

(i) The $\mathcal{H}_\infty$-control problem has a full-information feedback solution.

(ii) There exists a positive definite matrix $P \in \mathbf{F}^{n\times n}$ such that it satisfies the following LMI,

$$B_\perp^* \begin{bmatrix} APA^* - P + B_1B_1^* & APC_1^* + B_1D_{11}^* \\ C_1PA^* + D_{11}B_1^* & C_1PC_1^* + D_{11}D_{11}^* - I \end{bmatrix} B_\perp < 0, \tag{2.31}$$

where $B_\perp : \mathbf{\Omega} \rightarrow \mathbf{F}^{(n+q_1)\times(n+q_1-p_2)}$ *is such that* $\mathrm{NULL}(B) = \mathrm{SPAN}(B_\perp)$.

(iii) For each fixed $\omega \in \mathbf{\Omega}$*, there exist a positive definite matrix* $P \in \mathbf{F}^{n\times n}$ *and a real number* $\alpha > 0$ *such that they satisfy the following LMI*[2]*,*

$$\bar{A}\bar{P}\bar{A}^* - \bar{P} - BB^* < 0, \quad \bar{P} := \begin{bmatrix} P & 0 \\ 0 & \alpha I \end{bmatrix}. \tag{2.32}$$

It is noted that if (i) is satisfied, then the solutions can be chosen as static feedbacks. The LMI characterization to the full-information feedback $\mathcal{H}_\infty$-control problem is obtained for the first time in [119, 133].

Proof. **(i)** $\Leftrightarrow$ **(ii)** is verified in [69]; **(ii)** $\Leftrightarrow$ **(iii)** follows from the technique used in the proof of theorem 2.2.3 ((ii) $\Leftrightarrow$ (iii)). □

For the state-feedback solution, we have the following characterization [86].

Theorem 2.2.13 *Consider the system (2.29) with* $t \in \mathbf{Z}^+$*. Then the* $\mathcal{H}_\infty$*-control problem has a static state feedback solution if and only if there exists a positive definite matrix* $P \in \mathbf{F}^{n\times n}$ *such that it satisfies the following two LMIs,*

$$B_\perp^* \begin{bmatrix} APA^* - P + B_1B_1^* & APC_1^* + B_1D_{11}^* \\ C_1PA^* + D_{11}B_1^* & C_1PC_1^* + D_{11}D_{11}^* - I \end{bmatrix} B_\perp < 0,$$

$$\begin{bmatrix} B_1B_1^* - P & B_1D_{11}^* \\ D_{11}B_1^* & D_{11}D_{11}^* - I \end{bmatrix} < 0,$$

where $B_\perp : \mathbf{\Omega} \rightarrow \mathbf{F}^{(n+q_1)\times(n+q_1-p_2)}$ *is such that* $\mathrm{NULL}(B) = \mathrm{SPAN}(B_\perp)$.

The following result characterizes a dynamic output feedback solution to the $\mathcal{H}_\infty$-control problem.

[2]or equivalently,

$$\begin{bmatrix} APA^* - P + \alpha B_1B_1^* - B_2B_2^* & APC_1^* + \alpha B_1D_{11}^* - B_2D_{12}^* \\ C_1PA^* + \alpha D_{11}B_1^* - D_{12}B_2^* & \alpha(C_1PC_1^* + D_{11}D_{11}^* - I) - D_{12}D_{12}^* \end{bmatrix} < 0.$$

Theorem 2.2.14 *Consider the system (2.29) with $t \in \mathbf{Z}^+$. The $\mathcal{H}_\infty$-control problem has solution if and only if there are two positive definite matrices $P, Q \in \mathbf{F}^{n\times n}$ such that they satisfy the following LMIs,*

$$B_\perp^* \begin{bmatrix} APA^* - P + B_1B_1^* & APC_1^* + B_1D_{11}^* \\ C_1PA^* + D_{11}B_1^* & C_1PC_1^* + D_{11}D_{11}^* - I \end{bmatrix} B_\perp < 0, \tag{2.33}$$

$$C_\perp^* \begin{bmatrix} A^*QA - Q + C_1^*C_1 & A^*QB_1 + C_1^*D_{11} \\ B_1^*QA + D_{11}^*C_1 & B_1^*QB_1 + D_{11}^*D_{11} - I \end{bmatrix} C_\perp < 0, \tag{2.34}$$

$$\begin{bmatrix} P & I \\ I & Q \end{bmatrix} \geq 0, \tag{2.35}$$

where $B_\perp : \mathbf{\Omega} \to \mathbf{F}^{(n+q_1)\times(n+q_1-p_2)}$ and $C_\perp : \mathbf{\Omega} \to \mathbf{F}^{(n+p_1)\times(n+p_1-q_2)}$ are such that $\mathrm{N{\scriptsize ULL}}(B) = \mathrm{S{\scriptsize PAN}}(B_\perp)$ *and* $\mathrm{N{\scriptsize ULL}}(C) = \mathrm{S{\scriptsize PAN}}(C_\perp)$.

Continuous-Time Systems

The results for the continuous-time systems are similar to the ones for discrete-time systems.

Theorem 2.2.15 *Consider the system (2.29) with $t \in \mathbf{R}^+$. Then the $\mathcal{H}_\infty$-control problem has full-information feedback solution if and only if there exists a positive definite matrix $P \in \mathbf{F}^{n\times n}$ such that it satisfies the following LMI,*

$$B_\perp^* \begin{bmatrix} AP + PA^* + B_1B_1^* & PC_1^* + B_1D_{11}^* \\ C_1P + D_{11}B_1^* & D_{11}D_{11}^* - I \end{bmatrix} B_\perp < 0, \tag{2.36}$$

where $B_\perp : \mathbf{\Omega} \to \mathbf{F}^{(n+q_1)\times(n+q_1-p_2)}$ is such that $\mathrm{N{\scriptsize ULL}}(B) = \mathrm{S{\scriptsize PAN}}(B_\perp)$.

It is noted that the feedback solution can be static if exists; the state-feedback solution additionally requires that $D_{11}D_{11}^* - I < 0$. The following result characterizes a dynamic output feedback solution to the $\mathcal{H}_\infty$-control problem.

Theorem 2.2.16 *Consider the system (2.29) with $t \in \mathbf{Z}^+$. The $\mathcal{H}_\infty$-control problem has solution if and only if there are two positive definite matrices $P, Q \in \mathbf{F}^{n\times n}$ such that they satisfy the following LMIs,*

$$B_\perp^* \begin{bmatrix} AP + PA^* + B_1B_1^* & PC_1^* + B_1D_{11}^* \\ C_1P + D_{11}B_1^* & D_{11}D_{11}^* - I \end{bmatrix} B_\perp < 0, \tag{2.37}$$

$$C_\perp^* \begin{bmatrix} A^*Q + QA + C_1^*C_1 & QB_1 + C_1^*D_{11} \\ B_1^*Q + D_{11}^*C_1 & D_{11}^*D_{11} - I \end{bmatrix} C_\perp < 0, \tag{2.38}$$

$$\begin{bmatrix} P & I \\ I & Q \end{bmatrix} \geq 0, \tag{2.39}$$

where $B_\perp : \mathbf{\Omega} \to \mathbf{F}^{(n+q_1)\times(n+q_1-p_2)}$ *and* $C_\perp : \mathbf{\Omega} \to \mathbf{F}^{(n+p_1)\times(n+p_1-q_2)}$ *are such that* $\mathrm{NULL}(B) = \mathrm{SPAN}(B_\perp)$ *and* $\mathrm{NULL}(C) = \mathrm{SPAN}(C_\perp)$.

2.3 Stabilization of Linear Parameter-Varying Systems

In this section, we will apply the characterization of stabilizability and detectability developed in the last section to the stabilization of linear systems whose coefficients are functions of parameters. Given fixed positive integers p, q, and n, a p-input-q-output n-th order linear time invariant (discrete-time or continuous time) system over $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$ is denoted by the following dynamical equations.

$$\begin{cases} x(t+1) &= A(\omega)x(t) + B(\omega)u(t) \\ y(t) &= C(\omega)x(t) + D(\omega)u(t) \end{cases} \tag{2.40}$$

with $t \in \mathbf{Z}^+$; or

$$\begin{cases} \dot{x}(t) &= A(\omega)x(t) + B(\omega)u(t) \\ y(t) &= C(\omega)x(t) + D(\omega)u(t) \end{cases} \tag{2.41}$$

with $t \in \mathbf{R}^+$; where the matrices A, B, C, and D have their entries in $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$. In each case, x, u, y denote the n, p, q-dimensional state, input, and output vectors, respectively. It is further assumed that $\mathrm{RANK}(B(\omega)) = p < n$, and $\mathrm{RANK}(C(\omega)) = q < n$ for all $\omega \in \mathbf{\Omega}$.

The system (2.40) (or (2.41)) is said to be **pointwise stable** if it is stable for all $\omega \in \mathbf{\Omega}$ with $u(t) = 0$. Equivalently, for all $\omega \in \mathbf{\Omega}$, $\rho(A(\omega)) < 1$ for system (2.40) or $\mathrm{RE}(\lambda) < 0$ for all $\lambda \in \mathbf{\Lambda}(A(\omega))$ for system (2.41). In the following, only the linear system (2.40) over $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$ is considered, the continuous-time LPV system (2.41) is treated similarly.

Given the linear system (2.40) over $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$. It is said to be **pointwise stabilizable** if for each $\omega \in \mathbf{\Omega}$ the system is stabilizable; i.e., there is a $p \times n$ matrix F_ω over $\mathbf{F}$ such that the closed loop system with system matrix $A(\omega) + B(\omega)F_\omega$ is stable; or equivalently, the following PBH rank condition is satisfied,

$$\mathrm{RANK}\begin{bmatrix} zI - A(\omega) & B(\omega) \end{bmatrix} = n,$$

for all $z \in \mathbf{C}$ with $|z| \geq 1$ and $\omega \in \mathbf{\Omega}$. It is stabilizable if there exists a state-feedback $u(t) = F(\omega)x(t)$ where the $p \times n$ feedback matrix F is over $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$ such that the closed loop system is pointwise stable. It is obvious that the stabilizability implies pointwise stabilizability; it is also known that if $\mathbf{\Omega}$ is a closed bounded subset of $\mathbf{F}^N$, then pointwise stabilizability implies stabilizability [97, Theorem 5.10]. In the following we will extend the latter statement, in which case the set $\mathbf{\Omega}$ is not required to be bounded or closed. To this end, we first have the following results about the pointwise stabilizability of the system (2.40) which extend the results in the previous section.

Proposition 2.3.1 *Consider system (2.40), it is pointwise stabilizable if and only if any of the following two conditions are satisfied.*

(i) For each $\omega \in \mathbf{\Omega}$, there exists a positive definite matrix $P_\omega \in \mathbf{F}^{n \times n}$, such that

$$A^*(\omega)P_\omega A(\omega) - P_\omega - B(\omega)B^*(\omega) < 0. \tag{2.42}$$

(ii) For each $\omega \in \mathbf{\Omega}$, there exists a positive definite matrix $P_\omega \in \mathbf{F}^{n \times n}$, such that

$$B_\perp^*(\omega)(A^*(\omega)P_\omega A(\omega) - P_\omega)B_\perp(\omega) < 0. \tag{2.43}$$

In both cases, a state-feedback is given by

$$F_\omega = -(B^*(\omega)P_\omega^{-1}B(\omega))^{-1}B^*(\omega)P_\omega^{-1}A(\omega). \tag{2.44}$$

Proposition 2.3.2 *Consider system (2.40) over $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$. If it is pointwise stabilizable then the corresponding LMIs (2.42) and (2.43) have positive definite solutions over $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$.*

Proof. We just consider LMI condition (2.42), the condition (2.43) follows similarly. Suppose system (2.40) is pointwise stable, then for each $\omega_o \in \mathbf{\Omega}$, there exists a positive definite matrix $P_o \in \mathbf{F}^{n \times n}$, such that the LMI (2.42) is satisfied, i.e.,

$$A^*(\omega_o)P_o A(\omega_o) - P_o - B(\omega_o)B^*(\omega_o) < 0. \tag{2.45}$$

Since matrices $A(\omega)$ and $B(\omega)$ are over $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$, they are continuously dependent on $\omega \in \mathbf{\Omega}$. Thus, there exists a neighborhood $\mathcal{B}(\omega_o)$ of ω_o, such that the above inequality holds for all $\omega \in \mathcal{B}(\omega_o) \cap \mathbf{\Omega}$, i.e.,

$$A^*(\omega)P_o A(\omega) - P_o - B(\omega)B^*(\omega) < 0, \quad \forall \omega \in \mathcal{B}(\omega_o).$$

Now, $\{\mathcal{B}(\omega_o)\}|_{\omega_o \in \mathbf{\Omega}}$ is an open covering of $\mathbf{\Omega}$, i.e.,

$$\mathbf{\Omega} = \bigcup_{\omega_o \in \mathbf{\Omega}} \mathcal{B}(\omega_o) \cap \mathbf{\Omega}.$$

Since $\mathbf{\Omega} \subset \mathbf{F}^N$ where $\mathbf{F}^N$ is paracompact, there is a locally finite open subcovering $\{\mathcal{B}_i\}|_{i \in \mathbf{I}}$ for some index set $\mathbf{I}$ which refines $\{\mathcal{B}(\omega_o)\}|_{\omega_o \in \mathbf{\Omega}}$. Now $P_i \in \mathbf{F}^{n \times n}$ is taken to be positive definite for each $i \in \mathbf{I}$ such that

$$A^*(\omega) P_i A(\omega) - P_i - B(\omega) B^*(\omega) < 0, \quad \forall \omega \in \mathcal{B}_i.$$

It is known from the standard argument of **partitions of unity** that there is $\mathcal{C}^\infty$ partition of unity $\{\psi_i\}|_{i \in \mathbf{I}}$ to $\mathbf{\Omega}$ subordinated to the covering $\{\mathcal{B}_i\}|_{i \in \mathbf{I}}$; i.e., ψ_i is $\mathcal{C}^\infty$ and non-negative with support $\textsc{Supp}(\psi_i) \subset \mathcal{B}_i$ for each $i \in \mathbf{I}$, and

$$\sum_{i \in \mathbf{I}} \psi_i(\omega) = 1, \quad \forall \omega \in \mathbf{\Omega}. \tag{2.46}$$

Define a matrix-valued function $P : \mathbf{\Omega} \rightarrow \mathbf{F}^{n \times n}$ as

$$P(\omega) = \sum_{i \in \mathbf{I}} \psi_i(\omega) P_i, \quad \forall \omega \in \mathbf{\Omega}, \tag{2.47}$$

which is positive definite and $\mathcal{C}^\infty$ since it is locally a finite sum of $\mathcal{C}^\infty$ positive definite matrix-valued functions. Since the inequality (2.42) is affine in the unknown, it follows that for all $\omega \in \mathbf{\Omega}$,

$$A^*(\omega) P(\omega) A(\omega) - P(\omega) - B(\omega) B^*(\omega)$$

$$= \sum_{i \in \mathbf{I}} \psi_i(\omega) (A^*(\omega) P_i A(\omega) - P_i - B(\omega) B^*(\omega)) < 0.$$

Therefore, the positive matrix defined by (2.47) is over $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$ and satisfies (2.42).

□

The following theorem is the main result on stabilizability of linear systems over $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$, which generalizes Theorem 5.10 in [97].

Theorem 2.3.3 *Consider the system (2.40) or (2.41) over $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$. Then it is stabilizable if and only if it is pointwise stabilizable.*

Proof. The necessity is obvious. We will show the sufficiency. Only system (2.40) is considered. The proof for system (2.41) follows almost the same arguments.

Suppose the system considered is pointwise stabilizable, then by the preceding proposition, it follows that there exists a positive definite matrix $P(\omega)$ over $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$, such that for all $\omega \in \mathbf{\Omega}$,

$$A^*(\omega)P(\omega)A(\omega) - P(\omega) - B(\omega)B^*(\omega) < 0,$$

and the corresponding state feedback control is given by

$$F(\omega) = -(B^*(\omega)P^{-1}(\omega)B(\omega))^{-1}B^*(\omega)P^{-1}(\omega)A(\omega), \tag{2.48}$$

which is over $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$. □

By Stone-Weierstrass Theorem [148], we have the following observation.

Corollary 2.3.4 *Consider the system (2.40) or (2.41) over $\mathcal{C}^k(\mathbf{\Omega}_R, \mathbf{F})$, where $\mathbf{\Omega}_R$ is compact real subset in $\mathbf{F}^N$. If it is pointwise stabilizable, then the solution to (2.42) or (2.43) and the stabilizing feedback matrix can be chosen such that their entries are polynomials (or rational functions) of the parameters on $\mathbf{\Omega}_R$.*

Next, we will consider the stabilization by output feedback. To this end, we need the notions of detectability which is dual to stabilizability. Consider system (2.40) over $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$, it is pointwise detectable, if for all $\omega \in \mathbf{\Omega}$, there exists an output injection matrix L_ω in $\mathbf{F}^{n\times q}$ such that $A(\omega) + L_\omega C(\omega)$ is stable; in addition, it is said to be detectable if there is an $n \times q$ matrix L over $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$ such that $A(\omega) + L(\omega)C(\omega)$ is pointwise stable. Dually, we have the following theorem.

Theorem 2.3.5 *Consider the system (2.40) or (2.41) over $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$. Then it is detectable if and only if it is pointwise detectable.*

The detectability of system (2.40) implies that there exists an observer over $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$ which has the following realization,

$$\tilde{x}(t+1) = A(\omega)\tilde{x}(t) + B(\omega)u(t) - L(\omega)(y(t) - C(\omega)\tilde{x}(t) - D(\omega)u(t)),$$

where $L(\omega)$ is over $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$ such that $A(\omega) + L(\omega)C(\omega)$ is pointwise stable for $\omega \in \mathbf{\Omega}$. Therefore, $\tilde{x}(t) - x(t) \to 0$ as $t \to \infty$ for all $\omega \in \mathbf{\Omega}$. Moreover, we have the following result about stabilization by output feedback controller over $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$. The proof is straightforward.

Theorem 2.3.6 *Consider system (2.40) over $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$. Then there exists a stabilizing feedback controller for the system if and only if it is pointwise stabilizable and detectable, and such a controller is given by*

$$\begin{cases} \tilde{x}(t+1) = A(\omega)\tilde{x}(t) + B(\omega)u(t) - L(\omega)(y(t) - C(\omega)\tilde{x}(t) - D(\omega)u(t)) \\ u(t) = F(\omega)\tilde{x}(t) \end{cases}$$

where F and L are matrices over $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$ such that both $A(\omega) + B(\omega)F(\omega)$ and $A(\omega) + L(\omega)C(\omega)$ are pointwise stable.

Moreover, any stabilizing controller over $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$ can be parameterized as $\mathcal{F}_l(J, Q)$, where

$$J = \left[\begin{array}{c|cc} A(\omega) + B(\omega)F(\omega) + L(\omega)C(\omega) + L(\omega)D(\omega)F(\omega) & -L(\omega) & B(\omega) + L(\omega)D(\omega) \\ \hline F(\omega) & 0 & I \\ -(C(\omega) + D(\omega)F(\omega)) & I & -D(\omega) \end{array}\right],$$

which is over $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$ and Q is a pointwise stable system over $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$ such that the resulting closed loop system is well-posed for all $\omega \in \mathbf{\Omega}$.

2.4 $\mathcal{H}_\infty$-Control of Linear Parameter-Varying Systems

In the following, we consider the performance synthesis problem for a linear system over $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$.

Given an n-dimensional linear system over $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$ as follows,

$$\begin{bmatrix} z(t) \\ y(t) \end{bmatrix} = \left[\begin{array}{c|cc} A(\omega) & B_1(\omega) & B_2(\omega) \\ \hline C_1(\omega) & D_{11}(\omega) & D_{12}(\omega) \\ C_2(\omega) & D_{21}(\omega) & D_{22}(\omega) \end{array}\right] \begin{bmatrix} w(t) \\ u(t) \end{bmatrix}, \tag{2.49}$$

where $t \in \mathbf{Z}^+$ (or $t \in \mathbf{R}^+$), w, u, z, y are p, p_2, q, q_2-dimensional disturbance input, control input, regulated output, and measured output vectors. It is assumed that the dimensions of external disturbance vector and regulated output vector are the same, i.e., $p = q$, without loss of generality. The matrices have entries in $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$. It is assumed that $n + p > q_2$ and $n + p > p_2$. Define

$$\bar{A}(\omega) := \begin{bmatrix} A(\omega) & B_1(\omega) \\ C_1(\omega) & D_{11}(\omega) \end{bmatrix}, \; B(\omega) := \begin{bmatrix} B_2(\omega) \\ D_{12}(\omega) \end{bmatrix}^*, \; C(\omega) := \begin{bmatrix} C_2(\omega) & D_{21}(\omega) \end{bmatrix}.$$

Suppose $\mathrm{RANK}(B(\omega)) = p_2$ and $\mathrm{RANK}(C(\omega)) = q_2$ for all $\omega \in \mathbf{\Omega}$.

The system (2.49) is said to have **pointwise $\mathcal{H}_\infty$-control solution** if for each fixed $\omega \in \mathbf{\Omega}$, there exists an output feedback controller

$$u(t) = \left[\begin{array}{c|c} \hat{A}_\omega & \hat{B}_\omega \\ \hline \hat{C}_\omega & \hat{D}_\omega \end{array}\right] y(t)$$

with the coefficient matrices having entries in $\mathbf{F}$, such that the closed loop system is well-posed and stable, and has $\mathcal{H}_\infty$-norm less than 1. The system (2.49) over $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$ is said to have **$\mathcal{H}_\infty$-control solution over $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$** if there exists an output feedback controller

$$u(t) = \left[\begin{array}{c|c} \hat{A}(\omega) & \hat{B}(\omega) \\ \hline \hat{C}(\omega) & \hat{D}(\omega) \end{array}\right] y(t)$$

which is over $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$, such that the closed loop system is well-posed and stable, has $\mathcal{H}_\infty$-norm less than 1 for all $\omega \in \mathbf{\Omega}$.

In the following, we first consider the solvability of the $\mathcal{H}_\infty$-control problem via state feedback , in which case $y(t) = x(t)$, and full-information feedback, where $y(t) = \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}$. Only discrete-time case ($t \in \mathbf{Z}^+$) is considered. We immediately have the following result on pointwise solvability.

Theorem 2.4.1 *Consider the LPV system (2.49) with $t \in \mathbf{Z}^+$. Then the $\mathcal{H}_\infty$-control problem is pointwise solvable by a full-information feedback if and only if for each fixed $\omega \in \mathbf{\Omega}$, there exists a positive definite matrix $P_\omega \in \mathbf{F}^{n\times n}$ such that it satisfies the following LMI:*

$$B_\perp^*(\omega) \begin{bmatrix} A(\omega)P_\omega A^*(\omega) - P_\omega + B_1(\omega)B_1^*(\omega) & A(\omega)P_\omega C_1^*(\omega) + B_1(\omega)D_{11}^*(\omega) \\ C_1(\omega)P_\omega A^*(\omega) + D_{11}(\omega)B_1^*(\omega) & C_1(\omega)P_\omega C_1^*(\omega) + D_{11}(\omega)D_{11}^*(\omega) - I \end{bmatrix} B_\perp(\omega) < 0, \tag{2.50}$$

where $B_\perp : \mathbf{\Omega} \to \mathbf{F}^{(n+p)\times(n+p-p_2)}$ is defined such that $\textsc{Null}(B(\omega)) = \textsc{Span}(B_\perp(\omega))$.

Similarly, the state-feedback solutions can also be characterized by two LMIs. Next we will give a PBH-like rank necessity test for state-feedback robust performance solutions.

Theorem 2.4.2 *Consider the LPV system (2.49) over $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$ with $t \in \mathbf{Z}^+$. The system has a state-feedback robust performance solution only if the following rank condition is satisfied,*

$$\textsc{Rank} \begin{bmatrix} zI - A(\omega) & -B_1(\omega)\Delta_a & B_2(\omega) \\ -C_1(\omega) & I - D_{11}(\omega)\Delta_a & D_{12}(\omega) \end{bmatrix} = n + p \tag{2.51}$$

for all $z \in \mathbf{C}$ with $|z| \geq 1$, $\omega \in \mathbf{\Omega}$, and $\Delta_a \in \mathbf{C}^{p\times p}$ with $\bar{\sigma}(\Delta_a) \leq 1$.

Proof. We first have the following lemma, which follows from Proposition 3.4.2.

Lemma 2.4.3 *Consider an p-input-p-output discrete-time LTI system* $z(t) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] w(t)$ *($t \in \mathbf{Z}^+$). It has $\mathcal{H}_\infty$-norm less than 1 if and only if*

$$\mathrm{DET}(I - \begin{bmatrix} A & B \\ C & D \end{bmatrix} \Delta_N) \neq 0, \quad \Delta_N := \begin{bmatrix} z^{-1}I & 0 \\ 0 & \Delta_a \end{bmatrix},$$

for all $z \in \mathbf{C}$ with $|z| \geq 1$ and $\Delta_a \in \mathbf{C}^{p \times p}$ with $\bar{\sigma}(\Delta_a) \leq 1$.

Now for fixed $\omega \in \mathbf{\Omega}$, there exists a state feedback F_ω such that the following closed loop system,

$$z(t) = \left[\begin{array}{c|c} A(\omega) + B_2(\omega)F_\omega & B_1(\omega) \\ \hline C_1(\omega) + D_{12}(\omega)F_\omega & D_{11}(\omega) \end{array}\right] w(t),$$

has $\mathcal{H}_\infty$-norm less than 1. Thus, from the previous lemma, the following rank condition is satisfied,

$$\mathrm{RANK}(I - \begin{bmatrix} A(\omega) + B_2(\omega)F_\omega & B_1(\omega) \\ C_1(\omega) + D_{12}(\omega)F_\omega & D_{11}(\omega) \end{bmatrix} \begin{bmatrix} z^{-1}I & 0 \\ 0 & \Delta_a \end{bmatrix}) = n + p.$$

It is observed that

$$I - \begin{bmatrix} A(\omega) + B_2(\omega)F_\omega & B_1(\omega) \\ C_1(\omega) + D_{12}(\omega)F_\omega & D_{11}(\omega) \end{bmatrix} \begin{bmatrix} z^{-1}I & 0 \\ 0 & \Delta_a \end{bmatrix} =$$

$$= \begin{bmatrix} I - A(\omega)z^{-1} & -B_1(\omega)\Delta_a & -B_2(\omega)z^{-1} \\ -C_1(\omega)z^{-1} & I - D_{11}(\omega)\Delta_a & -D_{12}(\omega)z^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I \\ F_\omega & 0 \end{bmatrix}.$$

So it follows that

$$\begin{aligned} n + p &= \mathrm{RANK}(I - \begin{bmatrix} A(\omega) + B_2(\omega)F_\omega & B_1(\omega) \\ C_1(\omega) + D_{12}(\omega)F_\omega & D_{11}(\omega) \end{bmatrix} \begin{bmatrix} z^{-1}I & 0 \\ 0 & \Delta_a \end{bmatrix}) \\ &\leq \mathrm{RANK} \begin{bmatrix} I - A(\omega)z^{-1} & -B_1(\omega)\Delta_a & -B_2(\omega)z^{-1} \\ -C_1(\omega)z^{-1} & I - D_{11}(\omega)\Delta_a & -D_{12}(\omega)z^{-1} \end{bmatrix} \\ &= \mathrm{RANK} \begin{bmatrix} zI - A(\omega) & -B_1(\omega)\Delta_a & B_2(\omega) \\ -C_1(\omega) & I - D_{11}(\omega)\Delta_a & D_{12}(\omega) \end{bmatrix} \leq n + p. \end{aligned}$$

Therefore, the (full row) rank condition (2.51) is satisfied. □

We have the following statement which reveals the relations between the pointwise solvability and solvability over $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$ for the state feedback $\mathcal{H}_\infty$-control problem.

Theorem 2.4.4 *Consider the system (2.49) over $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$. The state feedback $\mathcal{H}_\infty$-control problem is solvable over $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$ if and only if the corresponding pointwise $\mathcal{H}_\infty$-control problem is solvable.*

Proof. The necessity is obvious. We only need to show the sufficiency. Suppose the pointwise state feedback $\mathcal{H}_\infty$-control problem has a solution, then by Theorem 2.4.1, for each fixed $\omega \in \mathbf{\Omega}$, there exists positive definite matrix $P_\omega \in \mathbf{F}^{n\times n}$ which satisfies the LMI (2.50). Note that the left hand side of the LMI is affine in P_ω. Therefore, the argument used in the proof of Proposition 2.3.2 can also be applied here. We therefore can find a smooth positive definite matrix-valued function $P : \mathbf{\Omega} \rightarrow \mathbf{F}^{n\times n}$ such that $P(\omega) \in \mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$ satisfies the LMI (2.50). Furthermore, a static state-feedback controller over $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$ can be constructed in terms of $P(\omega)$. □

Finally, we have the following theorem on the solvability of output feedback $\mathcal{H}_\infty$-control problem for system (2.49).

Theorem 2.4.5 *Consider the system (2.49) over $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$. The $\mathcal{H}_\infty$-control problem is solvable over $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$ if and only if the corresponding pointwise $\mathcal{H}_\infty$-control problem is solvable.*

Proof. Only the discrete time case ($t \in \mathbf{Z}^+$) is considered. The continuous time case follows the similar arguments. The necessity is obvious. We only need to show the sufficiency. Basically, the proof is the same as in the previous theorem. We only give an outline here. Suppose the pointwise $\mathcal{H}_\infty$-control problem has a solution. Then it follows from Theorem 2.2.14 that for each fixed $\omega \in \mathbf{\Omega}$, there are two positive definite matrices $P_\omega, Q_\omega \in \mathbf{F}^{n\times n}$ such that they satisfy the following LMIs:

$$B_\perp^*(\omega) \begin{bmatrix} A(\omega)P_\omega A^*(\omega) - P_\omega + B_1(\omega)B_1^*(\omega) & A(\omega)P_\omega C_1^*(\omega) + B_1(\omega)D_{11}^*(\omega) \\ C_1(\omega)P_\omega A^*(\omega) + D_{11}(\omega)B_1^*(\omega) & C_1(\omega)P_\omega C_1^*(\omega) + D_{11}(\omega)D_{11}^*(\omega) - I \end{bmatrix} B_\perp(\omega) < 0, \tag{2.52}$$

$$C_\perp^*(\omega) \begin{bmatrix} A^*(\omega)Q_\omega A(\omega) - Q_\omega + C_1^*(\omega)C_1(\omega) & A^*(\omega)Q_\omega B_1(\omega) + C_1^*(\omega)D_{11}(\omega) \\ B_1^*(\omega)Q_\omega A(\omega) + D_{11}^*(\omega)C_1(\omega) & B_1^*(\omega)Q_\omega B_1(\omega) + D_{11}^*(\omega)D_{11}(\omega) - I \end{bmatrix} C_\perp(\omega) < 0, \tag{2.53}$$

$$\begin{bmatrix} P_\omega & I \\ I & Q_\omega \end{bmatrix} \geq 0, \tag{2.54}$$

where $B_\perp : \mathbf{\Omega} \to \mathbf{F}^{(n+p)\times(n+p-p_2)}$ is such that $\text{NULL}(B(\omega)) = \text{SPAN}(B_\perp(\omega))$; and $C_\perp : \mathbf{\Omega} \to \mathbf{F}^{(n+p)\times(n+p-q_2)}$ such that $\text{NULL}(C(\omega)) = \text{SPAN}(C_\perp(\omega))$. Moreover, the coefficients $\hat{A}_\omega, \hat{B}_\omega, \hat{C}_\omega$, and $\hat{C}_\omega$ of a pointwise $\mathcal{H}_\infty$-controller can be chosen to be smoothly dependent on P_ω, Q_ω.

Since the three LMIs are affine in P_ω and Q_ω. Then the argument used in the proof of Proposition 2.3.2 can be applied here. And we can find two smooth positive definite matrix-valued functions $P, Q : \mathbf{\Omega} \to \mathbf{F}^{n\times n}$ such that $P(\omega)$ and $Q(\omega)$, which are in $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$, satisfy the three LMIs (2.52) – (2.54). Then an output-feedback controller over $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$ can be constructed in terms of $P(\omega)$ and $Q(\omega)$. □

2.5 Notes and References

The control issues of of linear systems over some algebras were considered extensively in [65, 101, 97] and references therein. The consideration in this chapter basically follows the treatment in [97]. The basic definitions of pointwise stability, pointwise stabilizability, and stabilizability in this chapter extend those defined in [97]. However, the general results here, which justify the pointwise design is enough for synthesis, do not require the parameter sets to be compact as assumed in [97]. In addition, we considered the $\mathcal{H}_\infty$-control issue for LPV systems to reflect the recent development in the robust control areas.

The traditional PBH test and Ricatti characterization of stabilizability and detectability can be found in standard text books, such as [95]. The LMI characterizations are basically from [57, 119, 125]. The LMI conditions for $\mathcal{H}_\infty$-control solutions are derived by Lu *et al.* [119, 111], Packard *et al.* [133, 131], Gahinet-Apkarian [69], and Huang-Lu [86].

In practice, the implementation of the LPV controllers for LPV systems is carried out by gain/dynamic scheduling [164, 149, 145] or adaptive/supervisory control [176, 125]. The justification of the pointwise design for gain scheduling is made by Shamma [162, 163].

Chapter 3

Control of Uncertain Linear Systems with LTI Perturbations

3.1 Introduction

In this chapter, we will consider a class of uncertain dynamical systems, whose coefficients can be represented as **linear fractional transformations** (LFTs) [144, 73, 132] on some block structure. This description is very natural for systems where the uncertainty (unknown parameters or perturbations) enter the model in a feedback fashion. For a linear discrete-time system with structured uncertainty Δ_P, it can be alternatively represented as an LFT on an augmented block structure $\Delta = \mathrm{DIAG}[z^{-1}I, \Delta_P]$ where the first repeated scalar block $z^{-1}I$ is a delay operator; in this case, the LFT formulation is a direct generalization of the now standard notation for state-space realizations of transfer functions. Also, a linear shift invariant (LSI) **multidimensional** system can be represented by an LFT formula [119].

The robustness analysis and synthesis issues under structured uncertainty were extensively investigated by Doyle *et al.* [53, 63, 54, 56, 119, 133, 111, 132, 137]. This chapter is a continuation of such pursuits. We consider the problem of generalized robust stabilization under **linear time-invariant** (LTI) or parametric perturbations. The uncertain discrete-time systems are emphasized, while most of the results can be easily generalized to the case of uncertain continuous time systems. The stability notions employed in this chapter are quite standard; they are a natural generalization of the conventional notions of stability for linear discrete systems, $\mathcal{H}_\infty$ performance of linear discrete systems [63, 133, 111, 132], and robust stability or performances for linear uncertain systems with (structured) parametric/dynamic uncertainties [63, 119, 133, 132]. In particular, in the linear uncertain system case, if the uncertainty Δ_P represents complex norm-bounded parametric or LTI dynamical uncertainty, the robust stability test reduces to a μ-test, which is referred to as the μ-stability test. The treatment for the case where Δ_P represents nonlinear or linear time varying (LTV) dynamical uncertainty leads to the $\mathcal{Q}$-stability [119].

In this chapter, an algebraic structure (LFT algebras) is introduced, and the issue of stabilization by controllers over this algebra is considered. This consideration for synthesis implies that both plant and controller are described by LFTs on some block structures, and in particular, the controller is allowed to depend on the same block structure as the plant. The control scheme thus can be implemented as gain scheduled or dynamically scheduled [176, 145], provided the uncertainties can be measured or identified on line. It is remarked that the control synthesis of uncertain linear systems over some other general algebraic structures was pursued in [97, 101, 115]. Some other related work includes [131, 3].

The properties of stabilizability and detectability for uncertain linear systems, which are related to the solvability of the stabilization problem, are examined in the defined algebras. In particular, it is shown how these properties can be reduced to a PBH-like test, which originated the work in [137, 138] where stability robustness analysis was conducted in the behavioral setting, and which extends the notion of structured singular value.

The remainder of this chapter is organized as follows. In Section 2, a generalized notion of structured singular-value is introduced, it is used in robustness synthesis problem. In Section 3, the description of uncertain linear systems in terms of LFTs is proposed. In Section 4, the robustness analysis is conducted. The robust stabilization and robust performance synthesis are considered in Sections 5 and 6.

3.2 Preliminaries: Structured Singular Values

In this section, we review and generalize the standard notion of structured singular values developed by Doyle [53]. The more detailed material about this subject is considered in the expository article [132].

3.2.1 Structured Singular Values

Consider a matrix $M \in \mathbf{F}^{n\times n}$ and an underlying block structure $\mathbf{\Delta}$,

$$\mathbf{\Delta} := \{\mathrm{DIAG}[\delta_1 I_{r_1}, \cdots, \delta_s I_{r_s}, \Delta_1, \cdots, \Delta_f] : \delta_i \in \mathbf{F}, \Delta_j \in \mathbf{F}^{m_j\times m_j}\} \subset \mathbf{F}^{n\times n}, \tag{3.1}$$

where the full blocks are allowed to be repeated.

Definition 3.2.1 *The* **structured singular value** $\mu_\Delta(M)$ *of a matrix* M *with respect to structure* $\mathbf{\Delta}$ *is defined as*

$$\mu_\Delta(M) := \sup_{\Delta\in\mathbf{\Delta}} \{\frac{1}{\bar\sigma(\Delta)} : \mathrm{DET}[I - \Delta M] = 0\} \tag{3.2}$$

unless no $\Delta \in \mathbf{\Delta}$ *makes* $I - \Delta M$ *singular, in which case* $\mu_\Delta(M) := 0$.

From the definition, it follows immediately that $\mu_\Delta(M) = \rho(M)$ if $\mathbf{\Delta} = \{\delta I : \delta \in \mathbf{C}\}$ and $\mu_\Delta(M) = \bar\sigma(M)$ if $\mathbf{\Delta} = \mathbf{C}^{n\times n}$.

The definition of μ can be extended to the case where M is an LTI operator and the block structure $\mathbf{\Delta}$ is a set of structured LTI operators. Define

$$\mathbf{B\Delta} := \{\Delta \in \mathbf{\Delta} : \bar\sigma(\Delta) \le 1\}. \tag{3.3}$$

It is known from μ-definition that $\mu_\Delta(M) < 1$ if and only if

$$\mathrm{RANK}(I - M\Delta) = n,$$

i.e., $I - M\Delta$ is invertible for all $\Delta \in \mathbf{B\Delta}$. Thence the μ-value has a clear robust control interpretation.

The exact computation of μ is generally difficult. Usually, an alternative notion, known as $\mathcal{Q}$-values whose computation is a convex feasibility problem, is used to approximate it. To this end, we define the **commutative matrix set** $\mathcal{D}$ of $\mathbf{\Delta}$ as follows,

$$\mathcal{D} = \{D \in \mathbf{F}^{n\times n} : D\Delta = \Delta D, \mathrm{DET}[D] \ne 0, \Delta \in \mathbf{\Delta}\}. \tag{3.4}$$

Thus $\mathcal{D}$ depends only on the structure of Δ. In addition, if $D, D_1, D_2 \in \mathcal{D}$, then $D^{-1} \in \mathcal{D}$, $D^* \in \mathcal{D}$ and $D_1 D_2 \in \mathcal{D}$.

Definition 3.2.2 *The* $\mathcal{Q}$**-value** *of* M *with respect to structure* $\mathbf{\Delta}$ *is defined as*

$$\mathcal{Q}_\Delta(M) := \inf_{D\in\mathcal{D}} \bar\sigma(DMD^{-1}). \tag{3.5}$$

Note that $\mathcal{Q}_\Delta(M) < 1$ if and only if there is a positive definite matrix $P \in \mathcal{D}$ such that

$$MPM^* - P < 0. \tag{3.6}$$

$\mathcal{Q}_\Delta(M)$ is an upper bound of $\mu_\Delta(M)$ [53], i.e.,

$$\mu_\Delta(M) \leq \inf_{D\in\mathcal{D}} \bar{\sigma}(DMD^{-1}).$$

The relation of μ and its upper bound in $\mathbf{C}$ is discussed in detail in [53, 132]. Specially, the upper bound is achieved for the following special block structures:

1) $\mathbf{\Delta} = \{\text{Diag}[\delta I_r, \Delta] : \delta \in \mathbf{C}, \Delta \in \mathbf{C}^{(n-r)\times(n-r)}\}$.

2) $\mathbf{\Delta} = \{\text{Diag}[\Delta_1, \cdots, \Delta_f] : \Delta_i \in \mathbf{C}^{m_i \times m_i}\} \subset \mathbf{C}^{n\times n}, f \leq 3$, where no blocks are repeated.

Remark 3.2.3 *Both $\mu_\Delta(M)$ and $\mathcal{Q}_\Delta(M)$ are continuous functions of M for fixed block structure $\mathbf{\Delta}$ in the case $\mathbf{F} = \mathbf{C}$.*

To end this review, we state the following result known as **main loop theorem** in [132, Theorem 4.3].

Lemma 3.2.4 *Given a block structure $\Delta = \text{Diag}[\Delta_1, \Delta_2]$ and a $\mathbf{F}$-valued matrix M partitioned as $M := \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$, then $\mu_\Delta(M) < 1$ if and only if*

$$\mu_{\Delta_2}(M_{22}) < 1, \quad \max_{\bar{\sigma}(\Delta_2)\leq 1} \mu_{\Delta_1}(\mathcal{F}_l(M, \Delta_2)) < 1.$$

3.2.2 A Generalized Notion of the Structured Singular Values

In the robustness synthesis to be treated in the next few sections, we will make use of a generalized μ.

Given $M, N \in \mathbf{F}^{m\times n}$ with $m \leq n$ and $\text{Rank}(N) = m$, the underlying block structure $\mathbf{\Delta}$ is defined as (3.1), and the corresponding sets $\mathbf{B\Delta}$ and $\mathcal{D}$ are also defined in (3.3) and (3.4).

Definition 3.2.5 *The* **generalized structured singular value** *$\mu_\Delta(M, N)$ of an ordered matrix pair (M, N) with respect to structure $\mathbf{\Delta}$ is defined as*

$$\mu_\Delta(M, N) := \sup_{\Delta\in\mathbf{\Delta}} \{\frac{1}{\bar{\sigma}(\Delta)} : \text{Rank}[N - M\Delta] < m\} \tag{3.7}$$

unless no $\Delta \in \mathbf{\Delta}$ makes $N - M\Delta$ drop rank, in which case $\mu_\Delta(M, N) := 0$.

It is noted that for all nonsingular matrix $E \in \mathbf{C}^{m\times m}$, $Q \in \mathbf{\Delta}$ with $\bar{\sigma}(Q) = 1$, and $D \in \mathcal{D}$,

$$\mu_\Delta(M,N) = \mu_\Delta(EM,EN) = \mu_\Delta(MQ,N) = \mu_\Delta(MD,ND).$$

From this definition, $\mu_\Delta(M,N) < 1$ if and only if the following full row rank condition is satisfied,

$$\mathrm{R{\scriptsize ANK}}(N - M\Delta) = m$$

for all $\Delta \in \mathbf{B\Delta}$. Additionally, one has the following observation.

Proposition 3.2.6 *Define*

$$\mathcal{Q}_\Delta(M,N) := \inf_{0<P\in\mathcal{D}} \{\gamma : MPM^* - \gamma^2 NPN^* < 0, \gamma \geq 0\}. \tag{3.8}$$

Then $\mu_\Delta(M,N) \leq \mathcal{Q}_\Delta(M,N)$.

Therefore, the $\mathcal{Q}$-value defined by (3.8) is an upper bound for $\mu_\Delta(M,N)$; the computation of $\mathcal{Q}$-value is an inf-optimization problem (cf. [28]). In the following, let $\mathbf{F} = \mathbf{C}$, we will see under what condition, the above upper bound can be achieved. To this end, we define quadratic functions $Q_i : \mathbf{C}^n \to \mathbf{C}^{r_i\times r_i}$ and $q_j : \mathbf{C}^n \to \mathbf{C}$ $(i = 1,\cdots,s; j = 1,\cdots,f)$ as follows,

$$Q_i(\eta) = (\eta^* M)_i^*(\eta^* M)_i - (\eta^* N)_i^*(\eta^* N)_i, \quad i = 1,\cdots,s \tag{3.9}$$

$$q_j(\eta) = (\eta^* M)_{s+j}(\eta^* M)_{s+j}^* - (\eta^* N)_{s+j}(\eta^* N)_{s+j}^*, \quad j = 1,\cdots,f \tag{3.10}$$

where for a vector $v \in \mathbf{F}^n$, it is partitioned as

$$v^T := \begin{bmatrix} v_1^T & \cdots & v_s^T & v_{s+1}^T & \cdots & v_{s+f}^T \end{bmatrix},$$

where $v_i \in \mathbf{F}^{r_i}$ and $v_{s+j} \in \mathbf{F}^{m_j}$ for $i = 1,\cdots,s; j = 1,\cdots,f$.

Next, define two sets as follows,

$$\begin{aligned} \nabla := \{&\mathrm{D{\scriptsize IAG}}[Q_1(\eta),\cdots,Q_s(\eta),q_1(\eta),\cdots,q_f(\eta)] : \\ &Q_i(\eta) \text{ and } q_j(\eta) \text{ are defined in (3.9) and (3.10)}, \eta \in \mathbf{C}^n, \|\eta\| = 1\} \end{aligned} \tag{3.11}$$

$$\mathbf{X} := \{\mathrm{D{\scriptsize IAG}}[X_1,\cdots,X_s,x_1,\cdots,x_f] : 0 \leq X_i = X_i^* \in \mathbf{C}^{r_i\times r_i}, x_j \in \mathbf{R}^+\}. \tag{3.12}$$

Definition 3.2.7 *The block structure* $\mathbf{\Delta}$ *is said to be* μ*-simple with respect to* (M,N) *if the set* ∇ *defined in (3.11) is such that* $\nabla \cap \mathbf{X} = \emptyset$ *implies* $\mathrm{Co}(\nabla) \cap \mathbf{X} = \emptyset$.

We have the following theorem.

Theorem 3.2.8 *If the block structure* $\mathbf{\Delta}$ *is* μ*-simple with respect to* (M, N)*, then* $\mu_\Delta(M, N) < 1$ *if and only if* $\mathcal{Q}_\Delta(M, N) < 1$.

Proof. The sufficiency immediately follows from Proposition 3.2.6; only the necessity is proved. The ideas of the proof basically follow the treatments in [53, 132, 138], and is divided into two steps.

Claim 1: $\mu_\Delta(M, N) < 1$ implies $\nabla \cap \mathbf{X} = \emptyset$.

In fact, if $\nabla \cap \mathbf{X} \neq \emptyset$, then from the definition of ∇, it follows that for some $\eta \in \mathbf{C}^n$ with $\|\eta\| = 1$,

$$Q_i(\eta) = (\eta^* M)_i^*(\eta^* M)_i - (\eta^* N)_i^*(\eta^* N)_i \geq 0, \quad i = 1, \cdots, s$$

$$q_j(\eta) = (\eta^* M)_{s+j}(\eta^* M)_{s+j}^* - (\eta^* N)_{s+j}(\eta^* N)_{s+j}^* \geq 0, \quad j = 1, \cdots, f.$$

This implies that for $i = 1, \cdots, s$, there exists $\delta_i \in \mathbf{C}$ with $|\delta_i| \leq 1$ such that $(\eta^* M)_i \delta_i = (\eta^* N)_i$, and for $j = 1, \cdots, f$, there exists a matrix $\Delta_j \in \mathbf{C}^{m_j \times m_j}$ with $\bar{\sigma}(\Delta_j) \leq 1$, such that $(\eta^* M)_{s+j}\Delta_j = (\eta^* N)_{s+j}$. Define

$$\Delta := \mathrm{DIAG}[\delta_1 I_{r_1}, \cdots, \delta_s I_{r_s}, \Delta_1, \cdots, \Delta_f],$$

then $\Delta \in \mathbf{B\Delta}$, and

$$\eta^* M \Delta = \eta^* N,$$

or

$$\eta^*(N - M\Delta) = 0$$

with $\eta \neq 0$, which implies

$$\mathrm{RANK}(N - M\Delta) < m,$$

which contradicts $\mu_\Delta(M, N) < 1$. This confirms the claim.

Claim 2: If the block structure is μ-simple with respect to (M, N), then $Q_\Delta(M, N) < 1$.

Define a new matrix set

$$\bar{\mathcal{D}} := \{\mathrm{DIAG}[D_1, \cdots, D_s, d_1, \cdots, d_f] : D_i = D_i^* \in \mathbf{C}^{r_i \times r_i}, d_j \in \mathbf{R}\}.$$

Then $\bar{\mathcal{D}}$ is a real vector inner-product space whose inner product is defined as

$$\langle X, D \rangle = \sum_{i=1}^{s} \mathrm{TR}(X_i D_i) + \sum_{j=1}^{f} x_j d_j.$$

By the assumption, $\mathrm{Co}(\nabla) \cap \mathbf{X} = \emptyset$ in the inner product space $\bar{\mathcal{D}}$. From the definition of ∇, it follows that ∇ is a compact set in $\bar{\mathcal{D}}$, therefore $\mathrm{Co}(\nabla)$ is also compact. Also note that $\mathbf{X}$ is a closed and convex set in $\bar{\mathcal{D}}$. Thence, by Hahn-Banach theorem, there exists a positive definite matrix $X = \mathrm{DIAG}[X_1, \cdots, X_s, x_1, \cdots, x_f] \in \mathbf{X}$ such that

$$\langle X, D \rangle < 0$$

for all $D \in \nabla$. Therefore, for all $\eta \in \mathbf{C}^n$ with $\|\eta\| = 1$, one has

$$\langle X, \mathrm{DIAG}[Q_1(\eta), \cdots, Q_s(\eta), q_1(\eta), \cdots, q_f(\eta)] \rangle < 0,$$

or

$$\eta^*(MPN^* - NPN^*)\eta < 0,$$

where $P := \mathrm{DIAG}[X_1, \cdots, X_s, x_1 I, \cdots, x_f I] > 0$; this leads to

$$MPM^* - NPN^* < 0,$$

which implies $\mathcal{Q}_\Delta(M, N) < 1$. $\square$

Using the same argument as in [132], one also has the following result.

Corollary 3.2.9 *If the block structure is μ-simple with respect to (M, N), then $\mu_\Delta(M, N) = \mathcal{Q}_\Delta(M, N)$.*

Remark 3.2.10 *From the work of [53, 132, 138], we know that Δ is μ-simple w.r.t. (M, N) if Δ has any one of the following structures:*

(i) $\mathbf{\Delta} = \{\delta I : \delta \in \mathbf{C}\} \subset \mathbf{C}^{n \times n}$.

(ii) $\mathbf{\Delta} = C^{n \times n}$.

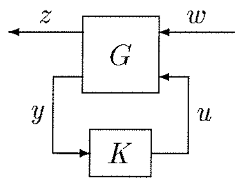
(iii) $\mathbf{\Delta} = \{\mathrm{DIAG}[\Delta_1, \Delta_2] : \Delta_i \in \mathbf{C}^{m_i \times m_i}\} \subset \mathbf{C}^{n \times n}$.

3.3 Uncertain Linear Systems in the LFT Framework

In this section, we review some standard material on descriptions of systems in terms of LFTs. For additional background material on both linear fractional transformations(LFTs), see [143, 144, 73], or the survey article [132].

3.3.1 Linear Fractional Transformations

The LFT formula arises naturally when we describe a well-posed feedback system as shown by the following block diagram.



The resulting input/output relation can be represented as $z = \mathcal{F}_l(G,K)w$, where $\mathcal{F}_l(G,K)$ is defined as the (lower) **linear fractional transformation** (LFT) on K with the coefficient matrix G. More explicitly, suppose G is partitioned as

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \tag{3.13}$$

with entries in $\mathbf{F}$, then for a $\mathbf{F}$-valued matrix K,

$$\mathcal{F}_l(G,K) := G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21} \tag{3.14}$$

provided the inverse is well defined [143, 144]. If G_{21} is square and nonsingular, then $\mathcal{F}_l(G,K) = (A + BK)(C + DK)^{-1}$ with $A = G_{11}G_{21}^{-1}, B = G_{12} - G_{11}G_{21}^{-1}G_{22}, C = G_{21}^{-1}$ and $D = -G_{21}^{-1}G_{22}$. Similarly, the (upper) LFT on Δ, which corresponds to the feedback Δ around upper loop, is defined as

$$\mathcal{F}_u(G,\Delta) = G_{22} + G_{21}\Delta(I - G_{11}\Delta)^{-1}G_{12}. \tag{3.15}$$

We have the following observation about the equivalent relation between the two LFT representations (3.14) and (3.15), which can be verified directly from their definitions.

Lemma 3.3.1 *Suppose matrices M and N are compatibly partitioned, and*

$$N = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} M \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix},$$

where the dimensions of the identity matrices are compatible with the partitions of M and N, then $\mathcal{F}_u(N,\Delta) = \mathcal{F}_l(M,\Delta)$.

The following observation is about the inverse property of an LFT. The proof can be found in [73].

Lemma 3.3.2 *Suppose G is partitioned as in (3.13).*

(i) Assume G_{12} and G_{21} have full column and row rank, respectively, if matrices K_1 and K_2 are such that $\mathcal{F}_l(G, K_1) = \mathcal{F}_l(G, K_2)$ then $K_1 = K_2$.

(ii) Let $P = \mathcal{F}_l(G, K)$. If G, G_{12} and G_{21} are square and invertible, and $\mathrm{DET}(G_{11} - P) \neq 0$, then $K = \mathcal{F}_u(G^{-1}, P)$.

The following properties about LFTs can be directly verified by the definitions (see [207]).

Lemma 3.3.3 *The LFT formula is closed under addition, multiplication, and inversion, i.e., if $T_1(\Delta_1) := \mathcal{F}_u(M_1, \Delta_1)$ and $T_2(\Delta_2) := \mathcal{F}_u(M_2, \Delta_2)$, then there exist suitable M_A, M_P, and M_I such that*

$$\begin{aligned} T_1(\Delta_1) + T_2(\Delta_2) &= \mathcal{F}_u(M_A, \Delta) \\ T_1(\Delta_1)T_2(\Delta_2) &= \mathcal{F}_u(M_P, \Delta) \\ T_1^{-1}(\Delta_1) &= \mathcal{F}_u(M_I, \Delta_1) \end{aligned}$$

with $\Delta := \mathrm{DIAG}[\Delta_1, \Delta_2]$, provided that the operations are well defined.

In the following, let $\mathcal{F}(\mathbf{\Delta}, \mathbf{F})$ denote the set of linear fractional transformations on some copies of $\Delta \in \mathbf{\Delta} \subset \mathbf{F}^{n \times n}$, e.g., if $g \in \mathcal{F}(\mathbf{\Delta}, \mathbf{F})$, then

$$g(\Delta) = \mathcal{F}_u(M, \mathrm{DIAG}[\Delta, \cdots, \Delta])$$

for some matrix M with entries in $\mathbf{F}$. In the following, we will view Δ in the definition as a symbol to represent the structure $\mathbf{\Delta}$. However, given $g \in \mathcal{F}(\mathbf{\Delta}, \mathbf{F})$, $g(\Delta)$ is a well defined $\mathbf{F}$-valued functions on some subset of $\mathbf{\Delta}$ where the corresponding LFT is well defined. To emphasize the set $\mathbf{\Omega} \subset \mathbf{\Delta}$, the notation $\mathcal{F}(\mathbf{\Omega}, \mathbf{F})$ is used.

Proposition 3.3.4 *The set $\mathcal{F}(\mathbf{\Delta}, \mathbf{F})$ is a commutative algebra over $\mathbf{F}$ with the following pointwise operations.*

$$\begin{aligned} (f+g)(\Delta) &= f(\Delta) + g(\Delta) \\ (fg)(\Delta) &= f(\Delta)g(\Delta) \\ (\alpha f)(\Delta) &= \alpha f(\Delta) \end{aligned}$$

where $f, g \in \mathcal{F}(\mathbf{\Delta}, \mathbf{F})$, $\alpha \in \mathbf{F}$, $\Delta \in \mathbf{\Delta}$.

Proof. The proof that the pointwise operations are closed follows from the above lemma. □

It is noted that if we view $\mathbf{F}^{n\times n}$ as the space $\mathbf{F}^N$ with $N = n^2$, and for each $g \in \mathcal{F}(\mathbf{\Omega}, \mathbf{F})$, $g(\Delta)$ is well defined for all $\Delta \in \mathbf{\Omega}$), then $\mathcal{F}(\mathbf{\Omega}, \mathbf{F})$ is a subalgebra of $\mathcal{C}^k(\mathbf{\Omega}, \mathbf{F})$. Also the polynomial algebra $\mathcal{P}(\mathbf{\Omega}, \mathbf{F})$ and rational algebra $\mathcal{R}(\mathbf{\Omega}, \mathbf{F})$ (cf. [97]) are subalgebras of $\mathcal{F}(\mathbf{\Delta}, \mathbf{F})$.

3.3.2 Linear Systems over LFT Algebras

In this section, we will consider a linear system over $\mathcal{F}(\mathbf{\Delta}_P, \mathbf{F})$ as follows,

$$G : \begin{cases} x(t+1) &= \Theta(\Delta_P)x(t) + \Gamma(\Delta_P)u(t) \\ y(t) &= \Phi(\Delta_P)x(t) + \Pi(\Delta_P)u(t) \end{cases} \tag{3.16}$$

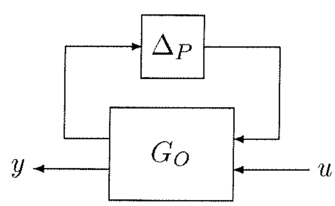
where $t \in \mathbf{Z}^+$; x, u, and y are n_o, p, and q-dimensional state, input, and output vectors, respectively; Δ_P is the symbol of the block structure $\mathbf{\Delta}_P$, which defined as follows,

$$\mathbf{\Delta}_P = \{\mathrm{DIAG}[\delta_1 I_{r_1}, \cdots, \delta_s I_{r_s}, \Delta_1, \cdots, \Delta_f] : \delta_i \in \mathbf{F}, \Delta_j \in \mathbf{F}^{m_j\times m_j}\} \subset \mathbf{F}^{m\times m}; \tag{3.17}$$

Θ, Γ, Φ, and Π are matrices with entries in $\mathcal{F}(\mathbf{\Delta}_P, \mathbf{F})$. As we agreed, the Δ_P appearing in the coefficients is just a symbol which indicates the "spatial" structure of the perturbation.

A Feedback Control Interpretation

The linear system over $\mathcal{F}(\mathbf{\Delta}_P, \mathbf{F})$ has a clear feedback interpretation. More concretely, if system (3.16) is over $\mathcal{F}(\mathbf{\Delta}_P, \mathbf{F})$, then the perturbation Δ_P enters the system in a feedback fashion by "pulling out Δ's" [27, 207] as follows[1]



for some nominal system G_O.

[1] In general, the block structure in the following block-diagram is not the same as the one (3.44) appears in the coefficient matrices in (3.16). However, they are assumed to be the same in the following discussion for simplicity and without loss of generality.

Conversely, the class of linear parameter-varying systems whose unknown parameters or perturbations enter the modeled system in a feedback fashion can be described in terms of LFTs on some specified perturbation structures.

In fact, since $G := \mathcal{F}_u(G_O, \Delta_P)$, suppose that the nominal system, where no uncertainty is imposed, is described by the following equations.

$$G_O : \begin{cases} x(t+1) &= M_{11}x(t) + M_{12}u_d(t) + M_{13}u(t) \\ y_d(t) &= M_{21}x(t) + M_{22}u_d(t) + M_{23}u(t) \\ y(t) &= M_{31}x(t) + M_{32}u_d(t) + M_{33}u(t) \end{cases}$$

with $t \in \mathbf{Z}^+$; define

$$M := \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}.$$

The input/output map G_O can be represented as LFT on $z^{-1}I$, i.e.,

$$G_O := \mathcal{F}_u(M, z^{-1}I),$$

where $z^{-1}I$ can be viewed as a symbol to represent the delay operator. Since the uncertainty Δ_P enters the system in an LFT way, the uncertain system can be represented as

$$\begin{aligned} G(\Delta_P) = \mathcal{F}_u(G_O, \Delta_P) &= \mathcal{F}_u(\mathcal{F}_u(M, z^{-1}I), \Delta_P) \\ &= \mathcal{F}_u(\begin{bmatrix} \Theta(\Delta_P) & \Gamma(\Delta_P) \\ \Phi(\Delta_P) & \Pi(\Delta_P) \end{bmatrix}, z^{-1}I). \end{aligned} \tag{3.18}$$

Therefore, $G(\Delta_P)$ has representation (3.16) where the coefficients matrices Θ, Γ, Φ, and Π are matrices with entries in $\mathcal{F}(\mathbf{\Delta}_P, \mathbf{F})$ which are defined as follows,

$$\Theta(\Delta_P) = \mathcal{F}_l(\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \Delta_P), \quad \Gamma(\Delta_P) = \mathcal{F}_l(\begin{bmatrix} M_{13} & M_{12} \\ M_{23} & M_{22} \end{bmatrix}, \Delta_P), \tag{3.19}$$

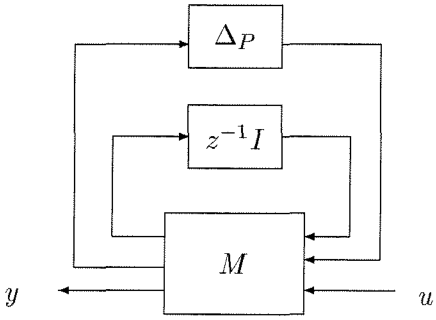
$$\Phi(\Delta_P) = \mathcal{F}_u(\begin{bmatrix} M_{22} & M_{21} \\ M_{32} & M_{31} \end{bmatrix}, \Delta_P), \quad \Pi(\Delta_P) = \mathcal{F}_u(\begin{bmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{bmatrix}, \Delta_P). \tag{3.20}$$

An Alternative Representation

The linear system (3.16) over $\mathcal{F}(\mathbf{\Delta}_P, \mathbf{F})$ has an alternative representation which is more convenient in some case. In fact, from (3.18), one has

$$G = \mathcal{F}_u(M, \Delta), \qquad \Delta := \begin{bmatrix} z^{-1}I & 0 \\ 0 & \Delta_P \end{bmatrix}, \tag{3.21}$$

i.e., G can be represented as following block diagram.



Therefore, the linear system over $\mathcal{F}(\mathbf{\Delta}_P, \mathbf{F})$ can be represented as an LFT on some block structure Δ, i.e.,

$$G = \mathcal{F}_u(\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \Delta) =: \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right](\Delta)^2, \quad \Delta := \begin{bmatrix} z^{-1} & 0 \\ 0 & \Delta_P \end{bmatrix}, \tag{3.22}$$

where $(A, B, C, D) \in \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times p} \times \mathbf{R}^{q\times n} \times \mathbf{R}^{q\times p}$ with $n = n_o + m$ are defined as follows,

$$A := \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \quad B := \begin{bmatrix} M_{13} \\ M_{23} \end{bmatrix}, \quad C := \begin{bmatrix} M_{31} & M_{32} \end{bmatrix}, \quad D := M_{33}, \tag{3.23}$$

where $z^{-1}I$ is a delay operator, and Δ_P is the symbol of $\mathbf{\Delta}_P \subset \mathbf{F}^{m\times m}$. In general, $\mathbf{\Delta}_P$ is taken as follows,

$$\mathbf{\Delta}_P = \{\text{DIAG}[\delta_1 I_{r_1}, \cdots, \delta_s I_{r_s}, \Delta_1, \cdots, \Delta_f] : \delta_i \in \mathbf{F}, \Delta_j \in \mathbf{F}^{m_j \times m_j}\} \subset \mathbf{F}^{m\times m},$$

which could include repeated full blocks. For simplicity, the system (3.22) is also called an **LFT system**[3]. By analogy with standard terminology, we will refer to the representation (3.22) as a "state-space realization" of the transfer function G.

As in the conventional linear systems, (non-singular) state variable transformations are useful in the analysis and synthesis of LFT systems. However, not all transformations are allowed in this setting, the admissible state variable transformations are therefore specified. Consider the LFT system (3.22) with block structure $\mathbf{\Delta}$ of dimension $n \times n$ and **commutative matrix set** $\mathcal{D}$ **of** $\mathbf{\Delta}$. If we think of the system as having "state" vector x, then a state variable transformation $x \mapsto x' := Tx$ is **admissible** if the transformation matrix

[2]In this notation, the structure symbol Δ might be ignored if it is clear from context.

[3]A class of linear shift-invariant multidimensional systems can also be represented as LFT systems, see Section 3.8.

$T \in \mathcal{D}$. The corresponding state-space realization transformation is

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right](\Delta) \mapsto \left[\begin{array}{c|c} TAT^{-1} & TB \\ \hline CT^{-1} & D \end{array}\right](\Delta).$$

Note that the transfer function after the transformation does not change. In the next section we will further see that some properties of LFT systems are also invariant under admissible state variable transformations.

3.4 Stability and Performances of Systems over LFT Algebras

In the next two sections, we introduce some basic notions for linear systems over $\mathcal{F}(\mathbf{\Delta}_P, \mathbf{F})$, including stability, stabilizability, and detectability. The system under consideration is given by

$$G : \left\{ \begin{array}{lcl} x(t+1) & = & \Theta(\Delta_P)x(t) + \Gamma(\Delta_P)w(t) \\ z(t) & = & \Phi(\Delta_P)x(t) + \Pi(\Delta_P)u(t) \end{array} \right. \tag{3.24}$$

with $t \in \mathbf{Z}^+$, where x, u, and y are n_o, p, and q-dimensional state, input, and output vectors, respectively; Θ, Γ, Φ, and Π are matrices with entries in $\mathcal{F}(\mathbf{\Delta}_P, \mathbf{F})$. Or equivalently

$$G(\Delta) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right](\Delta), \quad \Delta = \left[\begin{array}{cc} z^{-1}I & 0 \\ 0 & \Delta_P \end{array}\right], \tag{3.25}$$

where $z^{-1}I$ is a delay operator and Δ_P the symbol of $\mathbf{\Delta}_P$; The real matrices $(A, B, C, D) \in \mathbf{R}^{n \times n} \times \mathbf{R}^{n \times p} \times \mathbf{R}^{q \times n} \times \mathbf{R}^{q \times p}$ are defined in the previous section.

3.4.1 Robust Stability and Robust Performances

Given an uncertainty set $\mathbf{\Omega} \in \mathbf{\Delta}_P$, the system (3.24), or equivalently (3.25), over $\mathcal{F}(\mathbf{\Delta}_P, \mathbf{F})$ (or $\mathcal{F}(\mathbf{\Omega}, \mathbf{F})$) is said to be **robustly stable** if it is **pointwise stable**, i.e., for each $\Delta_P \in \mathbf{\Omega}$, the coefficient matrices in (3.24) are well-defined, and the system is stable for all $\Delta_P \in \mathbf{\Omega}$ with $w(t) = 0$, or equivalently,

$$\rho(\Theta(\Delta_P)) = \rho(\mathcal{F}_l(A, \Delta_P)) < 1 \tag{3.26}$$

for all $\Delta_P \in \mathbf{\Omega}$. From the definition of the robust stability, the following proposition follows.

Proposition 3.4.1 *Consider the system (3.24), or equivalently (3.25), over $\mathcal{F}(\mathbf{\Omega}, \mathbf{F})$, then the following statements are equivalent.*

(i) It is robustly stable.

(ii) The following rank condition is satisfied,

$$\text{Rank } (zI - \Theta(\Delta_P)) = n_o \tag{3.27}$$

for all $z \in \mathbf{C}$ with $|z| \geq 1$ and $\Delta_P \in \mathbf{\Omega}$.

(iii) The following rank condition is satisfied,

$$\text{Rank } (I - A\Delta) = n, \quad \Delta := \text{Diag}[z^{-1}I, \Delta_P] \tag{3.28}$$

for all $z \in \mathbf{C}$ with $|z| \geq 1$ and $\Delta_P \in \mathbf{\Omega}$.

(iv) For any $\Delta_P \in \mathbf{\Omega}$, there exists a positive matrix $Q_{\Delta_P} \in \mathbf{F}^{n\times n}$ such that the following LMI is satisfied,

$$\Theta(\Delta_P)Q_{\Delta_P}\Theta^*(\Delta_P) - Q_{\Delta_P} < 0. \tag{3.29}$$

The system, in which case it is assumed that $p = q$ without loss of generality, is said to have **robust performance** if for each $\Delta_P \in \mathbf{\Omega}$, the coefficient matrices in (3.24) are well-defined and the system has $\mathcal{H}_\infty$-norm < 1, or equivalently,

$$\bar{\sigma}\left(\mathcal{F}_u(M, \Delta)\right) < 1$$

for all $\Delta := \text{Diag}[z^{-1}I, \Delta_P]$ with $|z| \geq 1$ and $\Delta_P \in \mathbf{\Omega}$. The following proposition summarizes some characterizations for robust performance. The proof is omitted.

Proposition 3.4.2 *Consider the system (3.24), or equivalently (3.25), over $\mathcal{F}(\mathbf{\Omega}, \mathbf{F})$, then the following statements are equivalent.*

(i) It has robust performance.

(ii) The following rank condition is satisfied for all $z \in \mathbf{C}$ with $|z| \geq 1$, $\Delta_a \in \mathbf{C}^{p\times p}$ with $\bar{\sigma}(\Delta_a) \leq 1$, and $\Delta_P \in \mathbf{\Omega}$,

$$\text{Rank} \begin{bmatrix} zI - \Theta(\Delta_P) & -\Gamma(\Delta_P)\Delta_a \\ -\Phi(\Delta_P) & I - \Pi(\Delta_P)\Delta_a \end{bmatrix} = n_o + p. \tag{3.30}$$

(iii) The following rank condition is satisfied for all $z \in \mathbf{C}$ with $|z| \geq 1$, $\Delta_a \in \mathbf{C}^{p\times p}$ with $\bar{\sigma}(\Delta_a) \leq 1$, and $\Delta_P \in \mathbf{\Omega}$,

$$\text{Rank}(I - M\Delta_N) = n + p, \quad \Delta_N := \text{Diag}[z^{-1}I, \Delta_P, \Delta_a]. \tag{3.31}$$

(iv) For any $\Delta_P \in \mathbf{\Omega}$, there exists a positive matrix $Q_{\Delta_P} \in \mathbf{F}^{n\times n}$ such that the following LMI is satisfied,

$$\begin{bmatrix} \Theta(\Delta_P) & \Gamma(\Delta_P) \\ \Phi(\Delta_P) & \Pi(\Delta_P) \end{bmatrix} \begin{bmatrix} Q_{\Delta_P} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Theta^*(\Delta_P) & \Phi^*(\Delta_P) \\ \Gamma^*(\Delta_P) & \Pi^*(\Delta_P) \end{bmatrix} - \begin{bmatrix} Q_{\Delta_P} & 0 \\ 0 & I \end{bmatrix} < 0. \tag{3.32}$$

3.4.2 Special Cases: μ-Stability

In the following, we will consider the case where $\mathbf{F} = \mathbf{C}$ and $\mathbf{\Omega} = \mathbf{B\Delta}$:

$$\mathbf{B\Delta} := \{\Delta_P \in \mathbf{\Delta}_P : \bar{\sigma}(\Delta_P) \leq 1\} \subset \mathbf{C}^{m \times m}, \tag{3.33}$$

which is a bounded and closed set in $\mathbf{C}^{n \times n}$, so it is compact. Therefore, in this case, robust stability is equivalent to the uniform stability, i.e.,

$$\sup_{\Delta_P \in \mathbf{B\Delta}} \rho(\mathcal{F}_l(A, \Delta_P)) < 1,$$

since $\rho(\mathcal{F}_l(A, \Delta_P))$ is a continuous function of $\Delta_P \in \mathbf{B\Delta}$. Now suppose $\rho(M_{11}) < 1$, which means that the nominal system is stable. Define

$$\mathbf{\Delta} := \{\Delta = \text{DIAG}[z^{-1}I, \Delta_P] : z \in \mathbf{C}, \Delta_P \in \mathbf{\Delta}_P\}.$$

From the Main-Loop Theorem (Lemma 3.2.4), it is known that the above condition is equivalent to the μ-test: $\mu_\Delta(A) < 1$.

Now suppose $\mu_\Delta(A) < 1$, which means that the system is robustly stable as discussed above. The test for robust performance in this case is equivalent to the requirement:

$$\sup_{|z| \geq 1, \Delta_P \in \mathbf{B\Delta}} \bar{\sigma}\left(\mathcal{F}_u(M, \Delta)\right) < 1.$$

Since the maximization is also over a compact set, and $\bar{\sigma}\left(\mathcal{F}_u(M, \Delta)\right)$ is a continuous function of z and $\Delta_P \in \mathbf{B\Delta}$. Define

$$\mathbf{\Delta}_N := \{\text{DIAG}[\Delta, \Delta_a] : \Delta \in \mathbf{\Delta}, \Delta_a \in \mathbf{C}^{p \times p}\}.$$

Again from the Main-Loop Theorem (Lemma 3.2.4), the robust performance test is equivalent to μ-test: $\mu_{\Delta_N}(M) < 1$.

Motivated by the above observation, we give the following definition.

Definition 3.4.3 *The (3.22) over $\mathcal{F}(\mathbf{B\Delta}, \mathbf{C})$ is μ-stable (with respect to Δ) if $\mu_\Delta(A) < 1$.*

Therefore, the μ-stability notion is an abstraction of robust stability and robust performance tests. Since the μ computation is generally hard, we usually use a sufficient condition, $\mathcal{Q}$-test, as follows,

Definition 3.4.4 *The system (3.25) over $\mathcal{F}(\mathbf{B\Delta}, \mathbf{C})$ is $\mathcal{Q}$-stable (with respect to Δ) if $\mathcal{Q}_\Delta(A) < 1$, i.e., there is a $D \in \mathcal{D}$ such that $\bar{\sigma}(DAD^{-1}) < 1$.*

It is noted that $\mathcal{Q}$-stability is just a sufficient condition for μ-stable in general. We have the following Lyapunov characterization of the $\mathcal{Q}$-stability.

Lemma 3.4.5 *System A with frequency structure Δ is $\mathcal{Q}$-stable if and only if there exists a $P \in \mathcal{D}$ with $P = P^* > 0$ such that $APA^* - P < 0$.*

The computational problem for solving the above Lyapunov inequality, which is a LMI, is a convex feasibility problem.

The following structural property of LFT systems follows immediately from the above definitions of μ-stability and $\mathcal{Q}$-stability and properties of μ.

Theorem 3.4.6 *The μ-stability and $\mathcal{Q}$-stability of LFT systems are invariant under the admissible state variable transformations.*

Another important structural property of LFT systems is expressed by the following theorem.

Theorem 3.4.7 *Let A_1 and A_2 be two system matrices with respect to the frequency structures Δ_1 and Δ_2, respectively. Then*

(i) *if the system matrix $\begin{bmatrix} A_1 & A_{12} \\ A_{21} & A_2 \end{bmatrix}$ with any compatibly dimensioned matrices A_{12} and A_{21} is μ ($\mathcal{Q}$)-stable with respect to the frequency structure $\Delta := \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}$, where Δ_1 and Δ_2 are independent, then A_1 and A_2 are also μ ($\mathcal{Q}$)-stable with respect to structures Δ_1 and Δ_2, respectively.*

(ii) *the system matrix $\begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix}$ with any compatibly dimensioned matrix A_{12} is μ ($\mathcal{Q}$)-stable with respect to the frequency structure $\Delta := \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}$ if and only if A_1 and A_2 are also μ ($\mathcal{Q}$)-stable with respect to structures Δ_1 and Δ_2, respectively.*

Note that the statement (i) holds only in the case where the structures Δ_1 and Δ_2 are independent, while (ii) holds even when Δ_1 and Δ_2 depend on each other.

Proof. Assume that the commutative matrix sets of Δ_1, Δ_2 and $\Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}$ are $\mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{D}$, respectively.

(i) For the μ-case, these properties can be checked easily via the basic properties of μ. We will now focus on the $\mathcal{Q}$-case. Note that system $\begin{bmatrix} A_1 & A_{12} \\ A_{21} & A_2 \end{bmatrix}$ is assumed to be $\mathcal{Q}$-stable, so there exists a positive definite $P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \in \mathcal{D}$ (thus, $P_1 \in \mathcal{D}_1$ and $P_2 \in \mathcal{D}_2$ are both positive definite) such that

$$0 > APA^* - P = \begin{bmatrix} A_1 & A_{12} \\ A_{21} & A_2 \end{bmatrix} \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} A_1^* & A_{21}^* \\ A_{12}^* & A_2^* \end{bmatrix} - \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$$

$$= \begin{bmatrix} A_1P_1A_1^* + A_{12}P_2A_{12}^* - P_1 & A_1P_1A_{21}^* + A_{12}P_2A_2^* \\ A_{21}P_1A_1^* + A_2P_2A_{12}^* & A_2P_2A_2^* + A_{21}P_1A_{21}^* - P_2 \end{bmatrix}.$$

This implies

$$A_1P_1A_1^* - P_1 \le A_1P_1A_1^* + A_{12}P_2A_{12}^* - P_1 < 0$$

and

$$A_2P_2A_2^* - P_2 \le A_2P_2A_2^* + A_{21}P_1A_{21}^* - P_2 < 0,$$

which are what we need.

(ii) We first claim that $A_N = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$ is μ ($\mathcal{Q}$)-stable with respect to Δ if and only if A_1 and A_2 are μ ($\mathcal{Q}$)-stable with respect to Δ_1 and Δ_2 respectively. In fact, in the μ case, it follows by observing

$$\mu_\Delta(A_N) = \max\{\mu_{\Delta_1}(A_1), \mu_{\Delta_2}(A_2)\}.$$

As for the $\mathcal{Q}$ case, the conclusion follows by using the Lyapunov inequality characterization in Lemma 3.4.5.

Now we return to the proof of (ii). We just give the proof in μ case. The arguments for $\mathcal{Q}$ case follow similarly. From the above discussion, it is known that A_1 and A_2 are μ-stable with respect to Δ_1 and Δ_2 respectively if and only if $A_N = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$ is μ-stable with respect to Δ, i.e., $\mu_\Delta(A_N) < 1$. It is known that a μ-value is a continuous function, therefore for the given matrix A_{12}, there is an $\alpha \in \mathbf{R}^+$, such that

$$\mu_\Delta(\begin{bmatrix} A_1 & \alpha A_{12} \\ 0 & A_2 \end{bmatrix}) < 1.$$

Observe that $\begin{bmatrix} A_1 & \alpha A_{12} \\ 0 & A_2 \end{bmatrix} = TAT^{-1}$ with $T = \begin{bmatrix} \alpha I & 0 \\ 0 & I \end{bmatrix} \in \mathcal{D}$, which is an admissible state transformation for system A. Since the transformation does not change the stability by the preceding theorem, it follows that A_N is μ stable with respect to Δ if and only if A is μ stable with respect to Δ. □

Remark 3.4.8 *Part (ii) of the above theorem also implies that a cascade system is μ (Q)-stable if and only if each subsystem is μ (Q)-stable.*

3.5 Stabilization of Linear Systems over LFT Algebras

Given the block structure $\mathbf{\Delta}_P$ (see (3.44)). Consider the linear system (3.24) over $\mathcal{F}(\mathbf{\Delta}_P, \mathbf{F})$.

$$G: \begin{cases} x(t+1) &= \Theta(\Delta_P)x(t) + \Gamma(\Delta_P)w(t) \\ z(t) &= \Phi(\Delta_P)x(t) + \Pi(\Delta_P)w(t) \end{cases} \quad \text{or } G := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right](\Delta)$$

with $t \in \mathbf{Z}^+$, where x, u, and y are n_o, p and q-dimensional state, input, and output vectors, respectively; $\Delta := \mathrm{DIAG}[z^{-1}I, \Delta_P]$; Θ, Γ, Φ, and Π are matrices with entries in $\mathcal{F}(\mathbf{\Delta}_P, \mathbf{F})$; A, B, C, D are real matrices defined by (3.23).

3.5.1 Robust Stabilizability and Detectability

Given an uncertainty set $\mathbf{\Omega} \subset \mathbf{\Delta}_P$, we consider next the properties of system (3.24), or equivalently (3.25), over $\mathcal{F}(\mathbf{\Delta}_P, \mathbf{F})$ (or $\mathcal{F}(\mathbf{\Omega}, \mathbf{F})$) related to stabilization, i.e., stabilizability and detectability. As explained before, we will assume that for all $\Delta_P \in \mathbf{\Omega}$, the coefficient matrices in (3.24) are well defined. The system (3.24) is said to be **robustly stabilizable** if it is pointwise stabilizable, i.e., for each fixed $\Delta_P \in \mathbf{\Omega}$, there is a state feedback matrix F_{Δ_P} with entries in $\mathbf{F}$ such that $\rho(\Theta(\Delta_P) + F_{\Delta_P}\Gamma(\Delta_P)) < 1$. The system (3.24) is said to be **stabilizable over** $\mathcal{F}(\mathbf{\Omega}, \mathbf{F})$ if there is a state feedback matrix $F(\Delta_P)$ over $\mathcal{F}(\mathbf{\Omega}, \mathbf{F})$ such that $\rho(\Theta(\Delta_P) + F(\Delta_P)\Gamma(\Delta_P)) < 1$ for all $\Delta_P \in \mathbf{\Omega}$. The detectability notions can be correspondingly defined in terms of output injection.

The robust stabilizability test for linear system over $\mathcal{F}(\mathbf{\Omega}, \mathbf{F})$ can be conducted by a generalized PBH test.

Theorem 3.5.1 *Consider system (3.24) over $\mathcal{F}(\mathbf{\Omega}, \mathbf{F})$, the following statements are equivalent.*

(i) It is robustly stabilizable.

(ii) For each $\Delta_P \in \mathbf{\Omega}$, there exists a positive definite matrix $Q_{\Delta_P} \in \mathbf{F}^{n_o \times n_o}$, such that

$$\Theta(\Delta_P)Q_{\Delta_P}\Theta^*(\Delta_P) - Q_{\Delta_P} - \Gamma(\Delta_P)\Gamma^*(\Delta_P) < 0.$$

(iii) The following rank condition is satisfied,

$$\textsc{Rank}\begin{bmatrix} zI - \Theta(\Delta_P) & \Gamma(\Delta_P) \end{bmatrix} = n_o$$

for all $z \in \mathbf{C}$ with $|z| \geq 1$ and $\Delta_P \in \mathbf{\Omega}$.

(iv) The following rank condition is satisfied,

$$\textsc{Rank}\begin{bmatrix} I - A\Delta & B \end{bmatrix} = n, \quad \Delta = \textsc{Diag}[z^{-1}I, \Delta_P]$$

for all $z \in \mathbf{C}$ with $|z| \geq 1$ and $\Delta_P \in \mathbf{\Omega}$.

(v) The following rank condition is satisfied,

$$\textsc{Rank}(B_\perp^*(I - A\Delta)) = n - p, \quad \Delta = \textsc{Diag}[z^{-1}I, \Delta_P]$$

for all $z \in \mathbf{C}$ with $|z| \geq 1$ and $\Delta_P \in \mathbf{\Omega}$; and $B_\perp \in \mathbf{R}^{n\times(n-p)}$ with $B_\perp^* B = 0$ and $\textsc{Rank}\begin{bmatrix} B & B_\perp \end{bmatrix} = n$.

The following lemma is needed to prove the above result.

Lemma 3.5.2 *Consider a block matrix $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ whose entries are in $\mathbf{F}$ and A_{22} is invertible. It has full row rank if and only if the matrix $A_{11} + A_{12}A_{22}^{-1}A_{21}$ has full row rank.*

Proof. **[Theorem 3.5.1] (i) $\Leftrightarrow$ (ii)** follows from Theorem 2.2.3. **(i) $\Leftrightarrow$ (iii)** straightforwardly follows from the PBH test. **(iv) $\Leftrightarrow$ (v)** follows from easy algebraic manipulations. It is sufficient to show **(iii) $\Leftrightarrow$ (iv)**.

In fact the full row rank condition in (iii) is equivalent to

$$\textsc{Rank}\begin{bmatrix} I - \Theta(\Delta_P)z^{-1} & \Gamma(\Delta_P)z^{-1} \end{bmatrix} = n_o \tag{3.34}$$

for all $|z| \geq 1$ and $\Delta_P \in \mathbf{\Omega}$.

From (3.19) in the previous section, one has

$$\Theta(\Delta_P) = \mathcal{F}_l(\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \Delta_P) = M_{11} + M_{12}\Delta_P(I - M_{22}\Delta_P)^{-1}M_{21},$$

$$\Gamma(\Delta_P) = \mathcal{F}_l(\begin{bmatrix} M_{13} & M_{12} \\ M_{23} & M_{22} \end{bmatrix}, \Delta_P) = M_{13} + M_{12}\Delta_P(I - M_{22}\Delta_P)^{-1}M_{23}.$$

Therefore,

$$\begin{bmatrix} I-\Theta(\Delta_P)z^{-1} & \Gamma(\Delta_P)z^{-1} \end{bmatrix}$$
$$= \begin{bmatrix} I-(M_{11}+M_{12}\Delta_P(I-M_{22}\Delta_P)^{-1}M_{21})z^{-1} & (M_{13}+M_{12}\Delta_P(I-M_{22}\Delta_P)^{-1}M_{23})z^{-1} \end{bmatrix}$$
$$= \begin{bmatrix} I-M_{11}z^{-1} & M_{13}z^{-1} \end{bmatrix} - M_{12}\Delta_P(I-M_{22}\Delta_P)^{-1}\begin{bmatrix} M_{21}z^{-1} & M_{23}z^{-1} \end{bmatrix}.$$

From Lemma 3.5.2, it follows that the rank condition (3.34), which implies that the last matrix has full row rank n_0, is equivalent to that the matrix

$$\begin{bmatrix} I-M_{11}z^{-1} & M_{13}z^{-1} & M_{12}\Delta_P \\ M_{21}z^{-1} & M_{23}z^{-1} & I-M_{22}\Delta_P \end{bmatrix}$$

has full row rank n; or

$$\text{Rank}\begin{bmatrix} I-A\Delta & B \end{bmatrix} = \text{Rank}\begin{bmatrix} I-A\Delta & Bz^{-1} \end{bmatrix}$$

$$= \text{Rank}\begin{bmatrix} I-M_{11}z^{-1} & M_{12}\Delta_P & M_{13}z^{-1} \\ M_{21}z^{-1} & I-M_{22}\Delta_P & M_{23}z^{-1} \end{bmatrix} = n$$

for all $|z| \geq 1$ and $\Delta_P \in \mathbf{\Omega}$. □

In the following, we consider the case over $\mathcal{F}(\mathbf{\Omega}_R, \mathbf{R})$, where $\mathbf{\Omega}_R$ is real, bounded and closed. And one has the following statement about robust stabilizability and stabilizability over $\mathcal{F}(\mathbf{\Omega}_R, \mathbf{R})$.

Theorem 3.5.3 *Suppose $\mathbf{\Omega}_R$ is a compact real subset of $\mathbf{\Delta}_P$. Consider the system (3.22) over $\mathcal{F}(\mathbf{\Omega}_R, \mathbf{R})$. It is stabilizable over $\mathcal{F}(\mathbf{\Omega}_R, \mathbf{R})$ if and only if it is robustly stabilizable.*

The technique used in the proof closely follows the one used in [97, Theorem 5.10].

Proof. **[Theorem 3.5.3]** The necessity is obvious. We just prove the sufficiency next. Suppose the system is robustly stabilizable. By Lemma 2.2.2, there is a positive integer N_Δ for each fixed $\Delta_P \in \mathbf{\Omega}_R$ such that

$$F_k(\Delta_P) = (\Gamma^*(\Delta_P)P_k(\Delta)\Gamma(\Delta_P)+I)^{-1}\Gamma^*(\Delta_P)P_k(\Delta_P)\Theta(\Delta_P)$$

is a stabilizing feedback for all $k \geq N_\Delta$, where P_k is defined recursively by the Riccati difference equation:

$$P_{k+1}(\Delta_P) = I + \Theta^*(\Delta_P)P_k(\Delta_P)\Theta(\Delta_P)$$
$$-\Theta^*(\Delta_P)P_k(\Delta_P)\Gamma(\Delta_P)(\Gamma^*(\Delta_P)P_k(\Delta_P)\Gamma(\Delta_P)+I)^{-1}\Gamma^*(\Delta_P)P_k(\Delta_P)\Theta(\Delta_P)$$

with $k \in \mathbf{Z}^+$ and $P_0 = I$. By induction, it can be shown by the use of Lemma 3.3.3 that $P_k(\Delta_P)$ has all its entries in $\mathcal{F}(\mathbf{\Omega}_R, \mathbf{R})$ for all $k \in \mathbf{Z}^+$, therefore $F_{N_\Delta}(\Delta_P)$ also has its entries in $\mathcal{F}(\mathbf{\Omega}_R, \mathbf{R})$.

By continuity of Θ, Γ, and F_{N_Δ}, there is an open neighborhood $\mathcal{B}(\Delta_P)$ of Δ_P such that $F_k(\Delta_P)$ is a stabilizing feedback for all $\mathbf{\Omega}_R \in \mathcal{B}(\Delta_P)$ for all $k \geq N_\Delta$. Now $\{\mathcal{B}(\Delta_P)\}|_{\Delta_P \in \mathbf{\Omega}_R}$ is an open cover of $\mathbf{\Omega}_R$; since $\mathbf{\Omega}_R$ is compact, there exists a finite subcover $\{\mathcal{B}(\Delta_{P_i})\}|_{i=1,2,\cdots,r}$. Taking $N := \max\{N_{\Delta_i} : i = 1, 2, \cdots, r\}$, then it follows that $F_k(\Delta_P)$ stabilizing the system for all $k \geq N$ and $\Delta_P \in \mathbf{\Omega}_R$, and it is over $\mathcal{F}(\mathbf{\Omega}_R, \mathbf{R})$. □

Next we just state the dual results for detectability.

Proposition 3.5.4 *Consider system (3.24) over $\mathcal{F}(\mathbf{\Omega}, \mathbf{F})$, the following statements are equivalent.*

(i) It is robustly detectable.

(ii) For fixed $\Delta_P \in \mathbf{\Omega}$, there exists a positive definite matrix $Q_{\Delta_P} \in \mathbf{F}^{n_o \times n_o}$, such that

$$\Theta^*(\Delta_P) Q_{\Delta_P} \Theta(\Delta_P) - Q_{\Delta_P} - \Phi^*(\Delta_P)\Phi(\Delta_P) < 0.$$

(iii) For all $z \in \mathbf{C}$ with $|z| \geq 1$ and $\Delta_P \in \mathbf{\Omega}$,

$$\mathrm{KER} \begin{bmatrix} zI - \Theta(\Delta_P) \\ \Phi(\Delta_P) \end{bmatrix} = \emptyset.$$

(iv) For all $z \in \mathbf{C}$ with $|z| \geq 1$ and $\Delta_P \in \mathbf{\Omega}$,

$$\mathrm{KER} \begin{bmatrix} I - \Delta A \\ C \end{bmatrix} = \emptyset, \qquad \Delta = \begin{bmatrix} z^{-1} I & 0 \\ 0 & \Delta_P \end{bmatrix}.$$

(v) For all $z \in \mathbf{C}$ with $|z| \geq 1$ and $\Delta_P \in \mathbf{\Omega}$,

$$\mathrm{KER}((I - \Delta A) C_\perp^*) = \emptyset, \qquad \Delta = \begin{bmatrix} z^{-1} I & 0 \\ 0 & \Delta_P \end{bmatrix}$$

where $C_\perp \in \mathbf{R}^{(n-q)\times n}$ with $CC_\perp^* = 0$ and $\mathrm{RANK} \begin{bmatrix} C \\ C_\perp \end{bmatrix} = n.$

Theorem 3.5.5 *Suppose $\mathbf{\Omega}_R$ is a compact real subset of $\mathbf{\Delta}_P$. Consider the system (3.22) over $\mathcal{F}(\mathbf{\Omega}_R, \mathbf{R})$. It is detectable over $\mathcal{F}(\mathbf{\Omega}_R, \mathbf{R})$ if and only if it is robustly detectable.*

3.5.2 μ-Stabilizability and μ-Detectability

In this section, we will assume $\mathbf{F} = \mathbf{C}$ and $\mathbf{\Omega} = \mathbf{B\Delta} \subset \mathbf{C}^{m\times m}$ which is defined in (3.33). We consider the system (3.25) over $\mathcal{F}(\mathbf{\Delta}_P, \mathbf{C})$ (or $\mathcal{F}(\mathbf{B\Delta}, \mathbf{C})$) as follows,

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right](\Delta), \quad \Delta := \begin{bmatrix} z^{-1}I & 0 \\ 0 & \Delta_P \end{bmatrix}. \tag{3.35}$$

We have the following conditions about robust stabilizability based on the generalized singular values.

Theorem 3.5.6 *Consider system (3.16) over $\mathcal{F}(\mathbf{B\Delta}_P, \mathbf{F})$, the following statements are equivalent.*

(i) It is robustly stabilizable.

(ii) $\mu_\Delta(B_\perp^* A, B_\perp^*) < 1$, where

$$\Delta = \mathrm{DIAG}[\delta I, \Delta_P],\ \delta \in C,\ |\delta| < 1,\ \Delta_P \in \mathbf{B\Delta}_P. \tag{3.36}$$

Proof. The equivalence is a restatement of the equivalence (i) $\Leftrightarrow$ (v) of Theorem 3.5.1, by virtue of Definition 3.2.5. □

Note that the above characterization in general does not constructively give the stabilizing feedback; a more conservative criterion with computational advantages is based on the $\mathcal{Q}$-value bounds.

Theorem 3.5.7 *Consider the given system over $\mathcal{F}(\mathbf{B\Delta}_P, \mathbf{F})$. Let $\mathbf{\Delta}$ be as in (3.36), $\mathcal{D}$ the commutator set. The following are equivalent,*

(i) $\mathcal{Q}_\Delta(B_\perp^* A, B_\perp^*) < 1$.

(ii) There exists a positive definite matrix $P \in \mathcal{D}$ satisfying the following LMI,

$$B_\perp^*(APA^* - P)B_\perp < 0. \tag{3.37}$$

(iii) There exists a constant "state-feedback" F such that $A + BF$ is $\mathcal{Q}_\Delta$-stable.

Furthermore, if P is a solution of (3.37), the corresponding F in (iii) is given by

$$F = -(B^* P^{-1} B)^{-1} B^* P^{-1} A.$$

Proof. (i)$\Leftrightarrow$(ii) follows from the definition (3.8) of $\mathcal{Q}$. A proof of (ii)$\Leftrightarrow$(iii) can be found in previous work [119] on $\mathcal{Q}$-stabilizability. $\square$

The conditions $(i), (ii), (iii)$ in Theorem 3.5.7 are **sufficient** conditions for robust stabilizability as in Theorem 3.5.6. This follows obviously from the fact that $\mu_\Delta \leq \mathcal{Q}_\Delta$.

Since the structure Δ in (3.36) is not generally μ-simple, the converse does not hold in general, and the conditions in Theorem 3.5.7 are therefore conservative for robust stabilizability. This conservatism is reflected in the fact that in Theorem 3.5.7 we obtain a **constant** stabilizing state feedback, whereas in Theorem 3.5.6 a Δ_P-dependent state feedback is allowed.

In the following, we consider the stabilizability and detectability over $\mathcal{F}(\mathbf{B\Delta}, \mathbf{C})$, which are defined in terms of the following two special structures, respectively,

$$G_{SF} = \left[\begin{array}{c|c} A & B \\ \hline I & 0 \end{array}\right](\Delta), \qquad G_{OI} = \left[\begin{array}{c|c} A & I \\ \hline C & 0 \end{array}\right](\Delta),$$

where the block structures Δ in both cases are the same as the one for G[4].

Definition 3.5.8 *The system G with block structure Δ is* **μ-stabilizable** *if there exists a dynamical controller for the corresponding system G_{SF},*

$$K = \mathcal{F}_u(F, \Delta_0) = \left[\begin{array}{c|c} F_{11} & F_{12} \\ \hline F_{21} & F_{22} \end{array}\right](\Delta_0), \;\; \Delta_0 := \text{DIAG}[\Delta, \cdots, \Delta],$$

such that the closed loop system is μ-stable with respect to the induced block structure.

It is noted that if $\mathcal{F}_u(F, \Delta_0)$ is well defined, then the system is μ-stabilizable if and only if $\rho((A + B\mathcal{F}_u(F, \Delta_0))\Delta) < 1$ for $\bar{\sigma}(\Delta) \leq 1$.

Definition 3.5.9 *The system G with block structure Δ is* **μ-detectable** *if there exists a dynamical controller for the corresponding system G_{OI},*

$$K = \mathcal{F}_u(L, \Delta_0) = \left[\begin{array}{c|c} L_{11} & L_{12} \\ \hline L_{21} & L_{22} \end{array}\right](\Delta_0), \;\; \Delta_0 := \text{DIAG}[\Delta, \cdots, \Delta],$$

such that the closed loop system is μ-stable with respect to the induced block structure.

[4]It is noted that in this case, the coefficient matrices for (3.24) are **not** assumed to be well defined for all $\Delta_P \in \mathbf{B\Delta}$.

Note that in both cases, the state feedback and output injection are allowed to be dynamical, i.e., they are allowed to depend on the delay operator $z^{-1}I$. The above two properties can be also characterized in terms of stabilizing some augmented systems by static state feedback and output injection by using the following lemma.

Lemma 3.5.10 *System G is μ-stabilized by some $K = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right](\Delta_0)$ with block structure Δ_0 related to Δ if and only if the augmented system*

$$G_a = \left[\begin{array}{cc|cc} A & 0 & B & 0 \\ 0 & 0 & 0 & I \\ \hline C & 0 & D & 0 \\ 0 & I & 0 & 0 \end{array}\right](\Delta_N), \quad \Delta_N = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_0 \end{bmatrix}$$

is μ-stabilized by static feedback $F = \begin{bmatrix} \hat{D} & \hat{C} \\ \hat{B} & \hat{A} \end{bmatrix}$ with respect to block structure Δ_N.

Proof. This follows from the feedback-interconnection properties of LFTs. □

The following theorem gives a characterization of the μ-stabilizability.

Theorem 3.5.11 *The system is μ-stabilizable only if the following rank condition is satisfied,*

$$\text{Rank}\begin{bmatrix} I - A\Delta & B \end{bmatrix} = n, \quad \Delta = \text{Diag}[z^{-1}I, \Delta_P] \tag{3.38}$$

for all $z \in \mathbf{C}$ with $|z| \geq 1$ and $\Delta_P \in \mathbf{B\Delta}$. If the uncertainty set $\mathbf{B\Delta}$ is real, the rank condition (3.38) is sufficient.

Proof. **[Necessity]** Suppose the system is μ-stabilizable, then there is a μ-stabilizing "state"-feedback:

$$K = \mathcal{F}_u(F, \Delta_0) = \left[\begin{array}{c|c} F_{11} & F_{12} \\ \hline F_{21} & F_{22} \end{array}\right](\Delta_0), \;\; \Delta_0 := \text{Diag}[\Delta, \cdots, \Delta].$$

Suppose l is such that $\Delta_0 \in \mathbf{C}^{l \times l}$. From Lemma 3.5.10, it follows that

$$\text{Rank}(I - (\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} F_{22} & F_{21} \\ F_{12} & F_{11} \end{bmatrix})\begin{bmatrix} \Delta & 0 \\ 0 & \Delta_0 \end{bmatrix}) = n + l$$

for all $z \in \mathbf{C}$ with $|z| \geq 1$ and $\Delta_P \in \mathbf{B\Delta}$. Therefore, one has

$$n + l = \text{Rank}(I - (\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} F_{22} & F_{21} \\ F_{12} & F_{11} \end{bmatrix}) \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_0 \end{bmatrix})$$

$$\leq \text{Rank} \begin{bmatrix} I - A\Delta & 0 & B & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \leq n + l.$$

Therefore, the above inequalities become equalities. Thence,

$$\text{Rank} \begin{bmatrix} I - A\Delta & B \end{bmatrix} = n$$

for all $z \in \mathbf{C}$ with $|z| \geq 1$ and $\Delta_P \in \mathbf{B\Delta}$.

[Sufficiency] It is noted that $\text{Rank} \begin{bmatrix} I - A\Delta & B \end{bmatrix} = n$ if and only if the system is robustly stabilizable, then the conclusion follows from Theorem 3.5.3. □

From the above theorem and Theorem 3.5.3, one has the following statement.

Corollary 3.5.12 *Suppose the set* $\mathbf{B\Delta}$ *is real. The system is* μ*-stabilizable if and only if there is a static state feedback controller* $K(\Delta_o) = \mathcal{F}_u(F, \Delta_o)$ *where* Δ_o *only depends on* Δ_P.

This corollary implies that if the perturbation is real, then μ-stabilizability, robust stabilizability, and stabilizability over $\mathcal{F}(\mathbf{B\Delta}_P, \mathbf{R})$ are equivalent. Note that latter two notions of stabilizability are defined in terms of static state feedback, while for μ-stabilizability it is allowed to have dynamic state feedback.

The following theorem characterizes a class of systems over $\mathcal{F}(\mathbf{B\Delta}, \mathbf{C})$ for which the μ-stabilizing controllers can be obtained by solving some LMIs.

Theorem 3.5.13 *Consider the given system over* $\mathcal{F}(\mathbf{B\Delta}, \mathbf{C})$*. The implications between following statements,* **(i)** $\Rightarrow$ **(ii)** $\Rightarrow$ **(iii)***, hold.*

(i) There exists a positive definite matrix $P \in \mathcal{D}$ *satisfying the following LMI,*

$$APA^* - P - BB^* < 0. \tag{3.39}$$

(ii) There exists a μ*-stabilizing "state-feedback" controller which is given by*

$$u(t) = -(B^*P^{-1}B)^{-1}B^*P^{-1}A\tilde{x}(t),$$

where $P \in \mathcal{D}$ *is the solution of LMI (3.39).*

(iii) The following rank condition is satisfied,

$$\text{RANK}\begin{bmatrix} I - A\Delta & B \end{bmatrix} = n, \quad \Delta = \text{DIAG}[z^{-1}I, \Delta_P]$$

for all $z \in \mathbf{C}$ with $|z| \geq 1$ and $\Delta_P \in \mathbf{B\Delta}$.

In addition, let $B_\perp \in \mathbf{R}^{n\times(n-p)}$ is such that $B_\perp^* B = 0$ and $\text{DET}\begin{bmatrix} B & B_\perp \end{bmatrix} \neq 0$. If the block structure $\mathbf{\Delta}$ defined by

$$\mathbf{\Delta} := \{\text{DIAG}[z^{-1}, \Delta_P] : z \in \mathbf{C}, \Delta_P \in \mathbf{\Delta}_P\}$$

is μ-simple with respect to $(B_\perp^* A, B_\perp^*)$, then the above statements are equivalent.

Proof. **(i)** $\Rightarrow$ **(ii)** follows from the discussion in the next section. **(ii)** $\Rightarrow$ **(iii)** follows from Theorem 3.5.11.

Finally, we are in the position to prove **(iii)** $\Leftrightarrow$ **(i)** if $\mathbf{\Delta}$ is μ-simple with respect to $(B_\perp^* A, B_\perp^*)$; however, it follows from Theorem 3.2.8. □

Dually, one has the following statement about μ-detectability.

Theorem 3.5.14 *The system is μ-detectable only if the following kernel condition is satisfied,*

$$\text{KER}\begin{bmatrix} I - \Delta A \\ C \end{bmatrix} = \emptyset, \qquad \Delta = \begin{bmatrix} z^{-1}I & 0 \\ 0 & \Delta_P \end{bmatrix}, \tag{3.40}$$

for all $z \in \mathbf{C}$ with $|z| \geq 1$ and $\Delta_P \in \mathbf{B\Delta}$. In addition, if $\mathbf{B\Delta}$ is real, then the above rank condition is sufficient, and moreover there is a (static) output injection controller $K(\Delta_o) = \mathcal{F}_u(L, \Delta_o)$ where Δ_o only depends Δ_P.

Remark 3.5.15 *The kernel condition first appears in [137, 138] when stability robustness analysis is conducted in the behavioral setting. It is a generalization of the PBH test for detectability.*

3.5.3 Output Feedback Stabilization of Systems over LFT Algebras

In this section we outline how the results in the last two subsections can be combined to provide an output feedback stabilizing controller for the system (3.16). The controller will itself be a system over $\mathcal{F}(\mathbf{\Omega}, \mathbf{F})$ which can be given a gain-scheduling interpretation: for each value of the parameter space $\mathbf{\Omega} \subset \mathbf{\Delta}_P$, we schedule a controller which must stabilize the corresponding plant.

Stabilization over $\mathcal{F}(\mathbf{\Omega}, \mathbf{F})$

The system over $\mathcal{F}(\mathbf{\Omega}, \mathbf{F})$ to be considered is given in (3.16), i.e.,

$$G : \begin{cases} x(t+1) &= \Theta(\Delta_P)x(t) + \Gamma(\Delta_P)u(t) \\ y(t) &= \Phi(\Delta_P)x(t) + \Pi(\Delta_P)u(t) \end{cases}$$

where $t \in \mathbf{Z}^+$; x, u, and y are n_o, p, and q-dimensional state, input, and output vectors, respectively; Δ_P is the symbol of the block structure $\mathbf{\Delta}_P$, Θ, Γ, Φ, and Π are matrices with entries in $\mathcal{F}(\mathbf{\Omega}, \mathbf{F})$.

In this section, we need to find an output feedback controller over $\mathcal{F}(\mathbf{\Omega}, \mathbf{F})$, i.e.,

$$K : \begin{cases} \xi(t+1) &= \hat{\Theta}(\Delta_P)\xi(t) + \hat{\Gamma}(\Delta_P)y(t) \\ u(t) &= \hat{\Phi}(\Delta_P)\xi(t) + \hat{\Pi}(\Delta_P)y(t) \end{cases}$$

such that the closed loop system is robustly stable.

A version of this problem in terms of $\mathcal{Q}$-stability was covered in [119]; in fact, the same "separation" structure can be used in the context of stabilization over $\mathcal{F}(\mathbf{\Omega}, \mathbf{F})$ given in this chapter. We state the result for completeness; the proof follows similar lines as the one in [119].

Theorem 3.5.16 *Consider the system G (3.16) over $\mathcal{F}(\mathbf{\Omega}, \mathbf{F})$, assume it is stabilizable and detectable over $\mathcal{F}(\mathbf{\Omega}, \mathbf{F})$. Let $F(\Delta_P)$ and $L(\Delta_P)$ be the corresponding state feedback and output injection over $\mathcal{F}(\mathbf{\Omega}, \mathbf{F})$, respectively. Then there exists a controller K over $\mathcal{F}(\mathbf{\Omega}, \mathbf{F})$ such that the feedback interconnection $\mathcal{F}_l(G, K)$, which is defined over $\mathcal{F}(\mathbf{\Omega}, \mathbf{F})$, is robustly stable. Such a controller K has the following structure,*

$$K = \left[\begin{array}{c|c} \Theta(\Delta_P) + \Gamma(\Delta_P)F(\Delta_P) + L(\Delta_P)\Phi(\Delta_P) + L(\Delta_P)\Pi(\Delta_P)F(\Delta_P) & -L(\Delta_P) \\ \hline F(\Delta_P) & 0 \end{array}\right]. \tag{3.41}$$

μ-Stabilization

Next, we consider the μ-stabilization via output feedback of the system over $\mathcal{F}(\mathbf{B\Delta}_P, \mathbf{F})$.

Consider the following system over $\mathcal{F}(\mathbf{B\Delta}_P, \mathbf{F})$ (3.22):

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right](\Delta), \quad \Delta := \begin{bmatrix} z^{-1}I & 0 \\ 0 & \Delta_P \end{bmatrix}.$$

The μ-stabilization problem via output feedback is to find a controller

$$K = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right](\Delta_K), \quad \Delta_K := \mathrm{D{\small IAG}}[\Delta, \cdots, \Delta]$$

such that the closed loop system is μ stable with respect to the newly induced block structure $\Delta_N := \mathrm{D{\small IAG}}[\Delta, \Delta_K]$. The main result is stated as in the next theorem.

Theorem 3.5.17 *Consider the system G (3.22) over $\mathcal{F}(\mathbf{B\Delta}_P, \mathbf{F})$, there exists a controller K over $\mathcal{F}(\mathbf{B\Delta}_P, \mathbf{F})$ such that the feedback interconnection $\mathcal{F}_l(G, K)$, which is defined over $\mathcal{F}(\mathbf{B\Delta}_P, \mathbf{F})$, is μ-stable if and only if the system (3.22) is μ-stabilizable and μ-detectable.*

Proof. The necessity is obvious. Next, we just show sufficiency; the proof is constructive. Suppose

$$K_F = \left[\begin{array}{c|c} F_{11} & F_{12} \\ \hline F_{21} & F_{22} \end{array}\right](\Delta_0), \quad \Delta_0 := \mathrm{D{\small IAG}}[\Delta, \cdots, \Delta]$$

is the μ-stabilizing "state feedback," and

$$K_L = \left[\begin{array}{c|c} L_{11} & L_{12} \\ \hline L_{21} & L_{22} \end{array}\right](\Delta_0), \quad \Delta_0 := \mathrm{D{\small IAG}}[\Delta, \cdots, \Delta]$$

is the μ-stabilizing "output injection." Without loss of generality, it can be assumed that both block structures (Δ_0) are the same. Now define

$$A_a := \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad B_a := \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix}, \quad C_a := \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}, \quad D_a := \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix},$$

where the increased dimensions are compatible with the dimension of Δ_0. Consider the augmented system

$$G_a = \left[\begin{array}{c|c} A_a & B_a \\ \hline C_a & D_a \end{array}\right](\Delta_a), \quad \Delta_a := \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_0 \end{bmatrix}. \tag{3.42}$$

Then by Lemma 3.5.10, it is μ-stabilizable by constant "state feedback" $F := \begin{bmatrix} F_{22} & F_{21} \\ F_{12} & F_{11} \end{bmatrix}$ or by constant "output injection" $L := \begin{bmatrix} L_{22} & L_{21} \\ L_{12} & L_{11} \end{bmatrix}$. Now we construct an "output feedback" controller for the augmented system (3.42) as follows,

$$K_a = \left[\begin{array}{c|c} A_a + B_aF + LC_a + LD_aF & -L \\ \hline F & 0 \end{array}\right](\Delta_a) =: \begin{bmatrix} K_{11}(\Delta_a) & K_{12}(\Delta_a) \\ K_{21}(\Delta_a) & K_{22}(\Delta_a) \end{bmatrix},$$

where the block sizes in the latter matrix are compatible with the structure of $\begin{bmatrix} \Delta & 0 \\ 0 & \Delta_0 \end{bmatrix}$. It can be shown by the argument in [119] that the augmented closed loop system, which is defined as

$$\mathcal{F}_l(G_a(\Delta_a), K_a(\Delta_a)) = \mathcal{F}_l(\begin{bmatrix} A_a & B_aF \\ -LC_a & A + B_aF + LC_a \end{bmatrix}, \begin{bmatrix} \Delta_a & 0 \\ 0 & \Delta_a \end{bmatrix}),$$

is μ-stable with respect to the block structure $\begin{bmatrix} \Delta_a & 0 \\ 0 & \Delta_a \end{bmatrix}$. Therefore, the controller

$$\begin{aligned} K &:= \mathcal{F}_u(\begin{bmatrix} K_{22}(\Delta_a) & K_{21}(\Delta_a) \\ K_{12}(\Delta_a) & K_{11}(\Delta_a) \end{bmatrix}, \Delta_0) \\ &=: \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right](\Delta_K), \quad \Delta_K := \text{DIAG}[\Delta_0, \Delta, \Delta_0], \end{aligned}$$

for some suitable constant matrices $\hat{A}, \hat{B}, \hat{C}, \hat{D}$ with entries in $\mathbf{F}$ μ-stabilizes the system (3.22). □

3.6 Robust Performance Synthesis of Systems over LFT Algebras

In this section, we consider the following linear system over $\mathcal{F}(\mathbf{\Delta}_P, \mathbf{F})$,

$$G_p : \begin{cases} x(t+1) &= \Theta(\Delta_P)x(t) + \Gamma_1(\Delta_P)w(t) + \Gamma_2(\Delta_P)u(t) \\ z(t) &= \Phi_1(\Delta_P)x(t) + \Pi_{11}(\Delta_P)w(t) + \Pi_{12}(\Delta_P)u(t) \\ y(t) &= \Phi_2(\Delta_P)x(t) + \Pi_{21}(\Delta_P)w(t) + \Pi_{22}(\Delta_P)u(t) \end{cases} \tag{3.43}$$

where $t \in \mathbf{Z}^+$; x, w, u, z, y are n_o, p, p_2, p, q_2-dimensional state, disturbance input, control input, regulated output, and measured output vectors, respectively; Δ_P is the symbol of block structure $\mathbf{\Delta}_P$, which is defined as follows,

$$\mathbf{\Delta}_P = \{\text{DIAG}[\delta_1 I_{r_1}, \cdots, \delta_s I_{r_s}, \Delta_1, \cdots, \Delta_f] : \delta_i \in \mathbf{F}, \Delta_j \in \mathbf{F}^{m_j \times m_j}\} \subset \mathbf{F}^{m \times m}; \tag{3.44}$$

Θ, Γ_i, Φ_j, and Π_{ij} $(i, j = 1, 2)$ are matrices with entries in $\mathcal{F}(\mathbf{\Delta}_P, \mathbf{F})$. An alternative representation is as follows,

$$G_p := \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right](\Delta), \quad \Delta := \begin{bmatrix} z^{-1} & 0 \\ 0 & \Delta_P \end{bmatrix} \tag{3.45}$$

for some real matrices A, B_i, C_j, D_{ij} with $i, j = 1, 2$.

We first consider the robust performance synthesis problem by full-information feedback in which both the state x and the disturbance w are available. Given an uncertainty set $\mathbf{\Omega} \subset \mathbf{\Delta}_P$, consider the system (3.43) over $\mathcal{F}(\mathbf{\Delta}, \mathbf{F})$, or $\mathcal{F}(\mathbf{\Omega}, \mathbf{F})$, whose coefficient matrices are assumed to be well defined for all $\Delta_P \in \mathbf{\Omega}$; the full-information feedback robust performance synthesis problem is said to have **pointwise solution** if for each $\Delta_P \in \mathbf{\Omega}$, there exists a full-information feedback $F_{\Delta_P}(z)$, which could be dynamic, such that the closed loop system has $\mathcal{H}_\infty$-norm < 1. The full-information feedback robust performance problem is said to have **solution over** $\mathcal{F}(\mathbf{\Omega}, \mathbf{F})$, if there exists a full-information feedback $u := F(\Delta_P)$, which could be dynamic and over $\mathcal{F}(\mathbf{\Omega}, \mathbf{F})$, such that the closed loop system has $\mathcal{H}_\infty$-norm < 1 for all $\Delta_P \in \mathbf{\Omega}$. Define

$$\bar{A}(\Delta_P) := \begin{bmatrix} \Theta(\Delta_P) & \Gamma_1(\Delta_P) \\ \Phi_1(\Delta_P) & \Pi_{11}(\Delta_P) \end{bmatrix}, \quad \bar{B}(\Delta_P) := \begin{bmatrix} \Gamma_2(\Delta_P) \\ \Pi_{12}(\Delta_P) \end{bmatrix}$$

$$\tilde{A} := \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix}, \quad \tilde{B} := \begin{bmatrix} B_2 \\ D_{12} \end{bmatrix}.$$

We first have the following result about pointwise solvability, which is just a re-statement of Theorems 2.4.1 and 2.4.4.

Theorem 3.6.1 *Consider system (3.43) over $\mathcal{F}(\mathbf{\Omega}, \mathbf{F})$. The following statements are equivalent.*

(i) The full-information feedback robust performance problem has pointwise solution.

(ii) For each fixed $\Delta_P \in \mathbf{\Omega}$, there exists a positive definite matrix $X_{\Delta_P} \in \mathbf{F}^{n_o \times n_o}$, and $\alpha > 0$, such that the following LMI is satisfied,

$$\bar{A}(\Delta_P) \begin{bmatrix} X_{\Delta_P} & 0 \\ 0 & \alpha I \end{bmatrix} \bar{A}^*(\Delta_P) - \begin{bmatrix} X_{\Delta_P} & 0 \\ 0 & \alpha I \end{bmatrix} - \bar{B}(\Delta_P)\bar{B}^*(\Delta_P) < 0. \tag{3.46}$$

(iii) The following (full row) rank condition is satisfied,

$$\text{RANK} \begin{bmatrix} zI - \Theta(\Delta_P) & -\Gamma_1(\Delta_P)\Delta_a & \Gamma_2(\Delta_P) \\ -\Phi_1(\Delta_P) & I - \Pi_{11}(\Delta_P)\Delta_a & \Pi_{12}(\Delta_P) \end{bmatrix} = n_o + p \tag{3.47}$$

for all $z \in \mathbf{C}$ with $|z| \geq 1$, $\Delta_P \in \mathbf{\Omega}$, and $\Delta_a \in \mathbf{C}^{p \times p}$ with $\bar{\sigma}(\Delta_a) \leq 1$.

(iv) The following (full row) rank condition is satisfied,

$$\text{RANK} \begin{bmatrix} I - \tilde{A}\Delta_N & \tilde{B} \end{bmatrix} = n_o + m + p, \quad \Delta_N = \text{DIAG}[z^{-1}I, \Delta_P, \Delta_a]$$

for all $z \in \mathbf{C}$ with $|z| \geq 1$, $\Delta_P \in \mathbf{\Omega}$, and $\Delta_a \in \mathbf{C}^{p \times p}$ with $\bar{\sigma}(\Delta_a) \leq 1$.

Proof. **(i)** $\Leftrightarrow$ **(ii)** $\Leftrightarrow$ **(iii)** follows from Theorem 2.4.2. **(iii)** $\Leftrightarrow$ **(iv)** is derived by the use of a similar argument in Theorem 3.5.1. □

In the following, we consider the special case where $\mathbf{\Omega} = \mathbf{B\Delta}$ and $\mathbf{F} = \mathbf{C}$. The following theorem characterizes a special class of systems over $\mathcal{F}(\mathbf{B\Delta}, \mathbf{C})$ for which the solutions can be obtained by solving some LMIs.

Theorem 3.6.2 *Consider the system (3.43), or equivalently (3.45), over $\mathcal{F}(\mathbf{B\Delta}, \mathbf{C})$. The implications between following statements,* **(i)** $\Rightarrow$ **(ii)** $\Rightarrow$ **(iii)** $\Rightarrow$ **(iv)**, *hold.*

(i) There exists a positive definite matrix $P \in \mathcal{D}$ satisfying the following LMI,

$$\tilde{B}_\perp^*(\tilde{A}\begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix}\tilde{A}^* - \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix})\tilde{B}_\perp < 0,$$

where $\tilde{B}_\perp \in \mathbf{F}^{(n+p)\times n}$ such that $\tilde{B}_\perp^*\tilde{B} = 0$ and $\mathrm{DET}\begin{bmatrix} \tilde{B} & \tilde{B}_\perp \end{bmatrix} \neq 0$.

(ii) There exists a (static) full-information feedback controller F solving the robust performance problem over $\mathcal{F}(\mathbf{B\Delta}, \mathbf{C})$, which is given by

$$u(t) = -(\tilde{B}^*\begin{bmatrix} P^{-1} & 0 \\ 0 & I \end{bmatrix}\tilde{B})^{-1}\tilde{B}^*\begin{bmatrix} P^{-1} & 0 \\ 0 & I \end{bmatrix}\tilde{A}\begin{bmatrix} \tilde{x}(t) \\ w(t) \end{bmatrix}.$$

(iii) For each fixed $\Delta_P \in \mathbf{\Omega}$, there exists a positive definite matrix $X_{\Delta_P} \in \mathbf{F}^{n_o\times n_o}$, and $\alpha > 0$, such that the following LMI is satisfied,

$$\bar{A}(\Delta_P)\begin{bmatrix} X_{\Delta_P} & 0 \\ 0 & \alpha I \end{bmatrix}\bar{A}^*(\Delta_P) - \begin{bmatrix} X_{\Delta_P} & 0 \\ 0 & \alpha I \end{bmatrix} - \bar{B}(\Delta_P)\bar{B}^*(\Delta_P) < 0. \tag{3.48}$$

(iv) The following (full row) rank condition is satisfied,

$$\mathrm{RANK}\begin{bmatrix} I - \tilde{A}\Delta_N & \tilde{B} \end{bmatrix} = n_o + m + p, \quad \Delta_N = \mathrm{DIAG}[z^{-1}I, \Delta_P, \Delta_a]$$

for all $z \in \mathbf{C}$ with $|z| \geq 1$, $\Delta_P \in \mathbf{\Omega}$, and $\Delta_a \in \mathbf{C}^{p\times p}$ with $\bar{\sigma}(\Delta_a) \leq 1$.

In addition, if the block structure $\mathbf{\Delta}_N$ defined by

$$\mathbf{\Delta}_N := \{\mathrm{DIAG}[z^{-1}, \Delta_P, \Delta_a] : z \in \mathbf{C}, \Delta_P \in \mathbf{\Delta}_P, \Delta_a \in \mathbf{C}^{p\times p}\}$$

is μ-simple with respect to $(\tilde{B}_\perp^*\tilde{A}, \tilde{B}_\perp^*)$, then the above statements are equivalent.

Proof. **(i)** $\Rightarrow$ **(ii)** follows from the discussion in the next section. **(ii)** $\Rightarrow$ **(iii)** follows from Theorem 3.6.1. We show next **(iii)** $\Rightarrow$ **(iv)**.

In fact, (iii) implies that for each fixed $\Delta_P \in \mathbf{B\Delta}$, there exists a pointwise state feedback F_{Δ_P} such that the closed loop system

$$z(t) = \left[\begin{array}{c|c} \Theta(\Delta_P) + \Gamma_2(\Delta_P)F_{\Delta_P} & \Gamma_1(\Delta_P) \\ \hline \Phi_1(\Delta_P) + \Pi_{12}(\Delta_P)F_{\Delta_P} & \Pi_{11}(\Delta_P) \end{array}\right]$$

has robust performance, i.e.,

$$\textsc{Rank}(I - \begin{bmatrix} \Theta(\Delta_P) + \Gamma_2(\Delta_P)F_{\Delta_P} & \Gamma_1(\Delta_P) \\ \Phi_1(\Delta_P) + \Pi_{12}(\Delta_P)F_{\Delta_P} & \Pi_{11}(\Delta_P) \end{bmatrix} \begin{bmatrix} z^{-1} & 0 \\ 0 & \Delta_a \end{bmatrix}) = n_o + p.$$

Therefore

$$n_o + p = \textsc{Rank}(I - \begin{bmatrix} \Theta(\Delta_P) + \Gamma_2(\Delta_P)F_{\Delta_P} & \Gamma_1(\Delta_P) \\ \Phi_1(\Delta_P) + \Pi_{12}(\Delta_P)F_{\Delta_P} & \Pi_{11}(\Delta_P) \end{bmatrix} \begin{bmatrix} z^{-1} & 0 \\ 0 & \Delta_a \end{bmatrix})$$

$$\leq \textsc{Rank} \begin{bmatrix} zI - \Theta(\Delta_P) & -\Gamma_1(\Delta_P)\Delta_a & \Gamma_2(\Delta_P) \\ -\Phi_1(\Delta_P) & I - \Pi_{11}(\Delta_P)\Delta_a & \Pi_{12}(\Delta_P) \end{bmatrix} \leq n_o + p.$$

Therefore the inequalities as above become equality.

On the other hand, from Theorem 3.6.1, it follows that (iv) holds.

Finally, we are in position to prove **(iv)** $\Leftrightarrow$ **(i)** if $\mathbf{\Delta}_N$ is μ-simple with respect to $(\tilde{B}_\perp^* \tilde{A}, \tilde{B}_\perp^*)$; however, it follows from Theorem 3.2.8 $\square$

3.7 Notes and References

The LPV systems where the unknown parameters enter the systems in feedback fashions are considered in this chapter. This class of systems are described in LFT framework. The idea of LFT is originated in [143, 144], the detailed discussion is carried out in [207]. The basic tool for analysis in LFT framework is structured singular value: μ, which is developed mainly by Doyle [53] (some of the ideas also appear in [151]). The connection of μ and robustness analysis is made in [63]. A complete exposition of this issue is made in [132].

The synthesis issue is also considered in a sequence of conference papers by Doyle [53, 54, 56, 57, 55]. See also [119, 133, 134, 111, 131]. However, in those synthesis considerations, only sufficient conditions are provided for LTI perturbation case. The robustness synthesis in LFT framework can be reduced to the PBH-like test, which is motivated by the work in [137] where the robustness analysis is conducted in a behavioral settings. This consideration results in a generalized notion of μ. The discussion of the generalized μ follows the techniques by Doyle [53, 132, 138].

The results in this chapter is reported in [118].

3.8 Appendix: Multidimensional Systems Described in LFT Framework

Consider a 2-dimensional LSI system of order (n_1, n_2) which is described by the Roesser state-space equations [147],

$$\begin{cases} x_1(k_1+1, k_2) &= A_{11}x_1(k_1,k_2) + A_{12}x_2(k_1,k_2) + B_1 u(k_1,k_2) \\ x_2(k_1, k_2+1) &= A_{21}x_1(k_1,k_2) + A_{22}x_2(k_1,k_2) + B_2 u(k_1,k_2) \\ y(k_1,k_2) &= C_1 x_1(k_1,k_2) + C_2 x_2(k_1,k_2) + D u(k_1,k_2) \end{cases}$$

where $x_1(k_1,k_2) \in \mathbf{R}^{n_1}$ and $x_2(k_1,k_2) \in \mathbf{R}^{n_2}$ denote the state vectors, $u(k_1,k_2) \in \mathbf{R}^p$ the input vector, and $y(k_1,k_2) \in \mathbf{R}^q$ the output vector. Define

$$A := \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B := \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C := \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$

and $\Delta = \begin{bmatrix} z_1^{-1} I_{n_1} & 0 \\ 0 & z_2^{-1} I_{n_2} \end{bmatrix}$, where $z_i^{-1} I$ can be interpreted as a backward shift operator. The transfer matrix for this system with zero initial conditions is

$$G(\Delta) = D + C\left(\begin{bmatrix} z_1 I & 0 \\ 0 & z_2 I \end{bmatrix} - A\right)^{-1} B$$

$$= D + C\Delta(I - \Delta A)^{-1} B = \mathcal{F}_u\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \Delta\right), \tag{3.49}$$

i.e., this system is represented as an LFT with respect to frequency structure Δ.

More generally, an N-dimensional discrete LSI system with order $(n_1, \cdots, n_N)$ can also be represented in terms of an LFT as (3.49) with respect to frequency structure

$$\Delta = \mathrm{DIAG}[z_1^{-1} I_{n_1}, \cdots, z_N^{-1} I_{n_N}].$$

Define

$$\bar{U}^N := \{(z_1, \cdots, z_N) : z_i \in \mathbf{C}, | z_i | \geq 1\}$$

and

$$T(z_1, \cdots, z_N) = \text{DET}[I - \Delta A].$$

It is known that the N-dimensional system with system matrix A defined above is internally stable if and only if $T(z_1, \cdots, z_N) \neq 0$ in $\bar{U}^N$ [25, 1]. Equivalently, the system is stable if and only if for any $z_{10}, \ldots, z_{N0}$ such that $T(z_{10}, \cdots, z_{N0}) = 0$, then $\max\{| z_{10}^{-1} |, \cdots, | z_{N0}^{-1} |\} < 1$. In addition, the system is stable if there exists $P = \text{DIAG}\{P_1, \cdots, P_N\}$ which is positive definite with $P_i \in \mathbf{R}^{n_i \times n_i}, i = 1, \cdots, N$, such that the following Lyapunov inequality holds [1],

$$APA^* - P < 0.$$

Define

$$\mathbf{\Delta} = \{\text{DIAG}[z_1^{-1} I_{n_1}, \cdots, z_N^{-1} I_{n_N}] : z_i \in \mathbf{C}\}.$$

It is noted that the above internally stability definition is equivalent to $\mu_{\Delta}(A) < 1$, and the Lyapunov condition is equivalent to $\mathcal{Q}_{\Delta}(A) < 1$.

Chapter 4

Synthesis of Uncertain Linear Systems with LTV Perturbations

4.1 Introduction

In the last chapter, we have seen that the solvability conditions of stabilization and performance synthesis are equivalent to some PBH-like tests, which reduced to some generalized μ-tests if the class of systems has LTI uncertainty. Moreover, the corresponding $\mathcal{Q}$-tests provide the sufficient conditions. It will be shown that if arbitrary **linear time-varying** (LTV) uncertainty is allowed, then the $\mathcal{Q}$-stability tests become necessary for robust stability and robust performance.

In this chapter, we further consider the class of linear uncertain systems which are described in terms of **linear fractional transformation** (LFT) formulas, and the bounded uncertainty is allowed to be LTV. In particular, we examine the corresponding robust stabilization problem in detail. This stabilization problem is treated in an axiomatic fashion. The notions of $\mathcal{Q}$-stabilizability and $\mathcal{Q}$-detectability are defined; they are characterized by the positive definite solutions of linear matrix inequalities, which result in constant full information and full control (feedback) solutions. For the general output feedback control problem, we particularly consider the dynamic feedback solution. It is shown that both $\mathcal{Q}$-stabilizability and $\mathcal{Q}$-detectability are necessary and sufficient for robust stabilization by output feedback, and the resulting controllers are represented by LFTs on the same block structures as the plants, which means that they are dynamical and depend on the perturbation; they therefore may be thought of as **gain scheduled** or **dynamically scheduled**

[176, 145]. The application implications of such control schemes are obvious when the uncertainties can be measured or identified on line [176, 145]. A disadvantage of the proposed approach is that unless the uncertainties are arbitrary time-varying operators, the stability analysis on which the synthesis is based is potentially conservative, and possibly highly so. This conservativeness may be most troublesome when Δ represents slowly varying parameters, since neither the slow variation nor the parametric nature of the uncertainty is exploited. Thus the methods in this chapter should be viewed as a possible aid in conventional scheduling rather than a replacement.

The techniques in this chapter are motivated by those in [60], and directly generalize standard stabilization results and state-space methods, while streamlining much of the development. The construction for the **output feedback** (OF) problem is achieved via a separation argument which involves the reduction of the OF problem to two special problems: **full information** (FI) and **full control** (FC). The FI and FC $\mathcal{Q}$-stabilization problems are solved in terms of the positive definite solutions of certain LMIs, and the controllers can be chosen as static feedbacks. The resulting dynamic controller for the OF problem has a separation structure; all stabilizing controllers are parameterized as an LFT on a free stable parameter. An appealing benefit of this approach is that the necessity portion of the controller parameterization relies more heavily on elegant LFT machinery and avoids the need for coprime factorizations [199, 55, 129]. In a further generalization it is noted that all that is required for the separation principle to hold for the LFT systems is that the stability is invariant under certain system transformations and cascade interconnections. Thus the μ-stabilization problem admits a separation structure as well. It is remarked that the elegant separation property can not be easily carried out for the performance synthesis problem, although the synthesis by full-information feedback, partial-information feedback is similar to the the stabilization problem. This is the essential difference between the approaches used here and in [133, 131].

The structure of this chapter is as follows: in Sections 2 and 3, background material is provided, the properties of $\mathcal{Q}$-stability, stabilizability and detectability are characterized. In Section 4, the main results on the synthesis problems are stated; in addition, the static output feedback problem is considered. In Section 5, $\mathcal{Q}$-stabilization of the different special problems are examined and the relationships among them are established. The output feedback problem is solved via separation arguments. In Section 6, the stabilizing controller characterization problem is considered, and the parameterization of all stabilizing controllers is obtained from the special problems via separation arguments. The structure of the parameterized closed-loop map is examined.

4.2 Robust Stability, Robust Performance, and $\mathcal{Q}$-Stability

In the following discussion, we will re-examine the linear system over the LFT algebra (3.16), and it has the following representation.

$$G(\Delta) = \mathcal{F}_u(\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \Delta) =: \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right](\Delta), \quad \Delta := \text{DIAG}[z^{-1}I, \Delta_P] \tag{4.1}$$

with $(A, B, C, D) \in \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times p} \times \mathbf{R}^{q\times n} \times \mathbf{R}^{q\times p}$. Δ_P is a symbol to represent the block structure $\mathbf{\Delta}_P$.

It is also known from the discussion in Section 3.3.2 that system (4.1) can be represented as

$$G(\Delta_P) = \mathcal{F}_u(G_O(z), \Delta_P) = \mathcal{F}_u(\mathcal{F}_u(M, z^{-1}I), \Delta_P) \tag{4.2}$$

with $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$.

In this section, our emphasis is on dealing with the uncertain structure $\mathbf{\Delta}_P$ which includes time-varying uncertainty. To describe this, let $\ell^{s\times t}$ denote the linear time-varying causal operators: $\ell_2^t(\mathbf{Z}^+) \to \ell_2^s(\mathbf{Z}^+)$. $\mathbf{\Delta}_P$ denotes the structure of the linear time-varying causal perturbations:

$$\mathbf{\Delta}_P := \{\text{DIAG}[\delta_1 I_{r_1}, \cdots, \delta_s I_{r_s}, \Delta_1, \cdots, \Delta_f] : \delta_i \in \ell, \Delta_j \in \ell^{m_j\times m_j}\} \subset \ell^{m\times m}, \tag{4.3}$$

which may include repeated full blocks. In addition, the permissible uncertainty is defined as

$$\mathbf{B\Delta} := \{\Delta_P \in \mathbf{\Delta}_P : \|\Delta_P\|_{\ell_2} \le 1\}. \tag{4.4}$$

The system (4.1) is said to be **robustly stable** if the system is asymptotically stable for each fixed $\Delta_P \in \mathbf{B\Delta}$. It has **robust performance** if the system is stable and has ℓ_2-induced norm < 1 for each $\Delta_P \in \mathbf{B\Delta}$.

Now define

$$\mathcal{D}_P := \{D \in \mathbf{C}^{m\times m} \text{ nonsingular} : \left\|D\Delta D^{-1}\right\|_{\ell_2} = \|\Delta\|_{\ell_2}, \Delta \in \mathbf{\Delta}_P\}, \tag{4.5}$$

$$\mathcal{D} := \{D := \text{DIAG}[D_1, D_2] \in \mathbf{C}^{(n_o+m)\times(n_o+m)} : D_1 \in \mathbf{C}^{n_o\times n_o} \text{ is nonsingular}, D_2 \in \mathcal{D}_P\}. \tag{4.6}$$

Let $G_O(z)$ be partitioned as $G_O(z) := \begin{bmatrix} G_{11}(z) & G_{12}(z) \\ G_{12}(z) & G_{22}(z) \end{bmatrix}$. The following proposition is well known (cf. [159, 123]).

Proposition 4.2.1 *Consider the system (4.1), or equivalently (4.2).*

(i) It is robustly stable if and only if the nominal system is stable, i.e., $\rho(M_{11}) < 1$, and there exists a positive definite matrix $D_P \in \mathcal{D}_P$ such that

$$\left\| D_P G_{11}(z) D_P^{-1} \right\|_\infty < 1.$$

(ii) It has robust performance if and only if the nominal system is stable, and there exists a positive definite matrix $D_Q \in \mathcal{D}_P$ such that

$$\left\| \begin{bmatrix} D_Q & 0 \\ 0 & I \end{bmatrix} G_O(z) \begin{bmatrix} D_Q^{-1} & 0 \\ 0 & I \end{bmatrix} \right\|_\infty < 1.$$

Moreover, we have the following LMI characterizations.

Theorem 4.2.2 *Consider the system (4.1), or equivalently (4.2).*

(i) It is robustly stable if and only if there exists a positive definite matrix $P \in \mathcal{D}$ such that

$$APA^* - P < 0.$$

(ii) It has robust performance if and only if there exists a positive definite matrix $Q \in \mathcal{D}$ such that

$$M \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} M^* - \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} < 0.$$

Proof. We will show (i), and (ii) is verified similarly. In fact, let $A =: \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$, then

$$G_{11}(z) = \mathcal{F}_u(A, z^{-1}I) = \left[\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{21} & M_{22} \end{array} \right].$$

From the previous proposition, the system is robustly stable if and only if there exists $D_P \in \mathcal{D}_P$ such that

$$\left\| D_P G_{11}(z) D_P^{-1} \right\|_\infty < 1. \tag{4.7}$$

Since

$$D_P G_{11}(z) D_P^{-1} = \mathcal{F}_u(\begin{bmatrix} M_{11} & M_{12} D_P^{-1} \\ D_P M_{21} & D_P M_{22} D_P^{-1} \end{bmatrix}, z^{-1}I),$$

by the main-loop theorem, (4.7) is equivalent to

$$\mu_{\Delta_v}(\begin{bmatrix} M_{11} & M_{12}D_P^{-1} \\ D_P M_{21} & D_P M_{22} D_P^{-1} \end{bmatrix}) < 1, \tag{4.8}$$

with $\Delta_v \in \mathbf{\Delta}_v$, where

$$\mathbf{\Delta}_v := \{\text{DIAG}[z^{-1}I_{n_0}, \Delta_0] | z^{-1} \in \mathbf{C}, \Delta_0 \in \mathbf{C}^{(n-n_0)\times(n-n_0)}\}. \tag{4.9}$$

With the above block structure, the μ-value is equal to its upper bound: $\mathcal{Q}$-value; therefore, there exist $P_0 \in \mathbf{R}^{n_0 \times n_0}$ nonsingular, and $\gamma > 0$, such that

$$\bar{\sigma}(\begin{bmatrix} P_0 & 0 \\ 0 & \gamma I \end{bmatrix} \begin{bmatrix} M_{11} & M_{12}D_P^{-1} \\ D_P M_{21} & D_P M_{22} D_P^{-1} \end{bmatrix} \begin{bmatrix} P_0^{-1} & 0 \\ 0 & \gamma^{-1} I \end{bmatrix}) < 1,$$

or

$$\bar{\sigma}(\begin{bmatrix} P_0 & 0 \\ 0 & \gamma D_P \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} P_0^{-1} & 0 \\ 0 & \gamma^{-1} D_P^{-1} \end{bmatrix}) < 1$$

where $\gamma D_P \in \mathcal{D}_P$; i.e., $\bar{\sigma}(DAD^{-1}) < 1$, where $D := \begin{bmatrix} P_0 & 0 \\ 0 & \gamma D_P \end{bmatrix} \in \mathcal{D}$. Thus, the conclusion follows by defining $P = (D^* D)^{-1}$ which belongs to $\mathcal{D}$. □

It is known that the system (3.25) over $\mathcal{F}(\mathbf{B\Delta}, \mathbf{C})$ (or the system matrix A) is **$\mathcal{Q}$-stable** (with respect to Δ) if $\mathcal{Q}_\Delta(A) < 1$, i.e., there is a $D \in \mathcal{D}$ such that $\bar{\sigma}(DAD^{-1}) < 1$. The $\mathcal{Q}$-stability is equivalent to there existing a $P \in \mathcal{D}$ with $P = P^* > 0$ such that $APA^* - P < 0$. Therefore, from Theorem 4.2.2 and the definition of $\mathcal{Q}$-stability, we can see that the Q-stability is an abstraction of robust stability and robust performance of linear system under LTV perturbations.

It is also known from the previous chapter that $\mathcal{Q}$-stability is invariant under the admissible state variable transformations. A cascade system is $\mathcal{Q}$-stable if and only if each subsystem is $\mathcal{Q}$-stable.

4.3 Characterizations of $\mathcal{Q}$-Stabilizability

In this section, we will consider Q-stabilization of the following system over $\mathcal{F}(\mathbf{B\Delta}, \mathbf{C})$,

$$G(\Delta) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right](\Delta)$$

with $(A, B, C, D) \in \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times p} \times \mathbf{R}^{q\times n} \times \mathbf{R}^{q\times p}$ and assume further that B and C are of full column and row ranks, respectively, i.e., $\text{RANK}(B) = p \leq n$ and $\text{RANK}(C) = q \leq n$. Denote the commutative matrix set of Δ by $\mathcal{D}$.

$\mathcal{Q}$-stabilizability and $\mathcal{Q}$-detectability are defined similarly to their μ-counterparts as in the last chapter. For example,

Definition 4.3.1 *The system $G(\Delta)$ with frequency structure Δ is $\mathcal{Q}$-stabilizable if there exists a dynamical controller for the corresponding system* $G_{SF}(\Delta) := \left[\begin{array}{c|c} A & B \\ \hline I & 0 \end{array}\right]$,

$$K(\Delta_0) = \mathcal{F}_u(F, \Delta_0) = \left[\begin{array}{c|c} F_{11} & F_{12} \\ \hline F_{21} & F_{22} \end{array}\right],$$

such that the closed loop system is $\mathcal{Q}$-stable with respect to the induced frequency structure.

Also stabilizability and detectability can be characterized in terms of stabilizing some augmented systems by static state feedback and output injection.

For a one-dimensional system, stabilizability (detectability) is equivalent to the fact that the system can be stabilized by static state-feedback (output-injection). An immediate question arises: is this property still true for $\mathcal{Q}$-stabilizability? To answer it, we shall first consider how to characterize static state-feedback $\mathcal{Q}$-stabilizability.

Suppose the system is $\mathcal{Q}$-stabilizable by a static state-feedback matrix $F \in \mathbf{R}^{p\times n}$, i.e., $\mathcal{Q}_\Delta(A + BF) < 1$, or there exists a matrix $P \in \mathcal{D}$ with $P = P^* > 0$ such that

$$(A + BF)P(A + BF)^* - P < 0.$$

If $\text{RANK}(B) = p < n$ we can find a $B_\perp \in \mathbf{R}^{n\times(n-p)}$ such that $B^* B_\perp = 0$ and $\text{RANK}(B_\perp) \leq n - p$, then we have

$$B_\perp^*(A + BF)P(A + BF)^* B_\perp - B_\perp^* P B_\perp < 0,$$

or

$$B_\perp^* A P A^* B_\perp - B_\perp^* P B_\perp < 0.$$

So the solvability of the last LMI is necessary for the system to be static-state-feedback $\mathcal{Q}$-stabilizable. But, surprisingly, this condition is also sufficient if $\text{RANK}(B_\perp) = n - p$ as stated in the following proposition.

Proposition 4.3.2 *Consider the system $G(\Delta)$ with block structure Δ and $\mathrm{RANK}(B) = p < n$. Let $B_\perp \in \mathbf{R}^{n\times(n-p)}$ be such that $B^*B_\perp = 0$ and $\begin{bmatrix} B & B_\perp \end{bmatrix}$ be invertible. There exists a static state feedback F such that $A + BF$ is $\mathcal{Q}$-stable with respect to the frequency structure Δ if and only if there exists a matrix $P \in \mathcal{D}$ with $P = P^* > 0$ such that*

$$B_\perp^* APA^* B_\perp - B_\perp^* P B_\perp < 0. \tag{4.10}$$

Moreover, if $P \in \mathcal{D}$ with $P = P^* > 0$ satisfies the above inequality, then a $\mathcal{Q}$-stabilizing static state feedback matrix can be chosen as

$$F = -(B^*P^{-1}B)^{-1}B^*P^{-1}A. \tag{4.11}$$

Proof. By the definition of $\mathcal{Q}$-stability, there exists a static feedback F such that the closed loop system matrix $A + BF$ is $\mathcal{Q}$-stable with respect to the frequency structure Δ if and only if there exists $D \in \mathcal{D}$ such that

$$1 > \inf_{F\in\mathbf{R}^{p\times n}} \bar{\sigma}(D(A+BF)D^{-1}) = \inf_{F\in\mathbf{R}^{p\times n}} \bar{\sigma}(DAD^{-1} + DBFD^{-1}).$$

Let $V_\perp^* = (B_\perp^*(D^*D)^{-1}B_\perp)^{-\frac{1}{2}}B_\perp^* D^{-1}$, it is easy to check that $V_\perp^* V_\perp = I$ and $V_\perp^*(DB) = 0$. By Lemma 2.2.4, we have

$$1 > \inf_{F\in\mathbf{R}^{p\times n}} \bar{\sigma}(D(A+BF)D^{-1}) = \bar{\sigma}(V_\perp^* DAD^{-1}),$$

or

$$(V_\perp^* DAD^{-1})(V_\perp^* DAD^{-1})^* < I.$$

Take $P = (D^*D)^{-1}$, then $P \in \mathcal{D}$ and $P = P^* > 0$, hence we have

$$(B_\perp^* P B_\perp)^{-\frac{1}{2}} B_\perp^* APA^* B_\perp (B_\perp^* P B_\perp)^{-\frac{1}{2}} - I < 0,$$

or

$$B_\perp^* APA^* B_\perp - B_\perp^* P B_\perp < 0.$$

Moreover, if some $P \in \mathcal{D}$ with $P = P^* > 0$ satisfies the above inequality, then we can construct a constant state feedback matrix F via Lemma 2.2.4 such that $A+BF$ is $\mathcal{Q}$-stable. Let $V_0^* = (B^*(D^*D)B)^{-1/2}B^*D^*$, then $[V_0, V_\perp]$ unitary; thus $FD^{-1} = -(V_0^*DB)^{-1}V_0^*DAD^{-1}$ by preceding lemma. Wherefore,

$$F = -(V_0^*DB)^{-1}V_0^*DA = -(B^*P^{-1}B)^{-1}B^*P^{-1}A.$$

$\square$

Using the above result we can easily get

Theorem 4.3.3 *The system $G(\Delta)$ is $\mathcal{Q}$-stabilizable if and only if there exists a static feedback matrix F such that $A+BF$ is $\mathcal{Q}$-stable with respect to the same frequency structure.*

Proof. If B is square and of full rank, then the result is straightforward. We thus consider the case where $\mathrm{RANK}(B) = p < n$.

The sufficiency is obvious. As for the necessity, assume that the system can be $\mathcal{Q}$-stabilized by a dynamical controller $K(\Delta_0) = \mathcal{F}_u(F_0, \Delta_0)$ where $F_0 = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}$ and Δ_0 depends on the system frequency structure Δ. By Lemma 3.5.10, this is equivalent to the fact that the augmented system

$$G_a(\Delta_N) = \left[\begin{array}{cc|cc} A & 0 & B & 0 \\ 0 & 0 & 0 & I \\ \hline I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{array}\right]$$

is $\mathcal{Q}$-stabilized by the static feedback $\begin{bmatrix} F_{22} & F_{21} \\ F_{12} & F_{11} \end{bmatrix}$ with respect to the frequency structure $\Delta_N = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_0 \end{bmatrix}$. Denote the commutative matrix set of Δ_N by $\mathcal{D}_N$, then by the above proposition, there exists a $P_N =: \begin{bmatrix} P & P_1 \\ P_1^T & P_0 \end{bmatrix} \in \mathcal{D}_N$, which is positive definite such that

$$\begin{bmatrix} B\perp \\ 0 \end{bmatrix}^* \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} P_N \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}^* \begin{bmatrix} B\perp \\ 0 \end{bmatrix} - \begin{bmatrix} B\perp \\ 0 \end{bmatrix}^* P_N \begin{bmatrix} B\perp \\ 0 \end{bmatrix} < 0,$$

i.e.,

$$B_\perp^* A P A^* B_\perp - B_\perp^* P B_\perp < 0.$$

So the above LMI has a solution $P > 0$. It can be verified that $P \in \mathcal{D}$ by using the assumptions on the frequency structures and their commutative matrix sets. Therefore, the system can be $\mathcal{Q}$-stabilized by a static feedback matrix via the previous proposition. □

The next theorem provides some alternative characterizations to the $\mathcal{Q}$-stabilizability.

Theorem 4.3.4 *Consider the system (4.1), the following statements are equivalent.*

(i) The system is $\mathcal{Q}$-stabilizable.

(ii) There exists a static feedback matrix F such that $A+BF$ is $\mathcal{Q}$-stable with respect to the same block structure.

(iii) There exists a matrix $P \in \mathcal{D}$ with $P = P^* > 0$ such that

$$B_\perp^* APA^* B_\perp - B_\perp^* P B_\perp < 0. \tag{4.12}$$

(iv) There exists a matrix $P \in \mathcal{D}$ with $P = P^* > 0$ such that

$$APA^* - P - BB^* < 0. \tag{4.13}$$

(v) The map $\begin{bmatrix} I - A\Delta & B \end{bmatrix}$ with $\Delta := \mathrm{Diag}[\delta I, \Delta_P]$, which maps $\ell_2^{n+p}(\mathbf{Z}^+)$ to $\ell_2^n(\mathbf{Z}^+)$, is surjective for all $\delta \in \ell$ with $\|\delta\|_{\ell_2} \leq 1$ and $\Delta_P \in \mathbf{B\Delta}$.

Moreover, if $P \in \mathcal{D}$ with $P = P^* > 0$ satisfies the above inequalities (4.12) or (4.13), then a $\mathcal{Q}$-stabilizing static state feedback matrix can be chosen as

$$F = -(B^* P^{-1} B)^{-1} B^* P^{-1} A. \tag{4.14}$$

Proof. It has been shown that **(i)** $\Leftrightarrow$ **(ii)** $\Leftrightarrow$ **(iii)** in Proposition 4.3.2 and Theorem 4.3.3. **(iii)** $\Leftrightarrow$ **(iv)** follows from the Schur complement argument (see Theorem 2.2.3). **(iii)** $\Leftrightarrow$ **(v)** follows from [138]. $\square$

The characterizations of $\mathcal{Q}$-detectability is derived by some dual arguments. We have the following theorem.

Theorem 4.3.5 *Consider the system (4.1), the following statements are equivalent.*

(i) The system is $\mathcal{Q}$-detectable.

(ii) There exists a static output-injection matrix L such that $A + LC$ is $\mathcal{Q}$-stable with respect to the same block structure.

(iii) There exists a matrix $P \in \mathcal{D}$ with $P = P^* > 0$ such that

$$C_\perp A^* PAC_\perp^* - C_\perp PC_\perp^* < 0. \tag{4.15}$$

(iv) There exists a matrix $P \in \mathcal{D}$ with $P = P^* > 0$ such that

$$A^* PA - P - C^* C < 0. \tag{4.16}$$

(v) The map $\begin{bmatrix} I - \Delta A \\ C \end{bmatrix}$ with $\Delta := \begin{bmatrix} \delta I & 0 \\ 0 & \Delta_P \end{bmatrix}$, which maps $\ell_2^n(\mathbf{Z}^+)$ to $\ell_2^{n+q}(\mathbf{Z}^+)$, is injective for all $\delta \in \ell$ with $\|\delta\|_{\ell_2} \leq 1$ and $\Delta_P \in \mathbf{B\Delta}$; i.e.,

$$\mathrm{Ker} \begin{bmatrix} I - \Delta A \\ C \end{bmatrix} = \emptyset, \quad \Delta := \begin{bmatrix} \delta I & 0 \\ 0 & \Delta_P \end{bmatrix}$$

for all $\delta \in \ell$ with $\|\delta\|_{\ell_2} \leq 1$ and $\Delta_P \in \mathbf{B\Delta}$.

In addition, if $P \in \mathcal{D}$ with $P = P^* > 0$ satisfying the LMI (4.15) or AMI (4.16), a static output injection matrix L such that $A + LC$ is $\mathcal{Q}$-stable with respect to the same block structure can be taken as

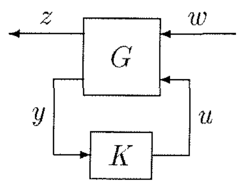
$$L = -AP^{-1}C^*(CP^{-1}C^*)^{-1}.$$

4.4 Stabilization and Controller Characterization

In this section, the main results for LFT system stabilization problems are stated; their constructive proofs will be given in the next two sections.

4.4.1 Problem Statements and Assumptions

Consider the control system with standard block diagram



where G is the plant with two sets of inputs: the exogenous inputs w and the control inputs u, and with two sets of outputs: the measured outputs y and the regulated outputs z. The control problem is to design a feedback controller K such that the resulting closed loop system has some prescribed properties. In this chapter, we only consider a generalized stabilization problem.

Suppose $(w, u, z, y) \in \mathbf{R}^{p_1} \times \mathbf{R}^{p_2} \times \mathbf{R}^{q_1} \times \mathbf{R}^{q_2}$, and $G(\Delta)$ with frequency/uncertainty structure $\mathbf{\Delta}$ has the following realization (with state $x \in \mathbf{R}^n$),

$$G(\Delta) = \begin{bmatrix} G_{11}(\Delta) & G_{12}(\Delta) \\ G_{21}(\Delta) & G_{22}(\Delta) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right], \tag{4.17}$$

where all matrices are real and have compatible dimensions with the related physical variables. We further assume that $\textsc{Rank}(B_2) = p_2 \leq n$ and $\textsc{Rank}(C_2) = q_2 \leq n$. We will mainly consider the non-trivial case where $p_2 < n$ and $q_2 < n$. In addition, let the state-space realization of $K(\Delta_0)$ be

$$K(\Delta_0) = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right], \tag{4.18}$$

which is assumed throughout to be $\mathcal{Q}$-stabilizable and $\mathcal{Q}$-detectable. In the following, it is assumed that $I - D_{22}\hat{D}$ is invertible to insure the **well-posedness** of the feedback system. The frequency/uncertainty structure $\mathbf{\Delta}_0$ of the controller is determined by $\mathbf{\Delta}$. In particular, the controller could have the same dependence on the frequency/uncertainty structure as the plant. As discussed in the introduction, in the linear uncertain system case, the controller can be given a "gain scheduling" or "dynamic scheduling" interpretation, as the controller depends on the same perturbations as does the plant; in the linear multidimensional system case, this means that a dynamical feedback controller is allowed.

From now on, we will concentrate on stabilization-related synthesis problems. We will mainly consider the case where there is no constraint on the controller's frequency structure Δ_0, so the controller can access all information about the plant's frequency structure Δ, i.e., Δ_0 can be some copies of Δ; hence $\Delta_0 = diag[\Delta, \Delta, \cdots]$; however, under the stabilizability and observability conditions, we will see it is sufficient that $\Delta_0 = \Delta$. In the static controller case, no information about the plant frequency structure Δ is available to the controller. On the other hand, the robust performance synthesis problem for a (uncertain) linear system can be viewed as a "stabilization" problem of an augmented systems with an extra block (see [133, 111]); in this case, only partial information about the plant block structure is available to its controllers; its solution is considered in [111, 133] in some detail.

A feedback controller $K(\Delta_0)$ is said to be **admissible** if it $\mathcal{Q}$-stabilizes $G(\Delta)$, (i.e., $\mathcal{F}_l(G(\Delta), K(\Delta_0))$ is $\mathcal{Q}$ stable). For convenience, this general synthesis problem is called the **output feedback** (OF) problem.

Next, we define the admissible controller set as $\mathcal{K}$, i.e.,

$$\mathcal{K} = \{K(\Delta_0) : \mathcal{F}_l(G(\Delta), K(\Delta_0)) \text{ is } \mathcal{Q} \text{ stable}\}.$$

And a subset $\mathcal{K}_s$ of $\mathcal{K}$ is defined as

$$\mathcal{K}_s = \{K \in \mathbf{R}^{p_2 \times q_2} : \mathcal{F}_l(G(\Delta), K) \text{ is } \mathcal{Q} \text{ stable}\}.$$

The following two synthesis problems are considered in this chapter,

- Find a static or dynamic output feedback $K(\Delta) \in \mathcal{K}$ which $\mathcal{Q}$-stabilizes $G(\Delta)$.
- Characterize all controllers $K \in \mathcal{K}$ that $\mathcal{Q}$-stabilize G, or more specifically, find J such that $\mathcal{K} = \{\mathcal{F}_l(J, Q) : Q(\Delta) \text{ is } \mathcal{Q}\text{-stable}\}$.

Note that G is $\mathcal{Q}$-stabilizable by K if and only if G_{22} can be $\mathcal{Q}$-stabilized by K. Thus, the input w and output z in the diagram do not affect the final stabilization results; they are reserved for some technical reasons, since they can also be viewed as some auxiliary input and output in the stabilization problem. They appear naturally when the external performance is considered [111, 133].

4.4.2 Solutions to Synthesis Problems: Static Controllers

In this subsection, we seek the conditions under which the system can be $\mathcal{Q}$-stabilized by static output-feedback controllers. The results in this subsection are essentially obtained in [55], and included here for completeness. The following lemma is key to our solutions (c.f. [139, 45, 55]).

Lemma 4.4.1 *(i)* **(Parrott's Theorem)** *Assume* $(X, B, C, A) \in \mathbf{R}^{n_1 \times m_1} \times \mathbf{R}^{n_1 \times m_2} \times \mathbf{R}^{n_2 \times m_1} \times \mathbf{R}^{n_2 \times m_2}$, *then*

$$\inf_{X \in \mathbf{R}^{n_1 \times m_1}} \bar{\sigma}\left(\begin{bmatrix} X & B \\ C & A \end{bmatrix}\right) = \max\left\{\bar{\sigma}\left(\begin{bmatrix} C & A \end{bmatrix}\right), \bar{\sigma}\left(\begin{bmatrix} B \\ A \end{bmatrix}\right)\right\} =: \gamma_0,$$

and the infimum can be achieved by $\hat{X} = -YA^*Z$, *where* Y *and* Z *solve the matrix equations* $Y(\gamma_0^2 I - A^*A)^{1/2} = B$ *and* $(\gamma_0^2 I - AA^*)^{1/2} Z = C$.

(ii) Suppose $\gamma > \gamma_0$. *The solutions* X *such that* $\bar{\sigma}\left(\begin{bmatrix} X & B \\ C & A \end{bmatrix}\right) < \gamma$ *are exactly those of the form*

$$X = -YA^*Z + \gamma(I - YY^*)^{1/2} W (I - Z^*Z)^{1/2},$$

where $Y = B(\gamma^2 I - A^*A)^{-1/2}$, $Z = (\gamma^2 I - AA^*)^{-1/2} C$, *and* W *is an arbitrary contraction* $(\bar{\sigma}(W) < 1)$.

As a corollary to the above lemma, we have the following result.

Lemma 4.4.2 *Consider the triple* $(A, B, C) \in \mathbf{R}^{n \times n} \times \mathbf{R}^{n \times p} \times \mathbf{R}^{q \times n}$ *with* $\mathrm{RANK}(B) = p < n$ *and* $\mathrm{RANK}(C) = q < n$. *Let* $B_\perp \in \mathbf{R}^{n \times (n-p)}$ *and* $B_0 \in \mathbf{R}^{p \times n}$ *be such that* $B_\perp^* B = 0$ *and* $\begin{bmatrix} B_0 & B_\perp \end{bmatrix}$ *is unitary, and let* $C_\perp \in \mathbf{R}^{n-q \times n}$ *and* $C_0 \in \mathbf{R}^{n \times q}$ *be such that* $C_\perp C^* = 0$ *and* $\begin{bmatrix} C_0 \\ C_\perp \end{bmatrix}$ *is unitary. Then*

$$\inf_{F \in \mathbf{R}^{p \times q}} \bar{\sigma}(A + BFC) = \max\{\bar{\sigma}(B_\perp^* A), \bar{\sigma}(AC_\perp^*)\}.$$

Proof. Note that

$$\bar{\sigma}(A+BFC) = \bar{\sigma}\left(\begin{bmatrix} B_0^*AC_0^* + B_0^*BFCC_0^* & B_0^*AC_\perp^* \\ B_\perp^*AC_0^* & B_\perp^*AC_\perp^* \end{bmatrix}\right),$$

then the results follow the preceding lemma. □

Given $\gamma > \max\{\bar{\sigma}(B_\perp^* A), \bar{\sigma}(AC_\perp^*)\}$, it is a routine adaptation of Parrott's theorem to get a parameterization of all matrices F such that $\bar{\sigma}(A+BFC) < \gamma$.

Consider the system $G_{22} = \left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right]$. We give the following theorem which can be proved similarly to proposition 4 by using the above lemma (see also [131]).

Theorem 4.4.3 *Consider the given system with* $\mathrm{RANK}(B_2) = p_2 < n$ *and* $\mathrm{RANK}(C_2) = q_2 < n$*. Assume that* $B_\perp \in \mathbf{R}^{n\times(n-p_2)}$ *is such that* $B_2^* B_\perp = 0$ *and* $\begin{bmatrix} B_2 & B_\perp \end{bmatrix}$ *is invertible, and* $C_\perp \in \mathbf{R}^{(n-q_2)\times n}$ *is such that* $C_\perp C_2^* = 0$ *and* $\begin{bmatrix} C_2 \\ C_\perp \end{bmatrix}$ *is invertible. Then there exists an admissible static controller, i.e.,* $\mathcal{K}_s \neq \emptyset$*, if and only if there exists a positive definite matrix* $X \in \mathcal{D}$ *such that the following two matrix inequalities hold,*

$$B_\perp^* A X A^* B_\perp - B_\perp^* X B_\perp < 0,$$

$$C_\perp A^* X^{-1} A C_\perp^* - C_\perp X^{-1} C_\perp^* < 0.$$

Note that by the same procedure in the proof of Proposition 4, we can constructively get a $\mathcal{Q}$-stabilizing static controller and the static controller characterization in terms of the solutions of the above two matrix inequalities. Note also that in the trivial cases, i.e., when $p_2 = n$ or $q_2 = n$, this problem is reduced to the state-feedback or output-injection problem.

As stated in Lemma 2, every stabilization problem with dynamic controllers can be transformed to a stabilization problem with static controllers; the solutions can therefore be obtained by statically $\mathcal{Q}$-stabilizing its augmented system by using the above theorem.

4.4.3 Solutions to Synthesis Problems: Dynamical Controllers

In this section, we give the main results about the stabilization of LFT systems. The controllers needn't be static, the constructive proofs will be given in the next two sections.

Theorem 4.4.4 *Consider the given system* G *with* $\mathrm{RANK}(B_2) = p_2 < n$ *and* $\mathrm{RANK}(C_2) = q_2 < n$*. Assume that* $B_\perp \in \mathbf{R}^{n\times(n-p_2)}$ *is such that* $B_2^* B_\perp = 0$ *and* $\begin{bmatrix} B_2 & B_\perp \end{bmatrix}$ *is invertible,* *and that* $C_\perp \in \mathbf{R}^{(n-q_2)\times n}$ *is such that* $C_\perp C_2^* = 0$ *and* $\begin{bmatrix} C_2 \\ C_\perp \end{bmatrix}$ *is invertible. Then there exists an admissible controller, i.e.,* $\mathcal{K} \neq \emptyset$*, if and only if there exist two positive definite matrices* $X \in \mathcal{D}$ *and* $Y \in \mathcal{D}$ *such that the following two LMIs hold:*

$$B_\perp^* AXA^* B_\perp - B_\perp^* X B_\perp < 0 \tag{4.19}$$

and

$$C_\perp A^* Y A C_\perp^* - C_\perp Y C_\perp^* < 0. \tag{4.20}$$

Moreover, when these conditions hold, such a controller is given by

$$K(\Delta) = \left[\begin{array}{c|c} A + B_2F + LC_2 + LD_{22}F & -L \\ \hline F & 0 \end{array}\right] \tag{4.21}$$

with the same frequency structure Δ *as the plant, where*

$$F = -(B_2^* X^{-1} B_2)^{-1} B_2^* X^{-1} A \qquad L = -AY^{-1}C_2^*(C_2 Y^{-1} C_2^*)^{-1}. \tag{4.22}$$

The controller given in this theorem has a separation structure, and is of the "observer form," we will discuss its structure in the next section. The next theorem gives a characterization of $\mathcal{K}$.

Theorem 4.4.5 *Assume that the conditions in the last theorem are satisfied, then the admissible controller set can be characterized by*

$$\mathcal{K} = \{\mathcal{F}_l(J(\Delta), Q(\Delta)) : Q(\Delta) \text{ is } \mathcal{Q}\text{-stable}\} \tag{4.23}$$

where

$$J = \left[\begin{array}{c|cc} A + B_2F + LC_2 + LD_{22}F & -L & B_2 + LD_{22} \\ \hline F & 0 & I \\ -(C_2 + D_{22}F) & I & -D_{22} \end{array}\right].$$

Remark 4.4.6 *If* $p_2 = n$ *or* $q_2 = n$*, then the corresponding LMI condition in the theorems disappears. In this case,* F *or* L *can be obtained easily without solving any LMI. For example, if* $p_2 = n$ *then a corresponding constant state-feedback matrix can be* $F = B_2^{-1}(A_F - A)$ *where* A_F *is chosen such that* $\mathcal{Q}_\Delta(A_F) < 1$*, say* $A_F = \alpha I$ *for some* $|\alpha| < 1$*.*

4.5 Stabilization Problem: Special Problems and Constructions

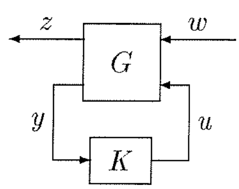
In this section we will consider the general stabilization problem which leads to a constructive proof of theorem 4.4.4. Since the necessity is obvious, we only need to consider the sufficiency. The LMI conditions (4.19) and (4.20) in Theorem 4.4.4 imply that there are constant matrices F and L such that $A+B_2F$ and $A+LC_2$ are $\mathcal{Q}$-stable, and they are given in (4.22) as in theorem 4.4.4, We therefore can assume this and do the constructions without being involved in explicitly solving an LMI at this stage. We first discuss four problems from which the solutions in Theorem 4.4.4 are constructed via a separation argument.

4.5.1 Duality, Equivalence, and Special Problems

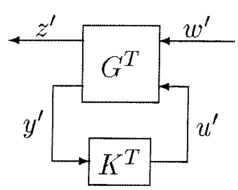
Duality of Linear Systems

Duality plays an important role in the analysis and synthesis of linear systems. We now examine this notion further in the setting employed in this chapter. Algebraically, the **dual system** of an LFT system represented by transfer function $G(\Delta)$ is defined as the system represented by the transpose of $G(\Delta)$, i.e., $G^T(\Delta)$. For concreteness, if $G(\Delta) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, then its dual system is realized as $G^T(\Delta) = \left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array}\right]$. It is noted that the frequency structures for both systems are the same.

Next, we consider a standard feedback system with block diagram



where the plant $G(\Delta)$ and the controller $K(\Delta_0)$ are assumed to be LFTs with respect to the frequency structures Δ and Δ_0. The dual structure of the above feedback system is shown as follows



whose plant and controller are the dual objects of $G(\Delta)$ and $K(\Delta_0)$, i.e., $G^T(\Delta)$ and $K^T(\Delta_0)$, respectively. It is routine to verify that $\mathcal{F}_l(G^T, K^T) = [\mathcal{F}_l(G,K)]^T$. In addition, K $\mathcal{Q}$-stabilizes G with respect to the induced frequency structure $\Delta_N = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_0 \end{bmatrix}$ if and only if K^T $\mathcal{Q}$-stabilizes G^T with respect to the frequency structure Δ_N. Whence, as far as stabilization or other synthesis problems are concerned, we can obtain the results for G^T from its dual object G if available.

Equivalence of Linear Systems

The notion of system equivalence will also play an important role in this chapter. Two systems are said to be **equivalent** if every achievable closed loop map of one system can be achieved by the other through some controller. More concretely, two LFT systems $G_1(\Delta)$ and $G_2(\Delta)$ are equivalent, if for all possible $K_1(\Delta_0)$, there exists a suitable $K_2(\Delta_0)$, such that $\mathcal{F}_l(G_1(\Delta), K_1(\Delta_0)) = \mathcal{F}_l(G_2(\Delta), K_2(\Delta_0))$; and also for all possible $K_2(\Delta_0)$, there is a suitable $K_1(\Delta_0)$, such that $\mathcal{F}_l(G_1(\Delta), K_1(\Delta_0)) = \mathcal{F}_l(G_2(\Delta), K_2(\Delta_0))$.

Special Structures

Structurally, we will define four special problems which are related to the general OF problem whose associated plant $G(\Delta)$ has the following structure (4.17),

$$G(\Delta) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]. \tag{4.24}$$

- **Full information** (FI) problem. The corresponding plant has the following structure,

$$G_{FI}(\Delta) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ \begin{bmatrix} I \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array}\right] \tag{4.25}$$

  where the "state" and the disturbance are directly measured.

- **Full control** (FC) problem. The associated plant is given by

$$G_{FC}(\Delta) = \left[\begin{array}{c|cc} A & B_1 & \begin{bmatrix} I & 0 \end{bmatrix} \\ \hline C_1 & D_{11} & \begin{bmatrix} 0 & I \end{bmatrix} \\ C_2 & D_{21} & \begin{bmatrix} 0 & 0 \end{bmatrix} \end{array}\right] \tag{4.26}$$

where two independent parts of the control input directly affect both the "state" and regulated output.

- **Disturbance feedforward** (DF) problem. The corresponding plant has the following structure,

$$G_{DF}(\Delta) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & I & 0 \end{array}\right] \tag{4.27}$$

where the disturbance enters the measurement directly.

- **Output estimation** (OE) problem. The corresponding plant has the following structure,

$$G_{OE}(\Delta) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & I \\ C_2 & D_{21} & 0 \end{array}\right] \tag{4.28}$$

where the control input enter the regulated output directly.

Note that all of these special systems have the same frequency structures as $G(\Delta)$. However, the parameters in the special structures do not necessarily refer to the same parameters of the above OF structure $G(\Delta)$. They are said to be special cases of the OF problem only in the sense that their structures are specified when compared with the general OF problems. The reader is referred to [60] for motivations of different problems.

Structurally, the FI and FC problems are dual, and so are the DF and OE problems. More precisely, $G_{FI}^T(\Delta)$ has an FC structure, $G_{FC}^T(\Delta)$ has an FI structure, and so on. In addition, FI and DF, and FC and OE are equivalent structures, respectively; this point will be made precise in the next subsection. These relationships therefore are shown in the following diagram:

$$\begin{tikzcd}
\mathrm{FI} \arrow[rr, leftrightarrow, "\text{Dual}"] \arrow[d, leftrightarrow, "\text{Equivalent}"'] & & \mathrm{FC} \arrow[d, leftrightarrow, "\text{Equivalent}"] \\
\mathrm{DF} \arrow[rr, leftrightarrow, "\text{Dual}"] & & \mathrm{OE}
\end{tikzcd}$$

4.5.2 Equivalence Relations between Special Problems

The equivalence relations between the DF and FI, and the OE and FC problems are examined in this subsection. The different structures $G_{FI}(\Delta)$, $G_{FC}(\Delta)$, $G_{DF}(\Delta)$, and $G_{OE}(\Delta)$ are given as in the preceding subsection. We first have the following observation about the DF and FI problems.

Proposition 4.5.1 *Consider the FI and DF structures as given in section 5.1. We have*

(i) $G_{DF}(\Delta) = \begin{bmatrix} I & 0 & 0 \\ 0 & C_2 & I \end{bmatrix} G_{FI}(\Delta),$

(ii) $G_{FI} = \mathcal{S}(G_{DF}, P_{DF})$, *where* $\mathcal{S}$ *denotes the* **Redheffer star product** *and*

$$P_{DF}(\Delta) = \left[\begin{array}{c|cc} A - B_1C_2 & B_1 & B_2 \\ \hline 0 & 0 & I \\ \begin{bmatrix} I \\ -C_2 \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array}\right]. \tag{4.29}$$

Proof. (i) is straightforward, we only prove (ii). Consider system $\mathcal{S}(G_{DF}, P_{DF})$, let x and $\hat{x}$ denote the state of G_{DF} and P_{DF}, respectively; take $e := x - \hat{x}$ and $\hat{x}$ as the states of the resulting interconnected system, then its realization is

$$\left[\begin{array}{cc|cc} A - B_1C_2 & 0 & 0 & 0 \\ B_1C_2 & A & B_1 & B_2 \\ \hline C_1 & C_1 & D_{11} & D_{12} \\ \begin{bmatrix} 0 \\ C_2 \end{bmatrix} & \begin{bmatrix} I \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array}\right]$$

with respect to the frequency structure $\begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix}$. The resulting transfer matrix is exactly G_{FI}, as claimed. □

The following theorem follows the above observation immediately:

Theorem 4.5.2 (i) $K_{FI} := K_{DF} \begin{bmatrix} C_2 & I \end{bmatrix}$ *Q-stabilizes* G_{FI} *if* K_{DF} *Q-stabilizes* G_{DF}. *Furthermore,* $\mathcal{F}_l(G_{FI}, K_{DF} \begin{bmatrix} C_2 & I \end{bmatrix}) = \mathcal{F}_l(G_{DF}, K_{DF})$.

(ii) *Suppose that* $A - B_1C_2$ *is Q-stable. Then* $K_{DF} := \mathcal{F}_l(P_{DF}, K_{FI})$ *Q-stabilizes* G_{DF} *if* K_{FI} *Q-stabilizes* G_{FI}. *Furthermore,* $\mathcal{F}_l(G_{DF}, \mathcal{F}_l(P_{DF}, K_{FI})) = \mathcal{F}_l(G_{FI}, K_{FI})$.

Proof. (i) it is easy. As for (ii), note that by Proposition 4.5.1, we have

$$\mathcal{F}_l(G_{FI}, K_{FI}) = \mathcal{F}_l(\mathcal{S}(G_{FI}, P_{DF}), K_{FI}) = \mathcal{F}_l(G_{DF}, \mathcal{F}_l(P_{DF}, K_{FI})).$$

The $\mathcal{Q}$-stability of the latter is confirmed by theorem 2.2.8 (ii), because of the structure of the closed loop system in terms of previous observation, the $\mathcal{Q}$-stability of $A - B_1C_2$, and the choice of K_{FI}. □

Remark 4.5.3 *This theorem shows that if $A - B_1C_2$ is $\mathcal{Q}$-stable, then problems FI and DF are equivalent, since the stabilizing controllers for the FI and DF structures can be obtained from each other such that the resulting closed-loop I/O properties are the same.*

Dually, we have the following results about the structures FC and OE.

Proposition 4.5.4 *Let the FC and OE structures be given as in section 5.1. We have*

(i) $G_{OE}(\Delta) = G_{FI}(\Delta) \begin{bmatrix} I & 0 \\ 0 & B_2 \\ 0 & I \end{bmatrix}.$

(ii) $G_{FC} = \mathcal{S}(G_{OE}, P_{OE})$, *where* P_{OE} *is*

$$P_{OE}(\Delta) = \left[\begin{array}{c|c|c} A - B_2C_1 & 0 & \begin{bmatrix} I & -B_2 \end{bmatrix} \\ \hline C_1 & 0 & \begin{bmatrix} 0 & I \end{bmatrix} \\ C_2 & I & \begin{bmatrix} 0 & 0 \end{bmatrix} \end{array}\right]. \tag{4.30}$$

Theorem 4.5.5 **(i)** $K_{FC} := \begin{bmatrix} B_2 \\ I \end{bmatrix} K_{OE}$ *$\mathcal{Q}$-stabilizes G_{FC} if K_{OE} $\mathcal{Q}$-stabilizes G_{OE}. Furthermore,* $\mathcal{F}_l(G_{FC}, \begin{bmatrix} B_2 \\ I \end{bmatrix} K_{OE}) = \mathcal{F}_l(G_{OE}, K_{OE})$.

(ii) *Suppose that $A - B_2C_1$ is $\mathcal{Q}$-stable. Then $K_{OE} := \mathcal{F}_l(P_{OE}, K_{FC})$ $\mathcal{Q}$-stabilizes G_{OE} if K_{FC} $\mathcal{Q}$-stabilizes G_{FC}. Furthermore, $\mathcal{F}_l(G_{OE}, \mathcal{F}_l(P_{OE}, K_{FC})) = \mathcal{F}_l(G_{FC}, K_{FC})$.*

Remark 4.5.6 *This theorem shows that if $A - B_2C_1$ is $\mathcal{Q}$-stable, then the FC and OE problems are input/output equivalent.*

4.5.3 OF Problem and Separation Property

In this section we constructively prove Theorem 7. Since the necessity is clear, we only consider the sufficiency. The construction essentially involves reducing the OF problem to the simpler FI and FC problems with the separation argument as a byproduct.

Consider System $G(\Delta)$ (4.17)

$$G(\Delta) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right].$$

We shall assume $D_{22} = 0$ without loss of generality. In fact, if $D_{22} \neq 0$, then the mapping

$$\hat{K}(\Delta) = K(\Delta)(I - D_{22}K(\Delta))^{-1} = \mathcal{F}_l(\begin{bmatrix} 0 & I \\ I & D_{22} \end{bmatrix}, K(\Delta))$$

is well defined by the assumption that the closed-loop system is well-posed. Define

$$\hat{G}(\Delta) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right].$$

Then $\mathcal{F}_l(G(\Delta), K(\Delta)) = \mathcal{F}_l(\hat{G}(\Delta), \hat{K}(\Delta))$. Thus, if $\hat{K}$ is designed for the above structure, then K can be obtained from Lemma 3.3.2 as

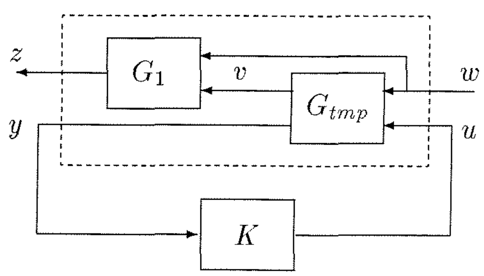
$$K(\Delta) = \mathcal{F}_u(\begin{bmatrix} -D_{22} & I \\ I & 0 \end{bmatrix}, \hat{K}(\Delta)) = \mathcal{F}_l(\begin{bmatrix} 0 & I \\ I & -D_{22} \end{bmatrix}, \hat{K}(\Delta)), \tag{4.31}$$

which justifies the simplification.

Next, we construct the controllers for the OF problem with $D_{22} = 0$. Let x denote the state of the system $G(\Delta)$. Since (A, B_2) is $\mathcal{Q}$-stabilizable, there is a constant matrix F such that $A + B_2F$ is $\mathcal{Q}$-stable. Note that $\begin{bmatrix} F & 0 \end{bmatrix}$ is actually a special FI stabilizing controller. Let

$$v = u - Fx,$$

then the system can be broken into two subsystems G_1 and G_{tmp} as shown pictorially below



with

$$G_1(\Delta) = \left[\begin{array}{c|cc} A + B_2F & B_1 & B_2 \\ \hline C_1 + D_{12}F & D_{11} & D_{12} \end{array}\right],$$

which is $\mathcal{Q}$-stable, and

$$G_{tmp}(\Delta) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline -F & 0 & I \\ C_2 & D_{21} & 0 \end{array}\right].$$

Since G_1 is $\mathcal{Q}$-stable, by Theorem 2.2.8, K $\mathcal{Q}$-stabilizes G if and only if K $\mathcal{Q}$-stabilizes G_{tmp}. Note that G_{tmp} is of OE structure. Let L be such that $A + LC_2$ is $\mathcal{Q}$-stable, then $\begin{bmatrix} L \\ 0 \end{bmatrix}$ is a $\mathcal{Q}$-stabilizing controller for the corresponding FC problem. Since $A + B_2F$ is $\mathcal{Q}$-stable by construction, by Theorem 4.5.5 (ii) we have a controller for G_{tmp} as follows,

$$K(\Delta) = \mathcal{F}_l(J, \begin{bmatrix} L \\ 0 \end{bmatrix}),$$

where

$$J(\Delta) = \left[\begin{array}{c|c|c} A + B_2F & 0 & \begin{bmatrix} I & -B_2 \end{bmatrix} \\ \hline -F & 0 & \begin{bmatrix} 0 & I \end{bmatrix} \\ C_2 & I & \begin{bmatrix} 0 & 0 \end{bmatrix} \end{array}\right].$$

Then we have

$$K(\Delta) = \left[\begin{array}{c|c} A + B_2F + LC_2 & -L \\ \hline F & 0 \end{array}\right].$$

Now we drop the assumption $D_{22} = 0$, by the transformation (4.31), we have the following result which restates Theorem 7.

Proposition 4.5.7 *Consider the general OF problem. Let F and L be such that $A + LC_2$ and $A + B_2F$ are $\mathcal{Q}$-stable, then the controller*

$$K(\Delta) = \left[\begin{array}{c|c} A + B_2F + LC_2 + LD_{22}F & -L \\ \hline F & 0 \end{array}\right]$$

with the frequency structure Δ $\mathcal{Q}$-stabilizes the given system.

Separation Principle for LFT Systems

The above construction was conducted by reducing the synthesis of OF problem to the independent synthesis of FI and OE problems. This reduction is based on a separation argument. And it also leads to a separation structure for the resulting closed loop system.

Let x and $\tilde{x}$ be the state vectors for the plant and controller. The controller has the following realization:

$$\begin{cases} \Delta^{-1}\tilde{x} = A\tilde{x} + B_2 u - L(y - C_2\tilde{x}) \\ u = F\tilde{x} \end{cases}$$

which has an observer structure. Let $\bar{x} = \begin{bmatrix} x \\ \tilde{x} \end{bmatrix}$ be the state vector of the closed loop system, the corresponding realization is

$$\left[\begin{array}{cc|c} A & B_2F & B_1 \\ -LC_2 & A + B_2F + LC_2 & -LD_{21} \\ \hline C_1 & D_{12}F & D_{11} \end{array}\right].$$

Next, we conduct state transformation $\bar{x} \mapsto T\bar{x} = \begin{bmatrix} x \\ \tilde{x} - x \end{bmatrix}$, i.e., $T = \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix}$. It is easy to see that the state transformation T is admissible. After the transformation, the realization is

$$\left[\begin{array}{cc|c} A + B_2F & B_2F & B_1 \\ 0 & A + LC_2 & B_1 - LD_{21} \\ \hline C_1 - D_{12}F & D_{12}F & D_{11} \end{array}\right],$$

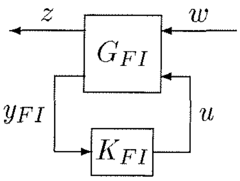
i.e., the system is decoupled into two separated $\mathcal{Q}$-stable subsystems, i.e., state-feedback system and output-injection system. Hence the closed-loop system after the admissible state variable transformation is also $\mathcal{Q}$-stable with respect to the new frequency structure $\Delta_N = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix}$ by Theorem 2.2.8, so is the original closed-loop system as desired.

4.6 Construction of the Stabilizing Controller Parameterization

This section is mainly devoted to the proof of Theorem 8, i.e., to construct the parameterization of all admissible controllers. We follow [60] to present a state-space-like approach to this problem without using any ideas from coprime factorization techniques. The techniques to be used are from the LFT theory, especially the inversion property of an LFT. The main idea of this approach is similar to the one for stabilization problem. That is, we will reduce the OF problem into the simpler FI and OE problems, then solve the output feedback problem by separation argument. To this end, we will first parametrize the equivalent classes of FI and FC stabilizing controllers which lead to all achievable closed loop maps for FI and FC problems. In this chapter, two controllers, K and K', are said to be **equivalent** if they produce the same input/output relationships for the corresponding closed loop systems, i.e., $\mathcal{F}_l(G,K) = \mathcal{F}_l(G,K')$, written as $K \cong K'$. The parameterizations for DF and OE controllers are then obtained from the results for FI and FC problems by the equivalence between related systems.

4.6.1 Admissible Controllers for FI and FC Problems

We first examine the FI structure. Consider plant G_{FI} (4.25) which is given in the last section, the feedback configuration is as follows,



We first have the following lemma about the structure of the FI stabilizing controllers.

Lemma 4.6.1 *The FI stabilizing controllers have the following general structure,*

$$K_{FI}(\Delta) = \begin{bmatrix} K_1(\Delta) & K_2(\Delta) \end{bmatrix}$$

with $K_1(\Delta)$ *Q-stabilizing* $\left[\begin{array}{c|c} A & B_2 \\ \hline I & 0 \end{array}\right]$ *and arbitrary Q-stable* $K_2(\Delta)$.

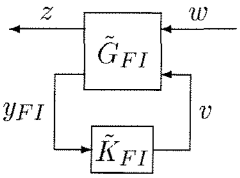
Proof. It follows from theorem 2.2.8. □

Proposition 4.6.2 *Let* F *be a constant matrix such that* $A + B_2F$ *is Q-stable. Then all admissible controllers for FI can be parameterized as*

$$K_{FI}(\Delta) \cong \begin{bmatrix} F & Q(\Delta) \end{bmatrix}$$

with Q-stable $Q(\Delta)$.

Proof. It is easy to see that the controller given in the above formula $\mathcal{Q}$-stabilizes the system $G_{FI}(\Delta)$. Hence we only need to show that the given set of controllers parameterizes all equivalence classes of $\mathcal{Q}$-stabilizing controllers. It is enough to show that there is a choice of $\mathcal{Q}$-stable $Q(\Delta)$ such that the transfer functions from w to u for any stabilizing controller $K_{FI}(\Delta) = \begin{bmatrix} K_1(\Delta) & K_2(\Delta) \end{bmatrix}$ and for $K^0_{FI}(\Delta) = \begin{bmatrix} F & Q(\Delta) \end{bmatrix}$ are the same, since this implies $\mathcal{F}_l(G_{FI}, K_{FI}) = \mathcal{F}_l(G_{FI}, K^0_{FI})$. To show that, make a change of control variable $v = u - Fx$, where x denotes the state of the system $G_{FI}(\Delta)$, then the system with the controller $K_{FI}(\Delta)$ is shown as in the following diagram,



where

$$\tilde{G}_{FI} := \left[\begin{array}{c|cc} A + B_2 F & B_1 & B_2 \\ \hline C_1 + D_{12} F & 0 & D_{12} \\ \begin{bmatrix} I \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array}\right], \qquad \tilde{K}_{FI} := K_{FI} - [F\ 0].$$

Let $Q(\Delta)$ be the transfer matrix from w to v; it is $\mathcal{Q}$-stable by the $\mathcal{Q}$-stability of the closed loop system. Then $u = Fx + v = Fx + Qw$, so $K_{FI}(\Delta) \cong \begin{bmatrix} F & Q(\Delta) \end{bmatrix}$. □

Next, the FC problem is considered. G_{FC} is given by (4.26) in the last section. We have the following dual result.

Proposition 4.6.3 *Let L be a constant matrix such that $A + LC_2$ is $\mathcal{Q}$-stable. Then the set of equivalent classes of all admissible controllers for FC in the above sense can be parameterized as*

$$K_{FC}(\Delta) \cong \begin{bmatrix} L \\ Q(\Delta) \end{bmatrix}$$

with any $\mathcal{Q}$-stable $Q(\Delta)$.

4.6.2 Admissible Controllers for Problems DF and OE

The DF and OE problems can be coped with by their equivalence relations to FI and FC problems, respectively. We first consider the DF structure. The plant G_{DF} is given by (4.27) in the last section. We will further assume that $A - B_1 C_2$ **is $\mathcal{Q}$-stable** in this subsection. It should be pointed out that this assumption is not necessary for DF problem to be solvable; however, it does simplify the solution.

Under the above assumption, the FI and DF problems are equivalent, as pointed out in the last section. It can be shown that if $K_{DF} \cong K'_{DF}$ in the DF structure, then $K_{DF} \begin{bmatrix} C_2 & I \end{bmatrix} \cong K'_{DF} \begin{bmatrix} C_2 & I \end{bmatrix}$ in the corresponding FI structure. Also if $K_{FI} \cong K'_{FI}$, then $\mathcal{F}_l(P_{DF}, K_{FI}) \cong \mathcal{F}_l(P_{DF}, K'_{FI})$, where P_{DF} is defined in (4.29).

Next, the parameterization of DF controllers is considered. Let $K_{DF}(\Delta)$ be an admissible controller for DF then $K_{FI}(\Delta) = K_{DF}(\Delta)\begin{bmatrix} C_2 & I \end{bmatrix}$ $\mathcal{Q}$-stabilizes the corresponding $G_{FI}(\Delta)$. Assume $K_{FI}(\Delta) \cong K'_{FI}(\Delta) = \begin{bmatrix} F & Q(\Delta) \end{bmatrix}$ for some $\mathcal{Q}$-stable $Q(\Delta)$, then $K'_{FI}(\Delta)$ $\mathcal{Q}$-stabilizes $G_{FI}(\Delta)$ and

$$\mathcal{F}_l(J_{DF}(\Delta), Q(\Delta)) = \mathcal{F}_l(P_{DF}(\Delta), K'_{FI}(\Delta)),$$

where

$$J_{DF}(\Delta) = \left[\begin{array}{c|cc} A + B_2F - B_1C_2 & B_1 & B_2 \\ \hline F & 0 & I \\ -C_2 & I & 0 \end{array}\right]$$

with F such that $A+B_2F$ is $\mathcal{Q}$-stable. Hence by Theorem 4.5.2, $K'_{DF}(\Delta) := \mathcal{F}_l(J_{DF}(\Delta), Q(\Delta))$ stabilizes $G_{DF}(\Delta)$ for any $\mathcal{Q}$-stable $Q(\Delta)$. Since $K_{FI}(\Delta) \cong K'_{FI}(\Delta)$, we have $K_{DF}(\Delta) \cong K'_{DF}(\Delta) = \mathcal{F}_l(J_{DF}(\Delta), Q(\Delta))$, which characterizes the equivalence classes of all controllers for DF problem by the equivalence of FI and DF structures.

Actually, the above construction of parameterization characterizes all admissible controllers (not just the equivalence classes) for the DF system.

Proposition 4.6.4 *All admissible controllers for the DF problem can be characterized by $K_{DF}(\Delta) = \mathcal{F}_l(J_{DF}(\Delta), Q_0(\Delta))$ with $\mathcal{Q}$-stable $Q_0(\Delta)$, where $J_{DF}(\Delta)$ is given as above.*

Proof. From the above construction, it is known that the controllers expressed in the given LFT formula do $\mathcal{Q}$-stabilize G_{DF}. Let K_{DF} be any admissible controller for G_{DF}, then $Q := \mathcal{F}_l(\hat{J}_{DF}, K_{DF})$ is $\mathcal{Q}$-stable, where

$$\hat{J}_{DF} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline -F & 0 & I \\ C_2 & I & 0 \end{array}\right],$$

since $\hat{J}_{DF}$ and G_{DF} have the same parameters (C_2, A, B_2) which are the only parameters affecting the stabilization results. We now claim that $K_{DF} = \mathcal{F}_l(J_{DF}, Q)$.

In fact, $\mathcal{F}_l(J_{DF}, Q) = \mathcal{F}_l(J_{DF}, \mathcal{F}_l(\hat{J}_{DF}, K_{DF})) =: \mathcal{F}_l(J_{tmp}, K_{DF})$, where J_{tmp} is defined as

$$J_{tmp} = \left[\begin{array}{cc|cc} A - B_1C_2 + B_2F & -B_2F & B_1 & B_2 \\ -B_1C_2 & A & B_1 & B_2 \\ \hline F & -F & 0 & I \\ -C_2 & C_2 & I & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} A - B_1C_2 & -B_2F & B_1 & B_2 \\ 0 & A + B_2F & 0 & 0 \\ \hline 0 & -F & 0 & I \\ 0 & C_2 & I & 0 \end{array}\right]$$

$$= \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

Hence, $\mathcal{F}_l(J_{DF}, Q) = \mathcal{F}_l(J_{tmp}, K_{DF}) = K_{DF}$. This shows that any admissible controller can be expressed in the form of $\mathcal{F}_l(J_{DF}, Q)$ for some $\mathcal{Q}$-stable Q. □

Next, we turn to the OE problem, G_{OE} is given by (4.28) in the last section. Similarly, we will assume that $A - B_2C_1$ is $\mathcal{Q}$-stable. The OE problem can be considered similarly to the above treatment. However, since OE problem is dual to DF problem, we have the following dual result.

Proposition 4.6.5 *All admissible controllers for the OE problem can be characterized as* $\mathcal{F}_l(J_{OE}, Q_0)$ *with any* $\mathcal{Q}$*-stable* Q_0*, where* J_{OE} *is defined as*

$$J_{OE} = \left[\begin{array}{c|cc} A - B_2C_1 + LC_2 & L & -B_2 \\ \hline C_1 & 0 & I \\ C_2 & I & 0 \end{array}\right]$$

with L *such that* $A + LC_2$ *is* $\mathcal{Q}$*-stable.*

4.6.3 All Admissible Controllers for Problem OF

Consider the system $G(\Delta)$ which is given by (4.17):

$$G(\Delta) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right].$$

As before, it is assumed that (A, B_2) is $\mathcal{Q}$-stabilizable and (C_2, A) is $\mathcal{Q}$-detectable with respect to the frequency structure Δ. We are now going to prove Theorem 8 which is restated as

Proposition 4.6.6 *Let F and L be such that $A + LC_2$ and $A + B_2F$ are $\mathcal{Q}$-stable, then all controllers which $\mathcal{Q}$-stabilize $G(\Delta)$ can be parameterized as $\mathcal{F}_l(J(\Delta), Q(\Delta)$, where*

$$J(\Delta) = \left[\begin{array}{c|cc} A + B_2F + LC_2 + LD_{22}F & -L & B_2 + LD_{22} \\ \hline F & 0 & I \\ -(C_2 + D_{22}F) & I & -D_{22} \end{array}\right]$$

and $Q(\Delta)$ is $\mathcal{Q}$-stable such that the resulting closed loop system is well-posed.

Proof. We will assume again $D_{22} = 0$ for simplicity. Let x denote the state of system G. Since (A, B_2) is $\mathcal{Q}$-stabilizable, there is a constant matrix F such that $A + B_2F$ is $\mathcal{Q}$-stable. Note that $\begin{bmatrix} F & 0 \end{bmatrix}$ is actually a special FI $\mathcal{Q}$-stabilizing controller. Let

$$v = u - Fx$$

as in the proof of Proposition 4.5.7, thus $K(\Delta)$ $\mathcal{Q}$-stabilizes $G(\Delta)$ if and only if it $\mathcal{Q}$-stabilizes

$$G_{tmp}(\Delta) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline -F & 0 & I \\ C_2 & D_{21} & 0 \end{array}\right].$$

However, $G_{tmp}(\Delta)$ is of the OE structure. Let L be such that $A + LC_2$ is $\mathcal{Q}$-stable. Then by Theorem 4.5.5 all controllers $\mathcal{Q}$-stabilizing $G_{tmp}(\Delta)$ are given by

$$K(\Delta) = \mathcal{F}_l(J(\Delta), Q(\Delta)),$$

where

$$J(\Delta) = \left[\begin{array}{c|cc} A + B_2F + LC_2 & L & -B_2 \\ \hline -F & 0 & I \\ C_2 & I & 0 \end{array}\right] = \left[\begin{array}{c|cc} A + B_2F + LC_2 & -L & B_2 \\ \hline F & 0 & I \\ -C_2 & I & 0 \end{array}\right].$$

This concludes our proof. □

This theorem shows that any admissible controller $K(\Delta)$ can be characterized as an LFT of a $\mathcal{Q}$-stable parameter matrix $Q(\Delta)$, i.e., $K(\Delta) = \mathcal{F}_l(J(\Delta), Q(\Delta))$. In fact, such a Q can be uniquely determined by K in some sense. To see this, by the inversion formulas for LFTs in Lemma 3.3.2, we can solve the equation $K(\Delta) = \mathcal{F}_l(J(\Delta), Q(\Delta))$ to uniquely give

$$Q = \mathcal{F}_u(J^{-1}, K) = \mathcal{F}_l(\hat{J}, K),$$

where

$$J^{-1} = \left[\begin{array}{c|cc} A & B_2 & L \\ \hline C_2 & D_{22} & I \\ -F & I & 0 \end{array}\right]$$

and

$$\hat{J} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} J^{-1} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = \left[\begin{array}{c|cc} A & L & B_2 \\ \hline -F & 0 & I \\ C_2 & I & D_{22} \end{array}\right] =: \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}.$$

Note that Q is stable if and only if K stabilizes $\hat{J}_{22}$. But $\hat{J}_{22} = G_{22}$, so Q is stable if and only if K stabilizes G, as desired. We summarize this observation as follows.

Theorem 4.6.7 *Any admissible controller $K(\Delta)$ can be characterized as an LFT of a $\mathcal{Q}$-stable parameter matrix $Q(\Delta)$, i.e., $K(\Delta) = \mathcal{F}_l(J(\Delta), Q(\Delta))$ with $Q(\Delta)$ realized by*

$$Q(\Delta) := \mathcal{F}_l(\hat{J}(\Delta), K(\Delta)),$$

where

$$\hat{J}(\Delta) = \left[\begin{array}{c|cc} A & L & B_2 \\ \hline -F & 0 & I \\ C_2 & I & D_{22} \end{array}\right]$$

and the realization for $K(\Delta)$ is $\mathcal{Q}$-stabilizable and $\mathcal{Q}$-detectable. Moreover, this characterization is unique for a given pair F and L satisfying the requirements stated in the above theorem.

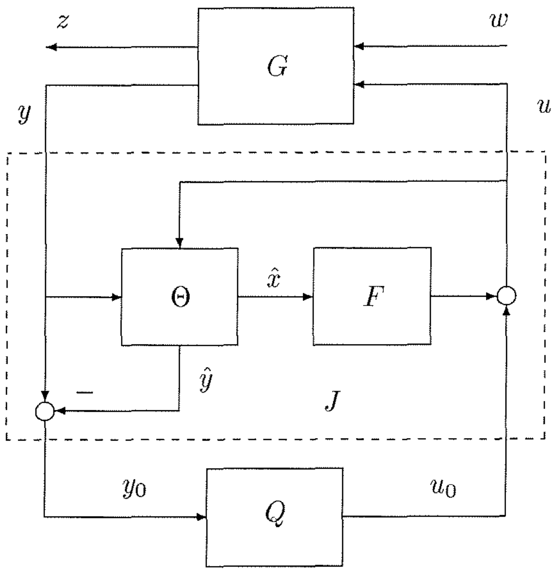
Remark 4.6.8 *Note that the key technique used in the stabilizing controller parameterization for both the disturbance feedforward and the output feedback problem is the inversion property of linear fractional transformation (Lemma 3.3.2).*

4.6.4 Structure of Closed Loop Systems

The central controller for this parameterization, i.e., the parameterized controller with the parameter $Q = 0$, is an observer-based controller, and the observer is as follows

$$\Theta : \begin{cases} \Delta^{-1}\hat{x} = A\hat{x} + Bu + L(y - C\hat{x}) \\ \hat{y} = C\hat{x} \end{cases}$$

where $\hat{x}$ is the estimated state x of the original system. Therefore, a parameterized controller has some separation structure, and the closed loop system is structured as follows



Next, consider the closed loop map which is also parameterized as

$$T_{zw} = \mathcal{F}_l(G, K) = \mathcal{F}_l(G, \mathcal{F}_l(J, Q)) =: \mathcal{F}_l(T, Q).$$

We have the following theorem about the structures of the closed loop maps, whose proof is straightforward and is omitted.

Theorem 4.6.9 *Let the state feedback and output injection matrices F and L be chosen as in the previous theorem. Then the closed loop maps are parameterized as $T_{zw} = \mathcal{F}_l(T, Q)$ with*

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} := \left[\begin{array}{cc|cc} A+BF & -BF & B_1 & B \\ 0 & A+LC & B_1+LD_{21} & 0 \\ \hline C_1+D_{12}F & -D_{12}F & D_{11} & D_{12} \\ 0 & C & D_{21} & 0 \end{array}\right] \tag{4.32}$$

and Q is stable. Moreover, T_{zw} is affine in Q, i.e.,

$$T_{zw} = T_{11} + T_{12}QT_{21}. \tag{4.33}$$

4.7 Notes and References

We have considered the problems of robust stabilization and parameterization of all stabilizing controllers for LFT systems with structured LTV perturbations. All of the manipulations have been based on some naturally defined $\mathcal{Q}$-stability notions. A separation principle is confirmed. It should be noted that most of the results, including the separation theory, also hold in the μ-stability case via simple change of notation. An exception is that the stabilization for FI structure by dynamic state feedback might not be equivalent to stabilization by constant state feedback.

The separation property discussed in this chapter holds in greater generality than for just the $\mathcal{Q}$ and μ stability problems. All that is required for the separation proof is that the notion of stability satisfy two requirements: 1) stability invariance under a sufficiently rich set of similarity transformations, as in Theorem 2.2.7, and 2) a certain structural property as given in Theorem 2.2.8. It would clearly be possible to develop a more abstract axiomatic stabilization theory using these 2 properties.

This chapter is largely based on the conference paper [119], where the main results in this chapter were presented. The solution in the context of $\mathcal{H}_\infty$-control was solved by Packard *et al.* at the same time [133]. Subsequently, many extensions and generalizations have appeared [111, 134, 131, 86]. The LMI characterization of the pure state-feedback solution and its application to an autopilot design example for a missile were given in [86]. More recently, the LMI treatment has been extended to cope with robust performance synthesis for continuous time systems [69, 94, 3]; see also [29] and references therein. The current discussion about relating robustness analysis of LPV system with LTV perturbation to $\mathcal{Q}$-stability follows from [159, 123, 124, 137, 138]. The synthesis approach, which leads to separation interpretation, basically follows from [60, 73].

PART II

Uncertain Nonlinear Systems

Chapter 5

Parameterization of Stabilizing Controllers for Nonlinear Systems

5.1 Introduction

Youla-parameterization for linear systems has two properties, i.e., (i) the free parameter set for the parameterized controllers is actually a linear space, and (ii) the stabilizing closed loop maps are also parameterized, and are affine in the free stable parameters. This fact therefore makes it possible to (exactly) solve various robust and optimal control problems (see for example, [199, 49, 204, 55, 192, 67, 27, 40] and references therein). As the basic requirement or constraint for feedback control design is that the designed controllers stabilize the feedback system, while Youla-parameterization provides a systematic way to choose the (optimal) stabilizing controllers. In the Youla-parameterization formula, each input-output (I/O) stabilizing controller can be characterized as a linear fractional transformation of some (I/O) stable parameter. The basic technique used in the derivation is coprime factorization. Due to the clear connections between the stability notions in both the I/O description and the state-space description for a linear system, a state-space formula has also been derived using the coprime factorization technique [101, 55, 129], and each internally stabilizing controller is characterized as a linear fractional transformation of some internally stable system.

When nonlinear systems are considered, it is expected that they could also enjoy the similar controller parameterizations and the properties which the linear parameterized closed-loop systems have. It is indeed the case for a special class of nonlinear systems

[48, 47, 66, 188]. However, for a more general class of nonlinear systems, the answer is not very straightforward. As far as the controller parameterization is concerned, a natural approach is to analogically use coprime factorization-like technique, although the parameterization formulas for some special cases can be derived without explicitly involving coprime factorization (see, e.g., [48, 47]). Nonetheless, the generalizations of the coprime factorization notion for nonlinear systems largely depend on how to define (I/O) stability and coprimeness of (I/O) (stable) operators. There have been a rich variety of versions of coprime factorization for nonlinear systems, because different stability and/or coprimeness notions have been used (*cf.* [80, 188, 168, 170, 180, 37, 135, 189] and references therein). The controller parameterizations can be more or less conducted based on these notions of coprime factorization [80, 188, 180, 135]. However, unlike in the linear system case, neither the computational implications of these results nor their implications in the state-space are clear. Some efforts have been made in this direction such that the coprime factorization can be conducted in terms of the state-space techniques [168, 169, 170, 189]. Contrary to the linear systems, one of the difficulties is that the state-space stability notion, i.e., asymptotic stability, doesn't imply any I/O stability notion in general. Thence, some concepts, such as the notion of input-to-state (I/S) stability proposed by Sontag in [168, 170], are needed to insure I/O stability by considering asymptotic stability. In particular, in the nice work by Sontag [168, 170], the finite I/O-gain-like stability notion is used to carry out the coprime factorization in the state space, where a finite I/S-gain-like notion of I/S stability is suggested as a bridge between asymptotic stability and I/O stability; it is concluded that if a nonlinear system is smoothly stabilizable, then there is a coprime factorization for the system; moreover, this coprime factorization can be constructed by using smooth state feedback. Verma and Hunt [189] use the similar technique to deal with the coprime factorization in the context of BIBO stability with a slightly different version of coprimeness, and another version of I/S stability, i.e. the bounded-input/bounded-state (BIBS) stability, is used. It is believed that the **potential** use of coprime factorization in the nonlinear control theory is to parameterize the stabilizing compensator laws (see [168]). So there comes up the question: do we really need to use the coprime factorization technique to get the stabilizing controller parameterization?

The answer to the above question is **NO.** In this chapter, we derive a parameterization formula of stabilizing controllers for time-invariant linear, input-affine nonlinear, and general nonlinear control systems directly in the state-space without using the coprime factorization. We use a state-space technique developed in Doyle *et al.* [60] and in Lu *et al.* [119] to deal with the controller parameterization problem. Basically, in this machinery, the general problem is decomposed into some simpler output-estimation and state-feedback problems by a technique of changing variables; the controller parameterization is constructed from the considerations of the simpler problems by the employment of a separation argument. (A separation principle follows from the construction.) In the resulting parameterization formula, the asymptotically stabilizing controllers are characterized as fractional transformations of some asymptotically stable parameters. From the state-space point of view, a parameterized controller is structured as an observer which estimates the state of the plant with zero input, a state feedback which uses the estimated state, and a free stable parameter. In the linear case, this formula is exactly the Youla-parameterization, which characterizes all internally stabilizing time-invariant linear controllers, and the parameterized closed-loop maps are affine in the free parameters. In the nonlinear case, in general, it just characterizes a class of asymptotically stabilizing controllers which have separation structures. This consideration is additionally motivated by some other work in which separation structures for some nonlinear feedback systems are confirmed [190, 167, 34, 122, 104, 181, 15, 113]. Unlike linear systems, the parameterized closed-loop maps do not have similar affine-like representation.

The Lyapunov technique is used in this chapter to deal with stability issue. The rest of this chapter is organized as follows. In section 2, the stabilizability and detectability of input-affine nonlinear systems are examined, they are characterized in terms of Hamilton-Jacobi inequalities. The parameterization of input-affine nonlinear system is considered in section 3, the formula of the parameterized input-affine locally stabilizing controllers is derived. In sections 4, the parameterization results extended to general nonlinear systems are briefly considered.

5.2 Preliminaries: Stabilizability and Detectability

The reader is referred to [79] for the basic Lyapunov stability theory (see also the survey article [171]). Consider a nonlinear system with a input-affine realization as follows,

$$G : \begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \tag{5.1}$$

where $x \in \mathbf{R}^n$ is the state vector, $u \in \mathbf{R}^p$ and $y \in \mathbf{R}^q$ are the input and output vectors, respectively. We will assume $f, g, h \in \mathbf{C}^2$, and $f(0) = 0, h(0) = 0$. Therefore, $0 \in \mathbf{R}^n$ is an equilibrium of both systems with $u = 0$.

Definition 5.2.1 *(i) The dynamical system G (5.1) (or $[f(x), g(x)]$) is said to be locally smoothly (or exponentially) stabilizable if there is a $\mathbf{C}^2$ function $F : \mathbf{R}^n \to \mathbf{R}^p$ such that $\dot{x} = f(x) + g(x)F(x)$ is locally asymptotically (or exponentially) stable about $x = 0$.*

(ii) The dynamical system G (5.1) (or $[h(x), g(x)]$) is said to be (locally) smoothly (or exponentially) detectable if there is a $\mathbf{C}^2$ matrix-valued function $L : \mathbf{R}^n \to \mathbf{R}^{n \times q}$ such that $\dot{x} = f(x) + L(x)h(x)$ is (locally) asymptotically (or exponentially) stable about $x = 0$.

The global versions of stabilizability and detectability can be defined similarly. The definition of stabilizability is quite standard. The detectability notion is defined in terms of output injection, which is analogical to the one in the linear case. However, the output injection depends on the state variable. This consideration is just of technical interests, since the implication for the detectability notion is that if the system is locally exponentially detectable, then there exists a local state observer for the original system, and the observer can be constructed by the output injection.

Remark 5.2.2 *The smooth stabilizability and the smooth detectability can be characterized in terms of Lyapunov functions. For instance, from the inverse Lyapunov theorem, it follows that system G (5.1) is (locally) smoothly stabilizable, if and only if there are a (locally) $\mathbf{C}^3$ positive definite function (i.e., Lyapunov function for the closed loop system) $V : \mathbf{R}^n \to \mathbf{R}^+$, a $\mathbf{C}^2$ function $F : \mathbf{R}^n \to \mathbf{R}^p$ with $F(0) = 0$, and $\mathbf{C}^2$ functions $\gamma_1, \gamma_2, \gamma_3$ of class $\mathcal{K}$ such that*

$$\gamma_1(\|x\|) \leq V(x) \leq \gamma_2(\|x\|) \tag{5.2}$$

$$\frac{\partial V(x)}{\partial x}(f(x) + g(x)F(x)) \leq -\gamma_3(\|x\|) \tag{5.3}$$

for $x \in \mathcal{B}_r$ with some $r > 0$.

Moreover, system (5.1) is locally exponentially stabilizable if and only if its linearized system around 0 is stabilizable, it is also noted that besides conditions (5.2) and (5.3), the necessary and sufficient conditions for local exponential stabilizability additionally require that

$$\lim_{s \to 0} \frac{\gamma_3(s)}{s^2} \in (0, \infty). \tag{5.4}$$

To conclude the review, we give a Hamilton-Jacobi Inequality (HJI) characterization for stabilizability and detectability.

Proposition 5.2.3 *Consider the system G (5.1).*

(i) It is (locally) smoothly stabilizable, if there exists a (locally) $\mathbf{C}^3$ *positive definite function* $V : \mathbf{R}^n \to \mathbf{R}^+$ *with* $\gamma_1(\|x\|) \leq V(x) \leq \gamma_2(\|x\|)$ *for some* $\mathbf{C}^2$ *functions* γ_1, γ_2 *of class* $\mathcal{K}$ *such that the following HJI is satisfied for* $x \in \mathcal{B}_r$ *with some* $r > 0$*,*

$$\frac{\partial V(x)}{\partial x} f(x) - \frac{\partial V(x)}{\partial x} g(x) g^T(x) \frac{\partial V^T(x)}{\partial x} \leq -\gamma_3(\|x\|) \tag{5.5}$$

with γ_3 *being of class* $\mathcal{K}$*. Moreover,* $F(x) = -g^T(x)\frac{\partial V^T(x)}{\partial x}$ *is such a stabilizing state feedback controller.*

(ii) It is (locally) smoothly detectable, if there exists a (locally) $\mathbf{C}^3$ *positive definite function* $U : \mathbf{R}^n \to \mathbf{R}^+$ *with* $\sigma_1(\|x\|) \leq U(x) \leq \sigma_2(\|x\|)$ *for some* $\mathbf{C}^2$ *functions* σ_1, σ_2 *of class* $\mathcal{K}$ *such that, for* $x \in \mathcal{B}_r$ *with some* $r > 0$*, the following HJI is satisfied,*

$$\frac{\partial U(x)}{\partial x} f(x) - h^T(x) h(x) \leq -\sigma_3(\|x\|) \tag{5.6}$$

with σ_3 *being of class* $\mathcal{K}$*, and there is a* $\mathbf{C}^2$ *matrix-valued function* $L(x)$ *such that*

$$\frac{\partial U(x)}{\partial x} L(x) = -h^T(x). \tag{5.7}$$

Moreover, $u = L(x)y$ *is such a stabilizing output injection.*

The proof is straightforward, so it is omitted. It is noted that the above characterizations are just sufficient in general (, system $\dot{x} = x^4 + x^2 u$ is such a counter-example for stabilizability condition (5.5)). However, they are also necessary for linear systems. Moreover, we have the following result about exponentially stability and detectability.

Proposition 5.2.4 *Consider the system G (5.1).*

(i) It is locally exponentially stabilizable if and only if there exists a (locally) $\mathbf{C}^3$ *positive definite function* $V : \mathbf{R}^n \to \mathbf{R}^+$ *with* $\gamma_1(\|x\|) \leq V(x) \leq \gamma_2(\|x\|)$ *for some* $\mathbf{C}^2$ *functions* γ_1, γ_2 *of class* $\mathcal{K}$ *such that the HJI (5.5) is satisfied for* $x \in \mathcal{B}_r$ *with some* $r > 0$ *and* γ_3 *being of class* $\mathcal{K}$ *and satisfying* $\lim_{s\to 0} \frac{\gamma_3(s)}{s^2} \in (0, \infty)$*. Moreover,* $F(x) = -g^T(x)\frac{\partial V^T(x)}{\partial x}$ *is a locally exponentially stabilizing state feedback controller.*

(ii) It is locally exponentially detectable if and only if there are a (locally) $\mathbf{C}^3$ *positive definite function* $U : \mathbf{R}^n \to \mathbf{R}^+$ *with* $\sigma_1(\|x\|) \leq U(x) \leq \sigma_2(\|x\|)$ *for some* $\mathbf{C}^2$ *functions*

σ_1, σ_2 of class $\mathcal{K}$ such that, for $x \in \mathcal{B}_r$ with some $r > 0$, the HJI (5.6) is satisfied with σ_3 being of class $\mathcal{K}$ and $\lim_{s \to 0} \frac{\sigma_3(s)}{s^2} \in (0, \infty)$. Moreover, $u = L(x)y$ with $L(x)$ being a solution to (5.7) is such a locally exponentially stabilizing output injection.

Proof. The sufficiency is straightforward. The necessity follows from the fact that the linearized system of G is stabilizable and detectable. In fact, let's consider part (i); the detectability of the linearized system implies that there exists a positive definite matrix P such that

$$PA^T + AP - BB^T < 0$$

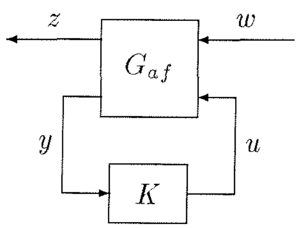
with $A = \frac{\partial f}{\partial x}(0), B = g(0)$ [125]. Now define $V(x) = x^T P^{-1} x$, it follows that V locally satisfies HJI (5.5) with some γ_3 being of class $\mathcal{K}$ and satisfying (5.4). Similar argument applies to part (ii). □

5.3 Stabilization of Input-Affine Nonlinear Systems

In this section, we will consider a simple class of nonlinear systems, i.e., the input-affine systems; the results can be extended to handle a broader class of systems, and the extension is the main issue of the next section. Basically, the techniques to be used are demonstrated in the last section. In this section, it is examined to what extent the treatments and results for linear systems in the last section can be extended to handle the input-affine nonlinear systems. All results are local, unless otherwise noted.

5.3.1 Controller Parameterization Problem Statement

In this section, the standard feedback configuration is as follows,



The plant has the following input-affine realization

$$G_{af} : \begin{cases} \dot{x} = f(x) + g_1(x)w + g(x)u \\ z = h_1(x) + k_{11}(x)w + k_{12}(x)u \\ y = h(x) + k_{21}(x)w + k(x)u \end{cases} \tag{5.8}$$

where $x \in \mathbf{R}^n$ is the state vector, $u \in \mathbf{R}^p$ and $w \in \mathbf{R}^{p_1}$ are input vectors, and $y \in \mathbf{R}^q$ and $z \in \mathbf{R}^{q_1}$ are output vectors, respectively. We will assume $f, g_1, g, h_1, h, k_{ij}, k \in \mathbf{C}^2$, and $f(0) = 0, h_1(0) = 0, h(0) = 0$. Therefore, $0 \in \mathbf{R}^n$ is an equilibrium of the system with $w = 0$ and $u = 0$. It is known that a controller locally stabilizes system G_{af} with $w = 0$ if and only if it stabilizes the following system,

$$G : \begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) + k(x)u \end{cases} . \tag{5.9}$$

In the following discussion, although we could guess the controller parameterization formula from the linear case, prove it directly using Lyapunov theory under some additional conditions, and develop a theory without involving the input w and output z, the development would be less appealing for the following reasons: (i) The alternative approach provides a constructive proof; (ii) The techniques used in the linear case could fail somewhere, the parallel treatments to the linear case would reveal this; (iii) It is natural to take the extra input w and output z into account to reveal some I/O properties for nonlinear systems. For example, with this (I/O) consideration, we shall see that, unlike in the linear case, the parameterized I/O maps are not affine in the parameters.

In this section, we assume that the system G is *locally smoothly stabilizable and locally smoothly detectable.* We are interested in finding a time-invariant controller $u = Ky$ which has the following input-affine realization,

$$K : \begin{cases} \dot{\tilde{x}} = a(\tilde{x}) + b(\tilde{x})y \\ u = c(\tilde{x}) + d(\tilde{x})y \end{cases}$$

with $a, b, c, d \in \mathbf{C}^2$ and $a(0) = 0, c(0) = 0$, such that the closed loop system $\mathcal{F}_l(G_{af}, K)$ is asymptotically stable with $w = 0$. It is assumed that $I - k(x)d(\tilde{x})$ is invertible to guarantee the well posedness of the feedback system. We shall assume $k(x) = 0$ for simplicity.

We are interested in the following problem: **To what extent, can the parameterization formula for linear systems be extended to handle the input-affine nonlinear systems?** Actually, in this section, we shall parameterize a class of input-affine time invariant controllers which locally asymptotically stabilize G such that the parameterized controllers are characterized as fractional transformation of some locally stable parameters. We denote $\mathcal{SP}_{af}$ as the class of input-affine nonlinear systems which have input-affine realizations like (5.9) and are locally asymptotically stable around 0 with zero inputs. Therefore, if $Q \in \mathcal{SP}_{af}$, by inverse Lyapunov theorem, it admits a $\mathbf{C}^3$ Lyapunov function $V_Q(\cdot)$.

Definition 5.3.1 *Consider a system G, two controllers K and K' are equivalent if their corresponding closed loop map are identical for zero initial conditions, i.e. $\mathcal{F}_l(G,K) = \mathcal{F}_l(G,K')$, written as $K \cong K'$.*

5.3.2 State-Feedback and Output-Injection

As in the linear case, the construction of controller parameterization is accomplished by decomposing the original output feedback problem into some simpler problems, which are known as full information (FI) and full control (FC) problems. In this subsection, those problems are considered.

We first deal with full information (FI) system, in which case both state and disturbance w are measured.

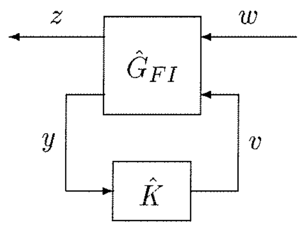
$$G_{FI}: \begin{cases} \dot{x} = f(x) + g_1(x)w + g(x)u \\ z = h_1(x) + k_{11}(x)w + k_{12}(x)u \\ y = \begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} w \end{cases}. \tag{5.10}$$

It is assumed that $[f(x), g(x)]$ is smoothly stabilizable, therefore, there exists a $\mathbf{C}^2$ function F such that $u = F(x)$ is a smooth stabilizing state feedback. Since both state x and disturbance w are available to the control input u, the control law $u = F(x) + Qw$ with $Q \in \mathcal{SP}_{af}$ is legal and it stabilizes the FI structure (5.10). Moreover, we have the following result about the parameterization of stabilizing controllers for FI structure.

Proposition 5.3.2 *Let $F : \mathbf{R}^n \to \mathbf{R}^p$ with $F(0) = 0$ be a smooth function such that $\dot{x} = f(x) + g(x)F(x)$ has an asymptotically stable equilibrium at $x = 0$. Then every input-affine stabilizing controller for FI structure (5.10) is equivalent to one of the controllers in the parameterized set as follows*

$$\mathcal{K}_{FI} = \left\{ \begin{bmatrix} F(\cdot) & Q \end{bmatrix} : Q \in \mathcal{SP}_{af} \right\}.$$

Proof. It can be easily verified that the control law $u = F(x) + Qw$ with $Q \in \mathcal{SP}_{af}$ stabilizes the FI structure by Vidyasagar's stability theorem for cascade systems [191]. Now, given a stabilizing controller $\mathcal{K}_{FI}$, we need to show that there is a $Q \in \mathcal{SP}_{af}$ such that $K_{FI} \cong \begin{bmatrix} F(\cdot) & Q \end{bmatrix}$. To this end, make a change of control variable as $v = u - F(x)$, where x denotes the state of the system G_{FI}, then the feedback system with the controller K_{FI} has the following block diagram,



where

$$\hat{G}_{FI}: \begin{cases} \dot{x} &= f(x) + g(x)F(x) + g_1(x)w + g(x)v \\ z &= h_1(x) + k_{12}(x)F(x) + k_{11}(x)w + k_{12}(x)v \\ y &= \begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} w \end{cases}$$

$$\hat{K} = K_{FI} - \begin{bmatrix} F(\cdot) & 0 \end{bmatrix}.$$

Let Q be the map from w to v; it belongs to $\mathcal{SP}_{af}$ by asymptotic stability of the closed loop system. Then $u = F(x)+v = F(x)+Qw$. It follows that $\mathcal{F}_l(G_{FI}, K_{FI}) = \mathcal{F}_l(G_{FI}, \begin{bmatrix} F(\cdot) & Q \end{bmatrix})$ provided that the initial states are zero, so $K_{FI} \cong \begin{bmatrix} F(\cdot) & Q \end{bmatrix}$. □

Remark 5.3.3 *If the system $[f(x), g(x)]$ is globally smoothly stabilizable, then there is a smooth $F_{I/O} : \mathbf{R}^n \to \mathbf{R}^p$, such that the system*

$$\dot{x} = f(x) + g(x)F_{I/O}(x) + g(x)v$$

with input v is I/S stable [168] (see Definition 5.4.7). Then by Sontag's theory [168] and the argument used in the above proof, it can be shown that every input-affine globally stabilizing controller is equivalent to $K_{FI} = F_{I/O}(x)+Qw$ with $Q \in \mathcal{SP}_{af}$ being globally asymptotically stable about 0.

As in the linear case, the stabilization problem where the control is directly injected to the state is considered next. Such structure is called full control (FC):

$$G_{FC}: \begin{cases} \dot{x} &= f(x) + g_1(x)w + \begin{bmatrix} I & 0 \end{bmatrix} u \\ z &= h_1(x) + k_{11}(x)w + \begin{bmatrix} 0 & I \end{bmatrix} u \\ y &= h(x) + k_{21}(x)w \end{cases}. \tag{5.11}$$

It is assumed that $[h(x), f(x)]$ is smoothly detectable. Thus, there exists a $\mathbf{C}^2$ smooth function $L : \mathbf{R}^n \to \mathbf{R}^{n\times q}$ such that $u = \begin{bmatrix} L(x) \\ 0 \end{bmatrix} y$ is a smooth stabilizing output injection. The following statement follows easily.

Proposition 5.3.4 *Let $L(\cdot)$ be a smooth matrix function such that $\dot{x} = f(x) + L(x)h(x)$ has an asymptotically stable equilibrium at $x = 0$. Then the following parameterized set characterizes a class of stabilizing controllers for FC structure (5.11),*

$$\mathcal{K}_{FC} = \left\{ \begin{bmatrix} L(x) \\ Q \end{bmatrix} : Q \in \mathcal{SP}_{af} \right\}.$$

It is noted that the controller is also allowed to depend on the state x. This consideration is mainly of technical interests as we will see soon.

Remark 5.3.5 *If the system $[h(x), f(x)]$ is globally smoothly detectable, let $L(\cdot)$ be a smooth matrix function such that $\dot{x} = f(x) + L(x)h(x)$ has a globally asymptotically stable equilibrium at $x = 0$. Then from Sontag's argument [168], it follows that $K_{FC} = \begin{bmatrix} L(x) \\ Q \end{bmatrix} y$ with $Q \in \mathcal{SP}_{af}$ being I/S stable globally stabilizes system G_{FC}.*

5.3.3 Locally Stabilizing Controller Parameterization

The main results of this section are given in this subsection. It will be shown that a class of input-affine (locally) stabilizing controllers are parameterized as fractional transformation of the parameters in $\mathcal{SP}_{af}$; the structures of the parameterized closed-loop maps are also examined. However, unlike linear systems, the closed loop maps are not affine in Q.

Controller Parameterization

We now consider the general output feedback stabilization problem. The solutions to this problem are based on the results in the last subsection. The nonlinear time-invariant plant is an input-affine system G_{af} (5.8) with $k_{22}(x) = 0$. It is assumed that $[f(x), g(x)]$ is locally smoothly stabilizable and $[h(x), f(x)]$ is locally smoothly detectable. So there are two $\mathbf{C}^3$ positive definite functions $V, U : \mathbf{R}^n \to \mathbf{R}^+$, two $\mathbf{C}^2$ functions $F : \mathbf{R}^n \to \mathbf{R}^p$, $L : \mathbf{R}^n \to \mathbf{R}^{n \times q}$, and two $\mathbf{C}^2$ functions γ and σ of class $\mathcal{K}$, such that

$$\mathcal{H}_{SF}(V, F, x) := \frac{\partial V(x)}{\partial x}(f(x) + g(x)F(x)) \leq -\gamma(\|x\|) \tag{5.12}$$

$$\mathcal{H}_{OI}(U, L, x) := \frac{\partial U(x)}{\partial x}(f(x) + L(x)h(x)) \leq -\sigma(\|x\|) \tag{5.13}$$

for $x \in \mathcal{B}_r$ with some $r > 0$.

As in the linear case, make a change of variable, i.e., let $v = u - F(x)$, then we get the following system,

$$G_{OE}: \begin{cases} \dot{x} &= f(x) + g_1(x)w + g(x)u \\ v &= -F(x) + \quad\quad + u \\ y &= h(x) + k_{21}(x)w \end{cases} \tag{5.14}$$

which has a constraint that $\dot{x} = f(x) + g(x)F(x)$ is asymptotically stable at $x = 0$. As far as the local asymptotic stabilization is concerned, $u = Ky$ stabilizes G_{OE} if and only if it stabilizes G_{af}. The above structure of G_{OE} is known as output estimation (OE).

Unlike the linear case, the two structures FC, which is discussed in the last subsection, and OE are not equivalent if it is just assumed that $[h(x), f(x)]$ is smoothly detectable. But there are indeed some close relations between this two structures. We can therefore take advantage of the FC results to deal with OE problem as in the linear case. Analogically, define a system P_{OE},

$$P_{OE}: \begin{cases} \dot{\tilde{x}} &= f(\tilde{x}) + g(\tilde{x})F(\tilde{x}) + \quad\quad \begin{bmatrix} I & -g(\tilde{x}) \end{bmatrix} u_0 \\ u &= -F(x) \quad\quad\quad\quad + \quad\quad \begin{bmatrix} 0 & I \end{bmatrix} u_0 \\ y_0 &= -h(\tilde{x}) \quad\quad\quad\quad + \; y \end{cases} .$$

Motivated by the linear treatment, we would expect that $G_{FC} = \mathcal{S}(G_{OE}, P_{OE})$. However, this conjecture generally fails in this case, since the internal dynamics for both systems with zero inputs are not identical in general. As we only consider local stabilization, there naturally arises a question: can we still use the FC local controllers to recover the OE controllers by $K_{OE} = \mathcal{F}_l(P_{OE}, K_{FC})$ as in the linear case? Or can the system $\mathcal{F}_l(G_{OE}, K_{OE})$ remain (locally) stable? The answer is positive if some stronger assumption about the detectability is imposed.

In this case, it is additionally assumed that $[h(x), f(x)]$ is locally exponentially detectable. Therefore, there are a $\mathbf{C}^3$ locally positive definite function $U : \mathbf{R}^n \to \mathbf{R}^+$, a locally smooth function $L : \mathbf{R}^n \to \mathbf{R}^p$, and a $\mathbf{C}^2$ function σ of class $\mathcal{K}$ such that (5.13) is satisfied, and in addition,

$$\lim_{s \to 0} \frac{\sigma(s)}{s^2} \in (0, \infty). \tag{5.15}$$

We have the following result about the stabilization, where the controller is recovered by $K = \mathcal{F}_l(P_{OE}, K_{FC})$ with $K_{FC} = \begin{bmatrix} L \\ Q \end{bmatrix}$.

Theorem 5.3.6 *Consider the system (5.8). Suppose that it is locally smoothly stabilizable and locally exponentially detectable. Let $F(\cdot)$ and $L(\cdot)$ be determined by the above characterizations (5.12), (5.13), and (5.15). Then the controller*

$$K : \begin{cases} \dot{\tilde{x}} = f(\tilde{x}) + g(\tilde{x})F(\tilde{x}) + L(\tilde{x})h(\tilde{x}) - L(\tilde{x})y \\ u = F(\tilde{x}) \end{cases} \tag{5.16}$$

(locally) asymptotically stabilizes system G_{af} around 0.

Moreover, the controller parameterized as $u = \mathcal{F}_l(M, Q)y$ with

$$M : \begin{cases} \dot{\tilde{x}} & = & f(\tilde{x}) + g(\tilde{x})F(\tilde{x}) + L(\tilde{x})h(\tilde{x}) & - & L(\tilde{x})y & + & g(\tilde{x})u_0 \\ u & = & F(\tilde{x}) & + & & & u_0 \\ y_0 & = & -h(\tilde{x}) & + & y & & \end{cases} \tag{5.17}$$

for all $Q \in \mathcal{SP}_{af}$ also (locally) asymptotically stabilizes system G around 0.

We use Lyapunov technique to prove the above theorem. First, we have the following observation.

Lemma 5.3.7 *Let $U(x) \geq 0$ and $L(x)$ satisfy (5.13) and (5.15), and $x, \tilde{x}$ be states of systems G_{OE} and K_{OE}, $e = \tilde{x} - x$. Define*

$$\mathcal{H}_E(e, x) := \frac{\partial U(e)}{\partial e}(f(e + x) - f(x) + L(e + x)(h(e + x) - h(x)))+$$
$$\frac{\partial U(e)}{\partial e}(g(x) - g(e + x))F(e + x).$$

Then for all $e, x \in \mathcal{B}_r$ with some $r > 0$, there exists a function π of class $\mathcal{K}$ with

$$\lim_{s \to 0} \frac{\pi(s)}{s^2} \in (0, \infty),$$

such that $\mathcal{H}_E(e, \tilde{x}) + \pi(\|e\|) \leq 0$.

Proof. Recall that

$$\mathcal{H}_{OI}(U, L, e) := \frac{\partial U(e)}{\partial e}(f(e) + L(e)h(e)) \leq -\sigma(\|e\|)$$

for a $\mathbf{C}^2$ function σ of Class $\mathcal{K}$. The conclusion follows by observing that the Hessian matrix of $\mathcal{H}_E(e, \tilde{x})$ with respect to e at 0 can be arbitrarily close to the one of $\mathcal{H}_{OI}(U, L, e)$ with respect to e at 0 if $\tilde{x} \in \mathcal{B}_r$ for r small enough. In this case, the Hessian matrix of $\mathcal{H}_E(e, \tilde{x})$ is negative definite. The conclusion follows. □

The following lemma, which is from [170, Corollary 5.1], is used in the proof.

Lemma 5.3.8 *Suppose system $\dot{x} = f(x,u)$ with $f \in \mathbf{C}^0$ has an asymptotically stable equilibrium at 0 when $u = 0$. Then there exists a continuous function $\sigma : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ with $\sigma(s) \neq 0$ if $s \neq 0$, such that given a number $r > 0$ there is $r_m > 0$, for each $r_0 \in (0, r_m)$, if initial state $x(0) \in \mathcal{B}_{r_0}$ and $u \in \mathcal{L}_\infty[0,\infty)$ for which $\|u\|_\infty \leq \sigma(r_0)$ and $u(t) \rightarrow 0$ as $t \rightarrow \infty$, then the solution $x(t)$ exists with $\|x(t)\| < r$ for all $t \in \mathbf{R}^+$, and it satisfies $x(t) \rightarrow 0$ as $t \rightarrow \infty$.*

Next, we give a proof of the stabilization result which closely follows the treatments of Sontag in [170].

Proof. [**Theorem 5.3.6**] Only the latter statement that $u = \mathcal{F}_l(M,Q)$ with $Q \in \mathcal{SP}_{af}$ locally stabilizes system (5.8) is proved, as the central controller (5.16) is obtained by letting $Q = 0$.

Consider $\mathcal{F}_l(G_{af}, \mathcal{F}_l(M,Q))$ for $Q \in \mathcal{SP}_{af}$ which has the following realization ($u_0 = Qy_0$),

$$\begin{cases} \dot{\xi} = a(\xi) + b(\xi)y_0 \\ u_0 = c(\xi) + d(\xi)y_0 \end{cases}.$$

So the dynamics of closed loop system with $w = 0$ is as follows,

$$\begin{cases} \dot{x} = f(x) + g(x)F(\tilde{x}) + g(x)(c(\xi) + d(\xi)(-h(\tilde{x}) + h(x))) \\ \dot{\tilde{x}} = f(\tilde{x}) + g(\tilde{x})F(\tilde{x}) + L(\tilde{x})(h(\tilde{x}) - h(x)) + g(\tilde{x})(c(\xi) + d(\xi)(-h(\tilde{x}) + h(x))) \\ \dot{\xi} = a(\xi) + b(\xi)(-h(\tilde{x}) + h(x)) \end{cases}.$$

Let $e = \tilde{x} - x$, the reorganization of the system yields

$$\begin{cases} \dot{e} = \eta(e,x) + (g(e+x) - g(x))(c(\xi) + d(\xi)(-h(e+x) + h(x))) \\ \dot{\xi} = a(\xi) + b(\xi)(-h(e+x) + h(x)) \\ \dot{x} = f(x) + g(x)F(e+x) + g(x)(c(\xi) + d(\xi)(-h(e+x) + h(x))) \end{cases}$$

where η is a function defined as

$$\eta(e,x) := f(e+x) - f(x) + L(x+e)(h(e+x) - h(x)) + (g(e+x) - g(x))F(e+x).$$

The proof of stability of the closed-loop system is divided into the following three steps.

Step 1. We first prove that $e(t) \rightarrow 0$ as $t \rightarrow \infty$ if $e(0) \in \mathcal{B}_{r_0}$, $\xi \in \mathcal{B}_{r_q}$ and $x \in \mathcal{B}_r$, for some $r_0, r, r_q > 0$. Consider the e-subsystem:

$$\dot{e} = \eta(e,x) + (g(e+x) - g(x))(c(\xi) + d(\xi)(-h(e+x) + h(x))). \tag{5.18}$$

Take the U as given in the theorem, it was shown in Lemma 5.3.7 that there exists $r > 0$, for $x \in \mathcal{B}_r, e \in \mathcal{B}_r$,

$$\frac{\partial U(e)}{\partial e}\eta(e,x) = \mathcal{H}_E(e,x), \leq -\pi(||e||)$$

where π is a function of of class $\mathcal{K}$ with $\lim_{s\to 0}\frac{\pi(s)}{s^2} \in (0,\infty)$. Therefore, there exists $r_q > 0$, and for all $\xi \in \mathcal{B}_{r_q}$, there is a function π_0 of class $\mathcal{K}$ such that for all $e, x \in \mathcal{B}_r$ with updated $r > 0$,

$$-\pi(||e||) + \frac{\partial U(e)}{\partial e}(g(e+x) - g(x))(c(\xi) + d(\xi)(-h(e+x) + h(x))) \leq -\pi_0(||e||).$$

Thus, for all $e, x \in \mathcal{B}_r$,

$$\begin{aligned}\dot{U}(e) &= \frac{\partial U(e)}{\partial e}\eta(e,t) + \frac{\partial U(e)}{\partial e}(g(e+x) - g(x))(c(\xi) + d(\xi)(-h(e+x) + h(x))) \\ &\leq -\pi(||e||) + \frac{\partial U(e)}{\partial e}(g(e+x) - g(x))(c(\xi) + d(\xi)(-h(e+x) + h(x))) \\ &\leq -\pi_0(||e||).\end{aligned}$$

Therefore, there is a function β_o of class $\mathcal{KL}$ such that

$$||e(t)|| \leq \beta_o(||e(0)||, t) \tag{5.19}$$

for all $t \in \mathbf{R}^+$, $e(0) \in \mathcal{B}_{r_0} \subset \mathbf{R}^n$ for some $r_0 > 0$ such that $\beta_o(r_0, 0) < r$, and $x \in \mathcal{B}_r, \xi \in \mathcal{B}_{r_q}$. Thus, $e(t)\to 0$ uniformly on $x \in \mathcal{B}_r, \xi \in \mathcal{B}_{r_q}$ as $t\to\infty$ if $e(0) \in \mathcal{B}_{r_0}$. Without loss of generality, it is assumed that r and r_q are chosen such that (5.19) holds for all $t \in \mathbf{R}^+$, $e(0) \in \mathcal{B}_{r_0}$, and $x \in \bar{\mathcal{B}}_r, \xi \in \bar{\mathcal{B}}_{r_q}$, where $\bar{\mathcal{B}}_r$ is the closure of $\mathcal{B}_r$.

Step 2. We will next show $\xi(t)\to 0$ as $t\to\infty$ for $\xi(0) \in \mathcal{B}_{q_0}$, $x(0) \in \mathcal{B}_{r_0}$, and $x(t) \in \mathcal{B}_r$ for some $q_0, r_0, r > 0$. Consider the ξ-subsystem:

$$\dot{\xi} = a(\xi) + b(\xi)(-h(e+x) + h(x)) \tag{5.20}$$

(with e as an input). If $e = 0$, then the system becomes $\dot{\xi} = a(\xi)$ which is locally stable. By Lemma 5.3.8, there exists a continuous function σ with $\sigma(s) \neq 0$ if $s \neq 0$, for the given $r_q > 0$, there is $q_0 > 0$ with $q_0 < r_q$ such that if $||e(t)|| < \sigma(q_0)$ with $e(t)\to 0$ as $t\to\infty$, then $||\xi(t)||_\infty < r_q$ and $x(t)\to 0$ as $t\to\infty$. Therefore, if the above conditions for $e(t)$ satisfying (5.19) are satisfied, then it is done.

We now verify that $e(t)$ satisfying (5.19) has the required properties for some suitable $r_0 > 0$. We first show that $\xi(t)$ satisfying (5.20) also satisfies $||\xi(t)||_\infty < r_q$ for all $\xi(0) \in \mathcal{B}_{q_0}$, and e evolving according to (5.18) with $e(0) \in \mathcal{B}_{r_0}$ for some $r_0 > 0$ and $x \in \mathcal{B}_r$. In fact, r_0 is adjusted such that $\beta_o(r_0, 0) \leq \sigma(q_0)$. Suppose there is a time T such that $\|\xi(T)\| \geq r_q$ with some $\xi(0) \in \mathcal{B}_{q_0}$ (here T is chosen to be the minimal such time). As $\|\xi(t)\| \leq r_q$ for $t \in [0, T]$, then for all $x(t) \in \mathcal{B}_r$, (5.19) is satisfied for $t \in [0, T]$, i.e., $\|e(t)\| \leq \beta_o(\|e(0)\|, t) \leq \beta_o(\|e(0)\|, 0) < \sigma(q_0)$ for $t \in [0, T]$. By the previous statement and the causality of system (5.20), we have $\xi(t) \in \mathcal{B}_{r_q}$ for $t \in [0, T]$ which contradicts the assumption $\|\xi(T)\| \geq r_q$. Therefore, $\xi(t) \in \mathcal{B}_{r_q}$ for $t \in [0, \infty)$. Thence, if $e(0) \in \mathcal{B}_{r_0}$ and $\|x\|_\infty < r$ then $\|e(t)\| \leq \beta_o(\|e(0)\|, t) \leq \beta_o(\|e(0)\|, 0) < \sigma(q_0)$ for $t \in [0, \infty)$. Therefore, by Lemma 5.3.8, it is concluded that $\xi(t) \to 0$ as $t \to \infty$ for $\xi(0) \in \mathcal{B}_{q_0}$, $x(0) \in \mathcal{B}_{r_0}$, and $x(t) \in \mathcal{B}_r$ for some $q_0, r_0, r > 0$.

Step 3. Finally, we will prove $x(t) \to 0$ if $x(0), e(0) \in \mathcal{B}_{r_0}$ and $\xi(0) \in q_0$. Consider the x-subsystem:

$$\dot{x} = f(x) + g(x)F(e + x) + g(x)(c(\xi) + d(\xi)(-h(e + x + h(x))) \tag{5.21}$$

Note that if $(e, \xi) = 0$, then the system becomes $\dot{x} = f(x) + g(x)F(x)$ which is asymptotically stable by assumption. It is also known from the above proofs that if $\xi(0) \in \mathcal{B}_{q_0}$, $x(0) \in \mathcal{B}_{r_0}$, and $x(t) \in \mathcal{B}_r$ for some $q_0, r_0, r > 0$, then $e(t) \to 0$ and $\xi(t) \to 0$ as $t \to \infty$. By the same argument as in Step 2, it is shown that $\|x(t)\| < r$ and $x(t) \to 0$ if $x(0), e(0) \in \mathcal{B}_{r_0}$ and $\xi(0) \in q_0$ for some suitably adjusted $r_0, q_0 > 0$.

This completes the proof. □

Remark 5.3.9 *(i) In Theorem 5.3.6, the parameters in the parameterization can also be chosen as general asymptotically stable systems whose realizations are not necessarily input-affine.*

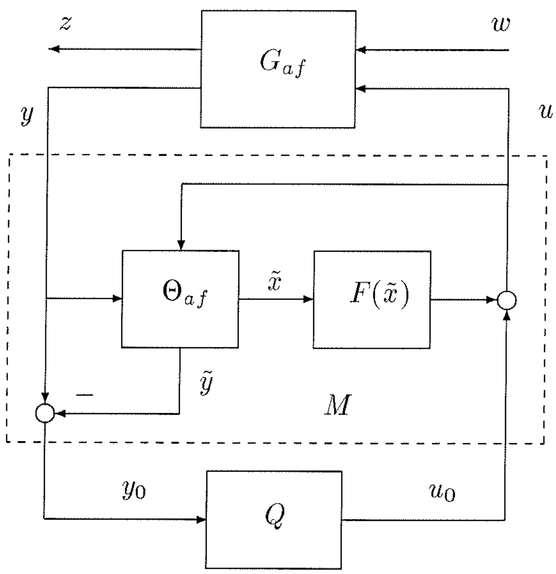
(ii) In general, the parameterization given in Theorem 5.3.6 only characterizes a class of stabilizing controllers. One reason is that the conditions of local asymptotic stabilizability and exponential detectability in the theorem are sufficient, but generally not necessary for output stabilization.

Separation Structures of Parameterized Controllers

The parameterized controller has a separation structure, and it is an observer-based controller. The observer is as follows.

$$\Theta_{af}:\begin{cases}\dot{\tilde{x}} = f(\tilde{x}) + g(\tilde{x})F(\tilde{x}) + L(\tilde{x})(h(\tilde{x}) - y)\\ \tilde{y} = h(\tilde{x})\end{cases}.$$

The estimated state is $\tilde{x}$ and $\tilde{x}(t) - x(t)\to 0$ as $t\to\infty$ for $w = 0$ because of the locally exponentially stability. The feedback system with a parameterized controller is thus structured as the following diagram.



5.3.4 Structures of Closed-Loop Maps

In this subsection, we will consider structures of the closed maps with the parameterized controllers. We first have the following definition of an I/O property.

Definition 5.3.10 *Consider an I/O operator* $P : \mathcal{L}^e_\infty[0,\infty)\to\mathcal{L}^e_\infty[0,\infty)$. *It is said to be locally I/O stable if there are* $k_I, k_O > 0$ *such that for all* $w \in \mathcal{L}_\infty[0,\infty)$ *with* $\|w\|_\infty \le k_I$, *then* $z := Pw \in \mathcal{L}_\infty[0,\infty)$ *and* $\|z\|_\infty \le k_O$.

We first have the following lemma about the relation between asymptotic stability and I/O stability for a nonlinear system, which follows from [170, Corollary 5.1] (see also [189, Lemma 4.1]).

Lemma 5.3.11 *Consider the following system,*

$$G : \begin{cases} \dot{x} = f_c(x) + g_c(x)w \\ z = h_c(x) + k_c(x)w \end{cases}$$

with $f_c, g_c, h_c, k_c \in \mathbf{C}^0$. It is assumed that $\dot{x} = f_c(x)$ is locally asymptotically stable around 0. Then given $\epsilon > 0$, there is a $\delta > 0$, such that for all $\|w\|_\infty \leq \delta$, $\|z\|_\infty \leq \epsilon$.

Next, consider the closed-loop map from w to z which is parameterized as follows,

$$T_{zw} = \mathcal{F}_l(G_{af}, \mathcal{F}_l(M, Q)) = \mathcal{F}_l(T, Q), \quad Q \in \mathcal{SP}_{af}$$

where T has the following realization,

$$\begin{cases} \dot{x} & = & f(x) + g(x)F(\tilde{x}) & + & g_1(x)w & + & g(x)u_0 \\ \dot{\tilde{x}} & = & f(\tilde{x}) + g(\tilde{x})F(\tilde{x}) + L(\tilde{x})(h(\tilde{x}) - h(x)) & - & L(\tilde{x})k_{21}(x)w & + & g(\tilde{x})u_0 \\ z & = & h_1(x) + k_{12}(x)F(\tilde{x}) & + & k_{11}(x)w & + & k_{12}(x)u_0 \\ y_0 & = & h(\tilde{x}) - h(x) & + & k_{21}(x)w & & \end{cases} \tag{5.22}$$

Now consider the structure of I/O map T which has zero initial conditions $x(0) = 0$ and $\tilde{x}(0) = 0$, the closed-loop map T_{zw} is locally I/O stable and parameterized as follows,

$$T_{zw}w = T_1(w, T_2(Q)w),$$

where T_1 and $T_2(Q)$ are locally I/O maps, and T_1 defines the map from $\begin{bmatrix} w \\ u_0 \end{bmatrix}$ to z as follows,

$$\begin{cases} \dot{x} & = & f(x) + g(x)F(\tilde{x}) & + & g_1(x)w & + & g(x)u_0 \\ \dot{\tilde{x}} & = & f(\tilde{x}) + g(\tilde{x})F(\tilde{x}) + L(\tilde{x})(h(\tilde{x}) - h(x)) & - & L(\tilde{x})k_{21}(x)w & + & g(\tilde{x})u_0 \\ z & = & h_1(x) + k_{12}(x)F(\tilde{x}) & + & k_{11}(x)w & + & k_{12}(x)u_0 \end{cases}$$

and $T_2(Q)$ is the map from w to u_0 defined as follows.

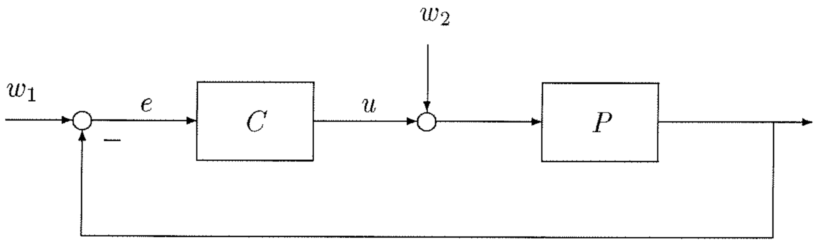
$$\begin{cases} \dot{x} = f(x) + g(x)F(\tilde{x}) + g_1(x)w + g(x)Q(h(\tilde{x}) - h(x) + k_{21}(x)w) \\ \dot{\tilde{x}} = f(\tilde{x}) + g(\tilde{x})F(\tilde{x}) + L(\tilde{x})(h(\tilde{x}) - h(x) - k_{21}(x)w) + g(\tilde{x})Q(h(\tilde{x}) - h(x) + k_{21}(x)w) \\ u_0 = Q(h(\tilde{x}) - h(x) + k_{21}(x)w) \end{cases}.$$

Both T_1 and $T_2(Q)$ are locally asymptotically stable with zero inputs as guaranteed from the development, then they are locally I/O stable by Lemma 5.3.11.

It is noted that that unlike in the linear case, **the closed loop maps have no affine-like relation with Q for nonlinear systems in general**. To conclude the I/O discussion, we consider an example from [48]. The parameterization of the closed-loop map is used in the nonlinear $\mathcal{H}_\infty$-optimal controller design in [66].

An Example

Consider a feedback system with the following block diagram,

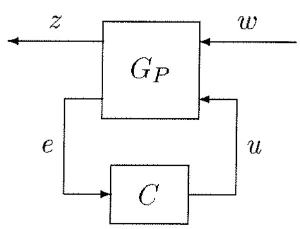


where P is a (locally) I/O stable plant. we need to parameterize a class of controllers C such that the resulting maps from $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ to $\begin{bmatrix} u \\ y \end{bmatrix}$ are (locally) I/O stable. This problem is considered in [48] in an I/O setting. In the following, we consider it in the state-space framework.

Suppose system P has the following input-affine realization,

$$P : \begin{cases} \dot{x} = f_P(x) + g_P(x) w_0 \\ y = h_P(x) \end{cases}$$

with $f_P, g_P, h_P \in \mathbf{C}^2$; and $\dot{x} = f_P(x)$ **is locally exponentially stable around** 0. Define $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ and $z = \begin{bmatrix} u \\ y \end{bmatrix}$, then the system block diagram is redrawn as follows,



with

$$G_P : \begin{cases} \dot{x} &= f_P(x) &+ \begin{bmatrix} 0 & g_P(x) \end{bmatrix} w &+ g_P(x) u \\ z &= \begin{bmatrix} 0 \\ h_P(x) \end{bmatrix} &+ & \begin{bmatrix} I \\ 0 \end{bmatrix} u \\ e &= -h_P(x) &+ \begin{bmatrix} I & 0 \end{bmatrix} w & \end{cases} .$$

As P is assumed to be locally exponentially stable around 0, the state feedback and output injection can be chosen as $F = 0$ and $L = 0$, respectively. Then by Theorem 5.3.6, a class of controllers C can be represented as $C = \mathcal{F}_l(M_P, Q)$, with

$$M_P : \begin{cases} \dot{\tilde{x}} = f_P(\tilde{x}) + g_P(\tilde{x})u_0 \\ u = u_0 \\ y_0 = h_P(\tilde{x}) + e \end{cases}$$

and $Q \in \mathcal{SP}_{af}$. Next, we examine the structure of the parameterized controller.

Lemma 5.3.12 *The controller $C = \mathcal{F}_l(M_P, Q)$ has structure $C = Q(I - PQ)^{-1}$.*

Proof. let $u_0 = Qy_0$, then

$$C : \begin{cases} \dot{\tilde{x}} = f_P(\tilde{x}) + g_P(\tilde{x})Q(h_P(\tilde{x}) + e) \\ u = Q(h_P(\tilde{x}) + e) \end{cases}$$

with $\tilde{x}(0) = 0$. Let $y_0 = h_P(\tilde{x}) + e$. Since $u = Qy_0$, it is sufficient to show $y_0 = (I - PQ)^{-1}e$, or $e = (I - PQ)y_0$.

In fact, consider $y_u := (I - PQ)y_0$, it can be written as

$$\begin{cases} \dot{x} = f_P(x) + g_P(x)Qy_0 \\ y_u = y_0 - h_P(x) \end{cases} .$$

Now replace $y_0 := h_P(\tilde{x}) + e$, then we have

$$\begin{cases} \dot{x} = f_P(x) + g_P(x)Q(h_P(\tilde{x}) + e) \\ y_u = h_P(\tilde{x}) + e - h_P(x) \end{cases} .$$

As $x(0) = 0$, by the uniqueness of the solution to differential equations, we have $x(t) = \tilde{x}(t)$. Then $y_u = e$, i.e., $e = (I - PQ)y_0$. □

We finally examine the structure of the closed loop map from w_1 to y provided $w_2 = 0$.

Lemma 5.3.13 *The closed loop map from w_1 to y with $w_2 = 0$ is parameterized as $T_{zw_1} = PQ$.*

Proof. Let $w_2 = 0$. then the closed loop map is parameterized as $T_{zw_1} = \mathcal{F}_l(T, Q)$ where T is as given in (5.22),

$$T : \begin{cases} \dot{x} = f_P(x) + g_P(x)u_0 \\ \dot{\tilde{x}} = f_P(\tilde{x}) + g_P(\tilde{x})u_0 \\ z = h_P(x) \\ y_0 = h_P(\tilde{x}) - h_P(x) + w \end{cases}$$

with $x(0) = \tilde{x}(0) = 0$, and $u_0 = Qy_0$. Therefore,

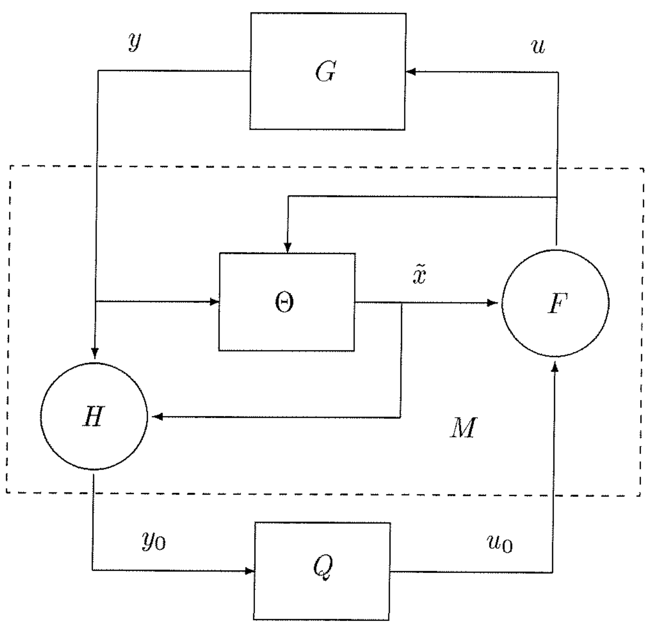
$$T_{zw_1} : \begin{cases} \dot{x} = f_P(x) + g_P(x)Q(h_P(\tilde{x}) - h_P(\tilde{x}) + w) \\ \dot{\tilde{x}} = f_P(\tilde{x}) + g_P(\tilde{x})Q(h_P(\tilde{x}) - h_P(x) + w) \\ z = h_P(x) \end{cases}$$

with $x(0) = \tilde{x}(0) = 0$. Therefore, $\tilde{x}(t) = x(t)$ for all $t \geq 0$, then $T_{zw_1} = PQ$ which is locally I/O stable. □

The interested reader can compare the above results with those in [48].

5.4 Stabilization of General Nonlinear Systems

In the last section, we considered the input-affine nonlinear systems, which have nice structures close to linear systems. The stabilizing control laws and stabilizing controller parameterizations are constructed based on observers. In this section, we will consider the parameterization problem for a more general class of nonlinear systems whose structures are not required to be input-affine. The treatment in this section is motivated by the results on input-affine systems. A set of stabilizing controllers for system G will also be characterized as fractional transformations of some stabilizing parameters, i.e., $\mathcal{F}_l(M, Q)$. Furthermore, the feedback systems will have the following separation structure.



5.4.1 Local Controller Parameterization

The plant considered in this subsection is

$$G_g : \begin{cases} \dot{x} = f(x, u) \\ y = h(x) \end{cases} \tag{5.23}$$

where $f(0,0) = 0, h(0) = 0$, $f, h \in \mathbf{C}^0$; x, u and y are assumed to have dimensions n, p, and q, respectively. Clearly, the origin 0 is an equilibrium of the system with $u = 0$. In this subsection, the locally stabilizing controller parameterization for system G_g (5.23) is considered. The parameterized controllers are represented as fractional transformations of some locally stable parameters. We first define the following a class of the local stable parameters.

Definition 5.4.1 *The class $\mathcal{SP}_{loc}$ of time-invariant nonlinear systems is so defined that each member has the following realization,*

$$Q : \begin{cases} \dot{x} = f_Q(x, u) \\ y = h_Q(x, u) \end{cases} \tag{5.24}$$

for some $f_Q, h_Q \in \mathbf{C}^0$, and is locally asymptotically stable at 0 with $u = 0$.

So if $Q \in \mathcal{SP}_{loc}$, by inverse Lyapunov theorem, it admits a locally $\mathbf{C}^1$ Lyapunov function $V_Q(\cdot)$.

Next, the notions of stabilizability and detectability for system G_g (5.23) are examined.

Definition 5.4.2 *G_g (5.23) is locally stabilizable around $x = 0$ if there is a continuous function $F : \mathbf{R}^n \rightarrow \mathbf{R}^p$ with $F(0) = 0$ such that $\dot{x} = f(x, F(x))$ is locally asymptotically stable around $x = 0$.*

The following technical definition is from [190].

Definition 5.4.3 *System G_g is said to be locally weakly detectable, if there are a $\mathbf{C}^0$ mapping $f_o : \mathbf{R}^n \times \mathbf{R}^q \times \mathbf{R}^p \rightarrow \mathbf{R}^n$ with $f_o(0,0,0) = 0$, a $\mathbf{C}^1$ locally positive definite function $W : \mathbf{R}^n \times \mathbf{R}^n \rightarrow \mathbf{R}^+$, and functions ϕ_1, ϕ_2, ϕ_3 of class $\mathcal{K}$ such that*

$$\phi_1(\|x - \tilde{x}\|) \leq W(x, \tilde{x}) \leq \phi_2(\|x - \tilde{x}\|) \tag{5.25}$$

$$\frac{\partial W(x, \tilde{x})}{\partial x} f(x, u) + \frac{\partial W(x, \tilde{x})}{\partial \tilde{x}} f_o(\tilde{x}, h(x), u) \leq -\phi_3(\|x - \tilde{x}\|) \tag{5.26}$$

for all $u \in \mathcal{B}_{r_u}$ and $x, \tilde{x} \in \mathcal{B}_r$ for some $r_u, r > 0$.

If the system G_g is (locally) weakly detectable, standard arguments show that there is a function β_0 of class $\mathcal{KL}$ such that the error state $e = \tilde{x} - x$ evolves according to the following dynamics,

$$\dot{e} = f_o(e + x, h(x), u) - f(x, u) =: \eta(e, x, u) \tag{5.27}$$

which satisfies

$$\|e(t)\| \leq \beta_0(\|e(0)\|, t) \tag{5.28}$$

for all $t \in \mathbf{R}^+$, $x \in \mathcal{B}_r$, and $u \in \mathcal{B}_{r_u}$. Therefore, $\tilde{x} \rightarrow x$ as $t \rightarrow \infty$, i.e., the system $\dot{\tilde{x}} = f_o(\tilde{x}, h(x), u)$ is a local observer for system G_g.

Remark 5.4.4 *A local observer for the general system (5.23) with $f, h \in \mathbf{C}^2$ can be constructed if there is a matrix-valued function $L(x)$ such that $\dot{x} = f(x, 0) + L(x)h(x)$ is locally exponentially stable. In fact, it can be shown that $\dot{x} = f_o(\tilde{x}, y, u)$ with*

$$f_o(\tilde{x}, y, u) = f(\tilde{x}, u) + L(\tilde{x})(h(\tilde{x}) - y)$$

is such a local observer. It is exactly the case for the observers constructed for the input-affine systems in the last section.

We have the following theorem about the local controller parameterization for system (5.23).

Theorem 5.4.5 *Suppose system G_g (5.23) is locally asymptotically stabilizable and locally weakly detectable. If in addition, there is a $\mathbf{C}^0$ function $F : \mathbf{R}^n \rightarrow \mathbf{R}^p$ such that $\dot{x} = f(x, F(x))$ is locally asymptotically stable at 0, and the function $f_o : \mathbf{R}^n \times \mathbf{R}^q \times \mathbf{R}^p \rightarrow \mathbf{R}^n$ is chosen as in the definition 5.4.3, then the controller parameterized as $u = \mathcal{F}_l(M_l, Q)y$ with M_l given by*

$$M_l : \begin{cases} \dot{\tilde{x}} = f_o(\tilde{x}, y, F(\tilde{x}) + u_0) \\ u = F(\tilde{x}) + u_0 \\ y_0 = h(\tilde{x}) - y \end{cases}$$

for all $Q \in \mathcal{SP}_{loc}$ also locally asymptotically stabilizes system G_g around 0.

The following proof basically follows the proof of Theorem 5.3.6, we just give a sketch here.

Proof. Assume $u_0 = Qy_0$ for $Q \in \mathcal{SP}_{loc}$ with the following realization

$$\begin{cases} \dot{\xi} = a(\xi, y_0) \\ u_0 = c(\xi, y_0) \end{cases}.$$

The dynamics of the closed loop system is described by

$$\begin{cases} \dot{x} = f(x, F(\tilde{x}) + c(\xi, h(\tilde{x}) - h(x))) \\ \dot{\tilde{x}} = f_o(\tilde{x}, h(x), F(\tilde{x}) + c(\xi, h(\tilde{x}) - h(x))) \\ \dot{\xi} = a(\xi, h(\tilde{x}) - h(x)) \end{cases}.$$

Take $e := \tilde{x} - x$ as the error state, then equivalently, the closed loop system can be represented with the state $x_c = [e^T \; \xi^T \; x^T]^T$ as

$$\begin{cases} \dot{e} = f_o(e + x, h(x), u) - f(x, u) \\ \dot{\xi} = a(\xi, h(x + e) - h(x)) \\ \dot{x} = f(x, F(x + e) + c(\xi, h(x + e) - h(x))) \end{cases}$$

where $u = F(x + e) + c(\xi, h(x + e) - h(x))$.

We first prove that $e(t) \to 0$ and $\xi(t) \to 0$ as $t \to \infty$ if $(e(0), \xi(0)) \in \mathcal{B}_{r_0} \times \mathcal{B}_{q_0}$ for some $r_0, q_0 > 0$.

Consider the e-subsystem. By the detectability, there is a function β_o of class $\mathcal{KL}$ such that

$$\|e(t)\| \leq \beta_o(\|e(0)\|, t) \tag{5.29}$$

for all $t \in \mathbf{R}^+$, $e(0) \in \mathcal{B}_{r_0} \subset \mathbf{R}^n$ and $\beta_o(r_0, 0) < r$, $x \in \mathcal{B}_r \subset \mathbf{R}^n$, and $u \in \mathcal{B}_{r_u}$ for some $r_0, r, r_u > 0$. So $e(t) \to 0$ as $t \to \infty$, for all $e(0) \in \mathcal{B}_{r_0}$. As $u = F(x+e) + c(\xi, h(x+e) - h(x))$ is continuous function of e, ξ, and x, it can be assumed that $u \in \mathcal{B}_{r_u}$ if $e, x \in \mathcal{B}_r$ and $\xi \in \mathcal{B}_{r_q}$ for some $r_q > 0$.

Next, consider the ξ-subsystem $\dot{\xi} = a(\xi, h(x + e) - h(x))$. If $e = 0$, then it becomes $\dot{\xi} = a(\xi, 0)$ which is locally asymptotically stable. By Lemma 5.3.8, there exists a continuous function σ with $\sigma(s) \neq 0$ if $s \neq 0$, for the given $r_q > 0$, there is $q_0 > 0$ with $q_0 < r_q$ such that if $\|e(t)\| < \sigma(q_0)$ with $e(t) \to 0$ as $t \to \infty$, then $\|\xi(t)\| < r_q$ and $x(t) \to 0$ as $t \to \infty$. However, from the similar arguments as in the proof of Theorem 5.3.6 (Step 2), it follows that $e(t)$, which satisfies (5.29) also satisfies the above conditions for $\xi(0) \in \mathcal{B}_{q_0}$, $e(0) \in \mathcal{B}_{r_0}$, and $x(t) \in \mathcal{B}_r$ for some $q_0, r_0, r > 0$. Therefore, by Lemma 5.3.8, it is concluded that $\xi(t) \to 0$ as $t \to \infty$.

The proof of this theorem is completed by showing that $x(t) \rightarrow 0$ as $t \rightarrow \infty$ if $x(0), e(0) \in \mathcal{B}_{r_0}$ and $\xi(0) \in q_0$ for some suitably adjusted $r_0, q_0 > 0$, but the latter follows the similar arguments as above.

$\square$

Remark 5.4.6 *The central controller for this parameterization can be recovered by letting $Q = 0$. By doing so, we have Vidyasagar's Theorem [190] as follows:*

Suppose that system G_g is locally asymptotically stabilizable and locally weakly detectable. If $\mathbf{C}^0$ function $F : \mathbf{R}^n \rightarrow \mathbf{R}^p$ is such that $\dot{x} = f(x, F(x))$ is locally asymptotically stable at 0, and $\mathbf{C}^0$ function $f_o : \mathbf{R}^n \times \mathbf{R}^q \times \mathbf{R}^p \rightarrow \mathbf{R}^n$ is chosen as in the definition 5.4.3, then the controller $u = Ky$ given by

$$K : \begin{cases} \dot{\tilde{x}} = f_o(\tilde{x}, y, F(\tilde{x})) \\ u = F(\tilde{x}) \end{cases}$$

locally asymptotically stabilizing the feedback system at $0 \in \mathbf{R}^n \times \mathbf{R}^n$.

5.4.2 Global Controller Parameterization

In this subsection, we generalize the local result in the previous subsection to get a global characterization. However, the conditions in general are very restrictive. The system considered is the same in (5.23), i.e.,

$$G_g : \begin{cases} \dot{x} = f(x, u) \\ y = h(x) \end{cases} \tag{5.30}$$

where $x \in \mathbf{R}^n$ is the state vector, $u \in \mathbf{R}^p$ and $y \in \mathbf{R}^q$ are the input and output vectors, respectively. $f, h \in \mathbf{C}^0$, $f(0,0) = 0, h(0) = 0$. Clearly, the origin 0 is an equilibrium of the system with $u = 0$. It is assumed that for all $u \in \mathcal{L}_\infty^e[0, \infty)$, $x(t)$ is defined for all initial state $x(0) \in \mathbf{R}^n$ and (almost) all $t \in \mathbf{R}^+$. We first have the following technical definition due to Sontag [168].

Definition 5.4.7 *Consider system $\dot{x} = f(x, u)$. It is input-to-state (I/S) stable if there exist functions β of class $\mathcal{KL}$ and γ of class $\mathcal{K}$ such that for each essentially bounded measurable control $u(\cdot)$ and each initial state $x(0)$, the solution $x(t)$ exists for each $t \geq 0$; and furthermore, it satisfies*

$$\|x(t)\| \leq \beta(\|x(0)\|, t) + \gamma(\|u\|_\infty). \tag{5.31}$$

Definition 5.4.8 *The class $\mathcal{SP}_{I/S}$ of nonlinear systems is defined such that each system in $\mathcal{SP}_{I/S}$ has a realization like (5.24) and is I/S stable.*

The following definition is due to Vidyasagar [190].

Definition 5.4.9 *System G_g (5.30) is said to be globally weakly detectable, if there are a $\mathbf{C}^0$ mapping $f_o : \mathbf{R}^n \times \mathbf{R}^q \times \mathbf{R}^p \to \mathbf{R}^n$ with $f_o(0,0,0) = 0$, a $\mathbf{C}^1$ positive definite function $W : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^+$, and functions ϕ_1, ϕ_2, ϕ_3 of class $\mathcal{K}_\infty$ such that*

$$\phi_1(\|x - \tilde{x}\|) \leq W(x, \tilde{x}) \leq \phi_2(\|x - \tilde{x}\|) \tag{5.32}$$

$$\frac{\partial W(x,\tilde{x})}{\partial x} f(x,u) + \frac{\partial W(x,\tilde{x})}{\partial \tilde{x}} f_o(\tilde{x}, h(x,u), u) \leq -\phi_3(\|x - \tilde{x}\|) \tag{5.33}$$

for all $u \in \mathbf{R}^p$ and $x, \tilde{x} \in \mathbf{R}^n$.

By Lyapunov Theorem, the above definition of detectability implies that there is a function β_0 of class $\mathcal{KL}$ such that the error state $e := \tilde{x} - x$ satisfies

$$\|e(t)\| \leq \beta_0(\|e(0)\|, t) \tag{5.34}$$

for all $t \in \mathbf{R}^+$, $x \in \mathbf{R}^n$, and $u \in R^p$. Therefore, system $\dot{\tilde{x}} = f_o(\tilde{x}, h(x,u), u)$ is an observer for system G.

Motivated by the construction of the stabilizing controller parameterization in the last subsection, we have the following result regarding the parameterization of a class of stabilizing controllers.

Theorem 5.4.10 *Suppose the system G_g (5.30) is globally stabilizable and globally weakly detectable. Let the function $f_o : \mathbf{R}^n \times \mathbf{R}^q \times \mathbf{R}^p \to \mathbf{R}^n$ be chosen as in the definition 5.4.9. If in addition, there are a $\mathbf{C}^0$ function $F : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^p$ for some integer $m > 0$ such that $\dot{x} = f(x, F(x+v, w))$ with input $\begin{bmatrix} v \\ w \end{bmatrix}$ is I/S stable, and a $\mathbf{C}^0$ function $H : \mathbf{R}^n \times \mathbf{R}^q \to \mathbf{R}^l$ for some integer $l > 0$ such that $\|H(x_1, h(x_2))\| \leq \kappa_h(\|x_1 - x_2\|)$ for some function κ_h of class $\mathcal{K}$ and for all $x_1, x_2 \in \mathbf{R}^n$. Then the controller parameterized as $u = \mathcal{F}_l(M_g, Q)y$ with M_g given by*

$$M_g : \begin{cases} \dot{\tilde{x}} = f_o(\tilde{x}, y, F(\tilde{x}, u_0)) \\ u = F(\tilde{x}, u_0) \\ y_0 = H(\tilde{x}, y) \end{cases}$$

for all $Q \in \mathcal{SP}_{I/S}$ also globally asymptotically stabilizes system G_g around 0.

Remark 5.4.11 *The central controller can be recovered by letting* $Q = 0$. *This theorem is reduced to the following statement about global stabilizability (see also [104, 181]).*

Suppose the system G_g *is globally asymptotically stabilizable and globally weakly detectable. Let the function* $f_o : \mathbf{R}^n \times \mathbf{R}^q \times \mathbf{R}^p \to \mathbf{R}^n$ *be chosen as in the definition 5.4.9. If in addition, A* $\mathbf{C}^0$ *function* $F : \mathbf{R}^n \to \mathbf{R}^p$ *is such that* $\dot{x} = f(x, F(x + v))$ *with input* v *is I/S stable, then the controller* $u = Ky$ *given by*

$$K : \begin{cases} \dot{\tilde{x}} = f_o(\tilde{x}, y, F(\tilde{x})) \\ u = F(\tilde{x}) \end{cases}$$

globally asymptotically stabilizes the feedback system at $0 \in \mathbf{R}^n \times \mathbf{R}^n$.

The above theorem is given in [108] in the case where the output y depends on both x and u, i.e., $y = h(x, u)$. In the following, we just give a sketch of the proof for completeness, and mainly emphasize on the discussion about the restrictiveness of the conditions.

Proof. **[Theorem 5.4.10: A Sketch]** The proof basically follows Sontag's arguments about global stability of cascade systems. Assume $u_0 = Qy_0$ for $Q \in \mathcal{SP}_{I/S}$ with the following realization,

$$\begin{cases} \dot{\xi} = a(\xi, y_0) \\ u_0 = c(\xi, y_0) \end{cases}.$$

The dynamics of the closed loop system is represented with the state $x_c = [e^T \ \xi^T \ x^T]^T$ (with $e := \tilde{x} - x$) as

$$\begin{cases} \dot{e} = f_o(e + x, h(x), u) - f(x, u) = \eta(e, x, u) \\ \dot{\xi} = a(\xi, H(x + e, h(x))) \\ \dot{x} = f(x, F(x + e, c(\xi, H(x + e, h(x))))) \end{cases}$$

with $u = F(x + e, c(\xi, H(x + e, h(x))))$, where $x, \tilde{x}, \xi$ are states of plant, observer, and the parameter, respectively.

By the detectability and the choice of f_o, there is a function β_o of class $\mathcal{KL}$ such that

$$\|e(t)\| \leq \beta_o(\|e(0)\|, t) \tag{5.35}$$

for all $t \in \mathbf{R}^+$, $e(0) \in \mathbf{R}^n$, $x \in \mathbf{R}^n$, and $u \in R^p$.

Consider the ξ-system, the I/S stability of $\dot{\xi} = a(\xi, y_0)$ with input $y_0 = H(x + e, h(x))$ and the growth condition on H imply that there exist a function β_q of class $\mathcal{KL}$ and a function γ_h of class $\mathcal{K}$ such that

$$\|\xi(t)\| \leq \beta_q(\|\xi(0)\|, t) + \gamma_h(\|e\|_\infty) \tag{5.36}$$

for all $t \in \mathbf{R}^+$, $x(0) \in \mathbf{R}^n$, and essentially bounded function $e : \mathbf{R}^+ \to \mathbf{R}^n$.

Now according to Sontag's argument in [170, 104], define

$$\beta_1(s,t) = \beta_q(\beta_q(s,t/2) + \gamma_h(\beta_o(s,0)), t/2) + \gamma_h(\beta_o(s,t/2)) + \beta_o(s,t).$$

Then it is easy to verify that β_1 is also of class $\mathcal{KL}$. Define $\hat{x} = \begin{bmatrix} e \\ \xi \end{bmatrix}$, from (5.35) and (5.36), it follows that

$$\|\hat{x}(t)\| \leq \beta_1(\|\hat{x}(0)\|, t), \forall t \in \mathbf{R}^+.$$

Finally, consider the x-system $\dot{x} = f(x, F(x+e, u_0))$ with $u_0 = c(\xi, H(x+e, h(x)))$. The I/S stability assumption about the system with $\begin{bmatrix} e \\ u_0 \end{bmatrix}$ as the input, the continuity of function c, and the growth condition on H together imply that there exist a function β_s of class $\mathcal{KL}$ and a function γ_κ of class $\mathcal{K}$ such that

$$\|x(t)\| \leq \beta_s(\|x(0)\|, t) + \gamma_\kappa(\|\hat{x}\|_\infty).$$

Now use the similar argument as the above, we can conclude that there is a function β of class $\mathcal{KL}$ such that

$$\|x_c(t)\| \leq \beta(\|x_c(0)\|, t), \forall t \in \mathbf{R}^+.$$

($x_c = \begin{bmatrix} \hat{x} \\ x \end{bmatrix}$). This concludes the globally asymptotic stability of the closed loop system. $\square$

Further Remarks

In general, the conditions in Theorem 5.4.10, i.e. the I/S stability condition and the growth rate condition on H, are restrictive. In the following, we will examine some examples that satisfy the conditions.

Remark 5.4.12 *A class of smoothly stabilizable (input-affine) nonlinear systems satisfy the I/S stability conditions in Theorem 5.4.10 [104, 108]. More generally, let's consider a globally (smoothly) stabilizable nonlinear system, $\dot{x} = f(x,u)$ with f smooth. By Sontag's* *arguments [170, 104], there exists a feedback law* $u = F_{I/O}(x)$ *and a smooth function* $G : \mathbf{R}^n \rightarrow \mathbf{R}^+$ *with* $0 < G(x) < C_m < \infty$ *for all* $x \in \mathbf{R}^n$ *such that the following system*

$$\dot{x} = f(x, F_{I/O}(x) + G(x)v)$$

with input v *is I/S stable, i.e., there are a function* β *of class* $\mathcal{KL}$ *and a function* γ_0 *of class* $\mathcal{K}$ *such that*

$$\|x(t)\| \leq \beta(\|x(0)\|, t) + \gamma_0(\|v\|_\infty)$$

for all $x(0) \in \mathbf{R}^n$ *and* $t \in \mathbf{R}^+$. *In addition, assume* $F_{I/O}$ *is a globally Lipschitz function with constant* $C > 0$, *and* $G(x) \geq C_g$ *with some* $C_g > 0$ *for all* $x \in \mathbf{R}^n$. *Define function* $F : \mathbf{R}^n \times \mathbf{R}^p \rightarrow \mathbf{R}^p$ *as* $F(x, w) = F_{I/O}(x) + w$. *Therefore, for* $x(t)$ *which satisfies*

$$\dot{x} = f(x, F(x + v, w)) = f(x, F_{I/O}(x) + (F_{I/O}(x + v) - F_{I/O}(x) + w)),$$

we have

$$\begin{aligned} \|x(t)\| &\leq \beta(\|x(0)\|, t) + \gamma_0(\left\|G^{-1}(x)(F_{I/O}(x + v) - F_{I/O}(x) + w)\right\|_\infty) \\ &\leq \beta(\|x(0)\|, t) + \gamma_0(\frac{1}{C_g}(C(\|v\|_\infty) + \|w\|_\infty)) \\ &\leq \beta(\|x(0)\|, t) + \gamma(\left\|\begin{bmatrix} v \\ w \end{bmatrix}\right\|_\infty) \end{aligned}$$

with $\gamma(s) := \gamma_0(\frac{1+C}{C_g}s)$ *being of class* $\mathcal{K}$, *which implies that the system* $\dot{x} = f(x, F(x + v, w))$ *with input* $\begin{bmatrix} v \\ w \end{bmatrix}$ *is I/S stable.*

Remark 5.4.13 *A class of feedback linearizable input-affine nonlinear systems satisfy the I/S stability conditions in Theorem 5.4.10 [68]. Suppose* $\dot{x} = f(x) + g(x)u$ *with* $x \in \mathbf{R}^n, u \in \mathbf{R}$ *and* f, g *smooth is exactly linearizable, i.e., there is a coordinate transformation* $z = \Phi(x)$ *(in fact,* $z_i = L_f^{i-1}(x)$*) with* $\Phi : \mathbf{R}^n \rightarrow \mathbf{R}^n$ *being a diffeomorphism such that the system under the new coordinate* $\dot{z} = \hat{f}(z) + \hat{g}(z)u$ *is as follows*

$$\begin{cases} \dot{z}_1 = z_2 \\ \cdots \\ \dot{z}_{n-1} = z_n \\ \dot{z}_n = b(z) + a(z)u \end{cases}.$$

Then by [68, Theorem 1], there is a control law $u = \mu(z)$, *such that* $\dot{z} = \hat{f}(z) + \hat{g}(z)\mu(z)$ *is globally asymptotically stable and in addition,* $\dot{z} = \hat{f}(z) + \hat{g}(z)(\mu(z + d) + w)$ *with input*

$\begin{bmatrix} d \\ w \end{bmatrix}$ *is I/S stable, i.e., there are a function* β *of class* $\mathcal{KL}$ *and a function* γ_0 *of class* $\mathcal{K}$ *such that*

$$\|z(t)\| \leq \beta(\|z(0)\|, t) + \gamma(\left\| \begin{bmatrix} d \\ w \end{bmatrix} \right\|_\infty)$$

for all $z(0) \in \mathbf{R}^n$ *and* $t \in \mathbf{R}^+$. *Now return to the original system, define* $F : \mathbf{R}^n \times \mathbf{R}^p \to \mathbf{R}^p$ $(p = 1)$ *as* $F(x, w) = \mu(\Phi(x)) + w$; *the original system becomes* $\dot{x} = f(x) + g(x)F(x + v, w)$ *and under the transformation* $z(t) = \Phi(x(t))$ *it becomes*

$$\dot{z} = \hat{f}(z) + \hat{g}(z)(\mu(z + ((\Phi(x + v) - \Phi(x))) + w).$$

As $\Phi^{-1} : \mathbf{R}^n \to \mathbf{R}^n$ *is still a diffeomorphism, it is continuous, therefore there is a function* κ *of class* $\mathcal{K}$ *such that* $\|\Phi^{-1}(z)\| \leq \kappa(\|z\|)$. *Now we additionally assume* $\Phi : \mathbf{R}^n \to \mathbf{R}^n$ *is globally Lipschitz with constant* $C > 0$, *then*

$$\begin{aligned} \|x(t)\| = \left\|\Phi^{-1}(z(t))\right\| \leq \kappa(\|z(t)\|) &\leq \kappa(\beta(\|\Phi(x(0))\|, t) + \gamma(\left\| \begin{bmatrix} \Phi(x + v) - \Phi(x) \\ w \end{bmatrix} \right\|_\infty)) \\ &\leq \kappa(\beta(C\|x(0)\|, t) + \gamma(C\|v\|_\infty + \|w\|_\infty)) \\ &\leq \beta_c(\|x(0)\|, t) + \gamma_c(\left\| \begin{bmatrix} v \\ w \end{bmatrix} \right\|_\infty), \end{aligned}$$

where $\beta_c(s, t) = \kappa(2\beta(Cs, t))$ *is again of class* $\mathcal{KL}$ *and* $\gamma_c(s) = \kappa(2\gamma(Cs + s))$ *is of class* $\mathcal{K}$.

Remark 5.4.14 *If the output function* h *is globally Lipschitz, then an* H, *which satisfies the requirement in the above theorem, can be taken as* $H(x, y) = h(x) - y$. *In fact,* $\|H(x_1, h(x_2))\| = \|h(x_1) - h(x_2)\| \leq C\|x_1 - x_2\|$ *for some* $C > 0$, *and* $\kappa_h(s) = Cs$ *is of class* $\mathcal{K}$.

From the above discussions, we see that the I/S stability conditions and growth conditions on H for the classes of systems we examined are reduced to the uniform continuity (or globally Lipschitz) conditions for some functions (as a reviewer pointed out). The global Lipschitz condition is a restrictive one.

5.5 Concluding Remarks

We have proposed a state-space approach to the parameterization of stabilizing controllers for time-invariant nonlinear systems without adopting coprime factorization technique. The central idea here is the decomposition of output feedback problem into simpler state feedback and state estimation problems. The stabilizing controllers are represented as fractional transformations of some stable parameters. Both local and global parameterizations are derived for the general nonlinear systems. These problems are treated under the assumption that the controllers have the same dimensions as the plants and have separation structures, and the observers are assumed to exist. However, in the general case, especially in the global case, the constructions of the observers are not provided.

The material in this chapter forms the papers [108, 109].

Chapter 6

Attenuation of Persistent $\mathcal{L}_\infty$-Bounded Disturbances for Nonlinear Systems

6.1 Introduction

The problem of **optimal rejection of persistent bounded disturbance** for a linear system was posed by Vidyasagar in [193]. It is a minimax optimization problem, i.e., the problem of minimization of the worst possible impact of a class of persistent bounded disturbances on the system. If the disturbance is denoted by w and the signal measuring the impact by z, and both signals are measured in $\mathcal{L}_\infty$ (in continuous time case), then the performance to be minimized is

$$J := \sup_{w \in \mathcal{L}_\infty[0,\infty), ||w||_\infty \leq 1} ||z||_\infty . \tag{6.1}$$

This problem is known as $\mathcal{L}^1$**-optimal control** problem [42, 43, 40], because the minimization (6.1) amounts to the minimization of the $\mathcal{L}_\infty$-induced norm, i.e., the $\mathcal{L}^1$-norm, of the linear system. The linear $\mathcal{L}^1$(or ℓ^1 in discrete-time case)-optimal control problem was extensively investigated in an input/output setting by using Youla-parameterization [42, 43, 40]. The relation between the $\mathcal{L}^1$-control of a linear continuous-time system and the ℓ^1-control of its Euler approximated discrete-time systems was established in [23]. Recently, for a linear discrete-time system, the corresponding ℓ^1-optimal control problem was solved in a state-space setting in terms of dynamical state-feedback [51, 40], continuous nonlinear static state-feedback [158, 160], and piece-wise linear static state-feedback [24]. Furthermore, Shamma [160] showed that if the linear ℓ^1-optimal control has any kind of solution, then there must exist a continuous (nonlinear) static state-feedback ℓ^1-controller. It is possible that by allowing the class of continuous nonlinear controllers, one can make the closed-loop $\mathcal{L}^1$-performance (6.1) strictly smaller than one can do using only linear controllers [44, 174]. Therefore, it is natural to consider such an optimal disturbance rejection problem in the nonlinear domain.

In this chapter, we will consider the problem of optimal rejection of $\mathcal{L}_\infty$-bounded disturbance for continuous-time nonlinear systems. However, in the nonlinear setting, the minimax optimization problem (6.1) is not equivalent to the minimization of the $\mathcal{L}_\infty$-induced gains of the corresponding nonlinear operators, while we will borrow the terminology **nonlinear $\mathcal{L}^1$-control** to refer to the corresponding nonlinear minimax optimization problem for convenience. The methods used in this chapter are greatly motivated by Shamma [158, 160], in which the ℓ_1-control problem for a linear discrete-time system is constructively solved in terms of continuous nonlinear static state-feedback. The main idea in the $\mathcal{L}^1(\ell^1)$-performance analysis and synthesis is to construct a certain invariant subset of the state-space such that achieving disturbance rejection is equivalent to restricting the state dynamics to this set. The techniques from viability theory, nonsmooth analysis, and set-valued analysis [9, 10, 39, 7] are extensively used; and the notion of **(controlled) invariance** [198, 7, 160] plays a central role. This treatment provides some geometrical insights into the robust ($\mathcal{L}^1$) control problem. It is remarked that the invariance notion has also been employed in other nonlinear contexts, such as the control synthesis with state and control constraints (see [78, 98, 22, 71] and references therein) and the zero dynamics [90, 8, 130].

The remainder of this chapter is organized as follows. Some mathematical preliminaries are provided in Appendix A, in which some concepts from set-valued analysis and nonsmooth analysis are reviewed, and the emphasis is on set-valued maps and contingent cones. In section 2, the $\mathcal{L}^1$-performance for a nonlinear system is analyzed. The $\mathcal{L}^1$-performance of a nonlinear system is characterized in terms of $\mathcal{L}^1$-performance domains. In section 3, the nonlinear $\mathcal{L}^1$-control synthesis problem is considered. The $\mathcal{L}^1$-control problem is characterized in terms of controlled $\mathcal{L}^1$-performance domains; a continuous static state-feedback $\mathcal{L}^1$-controller is constructed. In section 4, the (controlled) $\mathcal{L}^1$-performance domains are characterized in terms of the (controlled) invariance domains of some (controlled) differential inclusions. Some algorithms for computing the (controlled) invariance domains are provided. In section 5, the computation issues are considered, and some approximation methods are suggested. In particular, the relation between the $\mathcal{L}^1$-performance analysis and synthesis of a continuous-time system and and the ℓ^1-performance analysis and synthesis of its Euler approximated discrete-time systems is established. Those proofs that are relatively technical and less related are put in Appendix B.

6.2 $\mathcal{L}^1$-Performance Analysis of Nonlinear Systems

In this section, we will give some characterizations of the $\mathcal{L}^1$-performance for a nonlinear system. In the next section the synthesis problem is considered based on the analysis results in this section.

Consider a system with external disturbances as follows,

$$\begin{cases} \dot{x} = f(x, w) \\ z = h(x, w) \end{cases} \tag{6.2}$$

where $x \in \mathbf{R}^n$ is the state vector, $w \in \mathbf{BR}^p := \{v \in \mathbf{R}^p | \, \|v\| \leq 1\}$ and $z \in \mathbf{R}^q$ are the external disturbance input and the regulated output, respectively. Suppose that if $w(t) = 0$ and $x(0) = 0$, then $x(t) = 0$ and $z(t) = 0$. The performance which measures the property of disturbance rejection in $\mathcal{L}_\infty$ for system (6.2) was formulated by Vidyasagar in [193] as follows,

$$J = \sup_{w \in \mathcal{BL}_\infty[0,\infty)} \|z\|_\infty . \tag{6.3}$$

Note that, in the linear case, this performance J is $\|G\|_1$, i.e., the $\mathcal{L}_\infty$-induced norm of the input/output map $G : w \mapsto z$. We say that the system has a **disturbance attenuation** property if $J \leq 1$. This motivates the following definition.

Definition 6.2.1 *Consider the given system (6.2) with $x(0) = 0$. It has* $\mathcal{L}^1$-**performance** *if for all $w(t) \in \mathcal{BL}_\infty[0, \infty)$, $\|x(t)\|_\infty < \infty$ and $\|z(t)\|_\infty \leq 1$.*

Therefore, system (6.2) has $\mathcal{L}^1$-performance, if and only if it is **bounded-input-bounded-state** (BIBS) stable and $J \leq 1$. The above definition is a natural generalization of the $\mathcal{L}^1$-performance for a linear system. In the next few subsections, we will characterize the $\mathcal{L}^1$-performances.

6.2.1 $\mathcal{L}^1$-Performances and Reachable Sets

Consider system (6.2). We will assume f and h are continuous, and $f(0, 0) = 0, h(0, 0) = 0$. Therefore, $0 \in \mathbf{R}^n$ is an equilibrium of the system with $w = 0$. Moreover, we assume the admissible disturbance set is

$$\mathcal{W} := \{w : [0,\infty) \rightarrow \mathbf{BR}^p |\ w \text{ is measurable}\} = \mathcal{BL}_\infty[0,\infty). \tag{6.4}$$

We also define a subset $\mathcal{W}_c \subset \mathcal{W}$ as follows,

$$\mathcal{W}_c := \{w \in \mathcal{W} |\ w \text{ is continuous}\}. \tag{6.5}$$

It is assumed that system (6.2) has the BIBS property. Therefore, all possible solutions with the admissible inputs are in the space $\mathcal{C}(\mathbf{R}^+, \mathbf{R}^n)$. We assume that system (6.2) is **complete** in the sense that for each $w(t) \in \mathcal{W}$ and $x_0 \in \mathbf{R}^n$, the solution $x(t)$ to (6.2) starting at $x(0) = x_0$ is uniquely defined for almost every $t \in [0, \infty)$, and the solution continuously depends on the initial conditions with respect to the compact convergence topology in $\mathcal{C}(\mathbf{R}^+, \mathbf{R}^n)$. The **state transition function** $\phi : \mathbf{R}^+ \times \mathbf{R}^n \times \mathcal{W} \rightarrow \mathbf{R}^n$ is so defined that $x(T) = \phi(T, x_0, w^*)$ means that system (6.2) evolves from initial state x_0 to state x in time T under the input action w^*. Note that ϕ is well-defined and is continuous with respect to initial state because of completeness of system (6.2). We define the **reachable state maps** of system (6.2) with the admissible input set $\mathcal{W}$ and $\mathcal{W}_c$ in (6.4) as set-valued maps $\mathcal{R} : \mathbf{R}^n \rightsquigarrow \mathbf{R}^n$ and $\mathcal{R}_c : \mathbf{R}^n \rightsquigarrow \mathbf{R}^n$ with

$$\mathcal{R}(x) := \{\phi(t, x, w) | \forall w \in \mathcal{W}, t \in \mathbf{R}^+\}, \tag{6.6}$$

$$\mathcal{R}_c(x) := \{\phi(t, x, w) | \forall w \in \mathcal{W}_c, t \in \mathbf{R}^+\}. \tag{6.7}$$

Both $\mathcal{R}(x)$ and $\mathcal{R}_c(x)$ are bounded sets since system (6.2) is BIBS.

Definition 6.2.2 *A set $K \in \mathbf{R}^n$ is a* **weak invariant set** *for system (6.2) with respect to an admissible input set $\mathcal{W}$ if for all $x \in K$, and $w \in \mathcal{W}$, $\phi(t, x, w) \in K$ for almost all $t \geq 0$.*

The sets $\mathcal{R}(0)$ and $\mathcal{R}_c(0)$ have the weak invariance property. The case for $\mathcal{R}(0)$ is stated as in the following proposition.

Proposition 6.2.3 *For all $x \in \mathcal{R}(0)$, and $w \in \mathcal{W}$, $\phi(t, x, w) \in \mathcal{R}(0)$ for all $t \geq 0$. The closure $\overline{\mathcal{R}}(0)$ of $\mathcal{R}(0)$ has this weak invariance property.*

Proof. If $x \in \mathcal{R}(0)$, then by the definition of map $\mathcal{R}(0)$, there exist $w_1 \in \mathcal{W}$ and $T \in \mathbf{R}^+$ such that $x = \phi(T, 0, w_1)$. Now take $w_2 \in \mathcal{W}$, define $w : \mathbf{R}^+ \rightarrow \mathbf{BR}^p$ as

$$w(t) = \begin{cases} w_1(t) & \text{if } t \in [0, T] \\ w_2(t) & \text{if } t > T \end{cases}$$

and $w \in \mathcal{W}$. Therefore,

$$\phi(t, 0, w) \in \mathcal{R}(0), \quad \forall t \in \mathbf{R}^+.$$

In particular, if $t \geq T$,

$$\phi(t - T, x, w_2) = \phi(t - T, \phi(T, 0, w_1), w_2) = \phi(t, 0, w) \in \mathcal{R}(0).$$

To show $\overline{\mathcal{R}}(0)$ is invariant, we need to show that given $x \in \overline{\mathcal{R}}(0)$ and $w \in \mathcal{W}$, $\phi(t, x, w) \in \overline{\mathcal{R}}(0)$ for all $t \in \mathbf{R}^+$. In fact, suppose there exists a sequence $\{x_n\} \subset \mathcal{R}(0)$, such that $x_n \rightarrow x$ as $n \rightarrow \infty$. Therefore, $\phi(t, x_n, w) \in \mathcal{R}(0) \subset \overline{\mathcal{R}}(0)$ for all $t \in \mathbf{R}^+$. $\overline{\mathcal{R}}(0)$ is bounded since $\mathcal{R}(0)$ is bounded, therefore, by the completeness assumption, $\phi(t, x, w) = \lim_{n \rightarrow \infty} \phi(t, x_n, w) \in \overline{\mathcal{R}}(0)$ for all $t \in \mathbf{R}^+$. □

Next, we will characterize the $\mathcal{L}^1$-performance for system (6.2) in terms of the reachable set. We first give a weaker definition as follows,

Definition 6.2.4 *Consider the given system (6.2) with $x(0) = 0$. It has* **weak $\mathcal{L}^1$-performance** *if for all $w(t) \in \mathcal{W}_c$, $\|x(t)\|_\infty < \infty$ and $\|z(t)\|_\infty \leq 1$.*

Therefore, if system (6.2) has weak $\mathcal{L}^1$-performance, then

$$J_W = \sup_{w \in \mathcal{W}_c} \|z\|_\infty \leq 1. \tag{6.8}$$

As system (6.2) is BIBS, then there exists a compact set $\mathbf{X} \subset \mathbf{R}^n$, such that $\phi(t, 0, w) \in \mathbf{X}$ for all $w(t) \in \mathcal{W}$. Define a closed set as follows,

$$\Omega := \{x \in \mathbf{X} | \, \|h(x, w)\| \leq 1, \forall w \in \mathbf{BR}^p\}. \tag{6.9}$$

Then Ω is **bounded**. We immediately have the following assertion.

Theorem 6.2.5 *The system (6.2) has $\mathcal{L}^1$-performance $J \leq 1$ if and only if $\mathcal{R}(0) \subset \Omega$. It has weak $\mathcal{L}^1$-performance $J_W \leq 1$ if and only if $\mathcal{R}_c(0) \subset \Omega$.*

Furthermore, the optimal performance J is given by

$$J = \sup\{\|h(x, w)\| \, | x \in \overline{\mathcal{R}}(0), w \in \mathbf{BR}^p\}.$$

It is noted that in general the reachable set $\mathcal{R}(0)$ and $\mathcal{R}_c(0)$ are not easily computable by the definitions. In the next subsection, we will give some alternative characterizations in terms of the notion of invariance for differential inclusions.

6.2.2 $\mathcal{L}^1$-Performance Domains

The nonlinear systems with $\mathcal{L}^1$-performances can be described with the aid of a differential inclusion. Indeed, let's consider system (6.2), define a set-valued map $F : \mathbf{R}^n \rightsquigarrow \mathbf{R}^n$ as

$$F(x) := \{f(x,w)|w \in \mathbf{BR}^p\} \tag{6.10}$$

with the domain $\textsc{Dom}(F) = \Omega$. It is noted that all solutions to the differential equation $\dot{x} = f(x,w)$ with $w(t) \in \mathcal{W}$ are the solutions of the following differential inclusion,

$$\dot{x} \in F(x). \tag{6.11}$$

However, in general, not all solutions of differential inclusion $\dot{x} \in F(x)$ are the admissible solutions for the system (6.2) for some $w \in \mathcal{W}$; therefore, these two descriptions are not equivalent. We first have the following definition. The contingent cone of a set is defined in Section 6.7.2.

Definition 6.2.6 *Consider system (6.2); the bounded set Ω is defined as in (6.9). A closed set $K \in \Omega$ is an $\mathcal{L}^1$-**performance domain** for system (6.2) if $0 \in K$ and for all $x \in K$ and $w \in \mathbf{BR}^p$,*

$$f(x,w) \in T_K(x), \tag{6.12}$$

where $T_K(x)$ is the contingent cone of set K at x.

It will be seen that an $\mathcal{L}^1$-performance domain is a nonempty invariance domain of its corresponding differential inclusion (6.10)-(6.11) (see Section 6.4.1).

We first have the following theorem about the weak $\mathcal{L}^1$-performance.

Theorem 6.2.7 *Consider system (6.2). It has the weak $\mathcal{L}^1$-performance if and only if there exists an $\mathcal{L}^1$-performance domain for system (6.2).*

The proof of this theorem is given in the next subsection. As for the $\mathcal{L}^1$-performance, we first have the following assertion for a class of nonlinear systems.

Theorem 6.2.8 *Consider system (6.2).*

(i) It has $\mathcal{L}^1$-performance, then there exists an $\mathcal{L}^1$-performance domain.

(ii) If $f(x,w)$ is locally Lipschitz in $x \in \mathbf{R}^n$, then system (6.2) has the $\mathcal{L}^1$-performance if and only if there exists an $\mathcal{L}^1$-performance domain.

The proof of this theorem is given in the next subsection. In the following, we will mainly consider the case of interest in the sequel, where system (6.2) is affine in w. As a result, the Lipschitz property in Theorem 6.2.8 (ii) is not required. More concretely, we consider the following system,

$$\begin{cases} \dot{x} = f(x) + g(x)w \\ z = h(x,w) \end{cases}, \tag{6.13}$$

i.e., the function $f(x,w)$ in (6.2) is replaced by $f(x)+g(x)w$; the other assumptions on $f(x,w)$ are also imposed on $f(x)+g(x)w$.

Theorem 6.2.9 *Consider system (6.13) which is affine in w. The following statements are equivalent.*

(i) There exists an $\mathcal{L}^1$-performance domain for system (6.13).

(ii) System (6.13) has weak $\mathcal{L}^1$-performance.

(iii) System (6.13) has $\mathcal{L}^1$-performance.

The proof will be given in the next subsection. Theorems 6.2.5 and 6.2.9 imply that if $\overline{\mathcal{R}}(0) \subset \Omega$, then it is an $\mathcal{L}^1$-performance domain for system (6.13). In fact, it is the smallest $\mathcal{L}^1$-performance domain (Section 6.4.1). We next give an algorithm to compute the optimal performance J in (6.3) by using the bisection method (which is used in computing $\mathcal{H}_\infty$-performance [26]).

Given $\epsilon > 0$, one needs to find a $\gamma^* > 0$ such that $\gamma^* - \epsilon \leq J \leq \gamma^* + \epsilon$. Let $\gamma > 0$, define

$$D_\gamma := \{x \in \mathbf{R}^n | \, \|h(x,w)\| \leq \gamma, \forall w \in \mathbf{BR}^p\} \tag{6.14}$$

and let $\mathrm{D{\small INV}}(D_\gamma)$ be the largest invariance domain of the differential inclusion (6.10)-(6.11) in D_γ (see Section 6.4.1).

Algorithm 6.2.10 *Give $\gamma_M > \gamma_m \geq 0$ such that $\gamma_m \leq J \leq \gamma_M$.*

Step 1: *If $\gamma_M - \gamma_m \leq 2\epsilon$, let $\gamma^* = (\gamma_M + \gamma_m)/2$, then stop; otherwise go to step 2.*

Step 2: *Let $\gamma = (\gamma_M + \gamma_m)/2$ and compute $K_\gamma := \text{DINV}(D_\gamma)$.*

Step 3: *If $0 \in K_\gamma$, then redefine $\gamma_M := \gamma$; otherwise let $\gamma_m := \gamma$. Go to step 1.*

The above algorithm can be used to get an approximation of optimal $\mathcal{L}^1$-performance for system (6.13).

6.2.3 Proofs of the Main Theorems

Next, we will prove Theorems 6.2.7, 6.2.8, and 6.2.9. The techniques used in the proofs are basically from [7, 179, 201].

PROOF OF THEOREM 6.2.7

The following lemma from [9, 201] will be used in the following discussion.

Lemma 6.2.11 *Consider a differential equation $\dot{x} = \psi(x, t)$ with $\psi : \mathbf{R}^n \times \mathbf{R}^+ \rightarrow \mathbf{R}^n$ being continuous. Suppose a set $K \subset \mathbf{R}^n$ is closed. If $\psi(x, t) \in T_K(x)$ for all $x \in K$ and $t \in \mathbf{R}^+$, then for any $x_0 \in K$, there exists a solution $x(t)$ to the differential equation starting at x_0 which is viable in K, i.e. $x(t) \in K$ for almost all $t \in [0, \infty)$.*

Proof. **[Theorem 6.2.7]**

[Necessity] Let $K := \overline{\mathcal{R}}_c(0) \subset \Omega$, we now show it is an $\mathcal{L}^1$-performance domain. We need to show that for all $w_0 \in \mathbf{BR}^p$ and $x_0 \in K$, $f(x_0, w_0) \in T_K(x)$. In fact, given $T > 0$, one has $x(t) = \phi(t, x_0, w(t)) \in K$ for all $t \in [0, T]$ where $w(t) \in \mathcal{W}_c$ with $w(0) = w_0$ because of the weak invariance of the set $\overline{\mathcal{R}}_c(0)$. Therefore,

$$x(t) = x_0 + \int_0^t \dot{x}(s)ds = x_0 + t(\frac{1}{t}\int_0^t f(x(s), w(s))ds).$$

Notice that $f(x(s), w(s))$ is bounded in $[0, T]$, then by Lebesgue's differentiation theorem, one has

$$\lim_{t \rightarrow 0} \frac{1}{t}\int_0^t f(x(s), w(s))ds = f(x_0, w_0).$$

Therefore, one can find two sequences $\{t_n\}$ and $\{v_n\}$ with $t_n \rightarrow 0^+$ and $v_n \rightarrow f(x_0, w_0)$ as $n \rightarrow \infty$, such that $x_0 + t_n v_n \in K$ for all $n \in \mathbf{Z}^+$. Hence, $f(x_0, w_0) \in T_K(x_0)$, the conclusion then follows by Lemma 6.7.5.

[Sufficiency] Suppose $K \subset \Omega$ is an $\mathcal{L}^1$-performance domain. Given $w(t) \in \mathcal{W}_c$, consider the following time-varying differential equation,

$$\dot{x} = f(x, w(t)) =: f_w(x, t).$$

Note that the function $f_w : \mathbf{R}^n \times \mathbf{R}^+ \to \mathbf{R}^n$ is continuous; and by assumption, $f_w(x,t) \in T_K(x)$ for all $x \in K$ and $t \in \mathbf{R}^+$, then by Lemma 6.2.11, for all $x \in K$, $\phi(t, x, w) \in K$ for all $t \geq 0$. In particular, $\mathcal{R}_c(0) \subset K \subset \Omega$. □

PROOF OF THEOREM 6.2.8

Theorem 6.2.8 basically follows from [201]. We first restate a result from [201] which is used in the proof. The weak invariance is defined in Definition 6.2.3.

Lemma 6.2.12 ([201, Theorem 3.9]) *Consider system (6.2)*

(i) If K is a closed weak invariant set with respect to $\mathcal{W}$, then $f(x,w) \in T_K(x)$ for all $x \in K$ and $w \in \mathbf{BR}^p$.

(ii) If function $f(x,w)$ is locally Lipschitz in x, K is a closed set, and $f(x,w) \in T_K(x)$ for all $x \in K$ and $w \in \mathbf{BR}^p$, then both K and its interior are weak invariant sets.

Proof. **[Theorem 6.2.8]** (i) Suppose system (6.2) has $\mathcal{L}^1$-performance $J \leq 1$, by Theorem 6.2.5, one has that the closure $K := \bar{\mathcal{R}}(0)$ of $\mathcal{R}(0)$ belongs to Ω, since Ω is closed. By Proposition 6.2.3 and Lemma 6.2.12 (i), one has that for all $x \in K$,

$$f(x,w) \in T_K(x), \quad \forall w \in \mathbf{BR}^p,$$

or $F(x) \subset T_K(x)$. Thus, K is an invariance domain for F, which is closed, and $0 \in K$. Therefore $0 \in K \subset \textsc{Dinv}(\Omega)$.

(ii) The necessity is proved in (i), only the sufficiency is proved here. Suppose $K := \textsc{Dinv}(\Omega) \ni 0$, then $F(x) \subset T_K(x)$, or

$$f(x,w) \in T_K(x), \quad \forall w \in \mathbf{BR}^p.$$

Then by Lemma 6.2.12 (ii), one has that for all $x \in K$, and $w \in \mathcal{W}$, $\phi(t, x, w^*) \in K$ for all $t \geq 0$; in particular, $\mathcal{R}(0) \subset K \subset \Omega$. Then the assertion follows from Theorem 6.2.5. □

PROOF OF THEOREM 6.2.9

The following lemma is needed in the proof of Theorem 6.2.9.

Lemma 6.2.13 *Let continuous functions $f(x)$ and $g(x)$ be defined in (6.13). Given $\gamma > 0$ and $T > 0$ and $w(t) \in \mathcal{W}$, define*

$$\delta(x, w(t), \gamma) := \sup\{\|f(x) + g(x)w(t) - f(y) - g(y)w(t)\| \,|\forall x, y \in \Omega, \|x - y\| \leq \gamma\} \quad (6.15)$$

for almost every $t \in [0, T]$. Then map $t \mapsto \delta(x(t), w(t), \gamma)$ is measurable on $[0, T]$. Moreover, let $\{\gamma_n\}$ be a positive decreasing sequence converging to zero, and $\{\tau_n\}$ be a positive sequence converging to zero. Then

$$\lim_{n\to\infty} \frac{1}{\tau_n} \int_t^{t+\tau_n} \delta(x(s), w(s), \gamma_n) ds = 0 \tag{6.16}$$

almost everywhere in $[0, T]$.

Proof. It is obvious that the map $t \mapsto \delta(x(t), w(t), \gamma)$ is measurable on $[0, T]$. It is noted that for almost $t \in [0, T]$, and $\gamma_1 \geq \gamma_2 > 0$ $\delta(x(t), w(t), \gamma_1) \geq \delta(x(t), w(t), \gamma_2)$, and

$$\lim_{\gamma\to 0} \delta(x(t), w(t), \gamma) = 0.$$

Now let $\{\gamma_n\}$ is a decreasing sequence converging to zero. We will show (6.16) holds almost everywhere in $[0, T]$. In fact, by Lebesgue's differentiation theorem, one has

$$\lim_{\tau\to o^+} \frac{1}{\tau} \int_t^{t+\tau} \delta(x(s), w(s), \gamma_n) ds = \delta(x(t), w(t), \gamma_n) \tag{6.17}$$

almost everywhere in $[0, T]$ for each $n \in \mathbf{Z}^+$. Therefore, (6.17) holds almost everywhere in $[0, T]$ for all $n \in \mathbf{Z}^+$, since $\mathbf{Z}^+$ is countable.

Now take $t \in [0, T]$ such that (6.17) holds for all $n \in \mathbf{Z}^+$. Let $\epsilon > 0$ be given. Then we can find $n_1, n_2 \in \mathbf{Z}^+$, such that $\delta(x(t), w(t), \gamma_{n_1}) < \epsilon/2$, and

$$\frac{1}{\tau_n} \int_t^{t+\tau_n} \delta(x(s), w(s), \gamma_{n_1}) ds < \delta(x(t), w(t), \gamma_{n_1}) + \epsilon/2$$

for $n \geq n_2$. Therefore, if $n \geq \max\{n_1, n_2\}$, we have

$$\frac{1}{\tau_n} \int_t^{t+\tau_n} \delta(x(s), w(s), \gamma_n) ds < \frac{1}{\tau_n} \int_t^{t+\tau_n} \delta(x(s), w(s), \gamma_{n_1}) ds$$

$$< \delta(x(t), w(t), \gamma_{n_1}) + \epsilon/2 < \epsilon.$$

$\square$

Proof. **[Theorem 6.2.9]**

It is noted that the implication $(iii) \Rightarrow (ii)$ is obvious, and $(ii) \Rightarrow (i)$ follows from theorem 6.2.7. We only need to show $(i) \Rightarrow (iii)$. Suppose $K \subset \Omega$ is a compact $\mathcal{L}^1$-performance domain. It is sufficient to show that if $x_0 \in K$, then for all $w \in \mathcal{W}$, $\phi(t, x_0, w) \in K$ for all $t \in [0, \infty)$. The proof is divided into two steps.

Claim 1: Given $T > 0$, for the given $w \in \mathcal{W}$, $\phi(t, x, w) \in K$ for all $t \in [0, T]$.

Indeed, take $\tau > 0$, define a function $f_\tau : K \times [0, T] \rightarrow \mathbf{R}^n$ as follows,

$$f_\tau(x, t) = \frac{1}{\tau} \int_t^{t+\tau} (f(x) + g(x)w(s))ds = f(x) + g(x)w_\tau(t), \tag{6.18}$$

where

$$w_\tau(t) := \frac{1}{\tau} \int_t^{t+\tau} w(s)ds,$$

which is continuous on $[0, T]$. In fact, for all $t_1, t_2 \in [0, T]$,

$$\|w_\tau(t_2) - w_\tau(t_1)\| = \frac{1}{\tau} \left\| \int_{t_1+\tau}^{t_2+\tau} w(s)ds - \int_{t_1}^{t_2} w(s)ds \right\| \leq \frac{2}{\tau} \|t_2 - t_1\|,$$

since $w(s) \in \mathbf{BR}^p$ for almost all $s \in [0, T]$. Therefore, f_τ is continuous on the compact set $K \times [0, T]$. because of the continuity of f and g, we can assume $\|f(x) + g(x)w\| \leq \beta$ with some $\beta > 0$ for all $(x, w) \in K \times \mathbf{BR}^p$. Therefore, by (6.18), it follows that $\|f_\tau(x, t)\| \leq \beta$. Note that

$$\|w_\tau(t)\| \leq \frac{1}{\tau} \int_t^{t+\tau} \|w(s)\| ds \leq 1$$

for all $t \in [0, T]$. Therefore, $w_\tau \in \mathcal{W}_c \subset \mathcal{W}$. Then one has

$$f_\tau(x, t) = f(x) + g(x)w_\tau(t) \in T_K(x)$$

for all $x \in K$ $t \in \mathbf{R}^+$. By Lemma 6.2.11, the solution $x_\tau(t)$ to $\dot{x}_\tau = f_\tau(x_\tau, t)$ for $t \in [0, T]$ is viable in the compact set K, i.e., $x_\tau(t) \in K$ for $t \in [0, T]$.

On the other hand, one has that $\|\dot{x}_\tau\| = \|f_\tau(x, t)\| \leq \beta$. Take a sequence $\{x_n(\cdot)\} := \{x_{\tau_n}(\cdot)\}$, where $\tau_n \rightarrow 0$ as $n \rightarrow \infty$, then the sequence is equicontinuous. Then by Ascoli's Theorem, the sequence remains in a compact subset of the Banach space $\mathcal{C}([0, T], \mathbf{R}^n)$; therefore, there exists a subsequence, denoted as $\{x_n\}$ without loss of generality, which converges uniformly on [0,T] to an absolutely continuous function x which is viable in K since K is closed. Furthermore, the sequence $\{\dot{x}_n\}$ converges to $\dot{x}$ because $\dot{x}_n = f_\tau(x_n(t), t)$ and f is uniformly continuous on the compact set $K \times [0, T]$.

A sequence $\{\gamma_n\}$ is chosen as follows,

$$\gamma_n := \|x_n(t) - x(t)\| + \beta\tau_n$$

such that $\{\gamma_n\}$ is decreasingly converges to zero (otherwise, we can choose a decreasingly subsequence instead), and for all $s \in [t, t+\tau_n]$,

$$\|x_n(t) - x(s)\| \leq \|x_n(t) - x(t)\| + \beta\tau_n = \gamma_n.$$

Given $\epsilon > 0$, by Lemma 6.2.13, there exists an $n_0 \in \mathbf{Z}^+$, such that if $n \geq n_0$,

$$\begin{aligned} &\left\|\dot{x}_n(t) - \frac{1}{\tau_n}\int_t^{t+\tau_n}(f(x(s)) + g(x(s))w(s))ds\right\| \\ = &\frac{1}{\tau_n}\left\|\int_t^{t+\tau_n}(f(x_n(s)) + g(x_n(s))w(s))ds - \int_t^{t+\tau_n}(f(x(s)) + g(x(s))w(s))ds\right\| \\ \leq &\frac{1}{\tau_n}\int_t^{t+\tau_n}\delta(x(s), w(s), \gamma_n)ds < \epsilon. \end{aligned}$$

By Lebesgue's differentiation theorem, one has

$$\lim_{n\to\infty}\frac{1}{\tau_n}\int_t^{t+\tau_n}(f(x(s)) + g(x(s))w(s))ds = f(x(t)) + g(x(t))w(t),$$

since $f(x(s)) + g(x(s))w(s)$ is bounded, is thus integrable on [0,T]. Therefore,

$$\dot{x}(t) = f(x(t)) + g(x(t))w(t), \quad x(0) = x_0,$$

almost everywhere in $[0, T]$. By the completeness of system (6.13), $\phi(t, x_0, w) = x(t) \in K$ for all $t \in [0, T]$.

Claim 2: The viable solution $x(t)$ in K can be extended to $[0, \infty)$.

In fact, by Zorn's Lemma, one can extend the viable solution $x(t)$ in K to the interval $[0, T_{max})$ for some $T_{max} \geq T$ [201], i.e.,

$$x(t) \in K, \quad \forall t \in [0, T_{max})$$

and T_{max} is such a maximal number. Now, we show $T_{max} = \infty$. In fact, if not so, define

$$C := \limsup_{t\to T_{max}^-}\|x(t)\| < \infty,$$

as K is closed and bounded. Thence, $\|x(t)\| \leq C + 1$ for $t \in [T_{max} - \tau, T_{max}]$ with some $\tau > 0$. Since $\|w(t)\| \leq 1$, then $\|\dot{x}(t)\| = \|f(x(t)) + g(x(t))w(t)\| \leq \beta$. Therefore, for all $t_1, t_2 \in [T_{max} - \tau, T_{max}]$ with $t_1 \leq t_2$,

$$\|x(t_1) - x(t_2)\| \leq \int_{t_1}^{t_2}\|\dot{x}(s)\|\, ds \leq \beta(t_2 - t_1).$$

Therefore, the Cauchy criterion implies that $\lim_{t\to T_{max}^-} x(t)$ exists. Let $x(T_{max}) := \lim_{t\to T_{max}^-} x(t) \in K$. Since

$$x(t) = x_0 + \int_0^t \dot{x}(s)ds,$$

let $t\to T_{max}^-$, one has

$$x(T_{max}) = x_0 + \int_0^{T_{max}} \dot{x}(s)ds.$$

Then the solution can be extended to $[0, T_{max}]$. Now $x(T_{max}) \in K$, the same argument as in **Step 1** shows that there exists $T_0 > 0$ such that $x(t)$ can be extend to $[0, T_{max} + T_0)$. This leads to a contradiction about the maximality of T_{max}. Therefore, $T_{max} = \infty$. □

6.3 $\mathcal{L}^1$-Control of Nonlinear Systems

In this section, we will consider the nonlinear $\mathcal{L}^1$-control synthesis problem based on the characterizations of $\mathcal{L}^1$-performance in the previous section. A static state-feedback $\mathcal{L}^1$-controller is constructed for a nonlinear system.

Consider the following input-affine system,

$$\begin{cases} \dot{x} = f(x) + g_1(x)w + g_2(x)u \\ z = h(x) + k_1(x)w + k_2(x)u \end{cases} \tag{6.19}$$

where $x \in \mathbf{R}^n$ is the state vector, $w \in \mathbf{BR}^p := \{v \in \mathbf{R}^p | \, \|v\| \leq 1\}$, $u \in \mathbf{R}^m$, and $z \in \mathbf{R}^q$ are the external disturbance input, the control input, and the regulated output, respectively. We will assume f, g_1, g_2, h, k_1, and k_2 are continuous on $\mathbf{R}^n$, $\textsc{Rank}(g_1(x)) = n$ and $\textsc{Rank}(g_2(x)) = m$ for all $x \in \mathbf{R}^n$, and $f(0) = 0$ and $h(0) = 0$. Therefore, $0 \in \mathbf{R}^n$ is an equilibrium of the system with $w = 0$ and $u = 0$. Moreover, we assume the admissible disturbance set for system (6.19) is

$$\mathcal{W} := \{w \in \mathcal{L}_\infty[0,\infty) | \, \|w\|_\infty \leq 1\} = \mathcal{BL}_\infty[0,\infty). \tag{6.20}$$

The $\mathcal{L}^1$-control problem for system (6.19) is defined as follows,

Definition 6.3.1 *The state-feedback* $\mathcal{L}^1$**-control** *synthesis problem is to find a continuous state-feedback $u = \psi(x)$ for system (6.19) such that the resulting closed-loop system has the $\mathcal{L}^1$-performance.*

Define a set-valued map $U : \mathbf{R}^n \rightsquigarrow \mathbf{R}^m$ as follows,

$$U(x) := \{u \in \mathbf{R}^m | \, \|z\| = \|h(x) + k_1(x)w + k_2(x)u\| \leq 1, \forall w \in \mathbf{BR}^p\} \tag{6.21}$$

with domain

$$\text{Dom}(U) := \{x \in \mathbf{R}^n | U(x) \neq \emptyset\}.$$

Therefore, if $u = \psi(x)$ is an admissible $\mathcal{L}^1$-controller, then it necessarily satisfies $\psi(x) \in U(x)$ for all $x \in \text{Dom}(U)$. We thus define the set of admissible (state-feedback) controllers for the system G as follows,

$$\mathcal{K} := \{\psi : \mathbf{R}^n \rightarrow \mathbf{R}^m | \ \psi \text{ is continuous on } \text{Dom}(U) \text{ and}$$

$$\psi(x) \in U(x) \text{ for all } x \in \text{Dom}(U) \text{ with } \psi(0) = 0\}. \tag{6.22}$$

Let $F : \text{Graph}(U) \rightsquigarrow \mathbf{R}^n$ be another set-valued map defined as follows,

$$F(x, u) := \{f(x) + g_1(x)w + g_2(x)u | w \in \mathbf{BR}^p\}. \tag{6.23}$$

One immediately has the following observations. The upper semi-continuous (USC), lower semi-continuous (LSC), and Marchaud maps are defined in Appendix A.

Lemma 6.3.2 *(i) The set-valued map $U : \mathbf{R}^n \rightsquigarrow \mathbf{R}^m$ defined in (6.21) is locally bounded, USC, as well as LSC with closed values; and* $\text{Dom}(U)$ *is closed.*

(ii) The set-valued map $F : \text{Graph}(U) \rightsquigarrow \mathbf{R}^n$ defined in (6.23) is USC; if in addition $\text{Dom}(U)$ *is bounded, then it is Marchaud.*

Define $F_c : \mathbf{R}^n \rightsquigarrow \mathbf{R}^n$ by

$$F_c(x) := \bigcup_{u \in U(x)} F(x, u), \tag{6.24}$$

then the differential inclusion $\dot{x} \in F_c(x)$, which is derived from system (6.19), is a controlled differential inclusion (F, U) defined by (6.21) and (6.23). We first have the following definition.

Definition 6.3.3 *Consider system (6.19) and its corresponding controlled differential inclusion (F, U) defined by (6.21) and (6.23). Suppose $K \subset \text{Dom}(U)$ is closed, then K is a* **controlled $\mathcal{L}^1$-performance domain** *if $0 \in K$ and for each $x \in K$ there exists $u \in U(x)$ such that*

$$F(x, u) \subset T_K(x). \tag{6.25}$$

It will be shown in the next section that a controlled $\mathcal{L}^1$-performance domain for system (6.19) is a closed controlled invariance domain of (F, U). Next, we will characterize the solvability of $\mathcal{L}^1$-control synthesis problem in terms of the controlled $\mathcal{L}^1$-performance domains. It is assumed that system (6.19) is **complete** in the sense that for each $w(t) \in \mathcal{W}$, $u(t) \in \mathcal{L}_\infty[0,\infty)$ or $u = \psi(x) \in \mathcal{K}$, and $x_0 \in \mathbf{R}^n$, the solution $x(t)$ to (6.19) starting at $x(0) = x_0$ is uniquely defined for almost every $t \in [0,\infty)$, and the solutions are continuously dependent on the initial states. We first have the following assertion.

Theorem 6.3.4 (Necessary Conditions) *Consider system (6.19). If the $\mathcal{L}^1$-control problem has a static state-feedback solution, then there exists a controlled $\mathcal{L}^1$-performance domain.*

Proof. If the $\mathcal{L}^1$-control problem has a state-feedback solution, then there exists a state feedback $\psi \in \mathcal{K}$ such that the following closed-loop system

$$\begin{cases} \dot{x} = f(x) + g_2(x)\psi(x) + g_1(x)w \\ z = h(x) + k_2(x)\psi(x) + k_1(x)w \end{cases} \tag{6.26}$$

has the $\mathcal{L}^1$-performance. Define a set

$$\Omega_c := \{x \in \mathbf{R}^n |\, \|h(x) + k_2(x)\psi(x) + k_1(x)w\| \leq 1, \forall w \in \mathbf{BR}^p\}.$$

Let $\textsc{Dinv}(\Omega_c)$ be the largest $\mathcal{L}^1$-performance domain of the closed-loop system (6.26) contained in Ω. By Theorem 6.2.8, $0 \in \textsc{Dinv}(\Omega_c) \neq \emptyset$; and moreover, $\textsc{Dinv}(\Omega_c)$ is a controlled $\mathcal{L}^1$-performance domain for the original system by Definition 6.3.3. $\square$

It is noted that, the above theorem holds for more general class of nonlinear systems in addition to the class of input-affine systems. Next, we will give a sufficient condition for a modified $\mathcal{L}^1$-control problem to have a solution. Consider system (6.19). For $\epsilon \in [0,1)$, define the following performance,

$$J^\epsilon = \sup_{w \in \mathcal{L}_\infty[0,\infty), \|w\|_\infty \leq 1-\epsilon} \|z\|_\infty . \tag{6.27}$$

We will construct a state-feedback $u(\cdot) \in \mathcal{K}$ such that the closed-loop system satisfies $J^\epsilon \leq 1$ for any $\epsilon \in (0,1)$. We have the following theorem. The sleek sets are defined in Section 6.7.2.

Theorem 6.3.5 (Sufficient Conditions) *If there exists a sleek compact controlled $\mathcal{L}^1$-performance domain for system (6.19), then for all $\epsilon \in (0,1)$, there exists a continuous static state feedback such that the closed-loop system satisfies $J^\epsilon \leq 1$.*

Proof. Suppose $K \neq \emptyset$ is a sleek compact controlled $\mathcal{L}^1$-performance domain for system (6.19). It is sufficient to construct a state-feedback $\mathcal{L}^1$-controller $u = \phi \in \mathcal{K}$ such that the following modified system

$$\begin{cases} \dot{x} = f(x) + (1-\epsilon)g_1(x)w + g_2(x)u \\ z = h(x) + (1-\epsilon)k_1(x)w + k_2(x)u \end{cases} \tag{6.28}$$

with the constructed controller achieves $\mathcal{L}^1$-performance.

As $K \in \textsc{Dom}(U)$ is sleek, $T_K : K \rightsquigarrow \mathbf{R}^n$ is LSC with closed convex values. Define a set-valued map $T^\epsilon : K \rightsquigarrow \mathbf{R}^n$ as

$$T^\epsilon(x) := \{\xi | \xi + (1-\epsilon)g_1(x)w \in T_K(x), \forall w \in \mathbf{BR}^p\}. \tag{6.29}$$

It is easy to see that set-valued map T^ϵ is LSC with closed convex values on K. Define the (allowable control) set-valued map $C^\epsilon : K \rightsquigarrow \mathbf{R}^n$ as

$$C^\epsilon(x) := \{u \in U(x) | f(x) + g_2(x)u \in T^\epsilon(x)\}.$$

It can also be seen that the set-valued map C^ϵ has closed convex values on K.

On the other hand, we claim that there exists an $\alpha > 0$ such that for all $x \in K$, there exists a $u \in C^\epsilon(x)$ such that

$$f(x) + g_2(x)u + r \in T^\epsilon(x)$$

for all $r \in \mathbf{R}^n$ such that $\|r\| \leq \alpha$. In fact, since $T^0(x) \subset T^\epsilon(x)$ for $0 < \epsilon$, we have $C^0(x) \subset C^\epsilon(x)$. Also $K \neq \emptyset$ implies $C^0(x) \neq \emptyset$ for all $x \in K$. Therefore, there exists $u \in C^0(x) \subset C^\epsilon(x)$, such that

$$f(x) + g_2(x)u \in T^0(x),$$

or

$$f(x) + g_2(x)u + \epsilon g_1(x)w \in T^\epsilon(x), \quad \forall w \in \mathbf{BR}^p.$$

Then the claim is justified since $g_1(x)$ has rank n on the compact set K.

By employing Theorem 6.7.3, we can immediately deduce that the set-valued map C^ϵ is LSC. Furthermore, it can be verified that $0 \in C^\epsilon(0)$. We now use Michael's selection theorem (Proposition 6.7.4) to conclude that there exists a continuous selection $\psi : K \rightarrow \mathbf{R}^m$ of set-valued map $C^\epsilon : K \rightsquigarrow \mathbf{R}^n$ with $\psi(0) = 0$, then $\psi \in \mathcal{K}$.

Now we claim that the state feedback $u = \psi(x)$ is the desired controller. Indeed, the closed-loop system is

$$\begin{cases} \dot{x} = f(x) + g_2(x)\psi(x) + (1-\epsilon)g_1(x)w \\ z = h(x) + k_2(x)\psi(x) + (1-\epsilon)k_1(x)w \end{cases} . \tag{6.30}$$

From the construction, we know that for all $x \in K$ and $w \in \mathbf{BR}^p$,

$$f(x) + g_2(x)\psi(x) + (1-\epsilon)g_1(x)w \in T_K(x).$$

Thus the sleek set $K \neq \emptyset$ is an $\mathcal{L}^1$-performance domain for the above closed-loop system (6.30). Thus, Theorem 6.2.9 shows that the closed-loop system (6.30) has $\mathcal{L}^1$-performance. Therefore, the resulting controller for system (6.19) yields $J^\epsilon \leq 1$. □

6.4 $\mathcal{L}^1$-Control and (Controlled) Invariance

In this section, we will characterize the (controlled) $\mathcal{L}^1$-performance domains in terms of corresponding (controlled) differential inclusions. The notions of (controlled) invariance play a central role.

6.4.1 Differential Inclusions and Invariance Domains

Given a set-valued map $F : X \rightsquigarrow X$, we mainly consider, in this subsection, the following differential inclusion

$$\dot{x}(t) \in F(x(t)), \text{ for almost all } t \in [0, \infty). \tag{6.31}$$

A function $x : \mathbf{R}^+ \rightarrow X$ is said to be **viable** in a subset $K \subset X$ if $x(t) \in K$ for all $t \in \mathbf{R}^+$ [7]. We have the following definition.

Definition 6.4.1 *Consider differential inclusion (6.31). The subset $K \subset X$ is said to be* **invariant** *under F if for all $x_0 \in K$, any solution to (6.31) starting at x_o is viable in K. Given any closed subset $\Omega \subset \mathrm{DOM}(F)$. The largest closed subset of Ω which is invariant under F, denoted by $\mathrm{INV}_F(\Omega)$[1], is called the* **invariance kernel** *(IK) of Ω. The smallest closed subset of X invariant under F containing Ω is called* **invariant envelope** *(IE) of Ω, denoted as $\mathrm{ENV}_F(\Omega)$[1].*

We first have the following lemma.

[1] If clear from context, the subscription F will be dropped.

Lemma 6.4.2 *The class of invariance subsets under F is closed under the operation of subset union.*

The above lemma implies that the invariance kernel, if exists, is unique; The invariant envelope, which always exists by Zorn's lemma, is also unique. It is known that if F is Lipschitz, then there exists an invariance kernel for closed subset $\Omega \subset \mathrm{DOM}(F)$ [7, Theorem 5.4.2]. The following theorem gives another class of such nonlinear systems of interest in this chapter.

Theorem 6.4.3 *Suppose the set-valued map $F : \mathrm{DOM}(F) \rightsquigarrow X$ is Marchaud. Then, for any closed subset $\Omega \subset \mathrm{DOM}(F)$, there exists an IK (possibly empty) of Ω. It is the subset of initial points such that all solutions starting from them are viable in Ω.*

The proof of this theorem is given in Appendix B. Definition 6.4.1 can hardly be conveniently implemented for checking the invariance sets and computing the invariance kernels. We next give an alternative notion.

Definition 6.4.4 *Let $F : X \rightsquigarrow X$ be given. $K \subset \mathrm{DOM}(F)$ is an* **invariance domain** *(ID) of F if for all $x \in K$, $F(x) \subset T_K(x)$. Given any closed subset $\Omega \subset \mathrm{DOM}(F)$. We denote by $\mathrm{DINV}_F(\Omega)$ the largest closed invariance domain under F in Ω, and by $\mathrm{DENV}_F(\Omega)$ the smallest closed invariance domain of F containing Ω.*

Recall the definition of the $\mathcal{L}^1$-performance domains in Section 2, it is known from the above definition that any $\mathcal{L}^1$-performance domain is an invariance domain of the corresponding differential inclusion (6.10) and (6.11). Therefore, $\mathrm{DINV}_F(\Omega)$ exists for a class of parameterized set-valued maps $F : \mathbf{R}^n \rightsquigarrow \mathbf{R}^n$. From Theorem 6.2.9, we immediately have the following theorem which characterizes the $\mathcal{L}^1$-performance of system (6.2) in terms of the invariance domains.

Theorem 6.4.5 *Consider system (6.2) and its corresponding differential inclusion (6.10) and (6.11); the compact set Ω is defined in (6.9). Then the following statements are equivalent.*

(i) The system has weak $\mathcal{L}^1$-performance.

(ii) $0 \in \mathrm{DINV}_F(\Omega)$.

(iii) $\mathrm{DENV}_F(\{0\}) \subset \Omega$.

Furthermore, if any of the above statements holds, then for any $\mathcal{L}^1$-performance domain K, which is well-defined, one has

$$0 \in \text{DENV}_F(\{0\}) \subset K \subset \text{DINV}_F(\Omega) \subset \Omega.$$

Therefore, if the system has the weak $\mathcal{L}^1$-performance (or $\mathcal{L}^1$-performance for a system affine in w), then $\text{DENV}_F(\{0\})$ and $\text{DINV}_F(\Omega)$ are the smallest and the largest $\mathcal{L}^1$-performance domains, respectively. In the following, we will give some algorithms for computing the (closed) invariance domains in a given closed subset $\Omega \subset \text{DOM}(F)$. By modifying **viability kernel** algorithms in [7, pp.147–153], one has the following algorithms.

Algorithm 6.4.6 *Let $F : X \rightsquigarrow X$ and a closed subset $\Omega \subset \text{DOM}(F)$ be given. Define recursively the subsets K_n by*

$$K_0 := \Omega,$$

$$K_{n+1} := \{x \in K_n | F(x) \subset T_{K_n}(x)\}$$

where if K_n is empty in some step n stop there; otherwise define

$$K_\infty := \bigcap_{n=0}^{\infty} K_n. \tag{6.32}$$

It is observed that if $K_n = \emptyset$ for some $n \in \mathbf{Z}^+$, then $\text{DINV}_F(\Omega) = \emptyset$; otherwise, $\text{DINV}_F(\Omega) \subset K_\infty$ if exists, since $\text{DINV}_F(\Omega) \subset K_n$ for all n. However, in general, the inclusion can not be replaced by equality, i.e., the above algorithm does not yield the maximal invariance domain contained in Ω. Because the algorithm does not guarantee the subsets K_n to be closed; also in general the upper limit of the contingent cones $T_{K_n}(x)$ is not necessarily contained in the contingent cone to the upper limit of the subsets K_n [7]. In the following, an alternative algorithm yielding a closed invariance domain which is a subset of $\text{DINV}_F(\Omega)$ is provided. This algorithm is a modification of viability domain algorithm [7, p.151]; the set-valued map $T_K^c : X \rightsquigarrow X$ defined in Definition 6.7.8 is used.

Algorithm 6.4.7 *Let $F : X \rightsquigarrow X$ and a compact subset $\Omega \subset \text{DOM}(F)$ be given. Given a constant $c > 0$, define recursively the subsets K_n^c by*

$$K_0^c := \Omega,$$

$$K_{n+1}^c := \{x \in K_n^c | F(x) \subset T_{K_n^c}^c(x)\}.$$

If K_n^c is empty in some step n, then stop there; otherwise define

$$K_\infty^c := \bigcap_{n=0}^{\infty} K_n^c. \tag{6.33}$$

Theorem 6.4.8 *Let $F : X \rightsquigarrow X$ be LSC and $\Omega \subset \mathrm{Dom}(F)$ be a compact subset. In the above algorithm, if $K_n^c \neq \emptyset$ for all $n \in \mathbf{Z}^+$, then K_∞^c is a nonempty closed invariance domain of F.*

The proof is given in Appendix B.

6.4.2 Controlled Differential Inclusions and Controlled Invariance

Let X, Y, and Z be metric spaces. Given two set-valued maps, $U : X \rightsquigarrow Z$ and $F : \mathrm{Graph}(U) \rightsquigarrow Y$, Define a parameterized set-valued map $F_c : X \rightsquigarrow Y$ as follows,

$$F_c(x) := \bigcup_{u \in U(x)} F(x, u)$$

with $\mathrm{Dom}(F_c) = \mathrm{Dom}(U)$. Then the differential inclusion $\dot{x} \in F_c(x)$ is called a **controlled differential inclusion** (CDI), denoted as (F, U).

Definition 6.4.9 *Consider a CDI defined by (F, U). A subset $K \in \mathrm{Dom}(U)$ is* **controlled invariant** *under (F, U) if there exists a measurable function $u : \mathbf{R}^+ \to \mathbf{R}^m$ such that for all $x_o \in K$, the differential inclusion $\dot{x} \in F(x, u(t))$ has all solutions starting at x_o and viable in K, and $u(t) \in U(x(t))$ for all $t \in \mathbf{R}^+$. Given any closed subset $\Omega \subset \mathrm{Dom}(U)$, The largest closed subset of Ω which is controlled invariant under (F, U), denoted by $\mathrm{Cinv}(\Omega)$, is called the* **controlled invariance kernel** *(CIK) of Ω, and the* **controlled invariance envelope** *(CIE) $\mathrm{Cenv}(\Omega)$ of Ω is defined as the smallest closed controlled invariant subset containing Ω under (F, U).*

The notion of controlled invariance (or (A, B)-invariance for linear systems) was defined for linear systems to deal with disturbance decoupling in [198], and for nonlinear systems in the context of zero dynamics [90, 8]. The controlled invariance envelope exists and is unique. The controlled invariance kernel, if exists, is also unique, because of the following observation.

Lemma 6.4.10 *The class of controlled invariance subsets under (F, U) is closed under the operation of subset union.*

The following theorem characterizes the existence of CIK in a given closed set.

Theorem 6.4.11 *Consider the CDI defined by (F, U). Suppose $\Omega \subset \mathrm{Dom}(U) \subset X$ is compact, the set-valued maps $U : \mathrm{Dom}(\mathrm{U}) \rightsquigarrow Z$ is LSC with closed convex values, and $F : \mathrm{Graph}(U) \rightsquigarrow X$ is Marchaud. Then there exists a CIK (possibly empty) of Ω.*

Proof. Since $U : \textsc{Dom}(\mathrm{U}) \rightsquigarrow Z$ is LSC, by Michael's selection theorem (Proposition 6.7.4) there exists a continuous selection $u(x) \in U(x)$. Define a new set-valued map $F_u : \Omega \rightsquigarrow X$ as $F_u(x) := F(x, u(x))$. Since $F : \textsc{Graph}(U) \rightsquigarrow X$ is Marchaud, then there exists $C > 0$ such that

$$\|F_u(x)\| = \|F(x, u(x))\| \leq C(\|x\| + \|u(x)\| + 1) \leq C(C_u + 1)(\|x\| + 1)$$

with $C_u > 0$ being such that $\|u(x)\| \leq C_u$ as u is continuous on the compact set Ω, so F_u is also Marchaud. Therefore, by Theorem 6.4.3, there exists a maximal invariance kernel $\textsc{Inv}_{F_u}(\Omega)$, and it is controlled invariant under (F, U) by the definition. Zorn's Lemma implies that there exists a maximal controlled invariance subset, which is the CIK, of Ω. □

From the above theorem and Lemma 6.3.2, the controlled differential inclusion (F, U) defined by (6.21) and (6.23) has the CIK in $\textsc{Dom}(U)$ if $\textsc{Dom}(U)$ is compact.

Definition 6.4.12 *Consider the CDI defined by (F, U). A subset $K \in \textsc{Dom}(U)$ is a* **controlled invariance domain** *(CID) of (F, U) if for all $x \in K$, there exists a $u(x) \in U(x)$ such that $F(x, u(x)) \subset T_K(x)$. Given any closed subset $\Omega \subset \textsc{Dom}(U)$, $\textsc{Dci}(\Omega)$ is the largest closed controlled invariance domain in Ω under (F, U), and $\textsc{Dce}(\Omega)$ is the smallest closed controlled invariance domain containing Ω for (F, U).*

Recall the definition of the controlled $\mathcal{L}^1$-performance domains in the last section, it is known from the above definition that any controlled $\mathcal{L}^1$-performance domain of system (6.19) is a controlled invariance domain of the controlled differential inclusion (F, U) defined by (6.21) and (6.23). Therefore, $\textsc{Dci}(\Omega)$ exists for a class of parameterized controlled differential inclusions. We immediately have the following theorem which characterizes the controlled $\mathcal{L}^1$-performance of system (6.19) in terms of the controlled invariance domains.

Theorem 6.4.13 *Consider system (6.19) and its corresponding controlled differential inclusion (F, U) defined by (6.21) and (6.23). Suppose $\textsc{Dom}(U)$ is compact, and $K \in \textsc{Dom}(U)$ is a controlled $\mathcal{L}^1$-performance domain. Then*

$$0 \in \textsc{Dce}(\{0\}) \subset K \subset \textsc{Dci}(\textsc{Dom}(U)) \subset \textsc{Dom}(U).$$

Therefore, if the system has a controlled $\mathcal{L}^1$-performance domain, then $\textsc{Dce}(\{0\})$ and $\textsc{Dci}(\textsc{Dom}(U))$ are the smallest and the largest controlled $\mathcal{L}^1$-performance domains, respectively. In the following, we give some algorithms to compute the controlled invariance domains in some given closed set. Those algorithms are modifications of the (A, B)-invariance algorithm for linear systems [198], the **controlled invariance kernel** algorithm for controlled difference inclusions [160], and the **zero dynamics** algorithm [90, 130].

Algorithm 6.4.14 *Let $F : X \rightsquigarrow X$ and a closed subset $\Omega \subset \text{Dom}(F)$ be given. Define recursively the subsets K_n by*

$$K_0 := \Omega,$$

$$K_{n+1} := \{x \in K_n | F(x,u) \in T_{K_n}(x), \text{ for some } u \in U(x).\}$$

Define

$$K_\infty := \bigcap_{n=0}^{\infty} K_n. \tag{6.34}$$

It is observed that $\text{Dci}(\Omega) \subset K_n$ for all n, then $\text{Dci}(\Omega) \subset K_\infty$. However, in general, the inclusion can not be replaced by equality. In the following, we give a remedy to this problem as in Algorithm 6.4.7, however, instead of $\text{Dci}(\Omega)$ itself, only a closed invariance domain, which is a subset of $\text{Dci}(\Omega)$, is obtained.

Algorithm 6.4.15 *Let $F : X \times Z \rightsquigarrow X$ and a compact subset $\Omega \subset \text{Dom}(F)$ be given. Given a constant $c > 0$, define recursively the subsets K_n^c by*

$$K_0^c := \Omega,$$

$$K_{n+1}^c := \{x \in K_n^c | F(x,u) \in T_{K_n^c}^c(x), \text{ for some } u \in U(x)\}.$$

Then either K_n^c is empty in some step n, or

$$K_\infty^c := \bigcap_{n=0}^{\infty} K_n^c \tag{6.35}$$

is not empty.

Theorem 6.4.16 *Suppose Ω is compact and $F : X \times Z \rightsquigarrow X$ is LSC, $U : Z \rightsquigarrow X$ is locally bounded USC with closed values. In the above algorithm, either K_n^c is empty in some step n, or K_∞^c is a nonempty closed controlled invariance domain of F.*

The proof is given in the Appendix B.

6.5 Approximation Methods for $\mathcal{L}^1$-Performance Analysis and Synthesis

The $\mathcal{L}^1$-performance analysis and synthesis for nonlinear systems are reduced to the computations of (controlled) invariance domains for some (controlled) differential inclusions. However, unlike the discrete time systems, the algorithms given in the last section are not easy to implement. In this section, we will try to give some alternatives characterization for the (controlled) invariance domains and approximate them in terms of the (controlled) invariance domains of the corresponding Euler approximated discrete-time systems. To this end, we give the following definition [10, p.17].

Definition 6.5.1 *Let $\{K_n\}|_{n\in\mathbf{Z}^+}$ be a sequence of subsets of a metric space X. The* **upper limit** *of the sequence is a closed subset of X defined as*

$$\limsup_{n\to\infty} K_n := \{x \in X \,|\, \liminf_{n\to\infty} d(x, K_n) = 0\}.$$

Therefore, $\limsup_{n\to\infty} K_n$ is the set of cluster points of sequence $x_n \in K_n$, i.e., of limits of subsequence $x_{n_i} \in K_{n_i}$.

6.5.1 ℓ^1-Performance of Discrete-Time Nonlinear Systems

The material in this subsection is just the reformulation of some results from [160].

ℓ^1-PERFORMANCES

Consider the following discrete-time nonlinear system

$$\begin{cases} x(k+1) = f_d(x(k), w(k)) \\ z(k) = h_d(x(k), w(k)) \end{cases} \tag{6.36}$$

where f_d and h_d are continuous. The ℓ^1-performance for system (6.36) is defined similarly to that in the continuous times case (see Definition 6.2.1). Let a set-valued map $F_d : \mathbf{R}^n \rightsquigarrow \mathbf{R}^n$ be defined as

$$F_d(x) := \{f_d(x, w) | w \in \mathbf{BR}^p\}$$

with the domain $\textsc{Dom}(F_d) = \Omega$, where

$$\Omega := \{x \in \mathbf{R}^n | \, ||h_d(x, w)|| \leq 1, \forall w \in \mathbf{BR}^p\}$$

is assumed bounded. We can also similarly define the invariance and the invariance kernel of a closed set for the corresponding difference inclusion. We have the following result [160, Proposition 4.1].

Proposition 6.5.2 *The invariant kernel* $\mathrm{INV}_{F_d}(\Omega)$ *in* Ω *for difference inclusion* $x(k+1) \in F_d(x(k))$ *exists, and*

$$\mathrm{INV}_{F_d}(\Omega) = \bigcap_{j=1}^{\infty} K_j$$

where $K_0 = \Omega$, $K_{j+1} = \{x \in K_j : F_d(x) \subset K_j\}$. *Moreover, system (6.36) has* ℓ^1*-performance*

$$J := \sup_{w \in \ell_\infty, \|w\|_\infty \leq 1} \|z\|_\infty \leq 1,$$

if and only if $0 \in \mathrm{INV}_{F_d}(\Omega) \neq \emptyset$.

CONTROLLED INVARIANCE AND ℓ^1-CONTROL

Let X and Y be metric spaces. Given two set-valued map $U_d : X \rightsquigarrow Y$, $F_d : \mathrm{GRAPH}(U_d) \rightsquigarrow X$, then the difference inclusion

$$x(k+1) \in \bigcup_{u \in U_d(x(k))} F_d(x(k), u)$$

defines a controlled difference inclusion, denoted as (F_d, U_d), we can similarly define such concepts as controlled invariance and controlled invariance kernel of a closed set under (F_d, U_d) [160, Definitions 4.3 and 4.4]. The following result is due to Shamma [160, Proposition 4.2].

Proposition 6.5.3 *Consider a controlled difference inclusion defined by* (F_d, U_d). *Suppose* $\Omega \subset \mathrm{DOM}(U_d)$ *is compact,* $U_d : X \rightsquigarrow Y$ *is locally bounded USC with closed-values, and* $F_d : \mathrm{GRAPH}(U_d) \rightsquigarrow X$ *is LSC. Then the controlled invariance kernel* $\mathrm{CINV}(\Omega)$ *of* Ω *for* (F_d, U_d) *exists (possibly empty). And*

$$\mathrm{CINV}(\Omega) = \bigcap_{j=0}^{\infty} K_j$$

where K_j *is recursively defined,* $K_0 := \Omega$, $K_{j+1} = \{x \in K_j | F_d(x, u) \subset K_j, \exists u \in U_d(x)\}$.

Next, consider the following discrete-time control system

$$\begin{cases} x(k+1) = f_d(x(k)) + g_{d1}(x(k))w(k) + g_{d1}(x(k))u(k) \\ z(k) = h_d(x(k)) + k_{d1}(x(k)w(k) + k_{d2}(x(k))u(k) \end{cases} \tag{6.37}$$

with $f_d, g_{d1}, g_{d2}, h_d, k_{d1}$, and k_{d2} being continuous, and $\text{RANK}(g_1(x)) = n$ for all $x \in \mathbf{R}^n$. Similarly, the ℓ^1-control problem for system (6.37) can be defined as did for the continuous time case (see Definition 6.3.1). Define $U_d : \mathbf{R}^n \rightsquigarrow \mathbf{R}^m$

$$U_d(x) := \{u \in \mathbf{R}^m | \, ||h_d(x) + k_{d1}(x)w + k_{d2}(x)u|| \leq 1, \forall w \in \mathbf{BR}^p\}.$$

Suppose $\Omega := \text{DOM}(U_d)$ is bounded. Note that U_d is locally bounded USC with closed values. Let a set-valued map $F_d : \mathbf{R}^n \times \mathbf{R}^m \rightsquigarrow \mathbf{R}^n$ be defined as

$$F_d(x, u) := \{f_d(x) + g_{d1}(x)w + g_{d2}(x)u) | w \in \mathbf{BR}^p\}$$

with domain $\text{DOM}(F_d) = \Omega$. We have the following results about ℓ^1-control problem slightly generalizing Shamma's theorems [160, Theorem 5.1] and [158, Theorem 3.1].

Proposition 6.5.4 *Consider system (6.37). Then the following statements are true.*

(i) $\text{CINV}(\Omega)$ *exists.*

(ii) If the system has ℓ^1-control solution such that $J \leq 1$, then $0 \in \text{CINV}(\Omega) \neq \emptyset$.

(iii) If $0 \in \text{CINV}(\Omega) \neq \emptyset$ and $\text{CINV}(\Omega)$ is convex, then for all $\epsilon \in (0, 1)$, there exists a continuous static state feedback such that the closed-loop system satisfies:

$$J^\epsilon := \sup_{w \in \ell_\infty, ||w||_\infty \leq 1-\epsilon} ||z||_\infty \leq 1. \tag{6.38}$$

It is remarked that the results about ℓ^1-performance analysis and synthesis can also be characterized in terms of (controlled) invariance envelope. For example, we have the following version of Proposition 6.5.4.

Proposition 6.5.5 *Consider system (6.37). Then the following statements are true.*

(i) If the system has ℓ^1-control solution, then $\text{CENV}(\{0\}) \subset \Omega$.

(iii) If $\text{CENV}(\{0\}) \subset \Omega$ and $\text{CENV}(\{0\})$ is convex, then for all $\epsilon \in (0, 1)$, there exists a continuous static state feedback such that the closed-loop system satisfies (6.38).

6.5.2 Approximation of $\mathcal{L}^1$-Performance Domains

Consider system (6.13) which is rewritten as follows,

$$\begin{cases} \dot{x} = f(x, w) \\ z = h(x, w) \end{cases}$$

where $w \in \mathcal{W}$. Given $\tau > 0$, define a corresponding difference equation as

$$\begin{cases} x_\tau(k+1) = f_\tau(x_\tau(k), w_\tau(k)) \\ z_\tau(k) = h(x_\tau(k), w_\tau(k)) \end{cases} \tag{6.39}$$

where $w_\tau(k) := w(\tau k)$, and

$$f_\tau(x, w) := x + \tau f(x, w). \tag{6.40}$$

It is noted that the discrete-time system (6.39) is a Euler approximation of system (6.13). Let a set-valued map $F_\tau : \mathbf{R}^n \rightsquigarrow \mathbf{R}^n$ be defined as

$$F_\tau(x) := \{f_\tau(x, w) | w \in \mathbf{BR}^p\}$$

with the domain $\textsc{Dom}(F_\tau) = \Omega$, where

$$\Omega := \{x \in \mathbf{R}^n | \, \|h(x, w)\| \leq 1, \forall w \in \mathbf{BR}^p\}$$

is assumed bounded. Note that the map F_τ is LSC because of the continuity assumption on f for system (6.13). Consider the difference inclusion $x_\tau(k+1) \in F_\tau(x_\tau(k))$. Then by Proposition 6.5.2, we know that invariant kernel $\textsc{Inv}_{F_\tau}(\Omega)$ in Ω for difference inclusion $x_\tau(k+1) \in F_\tau(x_\tau(k))$ exists; and the discrete-time system (6.39) has ℓ^1-performance if and only if $\textsc{Inv}_{F_\tau}(\Omega) \neq \emptyset$.

We have the following result about the approximations of the $\mathcal{L}^1$-performance domains for system (6.13).

Theorem 6.5.6 *Consider system (6.13). Let $\{\tau_n\}$ be a decreasing sequence such that $\tau_n \to 0$ as $n \to \infty$, and $V_{\tau_n} \subset \Omega$ be closed and invariant under F_{τ_n} for each τ_n with $0 \in V_{\tau_n}$. Then $V_\infty := \limsup_{n\to\infty} V_{\tau_n}$ is a $\mathcal{L}^1$-performance domain for system (6.13).*

It is noted that in Theorem 6.5.6, for each τ_n, the corresponding Euler approximated discrete-time system has ℓ^1-performance $J \leq 1$. Possible choices for V_{τ_n} are $\textsc{Inv}_{F_{\tau_n}}(\Omega)$ and $\textsc{Env}_{F_{\tau_n}}(\{0\})$.

Proof. Choose $w \in \mathcal{W}_c$ and $x_0 \in V_\infty$, then the solution $\phi(t, x_0, w)$ to the differential equation $\dot{x} = f(x, w(t))$ with $x(0) = x_0$ is bounded. We first show that, for all $T > 0$, $\phi(t, x_0, w) \in V_\infty$ for all $t \in [0, T)$.

Consider the function $f_w(x, t) := f(x, w(t))$, which is continuous on compact set $\Omega \times [0, T]$; therefore, $\|f(x, w(t))\| < \beta$ for some $\beta > 0$, and it is uniformly continuous on

$\Omega \times [0,T]$. Given $\epsilon > 0$, there thus exists $\delta > 0$ such that for all $\tau \in (0,\delta]$,

$$\|f(x_1, w(t_1)) - f(x_2, w(t_2))\| < \epsilon \tag{6.41}$$

for all $(x_i, t_i) \in \Omega \times [0,T]$ $(i = 1,2)$ with $\|x_1 - x_2\| \le \delta$ and $|t_1 - t_2| < \delta$.

Since $x_0 \in V_\infty$, there exists $x_n^0 \in V_{\tau_n}$ such that x_0 is a cluster point of the sequence $\{x_n^0\}$; we assume $x_n^0 \to x_0$ as $n \to \infty$ without loss of generality. On the other hand, $\tau_n \to 0$ as $n \to \infty$, there exists $N > 0$ such that $\tau_n \in [0, \min\{\delta, \delta/\beta\})$ for all $n \ge N$. Take $n \ge N$, we consider a solution $x_{\tau_n}(k)$ for the difference equation defined in (6.39):

$$\dot{x}_{\tau_n}(k+1) = f_{\tau_n}(x_{\tau_n}(k), w_{\tau_n}(k))$$

with $x_{\tau_n}(0) = x_n^0$. Then $x_{\tau_n}(k) \in V_{\tau_n}$ for all $k \in \mathbf{Z}^+$ by the definition of V_{τ_n}. Now we associate with the solution a function $x_n \in \mathcal{C}([0,T), \mathbf{R}^n)$ as

$$x_n(t) := x_{\tau_n}(k) + \frac{x_{\tau_n}(k+1) - x_{\tau_n}(k)}{\tau_n}(t - k\tau_n)$$

for all $k \ge 0$ and $t \in [k\tau_n, (k+1)\tau_n)$ such that $t \in [0,T)$. Note that $x_{\tau_n}(k+1) - x_{\tau_n}(k) = \tau_n f(x_{\tau_n}(k), w_{\tau_n}(k)$; thus,

$$\|x_n(t) - x_{\tau_n}(k)\| \le \left\| \frac{x_{\tau_n}(k+1) - x_{\tau_n}(k)}{\tau_n} \right\| \tau_n < \beta\tau_n \le \delta$$

and $\dot{x}_n(t) = f(x_{\tau_n}(k), w_{\tau_n}(k))$ for $t \in [k\tau_n, (k+1)\tau_n)$ (hence $\|\dot{x}_n(t)\| < \beta$). From (6.41), we thus have

$$\|\dot{x}_n(t) - f(x_n(t), w(t))\| < \epsilon \tag{6.42}$$

for all $t \in [0,T)$. On the other hand, $\dot{x}_n(t)$ is bounded, then $x_n(t)$ is equicontinuous. Similar argument in terms of Ascoli's Theorem in the proof of Theorem 6.2.9 (i) yields that a subsequence of $\{x_n(t)\}$, still denoted as $\{x_n(t)\}$ without loss of generality, converges to an absolutely continuous function $x(t)$, and their derivatives $\dot{x}_n(t) \to \dot{x}(t)$ as $n \to \infty$. (6.42) implies that

$$\dot{x}(t) = f(x(t), w(t)).$$

Since $x_n(0) = x_{\tau_n}(0) \to x_0$ as $n \to \infty$; and each $t \ge 0$ is the limit of nodes $k_t\tau_n$, so $x(t)$ is the limit of $x_{\tau_n}(k_t) \in K_{\tau_n}$. Then $x(t) \in V_\infty$ for all $t \in [0,T)$. By the completeness of the given system, $\phi(t, x_0, w) = x(t) \in V_\infty$ for all $t \in [0,T)$.

Finally, from similar argument in Theorem 6.2.7, one can conclude that for all $x \in V_\infty$, $f(x,w) \in T_{V_\infty}(x)$ for all $w \in \mathbf{BR}^p$. Therefore, V_∞ is an $\mathcal{L}^1$-performance domain for system (6.13). $\square$

The following theorem, which generalizes [23, Theorem 2], characterizes the $\mathcal{L}^1$-performance domains for a class of special systems which include the linear systems.

Theorem 6.5.7 *Consider system (6.13). Suppose there exists $\tau > 0$ such that $V_\tau \subset \Omega$ is closed, convex, and invariant under F_τ with $0 \in V_\tau$. Then V_τ is an $\mathcal{L}^1$-performance domain for system (6.13).*

Proof. Since $V_\tau \subset \Omega$ is invariant under F_τ, one has that for all $x \in V_\tau$, $F_\tau(x) \in V_\tau$, or given $w \in \mathbf{BR}^p$,

$$x + \tau f(x, w) \in V_\tau.$$

By the assumption V_τ is convex, then

$$x + hf(x, w) \in V_\tau$$

for all $h \in [0, \tau]$. Now by Lemma 6.7.5, it follows that

$$f(x, w) \in T_{V_\tau}(x)$$

for all $w \in \mathbf{BR}^p$. Therefore, V_τ is an $\mathcal{L}^1$-performance domain for system (6.13). □

6.5.3 Approximation of Controlled $\mathcal{L}^1$-Performance Domains

Consider system (6.19), which is rewritten as follows,

$$\begin{cases} \dot{x} = f(x) + g_1(x)w + g_2(x)u \\ z = h(x) + k_1(x)w + k_2(x)u \end{cases}$$

where $w \in \mathcal{W}_c$. Given $\tau > 0$, define a corresponding difference equation, which is a Euler approximation of system (6.19), as follows,

$$\begin{cases} \dot{x}_\tau(k+1) = f_\tau(x_\tau(k), w_\tau(k), u_\tau(k)) \\ z_\tau(k) = h(x_\tau(k)) + k_1(x_\tau(k)w_\tau(k) + k_2(x_\tau(k))u_\tau(k) \end{cases} \tag{6.43}$$

where $w_\tau(k) := w(\tau k)$, and

$$f_\tau(x, w, u) := x + \tau(f(x) + g_1(x)w + g_2(x)u). \tag{6.44}$$

Define $U : \mathbf{R}^n \rightsquigarrow \mathbf{R}^m$ as

$$U(x) := \{u \in \mathbf{R}^m | \, \|h(x) + k_1(x)w + k_2(x)u\| \leq 1, \forall w \in \mathbf{BR}^p\}.$$

Let $\Omega := \textsc{Dom}(U)$ be bounded. Since U is locally bounded USC with closed values by Lemma 6.3.2. Then there exists a compact set $\mathcal{U} \subset \mathbf{R}^m$, such that

$$\bigcup_{x\in\Omega} U(x) \subset \mathcal{U}.$$

Let a set-valued map $F_\tau : \mathbf{R}^n \times \mathbf{R}^m \rightsquigarrow \mathbf{R}^n$ be defined as

$$F_\tau(x,u) := \{f_\tau(x,w,u) | w \in \mathbf{BR}^p\}$$

with the domain $\textsc{Dom}(F_\tau) = \Omega$. Consider the controlled difference inclusion defined by (F_τ, U). By Proposition 6.5.3, the controlled invariance kernel $\textsc{Cinv}(\Omega)$ of (F_τ, U) exists in Ω; and under some mild conditions, the discrete-time system (6.43) has ℓ^1-control solution if and only $\textsc{Cinv}(\Omega) \neq \emptyset$ (see Theorem 6.5.4).

We have the following result on the approximations of controlled $\mathcal{L}^1$-performance domains.

Theorem 6.5.8 *Consider system (6.19). Let $\{\tau_n\}$ be a decreasing sequence such that $\tau_n \to 0$ as $n \to \infty$, $V_{\tau_n} \subset \Omega$ be controlled invariant under (F_{τ_n}, U) for each τ_n with $0 \in V_{\tau_n}$. Then $V_\infty := \limsup_{n\to\infty} V_{\tau_n}$ is a controlled $\mathcal{L}^1$-performance domain for system (6.19).*

It is noted that, in Theorem 6.5.8, for each τ_n, the corresponding Euler approximated discrete-time system has a nonempty controlled ℓ^1-performance domain. Possible choices for V_{τ_n} are $\textsc{Cinv}(\Omega)$ and $\textsc{Cenv}(\{0\})$.

Proof. One needs to show that there exists $u \in U(x)$ such that $f(x,w,u) \in T_{V_\infty}(x)$ for all $x \in V_\infty$ and $w \in \mathbf{BR}^n$.

Suppose $w \in \mathcal{W}_c$ and $x_0 \in V_\infty$. We first show that, given $T > 0$, there exists a measurable function $u(t)$ such that the solution $\xi(t)$ to the differential equation $\dot{x} = f(x, w(t), u(t))$ with $x(0) = x_0$ is in V_∞ for all $t \in [0,T)$ and $u(t) \in U(\xi(t))$.

Consider the function $f_w(x,t,u) := f(x,w(t),u)$, which is continuous on compact set $\Omega \times [0,T] \times \mathcal{U}$; therefore, $\|f(x,w(t),u)\| < \beta$ for some $\beta > 0$, and it is uniformly continuous on $\Omega \times [0,T]$. Now given $\epsilon > 0$, there exists $\delta > 0$ such that for all $\tau \in (0,\delta]$,

$$\|f(x_1,w(t_1),u) - f(x_2,w(t_2),u)\| < \epsilon \tag{6.45}$$

for all $(x_i,t_i,u) \in \Omega \times [0,T] \times \mathcal{U}$ with $\|x_1 - x_2\| < \delta$ and $|t_1 - t_2| < \delta$.

Since $x_0 \in V_\infty$, therefore there exists $x_n^0 \in V_{\tau_n}$ such that x_0 is a cluster point of the sequence $\{x_n^0\}$; we assume $x_n^0 \to x_0$ as $n \to \infty$ without loss of generality. On the other hand, $\tau_n \to 0$ as $n \to \infty$, there exists $N > 0$ such that $\tau_n \in [0, \min\{\delta, \delta/\beta\})$ for all $n \geq N$. We consider a solution $x_{\tau_n}(k)$ for the difference equation defined in (6.39) for some $n \geq N$,

$$\dot{x}_{\tau_n}(k+1) = f_{\tau_n}(x_{\tau_n}(k), w_{\tau_n}(k), u_{\tau_n}(k))$$

with $x_{\tau_n}(0) = x_n^0$. Then by the definition of V_{τ_n}, there exists $u_{\tau_n}(k) \in U(x_{\tau_n}(k))$ such that $x_{\tau_n}(k) \in V_{\tau_n}$ for all $k \in \mathbf{Z}^+$. We define a function $x_n \in \mathcal{C}([0,T), \mathbf{R}^n)$ as

$$x_n(t) := x_{\tau_n}(k) + \frac{x_{\tau_n}(k+1) - x_{\tau_n}(k)}{\tau_n}(t - k\tau_n)$$

for all $k \geq 0$ and $t \in [k\tau_n, (k+1)\tau_n)$ such that $t \in [0,T)$. Note that $x_{\tau_n}(k+1) - x_{\tau_n}(k) = \tau_n f(x_{\tau_n}(k), w_{\tau_n}(k), u_{\tau_n}(k))$; thus,

$$\|x_n(t) - x_{\tau_n}(k)\| \leq \left\| \frac{x_{\tau_n}(k+1) - x_{\tau_n}(k)}{\tau_n} \right\| \tau_n < \beta\tau_n \leq \delta$$

and $\dot{x}_n(t) = f(x_{\tau_n}(k), w_{\tau_n}(k), u_{\tau_n}(k))$ for $t \in [k\tau_n, (k+1)\tau_n)$. Let the function $u_n : [0,T) \to \mathbf{R}^m$ be defined as

$$u_n(t) := u_{\tau_n}(k),$$

for $t \in [k\tau_n, (k+1)\tau_n) \subset [0,T)$. Therefore, from (6.45), we have

$$\|\dot{x}_n(t) - f(x_n(t), w(t), u_n(t))\| \leq \epsilon \tag{6.46}$$

for all $t \in [0,T)$.

It is noted that, $\dot{x}_n(t)$ is bounded, hence $x_n(t)$ is equicontinuous. Then the similar argument in terms of Ascoli's Theorem in the proof of Theorem 6.2.9 (i) yields that a subsequence of $\{x_n(t)\}$, still denoted as $\{x_n(t)\}$ without loss of generality, converges to an absolute continuous function $x(t)$, and their derivatives $\dot{x}_n(t) \to \dot{x}(t)$ as $n \to \infty$.

On the other hand, given $t \in [0,T)$, then $x(t)$ is a limit point of some $x_{\tau_n}(k)$; since for $t \in [k\tau_n, (k+1)\tau_n)$, $u_n(t) = u_{\tau_n}(k) \in U(x_{\tau_n}(k)) \in \mathcal{U}$, there exists a subsequence of $\{u_n(t))\}$, still denoted it as $\{u_n(t)\}$, converges to some $u(t) \in U(x(t))$ since U is USC with closed values. Note that $u : [0,T) \to \mathbf{R}^m$ can be chosen to be measurable by the construction[2].

Therefore, (6.46) implies that

$$\dot{x}(t) = f(x(t), w(t), u(t)).$$

[2]For example, $u(t) = (\limsup_{n\to\infty} u_n^1(t), \cdots, \limsup_{n\to\infty} u_n^m(t))^T$ where $u_n := (u_n^1, \cdots, u_n^m)^T$; $u(t)$ is measurable since its components are upper limits of simple functions.

Since $x_n(0) = x_{\tau_n}(0) \to x_0$ as $n \to \infty$; for each $t \geq 0$ is the limit of nodes $k_t\tau_n$, $x(t)$ is the limit of $x_{\tau_n}(k_t) \in K_{\tau_n}$. Then $x(t) \in V_\infty$ for all $t \in [0, T)$. The completeness of the given system implies $\xi(t) = x(t) \in V_\infty$ for all $t \in [0, T)$.

Finally, we will show that V_∞ is a controlled $\mathcal{L}^1$-performance domain for system (6.19). Take $x_0 \in V_\infty$. For given $w \in \mathcal{U}_c$, there exists an essentially bounded function $u : \mathbf{R}^+ \to \mathbf{R}^m$, such that the unique solution $x(t)$ for $\dot{x} = f(x) + g_1(x)w(t) + g_2(x)u(t)$ with $x(0) = x_0$ is viable in V_∞ and $u(t) \in U(x(t))$. One only needs to check that $\dot{x}(0) = f(x_0) + g_1(x_0)w(0) + g_2(x_0)u(0) \in T_{V_\infty}(x_0)$. In fact, for all $t \in [0, T]$, $x(t) \in V_\infty$, and

$$x(t) = x_0 + \int_0^t \dot{x}(s)ds$$

$$= x_0 + t(\frac{1}{t}\int_0^t (f(x(s)) + g_1(x(s))w(s) + g_2(x(s))u(s))ds).$$

Notice that $f(x(s)) + g_1(x(s))w(s) + g_2(x(s))u(s)$ is essentially bounded in $[0, T]$, then by Lebesgue's differentiation theorem, one has

$$\lim_{t \to 0} \frac{1}{t}\int_0^t (f(x(s)) + g_1(x(s))w(s) + g_2(x(s))u(s))ds = f(x_0) + g_1(x_0)w(0) + g_2(x_0)u(0).$$

Therefore, one can find two sequences $\{t_n\}$ and $\{v_n\}$ with $t_n \to 0^+$ and $v_n \to f(x_0) + g_1(x_0)w(0) + g_2(x_0)u(0)$ as $n \to \infty$, such that $x_0 + t_n v_n \in V_\infty$ for all $n \in \mathbf{Z}^+$. Hence, $f(x_0) + g_1(x_0)w(0) + g_2(x_0)u(0) \in T_{V_\infty}(x_0)$, the conclusion then follows by Lemma 6.7.5.

□

The following theorem characterizes the controlled $\mathcal{L}^1$-performance domains for a class of special systems which include the linear systems.

Theorem 6.5.9 *Consider system (6.19). Suppose there exists $\tau > 0$ such that $V_\tau \subset \Omega$ is closed, convex, and controlled invariant under (F_τ, U) with $0 \in V_\tau$. Then V_τ is a controlled $\mathcal{L}^1$-performance domain for system (6.19).*

Proof. The argument is similar to that in the proof of Theorem 6.5.7. □

6.6 Conclusions

In this chapter, the $\mathcal{L}^1$-control problems for nonlinear systems were investigated. The $\mathcal{L}^1$-performance analysis and the $\mathcal{L}^1$-control synthesis problems were characterized in terms of the $\mathcal{L}^1$-performance domains and the controlled $\mathcal{L}^1$-performance domains, which are the invariance domains and the controlled invariance domains of the corresponding differential inclusions, respectively. This treatment therefore provided some geometrical insights into the robust ($\mathcal{L}^1$) control problem. In addition, the relation between the $\mathcal{L}^1$-control of a continuous-time system and the ℓ^1-control of its Euler approximated discrete-time systems was established. Nonetheless, the computational implications of the results for general nonlinear systems in this chapter are not clear. The results in this chapter can serve for didactic purpose, and can be used to guide the design of nonlinear control systems with disturbance attenuation properties.

Another issue that was not explicitly addressed in this chapter is the asymptotic property of the nonlinear $\mathcal{L}^1$-control systems, i.e., when the initial states are not in any of the $\mathcal{L}^1$-performance domains, in which case the systems do not have $\mathcal{L}^1$-performance initially, do the systems eventually have $\mathcal{L}^1$-performance as they evolve? This issue can be investigated in the framework reported in [105]. Also the $\mathcal{L}^1$-optimal disturbance rejection problem in terms of minimizing the $\mathcal{L}_\infty$-induced gains of the corresponding nonlinear operators is addressed in [46].

6.7 Appendix A: Set-Valued Maps

In this section, we will review some basic notions from set-valued analysis. We refer to the books [9, 10, 7] for detailed account about the related issues.

6.7.1 Set-Valued Maps and Their Selections

Let X and Y be two normed spaces. A **set-valued map** F from X to Y is a map that associates with any $x \in X$ a subset $F(x)$ of Y. We denote it as

$$F : X \rightsquigarrow Y.$$

The subset $F(x)$ is called the **value** of F at $x \in X$. The **domain** and **graph** of F are defined as

$$\textsc{Dom}(F) := \{x \in X : F(x) \neq \emptyset\}.$$

$$\textsc{Graph}(F) := \{(x, y) \in X \times Y | y \in F(x)\}.$$

Definition 6.7.1 *Consider a set-valued map $F : X \rightsquigarrow Y$.*

(i) It is said to be **lower semi-continuous** *(LSC) if for all $x_0 \in X$, $y_0 \in F(x_0)$, and any sequence of elements $x_n \in \text{Dom}(F)$ converging to x_0, there exists a sequence of elements $y_n \in F(x_n)$ converging to y_0.*

(ii) It is said to be **upper semi-continuous** *(USC) if for all $x_0 \in X$, $y \in F(x_0)$, and for any open subset $\mathbf{N}$ of Y containing $F(x_0)$, there exists a neighborhood $\mathbf{N}(x_0)$ of x_0 such that $F(\mathbf{N}(x_0)) \subset \mathbf{N}$.*

Note that if F is USC with closed domain and closed values, then $\text{Graph}(F)$ is closed. Two special classes of continuous set-valued maps are defined as follows,

Definition 6.7.2 *Consider a set-valued map $F : X \rightsquigarrow Y$.*

(i) It is said to be **Marchaud**, *if it is USC, has compact convex images, and has* **linear growth property**, *i.e., there exists $C > 0$ such that for all $x \in \text{Dom}(F)$,*

$$\|F(x)\| \leq C(\|x\| + 1), \tag{6.47}$$

where $\|F(x)\| := \sup_{y \in F(x)} \|y\|$.

(ii) It is said to be **Lipschitz** *around $x \in X$ if there exist a positive constant L and a neighborhood $\mathcal{W} \subset \text{Dom}(F)$ of x such that, for all $x_1, x_2 \in \mathcal{W}$,*

$$F(x_1) \subset F(x_2) + L \|x_1 - x_2\| B_Y.$$

Note that, if $F : X \rightsquigarrow Y$ is bounded on $\text{Dom}(F)$, then it has the linear growth property (6.47).

The following technical result will be used [9, p.49].

Proposition 6.7.3 *Let X be a metric space and Y and Z be Banach spaces. Let $f : X \times Z \to Y$ be a continuous map such that for all $x \in X, u \mapsto f(x,u)$ is affine. Let set-valued maps $T : X \rightsquigarrow Y$ and $U : X \rightsquigarrow Z$ be LSC, and let U be locally bounded. Suppose there exists an $\alpha > 0$ such that for all $x \in X$, there exists a $u \in U(x)$ such that $f(x,u) + r \in T(x)$ for all $r \in Y$ with $\|r\| \leq \alpha$. Then the set-valued map $C : X \rightsquigarrow U$ defined by*

$$C(x) := \{u \in U(x) | f(x,u) \in T(x)\} \tag{6.48}$$

is LSC.

Given a set-valued map $F : X \rightsquigarrow Y$, there is map $f : X \to Y$ which is a selection of F, i.e. $f(x) \in F(x)$ for each $x \in X$. For a class of set-valued maps, we have the following lemma which is known as **Michael's selection theorem** (cf [10, p.355]).

Proposition 6.7.4 *Let X be a metric space, Y a Banach space, $F : X \rightsquigarrow Y$ which has the closed convex subsets as its values be LSC. Then there exists a continuous selection $f : X \to Y$ from F. In addition, if $y_0 \in F(x_0)$, then the continuous selection f of F can be chosen such that $f(x_0) = y_0$.*

6.7.2 Contingent Cones

Let X be a finite dimensional normed space, K be a nonempty subset of X, for each $x \in X$, define the distance of x to K as

$$d_K(x) := d(x, K) := \inf_{y \in K} \|x - y\| . \tag{6.49}$$

Define a set-valued map $T_K : X \rightsquigarrow X$,

$$T_K(x) := \{v | \liminf_{h \to 0^+} \frac{d_K(x + hv) - d_K(x)}{h} \leq 0\}. \tag{6.50}$$

For all $x \in X$, the value $T_K(x)$ is a closed cone, and is called the **contingent cone** to K at x. Note that if $\overline{K}$ denotes the closure of K, then $T_{\overline{K}} = T_K$; if $x \in \overline{K}$, then $T_K(x) = \{v | \liminf_{h \to 0^+} d_K(x + hv)/\tau = 0\}$, and if $x \in \text{INT}(K) \neq \emptyset$, then $T_K(x) = X$. Also if K is a manifold in X, then for any $x \in K$, $T_K(x)$ defines the tangent space of K at x. The following lemma characterizes contingent cones in terms of sequences [10, p.122].

Lemma 6.7.5 *Given a set $K \subset X$ and $x \in K$. $v \in T_K(x)$ if and only if there exist a nonincreasing sequence $h_n \to 0$ and a sequence $v_n \to v$, such that $x + h_n v_n \in K$ for all n.*

A subset K of X is said to be **sleek** if the set-valued map $T_K : K \rightsquigarrow X$ is LSC. The following result is from [7, p.161].

Proposition 6.7.6 *If K is sleek, then for all $x \in \overline{K}$,*

$$T_K(x) = \{v | \lim_{h \to 0^+, y \to_K x} d_K(y + hv)/h = 0\}$$

and $T_K(x)$ is a closed convex cone for all $x \in K$.

Convex sets are sleek. We next state a result about the computing the contingent cone of a set which is defined by some inequalities [10, p.123].

Proposition 6.7.7 *Given a $\mathcal{C}^1$ vector-valued function $g = (g_1, g_2, \cdots, g_p) : X \to \mathbf{R}^p$. Define a set*

$$K := \{x \in X | g_i(x) \geq 0, i = 1, 2, \cdots, p\}.$$

Given $x \in K$, define $I(x) := \{i = 1, 2, \cdots, p | g_i(x) = 0\}$, then under the regularity condition that there exists $v_0 \in X$ such that for all $i \in I(x)$, $\langle g_i'(x), v_0 \rangle > 0$. Then one has that for all $x \in K$ such that if $I(x) = \emptyset$, $T_K(x) = X$, otherwise

$$T_K(x) = \{v \in X | \langle g_i'(x), v \rangle \geq 0, \forall i \in I(x)\}.$$

Some alternatives to $T_K : X \rightsquigarrow X$ have some nice properties, one is the Clarke cone which has convex closed values [39]. In the following, we give another set-valued map $T_K^c : X \rightsquigarrow X$, whose values belong to the values of T_K [7, p.148].

Definition 6.7.8 *Let $K \subset X$ be closed, $c > 0$, and $x \in K$. The* **global contingent set***, denoted by $T_K^c(x)$, is the subset of all $v \in T_K(x)$ such that there exists a measurable function $\gamma(\cdot)$ bounded by c and satisfying*

$$x + tv + \int_0^t (t - \tau)\gamma(\tau) d\tau \in K.$$

It is noted that if $c_1 \geq c_2 > 0$, then $T_K^{c_1} \supset T_K^{c_2}$. And if $v \in T_K(x)$, then there exists $c > 0$ such that $v \in T_K^c(x)$. One of the nice properties about the global contingent set is that its graph is closed (see Lemma 6.8.2).

6.8 Appendix B: Proofs

PROOF OF THEOREM 6.4.3

Given $F : X \rightsquigarrow X$, consider the following differential inclusion,

$$\dot{x}(t) \in F(x(t)), \text{ for almost all } t \in [0, \infty). \tag{6.51}$$

Define a set-valued map

$$S_F : \mathrm{Dom}(F) \rightsquigarrow \mathcal{C}([0, \infty); X), \tag{6.52}$$

such that $S_F(x)$ is the set of all solutions to the differential inclusion (6.51) starting at $x(0) = x \in \mathrm{Dom}(F)$. S_F is called the **solution map** of differential inclusion (6.31) (see [7]). We have the following result about the solution map which follows from [7, Theorem 3.5.2].

Lemma 6.8.1 *Suppose $F : X \rightsquigarrow X$ is Marchaud, then the set-valued map S_F defined in (6.52) is USC with compact values supplied with the compact convergence topology. Moreover, the graph of the restriction $S_F|_K$ of S_F to any compact subset K of* $\textsc{Dom}(F)$ *is compact in $X \times \mathcal{C}([0,\infty); X)$.*

The proof of Theorem 6.4.3 follows the similar ideas in the proof of existence of viability kernel [7, Theorem 4.1.2].

Proof. **[Theorem 6.4.3]** Let $\mathcal{V}(\Omega) \subset \mathcal{C}([0,\infty); X)$ denote the subset of functions viable in $\Omega \subset \textsc{Dom}(F)$. Define

$$\textsc{Inv}_F(\Omega) := \{x \in \Omega | S_F(x) \subset \mathcal{V}(\Omega)\}.$$

We first show the set $\textsc{Inv}_F(\Omega)$ is closed. In fact, given $x \in \textsc{Inv}_F(\Omega)$, and let $\{x_n\} \subset \textsc{Inv}_F(\Omega)$ be a sequence such that $x_n \rightarrow x$ as $n \rightarrow \infty$. Therefore, there exists a compact set $K \subset X$ such that x and $x_n \in K$. Take a sequence $\{\xi_n\} \subset S_F(x_n)$, then the sequence $\{(x_n, \xi_n)\}$ belongs to $\textsc{Graph}(S_F)|_K$, which is compact by Lemma 6.8.1. Therefore, there exists a subsequence of $\{(x_n, \xi_n)\}$ converging to some $(x, \xi) \in \textsc{Graph}(S_F)|_K$. Therefore, $\xi \in S_F(x) \subset \mathcal{V}(\Omega)$. Hence, $x \in \textsc{Inv}_F(\Omega)$.

Next, we show $\textsc{Inv}_F(\Omega)$ is invariant under F. Indeed, take $x \in \textsc{Inv}_F(\Omega)$, we need to show that any $\xi \in S_F(x)$ is viable in $\textsc{Inv}_F(\Omega)$, i.e., $\xi(T) \in \textsc{Inv}_F(\Omega)$ for all $T > 0$. In fact, let $\xi_T \in S_F(\xi(T))$, define a function ξ_0 as follows,

$$\xi_0(t) := \begin{cases} \xi(t) & \text{if } t \in [0, T]; \\ \xi_T(t-T) & \text{if } t > T. \end{cases}$$

Then ξ_0 is a solution to the differential inclusion starting at x at time 0, and thus, is viable in Ω by the definition of $\textsc{Inv}_F(\Omega)$. Hence for all $t > T$, $\xi_T(t-T) \in \Omega$, therefore $S_F(\xi(T) \subset \mathcal{V}(\Omega)$, i.e., $\xi(T) \in \textsc{Inv}_F(\Omega)$.

Finally, we show $\textsc{Inv}_F(\Omega)$ is the largest invariance set contained in Ω. Indeed, let $K \subset \Omega$ is a closed invariance set of F, then for all $x \in K$, there exists a solution ξ to the differential inclusion starting at x which is viable in K, thus in Ω. Therefore, $x \in \textsc{Inv}_F(\Omega)$. □

PROOFS OF THEOREMS 6.4.8 AND 6.4.16

The set-valued map, $T_K^c : X \rightsquigarrow X$, is defined in Definition 6.7.8. The proofs of the two theorems make use of the following properties of T_K^c [7, Proposition 4.4.2]. The upper limit of a set sequence is defined in Definition 6.5.1.

Lemma 6.8.2 *The graph of the set-valued map $T_K^c : K \rightsquigarrow X$ is closed. In addition, let $K_\infty := \limsup_{n\to\infty} K_n$ denote the upper limit of a sequence of closed subsets K_n. Then*

$$\limsup_{n\to\infty} \text{GRAPH}(T_{K_n}^c) \subset \text{GRAPH}(T_{K_\infty}^c).$$

Proof. **[Theorem 6.4.8]** We first show that K_n^c defined in the Algorithm 6.4.7 is closed for each $n \in \mathbf{Z}^+$. In fact, K_0 is closed by definition. Suppose K_n^c is closed, it is sufficient to show K_{n+1}^c is closed. To this end, take a sequence $\{x_i\} \subset K_{n+1}^c$, such that $x_i \to x \in K_n^c$ as $n\to\infty$. Note that

$$F(x_i) \subset T_{K_n^c}^c(x_i). \tag{6.53}$$

We need to show $F(x) \subset T_{K_n^c}^c(x)$.

In fact, take any $y \in F(x)$, since F is LSC, then there exists a sequence $y_i \in F(x_i)$, such that $y_i \to y$ as $i\to\infty$. Note that from (6.53),

$$(x_i, y_i) \in \text{GRAPH}(T_{K_n^c}^c).$$

Since $\text{GRAPH}(T_{K_n^c}^c)$ is closed by Lemma 6.8.2, one has

$$\lim_{i\to\infty} (x_i, y_i) = (x, y) \in \text{GRAPH}(T_{K_n^c}^c).$$

Thus, $y \in T_{K_n^c}^c(x)$ as required. Therefore, $x_i \to x \in K_{n+1}^c$.

To show K_∞^c is invariant under F, take $x \in K_\infty^c$, we need to verify $F(x) \subset T_{K_\infty^c}^c(x)$.

Note that $x \in K_n^c$ for $n \in \mathbf{Z}^+$, then we have $F(x) \subset T_{K_n^c}^c(x)$. Now for all $y \in F(x)$, then $y \in T_{K_n^c}^c(x)$. Therefore, $(x, y) \subset \text{GRAPH}(T_{K_n^c}^c)$, or

$$(x, y) \subset \bigcap_{n=1}^{\infty} \text{GRAPH}(T_{K_n^c}^c) \subset \text{GRAPH}(T_{K_\infty^c}^c),$$

where the last inclusion is from Lemma 6.8.2. Thus, $y \in T_{K_\infty^c}^c(x)$. □

Next, we prove Theorem 6.4.16.

Proof. **[Theorem 6.4.16]** We first show that K_n^c defined in the Algorithm 6.4.15 is closed for each $n \in \mathbf{Z}^+$. In fact, K_0 is closed by definition. Suppose K_n^c is closed, it is sufficient to show K_{n+1}^c is closed. To this end, take a sequence $\{x_i\} \subset K_{n+1}^c$, such that $x_i \to x \in K_n^c$ as $n\to\infty$. Note that there exists $u_i \in U(x_i)$ such that

$$F(x_i, u_i) \subset T_{K_n^c}^c(x_i) \tag{6.54}$$

for each $i \in \mathbf{Z}^+$. Note that U is locally bounded and USC with closed value. Then there exists a subsequence, still denoted as $\{u_i\}$ without loss of generality, converging to some $u \in U(x)$. Now we show that $F(x, u) \subset T^c_{K^c_n}(x)$.

In fact, take any $y \in F(x, u)$, since F is LSC, then there exists a sequence $y_i \in F(x_i, u_i)$, such that $y_i \to y$ as $i \to \infty$. Note that from (6.54),

$$(x_i, y_i) \in \textsc{Graph}(T^c_{K^c_n}).$$

Since $\textsc{Graph}(T^c_{K^c_n})$ is closed by Lemma 6.8.2, one has

$$\lim_{i\to\infty}(x_i, y_i) = (x, y) \in \textsc{Graph}(T^c_{K^c_n}).$$

Thus, $y \in T^c_{K^c_n}(x)$ as required. Therefore, $x_i \to x \in K^c_{n+1}$.

We next show K^c_∞ is controlled invariant under (F, U). To this end, take $x \in K^c_\infty$, then $x \in K^c_n$ for $n \in \mathbf{Z}^+$, therefore there exists $u_n \in U(x)$ for each $n \in \mathbf{Z}^+$, $F(x, u_n) \subset T^c_{K^c_n}(x)$. Since U is locally bounded and USC with closed value, there exists a subsequence, $\{u_{n_i}\} \subset \{u_n\}$, converging to some $u \in U(x)$. We now show $F(x, u) \subset T^c_{K^c_\infty}(x)$.

In fact, for all $y \in F(x, u)$, there exists a sequence $\{y_{n_i}\}$, such that $y_{n_i} \in F(x, u_{n_i}) \subset T^c_{K^c_{n_i}}(x)$, and $y_{n_i} \to y \in F(x, u)$ as $i \to \infty$. On the other hand,

$$(x, y) = \lim_{i\to\infty}(x, y_{n_i}) \in \textsc{Graph}(T^c_{K^c_{n_i}}).$$

Then $y \in T^c_{K^c_{n_i}}(x)$. Note that

$$K^c_\infty = \bigcap_{n=1}^{\infty} K^c_n = \bigcap_{i=1}^{\infty} K^c_{n_i},$$

then

$$(x, y) \subset \bigcap_{i=1}^{\infty} \textsc{Graph}(T^c_{K^c_{n_i}}) \subset \textsc{Graph}(T^c_{K^c_\infty})$$

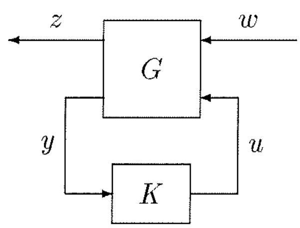
where the last inclusion is from Lemma 6.8.2. Therefore, $y \in T^c_{K^c_\infty}(x)$. $\square$

Chapter 7

$\mathcal{H}_\infty$-Control of Nonlinear Systems: A Class of Controllers

7.1 Introduction

An important issue in control system synthesis is to design a control system which attenuates the effects of external disturbances on some desired signals. In the last chapter, this problem was considered in the setting where the disturbances are persistently $\mathcal{L}_\infty$-Bounded. In this chapter, the disturbances are considered to have finite energy. The standard configuration we will consider is



where G is the generalized plant and K is the controller to be designed; w is the vector of exogenous disturbance inputs and u is the vector of control inputs; z is the the vector of outputs to be regulated; and y is the vector of measured outputs based on which the control action is generated. In this chapter and the next chapter, the performance of a system is measured by $\mathcal{L}_2$-gain. The synthesis problem, known as $\mathcal{H}_\infty$-control, is to find the controller(s) which stabilizes the closed loop system and its $\mathcal{L}_2$-gain is small enough. The general linear time invariant case was first solved by [55], but the resulting state-space formulas and derivations were substantially streamlined in [60]. The simplicity of this characterization together with its clear connections with traditional methods in optimal control have stimulated several attempts to generalize the $\mathcal{H}_\infty$ results in state space to nonlinear systems [183, 93, 15, 112, 185, 91].

Our goal in this chapter is to systematically examine the nonlinear $\mathcal{H}_\infty$-control problem in state space for a class of nonlinear systems and obtain an $\mathcal{H}_\infty$ controller parameterization. This investigation is carried out in a methodical fashion. We follow similar techniques used in the linear case [60], and propose a machinery for nonlinear $\mathcal{H}_\infty$-controller design. Four problems: full information (FI), disturbance feedforward (DF), full control (FC) and output estimation (OE), are also considered. These problems are essential for the constructions of the controllers, but structurally are not the special cases for the output feedback structure of the class of nonlinear systems considered in this paper; in addition, they are important in their own right. Any concept or result in this chapter is local unless otherwise noted. Specifically, problems DF, OE and the output feedback (OF) are treated locally. The sufficient conditions for solvability are obtained and a parameterized class of controllers are derived for each $\mathcal{H}_\infty$-control problems (note that the FI problem has been solved by van der Schaft [183, 93, 91]). Sufficient conditions for the output feedback $\mathcal{H}_\infty$-control problem to be locally solvable are also derived using this machinery. Like the conditions in the linear case, the solvability of the $\mathcal{H}_\infty$-control problem requires the positive definite solutions to two parallel decoupled HJIs with the same numbers of dependent parameters and these two solutions satisfy an additional condition. A class of $\mathcal{H}_\infty$-controllers are parameterized as a nonlinear fractional transformation on contractive, stable free nonlinear operators. It would be appropriate to emphasize at this point that similar sufficient conditions are also obtained by Isidori [91] with different machinery.

The remainder of this chapter is organized as follows: In section 2, some background material related to the $\mathcal{L}_2$-gains is given. In section 3, the $\mathcal{H}_\infty$-control problem is stated and the structure of the general system is simplified. We also give the four nonlinear structures of special problems related to the general system: FI, DF, FC and OE. In section 4, the $\mathcal{H}_\infty$-control problem for four special structures are considered, both the solvability conditions and controller parameterizations are given. In section 5, the main results of this paper, solutions to the output feedback $\mathcal{H}_\infty$-control problem, are given. The solvability of this problem requires the coupled positive definite solutions to two decoupled HJIs. The standard separation principle in this case is re-examined and a class of $\mathcal{H}_\infty$-controllers are parameterized. As an illustrative example, the $\mathcal{H}_\infty$-control design for a passive system is conducted.

7.2 $\mathcal{L}_2$-Gains of Nonlinear Systems

In this section, some background material about $\mathcal{L}_2$-gain analysis of nonlinear systems is provided. [197, 183].

Consider the following affine nonlinear time-invariant (NLTI) system,

$$G : \begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) + k(x)u \end{cases} \tag{7.1}$$

where $x \in \mathbf{R}^n$ is state vector, $u \in \mathbf{R}^p$ and $y \in \mathbf{R}^q$ are input and output vectors, respectively. We will assume $f, g, h, k \in \mathcal{C}^2$, and $f(0) = 0, h(0) = 0$. Therefore, $0 \in \mathbf{R}^n$ is the equilibrium of the system with $u = 0$. The state transition function $\phi : \mathbf{R}^+ \times \mathbf{R}^n \times \mathbf{R}^p \to \mathbf{R}^n$ is so defined that $x = \phi(T, x_0, u^*)$ means that system G evolves from initial state x_0 to state x in time T under the control action u^*.

Definition 7.2.1 *(i) A system G (or $[f(x), g(x)]$) is* **reachable** *from 0 if for all $x \in \mathbf{R}^n$, there exist $T \in R^+$ and $u^*(t) \in \mathcal{L}_2[0,T]$ such that $x = \phi(T, 0, u^*)$;*

(ii) A system G (or $[h(x), f(x)]$) is (zero-state) **detectable** *if for all $x \in \mathbf{R}^n$, $h(\phi(t,x,0)) = 0 \Rightarrow \phi(t,x,0) \rightarrow 0$ as $t \rightarrow \infty$; it is (zero-state)* **observable** *if for all $x \in \mathbf{R}^n$, $h(\phi(t,x,0)) = 0 \Rightarrow \phi(t,x,0) = 0$ for all $t \in \mathbf{R}^+$.*

Definition 7.2.2 *A system G is said to have $\mathcal{L}_2$-gain less than or equal to 1 if*

$$\int_0^T \|y(t)\|^2 \, dt \leq \int_0^T \|u(t)\|^2 \, dt \tag{7.2}$$

for all $T \geq 0$ and $u(t) \in \mathcal{L}_2[0,T]$, and $y(t) = h(x(t)) + k(x(t))u(t)$, as long as $x(t) := \phi(t, 0, u(t) \in \mathbf{X}$.

Note that in the above definition, we take the initial state $x(0) = 0$. Define

$$V_a(x) := \sup_{u \in \mathcal{L}_2[0,\tau], \tau \geq 0, x(0)=x} - \int_0^\tau (\|u(t)\|^2 - \|y(t)\|^2) dt. \tag{7.3}$$

Note that $V_a(x) \geq 0$ for all $x \in \mathbf{X}$, and if the system has $\mathcal{L}_2$-gain ≤ 1, then $V_a(0) = 0$. We will assume $V_a(0) = 0$ from now on.

As pointed out by Willems [197], $V_a(x) < \infty$ if and only if there exists a solution $V : \mathbf{X} \rightarrow \mathbf{R}^+$ with $V(0) = 0$ to the *integral dissipation inequality (IDI)*:

$$\mathcal{D}_I(V, x, u) := V(x) - V(x_0) - \int_0^T (\|u(t)\|^2 - \|y(t)\|^2) dt \leq 0 \tag{7.4}$$

where $x = \phi(T, x_0, u(t))$ and $u(t) \in \mathcal{L}_2[0, T]$, i.e. system G is *dissipative* with respect to supply rate $\|u(t)\|^2 - \|y(t)\|^2$, and $V_a(\cdot)$ is also a solution. Moreover, the solutions to IDI (7.4) form a convex set, and any solution $V(x) \geq 0$ for $x \in \mathbf{X}$ with $V(0) = 0$ satisfies $V(x) \geq V_a(x)$. The following lemma, which is from [196, 197], characterizes $\mathcal{L}_2$-gains.

Lemma 7.2.3 *(i) System G (7.1) has $\mathcal{L}_2$-gain ≤ 1* **if** *$V_a(x) < \infty$, for all $x \in \mathbf{X}$.*

(ii) If system G (7.1) is reachable from 0, then it has $\mathcal{L}_2$-gain ≤ 1 **only if** *$V_a(x) < \infty$, for all $x \in \mathbf{X}$.*

Proof. **(i)** $V_a(x) < \infty$ satisfies IDI (7.4), so

$$\int_0^T (\|u(t)\|^2 - \|y(t)\|^2)dt \geq V_a(x(T)) - V_a(0) = V_a(x(T)) \geq 0$$

for all $T \in \mathbf{R}^+$.

(ii) Take $x \in \mathbf{X}$, by the reachability assumption, there exist $T \in \mathbf{R}^+$ and $u_1(t) \in \mathcal{L}_2[-T, 0)$ such that $x(0) = x$ for $x(-T) = 0$. Now take any $u_2(t) \in \mathcal{L}_2[0, \infty)$. Define $u \in \mathcal{L}_2[-T, \infty)$ as

$$u(t) = \begin{cases} u_1(t), & \text{if } t \in [-T, 0); \\ u_2(t), & \text{if } t \in [0, \infty). \end{cases}$$

Since the NLTI system (7.1) has $\mathcal{L}_2$-gain ≤ 1, for all $\tau \geq 0$, then

$$\int_{-T}^{\tau} \|y(t)\|^2 \, dt \leq \int_{-T}^{\tau} \|u(t)\|^2 \, dt$$

by time-invariance of (7.1), i.e.,

$$-\int_0^{\tau} (\|u_2(t)\|^2 - \|y(t)\|^2)dt \leq \int_{-T}^{0} (\|u_1(t)\|^2 - \|y(t)\|^2)dt.$$

Therefore,

$$V_a(x) = \sup_{u_2 \in \mathcal{L}_2[0,\tau], \tau \geq 0, x(0)=x} -\int_0^{\tau} (\|u_2(t)\|^2 - \|y(t)\|^2)dt \leq \int_{-T}^{0} (\|u_1(t)\|^2 - \|y(t)\|^2)dt < \infty.$$

□

Thus, if the system is reachable from 0, then $\mathcal{L}_2$-gain ≤ 1 if and only if the system is dissipative with respect to supply rate $\|u(t)\|^2 - \|y(t)\|^2$; also $V_a(x) \geq 0$ is well-defined for all $x \in \mathbf{X}$, and there exists a solution $V : \mathbf{X} \to \mathbf{R}^+$ to IDI (7.4). Now define

$$\mathcal{H}(V, x) := \frac{\partial V}{\partial x}(x)(f(x) + g(x)R^{-1}(x)k^T(x)h(x)) +$$

$$\frac{1}{4}\frac{\partial V}{\partial x}(x)g(x)R^{-1}(x)g^T(x)\frac{\partial V^T}{\partial x}(x) + h^T(x)(I - k(x)k^T(x))^{-1}h(x). \tag{7.5}$$

The following lemma characterizes a class of nonlinear systems having $\mathcal{L}_2$-gain ≤ 1.

Lemma 7.2.4 *Consider a system G with $R(x) := I - k^T(x)k(x) > 0$ for all $x \in \mathbf{X}$, suppose G has $\mathcal{L}_2$-gain ≤ 1.*

i) If $V_a(x)$ defined by (7.3) is differentiable with respect to $x \in \mathbf{X}$, then it satisfies Hamilton-Jacobi equation (HJE)*: $\mathcal{H}(V, x) = 0$.*

ii) If $V : \mathbf{X} \to \mathbf{R}$ is differentiable on $\mathbf{X}$ and satisfies IDI (7.4), then it satisfies Hamilton-Jacobi inequality (HJI)*: $\mathcal{H}(V, x) \leq 0$.*

Proof. Part (i) follows from the standard argument (cf. [96]). As for Part (ii), since $V(x)$ is differentiable, the IDI (7.4) reduces to the following *differential dissipation inequality (DDI)*:

$$\begin{aligned} \mathcal{D}_D(V, x, u) &:= \frac{\partial V}{\partial x}(x)(f(x) + g(x)u) - \|u(t)\|^2 + \|y(t)\|^2 \\ &= \dot{V}(x) - \|u(t)\|^2 + \|y(t)\|^2 \leq 0. \end{aligned} \tag{7.6}$$

It follows that $\mathcal{D}_D(V, x, u) \leq 0$ for all $u \in \mathbf{R}^p$ if and only if

$$\sup_{u \in \mathbf{R}^p} \mathcal{D}_D(V, x, u) \leq 0.$$

Let

$$\frac{\partial \mathcal{D}_D(V, x, u)}{\partial u}\Big|_{u=u^*} = 0,$$

then

$$u^* = R^{-1}(x)k^T(x)h^T(x) + \frac{1}{2}R^{-1}(x)g^T(x)\frac{\partial V^T}{\partial x}(x).$$

Thus,

$$\mathcal{D}_D(V, x, u^*) = \sup_{u \in \mathbf{R}^p} \mathcal{D}_D(V, x, u) \leq 0 \qquad \Longleftrightarrow \qquad \mathcal{H}(V, x) \leq 0.$$

□

It is easy to see that the converse results in the above lemma are also true (see [96]). The following statement follows from the above proof.

Corollary 7.2.5 *Suppose $\pi(x)$ is a function defined on $\mathbf{X}$ with $\pi(0) = 0$, then $V(x)$ with $V(0) = 0$ satisfies HJI: $\mathcal{H}(V, x) + \pi(x) \leq 0$ for $x \in \mathbf{X}$ if and only if it satisfies DDI: $\mathcal{D}_D(V, x, u) + \pi(x) \leq 0$, or*

$$\frac{\partial V}{\partial x}(x)(f(x) + g(x)u) \leq \|u(t)\|^2 - \|y(t)\|^2 - \pi(x)$$

for all $u(t) \in \mathbf{R}^p$. Moreover, if $\pi(x) \geq 0$ and $V(x) \geq 0$ for all $x \in \mathbf{X}$, then the $\mathcal{L}_2$-gain ≤ 1.

The above discussion can be summarized as following Theorem which slightly generalizes [183, Theorem 2].

Theorem 7.2.6 *Consider system G (7.1) for $x \in \mathbf{X}$, suppose it is reachable from 0, then each of the following implications holds for all $x \in \mathbf{X}$ under the specified condition.*

$$
\begin{tikzcd}
 & \mathcal{L}_2\text{-}Gain < 1 \arrow[dl, shift left] \arrow[dr, "V_a \in \mathcal{C}^2"] & \\
\mathcal{D}_I(V,x,u) \leq 0 \arrow[ur, shift left, "V(x) \geq 0"] \arrow[d, shift right, "V \in \mathcal{C}^2"'] & & \mathcal{H}(V_a,x) = 0 \arrow[d, "V = V_a"] \\
\mathcal{D}_D(V,x,u) \leq 0 \arrow[u, shift right] \arrow[rr, shift left] & & \mathcal{H}(V,x) \leq 0 \arrow[ll, shift left]
\end{tikzcd}
$$

Recall that $V : \mathbf{R}^n \rightarrow \mathbf{R}^+$ is locally positive-definite if there exists $r > 0$ such that for $x \in \mathcal{B}_r$, $V(x) = 0 \Rightarrow x = 0$; it is globally positive-definite if $V(x) = 0 \Rightarrow x = 0$, and $\lim_{x\rightarrow\infty} V(x) = \infty$. The following lemma, which is due to Hill and Moylan [84] (see also [183]), partially establishes the relationship between finite gain (stability) and asymptotic stability.

Lemma 7.2.7 *(i) Suppose system G with $u = 0$ is asymptotically stable at 0, then any function $V(x)$ on $\mathbf{X}$ with $V(0) = 0$ satisfying IDI: $\mathcal{D}_I(V,x,u) \leq 0$ is non-negative. Specially, if $V(x)$ satisfies HJI: $\mathcal{H}(V,x) \leq 0$ with $V(0) = 0$, then $V(x) \geq 0$ for all $x \in \mathbf{X}$.*

(ii) Assume system G is zero-state detectable. If there is a positive definite solution $V(x)$ on $\mathbf{X}$ to HJI: $\mathcal{H}(V,x) \leq 0$ for all $x \in \mathbf{X}$, then the system G with $u = 0$ is asymptotically stable at 0.

Definition 7.2.8 *The class $\mathcal{FG}$ of (affine) NLTI systems is defined as*

$$\mathcal{FG} := \{G | G \text{ is asymptotically stable and related HJI has a positive definite solution}\}.$$

Therefore, if $G \in \mathcal{FG}$ with state x, then there exists a positive definite $V(x)$ such that $\dot{V}(x) \leq \|u\|^2 - \|y\|^2$ (here $y = Gu$), so it has $\mathcal{L}_2$-gain ≤ 1.

We conclude this section with an observation. Consider the system G, define a new system G_N as

$$G_N : \begin{cases} \dot{x} = f_N(x) + g_N(x)u \\ y_N = h_N(x) \end{cases}$$

with

$$f_N(x) = f(x) - g(x)(I - k^T(x)k(x))^{-1}k^T(x)h(x),$$
$$g_N(x) = g(x)(I - k^T(x)k(x))^{-1/2},$$

and

$$h_N(x) = (I - k(x)k^T(x))^{-1/2}h(x).$$

System G can be simplified as G_N in the following sense.

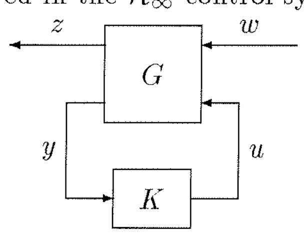
Theorem 7.2.9 *G_N is of $\mathcal{L}_2$-gain ≤ 1, and the related HJE (or HJI) has solution $V_a \geq 0$ (or $V \geq 0$) if and only if G has $\mathcal{L}_2$-gain ≤ 1, and its corresponding HJE (or HJI) also has the same solution. Moreover, G_N is zero-state detectable if and only if G is.*

Proof. Simple algebra shows both systems correspond to the same HJE(or HJI). It is obvious that G_N is detectable if and only if G is. □

7.3 $\mathcal{H}_\infty$-Control: Problem Statement and Special Problems

7.3.1 $\mathcal{H}_\infty$-Control Problem

The basic block diagram considered in the $\mathcal{H}_\infty$-control synthesis problem is



where G is the nonlinear plant with two sets of inputs: the exogenous disturbance inputs w and the control inputs u, and two sets of outputs: the measured outputs y and the regulated outputs z. And K is the controller to be designed. Both G and K are nonlinear time-invariant and can be realized as affine state-space equations:

$$G : \begin{cases} \dot{x} = f(x) + g_1(x)w + g_2(x)u \\ z = h_1(x) + k_{11}(x)w + k_{12}(x)u \\ y = h_2(x) + k_{21}(x)w + k_{22}(x)u \end{cases} \tag{7.7}$$

where $f, g_i, h_i, k_{ij} \in \mathcal{C}^2$, and $f(0) = 0, h_1(0) = 0, h_2(0) = 0$; x, w, u, z, and y are assumed to have dimensions n, p_1, p_2, q_1, and q_2, respectively, and

$$K : \begin{cases} \dot{\hat{x}} &= a(\hat{x}) &+ & b(\hat{x})y \\ u &= c(\hat{x}) &+ & d(\hat{x})y \end{cases} \tag{7.8}$$

with $a, b, c, d \in \mathcal{C}^2$ and $a(0) = 0, c(0) = 0$.

The initial states for both plant and controller are $x(0) = 0$ and $\hat{x}(0) = 0$. The closed loop system will be denoted as nonlinear operator $\Omega(G, K)$ which represents the input/output relation: $z = \Omega(G, K)w$. In the following, we consider the following local $\mathcal{H}_\infty$-control problem.

$\mathcal{H}_\infty$-CONTROL PROBLEM: Find an input-affine feedback controller K (or a class such controllers) for system G (7.7) if any, such that the closed-loop system $\Omega(G, K)$ is locally asymptotically stable with $w = 0$ and has $\mathcal{L}_2$-gain ≤ 1, i.e.,

$$\int_0^T (\|w(t)\|^2 - \|z(t)\|^2) dt \geq 0;$$

for all $T \in \mathbf{R}^+$, as long as the state of the closed loop system $(x, \hat{x}) \in \mathcal{B}_r \times \mathcal{B}_s$ for some $r, s > 0$.

The following assumptions on system structure are made:

[A1]: $k_{11}(x) = 0, k_{22}(x) = 0$;

[A2]: $k_{12}^T(x) \begin{bmatrix} h_1(x) & k_{12}(x) \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$;

[A3]: $\begin{bmatrix} g_1(x) \\ k_{21}(x) \end{bmatrix} k_{21}^T(x) = \begin{bmatrix} 0 \\ I \end{bmatrix}$.

It is known that a large class of nonlinear systems can be simplified to satisfy the above assumption when $\mathcal{H}_\infty$ control problem is considered (cf. [153]).

7.3.2 Special Problems

As in the linear case [60], we will consider four special problems which is essential to the construction of the $\mathcal{H}_\infty$-controllers for the output feedback problem considered in this paper, and will help us to examine the insights of the constructions and structures of nonlinear $\mathcal{H}_\infty$-controllers, especially to reveal the separation property for nonlinear $\mathcal{H}_\infty$-control systems. Structurally, those problem are not the special cases for the output feedback problem considered in this paper; they are also very important in their own right. The reader is referred to [60] for the motivations and interpretations about those problems.

- **Full Information (FI) Problem.**

In this case, both state x and disturbance w are directly available to controller. The plant is

$$G_{FI}:\begin{cases}\dot{x} &= f(x) + g_1(x)w + g_2(x)u\\ z &= h_1(x) + k_{12}(x)u\\ y &= \begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} w\end{cases}$$

where the structural assumption for this structure is

[A2]: $k_{12}^T(x)\begin{bmatrix} h_1(x) & k_{12}(x)\end{bmatrix} = \begin{bmatrix} 0 & I\end{bmatrix}$.

- **Full Control (FC) Problem**

The control action has full access to both state x through output injection and the regulated output. The plant is

$$G_{FC}:\begin{cases}\dot{x} &= f(x) + g_1(x)w + \begin{bmatrix} I & 0\end{bmatrix}u\\ z &= h_1(x) + \begin{bmatrix} 0 & I\end{bmatrix}u\\ y &= h_2(x) + k_{21}(x)w\end{cases}$$

with

[A3]: $\begin{bmatrix} g_1(x) \\ k_{21}(x)\end{bmatrix} k_{21}^T(x) = \begin{bmatrix} 0 \\ I\end{bmatrix}$.

- **Disturbance Feedforward (DF) Problem**

$$G_{DF}:\begin{cases}\dot{x} &= f(x) + g_1(x)w + g_2(x)u\\ z &= h_1(x) + k_{12}(x)u\\ y &= h_0(x) + w\end{cases}$$

where

[A2]: $k_{12}^T(x)\begin{bmatrix} h_1(x) & k_{12}(x)\end{bmatrix} = \begin{bmatrix} 0 & I\end{bmatrix}$.

- **Output Estimation (OE) Problem**

$$G_{OE}:\begin{cases}\dot{x} &= f(x) + g_1(x)w + g_0(x)u\\ z &= h_1(x) + u\\ y &= h_2(x) + k_{21}(x)w\end{cases}$$

where

[A3]: $\begin{bmatrix} g_1(x) \\ k_{21}(x) \end{bmatrix} k_{21}^T(x) = \begin{bmatrix} 0 \\ I \end{bmatrix}.$

In the linear case, the two pairs, FI and FC, DF and OE, are dual in structure. But it would be un-realistic to expect the same thing happens in the nonlinear case. In the next section, we will mainly consider those special problems.

7.4 $\mathcal{H}_\infty$-Control Synthesis: Solutions to Special Problems

Unlike linear case, the solutions for the special $\mathcal{H}_\infty$-control problems cannot be obtained by duality (if there is any). This section is devoted to the discussion of different special problems.

7.4.1 Full Information Problem

Consider

$$G_{FI}: \begin{cases} \dot{x} &= f(x) &+ & g_1(x)w &+ & g_2(x)u \\ z &= h_1(x) &+ & & & k_{12}(x)u \\ y &= \begin{bmatrix} x \\ 0 \end{bmatrix} &+ & \begin{bmatrix} 0 \\ I \end{bmatrix} w & & \end{cases} \tag{7.9}$$

with $x \in \mathbf{X}$. In addition to the structural constraints **[A2]**, we further assume

[A4]: $[h_1(x), f(x)]$ is zero-state detectable.

The $\mathcal{H}_\infty$-control problem for FI was first explicitly introduced and solved in [183] (see also [91]). The solutions to $\mathcal{H}_\infty$-control problem are related to the following HJI,

$$\mathcal{H}_{FI}(V,x) := \frac{\partial V}{\partial x}(x)f(x) + \frac{1}{4}\frac{\partial V}{\partial x}(x)(g_1(x)g_1^T(x) - g_2(x)g_2^T(x))\frac{\partial V^T}{\partial x}(x) + h_1^T(x)h_1(x) \leq 0. \tag{7.10}$$

The following theorem reveals more properties related to HJI for FI.

Theorem 7.4.1 *(i) $\mathcal{H}_{FI}(V,x) \leq 0$ ($x \in \mathbf{X}$) has a solution $V(x)$ with $V(0) = 0$ if and only if there is $F_0(x)$ such that*

$$\mathcal{H}_{SF}(V, F_0, x) := \frac{\partial V}{\partial x}(x)(f(x) + g_2(x)F_0(x)) + \frac{1}{4}\frac{\partial V}{\partial x}(x)g_1(x)g_1^T(x)\frac{\partial V^T}{\partial x}(x) +$$
$$+ (h_1(x) + k_{12}(x)F_0(x))^T(h_1(x) + k_{12}(x)F_0(x)) \leq 0.$$

Moreover, if $V(x)$ solves $\mathcal{H}_{FI}(V,x) \leq 0$ with $V(0)=0$, then $F_0(x)$ can be taken as

$$F_0(x) = -\frac{1}{2} g_2^T(x) \frac{\partial V^T}{\partial x}(x).$$

(ii) If $[h_1(x), f(x)]$ is assumed to be zero-state observable, then any solution $V(x) \geq 0$ to $\mathcal{H}_{FI}(V,x) \leq 0$ with $V(0)=0$ is positive definite.

Property (ii) insures that the HJI corresponding to FI has a positive definite solution under some mild assumption (observability). This justifies the assumption that HJI has positive definite solutions in the next Theorem.

Proof. (i) Note that there exists $F(x)$ such that

$$\mathcal{H}_{SF}(V, F_0, x) \leq 0,$$

if and only if

$$0 \geq \inf_{F_0(x)} \mathcal{H}_{SF}(V, F_0, x) = \mathcal{H}_{SF}(V, F_0, x)\big|_{F_0(x) = -\frac{1}{2} g_2^T(x) \frac{\partial V^T}{\partial x}(x)} = \mathcal{H}_{FI}(V, x).$$

The latter can be confirmed by taking

$$\frac{\partial \mathcal{H}_{SF}}{\partial F_0}(V, F_0, x) = 0$$

for fixed x.

(ii) Suppose $V(x) \geq 0$ is such that $\mathcal{H}_{FI}(V,x) \leq 0$. Then by the completion of square technique, it follows that

$$\dot{V}(x) = \frac{\partial V}{\partial x}(x)(f(x) + g_1(x)w + g_2(x)u)$$

$$\leq \left\| u + \frac{1}{2} g_2^T(x) \frac{\partial V^T}{\partial x}(x) \right\|^2 - \left\| w - \frac{1}{2} g_1^T(x) \frac{\partial V^T}{\partial x}(x) \right\|^2 - \|z\|^2 + \|w\|^2 .$$

If $u = -\frac{1}{2} g_2^T(x) \frac{\partial V^T}{\partial x}(x)$, then

$$\dot{V}(x) \leq -\|z\|^2 + \|w\|^2 .$$

Since $V(x) \geq 0$, the closed-loop system

$$\begin{cases} \dot{x} = f(x) - \frac{1}{2} g_2(x) g_2^T(x) \frac{\partial V^T}{\partial x}(x) + g_1(x) w \\ z = h_1(x) - \frac{1}{2} k_{12}(x) g_2^T(x) \frac{\partial V^T}{\partial x}(x) \end{cases}$$

is dissipative with respect to supply rate $\|w\|^2 - \|z\|^2$. Thus, for all $x \in \mathbf{R}^n$,

$$V(x) \geq \sup_{w \in \mathcal{L}_2[0,\tau], \tau \geq 0, x(0)=x} - \int_0^\tau (\|w(t)\|^2 - \|z(t)\|^2) dt$$

$$\geq - \int_0^\infty (0 - \|z(t)\|^2) dt = \int_0^\infty (\|z(t)\|^2) dt.$$

Therefore, $V(x) = 0 \Rightarrow \|z\|^2 = \|h_1(x)\|^2 + \frac{1}{4}\left\|g_2^T(x)\frac{\partial V^T}{\partial x}(x)\right\|^2 = 0$, so

$$\begin{cases} \dot{x} = f(x) - \frac{1}{2}g_2(x)g_2^T(x)\frac{\partial V^T}{\partial x}(x) \\ z = h_1(x) - \frac{1}{2}k_{12}(x)g_2^T(x)\frac{\partial V^T}{\partial x}(x) = 0 \end{cases} \Leftrightarrow \begin{cases} \dot{x} = f(x) \\ h_1(x) = 0 \end{cases}.$$

Thus, $x = 0$ by observability assumption. (Moreover, the closed-loop system is also observable.) $V(x)$ is (locally) positive definite. □

The solutions to the $\mathcal{H}_\infty$-control problem for FI in the above setting, which are given in [183, 91, 93], are included here for completeness.

Theorem 7.4.2 *Consider G_{FI} (7.9), suppose there exists a local positive definite function $V(x) \geq 0$ such that $\mathcal{H}_{FI}(V, x) \leq 0$, for $x \in \mathbf{B}_r$ with some $r > 0$, and $V(0) = 0$. Then the $\mathcal{H}_\infty$-control problem for FI is solvable, and such a FI $\mathcal{H}_\infty$-controller is provided by*

$$u = -\frac{1}{2}g_2^T(x)\frac{\partial V^T}{\partial x}(x).$$

Moreover, for all $Q \in \mathcal{FG}$, the controller

$$u = -\frac{1}{2}g_2^T(x)\frac{\partial V^T}{\partial x}(x) + Q(w - \frac{1}{2}g_1^T(x)\frac{\partial V^T}{\partial x}(x))$$

solves the FI $\mathcal{H}_\infty$ control problem.

Remark 7.4.3 *If $V(x)$ is globally positive definite, and $H_{FI}(V, x) \leq 0$ for all $x \in \mathbf{R}^n$, then Theorem 7.4.1 gives the global solution. The same thing happens for FC $\mathcal{H}_\infty$-control problem to be investigated later.*

7.4.2 Disturbance Feedforward Problem

Consider

$$G_{DF} : \begin{cases} \dot{x} &= f(x) + g_1(x)w + g_2(x)u \\ z &= h_1(x) + k_{12}(x)u \\ y &= h_0(x) + w \end{cases} \tag{7.11}$$

where $f, g_1, g_2, h_1, h_0, k_{12} \in \mathcal{C}^2$ under the constraints **[A2]**.

The additional assumptions relevant to DF problem are as follows,

[A4]: $[h_1(x), f(x)]$ is (locally) zero-state detectable;

[A6]: $\dot{x} = f(x) + g_1(x)h_0(x)$ is locally exponentially stable at 0.

By converse Lyapunov Theorem, assumption **[A6]** implies that there exists a locally positive definite function $U : \mathbf{R}^n \rightarrow \mathbf{R}^+$, such that

$$L_D(U, x) := \frac{\partial U}{\partial x}(x)(f(x) + g_1(x)h_0(x)) \tag{7.12}$$

is locally negative definite. Furthermore, the Hessian matrix of $L_D(U, x)$ is nonsingular at $x = 0$ by suitably choosing the Lyapunov function $U(x)$. And if $U(x)$ has these properties, so does $kU(x)$ for all constant $k > 0$.

Theorem 7.4.4 *Consider G_{DF} (7.11), suppose there exists a smooth solution $V(x) \geq 0$ with $V(0) = 0$ to $\mathcal{H}_{FI}(V, x) \leq 0$ for $x \in \mathbf{B}_r$. Then the $\mathcal{H}_\infty$-control problem for DF is solvable. Moreover, the controller given by*

$$\begin{cases} \dot{\tilde{x}} = f(\tilde{x}) - g_1(\tilde{x})h_0(\tilde{x}) + g_0(\tilde{x})F_0(\tilde{x}) + g_1(\tilde{x})y \\ u = F_0(\tilde{x}) \end{cases} \qquad F_0(\tilde{x}) := -\frac{1}{2}g_2^T(\tilde{x})\frac{\partial V^T}{\partial \tilde{x}}(\tilde{x})$$

solves the $\mathcal{H}_\infty$-control problem.

Consider the following system

$$P_{DF} : \begin{cases} \dot{\tilde{x}} &= f(\tilde{x}) - g_1(\tilde{x})h_0(\tilde{x}) &+ \quad g_1(\tilde{x})y &+ \quad g_2(\tilde{x})u \\ u &= & & u \\ y_0 &= \begin{bmatrix} \tilde{x} \\ -h_0(\tilde{x}) \end{bmatrix} &+ \begin{bmatrix} 0 \\ I \end{bmatrix} y & \end{cases}.$$

We have the following lemma.

Lemma 7.4.5 *Given two functions $V, U : \mathbf{R}^n \rightarrow \mathbf{R}$, where the function $V(x) \geq 0$ with $V(0) = 0$ additionally satisfies HJI: $\mathcal{H}_{FI}(V, x) \leq 0$ for $x \in \mathbf{B}_r$. Let $x, \tilde{x}$ be states of systems G_{DF} and P_{DF}, $e := x - \tilde{x}$. Define*

$$S(U, e, \tilde{x}) := \frac{\partial U}{\partial e}(e)[(f(x) - f(\tilde{x})) - g_1(\tilde{x})(h_0(x) - h_0(\tilde{x}))];$$

$$R_1(e, \tilde{x}) := h_1^T(x)h_1(x) - h_1^T(\tilde{x})h_1(\tilde{x});$$

$$R_2(e, \tilde{x}) := \frac{1}{4}\left\|(g_1^T(x) - g_1^T(\tilde{x}))\frac{\partial U^T}{\partial e}(e)\right\|^2 + \frac{1}{2}\frac{\partial V}{\partial \tilde{x}}(\tilde{x})g_1(\tilde{x})(g_1^T(x) - g_1^T(\tilde{x}))\frac{\partial U^T}{\partial e}(e)$$

$$-\frac{1}{2}\frac{\partial V}{\partial \tilde{x}}(\tilde{x})g_2(\tilde{x})(g_2^T(x) - g_2^T(\tilde{x}))\frac{\partial U^T}{\partial e}(e).$$

Then under the assumption **[A6]**, *there exists locally positive definite function $U(x)$ such that for all $x, \tilde{x} \in \mathcal{B}_r$ with some $r > 0$,*

(i) $S(U, e, \tilde{x}) \leq 0$,

(ii) $S(U, e, \tilde{x}) + R_1(e, \tilde{x}) + R_2(e, \tilde{x}) \leq 0$.

Proof. By assumption **[A6]**, there exists locally positive definite function $U_o(x)$ such that $L_D(U_o, x)$ defined by (7.12) locally negative definite and the Hessian matrix of $L_D(U_o, x)$ is nonsingular at $x = 0$. Note that $S(U_o, e, \tilde{x})$ has the same Hessian matrix with respect to e at $(e, \tilde{x}) = 0$ as does $L_D(U_o, e)$ at $e = 0$. Therefore, for $\tilde{x} \in \mathcal{B}_r$ for some $r > 0$ small enough, $S(U_o, e, \tilde{x})$ is dominated by its quadratic term of e which is the same as the one of $L_D(U_o, e)$, so $S(U_o, e, \tilde{x}) \leq 0$ for $x, \tilde{x} \in \mathcal{B}_r$ with the adjusted $r > 0$. This confirms (i). Also note that if $U_o(x)$ is such that $L_D(U_o, e)$ has negative definite Hessian matrix with respect to e at 0, so is $kU_o(x)$ for all constant $k > 0$. Thence, we can choose $k > 0$ large enough such that $S(kU_o, e, \tilde{x}) + R_1(e, \tilde{x})$ also has negative Hessian matrix with respect to e at $(e, \tilde{x}) = 0$; it is dominated by its quadratic term of e. By adding the higher order term $R_2(e, \tilde{x})$, we can conclude that $S(U, e, \tilde{x}) + R_1(e, \tilde{x}) + R_2(e, \tilde{x}) \leq 0$ with $U(x) := kU_o(x)$ for $x, \tilde{x} \in \mathcal{B}_r$ with adjusted $r > 0$. This confirms (ii). Note that the new U also works for (i). □

Proof. **[Theorem 7.4.4]** Consider $\Sigma(G_{DF}, P_{DF})$, which has following realization

$$\left\{ \begin{array}{rcllll} \dot{x} & = & f(x) & + & g_1(x)w & + \quad g_2(x)u \\ \dot{\tilde{x}} & = & (f(\tilde{x}) - g_1(\tilde{x})h_0(\tilde{x}) + g_1(\tilde{x})h_0(x)) & + & g_1(\tilde{x})w & + \quad g_2(\tilde{x})u \\ z & = & h_1(x) & + & & \quad k_{12}(x)u \\ y & = & \begin{bmatrix} \tilde{x} \\ h_0(x) - h_0(\tilde{x}) \end{bmatrix} & + & \begin{bmatrix} 0 \\ I \end{bmatrix} w & \end{array} \right. .$$

Let $e := x - \tilde{x}$, define

$$W(e, \tilde{x}) := U(e) + V(\tilde{x}),$$

where $U(x)$ is given in the above lemma, then $W(e, \tilde{x}) \geq 0$, and $W(0, 0) = U(0) + V(0) = 0$. For all $(x, \tilde{x}) \in \mathcal{B}_r$,

$$\dot{W}(e, \tilde{x}) = \dot{U}(e) + \dot{V}(\tilde{x})$$

$$= \frac{\partial U}{\partial e}(e)((f(x) - f(\tilde{x})) - g_1(\tilde{x})(h_0(x) - h_0(\tilde{x})))+$$

$$\frac{\partial U}{\partial e}(e)((g_1(x) - g_1(\tilde{x}))w + (g_2(x) - g_2(\tilde{x}))u)+$$

$$\frac{\partial V}{\partial \tilde{x}}(\tilde{x})(f(\tilde{x}) + g_1(\tilde{x})w + g_2(\tilde{x})u) + \frac{\partial V}{\partial \tilde{x}}(\tilde{x})g_1(\tilde{x})(h_0(x) - h_0(\tilde{x}))$$

$$\leq S(U, e, \tilde{x}) + \frac{\partial U}{\partial e}(e)(g_1(x) - g_1(\tilde{x}))w + \frac{\partial U}{\partial e}(e)(g_2(x) - g_2(\tilde{x}))u-$$

$$\frac{1}{4}\frac{\partial V}{\partial e}(e)(g_1(\tilde{x})g_1^T(\tilde{x}) - g_2(\tilde{x})g_2^T(\tilde{x}))\frac{\partial V^T}{\partial e}(e) - h_1^T(\tilde{x})h_1(\tilde{x})+$$

$$\frac{\partial V}{\partial \tilde{x}}(\tilde{x})(g_1(\tilde{x})w + g_2(\tilde{x})u) + \frac{\partial V}{\partial \tilde{x}}(\tilde{x})g_1(\tilde{x})(h_0(x) - h_0(\tilde{x}))$$

$$= S(U, e, \tilde{x}) + R_1(e, \tilde{x}) + R_2(e, \tilde{x})+$$

$$\left\| u + \frac{1}{2}g_2^T(\tilde{x})\frac{\partial V^T}{\partial \tilde{x}}(\tilde{x}) + \frac{1}{2}(g_2^T(x) - g_2^T(\tilde{x}))\frac{\partial U^T}{\partial e}(e)\right\|^2 - \frac{1}{4}\left\|(g_2^T(x) - g_2^T(\tilde{x}))\frac{\partial U^T}{\partial e}(e)\right\|^2 -$$

$$\left\| w - \frac{1}{2}g_1^T(\tilde{x})\frac{\partial V^T}{\partial \tilde{x}}(\tilde{x}) - \frac{1}{2}(g_1^T(x) - g_1^T(\tilde{x}))\frac{\partial U^T}{\partial e}(e)\right\|^2 - \|z\|^2 + \|w\|^2$$

$$\leq \left\| u + \frac{1}{2}g_2^T(\tilde{x})\frac{\partial V^T}{\partial \tilde{x}}(\tilde{x}) + \frac{1}{2}(g_2^T(x) - g_2^T(\tilde{x}))\frac{\partial U^T}{\partial e}(e)\right\|^2 - \frac{1}{4}\left\|(g_2^T(x) - g_2^T(\tilde{x}))\frac{\partial U^T}{\partial e}(e)\right\|^2 -$$

$$\left\| w - \frac{1}{2}g_1^T(\tilde{x})\frac{\partial V^T}{\partial \tilde{x}}(\tilde{x}) - \frac{1}{2}(g_1^T(x) - g_1^T(\tilde{x}))\frac{\partial U^T}{\partial e}(e)\right\|^2 - \|z\|^2 + \|w\|^2 .$$

If we take

$$u = -\frac{1}{2}g_2^T(\tilde{x})\frac{\partial V^T}{\partial \tilde{x}}(\tilde{x}) = F_0(\tilde{x}),$$

then

$$\dot{W}(e, \tilde{x}) \leq -\|z\|^2 + \|w\|^2 .$$

Thus,

$$\int_0^T (\|w\|^2 - \|z\|^2)dt \geq W(e(T), \tilde{x}(T)) - W(0,0) = W(e(T), \tilde{x}(T)) \geq 0.$$

for all $T \geq 0$ and $(x, \tilde{x}) \in \mathcal{B}_r$.

Next, consider the asymptotic stability of the closed-loop system. It is sufficient to show if $(x(0), \tilde{x}(0)) \in \mathcal{B}_r$, then $(x(t), \tilde{x}(t)) \to 0$ or $(e(t), x(t)) \to 0$ as $t \to \infty$. Take

$$u = -\frac{1}{2}g_2^T(\tilde{x})\frac{\partial V^T}{\partial \tilde{x}}(\tilde{x}) \quad \text{and} \quad w = 0.$$

$V(\tilde{x})$ and $U(e)$ is positive definite by assumptions. Therefore $W(e, \tilde{x}) = U(e) + V(\tilde{x})$ is also positive definite, and it can be used as a Lyapunov function.

We already have

$$\dot{W}(e,\tilde{x}) \leq -\|z\|^2 + \|w\|^2 = -\|z\|^2 \leq 0.$$

Now $\dot{W}(e,\tilde{x}) = 0 \quad \Rightarrow \quad z = 0$, then $x(t) \to 0$ as $t \to \infty$ by assumption **[A4]**. On the other hand, $\dot{U}(e) = T(U, e, \tilde{x})$ is negative definite for $(x, \tilde{x}) \in \mathcal{B}_r$ by previous lemma, then $e(t) \to 0$ as $t \to \infty$. By LaSalle's Theorem, the closed loop system is (locally) asymptotically stable.

Finally, the DF controller $u = \Omega(P_{DF}, K_{FI})y$ is recovered as

$$\begin{cases} \dot{\tilde{x}} = f(\tilde{x}) - \frac{1}{2}g_2(\tilde{x})g_2^T(\tilde{x})\frac{\partial V^T}{\partial \tilde{x}}(\tilde{x}) - g_1(\tilde{x})h_0(\tilde{x}) - g_1(\tilde{x})y \\ u = -\frac{1}{2}g_2^T(\tilde{x})\frac{\partial V^T}{\partial \tilde{x}}(\tilde{x}) \end{cases}.$$

□

Next, we consider the $\mathcal{H}_\infty$-controller parameterization. Denote

$$\epsilon_u(x,\tilde{x}) := \frac{1}{2}(g_2^T(x) - g_2^T(\tilde{x}))\frac{\partial U^T}{\partial e}(e), \quad \epsilon_w(x,\tilde{x}) := \frac{1}{2}(g_1^T(x) - g_1^T(\tilde{x}))\frac{\partial U^T}{\partial e}(e).$$

Define

$$u^* := -\frac{1}{2}g_2^T(\tilde{x})\frac{\partial V^T}{\partial \tilde{x}}(\tilde{x}) - \epsilon_u(x,\tilde{x}), \quad w^* := \frac{1}{2}g_1^T(\tilde{x})\frac{\partial V^T}{\partial \tilde{x}}(\tilde{x}) + \epsilon_u(x,\tilde{x}).$$

From the previous proof,

$$\dot{W}(e,\tilde{x}) \leq \|u - u^*\|^2 - \|w - w^*\|^2 - \|z\|^2 + \|w\|^2.$$

If

$$u = u^* + Q(w - w^*)$$

with $Q \in \mathcal{FG}$, let ξ be the state variable for Q, and U_Q be a solution to the HJI with respect to Q, then

$$\dot{U}_Q(\xi) \leq \|w - w^*\|^2 - \|u - u^*\|^2.$$

Therefore,

$$\dot{W}(e,\tilde{x}) + \dot{U}_Q(\xi) \leq -\|z\|^2 + \|w\|^2,$$

so

$$\int_0^T (\|w\|^2 - \|z\|^2)dt \geq W(e(T), \tilde{x}(T)) + U_Q(\xi(T)) \geq 0.$$

It seems that the controller recovered from $u = u^* + Q(w - w^*)$ solves the $\mathcal{H}_\infty$-control problem for DF. However, it can hardly be physically implemented, since (i) Q requires some extra high order terms $\epsilon_u(x,\tilde{x})$ and $\epsilon_w(x,\tilde{x})$ as parts of its input and output, but they can not be provided by the closed loop system; and (ii) the term $h_0(x) - h_0(\tilde{x})$, which is a part of the measured output, does not appear in the required input for Q. But, fortunately, the terms $\epsilon_u(x,\tilde{x})$ and $\epsilon_w(x,\tilde{x})$ can be eliminated by the assumption **[A6]**; and $h_0(x) - h_0(\tilde{x})$ is actually the measured noise introduced by the controller. It is reasonable to take $w_{DF} = w + h_0(x) - h_0(\tilde{x})$ as the total disturbance.

The DF $\mathcal{H}_\infty$-control problem can be modified as follows.

DF $\mathcal{H}_\infty$ CONTROL PROBLEM: Find a class of controllers such that the closed loop system is locally asymptotically stable and satisfies

$$\int_0^T (\|z\|^2 - \|w_{DF}\|^2)dt \le 0$$

for all $T \ge 0$, as long as the state of the closed loop system $x, \tilde{x} \in \mathcal{B}_r$; i.e., the controllers attenuate the external disturbance and the measured noise introduced by itself.

Theorem 7.4.6 *Under the assumptions of previous Theorem, define*

$$F_0(\tilde{x}) := -\frac{1}{2} g_2^T(\tilde{x}) \frac{\partial V^T}{\partial \tilde{x}}(\tilde{x}), \qquad F_1(\tilde{x}) := \frac{1}{2} g_1^T(\tilde{x}) \frac{\partial V^T}{\partial \tilde{x}}(\tilde{x}).$$

The controller $u = \Omega(M_{FD}, Q)y$ with M_{DF} given by

$$\begin{cases} \dot{\tilde{x}} &= f(\tilde{x}) + g_2(\tilde{x})F_0(\tilde{x}) - g_1(\tilde{x})h_0(\tilde{x}) &+& g_1(\tilde{x})y &+& g_2(\tilde{x})u_0 \\ u &= F_0(\tilde{x})) & & & & u_0 \\ y_0 &= -h_0(\tilde{x}) - F_1(\tilde{x}) &+& y & & \end{cases}$$

for all $Q \in \mathcal{FG}$ also solves the DF $\mathcal{H}_\infty$-control problem.

Proof. Consider system $\Omega(G_{DF}, \Omega(M_{DF}, Q))$, assume ξ is the state of Q. Take $W(e,\tilde{x})$ the same as in previous them, the same arguments in previous proof yield that, for all $(x, \tilde{x}, \xi) \in \mathcal{B}_r$,

$$\dot{W}(e,\tilde{x}) \le \left\| u + \frac{1}{2} g_2^T(\tilde{x}) \frac{\partial V^T}{\partial \tilde{x}}(\tilde{x}) \right\|^2 - \left\| w_{DF} - \frac{1}{2} g_1^T(\tilde{x}) \frac{\partial V^T}{\partial \tilde{x}}(\tilde{x}) \right\|^2 - \|z\|^2 + \|w_{DF}\|^2$$

Since $Q \in \mathcal{FG}$, then there is a positive definite $U_Q(\xi)$ related to Q such that

$$\dot{U}_Q(\xi) \le - \left\| u + \frac{1}{2} g_2^T(\tilde{x}) \frac{\partial V^T}{\partial \tilde{x}}(\tilde{x}) \right\|^2 + \left\| w_{DF} - \frac{1}{2} g_1^T(\tilde{x}) \frac{\partial V^T}{\partial \tilde{x}}(\tilde{x}) \right\|^2 .$$

So

$$\dot{W}(e,\tilde{x}) + \dot{U}_Q(\xi) \le - \|z\|^2 + \|w_{DF}\|^2 .$$

It can be concluded that

$$\int_0^T (\|w_{DF}\|^2 - \|z\|^2)dt \geq 0, \forall T \in R^+.$$

Next, consider the stability of the closed loop system whose state is $(e, \tilde{x}, \xi)$, set $w = 0$, then it is of the form

$$\begin{cases} \dot{e} = \alpha(t, e) \\ \dot{\tilde{x}} = \beta(t, e, \tilde{x}, \xi) \\ \dot{\xi} = a(\xi) + b(\xi)y_0 \end{cases}$$

where $y_0|_{(e,\tilde{x})=0} = 0$. Notice that the subsystems with states e, $\tilde{x}$ and ξ can be viewed as being hierarchically interconnected[1]. Assume the closed loop system evolves in $\mathcal{B}_r$. $e \to 0$ as $t \to \infty$ by the similar argument in the proof of previous Theorem. Consider the connected system with state $(\tilde{x}, \xi)$; $L_{DF}(\tilde{x}, \xi) := W(e, \tilde{x})|_{e=0} + U_Q(\xi)$ is positive definite, and can be used as the Lyapunov function. Let $w = 0$. Since $e = \tilde{x} - x = 0$, $w_{DF} = 0$, then

$$\dot{L}_{DF}(\tilde{x}, \xi) \leq -\|z\|^2 \leq 0,$$

thus $\dot{L}_{DF}(\tilde{x}, \xi) = 0$ implies $z = 0$, so $\tilde{x}(t) = x(t) \to 0$ as $t \to \infty$. On the other hand, $\tilde{x} = 0 \to y_0 = 0$, this also implies $\xi(t) \to 0$ as $t \to \infty$. LaSalle's Theorem implies interconnected system with states $(\tilde{x}, \xi)$ is asymptotically stable. By the stability Theorem of hierarchical systems [191], the closed loop system is locally asymptotically stable.

□

7.4.3 Full Control Problem

The main purpose of considering FC problem is to provide some required tools for the construction of output feedback $\mathcal{H}_\infty$-controllers. The system is as follows,

$$G_{FC} : \begin{cases} \dot{x} &= f(x) &+ g_1(x)w &+ \begin{bmatrix} I & 0 \end{bmatrix} u \\ z &= h_1(x) &+ & \begin{bmatrix} 0 & I \end{bmatrix} u \\ y &= h_2(x) &+ k_{21}(x)w & \end{cases} \tag{7.13}$$

with constraint **[A3]**. An additional assumption for this structure is as follows,

[A5]: $[h_2(x), f(x)]$ is zero-state detectable.

[1]Strictly speaking, the subsystems are not hierarchically interconnected; this issue is discussed in detail in [108].

The solvability of $\mathcal{H}_\infty$-control problem to FC is also related to the HJI:

$$\mathcal{H}_{FC}(U,x) := \frac{\partial U}{\partial x}(x)f(x) + \frac{1}{4}\frac{\partial U}{\partial x}(x)g_1(x)g_1^T(x)\frac{\partial U^T}{\partial x}(x) + h_1^T(x)h_1(x) - h_2^T(x)h_2(x) \le 0. \tag{7.14}$$

Theorem 7.4.7 *(i) If $U(x)$ with $U(0)=0$ satisfies*

$$\mathcal{H}_{OI}(U,L_0,x) := \frac{\partial U}{\partial x}(x)(f(x)+L_0(x)h_2(x)) +$$

$$\frac{1}{4}\frac{\partial U}{\partial x}(x)(g_1(x)+L_0(x)k_{21}(x))(g_1(x)+L_0(x)k_{21}(x))^T\frac{\partial U^T}{\partial x}(x) + h_1^T(x)h_1(x) \le 0$$

for some $L_0(x)$, then $U(x)$ satisfies $\mathcal{H}_{FC}(U,x)\le 0$ with $U(0)=0$ as well.

Conversely, if $U(x)$ satisfies $\mathcal{H}_{FC}(U,x)\le 0$ with $U(0)=0$, and $L_0(x)$ is such that

$$\frac{\partial U}{\partial x}(x)L_0(x) = -2h_2^T(x),$$

then $\mathcal{H}_{OI}(U,L_0,x)\le 0$.

(ii) If system $[h_2(x), f(x)]$ is zero-state observable, suppose $U(x)\ge 0$ solves $\mathcal{H}_{FC}(U,x)\le 0$ with $U(0)=0$ and

$$\frac{\partial U}{\partial x}(x)L_0(x) = -2h_2^T(x)$$

has a solution $L_0(x)$, then $U(x)$ is positive definite.

Proof. (i) Notice that there exists $L_0(x)$ such that

$$\mathcal{H}_{OI}(U,L_0,x)\le 0$$

only if

$$0 \ge \inf_{\frac{\partial U}{\partial x}(x)L_0(x)} \mathcal{H}_{OI}(U,L_0,x) = \mathcal{H}_{OI}(U,L_0,x)|_{\frac{\partial U}{\partial x}(x)L_0(x)=-2h_2^T(x)} = \mathcal{H}_{FC}(U,x).$$

It is also sufficient if $\frac{\partial U}{\partial x}(x)L_0(x) = -2h_2^T(x)$ has a solution $L_0(x)$.

(ii) Will be proved during the proof of the following Theorem. □

Theorem 7.4.8 *Suppose $U:\mathbf{R}^n\to\mathbf{R}^+$ is locally positive definite such that $\mathcal{H}_{FC}(U,x)\le 0$. If $L_0(x)$ satisfies*

$$\frac{\partial U}{\partial x}(x)L_0(x) = -2h_2^T(x), \tag{7.15}$$

then the controller given by "output injection"

$$u(x) = \begin{bmatrix} L_0(x) \\ 0 \end{bmatrix} y$$

solves the $\mathcal{H}_\infty$-control problem for FC.

Proof. (i) $\mathcal{L}_2$**-gain** ≤ 1. Just $U(x) \geq 0$ is assumed.

$$\dot{U}(x) = \frac{\partial U}{\partial x}(x)(f(x) + g_1(x)w + \begin{bmatrix} I & 0 \end{bmatrix} u)$$

$$\leq -\frac{1}{4}\frac{\partial U}{\partial x}(x)g_1(x)g_1^T(x)\frac{\partial U^T}{\partial x}(x) - h_1^T(x)h_1(x) + h_2^T(x)h_2(x) +$$

$$\frac{\partial U}{\partial x}(x)g_1(x)w + \begin{bmatrix} \frac{\partial U}{\partial x}(x) & 0 \end{bmatrix} u$$

$$= \left\| \begin{bmatrix} 0 & I \end{bmatrix} u \right\|^2 - \left\| w - \frac{1}{2}g_1^T(x)\frac{\partial U^T}{\partial x}(x) + k_{21}^T(x)h_2(x) \right\|^2 - \|z\|^2 + \|w\|^2 +$$

$$(\begin{bmatrix} \frac{\partial U}{\partial x}(x) & 2h_1^T(x) \end{bmatrix} u + 2h_2^T(x)(h_2(x) + k_{21}(x)w)).$$

Note that $k_{21}(x)k_{21}^T(x) = I$ for all $x \in \mathbf{R}^n$ by assumption **[A3]**, so $\|k_{21}(x)v\| \leq \|v\|$ for all $v \in \mathbf{R}^{p_1}$.

Observe that $y = h_2(x) + k_{21}(x)w = k_{21}(x)(w - \frac{1}{2}g_1^T(x)\frac{\partial U^T}{\partial x}(x) + k_{21}^T(x)h_2(x))$, then

$$\|y\|^2 = \|h_2(x) + k_{21}(x)w\|^2 \leq \left\| w - \frac{1}{2}g_1^T(x)\frac{\partial U^T}{\partial x}(x) + k_{21}^T(x)h_2(x) \right\|^2.$$

So

$$\dot{U}(x) \leq \left\| \begin{bmatrix} 0 & I \end{bmatrix} u \right\|^2 - \|y\|^2 - \|z\|^2 + \|w\|^2 + (\begin{bmatrix} \frac{\partial U}{\partial x}(x) & 2h_1^T(x) \end{bmatrix} u + 2h_2^T(x)y).$$

Note that

$$\dot{U}(x) \leq -\|y\|^2 - \|z\|^2 + \|w\|^2 \leq -\|z\|^2 + \|w\|^2$$

if u is such that $\begin{bmatrix} 0 & I \end{bmatrix} u = 0$ and $\begin{bmatrix} \frac{\partial U}{\partial x}(x) & 2h_1^T(x) \end{bmatrix} u + 2h_2^T(x)y = 0$; but the latter is guaranteed by taking the controller as the given "output injection:"

$$u(x) = \begin{bmatrix} L_0(x) \\ 0 \end{bmatrix} y,$$

where $L_0(x)$ solves

$$\frac{\partial U}{\partial x}(x)L_0(x) = -2h_2^T(x).$$

It follows that

$$\int_0^T (\|w\|^2 - \|z\|^2)dt \geq U(x(T)) - U(0) = U(x(T)) \geq 0$$

for all $T \geq 0$.

(ii) **Proof of Theorem 7.4.7(ii)**. It is assumed $[h_2(x), f(x)]$ is observable in this proof.

If the above controller is taken, then the closed loop system is

$$\begin{cases} \dot{x} = f(x) + L_0(x)h_2(x) + (g_1(x) + L_0(x)k_{21}(x))w \\ z = h_1(x) \\ y = h_2(x) + k_{21}(x)w \end{cases}.$$

From the previous proof,

$$\dot{U}(x) \leq -\|y\|^2 - \|z\|^2 + \|w\|^2 .$$

Since $U(x) \geq 0$, the closed loop system is dissipative with respect to $\|w(t)\|^2 - \|z(t)\|^2 - \|y(t)\|^2$. For all $x \in \mathbf{R}^n$,

$$U(x) \geq \sup_{w \in \mathcal{L}_2[0,\infty), x(0)=x} - \int_0^\infty (\|w(t)\|^2 - \|z(t)\|^2 - \|y(t)\|^2)dt$$

$$\geq - \int_0^\infty (0 - \|z(t)\|^2 - \|y(t)\|^2)dt = \int_0^\infty (\|z(t)\|^2 + \|y(t)\|^2)dt.$$

Therefore, $U(x) = 0 \Rightarrow y(t) = h_2(x(t)) = 0$ for $w(t) = 0$, thus

$$\begin{cases} \dot{x} = f(x) + L_0(x)h_2(x) \\ h_2(x) = 0 \end{cases} \Leftrightarrow \begin{cases} \dot{x} = f(x) \\ h_2(x) = 0 \end{cases}.$$

By the observability assumption $x = 0$, $U(x)$ is (locally) positive definite.

(iii) **Asymptotic Stability**

$U(x)$ is positive definite by assumption, it can be used as a Lyapunov function.
Set $w = 0$,

$$\dot{U}(x) \leq \|w(t)\|^2 - \|z(t)\|^2 - \|h_2(x(t))\|^2 = -\|z(t)\|^2 - \|h_2(x(t))\|^2 \leq 0.$$

So $\dot{U}(x) = 0 \quad \Rightarrow \quad h_2(x(t)) = 0 \quad \Rightarrow \quad x(t) \to 0$ as $t \to \infty$ by assumption **[A5]**. LaSalle's Theorem implies $\dot{x} = f(x) + L_0(x)h_2(x)$ is asymptotically stable.

□

From the above proof,

$$\dot{U}(x) \leq \|u_2\|^2 - \|y\|^2 - \|z\|^2 + \|w\|^2 + (\frac{\partial U}{\partial x}(x)u_1 + 2h_1^T(x)u_2 + 2h_2^T(x)y).$$

Assume $L_1(x)$ is such that

$$\frac{\partial U}{\partial x}(x)L_1(x) = -2h_1^T(x).$$

Take $u_1 = L_0(x)y + L_1(x)u_2$, then

$$\frac{\partial U}{\partial x}(x)u_1 + 2h_1^T(x)u_2 + 2h_2^T(x)y = 0.$$

Therefore,

$$\dot{U}(x) \le \|u_2\|^2 - \|y\|^2 - \|z\|^2 + \|w\|^2.$$

Let $u_2 = Qy$ with $Q \in \mathcal{FG}$ (so $u_1 = L_0(x)y + L_2(x)Qy$ then), Q can be assumed to have the following realization,

$$\begin{cases} \dot{\xi} = a(\xi) + b(\xi)y \\ u_2 = c(\xi) \end{cases}.$$

Then there exists $U_c(\xi) \ge 0$ positive definite such that

$$\dot{U}_c(\xi) \le \|y\|^2 - \|u_2\|^2$$

and $\dot{\xi} = a(\xi)$ is asymptotically stable.

Define $W(x,\xi) = U(x) + U_c(\xi)$ for $(x,\xi) \in \mathbf{R}^n \times \mathbf{R}^n$, then $W(x,\xi) \ge 0$ is positive definite. Notice that

$$\begin{aligned} \dot{W}(x,\xi) &= \dot{U}(x) + \dot{U}_c(\xi) \\ &\le (\|u_2\|^2 - \|y\|^2 - \|z\|^2 + \|w\|^2) + (\|y\|^2 - \|u_2\|^2) \\ &= \|w\|^2 - \|z\|^2. \end{aligned}$$

Thus,

$$\int_0^T (\|w\|^2 - \|z\|^2)dt \ge W(x(T),\xi(T)) - W(0,0) = W(x(T),\xi(T)) \ge 0$$

for all $T \ge 0$.

Thus, we motivated the characterization of a class of controllers.

Theorem 7.4.9 *The assumptions are the same as in the last Theorem. If in addition, U is such that $\mathcal{H}_{FC}(U,x)$ is negative definite, and $L_1(x)$ also satisfies*

$$\frac{\partial U}{\partial x}(x)L_1(x) = -2h_1^T(x), \tag{7.16}$$

then

$$u = \begin{bmatrix} L_0(x) + L_1(x)Q \\ Q \end{bmatrix} y$$

for all $Q \in \mathcal{FG}$ also solves the $\mathcal{H}_\infty$-control problem for FC.

Proof. We only need to consider the stability. Since $\mathcal{H}_{FC}(U,x)$ is negative definite, then there exist a positive definite $\pi(x)$, such that

$$\mathcal{H}_{FC}(U,x)+\pi(x)\leq 0.$$

By Corollary 7.2.5,

$$\dot{U}(x)\leq \|u_2\|^2-\|y\|^2-\|z\|^2+\|w\|^2-\pi(x).$$

$W(x,\xi)=U(x)+U_c(\xi)$ is positive definite, as $U(x)$ and $U_c(\xi)$ are. It can be taken as Lyapunov function, let $w=0$, then

$$\dot{W}(x,\xi)\leq (\|u_2\|^2-\|y\|^2-\|z\|^2-\pi(x))+\|y\|^2-\|u_2\|^2)=-\|z\|^2-\pi(x)\leq 0.$$

Note that $\dot{W}(x,\xi)=0$ implies $\pi(x)=0$, $x=0$ by assumption. If $x=0$, then $\dot{\xi}=a(\xi)+b(\xi)y=a(\xi)$ is asymptotically stable; so $\xi(t)\to 0$ as $t\to\infty$. LaSalle's Theorem implies the asymptotic stability. □

Remark 7.4.10 *Notice that the suggested $\mathcal{H}_\infty$-controllers for FC problems characterized in terms of HJI (7.14) in general depend also on the full state x. This is because the solutions $L_0(x)$ and $L_1(x)$ of equations (7.15) and (7.16) in general depend on the state x. The pure full control solution is possible when equations (7.15) and (7.16) both have constant solutions; however this condition is restrictive. At this point, we would satisfy with the way the controllers are constructed. It might be argued that the obvious controller u with u_1 stabilizes the system, say $u_1=-f(x)-x$, and $u_2=-h_1(x)$ can do this job with resulting $\mathcal{L}_2$-gain $=0$, since $z=0$ in this case. Nevertheless, the main purpose of considering the FC problem in terms of HJI (7.14) is to build up the required machinery for the output feedback problem. The current treatment indeed achieves this goal. This point will be clear soon.*

7.4.4 Output Estimation Problem

Consider

$$G_{OE}:\left\{\begin{array}{lllllll}\dot{x} & = & f(x) & + & g_1(x)w & + & g_0(x)u\\ z & = & h_1(x) & + & & & u\\ y & = & h_2(x) & + & k_{21}(x)w & & \end{array}\right. \tag{7.17}$$

where $g_1,g_0,h_1,h_2,k_{21}\in\mathcal{C}^2$ with constraint **[A3]**. An additional assumption for this structure is as follows,

[A5]: $[h_2(x),f(x)]$ is (locally) zero-state detectable.

Theorem 7.4.11 *Consider G_{OE} (7.17), suppose there exists a local positive definite solution $U(x)$ to HJI: $\mathcal{H}_{FC}(U,x) \leq 0$, with $U(0) = 0$; and $U(x)$ makes the Hessian matrix of $\mathcal{H}_{FC}(U,x)$ with respect to $x \in \mathbf{R}^n$ be negative definite at 0. If $L_0(x)$ satisfies*

$$\frac{\partial U}{\partial x}(x)L_0(x) = -2h_2^T(x),$$

then there exists a controller such that the closed loop system has $\mathcal{L}_2$-gain ≤ 1 as long as the state of the closed loop system does not leave $\mathcal{B}_r$. And such a controller can be given by

$$K_{OE}: \left\{ \begin{array}{rcl} \dot{\tilde{x}} & = & f(\tilde{x}) - g_0(\tilde{x})h_1(\tilde{x}) + L_0(\tilde{x})h_2(\tilde{x}) \quad - \quad L_0(\tilde{x})y \\ u & = & -h_1(\tilde{x}) \end{array} \right. .$$

Furthermore, the closed loop system is also locally asymptotically stable at 0, if in addition,

[A7]: *$\dot{x} = f(x) - g_0(x)h_1(x)$ is locally asymptotically stable.*

Lemma 7.4.12 *Suppose there exists a locally positive definite function $U(x) \geq 0$ such that $\mathcal{H}_{FC}(U,x)$ is locally negative definite. Let $x, \tilde{x}$ be states of systems G_{OE} and K_{OE}, $e = \tilde{x} - x$. Define*

$$\mathcal{H}_e(e,\tilde{x}) := \frac{\partial U}{\partial e}(e)(f(\tilde{x}) - f(x) + L_0(\tilde{x})(h_2(\tilde{x}) - h_2(x)))+$$

$$\frac{1}{4}\frac{\partial U}{\partial e}(e)(g_1(x) + L_0(\tilde{x})k_{12}(x))(g_1(x) + L_0(\tilde{x})k_{12}(x))^T\frac{\partial U^T}{\partial e}(e)+$$

$$(h_1^T(\tilde{x}) - h_1^T(x))(h_1(\tilde{x}) - h_1(x)) - \frac{\partial U}{\partial e}(e)(g_0(x) - g_0(\tilde{x}))h_1(\tilde{x})$$

with $L_0(\tilde{x})$ defined as in previous Theorem. Then for all $(x,\tilde{x}) \in \mathcal{B}_r$ with some $r > 0$, $\mathcal{H}_e(e,\tilde{x}) \leq 0$. Moreover, there exists a locally positive definite function $\pi(e)$ such that $\mathcal{H}_e(e,\tilde{x}) + \pi(e) \leq 0$.

Proof. Recall that $\mathcal{H}_{OI}(U, L_0, e) = \mathcal{H}_{FC}(U,e)$, where

$$\mathcal{H}_{OI}(U,L_0,e) := \frac{\partial U}{\partial e}(e)(f(e) + L_0(e)h_2(e))+$$

$$\frac{1}{4}\frac{\partial U}{\partial e}(e)(g_1(e) + L_0(e)k_{21}(e))(g_1(e) + L_0(e)k_{21}(e))^T\frac{\partial U^T}{\partial e}(e) + h_1^T(e)h_1(e).$$

Also note that the Hessian matrix of $\mathcal{H}_e(e,\tilde{x})$ with respect to e at 0 is the same as the one of $\mathcal{H}_{OI}(U,L_0,e) = \mathcal{H}_{FC}(U,e)$ with respect to e at 0. The conclusion follows by the similar argument in the proof of Lemma 7.4.5. □

Remark 7.4.13 *If we take* $u = \begin{bmatrix} L_0(\tilde{x}) \\ 0 \end{bmatrix} y$ *with* $y_0 = h_2(\tilde{x}) - h_2(x) - k_{12}(x)w$*, let* $z = h_1(x) - h_1(\tilde{x}) + \begin{bmatrix} 0 & I \end{bmatrix} u$*, then*

$$\dot{U}(e) = \frac{\partial U}{\partial e}(e)(f(\tilde{x}) - f(x)) + g_1(x)w + \begin{bmatrix} I & 0 \end{bmatrix} u)$$

$$\leq -\|y_0\|^2 + \|w\|^2 - \|z\|^2 - \frac{\partial U}{\partial e}(e)(g_0(x) - g_0(\tilde{x}))h_1(\tilde{x})$$

for all $(x, \tilde{x}) \in \mathcal{B}_r$.

Proof. **[Theorem 7.4.11]** Consider $\Omega(G_{OE}, K_{FC})$ which has following realization,

$$\begin{cases} \dot{x} &= f(x) - g_0(x)h_1(\tilde{x}) & + & g_1(x)w \\ \dot{\tilde{x}} &= (f(\tilde{x}) - g_0(\tilde{x})h_1(\tilde{x})) + L_0(\tilde{x})(h_2(\tilde{x}) - h_2(x)) & - & L_0(\tilde{x})k_{21}(x)w \\ z &= h_1(x) - h_1(\tilde{x}) \end{cases} .$$

Let $e = \tilde{x} - x$, for $(x, \tilde{x}) \in \mathcal{B}_r$,

$$\dot{U}(e) = \frac{\partial U}{\partial e}(e)(f(\tilde{x}) - f(x) + L_0(\tilde{x})(h_2(\tilde{x}) - h_2(x)) -$$

$$(L_0(\tilde{x})k_{21}(x) + g_1(x))w) - \frac{\partial U}{\partial e}(e)(g_0(\tilde{x}) - g_0(x))h_1(\tilde{x})$$

$$\leq \|w\|^2 - \|z\|^2 - \|y_0\|^2 \leq -\|z\|^2 + \|w\|^2$$

so

$$\int_0^T (\|w\|^2 - \|z\|^2)dt \geq U(e(T)) - U(0) = U(e(T)) \geq 0$$

for all $T \geq 0$.

Next, the asymptotic stability of the closed loop system is considered. It has the realization as follows

$$\begin{cases} \dot{\tilde{x}} = f(\tilde{x}) - g_0(\tilde{x})h_1(\tilde{x}) + L_0(\tilde{x})(h_2(\tilde{x}) - h_2(x)) \\ \dot{e} = \eta(t, e) \end{cases} .$$

Since $\dot{U}(e)$ is positive definite from the negative definiteness of (Hessian matrix of) $\mathcal{H}_{FC}(U, e)$. Therefore $e(t) \to 0$ as $t \to \infty$. Note also that $\dot{\tilde{x}} = f(\tilde{x}) - g_0(\tilde{x})h_1(\tilde{x})$ is asymptotically stable. Since the two systems are hierarchically interconnected, and asymptotically stable, the interconnected system is asymptotically stable [191]. □

Remark 7.4.14 *It is observed that the requirement for system $\dot{x} = f(x) - g_0(x)h_1(x)$ to be (locally) asymptotically stable is necessary for the zero-dynamics of the closed loop system to be (locally) asymptotically stable [186].*

Theorem 7.4.15 *Under the assumption of the previous Theorem, if in addition, $L_1(x)$ is such that*

$$\frac{\partial U}{\partial x}(x)L_1(x) = -2h_1^T(x),$$

then the controller $u = \Omega(M_{OE}, Q)y$ with M_{OE} given by

$$\begin{cases} \dot{\tilde{x}} &= f(\tilde{x}) - g_0(\tilde{x})h_1(\tilde{x}) + L_0(\tilde{x})h_2(\tilde{x}) & - & L_0(\tilde{x})y & + & (g_2(\tilde{x}) + L_1(\tilde{x}))u_2 \\ u &= -h_1(\tilde{x}) & + & & & u_2 \\ y_0 &= h_2(\tilde{x}) & - & y & & \end{cases}$$

for all $Q \in \mathcal{FG}$ also (locally) solves $\mathcal{H}_\infty$-control problem for OE.

Proof. Consider $\Omega(G_{OE}, \Omega(M_{OE}, Q))$ for $Q \in \mathcal{FG}$ which has following realization,

$$\begin{cases} \dot{\xi} = a(\xi) + b(\xi)y \\ u_2 = c(\xi) \end{cases}.$$

The similar argument shows that there exists $r > 0$, for $(x, \hat{x}, \xi) \in \mathcal{B}_r$,

$$\dot{U}(e) \le \|w\|^2 - \|z\|^2 - \|y_0\|^2 + \|u_2\|^2 - \pi(e)$$

for some locally positive definite $\pi(e)$.

And U_Q is a solution to the HJI with respect to Q with state ξ, then

$$\dot{U}_Q(\xi) \le \|y_0\|^2 - \|u_2\|^2.$$

So

$$\dot{U}(e) + \dot{U}_Q(\xi) \le -\|z\|^2 + \|w\|^2 - \pi(e) \le -\|z\|^2 + \|w\|^2.$$

Therefore,

$$\int_0^T (\|z\|^2 - \|w\|^2)dt \le U(0) - U(e(T)) = -U(e(T)) \le 0.$$

As for the stability, let $w = 0$, then the closed loop system has the following hierarchical structure[2],

$$\begin{cases} \dot{e} = \eta(t, e) \\ \dot{\xi} = a(\xi) + b(\xi)(h_2(\tilde{x}) - h_2(x)) \\ \dot{\tilde{x}} = f(\tilde{x}) - g_0(\tilde{x})h_1(\tilde{x}) + L_0(\tilde{x})(h_2(\tilde{x}) - h_2(x)) + (g_2(\tilde{x}) + L_1(\tilde{x}))c(\xi) \end{cases}.$$

[2]Strictly speaking, the closed-loop system is not hierarchically interconnected, see Section 5.3.

Take $L_{OE}(e,\xi) = U(e) + U_Q(\xi)$ as the Lyapunov function of the interconnected system with state (e,ξ), then $\dot{L}_{OE}(e,\xi) \leq -\|z\|^2 - \pi(e)$. Now $\dot{L}_{OE}(e,\xi) = 0 \Rightarrow \pi(e) = 0$, so $e = 0$. So in this case $\dot{\xi} = a(\xi)$; but it is asymptotically stable, so $\xi(t) \to 0$ as $t \to \infty$. The interconnected (e,ξ) is locally asymptotically stable by LaSalle's Theorem. Now if $(e,\xi) = 0$ then $\dot{\tilde{x}} = f(\tilde{x}) - g_0(\tilde{x})h_1(\tilde{x})$; but it is locally asymptotically stable at 0 by assumption **[A7]**. Thus, we can conclude that this closed loop system is asymptotically stable by the stability Theorem for hierarchical systems [191]. □

7.5 $\mathcal{H}_\infty$-Control: Output Feedback Problems

We now consider the output feedback $\mathcal{H}_\infty$ control problem. The solutions to this problem are based on the results in the last section.

7.5.1 Solutions to Output Feedback Problems

The NLTI plant is realized as the following input-affine state-space equation,

$$G : \begin{cases} \dot{x} = f(x) + g_1(x)w + g_2(x)u \\ z = h_1(x) + k_{12}(x)u \\ y = h_2(x) + k_{21}(x)w \end{cases} \tag{7.18}$$

where $f, g_i, h_i, k_{ij} \in \mathcal{C}^2$ and $f(0) = 0, h_1(0) = 0, h_2(0) = 0$; x, w, u, z, and y are assumed to have dimensions n, p_1, p_2, q_1, and q_2, respectively.

The following assumptions are made,

[A2]: $k_{12}^T(x) \begin{bmatrix} h_1(x) & k_{12}(x) \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$;

[A3]: $\begin{bmatrix} g_1(x) \\ k_{21}(x) \end{bmatrix} k_{21}^T(x) = \begin{bmatrix} 0 \\ I \end{bmatrix}$;

[A4]: $[h_1(x), f(x)]$ is zero-state detectable.

The main idea in the construction of $\mathcal{H}_\infty$-controllers is to convert the general problem OF into the simpler problems which have been solved.

Let $V(x) \geq 0$ be the solution of $\mathcal{H}_{FI}(V,x) \leq 0$, and

$$F_0(x) := -\frac{1}{2} g_2^T(x) \frac{\partial V^T}{\partial x}(x), \qquad F_1(x) := \frac{1}{2} g_1^T(x) \frac{\partial V^T}{\partial x}(x). \tag{7.19}$$

Define new variables $r := w - F_1(x)$ and $v := u - F_0(x)$. We get a new system

$$G_a : \begin{cases} \dot{x} &= f_a(x) + g_1(x)r + g_2(x)u \\ v &= h_a(x) + \qquad\qquad + u \\ y &= h_2(x) + k_{21}(x)r \end{cases} \tag{7.20}$$

where $f_a(x) := f(x) + g_1(x)F_1(x)$ and $h_a(x) := F_0(x)$.

We have the following lemma.

Lemma 7.5.1 *Consider systems G (7.18) and G_a (7.20). If the controller K makes $\Omega(G_a, K)$ have $\mathcal{L}_2$-gain ≤ 1, it also results in $\Omega(G, K)$ having $\mathcal{L}_2$-gain ≤ 1.*

Proof. Note that $z = \Omega(G, K)w$ and $r = \Omega(G_t, K)v$.

Since $V(x) \geq 0$ solves $\mathcal{H}_{FI}(V, x) + \psi(x) = 0$, then

$$\dot{V}(x) = \frac{\partial V}{\partial x}(x)(f(x) + g_1(x)w + g_2(x)u)$$

$$= -\|z\|^2 + \|w\|^2 - \left\| w - \frac{1}{2}g_1^T(x)\frac{\partial V^T}{\partial x}(x)\right\|^2 + \left\| u + \frac{1}{2}g_2^T(x)\frac{\partial V^T}{\partial x}(x)\right\|^2 - \psi(x)$$

$$\leq -\|z\|^2 + \|w\|^2 - \|r\|^2 + \|v\|^2 .$$

So for all $T \geq 0$,

$$\int_0^T (\|w\|^2 - \|z\|^2)dt \geq \int_0^T (\|r\|^2 - \|v\|^2)dt + V(x(T)) \geq \int_0^T (\|r\|^2 - \|v\|^2)dt.$$

$$\int_0^T (\|r\|^2 - \|v\|^2)dt \geq 0 \quad \Rightarrow \quad \int_0^T (\|w\|^2 - \|z\|^2)dt \geq 0.$$

□

Now define

$$\mathcal{H}_a(W, x) := \frac{\partial W}{\partial x}(x)f_a(x) + \frac{1}{4}\frac{\partial W}{\partial x}(x)g_1(x)g_1^T(x)\frac{\partial W^T}{\partial x}(x) + h_a^T(x)h_a(x) - h_2^T(x)h_2(x).$$

Take $W(x) = U(x) - V(x)$ with $W(0) = U(0) - V(0)$ where $V(x) \geq 0$ is given just now. Note that

$$\mathcal{H}_a(W, x) = \mathcal{H}_{FC}(U, x) - \mathcal{H}_{FI}(V, x) = \mathcal{H}_{FC}(U, x) + \psi(x),$$

where $\psi(x) \geq 0$ is such that $\mathcal{H}_{FI}(V, x) + \psi(x) = 0$. Thus, $\mathcal{H}_a(W, x) \leq 0$ if and only if $\mathcal{H}_{FC}(U, x) + \psi(x) \leq 0$. Assume $U(x)$ is such that $\mathcal{H}_{FC}(U, x) + \psi(x) \leq 0$ has a positive definite Hessian matrix at $x = 0$, then $\mathcal{H}_a(W, x)$ also has negative definite Hessian matrix at 0. Suppose $L_0(x)$ is such that $\frac{\partial W}{\partial x}(x)L_0(x) = -2h_2^T(x)$. The controller K for the new OE structure given by Theorem 7.4.11 is

$$K : \begin{cases} \dot{\hat{x}} = f_a(\hat{x}) + g_2(\hat{x})h_a(\hat{x}) + L_0(\hat{x})h_2(\hat{x}) - L(\hat{x})y \\ u = h_a(\hat{x}) \end{cases}$$

is such that system $\Omega(G_a, K)$ locally has $\mathcal{L}_2$-gain ≤ 1.

By lemma 7.5.1, $\Omega(G, K)$ has $\mathcal{L}_2$-gain ≤ 1. Next, we examine the stability of the closed loop system $\Omega(G, K)$ which has the following realization,

$$\begin{cases} \dot{x} = f(x) + g_2(x)F_0(\hat{x}) + g_1(x)w \\ \dot{\hat{x}} = f_K(\hat{x}) + L_0(\hat{x})(h_2(\hat{x}) - h_2(x)) + L_0(\hat{x})k_{21}(x)w \\ z = h_1(x) + k_{12}(x)F_0(\hat{x}) \end{cases}$$

where

$$f_K(\hat{x}) := f(\hat{x}) + g_1(\hat{x})F_1(\hat{x}) + g_2(\hat{x})F_0(\hat{x}). \tag{7.21}$$

Take $e = \hat{x} - x$. Note that $\mathcal{H}_a(W, \cdot)$ has negative definite Hessian matrix as does $\mathcal{H}_{FC}(U, \cdot)$. Using the same technique as in the proof of Theorem 7.4.8, it can be concluded that for some locally positive definite $\pi : \mathbf{R}^n \to \mathbf{R}^+$, such that if $(x, \hat{x}) \in \mathcal{B}_s$ for some $s > 0$,

$$\dot{W}(e) \leq \|r\|^2 - \|v\|^2 - \pi(e).$$

Let $L_{OF}(x, e) = V(x) + W(e)$ with $e = \hat{x} - x$. By assumption $V(x)$ and $W(e)$ are positive definite, so is $L(x, e)$, and it can be used as a Lyapunov function. Take $w = 0$,

$$\dot{V}(x) \leq -\|z\|^2 + \|v\|^2 - \|r\|^2,$$

$$\dot{L}_{OF}(x, e) = \dot{V}(x) + \dot{W}(e) \leq -\|z\|^2 - \pi(e) \leq 0.$$

Then $\dot{L}_{OF}(x, e) = 0 \Rightarrow z = 0$ and $\pi(e) = 0 \Rightarrow x = 0$ and $e = 0$. Therefore, $\dot{L}_{OF}(x, e)$ is locally negative definite, the closed loop system is locally asymptotically stable.

Therefore, we have the following results about output $\mathcal{H}_\infty$-control problem,

Theorem 7.5.2 *Consider G (7.18), if there is some $\psi(x) \geq 0$ with $\psi(0) = 0$ such that*

(i) there exists a locally positive definite $V(x)$ which solves the HJE: $\mathcal{H}_{FI}(V, x) + \psi(x) = 0$ with $V(0) = 0$.

(ii) there exists a locally positive definite $U(x)$ which satisfies the HJI: $\mathcal{H}_{FC}(U, x) + \psi(x) \leq 0$ with $U(0) = 0$. And $\mathcal{H}_{FC}(U, x) + \psi(x)$ has nonsingular Hessian matrix at 0.

(iii) $U(x) - V(x) \geq 0$ is locally positive definite. And

$$(\frac{\partial U}{\partial x}(x) - \frac{\partial V}{\partial x}(x))L_0(x) = -2h_2^T(x),$$

has a solution $L_0(x)$. Then the output feedback $\mathcal{H}_\infty$-control problem is solvable. Furthermore,

$$K : \begin{cases} \dot{\hat{x}} = f_K(\hat{x}) + L_0(\hat{x})h_2(\hat{x}) - L_0(\hat{x})y \\ u = F_0(\hat{x}) \end{cases}$$

is such a controller.

Similar results to the above theorem are also obtained by Isidori in [91]. Note that $\mathcal{H}_\infty$-controllers have separation structures. The separation principle for the $\mathcal{H}_\infty$-performance in nonlinear systems was first confirmed by Ball *et al.* [15] (see also [91]). Similar arguments to Theorems 7.4.15 and 7.5.2 can be also used to construct the controller parameterization as follows.

Theorem 7.5.3 *Consider a system G (7.18) satisfying the condition in Theorem 5.1. If in addition $L_1(x)$ satisfies*

$$(\frac{\partial U}{\partial x}(x) - \frac{\partial V}{\partial x}(x))L_1(x) = -2h_1^T(x),$$

then the controller $u = \Omega(M, Q)y$ with M given by

$$\begin{cases} \dot{\tilde{x}} & = & f_K(\tilde{x}) & - & L_0(\tilde{x})y & + & (g_2(\tilde{x}) + L_1(\tilde{x}))u_0 \\ u & = & F_0(\tilde{x}) & + & & & u_0 \\ y_0 & = & h_2(\tilde{x}) & - & y & & \end{cases}$$

for all $Q \in \mathcal{FG}$ also solves output feedback $\mathcal{H}_\infty$-control problem.

Proof. By lemma 7.5.1 and theorem 7.4.15 it follows that the closed loop system $\Omega(G, K)$ with $K = \Omega(M, Q)$ has the $\mathcal{L}_2$-gain ≤ 1. Now it is sufficient to consider the stability issue. Suppose Q has the following realization

$$\begin{cases} \dot{\xi} = a(\xi) + b(\xi)y_0 \\ u_0 = c(\xi) \end{cases}$$

and $U_Q(\xi)$ is such that $\dot{U}_Q(\xi) \leq \|u_0\|^2 - \|y_0\|^2$. Take $w = 0$, the closed loop system has following "hierarchical" structure,

$$\begin{cases} \dot{x} = f(x) + g_2(x)(F_0(\tilde{x}) + c(\xi)) \\ \dot{\xi} = a(\xi) + b(\xi)(h_2(\tilde{x}) - h_2(x)) \\ \dot{\tilde{x}} = f_K(\tilde{x}) + L_0(\tilde{x})(h_2(\tilde{x}) - h_2(x)) + (g_2(\tilde{x}) + L_1(\tilde{x}))c(\xi) \end{cases} .$$

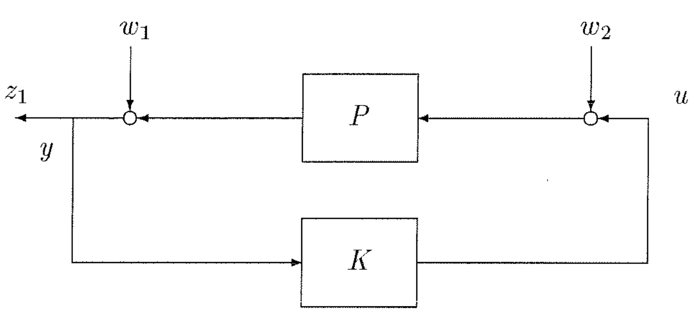
Let $V, W : \mathbf{R}^n \rightarrow \mathbf{R}^+$ positive definite be defined as in the preceding discussion. Denote $e = \tilde{x} - x$. Similar arguments to theorems 7.4.15 and 7.5.2 show that

$$\dot{W}(e) \leq \|r\|^2 - \|v\|^2 - \|y_0\|^2 + \|u_0\| - \pi(e)$$

for some positive definite $\pi : \mathbf{R}^n \rightarrow \mathbf{R}^+$. Define $L_{OF}(x, e, \xi) := V(x) + W(e) + U_Q(\xi)$ as the Lyapunov function of the closed loop system, then $\dot{L}_{OF}(x, e, \xi) \leq -\|z\|^2 - \pi(e)$. Now $\dot{L}_{OF}(e, \xi) = 0 \Rightarrow \pi(e) = 0$ and $\|z\| = 0$, so $e = 0$ and $z = 0$; the latter implies $x(t) \rightarrow 0$ as $t \rightarrow \infty$ by **[A4]**; on the other hand if $e = 0, x = 0$, then $\dot{\xi} = a(\xi)$, which is asymptotically stable and $\xi(t) \rightarrow 0$ as $t \rightarrow \infty$. The interconnected (e, ξ) is locally asymptotically stable by LaSalle's theorem and Vidyasagar's theorem [191]. □

7.5.2 Examples

In this subsection, we will examine an example from [59]. The basic block diagram is as follows,



Where P is a passive nonlinear plant; K is the controller to be designed such that the output z_1 is regulated; y is the measured output, based on which the control action u is produced; w_2 is the disturbance from the actuator; and w_1 is the noise from the sensor. The **control problem** is to design the controller K such that the influence of the noises w_1 and w_2 on the regulated output z_1 can be reduced to the minimal with the reasonable effort (control action should not be too large).

To formulate this problem, all the signals are considered in space $\mathcal{L}_2[0, \infty)$. The $\mathcal{H}_\infty$-control problem in this setting is to find a controller K (if any) for given $\gamma > 0$, such that

$$\int_0^T (\|z_1\|^2 + r\|u\|^2) dt \leq \gamma^2 \int_0^T (\|w_1\|^2 + \|w_2\|^2) dt, \forall T \in \mathbf{R}^+; w_1, w_2 \in \mathcal{L}_2[0, \infty),$$

where $r > 0$ is weighting factor. Let $\gamma^*(r)$ be the smallest γ satisfying the above inequality.

The case where P is a general lossless plant is considered in [185]. In the following, a passive plant which has the following realization is considered.

$$\begin{cases} \dot{x} = e^x(w_2 + u) \\ z_1 = x + w_1 \\ y = x + w_1 \end{cases}$$

We will consider two cases with different weighting factors $r = 0$ and $r = 1$. In both cases, since the stability of the resulting closed loop systems can be easily checked by using the corresponding Theorems, we just consider the $\mathcal{H}_\infty$-performance problem in a local sense.

Case I: $r = 0$

Consider the control problem that a controller K is designed such that:

$$\int_0^T \|z_1\|^2 \, dt \leq \gamma_0^2 \int_0^T (\|w_1\|^2 + \|w_2\|^2) dt, \forall T \in \mathbf{R}^+$$

where $\gamma_0 = 1/(1-\epsilon)$ for some $0 < \epsilon < 1$.

To standardize the problem, take

$$w := \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \qquad z := (1-\epsilon)z_1.$$

Thus, the state-space realization is

$$\begin{cases} \dot{x} &= & & \begin{bmatrix} 0 & e^x \end{bmatrix} w & + & e^x u \\ z &= (1-\epsilon)x & + & \begin{bmatrix} 1-\epsilon & 0 \end{bmatrix} w \\ y &= x & + & \begin{bmatrix} 1 & 0 \end{bmatrix} w \end{cases} .$$

Change the variable $u' = e^x u$. Now it looks like the output-injection control problem. But it is not standard, since we need to get ride of the term $\begin{bmatrix} 1-\epsilon & 0 \end{bmatrix} w$ in the regulated output z. By using the simplification method by Safonov *et al.* [153], we have the following simplified system,

$$\begin{cases} \dot{x} &= & & \begin{bmatrix} 0 & e^x \end{bmatrix} w & + & u' \\ z &= \frac{1-\epsilon}{\sqrt{2\epsilon-\epsilon^2}} x \\ y_N &= \frac{1}{\sqrt{2\epsilon-\epsilon^2}} x & + & \begin{bmatrix} 1 & 0 \end{bmatrix} w \end{cases}$$

with $y_N = \sqrt{2\epsilon - \epsilon^2} y$. Now the system has a required output-injection structure.

Consider the HJI with respect to this structure:

$$\mathcal{H}_{FC}(U,x) = \frac{\partial U}{\partial x}(x)\cdot 0 + \frac{1}{4}e^{2x}(\frac{\partial U}{\partial x}(x))^2 + (\frac{1-\epsilon}{\sqrt{2\epsilon-\epsilon^2}}x)^2 - (\frac{1}{\sqrt{2\epsilon-\epsilon^2}}x)^2 \leq 0.$$

A class of local positive solutions $U(x)$ are such that

$$\frac{\partial U}{\partial x}(x) = 2\rho e^{-x}x$$

for $0 \leq \rho \leq 1$. Take $\rho = 1$, the solution $L(x)$ to

$$\frac{\partial U}{\partial x}(x)L(x) = \frac{2x}{\sqrt{2\epsilon-\epsilon^2}}$$

is

$$L(x) = -\frac{e^x}{\sqrt{2\epsilon-\epsilon^2}}.$$

It follows that the controller is

$$u' = L(x)y_N = -\frac{e^x}{\sqrt{2\epsilon-\epsilon^2}}\cdot\sqrt{2\epsilon-\epsilon^2}y = -e^x y,$$

or the output-injection can be recovered as $u = -y$. Note that it is independent of ϵ.

This $\mathcal{H}_\infty$ controller is identity ($K = -1$). Actually, we have following general result which appears in [59].

Theorem 7.5.4 *Consider the feedback system as shown. Suppose the plant P has the same number of inputs and outputs, and is passive, i.e.,*

$$\int_0^T (Pe)^T e dt \geq 0, \qquad \forall T \in \mathbf{R}^+,$$

and $K = -1$, then

$$\int_0^T \|z_1\|^2\, dt \leq \int_0^T (\|w_1\|^2 + \|w_2\|^2)dt, \forall T \in R^+.$$

Case II: $r = 1$

Consider the control problem that a controller K is designed such that:

$$\int_0^T (\|z_1\|^2 + \|u\|)dt \leq \gamma_0^2 \int_0^T (\|w_1\|^2 + \|w_2\|^2)dt, \forall T \in \mathbf{R}^+,$$

where $\gamma_0 = \sqrt{2}/(1-\epsilon)$ for some $0 < \epsilon < 1$.

Take

$$w := \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \qquad z := \frac{1-\epsilon}{\sqrt{2}}\begin{bmatrix} z_1 \\ u \end{bmatrix}.$$

Thus, the state-space realization is

$$\begin{cases} \dot{x} &= & & \begin{bmatrix} 0 & e^x \end{bmatrix} w & + & e^x u \\ z &= \begin{bmatrix} \frac{1-\epsilon}{\sqrt{2}} x \\ 0 \end{bmatrix} & + & \begin{bmatrix} \frac{1-\epsilon}{\sqrt{2}} & 0 \\ 0 & 0 \end{bmatrix} w & + & \begin{bmatrix} 0 \\ \frac{1-\epsilon}{\sqrt{2}} \end{bmatrix} u \\ y &= x & + & \begin{bmatrix} 1 & 0 \end{bmatrix} w \end{cases} .$$

To standardize this structure, we can get the simplified system by the method introduced in [153] as follows,

$$\begin{cases} \dot{x} &= & & \begin{bmatrix} 0 & e^x \end{bmatrix} w & + & \frac{\sqrt{2}}{1-\epsilon} e^x u_N \\ z &= \begin{bmatrix} \frac{1-\epsilon}{\sqrt{1+2\epsilon-\epsilon^2}} x \\ 0 \end{bmatrix} & + & & & \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_N \\ y_N &= \sqrt{\frac{2}{1+2\epsilon-\epsilon^2}} x & + & \begin{bmatrix} 1 & 0 \end{bmatrix} w \end{cases}$$

with $u_N = \frac{1-\epsilon}{\sqrt{2}} u$ and $y_N = \sqrt{\frac{1+2\epsilon-\epsilon^2}{2}} y$. Now the system has a required output feedback structure.

Now take $\psi(x) = 0$ as in Theorem 7.5.2, consider the following HJE,

$$\mathcal{H}_{FI}(V, x) = \frac{\partial V}{\partial x}(x) \cdot 0 + \frac{1}{4}(e^{2x} - \frac{2e^{2x}}{(1-\epsilon)^2})(\frac{\partial V}{\partial x}(x))^2 + (\frac{1-\epsilon}{\sqrt{1+2\epsilon-\epsilon^2}} x)^2 = 0.$$

The positive solution $V(x)$ is such that

$$\frac{\partial V}{\partial x}(x) = \frac{2(1-\epsilon)^2}{1+2\epsilon-\epsilon^2} e^{-x} x.$$

Also consider an HJI as follows,

$$\mathcal{H}_{FC}(U, x) = \frac{\partial U}{\partial x}(x) \cdot 0 + \frac{1}{4} e^{2x} (\frac{\partial U}{\partial x}(x))^2 + (\frac{1-\epsilon}{\sqrt{1+2\epsilon-\epsilon^2}} x)^2 - (\sqrt{\frac{2}{1+2\epsilon-\epsilon^2}} x)^2 \leq 0.$$

A class of positive definite solutions $U(x)$ are such that

$$\frac{\partial U}{\partial x}(x) = 2\rho e^{-x} x$$

for $0 \leq \rho < 1$. They make $\mathcal{H}_{FC}(U, x)$ have negative Hessian matrix at 0.

Now it can be easily checked that $U(x) - V(x)$ is positive definite if ρ is taken to be close enough to 1. And $L_0(x)$ can be solved by

$$(\frac{\partial U}{\partial x}(x) - \frac{\partial V}{\partial x}(x)) L_0(x) = -2\sqrt{\frac{2}{1+2\epsilon-\epsilon^2}} x.$$

Then the controller can be constructed by Theorem 7.5.2.

7.6 Notes and References

The material in this section forms the paper [112]. The generalization of the results in this chapter to deal with the finite-time horizon and time-varying systems is given in [107], and the one for global solution is derived in [110].

The first state-space effort might be the relating of the nonlinear $\mathcal{H}_\infty$-control to differential game theory by several researchers [12, 16, 52]. The state-feedback nonlinear $\mathcal{H}_\infty$-control problem was first solved by Van der Schaft [183]. Isidori and Astolfi [93, 91] developed some sufficient HJI characterizations for the output control problem. They also parameterized a class of controllers for the full information (FI) structure [93].

Ball-Helton-Walker [15] derived necessary conditions for some $\mathcal{H}_\infty$-control problem to be solvable. These conditions are that two HJIs have positive solutions and the solutions are coupled locally. They confirmed the separation principle for the nonlinear $\mathcal{H}_\infty$-control system, and also provided a recipe to construct the controllers from the necessary conditions. Van der Schaft and Isidori also similarly considered the necessity part [185, 92]. In the next chapter, we will derive the necessary conditions in terms of NLMIs (see also [113]).

Chapter 8

$\mathcal{H}_\infty$-Control of Nonlinear Systems: Convex Characterizations

8.1 Introduction

The simplicity of the state space characterization of $\mathcal{H}_\infty$-control theory together with its clear connections with traditional methods in optimal control [60] have stimulated several attempts to generalize the linear $\mathcal{H}_\infty$ results in state space to nonlinear systems [183, 93, 15, 113]. Basically, in those treatments, the (dynamic) output feedback $\mathcal{H}_\infty$-controllers have separation structures; necessary and sufficient conditions for the $\mathcal{H}_\infty$-control problem to be (locally or globally) solvable are characterized in terms of Hamilton-Jacobi equations (HJE)s or inequalities (HJIs) [183, 93, 15, 113, 92, 185, 52]. Specifically, a local output feedback $\mathcal{H}_\infty$-controller, and a class of parameterized local $\mathcal{H}_\infty$ controllers are designed based on the required local solutions of some Hamilton-Jacobi equations or inequalities [93, 113]; also the fact that there exist output feedback $\mathcal{H}_\infty$-controllers (with separation structures) implies the solvability of two Hamilton-Jacobi equations or inequalities [15, 185]. Some efforts have been made to characterize the global solutions; a one-inequality sufficient and necessary condition for global solutions is given by Helton and Zhan in [82]; the necessary conditions can be further refined by two Hamilton-Jacobi Inequalities [15]. Whence, one of the major concerns in the state-space nonlinear $\mathcal{H}_\infty$-control theory is the computation issue involving in solving these Hamilton-Jacobi (partial differential) equations (HJEs) or inequalities (HJIs); progress along this line would be beneficial to applications of nonlinear $\mathcal{H}_\infty$-control theory. For example, Huang and Lin proposed a systematic procedure to find

Taylor series approximations to the solutions of the HJEs [86] (see also [120, 183]).

In this chapter, we propose an alternative approach with promising computational properties to the nonlinear $\mathcal{H}_\infty$-control problem. This is motivated by the fact that, essentially, the linear $\mathcal{H}_\infty$-control problem can be characterized as a convex problem which has some appealing computational properties [132, 29] (see also [119, 133, 111, 69, 94] for the treatments in linear case in terms of **linear matrix inequalities** (LMIs), which result in convex problems). We therefore examine the convexity of the nonlinear $\mathcal{H}_\infty$-control problem, and characterize the solutions in terms of **nonlinear matrix inequalities** (NLMIs) instead of the Hamilton-Jacobi equations or inequalities. Both state feedback and output feedback solutions are derived. In the output feedback case, the $\mathcal{H}_\infty$-controllers are not required to have separation structures; some necessary conditions are characterized in terms of three algebraic NLMIs. It is also confirmed that the three-NLMI characterization is sufficient for local solutions. It is noted that the algebraic NLMIs are in fact the state-dependent LMIs, therefore, some convex optimization methods for solving LMIs can be possibly used in the practical computation for solving NLMIs. Unfortunately, unlike the linear case, the solution of the NLMIs by themselves are not sufficient to guarantee the existence of the required controller, some additional condition is required, and the computational implications of the required additional constraints on the NLMI solutions are not totally clear at this moment. This issue is discussed more in the body of the chapter.

The remainder of this chapter is organized as follows: In section 2, some background material related to the $\mathcal{L}_2$-gains analysis is provided; the NLMI characterization of $\mathcal{L}_2$-gains is given. In section 3, the $\mathcal{H}_\infty$-control problem is stated; some assumptions on the system structures are made. In section 4, the main results of this chapter, i.e., solutions to the output feedback $\mathcal{H}_\infty$-control problem, are given; the solvability of this problem is characterized by three NLMIs. In section 5, it is further shown that under some (weak) separation structure assumptions for the $\mathcal{H}_\infty$-controllers, the solvability of the output-feedback $\mathcal{H}_\infty$-control problem implies the solvability by static state-feedback and output-injection. In section 6, the conditions for the existence of the solutions to these NLMIs which yield the $\mathcal{H}_\infty$-control solutions is examined. Some required technical material is reviewed in the appendix.

8.2 Stability and $\mathcal{H}_\infty$-Performances

In this section, some background material about stability and $\mathcal{L}_2$-gain analysis of nonlinear systems is provided. The reader is referred to Willems [197], van der Schaft [183], and Lu and Doyle [112] for more characterizations. We will reformulate these characterizations in terms of **nonlinear matrix inequalities** (NLMIs).

Consider the following input-affine nonlinear time-invariant (NLTI) system,

$$G : \begin{cases} \dot{x} = f(x) + g(x)w \\ z = h(x) + k(x)w \end{cases} \tag{8.1}$$

where $x \in \mathbf{R}^n$ is state vector, $w \in \mathbf{R}^p$ and $z \in \mathbf{R}^q$ are input and output vectors, respectively. $f, g, h, k \in \mathbf{C}^0$ are vector or matrix valued function, and $f(0) = 0, h(0) = 0$. From now on we will assume the system evolves on a convex open bounded subset $\mathbf{X} \subset \mathbf{R}^n$ containing the origin. Thus, $0 \in \mathbf{R}^n$ is the equilibrium of the system with $w = 0$. The corresponding state transition function is denoted as $\psi : \mathbf{R}^+ \times \mathbf{R}^+ \times \mathbf{R}^n \times \mathcal{L}_2^p(\mathbf{R}^+) \rightarrow \mathbf{R}^n$. It is assumed that the system (8.1) is **reachable** from 0 in the sense that for all $x \in \mathbf{X}$, there exist $T \in R^+$ and $w^*(t) \in \mathcal{L}_2[0, T]$ such that $x = \psi(T, 0, 0, w^*)$.

Note that in many cases system (8.1) can be rewritten (nonuniquely) as the following form which is also used in this chapter,

$$G : \begin{cases} \dot{x} = A(x)x + B(x)w \\ z = C(x)x + D(x)w \end{cases} \tag{8.2}$$

where $x \in \mathbf{R}^n$ is state vector, $w \in \mathbf{R}^p$ and $z \in \mathbf{R}^q$ are input and output vectors, respectively. We will assume A, B, C, D are $\mathbf{C}^0$ matrix-valued functions of suitable dimensions.

We first consider the Lyapunov stability of system (8.1). The Lyapunov theorem can be explicitly applied to the system (8.1) as follows.

Theorem 8.2.1 *Consider system (8.1). It is asymptotically stable around* 0 *if and only if there is a* $\mathbf{C}^1$ *positive definite function* $V : \mathbf{X} \rightarrow \mathbf{R}^+$ *such that*

$$\frac{\partial V}{\partial x}(x) f(x) < 0 \tag{8.3}$$

for all $x \in \mathbf{X} \setminus \{0\}$.

In the sequel, we mainly consider the performance of the system (8.1) in terms of its $\mathcal{L}_2$-gain.

Definition 8.2.2 *The system (8.1) or (8.2) with initial state* $x(0) = 0$ *is said to have* **$\mathcal{L}_2$-gain less than or equal to** γ *for some* $\gamma > 0$ *if*

$$\int_0^T \|z(t)\|^2 \, dt \leq \gamma^2 \int_0^T \|w(t)\|^2 \, dt \tag{8.4}$$

for all $T \geq 0$ *and* $w(t) \in \mathcal{L}_2[0, T]$, *as long as the state* $x(t) \in \mathbf{X}$ *for* $t \in [0, T]$.

In the following discussion, we only consider the case $\gamma = 1$ without loss of generality. As pointed out by Willems [197], if the system has $\mathcal{L}_2$-gain ≤ 1, if and only if there exists a function $V : \mathbf{X} \to \mathbf{R}^+$ with $V(0) = 0$, which is called **storage function** in [197], such that

$$V(x(T)) - V(x(0)) - \int_0^T (\|w(t)\|^2 - \|z(t)\|^2) dt \leq 0, \tag{8.5}$$

where $w(t) \in \mathcal{L}_2[0, T]$, and $x(t) \in \mathbf{X}$ for $t \in [0, T]$. Moreover, the functions $V(x)$ satisfying (8.5) form a convex set.

The following proposition characterizes $\mathcal{L}_2$-gains for a class of nonlinear systems in terms of NLMIs.

Theorem 8.2.3 *Consider system G given by (8.1) with $R(x) = I - k^T(x)k(x) > 0$, it is asymptotically stable and has $\mathcal{L}_2$-gain ≤ 1 if there exist a $\mathbf{C}^1$ positive definite function $V : \mathbf{X} \to \mathbf{R}^+$ such that*

$$\mathcal{M}_{Gain}(\frac{\partial V}{\partial x}, x) := \begin{bmatrix} \frac{\partial V}{\partial x}(x) f(x) + h^T(x)h(x) & \frac{1}{2}\frac{\partial V}{\partial x}(x) g(x) + h^T(x)k(x) \\ \frac{1}{2} g^T(x) \frac{\partial V^T}{\partial x}(x) + k^T(x)h(x) & k^T(x)k(x) - I \end{bmatrix} < 0 \tag{8.6}$$

for all $x \in \mathbf{X} \setminus \{0\}$.

Proof. By Schur's complement argument, we have that (8.6) is equivalent to

$$\mathcal{H}(\frac{\partial V}{\partial x}, x) := \frac{\partial V}{\partial x}(x) f(x) + h^T(x)h(x) +$$

$$(\frac{1}{2}\frac{\partial V}{\partial x}(x) g(x) + h^T(x)k(x))(I - k^T(x)k(x))^{-1}(\frac{1}{2} g^T(x) \frac{\partial V^T}{\partial x}(x) + k^T(x)h(x)) < 0 \tag{8.7}$$

for all $x \in \mathbf{X} \setminus \{0\}$. Thus,

$$\dot{V}(x) = \frac{\partial V}{\partial x}(x)(f(x) + g(x)w)$$

$$= \frac{\partial V}{\partial x}(x) g(x) w + \mathcal{H}(\frac{\partial V}{\partial x}, x) - h^T(x)h(x) -$$

$$(\frac{1}{2}\frac{\partial V}{\partial x}(x) g(x) + h^T(x)k(x))(I - k^T(x)k(x))^{-1}(\frac{1}{2} g^T(x) \frac{\partial V^T}{\partial x}(x) + k^T(x)h(x)) \tag{8.8}$$

$$= \|w(t)\|^2 - \|z(t)\|^2 -$$

$$\left\| R^{1/2}(x) w(t) - R^{-1/2}(x) k^T(x) h(x) - \frac{1}{2} R^{-1/2}(x) g^T(x) \frac{\partial V^T}{\partial x}(x) \right\|^2 + \mathcal{H}(\frac{\partial V}{\partial x}, x)$$

$$\leq \|w(t)\|^2 - \|z(t)\|^2 + \mathcal{H}(\frac{\partial V}{\partial x}, x) \tag{8.9}$$

$$\leq \|w(t)\|^2 - \|z(t)\|^2, \tag{8.10}$$

which implies that

$$\int_0^T \|z(t)\|^2 \, dt \le \int_0^T \|w(t)\|^2 \, dt - V(x(T)) \le \int_0^T \|w(t)\|^2 \, dt$$

for all $T \in \mathbf{R}^+$, since $V(0) = 0$ and $V(x) \ge 0$; i.e., the system has $\mathcal{L}_2$-gain ≤ 1.

On the other hand, take $w(t) = 0$, (8.9) becomes

$$\dot{V}(x) \le -\|z(t)\|^2 + \mathcal{H}(\frac{\partial V}{\partial x}, x).$$

Therefore, $\dot{V}(x) = 0$ implies $\mathcal{H}(\frac{\partial V}{\partial x}, x) = 0$, which in turn implies $x = 0$. Therefore, $V : \mathbf{X} \rightarrow \mathbf{R}^+$ is a Lyapunov function, and system G is asymptotically stable. □

Remark 8.2.4 *It is remarked that the above theorem on the $\mathcal{L}_2$-gain analysis exhibits some appealing computational properties. It is noted that the left-hand side of inequality (8.6) in Theorems 8.2.3, as well as (8.3) in Theorem 8.2.1, is affine in $V(x)$, and all positive definite solutions form convex sets, i.e., the characterization is a convex condition. This trivial fact has only been exploited systematically in the linear case, but we hope that numerical techniques may be developed to exploit it in the nonlinear case as well. The inequalities (8.3) and (8.6) are actually differential linear (or affine) matrix inequalities, but we will refer to them as* **nonlinear matrix inequalities** *(NLMIs) to emphasize their use in nonlinear problems. All of the conditions that are derived for the analysis problems in the remainder of this chapter are similarly convex, and this property will not be discussed for each problem.*

Although (8.6) provides a convex characterization of the $\mathbf{C}^1$ positive definite function V which yields $\mathcal{L}_2$-gain ≤ 1, this fact has not been well exploited as in the linear case, where the corresponding conditions are also finite dimensional algebraic LMIs (see for instance, [132, 29]). It is possible to provide alternative characterizations which are more immediately comparable to the linear case, and which are useful in the synthesis problem, but at the price of increased conservatism. Consider system (8.2), suppose $V : \mathbf{X} \rightarrow \mathbf{R}^+$ satisfies (8.6). In addition, let $\frac{\partial V}{\partial x}(x) = 2x^T P^T(x)$ with some $\mathbf{C}^0$ matrix valued function $P : \mathbf{X} \rightarrow \mathbf{R}^{n \times n}$, then (8.6) becomes

$$\begin{bmatrix} x^T(A^T(x)P(x) + P^T(x)A(x) + C^T(x)C(x))x & x^T(P^T(x)B(x) + C^T(x)D(x)) \\ (B^T(x)P(x) + D^T(x)C(x))x & D^T(x)D(x) - I \end{bmatrix} \le 0.$$

It is clearly sufficient for the above NLMI to hold that

$$\mathcal{M}(P, x) := \begin{bmatrix} A^T(x)P(x) + P^T(x)A(x) + C^T(x)C(x) & P^T(x)B(x) + C^T(x)D(x) \\ B^T(x)P(x) + D^T(x)C(x) & D^T(x)D(x) - I \end{bmatrix} < 0 \tag{8.11}$$

for all $x \in \mathbf{X}$. This observation is summarized as following theorem which gives alternative characterizations of the $\mathcal{L}_2$-gain of the system.

Theorem 8.2.5 *Consider the system G given by (8.2), suppose $I - D^T(x)D(x) > 0$. Given any $\mathbf{C}^0$ matrix-valued function $P : \mathbf{X} \rightarrow \mathbf{R}^{n \times n}$, the following inequalities are equivalent.*

(i) P satisfies (8.11), i.e., $\mathcal{M}(P,x) < 0$ for all $x \in \mathbf{X}$.

(ii) P satisfies

$$\hat{\mathcal{M}}(P,x) := \begin{bmatrix} A^T(x)P(x) + P^T(x)A(x) & P^T(x)B(x) & C^T(x) \\ B^T(x)P(x) & -I & D^T(x) \\ C(x) & D(x) & -I \end{bmatrix} < 0 \qquad (8.12)$$

for all $x \in \mathbf{X}$.

In addition, if there are a positive definite $\mathbf{C}^0$ matrix-valued function $P : \mathbf{X} \rightarrow \mathbf{R}^{n \times n}$ satisfying any of the above inequalities and a function $V : \mathbf{X} \rightarrow \mathbf{R}$ such that $\frac{\partial V}{\partial x}(x) = 2xP(x)$, then the system has $\mathcal{L}_2$-gain ≤ 1 and is asymptotically stable.

Proof. The standard result of Schur complements yields $\mathcal{M}(P,x) \leq 0$ if and only if $\hat{\mathcal{M}}(P,x) \leq 0$, since $I - D^T(x)D(x) > 0$. The later statement is confirmed by Theorem 8.2.3. $\square$

In section 5, we shall examine the computational implications of the above NLMIs.

Remark 8.2.6 *It should be emphasized that the existence of a $\mathbf{C}^0$ matrix-valued function $P : \mathbf{X} \rightarrow \mathbf{R}^{n \times n}$ which satisfies any of the above NLMIs is not enough to guarantee the system to have $\mathcal{L}_2$-gain ≤ 1; it is additionally required that there exists a function $V : \mathbf{X} \rightarrow \mathbf{R}$ such that $\frac{\partial V}{\partial x}(x) = 2x^T P(x)$. (See Lemma 8.8.1 for a characterization of a class of matrix-valued function $P : \mathbf{X} \rightarrow \mathbf{R}^{n \times n}$ which satisfies this additional requirement.)*

Remark 8.2.7 *If there is a $\mathbf{C}^0$ matrix-valued matrix P_0 such that $\mathcal{M}(P_0, x) < 0$ for $x \in \mathbf{X}$, then by continuity of $\mathcal{M}$ with respect to x, there is another $\mathbf{C}^0$ matrix-valued matrix P such that $\mathcal{M}(P,x) < 0$ and $\frac{\partial V}{\partial x}(x) = 2x^T P^T(x)$ for some $\mathbf{C}^1$ function $V : \mathcal{B}_d \rightarrow \mathbf{R}^+$ for some $d > 0$. In fact, a natural choice is a constant matrix $P = P_0(0)$, and $V(x) = x^T P x$. The same observation for Hamilton-Jacobi characterizations is made in [183].*

It is noted that by Lemma 8.8.1, the $\mathbf{C}^1$ function $V : \mathbf{X} \rightarrow \mathbf{R}$ which satisfies $\frac{\partial V}{\partial x}(x) = 2x^T P(x)$ for some positive definite matrix-valued solutions $P^T(x) = P(x) > 0$ and $V(0) = 0$ is positive definite on $\mathbf{X}$. Now we conclude the above discussions by defining a stronger $\mathcal{H}_\infty$-performance.

Definition 8.2.8 *The concerned system (8.2) is said to have* **strong** $\mathcal{H}_\infty$**-performance** *if there is a* $\mathbf{C}^0$ *positive definite matrix-valued function* $P(x) = P^T(x) > 0$ *which satisfies any of inequalities (8.11) and (8.12) for all* $x \in \mathbf{X}$ *such that* $\frac{\partial V}{\partial x}(x) = 2x^T P(x)$ *for some* $\mathbf{C}^1$ *function* $V : \mathbf{X} \to \mathbf{R}$.

Therefore, if the system has a strong $\mathcal{H}_\infty$-performance, by Lemma 8.8.1 and Theorem 8.2.5, it has $\mathcal{L}_2$-gain ≤ 1.

Example 8.2.9 *Consider the following system.*

$$\begin{cases} \dot{x} = -\alpha(x)x + \begin{bmatrix} -\alpha(x) & \alpha(x) \end{bmatrix} w \\ z = \beta x + \begin{bmatrix} \beta & 0 \end{bmatrix} w \end{cases} \qquad w := \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

where $\alpha : \mathbf{R} \to \mathbf{R}$ *is* $\mathbf{C}^0$ *and* $\alpha(x) > 0$ *for all* $x \in \mathbf{R}$, *and* $\beta \in \mathbf{R}$. *Now we show that it has strong* $\mathcal{H}_\infty$ *performance for all* $|\beta| < 1$. *In fact, consider the NLMI (8.11),*

$$\begin{bmatrix} -2\alpha(x)P(x) + 1 & \beta^2 - \alpha(x)P(x) & \alpha(x)P(x) \\ \beta^2 - \alpha(x)P(x) & \beta^2 - 1 & 0 \\ \alpha P(x) & 0 & -1 \end{bmatrix} < 0,$$

all solutions $P : \mathbf{R} \to \mathbf{R}^+$, *are positive definite, satisfy* $P(x) > \alpha^{-1}(x)$. *The other conditions in Definition 8.2.8 are trivially satisfied. Then the system has strong* $\mathcal{H}_\infty$*-performance.*

To see the conservativeness of the strong $\mathcal{H}_\infty$-performance characterized by NLMI (8.11) or (8.11), it may be noted that even its point-wise solution is not necessary for either stability or performance, as is well-known. This is shown in the following example.

Example 8.2.10 *Consider the following system of order two,*

$$\begin{cases} \dot{x}_1 = -x_1 + x_1 x_2^2 \\ \dot{x}_2 = -x_1^2 x_2 - x_2 \end{cases}$$

which evolves on $\mathbf{R}^2$. *Take a positive definite (quadratic) function* $V(x_1, x_2) = x_1^2 + x_2^2$, *which is actually a Lyapunov function for the system and satisfies (8.3). In fact,* $\dot{V}(x_1, x_2) = -2(x_1^2 + x_2^2)$, *which is negative definite; the system is therefore asymptotically stable.*

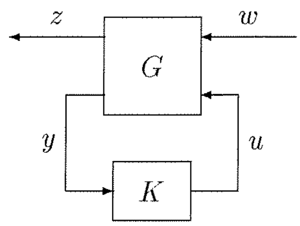
On the other hand, the system can be rewritten as the form (8.2), i.e., $\dot{x} = A(x)x$ *for* $x \in \mathbf{R}^2$. *There are two representations in which*

$$A(x) = \begin{bmatrix} -1 + x_2^2 & 0 \\ 0 & -1 - x_1^2 \end{bmatrix} \text{ and } A(x) = \begin{bmatrix} -1 & x_1 x_2 \\ -x_1 x_2 & -1 \end{bmatrix}.$$

Obviously, neither $A(x)$*'s are point-wise asymptotically stable on* $\mathbf{R}^2$. *This may be compared to the pointwise use of (8.6), which is discussed further in Section 6.3.*

8.3 $\mathcal{H}_\infty$-Control Problems

The feedback configuration for the $\mathcal{H}_\infty$-control synthesis problem is depicted as follows,



where G is the nonlinear plant with two sets of inputs: the exogenous disturbance inputs w and the control inputs u, and two sets of outputs: the measured outputs y and the regulated outputs z. K is the controller to be designed. It is required that the feedback configuration be **well-posed**. Both G and K are nonlinear time-invariant and can be realized as input-affine state-space equations:

$$G : \begin{cases} \dot{x} &= A(x)x + B_1(x)w + B_2(x)u \\ z &= C_1(x)x + D_{11}(x)w + D_{12}(x)u \\ y &= C_2(x)x + D_{21}(x)w + D_{22}(x)u \end{cases} \tag{8.13}$$

where $A, B_i, C_i, D_{ij} \in \mathbf{C}^0$ $(i, j = 1, 2)$; x, w, u, z, and y are assumed to have dimensions n, p_1, p_2, q_1, and q_2, respectively, without loss of generality, it is assumed that $n + p_1 \geq q_2$ and $n + q_1 \geq p_2$, and

$$K : \begin{cases} \dot{\xi} &= \hat{A}(\xi)\xi + \hat{B}(\xi)y \\ u &= \hat{C}(\xi)\xi + \hat{D}(\xi)y \end{cases} \tag{8.14}$$

with $\hat{A}, \hat{B}, \hat{C}, \hat{D} \in \mathbf{C}^0$. It is assumed that the feedback system (8.13)-(8.14) evolves on $\mathbf{X} \times \mathbf{X}_o$, where $\mathbf{X}$ and $\mathbf{X}_o$ are convex open bounded sets and contain the origins. The initial states for both plant and controller are $x(0) = 0$ and $\xi(0) = 0$.

In this chapter, we shall consider the following version of $\mathcal{H}_\infty$-control problem.

(STRONG) $\mathcal{H}_\infty$-CONTROL PROBLEM: Find a feedback controller K, if any, such that the closed-loop system has **strong $\mathcal{H}_\infty$-performance**, and is asymptotically stable with $w = 0$. In this case, the feedback system has $\mathcal{L}_2$-gain ≤ 1, i.e.,

$$\int_0^T (\|w(t)\|^2 - \|z(t)\|^2) dt \geq 0$$

with $x(0) = 0, \xi(0) = 0$, for all $T \in \mathbf{R}^+$ and $w \in \mathcal{L}_2(\mathbf{R}^+)$, as long as the states $(x(t), \xi(t)) \in \mathbf{X} \times \mathbf{X}_o$ for $t \in [0, T]$. The strong $\mathcal{H}_\infty$-control problem is said to accept **local solutions** if the above requirements for the closed-loop system hold for $(x(t), \xi(t)) \in \mathcal{B}_r \times \mathcal{B}_s$ with some $r, s > 0$ for $t \in [0, T]$. The controllers to be sought in solving the above $\mathcal{H}_\infty$-Control Problem is called **strong $\mathcal{H}_\infty$-controllers**.

The following assumptions are made for the $\mathcal{H}_\infty$-control problem.

Assumption 8.3.1 *Consider the given system (8.13) and controller (8.14).*

[A1] $\text{RANK}\begin{bmatrix} B_2(x) \\ D_{12}(x) \end{bmatrix} = p_2$ *for all* $x \in \mathbf{X}$.

[A2] $\text{RANK}\begin{bmatrix} C_2(x) & D_{21}(x) \end{bmatrix} = q_2$ *for all* $x \in \mathbf{X}$.

[A3] $D_{11}(x)D_{11}^T(x) < I$ *for all* $x \in \mathbf{X}$.

[A4] $I - \hat{D}(\xi)D_{22}(x)$ *is invertible for all* $(x, \xi) \in \mathbf{X} \times \mathbf{X}_o$.

The first three regularity assumptions are for technical reason. The last assumption assures the well-posedness of the feedback structure.

In the next few sections, we will characterize the solvability of the strong $\mathcal{H}_\infty$-control problem. Basically the treatment is divided into the following steps.

- Given a controller (8.14) for system (8.13) which yields a stable closed-loop system with strong $\mathcal{H}_\infty$-performance, characterize this closed-loop property in terms of NLMI (8.15) by Theorem 8.2.5. This NLMI depends on the coefficient (matrix-valued) functions of the controller.

- Further characterize the above NLMI such that the new characterizations are independent of the coefficient matrix-valued functions of the controller using Finsler's Theorem. The new characterizations are three NLMIs.

- Examine the conditions under which the three NLMIs derived in the last step have the solutions that yield strong $\mathcal{H}_\infty$-control solution.

In the next section, the first two steps are mainly covered. The last step is treated in detail in Section 6.

8.4 Solutions to $\mathcal{H}_\infty$-Control Problem

In this section, we will consider the general strong $\mathcal{H}_\infty$-control problem for the system given by (8.13) under assumptions **[A1]**, **[A2]**, and **[A3]**. The solvability conditions for the $\mathcal{H}_\infty$-control problem to have solutions are characterized in terms of NLMIs without assuming the controllers have separation structures.

Consider the system (8.13) which evolves on $\mathbf{X}$. Define

$$B(x) := \begin{bmatrix} B_2^T(x) & D_{12}^T(x) \end{bmatrix}, \quad C(x) := \begin{bmatrix} C_2(x) & D_{21}(x). \end{bmatrix}.$$

Let $\mathcal{N}(B(x))$ be the distribution on $\mathbf{X}$ which annihilates the row vectors of $B(x)$. The main theorem of this section is stated as follows.

Theorem 8.4.1 *Given the system (8.13), suppose there is a solution to the output feedback (strong) $\mathcal{H}_\infty$ control problem. Then under Assumption 8.3.1 there are two $\mathbf{C}^0$ positive definite matrix-valued functions $X, Y : \mathbf{X} \rightarrow \mathbf{R}^{n\times n}$ such that for all $x \in \mathbf{X} \subset \mathbf{R}^{n\times n}$, the following three NLMIs are satisfied,*

$$B_\perp^T(x) \begin{bmatrix} X(x)A^T(x) + A(x)X(x) + B_1(x)B_1^T(x) & X(x)C_1^T(x) + B_1(x)D_{11}^T(x) \\ C_1(x)X(x) + D_{11}(x)B_1^T(x) & D_{11}(x)D_{11}^T(x) - I \end{bmatrix} B_\perp(x) < 0 \tag{8.15}$$

with $B_\perp : \mathbf{X} \rightarrow \mathbf{R}^{(n+q_1)\times(n+q_1-p_2)}$ such that $\mathcal{N}(B(x)) = \text{SPAN}(B_\perp(x))$.

$$C_\perp^T(x) \begin{bmatrix} A^T(x)Y(x) + Y(x)A(x) + C_1^T(x)C_1(x) & Y(x)B_1(x) + C_1^T(x)D_{11}(x) \\ B_1^T(x)Y(x) + D_{11}^T(x)C_1(x) & D_{11}^T(x)D_{11}(x) - I \end{bmatrix} C_\perp(x) < 0 \tag{8.16}$$

with $C_\perp : \mathbf{X} \rightarrow \mathbf{R}^{(n+p_1)\times(n+p_1-q_2)}$ such that $\mathcal{N}(C(x)) = \text{SPAN}(C_\perp(x))$.

$$\begin{bmatrix} X(x) & I \\ I & Y(x) \end{bmatrix} \geq 0. \tag{8.17}$$

The proof of the main theorem is given next. The techniques used in the proof closely follows from [141, 15, 185, 119, 133, 69, 94].

Proof. Suppose there exists a strong $\mathcal{H}_\infty$-controller which is of control-affine form as follows,

$$K : \begin{cases} \dot{\xi} &= \hat{A}(\xi)\xi + \hat{B}(\xi)y \\ u &= \hat{C}(\xi)\xi + \hat{D}(\xi)y \end{cases}$$

with $\hat{A}, \hat{B}, \hat{C}, \hat{D} \in \mathbf{C}^0$. Suppose $\xi \in \mathbf{X}_o \subset \mathbf{R}^{n_d}$ for some integer $n_d > 0$, where $\mathbf{X}_o$ is a convex open subset containing the origin. The closed loop system evolves on $\mathbf{X} \times \mathbf{X}_o$. Now take $x_c = \begin{bmatrix} x \\ \xi \end{bmatrix}$ to be the state of the closed loop system; define

$$R(x_c) := (I - \hat{D}(\xi)D_{22}(x))^{-1}, \tag{8.18}$$

which is well defined for $(x,\xi) \in \mathbf{X} \times \mathbf{X}_o$ by assumption **[A4]**. The feedback system has the following description,

$$\begin{cases} \dot{x}_c &= A_c(x_c)x_c + B_c(x_c)w \\ z &= C_c(x_c)x_c + D_c(x_c)w \end{cases}$$

with

$$A_c(x_c) = A^a(x) + B_2^a(x_c)F_c(x_c)C_2^a(x), \quad B_c(x_c) = B_1^a(x) + B_2^a(x_c)F_c(x_c)D_{21}^a(x), \tag{8.19}$$

$$C_c(x_c) = C_1^a + D_{12}^a(x)F_c(x_c)C_2^a(x), \quad D_c(x_c) = D_{11}^a(x) + D_{12}^a(x)F_c(x_c)D_{21}^a(x), \tag{8.20}$$

where

$$A^a(x) := \begin{bmatrix} A(x) & 0 \\ 0 & 0 \end{bmatrix}, \quad B_1^a(x) := \begin{bmatrix} B_1(x) \\ 0 \end{bmatrix}, \quad B_2^a(x_c) := \begin{bmatrix} B_2(x) & 0 \\ \hat{B}(\xi)D_{22}(x) & I \end{bmatrix},$$

$$C_1^a(x) := \begin{bmatrix} C_1(x) & 0 \end{bmatrix}, \quad D_{11}^a(x) := D_{11}(x), \quad D_{12}^a(x) := \begin{bmatrix} D_{12}(x) & 0 \end{bmatrix},$$

$$C_2^a(x) := \begin{bmatrix} C_2(x) & 0 \\ 0 & I \end{bmatrix}, \quad D_{21}^a(x) := \begin{bmatrix} D_{21}(x) \\ 0 \end{bmatrix}, \quad D_{22}^a(x) := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and

$$F_c(x_c) := \begin{bmatrix} R(x_c)\hat{D}(\xi) & R(x_c)\hat{C}(\xi) \\ \hat{B}(\xi) & \hat{A}(\xi) \end{bmatrix}. \tag{8.21}$$

Since the feedback system has strong $\mathcal{H}_\infty$-performance, by Definition 8.2.8, there is a $\mathbf{C}^0$ positive definite matrix-valued function $P_c(x_c)$ on $\mathbf{X} \times \mathbf{X}_o$ such that

$$\mathcal{M}_c(P_c, x_c) := \begin{bmatrix} A_c^T(x_c)P_c(x_c) + P_c(x_c)A_c(x_c) & P_c(x_c)B_c(x_c) & C_c^T(x_c) \\ B_c^T(x_c)P_c(x_c) & -I & D_c^T(x_c) \\ C_c(x_c) & D_c(x_c) & -I \end{bmatrix} < 0. \tag{8.22}$$

Re-organizing the left-hand side of the above NLMI yields

$$\mathcal{M}_c(P_c, x_c) = \mathcal{M}_a(P_c, x_c) + \tilde{C}^T(x_c)F_c^T(x_c)\tilde{B}(x_c)T_c(x_c) + T_c^T(x_c)\tilde{B}^T(x_c)F_c(x_c)\tilde{C}(x_c) < 0 \tag{8.23}$$

where

$$\mathcal{M}_a(P_c, x_c) := \begin{bmatrix} (A^a(x))^T P_c(x_c) + P_c(x_c)A^a(x) & P_c(x_c)B_1^a(x) & (C_1^a(x))^T \\ (B_1^a(x))^T P_c(x_c) & -I & (D_{11}^a(x))^T \\ C_1^a(x) & D_{11}^a(x) & -I \end{bmatrix},$$

$$\tilde{B}(x_c) := \begin{bmatrix} (B_2^a(x_c))^T & 0 & (D_{12}^a(x))^T \end{bmatrix}, \quad \tilde{C}(x_c) := \begin{bmatrix} C_2^a(x) & D_{21}^a(x) & 0 \end{bmatrix},$$

and

$$T_c(x_c) = \begin{bmatrix} P_c(x_c) & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

It follows from Lemma 8.8.4 that (8.23) holds only if the following two inequalities hold (see Lemma 8.8.4),

$$\tilde{B}_\perp^T(x_c)T_c^{-T}(x_c)\mathcal{M}_a(P_c, x_c)T_c^{-1}(x_c)\tilde{B}_\perp(x_c) < 0, \tag{8.24}$$

$$\tilde{C}_\perp^T(x_c)\mathcal{M}_a(P_c, x_c)\tilde{C}_\perp(x_c) < 0 \tag{8.25}$$

for all $\tilde{B}_\perp(x_c)$ with $\text{SPAN}(\tilde{B}_\perp(x_c)) \in \mathcal{N}(\tilde{B}(x_c))$ and $\tilde{C}_\perp(x_c)$ with $\text{SPAN}(\tilde{C}_\perp(x_c)) \in \mathcal{N}(\tilde{C}(x_c))$.

Next, we consider (8.24), notice that $\mathcal{N}(\tilde{B}(x_c)) = \mathcal{N}(\check{B}(x))$ for

$$\check{B}(x) := \begin{bmatrix} B_2^T(x) & 0 & 0 & D_{12}^T(x) \\ 0 & I & 0 & 0 \end{bmatrix}.$$

Thence, (8.24) holds if and only if

$$\check{B}_\perp^T(x)T_c^{-T}(x_c)\mathcal{M}_a(P_c, x_c)T_c^{-1}(x_c)\check{B}_\perp(x) < 0 \tag{8.26}$$

for all $\check{B}_\perp(x)$ with $\text{SPAN}(\check{B}_\perp(x)) \in \mathcal{N}(\check{B}(x))$.

On the other hand, notice that

$$T_c^{-T}(x_c)\mathcal{M}_a(P_c, x_c)T_c^{-1}(x_c)$$

$$= \begin{bmatrix} P_c^{-1}(x_c)(A^a(x))^T + A^a(x)P_c^{-1}(x_c) & B_1^a(x) & P_c^{-1}(x)(C_{11}^a(x))^T \\ (B_1^a(x))^T & -I & D_{11}^T(x) \\ C_1^a(x)P_c^{-1}(x_c) & D_{11}(x) & -I \end{bmatrix}.$$

Since $P_c(x_c) = P_c(x, \xi)$ is invertible on $\mathbf{X} \times \mathbf{X}_o$, assume $X(x) = X^T(x) \in \mathbf{R}^{n \times n}$, which is positive definite and of class $\mathbf{C}^0$ on $\mathbf{X}$, is such that

$$P_c^{-1}(x, \phi(x)) = \begin{bmatrix} X(x) & X_1^T(x) \\ X_1(x) & X_0(x) \end{bmatrix}, \tag{8.27}$$

for some continuously differentiable function $\phi : x \mapsto \xi$ in $\mathbf{X}$ such that $\phi(\mathbf{X}) \subset \mathbf{X}_o$ (for example ϕ can be chosen as $\phi(x) = 0$). Therefore, by the arguments of Schur complements, (8.26), i.e. (8.24) implies (8.15). Thus, the first part is proved.

Next, consider (8.25), if we take $Y(x) \in \mathbf{R}^{n \times n}$, which is of class $\mathbf{C}^0$, such that

$$P_c(x, \phi(x)) = \begin{bmatrix} Y(x) & Y_1^T(x) \\ Y_1(x) & Y_0(x) \end{bmatrix}. \tag{8.28}$$

Notice that $\tilde{C}(x_c)$ just depends on $x \in \mathbf{X}$, (8.25) implies (8.16).

Finally, by Lemma 8.8.3, (8.27) and (8.28) hold if and only if

$$\begin{bmatrix} X(x) & I \\ I & Y(x) \end{bmatrix} \geq 0.$$

This concludes the proof. □

Remark 8.4.2 *It is noted that all couples $(X(x), Y(x))$ satisfying the NLMIs (8.15), (8.16), and (8.17) form a convex set. Therefore, Theorem 8.4.1 provides a convex characterization to the necessary conditions for the strong output feedback $\mathcal{H}_\infty$-control problem to be solvable.*

Remark 8.4.3 *From the above proof, we can conclude that if the strong $\mathcal{H}_\infty$-control problem has a static output feedback solution, then there are two $\mathbf{C}^0$ positive definite matrix-valued functions $X, Y : \mathbf{X} \rightarrow \mathbf{R}^{n \times n}$ such that they satisfy the conditions (8.15) and (8.16) in Theorem 8.4.1, and $X(x)Y(x) = I$ for all $x \in \mathbf{X}$. Note that the characterization is not convex in this case.*

It is noted that in general, the NLMI characterization in Theorem 8.4.1 is not sufficient, because on one hand, the strong $\mathcal{H}_\infty$-control characterization by NLMI (8.23) holds only if (8.24) and (8.25) hold, the converse implication in general is not true since the matrix-valued matrix function $F_c(x_c)$ has some special structure (8.21) which is not guaranteed to recover by Lemma 8.8.4; on the other hand, as noted in Remark 8.2.6, the existence of the positive definite matrix-valued function $P_c(x_c)$ satisfying the NLMI (8.22) is not enough to guarantee the closed loop system has storage function $V_c : \mathbf{X} \times \mathbf{X}_o \rightarrow \mathbf{R}^+$ such that $\frac{\partial V_c}{\partial x_c}(x_c) = 2x^T P_c(x_c)$, some additional constraints are required (see Lemma 8.8.1). Nontheless, the characterization is sufficient if the $\mathcal{H}_\infty$-control problem is considered locally as stated as follows.

Theorem 8.4.4 *Consider system (8.13), there exists a local solution to the output feedback (strong) $\mathcal{H}_\infty$ control problem if and only if there are two $\mathbf{C}^0$ positive definite matrix-valued functions $X, Y : \mathcal{B}_r \rightarrow \mathbf{R}^{n \times n}$ with $\mathcal{B}_r \subset \mathbf{R}^n$ for some $r > 0$ such that they satisfy the NLMIs (8.15), (8.16), and (8.17) for all $x \in \mathcal{B}_r$.*

Proof. The necessity follows from the previous theorem. The sufficiency follows from the continuity argument, we just give an outline for this part. We make a simplification assumption that $D_{22}(x) = 0$ without loss of generality.

Suppose two $\mathbf{C}^0$ positive definite matrix-valued functions $X, Y : \mathcal{B}_r \to \mathbf{R}^{n\times n}$ satisfy (8.15), (8.16), and (8.17) for all $x \in \mathcal{B}_r$. By the continuity of the coefficient matrix-valued functions A, B_i, C_j, D_{ij} $(i,j = 1,2)$, $B_\perp$, and $C_\perp$, it follows that the three NLMIs has local constant positive definite solutions $X_c := X(0), Y_c := Y(0)$ for all $x \in \mathcal{B}_r$ with some adjusted $r > 0$. From condition (8.17) and Lemma 8.8.3, it follows that we can find a constant positive definite matrix P_c such that

$$P_c^{-1} = \begin{bmatrix} X_c & X_1^T \\ X_1 & X_0 \end{bmatrix}, \quad P_c = \begin{bmatrix} Y_c & Y_1^T \\ Y_1 & Y_0 \end{bmatrix}.$$

Moreover, the constant matrix P_c locally satisfies (8.24) and (8.25). Now by Lemma 8.8.4, P_c locally satisfies (8.22) with the coefficient matrix-valued functions defined by (8.19) and (8.20) for some matrix valued function $F_c(x_c)$. Still by continuity argument, we can find a constant matrices $\hat{A}, \hat{B}, \hat{C}$, and $\hat{D}$, such that (8.22) locally holds by replacing $F_c(x_c)$ with $\hat{F}_c := \begin{bmatrix} \hat{D} & \hat{C} \\ \hat{B} & \hat{A} \end{bmatrix} := F_c(0)$. Define a positive definite function $V_c(x_c) := x_c^T P_c x_c$ with $x_c := \begin{bmatrix} x \\ \xi \end{bmatrix}$, then $\frac{\partial V_c}{\partial x_c}(x_c) = 2x_c^T P_c$. Therefore, by Theorem 8.2.5, the controller given by

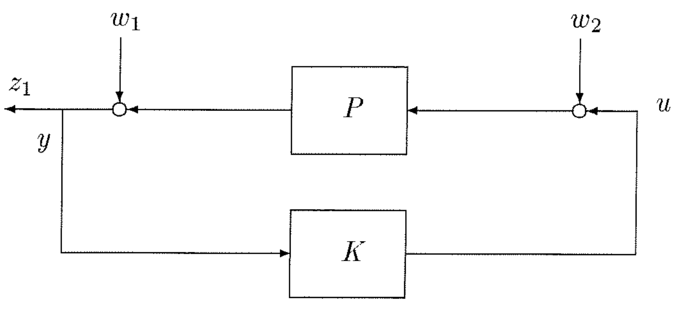
$$K_c : \begin{cases} \dot{\xi} & = & \hat{A}\xi & + & \hat{B}y \\ u & = & \hat{C}\xi & + & \hat{D}y \end{cases}$$

is a local strong $\mathcal{H}_\infty$-controller, and the resulting closed loop system is locally asymptotically stable.

□

To conclude this section, we examine an example which is from [59, 113].

Example 8.4.5 *The basic diagram is as follows.*



P is a passive nonlinear plant; K is the controller to be designed. The **control problem** *is to design the controller K such that the influence of the noises w_1 and w_2 on the regulated*

output z_1 can be reduced to the minimal with the reasonable effort (control action u should not be too large). This problem can be formulated in the $\mathcal{H}_\infty$-control framework as follows.

Given $\gamma > 0$, find a controller K (if exists) such that

$$\int_0^T (\|z_1\|^2 + \|u\|^2)dt \leq \gamma^2 \int_0^T (\|w_1\|^2 + \|w_2\|^2)dt, \forall T \in \mathbf{R}^+.$$

In this example, the plant P has the following realization:

$$\begin{cases} \dot{x} = e^x(w_2 + u) \\ z_1 = x + w_1 \\ y = x + w_1 \end{cases}$$

Take $\gamma > \sqrt{2}$, let $w := \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ and $z := \frac{1}{\gamma}\begin{bmatrix} z_1 \\ u \end{bmatrix}$ be input and output vectors, then the standard state-space realization is

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & e^x \end{bmatrix} w + e^x u \\ z = \begin{bmatrix} \gamma^{-1} \\ 0 \end{bmatrix} x + \begin{bmatrix} \gamma^{-1} & 0 \\ 0 & 0 \end{bmatrix} w + \begin{bmatrix} 0 \\ \gamma^{-1} \end{bmatrix} u \\ y = x + \begin{bmatrix} 1 & 0 \end{bmatrix} w \end{cases}.$$

From the Example 8.2.9, it is known that the above system accepts strong $\mathcal{H}_\infty$-solutions; i.e., there exists a controller K (actually K can be chosen as $K = -1$) such that the closed loop system has strong $\mathcal{H}_\infty$-performance. Whence, the three NLMIs (8.15), (8.16), and (8.17) in Theorem 8.4.1 should be satisfied. We now verify this.

We first consider NLMI (8.15), which is as follows

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & \gamma e^x \end{bmatrix} \begin{bmatrix} e^{2x} & \gamma^{-1}X(x) & 0 \\ \gamma^{-1}X(x) & \gamma^{-2}-1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & \gamma e^x \end{bmatrix} < 0. \tag{8.29}$$

All positive definite solutions satisfy $X(x) \leq (\gamma^2 - 1)e^x$.

The NLMI (8.16) is as follows

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \gamma^{-2} & \gamma^{-2} & e^x Y(x) \\ \gamma^{-2} & \gamma^{-2}-1 & 0 \\ e^x Y(x) & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -1 \\ 1 & 0 \end{bmatrix} < 0. \tag{8.30}$$

All positive definite solutions satisfy $Y(x) \leq e^{-x}$.

We then take the special solutions for the two NLMIs (8.29) and (8.30) as

$$X(x) = (\gamma^2 - 1)\alpha^{-1}e^{x}, \qquad Y(x) = \alpha^{-1}e^{-x} \tag{8.31}$$

for $\frac{1}{\sqrt{\gamma^2-1}} \leq \alpha < 1$. *Then* $X(x)Y(x) \geq 1$, *which implies condition (8.17). Therefore, the three NLMI conditions are satisfied. Moreover, since* $u = -y$ *is a static output feedback, by Remark 8.4.3, it is required that* $X(x)Y(x) = 1$, *which is however satisfied by taking* $\alpha = \frac{1}{\sqrt{\gamma^2-1}}$ *in (8.31).*

8.5 Separation Structures and State Feedback

In the last section, when the solvability conditions for the strong $\mathcal{H}_\infty$-control problem for system (8.13) are derived, the $\mathcal{H}_\infty$-controller is not assumed to have separation structures. The implications of NLMIs (8.15) and (8.16) in the last section have not been sufficiently revealed. This issue will be pursued further in this section. Actually, the NLMIs (8.15) and (8.16) are closely related to the state-feedback and output-injection conditions for nonlinear $\mathcal{H}_\infty$-control. In this section, it will be shown that under a weaker separation structure constraints, if the $\mathcal{H}_\infty$-control problem is solvable by output feedback, then it is also solvable by static state feedback.

We first state a theorem which justifies that the NLMI (8.15) characterizes state-feedback solution under additional constraints.

Theorem 8.5.1 *The strong* $\mathcal{H}_\infty$*-control problem is solvable by static state feedback if and only if there is a* $\mathbf{C}^0$ *matrix-valued function* $X(x) = X^T(x) > 0$ *with* $\frac{\partial V}{\partial x}(x) = 2x^T X^{-1}(x)$ *for some* $\mathbf{C}^1$ *function* $V : \mathbf{X} \to \mathbf{R}^+$ *such that for all* $x \in \mathbf{X}$, *the following NLMI holds,*

$$B_\perp^T(x)\begin{bmatrix} X(x)A^T(x) + A(x)X(x) + B_1(x)B_1^T(x) & X(x)C_1^T(x) + B_1(x)D_{11}^T(x) \\ C_1(x)X(x) + D_{11}(x)B_1^T(x) & D_{11}(x)D_{11}^T(x) - I \end{bmatrix} B_\perp(x) < 0 \tag{8.32}$$

with $B_\perp : \mathbf{X} \to \mathbf{R}^{(n+q_1)\times(n+q_1-p_2)}$ *such that* $\mathrm{S{\small PAN}}(B_\perp(x)) = \mathcal{N}(B(x))$.

Proof. The necessity basically follows the arguments in the proof of Theorem 8.4.1. The sufficiency also follows the proof of Theorem 8.4.1 by noting the converse direction in the proof goes through in this case, since a smooth static state-feedback can be constructed by using Lemma 8.8.4; then the conclusion follows by Theorem 8.2.5. $\square$

Next, we will find the relation between output feedback solutions and state-feedback solutions. Suppose the output feedback strong $\mathcal{H}_\infty$-control problem for the given system (8.13) is solvable, then there is a $\mathbf{C}^0$ positive definite matrix-valued function $P_c(x_c)$ such that (8.22) holds. Moreover, there is a positive definite function $V_c : \mathbf{X} \times \mathbf{X}_o \to \mathbf{R}^+$ such that

$$\frac{\partial V_c}{\partial x_c}(x_c) = 2x_c^T P_c(x_c). \tag{8.33}$$

Assumption 8.5.2 *Consider the positive definite function $V_c : \mathbf{X} \times \mathbf{X}_o \to \mathbf{R}^+$ satisfying (8.33). There is a $\mathbf{C}^1$ function $\phi : x \mapsto \xi$ with $\phi(0) = 0$ such that $\frac{\partial V_c}{\partial \xi}(x, \xi)|_{\xi=\phi(x)} = 0$ with $(x, \xi) \in \mathbf{X} \times \mathbf{X}_o$.*

Remark 8.5.3 *Note that the function V_c is a Lyapunov function of the closed loop system. Assumption 8.5.2 has a (weak) separation structure interpretation. In fact, many dynamical controllers have well-defined separation structures [15, 93, 113]. In such case, the states x, ξ of a plant and its controller satisfy that*

$$\xi(t) - \phi(x(t)) \to 0 \text{ as } t \to \infty$$

for some $\mathbf{C}^1$ function $\phi : x \mapsto \xi$ with $\phi(0) = 0$; in particular, if the initial states satisfy $\xi(0) = \phi(x(0))$, then $\xi(t) = \phi(x(t)$ for all $t \in \mathbf{R}^+$. A Lyapunov function U_c for the closed loop system is constructed as follows,

$$U_c(x, \xi) = V(x) + U(\xi - \phi(x))$$

where V and U are Lyapunov functions of the state-feedback system and the error system. Thence,

$$\frac{\partial U_c}{\partial \xi}(x, \xi) = \frac{\partial U}{\partial e}(e)|_{e=\xi-\phi(x)}.$$

If $e = 0$, i.e. $\xi = \phi(x)$, then

$$\frac{\partial U_c}{\partial \xi}(x, \xi)|_{\xi=\phi(x)} = \frac{\partial U}{\partial e}(e)|_{e=0} = 0.$$

Therefore, U_c satisfies the assumption.

From the proof of Theorem 8.4.1, it follows that (8.22) implies that there is a $\mathbf{C}^0$ positive definite matrix-valued function $X : \mathbf{X} \to \mathbf{R}^{n \times n}$ such that

$$P_c^{-1}(x, \phi(x)) = \begin{bmatrix} X(x) & X_1^T(x) \\ X_1(x) & X_0(x) \end{bmatrix}$$

for some continuously differentiable function $\phi : x \mapsto \xi$ on $\mathbf{X}$, and the NLMI (8.32) holds. Since $\frac{\partial V_c}{\partial x_c}(x_c) = 2x_c^T P_c(x_c)$ implies $\frac{\partial V_c}{\partial x_c}(x_c)P_c^{-1}(x_c) = 2x_c^T$, or

$$\left[\begin{array}{cc} \frac{\partial V_c}{\partial x}(x,\xi) & \frac{\partial V_c}{\partial \xi}(x,\xi) \end{array} \right] P_c(x,\xi) = 2 \left[\begin{array}{cc} x^T & \xi^T \end{array} \right]. \tag{8.34}$$

Take the function ϕ as in Assumption 8.5.2, define $V(x) := V_c(x, \phi(x))$, then (8.34) implies

$$\frac{\partial V_c}{\partial x}(x, \phi(x))X(x) = 2x^T.$$

Define $V(x) := V_c(x, \phi(x))$, then $V(x)$ is positive definite, and by Assumption 8.5.2, one has

$$\frac{\partial V}{\partial x}(x) = 2x^T X^{-1}(x). \tag{8.35}$$

Therefore, the $\mathcal{H}_\infty$-control problem is solvable in terms of static feedback. The above observation is summarized as follows.

Theorem 8.5.4 *If the strong $\mathcal{H}_\infty$-control problem is solvable (by the output feedback), then under Assumption 8.5.2, it can also be solved by static state feedback.*

Remark 8.5.5 *The above theorem also implies that under the separation structure assumption for the $\mathcal{H}_\infty$ controllers, the dynamic state-feedback can not do better than static state-feedback can as far as the strong $\mathcal{H}_\infty$ control problem is concerned.*

Similar argument applies to output injection problem.

Theorem 8.5.6 *The strong output injection $\mathcal{H}_\infty$-control problem is solvable if and only if there is a $\mathbf{C}^0$ matrix-valued function $Y(x) = Y^T(x) > 0$ with $\frac{\partial U}{\partial x}(x) = 2x^T Y(x)$ for some $\mathbf{C}^1$ function $U : \mathbf{X} \to \mathbf{R}^+$ such that for all $x \in \mathbf{X}$, the following NLMI holds,*

$$C_\perp^T(x) \left[\begin{array}{cc} A^T(x)Y(x) + Y(x)A(x) + C_1^T(x)C_1(x) & Y(x)B_1(x) + C_1^T(x)D_{11}(x) \\ B_1^T(x)Y(x) + D_{11}^T(x)C_1(x) & D_{11}^T(x)D_{11}(x) - I \end{array} \right] C_\perp(x) < 0 \tag{8.36}$$

with $C_\perp : \mathbf{X} \to \mathbf{R}^{(n+p_1)\times(n+p_1-q_2)}$ such that $\mathcal{N}(C(x)) = \textsc{Span}(C_\perp(x))$.

Let $\phi : x \mapsto \xi$ be $\phi(x) = 0$ for all $x \in \mathbf{X}$. Define a $\mathbf{C}^0$ positive definite matrix-valued function $Y : \mathbf{X} \to \mathbf{R}^{n\times n}$ such that

$$P_c(x, 0) = \left[\begin{array}{cc} Y(x) & Y_1^T(x) \\ Y_1(x) & Y_0(x) \end{array} \right].$$

Define $U(x) = V_c(x,0)$, which is positive definite, then $\frac{\partial V_c}{\partial x_c}(x_c) = 2x_c^T P_c(x_c)$ implies

$$\frac{\partial U}{\partial x}(x) = 2x^T Y(x). \tag{8.37}$$

Thus, by Theorem 8.5.6, we also have the following result.

Theorem 8.5.7 *If the strong $\mathcal{H}_\infty$-control problem is solvable (by output feedback), then it can also be solved by static output injection.*

8.6 Nonlinear Matrix Inequalities and Computational issues

In this section, we will address computational issue for strong $\mathcal{H}_\infty$-control performance analysis and synthesis. The material in this section is also of independent interest. We have known that, the $\mathcal{H}_\infty$-control performance analysis and synthesis involves solving some NLMIs, i.e., (8.11), (8.15), (8.16). We will show that under some regularity conditions, there exist positive-definite smooth solutions to NLMIs, and those solutions guarantee the solvability of $\mathcal{H}_\infty$-control problems.

8.6.1 Existence of Positive Definite Solutions

In this subsection, we examine under what condition, the solutions to NLMIs are positive definite. To be more concrete, we consider the NLMI (8.11) as follows,

$$\mathcal{M}(P,x) := \begin{bmatrix} A^T(x)P(x) + P^T(x)A(x) + C^T(x)C(x) & P^T(x)B(x) + C^T(x)D(x) \\ B^T(x)P(x) + D^T(x)C(x) & D^T(x)D(x) - I \end{bmatrix} \leq 0. \tag{8.38}$$

By Schur Complement argument, there exists a matrix valued function $P : \mathbf{X} \to \mathbf{R}^{n\times n}$ such that $\mathcal{M}(P,x) \leq 0$ if and only if there exists a matrix valued function $Q : \mathbf{X} \to \mathbf{R}^{n\times n}$ with $Q(x) \geq 0$ for all $x \in \mathbf{X}$ such that the following state-dependent Riccati equation has solution $P : \mathbf{X} \to \mathbf{R}^{n\times n}$,

$$\mathcal{H}(P,x) + Q(x) = 0,$$

where

$$\mathcal{H}(P,x) := A^T(x)P(x) + P^T(x)A(x) + C^T(x)C(x)$$

$$-(P^T(x)B(x) + C^T(x)D(x))(I - D^T(x)D(x))^{-1}(B^T(x)P(x) + D^T(x)C(x)).$$

Let $R(x) := I - D^T(x)D(x) > 0$, define a state-dependent Hamiltonian $H : \mathbf{X} \to \mathbf{R}^{2n\times 2n}$ as

$$H(x) := \begin{bmatrix} A(x) & 0 \\ -C^T(x)C(x) - Q(x) & -A^T(x) \end{bmatrix} +$$

$$\begin{bmatrix} B(x) \\ -C^T(x)D(x) \end{bmatrix} R^{-1}(x) \begin{bmatrix} D^T(x)C(x) & B^T(x) \end{bmatrix} \tag{8.39}$$

Using the terminology in [60, 75], we have the following result which is essentially from [75, Lemma 2.4].

Theorem 8.6.1 *$\mathcal{M}(P, x) \leq 0$ has non-negative definite solutions $P(x) \geq 0$ if and only if the state-dependent Hamiltonian $H : \mathbf{X} \to \mathbf{R}^{2n\times 2n}$ defined in (8.39) for some matrix-valued function $Q : \mathbf{X} \to \mathbf{R}^{n\times n}$ with $Q(x) \geq 0$ for all $x \in \mathbf{X}$ is in $dom(\bar{Ric})$, i.e. $H(x) \in dom(\bar{Ric})$ for each $x \in \mathbf{X}$, and $\bar{Ric}(H(x)) \geq 0$ is such a solution. In addition, if for each $x \in \mathbf{X}$,*

$$\bigcap_{i=0}^{n-1} \ker(\tilde{C}(x)A^i(x)) = \emptyset, \qquad \tilde{C}(x) = \begin{bmatrix} C(x) \\ Q^{1/2}(x) \end{bmatrix}$$

this solution is positive definite, i.e., $\bar{Ric}(H(x)) > 0$.

The above theorem implies that under the condition $H(x) \in dom(\bar{Ric})$ for each $x \in \mathbf{X}$, the NLMI $\mathcal{M}(P, x) \leq 0$ has non-negative definite solutions. To apply this theorem, the non-negative definite matrix valued function $Q(x)$ can be chosen on *ad hoc* basis, the verification of the state-dependent Hamiltonian $H(x) \in dom(\bar{Ric})$ can be conducted by invoking the linear result in [60, 75] and continuity argument. However, how to systematically choose $Q(x)$ and to verify $H(x) \in dom(\bar{Ric})$ are still practical issues.

8.6.2 Existence of Continuous Solutions

In the last subsection, we justified the existence of positive definite solutions to NLMIs under some regularity conditions. Since the solvability for each strong $\mathcal{H}_\infty$-control problem requires that the positive definite solutions to the corresponding NLMIs be **continuous**, a natural question is that, is there such a solution to a NLMI if it has pointwise positive definite solutions? In this section, we will justify this, i.e., we will consider the existence of continuous solutions to the given NLMIs provided that there are positive definite solutions for fixed $x \in \mathbf{X}$.

Let $\mathbf{X}$ be an open subset $\mathbf{R}^n$ with $0 \in \mathbf{X}$, consider a general matrix-valued map $\mathcal{M}$: $\mathbf{R}^{n\times n} \times \mathbf{X} \rightarrow \mathbf{R}^{m\times m}$, which is continuous and satisfies

$$\mathcal{M}(\sum_{k=1}^{N} \alpha_k P_k, x) = \sum_{k=1}^{N} \alpha_k \mathcal{M}(P_k, x) \tag{8.40}$$

for all $\alpha_k \geq 0$ with $\sum_{k=1}^{N} \alpha_k = 1$. Consider the following matrix inequality,

$$\mathcal{M}(P, x) < 0. \tag{8.41}$$

Note that **all of the NLMIs discussed in this chapter are in this matrix inequality class.**

The main result of this subsection is stated as follows:

Theorem 8.6.2 *Suppose the matrix inequality (8.41) has a positive definite solution P_x for each $x \in \mathbf{X}$, i.e., $\mathcal{M}(P_x, x) < 0$ for $x \in \mathbf{X}$, then there exists a $\mathbf{C}^0$ (in fact, $\mathbf{C}^\infty$) positive-definite matrix-valued function $P : \mathbf{X} \rightarrow \mathbf{R}^{n\times n}$, such that $\mathcal{M}(P(x), x) < 0$ for all $x \in \mathbf{X}$.*

Proof. Consider the NLMI (8.41): $\mathcal{M}(P, x) < 0$ for $x \in \mathbf{X}$. By assumption, for each $x \in \mathbf{X}$, there is a positive definite $P_x \in \mathbf{R}^{n\times n}$ such that $\mathcal{M}(P_x, x) < 0$. By continuity of $\mathcal{M}$ with respect to x, there is a $r_x > 0$ such that for all $y \in \mathbf{N}(x) := \{y : \|y - x\| < r_x\}$,

$$\mathcal{M}(P_x, y) < 0. \tag{8.42}$$

On the other hand, $\{\mathbf{N}(x)\}|_{x\in\mathbf{X}}$ is an open covering of X, i.e.,

$$X \subset \bigcup_{x\in\mathbf{X}} \mathbf{N}(x). \tag{8.43}$$

Then there is a locally finite open subcovering $\{\mathbf{N}_i\}|_{i\in\mathbf{I}}$ for some index set $\mathbf{I}$ which refines $\{\mathbf{N}(x)\}|_{x\in\mathbf{X}}$. By (8.42), $P_i \in \mathbf{R}^{n\times n}$ is taken to be positive definite for each $i \in \mathbf{I}$ such that

$$\mathcal{M}(P_i, x) < 0 \tag{8.44}$$

for all $x \in \mathbf{N}_i$.

It is known by the standard results of **partitions of unity** that there is a $\mathbf{C}^\infty$ partition of unity $\{\psi_i\}|_{i\in\mathbf{I}}$ to $\mathbf{X}$ subordinated to the covering $\{\mathbf{N}_i\}|_{i\in\mathbf{I}}$; i.e., ψ_i is $\mathbf{C}^\infty$ and non-negative with support $\mathrm{S{\scriptsize UPP}}(\psi_i) \subset \mathbf{N}_i$ for each $i \in \mathbf{I}$, and

$$\sum_{i\in\mathbf{I}} \psi_i(x) = 1, \forall x \in \mathbf{X}. \tag{8.45}$$

Define a matrix-valued function $P : \mathbf{X} \to \mathbf{R}^{n \times n}$ as

$$P(x) = \sum_{i \in \mathbf{I}} \psi_i(x) P_i, \forall x \in \mathbf{X}, \tag{8.46}$$

which is positive definite and $\mathbf{C}^\infty$ since it is locally a finite sum of $\mathbf{C}^\infty$ positive definite matrix-valued functions.

It follows from (8.45), (8.46) and (8.40) that

$$\mathcal{M}(P(x), x) = \mathcal{M}(\sum_{i \in \mathbf{I}} \psi_i(x) P_i, x) = \sum_{i \in \mathbf{I}} \psi_i(x) \mathcal{M}(P_i, x) < 0.$$

The last equality holds since the sum is finite for each $x \in \mathbf{X}$.

Thence, the constructed $\mathbf{C}^\infty$ matrix-valued function $P : \mathbf{X} \to \mathbf{R}^{n \times n}$ in (8.46) is positive definite and is a solution to $\mathcal{M}(P(x), x) < 0$. □

A nice convex property for NLMIs is stated by the following theorem, which is essentially from [197] and whose proof is easy and omitted here.

Theorem 8.6.3 *The $\mathbf{C}^0$ solutions $P : \mathbf{X} \to \mathbf{R}^{n \times n}$ to NLMI $\mathcal{M}(P, x) < 0$ form a convex set; the subset of all $\mathbf{C}^0$ non-negative definite solutions $P = P^T : \mathbf{X} \to \mathbf{R}^{n \times n}$ such that $\frac{\partial V}{\partial x}(x) = 2x^T P(x)$ for some function $V : \mathbf{X} \to \mathbf{R}$ is convex; the subset of all $\mathbf{C}^0$ positive definite solutions $P^T(x) = P(x) > 0$ such that $\frac{\partial V}{\partial x}(x) = 2x^T P(x)$ for some function V : $\mathbf{X} \to \mathbf{R}$ is also convex.*

8.6.3 Existence of Lyapunov Functions

As mentioned in Remark 8.2.6, the existence of positive definite matrix-valued function $P : \mathbf{X} \to \mathbf{R}^{n \times n}$ to NLMIs is not enough to guarantee the strong $\mathcal{H}_\infty$-control problem to have solution; some additional requirement is imposed in this chapter, i.e. there is a $\mathbf{C}^1$ storage function, $V : \mathbf{X} \to \mathbf{R}^+$, such that

$$\frac{\partial V}{\partial x}(x) = 2x^T P(x)$$

for all $x \in \mathbf{X}$. In this subsection, we will examine explicitly when it is the case for the solution constructed in the preceding subsection.

From the preceding subsection, a matrix-valued function $P : \mathbf{X} \to \mathbf{R}^{n \times n}$, which satisfies $\mathcal{M}(P, x) \leq 0$, is constructed as (8.46):

$$P(x) = \sum_{i \in \mathbf{I}} \psi_i(x) P_i, \forall x \in \mathbf{X},$$

for some index set $\mathbf{I}$, where $\{\psi_i\}|_{i\in\mathbf{I}}$ is a $\mathbf{C}^\infty$ partition of unity of $\mathbf{X}$ and $P_i = P_i^T \geq 0$. Notice that the above summation is locally finite. (If $\mathbf{X} \subset \mathbf{R^n}$ is bounded, then $\mathbf{I}$ can be chosen to be finite.)

Theorem 8.6.4 *Suppose the matrix valued function $P : \mathbf{X} \rightarrow \mathbf{R}^{n\times n}$ defined by*

$$P(x) = \sum_{i\in\mathbf{I}} \psi_i(x)P_i \tag{8.47}$$

with $\psi_i : \mathbf{X} \rightarrow \mathbf{R}^+$ being of class $\mathbf{C}^1$ and $P_i \in \mathbf{R}^{n\times n}$ for $i \in \mathbf{I}$ satisfies (8.41): $\mathcal{M}(P, x) \leq 0$ for all $x \in \mathbf{X}$; let $V_i(x) = x^T P_i x$ for all $i \in \mathbf{I}$. There exists a $\mathbf{C}^2$ function $V : \mathbf{X} \rightarrow \mathbf{R}$ such that $\frac{\partial V}{\partial x}(x) = 2x^T P(x)$ if and only if

$$\sum_{i\in\mathbf{I}} \frac{\partial \psi_i}{\partial x_j}(x) \cdot \frac{\partial V_i}{\partial x_l}(x) = \sum_{i\in\mathbf{I}} \frac{\partial \psi_i}{\partial x_l}(x) \cdot \frac{\partial V_i}{\partial x_j}(x) \tag{8.48}$$

for all $x \in \mathbf{X}$ and $j, l \in \{1, 2, \cdots, n\}$ with $j \neq l$.

Notice that the summations in (8.48) are finite for each $x \in \mathbf{X}$ by the construction. Next, we will give a direct proof to this theorem.

Proof. Define a 1-form ω on $\mathbf{X}$ as follows,

$$\omega := \sum_{i\in\mathbf{I}} \psi_i dV_i \tag{8.49}$$

which is well defined since the summations are locally finite. Since

$$\omega(x) = \sum_{i\in\mathbf{I}} 2\psi_i(x)x^T P_i dx = 2x^T P(x)dx, \quad dx := \begin{bmatrix} dx_1 \\ \vdots \\ dx_n \end{bmatrix},$$

there is a $\mathbf{C}^2$ function $V : \mathbf{X} \rightarrow \mathbf{R}$ such that $\frac{\partial V}{\partial x}(x) = 2x^T P(x)$ if and only if $\omega = dV$, i.e. the 1-form defined by (8.49) is **exact**. By Poincare Lemma (cf. [175]), the later statement is equivalent to that the 1-form ω is **closed**. i.e. the 2-form $d\omega$ on $\mathbf{X}$, which is the differential of ω, is 0.

On the other hand,

$$\begin{aligned} d\omega &= d(\sum_{i\in\mathbf{I}} \psi_i dV_i) = \sum_{i\in\mathbf{I}} d\psi_i \wedge dV_i \\ &= \sum_{i\in\mathbf{I}} (\sum_{j=1}^{n} \frac{\partial \psi_i}{\partial x_j} dx_j) \wedge (\sum_{l=1}^{n} \frac{\partial V_i}{\partial x_l} dx_l) \end{aligned}$$

$$= \sum_{i\in\mathbf{I}}\sum_{j<l}(\frac{\partial\psi_i}{\partial x_j}\cdot\frac{\partial V_i}{\partial x_l} - \frac{\partial\psi_i}{\partial x_l}\cdot\frac{\partial V_i}{\partial x_j})dx_j \wedge dx_l \tag{8.50}$$

$$= \sum_{j<l}\sum_{i\in\mathbf{I}}(\frac{\partial\psi_i}{\partial x_j}\cdot\frac{\partial V_i}{\partial x_l} - \frac{\partial\psi_i}{\partial x_l}\cdot\frac{\partial V_i}{\partial x_j})dx_j \wedge dx_l, \tag{8.51}$$

where (8.50) is derived by re-organization using that $dx_l \wedge dx_j = -dx_j \wedge dx_l$ (so $dx_j \wedge dx_j = 0$) for $j, l \in \{1, 2, \cdots, n\}$.

From (8.51) and the linear independence of 2-forms $\{dx_j \wedge dx_l\}|_{j<l}$, it follows that $d\omega = 0$ if and only if

$$\sum_{i\in\mathbf{I}}(\frac{\partial\psi_i}{\partial x_j}\cdot\frac{\partial V_i}{\partial x_l} - \frac{\partial\psi_i}{\partial x_l}\cdot\frac{\partial V_i}{\partial x_j}) = 0, \forall j, l \in \{1, 2, \cdots, n\}, j < l,$$

which is equivalent to (8.48). □

It is noted that Theorem 8.6.4 can also be proved using Lemma 8.8.1.

8.6.4 Existence of (Local) Constant Solutions to NLMIs

The above treatments about $\mathcal{H}_\infty$-performance analysis and synthesis are in terms of NLMIs, which are pointwise LMIs on state set $\mathbf{X}$, modulo some additional constraints on the solutions. We also know that if $\mathbf{X}$ is bounded, then we only need to solve a finite number of LMIs to get the solution for the NLMI on $\mathbf{X}$. In addition, if the set $\mathbf{X}$ is small enough, then we can get a constant solution to the NLMIs by using similar treatments in [29], which is motivated by the notion of global linearization of nonlinear systems developed by Liu *et al.* [106]. More concretely, we consider the following NLMI:

$$\begin{bmatrix} A^T(x)P(x) + P(x)A(x) & P(x)B(x) & C^T(x) \\ B^T(x)P(x) & -I & D^T(x) \\ C(x) & D(x) & -I \end{bmatrix} < 0,$$

where the coefficient matrix-valued functions $A(x), B(x), C(x), D(x)$ are assumed to be continuous on $\mathbf{X}$. The coefficient matrices are assumed in a convex set:

$$[A(x), B(x), C(x), D(x)] \in \mathrm{Co}\{[A_i, B_i, C_i, D_i]|_{i\in\{1,2,\cdots,L\}}\}, \forall x \in \mathbf{X},$$

for some A_i, B_i, C_i, D_i with $I - D_i^T D_i \leq 0$ for $i \in \{1, 2, \cdots, L\}$ with some integer $L > 0$, where Co stands for the **convex hull**.

If there is a constant (semi-)positive definite matrix $P \in \mathbf{R}^{n\times}$ such that

$$\begin{bmatrix} A_i^T P + PA_i & PB_i & C_i^T \\ B_i^T P & -I & D_i^T \\ C_i & D_i & -I \end{bmatrix} < 0, \ \forall i \in \{1, 2, \cdots, L\},$$

which are a set of linear matrix inequalities (LMIs) and can be solved in terms of convex optimization methods [29], then P also satisfies

$$\begin{bmatrix} A^T(x)P + PA(x) & PB(x) & C^T(x) \\ B^T(x)P & -I & D^T(x) \\ C(x) & D(x) & -I \end{bmatrix} < 0$$

for all $x \in \mathbf{X}$.

The solution automatically satisfies the condition (8.52), and the corresponding Lyapunov function is $V(x) = x^T Px$.

This treatment suggests a tractable approach to get local solutions, which can be used to find constant solutions on each partitioned state set $\mathbf{N}_i$ in the proof of Theorem 8.6.2. However, this approach generally leads to conservative results if the prescribed state set is large enough.

8.7 Notes and References

In this chapter, the $\mathcal{H}_\infty$-control problem for a class of nonlinear systems has been characterized in terms of nonlinear matrix inequalities which result in the convex problems. The solutions of NLMIs which insure the solutions of $\mathcal{H}_\infty$-control problems have been examined. Unfortunately, unlike the linear case, the solution of the NLMIs by themselves are not sufficient to guarantee the existence of the required controller. However, the proposed approach points out a new direction to make the nonlinear $\mathcal{H}_\infty$-control theory to be applicable.

The material in this chapter forms the paper [113].

8.8 Appendix: Some Technical Results

8.8.1 Equation $\frac{\partial V}{\partial x}(x) = 2x^T P(x)$

The following result is quite standard (see, for example [21, Lemma 2.22]).

Lemma 8.8.1 *Suppose a vector-valued function $p : \mathbf{X} \rightarrow \mathbf{R}^n$ is of class $\mathbf{C}^k$ for some integer $k \geq 1$; let*

$$p(x) = [p_1(x), \cdots, p_n(x)]^T \text{ for } x \in \mathbf{X}.$$

Then there exists a $\mathbf{C}^{k+1}$ function $V : \mathbf{X} \rightarrow \mathbf{R}$ such that

$$\frac{\partial V}{\partial x}(x) = 2p^T(x)$$

if and only if

$$\frac{\partial p_i}{\partial x_j}(x) = \frac{\partial p_j}{\partial x_i}(x) \tag{8.52}$$

for all $x \in \mathbf{X}$ *and* $i, j = 1, 2, \cdots, n$. *Moreover, if (8.52) holds, then an function* $V : \mathbf{X} \rightarrow \mathbf{R}$ *with* $V(0) = 0$ *is given by*

$$V(x) = 2x^T \int_0^1 p(tx)dt. \tag{8.53}$$

In addition, if $p(x) = P(x)x$ *for some* $\mathbf{C}^k$ *positive definite matrix-valued function* $P : \mathbf{X} \rightarrow \mathbf{R}^{n \times n}$, *then* $V(x)$ *is also positive definite function.*

8.8.2 Schur Complements

A reference for the material here is [85].

Lemma 8.8.2 *Suppose* $M = M^T \in \mathbf{R}^{(n+m)\times(n+m)}$ *is portioned as*

$$M = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

with $C \in \mathbf{R}^{m+m}$ *is non-singular, then* $M \geq 0$ *if and only if* $C > 0$ *and* $A - BC^{-1}B^T \geq 0$.

Lemma 8.8.3 *Let* $X = X^T, Y = Y^T \in \mathbf{R}^{n \times n}$ *be two positive definite matrices. Then there is a positive definite matrix* $P = P^T \in \mathbf{R}^{(n+m)\times(n+m)}$ *such that*

$$P = \begin{bmatrix} X & X_1^T \\ X_1 & X_0 \end{bmatrix}, \qquad P^{-1} = \begin{bmatrix} Y & Y_1^T \\ Y_1 & Y_0 \end{bmatrix}$$

if and only if $\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0.$

8.8.3 Characterization of a State-Dependent LMI

Given an integer $k \geq 0$ and a $\mathbf{C}^k$ matrix-valued function $B : \mathbf{X} \rightarrow \mathbf{R}^{m \times n}$, with $m \leq n$, and $\mathrm{RANK}(B(x)) = m$ for all $x \in \mathbf{X}$. Thus, there is an $(n - m)$-dimensional **distribution** $\mathcal{N}(B(x))$ on $\mathbf{X}$ which annihilates the row vectors in $B(x)$. Moreover, there is a $\mathbf{C}^k$ matrix-valued function $B_\perp : \mathbf{M} \rightarrow \mathbf{R}^{n \times (n-m)}$, such that its columns span the distribution $\mathcal{N}(B(x))$, i.e. $\mathcal{N}(B(x)) = \mathrm{SPAN}(B_\perp(x))$ for $x \in \mathbf{M}$ (cf. [90]). The following lemma generalizes its constant matrix version in [45] (see also for example, [55, 69, 94, 29]).

Lemma 8.8.4 *Given three* $\mathbf{C}^k$ *matrix-valued functions* $Q = Q^T : \mathbf{X} \rightarrow \mathbf{R}^{m\times m}$, $U : \mathbf{X} \rightarrow \mathbf{R}^{r\times m}$ *with* $\mathrm{RANK}(U(x)) = r < m$, *and* $V : \mathbf{X} \rightarrow \mathbf{R}^{s\times m}$ *with* $\mathrm{RANK}(V(x)) = s < m$. *There exists a* $\mathbf{C}^k$ *matrix-valued function* $F : \mathbf{X} \rightarrow \mathbf{R}^{s\times r}$ *such that the following matrix inequality is satisfied*

$$Q(x) + U^T(x)F^T(x)V(x) + V^T(x)F(x)U(x) < 0 \tag{8.54}$$

if and only if

$$U_\perp^T(x)Q(x)U_\perp(x) < 0, \quad V_\perp^T(x)Q(x)V_\perp(x) < 0 \tag{8.55}$$

for some $\mathbf{C}^k$ *matrix-valued functions* $U_\perp : \mathbf{X} \rightarrow \mathbf{R}^{m\times(m-r)}$ *with* $\mathrm{SPAN}(U_\perp(x)) = \mathcal{N}(U(x))$ *and* $V_\perp : \mathbf{X} \rightarrow \mathbf{R}^{m\times(m-s)}$ *with* $\mathrm{SPAN}(V_\perp(x)) = \mathcal{N}(V(x))$.

Proof. The necessity is obvious. As for the sufficiency, suppose the condition (8.55) is satisfied, from the constant matrix version of the above lemma, it follows that for each fixed $x \in \mathbf{X}$, there exists a matrix $F_x \in \mathbf{R}^{s\times r}$ such that

$$Q(x) + U^T(x)F_x^T V(x) + V^T(x)F_x U(x) < 0,$$

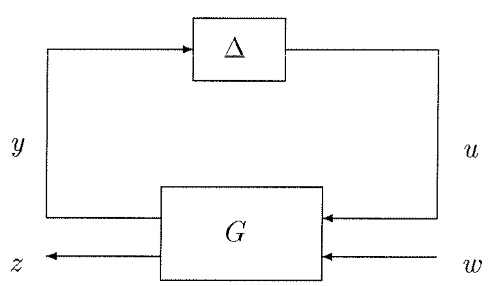
i.e., the NLMI (8.54) has a pointwise solution. Now by Theorem 8.6.2 and its proof, we can find a smooth matrix-valued function $F : \mathbf{X} \rightarrow \mathbf{R}^{s\times r}$ such that NLMI (8.54) is satisfied. □

Chapter 9

Robustness of Nonlinear Uncertain Systems

9.1 Introduction

In this chapter, a state-space characterization of robust stability and robust performances for a class of nonlinear systems subject to $\mathcal{L}_2$-bounded structured uncertainty is proposed; both analysis and synthesis problems are addressed. A standard representation for an uncertain system is depicted as in the following block diagram,



where G is the nominal system which is nonlinear time-invariant, Δ is the uncertainty which belongs to a designated uncertainty set, where in this chapter it is assumed to be $\mathcal{L}_2$-bounded strictly by 1. Both G and Δ are causal, and the interconnection for the uncertain system is well-posed for each admissible uncertainty; w is some external disturbance vector, and z is the regulated signal vector. The robustness analysis is to determine that under what conditions for nominal system G, the uncertain system is stable and/or satisfies some performance for each admissible uncertainty Δ; while the robustness synthesis problem is to decide under what conditions there are feedback control laws for the uncertain system such that the closed loop uncertain system has required robustness, and then design a control law.

The basic tool for robustness analysis of such uncertain systems is the small-gain theorem (see for example [203]). A general condition for robust stability analysis is that the $\mathcal{L}_2$-gain of the nominal system G is less than or equal to 1 [152, 62, 36, 156, 166]. When the uncertainty Δ is structured, i.e. it consists of multiple uncertainty blocks, the sufficient small-gain condition can be arbitrarily conservative for robust stability. By scaling treatment, a sufficient condition for robust stability in this case is that the nominal system has **scaled** small-gain: $\|DGD^{-1}\|_{\mathcal{L}_2} \leq 1$ for some scale D commuting with Δ (see for example [53, 151]). It is noted that under some additional time-varying, nonlinearity, or structural constraints on plant or uncertainty, the scaled small gain is necessary as well [159, 41, 155]. The scaling treatment not only reduces the conservatism, but also yields a convex characterization, which benefits computations. As for robust performance analysis problem, the performance robustness analysis problem can be transferred into a robust stability problem with structured uncertainty by adding an extra "uncertainty" block [63], the above small-gain arguments therefore still apply; see [63, 132, 142, 41] and references therein for this consideration.

In the above reviewed research, the systems are described in terms of their I/O behaviors, and robustness analysis is essentially reduced to gain analysis of the corresponding transfer matrices or I/O operators. The computational implications for the characterizations of robustness analysis and synthesis in this setting are not immediate, since the computational manipulations with symbols and transfer functions are not very numerically efficient. However, this problem can be remedied by taking a state-space treatment [60, 119, 127, 133, 134, 111, 132, 3, 29]. For the linear system, the small-gain conditions can be characterized in terms of **linear matrix inequalities** (LMIs) in state-space by the use of the KYP Lemma. Therefore, the robustness analysis and synthesis can be conducted by matrix manipulations which are inherently more numerically stable than symbolic or (I/O) transfer function manipulations; in addition, the LMI characterizations are convex conditions, which are computationally efficient (see [29] for a tutorial review of LMIs and their use). For the nonlinear systems, the state-space treatment enables the robustness analysis problems to be reduced to some computationally easier problems. In this chapter, it will be shown that the computation needed for robustness analysis and synthesis for nonlinear uncertain systems is not more difficult than that for checking the Lyapunov stability. The other advantage for the state-space treatment is that it captures the internal properties as well, besides the I/O-behaviors. Therefore, instead of the notion of I/O-stability, the notion of asymptotic stability is employed in this chapter; robust stability thus stands for robust asymptotic stability in the sequel. With the shifting of the emphasis, the robustness problems considered in this chapter are slightly different from the robustness problems in I/O-setting.

In this chapter, we give state-space characterizations of asymptotic stability and performance robustness for nonlinear uncertain systems, and consider both analysis and synthesis problems. By robust stability, we mean that the feedback systems are asymptotically stable for each admissible uncertainty; the robust performance means that the uncertain systems are asymptotically stable and has $\mathcal{L}_2$-gain ≤ 1 for each admissible uncertainty. The treatments of the robustness issues in this chapter are motivated by the work on the LMI characterization of robustness analysis and synthesis for linear systems [119, 133] (see [134, 111, 3, 29] for some extensions). The robustness results are characterized in terms of **nonlinear matrix inequalities** (NLMIs), which imply small-gain conditions. It is noted that the robustness results can be also represented in terms of Hamilton-Jacobi equations/inequalities. However, the NLMI characterizations offer potentially attractive computational features. In particular, like the linear case, the NLMIs trivially give convex conditions on the unknowns. The computation is not more difficult than that for checking Lyapunov stability; in other words, the provided approach reduces the robustness analysis and synthesis problems to the possible easiest problems. Unfortunately, the NLMI conditions involve neither a finite number of unknowns nor a finite number of constraints, so the computational advantages are far less immediate than for LMIs. Clearly much additional work will be needed on the computational aspects and sophisticated approximation techniques may be required to make the NLMI computation feasible.

The rest of the chapter is organized as follows. In Section 2, some standard results for robustness analysis are reviewed. In particular, the small-gain theorems are given for uncertain systems with structured uncertainty, and asymptotic stability and $\mathcal{L}_2$-gain analysis for nonlinear systems are reformulated as NLMIs. In Section 3 the stability robustness and performance robustness of nonlinear uncertain systems are characterized in state-space. The characterizations are in terms of NLMIs. In Section 4, we deal with the robustness synthesis problem; we just take the state feedback performance robustness synthesis problem as an example, and the solvability conditions are also in terms of NLMIs. We address some computational issues for robustness analysis and synthesis in Section 5.

9.2 Preliminaries: Small-Gain Theorems

To motivate the techniques to be used in this chapter, some standard results from robustness analysis are reviewed in this section. In particular, we will review the small-gain theorems for uncertain systems in input-output (I/O) setting, and a general description of the uncertainty of interest.

9.2.1 Robust I/O-Stability and Small-Gain Theorems

Consider a causal, nonlinear, and time-invariant operator[1] $G : \mathcal{L}_2^e[0,\infty) \rightarrow \mathcal{L}_2^e[0,\infty)$ with $G(0) = 0$. The **$\mathcal{L}_2$-gain** for G is defined as

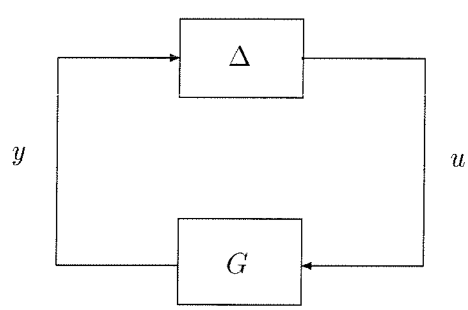
$$||G||_{\mathcal{L}_2} := \sup_{T \in (0,+\infty)} \sup_{u \in \mathcal{L}_2^e[0,\infty), ||P_T u||_2 \neq 0} \frac{||P_T G u||_2}{||P_T u||_2}, \tag{9.1}$$

where P_T denotes the truncation operator. Therefore, for the operator $G : u \mapsto y$, if it has $\mathcal{L}_2$-gain less than or equal to γ for some $\gamma \geq 0$ if and only if

$$\int_0^T ||y(t)||^2 \, dt \leq \gamma^2 \int_0^T ||u(t)||^2 \, dt \tag{9.2}$$

for all $T \in \mathbf{R}^+$. The system represented by the operator G is **I/O-stable** if $||G||_{\mathcal{L}_2} < \infty$.

Consider an uncertain system, denoted as (G, Δ), with the following block diagram,



where the nominal system G is time-invariant and causal, and the uncertainty Δ is causal and has structure defined as follows,

$$\Delta \in \mathbf{\Delta} := \{\text{block-diag}[\Delta_1, \cdots, \Delta_N] | \Delta_i : \mathcal{L}_2^p[0,\infty) \rightarrow \mathcal{L}_2^p[0,\infty) \text{ is causal}\}, \tag{9.3}$$

and has $\mathcal{L}_2$-gain < 1, i.e., it belongs to the following admissible uncertainty set.

$$\mathbf{B\Delta} := \{\Delta \in \mathbf{\Delta} | \; ||\Delta||_{\mathcal{L}_2} < 1\}. \tag{9.4}$$

---

[1]The notions of causality, time-invariance, and well-posedness etc. are extensively discussed in [195].

The feedback structure is assumed to be well-posed[1] for any uncertainty $\Delta \in \mathbf{B\Delta}$. The **robust I/O-stability** for the system (G, Δ) requires that $I - \Delta G$ is stably invertible for all $\Delta \in \mathbf{B\Delta}$.

The robust stability in the above sense can be checked by the use of small-gain theorem (see for example [203]), and a sufficient condition for the robust I/O-stability of the system (G, Δ) is

$$\|G\|_{\mathcal{L}_2} < 1.$$

Because of the block-diagonal structure (9.4), the above small-gain characterization can be arbitrarily conservative. We next define a non-conservative robustness measure for the plant G as follows.

Definition 9.2.1 *The* **structured gain** $\mu_{\mathbf{\Delta}}(G)$ *of operator* G *with respect to the structure* Δ *defined in (9.3) is defined as follows,*

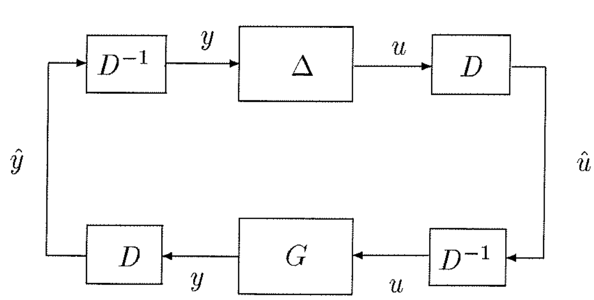
$$\mu_{\mathbf{\Delta}}(G) := \sup_{\Delta \in \mathbf{\Delta}} \{\|\Delta\|_{\mathcal{L}_2}^{-1} \,|I - \Delta G \text{ is not stably invertible}\}; \tag{9.5}$$

otherwise, if $I - \Delta G$ *stably invertible for all* $\Delta \in \mathbf{\Delta}$, *then* $\mu_{\mathbf{\Delta}}(G) := 0$.

With the above notion of structured gain, we immediately have the following assertion.

Proposition 9.2.2 *The uncertain system* (G, Δ) *with* $\Delta \in \mathbf{B\Delta}$ *is robustly I/O-stable if and only if* $\mu_{\mathbf{\Delta}}(G) \leq 1$.

However, the structured gain, $\mu_{\mathbf{\Delta}}(G)$, is not exactly computable in general. To get a more computable and less conservative condition, a scaling manipulation can be used to reduce the resulting conservatism.



where D is some real invertible matrix. Note that the above uncertain system $(DGD^{-1}, D\Delta D^{-1})$ is equivalent to the original system (G, Δ). Define a real valued matrix set $\mathcal{D}$ as

$$\mathcal{D} := \{\text{block-diag}\{d_1 I, d_2 I, \cdots, d_N I\} : d_i \in \mathbf{R}, d_i > 0\}, \tag{9.6}$$

where each of the identity matrices is compatible with the corresponding nonlinear uncertainty Δ_i. It is noted that $\|\Delta\|_{\mathcal{L}_2} = \|D\Delta D^{-1}\|_{\mathcal{L}_2}$ for all $D \in \mathcal{D}$ and $\Delta \in \mathbf{B\Delta}$; therefore, for each $D \in \mathcal{D}$, $\Delta \in \mathbf{B\Delta}$ if and only if $D\Delta D^{-1} \in \mathbf{B\Delta}$. The following result is the well-known scaled small-gain theorem, which follows directly by the application of the small-gain theorem to the scaled system $(DGD^{-1}, D\Delta D^{-1})$.

Proposition 9.2.3 (Scaled Small-Gain Theorem) *The uncertain system (G, Δ) is robustly I/O-stable if there is $D \in \mathcal{D}$ such that the following small scaled gain condition is satisfied,*

$$\left\|DGD^{-1}\right\|_{\mathcal{L}_2} \leq 1. \tag{9.7}$$

Remark 9.2.4 *(i) Under some additional assumptions, such as that the plant is linear or nonlinear with fading memories, the small scaled gain condition (9.7) is also necessary [159, 123, 155].*

(ii) For analysis of robust performances against structured perturbations, the similar scaling treatment can be conducted to reduce the conservatism.

In this chapter, we will mainly emphasize the internal properties of the uncertainty. Instead of I/O-stability, the asymptotic stability is considered in the sequel. The corresponding characterizations of stability and the $\mathcal{L}_2$-gains in the state-space setting are given in the last chapter. Next, we will give a general description of the uncertainty of interest.

9.2.2 Description of $\mathcal{L}_2$-Bounded Uncertainty

In the following, we follow Willems [195] to get a general internal description of the $\mathcal{L}_2$-bounded uncertainty. We first have the following definition for general dynamical systems.

Definition 9.2.5 *Given a causal I/O-stable operator $\Delta : \mathcal{L}_2^p[0,\infty) \rightarrow \mathcal{L}_2^q[0,\infty)$ with $\Delta(0) = 0$, its internal property is characterized by a triple $(\mathbf{X}, \psi, \eta)$, where*

- *$\mathbf{X}$ is an abstract set called state-space.*
- *$\psi : \mathbf{R}^+ \times \mathbf{R}^+ \times \mathbf{X} \times \mathcal{L}_2^p[0,\infty) \rightarrow \mathbf{X}$ is the state transition function; i.e., $x = \psi(t, t_0, x_0, w^*)$ means that system G is driven from initial state x_0 at time t_0 to state x at time t_1 by the input $w^* \in \mathcal{L}_2^p(\mathbf{R}^+)$. And ψ obeys the following axioms:*

  (i) $\psi(t_0, t_0, x_0, w) = x_0$ for all $t_0 \in \mathbf{R}^+, x_0 \in \mathbf{X}$, and $u \in \mathcal{L}_2^p(\mathbf{R}^+)$.

(ii) $\psi(t_1, t_0, x_0, u_1) = \psi(t_1, t_0, x_0, u_2)$ *for all* $(t_1, t_2) \in \mathbf{R}^+ \times \mathbf{R}^+$, $x_0 \in \mathbf{X}$, *and* $u_1, u_2 \in \mathcal{L}_2^p(\mathbf{R}^+)$ *satisfying* $u_1(t) = u_2(t)$ *for almost every* $t \in \mathbf{R}^+$.

(iii) $\psi(t_2, t_0, x_0, u) = \psi(t_2, t_1, \psi(t_1, t_0, x_0, u), u)$ *for almost all* $t_0 \leq t_1 \leq t_2$, $x_0 \in \mathbf{X}$, *and* $u \in \mathcal{L}_2^p(\mathbf{R}^+)$.

- $\eta : \mathbf{X} \times \mathbf{R}^p \to \mathbf{R}^q$ *is the read-out function, which satisfies* $\eta(0,0) = 0$. *And the function* $y(t) = \eta(\psi(t, t_0, x_0, u), u(t))$ *for all* $t \geq t_0$ *defines the output of the systems for* $t \geq t_0$.

In the above description, no explicit realization for the system Δ is required; so this description has some generality, e.g., the system can be infinite dimensional and can have time delay. In the following, we will impose the following constraints on the system.

Assumption 9.2.6 *Consider the system* Δ *as defined in Definition 9.2.5, the state-transition function* ψ *satisfy the following requirements,*

(i) $\psi(t_1, t_0, x_0, u(t - t_0)) = \psi(t_1 - t_0, 0, x_0, u(t))$ *for almost all* $t_1 \geq t_0$, $x_0 \in \mathbf{X}$, *and* $u \in \mathcal{L}_2^p(\mathbf{R}^+)$.

(ii) $\psi(t_1, t_0, 0, 0) = 0$ *for all* $t_1, t_0 \in \mathbf{R}^+$, *i.e., the origin is an equilibrium of the system.*

(iii) For all $x \in \mathbf{X}$ *and* $t_0 \in \mathbf{R}^+$, *there exist* $u \in \mathcal{L}_2^p(\mathbf{R}^+)$ *and* $t_1 \geq t_0$ *such that* $x = \psi(t_1, t_0, 0, u)$.

(iv) $\psi(t, t_0, x, 0)$ *is continuous in* t *for all* $t \geq t_0$ *and in* x *for* $x \in \mathbf{X}$.

The requirement (i) says that the system Δ is time-invariant. The requirement (ii) guarantees that there is an equilibrium point for the system with zero-input. (iii) says that the system is reachable from 0, the equilibrium point. (iv) is a regularity assumption; the systems represented by differential equations satisfy this condition.

We first state the general **Lyapunov Theorem** about asymptotic stability as follows:

Proposition 9.2.7 *Consider the dynamical system* Δ *defined in Definition 9.2.5 and satisfied Assumption 9.2.6. The following statements are equivalent.*

(i) The equilibrium 0 *of the system* Δ *with zero-input is asymptotically stable.*

(ii) There exists a $\mathbf{C}^1$ *positive definite function* $V : \mathbf{X} \to \mathbf{R}^+$ *such that* $\dot{V}(x) \leq 0$, *where* $\dot{V}(x) = 0$ *if and only if* $x = 0$.

(iii) There exists a $\mathbf{C}^1$ *positive definite function* $V : \mathbf{X} \to \mathbf{R}^+$ *such that* $\dot{V}(x) \leq 0$, *and those state trajectory* $x(t)$ *such that* $\dot{V}(x(t)) = 0$ *satisfy* $x(t) \to 0$ *as* $t \to \infty$.

The positive definite functions V in the above theorem is called **Lyapunov functions.** The stability test (iii) in the above proposition is LaSalle theorem.

It is known from [197, 84] that the system $\Delta : u \mapsto y$ has $\mathcal{L}_2$-gain ≤ 1 if and only if there exists a non-negative function, which is known as **storage function** [197], $V : \mathbf{X} \to \mathbf{R}^+$ such that for all $t_0, t_1 \in \mathbf{R}^+$, $x_0 \in \mathbf{X}$, and $u \in \mathcal{L}_2^p(\mathbf{R}^+)$, the following dissipation inequality holds,

$$V(x_1) - V(x_0) \leq \int_{t_0}^{t_1} (\|u(t)\|^2 - \|y(t)\|^2)dt, \tag{9.8}$$

where $x_1 = \psi(t_1, t_0, x_0, u)$ and $y(t) = \eta(\psi(t, t_0, x_0, u), u(t))$.

We further make the following assumptions:

Assumption 9.2.8 *Consider the system Δ which has $\mathcal{L}_2$-gain ≤ 1, the storage function $V : \mathbf{X} \to \mathbf{R}^+$ is a $\mathbf{C}^1$ defined as above is a $\mathbf{C}^1$ positive definite function.*

The above assumption on V guarantees the equilibrium of the system Δ is asymptotically stable by Lyapunov Theorem (see [197, Section 5]) with V being a Lyapunov function. In fact, the requirement that the storage function V is continuously differentiable can be weakened as that it is continuous [197, Theorem 6].

With the above preparation, we define the class $\mathbf{\Omega}$ of time-invariant structured uncertainties with N blocks as follows,

$$\mathbf{\Omega} := \{\text{block-diag}[\Delta_1, \cdot, \Delta_N] |\ \Delta_i : \mathcal{L}_2^{p_i}(\mathbf{R}^+) \to \mathcal{L}_2^{p_i}(\mathbf{R}^+) \text{ is defined as in Definition 9.2.5}$$

$$\text{and satisfies Assumptions 9.2.6 and 9.2.8 for each } i \in \{1, \cdots, N\}\ \}, \tag{9.9}$$

where p_i is some positive integer for each $i \in \{1, \cdots, N\}$.

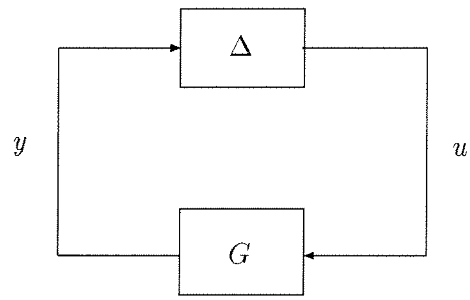
Thus, if $\Delta \in \mathbf{\Omega}$, then Δ has $\mathcal{L}_2$-gain ≤ 1, and Δ is globally asymptotically stable when the input is set to zero. Note that the case of time-varying uncertainty can be treated similarly [197], and the subsequent results can be generalized accordingly.

9.3 Stability and Performance Robustness of Uncertain Systems

In this problem, we will characterize robust stability and robust performances for nonlinear uncertain systems in state-space. The internal description for the uncertainty of interest is given in the previous section section.

9.3.1 Asymptotic Stability Robustness

Consider the uncertain system represented by the following standard feedback structure,



where the nominal system G has an input-affine realization, i.e.,

$$G : \begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x)x + k(x)u \end{cases} \tag{9.10}$$

and the uncertainty Δ belongs to the set $\mathbf{\Omega}$ defined in (9.9). Therefore, each admissible uncertainty Δ is a causal time-invariant operator having $\mathcal{L}_2$-gain ≤ 2 with continuously differentiable positive definite storage function. In the following, the uncertain system is denoted as (G, Δ).

Definition 9.3.1 *The uncertain system* (G, Δ) *is* **robustly stable** *if for each* $\Delta \in \mathbf{\Omega}$*, the feedback system is well-posed and asymptotically stable around* 0.

In the following, it is assume that $k(x) = 0$ for all $x \in \mathbf{X}$, therefore the nominal system is strictly causal in this case. The case where $k(x) \neq 0$ for some $x \in \mathbf{X}$ is considered in [113]. This assumption assures that each uncertainty $\Delta \in \mathbf{\Omega}$ the corresponding feedback configuration is well-posed. In the following, the robust stability is characterized.

To reduce the conservatism arising from the uncertainty structure, we also perform the following standard manipulation as in the I/O-setting. Define the scaling matrix set $\mathcal{D}$ as follows,

$$\mathcal{D} := \{\text{block-diag}\{d_1 I, d_2 I, \cdots, d_N I\} : d_i \in \mathbf{R}, d_i > 0\}, \tag{9.11}$$

where each of the identity matrices is compatible with the corresponding nonlinear uncertainty Δ_i. It is noted that if $\Delta \in \mathbf{\Omega}$, then for each $D \in \mathcal{D}$, $D\Delta D^{-1} \in \mathbf{\Omega}$. Therefore, $D\Delta D^{-1}$ is a legal (transformed) uncertainty structure. Hence, instead of (G, Δ), the new scaled uncertain system $(DGD^{-1}, D\Delta D^{-1})$ with some suitable scaling matrix is considered (see section 2 for the motivation). We have the following theorem about the robust stability for the structured uncertain systems, which gives a natural NLMI generalization to nonlinear systems of the LMI conditions for the linear case.

Theorem 9.3.2 *Consider the given uncertain system (G, Δ), it is robustly stable if there exist a positive definite $\mathbf{C}^1$ function $V : \mathbf{X} \rightarrow \mathbf{R}^+$ and a positive definite matrix $Q \in \mathcal{D}$ such that the following NLMI holds,*

$$\mathcal{M}_{RS}(\frac{\partial V}{\partial x}, Q, x) := \begin{bmatrix} \frac{\partial V}{\partial x}(x)f(x) + h^T(x)Qh(x) & \frac{1}{2}\frac{\partial V}{\partial x}(x)g(x) \\ \frac{1}{2}g^T(x)\frac{\partial V^T}{\partial x}(x) & -Q \end{bmatrix} < 0 \tag{9.12}$$

for all $x \in \mathbf{X} \setminus \{0\}$.

Proof. Consider (9.12). Note that

$$\begin{bmatrix} I & 0 \\ 0 & Q^{-1/2} \end{bmatrix} \mathcal{M}_{RS}(\frac{\partial V}{\partial x}, Q, x) \begin{bmatrix} I & 0 \\ 0 & Q^{-1/2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial V}{\partial x}(x)f(x) + h^T(x)Qh(x) & \frac{1}{2}\frac{\partial V}{\partial x}(x)g(x)Q^{-1/2} \\ \frac{1}{2}Q^{-1/2}g^T(x)\frac{\partial V^T}{\partial x}(x) & -I \end{bmatrix} < 0.$$

Define $\hat{g}(x) = g(x)Q^{-1/2}$ and $\hat{h}(x) = Q^{1/2}h(x)$. Using Schur complements argument, We have that the above inequality is equivalent to the following Hamilton-Jacobi inequality,

$$\hat{\mathcal{H}}(\frac{\partial V}{\partial x}, Q, x) := \frac{\partial V}{\partial x}(x)f(x) + \hat{h}^T(x)\hat{h}(x) + \frac{1}{4}\frac{\partial V}{\partial x}(x)\hat{g}(x)\hat{g}^T(x)\frac{\partial V^T}{\partial x}(x) \leq 0 \tag{9.13}$$

for all $x \in \mathbf{X}\setminus\{0\}$. Take V as defined in the statement, and define $\hat{u} = Q^{1/2}u$ and $\hat{y} = Q^{1/2}y$, then

$$\dot{V}(x) = \frac{\partial V}{\partial x}(x)(f(x) + \hat{g}(x)\hat{u})$$

$$= \|\hat{u}(t)\|^2 - \|\hat{y}(t)\|^2 - \left\|\hat{u}(t) + \frac{1}{2}\hat{g}^T(x)\frac{\partial V^T}{\partial x}(x)\right\|^2 + \hat{\mathcal{H}}(\frac{\partial V}{\partial x}, Q, x).$$

$$\dot{V}(x) \leq \|\hat{u}(t)\|^2 - \|\hat{y}(t)\|^2 + \hat{\mathcal{H}}(\frac{\partial V}{\partial x}, Q, x). \tag{9.14}$$

On the other hand, notice that $Q^{1/2} \in \mathcal{D}$; denote $Q^{1/2} = \text{block-diag}\{q_1 I, q_2 I, \cdots, q_N I\}$. Since for each $\Delta \in \mathbf{\Omega}$, $Q^{1/2}\Delta Q^{-1/2} \in \mathbf{\Omega}$; there therefore is a positive definite function $U_i : \mathbf{X}_i \rightarrow \mathbf{R}^+$ for nonlinear system $q_i \Delta_i q_i^{-1}$ for each $i \in \{1, 2, \cdots, N\}$ such that

$$\dot{U}_i(\xi_i) \leq \|\hat{y}_i(t)\|^2 - \|\hat{u}_i(t)\|^2, \tag{9.15}$$

where $\hat{u}_i = q_i u_i$, $\hat{y}_i = q_i y_i$ and ξ_i is the state vector of Δ_i on $\mathbf{X}_i$. Note that

$$\hat{u} = \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \vdots \\ \hat{u}_N \end{bmatrix}, \qquad \hat{y} = \begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \vdots \\ \hat{y}_N \end{bmatrix}.$$

Therefore,

$$\|\hat{u}\|^2 = \sum_{i=1}^{N} \|\hat{u}_i\|^2, \qquad \|\hat{y}\|^2 = \sum_{i=1}^{N} \|\hat{y}_i\|^2.$$

Next, define a positive definite function W on $\mathbf{X} \times \mathbf{X}_1 \times \cdots \times \mathbf{X}_N$ as

$$W(x, \xi_1, \cdots, \xi_N) = V(x) + \sum_{i=1}^{N} U_i(\xi_i). \tag{9.16}$$

So from (9.14) and (9.15), it follows that

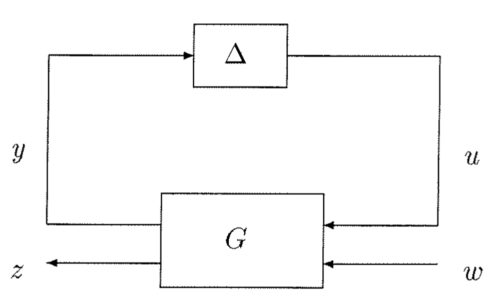
$$\begin{aligned} \dot{W}(x, \xi_1, \cdots, \xi_N) &\leq \|\hat{u}(t)\|^2 - \|\hat{y}(t)\|^2 + \hat{\mathcal{H}}(\frac{\partial V}{\partial x}, Q, x) + \sum_{i=1}^{N}(\|\hat{y}_i(t)\|^2 - \|\hat{u}_i(t)\|^2) \\ &\leq \hat{\mathcal{H}}(\frac{\partial V}{\partial x}, Q, x). \end{aligned}$$

Thence, if $\dot{W}(x, \xi_1, \cdots, \xi_N) = 0$, then $\mathcal{H}(\frac{\partial V}{\partial x}, Q, x) = 0$; it in turn implies $x = 0$.

On the other hand $x = 0$ implies $y = 0$. Moreover, for each uncertain system Δ_i with input $y = 0$, it is asymptotically stable by the discussion in Section 3.1, i.e., $\xi_i(t) \to 0$ as $t \to \infty$. By LaSalle's theorem, $W : \mathbf{X} \times \mathbf{X}_1 \times \cdots \times \mathbf{X}_N \to \mathbf{R}^+$ is a Lyapunov function for the feedback system, and the system is asymptotically stable. Therefore, the uncertain system is robustly stable. □

9.3.2 Performance Robustness

Consider the following feedback uncertain system which is represented by the following block diagram,



where w is some external disturbance vector, and it is assumed $w \in \mathcal{L}_2^p(\mathbf{R}^+)$; z is the regulated signal vector. The nominal plant G has the following realization,

$$G : \begin{cases} \dot{x} = f(x) + g_1(x)u + g_2(x)w \\ y = h_1(x) + k_{11}(x)u + k_{12}(x)w \\ z = h_2(x) + k_{21}(x)u + k_{22}(x)w \end{cases} \tag{9.17}$$

and the uncertainty Δ is structured which is assumed to belong to the set $\mathbf{\Omega}$ defined by (9.9). In the following, we denote the uncertain system as (G, Δ).

Definition 9.3.3 *The uncertain system (G, Δ) depicted above satisfies* **robust performance** *if for each $\Delta \in \mathbf{\Omega}$, the corresponding feedback system is well posed and has $\mathcal{L}_2$-gain ≤ 1, i.e.,*

$$\int_0^T (\|z(t)\|^2 - \|w(t)\|^2)dt \leq 0$$

for all $T \in \mathbf{R}^+$; in addition, it is asymptotically stable around 0 *for* $w = 0$.

In this section, we will examine under what conditions, the uncertain system depicted above has robust performance. We will assume $k_1 1(x) = 0$ for all $x \in k_{11}$ for simplicity, it is noted that the nominal system G is strictly causal with respect to the input u in this case. The case where $k_{11} \neq 0$ for some $x \in \mathbf{X}$ is considered in [115].

We take the similar scaling treatment for the nominal system G to reduce the conservatism arising from the structural constraints of the uncertainty. Instead of the system (G, Δ), the scaled system $(\begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix} G \begin{bmatrix} D^{-1} & 0 \\ 0 & I \end{bmatrix}, D\Delta D^{-1})$ is treated in the following, where $D \in \mathcal{D}$ with the scaling matrix set $\mathcal{D}$ defined in (9.11).

Let

$$g(x) := \begin{bmatrix} g_1(x) & g_2(x) \end{bmatrix}, h(x) := \begin{bmatrix} h_1(x) \\ h_2(x) \end{bmatrix}, k(x) := \begin{bmatrix} 0 & k_{12}(x) \\ k_{21}(x) & k_{22}(x) \end{bmatrix},$$

we have the following result about robust performance analysis.

Theorem 9.3.4 *Consider the uncertain system (G, Δ). It has robust performance if there exist a positive definite function $V : \mathbf{X} \rightarrow \mathbf{R}$ and a positive definite matrix $\hat{Q} \in \mathcal{D}$ such that the following NLMI holds,*

$$\mathcal{M}_{RP}(\frac{\partial V}{\partial x}, Q, x) := \begin{bmatrix} \frac{\partial V}{\partial x}(x)f(x) + h^T(x)Qh(x) & \frac{1}{2}\frac{\partial V}{\partial x}(x)g(x) + h^T(x)Qk(x) \\ \frac{1}{2}g^T(x)\frac{\partial V^T}{\partial x}(x) + k^T(x)Qh(x) & k^T(x)Qk(x) - Q \end{bmatrix} < 0$$

$$\tag{9.18}$$

with $Q := \begin{bmatrix} \hat{Q} & 0 \\ 0 & I \end{bmatrix}$ *for all* $x \in \mathbf{X} \setminus \{0\}$.

Proof. Consider (9.18). It is equivalent to the following inequality,

$$\begin{bmatrix} \frac{\partial V}{\partial x}(x)f(x) + h^T(x)Qh(x) & \frac{1}{2}\frac{\partial V}{\partial x}(x)g(x)Q^{-1/2} + h^T(x)Qk(x)Q^{-1/2} \\ \frac{1}{2}Q^{-1/2}g^T(x)\frac{\partial V^T}{\partial x}(x) + Q^{-1/2}k^T(x)Qh(x)Q^{-1/2} & Q^{-1/2}k^T(x)Qk(x)Q^{-1/2} - I \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 \\ 0 & Q^{-1/2} \end{bmatrix} \mathcal{M}_{RP}(\frac{\partial V}{\partial x}, Q, x) \begin{bmatrix} I & 0 \\ 0 & Q^{-1/2} \end{bmatrix} < 0. \tag{9.19}$$

Define $\hat{g}(x) := g(x)Q^{-1/2}$, $\hat{h}(x) := Q^{1/2}h(x)$ and $\hat{k}(x) := Q^{1/2}k(x)Q^{-1/2}$. Using Schur complements argument and the above inequality is equivalent to the following two inequalities,

$$\hat{R}(x) := I - \hat{k}^T(x)\hat{k}(x) > 0,$$

and

$$\hat{\mathcal{H}}(\frac{\partial V}{\partial x}, Q, x) := \frac{\partial V}{\partial x}(x)f(x) + \hat{h}^T(x)\hat{h}(x) +$$

$$(\frac{1}{2}\frac{\partial V}{\partial x}(x)\hat{g}(x) + \hat{h}^T(x)\hat{k}(x))(I - \hat{k}^T(x)\hat{k}(x))^{-1}(x)(\frac{1}{2}\hat{g}^T(x)\frac{\partial V^T}{\partial x}(x) + \hat{k}^T(x)\hat{h}(x)) < 0 \tag{9.20}$$

for all $x \in \mathbf{X} \setminus \{0\}$. Take V as defined in the statement; define $\hat{u} := Q^{1/2}\begin{bmatrix} u \\ w \end{bmatrix}$ and $\hat{y} = Q^{1/2}\begin{bmatrix} y \\ z \end{bmatrix}$ then

$$\dot{V}(x) \leq \|\hat{u}(t)\|^2 - \|\hat{y}(t)\|^2 + \hat{\mathcal{H}}(\frac{\partial V}{\partial x}, Q, x). \tag{9.21}$$

On the other hand, notice that $\hat{Q}^{1/2} \in \mathcal{D}$; denote $\hat{Q}^{1/2} = \text{block-diag}\{q_1 I, q_2 I, \cdots, q_N I\}$. Since $\Delta \in \mathbf{\Omega}$, $\hat{Q}^{1/2}\Delta\hat{Q}^{-1/2} \in \mathbf{\Omega}$; therefore there is a positive definite function $U_i : \mathbf{X}_i \rightarrow \mathbf{R}^+$ for nonlinear system $q_i\Delta_i q_i^{-1}$ for each $i \in \{1, 2, \cdots, N\}$ such that

$$\dot{U}_i(\xi_i) \leq \|\hat{y}_i(t)\|^2 - \|\hat{u}_i(t)\|^2, \tag{9.22}$$

where $\hat{u}_i = q_i u_i$, $\hat{y}_i = q_i y_i$ and ξ_i is the state vector of Δ_i on $\mathbf{X}_i$. Note that

$$\hat{u} = \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \vdots \\ \hat{u}_N \\ w \end{bmatrix}, \qquad \hat{y} = \begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \vdots \\ \hat{y}_N \\ z \end{bmatrix}.$$

Therefore,

$$\|\hat{u}\|^2 = \sum_{i=1}^{N} \|\hat{u}_i\|^2 + \|w\|^2, \qquad \|\hat{y}\|^2 = \sum_{i=1}^{N} \|\hat{y}_i\|^2 + \|z\|^2.$$

Next, define a positive definite function W on $\mathbf{X} \times \mathbf{X}_1 \times \cdots \times \mathbf{X}_N$ as

$$W(x, \xi_1, \cdots, \xi_N) = V(x) + \sum_{i=1}^{N} U_i(\xi_i). \tag{9.23}$$

So from (9.21) and (9.22), it follows that

$$\begin{aligned} \dot{W}(x, \xi_1, \cdots, \xi_N) &\leq \|\hat{u}(t)\|^2 - \|\hat{y}(t)\|^2 + \hat{\mathcal{H}}(\frac{\partial V}{\partial x}, Q, x) + \sum_{i=1}^{N}(\|\hat{y}_i(t)\|^2 - \|\hat{u}_i(t)\|^2) \\ &\leq \|w\|^2 - \|z\|^2 + \hat{\mathcal{H}}(\frac{\partial V}{\partial x}, Q, x) \\ &\leq \|w\|^2 - \|z\|^2. \end{aligned} \tag{9.24}$$

The latter inequality implies

$$\int_0^T (\|w\|^2 - \|z\|^2)dt \geq 0 \tag{9.25}$$

for all $T \in \mathbf{R}^+$, i.e., the feedback system has $\mathcal{L}_2$-gain ≤ 1.

Next, we consider the asymptotic stability for $w = 0$. In this case, (9.24) becomes

$$\dot{W}(x, \xi_1, \cdots, \xi_N) \leq -\|z\|^2 + \hat{\mathcal{H}}(\frac{\partial V}{\partial x}, Q, x).$$

Thence, if $\dot{W}(x, \xi_1, \cdots, \xi_N) = 0$, then $\hat{\mathcal{H}}(\frac{\partial V}{\partial x}, Q, x) = 0$, it in turn implies $x = 0$ by (9.20). But $x = 0$ implies $y = 0$, therefore $\xi_i(t) \to 0$ as $t \to \infty$, for Δ_i is asymptotically stable with zero-input (Section 3.1). By LaSalle's theorem, $W : \mathbf{X} \times \mathbf{X}_1 \times \cdots \times \mathbf{X}_N \to \mathbf{R}^+$ is a Lyapunov function for the given closed loop system, and the system is asymptotically stable.

Therefore, we conclude that the uncertain system is of robust performance. □

Next, we further relax the condition in the last theorem to get an alternative characterization for the robust performance of the depicted uncertain system.

Assumption 9.3.5 *Consider the nominal system G, define a new system:*

$$\begin{cases} \dot{x} = f(x) + g_1(x)u \\ z = h_2(x) + k_{21}(x)u \end{cases}$$

The state trajectory $x(t)$ of the system with all possible $u(t)$ such that $z(t) = 0$ satisfies $x(t) = 0$ for all $t \in \mathbf{R}^+$.

It is noted that in the linear case, the above assumption corresponds to the condition that the system has no transmission zero.

Theorem 9.3.6 *Under assumption 9.3.5, the uncertain system (G, Δ) has robust performance if there exist a positive definite $\mathbf{C}^1$ function $V : \mathbf{X} \rightarrow \mathbf{R}$ and a positive definite matrix $\hat{Q} \in \mathcal{D}$ such that the following NLMIs hold,*

$$\begin{bmatrix} \frac{\partial V}{\partial x}(x) f(x) + h^T(x) Q h(x) & \frac{1}{2}\frac{\partial V}{\partial x}(x) g(x) + h^T(x) Q k(x) \\ \frac{1}{2} g^T(x) \frac{\partial V^T}{\partial x}(x) + k^T(x) Q h(x) & k^T(x) Q k(x) - Q \end{bmatrix} \leq 0 \tag{9.26}$$

$$k^T(x) Q k(x) - Q < 0 \tag{9.27}$$

with $g(x), h(x), k(x)$ defined previously and $Q := \begin{bmatrix} \hat{Q} & 0 \\ 0 & I \end{bmatrix}$ for all $x \in \mathbf{X}$.

Proof. The proof is a combination of the one for the previous theorem and the one for theorem 8.2.3. We just give an outline here.

Take $V : \mathbf{X} \rightarrow \mathbf{R}^+$ as given, and $U_i : \mathbf{X}_i \rightarrow \mathbf{R}^+$ as in the proof of the last theorem. Define a positive definite function W on $\mathbf{X} \times \mathbf{X}_1 \times \cdots \times \mathbf{X}_N$ as

$$W(x, \xi_1, \cdots, \xi_N) = V(x) + \sum_{i=1}^{N} U_i(\xi_i). \tag{9.28}$$

It follows that

$$\dot{W}(x, \xi_1, \cdots, \xi_N) \leq \|w\|^2 - \|z\|^2,$$

which implies that the feedback system has $\mathcal{L}_2$-gain ≤ 1.

Next, we consider the asymptotic stability for $w = 0$. In this case,

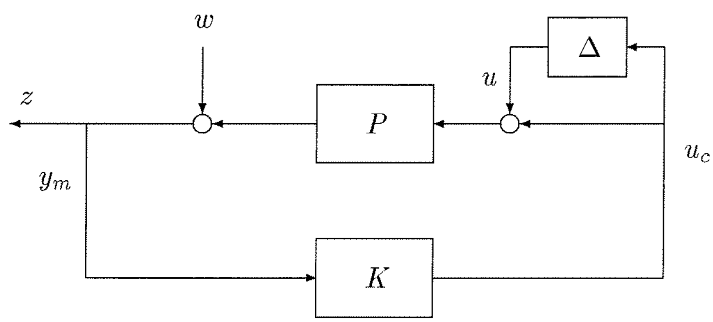
$$\dot{W}(x, \xi_1, \cdots, \xi_N) \leq -\|z\|^2.$$

Thence, if $\dot{W}(x, \xi_1, \cdots, \xi_N) = 0$, then $z = h_2(x) + k_{21}(x)u = 0$ which implies $x(t) = 0$ by assumption 9.3.5. But $x = 0$ implies $y = 0$, therefore $\xi_i(t) \rightarrow 0$ as $t \rightarrow \infty$, for Δ_i is asymptotically stable with zero-input. By LaSalle's theorem, $W : \mathbf{X} \times \mathbf{X}_1 \times \cdots \times \mathbf{X}_N \rightarrow \mathbf{R}^+$ is a Lyapunov function for the given closed loop system, and the system is asymptotically stable.

Therefore, the uncertain system satisfies robust performance. □

9.3.3 An Example

Consider an uncertain feedback system with block diagram as follows,



where P is the nonlinear plant; K is the controller such that the output z is supposed to be regulated; y_m is the measured output, based on which the control action u_c is produced; w is the disturbance from the actuator; and u is the disturbance from the sensor which is generated by $u = \Delta u_c$ with Δ being the bounded causal scalar uncertainties. The **robustness analysis problem** is to check that for a given controller K, whether or not the influence of the noises w on the regulated output z is reduced to the required degree for all possible Δ.

To formulate this problem, all the signals are considered in space $\mathcal{L}_2[0, \infty)$. We will check that given $\gamma > 0$ and K, does

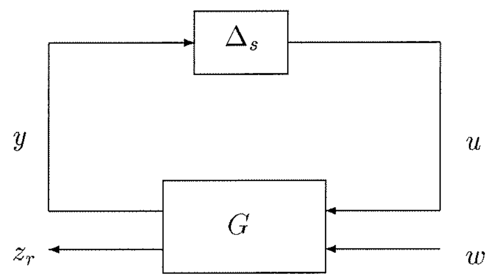
$$\int_0^T \|z\|^2 \, dt \le \gamma^2 \int_0^T \|w\|^2 \, dt, \forall T \in \mathbf{R}^+,$$

for all admissible Δ?

In this example, the plant has the following realization,

$$\begin{cases} \dot{x} = e^x(u + u_c) \\ z = x + w \\ y_m = x + w \end{cases}$$

and the controller $K = -1$, each admissible uncertainty $\Delta \in \frac{1}{\sqrt{2}}\mathbf{\Omega}$ (where the block number $N = 1$), which has $\mathcal{L}_2$-gain $\le \frac{1}{\sqrt{2}}$. We will check if the $\mathcal{L}_2$-gain of the feedback uncertain is less than or equal to $\frac{1}{\sqrt{2}}$. To this end, the standardized block diagram for the closed system is redrawn as follows



with

$$G: \begin{cases} \dot{x} = -e^x x - e^x u + e^x w \\ y = -\frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}u \\ z_r = \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}u \end{cases}.$$

Therefore, $z_r = \sqrt{2}z$. In this case, the admissible uncertainty $\Delta_s \in \mathbf{\Omega}$.

It is sufficient to check if the above feedback system has robust performance. We first consider $z_N := \epsilon z_r = \epsilon(\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}u)$ for some positive $\epsilon < 1$. Consider the NLMI (8.6) as follows,

$$\begin{bmatrix} -2e^x P(x) + \frac{1}{2}(Q+\epsilon^2) & -e^x P(x) + \frac{1}{2}(Q+\epsilon^2) & e^x P(x) \\ -e^x P(x) + \frac{1}{2}(Q+\epsilon^2) & \frac{1}{2}(\epsilon^2 - Q) & 0 \\ e^x P(x) & 0 & -1 \end{bmatrix} < 0 \qquad (9.29)$$

There exist positive solutions $(Q, P(x))$ to the above two inequalities, which satisfy $Q = 1$ and $\frac{1+\epsilon^2}{3-\epsilon^2}e^{-x} < P(x) < e^{-x}$. Hence,

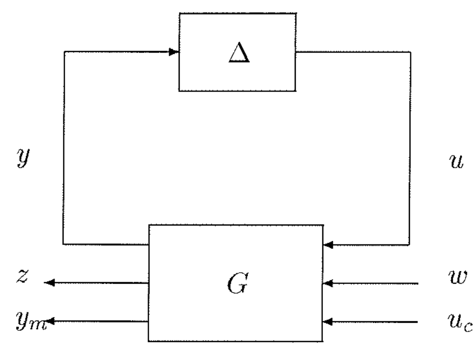
$$\int_0^T \|z_N\|^2 \, dt \leq \int_0^T \|w\|^2 \, dt, \forall T \in \mathbf{R}^+.$$

Therefore, the $\mathcal{L}_2$-gain for the closed loop system $\leq \frac{1}{\sqrt{2}\epsilon}$ for all $\epsilon < 1$, which in turn implies the $\mathcal{L}_2$-gain $\leq \frac{1}{\sqrt{2}}$.

9.4 Robustness Synthesis for Uncertain Systems

In the last two sections, the robustness conditions of uncertain systems are essentially characterized as the small-gain conditions for (scaled) nominal systems modulo some appropriate stabilizing conditions. So the robustness synthesis can be pursued by combining the robustness analysis results in the last two sections with the treatments of $\mathcal{H}_\infty$-control synthesis (see for example [183, 15, 113]). We just take the performances robustness synthesis problem as an example, the other problems can be done similarly. Technically, we closely follow the treatments in [113, 112], so we just take the state feedback case as an example. The output feedback case can be done similarly by just modifying the treatments in [113, 112]. It is remarked that the robust stabilization with unstructured uncertainty is also considered in [185].

Consider the following feedback uncertain system which is described as a feedback system set,



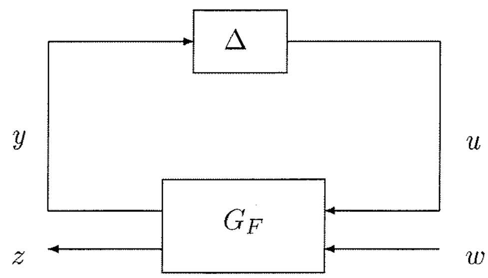
where $w \in \mathcal{L}_2^p(\mathbf{R}^+)$ is some external disturbance vector, z is the regulated signal vector, y_m is the measured output vector, and based on which the control input vector u_c is produced. The nominal plant G has the following realization,

$$G : \begin{cases} \dot{x} = f(x) + g_1(x)u + g_2(x)w + g_3(x)u_c \\ y = h_1(x) + k_{11}(x)u + k_{12}(x)w + k_{13}(x)u_c \\ z = h_2(x) + k_{21}(x)u + k_{22}(x)w + k_{23}(x)u_c \\ y_m = h_3(x) + k_{31}(x)u + k_{32}(x)w + k_{33}(x)u_c \end{cases} \tag{9.30}$$

where $f, g_i, h_j, k_{ij} \in \mathbf{C}^0$, and $f(0) = 0, h_j(0) = 0$, for $i, j = 1, 2, 3$. In this section, the state vector of the nominal system is directly measured, i.e., $y_m = x$; the uncertainty Δ is structured and belongs to the set $\mathbf{\Omega}$ defined by (9.9). The performance robustness synthesis problem by state feedback is defined as follows.

Definition 9.4.1 (State Feedback Synthesis Problem) *Find a state feedback law $u_c = K(x)$ with $K \in \mathbf{C}^0$ and $K(0) = 0$ for the uncertain system depicted above such that the closed loop uncertain system satisfies robust performance.*

If $u_c = K(x)$ is a state feedback law, then the closed loop uncertain system is as follows,



with

$$G_F : \begin{cases} \dot{x} = (f(x) + g_3(x)K(x)) + g_1(x)u + g_2(x)w \\ y = (h_1(x) + k_{13}(x)K(x)) + k_{11}(x)u + k_{12}(x)w \\ z = (h_2(x) + k_{23}(x)K(x)) + k_{21}(x)u + k_{22}(x)w \end{cases}$$

Define the scaling matrix set $\mathcal{D}$ is defined as before,

$$\mathcal{D} := \{\text{block-diag}\{d_1 I, d_2 I, \cdots, d_N I\} : \text{ for each i, } d_i \in \mathbf{R},\ d_i > 0\}. \tag{9.31}$$

Next, we consider two cases about robustness synthesis by state feedback.

9.4.1 State Feedback Solutions

Consider the uncertain system with the nominal plant as (9.30). Define

$$g(x) := \begin{bmatrix} g_1(x) & g_2(x) \end{bmatrix}, h(x) := \begin{bmatrix} h_1(x) \\ h_2(x) \end{bmatrix},$$

$$k_1(x) := \begin{bmatrix} k_{11}(x) & k_{12}(x) \\ k_{21}(x) & k_{22}(x) \end{bmatrix}, k_2(x) := \begin{bmatrix} k_{13}(x) \\ k_{23}(x) \end{bmatrix}.$$

The following structural constraints are imposed.

Assumption 9.4.2 *$k_1(x) = 0$, and $k_2^T(x) \begin{bmatrix} h(x) & k_2(x) \end{bmatrix} = \begin{bmatrix} 0 & R_0(x) \end{bmatrix}$ where $R_0(x) > 0$ for all $x \in \mathbf{X}$.*

We first have the following lemma.

Lemma 9.4.3 *Consider the system defined in (9.30) with the structural assumption 9.4.2. The following two statement are equivalent.*

(i) There exist a $\mathbf{C}^0$ vector-valued function $K(x)$ on $\mathbf{X}$, a $\mathbf{C}^1$ positive definite function $V : \mathbf{X} \to \mathbf{R}^+$, and a positive definite matrix $\hat{Q} \in \mathcal{D}$ such that the following NLMI holds,

$$\begin{bmatrix} \frac{\partial V}{\partial x}(x)(f(x) + g_3(x)K(x)) + (h(x) + k_2(x)K(x))^T Q(h(x) + k_2(x)K(x)) & \frac{1}{2}\frac{\partial V}{\partial x}(x)g(x) \\ \frac{1}{2}g^T(x)\frac{\partial V^T}{\partial x}(x) & -Q \end{bmatrix} < 0$$

for all $x \in \mathbf{X} \setminus \{0\}$.

(ii) There exist a $\mathbf{C}^1$ *positive definite function* $V : \mathbf{X} \rightarrow \mathbf{R}^+$ *and a positive definite matrix* $\hat{Q} \in \mathcal{D}$ *such that the following NLMI holds,*

$$\frac{\partial V}{\partial x}(x)f(x) + \frac{1}{4}\frac{\partial V}{\partial x}(x)(g(x)Q^{-1}g^T(x) - g_3(x)R_0^{-1}(x)g_3^T(x))\frac{\partial V^T}{\partial x}(x) + h^T(x)Qh(x) < 0$$

for all $x \in \mathbf{X} \setminus \{0\}$.

Moreover, if (ii) is true, then a state feedback function $K(x)$ *makes (i) true is as follows,*

$$K(x) = -\frac{1}{2}R_0^{-1}(x)g_3^T(x)\frac{\partial V}{\partial x}(x).$$

Proof. Note that the NLMI in statement (i) is equivalent to the following Hamilton-Jacobi inequality,

$$\frac{\partial V}{\partial x}(x)(f(x) + g_3(x)K(x)) + \frac{1}{4}\frac{\partial V}{\partial x}(x)g(x)Q^{-1}g^T(x)\frac{\partial V^T}{\partial x}(x) +$$

$$+(h(x) + k_2(x)K(x))^TQ(h(x) + k_2(x)K(x)) < 0$$

for all $x \in \mathbf{X} \setminus \{0\}$. By the same arguments as in [112], the conclusion follows. □

The main result in this subsection is stated as follows.

Theorem 9.4.4 *Consider the uncertain system with nominal plant as (9.30). Under assumption 9.4.2, the state feedback robust performance synthesis problem has a solution if there exist a positive definite* $\mathbf{C}^1$ *positive definite function* $V : \mathbf{X} \rightarrow \mathbf{R}^+$ *and a positive definite matrix* $\hat{Q} \in \mathcal{D}$ *such that the following NLMI holds,*

$$\frac{\partial V}{\partial x}(x)f(x) + \frac{1}{4}\frac{\partial V}{\partial x}(x)(g(x)Q^{-1}g^T(x) - g_3(x)R_0^{-1}(x)g_3^T(x))\frac{\partial V^T}{\partial x}(x) + h^T(x)Qh(x) < 0 \tag{9.32}$$

for all $x \in \mathbf{X} \setminus \{0\}$. *Moreover, if* $(V(x), Q)$ *is such a pair of solutions, then a state feedback function* $K(x)$ *makes the closed loop system has a robust performance is* $K(x) = -\frac{1}{2}R_0^{-1}(x)g_3^T(x)\frac{\partial V}{\partial x}(x)$.

Proof. Let $(V(x), Q)$ be as in the theorem. By the preceding lemma, there exists a $\mathbf{C}^0$ matrix valued function $K(x)$ on $\mathbf{X}$ defined as $K(x) = -\frac{1}{2}R_0^{-1}(x)g_3^T(x)\frac{\partial V}{\partial x}(x)$, such that

$$\begin{bmatrix} \frac{\partial V}{\partial x}(x)(f(x) + g_3(x)K(x)) + (h(x) + k_2(x)K(x))^TQ(h(x) + k_2(x)K(x)) & \frac{1}{2}\frac{\partial V}{\partial x}(x)g(x) \\ \frac{1}{2}g^T(x)\frac{\partial V^T}{\partial x}(x) & -Q \end{bmatrix} < 0 \tag{9.33}$$

for all $x \in \mathbf{X} \setminus \{0\}$. On the other hand, take $u_c = K(x)$ as a state feedback law, so the closed loop nominal system is as follows,

$$G_F : \begin{cases} \dot{x} = (f(x) + g_3(x)K(x)) + g_1(x)u + g_2(x)w \\ y = (h_1(x) + k_{13}(x)K(x)) \\ z = (h_2(x) + k_{23}(x)K(x)) \end{cases} .$$

By theorem 9.3.4, the closed loop uncertain system satisfies robust performance. □

Note that the above characterization is not convex in general. In the next subsection, we will give a convex characterization which have some computationally appealing property.

9.4.2 A Convex Characterization for State Feedback Solutions

In this section, instead of the nominal plant (9.30), the following nominal plant is examined,

$$G : \begin{cases} \dot{x} = A(x)x + B_1(x)u + B_2(x)w + B_3(x)u_c \\ y = C_1(x)x + D_{11}(x)u + D_{12}(x)w + D_{13}(x)u_c \\ z = C_2(x)x + D_{21}(x)u + D_{22}(x)w + D_{23}(x)u_c \\ y_m = x \end{cases} \tag{9.34}$$

where A, B_i, C_j, D_{ij} are $\mathbf{C}^0$ matrix-valued functions. It is assumed that $D_{11}(x) = 0$ for all $x \in \mathbf{X}$ for simplicity.

We now define

$$B(x) := \begin{bmatrix} B_1(x) & B_2(x) \end{bmatrix}, C(x) := \begin{bmatrix} C_1(x) \\ C_2(x) \end{bmatrix}, D(x) := \begin{bmatrix} 0 & D_{12}(x) \\ D_{21}(x) & D_{22}(x) \end{bmatrix},$$

and

$$\tilde{B}(x) := \begin{bmatrix} B_3^T(x) & D_{13}^T(x) & D_{23}^T(x) \end{bmatrix}.$$

Let $\mathcal{N}(\tilde{B}(x))$ be the distribution on $\mathbf{X}$ which annihilates all of the row vectors of $\tilde{B}(x)$.

We first have the following lemma.

Lemma 9.4.5 *The following two statements are equivalent.*

(i) There exist a $\mathbf{C}^0$ matrix valued function $F(x)$, a positive definite matrix-valued function $P : \mathbf{X} \rightarrow \mathbf{R}^{n \times n}$, and a positive definite matrix $\hat{Q} \in \mathcal{D}$ such that the following NLMI holds,

$$\begin{bmatrix} A_F^T(x)P(x) + P(x)A_F(x) + C_F^T(x)QC_F(x) & P(x)B(x) + C_F^T(x)QD(x) \\ B^T(x)P(x) + D^T(x)QC_F(x) & D^T(x)QD(x) - Q \end{bmatrix} < 0 \tag{9.35}$$

with $Q := \begin{bmatrix} \hat{Q} & 0 \\ 0 & I \end{bmatrix}$ *for all* $x \in \mathbf{X}$*, where*

$$A_F(x) = A(x) + B_3(x)F(x), \qquad C_F(x) = \begin{bmatrix} C_1(x) + D_{13}(x)F(x) \\ C_2(x) + D_{23}(x)F(x) \end{bmatrix}.$$

(ii) There exist a positive definite matrix-valued function $X : \mathbf{X} \to \mathbf{R}^{n \times n}$ *and a positive definite matrix* $\hat{Y} \in \mathcal{D}$ *such that the following NLMI holds,*

$$\mathcal{M}_{Syn}(X, Y, x) :=$$

$$B_\perp^T(x) \begin{bmatrix} A(x)X(x) + X(x)A^T(x) + B^T(x)YB(x) & X(x)C^T(x) + B(x)YD^T(x) \\ C(x)X(x) + D(x)YB^T(x) & D(x)YD^T(x) - Y \end{bmatrix} B_\perp(x) < 0 \tag{9.36}$$

with $Y := \begin{bmatrix} \hat{Y} & 0 \\ 0 & I \end{bmatrix}$*, and* $B_\perp(x)$ *is a* $\mathbf{C}^0$ *matrix-valued function on* $\mathbf{X}$ *such that* $span(B_\perp(x)) = \mathcal{N}(\tilde{B}(x))$ *for all* $x \in \mathbf{X}$*.*

Moreover, if any one of the above statements holds, then the solutions of the other NLMI can be chosen such that $P(x) = X^{-1}(x)$ *and* $Q = Y^{-1}$*.*

The proof of the above lemma, which uses Finsler's Theorem (cf. [29]), follows the arguments in [113] and [3]; it is omitted. It is noted that the NLMI (9.36) is affine in unknown $P(x)$ and Q. We have the following theorem which gives a convex characterizations for robust performance synthesis by state feedback.

Theorem 9.4.6 *Consider the uncertain system with nominal plant defined as (9.34). The state feedback robust performance synthesis problem has a solution if there exist a positive definite matrix-valued function* $X : \mathbf{X} \to \mathbf{R}^{n \times n}$ *and a positive definite matrix* $\hat{Y} \in \mathcal{D}$ *such that the NLMI (9.36) holds for all* $x \in \mathbf{X}$*, and* $\frac{\partial V}{\partial x}(x) = 2x^T X^{-1}(x)$ *for some* $\mathbf{C}^1$ *function* V *on* $\mathbf{X}$ *with* $V(0) = 0$*.*

Proof. Let $(X(x), Y)$ be as in the theorem. By the preceding lemma, there exists a $\mathbf{C}^0$ matrix valued function $F(x)$ on $\mathbf{X}$ such that

$$\begin{bmatrix} A_F^T(x)X^{-1}(x) + X^{-1}(x)A_F(x) + C_F^T(x)Y^{-1}C_F(x) & X^{-1}(x)B(x) + C_F^T(x)Y^{-1}D(x) \\ B^T(x)X^{-1}(x) + D^T(x)Y^{-1}C_F(x) & D^T(x)Y^{-1}D(x) - Y^{-1} \end{bmatrix} < 0. \tag{9.37}$$

On the other hand, take $u_c = F(x)x$ as a state feedback law, so the closed loop nominal system is as follows,

$$G_F : \begin{cases} \dot{x} = (A(x) + B_3(x)F(x))x + B_1(x)u + B_2(x)w \\ y = (C_1(x) + D_{13}(x)F(x))x + \qquad\qquad D_{12}(x)w \\ z = (C_2(x) + D_{23}(x)F(x))x + D_{21}(x)u + D_{22}(x)w \end{cases}.$$

We now claim the closed loop uncertain system satisfies robust performance. In fact, take $(P(x), Q) = (X^{-1}(x), Y^{-1})$. By the remarks in section 2.3, (9.37) implies

$$\begin{bmatrix} x^T(A_F^T(x)P(x) + P(x)A_F(x) + C_F^T(x)QC_F(x))x & x^T(P(x)B(x) + C_F^T(x)QD(x)) \\ (B^T(x)P(x) + D^T(x)QC_F(x))x & D^T(x)QD(x) - Q \end{bmatrix} < 0.$$

The conclusion therefore follows from theorem 9.3.4. □

9.5 NLMIs and Computational Considerations

We have examined different problems including Lyapunov stability, $\mathcal{L}_2$-gains, robust asymptotic stability, robust performance, and robustness synthesis. It is noted that the computation about robustness analysis and synthesis involves solving some NLMIs. We address computational issues for robustness analysis and synthesis in this section. The characterizations of different problems share a common structure, i.e., each of the solutions involves solving an algebraic NLMI and a **partial differential equation** (PDEs). The structures for different problems are summarized in the following table, where $p : \mathbf{X} \to \mathbf{R}^{1\times n}$ with $p(0) = 0$, $X : \mathbf{X} \to \mathbf{R}^{n\times n}$, and $Q, Y \in \mathbf{R}^{p\times p}$ are the unknowns.

Problems	**NLMIs**	**PDEs**
Lyapunov Stability	(8.3) $p(x)f(x) \leq 0$	$\frac{\partial V}{\partial x}(x) = p(x)$
$\mathcal{L}_2$-Gains	(8.6) $\mathcal{M}_{Gain}(p, x) \leq 0$	$\frac{\partial V}{\partial x}(x) = p(x)$
Robust Asymptotic Stability	(9.12) $\mathcal{M}_{RS}(p, Q, x) \leq 0$	$\frac{\partial V}{\partial x}(x) = p(x)$
Robust Performances	(9.18) $\mathcal{M}_{RP}(p, Q, x) \leq 0$	$\frac{\partial V}{\partial x}(x) = p(x)$
Robustness Synthesis	(9.36) $\mathcal{M}_{Syn}(X, Y, x) < 0$	$\frac{\partial V}{\partial x}(x) = 2x^TX^{-1}(x)$

Therefore, we have the following observation from the above discussion:

The computational effort needed for robustness analysis and synthesis is not more difficult than that for checking Lyapunov stability.

In other words, the provided approach reduces the robustness analysis and synthesis problems to the possible easiest problems, whose solutions involve solving NLMIs and PDEs. We will pursue these two issues in the next two subsections.

9.5.1 Solution Properties of NLMIs

Because of the similarity between the NLMIs (8.6), (9.12), (9.18), and (9.36). We just take (9.18) as an example to discuss the solutions. More explicitly, consider the following NLMI,

$$\mathcal{M}_{RP}(p,Q,x) := \begin{bmatrix} p(x)f(x) + h^T(x)Qh(x) & \frac{1}{2}p(x)g(x) + h^T(x)Qk(x) \\ \frac{1}{2}g^T(x)p^T(x) + k^T(x)Qh(x) & k^T(x)Qk(x) - Q \end{bmatrix} \leq 0, \quad (9.38)$$

which is actually a state-dependent LMI. Therefore,

In general if the state set $\mathbf{X}$ is not finite, then the computation for solving NLMIs is an infinitely dimensional LMI problem.

This is bad news for computational efforts to deal with nonlinear problems. We will examine some properties for the solutions of NLMIs, which will provide some guideline to computational efforts.

We first consider the solution about (9.38), we need to find a continuously differentiable function $p(x)$ on $\mathbf{X}$ and a constant positive definite matrix $Q \in \mathbf{R}^{p\times p}$. If we assume $f(x) = A(x)x, g(x) = B(x), h(x) = C(x)x$, $k(x) = D(x)$, and $p(x) = P(x)x$ in (9.38), then (9.38) is implied by the following NLMI (which is a more conservative characterization about robustness),

$$\mathcal{M}(P,Q,x) :=$$

$$\begin{bmatrix} A^T(x)P(x) + P^T(x)A(x) + C^T(x)QC(x) & P^T(x)B(x) + C^T(x)QD(x) \\ B^T(x)P(x) + D^T(x)QC(x) & D^T(x)QD(x) - Q \end{bmatrix} < 0. \quad (9.39)$$

Therefore, it is sufficient to find a $\mathbf{C}^0$ matrix-valued function $P : \mathbf{X} \to \mathbf{R}^{n\times n}$ and a positive definite matrix Q such that $\mathcal{M}(P,Q,x) < 0$. In other words, if $(P(x), Q)$ is a solution to (9.39), then $(P(x)x, Q)$ is a solution to (9.38). However, as discussed in [115], this treatment is conservative.

Next, we will justify the existence of continuous and positive definite solutions to the NLMI under some regularity conditions. Given a matrix valued function $S : \mathbf{X} \to \mathbf{R}^{n\times n}$ with $S(x) > 0$ for all $x \in \mathbf{X}$. Let $R(x) := Q - D^T(x)QD(x) > 0$, define a state-dependent Hamiltonian $H : \mathbf{X} \to \mathbf{R}^{2n\times 2n}$ as

$$H(x) := \begin{bmatrix} A(x) & 0 \\ -C^T(x)QC(x) - S(x) & -A^T(x) \end{bmatrix} +$$

$$+ \begin{bmatrix} B(x) \\ -C^T(x)QD(x) \end{bmatrix} R^{-1}(x) \begin{bmatrix} D^T(x)QC(x) & B^T(x) \end{bmatrix}. \tag{9.40}$$

The following result is essentially from [75, lemma 2.4].

Proposition 9.5.1 *$\mathcal{M}(P, Q, x) < 0$ has non-negative definite solutions $P(x) \geq 0$ and $Q > 0$ if and only if the state-dependent Hamiltonian $H : \mathbf{X} \to \mathbf{R}^{2n\times 2n}$ defined in (9.40) for some matrix-valued function $S : \mathbf{X} \to \mathbf{R}^{n\times n}$ with $S(x) > 0$ for all $x \in \mathbf{X}$ is in $dom(\bar{Ric})$, i.e. $H(x) \in dom(\bar{Ric})$ for each $x \in \mathbf{X}$. Moreover, $P(x) := \bar{Ric}(H(x)) \geq 0$ is such a solution with $Q > 0$ as given. In addition, if for each $x \in \mathbf{X}$,*

$$\bigcap_{i=0}^{n-1} \ker(\tilde{C}(x)A^i(x)) = \emptyset, \qquad \tilde{C}(x) = \begin{bmatrix} Q^{1/2}C(x) \\ S^{1/2}(x) \end{bmatrix},$$

this solution is positive definite, i.e., $\bar{Ric}(H(x)) > 0$.

The above theorem implies that under the condition $H(x) \in dom(\bar{Ric})$ for each $x \in \mathbf{X}$, the NLMI (9.39), $\mathcal{M}(P, Q, x) < 0$, has non-negative definite solutions $P(x) \geq 0$ and $Q > 0$. The following theorem further shows that such solutions can be chosen to be continuous in the case of interest in this chapter.

Theorem 9.5.2 *Suppose the matrix inequality $\mathcal{M}(P, Q, x) < 0$ has a positive definite solution P_x for each $x \in \mathbf{X}$ and $Q > 0$, then there exists a $\mathbf{C}^0$ matrix-valued function $P : \mathbf{X} \to \mathbf{R}^{n\times n}$ with $P(x) = P^T(x) > 0$, such that $\mathcal{M}(P(x), Q, x) < 0$ for all $x \in \mathbf{X}$.*

Proof. Consider the NLMI (9.39), $\mathcal{M}(P, Q, x) < 0$ with $x \in \mathbf{X}$. Then $\mathcal{M} : \mathbf{R}^{n\times n} \times \mathbf{R}^{p\times p} \times \mathbf{X} \to \mathbf{R}^{(n+p)\times(n+p)}$ is a continuous matrix-valued function and satisfies

$$\mathcal{M}(\sum_{k=1}^{N} \alpha_k P_k, \sum_{k=1}^{N} \alpha_k Q_k, x) = \sum_{k=1}^{N} \alpha_k \mathcal{M}(P_k, Q_k, x) \tag{9.41}$$

for all $\alpha_k \geq 0$ with $\sum_{k=1}^{N} \alpha_k = 1$.

By assumption, there exist a positive definite matrix $Q \in \mathbf{R}^{p\times p}$ and a positive definite matrix $P_x \in \mathbf{R}^{n\times n}$ and for each $x \in \mathbf{X}$ such that

$$\mathcal{M}(P_x, Q, x) < 0.$$

By continuity of M with respect to x, there is an $r_x > 0$ such that

$$\mathcal{M}(P_x, Q, x_0) < 0 \tag{9.42}$$

for all $x_0 \in \mathbf{N}(x) := \{x_0 : \|x_0 - x\| < r_x\}$.

On the other hand, $\{\mathbf{N}(x)\}|_{x\in\mathbf{X}}$ is an open covering of X, i.e.,

$$X \subset \bigcup_{x\in\mathbf{X}} \mathbf{N}(x). \tag{9.43}$$

Since the space $\mathbf{R}^n$ is *paracompact*, there is a locally finite open subcovering $\{\mathbf{N}_i\}|_{i\in\mathbf{I}}$ for some index set $\mathbf{I}$ which refines $\{\mathbf{N}(x)\}|_{x\in\mathbf{X}}$. By (9.42), $P_i \in \mathbf{R}^{n\times n}$ is taken to be positive definite for each $i \in \mathbf{I}$ such that

$$\mathcal{M}(P_i, Q, x) < 0 \tag{9.44}$$

for all $x \in \mathbf{N}_i$.

It is known by the standard argument of **continuous partitions of unity** that there is a smooth partition of unity $\{\phi_i\}|_{i\in\mathbf{I}}$ to $\mathbf{X}$ subordinated to the covering $\{\mathbf{N}_i\}|_{i\in\mathbf{I}}$; i.e., ϕ_i is smooth and non-negative with support $\textsc{Supp}(\phi_i) \subset \mathbf{N}_i$ for each $i \in \mathbf{I}$, and

$$\sum_{i\in\mathbf{I}} \phi_i(x) = 1, \forall x \in \mathbf{X}. \tag{9.45}$$

Define a matrix-valued function $P : \mathbf{X} \to \mathbf{R}^{n\times n}$ as

$$P(x) = \sum_{i\in\mathbf{I}} \phi_i(x) P_i, \forall x \in \mathbf{X}, \tag{9.46}$$

which is positive definite and smooth since it is locally a finite sum of smooth positive definite matrix-valued functions.

It follows from (9.45), (9.46) and (9.41) that

$$\mathcal{M}(P(x), Q, x) = \mathcal{M}(\sum_{i\in\mathbf{I}} \phi_i(x) P_i, Q, x) = \sum_{i\in\mathbf{I}} \phi_i(x) \mathcal{M}(P_i, Q, x) < 0.$$

The last equality holds since the summation is finite for each $x \in \mathbf{X}$.

Thence, the constructed smooth matrix-valued function $P : \mathbf{X} \to \mathbf{R}^{n\times n}$ in (9.46) is positive definite and is a solution to $\mathcal{M}(P(x), Q, x) < 0$. $\square$

Remark 9.5.3 *The similar technique can be used to directly examine the continuous solutions to the following NLMI (9.38),* $\mathcal{M}_{SP}(p, Q, x) < 0$ *for all* $x \in \mathbf{X} \setminus \{0\}$*. As a matter of fact, we can get a continuous solution* $p(x)$ *on* $\mathbf{X} \setminus \{0\}$ *to the above NLMI using the similar arguments as in the preceding proof. If* $\lim_{x \to 0} p(x) = 0$*, then the vector valued function* $p(x)$ *on* $\mathbf{X} \setminus \{0\}$ *can be continuously extended to* $\mathbf{X}$ *by defining* $p(0) = 0$*. The extension is a solution to (9.38) on* $\mathbf{X}$*.*

A nice convex property for NLMIs is stated by the following proposition whose proof is easy and omitted here.

Theorem 9.5.4 *The* $\mathbf{C}^0$ *solutions* $(P(x), Q)$ *to NLMI* $\mathcal{M}(P, Q, x) < 0$ *such that* $P : \mathbf{X} \to \mathbf{R}^{n \times n}$ *and* $Q > 0$ *form a convex set; the subset of solutions* $(P(x), Q)$ *such that* $P(x)$ *is* $\mathbf{C}^0$ *non-negative definite with* $\frac{\partial V}{\partial x}(x) = 2x^T P(x)$ *for some function* $V : \mathbf{X} \to \mathbf{R}$ *is convex; the subset of solutions* $(P(x), Q)$ *such that* $P(x)$ *is* $\mathbf{C}^0$ *positive definite with* $\frac{\partial V}{\partial x}(x) = 2x^T P(x)$ *for some function* $V : \mathbf{X} \to \mathbf{R}$ *is also convex.*

9.5.2 Existence of Lyapunov Functions

As mentioned earlier, the existence of solutions to NLMIs is not enough to give positive assertion about Lyapunov stability, $\mathcal{L}_2$-performance, robust stability, and robust performance; some additional requirement is imposed, i.e. there is a $\mathbf{C}^1$ Lyapunov function $V : \mathbf{X} \to \mathbf{R}^+$, such that

$$\frac{\partial V}{\partial x}(x) = 2p^T(x)$$

for all $x \in \mathbf{X}$. In this subsection, we will examine explicitly when it is the case for a class of such solutions.

The following result is quite standard, the reader is referred to [21] for the proofs (see also [113]).

Proposition 9.5.5 *Suppose a vector-valued function* $p : \mathbf{X} \to \mathbf{R}^n$ *is of class* $\mathbf{C}^1$*; let* $p(x) = [p_1(x), \cdots, p_n(x)]^T$ *for* $x \in \mathbf{X}$*. Then there exists* $V : \mathbf{X} \to \mathbf{R}$ *such that*

$$\frac{\partial V}{\partial x}(x) = 2p^T(x)$$

if and only if

$$\frac{\partial p_i}{\partial x_j}(x) = \frac{\partial p_j}{\partial x_i}(x) \tag{9.47}$$

for all $x \in \mathbf{X}$ and $i, j = 1, 2, \cdots, n$. Moreover, if (9.47) holds, then an function $V : \mathbf{X} \to \mathbf{R}$ with $V(0) = 0$ is given by

$$V(x) = 2x^T \int_0^1 p(tx)dt. \tag{9.48}$$

In addition, if $p(x) = P(x)x$ for some positive definite matrix-valued function $P(x)$, then $V(x)$ is also positive definite function.

For a class of solutions constructed in the proof in theorem 9.5.2, we can specially characterize the existence of the Lyapunov function in this case. The positive definite matrix-valued function $P : \mathbf{X} \to \mathbf{R}^{n \times n}$, which satisfies $\mathcal{M}(P, Q, x) < 0$, is constructed as (9.46)

$$P(x) = \sum_{i \in \mathbf{I}} \phi_i(x) P_i, \forall x \in \mathbf{X},$$

for some index set $\mathbf{I}$, where $\{\phi_i\}|_{i \in \mathbf{I}}$ is a partition of unity of $\mathbf{X}$ and $P_i = P_i^T \geq 0$. Notice that the above summation is locally finite. Similar argument leads to the following theorem, which can be verified by the previous proposition.

Theorem 9.5.6 *Suppose the matrix valued function $P : \mathbf{X} \to \mathbf{R}^{n \times n}$ defined by*

$$P(x) = \sum_{i \in \mathbf{I}} \phi_i(x) P_i \tag{9.49}$$

with $\phi_i : \mathbf{X} \to \mathbf{R}^+$ being of class $\mathbf{C}^1$ and $P_i \in \mathbf{R}^{n \times n}$ for $i \in \mathbf{I}$ satisfies: $\mathcal{M}(P, x) \leq 0$ for all $x \in \mathbf{X}$; let $V_i(x) = x^T P_i x$ for all $i \in \mathbf{I}$. There exists a $\mathbf{C}^2$ function $V : \mathbf{X} \to \mathbf{R}$ such that $\frac{\partial V}{\partial x}(x) = 2x^T P(x)$ if and only if

$$\sum_{i \in \mathbf{I}} \frac{\partial \phi_i}{\partial x_j}(x) \cdot \frac{\partial V_i}{\partial x_l}(x) = \sum_{i \in \mathbf{I}} \frac{\partial \phi_i}{\partial x_l}(x) \cdot \frac{\partial V_i}{\partial x_j}(x) \tag{9.50}$$

for all $x \in \mathbf{X}$ and $j, l \in \{1, 2, \cdots, n\}$ with $j \neq l$.

Notice that the summation in (9.50) is finite for each $x \in \mathbf{X}$.

9.5.3 Further Remarks

The above treatments about robustness analysis and synthesis are in terms of NLMIs, which are pointwise LMIs on state set $\mathbf{X}$, modulo some additional constraints on the solutions. From the proof of Theorem 9.5.2, we know that if $\mathbf{X}$ is bounded, then we only need to solve a finite number of LMIs to get the solution for the NLMI on $\mathbf{X}$. In the following, we will give an approach to obtaining a local solution.

Consider the NLMI (9.39). In the light of the notion of global linearization of nonlinear systems developed by Liu *et al.* [106], as in [29], the coefficient matrices in (9.39) are assumed in a convex set:

$$[A(x), B(x), C(x), D(x)] \in \mathrm{Co}\{[A_i, B_i, C_i, D_i]|_{i\in\{1,2,\cdots,L\}}\}, \forall x \in \mathbf{X},$$

where Co stands for the **convex hull.** In this case, a constant solution $(P, Q) \in \mathbf{R}^{n\times n} \times \mathbf{R}^{q\times q}$ to (9.39) is sought. Therefore, consider the following equations,

$$\begin{bmatrix} A_i^T P + P^T A_i + C_i^T Q C_i & P^T B_i + C_i^T Q D_i \\ B_i^T P + D_i^T Q C_i & D_i^T Q D_i - Q \end{bmatrix} < 0$$

for all $i \in \{1, 2, \cdots, L\}$, their common constant solutions (P, Q) can be obtained by the methods suggested in [29]. If such a solution (P, Q) exists, then (P, Q) is also a solution to (9.39), i.e.,

$$\begin{bmatrix} A^T(x)P + P^T A(x) + C^T(x)QC(x) & P^T B(x) + C^T(x)QD(x) \\ B^T(x)P + D^T(x)QC(x) & D^T(x)QD(x) - Q \end{bmatrix} < 0.$$

The solution automatically satisfies the condition (9.47), and the corresponding Lyapunov function is $V(x) = x^T Px$.

This treatment suggests a tractable algorithm to get local solutions, which can be used to seek constant solutions on each partitioned state set $\mathbf{N}_i$ in the proof of theorem 6.2. However, this approach generally leads to conservative results if the prescribed state set is large enough. This can be seen by the example in Section 3; it is obvious that there is no constant pair (P, Q) which satisfies the NLMI (9.29) for all $x \in \mathbf{R}$.

To summarize the above discussion, we suggest the following algorithm, based on Theorems 9.5.2 and 9.5.6, for the robustness analysis problem, which is characterized by the NLMI (9.38). The state set $\mathbf{X} \subset \mathbf{R}^n$ is assumed to be bounded.

Algorithm 9.5.7 *Consider the robustness analysis problem.*

(i) Find a covering $\{\mathbf{N}_i\}|_{i=1}^m$ *for state set* $\mathbf{X}$ *such that* $\mathbf{X} \subset \bigcup_{i=1}^m \mathbf{N}_i$*; construct a* $\mathbf{C}^1$ *partition of unity* $\{\phi_i\}$ *to the bounded set* $\mathbf{X}$ *subordinated to the covering such that*

$$\sum_{i=1}^m \phi_i(x) = 1, \quad \forall x \in \mathbf{X}.$$

(ii) Find constant solutions P_i *for* $i \in \{1, \cdots m\}$ *and* Q *such that for* $x \in \mathbf{X} \cap \mathbf{N}_i$,

$$\mathcal{M}(P_i, Q, x) < 0; \tag{9.51}$$

$$\sum_{i=1}^{m} \frac{\partial \phi_i}{\partial x_j}(x) \cdot \frac{\partial V_i}{\partial x_l}(x) = \sum_{i=1}^{m} \frac{\partial \phi_i}{\partial x_l}(x) \cdot \frac{\partial V_i}{\partial x_j}(x), \;\; V_i = x^T P_i x.$$

(iii) If there is a solution in Step (ii), then the robustness analysis problem has positive assertion; otherwise, go to Step (i).

In the above algorithm, the computation for finding P_i for $i \in \{1, \cdots m\}$ and Q which satisfy (9.51) can be done by the method similar to the local solution algorithm described above, it is a finite-dimensional LMI problem. Nonetheless, this algorithm is still problematic. It only provides a positive answer, i.e., if the above procedure can not be passed, the problem may still have solutions. Besides, in Step (i), to find a suitable partition for $\mathbf{X}$ is *ad hoc*.

9.6 Concluding Remarks

This chapter deals with the robustness analysis and synthesis for nonlinear systems in the state-space. The scaling treatment for the robust performance problem or robust stability analysis in the structured uncertainty case was used to reduce the possible conservatism arising from the structural constraints of the uncertainty. The characterizations are in terms of NLMIs which offer some computationally attractive properties. In an input-output point of view, the characterizations imply that both (structured) uncertainty and a (scaled) nominal plant have small $\mathcal{L}_2$-gain. From this characterization, it is concluded that the computation needed for robustness analysis and synthesis of nonlinear uncertain systems is not more difficult than that for checking Lyapunov stability of nonlinear systems; in other words, the provided approach reduces the robustness analysis and synthesis problems to the possible easiest problems. However, in general, the computation for solving such an easy problem is an infinitely dimensional problem.

Chapter 10

Conclusions

In the last two decades, much effort and attention have been devoted by control scientists to dealing with uncertainty and nonlinearity. While such efforts have been beset from the beginning with great difficulties, great interest in them was maintained. This dissertation is a continuation of such efforts. In this thesis, we deal with the uncertainty and nonlinearity in two cases where the nominal systems are linear time-invariant and the nominal systems are nonlinear time-invariant, respectively. Both robustness analysis and robustness synthesis under structured bounded uncertainty are treated; some systematic robustness design techniques are developed with the consideration of their computational properties. However, the development of efficient computational methods for the nonlinear problems is far from satisfactory. Much work remains to be done.

Since a control system is an artificial system, its description should capture its three functional elements: its goal, its internal character, and its outer environment. Formally, one can represent a control system as the triple $(G, \Delta; J)$, where G is the nominal model which describes the internal structure of the control system, and Δ can be viewed as the uncertainty which describes the interaction between the internal structure and the outer environment[1]; the interconnection of G and Δ is through feedback in this thesis. J represents the goal of the robust control systems, which in this dissertation is, roughly speaking, to attenuate the external disturbances and uncertainty. This consideration about modeling of control systems also manifests the relation between simplicity of the ideal and complexity of the reality. More precisely, the (structured) uncertainty description reflects the

[1]Strictly speaking, in this thesis, G does not exactly describe the internal structure, it is just an approximation, while Δ measures difference between the nominal model and the reality, and comes from both the internal perturbation and the interactions between the internal structure and the outer environment.

complexity of a control system in a simple, yet possibly conservative way.

Complexity is an essential issue for artificial systems. Although the consideration of uncertainty and nonlinearity in the modeling captures some aspects of the complexity, other powerful methodologies are still expected to more deeply understand the complexity and to more effectively synthesize control systems. Different development levels of control science reflect different depths of human understanding and ability to deal with the complexity. Nontheless, the difficulty inherent in the state of the development always exists. Since each level of concepts is connected with and in fact built upon the previous levels, when inadequacies manifest themselves, one must reach greater depth by penetrating previous levels and examining the whole structure of previous concepts. The difficulty of this task rapidly diverges with the depth of the examinations. Furthermore, the power of the (individual) human intellect is known to be limited. It is also noticed the fact that psychological and social limitations on the development of the creative ability of each individual may be even more stringent than natural limitations.

In addition, a dangerous side-effect of the development of control science is that the development might result in unfortunate separation between pure theories and the vital applications[2], while new concepts and new theories can not be created by free imagination without their justification in reality. This separation should be organically united so that the control science can regain its internal force and control scientists can have ever-increasingly clear comprehension of the control systems without losing sight of applications. At any rate, the only way for control science to remain vital and effectively face the methodological challenges, intellectual limitations, and the social reality is continuous reinjection of directly empirical ideas.

[2]However, it is possible that a good new theory does not have immediate applications (see the Preface).

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