Chapter 2

Discrete Routh Reduction

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Abstract

This chapter investigates the relationship between Routh symmetry reduction and time discretization for Lagrangian systems. Within the framework of discrete variational mechanics, a discrete Routh reduction theory is constructed for the case of abelian group actions, and extended to systems with constraints and non-conservative forcing or dissipation. Variational Runge–Kutta discretizations are considered in detail, including the extent to which symmetry reduction and discretization commute. In addition, we obtain the Reduced Symplectic Runge–Kutta algorithm, which can be considered a discrete analogue of cotangent bundle reduction. We demonstrate these techniques numerically for satellite dynamics about the Earth with a non-spherical J_2 correction, and the double spherical pendulum. The J_2 problem is interesting because in the unreduced picture, geometric phases inherent in the model and those due to numerical discretization can be hard to distinguish, but this issue does not appear in the reduced algorithm, and one can directly observe interesting dynamical structures. The main point of the double spherical pendulum is to provide an example with a nontrivial magnetic term in which our method is still efficient, but is challenging to implement using a standard method.

2.1 Introduction

Given a mechanical system with symmetry, we can restrict the flow on the phase space to a level set of the conserved momentum. This restricted flow induces a "reduced" flow on the quotient of this level set by the subgroup of the symmetry group that acts on it. Thus we obtain a reduced dynamical system on a reduced phase space. The process of reduction has been enormously important for many topics in mechanics such as stability and bifurcation of relative equilibria, integrable systems, etc.

The purpose of the present work is to contribute to the development of reduction theory for discrete time mechanical systems, using the variational formulation of discrete mechanics described in Marsden and West [2001]. We also explore the relationship between continuous time reduction and discrete time reduction, and discuss reduction for symplectic Runge–Kutta integration algorithms and its relationship to the theory of discrete reduction.

The discrete time mechanical systems used here are derived from a discrete variational principle on the discrete phase space $Q \times Q$. Properties such as conservation of symplectic structure and conservation of momentum follow in a natural way from the discrete variational principle, and the discrete evolution map can thus be regarded as a symplectic-momentum integrator for a continuous system.

The theory of discrete variational mechanics in the form we shall use it has its roots in the optimal control literature in the 1960's; see, for example, Jordan and Polak [1964], and Hwang and Fan [1967]. It was formulated in the context of mechanics by Maeda [1981], Veselov [1988, 1991] and Moser and Veselov [1991]. It was further developed by Wendlandt and Marsden [1997], and Marsden and Wendlandt [1997], including a constrained formulation, and by Marsden et al. [1998], who extended these ideas to multisymplectic partial differential equations. For a general overview and many more references we refer to Marsden and West [2001].

Although symplectic integrators have typically only been considered for conservative systems, in Kane et al. [2000] it was shown how the discrete variational mechanics can be extended to include forced and dissipative systems. This yields integrators for non-conservative systems which can demonstrate exceptionally good long-time behavior, and which correctly simulate the decay or growth in quantities such as energy and momentum. The discrete mechanics for non-conservative systems is discussed in §2.8.2, and it is shown how the discrete reduction theory can also handle forced and dissipative systems.

The formulation of discrete mechanics in this paper is best suited for constructing structure preserving integrators for mechanical systems that are specified in terms of a regular Lagrangian. Jalnapurkar and Marsden [2003], building on the work of Marsden and West [2001], show how to obtain structure-preserving variational integrators for mechanical systems specified in terms of a Hamiltonian. This method can be applied even if the Hamiltonian is degenerate.

A complementary approach to the Routh theory of reduction used in this paper is that of Lie–Poisson and Euler–Poincaré reduction, where the dynamics of an equivariant system on a Lie group can be reduced to dynamics on the corresponding Lie algebra. A discrete variational formulation of this was given in Marsden et al. [1999, 2000a], Bobenko et al. [1998], and Bobenko and Suris [1999]. Eventually one will need to merge that theory with the theory in the present paper.

We shall now briefly describe the contents of each section of this paper. In §2.2 we give a summary of some well known results on reduction for continuous-time mechanical systems with symmetry. Specifically, we discuss Routh reduction and its relationship with the theory of cotangent bundle reduction. In §2.3 we develop the theory of discrete reduction, which includes the derivation of a reduced variational principle, and proof of the symplecticity of the reduced flow. We also discuss in this section the relationship between continuous- and discrete-time reduction. In §2.4 we discuss a link between the theory of discrete mechanics and symplectic Runge–Kutta algorithms. In §2.5, we describe how our symplectic Runge–Kutta algorithm for a mechanical system with symmetry can be reduced to obtain a reduced symplectic Runge–Kutta algorithm. In §2.6 we put together in a coherent way the results of the previous sections. We also discuss how the original reduction procedure of Routh [1877, 1884] relates to our results. In §2.8 we extend the theory of discrete reduction to systems with constraints and external forces, and lastly, in §2.9 we present a numerical example of satellite dynamics about an oblate Earth.

2.2 Continuous Reduction

In this section we discuss reduction of continuous mechanical systems, in both the Lagrangian and Hamiltonian settings. Our purpose here is to fix notation and recall some basic results. For a more detailed exposition, see Marsden and Scheurle [1993a,b], Holm et al. [1998], Jalnapurkar and Marsden [2000], Marsden et al. [2000b], and Cendra et al. [2001] for Lagrangian reduction, and for Hamiltonian reduction, see, for example, Abraham and Marsden [1978] as well as Marsden [1992] for cotangent bundle reduction.

Suppose we have a mechanical system with configuration manifold Q, and let $L : TQ \to \mathbb{R}$ be a given Lagrangian. Let $q = (q^1, \ldots, q^n)$ be coordinates on Q. The Euler–Lagrange (EL) equations on TQ are

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0. \tag{2.2.1}$$

These equations define a flow on TQ if L is a regular Lagrangian, which we assume to be the case. Let X_E denote the vector field on TQ that corresponds to the flow. We have a Legendre transformation, $\mathbb{F}L : TQ \to T^*Q$, defined by

$$\mathbb{F}L : (q, \dot{q}) \mapsto \left(q, \frac{\partial L}{\partial \dot{q}}\right).$$

The Hamiltonian H on T^*Q is obtained by pushing forward the energy function E on TQ, which is defined by

$$E(q, \dot{q}) = \langle \mathbb{F}L(q, \dot{q}), \dot{q} \rangle - L(q, \dot{q}).$$

We have a canonical symplectic structure Ω_Q on T^*Q. From Ω_Q and H, we obtain the Hamiltonian vector field X_H on T^*Q. A basic fact is that X_H is the push-forward of X_E using $\mathbb{F}L$. Since the flow of X_H preserves Ω_Q, the flow of X_E, i.e., the flow derived from the EL equations, preserves $\Omega_L := (\mathbb{F}L)^* \Omega_Q$.

Suppose an abelian group G acts freely and properly on Q so that Q is a principal fibre bundle over ***shape space*** $S := Q/G$. Let $\pi_{Q,S} : Q \to S$ be the natural projection. Given $x \in S$, we can find an open set $U \subset S$, such that $\pi^{-1}_{Q,S}(U)$ is diffeomorphic to $G \times U$. Such a diffeomorphism is called a local trivialization. Given a local trivialization, we can use local coordinates on G and on S to obtain a set of local coordinates on Q. If $g = (g^1, \ldots, g^r)$ and $x = (x^1, \ldots, x^s)$ are local coordinates on G and $U \subset S$, respectively, then $q = (g, x) = (g^1, \ldots, g^r, x^1, \ldots, x^s)$ can be taken as local coordinates on Q.

The action of G on Q on be lifted to give actions of G on TQ and T^*Q. We also have a momentum map $J : T^*Q \to \mathfrak{g}^*$, defined by the equation $J(\alpha_q) \cdot \xi = \langle \alpha_q, \xi_Q(q) \rangle$, where $\alpha_q \in T^*_q Q$, $\xi \in \mathfrak{g}$, and $\xi_Q(q)$ is the infinitesimal generator corresponding to the action of G on Q evaluated at q. We can pull-back J to TQ using the Legendre transform $\mathbb{F}L$ to obtain a ***Lagrangian momentum map*** $J_L := \mathbb{F}L^* J : TQ \to \mathfrak{g}^*$.

If the Lagrangian L is invariant under the lifted action of G on TQ, the associated Hamiltonian H will be invariant under the action of G on T^*Q. In this situation, Noether's theorem tells us that the flows on TQ and on T^*Q preserve the momentum maps J_L and J, respectively.

Since locally, $Q \approx G \times S$, we also have the local representation $TQ \approx TG \times TS$. Thus, if $(g, \dot{g})$ are local coordinates on TG, and $(x, \dot{x})$ are local coordinates on TS, $(g, x, \dot{g}, \dot{x})$ are local coordinates on TQ. From the formula for the momentum map and freeness of the action, one sees that $\dot{g}$ is determined from $(g, x, \dot{x})$ and the value of the momentum. Thus, $J_L^{-1}(\mu)$ is locally diffeomorphic to $G \times TS$. If G is abelian (which is what we have assumed), it follows that G acts on $J_L^{-1}(\mu)$, and that $J_L^{-1}(\mu)/G$ is locally diffeomorphic to TS. Let the natural projection $J_L^{-1}(\mu) \to TS$ be denoted by $\pi_{\mu,L}$.

In a local trivialization, let $q \in Q$ correspond to $(g, x) \in G \times S$. Thus $T_q Q$ can be identified with $T_g G \times T_x S$.

Let $\mathfrak{A} : TQ \to \mathfrak{g}$ be a chosen principal connection. Using a local trivialization, the connection can be described by the equation

$$\mathfrak{A}(\dot{g}, \dot{x}) = A(x)\dot{x} + g^{-1} \cdot \dot{g}.$$

Here $A(x) : T_xS \to \mathfrak{g}$ is the restriction of $\mathfrak{A}$ to T_xS. (T_xS is identified with the subspace $T_xS \times \{0\}$ of $T_gG \times T_xS$, which is in turn identified with T_qQ.) Note that the map $A(x)$ depends upon the particular trivialization that we are using.

The connection gives us an intrinsic way of splitting each tangent space to Q into horizontal and vertical subspaces. The vertical space V_q at q is the tangent space to the group orbit through q. If $\mathfrak{A}_q : T_qQ \to \mathfrak{g}$ is the restriction of $\mathfrak{A}$, then the horizontal space H_q is defined as the kernel of $\mathfrak{A}_q$. The maps $\mathrm{hor} : T_qQ \to H_q$ and $\mathrm{ver} : T_qQ \to V_q$ are the horizontal and vertical projections obtained from the split $T_qQ = H_q \oplus V_q$.

If L is of the form kinetic minus potential energy, then $\mathfrak{A}$ can be chosen to be the *mechanical connection*, although we shall not insist on this choice. However, in this case one gets, for example, as in Marsden et al. [2000b], a *global diffeomorphism* $J_L^{-1}(\mu)/G \cong TS$.

Reduction on the Lagrangian Side. From the connection $\mathfrak{A}$ we obtain a 1-form $\mathfrak{A}_\mu$ on Q defined by

$$\mathfrak{A}_\mu(q)\dot{q} := \langle \mu, \mathfrak{A}(\dot{q}) \rangle.$$

The exterior derivative $\mathbf{d}\mathfrak{A}_\mu$ of $\mathfrak{A}_\mu$ is a 2-form on Q. It can be shown (see, for example, Marsden [1992] or Marsden et al. [2000b]) that $\mathbf{d}\mathfrak{A}_\mu$ is G-invariant and is zero on all vertical tangent vectors to Q. Thus, $\mathbf{d}\mathfrak{A}_\mu$ drops to a 2-form on S, which we shall call β_μ. It is often called the ***magnetic 2-form***.

If q is a curve that solves the Euler–Lagrange equations, then it is a solution of Hamilton's variational principle, which states that

$$\delta \int_a^b L(q, \dot{q})\, dt = 0,$$

for all variations δq of q that vanish at the endpoints. The curve x obtained by projecting this solution q onto the shape space also solves a variational principle on the shape space. This reduced variational principle has the form

$$\delta \int_a^b \hat{R}^\mu(x, \dot{x})\, dt = \int_a^b \beta_\mu(\dot{x}, \delta x)\, dt, \tag{2.2.2}$$

for all variations δx of x that vanish at the endpoints and for a function $\hat{R}^\mu$ that we shall now define.

To define the ***Routhian*** $\hat{R}^\mu$ on TS, we first define a function R^μ on TQ by

$$R^\mu(q,\dot{q}) = L(q,\dot{q}) - \mathfrak{A}_\mu(q)\dot{q},$$

where μ is the momentum of the solution q. The restriction of R^μ to $J_L^{-1}(\mu)$ is G-invariant and $\hat{R}^\mu$ is obtained by dropping $R^\mu|J_L^{-1}(\mu)$ to $J_L^{-1}(\mu)/G \approx TS$.

It is easy to check that the reduced variational principle above is equivalent to the equations

$$\frac{\partial \hat{R}^\mu}{\partial x} - \frac{d}{dt}\frac{\partial \hat{R}^\mu}{\partial \dot{x}} = \mathbf{i}_{\dot{x}}\beta_\mu(x), \tag{2.2.3}$$

where $\mathbf{i}_{\dot{x}}$ denotes interior product of the 2-form β_μ with the vector $\dot{x}$. We call Equation 2.2.3 the ***Routh equations***.

Reduction on the Hamiltonian Side. If the group G is abelian (which is what we have assumed), then from equivariance of the momentum map, we see that G acts on the momentum level set $J^{-1}(\mu) \subset T^*Q$. The quotient $J^{-1}(\mu)/G$ can be identified with T^*S. The projection $J^{-1}(\mu) \to T^*S$ called π_μ and can be defined as follows: If $\alpha_q \in J^{-1}(\mu)$, then the ***momentum shift*** $\alpha_q - \mathfrak{A}_\mu(q)$ annihilates all vertical tangent vectors at $q \in Q$, as shown by the following calculation:

$$\langle \alpha_q - \mathfrak{A}_\mu(q), \xi_Q(q)\rangle = J(\alpha_q)\cdot\xi - \langle \mu, \xi\rangle = \langle \mu,\xi\rangle - \langle\mu,\xi\rangle = 0.$$

Thus, $\alpha_q - \mathfrak{A}_\mu(q)$ induces an element of T_x^*S and $\pi_\mu(\alpha_q)$ is defined to be this element.

By Noether's theorem, the flow of the Hamiltonian vector field X_H leaves the set $J^{-1}(\mu)$ invariant and is equivariant, and so the restricted flow induces a reduced flow on T^*S. This reduced flow corresponds to a reduced Hamiltonian vector X_{H_μ} on T^*S, which can be obtained from a reduced Hamiltonian H_μ and a reduced symplectic form Ω_μ. The reduced energy at momentum level μ is denoted H_μ and is obtained by restricting H to $J^{-1}(\mu)$ and then, using its invariance, to drop it to a function on T^*S. Similarly, we get the reduced symplectic form Ω_μ by restricting Ω_Q to $J^{-1}(\mu)$ and then dropping to T^*S; namely, the reduced symplectic structure Ω_μ is related to Ω_Q by the equation

$$\pi_\mu^*\Omega_\mu = i_\mu^*\Omega_Q,$$

and is preserved by the reduced flow. An important result for cotangent bundles is that $\Omega_\mu = \Omega_S - \pi^*_{T^*S,S}\beta_\mu$, where Ω_S is the canonical symplectic form on T^*S, and $\pi_{T^*S,S} : T^*S \to S$ is the natural projection. See, for example, Marsden [1992] for the proof.

Relating Lagrangian and Hamiltonian Reduction. The projections $\pi_{\mu,L} : J_L^{-1}(\mu) \to TS$ and $\pi_\mu : J^{-1}(\mu) \to T^*S$ are related by the equation,

$$\pi_\mu \circ \mathbb{F}L = \mathbb{F}\hat{R}^\mu \circ \pi_{\mu,L},$$

where $\mathbb{F}\hat{R}^\mu : TS \to T^*S$ is the reduced Routh-Legendre transform and is defined by

$$\mathbb{F}\hat{R}^\mu : (x, \dot{x}) \mapsto \left(x, \frac{\partial \hat{R}^\mu}{\partial \dot{x}} \right).$$

Notice that the Routhian $\hat{R}^\mu$ has the momentum shift built into it as does the projection π_μ. It readily follows that the reduced dynamics on TS and on T^*S, given by the Routh equations and the vector field X_{H_μ}, respectively, are also related by the reduced Legendre transform $\mathbb{F}\hat{R}^\mu$. Thus, the relationships between the reduced and "unreduced" spaces and the reduced and unreduced dynamics can be depicted in the following commutative diagram:

$$\begin{array}{ccc} (J_L^{-1}(\mu), EL) & \xrightarrow{\mathbb{F}L} & (J^{-1}(\mu), X_H) \\ {\scriptstyle \pi_{\mu,L}} \downarrow & & \downarrow {\scriptstyle \pi_\mu} \\ (TS, R) & \xrightarrow{\mathbb{F}\hat{R}^\mu} & (T^*S, X_{H_\mu}) \end{array}$$

From the commutativity of this diagram, one sees that conservation of the symplectic 2-form $(\mathbb{F}\hat{R}^\mu)^*(\Omega_S - \pi^*_{T^*S,S}\beta_\mu)$ by the flow of the Routh equations follows from the conservation of the 2-form $\Omega_S - \pi^*_{T^*S,S}\beta_\mu$ by the flow of the reduced Hamiltonian vector field. Conservation of $(\mathbb{F}\hat{R}^\mu)^*(\Omega_S - \pi^*_{T^*S,S}\beta_\mu)$ can also be shown directly from the reduced variational principle (Equation 2.2.2).

Reconstruction. There is a general theory of reconstruction for both the Hamiltonian and Lagrangian sides of reduction. The problem is this: given an integral curve in the reduced space TS or T^*S, a value of μ and an initial condition in the μ-level set of the momentum map, how does one reconstruct the integral curve through that initial condition in TQ or T^*Q? This question involves the theory of geometric phases and of course is closely related to the classical constructions of solutions by quadratures given a set of integrals of motion. This is not a trivial procedure, even for abelian symmetry groups, although in this case things are somewhat more explicit. This procedure is discussed at length in, for example, Marsden et al. [1990], Marsden [1992] and Marsden et al. [2000b]. We shall need this theory at a couple of points in what follows.

2.3 Discrete Reduction

2.3.1 Discrete Variational Mechanics

In this paper, we will be using the theory of discrete mechanics as described in Marsden and West [2001]. In this subsection, we briefly describe the essential features of this theory and fix our notation.

Discrete Lagrangians. Given a configuration manifold Q, a discrete Lagrangian system consists of the ***discrete phase space*** $Q \times Q$ and a ***discrete Lagrangian*** $L_d : Q \times Q \to \mathbb{R}$. As we are interested in discrete systems which approximate a given continuous system, we will take discrete Lagrangians which depend on a ***timestep*** h, so that $L_d : Q \times Q \times \mathbb{R} \to \mathbb{R}$ should be thought of as approximating the action for time h,

$$L_d(q_0, q_1, h) \approx \int_0^h L(q(t), \dot{q}(t))\, dt, \tag{2.3.1}$$

where $q : [0, h] \to Q$ is a continuous trajectory solving the Euler–Lagrange equations for L with boundary conditions $q(0) = q_0$ and $q(h) = q_1$. When the timestep is fixed in a discussion, we often neglect the timestep dependence in L_d and write $L_d(q_0, q_1)$ for simplicity.

Discrete Euler–Lagrange Equations. Just as continuous trajectories are maps from $[0, T]$ to Q, we consider ***discrete trajectories***, which are maps from $\{0, h, 2h, \ldots, Nh = T\}$ to Q. This gives a set of points in Q which we denote $\mathbf{q} = \{q_k\}_{k=0}^N$.

Having defined a discrete Lagrangian, we define the ***discrete action*** to be a function mapping discrete trajectories $\mathbf{q} = \{q_k\}_{k=0}^N$ to the reals, given by

$$\mathfrak{G}_d(\mathbf{q}) = \sum_{k=0}^{N-1} L_d(q_k, q_{k+1}). \tag{2.3.2}$$

Hamilton's principle requires that the discrete action be stationary with respect to variations vanishing at $k = 0$ and $k = N$. Computing the variations gives

$$\begin{aligned}
\mathbf{d}\mathfrak{G}_d(\mathbf{q}) \cdot \delta\mathbf{q} &= \sum_{k=0}^{N-1} \left[D_1 L_d(q_k, q_{k+1}) \cdot \delta q_k + D_2 L_d(q_k, q_{k+1}) \cdot \delta q_{k+1} \right] \\
&= \sum_{k=1}^{N-1} \left[D_2 L_d(q_{k-1}, q_k) + D_1 L_d(q_k, q_{k+1}) \right] \cdot \delta q_k \\
&\qquad + D_1 L_d(q_0, q_1) \cdot \delta q_0 + D_2 L_d(q_{N-1}, q_N) \cdot \delta q_N.
\end{aligned}$$

The requirement that this be zero for all variations satisfying $\delta q_0 = \delta q_N = 0$ gives the ***discrete***

Euler–Lagrange (DEL) equations,

$$D_2L_d(q_{k-1}, q_k) + D_1L_d(q_k, q_{k+1}) = 0, \tag{2.3.3}$$

for each $k = 1, \ldots, N-1$. These implicitly define the ***discrete Lagrange map***, $F_{L_d} : Q\times Q \to Q\times Q$; $(q_{k-1}, q_k) \mapsto (q_k, q_{k+1})$. We also refer to this map as the ***discrete Lagrangian evolution operator***.

Discrete Lagrange Forms. The boundary terms in the expression for $\mathbf{d}\mathfrak{G}_d$ can be identified as the two ***discrete Lagrange*** 1***-forms*** on $Q \times Q$, which are

$$\Theta^+_{L_d}(q_0, q_1) = D_2L_d(q_0, q_1)\mathbf{d}q_1, \tag{2.3.4a}$$

$$\Theta^-_{L_d}(q_0, q_1) = -D_1L_d(q_0, q_1)\mathbf{d}q_0. \tag{2.3.4b}$$

In coordinates, note that

$$\Theta^+_{L_d} = \frac{\partial L_d}{\partial q^i_1} dq^i_1. \tag{2.3.5}$$

We define the ***discrete Lagrange*** 2***-form*** on $Q \times Q$ to be

$$\Omega_{L_d} = -\mathbf{d}\Theta^+_{L_d}, \tag{2.3.6}$$

which in coordinates is

$$\Omega_{L_d} = -\mathbf{d}\left(\frac{\partial L_d}{\partial q^i_1} dq^i_1\right) = \frac{\partial^2 L_d}{\partial q^i_1 \partial q^j_2} dq^i_1 \wedge dq^j_2. \tag{2.3.7}$$

A straightforward calculation shows that

$$\Omega_{L_d} = -\mathbf{d}\Theta^-_{L_d}. \tag{2.3.8}$$

The space of solutions of the discrete Euler–Lagrange equations can be identified with the space $Q\times Q$ of initial conditions (q_0, q_1). Restricting the free variations of the discrete action to this space shows that we have

$$\mathbf{d}\mathfrak{G}_d|_{Q\times Q} = -\Theta^-_{L_d} + (F^{N-1}_{L_d})^*(\Theta^+_{L_d}),$$

and so taking a second derivative and using the fact that $\mathbf{d}^2 = 0$ shows that

$$(F^{N-1}_{L_d})^*\Omega_{L_d} = \Omega_{L_d}.$$

In particular, taking $N = 2$, we see that *the discrete Lagrange evolution operator is symplectic*; that is,

$$(F_{L_d})^* \Omega_{L_d} = \Omega_{L_d}. \tag{2.3.9}$$

Discrete Legendre Transforms. Given a discrete Lagrangian we define the ***discrete Legendre transforms***, $\mathbb{F}^+ L_d, \mathbb{F}^- L_d : Q \times Q \to T^*Q$, by

$$\mathbb{F}^- L_d(q_0, q_1) = (q_0, -D_1 L_d(q_0, q_1)), \tag{2.3.10a}$$

$$\mathbb{F}^+ L_d(q_0, q_1) = (q_1, D_2 L_d(q_0, q_1)), \tag{2.3.10b}$$

and we observe that the discrete Lagrange 1- and 2-forms are related to the canonical 1- and 2-forms on T^*Q by pull-back under the discrete Legendre transforms; that is, $\Theta^{\pm}_{L_d} = (\mathbb{F}^{\pm} L_d)^*(\Theta)$ and $\Omega_{L_d} = (\mathbb{F}^{\pm} L_d)^*(\Omega)$.

Pushing the discrete Lagrange map forward to T^*Q with the discrete Legendre transform gives the ***push-forward discrete Lagrange map***, $\tilde{F}_{L_d} : T^*Q \to T^*Q$ by $\tilde{F}_{L_d} = \mathbb{F}^{\pm} L_d \circ F_{L_d} \circ (\mathbb{F}^{\pm} L_d)^{-1}$. One checks that one has the same map for the + case and the − case. In fact, the expression for the push-forward discrete Lagrange map can be seen to be determined as follows: $\tilde{F}_{L_d} : (q_0, p_0) \mapsto (q_1, p_1)$, where

$$p_0 = -D_1 L_d(q_0, q_1), \tag{2.3.11a}$$

$$p_1 = D_2 L_d(q_0, q_1). \tag{2.3.11b}$$

Note that by construction, the push-forward discrete Lagrange map preserves the canonical 2-form. The push-forward discrete Lagrange map is thus symplectic; that is, $(\tilde{F}_{L_d})^*(\Omega) = \Omega$.

Exact Discrete Lagrangians. The relationship between a discrete Lagrangian and the corresponding push-forward discrete Lagrange map is that of ***generating functions*** of the first kind. Generating function theory shows that for any symplectic map $T^*Q \to T^*Q$ (at least those near the identity), there is a corresponding generating function $Q \times Q \to \mathbb{R}$ which generates the map in the sense of Equation 2.3.11.

It is thus clear that there is a discrete Lagrangian for every symplectic map, including the exact flow $F^t_H : T^*Q \to T^*Q$ of the Hamiltonian system corresponding to the Lagrangian L. This is referred to as the ***exact discrete Lagrangian*** and Hamilton–Jacobi theory shows that it is equal to the action,

$$L^E_d(q_0, q_1, h) = \int_0^h L(q(t), \dot{q}(t))\, dt, \tag{2.3.12}$$

where $q : [0, h] \to Q$ solves the Euler–Lagrange equations for L with $q(0) = q_0$ and $q(h) = q_1$. This classical theorem of Jacobi is proved in, for example, Marsden and Ratiu [1999].

Using this exact discrete Lagrangian, the push-forward discrete Lagrange map will be exactly the Hamiltonian flow map for time h, so that $\tilde{F}_{L_d^E} = F_H^h$. That is, discrete trajectories $\mathbf{q} = \{q_k\}_{k=0}^N$ will exactly sample continuous trajectories $q(t)$, namely $q_k = q(kh)$.

Approximate Discrete Lagrangians. If we choose a discrete Lagrangian which only approximates the action, then the resulting push-forward discrete Lagrange map will only approximate the true flow. The orders of approximation are related, so that if the discrete Lagrangian is of order r,

$$L_d = \int_0^h L(q, \dot{q})\, dt, + \mathcal{O}(h^{r+1}), \tag{2.3.13}$$

then the push-forward discrete Lagrange map will also be of order r; that is,

$$\tilde{F}_{L_d}^h = F_H^h + \mathcal{O}(h^{r+1}). \tag{2.3.14}$$

By choosing discrete Lagrangians which are at least consistent ($r \geq 1$) we can regard the discrete Lagrange map as an ***integrator*** for the continuous system.

2.3.2 Discrete Mechanical Systems with Symmetry

Let G be an abelian Lie group that acts freely and properly on the configuration manifold Q. We will assume that our discrete Lagrangian L_d is invariant under the diagonal action of G on $Q \times Q$. Such a discrete Lagrangian could have been obtained by discretizing a continuous Lagrangian L that is invariant under the lifted action of G on TQ. For a discussion of how to construct G-invariant discrete Lagrangians from G-invariant Lagrangians using natural charts, please see §5.3.2.

Note that Q is a bundle over the shape space $S = Q/G$. Using a local trivialization, Q can be locally identified with $G \times S$. Thus $Q \times Q \approx G \times S \times G \times S$. With this identification, $(q_k, q_{k+1}) = (h_k, x_k, h_{k+1}, x_{k+1})$. We will use $\partial/\partial g_i$, $i = 1, 2$, to denote partial derivatives with respect to the first and second group variables, and $\partial/\partial s_i$, $i = 1, 2$, to denote partial derivatives with respect to the first and second shape space variables.

Given a discrete Lagrangian, the ***discrete momentum map***, $J_d : Q \times Q \to \mathfrak{g}^*$, is defined by

$$J_d(q_0, q_1) \cdot \xi = D_2 L_d(q_0, q_1) \cdot \xi_Q(q_1). \tag{2.3.15}$$

Since L_d is invariant under the action of G, we have

$$D_1 L_d(q_0, q_1) \cdot \xi_Q(q_0) + D_2 L_d(q_0, q_1) \cdot \xi_Q(q_1) = 0.$$

Thus,

$$\begin{aligned} J_d(q_0, q_1) \cdot \xi &= D_2 L_d(q_0, q_1) \cdot \xi_Q(q_1) = -D_1 L_d(q_0, q_1) \cdot \xi_Q(q_0) \\ &= \xi_Q(q_1) \;\lrcorner\; \Theta^+_{L_d}(q_0, q_1) = \xi_Q(q_0) \;\lrcorner\; \Theta^-_{L_d}(q_0, q_1), \end{aligned}$$

where $X \;\lrcorner\; \omega$ denotes the interior product of a vector X with a 1-form ω. Thus, if $\{q_0, q_1, q_2, \ldots\}$ solves the DEL equations, then

$$\begin{aligned} J_d(q_1, q_2) \cdot \xi &= D_2 L_d(q_1, q_2) \cdot \xi_Q(q_2) = -D_1 L_d(q_1, q_2) \cdot \xi_Q(q_1) \\ &= D_2 L_d(q_0, q_1) \cdot \xi_Q(q_1) = J_d(q_0, q_1) \cdot \xi. \end{aligned}$$

Thus, the discrete momentum is conserved by the discrete Lagrange map, $F_{L_d} : Q \times Q \to Q \times Q$, $F_{L_d} : (q_0, q_1) \mapsto (q_1, q_2)$. In other words, the discrete momentum is conserved along solutions of the DEL equations, which is referred to as the ***discrete Noether theorem***.

By definition of J_d,

$$J_d(q_0, q_1) \cdot \xi = J(D_2 L_d(q_0, q_1)) \cdot \xi,$$

where $J : T^*Q \to \mathfrak{g}^*$ is the momentum map on T^*Q. Thus,

$$J_d = J \circ \mathbb{F}L_d,$$

where $\mathbb{F}L_d = D_2 L_d : Q \times Q \to T^*Q$ is the discrete Legendre transform. (Note that in §2.3.1 we had two discrete Legendre transforms, $\mathbb{F}^+ L_d$ and $\mathbb{F}^- L_d$. For the remainder of this paper, we use the term discrete Legendre transform and the symbol $\mathbb{F}L_d$ to denote $\mathbb{F}^+ L_d$ to make a specific choice.) Thus, $\mathbb{F}L_d$ maps $J_d^{-1}(\mu)$, which is the μ-level set of the discrete momentum to $J^{-1}(\mu)$. Also, since J_d is conserved by the discrete evolution operator F_{L_d}, it follows that the push-forward discrete Lagrange map $\tilde{F}_{L_d} : T^*Q \to T^*Q$ preserves J.

In a local trivialization, where $q_1 = (\theta_1, x_1)$,

$$\xi_Q(q_1) = \frac{d}{dt}\bigg|_{t=0} (\exp(t\xi) \cdot \theta_1, x_1) = (TR_{\theta_1} \cdot \xi, 0),$$

where R_{θ_1} denotes right multiplication on G by θ_1. Thus,

$$J_d(q_0, q_1) \cdot \xi = \begin{bmatrix} \frac{\partial L_d}{\partial g_2} & \frac{\partial L_d}{\partial s_2} \end{bmatrix} \cdot \begin{bmatrix} TR_{\theta_1} \cdot \xi \\ \\ 0 \end{bmatrix} = \frac{\partial L_d}{\partial g_2} \circ TR_{\theta_1} \cdot \xi.$$

Hence,

$$J_d(q_0, q_1) = \frac{\partial L_d}{\partial g_2}(\theta_0, x_0, \theta_1, x_1) \circ TR_{\theta_1}.$$

The momentum map, $J_d : Q \times Q \to \mathfrak{g}^*$, is ***equivariant*** as the following calculation shows:

$$\begin{aligned} J_d(\theta \cdot q_0, \theta \cdot q_1) \cdot \xi &= D_1 L_d(\theta \cdot q_0, \theta \cdot q_1) \cdot \xi_Q(\theta \cdot q_0) \\ &= D_1 L_d(\theta \cdot q_0, \theta \cdot q_1) \cdot \theta \cdot (\mathrm{Ad}_{\theta^{-1}} \xi)_Q(q_0) \\ &= D_1 L_d(q_0, q_1) \cdot (\mathrm{Ad}_{\theta^{-1}} \xi)_Q(q_0) \\ &= J_d(q_0, q_1) \circ \mathrm{Ad}_{\theta^{-1}} \cdot \xi \\ &= (\mathrm{Ad}^*_{\theta^{-1}} J_d(q_0, q_1)) \cdot \xi. \end{aligned}$$

Thus, the coadjoint isotropy subgroup G_μ of G acts on $J_d^{-1}(\mu)$. Since G is abelian, $G_\mu = G$, and thus G acts on $J_d^{-1}(\mu)$.

If the value of the momentum is μ, the equation

$$\frac{\partial L_d}{\partial g_2}(\theta_0, x_0, \theta_1, x_1) \circ TR_{\theta_1} = \mu,$$

determines θ_1 implicitly as a function of θ_0, x_0, x_1 and μ. Thus the level set $J_d^{-1}(\mu)$ can be (locally) identified with $G \times S \times S$. The quotient $J_d^{-1}(\mu)/G$ is thus locally diffeomorphic to $S \times S$.

If we choose a momentum μ, it follows from the above discussion that there is a unique map $\psi_\mu : S \times S \to G$, such that,

$$J_d(e, x_k, \psi_\mu(x_k, x_{k+1}), x_{k+1}) = \mu.$$

Further, if $\theta_k \in G$, $\theta_k \cdot (e, x_k, \psi_\mu(x_k, x_{k+1}), x_{k+1}) = (\theta_k, x_k, \theta_k \cdot \psi_\mu(x_k, x_{k+1}), x_{k+1})$ is also in $J_d^{-1}(\mu)$. Thus for a given μ, the function giving θ_{k+1} in terms of θ_k, x_k and x_{k+1} is

$$\theta_{k+1} = \theta_k \cdot \psi_\mu(x_k, x_{k+1}). \tag{2.3.16}$$

Reconstruction. The following lemma gives a basic result on the reconstruction of discrete curves in the configuration manifold Q from those in the shape space S. The lemma is similar to its continuous counterpart, as in Lemma 2.2 of Jalnapurkar and Marsden [2000]. Recall that V_q denotes the vertical space at q, which is the space of all vectors at q that are infinitesimal generators $\xi_Q(q) \in T_qQ$. We say that the discrete Lagrangian L_d is ***group-regular*** if the bilinear map $D_2D_1L_d(q,q) : T_qQ \times T_qQ \to \mathbb{R}$ restricted to the subspace $V_q \times V_q$ is nondegenerate. Besides regularity, we shall make group-regularity a standing assumption in this chapter as well.

Lemma 2.1 (Reconstruction Lemma). *Let $\mu \in \mathfrak{g}^*$ be given, and $\mathbf{x} = \{x_0, \ldots, x_n\}$ be a sufficiently closely spaced discrete curve in S. Let $q_0 \in Q$ be such that $\pi_{Q,S}(q_0) = x_0$. Then, there is a unique closely spaced discrete curve $\mathbf{q} = \{q_0, \ldots, q_n\}$ such that $\pi_{Q,S}(q_k) = x_k$ and $J_d(q_k, q_{k+1}) = \mu$, for $k = 0, \ldots, n-1$.*

Proof. We must construct a point q_1 close to q_0 such that $\pi_{Q,S}(q_1) = x_1$ and $J_d(q_0, q_1) = \mu$. The construction of the subsequent points $q_2, \ldots, q_n$ will follow in an inductive fashion.

To do this, pick a local trivialization of the bundle $\pi_{Q,S} : Q \to Q/G$, where $Q \approx G \times S$ locally, and write points in this trivialization as $q_k = (\theta_k, x_k)$.

Given the point $q_0 = (\theta_0, x_0)$, we seek a near identity group element g, such that $q_1 := (g\theta_0, x_1)$ satisfies $J_d(q_0, q_1) = \mu$. By the definition of the discrete momentum map (Equation 2.3.15), this means that we must satisfy the condition

$$D_2L_d(q_0, q_1) \cdot \xi_Q(q_1) = \langle \mu, \xi \rangle$$

for all $\xi \in \mathfrak{g}$. In the local trivialization, this means that

$$D_2L_d((\theta_0, x_0), (g\theta_0, x_1)) \cdot (TR_{g\theta_0}\xi, 0) = \langle \mu, \xi \rangle ,$$

where R_g denotes right translation on the group by the element g.

Consider solving the above equation for $\theta_1 = g\theta_0$ as a function of θ_0, x_0, x_1, with μ fixed. We know the base solution corresponding to the case $x_1 = x_0$, namely $g = e$. The implicit function theorem tells us that when x_1 is moved away from x_0, there will be a unique solution for g near the identity, provided that the derivative of the defining relation with respect to g at the identity is invertible. But this condition is a consequence of group-regularity, so the result follows. □

Note that the above lemma makes no hypotheses about the sequence satisfying the discrete Euler–Lagrange equations.

To obtain the reconstruction equation in the continuous case, we require that the lifted curve is second-order, on the momentum surface, and that it projects down to the reduced curve. It is appropriate to consider the discrete analogue of the second-order curve condition, since it may not be apparent where we imposed such a condition.

We consider a discrete curve $\mathbf{x}$ as a sequence of points, $(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n)$ in $S \times S$. Lift each of the points in $S \times S$ to the momentum surface $J_d^{-1}(\mu) \subset Q \times Q$. This yields the sequence, $(q_0^0, q_1^0), (q_0^1, q_1^1), \ldots, (q_0^{n-1}, q_1^{n-1})$, which is unique up to a diagonal group action on $Q \times Q$. The discrete analogue of the second-order curve condition is that this sequence in $Q \times Q$ defines a discrete curve in Q, which corresponds to requiring that $q_1^k = q_0^{k+1}$, for $k = 0, \ldots, n-1$, which is clearly possible in the context of the reconstruction lemma.

This discussion of the discrete reconstruction equation naturally leads to issues of geometric phases, and it would be interesting to obtain an expression for the discrete geometric phase in terms of the discrete curve on shape space.

Reconstruction of Tangent Vectors. Let (q_0, q_1) be a lift of (x_0, x_1) to $J_d^{-1}(\mu)$, and $(\delta x_0, \delta x_1)$ be a tangent vector to $S \times S$ at (x_0, x_1). Given $\delta q_0 \in T_{q_0} Q$, with $T\pi_{Q,S} \cdot \delta q_0 = \delta x_0$, it is possible to find a $\delta q_1 \in T_{q_1} Q$, with $T\pi_{Q,S} \cdot \delta q_1 = \delta x_1$, such that $(\delta q_0, \delta q_1)$ is a tangent vector to $J_d^{-1}(\mu)$ at (q_0, q_1). Indeed, if in a local trivialization, $\delta q_0 = (\delta\theta_0, \delta x_0)$, then the required δq_1 is $(\delta\theta_1, \delta x_1)$, where $\delta\theta_1$ is obtained by differentiating Equation 2.3.16 as follows:

$$\delta\theta_1 = \delta\theta_0 \cdot \psi_\mu(x_0, x_1) + \theta_0 \cdot D_1\psi_\mu(x_0, x_1)\delta x_0 + \theta_0 \cdot D_2\psi_\mu(x_0, x_1)\delta x_1.$$

Discrete Connection. It should be noted that although our discussion of reconstruction is cast in terms of local trivializations, it is in fact intrinsic and can be thought of as a discrete horizontal lift in the sense of discrete connections developed in Chapter 4. The discrete connection associated with the reconstruction to the discrete μ-momentum surface is represented by the discrete connection 1-form $\mathcal{A}_d : Q \times Q \to G$, defined on a G-invariant neighborhood of the diagonal by $\mathcal{A}_d(q_0, q_1) = e$ iff $J_d(q_0, q_1) = \mu$, and extended to other points by

$$\mathcal{A}_d(g_0 q_0, g_1 q_1) = g_1 g_0^{-1}.$$

The reconstruction lemma (Lemma 2.1) may be viewed as providing the horizontal lift of this discrete connection.

The discrete connection given above is the natural choice of connection on $Q \times Q$ for the purpose of constructing a unified formulation of discrete, Lagrangian, and Hamiltonian reduction. Recall the following diagram,

$$\begin{array}{ccc} (TQ, J_L) & \xrightarrow{\mathbb{F}L} & (T^*Q, J) \\ & & \uparrow{\scriptstyle \mathbb{F}L_d} \\ & & (Q \times Q, J_d) \end{array}$$

and consider the horizontal space on TQ given by the μ-momentum surface, $J_L^{-1}(\mu)$. Since $J_L = (\mathbb{F}L)^*J$, and $J_d = (\mathbb{F}L_d)^*J$, it follows that $(\mathbb{F}L)_* J_L^{-1}(\mu) = (\mathbb{F}L_d)_* J_d^{-1}(\mu)$, and as a consequence, $(\mathbb{F}L_d)^*(\mathbb{F}L)_* J_L^{-1}(\mu) = J_d^{-1}(\mu)$. This implies that the discrete reconstruction equation is simply the horizontal lift with respect to the discrete connection on $Q \times Q$ that is consistent with the connection on TQ with respect to the fiber derivatives $\mathbb{F}L$ and $\mathbb{F}L_d$, and is therefore an intrinsic operation. The discrete connection obtained in this way is related to the discrete mechanical connection, and is given by the discrete connection 1-form introduced above.

Discrete connections also yield a semi-global isomorphism $(Q \times Q)/G \cong (S \times S) \oplus \tilde{G}$ (see §4.4.8) for neighborhoods of the diagonal, and this induces a semi-global isomorphism $J_d^{-1}(\mu)/G \cong S \times S$, which is a discrete analogue of the global diffeomorphism, $J_L^{-1}(\mu)/G \cong TS$, that was obtained in Marsden et al. [2000b] with the use of the mechanical connection.

2.3.3 Discrete Reduction

In this section, we start by assuming that we have been given a discrete Lagrangian, $L_d : Q \times Q \to \mathbb{R}$, that is invariant under the action of an abelian Lie group G on $Q \times Q$.

Let $\mathbf{q} := \{q_0, \ldots, q_n\}$ be a solution of the discrete Euler–Lagrange (DEL) equations. Let the value of the discrete momentum along this trajectory be μ. Let $x_i = \pi_{Q,S}(q_i)$, so that $\mathbf{x} := \{x_0, \ldots, x_n\}$ is a discrete trajectory on shape space. Since $\mathbf{q}$ satisfies the discrete variational principle, it is appropriate to ask if there is a reduced variational principle satisfied by $\mathbf{x}$.

An important issue in dropping the discrete variational principle to the shape space is whether we require that the varied curves are constrained to lie on the level set of the momentum map. The constrained approach is adopted in Jalnapurkar and Marsden [2000], and the unconstrained approach is used in Marsden et al. [2000b]. In the rest of this section, we will adopt the unconstrained approach of Marsden et al. [2000b], and will show that the variations in the discrete action sum evaluated at a solution of the discrete Euler–Lagrange equation depends only on the quotient variations, and therefore drops to the shape space without constraints on the variations.

By G-invariance of L_d, the restriction of L_d to $J_d^{-1}(\mu)$ drops to the quotient $J_d^{-1}(\mu)/G \approx S \times S$. The function obtained on the quotient is called the reduced Lagrangian and is denoted $\hat{L}_d$. Let $\pi_{\mu,d} : J_d^{-1}(\mu) \to S \times S$ be the projection. Let $(q_0, q_1) \in J_d^{-1}(\mu)$, and $(\delta q_0, \delta q_1) \in T_{(q_0,q_1)} J_d^{-1}(\mu)$. If $\pi_{Q,S} \cdot q_i = x_i$ and $T\pi_{Q,S} \cdot \delta q_i = \delta x_i$, $i = 0, 1$, then $\pi_{\mu,d}(q_0, q_1) = (x_0, x_1)$ and $T\pi_{\mu,d} \cdot (\delta q_0, \delta q_1) = (\delta x_0, \delta x_1)$. In this situation, we get $L_d(q_0, q_1) = \hat{L}_d(x_0, x_1)$, and so

$$DL_d(q_0, q_1) \cdot (\delta q_0, \delta q_1) = D\hat{L}_d(x_0, x_1) \cdot (\delta x_0, \delta x_1). \tag{2.3.17}$$

For $\mathbf{q}$ a solution of the DEL equations, and $\mathbf{x}$ the corresponding curve on the shape space

S, let $\delta\mathbf{x} = \frac{d}{d\varepsilon}\big|_{\varepsilon=0}\mathbf{x}_\varepsilon$ be a variation of $\mathbf{x}$. Let $\delta\mathbf{q} = \frac{d}{d\varepsilon}\big|_{\varepsilon=0}\mathbf{q}_\varepsilon$ be any variation of $\mathbf{q}$ such that $T\pi_{Q,S} \cdot \delta q_i = \delta x_i$. Then,

$$\begin{aligned}
\delta \sum_{k=0}^{n-1} L_d(q_k, q_{k+1}) &= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \sum_k L_d(q_{k\varepsilon}, q_{k+1\varepsilon}) \\
&= \sum_k DL_d(q_k, q_{k+1}) \cdot (\delta q_k, \delta q_{k+1}) \\
&= D_1 L_d(q_0, q_1) \cdot \delta q_0 \\
&\quad + \sum_{k=1}^{n-1} (D_2 L_d(q_{k-1}, q_k) + D_1 L_d(q_k, q_{k+1})) \cdot \delta q_k \\
&\quad + D_2 L_d(q_{n-1}, q_n) \cdot \delta q_n \\
&= D_1 L_d(q_0, q_1) \cdot \delta q_0 + D_2 L_d(q_{n-1}, q_n) \cdot \delta q_n,
\end{aligned} \tag{2.3.18}$$

where we have used the fact that $\mathbf{q}$ satisfies the discrete Euler–Lagrange equations.

Recall that the discrete momentum map is given by

$$J_d(q_k, q_{k+1}) \cdot \xi = D_2 L_d(q_k, q_{k+1}) \cdot \xi_Q(q_{k+1}) = -D_1 L_d(q_k, q_{k+1}) \cdot \xi_Q(q_k).$$

Given any connection $\mathfrak{A}$ on Q, we have a horizontal-vertical split of each tangent space to Q. Thus,

$$D_2 L_d(q_{n-1}, q_n) \cdot \delta q_n = D_2 L_d(q_{n-1}, q_n) \cdot \text{hor } \delta q_n + D_2 L_d(q_{n-1}, q_n) \cdot \text{ver } \delta q_n.$$

Now, ver $\delta q_n = \xi_Q(q_n)$, where $\xi = \mathfrak{A}(\delta q_n)$. Thus, $\mathfrak{A}(\delta q_n) = \mathfrak{A}(\text{ver } \delta q_n) = \mathfrak{A}(\xi_Q(q_n))$. So,

$$\begin{aligned}
D_2 L_d(q_{n-1}, q_n) \cdot \text{ver } \delta q_n &= D_2 L_d(q_{n-1}, q_n) \cdot \xi_Q(q_n) \\
&= J_d(q_{n-1}, q_n) \cdot \xi = \langle \mu, \xi \rangle = \langle \mu, \mathfrak{A}(\xi_Q(q_n)) \rangle \\
&= \langle \mu, \mathfrak{A}(\delta q_n) \rangle = \mathfrak{A}_\mu(q_n) \cdot \delta q_n.
\end{aligned} \tag{2.3.19}$$

Thus,

$$D_2 L_d(q_{n-1}, q_n) \cdot \delta q_n = D_2 L_d(q_{n-1}, q_n) \cdot \text{hor } \delta q_n + \mathfrak{A}_\mu(q_n) \cdot \delta q_n. \tag{2.3.20}$$

Similarly,

$$D_1 L_d(q_0, q_1) \cdot \delta q_0 = D_1 L_d(q_0, q_1) \cdot \text{hor } \delta q_0 - \mathfrak{A}_\mu(q_0) \cdot \delta q_0. \tag{2.3.21}$$

Thus, from Equation 2.3.18,

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \sum_k L_d(q_{k_\varepsilon}, q_{k+1_\varepsilon}) = D_1 L_d(q_0, q_1) \cdot \text{hor } \delta q_0 + D_2 L_d(q_{n-1}, q_n) \cdot \text{hor } \delta q_n + \mathfrak{A}_\mu(q_n) \cdot \delta q_n - \mathfrak{A}_\mu(q_0) \cdot \delta q_0. \tag{2.3.22}$$

Define a 1-form $\mathcal{A}$ on $Q \times Q$ by

$$\mathcal{A}(q_0, q_1)(\delta q_0, \delta q_1) = \mathfrak{A}_\mu(q_1) \cdot \delta q_1 - \mathfrak{A}_\mu(q_0) \cdot \delta q_0. \tag{2.3.23}$$

If $\pi_1, \pi_2 : Q \times Q \to Q$ are projections onto the first and the second components, respectively. Then,

$$\mathcal{A} = \pi_2^* \mathfrak{A}_\mu - \pi_1^* \mathfrak{A}_\mu.$$

Using G-invariance of $\mathfrak{A}_\mu$, it follows that $\mathcal{A}$ is G-invariant. Also,

$$\mathcal{A}(q_0, q_1)(\xi_Q(q_0), \xi_Q(q_1)) = \mathfrak{A}_\mu(q_1) \cdot \xi_Q(q_1) - \mathfrak{A}_\mu(q_0) \cdot \xi_Q(q_0) = \langle \mu, \xi \rangle - \langle \mu, \xi \rangle = 0.$$

Thus, $\mathcal{A}$ annihilates all vertical tangent vectors to $Q \times Q$. It is easy to check that the 1-form $\mathcal{A}|J_d^{-1}(\mu)$, obtained by restricting $\mathcal{A}$ to $J_d^{-1}(\mu)$ is also G-invariant and annihilates vertical tangent vectors to $J_d^{-1}(\mu)$. Therefore, $\mathcal{A}|J_d^{-1}(\mu)$ drops to a 1-form $\hat{\mathcal{A}}$ on $J_d^{-1}(\mu)/G \approx S \times S$.

If $\pi_{\mu,d} : J_d^{-1}(\mu) \to J_d^{-1}(\mu)/G$ is the projection, and $i_{\mu,d} : J_d^{-1}(\mu) \to Q \times Q$ is the inclusion, then $\hat{\mathcal{A}}$ and $\mathcal{A}$ are related by the equation

$$\pi_{\mu,d}^* \hat{\mathcal{A}} = i_{\mu,d}^* \mathcal{A}.$$

We define the 1-forms $\hat{\mathcal{A}}^+$ and $\hat{\mathcal{A}}^-$ on $S \times S$ and the maps $\hat{\mathcal{A}}_1, \hat{\mathcal{A}}_2 : S \times S \to T^*S$ by the relations

$$\hat{\mathcal{A}}^+(x_0, x_1) \cdot (\delta x_0, \delta x_1) = \hat{\mathcal{A}}_2(x_0, x_1) \cdot \delta x_1 = \hat{\mathcal{A}}(x_0, x_1) \cdot (0, \delta x_1),$$
$$\hat{\mathcal{A}}^-(x_0, x_1) \cdot (\delta x_0, \delta x_1) = \hat{\mathcal{A}}_1(x_0, x_1) \cdot \delta x_0 = \hat{\mathcal{A}}(x_0, x_1) \cdot (\delta x_0, 0).$$

Note that we have the relations $\hat{\mathcal{A}} = \hat{\mathcal{A}}^+ + \hat{\mathcal{A}}^-$, and

$$\hat{\mathcal{A}}(x_0, x_1) \cdot (\delta x_0, \delta x_1) = \hat{\mathcal{A}}_1(x_0, x_1) \cdot \delta x_0 + \hat{\mathcal{A}}_2(x_0, x_1) \cdot \delta x_1.$$

From Equation 2.3.23, it follows that,

$$\mathfrak{A}_\mu(q_n) \cdot \delta q_n - \mathfrak{A}_\mu(q_0) \cdot \delta q_0 = \sum_{k=0}^{n-1} \mathfrak{A}_\mu(q_{k+1}) \cdot \delta q_{k+1} - \mathfrak{A}_\mu(q_k) \cdot \delta q_k$$

$$= \sum_{k=0}^{n-1} \mathcal{A}(q_k, q_{k+1}) \cdot (\delta q_k, \delta q_{k+1}). \tag{2.3.24}$$

Thus, Equation 2.3.22 can be rewritten as

$$\begin{aligned} \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \sum_{k=0}^{n-1} L_d(q_{k_\varepsilon}, q_{k+1_\varepsilon}) =& D_1 L_d(q_0, q_1) \cdot \text{hor } \delta q_0 + D_2 L_d(q_{n-1}, q_n) \cdot \text{hor } \delta q_n \\ &+ \sum_{k=0}^{n-1} \mathcal{A}(q_k, q_{k+1}) \cdot (\delta q_k, \delta q_{k+1}), \end{aligned} \tag{2.3.25}$$

or equivalently,

$$\sum_{k=0}^{n-1} (DL_d - \mathcal{A})(q_k, q_{k+1}) \cdot (\delta q_k, \delta q_{k+1}) = D_1 L_d(q_0, q_1) \cdot \text{hor } \delta q_0 + D_2 L_d(q_{n-1}, q_n) \cdot \text{hor } \delta q_n. \tag{2.3.26}$$

The following lemma shows the sense in which the 1-form $(DL_d - \mathcal{A})$ on $Q \times Q$ drops to the quotient $J_d^{-1}(\mu)/G \approx S \times S$.

Lemma 2.2. *If* $(q_0, q_1) \in J_d^{-1}(\mu)$ *and* $(\delta q_0, \delta q_1) \in T_{(q_0,q_1)} Q \times Q$ *with* $\pi_{Q,S}(q_i) = x_i$ *and* $T\pi_{Q,S} \cdot \delta q_i = \delta x_i$, $i = 0, 1$, *then*

$$(DL_d - \mathcal{A})(q_0, q_1) \cdot (\delta q_0, \delta q_1) = (D\hat{L}_d - \hat{\mathcal{A}})(x_0, x_1) \cdot (\delta x_0, \delta x_1).$$

Proof. As we showed in the discussion at the end of §2.3.2, we can find $\delta q_1' \in T_{q_1} Q$ such that $T\pi_{Q,S} \cdot \delta q_1' = \delta x_1$ and $(\delta q_0, \delta q_1') \in T_{(q_0,q_1)} J_d^{-1}(\mu)$. Let $\delta q_1 = \delta q_1' + \delta q_1''$. Thus $\delta q_1'' \in T_{q_1} Q$ is vertical, i.e., $T\pi_{Q,S} \cdot \delta q_1'' = 0$. Now,

$$(DL_d - \mathcal{A})(q_0, q_1) \cdot (\delta q_0, \delta q_1) = (DL_d - \mathcal{A})(q_0, q_1) \cdot (\delta q_0, \delta q_1') + (DL_d - \mathcal{A})(q_0, q_1) \cdot (0, \delta q_1'').$$

Using Equation 2.3.17, and the fact that $\mathcal{A}|J_d^{-1}(\mu)$ drops to a 1-form $\hat{\mathcal{A}}$ on $S \times S$, we get

$$(DL_d - \mathcal{A})(q_0, q_1) \cdot (\delta q_0, \delta q_1') = (D\hat{L}_d - \hat{\mathcal{A}})(x_0, x_1) \cdot (\delta x_0, \delta x_1).$$

Also, by a calculation similar to that used to derive Equation 2.3.19, we have that

$$(DL_d - \mathcal{A})(q_0, q_1) \cdot (0, \delta q_1'') = D_2 L_d(q_0, q_1) \cdot \delta q_1'' - \mathfrak{A}_\mu(q_1) \delta q_1'' = 0. \qquad \square$$

With this lemma, and Equation 2.3.26, we conclude that

$$\sum_{k=0}^{n-1}(D\hat{L}_d - \hat{\mathcal{A}})(x_k, x_{k+1}) \cdot (\delta x_k, \delta x_{k+1}) = D_1 L_d(q_0, q_1) \cdot \text{hor } \delta q_0 + D_2 L_d(q_{n-1}, q_n) \cdot \text{hor } \delta q_n. \tag{2.3.27}$$

If $\delta \mathbf{x}$ is a variation of $\mathbf{x}$ that vanishes at the endpoints, then hor $\delta q_0 = 0$, and hor $\delta q_1 = 0$. Therefore,

$$\sum_{k=0}^{n-1}(D\hat{L}_d - \hat{\mathcal{A}})(x_k, x_{k+1}) \cdot (\delta x_k, \delta x_{k+1}) = 0.$$

Equivalently,

$$\delta \sum_{k=0}^{n-1} \hat{L}_d(x_k, x_{k+1}) = \sum_{k=0}^{n-1} \hat{\mathcal{A}}(x_k, x_{k+1}) \cdot (\delta x_k, \delta x_{k+1}). \tag{2.3.28}$$

Equating terms involving δx_k on the left-hand side of Equation 2.3.28 to the corresponding terms on the right, we get the ***discrete Routh (DR) equations*** giving dynamics on $S \times S$:

$$D_2 \hat{L}_d(x_{k-1}, x_k) + D_1 \hat{L}_d(x_k, x_{k+1}) = \hat{\mathcal{A}}_2(x_{k-1}, x_k) + \hat{\mathcal{A}}_1(x_k, x_{k+1}). \tag{2.3.29}$$

Note that these equations depend on the value of momentum μ.

Thus, we have shown that if $\mathbf{q}$ is a discrete curve satisfying the discrete Euler–Lagrange equations, the curve $\mathbf{x}$, obtained by projecting $\mathbf{q}$ down to S, satisfies the DR equations (Equation 2.3.29).

Now we shall consider the converse, the discrete reduction procedure: Given a discrete curve $\mathbf{x}$ on S that satisfies the DR equations, is $\mathbf{x}$ the projection of a discrete curve $\mathbf{q}$ on Q that satisfies the DEL equations?

Let the pair (q_0, q_1) be a lift of (x_0, x_1) such that $J_d(q_0, q_1) = \mu$. Let $\mathbf{q} = \{q_0, \ldots, q_n\}$ be the solution of the DEL equations with initial condition (q_0, q_1). Note that $\mathbf{q}$ has momentum μ. Let $\mathbf{x}' = \{x'_0, \ldots, x'_n\}$ be the curve on S obtained by projecting $\mathbf{q}$. By our arguments above, $\mathbf{x}'$ solves the DR equations. However $\mathbf{x}'$ has the initial condition (x_0, x_1), which is the same as the initial condition of $\mathbf{x}$. Thus, by uniqueness of the solutions of the DR equations, $\mathbf{x}' = \mathbf{x}$. Thus $\mathbf{x}$ is the projection of a solution $\mathbf{q}$ of the DEL equations with momentum μ. Also, for a given initial condition q_0, there is a unique lift of $\mathbf{x}$ to a curve with momentum μ. Such a lift can be constructed using the method described in §2.3.2. Thus, lifting $\mathbf{x}$ to a curve with momentum μ yields a solution of the discrete Euler–Lagrange equations, which projects down to $\mathbf{x}$.

We summarize the results of this section in the following Theorem.

Theorem 2.3. *Let* $\mathbf{x}$ *is a discrete curve on* S*, and let* $\mathbf{q}$ *be a discrete curve on* Q *with momentum*

μ that is obtained by lifting $\mathbf{x}$. Then the following are equivalent.

1. $\mathbf{q}$ *solves the DEL equations.*

2. $\mathbf{q}$ *is a solution of the discrete Hamilton's variational principle,*

$$\delta \sum_{k=0}^{n-1} L_d(q_k, q_{k+1}) = 0,$$

for all variations $\delta\mathbf{q}$ of $\mathbf{q}$ that vanish at the endpoints.

3. $\mathbf{x}$ *solves the DR equations,*

$$D_2\hat{L}_d(x_{k-1}, x_k) + D_1\hat{L}_d(x_k, x_{k+1}) = \hat{\mathcal{A}}_2(x_{k-1}, x_k) + \hat{\mathcal{A}}_1(x_k, x_{k+1}).$$

4. $\mathbf{x}$ *is a solution of the reduced variational principle,*

$$\delta \sum_{k} \hat{L}_d(x_k, x_{k+1}) = \sum_{k=0}^{n-1} \hat{\mathcal{A}}(x_k, x_{k+1}) \cdot (\delta x_k, \delta x_{k+1}),$$

for all variations $\delta\mathbf{x}$ of $\mathbf{x}$ that vanish at the endpoints.

2.3.4 Preservation of the Reduced Discrete Symplectic Form

The DR equations define a discrete flow map, $\hat{F}_k : S \times S \to S \times S$. We already know that the flow of the DEL equations preserves the symplectic form Ω_{L_d} on $Q \times Q$. In this section we show that the reduced flow $\hat{F}_k$ preserves a *reduced* symplectic form $\Omega_{\mu,d}$ on $S \times S$, and that this reduced symplectic form is obtained by restricting Ω_{L_d} to $J_d^{-1}(\mu)$ and then dropping to $S \times S$. In other words,

$$\pi_{\mu,d}^* \Omega_{\mu,d} = i_{\mu,d}^* \Omega_{L_d}.$$

The continuous analogue of this equation is

$$\pi_\mu^* \Omega_\mu = i_\mu^* \Omega_Q.$$

Since the projections $\pi_{\mu,d}$ and π_μ involve a momentum shift, the reduced symplectic forms $\Omega_{\mu,d}$ and Ω_μ include magnetic terms.

Recall that $\hat{L}_d : S \times S \to \mathbb{R}$ is the reduced Lagrangian, and $D\hat{L}_d$ is a 1-form on $S \times S$. Define 1-forms $D\hat{L}_d^+$ and $D\hat{L}_d^-$ on $S \times S$ by

$$D\hat{L}_d^+(x_0, x_1) \cdot (\delta x_0, \delta x_1) = D\hat{L}_d(x_0, x_1) \cdot (0, \delta x_1) = D_2\hat{L}_d(x_0, x_1) \cdot \delta x_1,$$

$$D\hat{L}_d^-(x_0, x_1) \cdot (\delta x_0, \delta x_1) = D\hat{L}_d(x_0, x_1) \cdot (\delta x_0, 0) = D_1\hat{L}_d(x_0, x_1) \cdot \delta x_0.$$

Note that the partial derivatives $D_1\hat{L}_d$ and $D_2\hat{L}_d$ are both maps $S \times S \to T^*S$.

Define 2-forms $\mathcal{B}$ and $\hat{\mathcal{B}}$ on $Q \times Q$ and $S \times S$ as follows:

$$\mathcal{B} = \mathbf{d}\mathcal{A}, \quad \hat{\mathcal{B}} = \mathbf{d}\hat{\mathcal{A}}.$$

Since $\pi_{\mu,d}^*\hat{\mathcal{A}} = i_{\mu,d}^*\mathcal{A}$, we get $\pi_{\mu,d}^*\hat{\mathcal{B}} = i_{\mu,d}^*\mathcal{B}$. Thus $\hat{\mathcal{B}}$ can be obtained by restricting $\mathcal{B}$ to $J_d^{-1}(\mu)$ and then dropping to $J_d^{-1}(\mu)/G \approx S \times S$.

Since $\mathcal{A} = \pi_2^*\mathfrak{A}_\mu - \pi_1^*\mathfrak{A}_\mu$, it follows that $\mathcal{B} = \pi_2^*B_\mu - \pi_1^*B_\mu$, where $B_\mu = \mathbf{d}\mathfrak{A}_\mu$ is a 2-form on Q.

Now B_μ drops to a 2-form β_μ on S. Using this fact, we find that $\hat{\mathcal{B}} = \hat{\pi}_2^*\beta_\mu - \hat{\pi}_1^*\beta_\mu$. Here, $\hat{\pi}_1, \hat{\pi}_2 : S \times S \to S$ are projections onto the first and second components, respectively. If we define $\hat{\mathcal{B}}^+ := \hat{\pi}_2^*\beta_\mu$, and $\hat{\mathcal{B}}^- := -\hat{\pi}_1^*\beta_\mu$, then $\hat{\mathcal{B}} = \hat{\mathcal{B}}^- + \hat{\mathcal{B}}^+$.

We define a function $\mathcal{S} : S \times S \to \mathbb{R}$ by

$$\mathcal{S}(x_0, x_1) := \sum_{k=0}^{n-1} \hat{L}_d(x_k, x_{k+1}),$$

where $\mathbf{x} = \{x_0, \ldots, x_n\}$ is a solution of the DR equations with initial condition (x_0, x_1). Thus,

$$\mathcal{S}(x_0, x_1) = \sum_{k=0}^{n-1} \hat{L}_d(\hat{F}_k(x_0, x_1)).$$

Our goal in this section will be to show that symplecticity of the reduced flow follows from the fact that $\mathbf{d}^2\mathcal{S} = 0$.

Recall that if we lift $\mathbf{x}$ to a discrete curve $\mathbf{q}$ on Q with momentum μ, then $\mathbf{q}$ is a solution of the DEL equations. Let $(\delta x_0, \delta x_1) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}(x_{0\varepsilon}, x_{1\varepsilon})$, and let $\mathbf{x}_\varepsilon = \{x_{0\varepsilon}, \ldots, x_{n\varepsilon}\}$ be a solution of the discrete Routh equations with initial condition $(x_{0\varepsilon}, x_{1\varepsilon})$. Let $\delta\mathbf{q}$ be any variation of $\mathbf{q}$ such that $T\pi_{Q,S} \cdot \delta q_i = \delta x_i$. Using Equation 2.3.27 in §2.3.3, we get

$$\begin{aligned}
\mathbf{d}\mathcal{S}(x_0, x_1)(\delta x_0, \delta x_1) &= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathcal{S}(x_{0\varepsilon}, x_{1\varepsilon}) \\
&= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \sum_{k=0}^{n-1} \hat{L}_d(x_{k\varepsilon}, x_{k+1\varepsilon}) \\
&= \sum_{k=0}^{n-1} D\hat{L}_d(x_k, x_{k+1}) \cdot (\delta x_k, \delta x_{k+1}) \\
&= D_1L_d(q_0, q_1) \cdot \text{hor } \delta q_0 + D_2L_d(q_{n-1}, q_n) \cdot \text{hor } \delta q_n
\end{aligned}$$

$$+\sum_{k=0}^{n-1}\hat{\mathcal{A}}(x_k,x_{k+1})\cdot(\delta x_k,\delta x_{k+1}).$$

Thus,

$$\begin{aligned}\mathbf{d}\mathcal{S}(x_0,x_1)(\delta x_0,\delta x_1)=&D_1L_d(q_0,q_1)\cdot\text{hor }\delta q_0+D_2L_d(q_{n-1},q_n)\cdot\text{hor }\delta q_n\\&+\sum_{k=0}^{n-1}(\hat{F}_k^*\hat{\mathcal{A}})(x_0,x_1)\cdot(\delta x_0,\delta x_1).\end{aligned}\tag{2.3.30}$$

We will eventually prove conservation of a reduced symplectic form by taking the exterior derivative of Equation 2.3.30. To do this, we need a number of preliminary calculations.

Lemma 2.4. $\mathbf{d}\left\{\sum_{k=0}^{n-1}(\hat{F}_k^*\hat{\mathcal{A}})\right\}=(\hat{F}_{n-1}^*\hat{\mathcal{B}}^+-\hat{\mathcal{B}}^+)-\hat{\mathcal{B}}$.

Proof. This is a straightforward verification using the facts that $\mathbf{d}\hat{\mathcal{A}}=\hat{\mathcal{B}}$, and that

$$\begin{aligned}(\hat{F}_k^*\hat{\mathcal{B}})(x_0,x_1)((\delta x_0,\delta x_1),(\delta x_0',\delta x_1'))&=\hat{\mathcal{B}}(x_k,x_{k+1})((\delta x_k,\delta x_{k+1}),(\delta x_k',\delta x_{k+1}'))\\&=\beta_\mu(x_{k+1})(\delta x_{k+1},\delta x_{k+1}')-\beta_\mu(x_k)(\delta x_k,\delta x_k').\end{aligned}$$

Here $\delta x_k,\delta x_k'$ are obtained by pushing forward $\delta x_0,\delta x_0'$, respectively, and $\delta x_{k+1},\delta x_{k+1}'$ are obtained by pushing forward $\delta x_1,\delta x_1'$, respectively. □

Lemma 2.5.

$$D_1L_d(q_0,q_1)\cdot\text{hor}\,\delta q_0=-D_2L_d(q_0,q_1)\cdot\text{hor}\,\delta q_1+(DL_d-\mathcal{A})(q_0,q_1)\cdot(\delta q_0,\delta q_1).$$

Proof.

$$\begin{aligned}D_1L_d(q_0,q_1)\cdot\text{hor }\delta q_0=&D_1L_d(q_0,q_1)\cdot\delta q_0-D_1L_d(q_0,q_1)\cdot\text{ver }\delta q_0\\=&DL_d(q_0,q_1)\cdot(\delta q_0,\delta q_1)-D_2L_d(q_0,q_1)\cdot\delta q_1-D_1L_d(q_0,q_1)\cdot\text{ver }\delta q_0\\=&DL_d(q_0,q_1)\cdot(\delta q_0,\delta q_1)-D_2L_d(q_0,q_1)\cdot\text{hor }\delta q_1\\&-D_2L_d(q_0,q_1)\cdot\text{ver }\delta q_1-D_1L_d(q_0,q_1)\cdot\text{ver }\delta q_0.\end{aligned}$$

As in Equation 2.3.19,

$$D_2L_d(q_0,q_1)\cdot\text{ver }\delta q_1=\mathfrak{A}_\mu(q_1)\cdot\delta q_1.$$

Similarly,

$$D_1 L_d(q_0, q_1) \cdot \text{ver } \delta q_0 = -\mathfrak{A}_\mu(q_0) \cdot \delta q_0.$$

Thus,

$$D_2 L_d(q_0, q_1) \cdot \text{ver } \delta q_1 + D_1 L_d(q_0, q_1) \cdot \text{ver } \delta q_0 = \mathcal{A}(q_0, q_1) \cdot (\delta q_0, \delta q_1).$$

The statement of the lemma now follows. □

Thus, Equation 2.3.30 can be rewritten as

$$\begin{aligned} \mathbf{d}\mathcal{S}(x_0, x_1)(\delta x_0, \delta x_1) =& (D_2 L_d(q_{n-1}, q_n) \cdot \text{hor } \delta q_n - D_2 L_d(q_0, q_1) \cdot \text{hor } \delta q_1) \\ &+ \sum_{k=0}^{n-1} (\hat{F}_k^* \hat{\mathcal{A}})(x_0, x_1) \cdot (\delta x_0, \delta x_1) + (DL_d - \mathcal{A})(q_0, q_1) \cdot (\delta q_0, \delta q_1) \\ =& (D_2 L_d(q_{n-1}, q_n) \cdot \text{hor } \delta q_n - D_2 L_d(q_0, q_1) \cdot \text{hor } \delta q_1) \\ &+ \sum_{k=0}^{n-1} (\hat{F}_k^* \hat{\mathcal{A}})(x_0, x_1) \cdot (\delta x_0, \delta x_1) + (D\hat{L}_d - \hat{\mathcal{A}})(x_0, x_1) \cdot (\delta x_0, \delta x_1). \end{aligned} \tag{2.3.31}$$

Lemma 2.6. $D_2 L_d(q_0, q_1) \cdot \text{hor}\, \delta q_1 = ((D\hat{L}_d)^+ - \hat{\mathcal{A}}^+)(x_0, x_1) \cdot (\delta x_0, \delta x_1)$.

Proof. Using Lemma 2.2, we get

$$\begin{aligned} D_2 L_d(q_0, q_1) \cdot \text{hor}\, \delta q_1 &= D_2 L_d(q_0, q_1) \cdot \delta q_1 - D_2 L_d(q_0, q_1) \cdot \text{ver}\, \delta q_1 \\ &= DL_d(q_0, q_1) \cdot (0, \delta q_1) - \mathfrak{A}_\mu(q_1) \cdot \delta q_1 \\ &= (DL_d - \mathcal{A})(q_0, q_1) \cdot (0, \delta q_1) \\ &= (D\hat{L}_d - \hat{\mathcal{A}})(x_0, x_1) \cdot (0, \delta x_1) \\ &= ((D\hat{L}_d)^+ - \hat{\mathcal{A}}^+)(x_0, x_1) \cdot (\delta x_0, \delta x_1). \end{aligned}$$

□

A consequence of this lemma is that

$$\begin{aligned} D_2 L_d(q_{n-1}, q_n) \cdot \text{hor } \delta q_n &= ((D\hat{L}_d)^+ - \hat{\mathcal{A}}^+)(x_{n-1}, x_n) \cdot (\delta x_{n-1}, \delta x_n) \\ &= (\hat{F}_{n-1}^*((D\hat{L}_d)^+ - \hat{\mathcal{A}}^+))(x_0, x_1) \cdot (\delta x_0, \delta x_1). \end{aligned}$$

Define the map $\hat{\mathbb{F}} : S \times S \to T^*S$ by

$$\hat{\mathbb{F}}(x_0, x_1) = D_2 \hat{L}_d(x_0, x_1) - \hat{\mathcal{A}}_2(x_0, x_1).$$

The map $\hat{\mathbb{F}}$ will play the role of a discrete Legendre transform. Let Θ_S be the canonical 1-form on T^*S.

Lemma 2.7. $(D\hat{L}_d)^+ - \hat{\mathcal{A}}^+ = \hat{\mathbb{F}}^*\Theta_S$.

Proof.

$$
\begin{aligned}
(\hat{\mathbb{F}}^*\Theta_S)(x_0,x_1)\cdot(\delta x_0,\delta x_1) &= \Theta_S(D_2\hat{L}_d(x_0,x_1) - \hat{\mathcal{A}}_2(x_0,x_1))\cdot T\hat{\mathbb{F}}\cdot(\delta x_0,\delta x_1)\\
&= (D_2\hat{L}_d(x_0,x_1) - \hat{\mathcal{A}}_2(x_0,x_1))\cdot T\pi_S\cdot T\hat{\mathbb{F}}\cdot(\delta x_0,\delta x_1),
\end{aligned}
$$

where $\pi_{T^*S,S} : T^*S \to S$ is the projection. Note that $\pi_{T^*S,S}\circ\hat{\mathbb{F}} = \hat{\pi}_2$, where $\hat{\pi}_2 : S\times S \to S$ is the projection onto the second component. Thus,

$$
\begin{aligned}
(\hat{\mathbb{F}}^*\Theta_S)(x_0,x_1)\cdot(\delta x_0,\delta x_1) &= (D_2\hat{L}_d(x_0,x_1) - \hat{\mathcal{A}}_2(x_0,x_1))\cdot T\hat{\pi}_2\cdot(\delta x_0,\delta x_1)\\
&= (D_2\hat{L}_d(x_0,x_1) - \hat{\mathcal{A}}_2(x_0,x_1))\cdot\delta x_1\\
&= ((D\hat{L}_d)^+ - \hat{\mathcal{A}}^+)(x_0,x_1)\cdot(\delta x_0,\delta x_1).
\end{aligned}
$$

□

Using Lemmas 2.6 and 2.7, Equation 2.3.31 can be rewritten as:

$$
\mathbf{d}\mathcal{S} = (\hat{F}_{n-1}^*(\hat{\mathbb{F}}^*\Theta_S) - \hat{\mathbb{F}}^*\Theta_S) + \sum_{k=0}^{n-1}(\hat{F}_k^*\hat{\mathcal{A}}) + D\hat{L}_d - \hat{\mathcal{A}}.
$$

Taking the exterior derivative on both sides of this equation and using Lemma 2.4 and the fact that $\mathbf{d}^2 = 0$ yields

$$
0 = \hat{F}_{n-1}^*(\hat{\mathbb{F}}^*\Omega_S - \hat{\mathcal{B}}^+) - (\hat{\mathbb{F}}^*\Omega_S - \hat{\mathcal{B}}^+),
$$

where $\Omega_S = -\mathbf{d}\Theta_S$ is the canonical 2-form on T^*S. Since $\pi_{T^*S,S}\circ\hat{\mathbb{F}} = \hat{\pi}_2$,

$$
\hat{\mathcal{B}}^+ = \hat{\pi}_2^*\beta_\mu = \hat{\mathbb{F}}^*\pi_{T^*S,S}^*\beta_\mu.
$$

Thus,

$$
\hat{\mathbb{F}}^*\Omega_S - \hat{\mathcal{B}}^+ = \hat{\mathbb{F}}^*(\Omega_S - \pi_{T^*S,S}^*\beta_\mu).
$$

We have thus proved the following Theorem.

Theorem 2.8. *The flow of the DR equations preserves the symplectic form*

$$
\begin{aligned}
\Omega_{\mu,d} &= \hat{\mathbb{F}}^*\Omega_S - \hat{\mathcal{B}}^+\\
&= (D_2\hat{L}_d - \hat{\mathcal{A}}_2)^*\Omega_S - \hat{\pi}_2^*\beta_\mu
\end{aligned}
$$

$$= \hat{\mathbb{F}}^*(\Omega_S - \pi^*_{T^*S,S}\beta_\mu).$$

We remark that the symplectic form $\Omega_{\mu,d}$ is just the pull-back by $\hat{\mathbb{F}}$ of the same symplectic form on T^*S that is obtained by the process of cotangent bundle reduction (see §2.2). The fact that $\Omega_{\mu,d}$ is closed follows from the closure of $(\Omega_S - \pi^*_{T^*S,S}\beta_\mu)$.

We will now complete our argument by showing that

$$\pi^*_{\mu,d}\Omega_{\mu,d} = i^*_{\mu,d}\Omega_{L_d}.$$

We showed in section §2.3.2 that the discrete Legendre transform $\mathbb{F}L_d : Q \times Q \to T^*Q$, $(q_0, q_1) \mapsto D_2L_d(q_0, q_1)$ maps $J_d^{-1}(\mu)$ to $J^{-1}(\mu)$, where J_d and J are the discrete and continuous momentum maps, respectively. For the rest of this section, let $\mathbb{F}' : J_d^{-1}(\mu) \to J^{-1}(\mu)$ be the restriction of $\mathbb{F}L_d$. Thus $\mathbb{F}' \circ i_\mu = i_{\mu,d} \circ \mathbb{F}L_d$, where $i_\mu : J^{-1}(\mu) \to T^*Q$ and $i_{\mu,d} : J_d^{-1}(\mu) \to Q \times Q$ are inclusions.

Recall that we had defined the map $\hat{\mathbb{F}} : S \times S \to T^*S$ as $D_2\hat{L}_d - \hat{\mathcal{A}}_2$.

Lemma 2.9. *The following diagram commutes.*

$$\begin{array}{ccc} J_d^{-1}(\mu) & \xrightarrow{\mathbb{F}'} & J^{-1}(\mu) \\ {\scriptstyle \pi_{\mu,d}}\downarrow & & \downarrow{\scriptstyle \pi_\mu} \\ S \times S & \xrightarrow{\hat{\mathbb{F}}} & T^*S \end{array}$$

Proof. Let $(q_0, q_1) \in J_d^{-1}(\mu)$. Thus $D_2L_d(q_0, q_1) \in J^{-1}(\mu)$, and

$$\pi_\mu(\mathbb{F}'(q_0, q_1)) = \pi_\mu(D_2L_d(q_0, q_1)).$$

Recall from §2.2 that $(D_2L_d(q_0, q_1) - \mathfrak{A}_\mu(q_1))$ annihilates all vertical tangent vectors and that $\pi_\mu(D_2L_d(q_0, q_1))$ is the element of $T^*_{x_1}S$ determined by $(D_2L_d(q_0, q_1) - \mathfrak{A}_\mu(q_1))$. For $\delta q_1 \in T_{q_1}Q$,

$$\langle D_2L_d(q_0, q_1) - \mathfrak{A}_\mu(q_1), \delta q_1\rangle = \langle D_2L_d(q_0, q_1) - \mathfrak{A}_\mu(q_1), \text{hor } \delta q_1\rangle.$$

Using the fact that $\mathfrak{A}_\mu(q_1)$ annihilates horizontal vectors, and Lemma 2.6, we obtain

$$\begin{aligned} \langle D_2L_d(q_0, q_1) - \mathfrak{A}_\mu(q_1), \delta q_1\rangle &= \langle D_2L_d(q_0, q_1), \text{hor } \delta q_1\rangle \\ &= D_2\hat{L}_d(x_0, x_1) \cdot \delta x_1 - \hat{\mathcal{A}}_2(x_0, x_1) \cdot \delta x_1 \,. \end{aligned}$$

Thus,

$$\pi_\mu(D_2L_d(q_0, q_1)) = D_2\hat{L}_d(x_0, x_1) - \hat{\mathcal{A}}_2(x_0, x_1),$$

which means $\hat{\mathbb{F}} \circ \pi_{\mu,d} = \pi_\mu \circ \mathbb{F}'$. □

Using this lemma, we get

$$\begin{aligned}\pi_{\mu,d}^* \Omega_{\mu,d} &= \pi_{\mu,d}^* \hat{\mathbb{F}}^* (\Omega_S - \pi_{T^*S,S}^* \beta_\mu) \\ &= (\mathbb{F}')^* \pi_\mu^* (\Omega_S - \pi_{T^*S,S}^* \beta_\mu) \\ &= (\mathbb{F}')^* i_\mu^* \Omega_Q = i_{\mu,d}^* \mathbb{F} L_d^* \Omega_Q \\ &= i_{\mu,d}^* \Omega_{L_d}.\end{aligned}$$

Here, we have used the fact that $\pi_\mu^*(\Omega_S - \pi^* \beta_\mu) = i_\mu^* \Omega_Q$, which comes from the theory of cotangent bundle reduction. We have thus proved the following Theorem.

Theorem 2.10. *The flow of the DR equations preserves the symplectic form*

$$\Omega_{\mu,d} = \hat{\mathbb{F}}^* (\Omega_S - \pi_{T^*S,S}^* \beta_\mu).$$

$\Omega_{\mu,d}$ *can be obtained by dropping to* $S \times S$ *the restriction of* Ω_{L_d} *to* $J_d^{-1}(\mu)$. *In other words,*

$$\pi_{\mu,d}^* \Omega_{\mu,d} = i_{\mu,d}^* \Omega_{L_d}.$$

In proving Theorem 2.10, we started from the reduced variational equation (Equation 2.3.30). There is also an alternate route to proving symplecticity of the reduced flow which relies on the fact that discrete flow on $Q \times Q$ preserves the symplectic form Ω_{L_d}. We will give an outline of the steps involved, without giving all the details. The idea is to first show that the restriction to $J_d^{-1}(\mu)$ of the symplectic form Ω_{L_d} drops to a 2-form $\Omega_{\mu,d}$ on $S \times S$. The fact that the discrete flow on $Q \times Q$ preserves the symplectic form Ω_{L_d} is then used to show that the reduced flow preserves $\Omega_{\mu,d}$.

The outline of the steps involved is as follows.

1. Consider the 1-form Θ_{L_d} on $Q \times Q$ defined by $\Theta_{L_d}(q_0, q_1) \cdot (\delta q_0, \delta q_1) = D_2 L_d(q_0, q_1) \cdot \delta q_1$. Θ_{L_d} is G-invariant, and thus the Lie derivative $\mathcal{L}_{\xi_{Q \times Q}} \Theta_{L_d}$ is zero.

2. Since $\Omega_{L_d} = -\mathbf{d}\Theta_{L_d}$, Ω_{L_d} is G-invariant. If $i_{\mu,d} : J_d^{-1}(\mu) \to Q \times Q$ is the inclusion, $\Theta'_{L_d} = i_{\mu,d}^* \Theta_{L_d}$ and $\Omega'_{L_d} = i_{\mu,d}^* \Omega_{L_d}$ are the restrictions of Θ_{L_d} and Ω_{L_d}, respectively, to $J_d^{-1}(\mu)$. It is easy to check that Θ'_{L_d} and Ω'_{L_d} are invariant under the action of G on $J_d^{-1}(\mu)$.

3. If $\xi_{J_d^{-1}(\mu)}$ is an infinitesimal generator on $J_d^{-1}(\mu)$, then

$$\xi_{J_d^{-1}(\mu)} \mathbin{\lrcorner} \Omega'_{L_d} = -\xi_{J_d^{-1}(\mu)} \mathbin{\lrcorner} \mathbf{d}\Theta'_{L_d} = -\mathcal{L}_{\xi_{J_d^{-1}(\mu)}} \Theta'_{L_d} + \mathbf{d}\xi_{J_d^{-1}(\mu)} \mathbin{\lrcorner} \Theta'_{L_d} = 0.$$

This follows from the G-invariance of Θ'_{L_d}, and the fact that $\Theta'_{L_d} \cdot \xi_{J_d^{-1}(\mu)} = \langle \mu, \xi \rangle$.

4. By steps 2 and 3, the form Ω'_{L_d} drops to a reduced form $\Omega_{\mu,d}$ on $J_d^{-1}(\mu)/G \approx S \times S$. Thus, if $\pi_{\mu,d} : J_d^{-1}(\mu) \to S \times S$ is the projection, then $\pi^*_{\mu,d}\Omega_{\mu,d} = \Omega'_{L_d}$. Note that the closure of $\Omega_{\mu,d}$ follows from the fact that Ω'_{L_d} is closed, which in turn follows from the closure of Ω_{L_d} and the relation $\Omega'_{L_d} = i^*_{\mu,d}\Omega_{L_d}$.

5. If $F_k : Q \times Q \to Q \times Q$ is the flow of the DEL equations, let F'_k be the restriction of this flow to $J_d^{-1}(\mu)$. We know that F'_k drops to the flow $\hat{F}_k$ of the DR equations on $S \times S$. Since F_k preserves Ω_{L_d}, F'_k preserves Ω'_{L_d}. Using this, it can be shown that $\hat{F}_k$ preserves $\Omega_{\mu,d}$. Note that it is sufficient to show that $\pi^*_{\mu,d}(\hat{F}^*_k\Omega_{\mu,d}) = \pi^*_{\mu,d}\Omega_{\mu,d}$.

6. It now remains to compute a formula for the reduced form $\Omega_{\mu,d}$. Using Lemma 2.9 (whose proof, in turn, relies on Lemma 2.6), it follows that

$$\begin{aligned}\pi^*_{\mu,d}\Omega_{\mu,d} = i^*_{\mu,d}\Omega_{L_d} = i^*_{\mu,d}\mathbb{F}L_d^*\Omega_Q &= (\mathbb{F}')^* i^*_\mu \Omega_Q \\ &= (\mathbb{F}')^*\pi^*_\mu(\Omega_S - \pi^*_{T^*S,S}\beta_\mu) \\ &= \pi^*_{\mu,d}\hat{\mathbb{F}}^*(\Omega_S - \pi^*_{T^*S,S}\beta_\mu).\end{aligned}$$

Thus $\pi^*_{\mu,d}\Omega_{\mu,d} = \pi^*_{\mu,d}\hat{\mathbb{F}}^*(\Omega_S - \pi^*_{T^*S,S}\beta_\mu)$, from which it follows that $\Omega_{\mu,d} = \hat{\mathbb{F}}^*(\Omega_S - \pi^*_{T^*S,S}\beta_\mu)$. Incidentally, this expression shows that $\Omega_{\mu,d}$ is nondegenerate provided the map $\hat{\mathbb{F}} = D_2\hat{L}_d - \hat{\mathcal{A}}_2$ is a local diffeomorphism.

2.3.5 Relating Discrete and Continuous Reduction

As we stated in §2.3.1, if the discrete Lagrangian L_d approximates the Jacobi solution of the Hamilton–Jacobi equation, then the DEL equations give us an integration scheme for the EL equations. In our commutative diagrams we will denote the relationship between the EL and DEL equations by a dashed arrow as follows:

$$(TQ, EL) \dashrightarrow (Q \times Q, DEL).$$

Thus, $\dashrightarrow$ can be read as "the corresponding discretization". By the continuous and discrete Noether theorems, we can restrict the flow of the EL and DEL equations to $J_L^{-1}(\mu)$ and $J_d^{-1}(\mu)$, respectively. We have seen that the flow on $J_L^{-1}(\mu)$ induces a reduced flow on $J_L^{-1}(\mu)/G \approx TS$, which is the flow of the Routh equations. Similarly, the discrete flow on $J_d^{-1}(\mu)$ induces a reduced discrete flow on $J_d^{-1}(\mu)/G \approx S \times S$, which is the flow of the discrete Routh equations. Since the DEL equations give us an integration algorithm for the EL equations, it follows that the DR equations give us an integration algorithm for the Routh equations.

Thus, to numerically integrate the Routh equations, we can follow either of the following approaches:

1. First solve the DEL equations to yield a discrete trajectory on Q, which can then be projected to a discrete trajectory on S.

2. Solve the DR equations to directly obtain a discrete trajectory on Q.

Either approach will yield the same result. We can express this situation by the following commutative diagram:

$$\begin{array}{ccc} (J_L^{-1}(\mu), EL) & \dashrightarrow & (J_d^{-1}(\mu), DEL) \\ \Big\downarrow{\scriptstyle \pi_{\mu,L}} & & \Big\downarrow{\scriptstyle \pi_{\mu,d}} \\ (TS, R) & \dashrightarrow & (S \times S, DR) \end{array} \tag{2.3.32}$$

The upper dashed arrow represents the fact that the DEL equations are an integration algorithm for the EL equations, and the lower dashed arrow represents the same relationship between the DR equations and the Routh equations. Note that for smooth group actions the order of accuracy will be equal for the reduced and unreduced algorithms. We will state this result precisely in the following corollary.

Corollary 2.11. *Given a discrete Lagrangian $L_d : Q \times Q \to \mathbb{R}$ of order r, and a smooth group action, the discrete Routh equations associated with the reduced discrete Lagrangian, $\hat{L}_d : S \times S \to \mathbb{R}$, obtained by dropping L_d to $S \times S$, is of order r as well.*

Proof. Recall that the order of the discrete Lagrangian is equal to the order of the push-forward discrete Lagrangian map, and as such, the discrete Euler–Lagrange equations yield a r-th order accurate approximation of the exact flow. When the group action is smooth, the projections $\pi_{\mu,L}$ and $\pi_{\mu,d}$ are smooth as well. Since the two projections agree when restricted to the position space, and the projections are smooth, the commutative diagram in Equation 2.3.32, together with the chain rule, implies that the discrete Routh equations yield a r-th order accurate approximation to the reduced flow. □

2.4 Relating the DEL Equations to Symplectic Runge–Kutta Algorithms

Symplectic Partitioned Runge–Kutta Methods. A well-studied class of numerical schemes for Hamiltonian and Lagrangian systems is the partitioned Runge–Kutta (PRK) algorithms (see

Hairer et al. [1993] and Hairer and Wanner [1996] for history and details). Stated for a regular Lagrangian system, a partitioned Runge–Kutta scheme is a map $F : T^*Q \to T^*Q$ defined by $F : (q_0, p_0) \mapsto (q_1, p_1)$, where

$$q_1 = q_0 + h \sum_{j=1}^{s} b_j \dot{Q}_j, \qquad p_1 = p_0 + h \sum_{j=1}^{s} \tilde{b}_j \dot{P}_j, \tag{2.4.1a}$$

$$Q_i = q_0 + h \sum_{j=1}^{s} a_{ij} \dot{Q}_j, \qquad P_i = p_0 + h \sum_{j=1}^{s} \tilde{a}_{ij} \dot{P}_j, \qquad i = 1, \dots, s, \tag{2.4.1b}$$

$$P_i = \frac{\partial L}{\partial \dot{q}}(Q_i, \dot{Q}_i), \qquad \dot{P}_i = \frac{\partial L}{\partial q}(Q_i, \dot{Q}_i), \qquad i = 1, \dots, s, \tag{2.4.1c}$$

where b_i, $\tilde{b}_i$, a_{ij} and $\tilde{a}_{ij}$ are real coefficients for $i, j = 1, \dots, s$ which define the method. Note that Equation 2.4.1c implicitly determined the Hamiltonian vector field $(\dot{Q}_i, \dot{P}_i)$ at the point $(Q_i, P_i) = \mathbb{F}L(Q_i, \dot{Q}_i)$.

The partitioned Runge–Kutta method, $F : T^*Q \to T^*Q$, approximates the flow map, $F_H^t : T^*Q \to T^*Q$, of the Hamiltonian system corresponding to the Lagrangian L, so that

$$F(q, p, h) = F_H^h(q, p) + \mathcal{O}(h^{r+1}),$$

where r, the order of the integration algorithm, is determined by the choice of the coefficients b_i, $\tilde{b}_i$, a_{ij} and $\tilde{a}_{ij}$.

As discussed in §2.2, the flow map F_H^t of the Hamiltonian system on T^*Q preserves the canonical symplectic form Ω on T^*Q. It can be shown that the partitioned Runge–Kutta method F preserves the canonical symplectic form if, and only if, the coefficients satisfy

$$b_i \tilde{a}_{ij} + \tilde{b}_j a_{ji} = b_i \tilde{b}_j, \qquad i, j = 1, \dots, s \tag{2.4.2a}$$

$$b_i = \tilde{b}_i, \qquad i = 1, \dots, s. \tag{2.4.2b}$$

Such schemes are known as symplectic partitioned Runge–Kutta (SPRK) methods.

Discrete Lagrangians for SPRK Methods. For any given time-step h, a symplectic partitioned Runge–Kutta method is a symplectic map $F : T^*Q \to T^*Q$. Therefore, as discussed in §2.3.1, there is a discrete Lagrangian L_d which generates it.

An explicit form for this discrete Lagrangian was found by Suris [1990], and is given by

$$L_d(q_0, q_1, h) = h \sum_{i=1}^{s} b_i L(Q_i, \dot{Q}_i),$$

where Q_i, $\dot{Q}_i$, P_i and $\dot{P}_i$ are such that Equations 2.4.1b and 2.4.1c are satisfied. It can then be shown, under assumptions (Equations 2.4.2a and 2.4.2b) on the coefficients, that the push-forward of the discrete Lagrangian map is exactly the symplectic partitioned Runge–Kutta method. The details of this calculation can be found in Suris [1990] or Marsden and West [2001].

For a partitioned Runge–Kutta method to be consistent, the coefficients must satisfy $\sum_{i=1}^{s} b_i = 1$. With this in mind, it can be readily seen that the L_d defined above is an approximation to the action over the interval $[0, h]$, as one would expect from §2.3.1.

Discrete Lagrangians from Polynomials and Quadrature. While the discrete Lagrangian given above generates any symplectic partitioned Runge–Kutta method, there is a subset of such methods for which the discrete Lagrangian has a particularly elegant form. These can be derived by approximating the action with polynomial trajectories and numerical quadrature.

As shown in §2.3.1, a discrete Lagrangian should be an approximation

$$L_d(q_0, q_1, h) \approx \operatorname*{ext}_{q \in \mathcal{C}(0,h)} S(q),$$

where $\mathcal{C}(0, h)$ is the space of trajectories $q : [0, h] \to Q$ with $q(0) = q_0$ and $q(h) = q_1$, and $S : \mathcal{C}(0, h) \to \mathbb{R}$ is the action $S(q) = \int_0^h L(q, \dot{q})dt$.

To approximate this, we take a finite-dimensional approximation $\mathcal{C}_d(0, h) \subset \mathcal{C}(0, h)$ of the trajectory space,

$$\mathcal{C}_d(0, h) = \{q \in \mathcal{C}(0, h) \mid q \text{ is a polynomial of degree } s\},$$

and we approximate the action integral by numerical quadrature to give an approximate action $S_d : \mathcal{C}(0, h) \to \mathbb{R}$,

$$S_d(q) = h \sum_{i=1}^{s} b_i L(q(c_i h), \dot{q}(c_i h)),$$

where (b_i, c_i) is the maximal-order quadrature rule on the unit interval with quadrature points c_i. We now set the discrete Lagrangian to be

$$L_d(q_0, q_1, h) = \operatorname*{ext}_{q_d \in \mathcal{C}_d(0,h)} S_d(q_d),$$

which can be explicitly evaluated. This procedure corresponds to the Galerkin projection of the weak form of the ODE onto the space of piecewise polynomial trajectories, an interpretation which is further discussed in Marsden and West [2001].

Theorem 2.12. *Take a set of quadrature points c_i and let L_d be the corresponding discrete Lagrangian as described above. Then the integrator generated by this discrete Lagrangian is equivalent* *to the partitioned Runge–Kutta scheme defined by the coefficients*

$$\begin{aligned} b_i = \tilde{b}_i &= \int_0^1 l_{i,s}(\tau)\mathbf{d}\tau, \\ a_{ij} &= \int_0^{c_i} l_{j,s}(\tau)\mathbf{d}\tau, \\ \tilde{a}_{ij} &= \tilde{b}_j\left(1 - \frac{a_{ji}}{b_i}\right), \end{aligned} \tag{2.4.3}$$

where the $l_{i,s}(\tau)$ *are the Lagrange polynomials associated with the* c_i*.*

Proof. Evaluating the conditions which imply that q_d extremizes S_d and combining this with the definition of the push-forward of the discrete Euler–Lagrange equations give the desired result. See Marsden and West [2001] for details. □

2.5 Reduction of the Symplectic Runge–Kutta Algorithm

Consider the SPRK algorithm for mechanical systems described in §2.4. The equations defining this algorithm are

$$(q_1, p_1) = (q_0, p_0) + h\sum_j (b_j\dot{Q}_j, \tilde{b}_j\dot{P}_j), \tag{2.5.1a}$$

$$(Q_i, P_i) = (q_0, p_0) + h\sum_j (a_{ij}\dot{Q}_j, \tilde{a}_{ij}\dot{P}_j), \tag{2.5.1b}$$

$$(\dot{Q}_j, \dot{P}_j) = X_H(Q_j, P_j), \tag{2.5.1c}$$

for some coefficients $b_j, \tilde{b}_j, a_{ij}, \tilde{a}_{ij}$ satisfying Equation 2.4.2. These equations specify the push-forward discrete Lagrange map for some discrete Lagrangian, as discussed in §2.4. We will assume that there is an abelian group G that acts freely and properly on the configuration manifold Q, and that the Lagrangian and the Hamiltonian functions are invariant under the lifted actions of G on TQ and T^*Q, respectively. Locally, $Q \approx G \times S$, where $S = Q/G$ is the shape space. Let $\theta = (\theta^1, \ldots, \theta^r)$ be local coordinates on G such that the group operation is addition, i.e., $\theta_1 \cdot \theta_2 = \theta_1 + \theta_2$. (Since the group is abelian, such coordinates can always be found.) Let $x = (x^1, \ldots, x^s)$ be coordinates on S. In a local trivialization, (θ, x) are coordinates on Q. Let $(\theta, x, p_\theta, p_x)$ be canonical cotangent bundle coordinates on T^*Q, and $(\theta, x, \dot{\theta}, \dot{x})$ be canonical tangent bundle coordinates on TQ. It is easy to show that in these canonical coordinates on T^*Q, elements of the set $J^{-1}(\mu) \subset T^*Q$ are of the form (θ, x, μ, p_x). Also, since the Hamiltonian H on T^*Q is group invariant, $H(\theta, x, p_\theta, p_x)$ is independent of θ. Note that here we are implicitly assuming that the vector space structure used to define the SPRK method is that in which the group action is addition.

For the remainder of this section, we will adopt a local trivialization to express the SPRK method in which the group action is addition. Rewriting the symplectic partitioned Runge–Kutta algorithm in terms of this local trivialization gives

$$\theta_1 = \theta_0 + h\sum_j b_j \dot{\Theta}_j, \qquad (p_\theta)_1 = (p_\theta)_0 + h\sum_j \tilde{b}_j (\dot{P}_\theta)_j, \tag{2.5.2}$$

$$x_1 = x_0 + h\sum b_j \dot{X}_j, \qquad (p_x)_1 = (p_x)_0 + h\sum_j \tilde{b}_j (\dot{P}_x)_j, \tag{2.5.3}$$

$$\Theta_i = \theta_0 + h\sum_j a_{ij} \dot{\Theta}_j, \qquad (P_\theta)_i = (p_\theta)_0 + h\sum_j \tilde{a}_{ij} (\dot{P}_\theta)_j, \tag{2.5.4}$$

$$X_i = x_0 + h\sum a_{ij} \dot{X}_j, \qquad (P_x)_i = (p_x)_0 + h\sum_j \tilde{a}_{ij} (\dot{P}_x)_j, \tag{2.5.5}$$

and further,

$$\dot{\Theta}_j = \frac{\partial H}{\partial p_\theta}, \qquad (\dot{P}_\theta)_j = -\frac{\partial H}{\partial \theta}, \tag{2.5.6}$$

$$\dot{X}_j = \frac{\partial H}{\partial p_x}, \qquad (\dot{P}_x)_j = -\frac{\partial H}{\partial x}. \tag{2.5.7}$$

By group invariance, H does not depend on θ, and so $\partial H/\partial\theta = 0$. Thus $(\dot{P}_\theta)_j = 0$, and therefore, $(p_\theta)_1 = (P_\theta)_i = (p_\theta)_0$. Hence, if $(q_0, p_0) \in J^{-1}(\mu)$, then (q_1, p_1) and (Q_i, P_i) also lie on $J^{-1}(\mu)$. (We already know from the theory in the previous sections that the symplectic partitioned Runge–Kutta algorithm preserves momentum; what we have verified here is that the intermediate points (Q_i, P_i) do not move off the momentum surface.)

If $\mathfrak{A}$ is a connection on Q, it can be represented in local coordinates as

$$\mathfrak{A}(\theta, x)(\dot{\theta}, \dot{x}) = A(x)\dot{x} + \dot{\theta}.$$

Thus, the 1-form $\mathfrak{A}_\mu$ on Q is given by

$$\mathfrak{A}_\mu(\theta, x)(\dot{\theta}, \dot{x}) = \langle \mu, A(x)\dot{x} + \dot{\theta}\rangle = \begin{bmatrix} \mu & \mu A(x) \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{x} \end{bmatrix}.$$

Thus, $\mathfrak{A}_\mu(\theta, x) = (\theta, x, \mu, \mu A(x))$.

As we have seen in §2.2, there is a projection $\pi_\mu : J^{-1}(\mu) \to T^*S$. If $\alpha_q \in J_q^{-1}(\mu)$, $(\alpha_q - \mathfrak{A}_\mu(q)) \in T_q^*Q$ annihilates all vertical tangent vectors at q, and $\pi_\mu(\alpha_q)$ is the element of T_x^*S determined by $(\alpha_q - \mathfrak{A}_\mu(q))$.

Suppose that in local coordinates, $\alpha_q = (\theta, x, \mu, p_x)$. Then, $(\alpha_q - \mathfrak{A}_\mu(q)) = (\theta, x, 0, p_x - \mu A(x))$.

Thus, $\pi_\mu(\theta, x, \mu, p_x) = (x, p_x - \mu A(x))$. Therefore, $T\pi_\mu : TJ^{-1}(\mu) \to T(T^*S)$ is given by

$$T\pi_\mu : (\dot\theta, \dot x, 0, \dot p_x) \mapsto (\dot x, \dot p_x - \mu \frac{\partial A}{\partial x}\dot x).$$

In components, $\mu A(x)$ can be represented as $\mu_a A_i^a(x)$ (sum over the repeated index a is implicit), and

$$\mu \frac{\partial A}{\partial x}\dot x = \mu_a \frac{\partial A_i^a}{\partial x^j}\dot x^j.$$

Let $(q, p) \in J^{-1}(\mu)$ and let $(\dot q, \dot p) = X_H(q, p)$. By Noether's theorem, we have that $(\dot q, \dot p) \in T_{(q,p)}(J^{-1}(\mu))$. In local coordinates,

$$(\dot\theta, \dot x, 0, \dot p_x) = X_H(\theta, x, \mu, p_x).$$

Now, by the theory of cotangent bundle reduction (see §2.2),

$$T\pi_\mu \cdot X_H(q, p) = X_{H_\mu}(\pi_\mu(q, p)),$$

i.e.,

$$(\dot x, \dot p_x - \mu \frac{\partial A}{\partial x}\dot x) = X_{H_\mu}(x, p_x - \mu A(x)).$$

If $(q_0, p_0) \in J^{-1}(\mu)$, we have seen how (Q_i, P_i) and (q_1, p_1) also lie in $J^{-1}(\mu)$. Let

$$\begin{aligned} \pi_\mu(q_0, p_0) &=: (x_0, s_0) = (x_0, (p_x)_0 - \mu A(x_0)), \\ \pi_\mu(Q_i, P_i) &=: (X_i, S_i) = (X_i, (P_x)_i - \mu A(X_i)), \\ \pi_\mu(q_1, p_1) &=: (x_1, s_1) = (x_1, (p_x)_1 - \mu A(x_1)). \end{aligned}$$

Then,

$$(\dot X_i, \dot S_i) := X_{H_\mu}(X_i, S_i) = (\dot X_i, (\dot P_x)_i - \mu \frac{\partial A}{\partial x}(X_i)\dot X_i). \tag{2.5.8}$$

Remark 2.1. *The Routh equations,*

$$\frac{\partial \hat R^\mu}{\partial x} - \frac{d}{dt}\frac{\partial \hat R^\mu}{\partial \dot x} = i_{\dot x}\beta_\mu(x),$$

define a vector field on TS which is related to the vector field X_{H_μ} by the reduced Legendre transform $\mathbb{F}\hat R^\mu$. The equations

$$s = \frac{\partial \hat R^\mu}{\partial \dot x}(x, \dot x), \tag{2.5.9}$$

and

$$\dot{s} = \frac{\partial \hat{R}^\mu}{\partial x}(x, \dot{x}) - i_{\dot{x}}\beta_\mu(x), \tag{2.5.10}$$

can be used to solve for $(\dot{x}, \dot{s})$ *in terms of* (x, s)*, and thereby implicitly define the vector field* X_{H_μ}*.*

Recall that

$$(p_x)_1 = (p_x)_0 + h\sum_j \tilde{b}_j(\dot{P}_x)_j.$$

Adding and subtracting terms, this becomes

$$(p_x)_1 - \mu A(x_1) = (p_x)_0 - \mu A(x_0) + h\sum_j \tilde{b}_j \left[(\dot{P}_x)_j - \mu\frac{\partial A}{\partial x}(X_j)\dot{X}_j\right]$$
$$+ \left[h\sum_j \left(\tilde{b}_j\mu\frac{\partial A}{\partial x}(X_j)\dot{X}_j\right) - (\mu A(x_1) - \mu A(x_0))\right]. \tag{2.5.11}$$

This can be rewritten as

$$s_1 = s_0 + h\sum_j \tilde{b}_j\dot{S}_j + \left[h\sum_j \left(\tilde{b}_j\mu\frac{\partial A}{\partial x}(X_j)\dot{X}_j\right) - (\mu A(x_1) - \mu A(x_0))\right]. \tag{2.5.12}$$

Similarly, it can be shown that

$$S_i = s_0 + h\sum_j \tilde{a}_{ij}\dot{S}_j + \left[h\sum_j \left(\tilde{a}_{ij}\mu\frac{\partial A}{\partial x}(X_j)\dot{X}_j\right) - (\mu A(X_i) - \mu A(x_0))\right]. \tag{2.5.13}$$

Putting the above equations together with the equations for x_1 and X_i, we get the following algorithm on T^*S:

$$x_1 = x_0 + h\sum b_j\dot{X}_j, \tag{2.5.14a}$$

$$s_1 = s_0 + h\sum_j \tilde{b}_j\dot{S}_j + \left[h\sum_j \left(\tilde{b}_j\mu\frac{\partial A}{\partial x}(X_j)\dot{X}_j\right) - (\mu A(x_1) - \mu A(x_0))\right], \tag{2.5.14b}$$

$$X_i = x_0 + h\sum a_{ij}\dot{X}_j, \tag{2.5.14c}$$

$$S_i = s_0 + h\sum_j \tilde{a}_{ij}\dot{S}_j + \left[h\sum_j \left(\tilde{a}_{ij}\mu\frac{\partial A}{\partial x}(X_j)\dot{X}_j\right) - (\mu A(X_i) - \mu A(x_0))\right], \tag{2.5.14d}$$

$$S_j = \frac{\partial \hat{R}^\mu}{\partial \dot{x}}(X_j, \dot{X}_j), \tag{2.5.14e}$$

$$\dot{S}_j = \frac{\partial \hat{R}^\mu}{\partial x}(X_j, \dot{X}_j) - i_{\dot{X}_j}\beta_\mu(X_j). \tag{2.5.14f}$$

We shall refer to this system of equations as the ***reduced symplectic partitioned Runge–Kutta (RSPRK)*** algorithm. Since we obtained this system by dropping the symplectic partitioned Runge–Kutta algorithm from $J^{-1}(\mu)$ to T^*S, it follows that this algorithm preserves the reduced symplectic form $\Omega_\mu = \Omega_S - \pi^*_{T^*S,S}\beta_\mu$ on T^*S.

Since the SPRK algorithm is an integration algorithm for the Hamiltonian vector field X_H on T^*Q, the RSPRK algorithm is an integration algorithm for the reduced Hamiltonian vector field X_{H_μ} on T^*S. The relationship between cotangent bundle reduction and the reduction of the SPRK algorithm can be represented by the following commutative diagram:

$$\begin{array}{ccc}
(J^{-1}(\mu), X_H) & \dashrightarrow & (J^{-1}(\mu), SPRK) \\
{\scriptstyle \pi_\mu}\downarrow & & \downarrow{\scriptstyle \pi_\mu} \\
(T^*S, X_{H_\mu}) & \dashrightarrow & (T^*S, RSPRK)
\end{array}$$

The dashed arrows here denote the corresponding discretization, as in Equation 2.3.32. We saw in §2.4 that the SPRK algorithm can be obtained by pushing forward the DEL equations by the discrete Legendre transform. By Lemma 2.9, this implies that the RSPRK algorithm can be obtained by pushing forward the DR equations by the reduced discrete Legendre transform $\hat{\mathbb{F}} = D_2\hat{L}_d - \hat{\mathcal{A}}_2$. These relationships are shown in the following commutative diagram:

$$\begin{array}{ccc}
(J_d^{-1}(\mu), DEL) & \xrightarrow{\mathbb{F}L_d} & (J^{-1}(\mu), SPRK) \\
{\scriptstyle \pi_{\mu,d}}\downarrow & & \downarrow{\scriptstyle \pi_\mu} \\
(S \times S, DR) & \xrightarrow{\hat{\mathbb{F}}} & (T^*S, RSPRK)
\end{array}$$

2.6 Putting Everything Together

Let us now recapitulate some of the main results of the previous sections.

We saw in §2.2 that the relationship between Routh reduction and cotangent bundle reduction can be represented by the following commutative diagram:

$$\begin{array}{ccc}
(J_L^{-1}(\mu), EL) & \xrightarrow{\mathbb{F}L} & (J^{-1}(\mu), X_H) \\
{\scriptstyle \pi_{\mu,L}}\downarrow & & \downarrow{\scriptstyle \pi_\mu} \\
(TS, R) & \xrightarrow{\mathbb{F}\hat{R}^\mu} & (T^*S, X_{H_\mu})
\end{array}$$

We saw in §2.3.5 that if L_d approximates the Jacobi solution of the Hamilton–Jacobi equation, the relationship between discrete and continuous Routh reduction is described by the following diagram:

$$\begin{array}{ccc} (J_L^{-1}(\mu), EL) & \dashrightarrow & (J_d^{-1}(\mu), DEL) \\ \downarrow{\scriptstyle \pi_{\mu,L}} & & \downarrow{\scriptstyle \pi_{\mu,d}} \\ (TS, R) & \dashrightarrow & (S \times S, DR) \end{array}$$

The dashed arrows mean that the DEL equations are an integration algorithm for the EL equations, and that the DR equations are an integration algorithm for the Routh equations.

If L_d is defined as in §2.4, we saw that the algorithm on T^*Q obtained by pushing forward the DEL equation using the discrete Legendre transform $\mathbb{F}L_d$ is the symplectic partitioned Runge–Kutta algorithm (Equation 2.4.1), which is an integration algorithm for X_H. This is depicted as follows:

$$\begin{array}{ccc} (J_L^{-1}(\mu), EL) & \dashrightarrow & (J_d^{-1}(\mu), DEL) \\ \downarrow{\scriptstyle \mathbb{F}L} & & \downarrow{\scriptstyle \mathbb{F}L_d} \\ (J^{-1}(\mu), X_H) & \dashrightarrow & (J^{-1}(\mu), SPRK) \end{array}$$

The SPRK algorithm on $J^{-1}(\mu) \subset T^*Q$ induces the RSPRK algorithm on $J^{-1}(\mu)/G \approx T^*S$. As we saw in §2.5, this reduction process is related to cotangent bundle reduction and to discrete Routh reduction as shown in the following diagram:

$$\begin{array}{ccccc} (J^{-1}(\mu), X_H) & \dashrightarrow & (J^{-1}(\mu), SPRK) & \xleftarrow{\mathbb{F}L_d} & (J_d^{-1}(\mu), DEL) \\ \downarrow{\scriptstyle \pi_\mu} & & \downarrow{\scriptstyle \pi_\mu} & & \downarrow{\scriptstyle \pi_{\mu,d}} \\ (T^*S, X_{H_\mu}) & \dashrightarrow & (T^*S, RSPRK) & \xleftarrow{\hat{\mathbb{F}}} & (S \times S, DR) \end{array}$$

Putting all the above commutative diagrams together into one diagram, we obtain Figure 2.1.

2.7 Links with the Classical Routh Equations

The Routhian function $\hat{R}^\mu$ that we have been using is not the same as the classical Routhian defined by Routh [1877]. The classical Routhian, which we shall denote $\hat{R}^\mu_c$, is a function on TS that is related to our Routhian by the equation

$$\hat{R}^\mu_c(x, \dot{x}) = \hat{R}^\mu(x, \dot{x}) + \langle \mu, A(x)\dot{x} \rangle.$$

Recall from §2.2 that the map $A(x) : T_xS \to \mathfrak{g}$ is the restriction of the connection $\mathfrak{A}$ to T_xS. (T_xS is identified with the subspace $T_xS \times \{0\}$ of $T_gG \times T_xS$, which in turn is identified with T_qQ.) Note

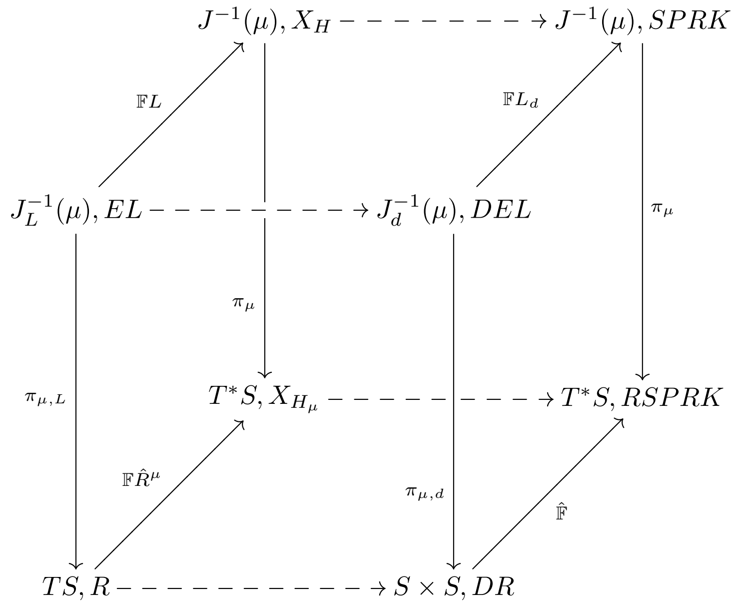

Figure 2.1: Complete commutative cube. Dashed arrows represent discretization from the continuous systems on the left face to the discrete systems on the right face. Vertical arrows represent reduction from the full systems on the top face to the reduced systems on the bottom face. Front and back faces represent Lagrangian and Hamiltonian viewpoints, respectively.

that the map $A(x)$ depends on our choice of local trivialization. Thus $\hat{R}_c^\mu$, too, depends on the trivialization.

The classical Routh equations are

$$\frac{\partial \hat{R}_c^\mu}{\partial x} - \frac{d}{dt}\frac{\partial \hat{R}_c^\mu}{\partial \dot{x}} = 0. \tag{2.7.1}$$

It can be verified (see, for example, Marsden and Ratiu [1999]) that these equations are equivalent to the modern Routh equations (Equation 2.2.3), which we restate here:

$$\frac{\partial \hat{R}^\mu}{\partial x} - \frac{d}{dt}\frac{\partial \hat{R}^\mu}{\partial \dot{x}} = i_{\dot{x}}\beta_\mu(x). \tag{2.7.2}$$

Thus the classical and the modern Routh equation define the same vector field X on TS.

To obtain dynamics on T^*S, we could use the fiber derivative of either the modern Routhian $\mathbb{F}\hat{R}^\mu$ or that of the classical Routhian $\mathbb{F}\hat{R}_c^\mu$. In coordinates on TS and T^*S, these fiber derivatives are

$$\mathbb{F}\hat{R}^\mu : (x, \dot{x}) \in TS \mapsto (x, \frac{\partial \hat{R}^\mu}{\partial \dot{x}}(x, \dot{x})) \in T^*S, \tag{2.7.3}$$

$$\mathbb{F}\hat{R}_c^\mu : (x, \dot{x}) \in TS \mapsto (x, \frac{\partial \hat{R}_c^\mu}{\partial \dot{x}}(x, \dot{x})) = (x, \frac{\partial \hat{R}^\mu}{\partial \dot{x}}(x, \dot{x}) + \mu A(x)) \in T^*S. \tag{2.7.4}$$

Note that the map $\mathbb{F}\hat{R}_c^\mu$ depends upon the trivialization.

We have seen in §2.2 that by pushing forward the dynamics on TS by $\mathbb{F}\hat{R}^\mu$, we obtain the vector field X_{H_μ} on T^*S. Recall that the restriction of the Hamiltonian vector field X_H to $J^{-1}(\mu)$ is π_μ-related to X_{H_μ}, where $\pi_\mu : J^{-1}(\mu) \to T^*S$ is the projection. Also recall that X_{H_μ} is Hamiltonian with respect to the non-canonical symplectic structure $\Omega_\mu = \Omega_S - \pi^*_{T^*S,S}\beta_\mu$ on T^*S.

If, on the other hand, we use $\mathbb{F}\hat{R}_c^\mu$ to push forward the dynamics from TS to T^*S, we obtain a vector field (which we shall call $X_{H'}$) that is Hamiltonian with respect to the *canonical* symplectic structure Ω_S on T^*S.

Consider the following symplectic partitioned Runge–Kutta scheme for integrating $X_{H'}$:

$$x_1 = x_0 + h\sum_{j=1}^s b_j \dot{X}_j, \qquad y_1 = y_0 + h\sum_{j=1}^s \tilde{b}_j \dot{Y}_j, \tag{2.7.5a}$$

$$X_i = x_0 + h\sum_{j=1}^s a_{ij} \dot{X}_j, \qquad Y_i = y_0 + h\sum_{j=1}^s \tilde{a}_{ij} \dot{Y}_j, \qquad i = 1, \ldots, s, \tag{2.7.5b}$$

$$Y_i = \frac{\partial \hat{R}_c^\mu}{\partial \dot{x}}(X_i, \dot{X}_i), \qquad \dot{Y}_i = \frac{\partial \hat{R}_c^\mu}{\partial x}(X_i, \dot{X}_i), \qquad i = 1, \ldots, s, \tag{2.7.5c}$$

for some coefficients $b_j, \tilde{b}_j, a_{ij}, \tilde{a}_{ij}$ satisfying Equation 2.4.2. It follows from that condition that this scheme preserves the canonical symplectic structure Ω_S. A particularly simple scheme of this form, that is second-order, was developed independently by Sanyal et al. [2003].

A natural question to ask at this point is how the above integration scheme for the reduced dynamics is related to the RSPRK scheme (Equation 2.5.14). To answer this question, consider the map $\sigma := \mathbb{F}\hat{R}_c^\mu \circ (\mathbb{F}\hat{R}^\mu)^{-1} : T^*S \to T^*S$. In coordinates, $\sigma : (x, s) \mapsto (x, y) = (x, s + \mu A(x))$. Note that the σ transforms X_{H_μ} to $X_{H'}$, i.e., $X_{H'} = \sigma_* X_{H_\mu}$. It can be verified that this map σ also transforms the RSPRK scheme (Equation 2.5.14) to the above SPRK scheme for $X_{H'}$. Thus, these two schemes for integrating the reduced dynamics are equivalent, and are related to each other by a momentum shift.

Though the derivation for the SPRK scheme for $X_{H'}$ (Equation 2.7.5) is shorter than the reduction process through which we obtained the RSPRK scheme (Equation 2.5.14), there are several reasons to prefer the RSPRK scheme. Firstly, the classical Routhian $\hat{R}_c^\mu$ and therefore the fiber derivative $\mathbb{F}\hat{R}_c^\mu$ and the vector field $X_{H'}$ are dependent on the trivialization. Consequently, the SPRK scheme for $X_{H'}$ is non-intrinsic. On the other hand, as we saw in §2.5, the RSPRK scheme (Equation 2.5.14) is derived by dropping the SPRK scheme (Equation 2.5.1) for X_H onto the quotient $T^*S = J^{-1}(\mu)/G_\mu$ in a manner that is independent of the trivialization. (Though the equations defining the RSPRK scheme have terms involving the map $A(x)$, which is trivialization dependent, the trivialization dependence "cancels out", causing the overall scheme to be trivialization independent.)

Secondly, since the vector field $X_{H'}$ and the SPRK scheme (Equation 2.7.5) are not derived by a reduction process, it is not possible to fit them in a natural way into a commutative diagram like that depicted in Figure 2.1.

Furthermore, the classical theory of Routh reduction does not generalize to the case of non-abelian symmetry groups, whereas the intrinsic, modern version does (see, for example, Marsden and Scheurle [1993a,b], Jalnapurkar and Marsden [2000], and Marsden et al. [2000b]). Thus, to develop numerical algorithms for the reduced dynamics of systems with non-abelian symmetry, one would need to build on the intrinsic approach developed in this paper.

2.8 Forced and Constrained Systems

2.8.1 Constrained Coordinate Formalism

It is often desirable for computational reasons to realize the configuration space as a constraint manifold Q in a containing space V.

Assume that the constraint manifold Q can be expressed as the preimage of a regular value of a G-invariant constraint function, $g : V \to \mathbb{R}^m$. Then, $g^{-1}(0) = Q \subset V$ is a constraint manifold of codimension m.

On the constraint manifold Q, the discrete Hamilton's variational principle states that

$$\delta \sum_{k=0}^{n-1} L_d(q_k, q_{k+1}) = 0$$

for all variations $\delta\mathbf{q}$ of $\mathbf{q}$ that vanish at the endpoints. By the Lagrange multiplier theorem, in the containing space V, this is equivalent to the ***discrete Hamilton's variational principle with constraints***,

$$\delta \left[\sum_{k=0}^{n-1} L_d(v_k, v_{k+1}) + \sum_{k=0}^{n} \lambda_k^T g(v_k) \right] = 0,$$

for all variations $\delta\mathbf{v}$ of $\mathbf{v}$ that vanish at the endpoints.

As the variations are arbitrary and vanish at the endpoints, this is equivalent to the ***discrete Euler–Lagrange equations with constraints***,

$$\begin{aligned} D_2 L_d(v_{k-1}, v_k) + D_1 L_d(v_k, v_{k+1}) + \lambda_k^T Dg(v_k) &= 0, \\ g(v_k) &= 0. \end{aligned}$$

In the case of higher-order discrete Lagrangians, one must be careful about the choice constrained discrete Lagrangians. In particular, the internal sample points used in defining the constrained discrete Lagrangian must lie on the constraint manifold Q. In practice, this corresponds to the inclusion of Lagrange multiplier terms for each of the internal sample points in the variational definition of the higher-order constrained discrete Lagrangian.

Consider the preshape space, $U = V/G$. As the constraint function $g : V \to \mathbb{R}^m$ is G-invariant, this induces the function $\hat{g} : U \to \mathbb{R}^m$.

In addition, we have a G-invariant discrete Lagrangian, $L_d^V : V \times V \to \mathbb{R}$, and the discrete Lagrangian on $Q \times Q$ is simply the restriction, i.e., $L_d^Q = L_d^V\big|_{Q\times Q}$. The discrete momentum maps are related by the following lemma.

Lemma 2.13. *The discrete momentum map, $J_d^Q : Q \times Q \to \mathfrak{g}^*$, is obtained from $J_d^V : V \times V \to \mathfrak{g}^*$ by restriction, i.e., $J_d^Q = J_d^V\big|_{Q\times Q}$.*

Proof. Since $q_0 \in Q \subset V$, $\xi_Q(q_0) \in T_{q_0}Q \hookrightarrow T_{q_0}V$. Q is G-invariant, thus, the group orbits lie on Q, and in particular, $\xi_Q(q_0) = \xi_V(q_0)$. The result then follows from the calculation:

$$\begin{aligned} J_d^Q(q_0, q_1) \cdot \xi &= D_1 L_d^Q(q_0, q_1) \cdot \xi_Q(q_0) \\ &= D_1 L_d^Q(q_0, q_1) \cdot \xi_V(q_0) \\ &= D_1 L_d^V(q_0, q_1) \cdot \xi_V(q_0) \\ &= J_d^V(q_0, q_1) \cdot \xi. \end{aligned}$$

□

Since the discrete momentum map on the constraint manifold is obtained by restriction, and in our subsequent discussion, all the forms are evaluated on the constraint manifold, we shall abuse notation and omit the superscripts denoting the spaces. We are thereby able to formulate the main theorem of this section.

Theorem 2.14. *Let $\mathbf{x}$ be a discrete curve on S, and let $\mathbf{y}$ be a discrete curve on U. Then, the following are equivalent.*

1. $\mathbf{x}$ *solves the discrete Routh equations,*

$$D_2\hat{L}_d(x_{k-1}, x_k) + D_1\hat{L}_d(x_k, x_{k+1}) = \hat{\mathcal{A}}_2(x_{k-1}, x_k) + \hat{\mathcal{A}}_1(x_k, x_{k+1}).$$

2. $\mathbf{x}$ *is a solution of the reduced variational principle,*

$$\delta \sum_{k=0}^{n-1} \hat{L}_d(x_k, x_{k+1}) = \sum_{k=0}^{n-1} \hat{\mathcal{A}}(x_k, x_{k+1}) \cdot (\delta x_k, \delta x_{k+1}),$$

for all variations $\delta\mathbf{x}$ *of* $\mathbf{x}$ *that vanish at the endpoints.*

3. $\mathbf{y}$ *solves the discrete (reduced) Routh equations with constraints,*

$$\begin{aligned} D_2\hat{L}_d\left(y_{k-1}, y_k\right) + D_1\hat{L}_d\left(y_k, y_{k+1}\right) + \lambda_k^T D\hat{g}\left(y_k\right) &= \hat{\mathcal{A}}_2\left(y_{k-1}, y_k\right) + \hat{\mathcal{A}}_1\left(y_k, y_{k+1}\right), \\ \hat{g}\left(y_k\right) &= 0. \end{aligned}$$

4. $\mathbf{y}$ *is a solution of the reduced discrete variational principle,*

$$\delta\left[\sum_{k=0}^{n-1}\hat{L}_d\left(y_k, y_{k+1}\right) + \sum_{k=0}^{n}\lambda_k^T\hat{g}\left(y_k\right)\right] = \sum_{k=0}^{n-1}\hat{\mathcal{A}}\left(y_k, y_{k+1}\right)\cdot\left(\delta y_k, \delta y_{k+1}\right),$$

for all variations $\delta\mathbf{y}$ *of* $\mathbf{y}$ *that vanish at the endpoints, and* $\hat{g}\left(y_k\right) = 0$.

Proof. If $\mathbf{q}$ is a lift of $\mathbf{x}$ onto the μ-momentum surface, then the first two statements are equivalent to the discrete Hamilton's variational principle, which states that

$$\delta\sum_{k=0}^{n-1}L_d\left(q_k, q_{k+1}\right) = 0,$$

for all variations $\delta\mathbf{q}$ of $\mathbf{q}$ that vanish at the endpoints. By the Lagrange multiplier theorem, this is equivalent to the discrete Hamilton's variational principle with constraints,

$$\delta\left[\sum_{k=0}^{n-1}L_d\left(v_k, v_{k+1}\right) + \sum_{k=0}^{n}\lambda_k^T g\left(v_k\right)\right] = 0,$$

for all variations $\delta\mathbf{v}$ of $\mathbf{v}$ that vanish at the endpoints.

As the variations are arbitrary and vanish at the endpoints, this is equivalent to the discrete Euler–Lagrange equations with constraints,

$$\begin{aligned} D_2L_d\left(v_{k-1}, v_k\right) + D_1L_d\left(v_k, v_{k+1}\right) + \lambda_k^T Dg\left(v_k\right) &= 0, \\ g\left(v_k\right) &= 0. \end{aligned}$$

Let $\mathbf{v}$ be a solution of the discrete Euler–Lagrange equations with constraints. Then,

$$\begin{aligned} &\delta\left[\sum_{k=0}^{n-1}L_d\left(v_k, v_{k+1}\right) + \sum_{k=0}^{n}\lambda_k^T g\left(v_k\right)\right] \\ &\quad = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0}\left[\sum_{k=0}^{n-1}L_d\left(v_{k_\epsilon}, v_{k+1_\epsilon}\right) + \sum_{k=0}^{n}\lambda_{k_\epsilon}^T g\left(v_{k_\epsilon}\right)\right] \end{aligned}$$

$$
\begin{aligned}
&= D_1 L_d\left(v_0, v_1\right) \cdot \delta v_0 + \sum_{k=1}^{n-1}\left(D_2 L_d\left(v_{k-1}, v_k\right) + D_1 L_d\left(v_k, v_{k+1}\right)\right) \cdot \delta v_k \\
&\quad + D_2 L_d\left(v_{n-1}, v_n\right) \cdot \delta v_n + \sum_{k=0}^{n} \underbrace{g\left(v_k\right)}_{0} \cdot \delta \lambda_k + \lambda_0^T D g\left(v_0\right) \cdot \delta v_0 \\
&\quad + \sum_{k=1}^{n-1}\left(\lambda_k^T D g\left(v_k\right)\right) \cdot \delta v_k + \lambda_n^T D g\left(v_n\right) \cdot \delta v_n \\
&= \left(D_1 L_d\left(v_0, v_1\right) + \lambda_0^T D g\left(v_0\right)\right) \cdot \delta v_0 + \left(D_2 L_d\left(v_{n-1}, v_n\right) + \lambda_n^T D g\left(v_n\right)\right) \cdot \delta v_n \\
&\quad + \sum_{k=1}^{n-1} \underbrace{\left[D_2 L_d\left(v_{k-1}, v_k\right) + D_1 L_d\left(v_k, v_{k+1}\right) + \lambda_k^T D g\left(v_k\right)\right]}_{0} \\
&= \left(D_1 L_d\left(v_0, v_1\right) + \lambda_0^T D g\left(v_0\right)\right) \cdot \delta v_0 + \left(D_2 L_d\left(v_{n-1}, v_n\right) + \lambda_n^T D g\left(v_n\right)\right) \cdot \delta v_n .
\end{aligned}
$$

Therefore, we have that $\mathbf{v}$ solves the discrete Euler–Lagrange equations with constraints if, and only if,

$$
\begin{aligned}
&\left.\frac{d}{d\epsilon}\right|_{\epsilon=0}\left[\sum_{k=0}^{n-1} L_d\left(v_{k_\epsilon}, v_{k+1_\epsilon}\right) + \sum_{k=0}^{n} \lambda_{k_\epsilon}^T g\left(v_{k_\epsilon}\right)\right] \\
&\quad = \left(D_1 L_d\left(v_0, v_1\right) + \lambda_0^T D g\left(v_0\right)\right) \cdot \delta v_0 + \left(D_2 L_d\left(v_{n-1}, v_n\right) + \lambda_n^T D g\left(v_n\right)\right) \cdot \delta v_n ,
\end{aligned}
$$

for all variations, including those that do not vanish at the endpoints.

Let $\mathbf{y}$ be the projection of $\mathbf{v}$, the solution of the DEL equations with constraints, onto the preshape space V/G, and $\delta \mathbf{y} = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \mathbf{y}_\epsilon$ be a variation of $\mathbf{y}$. By construction,

$$
\hat{g}\left(y_{k_\epsilon}\right) = g\left(v_{k_\epsilon}\right) .
$$

The terms $\lambda_0^T D g\left(v_0\right)$ and $\lambda_n^T D g\left(v_n\right)$ correspond to forces of constraint, and are therefore normal to the constraint manifold. Since the constraint manifold Q is G-invariant, the group orbits lie on the constraint manifold. As a consequence, the forces of constraint annihilate vertical variations, implying that

$$
\begin{aligned}
\lambda_0^T D g\left(v_0\right) \cdot \operatorname{ver} \delta v_0 &= 0, \\
\lambda_n^T D g\left(v_n\right) \cdot \operatorname{ver} \delta v_n &= 0.
\end{aligned}
$$

From which we conclude,

$$
\left.\frac{d}{d\epsilon}\right|_{\epsilon=0}\left[\sum_{k=0}^{n-1} L_d\left(v_{k_\epsilon}, v_{k+1_\epsilon}\right) + \sum_{k=0}^{n} \lambda_{k_\epsilon}^T g\left(v_{k_\epsilon}\right)\right]
$$

$$
\begin{aligned}
&= \left(D_1 L_d(v_0, v_1) + \lambda_0^T Dg(v_0)\right) \cdot \delta v_0 + \left(D_2 L_d(v_{n-1}, v_n) + \lambda_n^T Dg(v_n)\right) \cdot \delta v_n \\
&= \left(D_1 L_d(v_0, v_1) + \lambda_0^T Dg(v_0)\right) \cdot \text{hor } \delta v_0 + \left(D_2 L_d(v_{n-1}, v_n) + \lambda_n^T Dg(v_n)\right) \cdot \text{hor } \delta v_n \\
&\quad - \mathfrak{A}_\mu(v_0) \cdot \delta v_0 + \mathfrak{A}_\mu(v_n) \cdot \delta v_n \\
&= \left(D_1 L_d(v_0, v_1) + \lambda_0^T Dg(v_0)\right) \cdot \text{hor } \delta v_0 + \left(D_2 L_d(v_{n-1}, v_n) + \lambda_n^T Dg(v_n)\right) \cdot \text{hor } \delta v_n \\
&\quad + \sum_{k=0}^{n-1} \mathcal{A}(v_k, v_{k+1}) \cdot (\delta v_k, \delta v_{k+1}),
\end{aligned}
$$

where we used Equation 2.3.19 for the second to last equality. Then,

$$
\begin{aligned}
&\left(D_1 L_d(v_0, v_1) + \lambda_0^T Dg(v_0)\right) \cdot \text{hor } \delta v_0 + \left(D_2 L_d(v_{n-1}, v_n) + \lambda_n^T Dg(v_n)\right) \cdot \text{hor } \delta v_n \\
&\quad = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \left[\sum_{k=0}^{n-1} L_d(v_{k_\epsilon}, v_{k+1_\epsilon}) + \sum_{k=0}^{n} \lambda_{k_\epsilon}^T g(v_{k_\epsilon})\right] - \sum_{k=0}^{n-1} \mathcal{A}(v_k, v_{k+1}) \cdot (\delta v_k, \delta v_{k+1}).
\end{aligned}
$$

From Lemma 2.2, and the fact that $\hat{g}(y_{k_\epsilon}) = g(v_{k_\epsilon})$, this can be rewritten in terms of the reduced quantities,

$$
\begin{aligned}
&\left(D_1 L_d(v_0, v_1) + \lambda_0^T Dg(v_0)\right) \cdot \text{hor } \delta v_0 + \left(D_2 L_d(v_{n-1}, v_n) + \lambda_n^T Dg(v_n)\right) \cdot \text{hor } \delta v_n \\
&\quad = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \left[\sum_{k=0}^{n-1} \hat{L}_d(y_{k_\epsilon}, y_{k+1_\epsilon}) + \sum_{k=0}^{n} \lambda_{k_\epsilon}^T \hat{g}(y_{k_\epsilon})\right] - \sum_{k=0}^{n-1} \hat{\mathcal{A}}(y_k, y_{k+1}) \cdot (\delta y_k, \delta y_{k+1}).
\end{aligned}
$$

If the variations $\delta \mathbf{y}$ vanishes at the endpoints, i.e., $\delta y_0 = \delta y_n = 0$, then hor $\delta v_0 =$ hor $\delta v_n = 0$, and therefore

$$
\delta \left[\sum_{k=0}^{n-1} \hat{L}_d(y_k, y_{k+1}) + \sum_{k=0}^{n} \lambda_k^T \hat{g}(y_k)\right] = \sum_{k=0}^{n-1} \hat{\mathcal{A}}(y_k, y_{k+1}) \cdot (\delta y_k, \delta y_{k+1}),
$$

for all variations $\delta \mathbf{y}$ of $\mathbf{y}$ that vanish at the endpoints, and $\hat{g}(y_k) = 0$.

Since the variations are arbitrary and vanish at the endpoints, this is equivalent to the ***Discrete Routh equations with constraints***,

$$
\begin{aligned}
D_2 \hat{L}_d(y_{k-1}, y_k) + D_1 \hat{L}_d(y_k, y_{k+1}) + \lambda_k^T D\hat{g}(y_k) &= \hat{\mathcal{A}}_2(y_{k-1}, y_k) + \hat{\mathcal{A}}_1(y_k, y_{k+1}), \\
\hat{g}(y_k) &= 0.
\end{aligned}
$$

Conversely, if $\mathbf{y}$ satisfies the reduced variational principle, and $\mathbf{v}$ is its lift onto the μ-momentum surface, then a construction analogous to the derivation of the discrete Routh equations shows that $\mathbf{v}$ satisfies the discrete Hamilton's variational principle with constraints. □

2.8.2 Routh Reduction with Forcing

Mechanical systems with external forcing are governed by the ***Lagrange–d'Alembert variational principle***,

$$\delta \int L(q(t), \dot{q}(t))\, dt + \int F(q(t), \dot{q}(t)) \cdot \delta q dt = 0.$$

We define the ***discrete Lagrange–d'Alembert principle*** (Kane et al. [2000]) to be

$$\delta \sum_{k=0}^{n-1} L_d(q_k, q_{k+1}) + \sum_{k=0}^{n-1} F_d(q_k, q_{k+1}) \cdot (\delta q_k, \delta q_{k+1}) = 0,$$

for all variations $\delta \mathbf{q}$ of $\mathbf{q}$ that vanish at the endpoints. F_d is a 1-form on $Q \times Q$, and approximates the impulse integral between the points q_k and q_{k+1}, just as the discrete Lagrangian L_d approximates the action integral. We define the 1-forms F_d^+ and F_d^- on $Q \times Q$ and the maps $F_d^1, F_d^2 : Q \times Q \to T^*Q$ by the relations

$$\begin{aligned} F_d^+(q_0, q_1) \cdot (\delta q_0, \delta q_1) &= F_d^2(q_0, q_1) \cdot \delta q_1 = F_d(q_0, q_1) \cdot (0, \delta q_1), \\ F_d^-(q_0, q_1) \cdot (\delta q_0, \delta q_1) &= F_d^1(q_0, q_1) \cdot \delta q_0 = F_d(q_0, q_1) \cdot (\delta q_0, 0). \end{aligned}$$

The discrete Lagrange–d'Alembert principle may then be rewritten as

$$\delta \sum_{k=0}^{n-1} L_d(q_k, q_{k+1}) + \sum_{k=0}^{n-1} \left[F_d^1(q_k, q_{k+1}) \cdot \delta q_k + F_d^2(q_k, q_{k+1}) \cdot \delta q_{k+1} \right] = 0,$$

for all variations $\delta \mathbf{q}$ of $\mathbf{q}$ that vanish at the endpoints. This is equivalent to the ***forced discrete Euler–Lagrange equations***,

$$D_2 L_d(q_{k-1}, q_k) + D_1 L_d(q_k, q_{k+1}) + F_d^1(q_k, q_{k+1}) + F_d^2(q_{k-1}, q_k) = 0.$$

As we are concerned with mechanical systems with symmetry, we shall restrict our discussion to discrete forces that are invariant under the diagonal action of G on $Q \times Q$. In particular, for all $\xi \in \mathfrak{g}$, and all variations $(\delta q_0, \delta q_1)$ of (q_0, q_1),

$$F_d(\exp(t\xi) q_0, \exp(t\xi) q_1) \cdot (\delta q_0, \delta q_1) = F_d(q_0, q_1) \cdot (\delta q_0, \delta q_1).$$

Since the Routh reduction technique requires that the momentum map be conserved, we shall further restrict our discussion to G-invariant forcing that satisfies the discrete Noether theorem. This constrains our choice of forcing, as the following lemma illustrates.

Lemma 2.15. *Let* $\mathbf{q}$ *be a discrete curve on* Q *that solves the forced discrete Euler–Lagrange equa-*

tions. Then, the discrete Noether theorem is satisfied if, and only if,

$$\left(F_d^2\left(q_{k-1},q_k\right)+F_d^1\left(q_k,q_{k+1}\right)\right)\cdot \operatorname{ver}\delta q_k=0.$$

Proof. Given $\xi\in\mathfrak{g}$, consider the ξ-component of J_d, given by

$$J_d^\xi\left(q_0,q_1\right)=\left\langle J_d\left(q_0,q_1\right),\xi\right\rangle.$$

We compute the evolution of J_d^ξ along the flow of the forced discrete Euler–Lagrange equations:

$$\begin{aligned}
&J_d^\xi\left(q_1,q_2\right)-J_d^\xi\left(q_0,q_1\right)\\
&\quad=J_d\left(q_1,q_2\right)\cdot\xi-J_d\left(q_0,q_1\right)\cdot\xi\\
&\quad=-D_1L_d\left(q_1,q_2\right)\cdot\xi_Q\left(q_1\right)-D_2L_d\left(q_0,q_1\right)\cdot\xi_Q\left(q_1\right)\\
&\quad=-D_1L_d\left(q_1,q_2\right)\cdot\xi_Q\left(q_1\right)-D_2L_d\left(q_0,q_1\right)\cdot\xi_Q\left(q_1\right)\\
&\qquad+\underbrace{\left[D_2L_d\left(q_0,q_1\right)+D_1L_d\left(q_1,q_2\right)+F_d^2\left(q_0,q_1\right)+F_d^1\left(q_1,q_2\right)\right]}_{0}\cdot\xi_Q\left(q_1\right)\\
&\quad=\left[F_d^2\left(q_0,q_1\right)+F_d^1\left(q_1,q_2\right)\right]\cdot\xi_Q\left(q_1\right).
\end{aligned}$$

Since $J_d:Q\times Q\to\mathfrak{g}^*$, the discrete Noether theorem is satisfied if, and only if, $J_d^\xi\left(q_1,q_2\right)-J_d^\xi\left(q_0,q_1\right)=0$, for all $\xi\in\mathfrak{g}$. As the vertical space ver_{q_1} is given by

$$\operatorname{ver}_{q_1}=\left\{\xi_Q\left(q_1\right)\mid\xi\in\mathfrak{g}\right\},$$

this is equivalent to $F_d^2\left(q_0,q_1\right)+F_d^1\left(q_1,q_2\right)$ vanishing on all vertical vectors. □

For the rest of our discussion, we shall specialize to the case whereby $F_d^2\left(q_0,q_1\right)$ and $F_d^1\left(q_1,q_2\right)$ individually vanish on vertical vectors, which is a sufficient condition for momentum conservation.

The discrete forcing term F_d is an invariant 1-form under the diagonal action of G on $Q\times Q$, and vanishes on vertical vectors. By restricting F_d to $J_d^{-1}(\mu)$, it drops to $\hat{F}_d:S\times S\to T^*S\times T^*S$.

In this context, we may formulate a discrete Routh reduction theory for the discrete Lagrange–d'Alembert principle.

Theorem 2.16. *Let* $\mathbf{x}$ *be a discrete curve on* S*, and let* $\mathbf{q}$ *be a discrete curve on* Q *with momentum* μ *that is obtained by lifting* $\mathbf{x}$*. Then, the following are equivalent.*

1. $\mathbf{q}$ *solves the forced discrete Euler–Lagrange equations,*

$$D_2L_d\left(q_{k-1},q_k\right)+D_1L_d\left(q_k,q_{k+1}\right)+F_d^2\left(q_{k-1},q_k\right)+F_d^1\left(q_k,q_{k+1}\right)=0.$$

2. $\mathbf{q}$ *is a solution of the discrete Lagrange–d'Alembert variational principle,*

$$\delta \sum_{k=0}^{n-1} L_d\left(q_k, q_{k+1}\right) + \sum_{k=0}^{n-1} \left[F_d^1\left(q_k, q_{k+1}\right) \cdot \delta q_k + F_d^2\left(q_k, q_{k+1}\right) \cdot \delta q_{k+1}\right] = 0,$$

for all variations $\delta\mathbf{q}$ *of* $\mathbf{q}$ *that vanish at the endpoints.*

3. $\mathbf{x}$ *solves the Discrete Routh equations with forcing,*

$$\begin{aligned} D_2\hat{L}_d\left(x_{k-1}, x_k\right) + D_1\hat{L}_d\left(x_k, x_{k+1}\right) + \hat{F}_d^2\left(x_{k-1}, x_k\right) + \hat{F}_d^1\left(x_k, x_{k+1}\right) \\ = \hat{\mathcal{A}}_2\left(x_{k-1}, x_k\right) + \hat{\mathcal{A}}_1\left(x_k, x_{k+1}\right). \end{aligned}$$

4. $\mathbf{x}$ *is a solution of the reduced variational principle,*

$$\begin{aligned} \delta \sum_{k=0}^{n-1} \hat{L}_d\left(x_k, x_{k+1}\right) + \sum_{k=0}^{n-1} \left[\hat{F}_d^1\left(x_k, x_{k+1}\right) \cdot \delta x_k + \hat{F}_d^2\left(x_k, x_{k+1}\right) \cdot \delta x_{k+1}\right] \\ = \sum_{k=0}^{n-1} \hat{\mathcal{A}}\left(x_k, x_{k+1}\right) \cdot \left(\delta x_k, \delta x_{k+1}\right), \end{aligned}$$

for all variations $\delta\mathbf{x}$ *of* $\mathbf{x}$ *that vanish at the endpoints.*

Proof. We begin with the discrete Lagrange–d'Alembert variational principle,

$$\delta \sum_{k=0}^{n-1} L_d\left(q_k, q_{k+1}\right) + \sum_{k=0}^{n-1} \left[F_d^1\left(q_k, q_{k+1}\right) \cdot \delta q_k + F_d^2\left(q_k, q_{k+1}\right) \cdot \delta q_{k+1}\right] = 0,$$

for all variations $\delta\mathbf{q}$ of $\mathbf{q}$ that vanish at the endpoints.

Since the variations are arbitrary and vanish at the endpoints, this is equivalent to the forced discrete Euler–Lagrange equations,

$$D_2L_d\left(q_{k-1}, q_k\right) + D_1L_d\left(q_k, q_{k+1}\right) + F_d^2\left(q_{k-1}, q_k\right) + F_d^1\left(q_k, q_{k+1}\right) = 0.$$

Let $\mathbf{q}$ be a solution of the forced discrete Euler–Lagrange equations, then,

$$\begin{aligned} &\delta \sum_{k=0}^{n-1} L_d\left(q_k, q_{k+1}\right) + \sum_{k=0}^{n-1} \left[F_d^1\left(q_k, q_{k+1}\right) \cdot \delta q_k + F_d^2\left(q_k, q_{k+1}\right) \cdot \delta q_{k+1}\right] \\ &\quad = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \sum_{k=0}^{n-1} L_d\left(q_{k_\epsilon}, q_{k+1_\epsilon}\right) + \sum_{k=0}^{n-1} \left[F_d^1\left(q_k, q_{k+1}\right) \cdot \delta q_k + F_d^2\left(q_k, q_{k+1}\right) \cdot \delta q_{k+1}\right] \\ &\quad = D_1L_d\left(q_0, q_1\right) \cdot \delta q_0 + \sum_{k=1}^{n-1} \left(D_2L_d\left(q_{k-1}, q_k\right) + D_1L_d\left(q_k, q_{k+1}\right)\right) \cdot \delta q_k \end{aligned}$$

$$
\begin{aligned}
&+ D_2 L_d\left(q_{n-1}, q_n\right) \cdot \delta q_n + F_d^1\left(q_0, q_1\right) \cdot \delta q_0 \\
&+ \sum_{k=1}^{n-1} \big(F_d^1\left(q_k, q_{k+1}\right) + F_d^2\left(q_{k-1}, q_k\right)\big) \cdot \delta q_k + F_d^2\left(q_{n-1}, q_n\right) \cdot \delta q_n \\
={}& \big(D_1 L_d\left(q_0, q_1\right) + F_d^1\left(q_0, q_1\right)\big) \cdot \delta q_0 + \big(D_2 L_d\left(q_{n-1}, q_n\right) + F_d^2\left(q_{n-1}, q_n\right)\big) \cdot \delta q_n \\
&+ \sum_{k=1}^{n-1} \underbrace{\left[D_2 L_d\left(q_{k-1}, q_k\right) + D_1 L_d\left(q_k, q_{k+1}\right) + F_d^2\left(q_{k-1}, q_k\right) + F_d^1\left(q_k, q_{k+1}\right)\right]}_{0} \cdot \delta q_k \\
={}& \big(D_1 L_d\left(q_0, q_1\right) + F_d^1\left(q_0, q_1\right)\big) \cdot \delta q_0 + \big(D_2 L_d\left(q_{n-1}, q_n\right) + F_d^2\left(q_{n-1}, q_n\right)\big) \cdot \delta q_n.
\end{aligned}
$$

Conversely, for an arbitrary discrete curve $\mathbf{q}$ and an arbitrary variation $\delta\mathbf{q}$, the final equality only holds if $\mathbf{q}$ satisfies

$$
D_2 L_d\left(q_{k-1}, q_k\right) + D_1 L_d\left(q_k, q_{k+1}\right) + F_d^2\left(q_{k-1}, q_k\right) + F_d^1\left(q_k, q_{k+1}\right) = 0,
$$

which is the forced DEL equation.

Therefore, we have that $\mathbf{q}$ solves the forced discrete Euler–Lagrange equations if, and only if,

$$
\begin{aligned}
&\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \sum_{k=0}^{n-1} L_d\left(q_{k_\epsilon}, q_{k+1_\epsilon}\right) + \sum_{k=0}^{n-1} \left[F_d^1\left(q_k, q_{k+1}\right) \cdot \delta q_k + F_d^2\left(q_k, q_{k+1}\right) \cdot \delta q_{k+1}\right] \\
&\quad = \big(D_1 L_d\left(q_0, q_1\right) + F_d^1\left(q_0, q_1\right)\big) \cdot \delta q_0 + \big(D_2 L_d\left(q_{n-1}, q_n\right) + F_d^2\left(q_{n-1}, q_n\right)\big) \cdot \delta q_n,
\end{aligned}
$$

for all variations, including those that do not vanish at the endpoints.

Let $\mathbf{x}$ be the projection of $\mathbf{q}$, the solution of the forced DEL equations, onto the shape space S, and $\delta\mathbf{x} = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \mathbf{x}_\epsilon$ be a variation of $\mathbf{x}$. Since (q_k, q_{k+1}) is on the μ-momentum surface, and $(\delta q_k, \delta q_{k+1})$ is tangent to the momentum surface, we have by the construction of $\hat{F}_d$ the following relations

$$
\begin{aligned}
&\hat{F}_d^1\left(x_k, x_{k+1}\right) \cdot \delta x_k + \hat{F}_d^2\left(x_k, x_{k+1}\right) \cdot \delta x_{k+1} \\
&\quad = \hat{F}_d\left(x_k, x_{k+1}\right) \cdot \left(\delta x_k, 0\right) + \hat{F}_d\left(x_k, x_{k+1}\right) \cdot \left(0, \delta x_{k+1}\right) \\
&\quad = \hat{F}_d\left(x_k, x_{k+1}\right) \cdot \left(\delta x_k, \delta x_{k+1}\right) \\
&\quad = F_d\left(q_k, q_{k+1}\right) \cdot \left(\delta q_k, \delta q_{k+1}\right) \\
&\quad = F_d\left(q_k, q_{k+1}\right) \cdot \left(\delta q_k, 0\right) + F_d\left(q_k, q_{k+1}\right) \cdot \left(0, \delta q_{k+1}\right) \\
&\quad = F_d^1\left(q_k, q_{k+1}\right) \cdot \delta q_k + F_d^2\left(q_k, q_{k+1}\right) \cdot \delta q_{k+1}.
\end{aligned}
$$

This allows us to rewrite the sum over discrete forces in the discrete Lagrange–d'Alembert principle in terms of a sum over the reduced discrete forces,

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \sum_{k=0}^{n-1} L_d\left(q_{k_\epsilon}, q_{k+1_\epsilon}\right) + \sum_{k=0}^{n-1} \left[\hat{F}_d^1\left(x_k, x_{k+1}\right) \cdot \delta x_k + \hat{F}_d^2\left(x_k, x_{k+1}\right) \cdot \delta x_{k+1}\right]$$
$$= \left(D_1 L_d\left(q_0, q_1\right) + F_d^1\left(q_0, q_1\right)\right) \cdot \delta q_0 + \left(D_2 L_d\left(q_{n-1}, q_n\right) + F_d^2\left(q_{n-1}, q_n\right)\right) \cdot \delta q_n.$$

Splitting the variations into horizontal and vertical components, and using the assumption that the discrete forces vanish on vertical vectors, we have

$$\begin{aligned}
&\left(D_1 L_d\left(q_0, q_1\right) + F_d^1\left(q_0, q_1\right)\right) \cdot \delta q_0 + \left(D_2 L_d\left(q_{n-1}, q_n\right) + F_d^2\left(q_{n-1}, q_n\right)\right) \cdot \delta q_n \\
&\quad = \left(D_1 L_d\left(q_0, q_1\right) + F_d^1\left(q_0, q_1\right)\right) \cdot \left(\text{ver } \delta q_0 + \text{hor } \delta q_0\right) \\
&\qquad + \left(D_2 L_d\left(q_{n-1}, q_n\right) + F_d^2\left(q_{n-1}, q_n\right)\right) \cdot \left(\text{ver } \delta q_n + \text{hor } \delta q_n\right) \\
&\quad = \left(D_1 L_d\left(q_0, q_1\right) + F_d^1\left(q_0, q_1\right)\right) \cdot \text{hor } \delta q_0 + \left(D_2 L_d\left(q_{n-1}, q_n\right) + F_d^2\left(q_{n-1}, q_n\right)\right) \cdot \text{hor } \delta q_n \\
&\qquad - \mathfrak{A}_\mu\left(q_0\right) \cdot \delta q_0 + \mathfrak{A}_\mu\left(q_n\right) \cdot \delta q_n \\
&\quad = \left(D_1 L_d\left(q_0, q_1\right) + F_d^1\left(q_0, q_1\right)\right) \cdot \text{hor } \delta q_0 + \left(D_2 L_d\left(q_{n-1}, q_n\right) + F_d^2\left(q_{n-1}, q_n\right)\right) \cdot \text{hor } \delta q_n \\
&\qquad + \sum_{k=0}^{n-1} \mathcal{A}\left(q_k, q_{k+1}\right) \cdot \left(\delta q_k, \delta q_{k+1}\right),
\end{aligned}$$

where, as before, we used Equation 2.3.19 for the second to last equality. Then,

$$\begin{aligned}
&\left(D_1 L_d\left(q_0, q_1\right) + F_d^1\left(q_0, q_1\right)\right) \cdot \text{hor } \delta q_0 + \left(D_2 L_d\left(q_{n-1}, q_n\right) + F_d^2\left(q_{n-1}, q_n\right)\right) \cdot \text{hor } \delta q_n \\
&\quad = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \sum_{k=0}^{n-1} L_d\left(q_{k_\epsilon}, q_{k+1_\epsilon}\right) + \sum_{k=0}^{n-1} \left[\hat{F}_d^1\left(x_k, x_{k+1}\right) \cdot \delta x_k + \hat{F}_d^2\left(x_k, x_{k+1}\right) \cdot \delta x_{k+1}\right] \\
&\qquad - \sum_{k=0}^{n-1} \mathcal{A}\left(q_k, q_{k+1}\right) \cdot \left(\delta q_k, \delta q_{k+1}\right).
\end{aligned}$$

If the variations $\delta \mathbf{x}$ vanishes at the endpoints, i.e., $\delta x_0 = \delta x_n = 0$, then hor $\delta q_0 =$ hor $\delta q_n = 0$, and therefore,

$$\delta \sum_{k=0}^{n-1} \hat{L}_d\left(x_{k_\epsilon}, x_{k+1_\epsilon}\right) + \sum_{k=0}^{n-1} \left[\hat{F}_d^1\left(x_k, x_{k+1}\right) \cdot \delta x_k + \hat{F}_d^2\left(x_k, x_{k+1}\right) \cdot \delta x_{k+1}\right]$$
$$= \sum_{k=0}^{n-1} \hat{\mathcal{A}}\left(x_k, x_{k+1}\right) \cdot \left(\delta x_k, \delta x_{k+1}\right),$$

for all variations $\delta \mathbf{x}$ of $\mathbf{x}$ that vanish at the endpoints.

Since the variations are arbitrary and vanish at the endpoints, this is equivalent to the ***Discrete***

Routh equations with forcing,

$$D_2\hat{L}_d(x_{k-1},x_k)+D_1\hat{L}_d(x_k,x_{k+1})+\hat{F}_d^2(x_{k-1},x_k)+\hat{F}_d^1(x_k,x_{k+1})$$
$$=\hat{\mathcal{A}}_2(x_{k-1},x_k)+\hat{\mathcal{A}}_1(x_k,x_{k+1}).$$

Conversely, if $\mathbf{x}$ satisfies the reduced variational principle, and $\mathbf{q}$ is its lift onto the μ-momentum surface, then a construction analogous to the derivation of the discrete Routh equations show that $\mathbf{q}$ satisfies the discrete Lagrange–d'Alembert principle. □

2.8.3 Routh Reduction with Constraints and Forcing

By applying the techniques of the previous sections, we may synthesize the formalisms involving constraints and forcing. We shall state, without proof, the relevant equations in the following theorem.

Theorem 2.17. *Let* $\mathbf{x}$ *be a discrete curve on* S*, and let* $\mathbf{q}$ *be a discrete curve on* Q *with momentum* μ *that is obtained by lifting* $\mathbf{x}$*. Let* $\mathbf{y}$ *be a discrete curve on* U *obtained from* $\mathbf{x}$ *by the inclusion* $S=\hat{g}^{-1}(0)\hookrightarrow U$*, and let* $\mathbf{v}$ *be a discrete curve on* V *with momentum* μ *that is obtained by lifting* $\mathbf{y}$*. Then, the following are equivalent.*

1. $\mathbf{v}$ *solves the forced discrete Euler–Lagrange equations with constraints,*

$$D_2L_d(v_{k-1},v_k)+D_1L_d(v_k,v_{k+1})+F_d^2(v_{k-1},v_k)+F_d^1(v_k,v_{k+1})+\lambda_k^TDg(v_k)=0,$$
$$g(v_k)=0.$$

2. $\mathbf{v}$ *is a solution of the discrete Lagrange–d'Alembert variational principle with constraints,*

$$\delta\left[\sum_{k=0}^{n-1}L_d(v_k,v_{k+1})+\sum_{k=0}^{n}\lambda_k^Tg(v_k)\right]$$
$$+\sum_{k=0}^{n-1}\left[F_d^1(v_k,v_{k+1})\cdot\delta v_k+F_d^2(v_k,v_{k+1})\cdot\delta v_{k+1}\right]=0,$$

for all variations $\delta\mathbf{v}$ *of* $\mathbf{v}$ *that vanish at the endpoints, and* $g(v_k)=0$.

3. $\mathbf{y}$ *solves the Discrete Routh equations with forcing and constraints,*

$$D_2\hat{L}_d(y_{k-1},y_k)+D_1\hat{L}_d(y_k,y_{k+1})+\hat{F}_d^2(y_{k-1},y_k)+\hat{F}_d^1(y_k,y_{k+1})+\lambda_k^TD\hat{g}(y_k)$$
$$=\hat{\mathcal{A}}_2(y_{k-1},y_k)+\hat{\mathcal{A}}_1(y_k,y_{k+1}),$$
$$\hat{g}(y_k)=0.$$

4. $\mathbf{y}$ *is a solution of the reduced variational principle,*

$$\delta\left[\sum_{k=0}^{n-1}\hat{L}_d\left(y_k,y_{k+1}\right)+\sum_{k=0}^{n}\lambda_k^T\hat{g}\left(y_k\right)\right]$$
$$+\sum_{k=0}^{n-1}\left[\hat{F}_d^1\left(y_k,y_{k+1}\right)\cdot\delta y_k+\hat{F}_d^2\left(y_k,y_{k+1}\right)\cdot\delta y_{k+1}\right]=\sum_{k=0}^{n-1}\hat{\mathcal{A}}\left(y_k,y_{k+1}\right)\cdot\left(\delta y_k,\delta y_{k+1}\right),$$

for all variations $\delta\mathbf{y}$ *of* $\mathbf{y}$ *that vanish at the endpoints, and* $\hat{g}\left(y_k\right)=0$.

2.9 Example: J_2 Satellite Dynamics

2.9.1 Configuration Space and Lagrangian

An illustrative and important example of a system with an abelian symmetry group is that of a single satellite in orbit about an oblate Earth. The general aspects and background for this problem are discussed in Prussing and Conway [1993], and some interesting aspects of the geometry underlying it are discussed in Chang and Marsden [2003].

The configuration manifold Q is $\mathbb{R}^3$, and the Lagrangian for the system has the form, kinetic minus potential energy,

$$L(q,\dot{(q)})=\frac{1}{2}M_s\|\dot{q}\|^2-M_sV(q),$$

where M_s is the mass of the satellite and $V:\mathbb{R}^3\to\mathbb{R}$ is the gravitational potential due to the Earth, truncated at the first term in the expansion in the ellipticity,

$$V(q)=\frac{GM_e}{\|q\|}+\frac{GM_eR_e^2J_2}{\|q\|^3}\left(\frac{3}{2}\frac{(q^3)^2}{\|q\|^2}-\frac{1}{2}\right).$$

Here, G is the gravitational constant, M_e is the mass of the Earth, R_e is the radius of the Earth, J_2 is a small non-dimensional parameter describing the degree of ellipticity, and q^3 is the third component of q.

We will now assume that we are working in non-dimensional coordinates, so that

$$L(q,\dot{q})=\frac{1}{2}\|\dot{q}\|^2-\left[\frac{1}{\|q\|}+\frac{J_2}{\|q\|^3}\left(\frac{3}{2}\frac{(q^3)^2}{\|q\|^2}-\frac{1}{2}\right)\right]. \qquad (2.9.1)$$

This corresponds to choosing space and time coordinates in which the radius of the Earth is 1 and the period of orbit at zero altitude is 2π when $J_2=0$ (spherical Earth).

2.9.2 Symmetry Action

The symmetry of interest to us is that of rotation about the vertical (q^3) axis, so the symmetry group is the unit circle S^1. Using cylindrical coordinates, $q = (r, \theta, z)$, for the configuration, the symmetry action is $\phi : (r, \theta, z) \mapsto (r, \theta + \phi, z)$. Since $\|q\|$, $\|\dot{q}\|$, and $q^3 = z$ are all invariant under this transformation, so too is the Lagrangian.

This action is clearly not free on all of $Q = \mathbb{R}^3$, as the z-axis is invariant for all group elements. This is not a serious obstacle, however, as the lifted action is free on $T(Q \setminus (0, 0, 0))$ and this is enough to permit the application of the intrinsic Routh reduction theory outlined in §2.2. Alternatively, one can simply take $Q = \mathbb{R}^3 \setminus \{(0, 0, z) \mid z \in \mathbb{R}\}$ and then the theory literally applies.

The shape space, $S = Q/G$, is the half-plane $S = \mathbb{R}^+ \times \mathbb{R}$ and we will take coordinates (r, z) on S. In doing so, we are implicitly defining a global diffeomorphism $S \times G \to Q$ given by $((r, z), \theta) \mapsto (r, \theta, z)$.

The Lie algebra $\mathfrak{g}$ for $G = S^1$ is the real line $\mathfrak{g} = \mathbb{R}$, and we will identify the dual with the real line itself, $\mathfrak{g}^* \cong \mathbb{R}$. For a Lie algebra element $\xi \in \mathfrak{g}$, the corresponding infinitesimal generator is given by

$$\xi_Q : (r, \theta, z) \mapsto ((r, \theta, z), (0, \xi, 0)).$$

Recall that the Lagrange momentum map, $J_L : TQ \to \mathfrak{g}^*$, is defined by

$$J_L(v_q) \cdot \xi = \langle \mathbb{F}L(v_q), \xi_Q(q) \rangle,$$

so we have

$$J_L((r, \theta, z), (\dot{r}, \dot{\theta}, \dot{z})) \cdot \xi = \Big\langle (\dot{r}, r^2\dot{\theta}, \dot{z}), (0, \xi, 0) \Big\rangle = r^2\dot{\theta}\xi,$$

and

$$J_L((r, \theta, z), (\dot{r}, \dot{\theta}, \dot{z})) = r^2\dot{\theta}.$$

This momentum map is simply the vertical component of the standard angular momentum.

Consider the Euclidean metric on $\mathbb{R}^3$, which corresponds to the kinetic energy norm in the Lagrangian. From this metric we define the mechanical connection, $\mathfrak{A} : TQ \to \mathfrak{g}$, which is given by $\mathfrak{A}((r, \theta, z), (\dot{r}, \dot{\theta}, \dot{z})) = \dot{\theta}$. The 1-form $\mathfrak{A}_\mu$ on Q is thus given by $\mathfrak{A}_\mu = \mu \mathbf{d}\theta$. Taking the exterior derivative of this expression gives $\mathbf{d}\mathfrak{A}_\mu = \mu \mathbf{d}^2\theta = 0$, and so the reduced 2-form is $\beta_\mu = 0$.

2.9.3 Equations of Motion

Computing the Euler–Lagrange equations for the Lagrangian (Equation 2.9.1) gives the equations of motion,

$$\ddot{q} = -\nabla_q \left[\frac{1}{\|q\|} + \frac{J_2}{\|q\|^3} \left(\frac{3}{2} \frac{(q^3)^2}{\|q\|^2} - \frac{1}{2} \right) \right].$$

To calculate the reduced equations, we begin by calculating the Routhian,

$$\begin{aligned} R^\mu(r,\theta,z,\dot{r},\dot{\theta},\dot{z}) &= L(r,\theta,z,\dot{r},\dot{\theta},\dot{z}) - \mathfrak{A}_\mu(r,\theta,z)\cdot(\dot{r},\dot{\theta},\dot{z}) \\ &= \frac{1}{2}\|(\dot{r},\dot{\theta},\dot{z})\|^2 - \left[\frac{1}{r} + \frac{J_2}{r^3}\left(\frac{3}{2}\frac{z^2}{r^2} - \frac{1}{2} \right) \right] - \mu\dot{\theta}. \end{aligned}$$

We now choose a fixed value μ of the momentum and restrict ourselves to the space $J_L^{-1}(\mu)$, on which $\dot{\theta} = \mu$. The reduced Routhian, $\hat{R}^\mu : TS \to \mathbb{R}$, is the restricted Routhian dropped to the tangent bundle of the shape space. In coordinates, this is

$$\hat{R}^\mu(r,z,\dot{r},\dot{z}) = \frac{1}{2}\|(\dot{r},\dot{z})\|^2 - \left[\frac{1}{r} + \frac{J_2}{r^3}\left(\frac{3}{2}\frac{z^2}{r^2} - \frac{1}{2} \right) \right] - \frac{1}{2}\mu^2.$$

Recalling that $\beta_\mu = 0$, the Routh equations (Equation 2.2.3) can now be evaluated to give

$$(\ddot{r},\ddot{z}) = -\nabla_{(r,z)} \left[\frac{1}{r} + \frac{J_2}{r^3}\left(\frac{3}{2}\frac{z^2}{r^2} - \frac{1}{2} \right) \right],$$

which describes the motion on the shape space.

To recover the unreduced Euler–Lagrange equations from the Routh equations, one uses the procedure of reconstruction. This is covered in detail in Marsden et al. [1990], Marsden [1992] and Marsden et al. [2000b].

2.9.4 Discrete Lagrangian System

We now discretize this system with the discrete Lagrangian used in Theorem 2.12. Recall that the push-forward discrete Lagrange map associated with this discrete Lagrangian is a symplectic partitioned Runge–Kutta method with coefficients given by Equation 2.4.3.

Given a point $(q_0, q_1) \in Q \times Q$ we will take (q_0, p_0) and (q_1, p_1) to be the associated discrete Legendre transforms. As the discrete momentum map is the pull-back of the canonical momentum map, we have that

$$J_{L_d}(q_0, q_1) = (p_\theta)_0 = (p_\theta)_1.$$

Take a fixed momentum map value μ and restrict L_d to the set $J_{L_d}^{-1}(\mu)$. Dropping this to $S \times S$ now gives the reduced discrete Lagrangian, $\hat{L}_d : S \times S \to \mathbb{R}$.

As discussed in §2.5, the fact that we have taken coordinates in which the group action is addition in θ means that the push-forward discrete Lagrange map associated with the reduced discrete Lagrangian is the reduced method given by Equation 2.5.14. In fact, as the mechanical connection has $A(r,z) = 0$ and $\beta_\mu = 0$, the push-forward discrete Lagrange map is exactly a partitioned Runge–Kutta method with Hamiltonian equal to the reduced Routhian. As we saw in §2.7, these are generically related by a momentum shift, rather than being equal.

Given a trajectory of the reduced discrete system, we can reconstruct a trajectory of the unreduced discrete system by solving for the θ component of Equation 2.5.1. Correspondingly, a trajectory of the unreduced discrete system can be projected onto the shape space to give a trajectory of the reduced discrete system.

2.9.5 Example Trajectories

Solutions of the Spherical Earth System. Consider initially the system with $J_2 = 0$. This corresponds to the case of a spherical Earth, and so the equations reduce to the standard Kepler problem. As this is an integrable system, the trajectories consist of periodic orbits.

A slightly inclined circular trajectory is shown in Figure 2.2, in both the unreduced and reduced pictures. Note that the graph of the reduced trajectory is a quadratic, as $\|q\| = \sqrt{r^2 + z^2}$ is a constant.

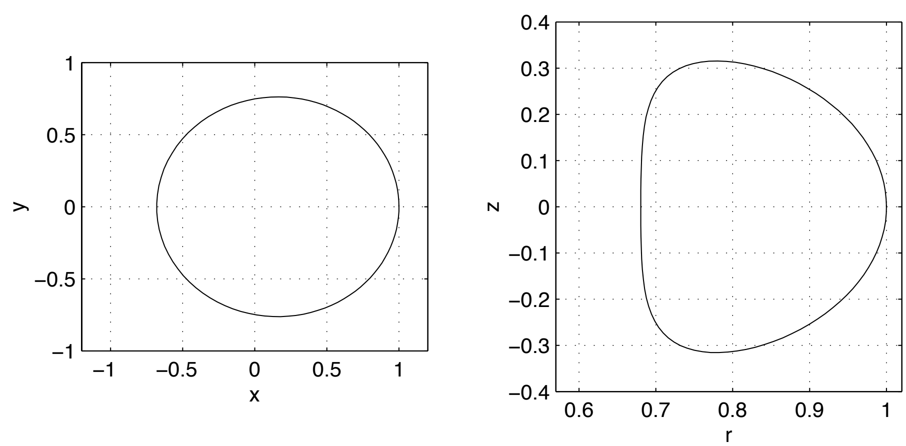

Figure 2.2: Unreduced (left) and reduced (right) views of an inclined elliptic trajectory for the continuous time system with $J_2 = 0$ (spherical Earth).

We will now investigate the effect of two different perturbations to the system, one due to taking non-zero J_2 and the other due to the numerical discretization.

The J_2 Effect. Taking $J_2 = 0.05$ (which is close the actual value for the Earth), the system becomes near-integrable and experiences breakup of the KAM tori. This can be seen in Figure 2.3, where the same initial condition is used as in Figure 2.2.

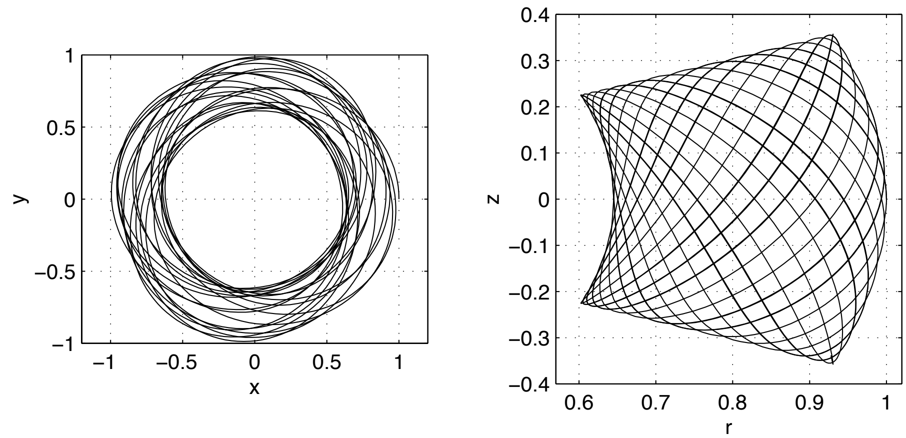

Figure 2.3: Unreduced (left) and reduced (right) views of an inclined elliptic trajectory for the continuous time system with $J_2 = 0.05$. Observe that the non-spherical terms introduce precession of the near-elliptic orbit in the symmetry direction.

Due to the fact that the reduced trajectory is no longer a simple curve, there is a geometric-phase-like effect which causes precession of the orbit. This precession can be seen in the thickening of the unreduced trajectory.

Solutions of the Discrete System for a Spherical Earth. We now consider the discrete system with $J_2 = 0$, for the second-order Gauss–Legendre discrete Lagrangian with timestep of $h = 0.3$. The trajectory with the same initial condition as above is given in Figure 2.4.

As can be seen from the reduced trajectory, the discretization has caused a similar breakup of the periodic orbit as was produced by the non-zero J_2. The effect of this is to, once again, induce precession of the orbit in the unreduced trajectory, in a way which is difficult to distinguish from the perturbation above due to non-zero J_2 when only the unreduced picture is considered. If the reduced pictures are consulted, however, then it is immediately clear that the system is much closer to the continuous time system with $J_2 = 0$ than to the system with non-zero J_2.

Solutions of the Discrete System with J_2 Effect. Finally, we consider the discrete system with non-zero $J_2 = 0.05$. The resulting trajectory is shown in Figure 2.5, and, clearly, it is not easy to determine from the unreduced picture whether the precession is due to the J_2 perturbation, the discretization, or some combination of the two.

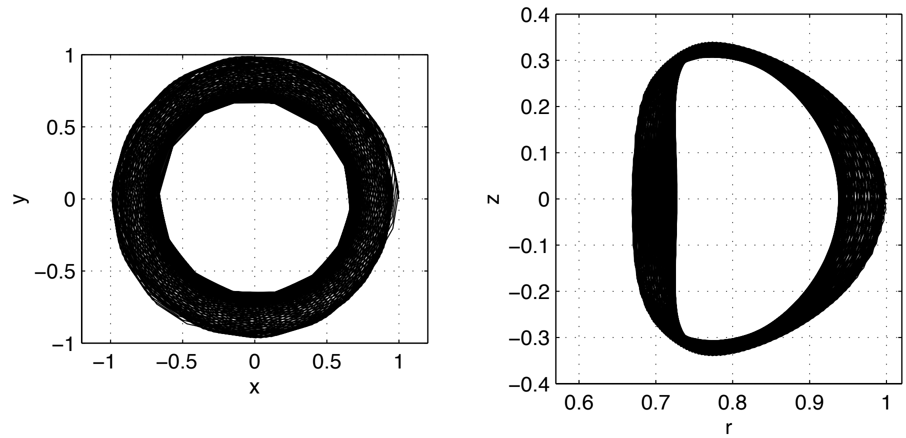

Figure 2.4: Unreduced (left) and reduced (right) views of an inclined trajectory of the discrete system with step-size $h = 0.3$ and $J_2 = 0$. The initial condition is the same as that used in Figure 2.2. The numerically introduced precession means that the unreduced picture looks similar to that of Figure 2.3 with non-zero J_2, whereas, by considering the reduced picture we can see the correct resemblance to the $J_2 = 0$ case of Figure 2.2.

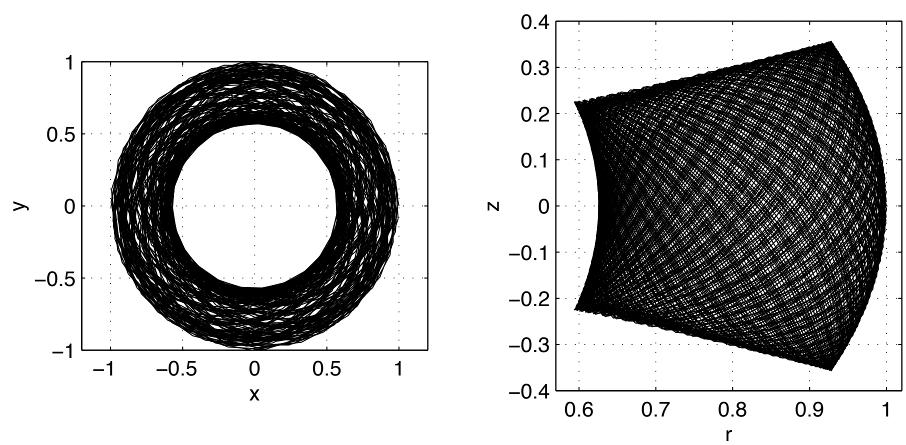

Figure 2.5: Unreduced (left) and reduced (right) views of an inclined trajectory of the discrete system with step-size $h = 0.3$ and $J_2 = 0.05$. The initial condition is the same as that used in Figure 2.3. The unreduced picture is similar to that of both Figures 2.3 and 2.4. By considering the reduced picture, we obtain the correct resemblance to Figure 2.3.

Taking the reduced trajectories, however, immediately shows that this discrete time system is structurally much closer to the non-zero J_2 system than to the original $J_2 = 0$ system. This confusion arises because both the J_2 term and the discretization introduce perturbations which act in the symmetry direction.

While this system is sufficiently simple that one can run simulations with such small timesteps that the discretization artifacts become negligible, this is not possible in general. This example demonstrates how knowledge of the geometry of the system can be important in understanding the discretization process, and how this can give insight into the behavior of numerical simulations. In particular, understanding how the discretization interacts with the symmetry action is extremely important.

2.9.6 Coordinate Systems

In this example, we have chosen cylindrical coordinates, thus making the group action addition in θ. One can always do this, as an abelian Lie group is isomorphic to a product of copies of $\mathbb{R}$ and S^1, but it may sometimes be preferable to work in coordinates in which the group action is not addition. For example, cartesian coordinates in the present example.

It may be easier, both in terms of computational expense, and in the simplicity of expressions, if we adopt a coordinate system in which the group action is not addition. We can still apply the Discrete Routh equations to obtain an integration scheme on $S \times S$. The push-forward of this under $\hat{\mathbb{F}}$ yields an integration scheme on T^*S. The trajectories on the shape space that we obtain in this manner could be different from those we would get with the RSPRK method. However, in both cases we would have conservation of symplectic structure, momentum, and the order of accuracy would be the same. One could choose whichever approach is cheaper and easier.

2.10 Example: Double Spherical Pendulum

2.10.1 Configuration Space and Lagrangian

We consider the example of the double spherical pendulum which has a non-trivial magnetic term and constraints. The configuration manifold Q is $S^2 \times S^2$, and the embedding linear space V is $\mathbb{R}^3 \times \mathbb{R}^3$.

The position vectors of each pendulum with respect to their pivot point are denoted by $\mathbf{q}_1$ and $\mathbf{q}_2$, as illustrated in Figure 2.6. These vectors are constrained to have lengths l_1 and l_2, respectively, and the pendula masses are denoted by m_1 and m_2.

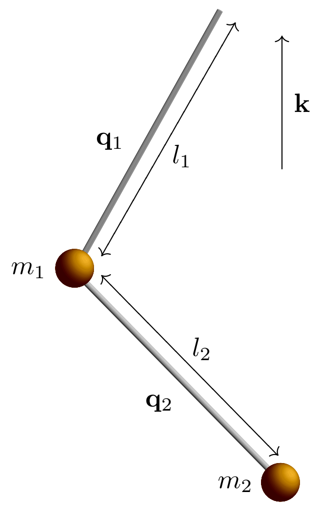

Figure 2.6: Double spherical pendulum.

The Lagrangian for the system has the form, kinetic minus potential energy,

$$L(\mathbf{q}_1, \mathbf{q}_2, \dot{\mathbf{q}}_1, \dot{\mathbf{q}}_2) = \frac{1}{2} m_1 \|\dot{\mathbf{q}}_1\|^2 + \frac{1}{2} m_2 \|\dot{\mathbf{q}}_1 + \dot{\mathbf{q}}_2\|^2 - m_1 g \mathbf{q}_1 \cdot \mathbf{k} - m_2 g (\mathbf{q}_1 + \mathbf{q}_2) \cdot \mathbf{k},$$

where g is the gravitational constant, and $\mathbf{k}$ is the unit vector in the z direction. The constraint function, $c : V \to \mathbb{R}^2$, is given by

$$c(\mathbf{q}_1, \mathbf{q}_2) = (\|\mathbf{q}_1\| - l_1, \|\mathbf{q}_2\| - l_2).$$

Using cylindrical coordinates, $\mathbf{q}_i = (r_i, \theta_i, z_i)$, for the configuration, we can express the Lagrangian as

$$\begin{aligned} L(q, \dot{q}) = {} & \frac{1}{2} m_1 \left(\dot{r}_1^2 + r_1^2 \dot{\theta}_1^2 + \dot{z}_1^2 \right) + \frac{1}{2} m_2 \Big\{ \dot{r}_1^2 + r_1^2 \dot{\theta}_1^2 + \dot{r}_2^2 + r_2^2 \dot{\theta}_2^2 \\ & + 2 \left(\dot{r}_1 \dot{r}_2 + r_1 r_2 \dot{\theta}_1 \dot{\theta}_2 \right) \cos\varphi + 2 \left(r_1 \dot{r}_2 \dot{\theta}_1 - r_2 \dot{r}_1 \dot{\theta}_2 \right) \sin\varphi + (\dot{z}_1 + \dot{z}_2)^2 \Big\} \\ & - m_1 g z_1 - m_2 g \left(z_1 + z_2 \right), \end{aligned}$$

where $\varphi = \theta_2 - \theta_1$. Furthermore, we can automatically satisfy the constraints by performing the following substitutions,

$$z_i = \sqrt{l_i^2 - r_i^2}, \qquad \dot{z}_i = -\frac{r_i \dot{r}_i}{\sqrt{l_i^2 - r_i^2}}.$$

2.10.2 Symmetry Action

The symmetry of interest to us is the simultaneous rotation of the two pendula about vertical (z) axis, so the symmetry group is the unit circle S^1. Using cylindrical coordinates, $\mathbf{q}_i = (r_i, \theta_i, z_i)$, for the configuration, the symmetry action is $\phi : (r_i, \theta_i, z_i) \mapsto (r_i, \theta_i + \phi, z_i)$. Since $\|\mathbf{q}_i\|$, $\|\dot{\mathbf{q}}_i\|$, $\|\dot{\mathbf{q}}_1 + \dot{\mathbf{q}}_2\|$, and $\mathbf{q}_i \cdot \mathbf{k}$ are all invariant under this transformation, so too is the Lagrangian.

This action is clearly not free on all of $V = \mathbb{R}^3 \times \mathbb{R}^3$, as the z-axis is invariant for all group elements. However, this does not pose a problem computationally, as long as the trajectories do not pass through the downward hanging configuration, corresponding to $r_1 = r_2 = 0$. To treat the downward handing configuration properly, we would need to develop a discrete Lagrangian analogue of the continuous theory of singular reduction described in Ortega and Ratiu [2001].

The Lie algebra $\mathfrak{g}$ for $G = S^1$ is the real line $\mathfrak{g} = \mathbb{R}$, and we will identify the dual with the real line itself $\mathfrak{g}^* \cong \mathbb{R}$. For a Lie algebra element $\xi \in \mathfrak{g}$, the corresponding infinitesimal generator is given by

$$\xi_Q : (r_1, \theta_1, z_1, r_2, \theta_2, z_2) \mapsto ((r_1, \theta_1, z_1, r_2, \theta_2, z_2), (0, \xi, 0, 0, \xi, 0)).$$

Recall that the Lagrange momentum map $J_L : TQ \to \mathfrak{g}^*$ is defined by

$$J_L(v_q) \cdot \xi = \langle \mathbb{F}L(v_q), \xi_Q(q) \rangle,$$

so we have

$$\begin{aligned}
&J_L((r_1, \theta_1, z_1, r_2, \theta_2, z_2), (\dot{r}_1, \dot{\theta}_1, \dot{z}_1, \dot{r}_2, \dot{\theta}_2, \dot{z}_2)) \cdot \xi \\
&\quad = \Big\langle \Big(m_1 \dot{r}_1 + m_2 \left[\dot{r}_1 + \dot{r}_2 \cos\varphi - r_2 \dot{\theta}_2 \sin\varphi \right], \\
&\qquad m_1 r_1^2 \dot{\theta}_1 + m_2 \left[r_1^2 \dot{\theta}_1 + r_1 r_2 \dot{\theta}_2 \cos\varphi + r_1 \dot{r}_2 \sin\varphi \right], \\
&\qquad m_1 \dot{z}_1 + m_2 \left[\dot{z}_1 + \dot{z}_2 \right], m_2 \left[\dot{r}_2 + \dot{r}_1 \cos\varphi + r_1 \dot{\theta}_1 \sin\varphi \right], \\
&\qquad m_2 \left[r_2^2 \dot{\theta}_2 + r_1 r_2 \dot{\theta}_1 \cos\varphi - r_2 \dot{r}_1 \sin\varphi \right], m_2 \left[\dot{z}_1 + \dot{z}_2 \right] \Big), (0, \xi, 0, 0, \xi, 0) \Big\rangle \\
&\quad = \Big((m_1 + m_2) r_1^2 \dot{\theta}_1 + m_2 r_2^2 \dot{\theta}_2 + m_2 r_1 r_2 \left(\dot{\theta}_1 + \dot{\theta}_2 \right) \cos\varphi + (r_1 \dot{r}_2 - r_2 \dot{r}_1) \sin\varphi \Big) \xi,
\end{aligned}$$

and

$$\begin{aligned}
&J_L((r_1, \theta_1, z_1, r_2, \theta_2, z_2), (\dot{r}_1, \dot{\theta}_1, \dot{z}_1, \dot{r}_2, \dot{\theta}_2, \dot{z}_2)) \\
&\quad = (m_1 + m_2) r_1^2 \dot{\theta}_1 + m_2 r_2^2 \dot{\theta}_2 + m_2 r_1 r_2 \left(\dot{\theta}_1 + \dot{\theta}_2 \right) \cos\varphi + (r_1 \dot{r}_2 - r_2 \dot{r}_1) \sin\varphi.
\end{aligned}$$

This momentum map is simply the vertical component of the standard angular momentum.

The locked inertia tensor is given by Marsden [1992],

$$\begin{aligned}\mathbb{I}(\mathbf{q}_1\mathbf{q}_2) &= m_1\|\mathbf{q}_1^\perp\|^2 + m_2\|(\mathbf{q}_1+\mathbf{q}_2)^\perp\|^2\\ &= m_1r_1^2 + m_2(r_1^2+r_2^2+2r_1r_2\cos\varphi).\end{aligned}$$

Furthermore, the mechanical connection is given by

$$\begin{aligned}\alpha(\mathbf{q}_1,\mathbf{q}_2,\dot{\mathbf{q}}_1,\dot{\mathbf{q}}_2) &= \mathbb{I}(\mathbf{q}_1,\mathbf{q}_2)^{-1}J_L(\mathbf{q}_1,\mathbf{q}_2,\dot{\mathbf{q}}_1,\dot{\mathbf{q}}_2)\\ &= \frac{(m_1+m_2)\,r_1^2\dot\theta_1 + m_2r_2^2\dot\theta_2 + m_2r_1r_2\left(\dot\theta_1+\dot\theta_2\right)\cos\varphi + (r_1\dot r_2 - r_2\dot r_1)\sin\varphi}{m_1r_1^2+m_2(r_1^2+r_2^2+2r_1r_2\cos\varphi)}.\end{aligned}$$

As a 1-form, it is given by

$$\begin{aligned}\alpha(\mathbf{q}_1,\mathbf{q}_2) &= \frac{1}{m_1r_1^2+m_2(r_1^2+r_2^2+2r_1r_2\cos\varphi)}\\ &\quad\times\left[(m_1+m_2)\,r_1^2\mathbf{d}\theta_1 + m_2r_2^2\mathbf{d}\theta_2 + m_2r_1r_2\left(\mathbf{d}\theta_1+\mathbf{d}\theta_2\right)\cos\varphi\right.\\ &\quad\left.+(r_1\mathbf{d}r_2 - r_2\mathbf{d}r_1)\sin\varphi\right].\end{aligned}$$

The μ-component of the mechanical connection is given by

$$\begin{aligned}\alpha_\mu(\mathbf{q}_1,\mathbf{q}_2) &= \frac{\mu}{m_1r_1^2+m_2(r_1^2+r_2^2+2r_1r_2\cos\varphi)}\\ &\quad\times\left\{\left[(m_1+m_2)\,r_1^2 + m_2r_1r_2\cos\varphi\right]\mathbf{d}\theta_1 + \left[m_2r_2^2+m_2r_1r_2\cos\varphi\right]\mathbf{d}\theta_2\right\}.\end{aligned}$$

Taking the exterior derivative of this 1-form yields a non-trivial magnetic term on the reduced space,

$$\begin{aligned}\mathbf{d}\alpha_\mu &= \frac{\mu}{\left[m_1r_1^2+m_2\left(r_1^2+r_2^2+2r_1r_2\cos\varphi\right)\right]^2}\\ &\quad\times\{m_2r_2\left[2\left(m_1+m_2\right)r_1r_2 + \left(m_1r_1^2+m_2(r_1^2+r_2^2)\right)\cos\varphi\right]\mathbf{d}r_1\wedge\mathbf{d}\theta_1\\ &\quad - m_2r_1\left[2\left(m_1+m_2\right)r_1r_2 + \left(m_1r_1^2+m_2(r_1^2+r_2^2)\right)\cos\varphi\right]\mathbf{d}r_2\wedge\mathbf{d}\theta_1\\ &\quad + m_2r_1r_2\sin\varphi[m_1r_1^2+m_2(r_1^2-r_2^2)]\mathbf{d}\theta_2\wedge\mathbf{d}\theta_1\\ &\quad - m_2r_2\left[2\left(m_1+m_2\right)r_1r_2 + \left(m_1r_1^2+m_2(r_1^2+r_2^2)\right)\cos\varphi\right]\mathbf{d}r_1\wedge\mathbf{d}\theta_2\\ &\quad + m_2r_1\left[2\left(m_1+m_2\right)r_1r_2 + \left(m_1r_1^2+m_2(r_1^2+r_2^2)\right)\cos\varphi\right]\mathbf{d}r_2\wedge\mathbf{d}\theta_2\\ &\quad + m_2r_1r_2\sin\varphi[m_1r_1^2+m_2(r_1^2-r_2^2)]\mathbf{d}\theta_1\wedge\mathbf{d}\theta_2\}.\end{aligned}$$

This 2-form drops to the quotient space to yield

$$
\begin{aligned}
\beta_\mu = & \frac{\mu}{\left[m_1 r_1^2 + m_2\left(r_1^2 + r_2^2 + 2 r_1 r_2 \cos\varphi\right)\right]^2} \\
& \times \left\{ m_2 r_2 \left[2\left(m_1 + m_2\right) r_1 r_2 + \left(m_1 r_1^2 + m_2 (r_1^2 + r_2^2)\right) \cos\varphi \right] \mathbf{d}\varphi \wedge \mathbf{d} r_1 \right. \\
& \qquad \left. - m_2 r_1 \left[2\left(m_1 + m_2\right) r_1 r_2 + \left(m_1 r_1^2 + m_2 (r_1^2 + r_2^2)\right) \cos\varphi \right] \mathbf{d}\varphi \wedge \mathbf{d} r_2 \right\} \\
= & \frac{\mu m_2 \left[2\left(m_1 + m_2\right) r_1 r_2 + \left(m_1 r_1^2 + m_2 (r_1^2 + r_2^2)\right) \cos\varphi \right]}{\left[m_1 r_1^2 + m_2\left(r_1^2 + r_2^2 + 2 r_1 r_2 \cos\varphi\right)\right]^2} \mathbf{d}\varphi \wedge (r_2 \mathbf{d} r_1 - r_1 \mathbf{d} r_2).
\end{aligned}
$$

The local representation of the connection can be computed from the expression

$$
\begin{aligned}
& \alpha(\theta_1, r_1, r_2, \varphi)(\dot\theta_1, \dot r_1, \dot r_2, \dot\varphi) \\
& \quad = A(r_1, r_2, \varphi) \begin{bmatrix} \dot r_1 \\ \dot r_2 \\ \dot\varphi \end{bmatrix} + \dot\theta_1 \\
& \quad = \frac{m_2}{m_1 r_1^2 + m_2 (r_1^2 + r_2^2 + 2 r_1 r_2 \cos\varphi)} \begin{bmatrix} -r_2 \sin\varphi & r_1 \sin\varphi & r_2^2 + r_1 r_2 \cos\varphi \end{bmatrix} \begin{bmatrix} \dot r_1 \\ \dot r_2 \\ \dot\varphi \end{bmatrix} + \dot\theta_1.
\end{aligned}
$$

From this, we observe that

$$
A(r_1, r_2, \varphi) = \frac{m_2}{m_1 r_1^2 + m_2 (r_1^2 + r_2^2 + 2 r_1 r_2 \cos\varphi)} \begin{bmatrix} -r_2 \sin\varphi & r_1 \sin\varphi & r_2^2 + r_1 r_2 \cos\varphi \end{bmatrix}.
$$

The amended potential V_μ is given by

$$
\begin{aligned}
V_\mu(q) &= V(q) + \frac{1}{2} \langle \mu, \mathbb{I}(q)^{-1} \mu \rangle \\
&= \left[m_1 g \mathbf{q}_1 + m_2 g (\mathbf{q}_1 + \mathbf{q}_2)\right] \cdot \mathbf{k} + \frac{1}{2} \cdot \frac{\mu^2}{m_1 \|\mathbf{q}_1^\perp\|^2 + m_2 \|(\mathbf{q}_1 + \mathbf{q}_2)^\perp\|^2} \\
&= -m_1 g \sqrt{l_1^2 - r_1^2} - m_2 g \left(\sqrt{l_1^2 - r_1^2} + \sqrt{l_2^2 - r_2^2} \right) + \frac{1}{2} \cdot \frac{\mu^2}{m_1 r_1^2 + m_2 (r_1^2 + r_2^2 + 2 r_1 r_2 \cos\varphi)}.
\end{aligned}
$$

The Routhian has the following expression on the momentum level set,

$$
R^\mu = \frac{1}{2} \| \operatorname{hor}(q, v) \|^2 - V_\mu.
$$

Recall that $\operatorname{hor}(v_q) = v_q - \xi_Q(v_q)$, where $\xi = \alpha(v_q)$, and $\xi_Q(v_q) = (0, \xi, 0, 0, \xi, 0)$. Then, we obtain

$$
\operatorname{hor}(v_q) = v_q - (0, \alpha(v_q), 0, 0, \alpha(v_q), 0)
$$

$$= (\dot r_1, \dot\theta_1 - \alpha(v_q), \dot z_1, \dot r_2, \dot\theta_2 - \alpha(v_q), \dot z_2).$$

From this, we conclude that

$$\begin{aligned}
&\frac{1}{2}\|\operatorname{hor}(q,v)\|^2 \\
&= \frac{1}{2}\begin{bmatrix} \dot r_1 \\ \dot\theta_1 - \alpha \\ \dot z_1 \\ \dot r_2 \\ \dot\theta_2 - \alpha \\ \dot z_2 \end{bmatrix}^T \begin{bmatrix} m_1+m_2 & 0 & 0 & m_2\cos\varphi & -m_2 r_2 \sin\varphi & 0 \\ 0 & (m_1+m_2)r_1^2 & 0 & m_2 r_1 \sin\varphi & m_2 r_1 r_2 \cos\varphi & 0 \\ 0 & 0 & m_1+m_2 & 0 & 0 & 0 \\ m_2\cos\varphi & m_2 r_1 \sin\varphi & 0 & m_2 & 0 & 0 \\ -m_2 r_2 \sin\varphi & m_2 r_1 r_2 \cos\varphi & 0 & 0 & m_2 r_2^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & m_2 \end{bmatrix} \begin{bmatrix} \dot r_1 \\ \dot\theta_1 - \alpha \\ \dot z_1 \\ \dot r_2 \\ \dot\theta_2 - \alpha \\ \dot z_2 \end{bmatrix} \\
&= \frac{1}{2}\left\{(m_1+m_2)r_1^2 + 2m_2 r_1 r_2 \cos\varphi + m_2 r_2^2\right\}\alpha^2 \\
&\quad - \left\{m_1 r_1^2 \dot\theta_1 + m_2\left[r_1 r_2(\dot\theta_1 + \dot\theta_2)\cos\varphi + (r_1 \dot r_2 - r_2 \dot r_1)\sin\varphi + (r_1^2\dot\theta_1 + r_2^2\dot\theta_2)\right]\right\}\alpha \\
&\quad + \frac{1}{2}m_1(\dot r_1^2 + r_1^2\dot\theta_1^2 + \dot z_1^2) \\
&\quad + \frac{1}{2}m_2\Big\{\dot r_1^2 + r_1^2\dot\theta_1^2 + \dot r_2^2 + r_2^2\dot\theta_2^2 + 2(\dot r_1 \dot r_2 + r_1 r_2 \dot\theta_1\dot\theta_2)\cos\varphi \\
&\qquad\qquad + 2(r_1 \dot r_2 \dot\theta_1 - r_2 \dot r_1 \dot\theta_2)\sin\varphi + (\dot z_1 + \dot z_2)^2\Big\},
\end{aligned}$$

where $\alpha = \frac{\mu}{\mathbb{I}}$.

These combine to yield an expression for the Routhian R^μ, which drops to TS to give $\hat{R}^\mu$, and allow us to apply the Reduced Symplectic Partitioned Runge–Kutta algorithm.

2.10.3 Example Trajectories

We have computed the reduced trajectory of the double spherical pendulum using the fourth-order RSPRK algorithm on the Routh equations, and the fourth-order SPRK algorithm on the classical Routh equations.

As discussed in §2.7, these two methods should yield equivalent reduced dynamics, related to each other by a momentum shift, and in particular, their trajectories in position space should agree. We first consider the evolution of r_1, r_2, and φ, using the RSPRK algorithm on the Routh equations, as well as the projection of the relative position of m_2 with respect to m_1 onto the xy plane as seen in Figure 2.7.

Figure 2.8 illustrates that the energy behavior of the trajectory is very good, as is typical of variational integrators, and does not exhibit a spurious drift. In comparison, when a non-symplectic fourth-order Runge–Kutta is applied to the unreduced dynamics, with time-steps that were a quarter

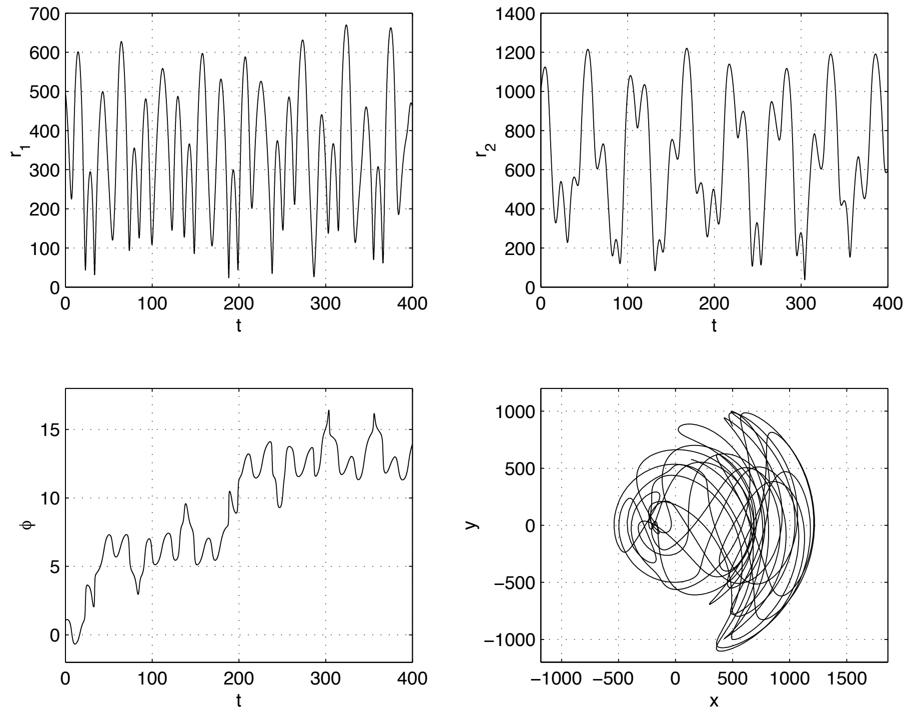

Figure 2.7: Time evolution of r_1, r_2, φ, and the trajectory of m_2, relative to m_1, using RSPRK.

of that used in the symplectic method, and we notice a systematic drift in the energy behavior.

Finally, we consider the relative error between the position trajectories and energy obtained from the RSPRK algorithm applied to the Routh equations as compared to the trajectories from the SPRK algorithm applied to the classical Routh equations. As Figure 2.9 clearly illustrates, these agree very well, as expected theoretically.

2.10.4 Computational Considerations

The choice of whether to compute in the unreduced space, and then project onto the shape space to obtain the reduced dynamics, or to compute the reduced dynamics directly using either the Discrete Routh equations, or the RSPRK algorithm, depends on the nature of the problem to be simulated.

Given a configuration space of dimension n, and a symmetry group of dimension m, we are faced with the option of implementing a conceptually simpler algorithm in $2n$ dimensions, as compared to a more geometrically involved algorithm in $2(n-m)$ dimensions. Whether the additional effort associated with implementing the reduced algorithm is justified depend on a number of factors, including the relative dimension of the configuration space and the symmetry group, the computational complexity of the iterative schemes used to solve the resulting implicit system of equations, and any additional structure that may arise in either the reduced or unreduced system.

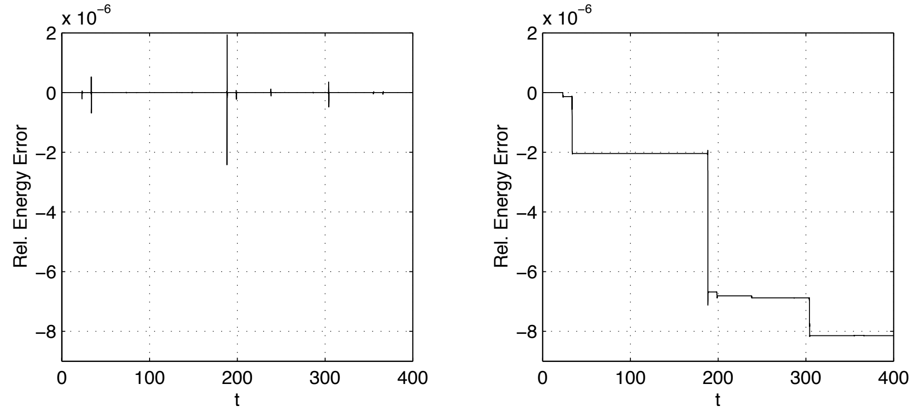

Figure 2.8: Relative energy drift $(E - E_0)/E_0$ using RSPRK (left) compared to the relative energy drift in a non-symplectic RK (right).

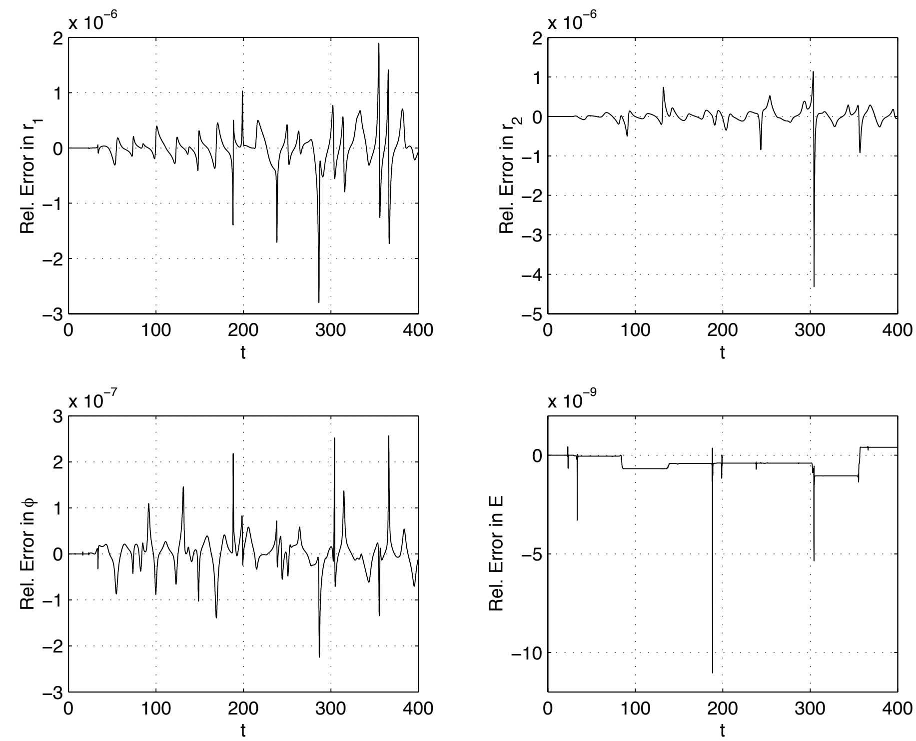

Figure 2.9: Relative error in r_1, r_2, φ, and E between RSPRK and SPRK.

For instance, if longtime or repeated simulations are desired of systems with high-dimensional symmetry groups, it can be advantageous to compute in the reduced space directly. An example of this situation, which is of current engineering interest, is simulating the dynamical behavior of connected networks of systems with their own internal symmetries.

If the systems to be connected are all identical, the geometric quantities that need to be computed, such as the mechanical connection, have a particularly simple repeated form, and the additional upfront effort in implementing the reduced algorithm can result in substantial computational savings.

Non-intrinsic numerical schemes such as the Symplectic Partitioned Runge–Kutta algorithm applied to the classical Routh equations can have undesirable numerical properties due to the need for coordinate-dependent local trivializations and the presence of coordinate singularities in these local trivializations, such as those encountered while using Euler angles for rigid-body dynamics. In the presence of non-trivial magnetic terms in the symplectic form, this can necessitate frequent changes of coordinate charts, as documented in Wisdom et al. [1984] and Patrick [1991]. In such instances, the coordinate changes can account for an overwhelming portion of the total computational effort. In contrast, intrinsic methods do not depend on a particular choice of coordinate system, and as such allow for the use of global charts through the use of containing vector spaces with constraints enforced using Lagrange multipliers.

Coordinate singularities can affect the quality of the simulation in subtle ways that may depend on the choice of numerical scheme. In the energy behavior of the simulation of the double spherical pendulum, we notice spikes in the energy corresponding to times when r_1 or r_2 are close to 0. While these errors accumulate in the non-symplectic method, the energy error in the symplectic method remains well-behaved. However, sharp spikes can be avoided altogether by evolving the equations as a constrained system with $V = \mathbb{R}^3 \times \mathbb{R}^3$, and constraint function $g(\mathbf{v}_1, \mathbf{v}_2) = (\|\mathbf{v}_1\| - l_1, \|\mathbf{v}_2\| - l_2)$ that is imposed using Lagrange multipliers, as opposed to choosing local coordinates that automatically satisfy the constraints. Here, the increased cost of working in the six-dimensional linear space V with constraints is offset by not having to transform between charts of $S^2_{l_1} \times S^2_{l_2}$, which can be significant if the trajectories are particularly chaotic.

While in simple examples, the effect of choosing local coordinates that allow the use of non-intrinsic schemes can be properly corrected for, this is not true in general for more complicated examples. Here, intrinsic schemes such as those we have developed in this paper for dealing with reduced dynamics and constrained systems are preferable, since they do not depend on a particular choice of local trivialization, and as such do not require frequent coordinate transformations.

2.11 Conclusions and Future Work

In summary, we have derived the Discrete Routh equations on $S \times S$, which are symplectic with respect to a non-canonical symplectic form, and retains the good energy behavior typically associated with variational integrators. Furthermore, when the group action can be expressed as addition, we obtain the Reduced Symplectic Partitioned Runge–Kutta algorithm on T^*S, that can be considered as a discrete analogue of cotangent bundle reduction. In addition, the theory has been extended to include constraints and forcing. By providing an understanding of how the reduced and unreduced formulations are related at a discrete level, we enable the user to freely choose whichever formulation is most appropriate, and provides the most insight into the problem at hand.

Certainly one of the obvious things to do in the future is to extend this reduction procedure to the case of nonabelian symmetry groups following the nonabelian version of Routh reduction given in Jalnapurkar and Marsden [2000] and Marsden et al. [2000b]. There are also several problems, including the averaged J_2 problem, in which one can also carry out discrete reduction by stages and in particular relate it to the semidirect product work of Bobenko and Suris [1999]. This is motivated by the fact that the semidirect product reduction theory of Holm et al. [1998] is a special case of reduction by stages (at least without the momentum map constraint), as was shown in Cendra et al. [1998]. In further developing discrete reduction theory, the discrete theory of connections on principal bundles developed in Leok et al. [2003] and Chapter 4 is particularly relevant, as it provides an intrinsic method of representing the reduced space $(Q \times Q)/G$ as $(S \times S) \oplus \tilde{G}$.

Another component that is needed in this work is a good discrete version of the calculus of differential forms. Note that in our work we found, being directed by mechanics, that the right discrete version of the magnetic 2-form is the difference of two connection 1-forms. It is expected that we could recover such a magnetic 2-form by considering the discrete exterior derivative of a discrete connection form in a finite discretization of space-time, and taking the continuum limit in the spatial discretization. Developing a discrete analogue of Stokes' Theorem would also provide insight into the issue of discrete geometric phases. Some work on a discrete theory of exterior calculus can be found in Desbrun et al. [2003a] and Chapter 3.

Of course, extensions of this work to the context of PDEs, especially fluid mechanics, would be very interesting.