Foundations of Computational Geometric Mechanics

Thesis by
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In Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy

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"Thought is only a flash between two long nights,
but this flash is everything"

Henri Poincaré, 1854–1912.

Preface

This thesis was submitted at the California Institute of Technology on April 19, 2004, in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Control and Dynamical Systems with a minor in Applied and Computational Mathematics. The thesis was defended on May 6, 2003, in Pasadena, CA, and was approved by the following thesis committee:

- Thomas Y. Hou, Applied and Computational Mathematics,
  California Institute of Technology,
- Jerrold E. Marsden (Chair), Control and Dynamical Systems,
  California Institute of Technology,
- Richard M. Murray, Control and Dynamical Systems, and Mechanical Engineering,
  California Institute of Technology,
- Michael Ortiz, Aeronautics, and Mechanical Engineering,
  California Institute of Technology,
- Alan D. Weinstein, Mathematics,
  University of California, Berkeley.

This thesis draws upon the following papers that have or will be submitted for publication, but an effort has been made to clarify the manner in which the research relates to each other by incorporating extended introductions and conclusions within each chapter, and through the use of examples that draw upon the various chapters.

- S.M. Jalnapurkar, M. Leok, J.E. Marsden, and M. West, *Discrete Routh Reduction*, J. FoCM, submitted.
- M. Desbrun, A.N. Hirani, M. Leok, and J.E. Marsden, *Discrete Exterior Calculus*, in preparation.
- M. Desbrun, A.N. Hirani, M. Leok, and J.E. Marsden, *Discrete Poincaré Lemma*, Appl. Numer. Math., submitted.

- M. Leok, J.E. Marsden, A.D. Weinstein, *A Discrete Theory of Connections on Principal Bundles*, in preparation.
- M. Leok, *Generalized Galerkin Variational Integrators*, in preparation.

A review article based on preliminary versions of the work on discrete exterior calculus, discrete Poincaré lemma, and discrete connections on principal bundles, received the *SIAM Student Paper Prize*, and the *Leslie Fox Prize in Numerical Analysis* (second prize), both in 2003.

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A special thanks to my parents, Belinda and James, for their unconditional love and support, and for the freedom to pursue my interests. I dedicate this thesis to Lina, who has made my life so much better in countless ways.

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Abstract

Geometric mechanics involves the study of Lagrangian and Hamiltonian mechanics using geometric and symmetry techniques. Computational algorithms obtained from a discrete Hamilton's principle yield a discrete analogue of Lagrangian mechanics, and they exhibit excellent structure-preserving properties that can be ascribed to their variational derivation.

We construct discrete analogues of the geometric and symmetry methods underlying geometric mechanics to enable the systematic development of computational geometric mechanics. In particular, we develop discrete theories of reduction by symmetry, exterior calculus, connections on principal bundles, as well as generalizations of variational integrators.

Discrete Routh reduction is developed for abelian symmetries, and extended to systems with constraints and forcing. Variational Runge–Kutta discretizations are considered in detail, including the extent to which symmetry reduction and discretization commute. In addition, we obtain the Reduced Symplectic Runge–Kutta algorithm, which is a discrete analogue of cotangent bundle reduction.

Discrete exterior calculus is modeled on a primal simplicial complex, and a dual circumcentric cell complex. Discrete notions of differential forms, exterior derivatives, Hodge stars, codifferentials, sharps, flats, wedge products, contraction, Lie derivative, and the Poincaré lemma are introduced, and their discrete properties are analyzed. In examples such as harmonic maps and electromagnetism, discretizations arising from discrete exterior calculus commute with taking variations in Hamilton's principle, which implies that directly discretizing these equations yield numerical schemes that have the structure-preserving properties associated with variational schemes.

Discrete connections on principal bundles are obtained by introducing the discrete Atiyah sequence, and considering splittings of the sequence. Equivalent representations of a discrete connection are considered, and an extension of the pair groupoid composition that takes into account the principal bundle structure is introduced. Discrete connections provide an intrinsic coordinatization of the reduced discrete space, and the necessary discrete geometry to develop more general discrete symmetry reduction techniques.

Generalized Galerkin variational integrators are obtained by discretizing the action integral through appropriate choices of finite-dimensional function space and numerical quadrature. Explicit expressions for Lie group, higher-order Euler–Poincaré, higher-order symplectic-energy-momentum, and pseudospectral variational integrators are presented, and extensions such as spatio-temporally adaptive and multiscale variational integrators are briefly described.

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