Appendix A

Review of Homological Algebra

For the reader's convenience, we will recall some basic definitions and results from homological algebra, which we have reproduced from Hungerford [1974].

Definition A.1. *A pair of homomorphisms,*

$$A \xrightarrow{f} B \xrightarrow{g} C \ ,$$

*is said to be **exact** at B if*

$$\operatorname{Im} f = \operatorname{Ker} g \,.$$

A finite sequence of homomorphisms,

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} A_2 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} A_{n-1} \xrightarrow{f_n} A_n \ ,$$

*is **exact** if*

$$\operatorname{Im} f_i = \operatorname{Ker} f_{i+1}, \qquad \text{for } i = 1, 2, \ldots, n-1.$$

An infinite sequence of homomorphisms,

$$\cdots \xrightarrow{f_{i-1}} A_{i-1} \xrightarrow{f_i} A_i \xrightarrow{f_{i+1}} A_{i+1} \xrightarrow{f_{i+2}} \cdots \ ,$$

*is **exact** if*

$$\operatorname{Im} f_i = \operatorname{Ker} f_{i+1}, \qquad \text{for all } i \in \mathbb{Z}.$$

Remark A.1. *We record below some of the properties of exact sequences.*

1. *The sequence* $0 \longrightarrow A \xrightarrow{f} B$ *is exact iff* f *is a monomorphism (one-to-one).*
2. *The sequence* $B \xrightarrow{g} C \longrightarrow 0$ *is exact iff* g *is a epimorphism (onto).*

3. *If* $A \xrightarrow{f} B \xrightarrow{g} C$ *is exact, then* $gf = 0$.

4. *If* $A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ *is exact, then*

$$\operatorname{Coker} f = B/\operatorname{Im} f = B/\operatorname{Ker} g = \operatorname{Coim} g \cong C.$$

5. *An exact sequence of the form* $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$, *is called a* ***short exact sequence****, and in particular,* f *is a monomorphism, and* g *is an epimorphism.*

6. *A short exact sequence is another way of presenting a submodule (*$A \cong \operatorname{Im} f$*) and its quotient module (*$B/\operatorname{Im} f = B/\ker g \cong C$*).*

We will now consider some results for short exact sequences, such as

$$0 \longrightarrow A_1 \underset{k}{\overset{f}{\rightleftarrows}} B \underset{h}{\overset{g}{\rightleftarrows}} A_2 \longrightarrow 0 \ ,$$

and their splittings.

Lemma A.1 (The Short Five Lemma). *Consider a commutative diagram,*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0 \end{array}$$

such that each row is a short exact sequence. Then,

1. α *and* γ *are monomorphisms, implies* β *is a monomorphism;*

2. α *and* γ *are epimorphisms, implies* β *is an epimorphism;*

3. α *and* γ *are isomorphisms, implies* β *is an isomorphism.*

Proof. The proof involves diagram chasing and the exactness of the rows. See, for example, page 176 of Hungerford [1974]. □

The short five lemma allows the following theorem to be proved. This theorem can be used to relate the various representations of a connection on a principal bundle, in both the continuous and discrete cases.

Theorem A.2. *Given a short exact sequence*

$$0 \longrightarrow A_1 \xrightarrow{f} B \xrightarrow{g} A_2 \longrightarrow 0 \ ,$$

the following conditions are equivalent.

1. *There is a homomorphism* $h : A_2 \to B$ *with* $g \circ h = 1_{A_2}$*;*

2. *There is a homomorphism* $k : B \to A_1$ *with* $k \circ f = 1_{A_1}$*;*

3. *The given sequence is isomorphic (with identity maps on* A_1 *and* A_2*) to the direct sum short exact sequence,*

$$0 \longrightarrow A_1 \xrightarrow{i_1} A_1 \oplus A_2 \xrightarrow{\pi_2} A_2 \longrightarrow 0\ ,$$

and in particular, $B \cong A_1 \oplus A_2$*.*

A short exact sequence that satisfies the equivalent conditions of Theorem A.2 is said to be ***split*** or a ***split exact*** sequence. The maps in Theorem A.2 are referred to as ***splittings*** of the short exact sequence.

Proof. We present the proof sketched on pages 177–178 of Hungerford [1974].

1 ⇒ 3. Consider the homomorphism $\varphi : A_1 \oplus A_2 \to B$, given by $(a_1, a_2) \mapsto f(a_1) + h(a_2)$, and verify that the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{i_1} & A_1 \oplus A_2 & \xrightarrow{\pi_2} & A_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1_{A_1}} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle 1_{A_2}} & & \\
0 & \longrightarrow & A_1 & \xrightarrow{f} & B & \underset{h}{\overset{g}{\rightleftarrows}} & A_2 & \longrightarrow & 0
\end{array}$$

is commutative. Use the short five lemma to conclude that φ is an isomorphism.

2 ⇒ 3. Consider the homomorphism $\psi : B \to A_1 \oplus A_2$, given by $b \mapsto (k(b), g(b))$, and verify that the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_1 & \underset{k}{\overset{f}{\rightleftarrows}} & B & \xrightarrow{g} & A_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1_{A_1}} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle 1_{A_2}} & & \\
0 & \longrightarrow & A & \xrightarrow{i_1} & A_1 \oplus A_2 & \xrightarrow{\pi_2} & A_2 & \longrightarrow & 0
\end{array}$$

is commutative. Use the short five lemma to conclude that ψ is an isomorphism.

3 ⇒ 1, 2. Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_1 & \underset{\pi_1}{\overset{i_1}{\rightleftarrows}} & A_1 \oplus A_2 & \underset{i_2}{\overset{\pi_2}{\rightleftarrows}} & A_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1_{A_1}} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle 1_{A_2}} & & \\
0 & \longrightarrow & A_1 & \xrightarrow{f} & B & \xrightarrow{g} & A_2 & \longrightarrow & 0
\end{array}$$

with exact rows, and where φ is an isomorphism. Let $h = \varphi i_2 : A_2 \to B$ and $k = \pi_1 \varphi^{-1} : B \to A_1$, and show using the commutativity of the diagram that $kf = 1_{A_1}$ and $gh = 1_{A_2}$. □

Appendix B

Geometry of the Special Euclidean Group

To allow the reader to apply the construction of the exact discrete connection using exponentials and logarithms to problems arising in geometric control, we review some of the basic geometry of the ***Special Euclidean Group*** in three dimensions, SE(3), which is the Lie group consisting of isometries of $\mathbb{R}^3$. A more detail discussion of the geometry of SE(3), and its applications to robotics can be found in Murray et al. [1994].

Representation of SE(3). The group SE(3) is a semidirect product of SO(3) and $\mathbb{R}^3$. Using homogeneous coordinates, we can represent SE(3) as follows,

$$\mathrm{SE}(3) = \left\{ \begin{pmatrix} R & p \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(4,\mathbb{R}) \,\middle|\, R \in \mathrm{SO}(3), p \in \mathbb{R}^3 \right\}$$

with the action on $\mathbb{R}^3$ given by the usual matrix-vector product when we identify $\mathbb{R}^3$ with the section $\mathbb{R}^3 \times \{1\} \subset \mathbb{R}^4$. In particular, given

$$g = \begin{pmatrix} R & p \\ 0 & 1 \end{pmatrix} \in \mathrm{SE}(3),$$

and $q \in \mathbb{R}^3$, we have

$$g \cdot q = Rq + p,$$

or as a matrix-vector product,

$$\begin{pmatrix} R & p \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q \\ 1 \end{pmatrix} = \begin{pmatrix} Rq + p \\ 1 \end{pmatrix}.$$

The Lie algebra of SE(3) is given by

$$\mathfrak{se}(3) = \left\{ \begin{pmatrix} \hat{\omega} & v \\ 0 & 0 \end{pmatrix} \in M_4(\mathbb{R}) \,\middle|\, \hat{\omega} \in \mathfrak{so}(3), v \in \mathbb{R}^3 \right\},$$

where $\hat{\cdot} : \mathbb{R}^3 \to \mathfrak{so}(3)$ is given by

$$\hat{\omega} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}.$$

Exponentials and Logarithms. The exponential map, $\exp : \mathfrak{se}(3) \to \mathrm{SE}(3)$, is given by

$$\exp \begin{pmatrix} \hat{\omega} & v \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \exp(\hat{\omega}) & Av \\ 0 & 1 \end{pmatrix},$$

where

$$A = I + \frac{1 - \cos\|\omega\|}{\|\omega\|^2}\hat{\omega} + \frac{\|\omega\| - \sin\|\omega\|}{\|\omega\|^3}\hat{\omega}^2,$$

and $\exp(\hat{\omega})$ is given by the Rodriguez' formula,

$$\exp(\hat{\omega}) = I + \frac{\sin\|\omega\|}{\|\omega\|}\hat{\omega} + \frac{1 - \cos\|\omega\|}{\|\omega\|^2}\hat{\omega}^2.$$

The logarithm, $\log : \mathrm{SE}(3) \to \mathfrak{se}(3)$, is given by

$$\log \begin{pmatrix} R & p \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \log(R) & A^{-1}p \\ 0 & 0 \end{pmatrix},$$

where

$$\log(R) = \frac{\phi}{2\sin\phi}(R - R^T) \equiv \hat{\omega},$$

and ϕ satisfies

$$\mathrm{Tr}(R) = 1 - 2\cos\phi, \qquad |\phi| < \pi,$$

and where

$$A^{-1} = I - \frac{1}{2}\hat{\omega} + \frac{2\sin\|\omega\| - \|\omega\|(1 + \cos\|\omega\|)}{2\|\omega\|^2 \sin\|\omega\|}\hat{\omega}^2.$$

Appendix C

Analysis of Multiscale Finite Elements in One Dimension

This appendix will analyze the discrete l_∞ error for multiscale finite elements (MsFEM) in one dimension when applied to a multiscale second-order elliptic equation with homogeneous boundary conditions. This will serve to motivate the use of multiscale shape functions in the construction of variational integrators, for problems with multiple temporal scales, as discussed in §5.7.

Let $a(y)$ be a smooth, periodic function in y, with period 1. Moreover, we assume that $a(y) \geq c_1 > 0$ for some positive constant c_1, and that $f(x)$ is a smooth function. Let $\epsilon > 0$ be a small parameter. Consider the following second-order elliptic PDE,

$$\frac{\partial}{\partial x}\left(a\left(\frac{x}{\epsilon}\right)\frac{\partial}{\partial x}u^\epsilon(x)\right) = f(x), \quad 0 < x < 1\,,$$

with homogeneous boundary conditions, $u^\epsilon(0) = 0 = u^\epsilon(1)$.

Analytical Solution. To obtain convergence estimates, it is relevant to consider the analytical solution of the above PDE. We have

$$\begin{aligned} \frac{\partial}{\partial x}\left(a\left(\frac{x}{\epsilon}\right)\frac{\partial}{\partial x}u^\epsilon(x)\right) &= f(x)\,, \\ a\left(\frac{x}{\epsilon}\right)\frac{\partial}{\partial x}u^\epsilon(x) - a(0)\frac{\partial}{\partial x}u^\epsilon(0) &= \int_0^x f(s)\,ds\,. \end{aligned}$$

Denoting $a(0)\frac{\partial}{\partial x}u^\epsilon(0)$ by c, we obtain

$$\begin{aligned} \frac{\partial}{\partial x}u^\epsilon(x) &= \frac{\int_0^x f(s)\,ds - c}{a\left(\frac{x}{\epsilon}\right)}\,, \\ u^\epsilon(x) &= \int_0^x \frac{\int_0^y f(s)\,ds - c}{a\left(\frac{y}{\epsilon}\right)}dy\,. \end{aligned}$$

We impose the boundary condition $u^\epsilon(1) = 0$, which yields

$$
\begin{aligned}
0 &= u^\epsilon(1) \\
&= \int_0^1 \frac{\int_0^y f(s)ds}{a\left(\frac{y}{\epsilon}\right)} dy - c \int_0^1 \frac{dy}{a\left(\frac{y}{\epsilon}\right)}\,, \\
c &= \frac{\int_0^1 \frac{\int_0^y f(s)ds}{a\left(\frac{y}{\epsilon}\right)} dy}{\int_0^1 \frac{dy}{a\left(\frac{y}{\epsilon}\right)}}\,.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
u^\epsilon(x) &= \int_0^x \frac{\int_0^y f(s)\, ds}{a\left(\frac{y}{\epsilon}\right)} dy - \frac{\int_0^1 \frac{\int_0^y f(s)ds}{a\left(\frac{y}{\epsilon}\right)} dy}{\int_0^1 \frac{dy}{a\left(\frac{y}{\epsilon}\right)}} \int_0^x \frac{dy}{a\left(\frac{y}{\epsilon}\right)} \\
&= \int_0^x \frac{F(y)}{a\left(\frac{y}{\epsilon}\right)} dy - \frac{\int_0^1 \frac{F(y)}{a\left(\frac{y}{\epsilon}\right)} dy}{\int_0^1 \frac{dy}{a\left(\frac{y}{\epsilon}\right)}} \int_0^x \frac{dy}{a\left(\frac{y}{\epsilon}\right)}.
\end{aligned}
$$

Analytical Expressions for the MsFEM Shape Functions. The MsFEM shape functions can be obtained analytically as follows,

$$
\begin{aligned}
\partial_x a\left(\frac{x}{\epsilon}\right) \partial_x \varphi_i^\epsilon &= 0\,, \\
a\left(\frac{x}{\epsilon}\right) \partial_x \varphi_i^\epsilon &= c_1\,, \\
\partial_x \varphi_i^\epsilon &= \frac{c_1}{a\left(\frac{x}{\epsilon}\right)}\,.
\end{aligned}
$$

For $x \in [x_{i-1}, x_i]$, we have

$$
\begin{aligned}
\varphi_i^\epsilon(x) &= c_1 \int_{x_{i-1}}^x \frac{ds}{a\left(\frac{s}{\epsilon}\right)}\,, \\
\varphi_i^\epsilon(x_i) &= c_1 \int_{x_{i-1}}^{x_i} \frac{ds}{a\left(\frac{s}{\epsilon}\right)} = 1\,, \\
c_1 &= \frac{1}{\int_{x_{i-1}}^{x_i} \frac{ds}{a\left(\frac{s}{\epsilon}\right)}}\,, \\
\varphi_i^\epsilon(x) &= \frac{\int_{x_{i-1}}^{x} \frac{ds}{a\left(\frac{s}{\epsilon}\right)}}{\int_{x_{i-1}}^{x_i} \frac{ds}{a\left(\frac{s}{\epsilon}\right)}}\,.
\end{aligned}
$$

For $x \in [x_i, x_{i+1}]$, we have

$$
\varphi_i^\epsilon(x_{i+1}) - \varphi_i^\epsilon(x) = c_1 \int_x^{x_{i+1}} \frac{ds}{a\left(\frac{s}{\epsilon}\right)}\,,
$$

$$
\begin{aligned}
0-1 &= \varphi_i^{\epsilon}(x_{i+1}) - \varphi_i^{\epsilon}(x_i) \\
&= c_1 \int_{x_i}^{x_{i+1}} \frac{ds}{a\left(\frac{s}{\epsilon}\right)}\,, \\
c_1 &= -\frac{1}{\int_{x_i}^{x_{i+1}} \frac{ds}{a\left(\frac{s}{\epsilon}\right)}}\,, \\
\varphi_i^{\epsilon}(x) &= \frac{\int_{x}^{x_{i+1}} \frac{ds}{a\left(\frac{s}{\epsilon}\right)}}{\int_{x_i}^{x_{i+1}} \frac{ds}{a\left(\frac{s}{\epsilon}\right)}}\,.
\end{aligned}
$$

Then, in general,

$$
\varphi_i^{\epsilon}(x) = \begin{cases}
\left[\int_{x_{i-1}}^{x_i} \frac{ds}{a\left(\frac{s}{\epsilon}\right)}\right]^{-1} \left[\int_{x_{i-1}}^{x} \frac{ds}{a\left(\frac{s}{\epsilon}\right)}\right], & x \in [x_{i-1}, x_i]\,; \\
\left[\int_{x_i}^{x_{i+1}} \frac{ds}{a\left(\frac{s}{\epsilon}\right)}\right]^{-1} \left[\int_{x}^{x_{i+1}} \frac{ds}{a\left(\frac{s}{\epsilon}\right)}\right], & x \in (x_i, x_{i+1}]\,; \\
0\,, & \text{otherwise}\,.
\end{cases}
$$

Discrete Error Analysis of the One-Dimensional MsFEM. Consider a uniform partition on the interval $I = [0,1]$,

$$
P : 0 = x_0 < x_1 < \ldots < x_N = 1\,,
$$

with mesh size $h = 1/N$. Further, let $I_i = [x_i, x_{i+1}]$. We define the discrete l_2 and l_∞ norms as follows,

$$
\begin{aligned}
\|f\|_{l_2} &= \left(\sum_{i=0}^{N} |f(x_i)|^2\right)^{1/2}\,, \\
\|f\|_{l_\infty} &= \max_{i=0,\ldots,N} |f(x_i)|\,.
\end{aligned}
$$

In this section, we will show that the l_∞ error for one-dimensional MsFEM is zero, and in particular, the l_2 error for one-dimensional MsFEM is zero as well.

The stiffness matrix is given by

$$
\begin{aligned}
A_{ij}^h &= a\left(\varphi_i^{\epsilon}, \varphi_j^{\epsilon}\right) \\
&= -\int_0^1 a\left(\frac{x}{\epsilon}\right) \nabla\varphi_i^{\epsilon} \nabla\varphi_j^{\epsilon} dx \\
&= \begin{cases}
-\int_{x_{i-1}}^{x_{i+1}} a\left(\frac{x}{\epsilon}\right) \partial_x \varphi_i^{\epsilon} \partial_x \varphi_j^{\epsilon} dx\,, & j = i-1, i, i+1\,; \\
0\,, & \text{otherwise}\,.
\end{cases}
\end{aligned}
$$

Since $\varphi_{i-1}^{\epsilon}+\varphi_i^{\epsilon}\equiv 1$ in I_{i-1} for all i, we have, $\partial_x\varphi_{i-1}^{\epsilon}+\partial_x\varphi_i^{\epsilon}\equiv 0$ in I_{i-1}, and hence,

$$\begin{aligned}
A_{ii}^h &= -\int_{I_{i-1}} a\left(\frac{x}{\epsilon}\right)\partial_x\varphi_i^{\epsilon}\partial_x\varphi_i^{\epsilon}dx - \int_{I_i} a\left(\frac{x}{\epsilon}\right)\partial_x\varphi_i^{\epsilon}\partial_x\varphi_i^{\epsilon}dx \\
&= -\int_{I_{i-1}} a\left(\frac{x}{\epsilon}\right)\partial_x\varphi_i^{\epsilon}\left(-\partial_x\varphi_{i-1}^{\epsilon}\right)dx - \int_{I_i} a\left(\frac{x}{\epsilon}\right)\partial_x\varphi_i^{\epsilon}\left(-\partial_x\varphi_{i+1}^{\epsilon}\right)dx \\
&= -\left(-\int_{I_{i-1}+I_{i-2}} a\left(\frac{x}{\epsilon}\right)\partial_x\varphi_i^{\epsilon}\partial_x\varphi_{i-1}^{\epsilon}dx - \int_{I_i+I_{i+1}} a\left(\frac{x}{\epsilon}\right)\partial_x\varphi_i^{\epsilon}\partial_x\varphi_{i+1}^{\epsilon}dx\right) \\
&= -\left(A_{ii-1}^h + A_{ii+1}^h\right),
\end{aligned}$$

where the second to last equality is because supp $\left(\partial_x\varphi_i^{\epsilon}\partial_x\varphi_{i-1}^{\epsilon}\right)\subset I_{i-1}$.

We note further that $a(\cdot,\cdot)$ is a symmetric bilinear form, and consequently,

$$A_{ij}^h = A_{ji}^h.$$

Let us define $B_i^h\equiv A_{ii-1}^h$, and therefore, $A_{ii+1}^h=A_{i+1,i}^h=B_{i+1}^h$. This allows us to conclude that

$$\begin{aligned}
\left(A^hU^h\right)_i &= A_{ij}^hU_j^h \\
&= A_{ii+1}^hU_{i+1}^h + A_{ii}^hU_i^h + A_{ii-1}^hU_{i-1}^h \\
&= A_{ii+1}^hU_{i+1}^h - \left(A_{ii+1}^h + A_{ii-1}^h\right)U_i^h + A_{ii-1}^hU_{i-1}^h \\
&= A_{ii+1}^h\left(U_{i+1}^h - U_i^h\right) - A_{ii-1}^h\left(U_i^h - U_{i-1}^h\right) \\
&= D^+\left(A_{ii-1}^h\left(U_i^h - U_{i-1}^h\right)\right) \\
&= D^+\left(B_i^h\left(U_i^h - U_{i-1}^h\right)\right) \\
&= D^+\left(B_i^hD^-U_i^h\right),
\end{aligned}$$

where the forward and backward difference operators D^+ and D^- are defined by

$$\begin{aligned}
D^+\left(f\left(u_i\right)\right) &= f\left(u_{i+1}\right) - f\left(u_i\right), \\
D^-\left(f\left(u_i\right)\right) &= f\left(u_i\right) - f\left(u_{i-1}\right).
\end{aligned}$$

The MsFEM equation is given by

$$A_{ij}^hU_j^h = f_i^h,$$

where $f_i^h=\int_0^1 f(x)\varphi_i(x)\,dx$. In particular,

$$f_i^h = \int_0^1 f(x)\varphi_i(x)\,dx$$

$$= \left[\int_{x_i}^{x_{i+1}} \frac{dx}{a\left(\frac{x}{\epsilon}\right)}\right]^{-1} \int_{x_i}^{x_{i+1}} f(x) \left[\int_{x}^{x_{i+1}} \frac{dx}{a\left(\frac{x}{\epsilon}\right)}\right] dx$$
$$+ \left[\int_{x_{i-1}}^{x_i} \frac{dx}{a\left(\frac{x}{\epsilon}\right)}\right]^{-1} \int_{x_{i-1}}^{x_i} f(x) \left[\int_{x_{i-1}}^{x} \frac{dx}{a\left(\frac{x}{\epsilon}\right)}\right] dx$$
$$= \left[\int_{x_i}^{x_{i+1}} \frac{dx}{a\left(\frac{x}{\epsilon}\right)}\right]^{-1} \left[\int_{x_i}^{x_{i+1}} \frac{F(x)}{a\left(\frac{x}{\epsilon}\right)} dx - \frac{\int_0^1 \frac{F(x)}{a\left(\frac{x}{\epsilon}\right)} dx}{\int_0^1 \frac{dx}{a\left(\frac{x}{\epsilon}\right)}} \int_{x_i}^{x_{i+1}} \frac{dx}{a\left(\frac{x}{\epsilon}\right)}\right]$$
$$- \left[\int_{x_{i-1}}^{x_i} \frac{dx}{a\left(\frac{x}{\epsilon}\right)}\right]^{-1} \left[\int_{x_{i-1}}^{x_i} \frac{F(x)}{a\left(\frac{x}{\epsilon}\right)} dx - \frac{\int_0^1 \frac{F(x)}{a\left(\frac{x}{\epsilon}\right)} dx}{\int_0^1 \frac{dx}{a\left(\frac{x}{\epsilon}\right)}} \int_{x_i-1}^{x_i} \frac{dx}{a\left(\frac{x}{\epsilon}\right)}\right],$$

where we performed an integration by parts to obtain the last equality. We will further rewrite the expression above using the difference operators.

$$f_i^h = D^+ \left(\left[\int_{x_{i-1}}^{x_i} \frac{dx}{a\left(\frac{x}{\epsilon}\right)}\right]^{-1} \left[\int_{x_{i-1}}^{x_i} \frac{F(x)}{a\left(\frac{x}{\epsilon}\right)} dx - \frac{\int_0^1 \frac{F(x)}{a\left(\frac{x}{\epsilon}\right)} dx}{\int_0^1 \frac{dx}{a\left(\frac{x}{\epsilon}\right)}} \int_{x_i-1}^{x_i} \frac{dx}{a\left(\frac{x}{\epsilon}\right)}\right] \right)$$
$$= D^+ \left(\left[\int_{x_{i-1}}^{x_i} \frac{dx}{a\left(\frac{x}{\epsilon}\right)}\right]^{-2} \left[\int_{x_{i-1}}^{x_i} \frac{1}{a\left(\frac{x}{\epsilon}\right)} dx\right] \left(u^\epsilon(x_i) - u^\epsilon(x_{i-1})\right) \right)$$
$$= D^+ \left(-\left(\int_{x_{i-1}}^{x_i} a\left(\frac{x}{\epsilon}\right) \left(\left[\int_{x_{i-1}}^{x_i} \frac{dx}{a\left(\frac{x}{\epsilon}\right)}\right]^{-1} \frac{1}{a\left(\frac{x}{\epsilon}\right)} \right) \left(\left[\int_{x_{i-1}}^{x_i} \frac{dx}{a\left(\frac{x}{\epsilon}\right)}\right]^{-1} \frac{-1}{a\left(\frac{x}{\epsilon}\right)} \right) dx \right) D^- u^\epsilon(x_i) \right) \tag{C.0.1}$$

Recall from our analytical expression for the MsFEM shape functions that

$$\varphi_i^\epsilon(x) = \begin{cases} \left[\int_{x_{i-1}}^{x_i} \frac{ds}{a\left(\frac{s}{\epsilon}\right)}\right]^{-1} \left[\int_{x_{i-1}}^{x} \frac{ds}{a\left(\frac{s}{\epsilon}\right)}\right], & x \in [x_{i-1}, x_i]; \\ \left[\int_{x_i}^{x_{i+1}} \frac{ds}{a\left(\frac{s}{\epsilon}\right)}\right]^{-1} \left[\int_{x}^{x_{i+1}} \frac{ds}{a\left(\frac{s}{\epsilon}\right)}\right], & x \in (x_i, x_{i+1}]; \\ 0, & \text{otherwise}, \end{cases}$$

and hence,

$$\partial_x \varphi_i^\epsilon(x) = \begin{cases} \left[\int_{x_{i-1}}^{x_i} \frac{ds}{a\left(\frac{s}{\epsilon}\right)}\right]^{-1} \frac{1}{a\left(\frac{x}{\epsilon}\right)}, & x \in [x_{i-1}, x_i]; \\ \left[\int_{x_i}^{x_{i+1}} \frac{ds}{a\left(\frac{s}{\epsilon}\right)}\right]^{-1} \frac{-1}{a\left(\frac{x}{\epsilon}\right)}, & x \in (x_i, x_{i+1}]; \\ 0, & \text{otherwise}. \end{cases}$$

We therefore recognize two of the terms in Equation C.0.1 as $\partial_x \varphi_i^\epsilon$ and $\partial_x \varphi_{i-1}^\epsilon$. Consequently, we have that

$$f_i^h = D^+ \left(- \left(\int_{x_{i-1}}^{x_i} a\left(\frac{x}{\epsilon}\right) \partial_x \varphi_i^\epsilon \partial_x \varphi_{i-1}^\epsilon dx \right) D^- u^\epsilon (x_i) \right)$$
$$= D^+ \left(A_{ii-1}^h D^- u^\epsilon (x_i) \right)$$
$$= D^+ \left(B_i^h D^- u^\epsilon (x_i) \right).$$

Hence,

$$A_{ij}^h u^\epsilon (x_j) = D^+ \left(B_i^h D^- u^\epsilon (x_i) \right) = f_i = A_{ij}^h U_j^h \,.$$

Since the matrix A^h is invertible, we can conclude that

$$U_i^h = u^\epsilon (x_i) \,.$$

This implies that one-dimensional MsFEM is exact at the nodal points, and in particular,

$$\|u^\epsilon - u_h\|_{l_2} = 0 \,,$$

and

$$\|u^\epsilon - u_h\|_{l_\infty} = 0 \,.$$