EDGE FUNCTION METHOD

APPLIED TO

THIN PLATES RESTING ON WINKLER FOUNDATION

Thesis by

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ABSTRACT

The Edge Function method formerly developed by Quinlan [25] is applied to solve the problem of thin elastic plates resting on spring supported foundations subjected to lateral loads the method can be applied to plates of any convex polygonal shapes, however, since most plates are rectangular in shape, this specific class is investigated in this thesis. The method discussed can also be applied easily to other kinds of foundation models (e.g. springs connected to each other by a membrane) as long as the resulting differential equation is linear.

In chapter VII, solution of a specific problem is compared with a known solution from literature. In chapter VIII, further comparisons are given. The problems of concentrated load on an edge and later on a corner of a plate as long as they are far away from other boundaries are also given in the chapter and generalized to other loading intensities and/or plates springs constants for Poisson's ratio equal to 0.2 .

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CHAPTER I. HISTORICAL BACKGROUND

The problem of deflection of an elastically supported elastic plate has become important since the introduction of concrete paving slabs for roads and aircraft runways. During World War II, considerable activity took place in northern Canada and Alaska so that the problem of transporting heavy equipment over frozen lakes became important. The above problems, as well as the necessity to design raft foundations for buildings, have resulted in a great deal of research activity on the problem of a loaded elastic plate on various kinds of supporting medium.

Soils are in general nonlinear, nonhomogeneous and anisotropic, but it is frequently assumed that a soil mass can be represented by a semi-infinite linearly elastic medium. However, even this model renders the problem too hard to solve in most cases. For the sake of mathematical expedience the simplest model, equivalent to a bed of springs, was suggested by Winkler[(1)] and many papers have been written on the plate problem using this model. The following is a brief review of the literature dealing with the problem of a loaded elastic plate on Winkler's model of a supporting medium. Later in this chapter a review will also be given concerning other models.

The problem of deflection of an elastic plate resting on some kind of medium was first discussed by Winkler.[(1)] He proposed the simplest relationship between the local reaction of the supporting medium p_s and the local deflection. He suggested that the pressure, p_s, is proportional to the local plate deflection. This is exactly equivalent to the condition of a buoyant plate resting on a liquid, as in the case, for example, of a floating ice sheet. The only reaction of the subgrade is an upward force proportional to the deflection. This description of a foundation reaction is frequently called a "Winkler Foundation." Winkler used this model to study the behavior of railroad rails resting on ties. It was also used by Hertz[40] in a study of a floating ice sheet.

Westergaard[2, 3] solved approximately the problem of an infinite or semi-infinite elastic thin plate resting on a Winkler Foundation when the load is uniformly distributed first over a circle and later an ellipse. He also gave some formulas for the maximum tensile stresses developed in the plate. In the infinite plate problem, the loading can be distributed uniformly over any area having both axes of x and y as axes of symmetry. Westergaard's investigation of tensile stresses had immediate application to the design of highways and airfield pavements.

Wyman[4] solved analytically the problem of a point load on an infinite thin plate resting on Winkler Foundation. The solution he obtained was expressed in terms of Bessel's functions. The point load solution can be generalized, as shown by Wyman, to obtain solutions for arbitrary loading condition. However, the solution is expressed in integral form and, in most cases, the integral is too hard to evaluate. Wyman, however, applied the idea to a uniform circular loading condition and evaluated the deflections.

R. K. Livesley[5] obtained solutions to the problem of arbitrary loading on an infinite, thin plate on a Winkler Foundation by using a double Fourier transform and expressed the solution as an integral. When an edge or edges are simply supported, the solution can be extended to semi-infinite plates or plates of the shape of an infinite quadrant. He also obtained solutions for a semi-infinite plate on a Winkler Foundation loaded by prescribed moments and shear stress normally along the edge of the plate. Dynamical loading is also discussed in the paper.

Other than the above papers, Kerr[6] solved the problem of a simply supported wedge-shaped plate, not supported by any foundation, subjected to uniform tension in the plane of the plate and loaded transversely by concentrated forces. The behavior of a loaded corner of the plate can be obtained from the results of the paper. In another paper, written by Kerr,[7] he tackled the problem of simply supported plates on a Winkler Foundation subjected to concentrated loads and obtained solutions for the following shapes of plates: (1) wedge-shaped; (2) infinite strip; (3) semi-infinite strip; (4) rectangular plate.

S. Timoshenko and Woinowsky-Krieger[8] included a chapter on the topic of plates on a Winkler Foundation. In that chapter, there are solutions to several problems, a very interesting one being that of a rectangular plate simply supported on all four edges and loaded by any arbitrary lateral loads. The solution is expressed as a double sine series. This is one of the very few exact solutions obtained for a finite plate on a Winkler Foundation.

Other than the above mentioned papers, Hetenyi[16] has solved the problem of elastic beams resting on a Winkler Foundation.

Many examples are given in the book. However, the book deals mainly with beam rather than plate problems.

The problems of plates on a Winkler Foundation that have been solved are mainly those of infinite, semi-infinite plates or those plates with the shape of an infinite quadrant and are listed in Table 1.1.

The problem of a finite plate on a Winkler Foundation is very complicated. There is only one solution in the literature (chapter 8).[8] From it, the solution of any arbitrary lateral load on a simply supported (hinged) rectangular plate can be obtained.

In addition to plates on the Winkler Foundations, many workers have proposed different kinds of models for the supporting medium. Hogg[9] solved the problem of a thin infinite plate, symmetrically loaded and resting on an elastic half-space. Then, in a later paper, Hogg[10] solved a similar problem in which the elastic foundation was of finite depth.

However, to make the problem more manageable, most investigations on a better foundation model have been carried out in one-dimension (a beam, instead of a plate). For example, most comparisons with experimental results have been made using a theoretical solution of an elastic beam on a variety of foundation models. The foundation pressure $p_s(x)$ is very often assumed to be proportional to the displacement and/or various derivatives of displacements. This model is popular because the Winkler's model of foundation is completely discontinuous. If other derivatives are taken into consideration, some of the continuous properties of the foundation are

TABLE 1.1

Solved Problems of Plates on a Winkler Foundation

Shape of Plate	Loading Condition	Boundary Conditions	Reference
Infinite	a) Point Load	Solution approaches zero as it is infinitely far away from the load	4
	b) Loads with Circular Symmetry		2, 3,
	c) Arbitrary		4, 5
Semi-Infinite	a) Elliptical Shapes	Free Edge	2, 3
	b) Circular and Elliptical Shapes		2,
Infinite Quadrant	c) Arbitrary	Simply Supported	5
	a) Arbitrary	Simply Supported	5, 6 7

taken into consideration. As mentioned earlier, in the Winkler Model $p_s(x)$ is assumed to be proportional to the displacement only. But in the most general linear case, it can be assumed to be expressed by the following equation:

$$p_s(x) = \sum_{n=0}^{N} \alpha_n \, w^{(n)}(x) \tag{1.1}$$

in which $w^{(n)}(x)$ denotes the n^{th} derivative of the deflection $w(x)$ with respect to the x-direction. An extensive review of the literature in this area is given in refs. (1), (11), (12), (13), (14), (15) and (16). For example: Pasternak[13] suggested the model:

$$p_s(x) = k_1 w(x) - k_2 w''(x), \quad \text{whereas} \tag{1.2}$$

Hetenyi[12] proposed the following model:

$$p_s(x) = k_1 w(x) - k_2 w^{(IV)}(x) \tag{1.3}$$

The Winkler Foundation[1] assumes complete lateral discontinuity in the elements in the foundation material, whereas the half-space of Hogg[9, 10] assumes complete continuity. Hetenyi's[17] equation (1.3) and chapter 10[16] used an arbitrary degree of continuity on the foundation and applied it successfully to the one-dimensional beam problem. However, it is doubtful whether this method could be applied to the two-dimensional plate equation. The reason for using this model, rather than the Winkler model, is that it offers more parameters that one can choose to approximate the actual elastic continuum model.

Most of the investigators who have proposed various models, however, have not given a rational method for choosing the values of the parameters in their models. (for example, k_1 and k_2 in the Pasternak and Hetenyi Model described in eq. (1.2) and (1.3)). Fletcher and Hermann[18] gave a systematic way to determine what values of the parameters (constants) in the model represented by eq. (1.2) and (1.3) should be used, if the elastic constants of the (linearized isotropic) foundation material are known. Timoshenko and Woinowsky-Krieger (p. 259)[8] give a table which gives engineers the values of the Winkler constant that should be used for various types of soil.

Other than the above-mentioned static models, Kerr[15] has suggested a time-dependent model in which a viscoelastic effect was introduced. However, Kerr applied the model to a plate which can only withstand transverse shear (a shear plate).

All of the solved problems involve a particular plate shape and boundary conditions generally simply-supported at the edges. For the general solution of slab problems even of moderate complexity, numerical procedures have to be adopted. So far in the literature, two methods have been used: the finite difference and the finite element techniques. Allen, and Severn[19, 20] used the finite difference method and reduced the solution to a system of linear equation. However, in this formulation it is sometimes difficult to introduce the boundary conditions.

Zienkiewicz and Cheung[21] and Severn[22] have applied the finite element technique to the plate problem. The finite element method is a versatile method that can deal with any boundary condition and different shapes of plates. The finite element problem usually uses a continuum model for foundation material, so it can model any combination of materials, layers, etc., but may be expensive.

Other investigators who have contributed to the problem are Vlasov and Leont'ev[23] and Holl.[24]

Among all the above-mentioned methods, only finite difference and finite element techniques can be used to handle the plate problem with some generality. However, they are not without fault. Their advantages and disadvantages will be compared and discussed in Chapter IX.

CHAPTER II. FORMULATION OF THE PROBLEM AND PARTICULAR SOLUTION ADOPTED

The foundation model that is adopted in this paper is the Winkler Foundation. In view of all the uncertainty connected with the soil property determination, it seems unjustifiable to use a more sophisticated model which would make the problem much more difficult. However, if other models for the soil reaction are used, as long as the differential equation involved is linear, the Edge Function technique can still be applied.

(a) Formulation

When the plate is thin compared to other dimensions (e.g. the radii of curvature of the surface) of the plate and the deflection w of the plate are small compared to the thickness of the plate, an approximate theory of bending of the plate by lateral loads can be developed by making the following assumptions:

1. There is no in-plane deformation in the middle surface of the plate.
2. Points in the plate lying initially on a normal to middle plane of the plate remain on the normal after bending.
3. The normal strains in the direction transverse to the plate are negligibly small.

Using these assumptions, all stress components can be expressed in terms of the deflection w of the plate, which is a function of the two coordinates in the plane of the plate. This function has to satisfy a linear partial differential equation, which, together with the boundary conditions, completely and uniquely define w. The solution of this equation gives all the information necessary for calculating stresses at any point of the plate. The equation of the plate for any of the model foundation material is given by:

$$\nabla^4 w(x, y) + \frac{p_s(x, y)}{D} = \frac{q(x, y)}{D} \tag{2.1}$$

where $$D = \frac{Eh^3}{12(1-\nu^2)}$$

E and ν are Young's modulus and Poisson's Ratio of the elastic plate.

h is the thickness of the plate.

q(x, y) is a function that describes the loading.

p_s is the foundation pressure

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$$

Adopting the Winkler model for the foundation, p_s is given by the following equation

$$p_s(x, y) = K w(x, y) \tag{2.2}$$

where K is a constant.

Substituting eq. (1.5) into (1.4) the differential equation of the deflected surface is obtained as follows:

$$\nabla^4 w(x, y) + \frac{K}{D} w(x, y) = \frac{q(x, y)}{D} \tag{2.3}$$

The above assumption and differential equations are basically adopted from ref. 8.

(b) Boundary Conditions

To make the solution unique, boundary conditions have to be specified. In most engineering problems, the properties of interest are among the following: displacement; slope; moment and shear.

The boundary conditions include specifying any two of the above properties along each boundary. To express the above properties mathematically involving derivatives of displacement w, the derivation in chapter 2 of ref. 8 is adopted and the results are listed as follows:

a) displacement is given by w

b) slope in the n direction is given by $\frac{\partial w}{\partial n}$

c) moment per unit length in the n direction is $-D\left(\frac{\partial^2 w}{\partial n^2} + \nu \frac{\partial^2 w}{\partial t^2}\right)$

d) shear force per unit length in the n direction is $-D\left(\frac{\partial^3 w}{\partial n^3} + (2-\nu)\frac{\partial^3 w}{\partial n \partial t^2}\right)$

(2.4)

where the coordinates are as given in Fig. 2.1.

To solve the problem, the equation is broken down into two parts to obtain: (1) The non-homogeneous solution; (2) The homogenous solution.

(1) The non-homogeneous solution (particular solution). From eq. (2.3) the particular solution is given by

$$\nabla^4 w_p + \frac{K}{D} w_p = \frac{q(x, y)}{D} \tag{2.5}$$

(2) The homogeneous solution (complementary solution) has to satisfy the following equation:

$$\nabla^4 w_c + \frac{K}{D} w_c = 0 \tag{2.6}$$

w_c is adjusted such that together with the particular solution it will give a total solution, w_t, that satisfies the loading and the boundary conditions. Details of the solution methods are given in the following chapters.

Then the total solution is given by

$$w_t = w_p + w_c \tag{2.7}$$

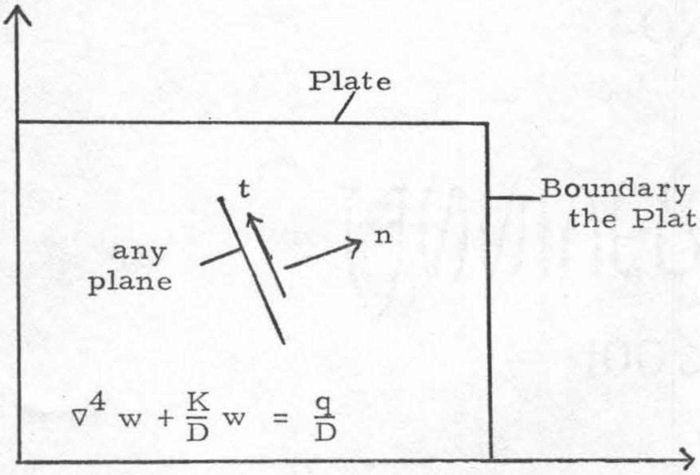


Fig. 2.1

in which n and t may refer to either a cartesian or a curvilinear co-ordinate system.

(c) Particular solutions for a rectangular plate

To find a particular solution w_p to eq. (2.5), the Navier solution method for Simply Supported Rectangular Plates (art. 28, in ref. 8) is adopted.

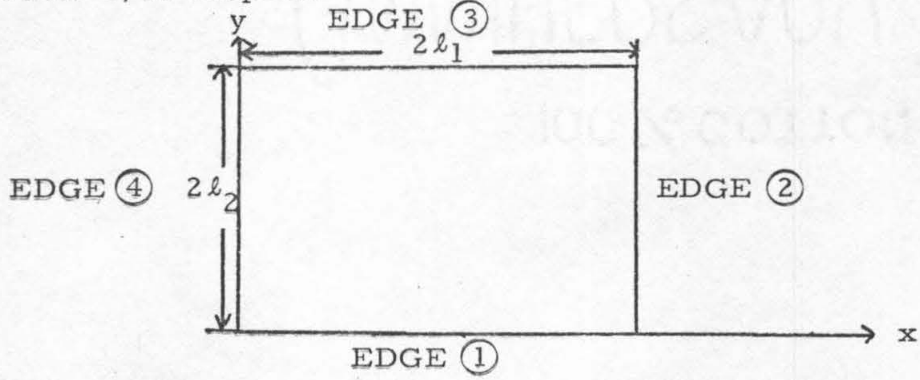


Fig. 2.2

A double Fourier series can be used to describe the loading function q(x, y) in eq. (2.5). Using the coordinate system in Fig. 2.2 the loading condition is described by the following equation:

$$\left.\begin{aligned}\frac{q(x,y)}{D} = &\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} Q1_{mn} \sin\frac{m\pi x}{\ell_1} \sin\frac{n\pi y}{\ell_2} \\ &+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} Q2_{mn} \cos\frac{m\pi x}{\ell_1} \cos\frac{n\pi y}{\ell_2} \\ &+\sum_{m=1}^{\infty}\sum_{n=0}^{\infty} Q3_{mn} \sin\frac{m\pi x}{\ell_1} \cos\frac{n\pi y}{\ell_2} \\ &+\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} Q4_{mn} \cos\frac{m\pi x}{\ell_1} \sin\frac{n\pi y}{\ell_2}\end{aligned}\right\} \qquad (2.8)$$

The standard procedures to calculate the Fourier coefficients $Q1_{mn}$, $Q2_{mn}$, $Q3_{mn}$ and $Q4_{mn}$ in eq. (2.8) can be used. In most loading conditions (point loads and uniformly distributed rectangular loads, like column loads) they can be calculated without resorting to numerical integration. However, for the most general loading conditions, numerical integration is needed.

The particular solution w_p also can be expressed in double Fourier series.

$$\left.\begin{aligned} w_p = & \sum_{m=1}\sum_{n=1} B1_{mn} \sin\frac{m\pi x}{\ell_1} \sin\frac{n\pi y}{\ell_2} \\ & + \sum_{m=0}\sum_{n=0} B2_{mn} \cos\frac{m\pi x}{\ell_1} \cos\frac{n\pi y}{\ell_2} \\ & + \sum_{m=1}\sum_{n=0} B3_{mn} \sin\frac{m\pi x}{\ell_1} \cos\frac{n\pi y}{\ell_2} \\ & + \sum_{m=0}\sum_{n=1} B4_{mn} \cos\frac{m\pi x}{\ell_1} \sin\frac{n\pi y}{\ell_2} \end{aligned}\right\} \tag{2.9}$$

Substituting eq. (2.9) and (2.8) in (2.5), each of the Fourier coefficients should be equated. The Fourier coefficients in eq. (2.9) can be calculated from the known Fourier coefficients in eq. (2.8). For the sake of clarity this is explicitly stated as follows:

$$BI_{mn} = \frac{QI_{mn}}{\left[\left(\frac{m\pi}{\ell_1}\right)^4 + 2\left(\frac{m\pi}{\ell_1}\right)^2\left(\frac{n\pi}{\ell_2}\right)^2 + \left(\frac{n\pi}{\ell_2}\right)^4 + \frac{K}{D}\right]} \tag{2.10}$$

where I denotes 1, 2, 3 or 4 as in eq. (2.9).

Several examples of particular solutions have been worked out in detail in Appendix D. They include:

(i) A point load inside the plate.

(ii) A point load on the corner of the plate

(iii) A point load on the edge of the plate.

(iv) A column load (uniformly distributed load over rectangular area).

In (ii) and (iii) some manipulation is employed to arrive at the correct answer.

Equation (2.9) only expresses the displacement of the particular solution. Properties of engineering interest other than displacement are: slope, moment and shear. Mathematical expressions of these properties of the particular solution are given in Appendix E. They are obtained by substituting wp (as expressed in eq. (2.9)) for w in eq. (2.4).

Now, it is presumed that the particular solution has been found satisfying the differential equation. However, this solution generally does not meet the required boundary conditions and is therefore not unique. (There are many solutions that will satisfy eq. (2.5)). Thus, the complementary solution has to be evaluated to give a total solution (eq. (2.7)), which will satisfy the differential equation and the required boundary conditions. To arrive at the necessary complementary solution, the Edge-Function Method developed by Quinlan[25] can be used. Chapter 3 introduces this concept and describes its application to the plate problem. Chapter 4 deals with the specific use obtained by applying the Edge-Function method to the rectangular plate problem.

This Fourier series representation of loading function given by eq. (2.8) can be used to describe any number of point loads, or distributed pressure, or even a combination of both types of loads any where on the plate.

CHAPTER III. EDGE FUNCTION METHOD

After studying the Edge-Function idea developed by Quinlan[25] to solve some very general plane strain or plane stress problems, R. F. Scott suggested the possibility of using this method to tackle the case of a slab on a Winkler Foundation loaded perpendicular to its plane. This has been examined and the following discussion describes the application of this method of solving the plate problem.

Quinlan's Edge Function idea is that, since the governing differential equation is isotropic, (see Fig. 3.1), its expression in terms of the x-y coordinate system is of the same form as that in terms of the x_j', y_j' system or x_q', y_q' system. Therefore, a solution of the homogeneous equation in any coordinate system (e.g. x_j', y_j') is also a solution to the differential equation with respect to other systems (e.g. x_q', y_q'), providing the appropriate change of variable is performed. (In this case, x_j', y_j' to x_q', y_q'.) Furthermore, since the equation is linear, the solutions with respect to various coordinate systems can be superimposed and the final solution will still satisfy the equation with respect to any coordinate system. This superimposed solution can be chosen such that the required boundary conditions are obtained.

The above idea is applied to the elastic plate on Winkler's Foundation problem as follows: Here, instead of two displacement components, we have only one displacement w. For the sake of generality, the following can be applied to plates of any polygonal shape, as long as they are convex.

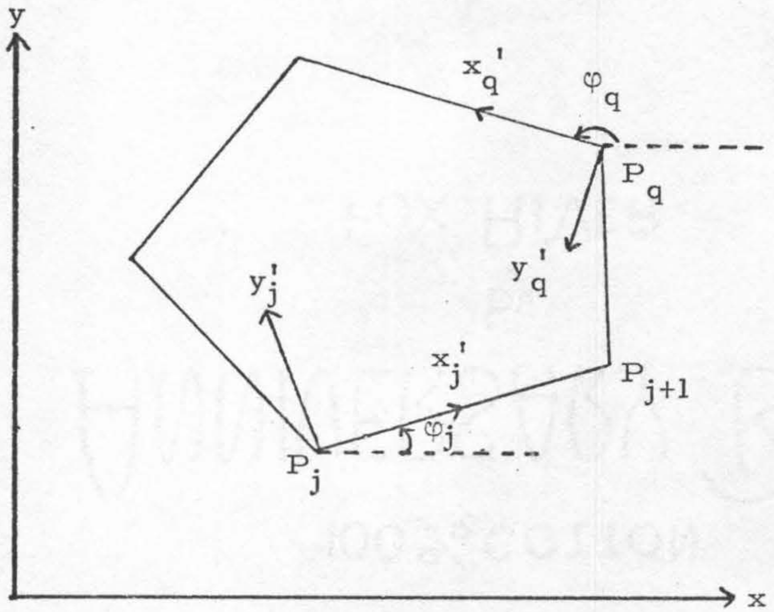


Fig. 3.1

Referring to Fig. 3.1, in which displacement is now normal to the plane and realizing now that the objective is to solve for the appropriate complementary solution as stated in eq. (2.6), let the j^{th} edge of the polygon be (x_j, y_j) in the x-y coordinate system. Then, any point having coordinates (x, y) with respect to the x-y system has coordinates (xj', yj') with respect to the xj', yj' system and the coordinates are related to each other by the transformation shown by eq. (3.1).

$$x = x_j + x_j' \cos\varphi_j - y_j' \sin\varphi_j$$
$$y = y_j + x_j' \sin\varphi_j + y_j' \cos\varphi_j \tag{3.1}$$

The differential equation with respect to the (x, y) coordinate system is given by eq. (2.6). If another coordinate system (x ', y ') is used, the differential equation has to be changed accordingly. To find out what the differential equation will look like in the new system, a coordinate transformation is performed on eq. (2.6). From (3.1) $\frac{\partial}{\partial x}$ can be expressed in the xj', yj' system.

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x_j'}\frac{\partial x_j'}{\partial x} + \frac{\partial}{\partial y_j'}\frac{\partial y_j'}{\partial x}$$

$$= \cos\varphi_j \frac{\partial}{\partial x_j'} - \sin\varphi_j \frac{\partial}{\partial y_j'} \tag{3.2}$$

Similarly $\frac{\partial}{\partial y} = \sin\varphi j \frac{\partial}{\partial x_j'} + \cos\varphi j \frac{\partial}{\partial y_j'}$.

Equation (3.2) is applied successively in eq. (2.6). Finally, the transformed differential equation with respect to the new x ', y ' system is

$$\frac{\partial^4 w_c^j (x_j', y_j')}{\partial x_j'^4} + 2\frac{\partial^4 w_c^j (x_j', y_j')}{\partial x_j'^2, \partial y_j'^2} + \frac{\partial^4 w_c^j (x_j', y_j')}{\partial y_j'^4} + \frac{K}{D} w_c^j (x_j', y_j') = 0 \tag{3.3}$$

It can be observed that eq. (3. 3) is of the same form as eq. (2. 6) if x_j' and y_j' is substituted for x and y respectively. From physical reasoning, since the plate and springs are isotropic and homogeneous, the differential equation must be the same with respect to any coordinate axes.

Furthermore, a solution $w_c^j(x_j', y_j')$, after changing x_j', y_j' into any cartesian coordinate system (e. g. x_q', y_q') using the appropriate form of equation (3. 1), will also satisfy the differential equation with respect to the x_q', y_q' system.

In other words, $w_c^j(x_j', y_j')$ will satisfy the following equation:

$$\frac{\partial^4 w_c^j(x_j', y_j')}{\partial x_q'^4} + 2\frac{\partial^4 w_c^j(x_j', y_j')}{\partial x_q'^2 \partial y_q'^2} + \frac{\partial^4 w_c^j(x_j', y_j')}{\partial y_q'^4} + \frac{K}{D} w_c^j(x_j', y_j') = 0 \qquad (3.4)$$

if x_j', y_j' is changed into x_q', y_q' appropriately following equation (3. 1).

Because of the above property and since the equation (2. 6) is linear, solutions with respect to different coordinate axes can be superimposed and the total solution will still satisfy the differential equation with respect to any particular set of axis. Thus, a very general form of solution to the complementary equation (2. 6) can be obtained as follows:

$$w_c(x, y) = \sum_{j=1}^{N} w_c^j(x_j', y_j') \qquad (3.5)$$

where (x_j', y_j') are related to (x, y) through eq. (3. 3), and are called the "Edge Functions" with respect to the j^{th} coordinate axis.

So far, the form of the complementary solution is known from eq. (3.5) and (3.3). However, there are still sets of constants in the complementary solution which remain unestablished. Applying the boundary conditions on each edge, these constants are determined. In Chapter 4, an example will be given and the complementary solution to a rectangular plate solved in detail.

Other research applying the Edge Function idea is given from refs. (28) to (39).

CHAPTER IV. EDGE FUNCTION METHOD FOR A RECTANGULAR PLATE ON WINKLER FOUNDATION

The following chapter deals with finding the complementary solution that will satisfy the proper boundary conditions to a problem of a rectangular plate on a Winkler Foundation. The differential equation for the homogeneous problem is, from Chapter II

$$\nabla^4 w_c + \frac{K}{D} w_c = 0 \tag{2.6}$$

For all rotated and translated coordinate systems, it will have the same form, as discussed in Chapter 3. One has to obtain solutions that satisfy the differential equation (3.3). Using the particular solution obtained in Chapter 2, the boundary conditions which arise from the particular solution will be easily given as a Fourier series. Thus, it is convenient to use Fourier series (sine and cosine) in the complementary solutions as well. First, let

$$w_c^j = \sum_{m=1}^{\infty} \sin\frac{m\pi x_j'}{\ell_j} f_m^j(y_j') + \sum_{m=0}^{\infty} \cos\frac{m\pi x_j'}{\ell_j} g_m^j(y_j') \tag{4.1}$$

Substituting equation (4.1) into equation (3.3) two differential equations arise as follows:

$$\left.\begin{aligned} \left(\frac{m\pi}{\ell_j}\right)^4 f_m^j - 2\left(\frac{m\pi}{\ell_j}\right)^2 f_m^{j''} + f_m^{j(IV)} + \frac{K}{D} f_m^j &= 0 \\ \left(\frac{m\pi}{\ell_j}\right)^4 g_m^j - 2\left(\frac{m\pi}{\ell_j}\right)^2 g_m^{j''} + g_m^{j(IV)} + \frac{K}{D} g_m^j &= 0 \end{aligned}\right\} \tag{4.2}$$

There are four independent solutions for each of equations (4.2): g_m^j and f_m^j can both be expressed as linear combinations of the following functions:

$$\left.\begin{array}{l} e^{\gamma_m^j y_j'} \sin \theta_m^j y_j' \\ e^{\gamma_m^j y_j'} \cos \theta_m^j y_j' \\ e^{-\gamma_m^j y_j'} \sin \theta_m^j y_j' \\ e^{-\gamma_m^j y_j'} \cos \theta_m^j y_j' \end{array}\right\} \qquad (4.3)$$

where θ_m^j and γ_m^j have to satisfy the following equation:

$$\left.\begin{array}{l} (\gamma_m^j)^2 - (\theta_m^j)^2 = (\frac{m\pi}{\ell_j})^2 \\ 2(\gamma_m^j)(\theta_m^j) = \sqrt{\frac{K}{D}} \end{array}\right\} \qquad (4.4)$$

One can neglect the part involving $e^{\gamma_m^j y_j'}$ in the solution for each Edge Function with respect to a particular coordinate system (e.g. $e^{\gamma_m' y_1'}(\sin \theta_m' y_1' + \cos \theta_m' y_1')$ with respect to the coordinate system (x_1', y_1'), as can be seen from the following discussion. The Edge Function method is basically a method of superposition. Each of the superimposed elements has some property which is governed by the boundary conditions. Their presence in the solution is to introduce the boundary conditions at each edge of the plate. Each of these elements has a particular form and a particular position. The different coordinate axes are set up in such a way that the bases $(y_j' = 0)$ are located along the edges of the plate. (e.g. for a rectangular plate, the coordinate axes are set up as shown in Fig. 4.2.) From Fig. 4.1, one can see that the Edge Function $w_c'(x_1', y_1')$ is introduced mainly to include the boundary conditions on edge 1. When $y_1' = 0$ is substituted into the solution, $w_c'(x_1', y_1' = 0)$ should constitute that part of the solution that makes the general solution behave according to the prescribed boundary conditions on edge 1. The function $w_c'(x_1', y_1')$ should become smaller with distance from edge 1 (y_1' getting bigger). At a point infinitely far away from the edge, the contribution from $w_c'(x_1', y_1')$ to the general solution should be zero. In effect, one is looking at a semi-infinite strip. When $y_1' = 0$ certain properties are prescribed and as $y_1' \to \infty$, the solution $\to 0$. So, obviously the $e^{\gamma_m^j y_j'}$ part of the solution should be omitted from each Edge Function. The same consideration holds for all the edges.

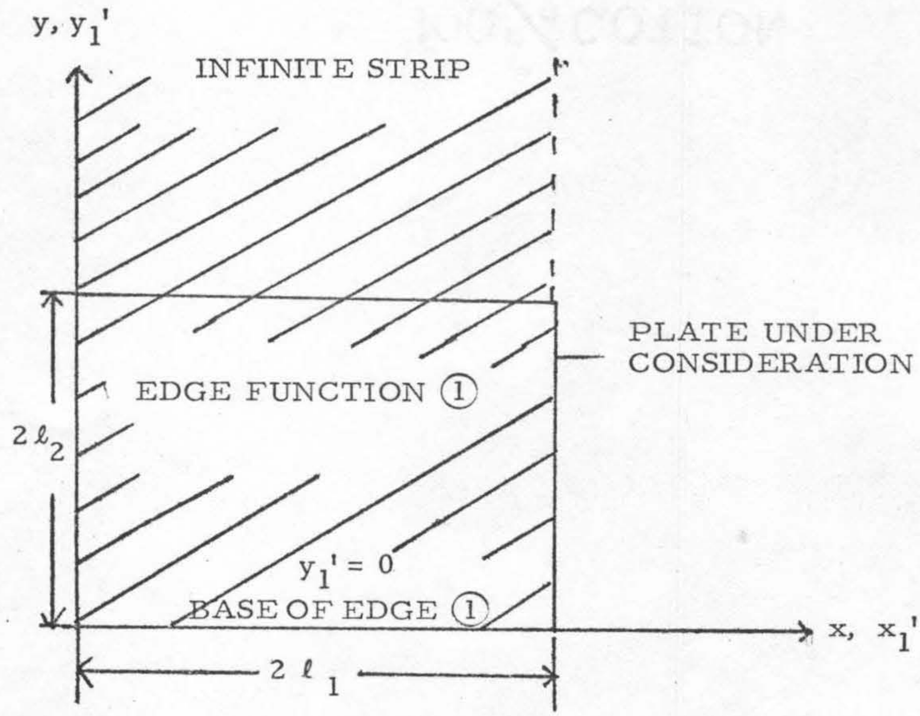


Fig. 4.1. Solution of Edge Function Associated with Edge 1 $\to 0$ as $y_1' \to \infty$

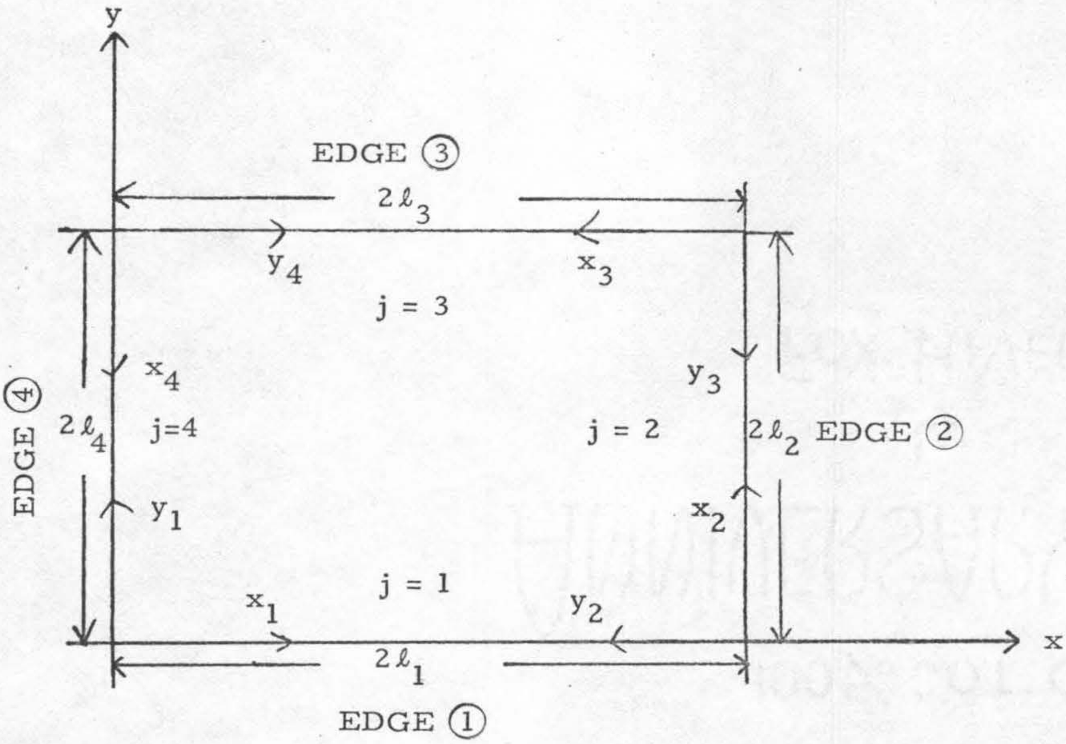


Fig. 4.2

The coordinate axes are set up as shown in Fig. 4.2. The x_1, y_1 axes are chosen such that they coincide with the x, y axes respectively. The different coordinates follow the following relationship:

$$\left.\begin{aligned} x_2 &= y_1 \\ y_2 &= 2\ell_1 - x_1 \\ x_3 &= 2\ell_1 - x_1 \\ y_3 &= 2\ell_2 - y_1 \\ x_4 &= 2\ell_2 - y_1 \\ y_4 &= x_1 \end{aligned}\right\} \qquad (4.5)$$

With the axes set up as shown in Fig. 4.2, the most general form of the complementary solution in the rectangular plate resting on a Winkler Foundation is as follows:

$$
\left.
\begin{aligned}
w_c(x,y) = &\sum_{j=1}^{4}\sum_{m=1}^{\infty} A_m^j \sin\frac{m\pi x_j}{\ell_j}\, e^{-\gamma_m^j y_j} \sin\theta_m^j y_j \\
&+ \sum_{j=1}^{4}\sum_{m=1}^{\infty} B_m^j \sin\frac{m\pi x_j}{\ell_j}\, e^{-\gamma_m^j y_j} \cos\theta_m^j y_j \\
&+ \sum_{j=1}^{4}\sum_{m=0}^{\infty} C_m^j \cos\frac{m\pi x_j}{\ell_j}\, e^{-\gamma_m^j y_j} \sin\theta_m^j y_j \\
&+ \sum_{j=1}^{4}\sum_{m=0}^{\infty} D_m^j \cos\frac{m\pi x_j}{\ell_j}\, e^{-\gamma_m y_j} \cos\theta_m^j y_j \\
\text{where} \qquad & (\gamma_m^j)^2 - (\theta_m^j)^2 = \left(\frac{m\pi}{\ell_j}\right)^2 \\
& 2(\gamma_m^j)\,(\theta_m^j) = \sqrt{\frac{K}{D}}
\end{aligned}
\right\}
\qquad (4.6)
$$

In equation (4.6) the only unknowns are the A_m^j, B_m^j, C_m^j and D_m^j. These coefficients are determined from the boundary conditions. With these calculated, the exact complementary solution is known. The particular solution is obtained from eq. (2.9) and the total solution can be calculated from eq. (2.7). Mathematical expressions of the complementary solution for slope, moment, and shear force are given in Appendix E.

The remaining part of Chapter 4 will be used mainly to illustrate how the A_m^j, B_m^j, C_m^j and D_m^j are chosen so that the boundary conditions are satisfied on all edges.

Recalling eq. (2.7), taking w_p to the left-hand side of the equation, the following is obtained:

$$w_t(x, y) - w_p(x, y) = \sum_{j=1}^{4} w_c^j(x_j, y_j) \tag{4.7}$$

Changing coordinates of the above equation into any coordinate axes (e.g. x_1, y_1) and introducing the coordinate of that edge 1, the equation is reduced to that involving only one independent variable (x_1, since $y_1 = 0$ is the coordinate of edge 1). The following equation is obtained:

$$w_t(x_1)-w_p(x_1) = \sum_{j=1}^{4} w_c^j \text{ (all coordinates changed to } x_1) \tag{4.8}$$

The function $w_t(x_1)$ in the above equation is the prescribed boundary condition on edge 1. It can easily be represented by a Fourier series. The displacement $w_p(x_1)$ is represented as a Fourier series also. As a result the left-hand side is reduced to a single Fourier series. (A sine and a cosine series involving x_1). The right-hand side of the equation, however, is more complicated. Referring to eq. (4.6), after changing all the (x_j, y_j) to (x_1, y_1) and then after substituting the coordinates of edge 1 ($y_1 = 0$), the following equation is obtained:

Equation (4.9) on next page

$$
\begin{aligned}
w_c(\text{on edge } ①) = & \sum_{m=1}^{\infty} B_m^1 \sin\frac{m\pi x_1}{\ell_1} + \sum_{m=0}^{\infty} D_m^1 \cos\frac{m\pi x_1}{\ell_1} \qquad j=1 \\
& + \sum_{m=0}^{\infty} C_m^2 e^{-\gamma_m^2(2\ell_1-x_1)} \sin\theta_m^2(2\ell_1-x_1) + \sum_{m=1}^{\infty} D_m^2 e^{-\gamma_m^2(2\ell_1-x_1)} \\
& \qquad \cos\theta_m^2(2\ell_1-x_1) \qquad j=2 \\
& \left.\begin{aligned}
& + \sum_{m=1}^{\infty} A_m^3\left(-\sin\frac{m\pi x_1}{\ell_3}\right) e^{-\gamma_m^3(2\ell_2)} \sin\theta_m^3(2\ell_2) + \sum_{m=1}^{\infty} B_m^3\left(-\sin\frac{m\pi x_1}{\ell_3}\right) e^{-\gamma_m^3(2\ell_2)} \cos\theta_m^3(2\ell_2) \\
& + \sum_{m=0}^{\infty} \cos\frac{m\pi x_1}{\ell_3}\left[C_m^3 e^{-\gamma_m^3(2\ell_2)} \sin\theta_m^3(2\ell_2) + D_m^3 e^{-\gamma_m^3(2\ell_2)} \cos\theta_m^3(2\ell_2)\right]
\end{aligned}\right\} j=3 \\
& + \sum_{m=0}^{\infty} C_m^4 e^{-\gamma_m^4 x_1} \sin\theta_m^4 x_1 + \sum_{m=0}^{\infty} D_m^4 e^{-\gamma_m^4 x_1} \cos\theta_m^4 x_1 \qquad j=4
\end{aligned}
\tag{4.9}
$$

In equation (4. 9) the part where $j = 1$ and $j = 3$ is a simple Fourier series. However, when $j = 2$ and $j = 4$, the part involving x_1 is not a Fourier series. In order to match the Fourier series, on the left-hand side of eq. (4. 8), the part involving x_1 has to be reduced to a Fourier series as well.

Letting

$$
\begin{aligned}
e^{-\gamma_m^2(2\ell_1-x_1)} \sin\theta_m^2(2\ell_1-x_1) &= \sum_{n=1}^{\infty} \tau_n \sin\frac{n\pi x_1}{\ell_1} + \sum_{n=0}^{\infty} \beta_n \cos\frac{n\pi x_1}{\ell_1} \\
e^{-\gamma_m^2(2\ell_1-x_1)} \cos\theta_m^2(2\ell_1-x_1) &= \sum_{n=1}^{\infty} \lambda_n \sin\frac{n\pi x_1}{\ell_1} + \sum_{n=0}^{\infty} \eta_n \cos\frac{n\pi x_1}{\ell_1} \\
e^{-\gamma_m^4 x_1} \sin\theta_m^4 x_1 &= \sum_{n=1}^{\infty} \rho_n \sin\frac{n\pi x_1}{\ell_1} + \sum_{n=0}^{\infty} \xi_n \cos\frac{n\pi x_1}{\ell_1} \\
e^{-\gamma_m^4 x_1} \cos\theta_m^4 x_1 &= \sum_{n=1}^{\infty} \sigma_n \sin\frac{n\pi x_1}{\ell_1} + \sum_{n=0}^{\infty} \omega_n \cos\frac{n\pi x_1}{\ell_1}
\end{aligned}
\tag{4.10}
$$

where τ_n, β_n, λ_n, η_n, ρ_n, ξ_n, σ_n and ω_n are Fourier coefficients and can be evaluated.

Substituting eq. (4.10) into eq. (4.9), the following equation involving only $\sin \frac{m\pi x_1}{\ell_1}$ and $\cos \frac{m\pi x_1}{\ell_1}$ is obtained.

w_c(on edge 1).

$$
\begin{aligned}
&= \sum_{m=1}^{\infty} B_m^1 \sin \frac{m\pi x_1}{\ell_1} + \sum_{m=0}^{\infty} D_m^1 \cos \frac{m\pi x_1}{\ell_1} \\
&+ \sum_{m=1}^{\infty} \left[\sum_{n=0}^{\infty} C_n^2\right] \tau_m \sin \frac{m\pi x_1}{\ell_1} + \sum_{m=0}^{\infty} \left[\sum_{n=0}^{\infty} C_n^2\right] \beta_m \cos \frac{m\pi x_1}{\ell_1} \\
&+ \sum_{m=1}^{\infty} \left[\sum_{n=0}^{\infty} D_n^2\right] \lambda_m \sin \frac{m\pi x_1}{\ell_1} + \sum_{m=0}^{\infty} \left[\sum_{n=0}^{\infty} D_n^2\right] \eta_m \cos \frac{m\pi x_1}{\ell_1} \\
&+ \sum_{m=1}^{\infty} - e^{-\gamma_m^3 (2\ell_2)} \sin \theta_m^3 (2\ell_2)\, A_m^3 \sin \frac{m\pi x_1}{\ell_1} \\
&+ \sum_{m=1}^{\infty} - e^{-\gamma_m^3 (2\ell_2)} \cos \theta_m^3 (2\ell_2)\, B_m^3 \sin \frac{m\pi x_1}{\ell_1} \\
&+ \sum_{m=0}^{\infty} e^{-\gamma_m^3 (2\ell_2)} \sin \theta_m^3 (2\ell_2)\, C_m^3 \cos \frac{m\pi x_1}{\ell_1} \\
&+ \sum_{m=0}^{\infty} e^{-\gamma_m^3 (2\ell_2)} \cos \theta_m^3 (2\ell_2)\, D_m^3 \cos \frac{m\pi x_1}{\ell_1} \\
&+ \sum_{m=1}^{\infty} \left[\sum_{n=0}^{\infty} C_n^4\right] \rho_m \sin \frac{m\pi x_1}{\ell_1} + \sum_{m=0}^{\infty} \left[\sum_{n=0}^{\infty} C_n^4\right] \xi_m \cos \frac{m\pi x_1}{\ell_1} \\
&+ \sum_{m=1}^{\infty} \left[\sum_{n=0}^{\infty} D_n^4\right] \sigma_m \sin \frac{m\pi x_1}{\ell_1} + \sum_{m=0}^{\infty} \left[\sum_{n=0}^{\infty} D_n^4\right] \omega_m \cos \frac{m\pi x_1}{\ell_1}
\end{aligned}
\tag{4.11}
$$

Equation (4.11) is simply a Fourier series, in which the only unknowns are the A_m^j, B_m^j, C_m^j and D_m^j's. Equating this Fourier series with that from the left-hand side of eq. (4.8) (the coefficients of the sine series with the sine series, and the coefficients of the cosine series with the cosine series) a simultaneous system of linear equations in the unknowns A_m^j, B_m^j, C_m^j and D_m^j is obtained.

For practical purposes, the index m in eqs. (4.9), (4.10), and (4.11) is truncated at $m = N$. Coefficients for $m > N$ are neglected. So there are $4 \times N$ A_m^j, $4 \times N$ B_m^j, and $4 \times (N+1)$ C_m^j, and $4 \times (N+1)$ D_m^j. As a result there are 16N+8 unknown A_m^j, B_m^j, C_m^j and D_m^j and the size of the matrix that has to be solved is (16N+8) by (16N+8). There are four edges and on each edge there are two boundary conditions. For each boundary condition there are N sine series and N+1 cosine series coefficients. Therefore, there are $8 \times (2N+1)$ equations that can be formed. Thus, there are equal numbers of unknowns and equations and the matrix is well-defined. For boundary conditions other than displacements, the various derivative of eqs. (2.9) and (4.6) have to be used. (See eq. (2.4)). After solving for the A_m^j, B_m^j, C_m^j and D_m^j from the matrix, they can be substituted back into eq. (4.6) to be used in eq. (2.7) for the complete solution.

CHAPTER V. CONVERGENCE

The convergence of the solution by the Edge-Function method to the correct solution depends mainly on two conditions: the smoothness of the prescribed boundary conditions and the smoothness of the boundary condition contribution from the particular solution. The particular solution is given by a double sine, cosine series (eq. 2.9). After inserting the coordinates of the edges in the particular solution, the function that describes the boundary contribution from the particular solution becomes simply a single sum Fourier series. Therefore, to simplify the problem, the Fourier transform of the prescribed boundary conditions from the total solution (eq. 2.7) is represented by a Fourier series (a sine and a cosine). Thus, the convergence of the solution depends mainly on the number of terms required to represent adequately the particular solution as well as the prescribed boundary conditions.

The displacement in the particular solution is represented by eq. (2.9). On substituting the coordinates of the edges of the plate into eq. (2.9), $w_p(x^*, y^*)$, where (x^*, y^*) represents the coordinates of an edge, the double sum series which originally has two independent variables can be reduced to a single sum Fourier series involving one independent variable if the coordinate system with respect to that particular edge is used. To be more specific the following example is given.

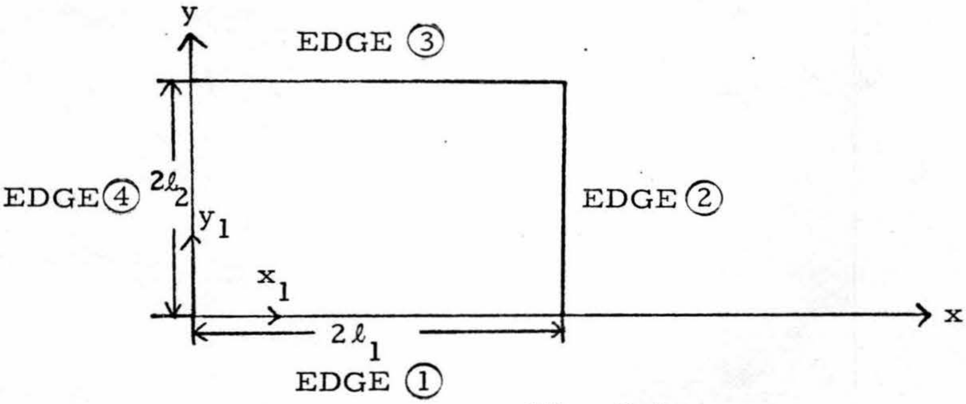


Fig. 5.1

Fig. 5.1 is a rectangular plate with coordinates set up as shown. On the first edge, the x_1, y_1 coordinate system coincides with the x-y coordinate system. Expressing the particular solution in the x_1, y_1 system and letting $y_1 = 0$ (coordinates of edge ①) the following expression for the displacement contribution from the particular solution at the boundary edge ① is obtained: (derived from eq. (2.9))

$$\left.\begin{aligned} w_p(x_1^*, y_1^*) &= \sum_{m=0}^{M}\left[\sum_{n=0}^{N} B2_{mn}\right]\cos\frac{m\pi x_1}{\ell_1} \\ &\quad + \sum_{m=1}^{M}\left[\sum_{n=0}^{N} B3_{mn}\right]\sin\frac{m\pi x_1}{\ell_1} \\ \text{where}\qquad BI_{mn} &= \frac{QI_{mn}}{\left[\left(\frac{m\pi}{\ell_1}\right)^4 + 2\left(\frac{m\pi}{\ell_1}\right)^2\left(\frac{n\pi}{\ell_2}\right)^2 + \left(\frac{n\pi}{\ell_2}\right)^2 + \frac{K}{D}\right]} \end{aligned}\right\} \qquad (5.1)$$

The coefficients in the resulting Fourier series (eq. (5.1)) are proportional to BI_{mn}, which are proportional to

$$\frac{1}{\left[\left(\frac{m\pi}{\ell_1}\right)^4 + 2\left(\frac{m\pi}{\ell_1}\right)^2\left(\frac{n\pi}{\ell_2}\right) + \left(\frac{n\pi}{\ell_2}\right)^4 + \frac{K}{D}\right]}$$ for certain loading conditions.

Thus, the particular solution for displacement on the boundary (Edge (1) in the example) converges like $\frac{1}{N^4}$.

Similarly, using the slope equation for the particular solution in Appendix E, the slope contribution of the particular solution on Edge (1) is given by the following:

$$\left.\begin{aligned} &\frac{\partial w_p(x^*, y^*)}{\partial y^1} \text{ on edge (1)} \\ &= \sum_{m=1}^{M} \left[\sum_{n=1}^{N} B1_{mn} \left(\frac{n\pi}{\ell_2}\right) \right] \sin \frac{n\pi x_1}{\ell_1} \\ &+ \sum_{m=0}^{M} \left[\sum_{n=1}^{N} B4_{mn} \left(\frac{n\pi}{\ell_2}\right) \right] \sin \frac{m\pi x_1}{\ell_1} \end{aligned}\right\} \quad (5.2)$$

where the BI_{mn} again behave like $\frac{1}{N^4}$.

The $(\frac{n\pi}{\ell_2})$ factor which appears on differentiating the series makes the coefficients in the Fourier series in eq. (5.2) converge slower and the slope in the particular solution converges as $\frac{1}{N^3}$.

Similarly, (see eq. (1.7)), we can apply the same procedures to the moment and shear expressions in the particular solution given in Appendix E. Since the moment basically involves the second derivatives and the shear involves the third derivatives, the moment converges as $\frac{1}{N^2}$ and the shear converges as $\frac{1}{N}$.

To summarize:

Displacement converges as $\frac{1}{N^4}$; Slope converges as $\frac{1}{N^3}$;
Moment converges as $\frac{1}{N^2}$; Shear converges as $\frac{1}{N}$

in the particular solution.

Thus, displacement converges very quickly, but convergence is less rapid as higher derivatives are considered. In an actual case, the double Fourier series have to be truncated at some term N. Depending on the degree of accuracy desired and which combination of displacement slope, moment or shear is of interest, one can decide on where the series should be truncated. If the behavior of the plate in terms of lower derivatives is desired, a smaller number of terms is needed in the series.

The convergence of the complete solution, other than depending on the convergence of the particular solution, also depends to a certain degree on the complementary solution. Referring back to Chapter IV, on trying to choose the correct coefficients to use in eq. (4.6) such that the complementary solution $w_c(x, y)$ will have the desired value on the boundaries of the plate, eq. (4.9) was derived. This equation has to be matched with the left-hand side of eq. (4.8) which is given as a single sum Fourier series. If eq. (4.9) is a single sum Fourier series, the coefficients can simply be matched term by term and the matrix is set up easily. However, eq. (4.9) is not a simple Fourier series, but involves some other functions. That is why, in eq. (4.9), the various functions have to be changed to a Fourier series. These functions vary as the θ_m^j and γ_m^j which depend on the shape and dimension of the plate, soil spring constant K and plate constant D (the last two equations of eq. (4.6)). All of these considerations will affect the smoothness of the various functions in eq. (4.9). Then, as these functions have to be represented by a Fourier series, eq. (4.10), the number of terms in the solution has to be large enough such that these functions can be represented adequately in eq. (4.10) by a Fourier series.

In general, it takes both a sine and a cosine series to expand any general function. In solving the matrix, the Fourier coefficients of all terms are coupled in each equation in the most general case. Thus, if one wants to include terms as high as those involving $\sin(\frac{N\pi x}{\ell})$ and $\cos(\frac{N\pi x}{\ell})$ terms, there are (16N+8) unknowns and a matrix of size (16N+8) by (16N+8) has to be solved.

CHAPTER VI. SYMMETRICAL PROPERTIES

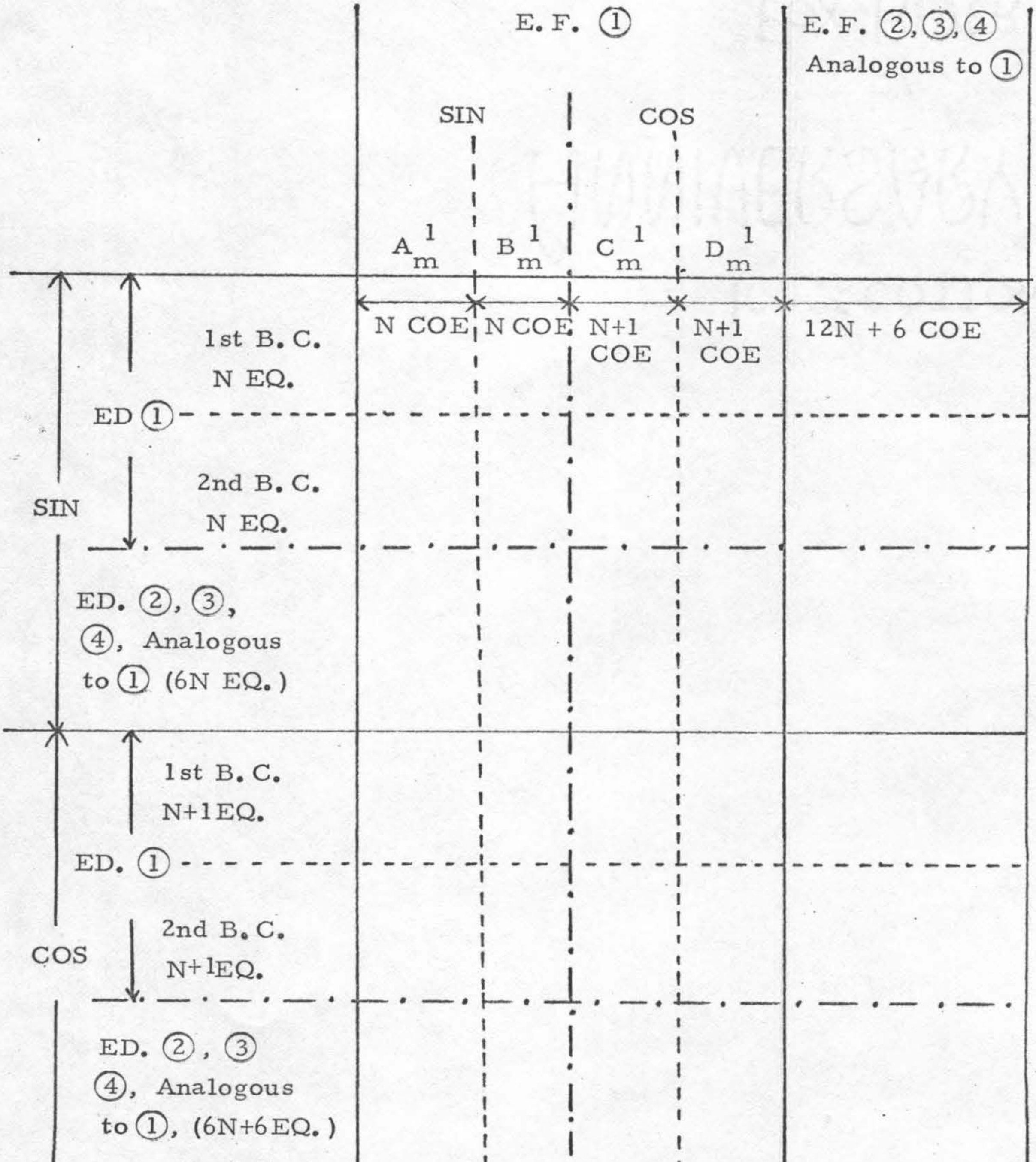
In the course of this research, a computer program has been set up to solve the rectangular plate problem; the matrix that is solved in the program is arranged as shown in Fig. 6.1.

As discussed in Chapter 5, in general a matrix of size (16N+8) by (16N+8) has to be solved, N being the term where the indices m in the series given in eq. (4.6) are terminated.

Symmetry in the problem of a thin plate on a Winkler Foundation as solved by the Edge Function method is basically of two types: (a) symmetrical boundary conditions, (b) symmetrical boundary and loading conditions. The remainder of this chapter deals with the two types of symmetries and how they can be used to reduce the computer time needed to solve the problems:

(a) Boundary condition symmetries:

The boundary conditions can be of four types: prescribed displacement, slope, moment or shear. Any two of the above can be prescribed on each edge. However, if the same type of properties is prescribed on each edge (e.g. displacement and moment are prescribed on all edges) the matrix that has to be solved can be reduced to one-fourth the size. This reduction can be made later if different functions of the same property (e.g. displacement) are prescribed on each edge. If the same type of boundary conditions are prescribed on all edges, the matrix as arranged in Fig. 6.2 will be cyclic. A cyclic matrix is one which has submatrices [A], [B], [C] and [D] arranged cyclically in the original matrix [M] as shown in Fig. 6.3.



Abbreviations: E. F. - Edge Function, E. D. - Edge, B. C. - Boundary Condition, COE. - Coefficients

Fig. 6.1

The Arrangements in the Matrix

From Fig. 6.1 one can shift the equations and partition the matrix into the form shown in Fig. 6.2.

EDGE	B. C.		1st EDGE $A_m^1\ B_m^1\ C_m^1\ D_m^1$	2nd EDGE Analogous to 1st	3rd EDGE Analogous to 1st	4th EDGE Analogous to 1st
①	1st 2nd 1st 2nd	SIN SIN COS COS	A_{11}	A_{12}	A_{13}	A_{14}
②	1st 2nd 1st 2nd	SIN SIN COS COS	A_{21}	A_{22}	A_{23}	A_{24}
③	1st 2nd 1st 2nd	SIN SIN COS COS	A_{31}	A_{32}	A_{33}	A_{34}
④	1st 2nd 1st 2nd	SIN SIN COS COS	A_{41}	A_{42}	A_{43}	A_{44}

Fig. 6.2

The big matrix can be partitioned into 16 submatrices. Each of the submatrices is now (4N+2) by (4N+2) or one-fourth the size of the original matrix.

$$[M] = \begin{array}{|c|c|c|c|} \hline A & B & C & D \\ \hline D & A & B & C \\ \hline C & D & A & B \\ \hline B & C & D & A \\ \hline \end{array}$$

The original matrix that has been set up. If it is cyclic as shown in the figure, computing time can be reduced.

Fig. 6.3

Therefore, the following equation with unknown vector $\underset{\sim}{x}$ has to be solved.

$$[M]\{\underset{\sim}{x}\} = \{\underset{\sim}{R}\} \tag{6.1}$$

Vector $\underset{\sim}{R}$ is the right-hand vector and can be derived from the left-hand side of eq. (4.8) which is the known function describing the reduced boundary condition which the complementary solution has to satisfy on the boundaries.

If the matrix [M] in eq. (6.1) is not cyclic, the original matrix [M] of size 16N + 8 by 16N + 8 has to be inverted. However, when [M] is cyclic, the following can be done.

Suppose

$$\{\underset{\sim}{x}\} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} \tag{6.2}$$

$$\{\underset{\sim}{R}\} \quad \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} \tag{6.3}$$

and [M] in eq. (6.1) is cyclic as shown in Fig. 6.3, then the following equation has to be solved.

$$\begin{array}{ccc} [M] & \{x\} & \{R\} \\ \begin{bmatrix} A & B & C & D \\ D & A & B & C \\ C & D & A & B \\ B & C & D & A \end{bmatrix} & \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = & \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} \end{array} \tag{6.4}$$

A transformation matrix [S] can be chosen to transform eq. (6.4). Pre-multiplying and pro-multiplying the cyclic matrix [M] in eq. (6.4) by $[S]^{-1}$ [S] (as shown in eq. 6.4), the equation should still be the same since $[S]^{-1}[S] = [I]$,

$$S^{-1}\left[SMS^{-1}\right]S\ \{\underset{\sim}{x}\} = \{\underset{\sim}{R}\} \tag{6.5}$$

S should be chosen such that SMS^{-1} in equation (6.5) is a matrix where there are only non-zero elements along the diagonal of the matrix. In other words

$$[M_D] = SMS^{-1} = \begin{bmatrix} [E] & 0 & 0 & 0 \\ 0 & [F] & 0 & 0 \\ 0 & 0 & [G] & 0 \\ 0 & 0 & 0 & [H] \end{bmatrix} \tag{6.6}$$

where [E], [F], [G] and [H] are submatrices. Letting $[M_D] = SMS^{-1}$ in eq. (6.5) the following is obtained:

$$[S]^{-1}[M_D][S]\{\underset{\sim}{x}\} = \{\underset{\sim}{R}\} \tag{6.7}$$

Thus $$\{\underset{\sim}{x}\} = S^{-1} M_D^{-1} S\ \{\underset{\sim}{R}\} \tag{6.8}$$

Since M_D is a matrix which has non-zero submatrices only along the diagonal and each sub-matrix is only one-fourth the size of the original matrix, the problem is reduced to just inverting four matrices each one-fourth the size of the original matrix. The cost of solving the matrix is proportional to N^2 where N is the size of the matrix. Thus, the computer time is only one-fourth of the original in this case.

Now, the matrix S and S^{-1} which diagonalizes [M] has to be discovered.

Let S be divided into 16 submatrices

$$[S] = \begin{bmatrix} S_{00} & S_{01} & S_{02} & S_{03} \\ S_{10} & S_{11} & S_{12} & S_{13} \\ S_{20} & S_{21} & S_{22} & S_{23} \\ S_{30} & S_{31} & S_{32} & S_{33} \end{bmatrix} \tag{6.9}$$

The sub matrix $$S_{mn} = [I] \frac{1}{\sqrt{N}} \exp\left(\frac{2\pi mn i}{N}\right) \tag{6.10}$$

where i is an imaginary number. N is the number of submatrices to a cycle, in this case, 4.

Thus, $$S_{mn}^{-1} = [I] \frac{1}{\sqrt{N}} \exp\left(\frac{-2\pi i mn}{N}\right) \tag{6.11}$$

As indicated by the equation, the matrix S related to our problem is

$$[S] = \frac{1}{2} \begin{bmatrix} [I] & [I] & [I] & [I] \\ [I] & i[I] & -[I] & -i[I] \\ [I] & -[I] & [I] & -[I] \\ [I] & -i[I] & -[I] & i[I] \end{bmatrix} \tag{6.12}$$

$$[S]^{-1} = \frac{1}{2} \begin{bmatrix} [I] & [I] & [I] & [I] \\ [I] & -i[I] & -[I] & i[I] \\ [I] & -[I] & [I] & -[I] \\ [I] & [i] & -[I] & -i[I] \end{bmatrix} \tag{6.13}$$

After performing the operation shown on eq. (6.6) using (6.12) and (6.13), the following is obtained (i being the imaginary number):

$$[M_D] = \begin{bmatrix} [A+B+C+D] & 0 & 0 & 0 \\ 0 & [A-C+iB-iD] & 0 & 0 \\ 0 & 0 & [A-B+C-D] & 0 \\ 0 & 0 & 0 & [A-C-iB+iD] \end{bmatrix} \tag{6.15}$$

From the notation used in eq. (6.6)

$$\begin{aligned} [E] &= [A] + [B] + [C] + [D] \\ [F] &= [A] - [C] + i[B] - i[D] \\ [G] &= [A] + [C] - [B] - [D] \\ [H] &= [A] - [C] - i[B] + i[D] \end{aligned} \tag{6.14}$$

Since $[M_D]$ is a complex matrix, $[M_D]^{-1}$ is complex also.

To solve for the unknown $\{\underset{\sim}{x}\}$, from eq. (6.8), one has to find M_D^{-1} and since M_D is cyclic, using the notation used in eq. (6.6), one has to invert [E], [F], [G] and [H]. From eq. (6.15) these submatrices are each one-fourth the size of the original matrix. Consequently, four submatrices each one-fourth the size of the original matrix have to be inverted. However, from eq. (6.15), since [F] and [H] are complex, some manipulation is needed to evaluate the inverse. The operation in inverting a complex matrix is well-known.

Suppose $[\alpha + i\beta]^{-1} = [P + i\,Q]$,

Then $[P] = [\alpha + \beta\,\alpha^{-1}\,\beta]^{-1}$

$$(6.16)$$

and $[Q] = -[\alpha^{-1}\,\beta\,P]$

Thus, if one has to invert a complex matrix, (say $[\alpha + i\,\beta]$), one has to evaluate [P] (the real part of the inverse) and then [Q], the imaginary part of the inverse) by eq. (6.16).

From equation (6.15), $[E]^{-1}$ and $[G]^{-1}$ can be evaluated easily since they are both real matrices. $[F]^{-1}$ and $[H]^{-1}$ can be calculated using eq. (6.16). To be more explicit let

$$\alpha = [A] + [C]$$

$$\beta = [B] - [D]$$

[A], [B], [C], [D] being submatrices from the original matrix as shown in Fig. 6.3.

So, $[F]^{-1} = [\alpha + i\,\beta]^{-1} = [P + i\,Q]$

and noting that from eq. (6.15) the real part of [H] and [F] are the same and their imaginary part are of opposite sign.

$$[H]^{-1} = [\alpha - i\,\beta]^{-1} = [P - i\,Q]$$

[P] and [Q] can be calculated from eq. (6.16).

$$[P] = [\alpha + \beta\,\alpha^{-1}\,\beta]^{-1}$$

$$[Q] = -[\alpha^{-1}\,\beta\,P]$$

Thus, inversion of [F] and [H] is reduced (from above equation) into calculating $[\alpha^{-1}]$ and $[\alpha + \beta\, \alpha^{-1}\beta]^{-1}$, both matrices being only one-fourth the size of the original matrix.

Since each inversion of matrix is only one-fourth of the original size and there are four inversions, computer time in inverting the matrices is only one-fourth of that required to solve the original matrix.

After solving $[M_D]^{-1}$ eq. (6.8) can be applied to solve for $\{x\}$.

(b) Symmetry in boundary conditions as well as loading condition:

Other than the above-mentioned symmetrical properties that give rise to a cyclic matrix, another course of symmetry can arise from boundary and loading conditions. If the prescribed boundary conditions as well as loading conditions are symmetrical with respect to all edges, then other than resulting in a cyclic matrix as discussed earlier in this chapter, the right-hand vectors $\underset{\sim}{R}_1$, $\underset{\sim}{R}_2$, $\underset{\sim}{R}_3$ and $\underset{\sim}{R}_4$ in eq. (6.4) are identical to each other. So, obviously $\underset{\sim}{x}_1$, $\underset{\sim}{x}_2$, $\underset{\sim}{x}_3$ and $\underset{\sim}{x}_4$ must also be equal to each other. Letting $\hat{\underset{\sim}{x}} = \underset{\sim}{x}_1 = \underset{\sim}{x}_2 = \underset{\sim}{x}_3 = \underset{\sim}{x}_4$ and $\hat{\underset{\sim}{R}} = \underset{\sim}{R}_1 = \underset{\sim}{R}_2 = \underset{\sim}{R}_3 = \underset{\sim}{R}_4$, then the following equation has to be solved

$$\Big[[A] + [B] + [C] + [D]\Big]\ \hat{\underset{\sim}{x}} = \hat{\underset{\sim}{R}} \qquad (6.17)$$

Thus, all that needs to be done is to solve a matrix $\Big[[A] + [B] + [C] + [D]\Big]$, which is one-fourth the original size and computing time is only one-sixteenth of the original case.

CHAPTER VII. CONCLUSION AND EXAMPLES

a) Conclusion

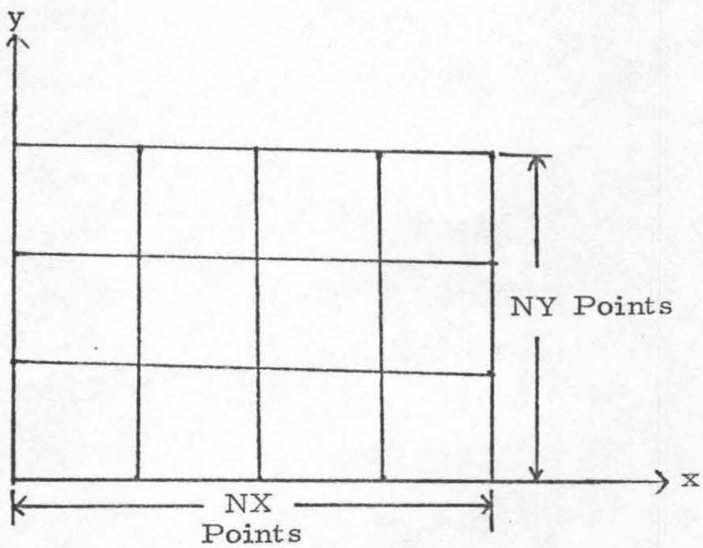
The previous chapters illustrate the application of the Edge Function method on plate problems on a Winkler Foundation. A computer program has been set up such that it can be used to solve the problem of a thin rectangular plate on a Winkler Foundation. To utilize the program, the following has to be done:

1) Using numerical process or otherwise, evaluate the Fourier coefficients of the series expressing the loading conditions as discussed in Chapter II (b).

2) On each of the edges, evaluate the Fourier coefficients of the series expressing the boundary conditions.

3) The Fourier coefficients evaluated in steps (1) and (2) are used as input data in the program already set up and the output will be the coefficients of Edge Functions on each edge (see eq. 4.6). Total solutions of displacements as well as slope, moment and shear in both x and y directions at points shown in Fig. 7.1 are also calculated and presented as computer output.

The program can easily be modified for point loads and column loads such that step (1) can be incorporated into the main program. The program can also be modified easily so that solutions on any points in the plate can be calculated instead of following the pattern shown in Fig. 7.1. A number of problems have been solved as examples. An example which has been solved using the computer program is presented in this chapter. The solution is compared to that from the literature. Further examples solved by the same programs are presented in the appendix.

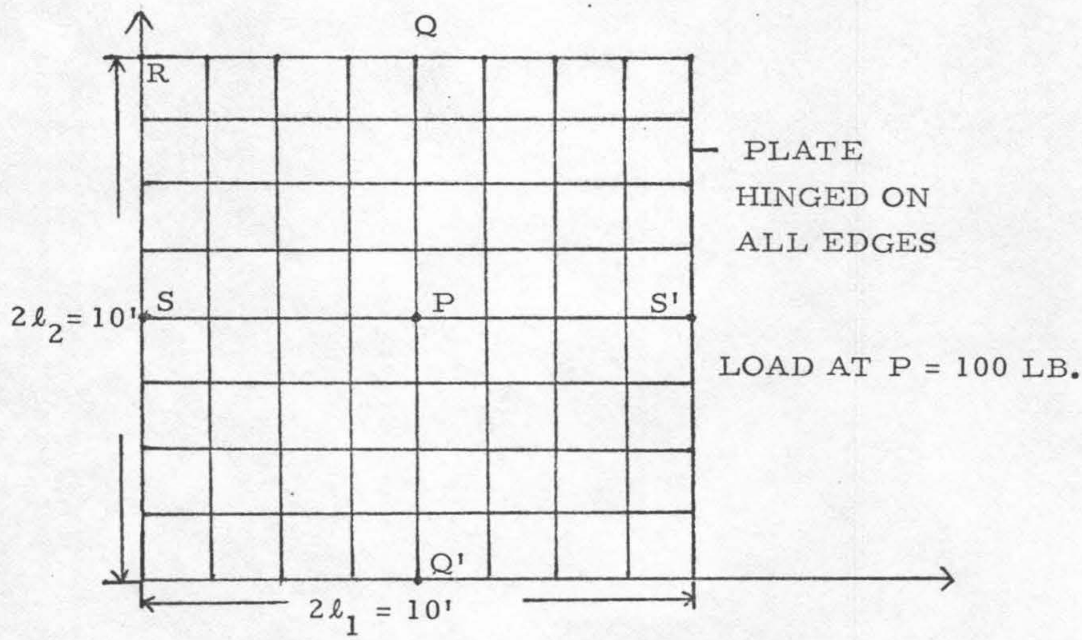


Solutions are Calculated on Each Nodal Point

Fig. 7.1

b) Example

The problem of a simply supported square plate subjected to central point load resting on a Winkler Foundation is solved. The plate shown is 10 ft by 10 ft with a point load of 100 lb on the center point of the plate and hinged on all edges (Displacement and moment equal to zero). (See Fig. 7.2.)



Solutions are Calculated at the Nodal Points

Fig. 7.2

Solutions are calculated at the nodal points as shown. In Tables 7.5 and 7.7, only those values at the nodal points in the rectangular area PQRS are presented.

The physical constants used in the problem are as follows:

K spring constant of Winkler Foundation is 1 lb/ft^3

E Young's modulus for the elastic plate is 10^4 lb/ft^2

ν Poisson's ratio of the plate is 0.2

h thickness of the plate is 0.1 ft

D the plate constant given by $\dfrac{Eh^3}{12(1-\nu^2)} = 0.86805 \text{ ft}^4$

ℓ the characteristic length of the plate spring system is

$\sqrt[4]{\dfrac{D}{K}} = 0.965 \text{ ft}$

P the point load is 100 lb

ℓ_1 $\frac{1}{2}$ length of the plate is 5 ft in the x-direction

ℓ_2 $\frac{1}{2}$ length of the plate is 5 ft in the y-direction

Tables 7.1, 7.3, 7.5 and 7.7 contain solutions calculated from Timoshenko's[(8)] solution (chapter 8). Tables 7.2, 7.4, 7.6 and 7.8 contain solutions calculated by the Edge Function method.

For equilibrium consideration for the Edge Function method as well as the Timoshenko solution, the total upward force is contributed from the spring force as well as the shearing force on the boundary. The following table gives the values of the upward force calculated. They are supposed to be equal to 100 lb when the total downward load is applied. The spring force was calculated from the displacements of the plate at each point and the shearing force calculated from the shearing force in the normal direction on each edge obtained from the 3rd derivative of the displacement along the edge.

COMPARISON OF EQUILIBRIUM

	Edge Function	Timoshenko's Solution
Force from springs	90.5 lb	87.5 lb
Shear force on the edges	2.3 lb	10.6 lb
Total upward force	92.8 lb	98.1 lb

From Tables 7.1 and 7.2 the values of displacement agree very closely (the maximum deflection is 13.368 in Timoshenko's solution and 13.384 in the Edge Function method; 3 significant figure of accuracy is achieved with 20 terms). The slope results in Tables 7.3 and 7.4 also agree fairly closely. However, from

Tables 7.5 to 7.8, the moment from both solutions does not agree so well and shearing forces differ even more. As higher degrees of differentiation are considered, the results from the Edge Function method diverge more from Timoshenko's solution. This is because the convergence in the Edge Function method deteriorates. To get a better solution more terms would have to be taken.

Solutions on axis SPS' (see Fig. 7.2), a line on the mid-plate, are given on Fig. 7.3 through Fig. 7.5. Shear on Q'PQ, a line on the mid-plate parallel to Y axis, is plotted and given on Fig. 7.6, as shear in the Y direction is calculated instead of shear in the X-direction.

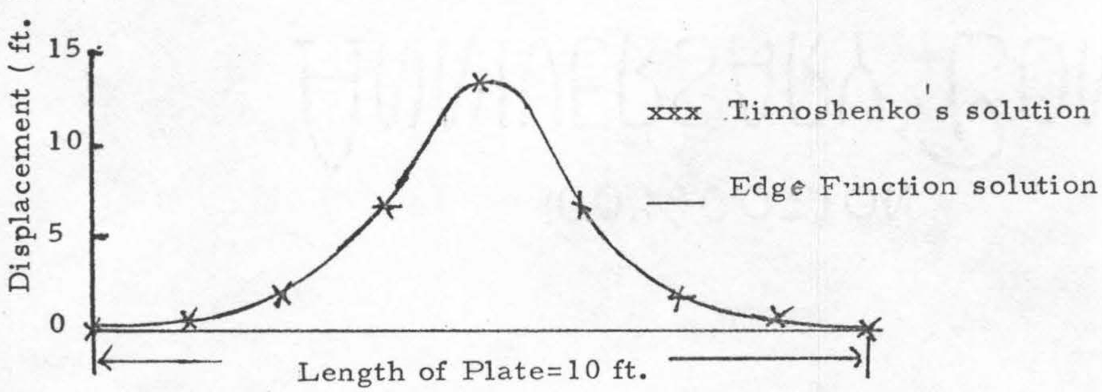


Fig. 7. 3 Plotting of Displacement along line SPS' in Fig. 7. 2

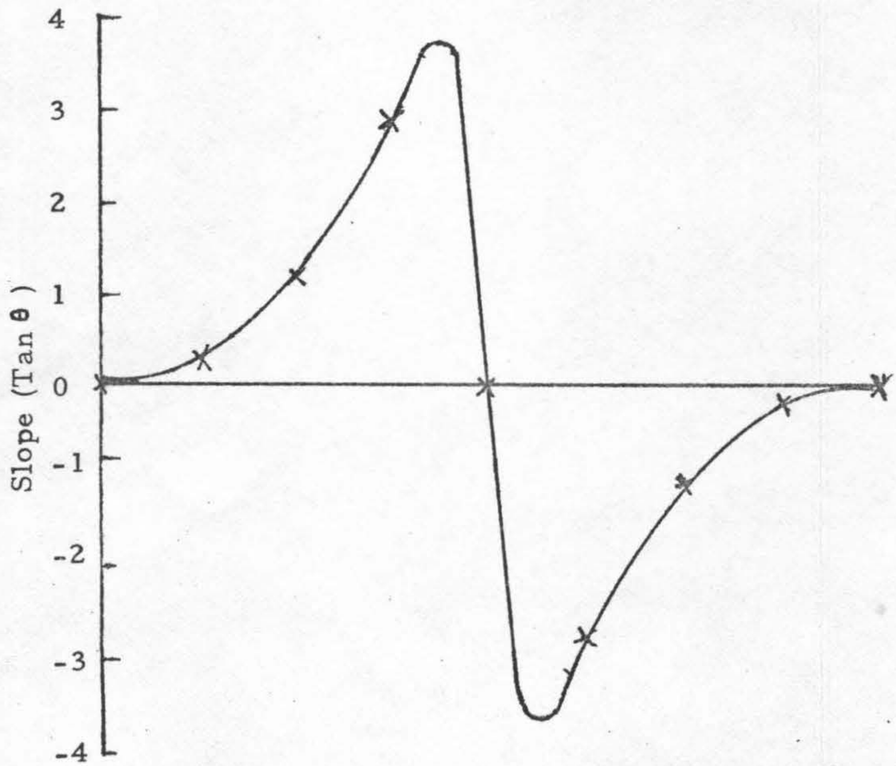


Fig. 7. 4 Plotting of Slope of Slope along line SPS' in Fig. 7. 2

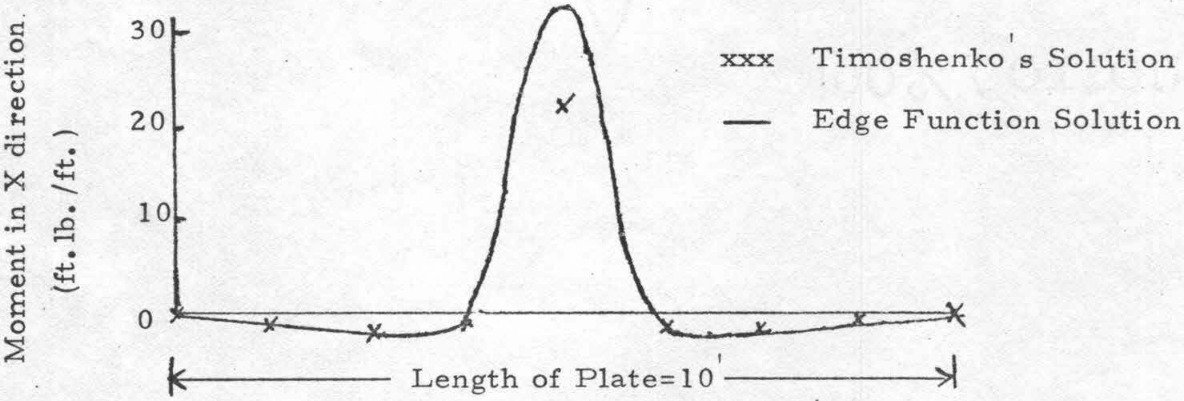


Fig. 7.5 Plotting of Moment along SPS' (Fig. 7.2)

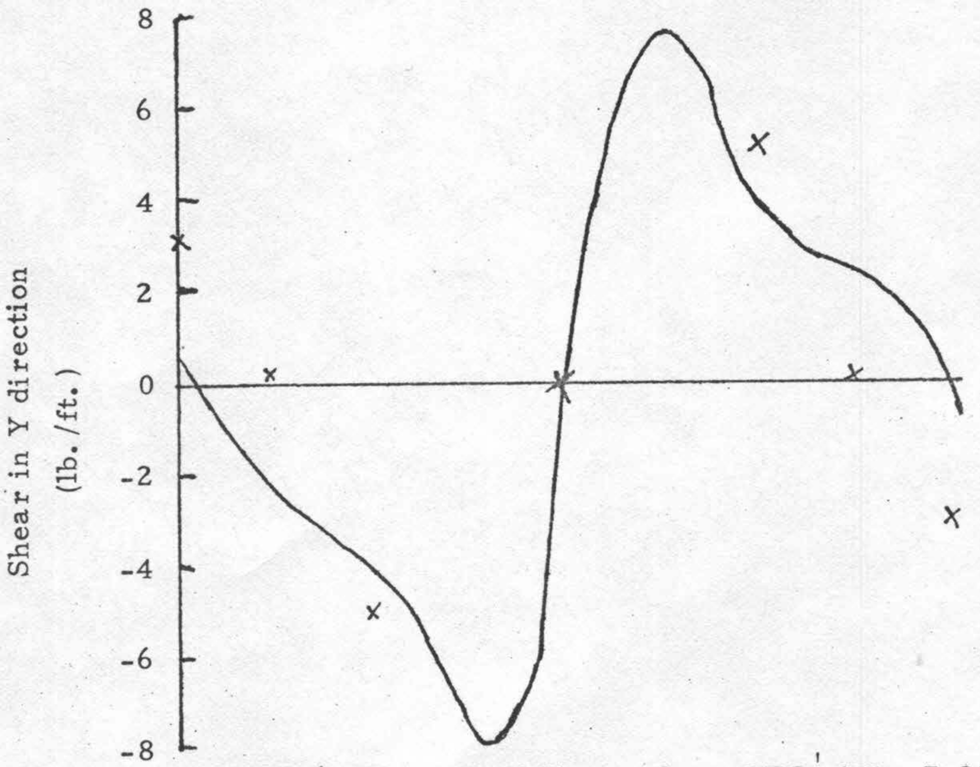


Fig. 7.6 Plotting of Shear along QPQ' (Fig. 7.2)

TABLE 7.1

Displacement from T. Solution (ft.)

0.0	-0.41468E-06	0.96328E-06	0.56694E-05	0.10750E-04	0.56694E-05	0.96327E-06	-0.41469E-06	0.71045E-11
0.0	-0.12098E 00	-0.13383E 00	-0.41895E-02	0.93728E-01	-0.41905E-02	-0.13383E 00	-0.12098E 00	-0.41468E-06
0.0	-0.13383E 00	0.15364E 00	0.10493E 01	0.16768E 01	0.10493E 01	0.15363E 00	-0.13384E 00	0.96324E-06
0.0	-0.42024E-02	0.10493E 01	0.41130E 01	0.67309E 01	0.41130E 01	0.10493E 01	-0.42043E-02	0.56694E-05
0.0	0.93719E-01	0.16768E 01	0.67309E 01	0.13368E 02	0.67309E 01	0.16768E 01	0.93716E-01	0.10750E-04
0.0	-0.42040E-02	0.10493E 01	0.41130E 01	0.67309E 01	0.41130E 01	0.10493E 01	-0.42069E-02	0.56694E-05
0.0	-0.13383E 00	0.15364E 00	0.10493E 01	0.16768E 01	0.10493E 01	0.15364E 00	-0.13383E 00	0.96326E-06
0.0	-0.12098E 00	-0.13383E 00	-0.41864E-02	0.93729E-01	-0.41873E-02	-0.13383E 00	-0.12098E 00	-0.41463E-05
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

TABLE 7.2

Displacement from E.F. Solution (ft.)

0.15684E 00	0.71038E-01	0.19589E-01	0.60144E-02	0.49823E-02	0.60155E-02	0.19592E-01	0.71040E-01	0.15684E 00
0.71040E-01	-0.79382E-01	-0.11361E 00	0.11419E-01	0.10974E 00	0.11420E-01	-0.11361E 00	-0.79380E-01	0.71040E-01
0.19590E-01	-0.11361E 00	0.16458E 00	0.10559E 01	0.16822E 01	0.10559E 01	0.16458E 00	-0.11361E 00	0.19590E-01
0.60117E-02	0.11390E-01	0.10559E 01	0.41120E 01	0.67269E 01	0.41120E 01	0.10559E 01	0.11399E-01	0.60115E-02
0.49772E-02	0.10969E 00	0.16822E 01	0.67268E 01	0.13384E 02	0.67269E 01	0.16821E 01	0.10970E 00	0.49770E-02
0.60104E-02	0.11390E-01	0.10559E 01	0.41120E 01	0.67268E 01	0.41120E 01	0.10559E 01	0.11398E-01	0.60101E-02
0.19588E-01	-0.11361E 00	0.16458E 00	0.10559E 01	0.16822E 01	0.10559E 01	0.16458E 00	-0.11361E 00	0.19587E-01
0.71038E-01	-0.79384E-01	-0.11361E 00	0.11409E-01	0.10972E 00	0.11410E-01	-0.11361E 00	-0.79381E-01	0.71037E-01
0.15684E 00	0.71037E-01	0.19588E-01	0.60141E-02	0.49821E-02	0.60154E-02	0.19592E-01	0.71040E-01	0.15684E 00

T. Solution Timoshenko's solution

E.F. Solution Edge Function's solution

TABLE 7.3

Slope in X direction from T. SOLUTION (Tan θ)

-0.53796E-06	0.12997E-06	0.23432E-05	0.49081E-05	0.12247E-11	-0.49081E-05	-0.23432E-05	-0.12998E-06	0.53793E-06
-0.11148E 00	-0.66957E-01	0.54169E-01	0.12677E 00	-0.80115E-07	-0.12677E 00	-0.54169E-01	0.66956E-01	0.11148E 00
-0.16179E 00	0.48110E-02	0.49632E 00	0.81387E 00	-0.74311E-06	-0.81387E 00	-0.49632E 00	-0.48107E-02	0.16180E 00
-0.12913E 00	0.25868E 00	0.15948E 01	0.30690E 01	-0.18669E-05	-0.30690E 01	-0.15948E 01	-0.25868E 00	0.12913E 00
-0.84384E-01	0.43619E 00	0.23993E 01	0.56969E 01	0.45985E-05	-0.56970E 01	-0.23992E 01	-0.43619E 00	0.84412E-01
-0.12913E 00	0.25868E 00	0.15948E 01	0.30690E 01	-0.18670E-05	-0.30690E 01	-0.15948E 01	-0.25868E 00	0.12914E 00
-0.16179E 00	0.48119E-02	0.49632E 00	0.81387E 00	-0.74310E-06	-0.81388E 00	-0.49632E 00	-0.48117E-02	0.16180E 00
-0.11148E 00	-0.66957E-01	0.54171E-01	0.12677E 00	-0.80131E-07	-0.12677E 00	-0.54170E-01	0.66957E-01	0.11148E 00
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

TABLE 7.4

Slope in x direction from E.F. SOLUTION (Tan θ)

-0.65806E-01	-0.60417E-01	-0.22693E-01	-0.29333E-02	0.80762E-07	0.29335E-02	0.22681E-01	0.60407E-01	0.65807E-01
-0.12813E 00	-0.91062E-01	0.44765E-01	0.12703E 00	0.29944E-06	-0.12703E 00	-0.44766E-01	0.91062E-01	0.12813E 00
-0.15001E 00	-0.21748E-02	0.49038E 00	0.81292E 00	0.21613E-05	-0.81291E 00	-0.49038E 00	0.21791E-02	0.15001E 00
-0.10812E 00	0.25533E 00	0.15868E 01	0.30652E 01	0.10010E-04	-0.30652E 01	-0.15867E 01	-0.25532E 00	0.10811E 00
-0.65245E-01	0.42787E 00	0.23927E 01	0.57082E 01	0.74993E-04	-0.57082E 01	-0.23926E 01	-0.42787E 00	0.65221E-01
-0.10812E 00	0.25533E 00	0.15868E 01	0.30652E 01	0.10012E-04	-0.30652E 01	-0.15867E 01	-0.25532E 00	0.10811E 00
-0.15001E 00	-0.21640E-02	0.49039E 00	0.81292E 00	0.21780E-05	-0.81292E 00	-0.49039E 00	0.21683E-02	0.15001E 00
-0.12813E 00	-0.91063E-01	0.44762E-01	0.12703E 00	0.34573E-06	-0.12703E 00	-0.44762E-01	0.91063E-01	0.12813E 00
-0.65807E-01	-0.60418E-01	-0.22693E-01	-0.29331E-02	-0.45898E-07	0.29336E-02	0.22682E-01	0.60408E-01	0.65806E-01

TABLE 7.5

Moment in x direction from T. Solution (ft. lb. /ft.)

0.0	-0.11334E-05	-0.23570E-05	-0.13924E-05	0.13962E-04
0.0	-0.72695E-01	-0.13569E 00	-0.93430E-01	-0.93846E-01
0.0	-0.25221E 00	-0.46911E 00	-0.12405E 00	0.32007E 00
0.0	-0.56329E 00	-0.12117E 01	-0.19447E 00	0.33866E 01
0.0	-0.92808E 00	-0.21507E 01	-0.16473E 01	0.22136E 02

TABLE 7.6

Moment in x direction from E.F. Solution (ft. lb. /ft.)

0.97234E-02	-0.20027E-01	-0.19422E-01	-0.14837E-02	-0.15762E-01	-0.14811E-02	-0.19422E-01	-0.20028E-01	0.97227E-02
0.13941E-01	-0.82576E-01	-0.14853E 00	-0.99347E-01	-0.33132E-01	-0.99346E-01	-0.14853E 00	-0.82576E-01	0.13941E-01
0.16594E-01	-0.25320E 00	-0.47995E 00	-0.13480E 00	0.47630E 00	-0.13480E 00	-0.47995E 00	-0.25320E 00	0.16594E-01
0.16974E-01	-0.55930E 00	-0.12223E 01	-0.15571E 00	0.36379E 01	-0.15570E 00	-0.12222E 01	-0.55929E 00	0.16974E-01
-0.95339E-02	-0.77453E 00	-0.17627E 01	-0.10304E 01	0.33515E 02	-0.10305E 01	-0.17625E 01	-0.77459E 00	-0.95341E-02
0.16975E-01	-0.55930E 00	-0.12223E 01	-0.15571E 00	0.36379E 01	-0.15570E 00	-0.12222E 01	-0.55929E 00	0.16974E-01
0.16595E-01	-0.25320E 00	-0.47995E 00	-0.13480E 00	0.47629E 00	-0.13480E 00	-0.47995E 00	-0.25320E 00	0.16594E-01
0.13941E-01	-0.82575E-01	-0.14853E 00	-0.99347E-01	-0.33097E-01	-0.99346E-01	-0.14853E 00	-0.82575E-01	0.13941E-01
0.97236E-02	-0.20027E-01	-0.19422E-01	-0.14837E-02	-0.15762E-01	-0.14813E-02	-0.19422E-01	-0.20027E-01	0.97236E-02

TABLE 7.7

Shear in y direction from T. Solution (lb. /ft.)

0.00000E+00	0.87971E-01	0.27295E+00	0.46323E+00	0.33067E+01
0.00000E+00	0.19174E+00	0.41642E+00	0.47467E+00	0.38596E+00
0.00000E+00	0.39512E+00	0.71944E+00	0.74918E-01	-0.50765E+01
0.00000E+00	0.37377E+00	0.82054E+00	-0.10773E+01	-0.10615E+02
0.00000E+00	0.14121E-06	-0.72352E-09	-0.34863E-06	-0.21495E-04

STOP

TABLE 7.8

Shear in y direction from E.F. Solution (lb. /ft)

-0.53190E 00	0.12819E 00	0.24167E 00	0.41199E 00	0.57487E 00	0.41199E 00	0.24167E 00	0.12818E 00	-0.53189E 00
0.22432E-01	0.19086E 00	0.39242E 00	0.43053E 00	-0.24554E 01	0.43054E 00	0.39241E 00	0.19086E 00	0.22430E-01
0.31815E-01	0.35491E 00	0.69319E 00	0.10884E 00	-0.38582E 01	0.10883E 00	0.69319E 00	0.35492E 00	0.31815E-01
-0.79636E-01	0.35413E 00	0.83973E 00	-0.10222E 01	-0.77601E 01	-0.10222E 01	0.83975E 00	0.35410E 00	-0.79636E-01
-0.95787E-04	-0.35483E-04	0.83827E-04	0.20459E-03	0.15716E-01	0.20455E-03	0.83823E-04	-0.35431E-04	-0.74358E-04
0.79638E-01	-0.35412E 00	-0.83973E 00	0.10222E 01	0.77603E 01	0.10222E 01	-0.83975E 00	-0.35410E 00	0.79637E-01
-0.31815E-01	-0.35491E 00	-0.69319E 00	-0.10884E 00	0.38580E 01	-0.10883E 00	-0.69318E 00	-0.35492E 00	-0.31816E-01
-0.22424E-01	-0.19085E 00	-0.39243E 00	-0.43055E 00	0.24538E 01	-0.43055E 00	-0.39242E 00	-0.19086E 00	-0.22426E-01
0.53190E 00	-0.12818E 00	-0.24168E 00	-0.41201E 00	-0.57631E 00	-0.41201E 00	-0.24168E 00	-0.12818E 00	0.53191E 00

CHAPTER VIII

ITERATIVE METHOD AND FURTHER USEFUL EXAMPLES

a) Iterative Method

To recapitulate, Chapter III discusses the general concepts of the Edge Function method and Chapter IV applies the concepts for the problem of a rectangular plate resting on a Winkler Foundation subjected to lateral loads. As a result of the application, a very big matrix is set up which leads to four sets of coefficients for the Edge Functions (see eq. 4.6). Solving the matrix leads to a solution for the coefficients simultaneously. This simultaneous approach of solving the coefficients of the Edge Functions requires a great deal of storage space in the computer for the matrix as well as requiring a lot of computer time. This disadvantage is further magnified when solutions of higher directives of displacement are required as more terms are needed for a good answer.

Another approach is that an iterative process can also be used which basically eliminates the necessity of solving a very large matrix and which can easily be applied to plates of any convex polygonal shapes. This iterative approach will be illustrated as follows:

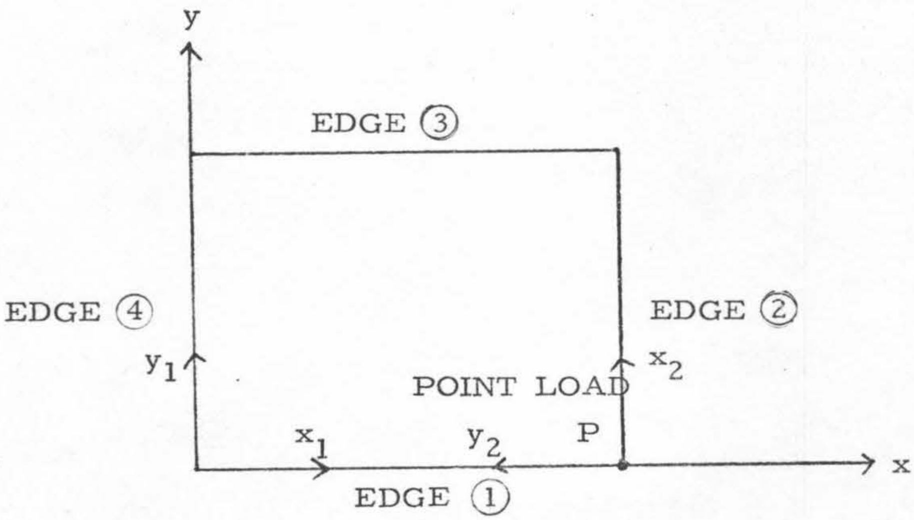


Fig. 8.1

Referring to Fig. 8.1, firstly, approximate coefficients for the Edge Function on edge 1 are evaluated such that the sum ($\widetilde{w}_{t1}$) of Edge Function 1, ($\widetilde{w}_c^1$) and the particular solution (w_p) will meet the necessary boundary conditions on edge 1. (The work involved in finding this approximate Edge Function on edge 1 is very small and will be discussed later.) However, anywhere else, such as along edges 2 and 3, this solution $\widetilde{w}_{t1}$ does not converge to the required boundary values. The boundary values on other edges can be accounted for one by one as opposed to the simultaneous approach. From the prescribed boundary values on edge 2 ($w_t(x_2)$) and this approximate solution ($\widetilde{w}_{t1}$) a new set of boundary values ($w_t(x_2) - \widetilde{w}_{t1}(x_2)$) can be calculated on edge 2 numerically or otherwise. Then an approximate Edge Function on edge (2) is chosen according to these new boundary conditions. Next, a new approximate solution $\widetilde{w}_{t2}$ can be evaluated as the sum of the approximate Edge Function on edge (2) ($\widetilde{w}_c^2$) and the original approximate solution $\widetilde{w}_{t1}$. The resulting solution is called $\widetilde{w}_{t2}$. This solution will satisfy the boundary conditions on edge (2) ; however, it will not meet the required boundary conditions on the other edges except on edge (1) where the solution will almost converge to the required boundary conditions. Similarly, a new set of boundary conditions can be calculated from $\widetilde{w}_{t2}$ on other edges and Edge Functions on other edges chosen accordingly. This will lead to an approximate solution after going around the edges of the plate once. If a better solution is needed it can be calculated by another sweep around the plate.

The following paragraph discusses what is involved in calculating each Edge Function using the iterative approach. Basically, the coefficients of the Edge Functions (see eq. 4.6) have to be calculated to satisfy some reduced boundary conditions. A numerical procedure or other method is used to reduce the boundary condition to a Fourier series. This series has to be equated with the Edge Function on that edge (say edge (1)). On substituting the coordinates of edge (1) in the Edge Function (1), the part involving y_1 is reduced to a known coefficient (with $y_1 = 0$). So, the Edge Function itself is reduced into a Fourier series with variables x_1 in sine and cosine series only and unknown coefficients A_m^1, B_m^1, C_m^1 and D_m^1 associated with them. For displacement, for instance, it is given in the form $\sum_{m=1}^{N} (\alpha_1 A_m^1 + \alpha_2 B_m^1) \sin \frac{m\pi x_1}{\ell_1}$ + $\sum_{m=0}^{N} (\alpha_3 C_m^1 + \alpha_4 D_m^1) \cos \frac{m\pi x_1}{\ell_1}$. Thus, all that is left is to equate the coefficients of the sine and cosine series with the sine and cosine series respectively of the reduced boundary conditions for each m. In this way, the problem is reduced to solving a 2 x 2 matrix twice for each m. Most of the computer time required in the iterative process will be used only in calculating the new boundary conditions on each edge and performing a Fourier analysis of the reduced boundary conditions. The cost in calculating further terms is just proportional to the terms rather than to N^2 in the simultaneous method.

Another advantage of this iterative process is that engineering judgement can be readily applied to reduce computing time. To illustrate this the example in Fig. 8.1 is given. Suppose one is interested in a rectangular plate problem with a load at a corner B as shown in this figure.

If the plate is big enough, the contribution from the Edge Functions (3) and (4) around B will be very small because the Edge Functions decay exponentially with distance from the edges. Realizing this fact, one can simply calculate the Edge Functions on edge (1) and edge (2) if one is only interested in the solution around the load. This economizing technique can also be applied to the simultaneous approach discussed earlier in previous chapters. In sections b) and c) in the latter part of this chapter, the iterative idea is used to solve the problem of a point load on an edge and on a corner respectively.

The reactions of the plate to a concentrated load on a plate on a Winkler Foundation are usually fairly well concentrated around the load. So, a few very useful examples can be calculated and applied to designs of slabs for airfield runways and paving slabs for roads. These examples are: (1) the load is at a considerable distance from the edges; (2) the load is at an edge but far away from any corners and (3) the load is at a rectangular corner of a large slab. For case (1) a fairly thorough investigation is done by Westergaard[2] and the readers are referred to the reference for more detailed information.

b) <u>Further examples: Concentrated load on an edge far away from any corner.</u>

Originally, the intention was to use the Edge Function method to calculate the solution of a concentrated load on an edge and at a corner far away from any other boundaries. The example presented in Appendices B and C are aimed at the above problems. However, because of the high derivatives involved, the original attempt presented in Appendices B and C is inadequate. Higher terms are needed. Since there is only one edge in the edge problem and two in the corner problem, the iterative idea can be applied easily. Results of the iterative method applied to solve this problem are presented in sections b and c in this chapter.

Westergaard[2, 3] calculated the problem of a semi-infinite plate resting on a Winkler Foundation with a load of uniform pressure on an ellipse tangential to the plate boundary (see Fig. 8.2). He gave an approximate formula of displacement calculated along the line P Q shown in Fig. 8.2. He also gave an estimate of bending stress at the edge of the plate on point P. This problem is an interesting problem with a lot of engineering application. The exact same problem is calculated using the iterative idea of the Edge Function method. A point load of the same magnitude at the edge of the plate is also calculated and compared with Westergaard's approximating formulas.[3]

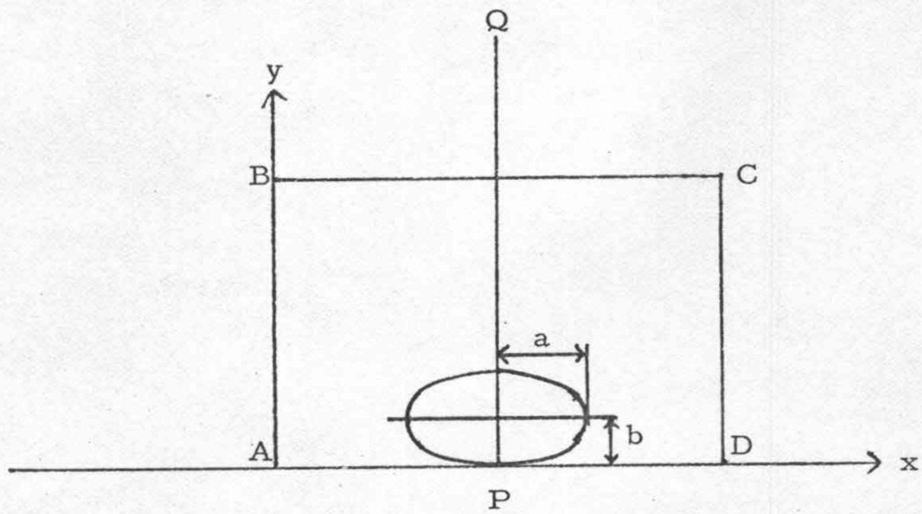


Fig. 8.2

The Edge Function method which usually applies to a finite plate is used to compare the solution of a semi-infinite plate by the following argument. If a plate ABCD (shown in Fig. 8.2) is big enough the edge effects along other edges to solutions close to the load must be very small and can be neglected. So, the behavior of the solution of a large enough plate should be very close to the same problem for the semi-infinite plate around the load. Thus, the Edge Function method can be applied to the same problem. Also, noting that the Edge Function's contribution around the load is very small from other edges (AB, BC and CD), only the edge function on edge AD is needed.

Referring to Fig. 8.2 and Fig. 8.5 the Edge Function method gives a solution very close to that given by the approximating formula given by Westergaard[3] in the region close to the load. The point load gives the limit of the solution as the ellipse or distributed load → 0 in dimension. The values given in the x and y axes are dimensionless and the displacement is virtually equal to zero at a distance larger than 3.5 of characteristic length of the plate spring system.

From Fig. 8.6, it can be seen that for points just a little distance away from the load, the solution for the point load and the elliptic for M_x is almost identical. The oscillating behavior of the moment M_y for the point load (given by curve A-2) may be due to errors accumulated in the computer due to large numbers of iterations. The tensile bending stress can be calculated from the formula $-\frac{Eh}{2} \frac{\partial^2 w}{\partial x^2}$ at P (Fig. 8.2), where h is the thickness and E the Young's modulus of the plate. Applying this formula (used by Westergaard[3]), the tensile bending stress calculated for the same elliptic load problem is 20.91 units compared to 23.57 units by Westergaard's approximate formula in the same paper.

In the graph, Fig. 8.7, the moment in the y-direction is plotted against the moment in the x-direction, for the problem of constant pressure over a small ellipse. The shape of the curve remains constant for any loading intensity as the values along the X-axis are magnified by the same factor as the values along the Y-axis. Fig. 8.8 contains the plot of M_y against M_x for a point load.

In most engineering practice the plate is reinforced in the X and Y direction and at the bottom and top of the plate in such a way that the yield envelope has a shape shown below.

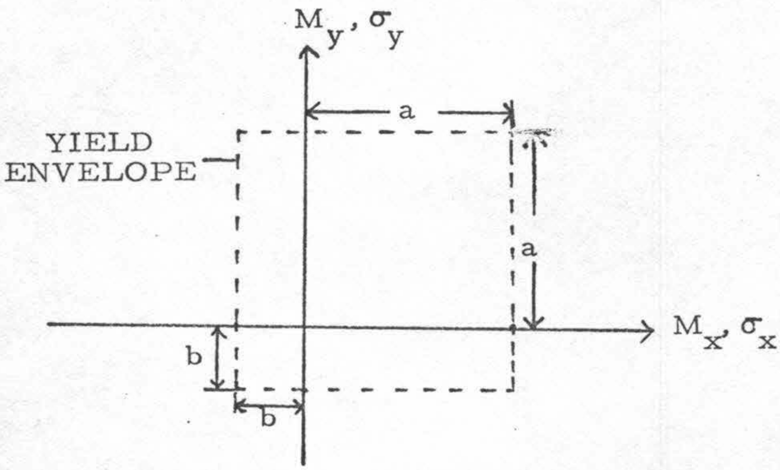


Fig. 8.3

From the M_x against M_y plot shown in Fig. 8.7 and Fig. 8.8, the curve has the shape like the figure shown below.

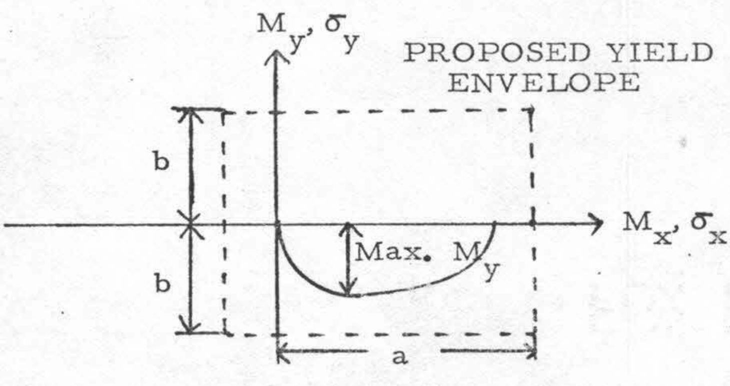


Fig. 8.4

If b in Fig. 8.3 is too small (because the positive M_x is usually the basic concern, the maximum negative M_y might exceed the envelope and the plate will break along lines parallel to the x-axis and cracking starts on the top of the plate. A more efficient envelope, it would seem, is of the form shown in Fig. 8.4.

In Fig. 8.9 the displacement along PD of a point load problem is plotted and in Fig. 8.10 the moment in the x-direction along PD of the same problem is plotted.

Because the problem is linear, the curves plotted in Figs. 8.5, 8.6 8.9 and 8.10 can be generalized for concentrated force of any magnitude. The dimensionless values of the graphs can be read off the axis on the right-hand side for any point x from the P in Fig. 8.2. Then, for displacement, the coefficient is multiplied by a factor of $\frac{P}{K\ell^2}$ where P is the magnitude of the point load in Force units, K is the spring constant in Force/(unit length)^3 and ℓ is the characteristic length of the problem in length units. For moment, the coefficient is simply multiplied by P.

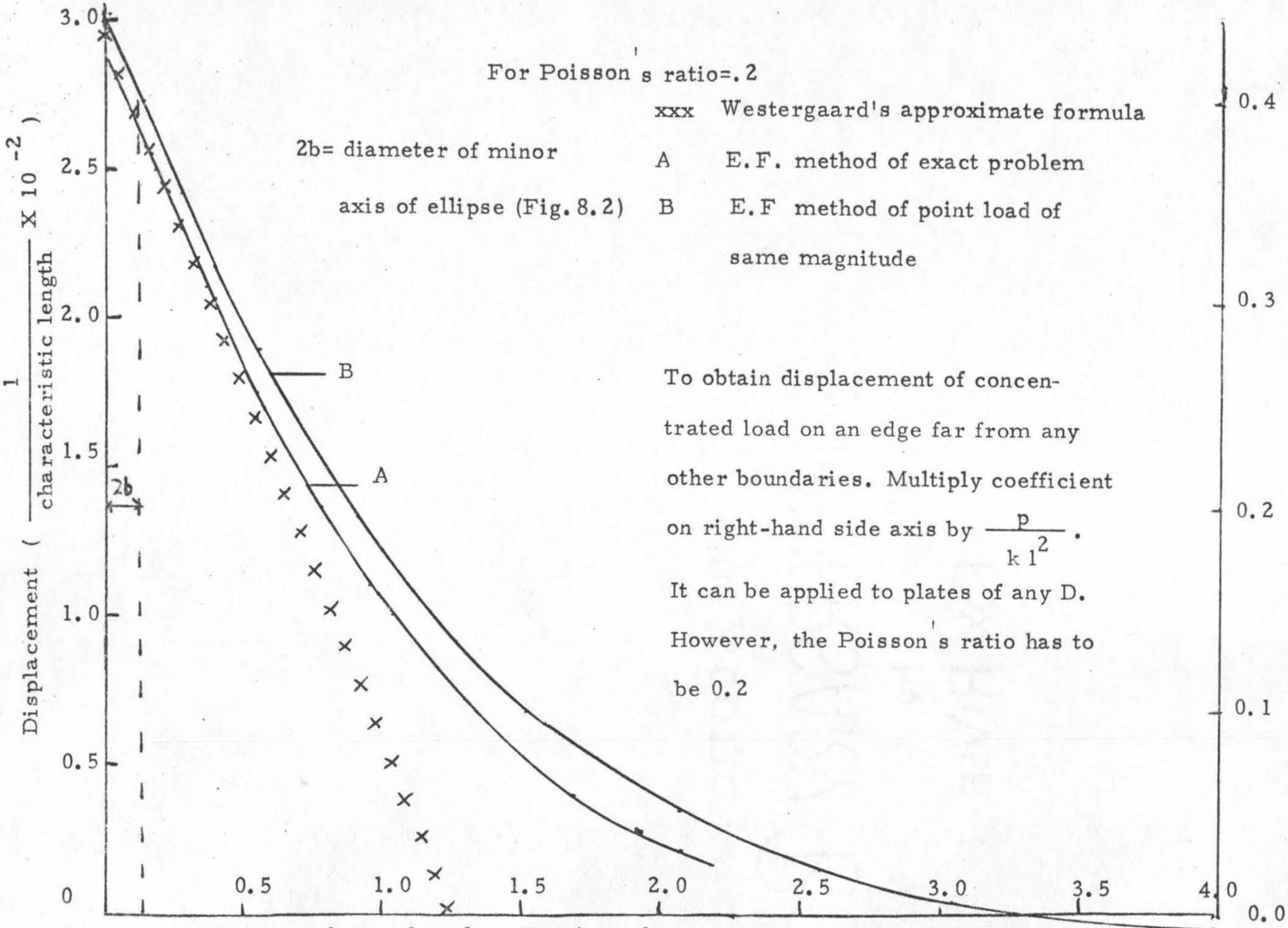


Fig. 8.5 Plotting of displacement along PQ (Fig. 8.2)

Abreviations: E.F. - Edge Function; p -load in force units; K - spring constant; l - characteristic length of spring plate system.

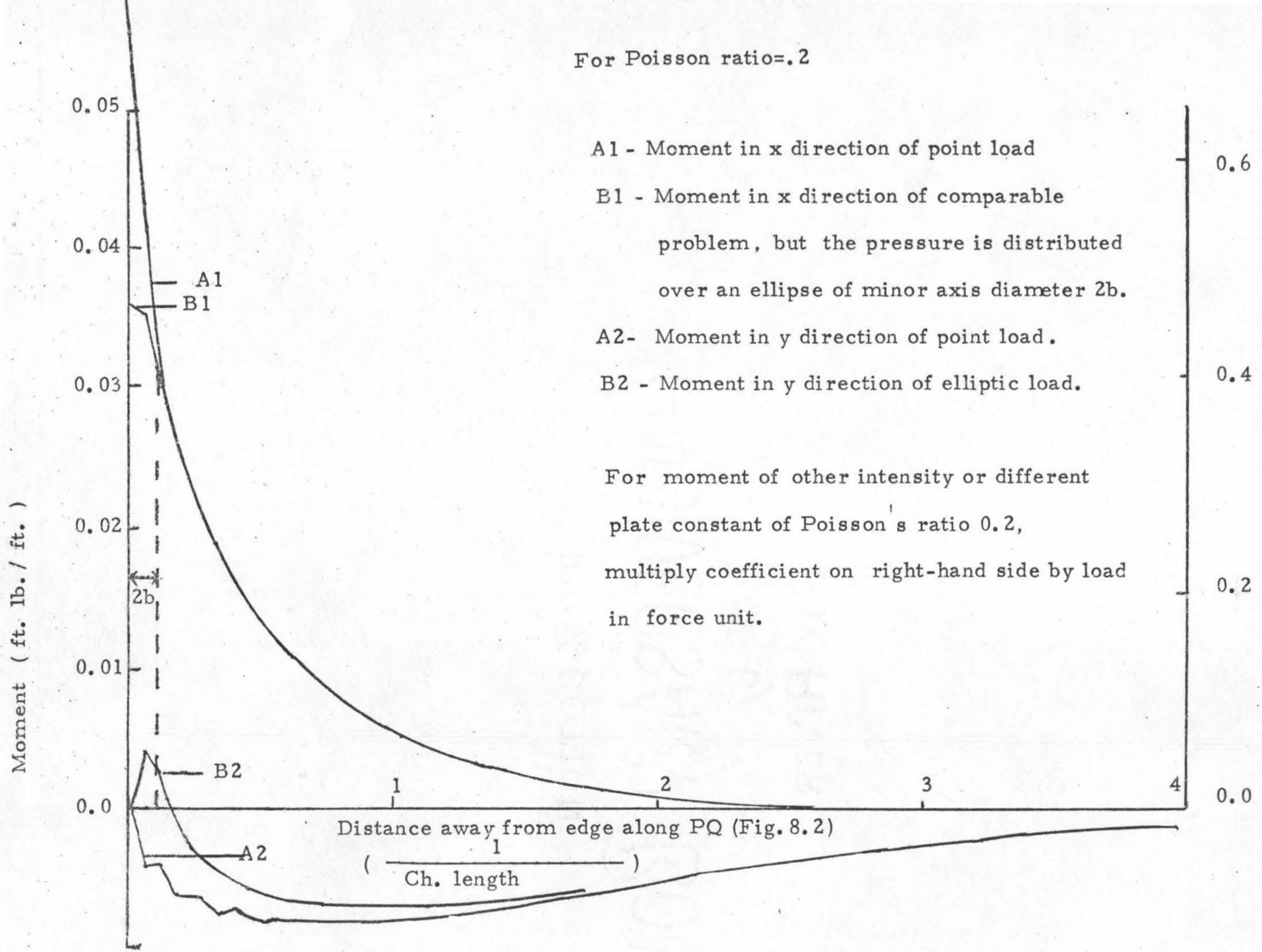


Fig. 8.6 Plotting of Moments along PQ (Fig. 8.2)

Abreviations: Ch. length - characteristic length

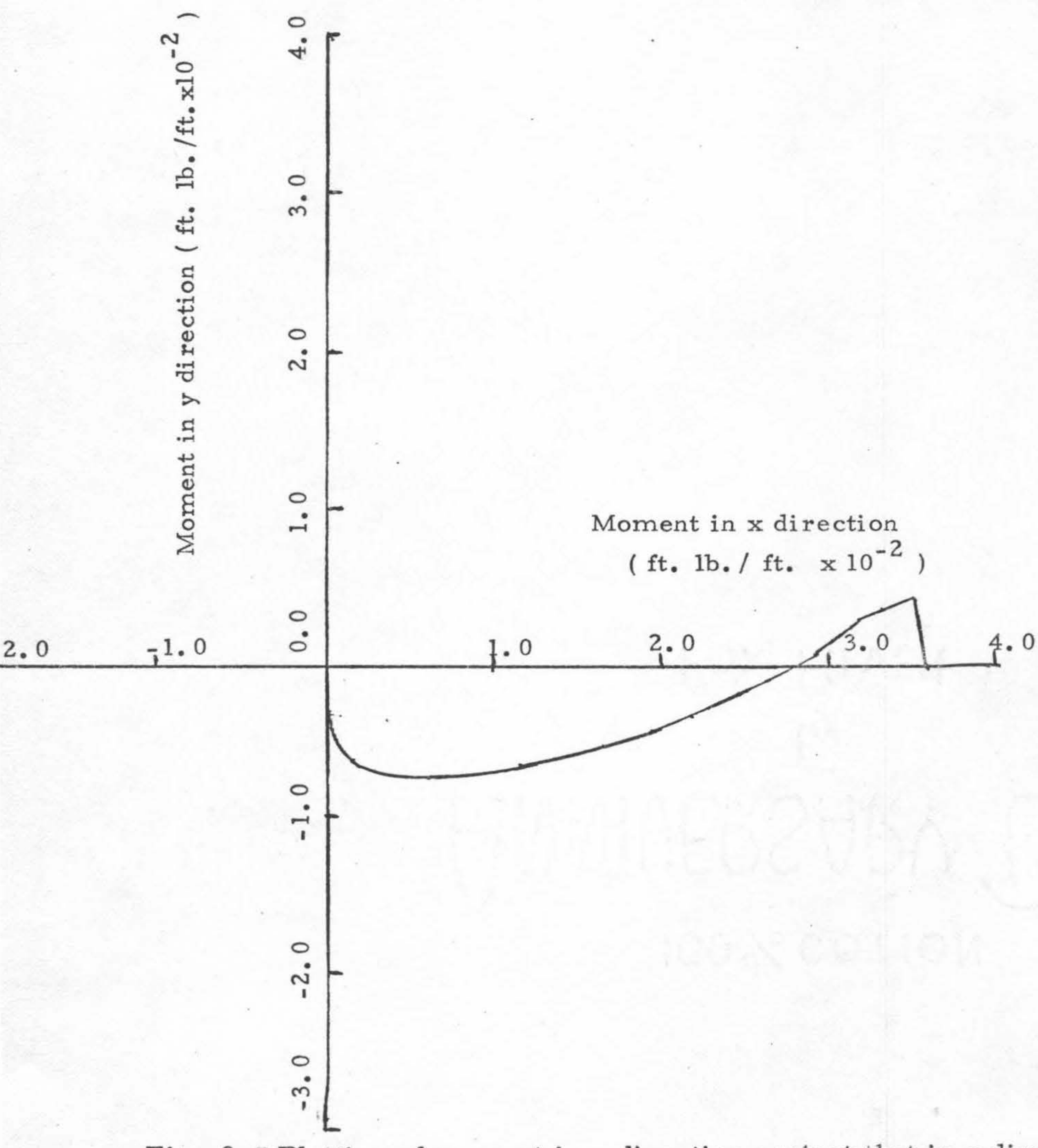


Fig. 8. 7 Plotting of moment in x direction against that in y direction along P Q (Fig. 8.2) of uniformily distributed load over an ellipse.

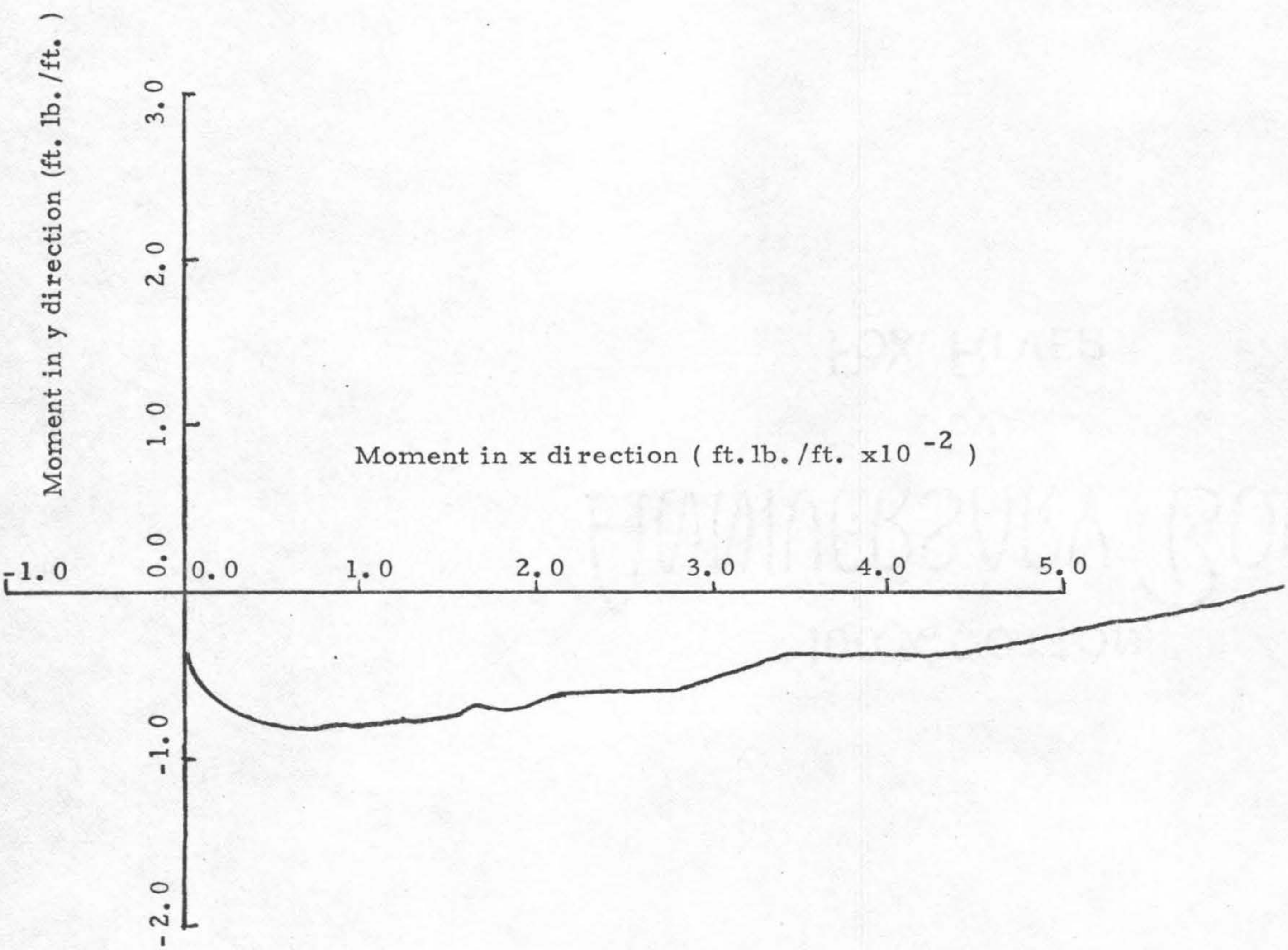


Fig. 8.8 Plotting of moment in x direction against that in y direction along P Q (Fig. 8.2) of point load.

For Poisson's ratio= 0.2

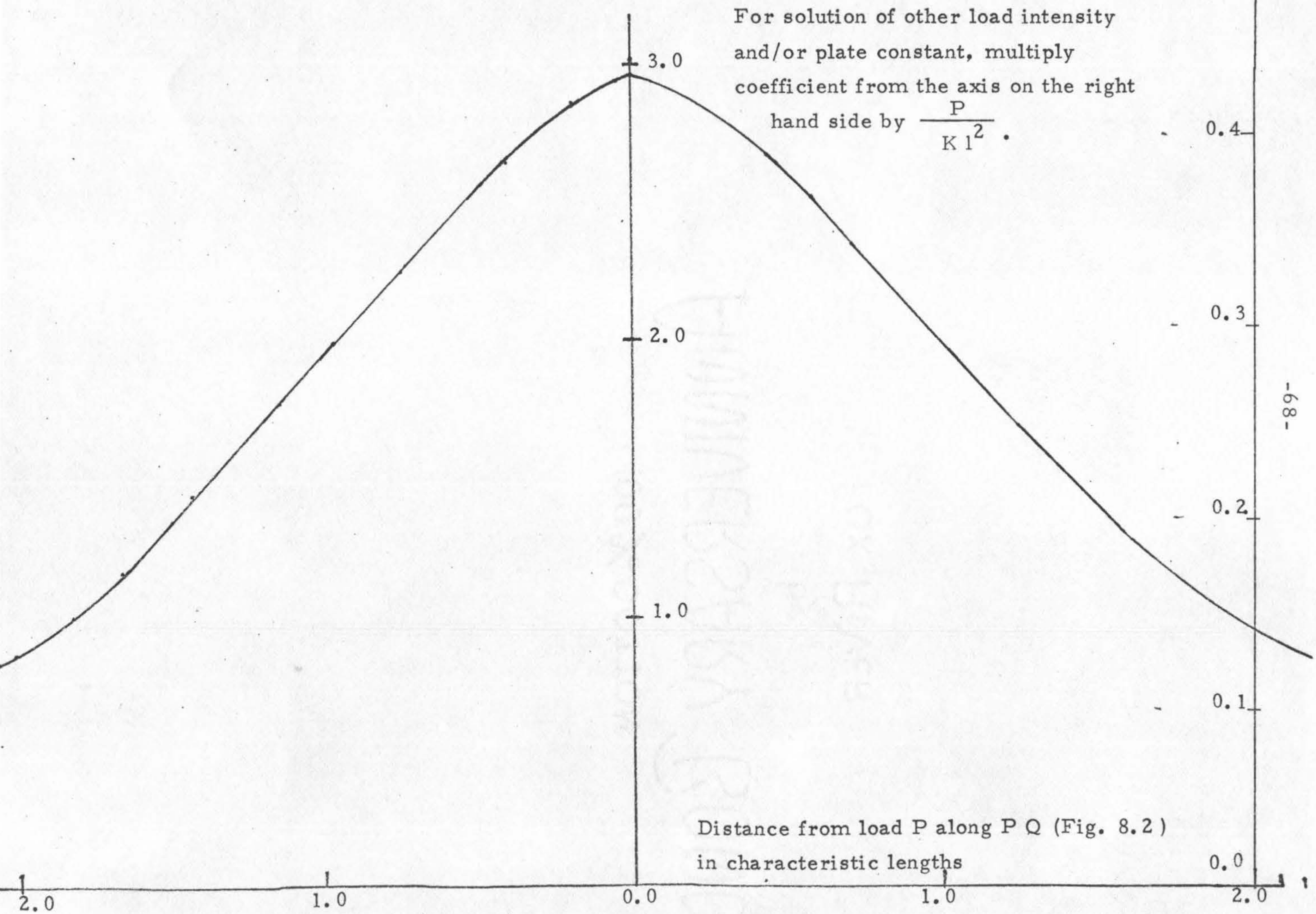


Fig. 8.9 Displacement along the edge of a point load very close to the edge.

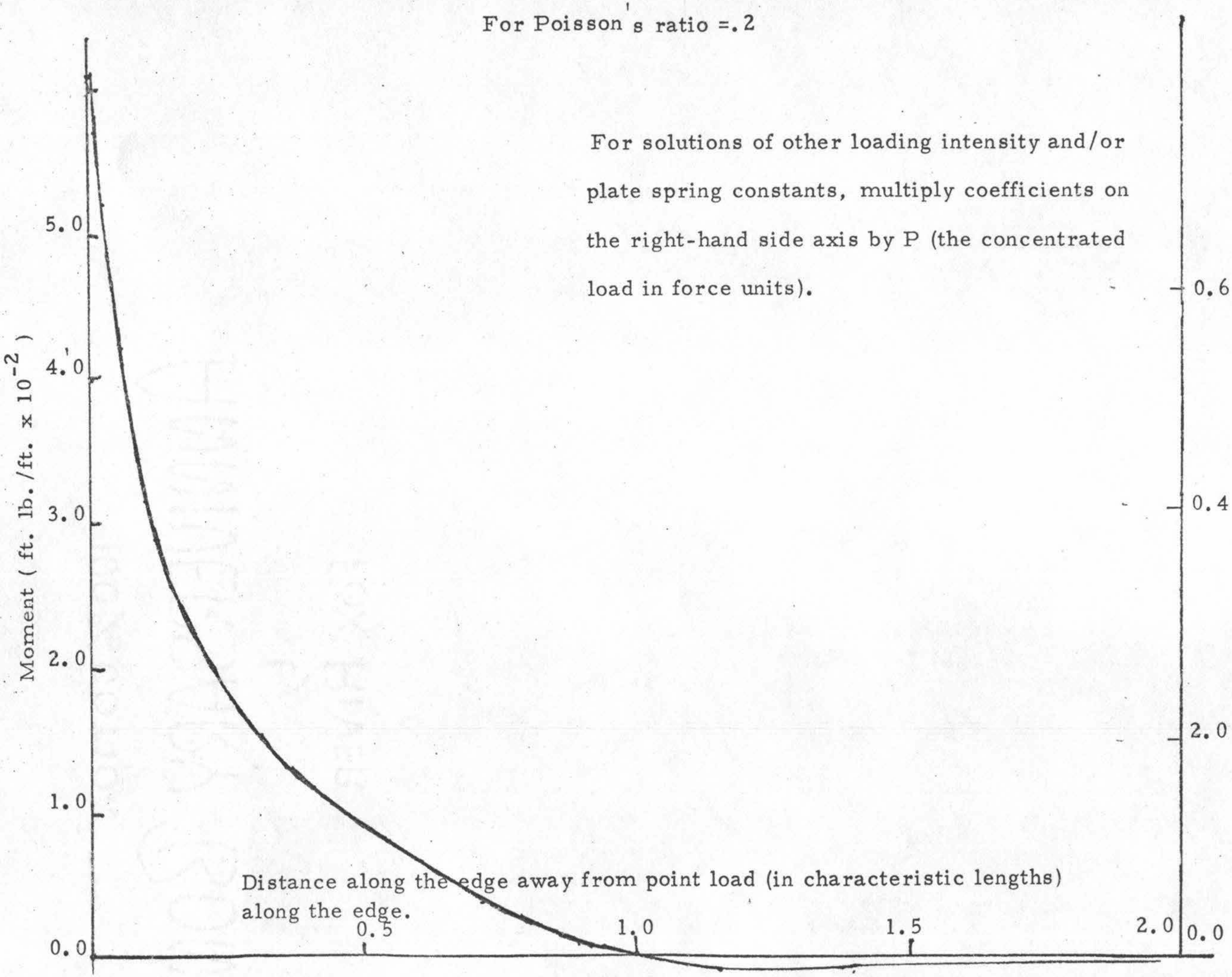


Fig. 8.10 Plotting of moment in the tangential direction of a pt. load on the edge of a plate

c) Further Example: A point load at corner B

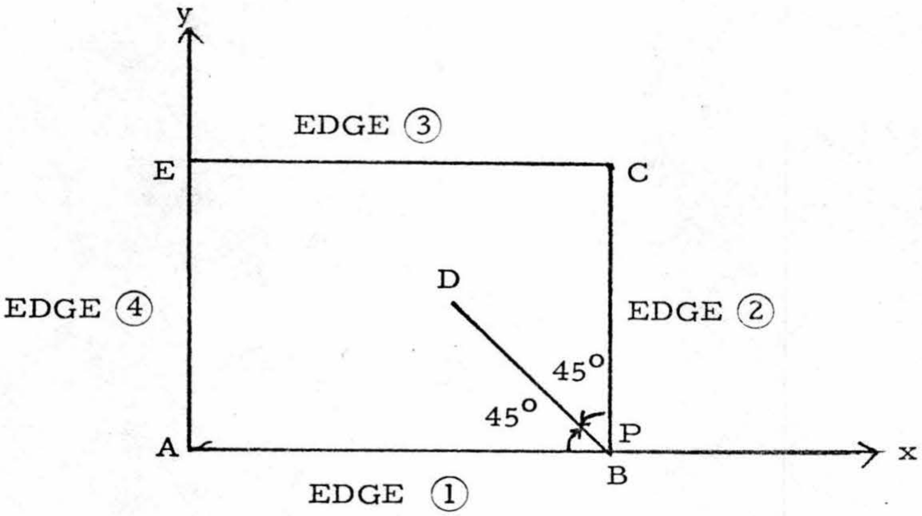


Fig. 8.11

From Fig. 8.11 the plate is chosen large enough so that the Edge Functions from 4 and 3 are small around the load P. In Fig. 8.12, displacement away from the load along line PD is plotted. In Fig. 8.13, the moment in the direction along the edge is plotted at points along the edge away from the corner. In Fig. 8.14 the moment in the PD direction (see Fig. 8.11) is plotted along PD.

As in section b) the results can be generalized for concentrated loads of other magnitudes.

For Poisson's ratio=.2

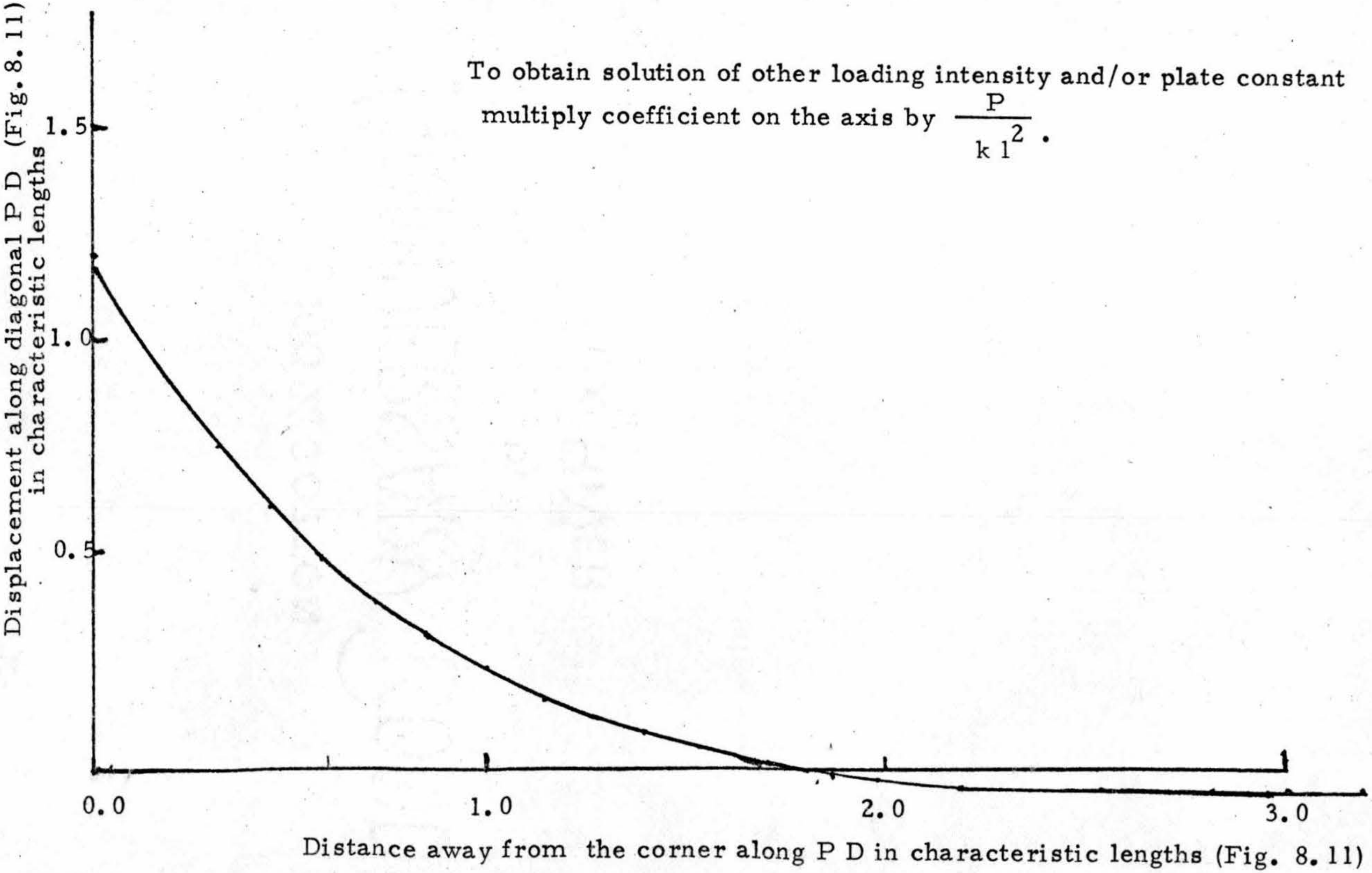


Fig. 8.12 Displacement along the diagonal PD for a point load on a rectangular corner far away from other boundaries.

For Poisson s ratio=.2

For concentrated load P units, multiply coefficient read of the axis by P.

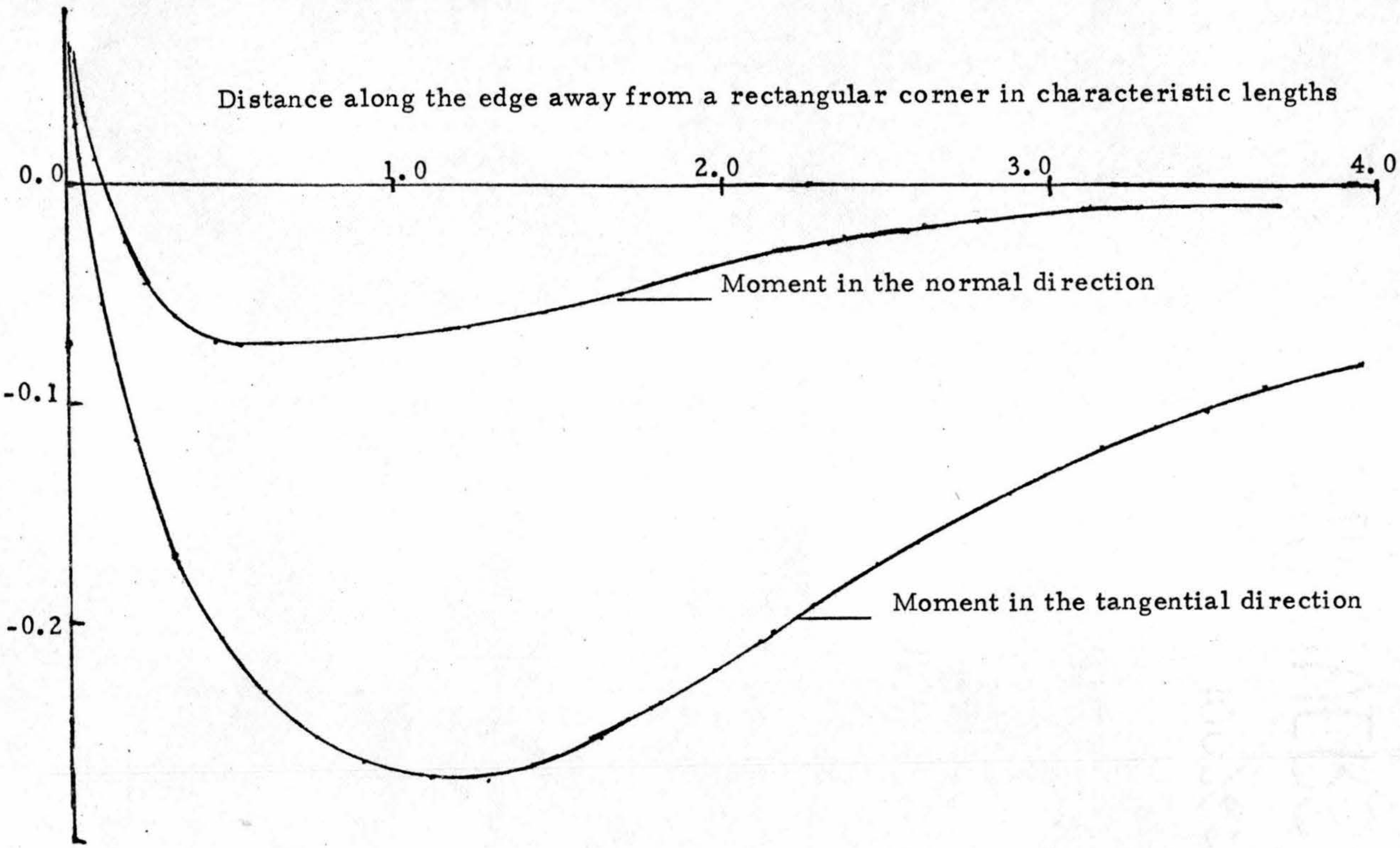


Fig. 8.13 Plotting of moment in both the normal and tangential directions on a line on the edge away from a corner where there is a point load.

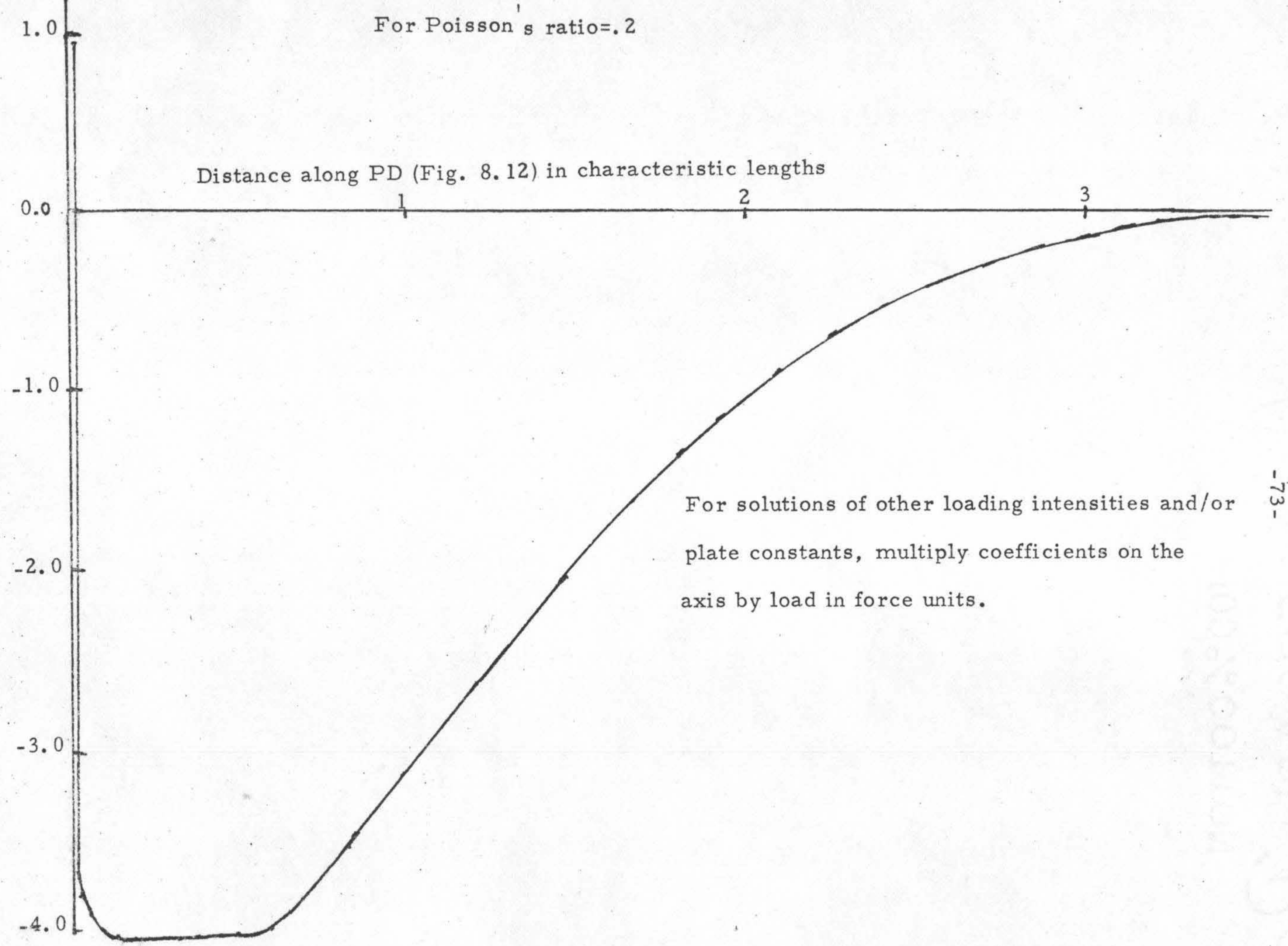
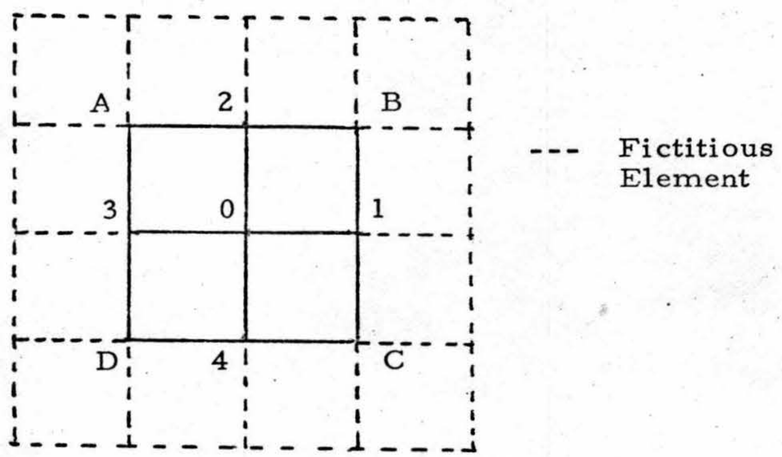


Fig. 8.14 Plotting of moment in direction PD along PD(Fig. 8.11) of a point load at a corner.

CHAPTER IX. COMPARISON WITH OTHER AVAILABLE METHODS

So far, there are only two methods which can be used to solve the plate problem with any generality as mentioned in Chapter 1: the finite difference method and finite element method. Before the finite element method was introduced, the finite difference method was commonly used to solve the plate problem. There are advantages and disadvantages to each of the above methods as well as to the methods described here. The remainder of this chapter will compare the two numerical methods with the Edge Function method.

The finite difference method[19, 20, 26] is one which uses the finite difference approximation to the plate equation. To solve this equation, the slab is imagined to be separated into a finite number of elements (in most cases, these elements are taken to be quadrilateral). These elements are connected only at the corners or 'nodal points,' and solutions are calculated at these nodal points (see Fig. 9.1



The plate ABCD is divided into 4 elements (usually many more). Fictitious elements are introduced to apply the boundary conditions.

Fig. 9.1

Applying the finite difference approximation to the governing differential equation, solutions at each point (e. g. 0) is given an expression of solutions at neighboring points (in this case 1, 2, 3, 4). The boundary condition consideration is brought into the system by introducing fictitious points outside the boundaries (shown in dotted line in Fig. 7. The same approximation relation holds for points at the boundaries (relating solutions at these points to neighboring connected points including those fictitious points outside the plate). Thus, a linear system of equations is set up involving the solutions at the nodal points as unknowns. Known solutions of boundary points are also involved in this system of linear equations. When boundary solutions are known they can be taken to the right-hand side of the system of linear equations and appear as the right-hand vector. Solutions of points other than those on the boundary are unknowns in the linear equations. Solving the systems of equations will give rise to the solutions on nodal points in the plate.

The finite element method is a very versatile method and has some similarity to the finite difference method. The slab is also partitioned into small elements.[21, 22] Instead of using the finite difference equation, it assumes a particular form for the solution of displacement in each element. In most cases, a polynomial form is assumed. Then equilibrium consideration for each element is established by calculating a stiffness matrix. Then the matrix for the whole system is assembled by combining the stiffness matrix according to the geometrical arrangement of the elements. Thus, equilibrium of the whole plate is established with the coefficients in the element polynomial as unknowns. There is the same number of unknown polynomials as the number of elements. Compatibility relation is also introduced to obtain continuity. To solve for the unknowns, the composite stiffness matrix has to be inverted, the loading being the right-hand vector in the system. The finite element method is versatile because each element stiffness matrix can be different from the others. So, it is easy to incorporate inhomogeneous and/or anisotropic problems. Also, any model for the foundation material can be used. Different foundation models give rise to different element stiffness matrices, which are also affected by different boundary conditions. The element stiffness matrix is usually obtained by a variational technique. The composite matrix can be arranged into a banded matrix and this property can be applied to reduce computing time. The matrix is also symmetric about the leading diagonal and is positive definite. Thus, further reduction on computing time can be applied.

The previous two paragraphs give brief introduction to the finite difference and finite element methods. The following paragraphs briefly compare their advantages and disadvantages.

This paragraph discusses the economics of the three methods. A major portion of computer time is used to solve the matrix in both the finite difference and the finite element method. For N nodal points, the finite difference will have 2N unknowns and thus it is required to solve a 2N by 2N matrix. For the finite element method, each finite element will introduce (for most forms of displacement found in most literature) 12 unknowns in the functional form. Thus, for even a moderate number of finite elements, the matrix is very large. However, a lot of manipulation can be applied to reduce computer time.

For the Edge Function method, the simultaneous approach on problems involving lower derivatives probably is not as economical as the finite difference method, but is comparable to the finite element method with all the "well conditioned" properties of the finite elements matrix taken into account. However, since higher derivatives reduce convergence considerably in the Edge Function method and a much larger number of terms is required for similar problems of prescribed shear and moment, the simultaneous approach probably is not even as economical as the finite element method. However, as indicated by the iterative approach at the beginning of this chapter, the finite element method is much more expensive to use compared with the Edge Function method with this approach.

The finite element method is, no doubt, the most versatile method. It can be used to solve anisotropic and non-homogeneous problems. It is also more readily applied to other foundation models. Besides, it handles re-entrant corners and openings without difficulty. The Edge Function method does have one advantage in that it handles convex plates of any polygonal shapes readily, while in both the finite difference and finite elements, difficulties might arise because elements of shapes (triangular) other than rectangular have to be used.

The accuracy in the finite element and finite difference method is not as good as that of the Edge Function method generally. Most investigations in the Finite Element method only have continuity established across the nodal points in the x and y directions. Along element boundaries between nodal points, continuity is usually not established. Also, equilibrium is maintained only in the global sense; it need not be observed everywhere in the Finite Element method. Accuracy in the Edge Function method is enhanced by taking further terms in the Edge Functions while in the two finite methods smaller elements are required. So, if enough terms are taken in the Edge Function method, its solution will be the limit of the Finite method if the finite elements are infinitely small.

After obtaining the coefficients in the Edge Functions, solutions at any point in the plate can be calculated easily. Also, directional properties like slope, moment and shear in any direction can be evaluated in the Edge Function method. The two finite methods, however, give solutions at nodal points in x and y directions only.

Another major advantage of the Edge Function method is that after a relatively simple program is written, further problems can be solved with a minimum of preliminary work. All that is required is to enter the computer with the loading and boundary conditions, the geometry of the plate and the points where the solution is to be calculated. In the two finite methods, the slab has to be partitioned into finite numbers of elements, the loading has to be evaluated and portions of it assigned at certain nodal points.

So, some of the work is involved in each application, and experience is needed before one can use the finite method to its full extent. By comparison in the Edge Function method, the user does not have to have much experience.

The major disadvantages of the Edge Function method is that its versatility is no match for the finite element method and possibly its slow convergence when higher derivatives have to be calculated.

APPENDIX

The appendix contains solutions to several selected problems. All problems deal with plates on a Winkler Foundation.

The following are data common to all the problems calculated and presented in the Appendix.

K spring constant used for Winkler Foundation is 1 lb/ft^3

E Young's modulus for the plate is 10^4 lb/ft^2

ν Poisson's ratio of the plate is 0.2

h thickness of the plate is 0.1 ft

D the plate constant given by $\dfrac{Eh^3}{12(1-\nu^2)} = 0.86805 \text{ ft}^4$

Therefore the characteristic length of the plate-spring system is $\sqrt[4]{\dfrac{D}{K}} = 0.965$ ft.

APPENDIX A

The problem is a square plate with a point load in the middle of the plate and the edges are all free. The plate is resting on a Winkler Foundation.

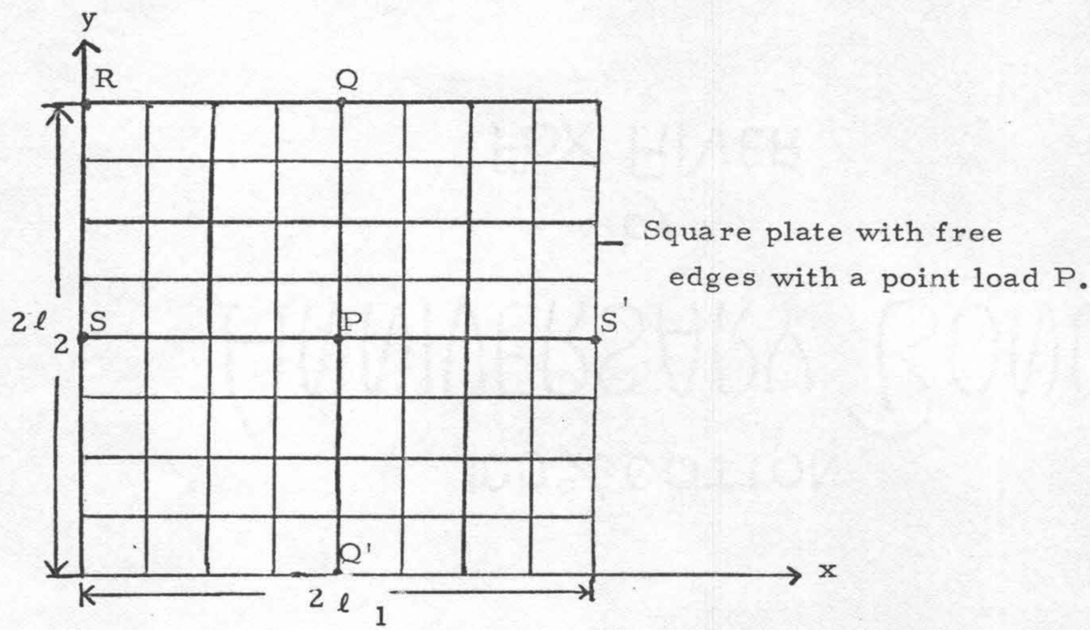


Fig. A-1

Solutions are calculated at the nodal points as shown. Table A-1 contains solutions of displacements, A-2 and A-3 are slopes in the 2 directions, A-4 and A-5, moments, and A-6 shear in the x-direction. For equilibrium the force calculated from the spring reaction is 72.6 lb, the shear force contribution being zero. The total load is 100 lb. If a finer network is calculated the upward force would approach 100. The displacement and slope calculated is probably quite adequate. The shear and moment calculated, however, might need further improvement at the point directly under the load. The solutions calculated at the middle portion of the plate are very close to those of the hinged plate shown in Chapter 7. This demonstrates that the edge function effect decreases rapidly away from the edges.

TABLE A-1

Displacement of a free plate on Winkler Foundation of a point load in the center

(in ft.)

-0.54948E-01	-0.16416E 00	-0.34620E 00	-0.51623E 00	-0.58450E 00	-0.51623E 00	-0.34619E 00	-0.16416E 00	-0.54948E-01
-0.16416E 00	-0.19665E 00	-0.21355E 00	-0.12916E 00	-0.54022E-01	-0.12915E 00	-0.21355E 00	-0.19665E 00	-0.16416E 00
-0.34620E 00	-0.21355E 00	0.17473E 00	0.10789E 01	0.17001E 01	0.10789E 01	0.17473E 00	-0.21355E 00	-0.34620E 00
-0.51623E 00	-0.12918E 00	0.10789E 01	0.41664E 01	0.67807E 01	0.41664E 01	0.10789E 01	-0.12918E 00	-0.51624E 00
-0.58450E 00	-0.54074E-01	0.17001E 01	0.67807E 01	0.13438E 02	0.67807E 01	0.17001E 01	-0.54059E-01	-0.58450E 00
-0.51624E 00	-0.12918E 00	0.10789E 01	0.41664E 01	0.67807E 01	0.41664E 01	0.10789E 01	-0.12918E 00	-0.51624E 00
-0.34620E 00	-0.21355E 00	0.17473E 00	0.10789E 01	0.17001E 01	0.10789E 01	0.17473E 00	-0.21355E 00	-0.34620E 00
-0.16417E 00	-0.19665E 00	-0.21355E 00	-0.12916E 00	-0.54038E-01	-0.12916E 00	-0.21355E 00	-0.19665E 00	-0.16417E 00
-0.54948E-01	-0.16416E 00	-0.34620E 00	-0.51623E 00	-0.58450E 00	-0.51623E 00	-0.34619E 00	-0.16416E 00	-0.54948E-01

TABLE A-2

Slope (tan θ) in the X direction

-0.60831E-01	-0.12165E 00	-0.15587E 00	-0.10336E 00	0.12873E-06	0.10336E 00	0.15586E 00	0.12164E 00	0.60831E-01
-0.17353E-01	-0.33895E-01	0.23836E-01	0.93874E-01	0.21356E-06	-0.93875E-01	-0.23836E-01	0.33894E-01	0.17353E-01
0.98129E-01	0.14932E 00	0.52646E 00	0.80855E 00	0.21570E-05	-0.80856E 00	-0.52645E 00	-0.14932E 00	-0.98130E-01
0.25391E 00	0.47238E 00	0.16476E 01	0.30686E 01	0.99909E-05	-0.30686E 01	-0.16476E 01	-0.47238E 00	-0.25391E 00
0.34141E 00	0.66887E 00	0.24607E 01	0.57128E 01	0.74966E-04	-0.57127E 01	-0.24607E 01	-0.66387E 00	-0.34143E 00
0.25391E 00	0.47238E 00	0.16476E 01	0.30686E 01	0.99939E-05	-0.30686E 01	-0.16476E 01	-0.47238E 00	-0.25391E 00
0.98129E-01	0.14933E 00	0.52646E 00	0.80857E 00	0.21634E-05	-0.80857E 00	-0.52646E 00	-0.14933E 00	-0.98130E-01
-0.17353E-01	-0.33895E-01	0.23833E-01	0.93875E-01	0.30835E-06	-0.93875E-01	-0.23833E-01	0.33895E-01	0.17353E-01
-0.60831E-01	-0.12165E 00	-0.15587E 00	-0.10336E 00	-0.14997E-06	0.10336E 00	0.15586E 00	0.12164E 00	0.60832E-01

TABLE A- 3

Slope in the Y direction (tan θ)

0.60831E-01	0.17353E-01	-0.98130E-01	-0.25391E 00	-0.34143E 00	-0.25391E 00	-0.98130E-01	0.17353E-01	0.60832E-01
0.12164E 00	0.33897E-01	-0.14933E 00	-0.47239E 00	-0.66885E 00	-0.47239E 00	-0.14932E 00	0.33896E-01	0.12164E 00
0.15586E 00	-0.23832E-01	-0.52646E 00	-0.16476E 01	-0.24607E 01	-0.16476E 01	-0.52645E 00	-0.23835E-01	0.15586E 00
0.10338E 00	-0.93858E-01	-0.80855E 00	-0.30686E 01	-0.57127E 01	-0.30686E 01	-0.80854E 00	-0.93858E-01	0.10338E 00
0.13249E-06	0.31522E-06	0.21583E-05	0.10007E-04	0.74987E-04	0.10010E-04	0.21646E-05	0.30976E-06	-0.14618E-06
-0.10338E 00	0.93858E-01	0.80855E 00	0.30686E 01	0.57128E 01	0.30686E 01	0.80854E 00	0.93858E-01	-0.10338E 00
-0.15587E 00	0.23831E-01	0.52646E 00	0.16476E 01	0.24608E 01	0.16476E 01	0.52646E 00	0.23834E-01	-0.15587E 00
-0.12165E 00	-0.33897E-01	0.14933E 00	0.47240E 00	0.66889E 00	0.47240E 00	0.14932E 00	-0.33896E-01	-0.12165E 00
-0.60831E-01	-0.17353E-01	0.98129E-01	0.25391E 00	0.34141E 00	0.25391E 00	0.98129E-01	-0.17353E-01	-0.60831E-01

TABLE A - 4

Moment in the X direction (ft. lb. / ft.)

0.14035E-01	0.47843E-01	-0.63720E-02	-0.57445E-01	-0.10797E 00	-0.57442E-01	-0.63723E-02	0.47842E-01	0.14036E-01
0.72353E-02	-0.31552E-02	-0.95988E-01	-0.84202E-01	-0.21205E-01	-0.84201E-01	-0.95988E-01	-0.31556E-02	0.72355E-02
0.49930E-02	-0.14785E 00	-0.41558E 00	-0.11321E 00	0.48414E 00	-0.11320E 00	-0.41558E 00	-0.14785E 00	0.49933E-02
0.19693E-02	-0.42498E 00	-0.11488E 01	-0.18249E 00	0.36376E 01	-0.18248E 00	-0.11487E 01	-0.42497E 00	0.19695E-02
-0.27012E-01	-0.62784E 00	-0.16849E 01	-0.10149E 01	0.33511E 02	-0.10150E 01	-0.16847E 01	-0.62790E 00	-0.27013E-01
0.19700E-02	-0.42497E 00	-0.11488E 01	-0.18248E 00	0.36376E 01	-0.18247E 00	-0.11487E 01	-0.42497E 00	0.19694E-02
0.49939E-02	-0.14785E 00	-0.41553E 00	-0.11321E 00	0.48414E 00	-0.11321E 00	-0.41558E 00	-0.14785E 00	0.49935E-02
0.72358E-02	-0.31544E-02	-0.95985E-01	-0.84202E-01	-0.21171E-01	-0.84201E-01	-0.95986E-01	-0.31549E-02	0.72356E-02
0.14035E-01	0.47843E-01	-0.63720E-02	-0.57445E-01	-0.10797E 00	-0.57443E-01	-0.63723E-02	0.47842E-01	0.14035E-01

TABLE A - 5

Moment in the Y direction (ft. lb. / ft.)

0.20449E 00	0.38533E-02	-0.15516E-01	-0.95072E-02	-0.34734E-01	-0.95082E-02	-0.15517E-01	0.38530E-02	0.20449E 00
0.17243E 00	0.11558E-01	-0.11052E 00	-0.40649E 00	-0.61814E 00	-0.40645E 00	-0.11052E 00	0.11558E-01	-0.17243E 00
0.30423E-01	-0.11515E 00	-0.43841E 00	-0.11564E 01	-0.16857E 01	-0.11564E 01	-0.43841E 00	-0.11515E 00	0.30425E-01
0.21973E 00	-0.12170E 00	-0.15966E 00	-0.20132E 00	-0.10232E 01	-0.20132E 00	-0.15966E 00	-0.12170E 00	0.21972E 00
0.60696E 00	-0.11611E-01	0.55314E 00	0.36704E 01	0.33526E 02	0.36704E 01	0.55314E 00	-0.11645E-01	-0.60695E 00
0.21972E 00	-0.12170E 00	-0.15966E 00	-0.20133E 00	-0.10231E 01	-0.20133E 00	-0.15966E 00	-0.12170E 00	0.21972E 00
0.30426E-01	-0.11515E 00	-0.43841E 00	-0.11565E 01	-0.16858E 01	-0.11565E 01	-0.43841E 00	-0.11515E 00	0.30423E-01
0.17243E 00	0.11553E-01	-0.11052E 00	-0.40650E 00	-0.61807E 00	-0.40650E 00	-0.11052E 00	0.11558E-01	-0.17243E 00
0.20449E 00	0.38530E-02	-0.15516E-01	-0.95079E-02	-0.34735E-01	-0.95081E-02	-0.15516E-01	0.38533E-02	0.20449E 00

TABLE A - 6

Shear force in the X direction (lb. / ft.)

0.18239E-05	-0.33339E-01	-0.75800E-01	0.41897E-01	-0.93522E-04	-0.41898E-01	0.75799E-01	0.33348E-01	0.81046E-05
-0.71516E-07	-0.16250E 00	-0.33655E 00	-0.33732E 00	-0.35551E-04	0.33732E 00	0.33655E 00	0.16256E 00	0.47046E-05
-0.56435E-07	-0.37072E 00	-0.70900E 00	-0.84450E 00	0.83830E-04	0.84450E 00	0.70900E 00	0.37072E 00	-0.47710E-05
0.21098E-06	-0.40606E 00	-0.15244E 00	0.99823E 00	0.20459E-03	-0.99823E 00	0.15244E 00	0.40605E 00	-0.16778E-04
-0.91895E-06	0.24818E 01	0.38064E 01	0.77243E 01	0.15717E-01	-0.77245E 01	-0.38066E 01	-0.24834E 01	-0.14443E-02
0.21114E-06	-0.40606E 00	-0.15245E 00	0.99821E 00	0.20459E-03	-0.99821E 00	0.15244E 00	0.40605E 00	-0.16779E-04
-0.10796E-06	-0.37073E 00	-0.70900E 00	-0.84449E 00	0.83824E-04	0.84449E 00	0.70900E 00	0.37073E 00	-0.46205E-05
0.31312E-06	-0.16255E 00	-0.33654E 00	-0.33734E 00	-0.35341E-04	0.33734E 00	0.33654E 00	0.16255E 00	0.44587E-05
0.89737E-07	-0.33343E-01	-0.75800E-01	0.41896E-01	-0.76583E-04	-0.41899E-01	0.75799E-01	0.33344E-01	0.63703E-05

APPENDIX B

This problem is that of a free elastic plate resting on a Winkler Foundation with four point loads at the mid-point of each edge. Four point loads are chosen instead of one point load because with the loading and boundary condition symmetry a lot of computing time can be saved. Besides, some manipulation is required to obtain a particular solution of a point load on only one edge (see Appendix D).

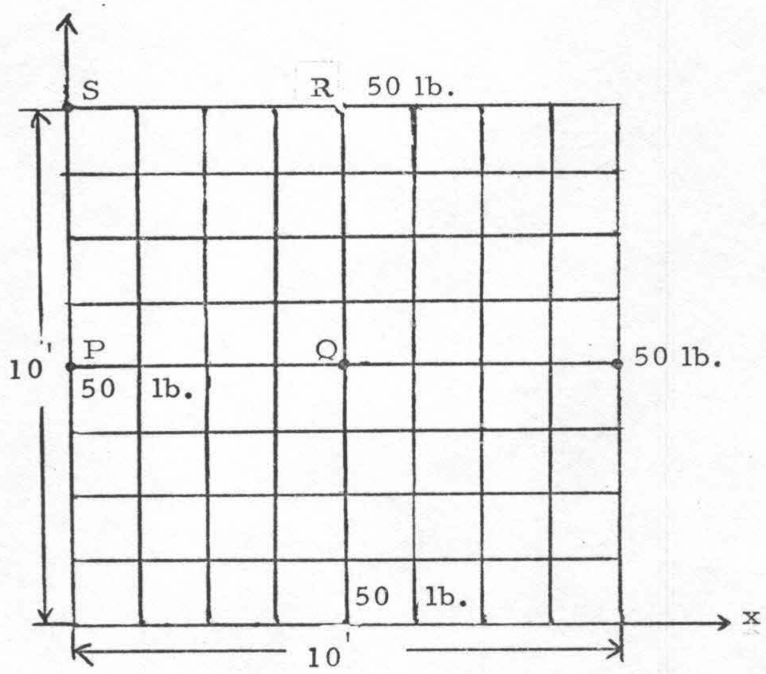


Fig. B-1

Solutions are calculated for the nodal points shown in Fig. B-1.

Only points in rectangle PQRS are calculated to save computing time. Other points where solutions are needed can be deduced from those calculated in PQRS. The total spring force calculated by direct integration of the Fourier coefficients of the particular solution and the coefficients of the Edge Functions is 200 lb as compared to 200 lb downward force. Plots of solutions on line PQ are given in the Figs. B-2 to B-5.

TABLE B - 1

Displacement of 4 point load on the edges of a square plate (ft.)

0.22957E 00	0.48577E 00	0.39592E 01	0.13047E 02	0.23813E 02
0.48579E 00	-0.53090E 00	0.29952E 00	0.35084E 01	0.61152E 01
0.39592E 01	0.29953E 00	-0.11066E 01	-0.67072E 00	-0.14818E 00
0.13047E 02	0.35084E 01	-0.67072E 00	-0.13729E 01	-0.12977E 01
0.23813E 02	0.61153E 01	-0.14817E 00	-0.12977E 01	-0.13043E 01

TABLE B- 2

Slope in **x** direction (tan θ)

-0.44582E 00	0.11686E 01	0.47522E 01	0.56967E 01	0.48391E-04
-0.10946E 01	-0.24840E 00	0.16682E 01	0.21346E 01	0.11072E-04
-0.32590E 01	-0.21476E 01	-0.19711E 00	0.64347E 00	0.36175E-05
-0.87377E 01	-0.55836E 01	-0.14996E 01	-0.16413E-02	0.35246E-05
-0.19551E 02	-0.87424E 01	-0.22140E 01	-0.15201E 00	0.40835E-05

TABLE B - 3

Moment in **x** direction (ft. lb. /ft.)

-0.63809E 00	-0.18107E 01	-0.30440E 01	-0.23928E 01	0.38200E 02
-0.16279E 00	-0.13051E 01	-0.17434E 01	-0.69276E 00	0.26218E 01
-0.23930E 00	-0.17309E 01	-0.12875E 01	-0.41602E 00	0.22114E 00
0.97456E-01	-0.34538E 01	-0.19146E 01	-0.41067E 00	0.55230E-01
0.30599E 01	-0.55879E 01	-0.24697E 01	-0.49778E 00	0.75906E-02

TABLE B - 4

Shear in x direction (lb. / ft.)

-0.20831E-03	0.12260E 01	0.30658E 01	-0.70488E 01	0.17865E-01
0.13152E-04	-0.15689E 00	-0.13540E 01	-0.18851E 01	0.21404E-03
-0.83725E-05	0.13934E 01	0.19311E 00	-0.78254E 00	0.81954E-04
-0.28899E-04	0.10094E 01	0.11138E 01	0.38179E 00	-0.38387E-04
-0.64432E-04	-0.33847E 01	-0.20356E 01	-0.21222E 01	-0.39021E-03

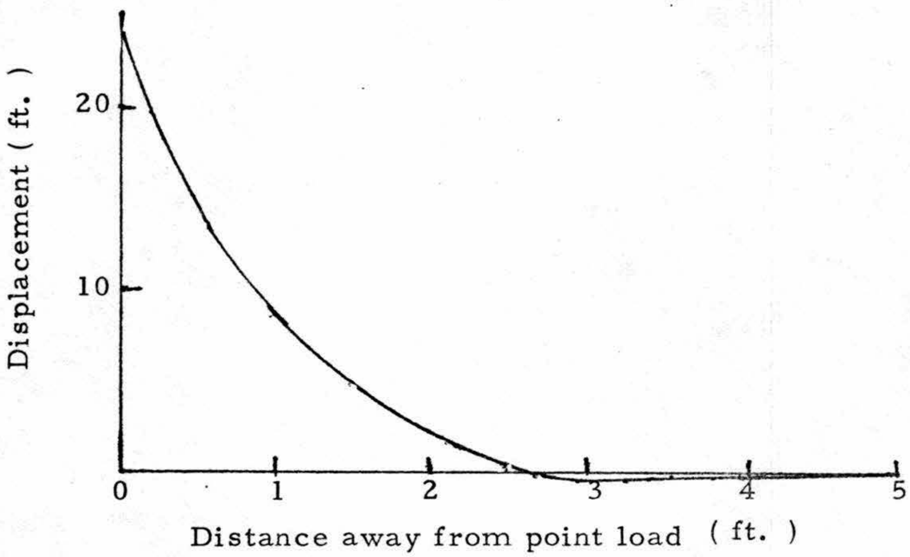


Fig. B - 2 Plotting of displacement along P Q (Fig. B-1)

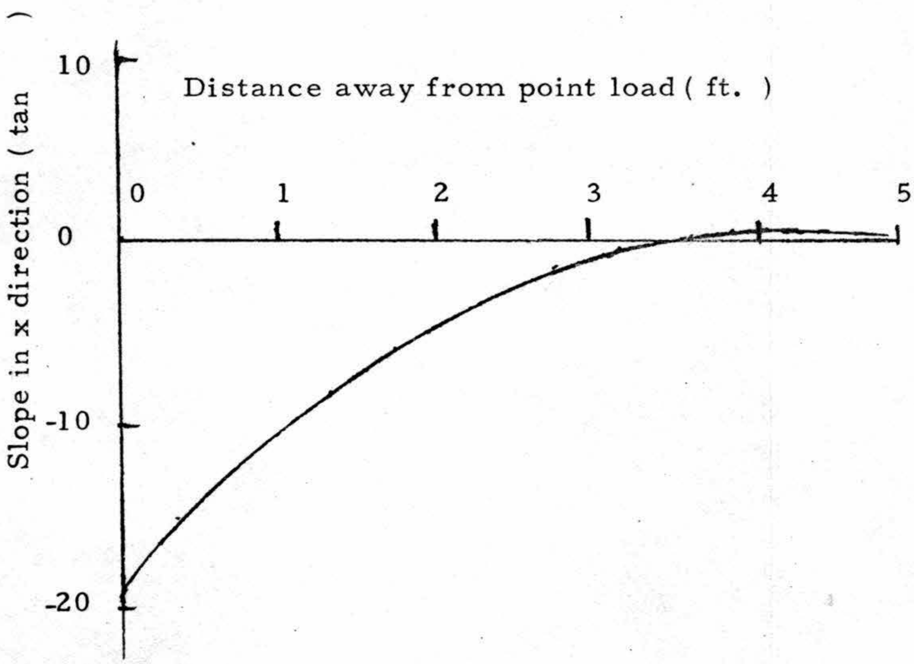


Fig. B-3 Plotting of slope in x direction along P Q

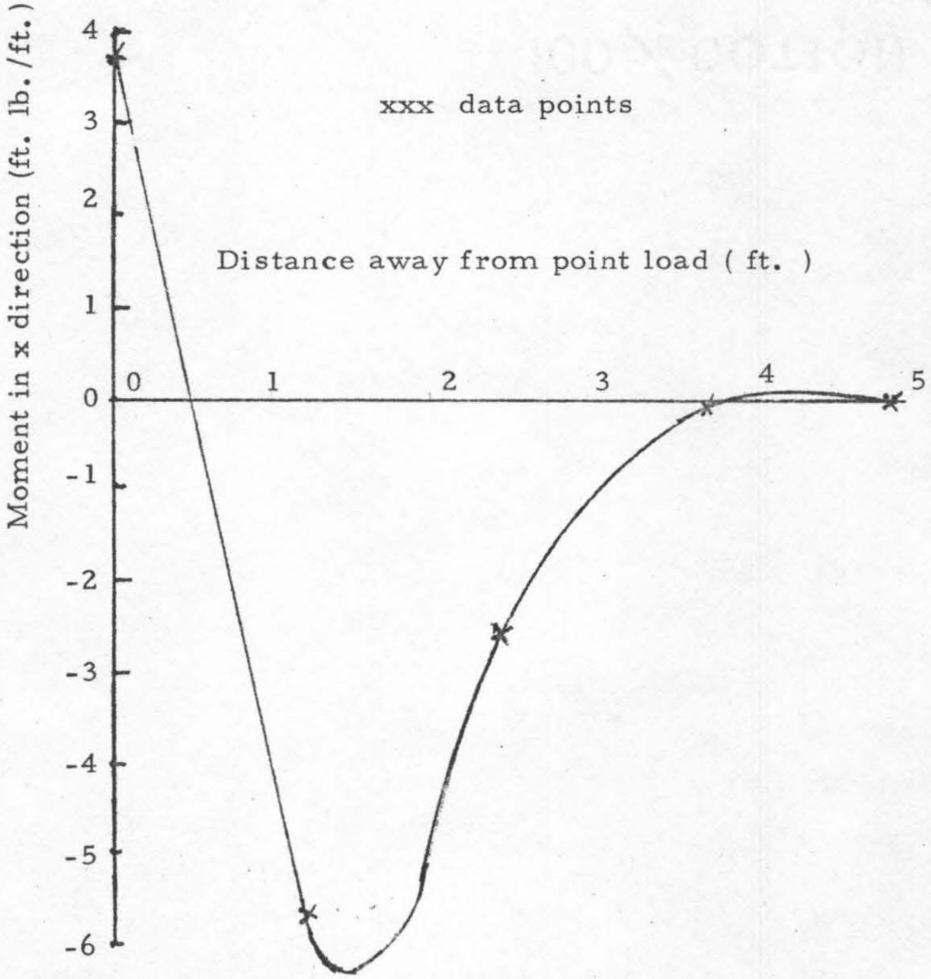


Fig. B-4 Plotting of Moment in x direction along PQ (Fig. B-1)

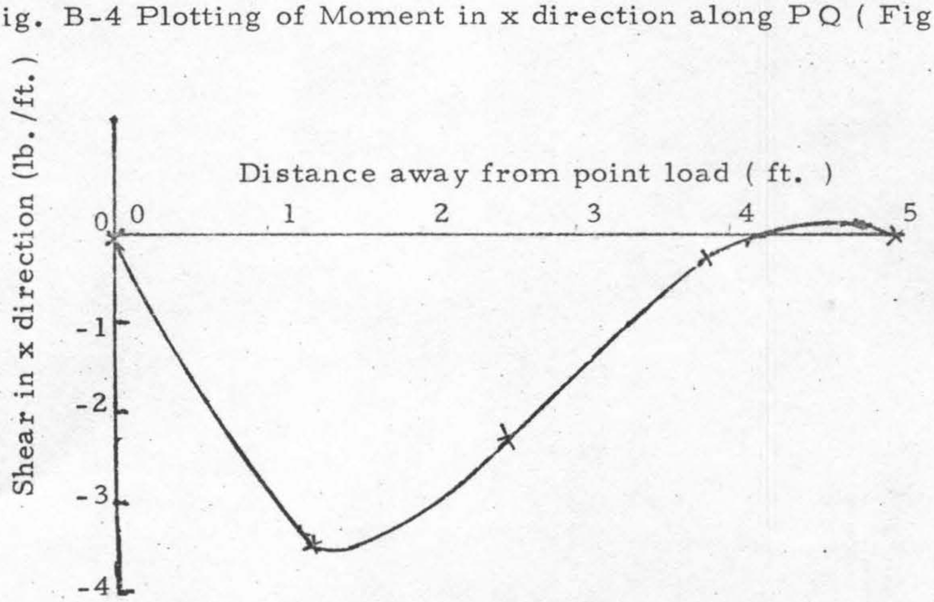


Fig. B-5 Plotting of shear in x direction along PQ.

APPENDIX C

Solutions are calculated to the problem of a point load on each corner of the plate on a Winkler Foundation (Fig. C-1). The edges are free on all edges.

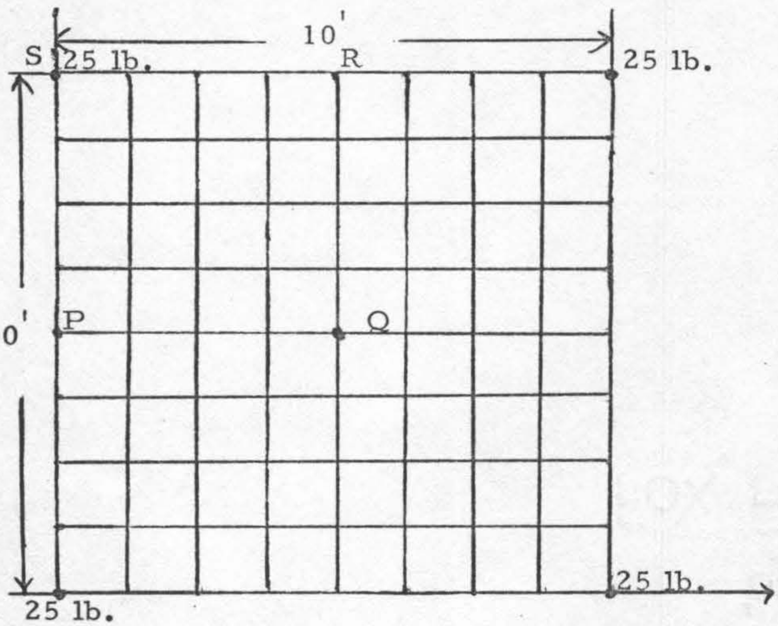


Fig. C-1

Solutions are calculated at the nodal point in the rectangle PQRS. Other point solutions can be deduced from those calculated.

The upward spring force calculated from integrating the total solutions is 100 lb. Comparing to the downward load of 100 lb Fig. C-2 to C-5 gives plottings of solutions calculated on line SR.

TABLE C- 1

(in ft.)

Displacement of a point load on each corner on a square plate

0.32763E 02	0.11770E 02	0.22796E 01	-0.27930E 00	-0.62531E 00
0.11770E 02	0.36690E 01	-0.55170E-01	-0.68770E 00	-0.62112E 00
0.22797E 01	-0.55177E-01	-0.90799E 00	-0.68202E 00	-0.47153E 00
-0.27976E 00	-0.68768E 00	-0.68202E 00	-0.35437E 00	-0.18168E 00
-0.62531E 00	-0.62112E 00	-0.47153E 00	-0.18168E 00	-0.43487E-01

TABLE C-2

(in tan θ)

Slope in x direction of pt. loads on corners of a sq. plate.

-0.19039E 02	-0.12058E 02	-0.39863E 01	-0.75473E 00	0.19735E-05
-0.69926E 01	-0.49119E 01	-0.13523E 01	-0.11046E-02	0.38217E-05
-0.16680E 01	-0.13931E 01	-0.82063E-01	0.28352E 00	0.14516E-05
-0.20835E 00	-0.22712E 00	0.20204E 00	0.25074E 00	0.39972E-06
0.10547E 00	0.11713E-01	0.21490E 00	0.20401E 00	0.19072E-06

TABLE C-3

(in ft, lb. /ft.)

Moment in x direction of pt. loads on corners of a sq. plate

0.61105E 01	-0.67156E 01	-0.34401E 01	-0.10354E 01	-0.19914E 00
0.36175E 00	-0.34748E 01	-0.19201E 01	-0.36853E 00	0.12078E 00
0.12815E 00	-0.13717E 01	-0.73323E 00	0.91822E-02	0.25251E 00
0.49352E 00	-0.37710E 00	-0.18039E 00	0.12352E 00	0.22850E 00
0.12114E 00	-0.10251E 00	-0.34200E-01	0.13652E 00	0.19865E 00

TABLE C-4

(in lb. / ft.)

Shear in x direction of pt. loads on corners of a sq. plate

0.66737E-02	0.12777E 02	-0.44624E 01	-0.48359E 00	-0.18730E-03
-0.20527E-04	0.39054E 01	0.29763E 01	0.12610E 01	-0.15464E-06
-0.13935E-04	0.20482E 01	0.15806E 01	0.66237E 00	-0.21278E-05
0.32969E-04	0.64682E 00	0.49064E 00	0.17551E 00	0.67065E-06
-0.98475E-04	0.23116E 00	0.15735E 00	0.24556E-01	0.19645E-05

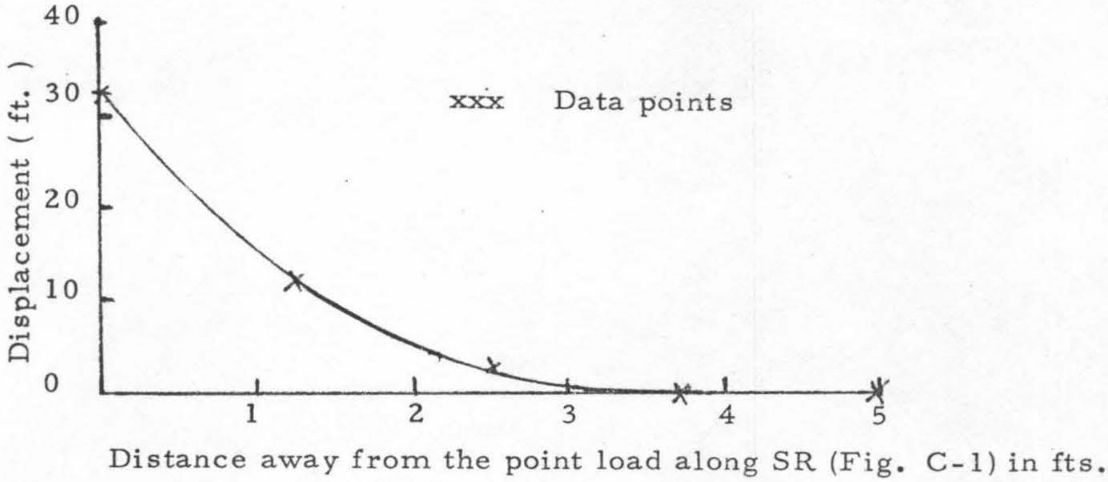


Fig. C-2 Plotting of displacement on line SR for a corner load on each corner of a rectangular plate.

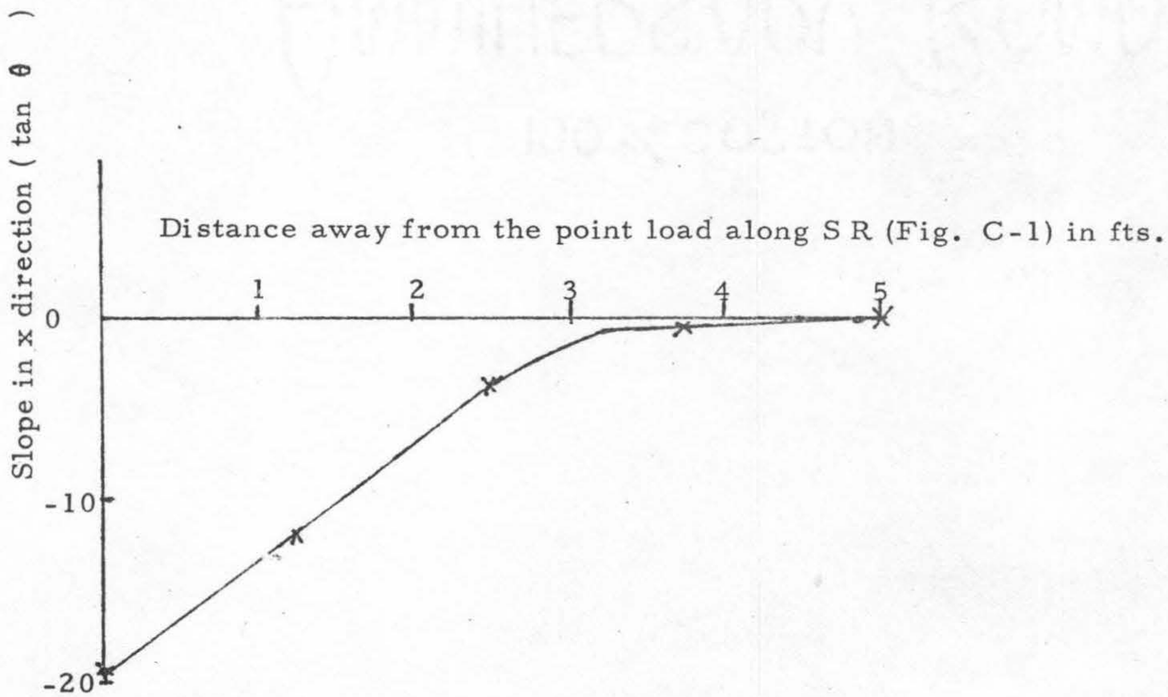


Fig. C-3 Plotting of slope in x direction along SR (in Fig. C-1) for a corner load on each corner of a rectangular plate.

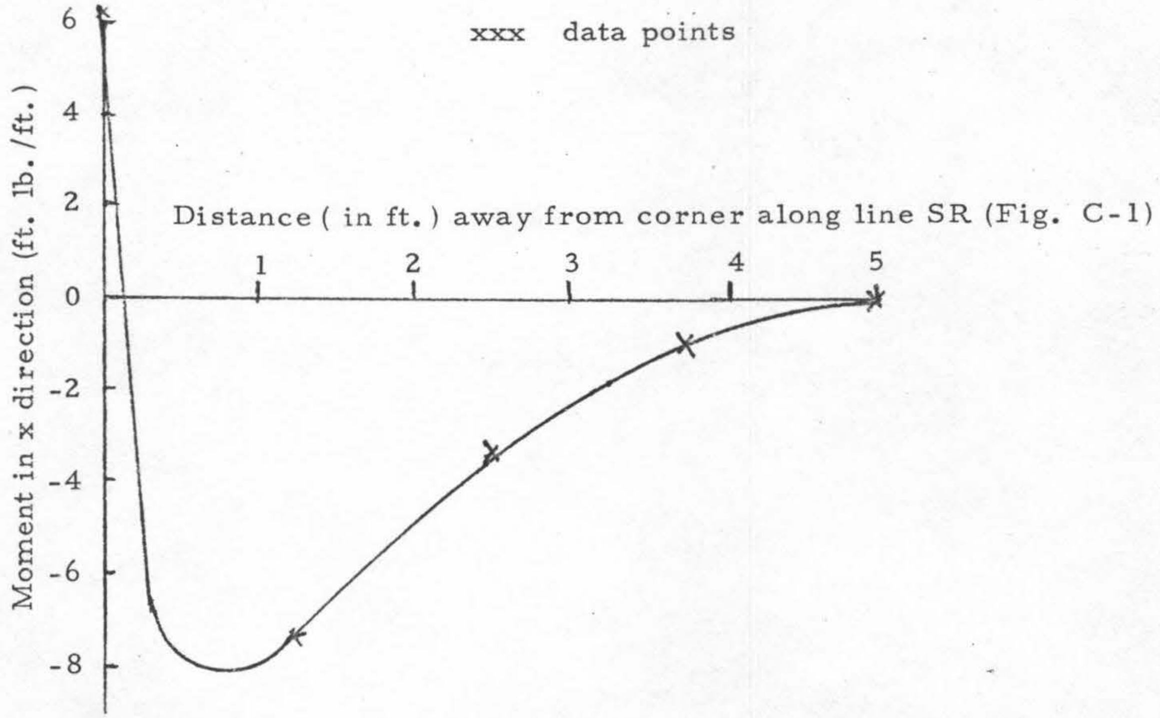


Fig. C-4 Plotting of moment in x direction along line SR (Fig. C-1) of problem of a point load on each corner of a sq. plate.

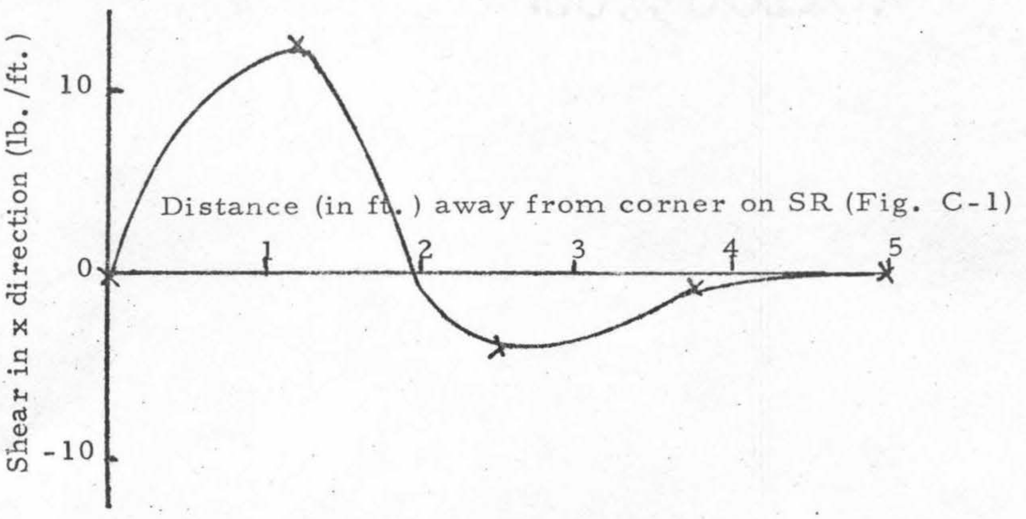


Fig. C-5 Plotting of shear in x direction along SR (Fig. C-1) of problem of a point load on each corner of a sq. plate.

APPENDIX D

A few examples illustrating the method used to express the loading conditions discussed in Chapter II are presented in Appendix D. We include

(i) A point load inside the plate.

(ii) A column load (a distributed load).

(iii) A point load on the edge of the plate.

(iv) A point load on the corner of the plate.

These are the solutions used in the calculations relating to the examples given throughout this thesis.

To describe a point load inside the plate, the properties of the point load have to be defined first. The following is the definition of a point load in this thesis:

(1) The pressure caused by a point load P is zero everywhere on the plate except at the point under the load.

(2) The pressure caused by the point load P on that particular point is infinite.

(3) The integral of the pressure times the infinitesimal area under it over the whole plate $\iint_{\text{whole plate}} q(x, y)dxdy$ is equal to P. This is required from equilibrium considerations.

All these properties are met by a delta function. For a point load P units in wt at x = a, y = b, the load can be represented by

$$q(x, y) = P\delta(x-a)(y-b) \qquad \text{(D-1)}$$

where P is the magnitude of the point load. This expression in q(x, y) satisfies all the above definitions of a point load.

(i) For a point load inside the plate:

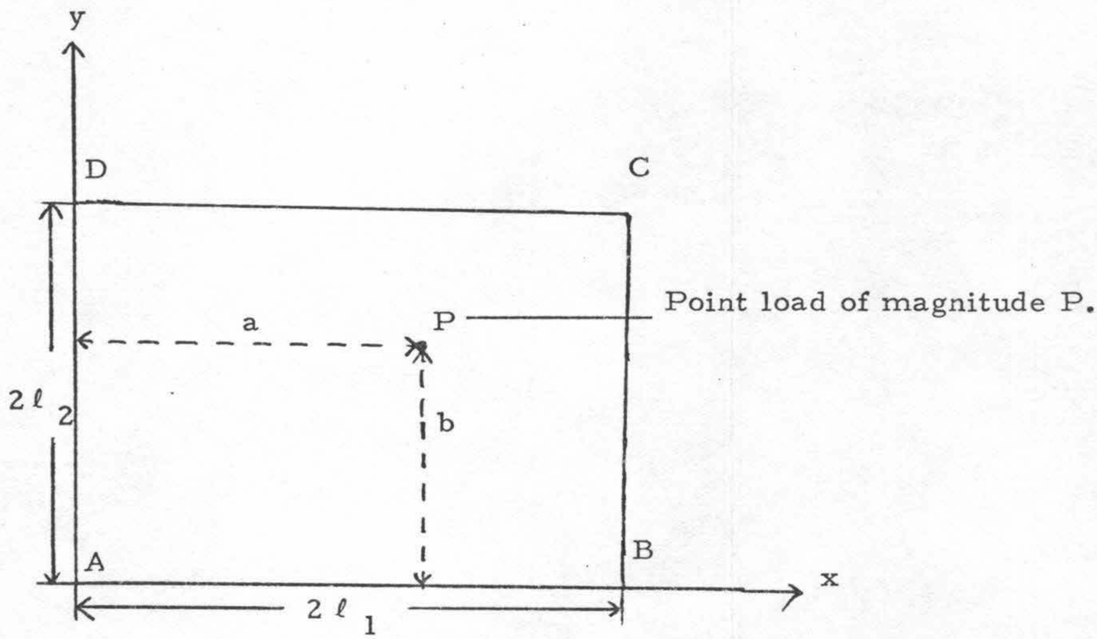


Fig. D-1

The following illustrates the way to express a point load as discussed in Chapter II, eq. (2.1). A similar method of expressing a point load has been used by Timoshenko and Woinowsky-Krieger[8] in article 34.

Using eq. (2.1), and evaluating the Fourier coefficients in it as in the standard Fourier series theory,

$$Q1_{mn} = \frac{1}{\ell_1 \ell_2} \int_0^{2\ell_2} \int_0^{2\ell_1} \frac{P}{D} \delta(x-a)\delta(y-b) \sin\frac{m\pi x}{\ell_1} \sin\frac{n\pi y}{\ell_2} \, dxdy$$

$$= \frac{P}{\ell_1 \ell_2 D} \sin\frac{m\pi a}{\ell_1} \sin\frac{n\pi b}{\ell_2} \qquad m \,\&\, n \text{ from 1 to } \infty$$

$$Q2_{mn} = \frac{P}{\ell_1 \ell_2 D} \cos\frac{m\pi a}{\ell_1} \cos\frac{n\pi b}{\ell_2} \quad \text{when m and n are from 1 to } \infty$$

$$Q2_{mn} = \frac{P}{2\ell_1 \ell_2 D} \cos\frac{m\pi a}{\ell_1} \cos\frac{n\pi b}{\ell_2} \quad \text{when one of m or n} = 0$$

$$Q2_{oo} = \frac{P}{4\ell_1 \ell_2 D} \cos\frac{m\pi a}{\ell_1} \cos\frac{n\pi b}{\ell_2}$$

$$Q3_{mn} = \frac{P}{D\ell_1 \ell_2} \sin\frac{m\pi a}{\ell_1} \cos\frac{n\pi b}{\ell_2} \quad \text{when n and m are from 1 to } \infty$$

$$Q3_{mo} = \frac{P}{2D\ell_1 \ell_2} \sin\frac{m\pi a}{\ell_1} \quad \text{when m is from 1 to } \infty$$

$$Q4_{mn} = \frac{P}{D\ell_1 \ell_2} \cos\frac{m\pi a}{\ell_1} \sin\frac{n\pi b}{\ell_2} \quad \text{when n and m are from 1 to } \infty$$

$$Q4_{on} = \frac{P}{2D\ell_1 \ell_2} \sin\frac{n\pi b}{\ell_2} \quad \text{when n is from 1 to } \infty \qquad \text{(D-1)}$$

(ii) For a column load (A uniformly distributed load on a rectangular area)

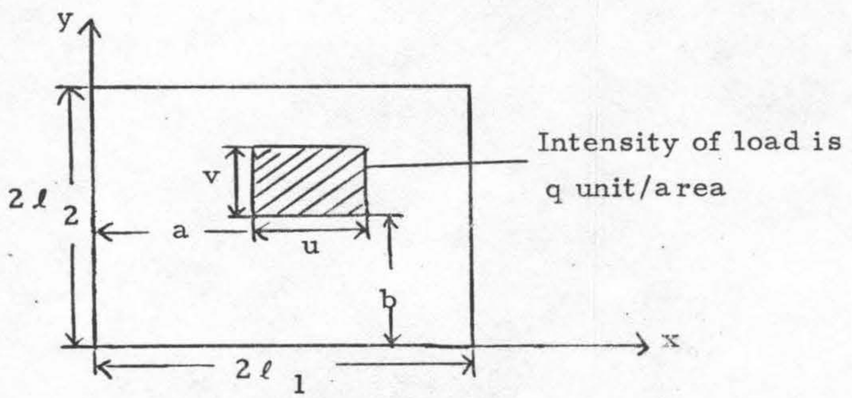


Fig. D-2

For a column load as shown in Fig. D-2, the Fourier coefficients in eq. (2.1) are as follows:

$$Q1_{mn} = \frac{q}{D(mn\pi^2)}\left[\cos\frac{m\pi a}{\ell_1} - \cos\frac{m\pi(a+u)}{\ell_1}\right]\left[\cos\frac{n\pi b}{\ell_2} - \cos\frac{n\pi(b+v)}{\ell_2}\right]$$

$$Q2_{mn} = \frac{q}{D(mn\pi^2)}\left[\sin\frac{m\pi(a+u)}{\ell_1} - \sin\frac{m\pi a}{\ell_1}\right]\Big[\sin\frac{n\pi(h+v)}{\ell_2} - \sin\frac{n\pi b}{\ell_2} \quad \text{for } m\&n \geq 1$$

$$Q2_{on} = \frac{qu}{D(2\ell_1)n\pi}\left[\sin\frac{n\pi(b+u)}{\ell_2} - \sin\frac{n\pi b}{\ell_2}\right] \quad \text{when } n \geq 1$$

$$Q2_{mo} = \frac{qv}{2D\ell_2 m\pi}\left[\sin\frac{m\pi(a+u)}{\ell_2} - \sin\frac{m\pi a}{\ell_2}\right] \quad \text{when } m \geq 1$$

$$Q2_{oo} = \frac{quv}{D(4\ell_1\ell_2)}$$

$$Q3_{mn} = \frac{q}{Dmn\pi^2}\left[\cos\frac{m\pi a}{\ell_1} - \cos\frac{m\pi(a+u)}{\ell_1}\right]\left[\sin\frac{n\pi(b+u)}{\ell_2} - \sin\frac{n\pi b}{\ell_2}\right]$$

$$m\&n \geq 1$$

$$Q3_{mo} = \frac{qv}{Dm\pi(2\ell_2)}\left[\cos\frac{m\pi a}{\ell_1} - \cos\frac{m\pi(a+u)}{\ell_1}\right] \quad m \geq 1$$

$$Q4_{mn} = \frac{q}{Dmn\pi^2}\left[\sin\frac{m\pi(a+u)}{1} - \sin\frac{(m\pi a)}{\ell_1}\right]\left[\cos\frac{n\pi b}{\ell_2} - \cos\frac{n\pi(b+v)}{\ell_2}\right]$$

$$m\&n \geq 1$$

$$Q4_{on} = \frac{qu}{2Dn\pi\ell_1}\left[\cos\frac{n\pi b}{\ell_2} - \cos\frac{n\pi(b+v)}{\ell_2}\right] \quad n \geq 1$$

(D-2)

(iii) For a point load on the edge.

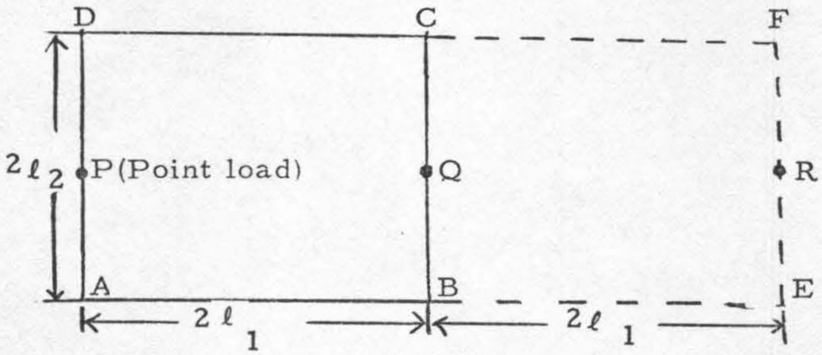


Fig. D-3

If eq. (D-1) is used to express a point load P on the edge, because of the periodicity of the Fourier series, the calculated loading function (a double sum Fourier series) will give rise to a point load at Q also. Other than this, because of the discontinuity of the delta function, the resultant Fourier series only has the effect of half of the load inside the plate around point P. The other half is on the other side of the plate. Thus, to obtain the solution of required intensity inside the plate, the Fourier coefficients calculated from eq. (D-1) have to be doubled. To obtain a true effect of only one point load inside the plate, the following has to be done. To obtain a solution of a single point load P in plate ABCD, calculation has to be carried out on an extended plate AEFD. Then, the resultant Fourier series will have the effect of a point load P and a point load R (see Fig. D-3). However, solutions of the particular solution in plate ABCD are used only.

(iv) For a point load on the corner.

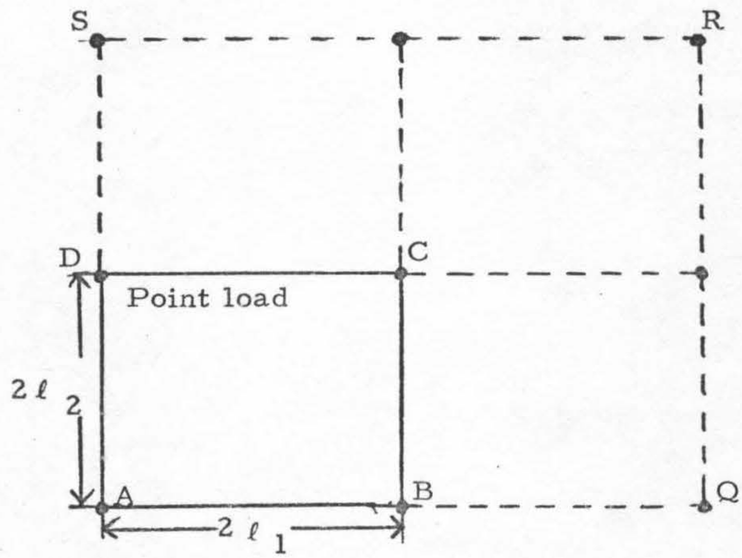


Fig. D-4

By the same reasoning as for a point on the edge, if the Fourier series is used to express a point load on the corner of the plate using eq. (D-1) the result is the effect of one-fourth of the load on all four corners. To get around the above, an extended plate AQRS is used. Fourier series approximated on a corner load is calculated on this extended plate; however, only the area of the plate inside ABCD is used in the particular calculation of the solution. Also, four times the intensity of the load on the corner is needed.

APPENDIX E

The displacement in the total solution $w_t(x, y)$ given by the Edge Function method is composed of components from a particular solution and a complementary solution as shown in equation (2.7). If the expressions for slope, moment, and shear in the solution are wanted, equation (2.4) has to be evaluated using $w_t(x, y)$. The following are equations obtained for these variables from the particular and complementary solutions. They are also needed in setting up the matrix to evaluate the coefficients in the edge functions.

<u>Particular solution for the rectangular plate given in Chapter II</u>

After operating equation (2.4) on equation (2.9), the following is obtained. The coefficients BI_{mn} in the following are given by equation (2.3) (referring back to Chapter II, Fig. 2-1).

(i) Displacement is given by (1.7).

(ii) Slope in the Y direction $\left(\frac{\partial w_p}{\partial y} \right)$

$$
\begin{aligned}
&= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B1_{mn} \left(\frac{n\pi}{\ell_2} \right) \sin \frac{m\pi x}{\ell_1} \cos \frac{n\pi y}{\ell_2} \\
&+ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} - B2_{mn} \left(\frac{n\pi}{\ell_2} \right) \cos \frac{m\pi x}{\ell_1} \sin \frac{n\pi y}{\ell_2} \\
&+ \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} - B3_{mn} \left(\frac{n\pi}{\ell_2} \right) \sin \frac{m\pi x}{\ell_1} \sin \left(\frac{n\pi y}{\ell_2} \right) \\
&+ \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} B4_{mn} \left(\frac{n\pi}{\ell_2} \right) \cos \frac{m\pi x}{\ell_1} \cos \frac{n\pi y}{\ell_2}
\end{aligned}
\qquad \text{(E-1)}
$$

(iii) Moment in the y direction $\left(-D \left(\frac{\partial^2 w_p}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \right)$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} -D\left[-\left(\frac{n\pi}{\ell_2}\right)^2 - \nu\left(\frac{m\pi}{\ell_1}\right)^2\right]\left[B1_{mn}\sin\frac{m\pi x}{\ell_1}\sin\frac{n\pi y}{\ell_2}\right]$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} -D\left[-\left(\frac{n\pi}{\ell_2}\right)^2 - \nu\left(\frac{m\pi}{\ell_1}\right)^2\right]\left[B2_{mn}\cos\frac{m\pi x}{\ell_1}\cos\frac{n\pi y}{\ell_2}\right]$$

$$+\sum_{m=1}^{\infty}\sum_{n=0}^{\infty} -D\left[-\left(\frac{n\pi}{\ell_2}\right)^2 - \nu\left(\frac{m\pi}{\ell_1}\right)^2\right]\left[B3_{mn}\sin\frac{m\pi x}{\ell_1}\cos\frac{n\pi y}{\ell_2}\right] \qquad \text{(E-2)}$$

$$+\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} -D\left[-\left(\frac{n\pi}{\ell_2}\right)^2 - \nu\left(\frac{m\pi}{\ell_1}\right)^2\right]\left[B4_{mn}\cos\frac{m\pi x}{\ell_1}\sin\frac{n\pi y}{\ell_2}\right]$$

(iv) Shearing force in the y direction $\left(-D\left(\frac{\partial^3 w_p}{\partial y^3} + (2-\nu)\frac{\partial^3 w_p}{\partial y \partial x^2}\right)\right)$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} -D\left[-\left(\frac{n\pi}{\ell_3}\right)^3 - (2\nu)\left(\frac{m\pi}{\ell_1}\right)^2\left(\frac{n\pi}{\ell_2}\right)\right] B1_{mn}\sin\frac{m\pi x}{\ell_1}\cos\frac{n\pi y}{\ell_2}$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} -D\left[\left(\frac{n\pi}{\ell_2}\right)^3 + (2-\nu)\left(\frac{m\pi}{\ell_1}\right)^2\left(\frac{n\pi}{\ell_2}\right)\right] B2_{mn}\cos\frac{m\pi x}{\ell_1}\sin\frac{n\pi y}{\ell_2}$$

$$+\sum_{m=1}^{\infty}\sum_{n=0}^{\infty} -D\left[\left(\frac{n\pi}{\ell_2}\right)^3 + (2-\nu)\left(\frac{n\pi}{\ell_1}\right)\left(\frac{m\pi}{\ell_2}\right)^2\right] B3_{mn}\sin\frac{m\pi x}{\ell_1}\sin\frac{n\pi y}{\ell_2} \qquad \text{(E-3)}$$

$$+\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} -D\left[-\left(\frac{n\pi}{\ell_2}\right)^3 - (2-\nu)\left(\frac{m\pi}{\ell_1}\right)^2\left(\frac{n\pi}{\ell_2}\right)\right] B4_{mn}\cos\frac{m\pi x}{\ell_1}\sin\frac{n\pi y}{\ell_2}$$

Analogous expressions for slope, moment, and shear in other directions can be evaluated.

Complementary solution for the rectangular plate given in Chapter IV

After operating eq. (2.4) on eq. (4.6), the following is obtained. θ_m^j and γ_m^j satisfy the last two expressions in eq. (4.6). The following is also based on reference to Fig. 4-3 and relations given by eq. (4.5).

(i) Displacement $w_c(x, y)$ is given by eq. (4.6). For the sake of compactness, the following notations are used:

$$G_1(m, y_j) = e^{-\gamma_m^j y_j} \sin(\theta_m^j y_j)$$

$$G_2(m, y_j) = e^{-\gamma_m^j y_j} \cos(\theta_m^j y_j)$$

$$G_3(m, y_j) = e^{-\gamma_m^j y_j} [-\gamma_m^j \sin(\theta_m^j y_j) - \theta_m^j \sin(\theta_m^j y_j)]$$

$$G_4(m, y_j) = e^{-\gamma_m^j y_j} [-\gamma_m^j \cos(\theta_m^j y_j) - \theta_m^j \sin(\theta_m^j y_j)]$$

$$G_5(m, y_j) = e^{-\gamma_m^j y_j} [(\gamma_m^j)^2 \sin(\theta_m^j y_j) - 2(\gamma_m^j)(\theta_m^j)\cos(\theta_m^j y_j) - (\theta_m^j)^2 \sin(\theta_m^j y_j)]$$

$$G_6(m, y_j) = e^{-\gamma_m^j y_j} [(\gamma_m^j)^2 \cos(\theta_m^j y_j) + 2(\gamma_m^j)(\theta_m^j)\sin(\theta_m^j y_j) - (\theta_m^j)^2 \cos(\theta_m^j y_j)]$$

$$G_7(m, y_j) = e^{-\gamma_m^j y_j} [-(\gamma_m^j)^3 \sin(\theta_m^j y_j) + 3(\gamma_m^j)^2(\theta_m^j)\cos(\theta_m^j y_j) + 3(\gamma_m^j)(\theta_m^j)^2 \sin(\theta_m^j y_j) - (\theta_m^j)^3 \cos(\theta_m^j y_j)]$$

$$G_8(m, y_j) = e^{-\gamma_m^j y_j} [-(\gamma_m^j)^3 \cos(\theta_m^j y_j) - 3(\gamma_m^j)^2(\theta_m^j)\sin(\theta_m^j y_j) + 3(\gamma_m^j)(\theta_m^j)^2 \cos(\theta_m^j y_j) + (\theta_m^j)^3 \sin(\theta_m^j y_j)]$$

(ii) Slope in the y_1 direction

$$\frac{\partial w_c}{\partial y_1} = \sum_m \left[A_m^1 \sin(\frac{m\pi x_1}{\ell_1}) + C_m^1 \cos(\frac{m\pi x_1}{\ell_1}) \right] G_3(m, y_1)$$

$$+ \sum_m \left[B_m^1 \sin(\frac{m\pi x_1}{\ell_1}) + D_m^1 \cos(\frac{m\pi x_1}{\ell_1}) \right] G_4(m, y_1) +$$

$$+\sum_m \left[A_m^2 \left(\frac{m\pi}{\ell_1}\right)\cos\left(\frac{m\pi x_2}{\ell_2}\right) - C_m^2\left(\frac{m\pi}{\ell_2}\right)\sin\left(\frac{m\pi x_2}{\ell_2}\right)\right] G_1(m, y_2)$$

$$+\sum_m \left[B_m^2 \left(\frac{m\pi}{\ell_2}\right)\cos\left(\frac{m\pi x_2}{\ell_2}\right) - D_m^2\left(\frac{m\pi}{\ell_2}\right)\sin\left(\frac{m\pi x_2}{\ell_2}\right)\right] G_2(m, y_2)$$

$$+\sum_m -\left[A_m^3 \sin\left(\frac{m\pi x_3}{\ell_3}\right) + C_m^3\cos\left(\frac{m\pi x_3}{\ell_3}\right)\right] G_3(m, y_3)$$

$$+\sum_m -\left[B_m^3 \sin\left(\frac{m\pi x_3}{\ell_3}\right) + D_m^3\cos\left(\frac{m\pi x_3}{\ell_3}\right)\right] G_4(m, y_3)$$

$$+\sum_m -\left[A_m^4 \left(\frac{m\pi}{\ell_4}\right)\cos\left(\frac{m\pi x_4}{\ell_4}\right) - C_m^4\left(\frac{m\pi}{\ell_4}\right)\sin\left(\frac{m\pi x_4}{\ell_4}\right)\right] G_1(m, y_4)$$

$$+\sum_m -\left[B_m^4 \left(\frac{m\pi}{\ell_4}\right)\cos\left(\frac{m\pi x_4}{\ell_4}\right) - D_m^4\left(\frac{m\pi}{\ell_4}\right)\sin\left(\frac{m\pi x_4}{\ell_4}\right)\right] G_2(m, y_4) \qquad \text{(E. 4)}$$

(iii) Moment in y_1 direction of w_c: $\left(-D\left(\frac{\partial^2 w_c}{\partial y_1^2} + \nu \frac{\partial^2 w_c}{\partial x_1}\right)\right)$

$$= -D\left\{\sum_m \left[A_m^1 \sin\left(\frac{m\pi x_1}{\ell_1}\right) + C_m^1\cos\left(\frac{m\pi x_1}{\ell_1}\right)\right] G_5(m, y_1)\right.$$

$$\left. + \sum_m \left[B_m^1 \sin\left(\frac{m\pi x_1}{\ell_1}\right) + D_m^1\cos\left(\frac{m\pi x_1}{\ell_1}\right)\right] G_6(m, y_1)\right\}$$

$$-D\nu\left\{\sum_m \left[A_m^1 \left(\frac{m\pi}{\ell_1}\right)^2 \sin\left(\frac{m\pi x_1}{\ell_1}\right) + C_m^1\left(\frac{m\pi}{\ell_1}\right)^2\cos\left(\frac{m\pi x_1}{\ell_1}\right)\right] G_1(m, y_1)\right\}$$

$$-D\nu\left\{\sum_m \left[B_m^1 \left(\frac{m\pi}{\ell_1}\right)^2 \sin\left(\frac{m\pi x_1}{\ell_1}\right) + D_m^1\left(\frac{m\pi}{\ell_1}\right)^2\cos\left(\frac{m\pi x_1}{\ell_1}\right)\right] G_2(m, y_1)\right\}$$

$$-D\left\{\sum_m -\left(\frac{m\pi}{\ell_2}\right)^2 \left[A_m^2 \sin\left(\frac{m\pi x_2}{\ell_2}\right) + C_m^2\cos\left(\frac{m\pi x_2}{\ell_2}\right)\right] G_1(m, y_2)\right.$$

$$\left. + \sum_m -\left(\frac{m\pi}{\ell_2}\right)^2 \left[B_n^2 \sin\left(\frac{m\pi x_2}{\ell_2}\right) + D_m^2\cos\left(\frac{m\pi x_2}{\ell_2}\right)\right] G_2(m, y_2)\right\}$$

$$-D\nu\left\{\sum_m \left[A_m^2 \sin\left(\frac{m\pi x_2}{\ell_2}\right) + C_m^2\cos\left(\frac{m\pi x_2}{\ell_2}\right)\right] G_5(m, y_2)\right.$$

$$+\sum_m \left[B_m^2 \sin(\frac{m\pi x_2}{\ell_2}) + D_m^2 \cos(\frac{m\pi x_2}{\ell_2})\right] G_6(m, y_2)\Big\}$$

$$-D\Big\{\sum_m \left[A_m^3 \sin(\frac{m\pi x_3}{\ell_3}) + C_m^3 \cos(\frac{m\pi x_3}{\ell_3})\right] G_5(m, y_3)$$

$$+\sum_m \left[B_m^3 \sin(\frac{m\pi x_3}{\ell_3}) + D_m^3 \cos(\frac{m\pi x_3}{\ell_3})\right] G_6(m, y_3)\Big\}$$

$$-\nu D\Big\{\sum_m (\frac{m\pi}{\ell_3})^2 \left[A_m^3 \sin(\frac{m\pi x_3}{\ell_3}) + C_m^3 \cos(\frac{m\pi x_3}{\ell_3})\right] G_1(m, y_3)$$

$$+\sum_m (\frac{m\pi}{\ell_3})^2 \left[B_m^3 \sin(\frac{m\pi x_3}{\ell_3}) + D_m^3 \cos(\frac{m\pi x_3}{\ell_3})\right] G_2(m, y_3)\Big\}$$

$$-D\Big\{\sum_m -(\frac{m\pi}{\ell_4})^2 \left[A_m^4 \sin(\frac{m\pi x_4}{\ell_4}) + C_m^4 \cos(\frac{m\pi x_4}{\ell_4})\right] G_1(m, y_3)$$

$$+\sum_m -(\frac{m\pi}{\ell_4})^2 \left[B_m^4 \sin(\frac{m\pi x_4}{\ell_4}) + D_m^4 \cos(\frac{m\pi x_4}{\ell_4})\right] G_2(m, y_3)\Big\}$$

$$-D\nu\Big\{\sum_m \left[A_m^4 \sin(\frac{m\pi x_4}{\ell_4}) + C_m^4 \cos(\frac{m\pi x_4}{\ell_4})\right] G_5(m, y_4)$$

$$+\sum_m \left[B_m^4 \sin(\frac{m\pi x_4}{\ell_4}) + D_m^4 \cos(\frac{m\pi x_4}{\ell_4})\right] G_6(m, y_4)\Big\} \quad . \qquad \text{(E. 5)}$$

(iv) Shear in y_1 direction of complementary solution:

$$\left(-D(\frac{\partial^3 w_c}{\partial y_1^3} + (2-\nu)\frac{\partial^3 w_c}{\partial y_1 \partial x_1^2}\right)$$

$$= -D\Big\{\sum_m \left[A_m^1 \sin(\frac{m\pi x_1}{\ell_1}) + C_m^1 \cos(\frac{m\pi x_1}{\ell_1})\right] G_7(m, y_1)$$

$$+\sum_m \left[B_m^1 \sin(\frac{m\pi x_1}{\ell_1}) + D_m^1 \cos(\frac{m\pi x_1}{\ell_1})\right] G_8(m, y_1)$$

$$-D(2-\nu)\Big\{\sum_m -(\frac{m\pi}{\ell_1})^2 \left[A_m^1 \sin(\frac{m\pi x_1}{\ell_1}) + C_m^1 \cos(\frac{m\pi x_1}{\ell_1})\right] G_3(m, y_1)$$

$$+\sum_m -\left(\frac{m\pi}{\ell_1}\right)^2 \left[B_m^1 \sin\left(\frac{m\pi x_1}{\ell_1}\right) + D_m^1 \cos\left(\frac{m\pi x_1}{\ell_1}\right)\right] G_4(m, y_1)$$

$$-D\left\{\sum_m \left(\frac{m\pi}{\ell_2}\right)^3 \left[-A_m^2 \cos\left(\frac{m\pi x_2}{\ell_2}\right) + C_m^2 \sin\left(\frac{m\pi x_2}{\ell_2}\right)\right] G_1(m, y_2)\right.$$

$$\left.+\sum_m \left(\frac{m\pi}{\ell_2}\right)^3 \left[-B_m^2 \cos\left(\frac{m\pi x_2}{\ell_2}\right) + D_m^2 \sin\left(\frac{m\pi x_2}{\ell_2}\right)\right] G_2(m, y_2)\right.$$

$$-D(2-\nu)\left\{\sum_m \left(\frac{m\pi}{\ell_2}\right) \left[A_m^2 \cos\left(\frac{m\pi x_2}{\ell_2}\right) - C_m^2 \sin\left(\frac{m\pi x_2}{\ell_2}\right)\right] G_5(m, y_2)\right.$$

$$\left.+\sum_m \left(\frac{m\pi}{\ell_2}\right) \left[B_m^2 \cos\left(\frac{m\pi x_2}{\ell_2}\right) - D_m^2 \sin\left(\frac{m\pi x_2}{\ell_2}\right)\right] G_6(m, y_2)\right.$$

$$+D\left\{\sum_m \left[A_m^3 \sin\left(\frac{m\pi x_3}{\ell_3}\right) + C_m^3 \cos\left(\frac{m\pi x_3}{\ell_3}\right)\right] G_7(m, y_3)\right.$$

$$\left.+\sum_m \left[B_m^3 \sin\left(\frac{m\pi x_3}{\ell_3}\right) + D_m^3 \cos\left(\frac{m\pi x_3}{\ell_3}\right)\right] G_8(m, y_3)\right\}$$

$$+D(2-\nu)\left\{\sum_m -\left(\frac{m\pi}{\ell_3}\right)^2 \left[A_m^3 \sin\left(\frac{m\pi x_3}{\ell_3}\right) + C_m^3 \cos\left(\frac{m\pi x_3}{\ell_3}\right)\right] G_3(m, y_3)\right.$$

$$\left.+\sum_m -\left(\frac{m\pi}{\ell_3}\right)^2 \left[B_m^3 \sin\left(\frac{m\pi x_3}{\ell_3}\right) + D_m^3 \cos\left(\frac{m\pi x_3}{\ell_3}\right)\right] G_4(m, y_3)\right\}$$

$$+D\left\{\sum_m \left(\frac{m\pi}{\ell_4}\right)^3 \left[-A_m^4 \cos\left(\frac{m\pi x_4}{\ell_4}\right) + C_m^4 \sin\left(\frac{m\pi x_4}{\ell_4}\right)\right] G_1(m, y_4)\right.$$

$$\left.+\sum_m \left(\frac{m\pi}{\ell_4}\right)^3 \left[-B_m^4 \cos\left(\frac{m\pi x_4}{\ell_4}\right) + D_m^4 \sin\left(\frac{m\pi x_4}{\ell_4}\right)\right] G_2(m, y_4)\right\}$$

$$+D(2-\nu)\left\{\sum_m \left(\frac{m\pi}{\ell_4}\right) \left[A_m^4 \cos\left(\frac{m\pi x_4}{\ell_4}\right) - C_m^4 \sin\left(\frac{m\pi x_4}{\ell_4}\right)\right] G_5(m, y_4)\right.$$

$$\left.+\sum_m \left(\frac{m\pi}{\ell_4}\right) \left[B_m^4 \cos\left(\frac{m\pi x_4}{\ell_4}\right) - D_m^4 \sin\left(\frac{m\pi x_4}{\ell_4}\right)\right] G_6(m, y_4)\right\} \quad . \qquad (E.6)$$

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